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A coefficient-six bound for the Erdős–Surányi interval product problem

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Abstract

We prove that the Erdős–Surányi interval product-cover function satisfies $g(n) \le 6n + \pi(12n)$. The main step is a weighted hinge inequality on every shifted interval. We assign source excess to sets of exact prime-power valuations, then use a shared random choice at each weight scale to bound the sum of the signed payments. Five initial bands are certified by finite rational inequalities, and a geometric majorant handles all finer scales. Both the universal hinge and the original product-cover theorem have Lean proofs. The conjectural coefficient two is not established.

Problem and status. Erdős and Surányi asked how many sites in a translated interval are needed to supply a given product [1, 2]. We use the at-most convention: $g(n)$ is the worst-case minimum for n distinct originals and an interval as long as their maximum. Chen's weighted hinge and LP method, and Bergström's shifted-hinge and rounding work, are antecedents [3, 4, 5]. The unrestricted leading coefficient reported in Wang's public artifact is $56501/6400 = 8.82828125$ [6]; recent near-two bounds impose restrictions on the interval length and do not give an unrestricted bound [8]. Our earlier finite-case paper has a different quantifier on n [7]. The new step here is the weighted threshold-six inequality for all admissible real prime-power profiles and every phase of the interval.

Machine verification. The Lean companion proves the theorem below and the original interval cover for all positive original labels at most M. It uses exact finite checks, an all-scale rational estimate, and the full integral transfer. The supplement distinguishes the written estimates from the rational variants used in Lean. The proof declarations use only `propext`, `Classical.choice`, and `Quot.sound`. The public Lean companion is pinned to commit `604aed341e52`. The interval-cover theorem `Erdos708.cover_six` is registered as Palomar record `PALOMAR-2026-11-28-000080`.

Theorem 1 (Coefficient-six hinge)**.** *Let $M \ge 1$ be an integer and let I be any M consecutive signed integers. For finitely many primes and positive exponents take $a_{p,e} \ge 0$ with $\sum_e a_{p,e} \le 1$ for each p, and put*

$$f_p(x) = \sum_e a_{p,e} \mathbf{1}_{p^e \mid x}, \qquad S(x) = \sum_p f_p(x).$$

Then

$$\sum_{k=1}^{M} \big(S(k) - 6\big)_+ \;\le\; \sum_{x \in I} \big(S(x) - 1\big)_+.$$

Divisibility at zero has its ordinary meaning; $S(0)$ is a finite sum. There are no other restrictions on the support, exponents, coefficients, or phase.

Covering theorem. *For every set A of n distinct positive integers at most M, every signed M-term interval contains a* set *of at most $6n + \pi(12n)$ sites whose product is divisible by $\prod A$. Thus $g(n) \le 6n + O(n/\log n)$, in the at-most convention.*

Proof idea. In the low-mass case every hot source chooses one small set of prime powers, its carrier. The exact valuation of that carrier identifies a set of possible payments in the shifted interval. Charges are assigned only once at the source; at the target, a single random subset and a monotone eligibility cutoff prevent two carriers from claiming the same event. The finite arithmetic verifies that the combined payment fits inside the available target hinge.

1 Reduction and assignment of source charges

Call $\sum_{p,e} a_{p,e}/p^e$ the reciprocal mass. Put $Q = 30030$, the product of the first six primes, and discard all increments with activation $p^e > M/Q$. This does not change any source excess: if a discarded increment is active at $k \le M$, then $k/p^e < Q$ has at most five prime bases and $S(k) \le 6$. On the target, deletion can only decrease the hinge. Relabel the remaining system S and put $H_* = 1025/1024$.

