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UNIMODALITY OF PARKING FUNCTIONS COUNTED BY MAXIMUM DISPLACEMENT

Abstract. We prove that for every $n \geq 1$ the numbers of parking functions of length n with maximum displacement $0, 1, \ldots, n-1$ form a unimodal sequence, as conjectured by Celano, Elder, Hadaway, Harris, Martin, Priestley and Udell. More precisely, for $n \geq 2$ the sequence rises strictly up to a term, its peak, and falls strictly from there on, except that the term after the peak may be equal to it. The position of the peak in the sequence for $n+1$ is the same as in the sequence for n or larger by one, and in the second case no two neighbouring terms of the sequence for $n+1$ are equal. The proof starts from a recursion for the numbers of parking functions whose maximum displacement is bounded, and it combines analytic estimates with exact computations of these numbers for $n \leq 300$.

1. Introduction

1.1. **Parking functions and their maximum displacement.** Konheim and Weiss [12, Sections 3 and 6] studied an occupancy discipline in which balls are placed one after another into cells that are arranged in a circle, each ball into the first free cell at or after a cell that is assigned to it, and they applied it to cars that park along a street. The sequences of assignments under which every car finds a spot in the street are the parking functions. Hashing with linear probing places the items of a table by the same discipline, and the distance between the place that a car or an item takes and the place assigned to it, its displacement, has been studied in both settings [5, 13, 14, 16]. This paper is about the largest displacement of a car in a parking function, and about the way in which the parking functions of a given length are distributed over its values.

Let $n \geq 1$ be an integer. A *preference sequence* of length n is a sequence $\alpha = (\alpha_1, \ldots, \alpha_n) \in \{1, \ldots, n\}^n$. It is read as follows. Spots $1, \ldots, n$ lie in a row along a street and are all empty at the start. Cars $1, \ldots, n$ arrive in this order, and car i prefers spot α_i: it looks from spot α_i onward, in the direction of increasing spot numbers, and parks in the first empty spot it meets. If the spots $\alpha_i, \alpha_i+1, \ldots, n$ are all occupied, car i cannot park. This is the *parking process*. If every car parks, α is called a *parking function* of length n, and the set of all parking functions of length n is written PF_n. For example, with the preference sequence $(1, 2, 1)$ the three cars park in spots 1, 2 and 3, whereas with $(2, 2, 3)$ the third car finds spot 3 occupied and cannot park. The number of parking functions of length n is $(n+1)^{n-1}$; this count follows from Lemma 2 and formula (6.1) of the article of Konheim and Weiss [12], and Foata and Riordan [9, Section 2] give a proof by an argument of Pollak.

For $\alpha \in \mathrm{PF}_n$ let $\mathrm{pos}_i(\alpha)$ be the spot in which car i parks. The *displacement* of car i is the nonnegative integer $\mathrm{pos}_i(\alpha) - \alpha_i$, the distance between the spot that

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the car takes and the spot that it prefers, and the *maximum displacement* of α is

$$M(\alpha) = \max_{1\le i\le n} \big(\mathrm{pos}_i(\alpha) - \alpha_i\big) \in \{0, 1, \dots, n-1\}.$$

For $0 \le \ell \le n-1$ we put

$$b_{n,\ell} = \#\{\alpha \in \mathrm{PF}_n : \ M(\alpha) = \ell\}.$$

The numbers $b_{n,0}, \dots, b_{n,n-1}$ add up to $(n+1)^{n-1}$. The two ends of this sequence are known [5, Example 2.8 and Section 6]: the parking functions with maximum displacement 0 are the permutations, so that $b_{n,0} = n!$, and $b_{n,n-1} = n^{n-2}$ for $n \ge 2$. The sequences for $n \le 7$ are as follows [5, Table 1].

(1.1)

n	$b_{n,0},\ b_{n,1},\ \dots,\ b_{n,n-1}$
1	1
2	2, 1
3	6, 7, 3
4	24, 51, 34, 16
5	120, 421, 377, 253, 125
6	720, 3963, 4594, 3688, 2546, 1296
7	5040, 42253, 62145, 57398, 46142, 32359, 16807

Each of these sequences rises to a largest term and falls after it. A finite sequence $(v_0, \dots, v_m)$ of numbers is *unimodal* if there is $j^* \in \{0, \dots, m\}$ such that $v_{j-1} \le v_j$ for $1 \le j \le j^*$ and $v_j \ge v_{j+1}$ for $j^* \le j \le m-1$; equality is allowed. For the total displacement, the sum of the displacements of the cars, a result on the shape of a sequence of counts is known. The total displacement is essentially equivalent to the sum $\alpha_1 + \cdots + \alpha_n$ of the preferences, as is noted in [13], and the generating polynomial of this sum over the parking functions of length n has log-concave coefficients [13, Theorem 3.1]. For the maximum displacement, Celano, Elder, Hadaway, Harris, Martin, Priestley and Udell conjectured that the sequence $(b_{n,0}, \dots, b_{n,n-1})$ is unimodal for every $n \ge 1$ [5, Conjecture 6.5]. In this paper we prove their conjecture.

Theorem 1.1. *For every integer $n \ge 1$ the sequence*

$$(b_{n,0}, b_{n,1}, \dots, b_{n,n-1})$$

is unimodal.

Our second result describes these sequences more precisely and relates each of them to the next: for $n \ge 2$ the sequence rises strictly up to a term $b_{n,p}$ and falls strictly from there on, except that $b_{n,p+1}$ may be equal to $b_{n,p}$, and the index p stays the same or increases by one when n is replaced by $n+1$. To state it we arrange the numbers $b_{n,k}$ in an array $(b_{n,k})$ whose rows are indexed by the length n and whose columns are indexed by the displacement, for which we write k when it is the index of a column: *row n* of the array is $(b_{n,0}, \dots, b_{n,n-1})$, and *column k* consists of the entries $b_{n,k}$ with $n \ge k+1$.

It will be convenient to have the notions involved for arbitrary arrays. By an *array of natural numbers* we mean a family f of nonnegative integers $f(n,k)$, indexed by the integers $n \ge 0$ and $k \ge 0$; its row n is $(f(n,0), \dots, f(n,n-1))$. Let $n \ge 1$ and $p \ge 0$ be integers. Row n of f *has its peak at position p* if $p \le n-1$ and

$$f(n,0) < \cdots < f(n,p) \ \ge \ f(n,p+1) > \cdots > f(n,n-1),$$

and it *has a strict peak at position* p if $p \le n-1$ and

$$f(n,0) < \cdots < f(n,p) > f(n,p+1) > \cdots > f(n,n-1),$$

where in both cases the terms after $f(n,p)$ are absent if $p = n-1$.

Theorem 1.2. *For every integer $n \ge 2$ there is an integer $p \ge 0$ with the following two properties.*

(1) *Row n of the array $(b_{n,k})$ has its peak at position p.*
(2) *Row $n+1$ has its peak at position p as well, or it has a strict peak at position $p+1$.*

For $n \ge 2$ we write p_n for the least integer p such that row n has its peak at position p; by Theorem 1.2 there is one. In (1.1) one reads off $p_2 = 0$, $p_3 = p_4 = p_5 = 1$ and $p_6 = p_7 = 2$, so that both alternatives of the second property occur. Theorem 1.1 follows from Theorem 1.2, and both theorems are proved in Section 23.

Theorem 1.2 leaves two questions about the rows open. It does not give p_n as a function of n. And it does not decide whether the equality $b_{n,p_n+1} = b_{n,p_n}$ ever occurs.

1.2. **Earlier work.** The paper [5] in which the conjecture is posed calls a parking function an ℓ-interval parking function when its maximum displacement is at most ℓ (Definition 2.7 there). The number $b_{n,\ell}$ is therefore the number of ℓ-interval parking functions of length n that are not $(\ell-1)$-interval. The authors tabulate these numbers for $n \le 9$ (Table 1) and give the two end values quoted above. Before the conjecture they state (Section 6), as consequences of their enumerative results, inequalities between the first three entries and between the last two entries of a row, and they add that experimentally the pattern seems to continue. Throughout, we cite this paper in its second version, and we refer to its statements, sections and tables by the numbers that they have in that version.

Exact descriptions are known for the *cumulative counts*, which are also the numbers that our proof works with. For integers $n \ge 0$ and $\ell \ge 0$ let $A_{n,\ell}$ be the number of parking functions of length n whose maximum displacement is at most ℓ. Here the length $n = 0$ is admitted: the empty sequence is the one parking function of length 0, and its maximum displacement is 0. We extend the definition of $b_{n,\ell}$ accordingly to all integers $n \ge 0$ and $\ell \ge 0$, as the number of parking functions of length n whose maximum displacement is ℓ. Then $b_{n,\ell} = 0$ for $\ell \ge n \ge 1$, while $b_{0,0} = 1$ and $b_{0,\ell} = 0$ for $\ell \ge 1$, and

$$A_{n,\ell} = b_{n,0} + b_{n,1} + \cdots + b_{n,\ell}, \qquad A_{0,\ell} = 1.$$

Celano et al. write $A_{n,\ell}$ as a sum over outcomes. The outcome of a parking function of length n is the permutation σ of $\{1,\dots,n\}$ whose i-th entry σ_i is the number of the car that stands in spot i. For a permutation σ and a spot i let $L_i(\sigma)$ be the length of the longest run of consecutive spots that ends at spot i, spot i included, and whose cars all have numbers at most σ_i. Then the number of parking functions with outcome σ and maximum displacement at most ℓ is

$$\prod_{i=1}^{n} \min\bigl(\ell+1,\, L_i(\sigma)\bigr),$$

and $A_{n,\ell}$ is the sum of these products over all permutations σ [5, Section 4.1, (6)–(8)]. Aguilar-Fraga et al. obtain a recursion in n by fixing the spot of the car that

arrives last [3, Theorem 3.7]. With n replaced by $n+1$ it reads

$$A_{n+1,\ell} = \sum_{j=0}^{n} \min(j+1, \ell+1) \binom{n}{j} A_{j,\ell}\, A_{n-j,\ell} \tag{1.2}$$

for all $n \geq 0$ and $\ell \geq 0$. In the term of j the last car parks in spot $j+1$: the factor $\binom{n}{j}$ chooses the j cars that stand to its left, the factors $A_{j,\ell}$ and $A_{n-j,\ell}$ count the ways in which the cars to its left and to its right park, and $\min(j+1, \ell+1)$ is the number of preferences from which the last car reaches spot $j+1$ with a displacement of at most ℓ. We refer to (1.2) as the *last-car recursion*. It is the starting point of our proof, and it is quoted in Section 3 as Proposition 3.1.

A limit theorem for the maximum displacement is due to Petersson [14, Theorem 2.1]. It is stated for hashing with linear probing, for a table with m cells and $m-1$ items, conditioned on its last cell remaining empty: if M_m denotes the maximum displacement of the items, then for every real number $u > 0$ the probability that $(m - M_m)/m^{2/3} \leq u$ converges to $1 - \psi(u^{3/2})$ as m tends to infinity. Here ψ is an explicit power series, and it is the function $\psi(s) = \mathbb{E}\, e^{-sJ}$, where J is the integral over the interval $[0,1]$ of the running maximum of a standard Brownian motion [11, Corollary 2.4]. For $m = n+1$ such a table is a parking function of length n chosen uniformly at random. In terms of the numbers $b_{n,\ell}$ the theorem therefore says that all but an arbitrarily small fraction of the sum $(n+1)^{n-1}$ of row n lies at a distance of order $n^{2/3}$ from the end $\ell = n-1$ of the row.

There are also results on subfamilies of the parking functions. Selig and Zhu showed that for non-decreasing parking functions the maximum displacement is the height of the corresponding Łukasiewicz path [16, Theorem 2.1]. For arbitrary parking functions the two quantities can differ [16, Remark 2.3]. Ager-Hart, Beerbower, Harris, Jakovleski and McClinton count the ℓ-interval Fubini rankings, the Fubini rankings in which at most $\ell+1$ competitors tie at a rank [2, Corollary 4.1]; as they recall, Fubini rankings are parking functions.

1.3. **Outline of the proof.** Of the results described in the preceding subsection, our proof uses one, the last-car recursion (1.2). We do not prove Theorem 1.1 directly. The shape of the rows is obtained by an induction on n, and what this induction passes from row n to row $n+1$ is the signs of the differences of neighbouring entries. The statement that it yields is therefore Theorem 1.2, which describes these signs in a row and relates each row to the next.

For integers $n \geq 1$ and $k \geq 1$ let $d_{n,k} = b_{n,k} - b_{n,k-1}$ be the *step* of column k in row n; we say that column k is *positive* in row n if $d_{n,k} > 0$, and *negative* if $d_{n,k} < 0$. A *position* is a pair (n,k) of integers with $1 \leq k \leq n-1$, and the *diagonal neighbour* of position (n,k) is position $(n+1, k+1)$. Theorem 1.2 follows from two rules about the signs of the steps when the length grows by one. The *column rule* says that $d_{n,k} > 0$ implies $d_{n+1,k} > 0$, and the *diagonal rule* says that $d_{n,k} \leq 0$ implies $d_{n+1,k+1} < 0$, in both cases for all integers $n \geq 2$ and $1 \leq k \leq n-1$.

We use the words column, step and position, and the names of the two rules, for an arbitrary array f of natural numbers as well, with $d_{n,k}$ replaced by $f(n,k) - f(n,k-1)$. For every such array the two rules imply the conclusion of Theorem 1.2, by an induction on the row (Proposition 23.4): the column rule carries the rising part of row n to row $n+1$, the diagonal rule carries the rest one column to the

right and makes it strictly falling, and the one step of row $n+1$ that neither rule reaches decides whether the peak stays or moves.

The parking process enters the proof of the two rules only through the last-car recursion (1.2). The recursion holds for the cumulative counts because the condition $M(\alpha) \le \ell$ is a condition on each car separately, which is inherited by the two groups of cars into which the last car divides a parking function; the condition $M(\alpha) = \ell$ is not of this kind. The steps are recovered from the cumulative counts as second differences in the bound on the displacement,

$$d_{n,k} = A_{n,k} - 2A_{n,k-1} + A_{n,k-2} \qquad (k \ge 2),$$

so that the two rules are statements on the sign of a second difference of the solutions of a family of quadratic recursions. We write the recursion as a differential equation for the generating series of the sequence $(A_{n,\ell}/n!)_{n\ge 0}$ and derive from it the linear relations on which the rest of the proof is built.

One of these relations divides the array into two regions. Rows $k+1 \le n \le 2k-1$ are called the *first layer* of column k; it is empty for $k = 1$. Rows $n \ge 2k$ lie *beyond the first layer*. In the first layer the step satisfies a linear recursion along the column (Proposition 4.12), in which $d_{n,k}$ is a combination, with positive coefficients, of the entry $b_{n-1,k-1}$ of the neighbouring column and of the steps of column k in the rows from k to $n-1$. In every row $n \ge k+1$, and so also beyond the first layer, an inequality of this form holds (Proposition 5.3).

The two regions behave differently. Beyond the first layer the step is positive at every position except eleven, the positions $(2k, k)$ with $1 \le k \le 7$ and $(2k+1, k)$ with $2 \le k \le 5$ (Proposition 19.6). At each of these eleven positions the step is negative, at $(2, 1)$ by row 2 of (1.1) and at the other ten by Lemma 6.1, and both rules follow at once in this region when one knows in addition that the step is negative at certain positions next to the eleven. In the first layer a column $k \ge 9$ turns positive in some row of the layer and stays positive from the first such row to the end of the layer, whereas columns $2 \le k \le 8$ are negative in their whole first layer. We call the least row $n \ge k+1$ with $d_{n,k} > 0$ the *first positive row* of column k. In the first layer the column rule says that a positive step in a row of the layer is followed by a positive step in the next row of the layer (Proposition 16.5). The diagonal rule needs more there, namely that a step of column k which is not positive is followed by a negative step of column $k+1$ one row below.

Each rule is proved region by region. Within a region the small columns are settled by exact computation, entirely or up to a certain row from which an analytic argument propagates the positive sign down the column, and the large columns by analytic estimates; in the columns between them the two are combined. The dividing columns are 82, for the column rule in the first layer and for the signs beyond the first layer, and 265, for the diagonal rule in the first layer. The results of the computations are stated as five lemmas in Section 6; they are obtained from the integers $A_{n,\ell}$ in rows $n \le 300$.

The difficulty of the proof lies in three places. The first is the diagonal rule in the first layer of columns $k \ge 265$. We call $r = n - k$ the *deficit* of position (n, k), and we regard the position as the point (k, r) of the plane. By *the curve* we mean the curve $k^2 = r^3$ in this plane, for real numbers k and r. A position with deficit r lies *behind the curve* if $r^3 \le k^2$, and *beyond the curve* if $r^3 > k^2$; it lies *at most one row beyond the curve* if $(r-1)^3 \le k^2$. The diagonal rule in this region is obtained from two analytic statements that have to meet. Proposition 22.6 is the diagonal

rule at the positions behind the curve in columns $k \geq 265$. Proposition 16.2 says that every column $k \geq 265$ is positive in a row $j > k$ with $(j-k-1)^3 \leq k^2$, that is, at a position (j,k) that lies at most one row beyond the curve. In Section 23 the two propositions are combined with the column rule in the first layer, which places every step of the first layer of such a column that is not positive in a row before row j, and so behind the curve, where Proposition 22.6 applies.

Both propositions start from Proposition 13.6, which writes the number $d_{n,k}/n!$ at every position of the first layer as the difference of two terms: a second difference of a polynomial in a real variable y, the curvature term, and a multiple of a sum over compositions, the boundary term. The proposition also gives, in the same row and with the same polynomial, a formula of this form for the step of column $k+1$. Read for column $k-1$, that formula is a second decomposition of $d_{n,k}/n!$ into a curvature term and a boundary term, with the polynomial of column $k-1$, and it holds for $k \geq 3$ in rows $k \leq n \leq 2k-3$. The sums in question are the chain sums of Section 11. For an integer $m \geq 1$ and real numbers y and N with $N \geq m$ they are

$$\Lambda_m(y;N) = \sum_{q \geq 1} \sum_{0=s_0<s_1<\cdots<s_q=m} \prod_{i=1}^{q} a_{s_i - s_{i-1}} \prod_{i=1}^{q-1} \frac{y}{N-s_i},$$

where $a_i = i^{i-1}/i!$: a composition of m is weighted by the factor a_i for each part of size i and by the factor $y/(N-s)$ for each cut at position s.

In the proof of Proposition 22.6 the first of the two decompositions is used at a position behind the curve and at its diagonal neighbour, and the growth of its two terms along the diagonal is compared. The curvature term is a sum, and each of its terms is the product of a coefficient formed from the numbers a_i and of a sum over compositions. The sum over compositions grows by at most an explicit factor, by Hölder's and Jensen's inequalities (Proposition 21.5), and so does the coefficient, by Stirling's formula and the fact that the numbers $e^{-i}a_i$, $i \geq 1$, which are the probabilities of the Borel distribution with parameter 1, add up to one (Proposition 22.2). Each term of the curvature term therefore grows by at most an explicit factor, the product of these two, and the boundary term grows by at least an explicit factor, by an instance of the correlation inequality of Fortuin, Kasteleyn and Ginibre [10, Proposition 1] on the sets of cuts of the compositions. That the first factor is smaller than the second is an inequality between closed forms in k, r and the index of the term (Lemma 22.3). The two conditions $r^3 \leq k^2$ and $k \geq 265$ of Proposition 22.6 are the conditions under which we prove this inequality, and they enter it in different ways. After logarithms, the inequality compares the bounds for the growth of a term of the curvature term with the sum of $\log(1+1/k)$, which comes from a factor $k/(k+1)$ in the first factor, and of the logarithm of the gain of the chain sum in the boundary term. Its proof is a comparison of explicit constants. The condition $r^3 \leq k^2$, under which $r^{3/2}$ is at most k, is the range in which we carry out this comparison, and this is why the proposition is restricted to the positions behind the curve. The bound $k \geq 265$ enters through the numerical constants in the proof of the inequality, which are chosen for the columns from 265 on; for the columns up to 264 the diagonal rule in the first layer is proved with the help of the computations of Section 6 instead (Proposition 23.2).

Proposition 16.2 is a separate estimate, of the step at a single position: the curvature term of the second decomposition is bounded from below and its boundary

term from above, and at the positions near the curve the two bounds are compared with explicit numerical constants, for which we use Stirling's formula.

The second place of difficulty is the column rule in the first layer of columns $k \geq 82$, and with it the positive sign of these columns beyond the first layer. Both start from a positive row early in the first layer, which is computed for $k \leq 264$ and is the row j of Proposition 16.2 for $k \geq 265$; the proof of that proposition does not use the column rule. The obstacle is what the rows before the first positive row contribute to the recursion along the column at a later row: in these rows the step is not positive, and nothing is known about its size. In the first layer we meet it by two arguments, each for the rows that the other cannot reach. On a little less than the first half of the layer, the recursion at row n is compared with the recursion at the first positive row. On the remaining rows the step is bounded from below by the entries of column $k-1$, and the terms of this bound whose coefficients may be negative are controlled by a comparison with the renewal sequence of a positive log-convex kernel. This comparison rests on a theorem of de Bruijn and Erdős [6, Theorem 1], by which such a renewal sequence is log-convex. Their theorem is stated for a kernel whose sum is one, which is not the case for the kernel of the comparison, and we include a proof of the form that we need. Beyond the first layer the positive sign is carried from row to row by an induction, in which the contributions of the rows before the first positive row are not bounded one by one; we bound a weighted sum of them, by a summation by parts that rests on a convexity along the diagonal of the cumulative counts, taken with suitable explicit weights.

The third place of difficulty is strictness. Both rules conclude with a strict inequality, and the diagonal rule starts from a step that may be zero. This is more than Theorem 1.1 asks, since unimodality allows equal neighbours everywhere in a row, while Theorem 1.2 allows them in one place only. Wherever a rule is read off from the signs of the steps at certain positions, a step equal to zero has to be excluded there, and where no argument excludes it, the signs are computed. This is the reason why every sign of a step that Lemmas 6.1, 6.3, 6.6 and 6.7 state is a strict inequality, $d_{n,k} < 0$ or $d_{n,k} > 0$.

The inequalities of Hölder and Jensen, Stirling's formula, the total mass of the Borel distribution, the correlation inequality of Fortuin, Kasteleyn and Ginibre and the theorem of de Bruijn and Erdős, which the arguments for the first two places of difficulty use, are classical. What is particular to the proof is where they are applied: to the two terms of the formulas of Proposition 13.6, which are followed along a diagonal and compared near the curve, and to the renewal sequence of the second place. Proposition 16.2 is the statement on which both rules depend throughout columns $k \geq 265$: the diagonal rule in their first layer needs its bound for the first positive row in the form in which it is stated, and the column rule in the first layer and the positive signs beyond the first layer are derived from the positive row that it supplies.

The paper is organised as follows. Section 2 fixes conventions and collects the facts about the numbers a_i and related sequences that several later sections use. Section 3 quotes the last-car recursion, and Section 4 derives from it the recursion along a column in its first layer. Section 5 proves two inequalities for a column that hold in every row and the propagation of a positive step down a column, and Section 6 states the results of the exact computations. Sections 7 to 10 prove that a

column $k \geq 82$ which turns positive early in its first layer stays positive to the end of the layer (Proposition 10.5). Sections 11 to 16 prove Proposition 16.2 and draw its consequences for the first layer. Sections 17 to 19 determine the signs beyond the first layer. Sections 20 to 22 prove Proposition 22.6. In Section 23 the two rules are assembled from these statements, and Theorems 1.2 and 1.1 are proved.

2. Preliminaries

Throughout the paper an empty sum is 0, an empty product is 1, and $0^0 = 1$. Later proofs use the following elementary inequalities. For every real number $u \geq 0$ the Taylor polynomials of $\log(1+u)$ bound it alternately, those of even degree from below and those of odd degree from above; we use this in the forms

$$u - \frac{u^2}{2} \leq \log(1+u) \leq u - \frac{u^2}{2} + \frac{u^3}{3},$$
$$u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} \leq \log(1+u).$$

The exponential function satisfies

$$e^y \leq 1 + y + y^2 \qquad (0 \leq y \leq 1.7),$$

as is easily verified from the exponential series. By comparison with integrals, for integers n and real numbers y,

$$\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2\sqrt{n} - 1 \quad (n \geq 1), \qquad \sum_{i=0}^{n-1} \sqrt{i} \leq \frac{2}{3}\, n\sqrt{n} \quad (n \geq 0),$$
$$\frac{1}{(y+1)\sqrt{y+1}} \leq 2\Big(\frac{1}{\sqrt{y}} - \frac{1}{\sqrt{y+1}}\Big) \leq \frac{1}{y\sqrt{y}} \quad (y > 0).$$

Finally, we use Jensen's inequality for the convex function $t \mapsto \theta^t$, where $\theta > 0$ is a real number, with weights that need not be normalised: if I is a finite set, if $\rho_i \geq 0$ and t_i are real numbers for every $i \in I$, and if $\bar{t}$ is a real number such that $\bar{t} \sum_{i\in I} \rho_i = \sum_{i\in I} \rho_i\, t_i$, then

$$\theta^{\bar{t}} \sum_{i\in I} \rho_i \leq \sum_{i\in I} \rho_i\, \theta^{t_i}.$$

2.1. The Cayley coefficients.

The *Cayley coefficients* are the real numbers

$$a_0 = 0, \qquad a_j = \frac{j^{\,j-1}}{j!} \quad (j \geq 1).$$

By Cayley's formula, j^{j-1} is the number of rooted labelled trees on j vertices. The first value is $a_1 = 1$, and $a_j > 0$ for every $j \geq 1$. The value $a_0 = 0$ is chosen so that a sum over the pairs of indices with a given sum, such as $\sum_{i=0}^{t} a_i\, a_{t-i}$, can run over all i with $0 \leq i \leq t$: the two extreme terms contain a_0 and vanish.

Beside the Cayley coefficients we use the same numbers with the index shifted by one,

$$c_n = a_{n+1} = \frac{(n+1)^n}{(n+1)!} = \frac{(n+1)^{n-1}}{n!} \qquad (n \geq 0).$$

Thus $c_0 = 1$. By the count of the parking functions recalled in Section 1, there are $(n+1)^{n-1}$ parking functions of length n for every $n \geq 1$. Since the empty sequence is the one parking function of length 0, the number c_n is, for every $n \geq 0$, the number of parking functions of length n divided by $n!$.

Consecutive Cayley coefficients satisfy, for every integer $j \geq 1$,

$$a_{j+1} = \Big(1 + \frac{1}{j}\Big)^{j-1} a_j, \tag{2.1}$$

since both sides are equal to $(j+1)^{j-1}/j!$. The next lemma shows that the ratio $(1+1/j)^{j-1}$ does not decrease in j. It is derived from the classical fact that the sequence $(1+1/i)^i$, whose limit is e, does not decrease: for all integers i and j with $1 \leq i \leq j$,

$$\Big(1 + \frac{1}{i}\Big)^{i} \leq \Big(1 + \frac{1}{j}\Big)^{j}. \tag{2.2}$$

Lemma 2.1. *For all integers i and j with $1 \leq i \leq j$,*

$$\Big(1 + \frac{1}{i}\Big)^{i-1} \leq \Big(1 + \frac{1}{j}\Big)^{j-1}.$$

Proof. By induction on j it suffices to prove the inequality for $j = i+1$. In this case it follows from (2.2) on dividing the left-hand side by $1 + 1/i$ and the right-hand side by the smaller number $1 + 1/(i+1)$. □

Together with Lemma 2.1, the identity (2.1) gives the log-convexity of the Cayley coefficients on indices $j \geq 1$, in the following form; for $m = 1$ and $j = i+1$ it is the usual inequality $a_{i+1}^2 \leq a_i\, a_{i+2}$.

Lemma 2.2. *For all integers i, j and m with $1 \leq i \leq j$ and $m \geq 0$, the Cayley coefficients satisfy*

$$a_{i+m}\, a_j \leq a_i\, a_{j+m}.$$

Proof. We argue by induction on m, for fixed $i \leq j$; the case $m = 0$ is trivial. Suppose that $a_{i+m}\, a_j \leq a_i\, a_{j+m}$. Since $1 \leq i+m \leq j+m$, (2.1) and Lemma 2.1 give

$$\frac{a_{i+m+1}}{a_{i+m}} = \Big(1 + \frac{1}{i+m}\Big)^{i+m-1} \leq \Big(1 + \frac{1}{j+m}\Big)^{j+m-1} = \frac{a_{j+m+1}}{a_{j+m}}.$$

Multiplying this inequality by the induction hypothesis gives $a_{i+m+1}\, a_j \leq a_i\, a_{j+m+1}$. □

For the Cayley coefficients the inequality with a general shift came directly from the explicit ratio. For other sequences, in Sections 8 and 9, only the inequality between three consecutive terms will be known, and the following lemma passes from it to the inequality with a general shift.

Lemma 2.3. *Let $(v_n)_{n \geq 0}$ be a sequence of real numbers such that $v_n > 0$ and $v_{n+1}^2 \leq v_n\, v_{n+2}$ for every integer $n \geq 0$. Then for all integers i, j and m with $0 \leq i \leq j$ and $m \geq 0$,*

$$v_{i+m}\, v_j \leq v_i\, v_{j+m}.$$

Proof. Since the numbers v_n are positive, the hypothesis says that the ratio v_{n+1}/v_n does not decrease in n. Hence, for $i \leq j$,

$$\frac{v_{i+m}}{v_i} = \prod_{t=0}^{m-1} \frac{v_{i+t+1}}{v_{i+t}} \leq \prod_{t=0}^{m-1} \frac{v_{j+t+1}}{v_{j+t}} = \frac{v_{j+m}}{v_j},$$

which is the claim. □

2.2. **Abel's identity and a recursion for the sequence (c_n).** The recursion for the numbers c_n that we need is a case of a classical identity, Abel's generalisation of the binomial theorem. For a real number x and an integer $k \geq 0$ let

$$w_0(x) = 1, \qquad w_k(x) = x\,(x+k)^{k-1} \quad (k \geq 1).$$

In Abel's identity these polynomials in x play the role of the powers x^k of the binomial theorem, and the powers $(y-k)^{n-k}$ that of y^{n-k}.

Lemma 2.4 (Abel [1, p. 159])**.** *For every integer $n \geq 0$ and all real numbers x and y,*

$$\sum_{k=0}^{n} \binom{n}{k} w_k(x)\,(y-k)^{n-k} = (x+y)^n.$$

Abel states his identity for three arbitrary quantities x, α and β and a positive integer exponent, and his proof covers the exponent 0 as well; Lemma 2.4 is its case $\beta = -1$, with Abel's α and x replaced by our x and y.

For $x = 1$ the weights of Abel's identity are the numbers c_k up to a factorial: for every integer $k \geq 0$,

$$c_k = \frac{w_k(1)}{k!}. \tag{2.3}$$

Indeed, $w_k(1) = (k+1)^{k-1}$ for every $k \geq 0$. With (2.3), Abel's identity becomes a recursion for the numbers c_n, with which Section 3 compares the last-car recursion of Proposition 3.1.

Lemma 2.5. *For every integer $n \geq 0$, the numbers c_n satisfy*

$$(n+1)\,c_{n+1} = \sum_{j=0}^{n} (j+1)\,c_j\,c_{n-j}.$$

Proof. Since $(j+1)\,c_j = (j+1)^j/j!$ for every $j \geq 0$, the term of index $j = n-k$ of the sum is, by (2.3),

$$(n-k+1)\,c_{n-k}\,c_k = \frac{(n+1-k)^{n-k}}{(n-k)!} \cdot \frac{w_k(1)}{k!} = \frac{1}{n!}\binom{n}{k} w_k(1)\,(n+1-k)^{n-k}.$$

By Lemma 2.4 with $x = 1$ and $y = n+1$, the sum of these terms over $0 \leq k \leq n$ is $(n+2)^n/n!$, which is $(n+1)\,c_{n+1}$. □

Written for the Cayley coefficients themselves, the recursion of Lemma 2.5 becomes the following convolution identity, which is used in Section 13.