For reciprocal mass at least H_*, scale the remaining system down to exactly H_*, obtaining V. Then $\sum_{k\le M} V(k) \ge M(1-1/Q)H_*$. For coordinates $u_p \in [0,1]$, $\big(\sum u_p - 6\big)_+ \le e_7(u)$, where e_j is the elementary symmetric polynomial. This follows on the Boolean cube and then by separate concavity of the difference. Positive expansion over distinct primes gives $\sum_{k\le M} e_7(V_p(k)) \le MH_*^7/7!$. Hence

$$\sum_{k\le M} \min(S(k),6) \;\ge\; M\big((1-1/Q)H_* - H_*^7/7!\big) > M.$$

The target sum of the weights is at least the prefix sum, since an M-term interval contains at least $\lfloor M/q \rfloor$ multiples of every q. Now use $\big(S(x)-1\big)_+ \ge S(x) - 1$.

Henceforth the reciprocal mass is less than H_*. Round each positive component value down to

$$\mathcal{D} = \{j2^{-d} : j, d \in \mathbb{Z},\ 16 \le j \le 31,\ j \le 2^d\}, \qquad \kappa = 16/17, qquadT = 6\kappa.$$

Write the rounded components b_p and $B = \sum b_p$. Then $\kappa f_p \le b_p \le f_p$, and $\big(S-6\big)_+ \le \big(B-T\big)_+/\kappa$. Rounding preserves nondecreasing valuation profiles. The reciprocal mass does not increase: for any such component b, the sum of reciprocal-weighted increments equals $\sum_{v\ge 0} b(v)(p^{-v} - p^{-v-1})$.

At $k \le M$ with $B(k) > T$, put aside the numerically largest full prime power of k, regardless of whether its base is weighted. From the remaining positive components take the shortest decreasing prefix whose total exceeds one, breaking ties by a fixed order. This is possible since removing one component loses at most one. Denote its bases by C, and label each with its *full* factor $q_p = p^{v_p(k)}$. Assign the entire charge $B(k) - T$ to this labeled carrier. Let M_C be the resulting total, and set

$$\theta = \min_{p\in C} b_p(q_p), \quad s = \sum_{p\in C} b_p(q_p), \quad h = |C|, \qquad 1 < s \le 1+\theta, \quad 2 \le h \le 1/\theta + 1,$$

$$P = \prod_{p\in C} q_p, \qquad \nu = \lfloor M/P \rfloor, \qquad \rho = \prod_{p\in C}(1-1/p), \qquad c_C = M_C/(\nu\rho).$$

2 Localizing the charges and paying an annulus

Let $J_j(C)$ be the product of the j smallest primes outside C ($J_0 = 1$). Put

$$a = \frac{T - 1 + \theta - s}{\theta}, \quad m = \lfloor a \rfloor + 1, \quad m_0 = \lfloor (T-2)/\theta \rfloor + 1, \quad R = J_{m_0-1}(C), \quad L = \lfloor M/(PR) \rfloor.$$

Cap each outside component at θ; let U_C be their sum after also discarding increments with activation power above L. At an assigned source, before truncation the capped outside sum U obeys $B(k) \le s + U(k) + 1 - \theta$. Thus $U(k)/\theta > a \ge (T-2)/\theta$, and at least m_0 outside bases are positive. Any active outside increment leaves at least $m_0 - 1$ other outside factors in k/P; their product is at least R. No such increment is discarded. Also $q_p \le L$ for every $p \in C$: the set-aside full factor is at least q_p and leaves at least $m_0 - 1$ other positive outside factors. On the assigned sources we therefore have

$$B(k) - T \le \theta\,(U_C(k/P)/\theta - a)\,, \qquad \nu \ge J_{m_0}(C).$$

For primes $r_1 < r_2 < \cdots$ set

$$D_{h,j} = \left(\prod_{i \le j} r_i\right) \prod_{j < i \le j+h} (1 - 1/r_i).$$

Then $\rho J_j(C) \ge D_{h,j}$: exchange a smaller member p of C with a larger member q of the first j outside primes. The product changes by the factor $p^2(q-1)/(q^2(p-1)) \le 1$. Once the first j primes are outside, the next h minimize the remaining density product.

Lemma 2 (Source coefficient)**.** *For $2 \le r \le m$ and $v = \lfloor ra/(r-1) \rfloor$, the entire normalized source charge satisfies*

$$c_C \le \epsilon_r := \theta(v-a)(H_*/\theta)^r \frac{(v-r)!}{v!} \left(1 + \frac{2^{h-1}}{D_{h,m-r}}\right).$$

Proof. For $u_i \in [0,1]$, the bound $\left(\sum_i u_i - a\right)_+ \le (v-a)e_r(u)/\binom{v}{r}$ follows by separate concavity of the difference and the maximum of $(w-a)/\binom{w}{r}$ at the Boolean vertices with $w > a$. The consecutive ratio is at least one exactly when $w+1 \le ra/(r-1)$. Use the outside components of U_C/θ for the u_i. On the source cofactor k/P these meet the cutoff $\sum u_i > a$ and there are at least m positive bases. In a term of $e_r(u)$ with activation product d, the other $m-r$ bases give $d \le \nu/J_{m-r}(C)$. Among cofactors at most ν, at most $(\nu/d)\rho + 2^{h-1}$ are divisible by d and coprime to P; only negative terms of inclusion–exclusion contribute positive floor errors. Using the bound on d before summing, the coprime count is at most $(\nu/d)\rho(1 + 2^{h-1}/D_{h,m-r})$. The distinct-base reciprocal expansion costs at most $(H_*/\theta)^r/r!$, establishing the formula after division by $\nu\rho$. □

On the target define the indicator

$$A_C(x) = \mathbf{1}_{P|x} \prod_{p \in C} (1 - \mathbf{1}_{pq_p|x}),$$

which requires all selected full valuations to match. Put $t = M/P$ and let $H_C \le H_*$ be the reciprocal mass of U_C. Inclusion–exclusion with one unit of discrepancy per modulus yields

$$\sum_I A_C \ge t\rho - 2^h, \qquad \sum_I A_C U_C \le t H_C(\rho + 2^h/R).$$

For the second bound use the upper count $t\rho/q + 2^h$ for an outside increment at q, then $q \le L \le t/R$. With $K > 0$ define

$$d_C = 1 - \frac{2^h}{D_{h,m_0}} - \left(1 + \frac{2^h}{D_{h,m_0-1}}\right)\frac{H_*}{K}.$$

If $d_C > 0$ and $\lambda_C \ge 1/(\kappa d_C)$, the signed payment

$$F_C(x) = \lambda_C c_C A_C(x)(1 - U_C(x)/K) \quad \text{satisfies} \quad \sum_I F_C \ge M_C/\kappa. \tag{P}$$

This uses $t \ge \nu \ge J_{m_0}(C)$. Negative values of F_C are retained in (P).

3 One probability space per band

For one band of terminal values $\theta \geq b > 0$, fix $0 < \alpha < 1$. At $x \neq 0$, independently retain each prime with probability $\alpha \min(b_p(x)/b, 1)$, using the *same* subset for every carrier in the band. For a carrier with $A_C(x) = 1$ let Q_C be the positive supported bases whose full factor at x is at most L, and E_C the event that the retained subset meets Q_C in exactly C. The preceding source localization ensures $C \subseteq Q_C$.

Lemma 3 (Disjoint carrier events)**.** *The events E_C are disjoint in one band, even if their terminal values differ.*

Proof. If two events occur and their cutoffs satisfy $L_C \leq L_D$, then $Q_C \subseteq Q_D$, so $C \subseteq D$. Along this inclusion, P and m_0 cannot decrease, and the complementary primorial J_{m_0-1} cannot decrease either. Thus $L_D \leq L_C$. Equal cutoffs give equal eligible sets and hence equal carriers. The exact valuation requirement rules out two different labelings of the same bases at x. □