Lemma 2.6. *For every integer $t \geq 0$,*

$$\sum_{i=0}^{t-1} i\,a_i\,a_{t-i} = (t-1)\,a_t.$$

Proof. For $t \leq 1$ both sides are zero, since $a_0 = 0$. Let $t = n+2$ with $n \geq 0$. The term $i = 0$ vanishes, and with $i = j+1$ the other terms are the numbers $(j+1)\,c_j\,c_{n-j}$ with $0 \leq j \leq n$, whose sum is $(n+1)\,c_{n+1} = (t-1)\,a_t$ by Lemma 2.5. □

2.3. **The Poisson mass and Stirling's formula.** For an integer $n \geq 0$ let

$$\nu_n = \frac{n^n}{n!}, \qquad P(n) = e^{-n}\,\nu_n = e^{-n}\,\frac{n^n}{n!},$$

so that $\nu_0 = P(0) = 1$. The number $P(n)$, the *Poisson mass*, is the probability that a Poisson variable with mean n takes the value n. The sequences of this section are estimated through it, and by Stirling's formula $P(n)$ is close to $1/\sqrt{2\pi n}$. Note that $P(n) = (n/e)^n/n!$ for every $n \geq 0$, and that in a product of two Poisson masses whose arguments add up to m the exponential factors combine to e^{-m}, whatever the splitting:

$$P(g)\,P(m-g) = e^{-m}\,\nu_g\,\nu_{m-g} \qquad (0 \leq g \leq m).$$

We need the approximation of $P(n)$ by $1/\sqrt{2\pi n}$ with explicit bounds on both sides that hold for every $n \geq 1$. Both bounds come from Stirling's formula in the form that Robbins gave it [15, p. 26, (1) and (2)]: for every integer $n \geq 1$,

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{1/(12n+1)} < n! < \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{1/(12n)}. \tag{2.4}$$

Here and for the rest of the paper we write

$$c_{\mathrm{S}} = \frac{1}{\sqrt{2\pi}}$$

for the constant of Stirling's formula. Since $P(n) = (n/e)^n/n!$, the bounds (2.4) give, for every integer $n \geq 1$,

$$c_{\mathrm{S}}\Big(1 - \frac{1}{12n}\Big)\frac{1}{\sqrt{n}} \leq P(n) \leq \frac{c_{\mathrm{S}}}{\sqrt{n}}. \tag{2.5}$$

The upper bound follows from the left inequality of (2.4), in which $e^{1/(12n+1)} \geq 1$. For the lower bound, the right inequality is used together with $e^{-1/(12n)} \geq 1 - 1/(12n)$, which holds because $e^t \geq 1 + t$ for every real number t.

Where a numerical comparison needs the value of the constant, we use the enclosure

$$0.39894 \leq c_{\mathrm{S}} \leq 0.39894229.$$

3. The cumulative counts and the last-car recursion

The proof starts from the last-car recursion for the cumulative counts $A_{n,\ell}$, which we recall from the literature (Proposition 3.1); it is through this recursion that the parking process enters the later arguments. We then pass to the numbers $A_{n,\ell}/n!$ and to the normalised counts and steps formed from them, with which the rest of the paper works.

3.1. **The last-car recursion.** Recall from Section 1 that the cumulative counts $A_{n,\ell}$ are defined for all integers $n \geq 0$ and $\ell \geq 0$, and that $A_{0,\ell} = 1$. The last-car recursion expresses $A_{n+1,\ell}$ by the numbers $A_{j,\ell}$ with $j \leq n$; it results from sorting the parking functions of length $n+1$ by the spot in which the last car parks. It is stated by Aguilar-Fraga et al. [3, Theorem 3.7], with n instead of $n+1$, for the number of parking functions of length n in which every car parks at most ℓ spots after the spot that it prefers, which is $A_{n,\ell}$.

Proposition 3.1 ([3, Theorem 3.7]). *For all integers $n \geq 0$ and $\ell \geq 0$, the cumulative counts satisfy*

$$A_{n+1,\ell} = \sum_{j=0}^{n} \min(j+1, \ell+1) \binom{n}{j} A_{j,\ell}\, A_{n-j,\ell}.$$

3.2. **The steps as second differences.** The numbers $b_{n,\ell}$ enter the proof only through the cumulative counts. For all integers $n \geq 0$ and $\ell \geq 0$,

$$b_{n,0} = A_{n,0}, \qquad b_{n,\ell+1} = A_{n,\ell+1} - A_{n,\ell}, \tag{3.1}$$

and $A_{n,\ell} \leq A_{n,\ell'}$ whenever $\ell \leq \ell'$. These relations are immediate from the definitions.

The theorems of this paper are statements on the signs of the steps $d_{n,k} = b_{n,k} - b_{n,k-1}$, which Section 1 defines for integers $n \geq 1$ and $k \geq 1$. The numbers $b_{0,\ell}$ are defined as well, and we extend the definition of the step to length $n = 0$ by the same formula, so that $d_{n,k}$ is defined for all integers $n \geq 0$ and $k \geq 1$. Since $b_{n,k}$ is a first difference of the cumulative counts in the bound on the displacement, a step is a second difference: for all integers $k \geq 2$ and $n \geq 0$, the relations (3.1), applied to $b_{n,k}$ and, since $k - 1 \geq 1$, to $b_{n,k-1}$, give

$$b_{n,k} - b_{n,k-1} = A_{n,k} - 2A_{n,k-1} + A_{n,k-2}. \tag{3.2}$$

It is from this second difference that the exact computations of Section 6 read the sign of a step.

3.3. **The normalised counts and their recursion.** From now on we work with the counts of length n divided by $n!$. The reason is that the last-car recursion contains a binomial coefficient, which this division removes, since

$$\frac{1}{n!}\binom{n}{j} = \frac{1}{j!\,(n-j)!} \qquad (0 \leq j \leq n).$$

The recursion that results can be written as a differential equation for a generating series, and it can be compared with the recursion of Lemma 2.5 for the sequence (c_n). For integers $y \geq 1$ and $n \geq 0$ we put

$$z_n^{(y)} = \frac{A_{n,y-1}}{n!},$$

the *normalised count of threshold y*: it is the number of parking functions of length n all of whose displacements are smaller than y, divided by $n!$. The index is the threshold y and not the bound $y - 1$ on the maximum displacement. With this choice the factor $\min(j+1, \ell+1)$ of the last-car recursion is $\min(j+1, y)$, and $z_n^{(y)} = c_n$ for the lengths $n \leq y$ (Proposition 3.4). For integers $k \geq 0$ and $n \geq 0$ we put

$$\beta_n^{(k)} = \frac{b_{n,k}}{n!},$$

the *normalised count of column k*, and for integers $k \geq 1$ and $n \geq 0$

$$\delta_n^{(k)} = \beta_n^{(k)} - \beta_n^{(k-1)} = \frac{d_{n,k}}{n!},$$

the *normalised step* of column k in row n. When column k is fixed, we write δ_n for $\delta_n^{(k)}$.

Clearly the numbers $z_n^{(y)}$ and $\beta_n^{(k)}$ are nonnegative, $z_n^{(y)} \le z_n^{(y')}$ whenever $y \le y'$, and $z_0^{(y)} = 1$. Dividing (3.1) by $n!$, we obtain $\beta_n^{(0)} = z_n^{(1)}$ and

$$\beta_n^{(k)} = z_n^{(k+1)} - z_n^{(k)} \qquad (k \ge 1), \tag{3.3}$$

and (3.2) becomes

$$\delta_n^{(k)} = z_n^{(k+1)} - 2\, z_n^{(k)} + z_n^{(k-1)} \qquad (k \ge 2,\ n \ge 0). \tag{3.4}$$

The quantities of one column k are therefore formed from the normalised counts of the three thresholds $k+1$, k and $k-1$. Finally, the real number $\delta_n^{(k)}$ is positive, nonnegative, negative or nonpositive exactly when the integer $d_{n,k}$ is. The analytic sections prove inequalities for the real numbers $\delta_n^{(k)}$, whereas the theorems and the exact computations speak of the integers $d_{n,k}$, and these equivalences are the passage between the two.

The recursion that the numbers $z_n^{(y)}$ satisfy belongs to the following family of recursions, which is used with a general weight in Section 4.

Definition 3.2. Let $\rho = (\rho(t))_{t\ge1}$ and $W = (W_n)_{n\ge0}$ be sequences of real numbers. The sequence W is a *solution of the weighted last-car recursion with weight* ρ if $W_0 = 1$ and

$$(n+1)\, W_{n+1} = \sum_{j=0}^{n} \rho(j+1)\, W_j\, W_{n-j} \qquad \text{for every integer } n \ge 0.$$

The recursion gives W_{n+1} in terms of $W_0, \dots, W_n$ and of $\rho(1), \dots, \rho(n+1)$. The sequence $c = (c_n)_{n\ge0}$ is a solution for the weight $\rho(t) = t$: we have $c_0 = 1$, and the recursion is Lemma 2.5.

Corollary 3.3. *Let $y \ge 1$ be an integer. Then the sequence $z^{(y)} = (z_n^{(y)})_{n\ge0}$ of the normalised counts of threshold y is a solution of the weighted last-car recursion with the weight $\rho(t) = \min(t, y)$, in the sense of Definition 3.2.*

Proof. We have $z_0^{(y)} = 1$. For every integer $n \ge 0$, dividing the recursion of Proposition 3.1 for $\ell = y - 1$ by $n!$ gives

$$(n+1)\, z_{n+1}^{(y)} = \sum_{j=0}^{n} \min(j+1, y)\, z_j^{(y)}\, z_{n-j}^{(y)}, \tag{3.5}$$

which is the recursion of Definition 3.2 for the weight $\rho(t) = \min(t, y)$. □

3.4. **The free range of a threshold.** For $t \le y$ the weight $\min(t, y)$ is the weight t of the sequence c, and so the two solutions agree up to index y. The lengths $n \le y$ form the *free range* of threshold y.

Proposition 3.4. *Let $y \ge 1$ and $n \ge 0$ be integers with $n \le y$. Then $z_n^{(y)} = c_n$.*

Proof. We prove the claim by induction on n. For $n = 0$ we have $z_0^{(y)} = 1 = c_0$. Let $n+1 \le y$, and suppose that $z_j^{(y)} = c_j$ for $0 \le j \le n$. Since $\min(j+1, y) = j+1$ for these j, Corollary 3.3 and Lemma 2.5 give

$$(n+1)\, z_{n+1}^{(y)} = \sum_{j=0}^{n} (j+1)\, c_j\, c_{n-j} = (n+1)\, c_{n+1},$$

and therefore $z_{n+1}^{(y)} = c_{n+1}$. □

In terms of the counts, Proposition 3.4 says that

$$A_{n,\ell} = (n+1)^{n-1} \qquad \text{whenever } n \le \ell + 1,$$

which is the classical count of parking functions of length n given in Section 1.

The free range is the reason why a column $k \ge 1$ begins in row $k+1$. Indeed, a length $n \le k$ lies in the free range of both thresholds k and $k+1$, and (3.3) and Proposition 3.4 give

$$\beta_n^{(k)} = c_n - c_n = 0 \qquad (k \ge 1,\ n \le k).$$

Consequently

$$\delta_n^{(k)} = 0 \qquad (k \ge 2,\ n < k).$$

In row $n = k$ the step of a column $k \ge 2$ is $\delta_k^{(k)} = -a_k/k$, by Lemma 4.11 of Section 4, with the Cayley coefficient a_k of Section 2.

3.5. **The tilted recursion.** Several later arguments multiply the term of index n of a sequence by the n-th power of a real number. Such a number x is called a *tilt*, and to *tilt* a sequence $(s_n)_{n\ge0}$ of real numbers by x is to pass to the sequence $(s_n\, x^n)_{n\ge0}$. The numbers $c_N\, x^N$ are called the *tilted Cayley masses*, and $z_N^{(y)}\, x^N$ is the *tilted normalised count* of threshold y. By Proposition 3.4 the tilted normalised count of threshold y is the tilted Cayley mass $c_N\, x^N$ for $N \le y$, and for a tilt $x \ge 0$ the tilted normalised counts are nonnegative and do not decrease in the threshold.

Multiplying the recursion (3.5), taken for $n = N$, by x^{N+1} and writing $x^{N+1} = x \cdot x^i \cdot x^{N-i}$ in the term of index i, we obtain the tilted recursion

$$(N+1)\, z_{N+1}^{(y)}\, x^{N+1} = x \sum_{i=0}^{N} \min(i+1, y)\, \bigl(z_i^{(y)}\, x^i\bigr)\bigl(z_{N-i}^{(y)}\, x^{N-i}\bigr) \tag{3.6}$$

for every integer $y \ge 1$, every real number x and every integer $N \ge 0$. The later sections apply it at the critical tilt of a column, which is defined in Section 5.

4. Generating series, the prime kernel and the column equation

We now write the normalised last-car recursion of Corollary 3.3 as a differential equation for a formal power series and derive from it, by the algebra of power series, the linear relations on which the later sections rest: one for the solutions of a weighted last-car recursion (Propositions 4.9 and 4.10), and the *column equation*, which is the recursion that the normalised step satisfies along a column in its first layer (Proposition 4.12).

4.1. **The series of a threshold.** All series in this section are formal power series in one variable X with real coefficients. For a series ϕ we write $[X^n]\phi$ for the coefficient of X^n in ϕ and ϕ' for the formal derivative of ϕ, so that $[X^n]\phi' = (n+1)\,[X^{n+1}]\phi$; the derivative obeys the product rule. For two series ϕ and ψ with the coefficients $\phi_j = [X^j]\phi$ and $\psi_j = [X^j]\psi$ the coefficients of the product are

$$[X^n](\phi\psi) = \sum_{j=0}^{n} \phi_j\, \psi_{n-j} = \phi_0\, \psi_n + \sum_{m=1}^{n} \phi_m\, \psi_{n-m}. \tag{4.1}$$

A series ϕ with $\phi_0 \ne 0$ has an inverse ϕ^{-1}, which we also write $1/\phi$, and the constant coefficient of ϕ^{-1} is $1/\phi_0$.

Let $y \geq 1$ be an integer, and let $z_n^{(y)} = A_{n,y-1}/n!$ be the normalised count of threshold y (Section 3). By Corollary 3.3 the sequence $z^{(y)}$ is the solution of the weighted last-car recursion with the weight $\min(t, y)$: we have $z_0^{(y)} = 1$ and

$$(n+1)\, z_{n+1}^{(y)} = \sum_{j=0}^{n} \min(j+1, y)\, z_j^{(y)}\, z_{n-j}^{(y)} \qquad (n \geq 0). \tag{4.2}$$

The *series of the threshold* y is the formal power series

$$Z_y = \sum_{n \geq 0} z_n^{(y)}\, X^n,$$

whose constant coefficient is 1. Beside it we consider the series

$$\Gamma_y = \sum_{t \geq 1} \frac{\min(t, y)}{t}\, z_{t-1}^{(y)}\, X^t,$$

which is made so that the coefficients of its derivative are the first factors on the right of (4.2):

$$[X^j]\, \Gamma_y' = \min(j+1, y)\, z_j^{(y)} \qquad (j \geq 0). \tag{4.3}$$

The same two series can be formed for an arbitrary weight, and we need them in this generality in Section 4.4. For real sequences $\rho = (\rho(t))_{t \geq 1}$ and $W = (W_n)_{n \geq 0}$ we put

$$\mathbf{W} = \sum_{n \geq 0} W_n\, X^n, \qquad \Gamma = \sum_{t \geq 1} \frac{\rho(t)}{t}\, W_{t-1}\, X^t,$$

so that $[X^j]\, \Gamma' = \rho(j+1)\, W_j$ for $j \geq 0$. The notation does not show the two sequences from which $\mathbf{W}$ and Γ are formed; they will be named wherever these series are used. For the weight $\rho(t) = \min(t, y)$ and the sequence $W = z^{(y)}$ the series $\mathbf{W}$ and Γ are Z_y and Γ_y.

The next lemma is the last-car recursion in the language of series.

Lemma 4.1. *For every integer $y \geq 1$, the series Z_y of threshold y satisfies*

$$Z_y' = \Gamma_y'\, Z_y.$$

Proof. Let $\rho = (\rho(t))_{t \geq 1}$ and $W = (W_n)_{n \geq 0}$ be real sequences such that W is a solution of the weighted last-car recursion with weight ρ in the sense of Definition 3.2, so that

$$(n+1)\, W_{n+1} = \sum_{j=0}^{n} \rho(j+1)\, W_j\, W_{n-j}$$

for every $n \geq 0$, and let $\mathbf{W}$ and Γ be the series formed from them. Since $[X^j]\, \Gamma' = \rho(j+1)\, W_j$, the right-hand side of the recursion is the coefficient of X^n in $\Gamma'\, \mathbf{W}$, by (4.1), and the left-hand side is the coefficient of X^n in $\mathbf{W}'$. Hence

$$\mathbf{W}' = \Gamma'\, \mathbf{W}. \tag{4.4}$$

By Corollary 3.3 this applies to the sequence $W = z^{(y)}$ and the weight $\rho(t) = \min(t, y)$, for which $\mathbf{W} = Z_y$ and $\Gamma = \Gamma_y$. □

The proof gives more than the lemma states: the equation (4.4) holds whenever W is a solution of the weighted last-car recursion with weight ρ in the sense of Definition 3.2.

Since the constant coefficient of Z_y is 1, the series Z_y has an inverse. The *prime kernel series* of threshold y is

$$K_y = 1 - Z_y^{-1},$$

and the *prime kernel* of threshold y is the sequence of its coefficients, $\kappa_m^{(y)} = [X^m]K_y$ for $m \geq 0$. By definition $Z_y^{-1} = 1 - K_y$, so that

$$Z_y\,(1 - K_y) = 1. \tag{4.5}$$

Since the constant coefficient of Z_y^{-1} is 1, we have $\kappa_0^{(y)} = 0$, and by (4.1), for every series ϕ and every $n \geq 0$,

$$[X^n]\big(\phi\,(1 - K_y)\big) = [X^n]\phi - \sum_{m=1}^{n} \kappa_m^{(y)}\,[X^{n-m}]\phi. \tag{4.6}$$

For $\phi = Z_y$ and $n \geq 1$ the left-hand side is zero by (4.5), and we obtain

$$z_n^{(y)} = \sum_{m=1}^{n} \kappa_m^{(y)}\, z_{n-m}^{(y)} \qquad (n \geq 1). \tag{4.7}$$

An equation of this form, in which a sequence is expressed by its own earlier terms through a convolution, is called a renewal equation, and the sequence $\kappa^{(y)}$ is its kernel. As $z_0^{(y)} = 1$, the equation (4.7) determines $\kappa_n^{(y)}$ from the normalised counts up to index n.

The name prime kernel refers to the prime parking functions, a notion due to Gessel, for which we refer to the survey of Yan [17]. The properties of the kernel that we need follow from the differential equation of Lemma 4.1, which we first carry over to the series K_y.

Corollary 4.2. *For every integer $y \geq 1$, the prime kernel series K_y of threshold y satisfies*

$$K_y' = (1 - K_y)\,\Gamma_y'.$$

Proof. Differentiating (4.5) by the product rule gives $Z_y'\,(1 - K_y) = Z_y\,K_y'$. By Lemma 4.1 and (4.5) the left-hand side is $\Gamma_y'\,Z_y\,(1 - K_y) = \Gamma_y'$. Hence $\Gamma_y' = Z_y\,K_y'$, and multiplying by $1 - K_y$ gives $(1 - K_y)\,\Gamma_y' = K_y'$, again by (4.5). □

4.2. **The series of all parking functions and the free prime kernel.** When the displacement is not bounded, the inverse of the series can be computed. Recall from Section 2 the Cayley coefficients a_j and the numbers $c_n = a_{n+1}$; in particular $a_0 = 0$ and $c_0 = 1$. We put

$$F = \sum_{n \geq 0} c_n\, X^n = 1 + X + \tfrac{3}{2}X^2 + \tfrac{8}{3}X^3 + \cdots.$$

We write $E = 1/F$ for the inverse of F; thus $EF = 1$, and the constant coefficient of E is 1. The *tree series* is $T = XF$. Since $c_n = a_{n+1}$ and $a_0 = 0$, we have $[X^n]\,T = a_n$ for every $n \geq 0$.

The sequence c is the solution of the weighted last-car recursion with the weight $\rho(t) = t$: we have $c_0 = 1$, and the recursion is Lemma 2.5. For this weight and the sequence $W = c$ the series $\mathbf{W}$ is F, and the series Γ is T, because its coefficient of X^t is $c_{t-1} = a_t$ for $t \geq 1$. Thus (4.4) reads

$$F' = T'F. \tag{4.8}$$

The inverse E satisfies a similar equation, which is used in Section 4.4. Differentiating $EF = 1$ gives $E'F = -EF'$, and $EF' = E\,T'F$ by (4.8). Multiplying by E and using $EF = 1$, we obtain

$$E' = -E\,T'. \tag{4.9}$$

Since the constant terms of F and T are 1 and 0, the equation (4.8) says that $F = e^T$, and since $T = XF$ it follows that $T = Xe^T$. The series T is thus the Cayley tree function, the exponential generating series of the rooted labelled trees [8, Section II.5.1].

The *free prime kernel* is the sequence $(\varepsilon_m)_{m\ge0}$ with

$$\varepsilon_0 = 0, \qquad \varepsilon_m = \frac{(m-1)^{m-1}}{m!} \quad (m \ge 1),$$

so that $\varepsilon_1 = 1$; the next values are $\varepsilon_2 = 1/2$, $\varepsilon_3 = 2/3$ and $\varepsilon_4 = 9/8$. Clearly $\varepsilon_m > 0$ for every $m \ge 1$, and

$$m\,\varepsilon_m = \nu_{m-1} \qquad (m \ge 1),$$

where $\nu_n = n^n/n!$ (Section 2). The next lemma shows that the coefficient of X^m in E is $-\varepsilon_m$ for every $m \ge 1$, and it expresses the coefficients of a quotient $\phi/F = E\phi$ through the kernel ε.

Lemma 4.3. *For every formal power series ϕ with real coefficients $\phi_j = [X^j]\phi$ and every integer $n \ge 0$, the inverse E of the series F satisfies*

$$[X^n](E\phi) = \phi_n - \sum_{m=1}^{n} \varepsilon_m\,\phi_{n-m}.$$

Proof. Since $F = e^T$, we have $E = e^{-T}$, and T satisfies the equation $T = Xe^T$. For this equation and a power series $H(u)$ the Lagrange inversion formula [8, Theorem A.2] reads

$$[X^m]\,H(T) = \frac{1}{m}\,[u^{m-1}]\big(H'(u)\,e^{mu}\big) \qquad (m \ge 1).$$

We apply it with $H(u) = e^{-u}$. Then $H(T) = E$ and $H'(u)\,e^{mu} = -e^{(m-1)u}$, and the coefficient of u^{m-1} in $e^{(m-1)u}$ is $(m-1)^{m-1}/(m-1)!$. Hence

$$[X^m]E = -\frac{1}{m}\cdot\frac{(m-1)^{m-1}}{(m-1)!} = -\varepsilon_m$$

for every $m \ge 1$. Since $[X^0]E = 1$, the assertion is the second form of (4.1) for the product $E\phi$. □

For $\phi = F$ the lemma gives a renewal equation for the sequence (c_n), with the kernel ε.

Corollary 4.4. *For every integer $n \ge 1$,*

$$\sum_{m=1}^{n} \varepsilon_m\,c_{n-m} = c_n.$$

Proof. By Lemma 4.3 with $\phi = F$, the difference of the two sides of the asserted equation is the coefficient of X^n in EF. Since $EF = 1$, this coefficient is zero for $n \ge 1$. □

We shall also need a renewal equation for the numbers ν_n, with the Cayley coefficients as the kernel; the term $j = 0$ of its sum is zero, because $a_0 = 0$.

Lemma 4.5. *For every integer $n \geq 1$,*

$$\sum_{j=0}^{n} a_j \, \nu_{n-j} = \nu_n.$$

Proof. For the tree function T, the series $1/(1-T)$ is the exponential generating series of the mappings of a finite set into itself, of which there are i^i on i points [8, Section II.5.2, (48) and (49)]. Thus

$$\frac{1}{1-T} = \sum_{i \geq 0} \nu_i \, X^i.$$

The coefficients of T are the numbers a_j. By (4.1) the difference of the two sides of the asserted equation is therefore the coefficient of X^n in the product of $1-T$ and $1/(1-T)$, which is 1, and this coefficient is zero because $n \geq 1$. □

4.3. **The prime kernel in and beyond the free range.** We return to a threshold $y \geq 1$. In its free range $n \leq y$ the normalised counts are those of all parking functions, $z_n^{(y)} = c_n$ (Proposition 3.4), so that the series Z_y and F agree up to degree y. The same then holds for their inverses.

Lemma 4.6. *Let $y \geq 1$ and m be integers with $1 \leq m \leq y$. Then the prime kernel of threshold y satisfies*

$$\kappa_m^{(y)} = \varepsilon_m.$$

Proof. We argue by induction on m and suppose that $\kappa_i^{(y)} = \varepsilon_i$ for $1 \leq i < m$. In the renewal equation (4.7) for $n = m$ all the indices m and $m - i$ are at most y, so that by Proposition 3.4

$$c_m = \sum_{i=1}^{m} \kappa_i^{(y)} \, c_{m-i}.$$

Subtracting the equation of Corollary 4.4, we obtain

$$\sum_{i=1}^{m} \bigl(\kappa_i^{(y)} - \varepsilon_i\bigr) \, c_{m-i} = 0.$$

The terms with $i < m$ vanish by the induction hypothesis, and the term $i = m$ is $(\kappa_m^{(y)} - \varepsilon_m) \, c_0 = \kappa_m^{(y)} - \varepsilon_m$. □

Beyond the free range, for $y \geq 2$, the kernel satisfies a linear recursion with explicit coefficients. The recursion comes from the fact that the weight $\min(j+1, y)$ of the last-car recursion differs from the constant y only for $j \leq y-2$, where the normalised counts are the free ones.

Lemma 4.7. *Let $y \geq 2$ and $N \geq y$ be integers. Then the prime kernel of threshold y satisfies*

$$N \, \kappa_N^{(y)} = \sum_{\gamma=1}^{y-1} (y - \gamma) \, a_\gamma \, \kappa_{N-\gamma}^{(y)}.$$

Proof. Let C_y be the polynomial

$$C_y = \sum_{j=0}^{y-2} (y-1-j)\, c_j\, X^j.$$

Then $\Gamma'_y = y\, Z_y - C_y$. Indeed, by (4.3) the coefficient of X^j in Γ'_y is $y\, z_j^{(y)}$ for $j \geq y-1$, and for $j \leq y-2$ it is $(j+1)\, c_j = y\, c_j - (y-1-j)\, c_j$ by Proposition 3.4. By Corollary 4.2 and (4.5) we therefore have

$$K'_y = (1-K_y)\,(y\, Z_y - C_y) = y - (1-K_y)\, C_y.$$

We compare the coefficients of X^{N-1}. On the left it is $N\, \kappa_N^{(y)}$. On the right the constant y contributes nothing, because $N-1 \geq 1$, and the coefficient of X^{N-1} in C_y is zero, because $N-1 > y-2$. By (4.6) with $\phi = C_y$ we obtain

$$N\, \kappa_N^{(y)} = \sum_{m=1}^{N-1} \kappa_m^{(y)}\, [X^{N-1-m}]\, C_y.$$

We substitute $\gamma = N-m$. The coefficient of $X^{\gamma-1}$ in C_y is $(y-\gamma)\, c_{\gamma-1} = (y-\gamma)\, a_\gamma$ for $1 \leq \gamma \leq y-1$ and zero for $\gamma \geq y$, and since $N-1 \geq y-1$ this is the assertion. □

For $y = 2$ the recursion is $N\, \kappa_N^{(2)} = \kappa_{N-1}^{(2)}$ for $N \geq 2$.

Let x be a tilt (Section 3). Multiplying the recursion by $x^N = x^\gamma x^{N-\gamma}$, we see that the tilted prime kernel $\kappa_m^{(y)}\, x^m$ satisfies the same recursion with the tilted weights: for $N \geq y \geq 2$,

$$N\, \kappa_N^{(y)}\, x^N = \sum_{\gamma=1}^{y-1} (y-\gamma)\, a_\gamma x^\gamma \cdot \kappa_{N-\gamma}^{(y)}\, x^{N-\gamma}. \tag{4.10}$$

For $x \geq 0$ the tilted weights $(y-\gamma)\, a_\gamma x^\gamma$ are nonnegative. In the free range, Lemma 4.6 gives $\kappa_m^{(y)}\, x^m = \varepsilon_m\, x^m$ for $1 \leq m \leq y$, where $\varepsilon_m\, x^m$ is the tilted free prime kernel.

The coefficients of the inverse of a series have no evident sign. For the prime kernel, tilted or not, the sign follows from the recursion.

Lemma 4.8. *For every integer $y \geq 2$, every real number $x \geq 0$ and every integer $m \geq 0$, the prime kernel of threshold y satisfies*

$$\kappa_m^{(y)}\, x^m \geq 0.$$

Proof. We argue by induction on m. For $m = 0$ we have $\kappa_0^{(y)} = 0$. For $1 \leq m \leq y$ we have $\kappa_m^{(y)}\, x^m = \varepsilon_m\, x^m \geq 0$ by Lemma 4.6. For $m > y$ every term on the right-hand side of (4.10) with $N = m$ is nonnegative, by the induction hypothesis and because the tilted weights are nonnegative. Hence $m\, \kappa_m^{(y)}\, x^m \geq 0$. □

For $x = 1$ the lemma gives $\kappa_m^{(y)} \geq 0$ for every $y \geq 2$ and every $m \geq 0$.

4.4. **A linear relation for the weighted last-car recursion.** The last-car recursion is quadratic. We now show that the solution of a weighted last-car recursion satisfies a linear relation over a range of indices, as soon as its weight agrees on an initial range with the weight $\rho(t) = t$, whose solution is the sequence c. The column equations of Section 4.5 are obtained from this relation.

Let W be a solution of the weighted last-car recursion with weight ρ, let $\mathbf{W}$ and Γ be the series formed from ρ and W, and suppose that $\rho(t) = t$ for $1 \le t \le k$. The proofs below compare the quotient $\mathbf{W}/F$ with the series $T - \Gamma$, whose coefficients are linear in the numbers W_{t-1}, and they determine the coefficients of the quotient up to degree $2k+1$ from the differential equation of the quotient.

The resulting relation has two forms. The first is the relation multiplied by F.

Proposition 4.9. *Let $\rho = (\rho(t))_{t\ge1}$ and $W = (W_n)_{n\ge0}$ be real sequences such that W is a solution of the weighted last-car recursion with weight ρ in the sense of Definition 3.2, and let $k \ge 0$ be an integer such that $\rho(t) = t$ for every t with $1 \le t \le k$. Then for every integer n with $0 \le n \le 2k+1$,*

$$c_n - W_n = \sum_{t=k+1}^{n} c_{n-t} \Big(c_{t-1} - \frac{\rho(t)}{t}\, W_{t-1} \Big).$$

For $n \le k$ the sum is empty and the statement is $W_n = c_n$.