Carrier bases have selection probability α. For an eligible outside base, all its capped increments survive in U_C; its exclusion probability is at least $(1-\alpha)^{\min(b_p(x),\theta)/b}$. For $b_p(x) \leq b$ this is log concavity of $1 - \alpha y$ on $0 \leq y \leq 1$, applied to $y = b_p(x)/b$; for $b_p(x) > b$, use the exclusion probability $1 - \alpha$ and exponent at least one. Ineligible outside bases only weaken the resulting lower bound. Hence

$$\Pr(E_C) \geq \alpha^h (1-\alpha)^{U_C(x)/b}.$$

Also $B(x) - 1 \geq s - 1 + U_C(x) > 0$ on the annulus. For a band it therefore suffices to bound the following quantity by γ_b, over all legal θ, s, h and $0 \leq u < K$:

$$\lambda_C \min_{2 \leq r \leq m} \epsilon_r \; \alpha^{-h} \frac{1 - u/K}{s - 1 + u} (1-\alpha)^{-u/b}. \tag{B}$$

Indeed each positive payment is bounded by $\gamma_b(B(x) - 1)\Pr(E_C)$. Discard nonpositive payments *only* for this pointwise upper bound. Lemma 3 gives $\sum_{C \text{ in band}} F_C(x) \leq \gamma_b \big(B(x) - 1\big)_+$. At zero all exact annuli vanish.

Appendix A derives $\sum_b \gamma_b \leq 963589587/10^9 + 1/2500 < 1$, including every smaller dyadic band. Together with (P) and the exact assignment of source charges, this proves the hinge inequality. Each input has only finitely many actual carriers; the infinite band sum is only a positive universal majorant.

4 From the hinge to distinct actual sites

First take a positive interval. For bounded dual weights $z_p \in [0, 1]$ write $W = \sum_p z_p v_p$ as $W = S + E$, with $S = \sum_p \min(z_p v_p, 1)$. Here S is in the capped nested class on the finite evaluation sets. The excess E is a nonnegative combination of prime-power divisibility indicators, and has target sum at least its prefix sum. Since $E > 0$ forces $S \geq 1$, $\big(W - 1\big)_+ = \big(S - 1\big)_+ + E$, while $\big(W - 6\big)_+ \leq \big(S - 6\big)_+ + E$. Thus the hinge passes to additive valuations with the same coefficient.

Put $L_* = 12n$ and $H = 6n + \pi(L_*)$. Among primes at most $12n$, only two is even, so $\pi(12n) \leq 6n$ and $H \leq L_*$. If $M \leq H$ choose the whole interval. Otherwise let F be the $n - r$ originals whose prime bases are all at most L_*. On the full ambient I, minimize $\sum_{x \in I} y_x$ over $0 \leq y_x \leq 1$ with per-prime additive valuation demands from F. This LP is feasible by prime-power multiple counts. Its dual objective is

$$\sum_{a \in F} W(a) - \sum_{x \in I} \big(W(x) - 1\big)_+, \qquad z_p \geq 0.$$

For $z_p > 1$, every site with positive p-valuation has $W > 1$, so the slope in z_p is demand minus total available p-valuation. It is nonpositive, and we may clamp the dual price to one. The additive hinge bounds the optimum by $6|F|$: indeed, $\sum_{a\in F} W(a) \le 6|F| + \sum_{k\le M}\big(W(k)-6\big)_+$. An optimal vertex has at most $\pi(L_*)$ fractional sites, because more would give a null perturbation on the active demand rows. Round them upward to obtain a set C_* of at most $6(n-r)+\pi(L_*)$ actual sites.

Each of the r removed originals supplies one full factor q_a on a prime base greater than L_*. For any such base separately, sort its selected powers decreasingly. The first j requests have j distinct original labels divisible by the jth power, and thus at least j target multiples. Greedy matching gives distinct sites within that prime. The resulting set of at most r sites can overlap C_*, since the two families of required prime valuations are disjoint. The remaining cofactor $a/q_a < M/L_*$ has at least H eligible multiples. Pay each nonunit cofactor on a genuinely new site: at most $6(n-r)+\pi(L_*)+2r-1 \le H-1$ sites are forbidden. The final set meets every valuation, and has at most H sites. This does not approximate the full-instance fractional optimum, so the known full-instance rounding obstruction is immaterial. Intervals containing zero use $\{0\}$; negative intervals reflect. This establishes the covering theorem.

A Numerical budget for the annulus proof

This supplement gives a written route to the numerical budget. The signed carrier argument and the integral transfer are in the paper. The Lean development proves both theorem statements using the same carrier model, with rational product bounds in place of the exponential estimates below.