Proof. Let $\mathbf{W}$ and Γ be the series formed from ρ and W as in Section 4.1, so that $\mathbf{W}' = \Gamma' \mathbf{W}$ by (4.4), and put

$$R = T - \Gamma, \qquad Q = E\,\mathbf{W}.$$

We write R_i and Q_i for the coefficients of X^i in R and in Q. The coefficient of X^t in T is $a_t = c_{t-1}$ for $t \ge 1$, and that in Γ is $(\rho(t)/t)\, W_{t-1}$. Hence $R_0 = 0$ and

$$R_t = c_{t-1} - \frac{\rho(t)}{t}\, W_{t-1} \qquad (t \ge 1). \tag{4.11}$$

We first note that $W_n = c_n$ for $0 \le n \le k$. This follows easily by induction on n: we have $W_0 = 1 = c_0$, and as long as $n+1 \le k$ the recursion that gives W_{n+1} and the recursion that gives c_{n+1} (Lemma 2.5) have the same coefficients, because $\rho(j+1) = j+1$ for $j+1 \le n+1 \le k$. Consequently, for $1 \le t \le k$ we have $\rho(t)/t = 1$ and $W_{t-1} = c_{t-1}$, and (4.11) shows that

$$R_t = 0 \qquad (0 \le t \le k). \tag{4.12}$$

Next, by the product rule, (4.9) and (4.4),

$$Q' = E'\,\mathbf{W} + E\,\mathbf{W}' = -E\,T'\,\mathbf{W} + E\,\Gamma'\,\mathbf{W} = -R'\,Q.$$

Clearly $Q_0 = 1$, and the coefficient of X^{i-1} in the last equation is

$$i\,Q_i = -\sum_{j=0}^{i-1} (j+1)\, R_{j+1}\, Q_{i-1-j} \qquad (i \ge 1). \tag{4.13}$$

For $1 \le i \le k$ every R_{j+1} on the right has $j+1 \le i \le k$ and vanishes by (4.12); hence $Q_i = 0$ for $1 \le i \le k$. Now let $1 \le i \le 2k+1$, and consider a term of (4.13) with $j < i-1$. If $j+1 \le k$, then $R_{j+1} = 0$. If $j \ge k$, then $1 \le i-1-j \le k$, and $Q_{i-1-j} = 0$ by what was just shown. So only the term $j = i-1$ remains, and it is $i\,R_i\,Q_0 = i\,R_i$. Altogether,

$$Q_0 = 1 \qquad \text{and} \qquad Q_i = -R_i \quad (1 \le i \le 2k+1). \tag{4.14}$$

Finally, $Q\,F = E\,F\,\mathbf{W} = \mathbf{W}$. Let $n \le 2k+1$. By (4.1), (4.14) and (4.12), the coefficient of X^n in this identity is

$$W_n = Q_0\, c_n + \sum_{t=1}^{n} Q_t\, c_{n-t} = c_n - \sum_{t=k+1}^{n} R_t\, c_{n-t},$$

and with (4.11) this is the assertion. □

Proposition 4.9 gives W_n through one earlier value at a time, whereas the recursion of W gives a value through products of two earlier values. In Section 13 it is applied repeatedly to the normalised counts of the thresholds, which leads to the chain sums of Section 11.

For a threshold $y \geq 1$ the proposition applies to the sequence $W = z^{(y)}$ and the weight $\rho(t) = \min(t, y)$, by Corollary 3.3, with $k = y$. Multiplied by $n!$, it then reads, for $n \leq 2y + 1$,

$$(n+1)^{n-1} - A_{n,y-1} = \sum_{t=y+1}^{n} \binom{n}{t} (n-t+1)^{n-t-1} \left(t^{t-1} - y\, A_{t-1,y-1}\right).$$

The left-hand side is the number of parking functions of length n in which some car has a displacement of at least y, and on the right the cumulative counts occur for smaller lengths only.

The second form is the same relation before the multiplication by F. Here the kernel ε appears, through Lemma 4.3.

Proposition 4.10. *Let $\rho = (\rho(t))_{t\geq 1}$ and $W = (W_n)_{n\geq 0}$ be real sequences such that W is a solution of the weighted last-car recursion with weight ρ in the sense of Definition 3.2, and let $k \geq 0$ be an integer such that $\rho(t) = t$ for every t with $1 \leq t \leq k$. Then for every integer n with $1 \leq n \leq 2k + 1$,*

$$W_n - \sum_{m=1}^{n} \varepsilon_m\, W_{n-m} = \begin{cases} 0, & n \leq k, \\ \dfrac{\rho(n)}{n}\, W_{n-1} - c_{n-1}, & n > k. \end{cases}$$

Proof. The hypotheses are those of Proposition 4.9, and we use the notation of its proof. By Lemma 4.3, applied to the series $\mathbf{W}$, the left-hand side is the coefficient Q_n of X^n in $Q = E\,\mathbf{W}$, and $Q_n = -R_n$ by (4.14). For $n \leq k$ this is zero by (4.12), and for $n > k$ it is $(\rho(n)/n)\, W_{n-1} - c_{n-1}$ by (4.11). □

For the thresholds Proposition 4.10 reads as follows. Let $y \geq 1$ and n be integers with $y < n \leq 2y + 1$. Then

$$z_n^{(y)} - \sum_{m=1}^{n} \varepsilon_m\, z_{n-m}^{(y)} = \frac{y}{n}\, z_{n-1}^{(y)} - c_{n-1}. \tag{4.15}$$

Indeed, by Corollary 3.3 the sequence $z^{(y)}$ is the solution of the weighted last-car recursion with the weight $\min(t, y)$, which is equal to t for $t \leq y$, so that Proposition 4.10 applies with $k = y$. It gives (4.15), because $\min(n, y) = y$.

4.5. **The column equations.** We now fix a column and combine the relations (4.15) of neighbouring thresholds. Recall from Section 3 the normalised count $\beta_n^{(k)} = b_{n,k}/n!$ of column k and the normalised step $\delta_n^{(k)} = \beta_n^{(k)} - \beta_n^{(k-1)} = d_{n,k}/n!$. They are a first and a second difference of the normalised counts in the threshold, the latter being the identity (3.4), which holds for $k \geq 2$, and they vanish in the

free range of the thresholds involved: for all $n \geq 0$,

$$\beta_n^{(k)} = z_n^{(k+1)} - z_n^{(k)} \qquad (k \geq 1), \tag{4.16}$$

$$\beta_n^{(k)} = 0 \qquad (k \geq 1,\ n \leq k), \tag{4.17}$$

$$\delta_n^{(k)} = 0 \qquad (k \geq 2,\ n < k). \tag{4.18}$$

The step of a column $k \geq 2$ in row $n = k$ is given by the next lemma.

Lemma 4.11. *For every integer $k \geq 2$, the normalised step of column k in row k is*

$$\delta_k^{(k)} = -\frac{a_k}{k}.$$

Proof. By Corollary 3.3 the sequence $z^{(k-1)}$ is the solution of the weighted last-car recursion with the weight $\min(t, k-1)$, which is equal to t for $t \leq k-1$. We apply Proposition 4.9 to it, with k replaced by $k-1$ and with $n = k$, which is allowed because $k \leq 2(k-1)+1$. The sum has the single term $t = k$, and since $z_{k-1}^{(k-1)} = c_{k-1}$ by Proposition 3.4, we obtain

$$c_k - z_k^{(k-1)} = c_{k-1} - \frac{k-1}{k}\, c_{k-1} = \frac{c_{k-1}}{k} = \frac{a_k}{k}.$$

On the other hand, $\beta_k^{(k)} = 0$ by (4.17), and $\beta_k^{(k-1)} = z_k^{(k)} - z_k^{(k-1)} = c_k - z_k^{(k-1)}$ by (4.16) and Proposition 3.4. Therefore

$$\delta_k^{(k)} = \beta_k^{(k)} - \beta_k^{(k-1)} = -\bigl(c_k - z_k^{(k-1)}\bigr) = -\frac{a_k}{k}. \qquad \square$$

Multiplied by $k!$, the lemma says that $d_{k,k} = -k^{k-2}$.

In the first layer of the column, from row $k+1$ to row $2k-1$, the normalised step satisfies a linear recursion along the column, the column equation, which starts from the value of Lemma 4.11. Its proof takes the second difference in the threshold of the relations (4.15), whose left-hand sides are linear in the normalised counts with coefficients that do not depend on the threshold.

Proposition 4.12. *Let $k \geq 2$ and n be integers with $k+1 \leq n \leq 2k-1$, and write $\delta_i = \delta_i^{(k)}$ for the normalised step of column k in row i. Then*

$$\delta_n = \frac{k+1}{n}\, \delta_{n-1} + \frac{2}{n}\, \beta_{n-1}^{(k-1)} + \sum_{m=1}^{n-k} \varepsilon_m\, \delta_{n-m}.$$

Proof. The hypothesis $k+1 \leq n \leq 2k-1$ gives $y < n \leq 2y+1$ for the two thresholds $y = k$ and $y = k-1$, and $k-1 \geq 1$; the conditions that bind are $k < n$ for $y = k$ and $n \leq 2(k-1)+1$ for $y = k-1$. For $y = k+1$ it gives $y < n \leq 2y+1$ unless $n = k+1$, and for $n = k+1$ the relation (4.15) holds with $y = k+1$ as well: by Corollary 3.3 and Proposition 4.10, applied to the sequence $z^{(k+1)}$ and the weight $\min(t, k+1)$ with k replaced by $k+1$, its left-hand side is zero, and its right-hand side is $z_k^{(k+1)} - c_k$, which is zero by Proposition 3.4. We take (4.15) for $y = k+1$, subtract twice (4.15) for $y = k$, and add (4.15) for $y = k-1$. The result is

$$\delta_n - \sum_{m=1}^{n} \varepsilon_m\, \delta_{n-m} = \frac{k+1}{n}\, \delta_{n-1} + \frac{2}{n}\, \beta_{n-1}^{(k-1)}. \tag{4.19}$$

Indeed, the left-hand sides combine to the left-hand side of (4.19) by (3.4). On the right the terms c_{n-1} cancel, and what remains is $1/n$ times

$$(k+1)\, z_{n-1}^{(k+1)} - 2k\, z_{n-1}^{(k)} + (k-1)\, z_{n-1}^{(k-1)} = (k+1)\, \delta_{n-1} + 2\, \beta_{n-1}^{(k-1)},$$

where the equality holds by (3.4) and by (4.16) for column $k-1$. Finally, for $m > n-k$ we have $n-m < k$ and $\delta_{n-m} = 0$ by (4.18), so that the sum in (4.19) is the sum over $1 \le m \le n-k$. Moving it to the right gives the assertion. □

Multiplied by $n!$, the column equation is an identity between integers,

$$d_{n,k} = (k+1)\, d_{n-1,k} + 2\, b_{n-1,k-1} + \sum_{m=1}^{n-k} \binom{n}{m} (m-1)^{m-1}\, d_{n-m,k},$$

which can be checked against the rows displayed in Section 1: for $k=3$ and $n=5$ it reads $-124 = 4\cdot(-18) + 2\cdot 34 + 5\cdot(-18) + 10\cdot(-3)$. Beyond the first layer the equation is replaced by the inequality of Proposition 5.3, which has the prime kernel $\kappa^{(k)}$ of threshold k instead of ε. In the first layer the two kernels are evaluated only at indices $m \le n-k \le k-1$, where they agree by Lemma 4.6.

The normalised counts of a column satisfy a column equation of their own, of the same kind as that of the normalised step and on a slightly longer range of rows. It is used in Section 10, where it is applied to column $k-1$, whose normalised counts $\beta_{n-1}^{(k-1)}$ occur in Proposition 4.12.

Lemma 4.13. *Let $k \ge 1$ and n be integers with $k+1 \le n \le 2k+1$. Then*

$$\beta_n^{(k)} = \frac{z_{n-1}^{(k+1)}}{n} + \frac{k}{n}\, \beta_{n-1}^{(k)} + \sum_{m=1}^{n-k-1} \varepsilon_m\, \beta_{n-m}^{(k)}.$$

Proof. We apply (4.15) with $y = k+1$, which is allowed because $k+1 < n \le 2k+3$ when $n > k+1$, and with $y = k$, which is allowed because $k < n \le 2k+1$. For $n = k+1$ the relation holds with $y = k+1$ as well: by Corollary 3.3 and Proposition 4.10, applied to the sequence $z^{(k+1)}$ and the weight $\min(t, k+1)$ with k replaced by $k+1$, its left-hand side is zero, and its right-hand side is $z_k^{(k+1)} - c_k$, which is zero by Proposition 3.4. We subtract the second relation from the first. The terms c_{n-1} cancel, and by (4.16), used on the left for every index and on the right for index $n-1$, we obtain

$$\beta_n^{(k)} - \sum_{m=1}^{n} \varepsilon_m\, \beta_{n-m}^{(k)} = \frac{k+1}{n}\, z_{n-1}^{(k+1)} - \frac{k}{n}\, z_{n-1}^{(k)} = \frac{z_{n-1}^{(k+1)}}{n} + \frac{k}{n}\, \beta_{n-1}^{(k)}.$$

For $m > n-k-1$ we have $n-m \le k$ and $\beta_{n-m}^{(k)} = 0$ by (4.17), so that the sum on the left is the sum over $1 \le m \le n-k-1$. Moving it to the right gives the assertion. □

The equation gives the normalised counts of column k row by row. It starts in row $n = k+1$, where the sum is empty.

5. Two inequalities for a column and the propagation of a positive step

In the first layer of a column the normalised step satisfies the column equation of Proposition 4.12, which is an identity. Here we prove two inequalities for a column $k \geq 2$ that are not restricted to the first layer and that replace this identity beyond it. The first, Proposition 5.2, holds in every row. The second, Proposition 5.3, holds in the rows of the first layer and in all rows beyond it. From Proposition 5.2 we derive the propagation of a positive step down a column (Proposition 5.5), which gives the positive signs beyond the first layer for columns $2 \leq k \leq 81$ in Section 19. The propagation requires a tilt x with $0 < x < 1$ at which the tilted normalised counts of threshold k do not increase. In the last subsection we define the critical tilt of a column and show that at the critical tilt of a column $k \geq 2$ the tilted normalised counts of threshold k do not increase (Proposition 5.9).

5.1. **The tree curvature of a column.** We work with the normalised quantities of Section 3: the normalised counts $z_n^{(y)}$ of a threshold $y \geq 1$, the normalised counts $\beta_n^{(\ell)}$ of a column $\ell \geq 0$, and the normalised step $\delta_n^{(k)}$ of a column $k \geq 1$, which is positive, nonnegative, negative or nonpositive exactly when the integer $d_{n,k}$ is. Recall that the numbers $z_n^{(y)}$ are nonnegative and do not decrease in the threshold y, that $z_0^{(y)} = 1$, and that $z_n^{(y)} = c_n$ in the free range $n \leq y$ of the threshold (Proposition 3.4). Recall also that $\beta_n^{(\ell)} = z_n^{(\ell+1)} - z_n^{(\ell)}$ for a column $\ell \geq 1$ and that, by (3.4), the normalised step $\delta_n^{(k)}$ of a column $k \geq 2$ is the second difference, in the threshold, of the normalised counts of the three thresholds $k+1$, k and $k-1$, and that this step vanishes for $n < k$. The following sequence is formed from the same three thresholds.

Definition 5.1. Let $k \geq 2$ be an integer. The *tree curvature* of column k is the sequence of real numbers $\xi_t^{(k)}$, $t \geq 0$, defined by $\xi_0^{(k)} = 0$ and, for $t \geq 1$,

$$\xi_t^{(k)} = \frac{1}{t}\Big(\min(t, k+1)\, z_{t-1}^{(k+1)} - 2\,\min(t,k)\, z_{t-1}^{(k)} + \min(t, k-1)\, z_{t-1}^{(k-1)}\Big).$$

When column k is fixed, we write ξ_t for $\xi_t^{(k)}$.

In terms of the series of Section 4 the definition says that $\xi_t^{(k)}$ is the coefficient of X^t in $\Gamma_{k+1} - 2\,\Gamma_k + \Gamma_{k-1}$, where

$$\Gamma_y = \sum_{t \geq 1} \frac{\min(t,y)}{t}\, z_{t-1}^{(y)}\, X^t.$$

The series Z_y of the threshold, whose coefficients are the numbers $z_n^{(y)}$, satisfies $Z_y' = \Gamma_y' Z_y$ (Lemma 4.1). By (3.4) the step $\delta_n^{(k)}$ is the coefficient of X^n in $Z_{k+1} - 2Z_k + Z_{k-1}$. Thus the step is the second difference, in the threshold, of the series Z_y, and the tree curvature is the same second difference of the series Γ_y.

Let $k \geq 2$. In the rest of this section we also write δ_n for $\delta_n^{(k)}$ when column k is fixed. Below index k the tree curvature vanishes:

$$\xi_t = 0 \qquad (0 \leq t < k). \tag{5.1}$$

Indeed, for $1 \leq t \leq k-1$ the three terms of the definition cancel: the three minima are equal to t, and the three normalised counts are equal to c_{t-1}, because index

$t-1$ lies in the free range of each of the thresholds $k+1$, k and $k-1$. Beyond index k the minima are the thresholds themselves: for every $n \geq k$,

$$\begin{aligned}(n+1)\,\xi_{n+1} &= (k+1)\,z_n^{(k+1)} - 2k\,z_n^{(k)} + (k-1)\,z_n^{(k-1)} \\ &= (k+1)\,\delta_n + 2\,\beta_n^{(k-1)},\end{aligned} \tag{5.2}$$

where the second equality follows from (3.4) and from $\beta_n^{(k-1)} = z_n^{(k)} - z_n^{(k-1)}$.

The right-hand side of (5.2), with n replaced by $n-1$, is n times the sum of the first two terms on the right of the column equation. It is also the left-hand side of the inequality of Proposition 5.3 below, which holds in every row $n \geq k+1$ and in which the prime kernel of threshold k stands where the column equation has the free prime kernel.

5.2. **Two inequalities between the step and the tree curvature.** The first inequality bounds the step of column k from below by a sum over the tree curvature of the column, in which the weights are normalised counts of threshold k.

Proposition 5.2. *Let $k \geq 2$ and $n \geq 0$ be integers. Then the tree curvature of column k satisfies*

$$\sum_{t=0}^{n} \xi_t^{(k)}\, z_{n-t}^{(k)} \leq \delta_n^{(k)}.$$

The second inequality is the one announced at the end of Section 5.1. From it the positive sign of columns $k \geq 82$ beyond the first layer is derived in Section 19.

Proposition 5.3. *Let $k \geq 2$ and $n \geq k+1$ be integers. Then*

$$(k+1)\,\delta_{n-1}^{(k)} + 2\,\beta_{n-1}^{(k-1)} \leq n\left(\delta_n^{(k)} - \sum_{m=1}^{n-k} \kappa_m^{(k)}\,\delta_{n-m}^{(k)}\right),$$

where $\kappa^{(k)}$ is the prime kernel of threshold k.

The two propositions are proved together, since both are obtained from one relation between power series. Let $k \geq 2$. For a threshold $y \geq 1$ we use the series Z_y, K_y and Γ_y of Section 4. The prime kernel series $K_y = 1 - Z_y^{-1}$ satisfies the relation $Z_y\,(1-K_y) = 1$ of (4.5); the coefficient of X^0 in $1-K_y$ is 1, and the coefficient of X^m is $-\kappa_m^{(y)}$ for $m \geq 1$. The derivative of Γ_y has coefficients $[X^j]\,\Gamma_y' = \min(j+1, y)\,z_j^{(y)}$, and

$$Z_y' = \Gamma_y'\,Z_y, \qquad K_y' = (1-K_y)\,\Gamma_y' \tag{5.3}$$

by Lemma 4.1 and Corollary 4.2. We put

$$\Delta = Z_{k+1} - 2\,Z_k + Z_{k-1}, \qquad \Xi = \Gamma_{k+1} - 2\,\Gamma_k + \Gamma_{k-1},$$

so that the coefficient of X^n in Δ is the step δ_n and the coefficient of X^t in Ξ is the tree curvature ξ_t, as was said in Section 5.1. For formal power series S and S^* with real coefficients we write $S \succeq 0$ if every coefficient of S is nonnegative, and $S \succeq S^*$ if $S - S^* \succeq 0$. Sums and products of series that are $\succeq 0$ are $\succeq 0$. With this notation the relation is $\Delta\,(1-K_k) \succeq \Xi$. The proof establishes it and then reads it at the coefficient of X^n, for Proposition 5.2 after multiplication by Z_k.

To see where this relation comes from, we compare thresholds $k+1$ and $k-1$ with threshold k twice, through the series Z_y and through the series Γ_y. For $y \geq 2$ let $L_y = \Gamma_y - \Gamma_{y-1}$, and put

$$\begin{aligned} L_+ &= L_{k+1}, \qquad & L_- &= L_k, \\ Y_+ &= (Z_{k+1} - Z_k)\,(1 - K_k), \qquad & Y_- &= (Z_k - Z_{k-1})\,(1 - K_k). \end{aligned}$$

Then $\Xi = L_+ - L_-$ and $\Delta\,(1 - K_k) = Y_+ - Y_-$, so that the relation follows from the two comparisons $Y_+ \succeq L_+$ and $L_- \succeq Y_-$. By (4.5), Y_+ and Y_- are the quotients $Z_{k+1}/Z_k - 1$ and $1 - Z_{k-1}/Z_k$. The proof obtains the two comparisons from the differential equations (5.3) and from a principle on series with nonnegative coefficients, with which it begins.

Proof of Propositions 5.2 and 5.3. Let S, S_1 and S_2 be formal power series with real coefficients such that

$$S' = S_1\,S + S_2, \qquad S_1 \succeq 0, \qquad S_2 \succeq 0, \qquad S(0) \geq 0, \tag{5.4}$$

where $S(0)$ denotes the constant term of S. Then $S \succeq 0$. Indeed, the coefficient of X^n in the equation is

$$(n+1)\,[X^{n+1}]\,S = \sum_{j=0}^{n} [X^j]\,S_1 \cdot [X^{n-j}]\,S + [X^n]\,S_2,$$

and the claim follows by induction on the index of the coefficient. For $S_1 = 0$ this principle says that a series with $S' \succeq 0$ and $S(0) \geq 0$ is $\succeq 0$.

Let $y \geq 2$, and let

$$J_y = \sum_{j \geq y-1} z_j^{(y-1)}\, X^j.$$

We show that

$$L_y' = y\,(Z_y - Z_{y-1}) + J_y, \qquad L_y' \succeq 0, \qquad L_y \succeq 0. \tag{5.5}$$

The coefficient of X^j in $L_y' = \Gamma_y' - \Gamma_{y-1}'$ is

$$\min(j+1, y)\, z_j^{(y)} - \min(j+1, y-1)\, z_j^{(y-1)}.$$

For $j \geq y-1$ this is the sum of $y\,(z_j^{(y)} - z_j^{(y-1)})$ and $z_j^{(y-1)}$. For $j \leq y-2$ it is 0, as is the coefficient of X^j on the right-hand side, because both minima are $j+1$ and $z_j^{(y)} = z_j^{(y-1)} = c_j$ in the free range of both thresholds. This proves the identity. The series $Z_y - Z_{y-1}$ and J_y are $\succeq 0$, because the normalised counts are nonnegative and do not decrease in the threshold. Hence $L_y' \succeq 0$, and since L_y has constant term 0, the principle (5.4) with $S_1 = 0$ gives $L_y \succeq 0$. We use (5.5) for $L_+ = L_{k+1}$ and for $L_- = L_k$.

Next we compute the derivatives of Y_+ and Y_-. By (4.5), $Y_+ = Z_{k+1}\,(1-K_k) - 1$ and $Y_- = 1 - Z_{k-1}\,(1 - K_k)$, and both series have constant term 0. By (5.3),

$$\begin{aligned} Y_+' &= Z_{k+1}'\,(1 - K_k) - Z_{k+1}\,K_k' \\ &= Z_{k+1}\,(1 - K_k)\,(\Gamma_{k+1}' - \Gamma_k') = (1 + Y_+)\,L_+', \\ Y_-' &= -Z_{k-1}'\,(1 - K_k) + Z_{k-1}\,K_k' \\ &= Z_{k-1}\,(1 - K_k)\,(\Gamma_k' - \Gamma_{k-1}') = (1 - Y_-)\,L_-'. \end{aligned}$$

The first equation gives

$$(Y_+ - L_+)' = Y_+ L'_+ = L'_+ (Y_+ - L_+) + L_+ L'_+.$$

Here $L'_+ \succeq 0$ and $L_+ L'_+ \succeq 0$ by (5.5), and $Y_+ - L_+$ has constant term 0. By (5.4), $Y_+ \succeq L_+$.

The second equation gives $(L_- - Y_-)' = Y_- L'_-$, so that the comparison $L_- \succeq Y_-$ will follow once we know that $Y_- \succeq 0$. The equation $Y'_- = (1 - Y_-) L'_-$ does not show this as it stands: written as $Y'_- = -L'_- Y_- + L'_-$, it multiplies Y_- by the series $-L'_-$, whose coefficients are not positive, and (5.4) does not apply. We rewrite it. Multiplying $Y_- = (Z_k - Z_{k-1})(1 - K_k)$ by Z_k we get $Z_k - Z_{k-1} = Y_- Z_k$ by (4.5), and therefore

$$(1 - Y_-)(Z_k - Z_{k-1}) = Z_{k-1}(1 - K_k) \cdot Y_- Z_k = Z_{k-1} Y_-.$$

Inserting the identity of (5.5) for $y = k$ into $Y'_- = (1 - Y_-) L'_-$, we obtain

$$Y'_- = k(1 - Y_-)(Z_k - Z_{k-1}) + (1 - Y_-) J_k = (k Z_{k-1} - J_k) Y_- + J_k.$$

In this equation the series that multiplies Y_- is $\succeq 0$: its coefficient of X^j is $k z_j^{(k-1)}$ for $j < k - 1$ and $(k-1) z_j^{(k-1)}$ for $j \geq k - 1$. Since $J_k \succeq 0$ and Y_- has constant term 0, the principle (5.4) gives $Y_- \succeq 0$. Hence $(L_- - Y_-)' = Y_- L'_- \succeq 0$, and as $L_- - Y_-$ has constant term 0, we conclude that $L_- \succeq Y_-$.

Since $\Xi = L_+ - L_-$ and $\Delta(1 - K_k) = Y_+ - Y_-$, the two comparisons $Y_+ \succeq L_+$ and $L_- \succeq Y_-$ give

$$\Delta(1 - K_k) \succeq \Xi. \tag{5.6}$$

For Proposition 5.2, let $n \geq 0$. Multiplying the series $\Delta(1 - K_k) - \Xi \succeq 0$ by $Z_k \succeq 0$ and using (4.5), we find that $\Delta \succeq \Xi Z_k$. The coefficient of X^n in Δ is δ_n, and the coefficient of X^n in ΞZ_k is the left-hand side of the inequality of Proposition 5.2.

For Proposition 5.3, let $n \geq k + 1$. By (5.6) the coefficient of X^n in $\Delta(1 - K_k)$ is at least the coefficient of X^n in Ξ, which is ξ_n. The former is

$$\delta_n - \sum_{m=1}^{n} \kappa_m^{(k)} \delta_{n-m} = \delta_n - \sum_{m=1}^{n-k} \kappa_m^{(k)} \delta_{n-m},$$

where the terms with $m > n - k$ vanish because $\delta_j = 0$ for $j < k$. By (5.2), applied with $n - 1 \geq k$ instead of n, we have $n \xi_n = (k+1) \delta_{n-1} + 2 \beta_{n-1}^{(k-1)}$, and the inequality of Proposition 5.3 follows on multiplying by n. □

Divided by n, the inequality of Proposition 5.3 is a lower bound for the step of row n by quantities of the earlier rows,

$$\delta_n \geq \frac{k+1}{n} \delta_{n-1} + \frac{2}{n} \beta_{n-1}^{(k-1)} + \sum_{m=1}^{n-k} \kappa_m^{(k)} \delta_{n-m}.$$

The term with $\beta_{n-1}^{(k-1)}$ is a nonnegative source, and the other terms feed the earlier steps of the column back with nonnegative coefficients, since $\kappa_m^{(k)} \geq 0$ (Lemma 4.8). On rows $k + 1 \leq n \leq 2k - 1$ of the first layer the kernel of the sum is the free prime kernel: there $m \leq n - k \leq k - 1$, so that $\kappa_m^{(k)} = \varepsilon_m$ by Lemma 4.6.

5.3. **The propagation of a positive step.** We now turn Proposition 5.2 into a statement on the sign of the step. The two rules from which Theorem 1.2 is derived conclude with strict inequalities, as was explained in Section 1, so that we need a lower bound for the step that can be positive. The following lemma gives such a bound at a tilt $x > 0$ at which the tilted normalised counts $z_N^{(k)} x^N$ do not increase in N, under a condition on sums of the numbers $\xi_t x^t$. Its proof is a summation by parts in the inequality of Proposition 5.2, after the weight that the term $t = n$ of the sum carries has been lowered.

Lemma 5.4. *Let $k \geq 2$ and $n \geq k$ be integers and $x > 0$ a real number. Assume that*

(1) $z_{N+1}^{(k)}\, x^{N+1} \leq z_N^{(k)}\, x^N$ *for every integer $N \geq 0$;*
(2) *for every integer s with $k \leq s \leq n$,*

$$\sum_{t=s}^{n} \xi_t^{(k)}\, x^t \geq 0.$$

Then

$$(1-x)\, \xi_n^{(k)} \leq \delta_n^{(k)}.$$

Proof. We multiply the inequality of Proposition 5.2 by x^n. Since the terms with $t < k$ vanish by (5.1), this gives

$$\sum_{t=k}^{n} \xi_t x^t \cdot z_{n-t}^{(k)}\, x^{n-t} \leq \delta_n\, x^n. \tag{5.7}$$

For $k \leq t \leq n$ put $\sigma_t = \xi_t x^t$, and let $\rho_t = z_{n-t}^{(k)}\, x^{n-t}$ for $k \leq t < n$ and $\rho_n = z_1^{(k)}\, x$. Thus ρ_t is the weight of σ_t in (5.7), except for $t = n$, where the weight $z_0^{(k)} = 1$ has been replaced by $z_1^{(k)}\, x$. Since $z_1^{(k)} = c_1 = 1$, we have

$$\sum_{t=k}^{n} \sigma_t\, z_{n-t}^{(k)}\, x^{n-t} = \sum_{t=k}^{n} \sigma_t\, \rho_t + (1-x)\, \xi_n\, x^n. \tag{5.8}$$

The numbers ρ_t are nonnegative, and they do not decrease in t: for $k \leq s < n-1$, hypothesis (1) with $N = n-s-1$ is the inequality $\rho_s \leq \rho_{s+1}$, and if $n > k$, then $\rho_{n-1} = z_1^{(k)}\, x = \rho_n$.

Summation by parts gives

$$\sum_{t=k}^{n} \sigma_t\, \rho_t = \rho_k \sum_{t=k}^{n} \sigma_t + \sum_{s=k+1}^{n} (\rho_s - \rho_{s-1}) \sum_{t=s}^{n} \sigma_t.$$

The right-hand side is nonnegative by hypothesis (2), because $\rho_k \geq 0$ and $\rho_{s-1} \leq \rho_s$. With (5.7) and (5.8) we get $(1-x)\, \xi_n\, x^n \leq \delta_n\, x^n$, and the claim follows. □

Lemma 5.4 makes the step δ_n positive when the tree curvature ξ_n is positive and $x < 1$. In the other direction, (5.2) expresses the next tree curvature ξ_{n+1} by the step δ_n of a row $n \geq k$ and by the number $\beta_n^{(k-1)}$, which is nonnegative. Applied alternately, the two relations turn finitely many inequalities at one row into a sign in infinitely many rows.

Proposition 5.5. *Let $k \geq 2$ and $m \geq k$ be integers and x a real number with $0 < x < 1$. Assume that*

(1) $z_{N+1}^{(k)}\, x^{N+1} \le z_N^{(k)}\, x^N$ *for every integer* $N \ge 0$*;*
(2) *for every integer* s *with* $k \le s \le m+1$*,*

$$\sum_{t=s}^{m+1} \xi_t^{(k)}\, x^t \ge 0;$$

(3) $\delta_m^{(k)} > 0$.

Then $\delta_n^{(k)} > 0$ *for every integer* $n \ge m$.

Proof. We say that the sums are nonnegative at a row $n \ge k$ if hypothesis (2) of Lemma 5.4 holds for this n, and we show by induction on $n \ge m$ that $\delta_n > 0$ and that the sums are nonnegative at row $n+1$. For $n = m$ these are the hypotheses (3) and (2).

Let $n \ge m$, and suppose that $\delta_n > 0$ and that the sums are nonnegative at row $n+1$. Since $n \ge k$ and $\beta_n^{(k-1)} \ge 0$, the right-hand side of (5.2) is positive, and hence $\xi_{n+1} > 0$. We apply Lemma 5.4 at row $n+1$; its hypotheses are hypothesis (1) and the nonnegativity of the sums at row $n+1$. As $1 - x > 0$, it gives

$$\delta_{n+1} \ge (1-x)\, \xi_{n+1} > 0.$$

By (5.2) again, now for row $n+1$, we have $\xi_{n+2} > 0$. Hence the sums are nonnegative at row $n+2$ as well: each of them is one of the sums at row $n+1$, or the empty sum, plus the positive term $\xi_{n+2}\, x^{n+2}$. This completes the induction. □

In Section 19 the proposition is applied to columns $2 \le k \le 81$; hypotheses (2) and (3) are then finitely many inequalities, which Lemmas 6.4 and 6.3 supply.

5.4. **The critical tilt.** It remains to find a tilt $x < 1$ at which the tilted normalised counts $z_N^{(k)} x^N$ of threshold k do not increase in N. In the free range $N \le k$ these are the tilted Cayley masses $c_N x^N$, and the ratio of two consecutive masses is explicit: for every real number x and every integer $j \ge 0$,

$$c_{j+1}\, x^{j+1} = x \left(1 + \frac{1}{j+1}\right)^j c_j\, x^j. \tag{5.9}$$

This follows from $c_j = a_{j+1}$ and from (2.1). By Lemma 2.1 the factor $(1+1/(j+1))^j$ does not decrease in j. Hence, among the factors $(1+1/(j+1))^j$ with $j+1 \le k$, the largest is the factor $(1+1/k)^{k-1}$ of $j = k-1$. We take the tilt x for which $x\,(1+1/k)^{k-1} = 1$.

For an integer $k \ge 1$ the *critical tilt* of column k is

$$x_k = \left(\frac{k}{k+1}\right)^{k-1}.$$

It is the real number determined by

$$x_k \left(1 + \frac{1}{k}\right)^{k-1} = 1. \tag{5.10}$$

Clearly $0 < x_k \le 1$, with $x_1 = 1$. The first values are $x_2 = 2/3$, $x_3 = 9/16$ and $x_4 = 64/125$. By (5.9) for $j = k-1$, the equation (5.10) says that at the critical tilt the tilted Cayley masses of indices $k-1$ and k are equal. When a column k is fixed, we write

$$\omega_n = c_n\, x_k^n \qquad \text{and} \qquad u_N^{(y)} = z_N^{(y)}\, x_k^N$$

for the tilted Cayley masses and for the tilted normalised counts of a threshold $y \geq 1$ at the critical tilt of this column, so that $u_N^{(y)} = \omega_N$ for $N \leq y$ by Proposition 3.4.

The first lemma is the monotonicity of the masses for which the critical tilt was chosen.

Lemma 5.6. *Let k and n be integers with $n \geq 0$ and $n+1 \leq k$. Then the critical tilt x_k of column k satisfies*

$$c_{n+1}\, x_k^{n+1} \leq c_n\, x_k^n.$$

Proof. By (5.9) the left-hand side is the right-hand side multiplied by $x_k\,(1+1/(n+1))^n$, and this factor is at most $x_k\,(1+1/k)^{k-1} = 1$, by Lemma 2.1 applied to the integers $n+1 \leq k$ and by (5.10). □

Since $u_N^{(k)} = \omega_N$ for $N \leq k$, the lemma says for a column $k \geq 2$ that at the critical tilt the tilted normalised counts of threshold k do not increase in their free range. Proposition 5.9 below states that at this tilt they do not increase anywhere. Besides Lemma 5.6, its proof needs one further inequality in the free range, namely

$$x_k \sum_{j=1}^{k} c_j\, x_k^j \leq 1. \tag{5.11}$$

Since $x \cdot c_j\, x^j = a_{j+1}\, x^{j+1}$ for every real number x, the left-hand side can also be written with the Cayley coefficients:

$$x_k \sum_{j=1}^{k} c_j\, x_k^j = \sum_{n=2}^{k+1} a_n\, x_k^n. \tag{5.12}$$

For $k \geq 12$ we prove (5.11) by an estimate (Lemma 5.8); for $2 \leq k \leq 11$ the sum is evaluated, in the proof of Proposition 5.9. The estimate rests on Stirling's formula, which controls the numbers $a_n e^{-n}$, the terms $a_n\, x^n$ at the tilt $x = 1/e$. The next lemma bounds the quotient of the critical tilt by this tilt. It will be used in the form $(e\, x_k)^n \leq e^{3n/(2k)}$, a bound that stays below a fixed number as long as n is at most a fixed multiple of k.

Lemma 5.7. *For every integer $k \geq 1$ the critical tilt x_k satisfies*

$$e\, x_k \leq \exp\Big(\frac{3}{2k}\Big).$$

Proof. For every real number $t \geq 0$ one has $\log(1+t) \geq 2t/(2+t)$, since the two sides agree at $t = 0$ and the derivative of their difference is $t^2/((1+t)(2+t)^2) \geq 0$. For $t = 1/k$ this gives

$$(k-1)\,\log\Big(1+\frac{1}{k}\Big) \geq \frac{2(k-1)}{2k+1} \geq 1 - \frac{3}{2k},$$

where the second inequality is equivalent to $(2k-3)(2k+1) \leq 4k(k-1)$, and the left-hand side of this is $4k(k-1) - 3$. Hence $(1+1/k)^{k-1} \geq \exp(1-3/(2k))$, and the claim follows from (5.10). □

We now prove (5.11) for $k \geq 12$.

Lemma 5.8. *For every integer $k \geq 12$ the critical tilt x_k satisfies*

$$x_k \sum_{j=1}^{k} c_j\, x_k^j \leq 1.$$

Proof. Write $x = x_k$. By (5.12) the left-hand side is the sum of $a_n\, x^n$ over $2 \le n \le k+1$. We first bound a single term. For $n \ge 1$,

$$a_n\, x^n = \frac{n^{n-1}}{n!}\, x^n = \frac{(e\,x)^n}{n}\, P(n) \le \frac{c_{\mathrm{S}}\,(e\,x)^n}{n\sqrt{n}},$$

by the definition of the Poisson mass and by the upper bound of (2.5). Lemma 5.7 gives $(e\,x)^n \le e^{3n/(2k)}$. For $2 \le n \le k+1$ the exponent satisfies $0 \le 3n/(2k) \le 3(k{+}1)/(2k) \le 1.625$, because $k \ge 12$, and the inequality $e^\theta \le 1{+}\theta{+}\theta^2$, which holds for $0 \le \theta \le 1.7$ (Section 2), applies with $\theta = 3n/(2k)$. Hence, for $2 \le n \le k+1$,

$$a_n\, x^n \le c_{\mathrm{S}} \Big(\frac{1}{n\sqrt{n}} + \frac{3}{2k} \cdot \frac{1}{\sqrt{n}} + \frac{9}{4k^2}\, \sqrt{n} \Big). \tag{5.13}$$

Next we sum the three terms over $2 \le n \le k+1$. For the first term we use that for every real number $t \ge 1$,

$$\frac{1}{t\sqrt{t}} \le \int_{t-1/2}^{t+1/2} \frac{d\sigma}{\sigma\sqrt{\sigma}} = \frac{2}{\sqrt{t-1/2}} - \frac{2}{\sqrt{t+1/2}},$$

because the function $\sigma \mapsto \sigma^{-3/2}$ is convex. The sum of the right-hand side over $t = 2, \dots, k+1$ telescopes, so that

$$\sum_{n=2}^{k+1} \frac{1}{n\sqrt{n}} \le \frac{2}{\sqrt{3/2}} - \frac{2}{\sqrt{k+3/2}}.$$

For the other two terms we use the bounds

$$\sum_{i=1}^{m} \frac{1}{\sqrt{i}} \le 2\sqrt{m} - 1 \quad (m \ge 1) \qquad \text{and} \qquad \sum_{i=0}^{m-1} \sqrt{i} \le \frac{2}{3}\, m\sqrt{m} \quad (m \ge 0)$$

of Section 2, the first with $m = k+1$ and the second with $m = k+2$. Leaving out the term $i = 1$ of the first sum and the terms $i = 0$ and $i = 1$ of the second, we get

$$\sum_{n=2}^{k+1} \frac{1}{\sqrt{n}} \le 2\sqrt{k+1} - 2, \qquad \sum_{n=2}^{k+1} \sqrt{n} \le \frac{2}{3}\,(k+2)\sqrt{k+2}.$$

Let $r = \sqrt{k+2}$. Then $\sqrt{k+1} \le r$ and $\sqrt{k+3/2} \le r$, and the last bound, multiplied by $9/(4k^2)$, is $3r^3/(2k^2)$. Summing (5.13) over $2 \le n \le k+1$, we therefore obtain

$$\sum_{n=2}^{k+1} a_n\, x^n \le c_{\mathrm{S}} \Big(\frac{2}{\sqrt{3/2}} - \frac{2}{r} + \frac{3\,(r-1)}{k} + \frac{3\,r^3}{2k^2} \Big). \tag{5.14}$$

We claim that the last three terms in the parenthesis add up to at most 0.87. After multiplication by $2rk^2$, and with $r^2 = k+2$, the claim is equivalent to

$$2.5\,k^2 + 12\,k + 6 \le r\,k\,(0.87\,k + 3).$$

Since $k \ge 12$, we have $r \ge \sqrt{14} \ge 3.74$, so that the right-hand side is at least $3.2538\,k^2 + 11.22\,k$, and the inequality holds because $0.7538\,k^2 \ge 9\,k \ge 0.78\,k + 6$ for $k \ge 12$. Moreover $2/\sqrt{3/2} \le 1.6331$, because $1.5 \cdot 1.6331^2 \ge 4$, and $c_{\mathrm{S}} \le 0.39894229$. The right-hand side of (5.14) is thus at most $0.39894229 \cdot (1.6331 + 0.87) < 1$, which proves the lemma. □

With Lemmas 5.6 and 5.8 we can prove the monotonicity at the critical tilt, which is hypothesis (1) of Proposition 5.5 for $x = x_k$.

Proposition 5.9. *Let $k \geq 2$ be an integer and x_k the critical tilt of column k. Then for every integer $N \geq 0$,*

$$z_{N+1}^{(k)}\, x_k^{N+1} \leq z_N^{(k)}\, x_k^N .$$

Proof. We first prove a statement on real sequences. Let $y \geq 1$ be an integer, $x \geq 0$ a real number, and $(v_N)_{N\geq 0}$ a sequence of real numbers with $v_0 = 1$ and

$$(N+1)\, v_{N+1} = x \sum_{i=0}^{N} \min(i+1, y)\, v_i\, v_{N-i} \qquad (N \geq 0). \tag{5.15}$$

Suppose that

$$v_{N+1} \leq v_N \quad \text{for } 0 \leq N < y, \qquad \text{and} \qquad x \sum_{j=1}^{y} v_j \leq 1. \tag{5.16}$$

We claim that $v_{N+1} \leq v_N$ for every $N \geq 0$.

By (5.15) and induction, $v_N \geq 0$ for every N. Put $\phi_N = v_N - v_{N+1}$. For $N \geq y$ we have

$$(N+1)\, \phi_N = (1-xy)\, v_N + x \sum_{i=0}^{N-1} \min(i+1, y)\, v_i\, \phi_{N-1-i}. \tag{5.17}$$

To see this, write $(N+1)\,\phi_N = v_N + N\, v_N - (N+1)\, v_{N+1}$ and subtract (5.15) from the same equation with N replaced by $N-1$. The term $i = N$ of (5.15) is $xy\, v_N$, because $N+1 > y$ and $v_0 = 1$, and the terms with $i \leq N-1$ of the two equations combine to the sum in (5.17).

We prove that $\phi_N \geq 0$ by induction on N. For $N < y$ this is the first condition in (5.16). Let $N \geq y$, and suppose that $\phi_j \geq 0$ for all $j < N$. Then $v_N \leq v_i$ for every $i \leq N$, and every term of the sum in (5.17) is nonnegative. We keep only the terms with $i \geq N - y$ and index them by $j = N - 1 - i$, so that $0 \leq j \leq y-1$; these terms occur in the sum because $y \leq N$. For them $\min(N-j, y) \geq y - j$, as $N \geq y$, and $v_{N-1-j} \geq v_N$. Hence

$$\sum_{i=0}^{N-1} \min(i+1, y)\, v_i\, \phi_{N-1-i} \geq v_N \sum_{j=0}^{y-1} (y-j)\, \phi_j .$$

The last sum is evaluated by summation by parts: since $\phi_j = v_j - v_{j+1}$ and $v_0 = 1$,

$$\sum_{j=0}^{y-1} (y-j)\, \phi_j = y\, v_0 - \sum_{j=1}^{y} v_j = y - \sum_{j=1}^{y} v_j .$$

We insert this into (5.17), where the terms $xy\, v_N$ cancel, and find that

$$(N+1)\, \phi_N \geq (1-xy)\, v_N + x\, v_N \left(y - \sum_{j=1}^{y} v_j \right) = v_N \left(1 - x \sum_{j=1}^{y} v_j \right) \geq 0$$

by the second condition in (5.16). This proves the claim.

We apply the claim with $y = k$, with $x = x_k$ and with $v_N = u_N^{(k)} = z_N^{(k)}\, x_k^N$. This sequence has $v_0 = z_0^{(k)} = 1$, and it satisfies (5.15) by (3.6). For $N \leq k$ we have $v_N = c_N\, x_k^N$ by Proposition 3.4, so that the first condition in (5.16) is Lemma 5.6 and the second is the inequality (5.11). For $k \geq 12$ this inequality is Lemma 5.8. For $2 \leq k \leq 11$ its left-hand side, in the form (5.12), is a sum of k

rational numbers, and its exact value is smaller than 1 in each of the ten cases. For $k = 2$ it is $x_2^2 + (3/2)\, x_2^3 = 8/9$, and rounded to three decimal places the values for $k = 2, 3, \ldots, 11$ are 0.889, 0.850, 0.830, 0.817, 0.807, 0.799, 0.793, 0.788, 0.783 and 0.779. Hence $u_{N+1}^{(k)} \le u_N^{(k)}$ for every $N \ge 0$, which is the assertion. □

Because of Proposition 5.9 the arguments for the rows beyond the first layer are carried out at the critical tilt. Without the threshold the corresponding sequence behaves in the opposite way beyond the free range: by (5.9) and Lemma 2.1 the numbers $c_N\, x_k^N$ do not decrease from index $k - 1$ on.

6. Exact computations

The signs of the steps in the small columns, which the estimates of the later sections do not reach, are obtained by exact computation with the numbers of parking functions. The results are stated in the five lemmas of this section. They concern the steps $d_{n,k}$ at finitely many positions of the array, all in rows $n \le 300$.

Most of what the lemmas state is the sign of the step at each position of a finite set, as a strict inequality $d_{n,k} < 0$ or $d_{n,k} > 0$. The inequalities are strict because the later sections need them in this form: the column rule and the diagonal rule conclude with strict inequalities, and the diagonal rule starts from a step that may be zero. A step equal to zero therefore has to be excluded at the positions from which the rules are read, and where no argument excludes it, the sign is computed. Lemma 6.4, on columns $2 \le k \le 81$, states facts of other kinds. It says that a positive step in a row of the first layer is followed by a positive step in the next row of the layer, and that finitely many sums formed from the cumulative counts are nonnegative.

6.1. **The integers that are computed.** All five lemmas concern columns $k \ge 2$, and they are proved through the cumulative counts $A_{n,\ell}$. By (3.2) the sign of a step is decided by a comparison between integers: for $k \ge 2$ the step $d_{n,k}$ is positive if and only if $2A_{n,k-1} < A_{n,k} + A_{n,k-2}$, and it is negative if and only if $A_{n,k} + A_{n,k-2} < 2A_{n,k-1}$.

The numbers $A_{n,\ell}$ are computed from the last-car recursion of Proposition 3.1, together with the value $A_{0,\ell} = 1$. For a fixed ℓ the recursion determines $A_{1,\ell}, A_{2,\ell}, \ldots$ one after the other, each from all the numbers before it.

The signs and the sums are obtained by computing the integers concerned and comparing them. We carry out these computations by computer, in exact integer arithmetic. So that the reader can repeat the computations, each proof says which integers are computed, by which recursion, and which comparisons are made. The numbers $n_0(k)$ and $\zeta(k)$ that enter the statements of four of the lemmas are listed in full (Definitions 6.2 and 6.5).

A repetition of the computations can be tested against the following. By Proposition 3.4 we have $A_{n,\ell} = n!\, c_n = (n+1)^{n-1}$ whenever $n \le \ell + 1$, so that for each ℓ the numbers $A_{0,\ell}, \ldots, A_{\ell+1,\ell}$, which the recursion produces first, are known in closed form. The two end values of a row that are recalled in Section 1 say that $A_{n,0} = n!$ for every $n \ge 1$ and that $A_{n,n-1} - A_{n,n-2} = n^{n-2}$ for every $n \ge 2$. Rows $n \le 7$ of the array are displayed in Section 1, and the numbers $b_{n,\ell}$ with $n \le 9$ are tabulated in [5, Table 1]; the cumulative counts are their partial sums.

The first rows show how the recursion runs. The recursion gives $A_{1,\ell} = 1$ and $A_{2,\ell} = 1 + \min(2, \ell + 1)$, so that $A_{2,0} = 2$ and $A_{2,\ell} = 3$ for $\ell \geq 1$, and then

$$A_{3,\ell} = A_{2,\ell} + 2\min(2, \ell + 1) + \min(3, \ell + 1)\, A_{2,\ell}.$$

Besides the results of the five lemmas, a further fact about small columns is obtained by evaluation and not by an estimate; it concerns explicit rational numbers and not the numbers $A_{n,\ell}$. For each of the ten columns $2 \leq k \leq 11$ the proof of Proposition 5.9 evaluates a sum of k rational numbers and compares it with 1. The arithmetic with explicit constants that the estimates of the later sections require is part of their proofs; this includes the expansion of a polynomial of degree 15 in the proof of Proposition 16.1.

6.2. **Negative steps in columns 2 to 9.** The first lemma gives negative signs, and three statements of the later sections take such signs from it. Beyond the first layer the step is positive except at eleven positions, the positions $(2k, k)$ with $1 \leq k \leq 7$ and $(2k + 1, k)$ with $2 \leq k \leq 5$ (Proposition 19.6). We call these eleven positions the *exceptional positions*. The two rules in this region, Corollaries 19.7 and 19.8, need a negative step one row above each exceptional position other than $(2, 1)$ and at the diagonal neighbour of each exceptional position. In the first layer of columns $2 \leq k \leq 8$ the diagonal rule needs the negative sign of column $k + 1$ in rows $k + 2$ to $2k$ (Proposition 23.2). All these positions are among the positions (n, k) of the following table, of which there are $3 + 4 + 5 + 6 + 6 + 7 + 7 + 7 = 45$.

(6.1)

column k	2	3	4	5	6	7	8	9
rows n	3–5	4–7	5–9	6–11	7–12	8–14	9–15	10–16

In this table each column $2 \leq k \leq 8$ has every row of its first layer, and in addition row $2k$ when $k \leq 7$ and row $2k + 1$ when $k \leq 5$. These ten positions beyond the first layer are the exceptional positions other than $(2, 1)$. Column 9 has rows $10 \leq n \leq 16$, which are its first layer without the last row; in row 17, column 9 is positive, by Lemma 6.6 below and $\zeta(9) = 7$. The lemma describes the positions of the table by four conditions.

Lemma 6.1. *Let $k \geq 2$ and $n \geq k + 1$ be integers such that one of the following four conditions holds:*

(1) *$n \leq 2k + 1$ and $k \leq 5$;*
(2) *$n \leq 2k$ and $k \leq 7$;*
(3) *$n \leq 2k - 1$ and $k \leq 8$;*
(4) *$n \leq 2k - 2$ and $k \leq 9$.*

Then $d_{n,k} < 0$.

Proof. By (3.2), the inequality $d_{n,k} < 0$ is equivalent to

$$A_{n,k} + A_{n,k-2} < 2A_{n,k-1}.$$

We compute the integers $A_{n,\ell}$ with $n \leq 16$ and $\ell \leq 9$ from the last-car recursion; they have at most 19 decimal digits. We find that this inequality holds at each of the forty-five positions of (6.1); at the thirteen of them that lie in rows $n \leq 7$ it can also be read from the rows of the array that are displayed in Section 1. Every pair (n, k) as in the lemma is one of these positions, because the last row of column k in (6.1) is $2k + 1$ for $k \leq 5$, at least $2k$ for $k \leq 7$, at least $2k - 1$ for $k \leq 8$, and at least $2k - 2$ for $k \leq 9$. □

6.3. **Columns 2 to 81.** For a column $2 \le k \le 81$ the sign of the step has to be determined in every row $n \ge 2k$ (Proposition 19.6), and these are infinitely many rows. They are divided at a row $n_0(k)$ that is listed for the column. In the rows below $n_0(k)$ the signs are computed (Lemma 6.3). At row $n_0(k)$ a condition is verified under which Proposition 5.5 carries the positive step of row $n_0(k)-1$ to all later rows (Lemma 6.4, part 4). The columns end at 81 because from column 82 on the signs beyond the first layer and the column rule in the first layer are proved by analysis (Propositions 19.5 and 16.4).

Definition 6.2. For $2 \le k \le 81$ the number $n_0(k)$ is given by the following table.

k	2	3	4	5	6	7	8	9	10	11
$n_0(k)$	7	10	13	15	18	20	23	25	28	30
k	12	13	14	15	16	17	18	19	20	21
$n_0(k)$	32	35	37	39	41	44	46	48	50	52
k	22	23	24	25	26	27	28	29	30	31
$n_0(k)$	55	57	59	61	63	66	68	70	72	74
k	32	33	34	35	36	37	38	39	40	41
$n_0(k)$	76	78	80	82	85	87	89	91	93	95
k	42	43	44	45	46	47	48	49	50	51
$n_0(k)$	97	99	101	103	105	107	109	111	114	116
k	52	53	54	55	56	57	58	59	60	61
$n_0(k)$	118	120	122	124	126	128	130	132	134	136
k	62	63	64	65	66	67	68	69	70	71
$n_0(k)$	138	140	142	144	146	148	150	152	154	156
k	72	73	74	75	76	77	78	79	80	81
$n_0(k)$	158	160	162	164	166	168	170	171	173	175

Rows $2k \le n < n_0(k)$, which lie between the first layer and row $n_0(k)$, are called the *junction* of column k. Comparing the entries of the table with $2k$, one finds the following values of the difference $n_0(k) - 2k$.

$n_0(k)-2k$	3	4	5	6	7	8	9
columns k	2	3	4, 5	6, 7	8, 9	10–12	13–16
$n_0(k)-2k$	10	11	12	13	14	13	
columns k	17–21	22–26	27–35	36–49	50–78	79–81	

In particular $2k+3 \le n_0(k) \le 2k+14$, so that a junction has between 3 and 14 rows.

The numbers $n_0(k)$ are given by the table and not by a formula. All that the later sections use about $n_0(k)$ is what Lemma 6.3 and part 4 of Lemma 6.4 below state, and any row for which the two statements hold could be listed instead of $n_0(k)$.

The first lemma on these columns gives the signs in the junctions.

Lemma 6.3. *Let k be an integer with $2 \le k \le 81$, and let $n_0(k)$ be the number of Definition 6.2. Then the following two statements hold.*

(1) *One has $2k+3 \le n_0(k)$.*
(2) *For every integer n with $2k \le n < n_0(k)$, other than $n = 2k$ when $k \le 7$ and $n = 2k+1$ when $k \le 5$, one has $d_{n,k} > 0$.*

Proof. Part 1 is read off from the table of Definition 6.2. For part 2, the inequality $d_{n,k} > 0$ is equivalent, by (3.2), to

$$2A_{n,k-1} < A_{n,k} + A_{n,k-2}. \tag{6.2}$$

The rows of part 2 are rows $2k+2 \le n < n_0(k)$ for $2 \le k \le 5$, rows $2k+1 \le n < n_0(k)$ for $k = 6, 7$, and rows $2k \le n < n_0(k)$ for $8 \le k \le 81$. These are 933 positions, of which 26 lie in columns 2 to 8 and 907 in columns 9 to 81, and their rows range from 6 to 174. We compute the integers $A_{n,\ell}$ with $n \le 174$ and $\ell \le 81$ from the last-car recursion; the largest of them, $A_{174,81}$, has 387 decimal digits. We find that (6.2) holds at each of the 933 positions. □

The rows that part 2 of Lemma 6.3 excepts are those of the ten exceptional positions $(2k, k)$ with $2 \le k \le 7$ and $(2k+1, k)$ with $2 \le k \le 5$, at which the step is negative by Lemma 6.1. Part 1 is recorded for the application of Proposition 5.5, which starts from a positive step in row $n_0(k) - 1$: this row has to be a row of the junction that lies after the excepted rows $2k$ and $2k+1$.

The second lemma on columns $2 \le k \le 81$ concerns a column in the rows from its first layer to row $n_0(k)$. The later sections take from it the two facts about these columns that are not single signs. Part 2 is the column rule for two consecutive rows of the first layer, which is Proposition 16.5 for columns $k \le 81$, and part 4 is the second hypothesis of Proposition 5.5 with $x = x_k$ and $m = n_0(k) - 1$. Parts 1 and 3 are weaker forms of the two parts of Lemma 6.3 and follow from it.

Lemma 6.4. *Let k be an integer with $2 \le k \le 81$, let $n_0 = n_0(k)$ be the number of Definition 6.2, let x_k be the critical tilt of column k, and let $\xi_t = \xi_t^{(k)}$ be the tree curvature of column k (Definition 5.1). Then the following four statements hold.*

1. *One has $2k \le n_0$.*
2. *For every integer n with $k+1 \le n \le 2k-2$, if $d_{n,k} > 0$, then $d_{n+1,k} > 0$.*
3. *For every integer n with $2k \le n$, $17 \le n$ and $n < n_0$, one has $d_{n,k} \ge 0$.*
4. *For every integer s with $k \le s \le n_0$,*

$$\sum_{t=s}^{n_0} \xi_t \, x_k^t \ge 0.$$

Proof. Part 1 holds because $2k+3 \le n_0$ by part 1 of Lemma 6.3. For part 3, let n be an integer with $2k \le n$, $17 \le n$ and $n < n_0$. The rows that part 2 of Lemma 6.3 excepts are row $2k$ of a column $k \le 7$ and row $2k+1$ of a column $k \le 5$, and these are rows $n \le 14$. Since $n \ge 17$, row n is not excepted, and that lemma gives $d_{n,k} > 0$.

For parts 2 and 4 we compute, for the three values $\ell = k, k-1, k-2$, the integers $A_{m,\ell}$ with $0 \le m \le n_0 - 1$ from the last-car recursion. By (3.2), part 2 asserts that for every n with $k+1 \le n \le 2k-2$ at least one of the two inequalities

$$A_{n,k} + A_{n,k-2} \le 2A_{n,k-1}, \qquad 2A_{n+1,k-1} < A_{n+1,k} + A_{n+1,k-2}$$

holds; the first of them says that $d_{n,k} \le 0$, and the second that $d_{n+1,k} > 0$. Rows n and $n+1$ are among the rows computed, since $n+1 \le 2k-1 < n_0$ by part 1. For the eighty columns together these are

$$\sum_{k=2}^{81} (k-2) = 3160$$

pairs (k,n), and we find that for each of them one of the two inequalities holds.

For part 4 we write the tree curvature with the cumulative counts. By Definition 5.1 and since $z_n^{(y)} = A_{n,y-1}/n!$, we have, for $t \geq 1$,

$$\xi_t = \frac{1}{t!}\Big(\min(t,k+1)\,A_{t-1,k} - 2\min(t,k)\,A_{t-1,k-1} + \min(t,k-1)\,A_{t-1,k-2}\Big). \tag{6.3}$$

Let $I_t = t!\,\xi_t$ for $1 \leq t \leq n_0$. This is the integer in the bracket of (6.3). It is formed from the numbers $A_{t-1,\ell}$ of the three values of ℓ, which lie in a row $t-1 \leq n_0 - 1$. For $k \leq s \leq n_0$ let

$$\Sigma_s = \sum_{t=s}^{n_0} \frac{n_0!}{t!}\; k^{(k-1)(t-s)}\,(k+1)^{(k-1)(n_0-t)}\; I_t,$$

which is an integer as well. Since $x_k = k^{k-1}/(k+1)^{k-1}$, one has

$$\Sigma_s = n_0!\,(k+1)^{(k-1)n_0}\,k^{-(k-1)s} \sum_{t=s}^{n_0} \xi_t\, x_k^t,$$

and the factor in front of the sum is positive. Part 4 therefore asserts that $\Sigma_s \geq 0$ for every $k \leq s \leq n_0$. The integers Σ_s are computed for $s = n_0, n_0-1, \ldots, k$ in this order, from $\Sigma_{n_0+1} = 0$ and

$$\Sigma_s = \frac{n_0!}{s!}\,(k+1)^{(k-1)(n_0-s)}\,I_s + k^{k-1}\,\Sigma_{s+1}.$$

For example, let $k = 2$, so that $n_0 = 7$, $k^{k-1} = 2$ and $(k+1)^{k-1} = 3$. The numbers $A_{m,\ell}$ with $1 \leq m \leq 6$ and $\ell \leq 2$ are partial sums of the rows displayed in Section 1, and with them

$$(I_2, I_3, \ldots, I_7) = (-1,\ -1,\ 2,\ 51,\ 710,\ 9819).$$

The recursion then gives

$$(\Sigma_7, \Sigma_6, \ldots, \Sigma_2) = (9819,\ 34548,\ 88374,\ 188088,\ 308136,\ 3912),$$

and these six integers are positive.

For the eighty columns together there are

$$\sum_{k=2}^{81} \big(n_0(k) - k + 1\big) = 4343$$

integers Σ_s. They have up to 14737 decimal digits, a size that is reached by Σ_{81} for $k = 81$. We find that $\Sigma_s \geq 0$ for every $2 \leq k \leq 81$ and every $k \leq s \leq n_0(k)$. □

6.4. **Columns 9 to 264.** By Lemma 6.1 each column $2 \leq k \leq 8$ is negative in every row of its first layer, and by Lemma 6.3 columns $2, 3, \ldots, 8$ are positive in rows 6, 8, 10, 12, 13, 15 and 16, respectively. For each column $9 \leq k \leq 264$, Lemma 6.6 gives a positive step in a row of its first layer, and Lemma 6.7 gives the negative signs of column $k+1$ in the rows from $k+2$ up to that row. These two lemmas end at column 264 because the statements that replace them, Propositions 16.2 and 22.6, are proved from column 265 on.