A.1 Finite bands

For the manuscript's rounded values $\theta \ge 1/32$, use the five bands and parameters below. The rightmost column is an outward integer numerator over 10^9.

terminal values θ	K	α	$10^9\,\gamma_b$ upper
$[1/2, 1]$	5/2	9/20	703992838
$[1/4, 1/2)$	7/4	1/2	213089445
$[1/8, 1/4)$	4/3	2/5	37775080
$[1/16, 1/8)$	7/6	9/25	8004610
$[1/32, 1/16)$	13/12	7/20	727614

Here b is the left endpoint. The following steps suffice to recover the bound independently of any optimization routine.

Integer carrier masses. For each $\theta = j\Delta$ in the band, all rounded values at least θ are multiples of its dyadic unit Δ. Define $r(0) = 0$ and

$$r(n) = 1 + \max\{r(n-k) : k\Delta \text{ is an allowed level} \ge \theta,\ k \le n\}$$

for reachable positive n, skipping unreachable states. The last (minimum-valued) component is θ, so a reachable $n\Delta \le 1$ with $(n+j)\Delta > 1$ gives a mass $s = (n+j)\Delta$ whose maximum rank is $1 + r(n)$. Smaller ranks cannot make the bound larger: for fixed θ, s, $D_{h,j}$ decreases with h, while the payment multiplier, the source-moment bounds (and their minimum over moment orders) and α^{-h} do not decrease with h. Checking that the multiplier is legal at the maximum rank also covers every smaller one.

The outside real variable. Set $\delta = b/16$. Let r_0 be the largest multiple of 10^{-8} with $r_0^{16} \le 1-\alpha$. For every integer $j \ge 0$ with $j\delta < K$ and each $u \in [j\delta, (j+1)\delta) \cap [0, K)$,

$$\frac{1-u/K}{s-1+u}(1-\alpha)^{-u/b} \;\le\; \frac{1-j\delta/K}{s-1+j\delta}\, r_0^{-j-1}.$$

The first factor on the left is decreasing; the exponential factor is increasing and bounded using the right endpoint. Thus a literal scan of the right-hand side over every legal j is a complete exact test, not sampling the real variable. For the faster production search, the ratio between consecutive positive bounds is at least one exactly when

$$(K-(j+1)\delta)(s-1+j\delta) - r_0(K-j\delta)(s-1+(j+1)\delta) \ge 0.$$

This is a concave quadratic in j. Its nonnegative region is an interval, so the worst value is at the left endpoint or the peak just after the last increase. The independent literal scan avoids reliance on this shortcut.

Recorded arithmetic. The check evaluates 81 terminal levels and 1,581 cells at their maximum ranks. It tests every feasible moment order in the source coefficient. It gives the sum bound

$$\sum_{b \text{ finite}} \gamma_b \le 963589587/10^9.$$

The table displays the conservatively rounded final bounds only; the exact rational rows and implementation remain frozen. This argument does not use a numerical fit or a density ensemble.

A.2 Every smaller scale

For $\theta = j2^{-d} < 1/32$, take $d \ge 10$ and group j as 16:21, 22:27, 28:31. Each group has one common selection law, with $\theta/b < 4/3$, $K = 9/8$, $\alpha = 2/5$, and $\lambda = 10$. We check first that this multiplier is legal, then bound the source and pointwise target quantities.

Signed interval cost. Write $j_0 = m_0 - 1 = \lfloor (62/17)/\theta \rfloor$. Then $j_0 \ge 58$ and $j_0 \ge 3(h-1)$, giving $j_0 \ge h+8$. Also $\rho \ge 1/(h+1)$ by the telescoping bound from the ith prime being at least $i+1$, and every product of j_0 distinct primes is at least $(j_0+1)!$. The same density estimate applies to the comparison products D_{h,j_0} and D_{h,j_0+1}. Both interval error terms are therefore bounded by

$$(h+1)2^h/(h+9)! \le 12/11! < 10^{-4}.$$

The sequence decreases from $h = 2$. The exact inequality

$$10(16/17)\left(1 - 10^{-4} - (1+10^{-4})\frac{1025/1024}{9/8}\right) > 1$$

then gives the stated multiplier in the signed-payment inequality.