The positions that these two lemmas concern lie in the rows up to 300 and in the columns up to 265, but each of them lies in the first layer of its column, and for such positions the cumulative counts need not be computed from row 0 on. Let

$|\mathrm{PF}_n|$ denote the number of parking functions of length $n \geq 0$. No displacement of a parking function of length n exceeds $n-1$, so that $A_{n,\ell} = |\mathrm{PF}_n|$ for $\ell \geq n-1$, and Proposition 3.4 gives $|\mathrm{PF}_n| = n!\, c_n = (n+1)^{n-1}$; in particular $|\mathrm{PF}_0| = 1$. For integers $n \geq 0$ and $\ell \geq 0$ let

$$\bar{A}_{n,\ell} = |\mathrm{PF}_n| - A_{n,\ell}$$

be the number of parking functions of length n whose maximum displacement is greater than ℓ. Then $\bar{A}_{n,\ell} = 0$ for $n \leq \ell + 1$, and (3.2) becomes

$$d_{n,k} = 2\bar{A}_{n,k-1} - \bar{A}_{n,k} - \bar{A}_{n,k-2} \tag{6.4}$$

for all integers $k \geq 2$ and $n \geq 0$.

The proofs of the two lemmas below compute the numbers $\bar{A}_{n,\ell}$ by a recursion in n, which the proof of Lemma 6.6 derives from Proposition 4.9.

Definition 6.5. For $9 \leq k \leq 264$ the number $\zeta(k)$ is given by the following table: $\zeta(k)$ is the value ζ in whose range of columns k lies.

ζ	columns k	ζ	columns k	ζ	columns k
7	9	17	67–74	27	163–174
8	10–14	18	75–83	28	175–185
9	15–20	19	84–91	29	186–197
10	21–25	20	92–101	30	198–210
11	26–31	21	102–110	31	211–222
12	32–37	22	111–120	32	223–235
13	38–44	23	121–130	33	236–248
14	45–51	24	131–141	34	249–261
15	52–58	25	142–151	35	262–264
16	59–66	26	152–162		

Like the numbers $n_0(k)$, the numbers $\zeta(k)$ are given by the table and not by a formula. All that the later sections use about $\zeta(k)$ is what the two lemmas below state and, for $82 \leq k \leq 264$, the bound $100\,(\zeta(k)+1) \leq 27\,k$, which the proof of Corollary 16.3 reads from the table; any number for which these hold could be listed instead of $\zeta(k)$.

The next lemma says what the numbers mean: the number $\zeta(k)$ locates a positive step of column k and a negative step of column $k+1$ in two consecutive rows. The negative step, which is part 3 of the lemma, is the case $n = k + \zeta(k)$ of Lemma 6.7 below; what the later sections use is the positive step of part 2.

Lemma 6.6. *Let k be an integer with $9 \leq k \leq 264$, and let $\zeta = \zeta(k)$ be the number of Definition 6.5. Then the following three statements hold.*

(1) *One has $\zeta + 2 \leq k$.*
(2) *One has $d_{k+\zeta+1,\,k} > 0$.*
(3) *One has $d_{k+\zeta,\,k+1} < 0$.*

Proof. Part 1 is read off from the table of Definition 6.5: in each of its lines the value of ζ plus two is at most the first column of the range.

Let $\ell \geq 0$ and put $y = \ell + 1$, so that $A_{n,\ell} = n!\, z_n^{(y)}$. By Corollary 3.3 the sequence $z^{(y)}$ is a solution of the weighted last-car recursion with the weight $\rho(t) = \min(t, y)$, and this weight is t for $1 \leq t \leq y$ and y for $t \geq y+1$. Proposition 4.9 therefore

applies to $z^{(y)}$ and gives, for every $n \le 2y+1$,

$$c_n - z_n^{(y)} = \sum_{t=y+1}^{n} c_{n-t}\Big(c_{t-1} - \frac{y}{t}\, z_{t-1}^{(y)}\Big).$$

We multiply this identity by $n!$. The left-hand side becomes $\bar{A}_{n,\ell}$. Since $m!\, c_m = |\mathrm{PF}_m|$ for every $m \ge 0$ and $(t-1)!\, z_{t-1}^{(y)} = A_{t-1,\ell} = |\mathrm{PF}_{t-1}| - \bar{A}_{t-1,\ell}$, and since $n!/((n-t)!\,(t-1)!) = t\binom{n}{t}$, a direct computation gives, for all integers $\ell \ge 0$ and $0 \le n \le 2y+1$, where $y = \ell + 1$,

$$\bar{A}_{n,\ell} = \sum_{t=y+1}^{n} \binom{n}{t} |\mathrm{PF}_{n-t}| \Big((t-y)\,|\mathrm{PF}_{t-1}| + y\,\bar{A}_{t-1,\ell}\Big). \tag{6.5}$$

The right-hand side of (6.5) contains the numbers $\bar{A}_{m,\ell}$ with $y \le m \le n-1$ and no others.

By part 1, $k+\zeta+1 \le 2k-1$, so that rows $n = k+\zeta$ and $n = k+\zeta+1$ satisfy $n \le 2\ell+3$ for each of the four values $\ell = k-2, k-1, k, k+1$, and (6.5) holds in these rows and in the rows before them. For these four values of ℓ we compute the integers $\bar{A}_{m,\ell}$ with $\ell+1 \le m \le k+\zeta+1$ from (6.5), starting from $\bar{A}_{\ell+1,\ell} = 0$. Their rows are rows $m \le 300$, so that each of them is smaller than $|\mathrm{PF}_{300}| = 301^{299}$, a number of 742 decimal digits. We find that for every $9 \le k \le 264$

$$\bar{A}_{k+\zeta+1,\,k} + \bar{A}_{k+\zeta+1,\,k-2} < 2\,\bar{A}_{k+\zeta+1,\,k-1},$$
$$2\,\bar{A}_{k+\zeta,\,k} < \bar{A}_{k+\zeta,\,k+1} + \bar{A}_{k+\zeta,\,k-1}.$$

These are two comparisons for each of the 256 columns. By (6.4) for column k in row $k+\zeta+1$, the first inequality is part 2. By (6.4) for column $k+1$ in row $k+\zeta$, the second inequality is part 3. □

Since every value of ζ is at least 7, part 1 of Lemma 6.6 places row $k+\zeta(k)+1$ in the first layer of column k and row $k+\zeta(k)$ in the first layer of column $k+1$. Proposition 23.2 uses the positive row of part 2 together with Corollary 16.6. For $82 \le k \le 264$ it is the positive row that Corollary 16.3 supplies, from which the arguments for columns $k \ge 82$ start.

The last lemma serves the diagonal rule in the first layer of columns $9 \le k \le 264$ (Proposition 23.2): for a step of the first layer of column k in a row $n \le k+\zeta(k)$ that is not positive, the rule asks for a negative step of column $k+1$ in row $n+1$, which is one of the rows $k+2, \dots, k+\zeta(k)+1$. The sign is computed in each of these rows: from the negative step in row $k+\zeta(k)$, Corollary 16.6, applied to column $k+1$, shows only that the steps in the earlier rows of the first layer are not positive.

Lemma 6.7. *Let k be an integer with $9 \le k \le 264$, and let $\zeta(k)$ be the number of Definition 6.5. Then $d_{n,k+1} < 0$ for every integer n with $k+2 \le n \le k+\zeta(k)+1$.*

Proof. By (6.4) for column $k+1$, the inequality $d_{n,k+1} < 0$ is equivalent to

$$2\,\bar{A}_{n,k} < \bar{A}_{n,k+1} + \bar{A}_{n,k-1}.$$

The lemma asserts this in the $\zeta(k)$ rows $k+2 \le n \le k+\zeta(k)+1$ of column $k+1$, for every $9 \le k \le 264$. These are

$$\sum_{k=9}^{264} \zeta(k) = 5934$$

positions, all of them in rows $n \le 300$. Row $n = k + \zeta(k)$ is one of these rows of column $k+1$, because $\zeta(k) \ge 7$, and in this row the claim is part 3 of Lemma 6.6. This settles 256 of the positions, one in each column, and 5678 remain. By part 1 of Lemma 6.6, every row $n \le k + \zeta(k) + 1$ satisfies $n \le 2k - 1$, hence $n \le 2\ell + 3$ for the three values $\ell = k-1, k, k+1$, so that (6.5), which was shown in the proof of Lemma 6.6, holds in the rows concerned. For these three values of ℓ we compute the integers $\bar{A}_{m,\ell}$ with $\ell + 1 \le m \le k + \zeta(k) + 1$ from (6.5); as in the proof of Lemma 6.6, they are smaller than 301^{299}. We find that the inequality holds at each of the 5678 remaining positions. □

7. Monotonicity along the diagonal and the diagonal lower bound

The arguments of Sections 8, 10 and 19 compare the entry of the array at a position with the entry at its diagonal neighbour. The comparison that they use is Proposition 7.2: for integers $k \ge 2$ and $n \ge 1$ the normalised counts satisfy $\gamma_k \, \beta_{n-1}^{(k-1)} \le \beta_n^{(k)}$ with the factor $\gamma_k = (1+1/k)^{k-2}$. We derive it from a monotonicity along the diagonal for the cumulative counts (Proposition 7.1).

7.1. **Monotonicity along the diagonal.** Let $y \ge 1$ and $n \ge 0$ be integers. Since there are $(n+1)^{n-1}$ parking functions of length n, the quotient

$$\pi_n^{(y)} = \frac{z_n^{(y)}}{c_n} = \frac{A_{n,y-1}}{(n+1)^{n-1}}$$

is the probability that the maximum displacement of a parking function of length n, chosen uniformly at random, is smaller than y. By Section 3, the number $z_n^{(y)}$ is nonnegative and does not decrease in the threshold y, and $z_n^{(y)} = c_n$ in the free range $n \le y$ (Proposition 3.4). Hence $\pi_n^{(y)}$ is nonnegative, it does not decrease in y, and $\pi_n^{(y)} = 1$ for $n \le y$.

The last-car recursion expresses these probabilities as averages. For integers $0 \le j \le n$ put

$$\theta_n(j) = \frac{(j+1)\, c_j \, c_{n-j}}{(n+1)\, c_{n+1}}, \qquad \psi_n^{(y)}(j) = \frac{\min(j+1, y)}{j+1} \, \pi_j^{(y)} \, \pi_{n-j}^{(y)}.$$

With these numbers the recursion of Corollary 3.3, divided by $(n+1)\, c_{n+1}$, reads

$$\pi_{n+1}^{(y)} = \sum_{j=0}^{n} \theta_n(j) \, \psi_n^{(y)}(j). \tag{7.1}$$

The numbers $\theta_n(0), \dots, \theta_n(n)$ are positive, and by Lemma 2.5 they add up to one:

$$\sum_{j=0}^{n} \theta_n(j) = 1. \tag{7.2}$$

In terms of the parking process, the term of index j of the last-car recursion (Proposition 3.1) belongs to the parking functions in which the last car parks in spot $j+1$.

The following proposition states that the probabilities $\pi_n^{(y)}$ do not decrease along the diagonals of their array, $\pi_n^{(y)} \le \pi_{n+1}^{(y+1)}$. With n replaced by $n+1$, the two sides of this inequality are averages of the form (7.1) in which both the weights and the values differ. The proof compares the two averages through their weights and their values separately: it shows that the weights θ_{n+1} arise from the weights θ_n by moving, for every j, a part of the mass at the point j to the point $j+1$, and

that the value $\psi_n^{(y)}(j)$ is at most each of the two values $\psi_{n+1}^{(y+1)}(j)$ and $\psi_{n+1}^{(y+1)}(j+1)$ once the proposition is known for the shorter lengths j and $n-j$.

Proposition 7.1. *Let $y \geq 1$ and $n \geq 0$ be integers. Then the normalised counts of the thresholds y and $y+1$ satisfy*

$$\frac{z_n^{(y)}}{c_n} \leq \frac{z_{n+1}^{(y+1)}}{c_{n+1}}.$$

Proof. In the notation introduced above the claim is $\pi_n^{(y)} \leq \pi_{n+1}^{(y+1)}$, and we prove it for all $y \geq 1$ by induction on n. For $n = 0$ both sides are equal to 1, because the lengths 0 and 1 lie in the free ranges of thresholds y and $y+1$. Let $n \geq 0$, suppose that $\pi_i^{(y)} \leq \pi_{i+1}^{(y+1)}$ holds for all integers $0 \leq i \leq n$ and $y \geq 1$, and fix $y \geq 1$. In the induction step we write

$$\theta = \theta_n, \qquad \theta^+ = \theta_{n+1}, \qquad \psi = \psi_n^{(y)}, \qquad \psi^+ = \psi_{n+1}^{(y+1)}.$$

By (7.1), the inequality $\pi_{n+1}^{(y)} \leq \pi_{n+2}^{(y+1)}$ that is to be proved reads

$$\sum_{j=0}^{n} \theta(j)\,\psi(j) \leq \sum_{j=0}^{n+1} \theta^+(j)\,\psi^+(j). \tag{7.3}$$

We first compare the values: for $0 \leq j \leq n$,

$$\psi(j) \leq \psi^+(j) \qquad \text{and} \qquad \psi(j) \leq \psi^+(j+1). \tag{7.4}$$

By definition,

$$\psi(j) = \frac{\min(j+1, y)}{j+1}\,\pi_j^{(y)}\,\pi_{n-j}^{(y)},$$
$$\psi^+(j) = \frac{\min(j+1, y+1)}{j+1}\,\pi_j^{(y+1)}\,\pi_{n+1-j}^{(y+1)},$$
$$\psi^+(j+1) = \frac{\min(j+2, y+1)}{j+2}\,\pi_{j+1}^{(y+1)}\,\pi_{n-j}^{(y+1)}.$$

Each of these values is a product of three nonnegative factors, which we compare one by one. For the first inequality of (7.4), the first factor does not decrease when y is replaced by $y+1$; then $\pi_j^{(y)} \leq \pi_j^{(y+1)}$ by the monotonicity in the threshold; and $\pi_{n-j}^{(y)} \leq \pi_{n-j+1}^{(y+1)}$ by the induction hypothesis for the length $n-j$. For the second inequality, the first factor does not decrease because $\min(j+2, y+1) = \min(j+1, y) + 1$ and $\min(j+1, y) \leq j+1$; then $\pi_j^{(y)} \leq \pi_{j+1}^{(y+1)}$ by the induction hypothesis for the length j; and $\pi_{n-j}^{(y)} \leq \pi_{n-j}^{(y+1)}$ by the monotonicity in the threshold.

Next we compare the weights: for $0 \leq j \leq n$,

$$\theta^+(j) \leq \theta(j) \qquad \text{and} \qquad \theta^+(j+1) \leq \theta(j). \tag{7.5}$$

Cancelling the factor $(j+1)\,c_j$ and clearing the denominators, we see that the first inequality reads

$$(n+1)\,c_{n+1-j}\,c_{n+1} \leq (n+2)\,c_{n-j}\,c_{n+2}.$$

It holds because $c_{n+1-j}\,c_{n+1} \leq c_{n-j}\,c_{n+2}$; since $c_i = a_{i+1}$, this is Lemma 2.2 for the two indices $n-j+1 \leq n+2$ and the shift $m = 1$. Similarly, the second inequality reads

$$\frac{(j+2)\,c_{j+1}}{(j+1)\,c_j} \leq \frac{(n+2)\,c_{n+2}}{(n+1)\,c_{n+1}}.$$

By (2.1) we have $c_i = (1+1/i)^{i-1}\, c_{i-1}$ for $i \ge 1$. The left-hand side is therefore the first term, and the right-hand side is the last term, of the chain

$$\Big(1+\frac{1}{j+1}\Big)^{j+1} \le \Big(1+\frac{1}{n+1}\Big)^{n+1} = \frac{n+2}{n+1}\Big(1+\frac{1}{n+1}\Big)^{n} \le \frac{n+2}{n+1}\Big(1+\frac{1}{n+2}\Big)^{n+1}.$$

Here the first step holds because $(1+1/i)^i$ does not decrease in $i \ge 1$ (Section 2) and $j \le n$, and the last step is Lemma 2.1.

It remains to derive (7.3) from (7.2), (7.4) and (7.5). Let

$$\sigma_t = \sum_{j=0}^{t-1} \theta(j) \qquad \text{and} \qquad \sigma_t^+ = \sum_{j=0}^{t-1} \theta^+(j)$$

be the partial sums of the two families of weights, the first for $0 \le t \le n+1$ and the second for $0 \le t \le n+2$, so that $\sigma_{n+1} = \sigma_{n+2}^+ = 1$ by (7.2). They interlace:

$$\sigma_t^+ \le \sigma_t \le \sigma_{t+1}^+ \qquad (0 \le t \le n+1). \tag{7.6}$$

The first inequality follows term by term from the first inequality of (7.5). The second inequality of (7.5) gives

$$1-\sigma_t = \sum_{j=t}^{n} \theta(j) \ge \sum_{j=t}^{n} \theta^+(j+1) = 1-\sigma_{t+1}^+,$$

which is the second inequality of (7.6). We now show by induction on t that, for $0 \le t \le n+1$,

$$\sum_{j=0}^{t-1} \theta(j)\,\psi(j) \le \sum_{j=0}^{t-1} \theta^+(j)\,\psi^+(j) + \big(\sigma_t - \sigma_t^+\big)\,\psi^+(t). \tag{7.7}$$

This is trivial for $t=0$. For the step from t to $t+1 \le n+1$ we split the weight $\theta(t) = \sigma_{t+1} - \sigma_t$ into the two parts $\sigma_{t+1}^+ - \sigma_t$ and $\sigma_{t+1} - \sigma_{t+1}^+$, which are nonnegative by (7.6), and we use the first inequality of (7.4) on the first part and the second inequality on the second part:

$$\theta(t)\,\psi(t) \le \big(\sigma_{t+1}^+ - \sigma_t\big)\,\psi^+(t) + \big(\sigma_{t+1} - \sigma_{t+1}^+\big)\,\psi^+(t+1).$$

When this is added to (7.7), the terms with $\psi^+(t)$ combine to $(\sigma_{t+1}^+ - \sigma_t^+)\,\psi^+(t) = \theta^+(t)\,\psi^+(t)$, and the result is (7.7) for $t+1$. For $t = n+1$ we have $\sigma_{n+1} = \sigma_{n+2}^+$, and the last term of (7.7) is

$$\big(\sigma_{n+2}^+ - \sigma_{n+1}^+\big)\,\psi^+(n+1) = \theta^+(n+1)\,\psi^+(n+1).$$

So (7.7) for $t = n+1$ is (7.3), which completes the induction step. □

7.2. **A lower bound along the diagonal.** For an integer $k \ge 2$ the *diagonal factor* is

$$\gamma_k = \Big(1+\frac{1}{k}\Big)^{k-2}.$$

We have $\gamma_2 = 1$ and

$$\Big(1+\frac{1}{k}\Big)^{k-1} = \gamma_k\Big(1+\frac{1}{k}\Big), \tag{7.8}$$

where the left-hand side is the quotient $a_{k+1}/a_k = c_k/c_{k-1}$ of consecutive Cayley coefficients, by (2.1). The diagonal factor does not decrease in k: for integers $2 \le i \le j$, Lemma 2.1 and (7.8) give

$$\gamma_i \left(1 + \frac{1}{i}\right) \le \gamma_j \left(1 + \frac{1}{j}\right) \le \gamma_j \left(1 + \frac{1}{i}\right),$$

and hence $\gamma_i \le \gamma_j$.

The following proposition is a statement about the differences $\beta_n^{(k)} = z_n^{(k+1)} - z_n^{(k)}$ and $\beta_{n-1}^{(k-1)} = z_{n-1}^{(k)} - z_{n-1}^{(k-1)}$ of the cumulative counts. Its proof is an induction on n that uses Proposition 7.1 together with the two comparisons of power series that are established in the proof of Propositions 5.2 and 5.3.

Proposition 7.2. *Let $k \ge 2$ and $n \ge 1$ be integers. Then the normalised counts of columns $k-1$ and k satisfy*

$$\gamma_k \, \beta_{n-1}^{(k-1)} \le \beta_n^{(k)},$$

where γ_k is the diagonal factor.

Proof. Fix $k \ge 2$, and let

$$\mathfrak{D} = (Z_{k+1} - Z_k) - \gamma_k \, X \, (Z_k - Z_{k-1}),$$

where Z_y is the series of threshold y (Section 4). We write $\mathfrak{D}_n$ for the coefficient of X^n in $\mathfrak{D}$. Since $z_0^{(y)} = 1$ for every threshold $y \ge 1$ and $\beta_n^{(\ell)} = z_n^{(\ell+1)} - z_n^{(\ell)}$ for $\ell \ge 1$ (Section 3), we have $\mathfrak{D}_0 = 0$ and

$$\mathfrak{D}_n = \beta_n^{(k)} - \gamma_k \, \beta_{n-1}^{(k-1)} \qquad (n \ge 1),$$

so that the proposition asserts that $\mathfrak{D}_n \ge 0$ for every $n \ge 1$.

As in Section 5, for formal power series S and S^* with real coefficients we write $S \succeq S^*$ if every coefficient of $S - S^*$ is nonnegative, and we put

$$L_+ = \Gamma_{k+1} - \Gamma_k, \qquad L_- = \Gamma_k - \Gamma_{k-1},$$
$$Y_+ = (Z_{k+1} - Z_k)\,(1 - K_k), \qquad Y_- = (Z_k - Z_{k-1})\,(1 - K_k),$$

with the prime kernel series K_k and the series Γ_y of Section 4. These are the four series that the proof of Propositions 5.2 and 5.3 compares. That proof establishes the two comparisons

$$Y_+ \succeq L_+ \qquad \text{and} \qquad L_- \succeq Y_- \tag{7.9}$$

as relations between power series, with no hypothesis other than $k \ge 2$.

By the definitions of $\mathfrak{D}$, Y_+ and Y_- we have $\mathfrak{D}\,(1 - K_k) = Y_+ - \gamma_k \, X \, Y_-$. Let

$$\mathfrak{L} = L_+ - \gamma_k \, X \, L_-,$$

and write $\mathfrak{L}_n$ for the coefficient of X^n in $\mathfrak{L}$. Then

$$\mathfrak{D}\,(1 - K_k) - \mathfrak{L} = (Y_+ - L_+) + \gamma_k \, X \, (L_- - Y_-),$$

and every coefficient of the right-hand side is nonnegative, by (7.9) and because $\gamma_k > 0$. Hence

$$\mathfrak{D}\,(1 - K_k) \succeq \mathfrak{L}.$$

The coefficients of $1 - K_k$ are 1 at X^0 and $-\kappa_m^{(k)}$ at X^m for $m \ge 1$ (Section 4). Read at the coefficient of X^n, the last relation is therefore

$$\mathfrak{D}_n \ge \sum_{m=1}^{n} \kappa_m^{(k)} \, \mathfrak{D}_{n-m} + \mathfrak{L}_n \qquad (n \ge 0). \tag{7.10}$$

Next we show that

$$(i+1)\,\mathfrak{L}_{i+1} \ge k\,\mathfrak{D}_i \qquad \text{for every integer } i \ge k. \tag{7.11}$$

By the definition of Γ_y, for $t \ge y$ the coefficient of X^t in Γ_y is $y\,z_{t-1}^{(y)}/t$. Since $i \ge k$, this applies to the four coefficients that occur in $\mathfrak{L}_{i+1}$, and

$$\begin{aligned}(i+1)\,\mathfrak{L}_{i+1} &= (k+1)\,z_i^{(k+1)} - k\,z_i^{(k)} - \gamma_k\,\frac{i+1}{i}\,\bigl(k\,z_{i-1}^{(k)} - (k-1)\,z_{i-1}^{(k-1)}\bigr),\\ k\,\mathfrak{D}_i &= k\,\bigl(z_i^{(k+1)} - z_i^{(k)}\bigr) - k\,\gamma_k\,\bigl(z_{i-1}^{(k)} - z_{i-1}^{(k-1)}\bigr).\end{aligned}$$

Subtracting and collecting the terms, we find

$$\begin{aligned}(i+1)\,\mathfrak{L}_{i+1} - k\,\mathfrak{D}_i = &\Bigl(z_i^{(k+1)} - \gamma_k\,\frac{i+1}{i}\,z_{i-1}^{(k)}\Bigr)\\ &+ \gamma_k\,\frac{i+1-k}{i}\,\bigl(z_{i-1}^{(k)} - z_{i-1}^{(k-1)}\bigr).\end{aligned}$$

The second term on the right is nonnegative, because $i+1-k > 0$ and $z_{i-1}^{(k-1)} \le z_{i-1}^{(k)}$. By (2.1) and (7.8) we have $c_n = \gamma_n\,(1+1/n)\,c_{n-1}$ for $n \ge 2$, so that Proposition 7.1, applied with n replaced by $n-1$, says that

$$\gamma_n\,\Bigl(1+\frac{1}{n}\Bigr)\,z_{n-1}^{(y)} \le z_n^{(y+1)} \qquad (y \ge 1,\ n \ge 2). \tag{7.12}$$

The first term is nonnegative by (7.12) with $y = k$ and $n = i$, because $\gamma_i \ge \gamma_k$ and $z_{i-1}^{(k)} \ge 0$:

$$z_i^{(k+1)} \ge \gamma_i\,\frac{i+1}{i}\,z_{i-1}^{(k)} \ge \gamma_k\,\frac{i+1}{i}\,z_{i-1}^{(k)}.$$

This proves (7.11).

We can now prove by induction on n that $\mathfrak{D}_n \ge 0$ for every $n \ge 0$. For $n \le k$ we have $\mathfrak{D}_n = 0$: this was noted for $n = 0$, and for $1 \le n \le k$ it holds because $z_n^{(k+1)} = z_n^{(k)}$ and $z_{n-1}^{(k)} = z_{n-1}^{(k-1)}$ by Proposition 3.4. Let $n = i+1$ with $i \ge k$, and suppose that $\mathfrak{D}_j \ge 0$ for all $j \le i$. On the right-hand side of (7.10) the sum is nonnegative, because $\kappa_m^{(k)} \ge 0$ by Lemma 4.8 with $x = 1$ and $\mathfrak{D}_{i+1-m} \ge 0$ by the induction hypothesis. Moreover $\mathfrak{L}_{i+1} \ge k\,\mathfrak{D}_i/(i+1) \ge 0$ by (7.11) and the induction hypothesis. Hence $\mathfrak{D}_{i+1} \ge 0$. □

For $n \le k$ both sides of the inequality of Proposition 7.2 are zero, because $\beta_m^{(\ell)} = 0$ for $m \le \ell$ and $\ell \ge 1$ (Section 3).

Section 19 needs Proposition 7.2 at the critical tilt. Recall from Section 5 that the critical tilt of a column $k \ge 1$ is $x_k = (k/(k+1))^{k-1}$; it is positive and satisfies $x_k\,(1+1/k)^{k-1} = 1$. For a fixed column $k \ge 2$, in terms of the tilted normalised counts $u_N^{(y)} = z_N^{(y)}\,x_k^N$ of Section 5, we define the *tilted column counts*

$$U_N = u_N^{(k)} - u_N^{(k-1)}, \qquad U_N^+ = u_N^{(k+1)} - u_N^{(k)} \qquad (N \ge 0).$$

Since $\beta_N^{(\ell)} = z_N^{(\ell+1)} - z_N^{(\ell)}$ for $\ell \ge 1$, they are the normalised counts of columns $k-1$ and k, both tilted by the critical tilt of column k:

$$U_N = \beta_N^{(k-1)}\,x_k^N, \qquad U_N^+ = \beta_N^{(k)}\,x_k^N, \tag{7.13}$$

and their difference $U_N^+ - U_N = \delta_N^{(k)}\,x_k^N$ is the normalised step of the column tilted by x_k, for which we write η_N. Inserting (7.8) into the equation $x_k\,(1+1/k)^{k-1} = 1$

and multiplying by k, we obtain

$$(k+1)\,x_k\,\gamma_k = k. \tag{7.14}$$

Corollary 7.3. *Let $k \geq 2$ and $n \geq 1$ be integers. Then the tilted column counts U_N and U_N^+ of column k satisfy*

$$k\,U_{n-1} \leq (k+1)\,U_n^+.$$

Proof. By (7.13), (7.14) and Proposition 7.2,

$$\begin{aligned} k\,U_{n-1} = k\,x_k^{\,n-1}\,\beta_{n-1}^{(k-1)} &= (k+1)\,x_k^{\,n}\,\gamma_k\,\beta_{n-1}^{(k-1)} \\ &\leq (k+1)\,x_k^{\,n}\,\beta_n^{(k)} = (k+1)\,U_n^+. \end{aligned}$$

□

8. Persistence on the near rows of the first layer

This section and Section 10 prove Proposition 10.5: in a column $k \geq 82$, a step that has turned positive early in the first layer stays positive up to the end of the layer. The argument for a row n depends on the distance of n from row k, and here we treat the near rows, $n \leq k + 0.47k$, for which the hypothesis $k \geq 42$ suffices (Proposition 8.6). The obstacle is what the rows before the first positive row q contribute to the column equation of row n (Proposition 4.12): the step is not positive in these rows, and nothing is known about its size. We meet it by comparing the equation of row n with the equation of row q, in which this contribution is known to be outweighed, because the step in row q is positive.

8.1. **The diagonal tilt.** Let $k \geq 2$ be an integer. In this subsection column k is fixed, and we write δ_n for its normalised step $\delta_n^{(k)}$. The *diagonal tilt* of column k is the reciprocal of the diagonal factor γ_k of Section 7,

$$\hat{x}_k = \frac{1}{\gamma_k} = \Big(\frac{k}{k+1}\Big)^{k-2},$$

so that $\hat{x}_k > 0$ and $\hat{x}_k\,\gamma_k = 1$. The diagonal tilt is to be distinguished from the critical tilt $x_k = (k/(k+1))^{k-1}$ of Section 5. We tilt by $\hat{x}_k$ the three sequences that occur in the column equation of column k, namely the free prime kernel of Section 4, the normalised counts of column $k-1$ and the normalised step of column k: for integers $m \geq 0$, $j \geq 0$ and $n \geq 0$ let

$$\hat{\varepsilon}_m = \hat{x}_k^{\,m}\,\varepsilon_m, \qquad \hat{u}_j = \hat{x}_k^{\,j}\,\beta_j^{(k-1)}, \qquad \hat{\eta}_n = \hat{x}_k^{\,n}\,\delta_n.$$

We refer to $\hat{\varepsilon}$ as the *kernel at the diagonal tilt*, to $\hat{u}$ as the *column count at the diagonal tilt* and to $\hat{\eta}$ as the *step at the diagonal tilt*; the column is not shown in the notation.

Since $\hat{x}_k > 0$, the number $\hat{\eta}_n$ has the sign of δ_n and hence of the step $d_{n,k}$. Moreover, $\hat{u}_j \geq 0$ for every $j \geq 0$, and $\hat{\varepsilon}_m > 0$ for every $m \geq 1$.

Multiplying the equation of Proposition 4.12 by $\hat{x}_k^{\,n}$, we obtain for $k+1 \leq n \leq 2k-1$

$$\hat{\eta}_n = \frac{2\hat{x}_k}{n}\,\hat{u}_{n-1} + \frac{(k+1)\,\hat{x}_k}{n}\,\hat{\eta}_{n-1} + \sum_{m=1}^{n-k} \hat{\varepsilon}_m\,\hat{\eta}_{n-m}. \tag{8.1}$$

The first term on the right is the source, which comes from column $k-1$ and is nonnegative; the second is a multiple of the value $\hat{\eta}_{n-1}$ in the row before; and the

sum is the feedback of rows $k, \dots, n-1$, with the kernel $\hat{\varepsilon}$. In row k itself we have

$$\hat{\eta}_k = -\hat{u}_k, \tag{8.2}$$

because $\delta_k = \beta_k^{(k)} - \beta_k^{(k-1)}$ and $b_{k,k} = 0$.