Source charge. Let $z = 7/2$, and note that $\theta a \ge 62/17$ for every tail carrier. Use $y \le z^y/(e \log z)$ for $y \ge 0$ and $z^u \le 1 + (z-1)u$ for $0 \le u \le 1$. After bounding the source charge by $\theta(U_C/\theta - a)$, drop source coprimality for this upper bound only. Expanding the product of the positive affine bounds and counting prefix multiples gives

$$\sum_{n \le \nu} z^{U_C(n)/\theta} \le \nu \exp((z-1)H_*/\theta).$$

Consequently

$$c_C \le \frac{(h+1)\theta}{e\log(7/2)}\left(\frac{\exp((5/2)H_*)}{(7/2)^{62/17}}\right)^{1/\theta} \le \frac{45}{128}(5/39)^{1/\theta}.$$

For the last step, $\theta(h+1) \le 9/8$ and $e\log(7/2) > 16/5$, by $e > 8/3$, $e < 87/32$, and $(87/32)^6 < (7/2)^5$. The bound on the exponential base follows from $\exp(5/2048) \le 2048/2043$ and

$$(39/5)^{34}(87/32)^{85}(2048/2043)^{34} < (7/2)^{124}.$$

Pointwise target capacity. For a carrier in the band, the inverse selection factor is at most $(5/2)^h(5/3)^{u/b}$. When $u \ge K/2 = 9/16$, use $s - 1 + u \ge 9/16$ and $(5/2)^2(5/3)^3 < (27/5)^2$. When $u \le K/2$, use the lattice gap $s - 1 \ge \theta/31$ and $(5/2)^4(5/3)^3 < (15/4)^4$. The corresponding bounds on the budget (B) in the manuscript are

$$\frac{125}{8}(9/13)^{1/\theta} \qquad \text{and} \qquad \frac{34875}{128\theta}(25/52)^{1/\theta}.$$

For example, in the larger-u regime $u\theta/b \le (9/8)(4/3) = 3/2$ and $h \le 1/\theta + 1$, giving the first exponential base after the source factor. The smaller regime instead has $u\theta/b \le 3/4$. The full band maximum may be safely bounded by the sum of both bounds.

Summing all scales. On dyadic denominator $d = 10 + v$, $1/\theta > 32\,2^v$. Both expressions decrease in $1/\theta \ge 32$. Put $Z = (9/13)^{32}$ and $W = (25/52)^{32}$. There are three bands per denominator, so using $2^v \ge v + 1$ gives the upper bound

$$3\left[\frac{125}{8}\frac{Z}{1-Z} + \frac{34875}{128}\frac{32W}{1-2W}\right] < \frac{1}{2500}.$$

This final comparison is rational, and covers infinitely many possible levels by an analytic majorant. The actual input always has finitely many carriers.

A.3 The rational route used in Lean

The kernel-checked route replaces the exponential probability comparison by

$$\prod_i (1 - \alpha x_i) \ge (1-\alpha)^m(1-\alpha f), \quad 0 \le x_i \le 1, \quad \sum_i x_i = m + f, \quad 0 \le f < 1.$$

Pairwise transfers to the endpoints prove this directly. A finite auxiliary Bernoulli expansion and a geometric remainder then bound the entire tail by $1/4500$, without logs or noninteger powers in the certificate. The 81 terminal levels, 1,581 cells, and finite bound are unchanged. The module `H6PriceTags` combines this tail with the finite budget. The module `H6Unconditional` then integrates the signed payments over the literal carrier keys and proves the original cover.

Reproduction

The companion repository will contain the complete Lean source closure and the pinned toolchain. An independently implemented arithmetic checker additionally recovers the historical table at 81 terminal levels and 1,581 cells from the frozen inputs. This check is corroboration, not a substitute for the proof of the unrestricted theorem.

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