The tilt has the value $1/\gamma_k$ so that the factor γ_k disappears from the lower bound of Proposition 7.2, by which $\gamma_k \, \beta_{n-1}^{(k-1)} \le \beta_n^{(k)}$ for every integer $n \ge 1$. Multiplying this inequality by $\hat{x}_k^n$ and using $\hat{x}_k \, \gamma_k = 1$ and $\beta_n^{(k)} = \beta_n^{(k-1)} + \delta_n$, we obtain

$$\hat{u}_{n-1} - \hat{u}_n \le \hat{\eta}_n \qquad \text{for every integer } n \ge 1. \tag{8.3}$$

This inequality bounds the step at the diagonal tilt from below by the decrease of the column count at this tilt, and together with (8.2) this is how the size of the steps that are not positive will be controlled.

8.2. **Three properties of the kernel at the diagonal tilt.** The comparison of the equation (8.1) at two rows requires that the kernel $\hat{\varepsilon}$ is log-convex, that it does not increase on the range of indices that occur, and that it satisfies a condition which relates its decrease between the two rows to the decrease of the coefficient $2\hat{x}_k/n$ of the source. All three properties are read off from the *step ratio* of the free prime kernel, which is defined for integers $m \ge 1$ by

$$g_1 = \frac{1}{2}, \qquad g_m = \frac{m}{m+1}\Big(1 + \frac{1}{m-1}\Big)^{m-1} \quad (m \ge 2).$$

It is the ratio of consecutive values of the kernel,

$$\varepsilon_{m+1} = g_m \, \varepsilon_m \qquad \text{for every integer } m \ge 1, \tag{8.4}$$

as one checks directly from $\varepsilon_1 = 1$, $\varepsilon_2 = 1/2$ and $\varepsilon_m = (m-1)^{m-1}/m!$. The step ratio does not decrease:

$$g_m \le g_{m+1} \qquad \text{for every integer } m \ge 1. \tag{8.5}$$

Indeed, $g_1 = 1/2$ and $g_2 = 4/3$, and for $m \ge 2$ neither factor of g_m decreases when m is replaced by $m+1$, the second by the monotonicity of $(1+1/i)^i$ in $i \ge 1$, which is recalled in Section 2.

The first property is the log-convexity of the kernel on indices $m \ge 1$.

Lemma 8.1. *For every integer $m \ge 1$ the free prime kernel satisfies*

$$\varepsilon_{m+1}^2 \le \varepsilon_m \, \varepsilon_{m+2}.$$

Proof. By (8.4) and (8.5),

$$\varepsilon_{m+1}^2 = g_m \, \varepsilon_m \, \varepsilon_{m+1} \le g_{m+1} \, \varepsilon_m \, \varepsilon_{m+1} = \varepsilon_m \, \varepsilon_{m+2}. \qquad \square$$

The lemma compares neighbouring indices, and we need to compare two indices at an arbitrary distance. Lemma 2.3, applied to the positive sequence $(\varepsilon_{n+1})_{n \ge 0}$, which satisfies its hypothesis by Lemma 8.1, gives

$$\varepsilon_{m+t} \, \varepsilon_r \le \varepsilon_m \, \varepsilon_{r+t} \qquad \text{for all integers } 1 \le m \le r \text{ and } t \ge 0. \tag{8.6}$$

A tilt multiplies both sides of (8.6) by the same power of the tilt, so the kernel $\hat{\varepsilon}$ at the diagonal tilt satisfies the same inequality, whatever the column.

The other two properties depend on the tilt, and for them we have to know its size, which the next lemma bounds from both sides. Here and below we use the Taylor bounds of the logarithm recalled in Section 2,

$$u - \frac{u^2}{2} \le \log(1+u) \le u - \frac{u^2}{2} + \frac{u^3}{3} \qquad (u \ge 0). \tag{8.7}$$

Lemma 8.2. *For every integer $k \ge 2$ the diagonal tilt $\hat{x}_k$ of column k satisfies*

$$e^{-1} \le \hat{x}_k \le e^{-1}\, e^{5/(2k)}.$$

Proof. We have $1 + \log \hat{x}_k = 1 - (k-2)\log(1+1/k)$, and $k - 2 \ge 0$. The lower bound of (8.7) with $u = 1/k$ gives

$$1 + \log \hat{x}_k \le 1 - (k-2)\Big(\frac{1}{k} - \frac{1}{2k^2}\Big) = \frac{5}{2k} - \frac{1}{k^2} \le \frac{5}{2k},$$

and the upper bound gives

$$1 + \log \hat{x}_k \ge 1 - (k-2)\Big(\frac{1}{k} - \frac{1}{2k^2} + \frac{1}{3k^3}\Big) = \frac{5}{2k} - \frac{4}{3k^2} + \frac{2}{3k^3} \ge \frac{5}{2k} - \frac{4}{3k^2}.$$

Since $5/(2k) - 4/(3k^2) = (15k-8)/(6k^2) \ge 0$, it follows that $0 \le 1 + \log \hat{x}_k \le 5/(2k)$, which is equivalent to the claim. □

Since $\hat{x}_k\, \gamma_k = 1$, the upper bound of the lemma is equivalent to

$$\gamma_k \ge \exp\Big(1 - \frac{5}{2k}\Big) \qquad \text{for every integer } k \ge 2. \tag{8.8}$$

For $m \ge 1$ we have $\hat{\varepsilon}_{m+1} = \hat{x}_k^{\,m+1}\, g_m\, \varepsilon_m$ by (8.4), and $\hat{\varepsilon}_m = \hat{x}_k^{\,m+1}\, \gamma_k\, \varepsilon_m$. By Lemma 8.3 below, $\varepsilon_{m+1} \le \gamma_k\, \varepsilon_m$, and with it the second property, for integers $k \ge 2$ and $m \ge 1$ with $5\,(m+1) \le 3\,k$. Its proof compares the step ratio g_m with γ_k by an upper bound for g_m and by (8.8).

Lemma 8.3. *Let $k \ge 2$ and $m \ge 1$ be integers with $5\,(m+1) \le 3\,k$. Then the free prime kernel ε and the diagonal factor $\gamma_k = (1+1/k)^{k-2}$ satisfy*

$$\varepsilon_{m+1} \le \gamma_k\, \varepsilon_m.$$

Proof. By (8.4) it suffices to show that $g_m \le \gamma_k$. For $m = 1$ we have $g_1 = 1/2 \le 1 \le \gamma_k$. For $m \ge 2$ we use the bound

$$g_m \le \exp\Big(1 - \frac{3}{2(m+1)}\Big) \qquad \text{for every integer } m \ge 2, \tag{8.9}$$

which we prove first. Let $m \ge 2$ be an integer and put $i = m - 1$, so that

$$\log g_m = i\, \log\Big(1 + \frac{1}{i}\Big) - \log\Big(1 + \frac{1}{i+1}\Big).$$

We bound the first logarithm from above and the second from below by (8.7), with $u = 1/i$ and with $u = 1/(i+1)$, and obtain

$$\log g_m \le 1 - \frac{1}{2i} + \frac{1}{3i^2} - \frac{1}{i+1} + \frac{1}{2(i+1)^2}.$$

A straightforward computation shows that the difference between $1 - 3/(2(i+2))$ and the right side is

$$\frac{7i^3 + 4i^2 - 4i - 4}{6\, i^2 (i+1)^2 (i+2)},$$

which is positive for $i \geq 1$. Since $i+2 = m+1$, this proves (8.9). Now let $m \geq 2$ be as in the lemma. The hypothesis says that $5/(2k) \leq 3/(2(m+1))$, and (8.9) and (8.8) give

$$g_m \leq \exp\Big(1 - \frac{3}{2(m+1)}\Big) \leq \exp\Big(1 - \frac{5}{2k}\Big) \leq \gamma_k. \qquad \square$$

The third property concerns the kernel together with the coefficient of the source. In the equation (8.1) of a row $n = k + r$ the source has the coefficient $2\hat{x}_k/n$, and row k, which lies at distance r, has the weight $\hat{\varepsilon}_r$. The comparison of this equation with the equation of an earlier row will require that between the two rows the weight of row k has fallen at least as much as the coefficient of the source, and this holds if the product $(k+m)\,\hat{\varepsilon}_m$ does not increase with m. The next lemma shows that this is the case for indices $m \leq 0.47k$.

Lemma 8.4. *Let $k \geq 42$ be an integer and let $i \leq j$ be integers with $i \geq 1$ and $100\,j \leq 47\,k$. Then the free prime kernel at the diagonal tilt of column k, $\hat{\varepsilon}_m = \hat{x}_k^m\,\varepsilon_m$, satisfies*

$$(k+j)\,\hat{\varepsilon}_j \leq (k+i)\,\hat{\varepsilon}_i.$$

Proof. It suffices to prove that

$$(k+m+1)\,\hat{\varepsilon}_{m+1} \leq (k+m)\,\hat{\varepsilon}_m \tag{8.10}$$

for every integer $m \geq 1$ with $100\,(m+1) \leq 47\,k$; the lemma follows by applying this for $m = i, \ldots, j-1$, since $m+1 \leq j$ for these m. Since $\hat{\varepsilon}_{m+1} = \hat{x}_k^{m+1}\,g_m\,\varepsilon_m$ and $\hat{\varepsilon}_m = \hat{x}_k^{m+1}\,\gamma_k\,\varepsilon_m$, the inequality (8.10) follows from

$$(k+m+1)\,g_m \leq (k+m)\,\gamma_k. \tag{8.11}$$

For $m = 1$ this holds because $g_1 = 1/2$ and $\gamma_k \geq 1$.

Let $m \geq 2$. By (8.9), which was shown in the proof of Lemma 8.3, and by (8.8), the inequality (8.11) holds as soon as

$$1 + \frac{1}{k+m} \leq \exp\Big(\frac{3}{2(m+1)} - \frac{5}{2k}\Big),$$

and since $1 + y \leq e^y$ for every real y, this follows from

$$\frac{1}{k+m} \leq \frac{3}{2(m+1)} - \frac{5}{2k}.$$

Bringing the three fractions to a common denominator, we find

$$\frac{3}{2(m+1)} - \frac{5}{2k} - \frac{1}{k+m} = \frac{3k^2 - 5m^2 - (4k+5)\,m - 7k}{2k\,(m+1)(k+m)},$$

so that it remains to show that

$$5m^2 + (4k+5)\,m + 7k \leq 3k^2.$$

The left side increases with m, and by hypothesis $m \leq 0.47k - 1$. At $m = 0.47k - 1$ the left side equals

$$5\,(0.47k-1)^2 + (4k+5)(0.47k-1) + 7k = 2.9845\,k^2 + 0.65\,k,$$

and $2.9845\,k^2 + 0.65\,k \leq 3k^2$ is equivalent to $k \geq 0.65/0.0155$, which holds because $k \geq 42$. $\square$

The rows beyond $k + 0.47k$ are treated by a different argument in Section 10.

8.3. **The comparison at two rows.** We can now say how the equation (8.1) is compared at two rows. Let q be the first positive row of a column $k \geq 42$, and let n be a later row with $n \leq k + 0.47k$. In the equations of rows q and n the feedback of rows $k, \dots, q-1$ is not positive, and the question is whether at row n it can outweigh the source although it did not at row q. Three facts work against this. By (8.2) and (8.3), the proof of Proposition 8.5 bounds the sum of the values of $\hat{\eta}$ in these rows from below in terms of the column count at the diagonal tilt. By the log-convexity of the kernel, in the form (8.6), the weight that one of these rows has in the equation of row n is at most $\hat{\varepsilon}_{n-k}/\hat{\varepsilon}_{q-k}$ times its weight in the equation of row q. And by Lemma 8.4 the coefficient of the source at row n is at least $\hat{\varepsilon}_{n-k}/\hat{\varepsilon}_{q-k}$ times its coefficient at row q.

This comparison is a statement on arbitrary real sequences, and we prove it as such. In the application of Section 8.4 the sequences v, π and ϕ are the step, the column count and the kernel at the diagonal tilt of a column k, that is, $\hat{\eta}$, $\hat{u}$ and $\hat{\varepsilon}$; the numbers σ_q, σ_n and ρ_q, ρ_n are the coefficients of the source and of the preceding row in the equation (8.1) of rows q and n. We keep the language of the application and refer to the index j of v_j and of π_j as a row.

Proposition 8.5. *Let $(v_j)_{j\geq 0}$, $(\pi_j)_{j\geq 0}$ and $(\phi_m)_{m\geq 0}$ be sequences of real numbers, let σ_q, σ_n, ρ_q and ρ_n be real numbers with $\sigma_q \geq 0$, $\rho_q \geq 0$ and $\rho_n > 0$, and let $k \geq 0$, $r_q \geq 1$ and $t \geq 1$ be integers. Put $q = k + r_q$ and $n = q + t$, and assume the following.*

(1) *The sequence v satisfies the two equations*

$$v_q = \sigma_q \, \pi_{q-1} + \rho_q \, v_{q-1} + \sum_{m=1}^{r_q} \phi_m \, v_{q-m}$$

and

$$v_n = \sigma_n \, \pi_{n-1} + \rho_n \, v_{n-1} + \sum_{m=1}^{r_q+t} \phi_m \, v_{n-m}.$$

(2) *One has $v_k = -\pi_k$.*
(3) *For every j with $k+1 \leq j \leq n-1$ one has $\pi_{j-1} - \pi_j \leq v_j$.*
(4) *For every $j \geq 0$ one has $\pi_j \geq 0$.*
(5) *One has $\phi_m > 0$ for $1 \leq m \leq r_q + t$, and $\phi_{m+1} \leq \phi_m$ for $1 \leq m \leq r_q + t - 1$.*
(6) *For every m with $1 \leq m \leq r_q$ one has $\phi_{m+t} \, \phi_{r_q} \leq \phi_{r_q+t} \, \phi_m$.*
(7) *One has $\phi_{r_q+t} \, \sigma_q \leq \phi_{r_q} \, \sigma_n$.*
(8) *One has $v_j \leq 0$ for $k \leq j < q$, and $v_j > 0$ for $q \leq j < n$.*

Then $v_n > 0$.

Proof. Suppose, for a contradiction, that $v_n \leq 0$. We show that then $\phi_t < \sigma_n < \phi_{t+1}$, which is impossible, because $\phi_{t+1} \leq \phi_t$ by hypothesis (5).

In the equation of row n we separate rows $q, \dots, n-1$, in which v is positive, from rows $k, \dots, q-1$, in which it is not. Let

$$I = \sum_{m=1}^{t} \phi_m \, v_{n-m}, \qquad J_n = \sum_{m=1}^{r_q} \phi_{m+t} \, v_{q-m}, \qquad J_q = \sum_{m=1}^{r_q} \phi_m \, v_{q-m}.$$

The sum I is the feedback of rows $q, \dots, n-1$ at row n, and J_n and J_q are the feedback of rows $k, \dots, q-1$ at row n and at row q. Splitting the sum of the second

equation of hypothesis (1) at $m = t$ and using $n - (m+t) = q - m$, we see that the two equations read

$$\begin{aligned} v_n &= \sigma_n \, \pi_{n-1} + \rho_n \, v_{n-1} + I + J_n, \\ v_q &= \sigma_q \, \pi_{q-1} + \rho_q \, v_{q-1} + J_q. \end{aligned} \tag{8.12}$$

Let also

$$\Sigma^+ = \sum_{j=q}^{n-1} v_j, \qquad \Sigma^- = \sum_{j=k}^{q-1} v_j,$$

so that $\Sigma^+ > 0$ and $\Sigma^- \le 0$ by hypothesis (8).

By hypothesis (5) the kernel is positive and does not increase on indices $1, \dots, r_q + t$. In the sum I we have $\phi_m \ge \phi_t$ and $v_{n-m} > 0$, and in the sum J_n we have $\phi_{m+t} \le \phi_{t+1}$ and $v_{q-m} \le 0$. Hence

$$I \ge \phi_t \, \Sigma^+, \qquad J_n \ge \phi_{t+1} \, \Sigma^-. \tag{8.13}$$

Multiplying the inequality of hypothesis (6) by $v_{q-m} \le 0$ and summing over $1 \le m \le r_q$, we obtain

$$\phi_{r_q+t} \, J_q \le \phi_{r_q} \, J_n. \tag{8.14}$$

Summing the inequality of hypothesis (3) over rows $q \le j \le n-1$ gives $\pi_{q-1} - \pi_{n-1} \le \Sigma^+$. Summing it over rows $k+1 \le j \le q-1$ gives $\pi_k - \pi_{q-1} \le \Sigma^- - v_k$, and by hypothesis (2) this is $-\pi_{q-1} \le \Sigma^-$. Hence

$$\pi_{q-1} \le \pi_{n-1} + \Sigma^+, \qquad -(\Sigma^+ + \Sigma^-) \le \pi_{n-1}. \tag{8.15}$$

We turn to the upper bound for σ_n. Since $v_n \le 0$ and $\rho_n \, v_{n-1} > 0$, the first equation of (8.12) gives

$$\sigma_n \, \pi_{n-1} + I < -J_n. \tag{8.16}$$

Hence

$$\sigma_n \, \pi_{n-1} < -J_n - I \le -\phi_{t+1} \, \Sigma^- - \phi_t \, \Sigma^+ \le -\phi_{t+1} \, (\Sigma^+ + \Sigma^-) \le \phi_{t+1} \, \pi_{n-1}.$$

Here the second step is (8.13), the third uses $\phi_{t+1} \le \phi_t$ and $\Sigma^+ > 0$, and the fourth is (8.15), multiplied by $\phi_{t+1} > 0$. If π_{n-1} were 0, this would read $0 < 0$. Hence $\pi_{n-1} > 0$ by hypothesis (4), and $\sigma_n < \phi_{t+1}$.

For the lower bound we start at row q. There $v_q > 0$, $v_{q-1} \le 0$ and $\rho_q \ge 0$, so that the second equation of (8.12) gives $-J_q < \sigma_q \, \pi_{q-1}$. We transfer this inequality to row n by the chain

$$\begin{aligned} -\phi_{r_q} \, J_n &\le -\phi_{r_q+t} \, J_q < \phi_{r_q+t} \, \sigma_q \, \pi_{q-1} \\ &\le \phi_{r_q+t} \, \sigma_q \, (\pi_{n-1} + \Sigma^+) \le \phi_{r_q} \, \sigma_n \, (\pi_{n-1} + \Sigma^+). \end{aligned}$$

Its first step is (8.14). The second is the inequality just obtained, multiplied by $\phi_{r_q+t} > 0$. The third is (8.15), with $\phi_{r_q+t} \, \sigma_q \ge 0$, and the fourth is hypothesis (7), with $\pi_{n-1} + \Sigma^+ \ge 0$. Dividing by $\phi_{r_q} > 0$, we get $-J_n < \sigma_n \, (\pi_{n-1} + \Sigma^+)$, and with (8.16) this gives $I < \sigma_n \, \Sigma^+$. By (8.13) we have $\phi_t \, \Sigma^+ \le I < \sigma_n \, \Sigma^+$, and since $\Sigma^+ > 0$, we conclude that $\phi_t < \sigma_n$. This completes the proof. □

8.4. **The near rows.** We now apply Proposition 8.5 to the step of a column at its diagonal tilt and obtain the result of this section.

Proposition 8.6. *Let $k \geq 42$, $r_q \geq 1$ and $t \geq 1$ be integers with $100\,(r_q+t) \leq 47\,k$. Assume that $d_{j,k} \leq 0$ for every j with $k \leq j < k + r_q$, and that $d_{j,k} > 0$ for every j with $k + r_q \leq j < k + r_q + t$. Then*

$$d_{k+r_q+t,\,k} > 0.$$

Proof. Put $q = k + r_q$ and $n = q + t$, and let $\hat{\varepsilon}$, $\hat{u}$ and $\hat{\eta}$ be the kernel, the column count and the step at the diagonal tilt $\hat{x}_k$ of column k, as in Section 8.1. Since $\hat{\eta}_j$ has the sign of $d_{j,k}$, it suffices to show that $\hat{\eta}_n > 0$. We obtain this from Proposition 8.5, applied to the integers k, r_q and t, to the sequences

$$v_j = \hat{\eta}_j, \qquad \pi_j = \hat{u}_j, \qquad \phi_m = \hat{\varepsilon}_m,$$

and to the numbers $\sigma_j = 2\hat{x}_k/j$ and $\rho_j = (k+1)\,\hat{x}_k/j$ for $j = q$ and $j = n$, which are positive. It remains to verify the eight hypotheses of that proposition.

Hypothesis (1) holds because $100\,(r_q + t) \leq 47\,k$ gives $n \leq k + 0.47k \leq 2k - 1$. Hence $k + 1 \leq q < n \leq 2k - 1$, and the two equations are (8.1) for rows q and n, in which the sum runs up to $q - k = r_q$ and up to $n - k = r_q + t$. Hypotheses (2), (3) and (4) are (8.2), the inequality (8.3), which holds for every $j \geq 1$, and the nonnegativity of $\hat{u}_j$.

For hypothesis (5), the numbers $\hat{\varepsilon}_m$ with $m \geq 1$ are positive. For $1 \leq m \leq r_q + t - 1$ we have $m + 1 \leq 0.47k$, hence $5\,(m+1) \leq 3\,k$, and the inequality of Lemma 8.3, multiplied by $\hat{x}_k^{\,m+1}$, is $\hat{\varepsilon}_{m+1} \leq \hat{\varepsilon}_m$, because $\hat{x}_k\,\gamma_k = 1$. Hypothesis (6) is (8.6) with $r = r_q$, multiplied by $\hat{x}_k^{\,m+t+r_q}$.

After multiplication by $qn/(2\hat{x}_k)$, hypothesis (7) reads

$$(k + r_q + t)\,\hat{\varepsilon}_{r_q+t} \leq (k + r_q)\,\hat{\varepsilon}_{r_q}.$$

This is Lemma 8.4 with $i = r_q$ and $j = r_q + t$, whose hypothesis $100\,j \leq 47\,k$ is the one of the proposition. Finally, the two conditions of hypothesis (8) are the two assumptions on the steps $d_{j,k}$, because $\hat{\eta}_j$ has the sign of $d_{j,k}$. □

9. Renewal sequences and the frozen kernel

The far rows of the first layer, which are treated in Section 10, require a bound for a weighted sum of terms of a sequence v against the far part of a kernel, in terms of the last value of v alone, when all that is known about v is that it dominates its convolution with the kernel. We obtain such a bound by comparing v with the renewal sequence of the kernel (Proposition 9.4); the comparison rests on the log-convexity of that sequence, a theorem of de Bruijn and Erdős (Proposition 9.3). We then treat the kernel to which the comparison is applied, the frozen kernel, and its renewal sequence.

9.1. **Renewal sequences and log-convex kernels.**

Definition 9.1. Let $\varphi = (\varphi_i)_{i \geq 1}$ and $u = (u_n)_{n \geq 0}$ be sequences of real numbers. The sequence u is *the renewal sequence of the kernel* φ if

$$u_0 = 1 \qquad \text{and} \qquad u_n = \sum_{i=1}^{n} \varphi_i\, u_{n-i} \quad \text{for every integer } n \geq 1.$$

The second condition, for a given n, is the *renewal equation* at n. For a given kernel the two conditions determine u term by term: $u_1 = \varphi_1$, $u_2 = \varphi_1^2 + \varphi_2$, and so on. In terms of formal power series they say that

$$\sum_{n\geq 0} u_n X^n = \Big(1 - \sum_{i\geq 1} \varphi_i X^i\Big)^{-1}.$$

When the numbers φ_i are the probabilities of the lengths of the steps of a renewal process, u_n is the probability that a renewal occurs at time n. We make no assumption of this kind, and the sum of the φ_i need not be one. Two instances have occurred already: by Corollary 4.4 the sequence $(c_n)_{n\geq 0}$ is the renewal sequence of the kernel $(\varepsilon_m)_{m\geq 1}$, and by Lemma 4.5 the sequence $(\nu_n)_{n\geq 0}$ is the renewal sequence of the kernel $(a_j)_{j\geq 1}$; recall that $c_0 = \nu_0 = 1$.

Let u be the renewal sequence of a kernel φ. If $\varphi_i > 0$ for every $i \geq 1$, then $u_n > 0$ for every $n \geq 0$; this follows from the renewal equation by induction on n.

We think of the index of a sequence as a time. Let φ be a kernel and $v = (v_s)_{s\geq 0}$ a sequence of real numbers. The sum $\varphi_1 v_{s-1} + \varphi_2 v_{s-2} + \cdots + \varphi_s v_0$ is the *feedback* that v receives at time s from its earlier values through the kernel, and the difference ι_s between v_s and this feedback is the *innovation* of v at time s with respect to φ. The renewal sequence of φ is the sequence whose innovations are $\iota_0 = 1$ and $\iota_s = 0$ for $s \geq 1$. The following lemma recovers an arbitrary sequence from its innovations: each innovation is carried to the later times by the renewal sequence, which is in this sense the resolvent of the kernel. We need it in the proof of Proposition 9.4, where it writes a sequence with nonnegative innovations as a combination of shifted copies of the renewal sequence with nonnegative coefficients.

Lemma 9.2. *Let $\varphi = (\varphi_i)_{i\geq 1}$, $u = (u_n)_{n\geq 0}$, $v = (v_s)_{s\geq 0}$ and $\iota = (\iota_s)_{s\geq 0}$ be sequences of real numbers. Suppose that u is the renewal sequence of the kernel φ in the sense of Definition 9.1, and that*

$$v_s = \iota_s + \sum_{i=1}^{s} \varphi_i \, v_{s-i} \qquad \text{for every integer } s \geq 0.$$

Then

$$v_s = \sum_{j=0}^{s} u_{s-j} \, \iota_j \qquad \text{for every integer } s \geq 0.$$

Proof. We argue by induction on s and assume the formula for all indices smaller than s. Inserting it for each v_{s-i} with $1 \leq i \leq s$ and exchanging the order of summation, we obtain

$$\sum_{i=1}^{s} \varphi_i \, v_{s-i} = \sum_{j=0}^{s-1} \Big(\sum_{i=1}^{s-j} \varphi_i \, u_{s-j-i} \Big) \iota_j = \sum_{j=0}^{s-1} u_{s-j} \, \iota_j,$$

where the inner sum is u_{s-j} by the renewal equation at $s-j \geq 1$. Adding $\iota_s = u_0 \, \iota_s$ gives the formula at s. □

The next statement transfers log-convexity from a kernel to its renewal sequence. It is a theorem of de Bruijn and Erdős [6, Theorem 1]. There it is stated for a kernel with nonnegative terms whose sum is one, with strict inequalities in the hypothesis and in the conclusion, and the remarks that follow the theorem give the form with non-strict inequalities. We need the theorem for a kernel with positive terms and with no condition on its sum, in the non-strict form, and we prove it in this form.

Proposition 9.3. *Let $\varphi = (\varphi_i)_{i\ge 1}$ and $u = (u_n)_{n\ge 0}$ be sequences of real numbers such that u is the renewal sequence of the kernel φ in the sense of Definition 9.1. Suppose that $\varphi_i > 0$ for every $i \ge 1$ and that $\varphi_{i+1}^2 \le \varphi_i\,\varphi_{i+2}$ for every $i \ge 1$. Then*

$$u_{n+1}^2 \le u_n\,u_{n+2} \qquad \text{for every integer } n \ge 0.$$

Proof. All u_n are positive, as was noted after Definition 9.1. We prove the inequality by induction on n. For $n = 0$ the renewal equations give $u_1 = \varphi_1$ and $u_2 = \varphi_1 u_1 + \varphi_2 = \varphi_1^2 + \varphi_2$, so that $u_1^2 = \varphi_1^2 \le \varphi_1^2 + \varphi_2 = u_0\,u_2$.

Let $n \ge 1$, and suppose that $u_{j+1}^2 \le u_j\,u_{j+2}$ for $0 \le j \le n-1$. Then the quotients u_{j+1}/u_j do not decrease as j runs from 0 to n, and by the hypothesis on the kernel the quotients $\varphi_{i+2}/\varphi_{i+1}$ do not decrease in $i \ge 0$. Consequently, for all integers i and j with $0 \le i \le n$ and $0 \le j \le n$,

$$\varphi_{i+2}\,\varphi_{n+1} \le \varphi_{i+1}\,\varphi_{n+2} \qquad \text{and} \qquad u_{j+1}\,u_n \le u_j\,u_{n+1}. \tag{9.1}$$

Consider the sum over the pairs (i, j) of nonnegative integers with $i + j = n - 1$,

$$S = \sum_{i+j=n-1} \big(\varphi_{n+2}\,\varphi_{i+1} - \varphi_{n+1}\,\varphi_{i+2}\big)\big(u_{n+1}\,u_j - u_n\,u_{j+1}\big).$$

By (9.1) both factors of every term are nonnegative, so that $S \ge 0$. On the other hand, S can be evaluated by the renewal equations at n, $n+1$ and $n+2$. The first of them, the second without its last term $\varphi_{n+1}u_0$, the second without its first term $\varphi_1 u_n$, and the third without its first term $\varphi_1 u_{n+1}$ and its last term $\varphi_{n+2}u_0$ read

$$\begin{aligned}
\sum_{i+j=n-1} \varphi_{i+1}\,u_j &= u_n,\\
\sum_{i+j=n-1} \varphi_{i+1}\,u_{j+1} &= u_{n+1} - \varphi_{n+1},\\
\sum_{i+j=n-1} \varphi_{i+2}\,u_j &= u_{n+1} - \varphi_1\,u_n,\\
\sum_{i+j=n-1} \varphi_{i+2}\,u_{j+1} &= u_{n+2} - \varphi_1\,u_{n+1} - \varphi_{n+2}.
\end{aligned}$$

We expand the product in S and insert these four sums. All terms cancel except two, and we find

$$S = \varphi_{n+1}\big(u_n\,u_{n+2} - u_{n+1}^2\big).$$

Since $S \ge 0$ and $\varphi_{n+1} > 0$, it follows that $u_{n+1}^2 \le u_n\,u_{n+2}$. □

We come to the comparison with the renewal sequence. Let u be the renewal sequence of a positive log-convex kernel φ, and let v be a sequence whose innovations with respect to φ are nonnegative up to a time ρ. Given an integer $M \ge 1$ and weights θ_m, the sum

$$\sum_{m=M}^{\rho} \theta_m\,\varphi_m\,v_{\rho-m}$$

is the part of the feedback of v at time ρ that passes through the kernel indices $m \ge M$, taken with the weights θ_m; this is the *far feedback* of v at time ρ. Proposition 9.4 below bounds the far feedback of v, relative to the value v_ρ, by the far feedback of the renewal sequence u relative to u_ρ, for weights that are nonnegative and nondecreasing from index M to index ρ.

The reason for the bound is the following. By Lemma 9.2, v is a combination of shifted copies of u, one for each innovation, with nonnegative coefficients. An innovation injected at time $\rho - s$ contributes u_s times its size to v_ρ and

$$\sum_{m=M}^{s} \theta_m \, \varphi_m \, u_{s-m}$$

times its size to the far feedback. It is therefore enough to see that the quotient of this sum by u_s is largest for $s = \rho$. By the renewal equation at s, the numbers $\varphi_m \, u_{s-m}/u_s$ with $1 \le m \le s$ are positive and add up to one; in terms of a renewal process, the m-th of them is the share of u_s that comes from the histories whose last step has length m. For a log-convex sequence u every quotient u_{s-m}/u_s does not increase when s grows. Hence the share of the last steps shorter than M does not increase with s, and the share of the last steps of length at least M does not decrease. Since the weights are nondecreasing, a shift of mass towards longer last steps can only increase the weighted share.

Proposition 9.4. *Let $\varphi = (\varphi_i)_{i\ge1}$, $u = (u_n)_{n\ge0}$, $v = (v_n)_{n\ge0}$ and $\theta = (\theta_m)_{m\ge1}$ be sequences of real numbers, and let $\rho \ge 0$ and $M \ge 1$ be integers. Assume the following.*

1. *One has $\varphi_i > 0$ for every $i \ge 1$ and $\varphi_{i+1}^2 \le \varphi_i \, \varphi_{i+2}$ for every $i \ge 1$.*
2. *The sequence u is the renewal sequence of the kernel φ in the sense of Definition 9.1.*
3. *One has $\theta_M \ge 0$, and $\theta_a \le \theta_b$ for all integers a and b with $M \le a \le b \le \rho$.*
4. *For every integer s with $0 \le s \le \rho$,*

$$\sum_{i=1}^{s} \varphi_i \, v_{s-i} \le v_s.$$

Then

$$\Big(\sum_{m=M}^{\rho} \theta_m \, \varphi_m \, v_{\rho-m} \Big) \, u_\rho \le \Big(\sum_{m=M}^{\rho} \theta_m \, \varphi_m \, u_{\rho-m} \Big) \, v_\rho.$$

Proof. If $M > \rho$, both sums are empty and there is nothing to prove, so let $M \le \rho$. By hypotheses (1) and (2), $u_n > 0$ for every $n \ge 0$, as was noted after Definition 9.1, and $u_{n+1}^2 \le u_n \, u_{n+2}$ for every $n \ge 0$ by Proposition 9.3. Hence Lemma 2.3, applied to the sequence u with the indices $s - m \le \rho - m$ and the shift m, gives

$$u_s \, u_{\rho-m} \le u_{s-m} \, u_\rho \qquad (0 \le m \le s \le \rho). \tag{9.2}$$

We first compare the tails of the renewal equations at two times: for every integer s with $M \le s \le \rho$,

$$\Big(\sum_{m=M}^{s} \varphi_m \, u_{s-m} \Big) \, u_\rho \le \Big(\sum_{m=M}^{\rho} \varphi_m \, u_{\rho-m} \Big) \, u_s. \tag{9.3}$$

Indeed, by the renewal equations at s and at ρ the two sums in parentheses are

$$u_s - \sum_{m=1}^{M-1} \varphi_m \, u_{s-m} \qquad \text{and} \qquad u_\rho - \sum_{m=1}^{M-1} \varphi_m \, u_{\rho-m}.$$

After multiplication by u_ρ and by u_s the leading terms agree, and (9.3) becomes

$$\sum_{m=1}^{M-1} \varphi_m \, u_{\rho-m} \, u_s \le \sum_{m=1}^{M-1} \varphi_m \, u_{s-m} \, u_\rho,$$

which holds term by term by (9.2).

Next we bring in the weights. Put

$$W = \sum_{m=M}^{\rho} \theta_m \, \varphi_m \, u_{\rho-m}.$$

By hypothesis (3), $\theta_m \ge \theta_M \ge 0$ for $M \le m \le \rho$, so that $W \ge 0$. We claim that for every integer s with $M \le s \le \rho$,

$$\Big(\sum_{m=M}^{s} \theta_m \, \varphi_m \, u_{s-m} \Big) \, u_\rho \le W \, u_s. \tag{9.4}$$

We split W into its terms with $m \le s$ and its terms with $m > s$. Then (9.4) is equivalent to

$$\sum_{m=M}^{s} \theta_m \, \varphi_m \left(u_{s-m} \, u_\rho - u_s \, u_{\rho-m} \right) \le u_s \sum_{m=s+1}^{\rho} \theta_m \, \varphi_m \, u_{\rho-m}. \tag{9.5}$$

On the left side every difference in parentheses is nonnegative by (9.2), and $\theta_m \le \theta_s$ for $M \le m \le s$ by hypothesis (3). The left side is therefore at most

$$\theta_s \sum_{m=M}^{s} \varphi_m \left(u_{s-m} \, u_\rho - u_s \, u_{\rho-m} \right) \le \theta_s \, u_s \sum_{m=s+1}^{\rho} \varphi_m \, u_{\rho-m},$$

where the inequality is (9.3), with the terms $m \le s$ of its right side moved to the left, multiplied by $\theta_s \ge 0$. On the right side of (9.5) we have $\theta_m \ge \theta_s$ for $s < m \le \rho$, again by hypothesis (3), so the last expression is at most the right side of (9.5). This proves (9.4).

Finally we decompose v into its innovations. For $s \ge 0$ let

$$\iota_s = v_s - \sum_{i=1}^{s} \varphi_i \, v_{s-i},$$

so that $\iota_s \ge 0$ for $s \le \rho$ by hypothesis (4). By Lemma 9.2,

$$v_s = \sum_{j=0}^{s} u_{s-j} \, \iota_j \qquad \text{for every } s \ge 0.$$

We insert this for $v_{\rho-m}$ and exchange the two sums:

$$\sum_{m=M}^{\rho} \theta_m \, \varphi_m \, v_{\rho-m} = \sum_{j=0}^{\rho-M} \iota_j \sum_{m=M}^{\rho-j} \theta_m \, \varphi_m \, u_{(\rho-j)-m}.$$

We multiply by u_ρ and apply (9.4) with $s = \rho - j$ to each inner sum. Since $\iota_j \ge 0$, this gives

$$\Big(\sum_{m=M}^{\rho} \theta_m \, \varphi_m \, v_{\rho-m} \Big) \, u_\rho \le W \sum_{j=0}^{\rho-M} u_{\rho-j} \, \iota_j \le W \sum_{j=0}^{\rho} u_{\rho-j} \, \iota_j = W \, v_\rho.$$

The second inequality adds the terms with $\rho - M < j \le \rho$, which are nonnegative because $W \ge 0$, $u_{\rho-j} > 0$ and $\iota_j \ge 0$, and the last equality is Lemma 9.2 at $s = \rho$. □

9.2. **The frozen kernel and its renewal sequence.** We now introduce the kernel to which the statements of Section 9.1 are applied. Let ε be the free prime kernel of Section 4. The *frozen kernel* is the sequence $f = (f_i)_{i\ge1}$ with

$$f_1 = \frac{3}{2}\,e^{-1}, \qquad f_i = e^{-i}\,\varepsilon_i \quad \text{for } i \ge 2. \tag{9.6}$$

Since $\varepsilon_1 = 1$, it is the sequence $(e^{-i}\varepsilon_i)_{i\ge1}$ with $1/(2e)$ added to its first term. We also put $f_0 = 0$.

The kernel is made for one purpose, which explains its form; the argument itself is part of the proof of Proposition 10.4. Let $k \ge 2$. By Lemma 4.13 the normalised counts of column $k-1$ satisfy, in rows $k \le n \le 2k-1$,

$$\beta_n^{(k-1)} = \frac{z_{n-1}^{(k)}}{n} + \frac{k-1}{n}\,\beta_{n-1}^{(k-1)} + \sum_{m=1}^{n-k} \varepsilon_m\,\beta_{n-m}^{(k-1)},$$

and all terms on the right are nonnegative. The coefficient $(k-1)/n$ of the preceding row depends on the row, and it is at least $1/2$ as long as $n \le 2k-2$. If this coefficient is replaced by the constant $1/2$ and the first term is dropped, the equation becomes an inequality. For the numbers $v_s = e^{-s}\beta_{k+s}^{(k-1)}$ this inequality is

$$\sum_{i=1}^{s} f_i\,v_{s-i} \le v_s \qquad (1 \le s \le k-2),$$

where the summand $1/(2e)$ of f_1 comes from the constant $1/2$. The sequence v thus has nonnegative innovations with respect to a kernel that depends neither on the column nor on the row, as hypothesis (4) of Proposition 9.4 requires. The kernel is called frozen because the coefficient $(k-1)/n$ of the preceding row, which varies with the row, has been frozen at one value.

The factors e^{-i} tilt the kernel by $1/e$, and this tilt is chosen with a view to the estimates by which Section 10 evaluates the comparison with the renewal sequence. The inequality itself could be written at any tilt $x > 0$, for the numbers $x^s\beta_{k+s}^{(k-1)}$: it suffices to multiply it by $(e\,x)^s$. The terms of the kernel are then those of f multiplied by the geometric factors $(e\,x)^i$. Section 10 bounds the kernel terms with their weights uniformly in m (Lemma 10.3) and the terms of the renewal sequence separately, through a partial sum of the sequence and a lower bound for a single term.

At the tilt $1/e$ the sum of the kernel ε is $1 - 1/e$ (Corollary 9.9), and f adds $1/(2e)$ to its first term. The partial sums of the renewal sequence of f are bounded (Lemma 9.10), while its terms decay no faster than $(s+1)^{-3/2}$ (Lemma 9.7). These are the two bounds for the renewal sequence that Section 10 uses.

The frozen kernel is positive: $f_i > 0$ for every $i \ge 1$, because $\varepsilon_i > 0$ for $i \ge 1$. It is also log-convex; the next lemma derives this from the log-convexity of ε (Lemma 8.1).

Lemma 9.5. *For every integer $i \ge 1$, the frozen kernel f of* (9.6) *satisfies*

$$f_{i+1}^2 \le f_i\,f_{i+2}.$$

Proof. Since $i+1 \geq 2$, we have $f_{i+1} = e^{-(i+1)}\varepsilon_{i+1}$ and $f_{i+2} = e^{-(i+2)}\varepsilon_{i+2}$. By Lemma 8.1, $\varepsilon_{i+1}^2 \leq \varepsilon_i\,\varepsilon_{i+2}$, and therefore

$$f_{i+1}^2 = e^{-i}\,e^{-(i+2)}\,\varepsilon_{i+1}^2 \leq e^{-i}\varepsilon_i \cdot e^{-(i+2)}\varepsilon_{i+2} \leq f_i\,f_{i+2},$$

because $e^{-i}\varepsilon_i \leq f_i$. □

The hypotheses of Propositions 9.3 and 9.4 on the kernel thus hold for f. Because f depends neither on the column nor on the row, its renewal sequence is one sequence, and it can be written down. Let F be the power series with coefficients c_n and $E = 1/F$ its inverse, as in Section 4. For integers $j \geq 0$ and $s \geq 0$ let $h_{j,s} = [X^s]\,X^jF^{j+1}$ be the coefficient of X^s in the power series X^jF^{j+1}, and put

$$H_s = \sum_{j=0}^{s} 2^{-j}\,h_{j,s}, \qquad G_s = e^{-s}\,H_s \qquad (s \geq 0). \tag{9.7}$$

The sequence G is the *resolvent sequence* of the frozen kernel.

The formula is found as follows. The tilt by $1/e$ turns the sequence H into G, and it turns into f the kernel whose first term is $\varepsilon_1 + 1/2$ and whose m-th term is ε_m for $m \geq 2$. Since E is the series with constant term 1 and coefficient $-\varepsilon_m$ of X^m for $m \geq 1$, the renewal sequence of the latter kernel has the generating series $1/(E - X/2)$, and

$$\frac{1}{E - X/2} = \frac{F}{1 - XF/2} = \sum_{j\geq 0}\Big(\frac{X}{2}\Big)^j F^{j+1}.$$

The coefficient of X^s in this series, to which only the terms with $j \leq s$ contribute, is the number H_s of (9.7). The following lemma confirms that G is the renewal sequence of f. The sequence G is defined by its coefficients and not by its renewal equation, because the coefficients $h_{j,s}$ have a closed form, from which a lower bound for G can be read off.

Lemma 9.6. *Let f be the frozen kernel of* (9.6) *and G the sequence of* (9.7)*. Then G is the renewal sequence of the kernel f in the sense of Definition 9.1.*

Proof. Clearly $h_{j,s} = 0$ for $s < j$, so that H_s is the sum of $2^{-j}h_{j,s}$ over $0 \leq j \leq J$, for every $J \geq s$, and $h_{0,s} = c_s$. In particular $H_0 = c_0 = 1$, and hence $G_0 = 1$.

Fix integers $j \geq 0$ and $s \geq 1$. Lemma 4.3 expresses the coefficients of the product of E with an arbitrary power series; for the series X^jF^{j+1} it gives

$$h_{j,s} - \sum_{m=1}^{s} \varepsilon_m\,h_{j,s-m} = [X^s]\big(E \cdot X^jF^{j+1}\big) = [X^s]\,X^jF^j,$$

because $EF = 1$. The right side is 0 for $j = 0$, and it is $h_{j-1,s-1}$ for $j \geq 1$. We multiply by 2^{-j} and sum over $0 \leq j \leq s$. Since $h_{j,s-m} = 0$ for $j > s - m$, the left side becomes H_s minus the sum of $\varepsilon_m H_{s-m}$ over $1 \leq m \leq s$, and the right side becomes

$$\sum_{j=1}^{s} 2^{-j}\,h_{j-1,s-1} = \frac{1}{2}\sum_{j=0}^{s-1} 2^{-j}\,h_{j,s-1} = \frac{1}{2}\,H_{s-1}.$$

Hence

$$H_s = \sum_{m=1}^{s} \varepsilon_m\,H_{s-m} + \frac{1}{2}\,H_{s-1} \qquad \text{for every integer } s \geq 1.$$

Together with $H_0 = 1$ this says that H is the renewal sequence of the kernel whose first term is $\varepsilon_1 + 1/2 = 3/2$ and whose m-th term is ε_m for $m \geq 2$. We multiply the last display by e^{-s}. Since $e^{-s} = e^{-m}e^{-(s-m)}$, the term $e^{-s}\varepsilon_m H_{s-m}$ is $e^{-m}\varepsilon_m\, G_{s-m}$, and $e^{-s}H_{s-1}/2 = G_{s-1}/(2e)$. The contributions at each index i add up to $f_i G_{s-i}$, because f_i is $e^{-i}\varepsilon_i$, increased by $1/(2e)$ for $i = 1$. Hence $G_s = \sum_{i=1}^{s} f_i\, G_{s-i}$ for every integer $s \geq 1$, and together with $G_0 = 1$ this says that G is the renewal sequence of f. □

Since $f_i > 0$ for every $i \geq 1$, Lemma 9.6 and the positivity of the renewal sequence of a positive kernel give

$$G_s > 0 \qquad \text{for every integer } s \geq 0. \tag{9.8}$$

With Lemmas 9.5 and 9.6, Proposition 9.4 can be applied to the frozen kernel and its renewal sequence G, and it bounds a far feedback by an expression in the values of G. We now prove the two bounds for G by which Section 10 evaluates this expression. The lower bound is the next lemma.

Lemma 9.7. *Let G be the sequence of* (9.7)*. For every integer $s \geq 6$,*

$$G_s \geq \frac{e}{s+1} \cdot \frac{1 - \dfrac{1}{12(s+1)}}{\sqrt{2\pi(s+1)}} \cdot \Big(4 - \frac{21}{s+1}\Big).$$

Proof. For $0 \leq j \leq s$ the number $h_{j,s}$ is the coefficient of X^{s-j} in F^{j+1}. Since $F = e^T$ and $T = Xe^T$ (Section 4), the Lagrange inversion formula [8, Theorem A.2], applied to the function $u \mapsto e^{(j+1)u}$, gives for every integer $n \geq 1$

$$[X^n]\, F^{j+1} = \frac{1}{n}\,[u^{n-1}]\big((j+1)\, e^{(j+1+n)u}\big) = \frac{(j+1)\,(j+1+n)^{n-1}}{n!},$$

and the constant term of F^{j+1} is 1. For $n = s - j$ this shows that $h_{j,s} = (j+1)\,(s+1)^{s-j-1}/(s-j)!$, also for $j = s$. Dividing by $h_{0,s} = c_s = (s+1)^{s-1}/s!$ we obtain, for $0 \leq j \leq s$,

$$h_{j,s} = c_s\,(j+1)\,\frac{s!}{(s-j)!\,(s+1)^j} = c_s\,(j+1)\prod_{t=1}^{j}\Big(1 - \frac{t}{s+1}\Big).$$

A product of factors $1 - x_t$ with numbers x_t in the interval $[0,1]$ is at least one minus the sum of the x_t, and the numbers $t/(s+1)$ with $1 \leq t \leq j$ add up to $j(j+1)/(2(s+1))$. Therefore

$$h_{j,s} \geq c_s\,(j+1)\Big(1 - \frac{j(j+1)}{2(s+1)}\Big) \qquad (0 \leq j \leq s).$$

We multiply by 2^{-j} and sum over $0 \leq j \leq s$:

$$H_s \geq c_s \sum_{j=0}^{s} \frac{j+1}{2^j} - \frac{c_s}{2(s+1)} \sum_{j=0}^{s} \frac{j\,(j+1)^2}{2^j}.$$

It is easy to check by induction on s that

$$\sum_{j=0}^{s} \frac{j+1}{2^j} = 4 - \frac{s+3}{2^s},$$
$$\sum_{j=0}^{s} \frac{j\,(j+1)^2}{2^j} = 40 - \frac{s^3+8s^2+27s+40}{2^s}.$$

In particular the second sum is at most 40. Moreover $(s+1)(s+3) \le 2^s$ for $s \ge 6$, as follows by induction from the case $s = 6$, which is $63 \le 64$. Hence $(s+3)/2^s \le 1/(s+1)$ for $s \ge 6$, and

$$H_s \ge c_s\Big(4 - \frac{1}{s+1} - \frac{20}{s+1}\Big) = c_s\Big(4 - \frac{21}{s+1}\Big) \qquad (s \ge 6).$$

It remains to bound $e^{-s}c_s$. Since $c_s = (s+1)^s/(s+1)!$, we have $e^{-s}c_s = e\,P(s+1)/(s+1)$ with the Poisson mass P, and the lower bound of (2.5) gives

$$P(s+1) \ge \Big(1 - \frac{1}{12(s+1)}\Big)\frac{1}{\sqrt{2\pi(s+1)}}.$$

For $s \ge 6$ the number $4 - 21/(s+1)$ is nonnegative, so that

$$\begin{aligned} G_s = e^{-s}H_s &\ge e^{-s}c_s\Big(4 - \frac{21}{s+1}\Big) \\ &\ge \frac{e}{s+1} \cdot \frac{1 - \dfrac{1}{12(s+1)}}{\sqrt{2\pi(s+1)}} \cdot \Big(4 - \frac{21}{s+1}\Big). \end{aligned}$$

□

9.3. **Total masses.** It remains to bound the partial sums of G. If one sets $X = 1$ in the identity between generating series of Section 9.1, the sum of a renewal sequence appears as $1/(1-\sigma)$, where σ is the sum of its kernel. A bound for the sum of G therefore amounts to a bound below one for the sum of the frozen kernel, and by (9.6) this requires the sum of $e^{-m}\varepsilon_m$ over $m \ge 1$. The proof of Corollary 9.9 obtains this sum from the sum of the sequence $(e^{-n}c_n)_{n\ge0}$, which is given by the total mass of the Borel distribution (Lemma 9.8), and from the fact that the sequence $(c_n)_{n\ge0}$ is the renewal sequence of the kernel $(\varepsilon_m)_{m\ge1}$ (Corollary 4.4). It does not substitute into an identity between generating series, and it argues with partial sums instead.

Let u be the renewal sequence of a kernel φ. For integers $N \ge 0$ we write

$$\sigma_N = \sum_{i=1}^{N} \varphi_i, \qquad \Sigma_N = \sum_{n=0}^{N} u_n$$

for the partial sums of the kernel and of its renewal sequence.

The numbers $e^{-j}a_j = e^{-j}j^{j-1}/j!$, $j \ge 1$, are the probabilities of the Borel distribution with parameter 1, and the next lemma is the statement that this distribution has total mass one.

Lemma 9.8. *The series of the nonnegative numbers $e^{-j}a_j$, $j \ge 1$, converges, and*

$$\sum_{j=1}^{\infty} e^{-j}\,\frac{j^{j-1}}{j!} = 1.$$

Proof. Aldous and Pitman state two facts about the total progeny of a Galton–Watson process [4, Section 2.2 and (10)], which we use for the Poisson offspring law with mean 1: the total progeny is then finite with probability one, and it equals j with probability $e^{-j}j^{j-1}/j!$, for every integer $j \geq 1$. From these two facts we deduce the lemma. When the total progeny is finite, it is one of the integers $j \geq 1$, so that the probability that it is finite is the sum of the numbers $e^{-j}j^{j-1}/j!$ over $j \geq 1$. Hence this series converges, and its sum is 1. □

As a consequence we obtain the sum of the kernel ε at the tilt $1/e$.

Corollary 9.9. *The series of the nonnegative numbers $e^{-m}\varepsilon_m$, $m \geq 1$, converges, and*

$$\sum_{m=1}^{\infty} e^{-m}\,\frac{(m-1)^{m-1}}{m!} = 1 - \frac{1}{e}.$$

Proof. Let u be the renewal sequence of a kernel φ with $\varphi_i > 0$ for every $i \geq 1$, so that $u_n > 0$ for every $n \geq 0$, as was noted after Definition 9.1. Adding the renewal equations at $n = 1, \ldots, N$ and collecting the terms with the same i, we obtain

$$\Sigma_N - 1 = \sum_{n=1}^{N}\sum_{i=1}^{n} \varphi_i\, u_{n-i} = \sum_{i=1}^{N} \varphi_i\, \Sigma_{N-i} \qquad (N \geq 0). \tag{9.9}$$

The numbers Σ_N do not decrease in N, and $\Sigma_0 = 1$. If every Σ_{N-i} in (9.9) is replaced by the larger number Σ_N, we get $\Sigma_N - 1 \leq \sigma_N\,\Sigma_N$, and dividing by $\Sigma_N \geq 1$ gives

$$1 - \frac{1}{\Sigma_N} \leq \sigma_N \qquad (N \geq 0). \tag{9.10}$$

If, in (9.9) with N replaced by $N + L$, the terms with $i > N$ are discarded and Σ_{N+L-i} is replaced by the smaller number Σ_L in the terms with $i \leq N$, we get

$$\sigma_N\,\Sigma_L \leq \Sigma_{N+L} - 1 \qquad (N \geq 0,\ L \geq 0). \tag{9.11}$$

We now take $\varphi_m = e^{-m}\varepsilon_m$ for $m \geq 1$ and $u_n = e^{-n}c_n$ for $n \geq 0$. By Corollary 4.4 and $c_0 = 1$, the sequence (c_n) is the renewal sequence of the kernel $(\varepsilon_m)_{m\geq 1}$. We multiply its renewal equation at $n \geq 1$ by e^{-n}; since $e^{-n} = e^{-m}e^{-(n-m)}$, the term $e^{-n}\varepsilon_m c_{n-m}$ is $\varphi_m\, u_{n-m}$, so that $u_n = \sum_{m=1}^{n} \varphi_m\, u_{n-m}$. Together with $u_0 = c_0 = 1$ this says that u is the renewal sequence of φ. Since $\varepsilon_m > 0$ for $m \geq 1$, the inequalities (9.10) and (9.11) hold for the partial sums σ_N of φ and the partial sums Σ_N of u.

Since $c_n = a_{n+1}$, the number $e^{-n}c_n$ is e times the term $e^{-(n+1)}a_{n+1}$ of the series of Lemma 9.8. Hence $\Sigma_N \leq e$ for every N, and Σ_N tends to e. By (9.11), $\sigma_N\,\Sigma_L \leq e - 1$ for all N and L, and letting L tend to infinity we get $\sigma_N \leq 1 - 1/e$. By (9.10), $\sigma_N \geq 1 - 1/\Sigma_N$, and the right side tends to $1 - 1/e$. Hence σ_N tends to $1 - 1/e$. □

We can now bound the partial sums of the resolvent sequence. The bound of the next lemma is the value of $1/(1-\sigma)$ at $\sigma = (1 - 1/e) + 1/(2e) = 1 - 1/(2e)$, the sum of the kernel ε at the tilt $1/e$ (Corollary 9.9) increased by the summand $1/(2e)$ of f_1.

Lemma 9.10. *Let G be the sequence of* (9.7)*. For every integer $N \geq 0$,*

$$\sum_{s=0}^{N-1} G_s \leq 2e.$$

Proof. By Lemma 9.6, G is the renewal sequence of the frozen kernel f, and $f_i > 0$ for every $i \geq 1$. Hence (9.10), which was shown in the proof of Corollary 9.9 for the renewal sequence of every kernel with positive terms, holds for the partial sums σ_N of f and the partial sums Σ_N of G. One has $\sigma_0 = 0$, and by (9.6), σ_N is for $N \geq 1$ the sum of $e^{-i}\varepsilon_i$ over $1 \leq i \leq N$ plus $1/(2e)$, where this sum is at most $1 - 1/e$ by Corollary 9.9. Hence

$$1 - \frac{1}{\Sigma_N} \leq \sigma_N \leq 1 - \frac{1}{e} + \frac{1}{2e} = 1 - \frac{1}{2e} \qquad (N \geq 0),$$

and therefore $\Sigma_N \leq 2e$ for every $N \geq 0$. The sum of the lemma is empty for $N = 0$, and it is Σ_{N-1} for $N \geq 1$. □

10. Persistence on the far rows of the first layer

We return to the first layer of a column $k \geq 82$. Proposition 8.6 shows that a column which has turned positive stays positive on the near rows $n \leq k + 0.47k$. Here we prove that the step in a far row n, with $k + 0.47k < n \leq 2k - 1$, is positive if the steps in rows $k + r_q, \ldots, n - 1$ are positive for an integer $r_q \geq 0$ with $3\,r_q \leq k$ (Proposition 10.4). We then combine the two statements. Call a row $k + r_q$ early if $r_q \geq 1$ and $3\,r_q \leq k$. A column $k \geq 82$ whose steps in rows $k, \ldots, k + r_q - 1$ are not positive and whose step in an early row $k + r_q$ is positive has a positive step in every row from $k + r_q$ to $2k - 1$, the end of its first layer (Proposition 10.5). In Section 16 this proposition is applied together with Corollary 16.3, which provides in every column $k \geq 82$ a positive row j with $k + 1 \leq j \leq k + 27k/100$, so that the first positive row of the column is early in this sense.

The comparison of the kernel on which the argument of Section 8 rests is established only up to row $k + 0.47k$. On the far rows we use instead that the first positive row is early, so that a long run of rows directly before row n is known to be positive; with such a run the step in row n can be bounded from below by the entries of column $k - 1$ alone.

10.1. **A lower bound for the step after a run of nonnegative steps.** Let $k \geq 2$. Throughout the section, $\delta_n = \delta_n^{(k)}$ is the normalised step of column k, which has the sign of $d_{n,k}$, and $\beta_n^{(k-1)}$ is the normalised count of column $k-1$ (Section 3). For a row $k + 1 \leq n \leq 2k - 1$ of the first layer of column k, the column equation of Proposition 4.12 reads

$$\delta_n = \frac{k+1}{n}\,\delta_{n-1} + \frac{2}{n}\,\beta_{n-1}^{(k-1)} + \sum_{m=1}^{n-k} \varepsilon_m\,\delta_{n-m}.$$

It expresses δ_n in terms of the source $(2/n)\,\beta_{n-1}^{(k-1)}$ and of the steps $\delta_{n-1}, \ldots, \delta_k$ of the earlier rows, and nothing is known about the size of the steps before the first positive row. Every step, however, is bounded from below in terms of column $k - 1$. Subtracting $\beta_j^{(k-1)}$ from both sides of the inequality $\gamma_k\,\beta_{j-1}^{(k-1)} \leq \beta_j^{(k)}$ of

Proposition 7.2, in which $\gamma_k = (1+1/k)^{k-2}$ is the diagonal factor of Section 7, we get

$$\gamma_k\,\beta_{j-1}^{(k-1)} - \beta_j^{(k-1)} \le \delta_j \qquad (j \ge 1). \tag{10.1}$$

The kernel of the column equation is the free prime kernel ε_m, which is positive. So (10.1) may be inserted for every step δ_{n-m} in the sum of the column equation, the feedback of the earlier rows, whatever the sign of that step. The entry $\beta_{n-1-m}^{(k-1)}$ then occurs twice, with the coefficient $\gamma_k\varepsilon_m$ from row $n-m$ and with the coefficient $-\varepsilon_{m+1}$ from row $n-m-1$, so that the insertion leaves $(\gamma_k\varepsilon_m - \varepsilon_{m+1})\,\beta_{n-1-m}^{(k-1)}$ at distance m. This is a summation by parts, and by itself it is of no use: the entry $\beta_{n-1}^{(k-1)}$ receives the coefficient $-\varepsilon_1 = -1$ from row $n-1$, against the coefficient $2/n$ that it has in the source.

The rows directly before n can be treated better when their steps are known to be nonnegative. For such a row the weight ε_m may be replaced by any smaller nonnegative weight before (10.1) is inserted. Suppose that the steps in the $\tau-1$ rows before n are nonnegative. On these rows we replace ε_m by the geometric weight $\psi\,\gamma_k^{m-1}$, where $\psi = \varepsilon_\tau/\gamma_k^{\tau-1}$. These weights are at most ε_m as soon as $\varepsilon_{m+1} \le \gamma_k\,\varepsilon_m$ for $m < \tau$; the insertions of (10.1) telescope exactly for them; and the term that the telescoping leaves at the far end of the run cancels against the first term of the summation by parts over the remaining rows. What the run costs is the single term $-\psi\,\beta_{n-1}^{(k-1)}$, and ψ is small when the run is long. The following lemma carries this out for arbitrary sequences, with ϕ, u, Δ and χ in the roles of ε, $\beta^{(k-1)}$, δ and γ_k.

Lemma 10.1. *Let $(\phi_m)_{m\ge0}$, $(u_j)_{j\ge0}$ and $(\Delta_j)_{j\ge0}$ be sequences of real numbers, let $\chi > 0$ be a real number, and let $k \ge 1$, n and τ be integers with $k+1 \le n$ and $2 \le \tau \le n-k$. Suppose that the following six conditions hold.*

(1) *$\phi_m > 0$ for every $m \ge 1$.*
(2) *$u_{k-1} = 0$.*
(3) *The sequences satisfy*

$$\frac{2}{n}\,u_{n-1} + \frac{k+1}{n}\,\Delta_{n-1} + \sum_{m=1}^{n-k}\phi_m\,\Delta_{n-m} \le \Delta_n.$$

(4) *$\chi\,u_{j-1} - u_j \le \Delta_j$ for every j with $k \le j \le n-1$.*
(5) *$\phi_{m+1} \le \chi\,\phi_m$ for every m with $1 \le m < \tau$.*
(6) *$\Delta_j \ge 0$ for every j with $n-\tau+1 \le j \le n-1$.*

Then

$$\Big(\frac{2}{n} - \frac{\phi_\tau}{\chi^{\tau-1}}\Big)\,u_{n-1} + \sum_{m=\tau}^{n-k-1}\big(\chi\,\phi_m - \phi_{m+1}\big)\,u_{n-1-m} \le \Delta_n.$$

Proof. Put $\psi = \phi_\tau/\chi^{\tau-1}$, which is positive by (1). We have

$$\psi\,\chi^{m-1} \le \phi_m \qquad (1 \le m \le \tau). \tag{10.2}$$

Indeed, for $m = \tau$ this is an equality, and the other cases follow from (5) by downward induction on m.

Since $\tau \ge 2$, we have $\Delta_{n-1} \ge 0$ by (6), and (3) gives

$$\frac{2}{n}\,u_{n-1} + \sum_{m=1}^{n-k}\phi_m\,\Delta_{n-m} \le \Delta_n. \tag{10.3}$$

For $1 \le m \le n-k$, condition (4) with $j = n-m$ reads $\chi\, u_{n-1-m} - u_{n-m} \le \Delta_{n-m}$.

For $1 \le m \le \tau - 1$ we have $\Delta_{n-m} \ge 0$ by (6), so that by (10.2) and (4)

$$\phi_m\, \Delta_{n-m} \ge \psi\, \chi^{m-1}\, \Delta_{n-m} \ge \psi\, \big(\chi^m\, u_{n-1-m} - \chi^{m-1}\, u_{n-m}\big).$$

The right-hand sides telescope, and their sum over these m is $\psi\, \chi^{\tau-1} u_{n-\tau} - \psi\, u_{n-1}$. As $\psi\, \chi^{\tau-1} = \phi_\tau$, this gives

$$\sum_{m=1}^{\tau-1} \phi_m\, \Delta_{n-m} \ge \phi_\tau\, u_{n-\tau} - \psi\, u_{n-1}. \tag{10.4}$$

For $\tau \le m \le n-k$ we use only that $\phi_m > 0$, which gives $\phi_m\, \Delta_{n-m} \ge \phi_m\, (\chi\, u_{n-1-m} - u_{n-m})$. We sum the right-hand sides over these m and collect the terms that contain the same entry of u. The entry u_{k-1}, which occurs with the coefficient $\chi\, \phi_{n-k}$, vanishes by (2), and we obtain

$$\sum_{m=\tau}^{n-k} \phi_m\, \Delta_{n-m} \ge -\phi_\tau\, u_{n-\tau} + \sum_{m=\tau}^{n-k-1} \big(\chi\, \phi_m - \phi_{m+1}\big)\, u_{n-1-m}. \tag{10.5}$$

When (10.4) and (10.5) are added, the terms $\phi_\tau\, u_{n-\tau}$ and $-\phi_\tau\, u_{n-\tau}$ cancel. Inserting the result into (10.3), we obtain the assertion. □

10.2. **The cost of the run and the far coefficients.** When Lemma 10.1 is applied to a column k, we have $\phi = \varepsilon$ and $\chi = \gamma_k$, and we write $t = \tau - 1$ for the length of the run. The run then costs the term $-\psi\, \beta_{n-1}^{(k-1)}$ with $\psi = \varepsilon_{t+1}/\gamma_k^t = \varepsilon_{t+1}\, \hat{x}_k^t$, where $\hat{x}_k = 1/\gamma_k$ is the diagonal tilt of column k (Section 8). The proof of Proposition 10.4 compares $n\psi/2$ with 1. The next lemma bounds $n\psi/2$, for a run of about σk rows and a row $n \le Lk$, by a quantity that decreases like $k^{-1/2}$. In it Y stands for n, and the factor C accounts for the facts that t is an integer between σk and $\sigma k + 1$ and that $\hat{x}_k$ exceeds e^{-1} by a factor of at most $e^{5/(2k)}$.

Lemma 10.2. *Let $k \ge 15$ and $t \ge 1$ be integers, and let σ, C, L and Y be real numbers such that $\sigma > 0$, $C > 0$ and*

$$\sigma\, (k+3) \le C\, (\sigma k + 1), \qquad \sigma k \le t \le \sigma k + 1, \qquad 0 \le Y \le L\, k.$$

Then the free prime kernel ε and the diagonal tilt $\hat{x}_k$ of column k satisfy

$$\frac{Y}{2}\, \varepsilon_{t+1}\, \hat{x}_k^t \le \frac{C\, L\, e^{5\sigma/2}}{2\sigma\sqrt{2\pi\sigma k}}.$$

Proof. Put $\psi = \varepsilon_{t+1}\, \hat{x}_k^t$. Since $(t+1)\, \varepsilon_{t+1} = \nu_t$ and $e^{-t}\nu_t = P(t)$, where $t \ge 1$, the upper bound of (2.5) gives $(t+1)\, \varepsilon_{t+1} \le e^t/\sqrt{2\pi t}$, and Lemma 8.2 gives $\hat{x}_k^t \le e^{-t} e^{5t/(2k)}$. As $t \le \sigma k + 1$, we obtain

$$\psi\, (t+1)\sqrt{2\pi t} \le e^{5t/(2k)} \le e^{5\sigma/2}\, e^{5/(2k)}.$$

It is easy to check that $e^h \le 1/(1-h) \le 1 + 6h/5$ for every real number h with $0 \le h \le 1/6$. As $5/(2k) \le 1/6$ for $k \ge 15$, this gives $e^{5/(2k)} \le 1 + 3/k$. By the hypothesis on C and by $\sigma k \le t$,

$$\sigma k \left(1 + \frac{3}{k}\right) = \sigma\, (k+3) \le C\, (\sigma k + 1) \le C\, (t+1).$$

We multiply the first display by σk, insert these two bounds, and divide by $t+1$. This gives $\psi\,\sigma k\sqrt{2\pi t} \le C\,e^{5\sigma/2}$, and here $\sqrt{2\pi t} \ge \sqrt{2\pi\sigma k}$. Finally $0 \le Y \le L\,k$ and $\psi \ge 0$, so that

$$\frac{Y}{2}\,\psi \le \frac{L\,k}{2}\cdot\frac{C\,e^{5\sigma/2}}{\sigma k\sqrt{2\pi\sigma k}} = \frac{C\,L\,e^{5\sigma/2}}{2\sigma\sqrt{2\pi\sigma k}}. \qquad \square$$

We turn to the sum in the conclusion of Lemma 10.1. Recall the step ratio g_m of the free prime kernel, for which $\varepsilon_{m+1} = g_m\,\varepsilon_m$ (Section 8). The coefficient that belongs to distance m is $\gamma_k\,\varepsilon_m - \varepsilon_{m+1} = -(g_m - \gamma_k)\,\varepsilon_m$. It is nonnegative as long as $m+1 \le 3k/5$, by Lemma 8.3, and it may be negative beyond. In the proof of Proposition 10.4 the coefficients with $m+1 > 3k/5$ occur with ε_m replaced by $e^{-m}\varepsilon_m$. For every $m \ge 2$ the number $e^{-m}\varepsilon_m$ is the term f_m of the frozen kernel of Section 9. What has to be bounded is therefore $(g_m - \gamma_k)\,f_m$, and the next lemma bounds it, uniformly in m, by a quantity of the order $k^{-5/2}$.

Lemma 10.3. *Let k and $m \ge 49$ be integers with $m+1 \le k$ and $3k \le 5(m+1)$. Then the step ratio g_m of the free prime kernel, the diagonal factor γ_k and the frozen kernel f satisfy*

$$(g_m - \gamma_k)\,f_m \le \frac{105}{100}\cdot\frac{c_{\mathrm{S}}}{k^2\sqrt{k}}.$$

Proof. Put $j = m+1$, so that $50 \le j \le k$ and $3k \le 5j$. We first show that

$$g_m - \gamma_k \le e\cdot\frac{5j-3k}{2kj}. \tag{10.6}$$

Since $m \ge 2$ and $m+1 = j$, the bound (8.9) for the step ratio reads $g_m \le e^{1-3/(2j)}$. On the other hand (8.8) gives $\gamma_k \ge e^{1-5/(2k)}$, because $k \ge 2$. As $5/(2k) - 3/(2j) = (5j-3k)/(2kj)$ and $e^{-h} \ge 1-h$ for every real h,

$$\gamma_k \ge e^{1-3/(2j)}\exp\Bigl(-\frac{5j-3k}{2kj}\Bigr) \ge e^{1-3/(2j)}\Bigl(1-\frac{5j-3k}{2kj}\Bigr).$$

Together with the bound for g_m this yields (10.6), because $5j-3k \ge 0$ and $e^{1-3/(2j)} \le e$.

Next, $m\,\varepsilon_m = \nu_{m-1}$ and $e^{-(m-1)}\nu_{m-1} = P(m-1)$, so that by the upper bound of (2.5)

$$f_m = e^{-m}\varepsilon_m = \frac{e^{-1}P(m-1)}{m} \le \frac{e^{-1}c_{\mathrm{S}}}{m\sqrt{m-1}} = \frac{e^{-1}c_{\mathrm{S}}}{(j-1)\sqrt{j-2}}.$$

Since $f_m \ge 0$ and the right-hand side of (10.6) is nonnegative, multiplying the two bounds gives

$$(g_m - \gamma_k)\,f_m \le \frac{(5j-3k)\,c_{\mathrm{S}}}{2kj\,(j-1)\sqrt{j-2}}.$$

It remains to show that $(5j-3k)\,k^{3/2} \le 1.05\cdot 2j\,(j-1)\sqrt{j-2}$. For every real $\rho \ge 0$,

$$2\rho^5 - 5\rho^2 + 3 = (\rho-1)^2\,(2\rho^3 + 4\rho^2 + 6\rho + 3) \ge 0.$$

For $\rho = \sqrt{j/k}$ and after multiplication by $k^{5/2}$ this is $(5j-3k)\,k^{3/2} \le 2j^{5/2}$. Moreover $j^{3/2} \le 1.05\,(j-1)\sqrt{j-2}$: after squaring and division by j^3 this is the inequality $1 \le 1.1025\,(1-1/j)^2(1-2/j)$, and for $j \ge 50$ its right-hand side is at least $1.1025\cdot 0.98^2\cdot 0.96 > 1$. Multiplying the last inequality by $2j$ and combining it with the one before, we obtain the claim. $\square$

10.3. **The far rows.** The following proposition treats a far row $k+r$ of the first layer of a column $k \geq 82$. The proof applies Lemma 10.1 to a run of positive rows before row $k+r$ whose length is proportional to k, and it bounds the cost of the run by Lemma 10.2. For $r \leq 3k/5$ this suffices, because the sum in the conclusion of Lemma 10.1 then has no negative term. For $r > 3k/5$ the terms of the sum that may be negative are bounded by Proposition 9.4, applied to column $k-1$ and to the renewal sequence G of the frozen kernel, and by Lemma 10.3 and the two bounds of Section 9 for G; as the bound that results grows with r, these rows are divided once more at $4k/5$, beyond which the run is longer and the bound for its cost is smaller.

Proposition 10.4. *Let $k \geq 82$, $r \geq 0$ and $r_q \geq 0$ be integers such that*

$$47\,k < 100\,r, \qquad r+1 \leq k, \qquad 3\,r_q \leq k,$$

and suppose that $d_{j,k} > 0$ for every j with $k + r_q \leq j < k + r$. Then $d_{k+r,k} > 0$.

Proof. Let $n = k + r$; the hypotheses on r say that $k + 0.47k < n \leq 2k - 1$. We write $\delta_j = \delta_j^{(k)} = d_{j,k}/j!$, and we have to show that $\delta_n > 0$.

Let $t_0 \geq 1$ be an integer such that $\delta_j \geq 0$ for every j with $n - t_0 \leq j \leq n-1$ and

$$t_0 + 1 \leq r, \qquad 5\,(t_0+1) \leq 3k, \tag{10.7}$$

and put $\psi = \varepsilon_{t_0+1}\,\hat{x}_k^{\,t_0}$, where $\hat{x}_k = 1/\gamma_k$. Then

$$\Big(\frac{2}{n} - \psi\Big)\,\beta_{n-1}^{(k-1)} + \sum_{m=t_0+1}^{r-1} \big(\gamma_k\,\varepsilon_m - \varepsilon_{m+1}\big)\,\beta_{n-1-m}^{(k-1)} \leq \delta_n. \tag{10.8}$$

This is the conclusion of Lemma 10.1 for the sequences $\phi_m = \varepsilon_m$, $u_j = \beta_j^{(k-1)}$ and $\Delta_j = \delta_j$, with $\chi = \gamma_k$ and $\tau = t_0 + 1$. Here $n - k = r$, so that the requirement $2 \leq \tau \leq n - k$ of the lemma is the first part of (10.7), and $\phi_\tau/\chi^{\tau-1} = \psi$. The six conditions of the lemma hold. Conditions (1) and (2) are clear, since $\varepsilon_m > 0$ for $m \geq 1$ and $\beta_{k-1}^{(k-1)} = b_{k-1,k-1}/(k-1)! = 0$. Condition (3) holds with equality: it is the column equation of Proposition 4.12, which is available because $k + 1 \leq n \leq 2k - 1$. Condition (4) is (10.1). Condition (5) follows from Lemma 8.3, since $5\,(m+1) \leq 5\,(t_0+1) \leq 3k$ for $m \leq t_0$ by the second part of (10.7), and condition (6) is the assumption on rows $n - t_0, \dots, n - 1$.

Next we fix the length t_0 of the run. We put

$$\sigma = \begin{cases} \dfrac{47}{100} - \dfrac{1}{3} = \dfrac{41}{300} & \text{if } r \leq \dfrac{3k}{5}, \\[2mm] \dfrac{3}{5} - \dfrac{1}{3} = \dfrac{4}{15} & \text{if } \dfrac{3k}{5} < r \leq \dfrac{4k}{5}, \\[2mm] \dfrac{4}{5} - \dfrac{1}{3} = \dfrac{7}{15} & \text{if } r > \dfrac{4k}{5}, \end{cases} \qquad t_0 = \lfloor \sigma k \rfloor + 1.$$

Then $\sigma k \leq t_0 \leq \sigma k + 1$, and in each of the three cases $r > (\sigma + 1/3)\,k$. Together with $3r_q \leq k$ this gives $r - r_q > \sigma k$, and hence $r - r_q \geq t_0$, because $r - r_q$ is an integer. Rows $n - t_0, \dots, n - 1$ are therefore among rows $k + r_q, \dots, n - 1$, in which the step is positive by hypothesis. Moreover $t_0 + 1 \leq \sigma k + 2 < (\sigma + 1/3)\,k < r$ and $5\,(t_0 + 1) \leq 5\sigma k + 10 \leq 7k/3 + 10 \leq 3k$, both because $k \geq 15$. Thus t_0 satisfies (10.7), and (10.8) holds with this t_0.

Let L and $C > 0$ be real numbers with $n \le L\,k$ and $\sigma\,(k+3) \le C\,(\sigma k + 1)$. Lemma 10.2, applied with $t = t_0$ and $Y = n$, bounds the nonnegative number $n\psi/2$ by $C\,L\,e^{5\sigma/2}/\big(2\sigma\sqrt{2\pi\sigma k}\big)$. Squaring, we get

$$\Big(\frac{n\psi}{2}\Big)^2 \le \frac{C^2 L^2\, e^{5\sigma}}{8\pi\,\sigma^3\,k}. \tag{10.9}$$

We shall evaluate the right-hand side with $\pi > 3.14$ and with the bounds

$$e^{41/60} \le 2, \qquad e^{4/3} \le 3.8, \qquad e^{7/3} \le 10.32$$

for the three values of $e^{5\sigma}$. These bounds hold because $e < 2.7183$ and $2.7183^{41} < 2^{60}$, $2.7183^4 < 3.8^3$, $2.7183^7 < 10.32^3$.

We also note that $\beta_{n-1}^{(k-1)} > 0$, because the preference sequence $(1, 2, \ldots, n-2,\, n-k)$ of length $n-1$ is clearly a parking function with maximum displacement $k-1$: its last car prefers spot $n-k$ and parks in spot $n-1$.

It remains to derive $\delta_n > 0$ from (10.8) in each of the three ranges of r that occur in the definition of σ. Suppose first that $r \le 3k/5$. Then every index m of the sum in (10.8) satisfies $5\,(m+1) \le 5r \le 3k$, so that its coefficient is nonnegative by Lemma 8.3; the entries $\beta_{n-1-m}^{(k-1)}$ are nonnegative as well. Dropping the sum, we get

$$\frac{2}{n}\Big(1 - \frac{n\psi}{2}\Big)\,\beta_{n-1}^{(k-1)} \le \delta_n. \tag{10.10}$$

Here $n \le 8k/5$ and $3\sigma \le 1$, so that (10.9) holds with $L = 8/5$ and $C = 1$:

$$\Big(\frac{n\psi}{2}\Big)^2 \le \frac{(8/5)^2 \cdot 2}{8 \cdot 3.14 \cdot (41/300)^3\,k} \le \frac{79.85}{k} < 1,$$

where the last step uses $k \ge 82$. Hence $n\psi/2 < 1$, and (10.10) gives $\delta_n > 0$, because $\beta_{n-1}^{(k-1)} > 0$.

Now let $r > 3k/5$. In this case the sum in (10.8) may have negative terms, and we bound it from below. Let $m_0 = \lfloor 3k/5 \rfloor$. Then $49 \le m_0 \le r-1 \le k-2$, the first inequality because $k \ge 82$. For $m \ge 1$ put $\theta_m = \max(g_m - \gamma_k,\, 0)$. As $\varepsilon_{m+1} = g_m\,\varepsilon_m$, we have

$$\gamma_k\,\varepsilon_m - \varepsilon_{m+1} = -(g_m - \gamma_k)\,\varepsilon_m \ge -\theta_m\,\varepsilon_m \qquad (m \ge 1).$$

Moreover $\theta_m = 0$ for $m < m_0$, because then $5\,(m+1) \le 5m_0 \le 3k$ and Lemma 8.3 gives $g_m \le \gamma_k$. As the entries $\beta_{n-1-m}^{(k-1)}$ are nonnegative, the sum in (10.8) is therefore at least $-\Sigma$, where

$$\Sigma = \sum_{m=m_0}^{r-1} \theta_m\,\varepsilon_m\,\beta_{n-1-m}^{(k-1)},$$

and (10.8) implies

$$\Big(\frac{2}{n} - \psi\Big)\,\beta_{n-1}^{(k-1)} - \Sigma \le \delta_n. \tag{10.11}$$

To bound Σ we compare column $k-1$ with the renewal sequence of the frozen kernel. Let $v_i = e^{-i}\beta_{k+i}^{(k-1)}$ for $i \ge 0$. We claim that

$$\sum_{m=1}^{i} f_m\,v_{i-m} \le v_i \qquad (0 \le i \le k-2). \tag{10.12}$$

For $i = 0$ this says that $v_0 \geq 0$. For $1 \leq i \leq k-2$ we apply Lemma 4.13 to column $k-1$ and row $k+i$, which is at most $2k-1$:

$$\beta_{k+i}^{(k-1)} = \frac{z_{k+i-1}^{(k)}}{k+i} + \frac{k-1}{k+i}\,\beta_{k+i-1}^{(k-1)} + \sum_{m=1}^{i} \varepsilon_m\,\beta_{k+i-m}^{(k-1)}.$$

The first term on the right is nonnegative, and $(k-1)/(k+i) \geq 1/2$ because $i \leq k-2$. Dropping the first term and multiplying by e^{-i}, we get

$$v_i \geq \frac{e^{-1}}{2}\,v_{i-1} + \sum_{m=1}^{i} e^{-m}\varepsilon_m\,v_{i-m}.$$

The right-hand side is the left-hand side of (10.12), since $f_1 = e^{-1}\varepsilon_1 + e^{-1}/2$ and $f_m = e^{-m}\varepsilon_m$ for $m \geq 2$. This proves (10.12). We need it for $i \leq r-1$, and these i are at most $k-2$ by the hypothesis $r \leq k-1$.

The frozen kernel is positive, and it is log-convex by Lemma 9.5. The sequence G is its renewal sequence by Lemma 9.6. The weights θ_m are nonnegative, and they do not decrease when m grows, because g_m does not (Section 8). With (10.12) for $i \leq r-1$, these are the hypotheses of Proposition 9.4 for the kernel f, its renewal sequence G, the sequence v and the weights θ_m with $m_0 \leq m \leq r-1$. The proposition gives

$$\Big(\sum_{m=m_0}^{r-1} \theta_m\,f_m\,v_{r-1-m}\Big)\,G_{r-1} \leq \Big(\sum_{m=m_0}^{r-1} \theta_m\,f_m\,G_{r-1-m}\Big)\,v_{r-1}.$$

For $m \geq m_0$ we have $f_m = e^{-m}\varepsilon_m$, so that $f_m\,v_{r-1-m} = e^{-(r-1)}\,\varepsilon_m\,\beta_{n-1-m}^{(k-1)}$ by the definition of v, and $v_{r-1} = e^{-(r-1)}\beta_{n-1}^{(k-1)}$. After the factor $e^{-(r-1)}$ has been cancelled, the inequality reads

$$\Sigma\,G_{r-1} \leq \Big(\sum_{m=m_0}^{r-1} \theta_m\,f_m\,G_{r-1-m}\Big)\,\beta_{n-1}^{(k-1)}. \tag{10.13}$$

For $m_0 \leq m \leq r-1$ we have $m \geq 49$, $m+1 \leq k$ and $3k < 5\,(m_0+1) \leq 5\,(m+1)$, so that Lemma 10.3 gives $\theta_m\,f_m \leq 1.05\,c_{\mathrm{S}}\,k^{-5/2}$. The numbers G_i are positive by (9.8). Hence the sum in (10.13) is at most $1.05\,c_{\mathrm{S}}\,k^{-5/2}$ times $G_0 + G_1 + \cdots + G_{r-1-m_0}$. We multiply (10.11) by G_{r-1}, insert (10.13) with this bound, and divide by G_{r-1} again. With

$$\Omega = \frac{n}{2} \cdot \frac{1.05\,c_{\mathrm{S}}}{k^{5/2}} \cdot \frac{1}{G_{r-1}} \sum_{i=0}^{r-1-m_0} G_i$$

the result is

$$\frac{2}{n}\Big(1 - \frac{n\psi}{2} - \Omega\Big)\,\beta_{n-1}^{(k-1)} \leq \delta_n. \tag{10.14}$$

Since $\beta_{n-1}^{(k-1)} > 0$, it remains to show that $n\psi/2 + \Omega < 1$.

For this we bound G_{r-1} from below. For an integer $j \geq 6$ let $\upsilon(j) = \big(1 - 1/(12j)\big)\big(4 - 21/j\big)$; both factors are positive and increase with j. As $r-1 \geq m_0 \geq 6$, Lemma 9.7 gives $G_{r-1} \geq e\,c_{\mathrm{S}}\,r^{-3/2}\,\upsilon(r)$. As $n = k + r$, it follows that, if $r_0 \geq 6$ is

an integer with $r \ge r_0$, so that $\upsilon(r) \ge \upsilon(r_0)$, then

$$\Omega \le \frac{1.05}{2} \cdot \frac{(k+r)\, r^{3/2}}{k^{5/2}} \cdot \frac{1}{e\, \upsilon(r_0)} \sum_{i=0}^{r-1-m_0} G_i. \tag{10.15}$$

In the range $3k/5 < r \le 4k/5$ we have $n \le 9k/5$ and $3\sigma \le 1$, so that (10.9) holds with $L = 9/5$ and $C = 1$:

$$\Big(\frac{n\psi}{2}\Big)^2 \le \frac{(9/5)^2 \cdot 3.8}{8 \cdot 3.14 \cdot (4/15)^3\, k} \le \frac{25.85}{k},$$

and hence $n\psi/2 \le 0.57$, because $k \ge 82$. In (10.15) we may take $r_0 = 50$, because $r > 3k/5 \ge 49.2$, and $\upsilon(50) = (599/600) \cdot (179/50) > 3.574$. The sum of the G_i is at most $2e$ by Lemma 9.10, and

$$(k+r)\, r^{3/2} \le \frac{9}{5} \Big(\frac{4}{5}\Big)^{3/2} k^{5/2} \le 1.2881\, k^{5/2},$$

since $(4/5)^3 \le 0.7156^2$. Therefore $\Omega \le 1.05 \cdot 1.2881/3.574 < 0.385$, and $n\psi/2 + \Omega < 0.57 + 0.385 < 1$.

Finally, in the range $r > 4k/5$ we have $n \le 2k$, and (10.9) holds with $L = 2$ and $C = 1.011$: for $\sigma = 7/15$ the condition $\sigma\,(k+3) \le C\,(\sigma k + 1)$ is equivalent to $7/5 - 1.011 \le 0.011 \cdot 7k/15$, which holds for $k \ge 76$. Thus

$$\Big(\frac{n\psi}{2}\Big)^2 \le \frac{1.011^2 \cdot 4 \cdot 10.32}{8 \cdot 3.14 \cdot (7/15)^3\, k} \le \frac{16.53}{k}.$$

In (10.15) we have $(k+r)\, r^{3/2} \le 2k^{5/2}$, since $r \le k$.

Suppose first that $k \ge 110$. Then $(n\psi/2)^2 \le 16.53/110$, hence $n\psi/2 \le 0.395$. We may take $r_0 = 89$, because $r > 4k/5 \ge 88$, and $\upsilon(89) = (1067/1068) \cdot (335/89) > 3.76$. With the bound $2e$ of Lemma 9.10 for the sum of the G_i we get $\Omega \le 2.1/3.76 < 0.56$, and $n\psi/2 + \Omega < 0.395 + 0.56 < 1$.

Suppose now that $82 \le k \le 109$. Then $(n\psi/2)^2 \le 16.53/82$, hence $n\psi/2 \le 0.455$. We may take $r_0 = 66$, because $r > 4k/5 \ge 65.6$, and $\upsilon(66) = (791/792) \cdot (81/22) > 3.677$. The bound $2e$ for the sum of the G_i is too weak here, and we bound the sum more carefully. Let $J = r - m_0 + 1$, so that the sum in question is $G_0 + G_1 + \cdots + G_{J-2}$. From $4k/5 < r \le k - 1$ and $3k/5 - 1 < m_0 \le 3k/5$ we get $k/5 < r - m_0 < 2k/5$, and hence $18 \le J \le 44$, because $82 \le k \le 109$. For $i \ge 17$ we have $1 - 1/(12(i+1)) \ge 215/216$ and $4 - 21/(i+1) \ge 17/6$, so that Lemma 9.7 and the bound $c_{\mathrm{S}} \ge 0.39894$ give

$$G_i \ge \frac{e\, c_{\mathrm{S}}}{(i+1)^{3/2}} \cdot \frac{215}{216} \cdot \frac{17}{6} \ge \frac{1.12\, e}{(i+1)^{3/2}} \qquad (i \ge 17).$$

Inserting here the bound $1/(j\sqrt{j}) \ge 2\,\big(1/\sqrt{j} - 1/\sqrt{j+1}\big)$ of Section 2 with $j = i+1$ and summing over $J - 1 \le i \le 16J - 2$, where $i \ge 17$, we get

$$\sum_{i=J-1}^{16J-2} G_i \ge 2 \cdot 1.12\, e \Big(\frac{1}{\sqrt{J}} - \frac{1}{\sqrt{16J}}\Big) = \frac{1.68\, e}{\sqrt{J}}.$$

The right-hand side is at least $1.68\, e/6.64 \ge 0.253\, e$, because $J \le 44 \le 6.64^2$. On the other hand $G_0 + G_1 + \cdots + G_{16J-2} \le 2e$ by Lemma 9.10. Subtracting, we obtain $G_0 + G_1 + \cdots + G_{J-2} \le (2 - 0.253)\, e = 1.747\, e$. Now (10.15) gives $\Omega \le 1.05 \cdot 1.747/3.677 < 0.5$, and $n\psi/2 + \Omega < 0.455 + 0.5 < 1$.

In each of the last two ranges of r we have shown that $n\psi/2 + \Omega < 1$, so that (10.14) gives $\delta_n > 0$. □

10.4. **Columns with an early first positive row.** We now combine Propositions 8.6 and 10.4 by an induction along the column.

Proposition 10.5. *Let $k \geq 82$ and $r_q \geq 1$ be integers with $3\,r_q \leq k$. Suppose that $d_{j,k} \leq 0$ for every j with $k \leq j < k + r_q$, and that $d_{k+r_q,k} > 0$. Then $d_{n,k} > 0$ for every n with $k + r_q \leq n \leq 2k - 1$.*

Proof. We prove $d_{n,k} > 0$ by induction on n. For $n = k + r_q$ this is a hypothesis. Let $k + r_q < n \leq 2k - 1$, and suppose that $d_{j,k} > 0$ for every j with $k + r_q \leq j < n$. Write $n = k + r$, so that $r_q < r \leq k - 1$. If $100\,r > 47\,k$, then Proposition 10.4, applied with these r and r_q, gives $d_{n,k} > 0$. If $100\,r \leq 47\,k$, we apply Proposition 8.6 with $t = r - r_q \geq 1$. Its hypotheses hold: we have $100\,(r_q + t) \leq 47\,k$ and $d_{j,k} \leq 0$ for $k \leq j < k + r_q$, and the positivity of $d_{j,k}$ for $k + r_q \leq j < k + r_q + t = n$ is the induction hypothesis. It gives $d_{k+r_q+t,k} = d_{n,k} > 0$. □

11. Weighted sums over compositions

In Section 13 the normalised step at a position of the first layer is expressed through the chain sums $\Lambda_m(y; N)$ of the introduction. A chain sum is a sum over the compositions of the integer m in which a part of size p has weight a_p and a cut at position s has weight $y/(N - s)$. Here we define such sums for an arbitrary weight $f(s)$ of the cuts, since the later sections compare the weight $y/(N - s)$ with constant weights, for which the sums can be evaluated, and we prove the recursions and identities for them on which the proofs of Propositions 16.2 and 22.6 rest.

11.1. **Compositions and cut profiles.** Let $m \geq 1$ be an integer. A *composition* of m is a finite sequence $\mathbf{p} = (p_1, \dots, p_q)$ of positive integers with $p_1 + \cdots + p_q = m$. The numbers p_i are its *parts*, and $q \geq 1$ is the number of its parts. The *cut points* of $\mathbf{p}$ are the $q - 1$ partial sums $s_i = p_1 + \cdots + p_i$ with $1 \leq i \leq q - 1$. With $s_0 = 0$ and $s_q = m$ we have $0 = s_0 < s_1 < \cdots < s_q = m$ and $p_i = s_i - s_{i-1}$. We think of $\mathbf{p}$ as a way of cutting a segment of length m at some of its interior integer points $1, \dots, m - 1$ into pieces of the lengths $p_1, \dots, p_q$. A composition is determined by the set of its cut points, and every subset of $\{1, \dots, m - 1\}$ is the set of cut points of exactly one composition of m. The integer 0 has no composition.

A *cut profile* is a function $f\colon \mathbb{N} \to \mathbb{R}$, where $\mathbb{N} = \{0, 1, 2, \dots\}$; the number $f(s)$ is the weight of a cut at position s. The *weight* of the composition $\mathbf{p}$ for the cut profile f is

$$\mathrm{wt}_f(\mathbf{p}) = \prod_{i=1}^{q} a_{p_i} \prod_{i=1}^{q-1} f(s_i),$$

the product of the Cayley coefficients of its parts and of the values of f at its cut points. These values, one for each cut point of $\mathbf{p}$, are called the *cut factors* of $\mathbf{p}$. For integers $q \geq 0$ and $m \geq 0$ we write $V_q(m; f)$ for the sum of the weights $\mathrm{wt}_f(\mathbf{p})$ of the compositions $\mathbf{p}$ of m with exactly q parts, so that

$$V_q(m; f) = \sum_{0 = s_0 < s_1 < \cdots < s_q = m} \prod_{i=1}^{q} a_{s_i - s_{i-1}} \prod_{i=1}^{q-1} f(s_i) \qquad (q \geq 1),$$

and we write

$$\Lambda_m[f] = \sum_{q=1}^{m} V_q(m; f)$$

for the total weight of all compositions of m. We have $V_q(m; f) = 0$ for $q = 0$, for $m = 0$ and for $q > m$, since a composition of m has at least one part and at most m parts, and $\Lambda_0[f] = 0$. The only composition with one part is (m), which has no cut point, so that $V_1(m; f) = a_m$ for every $m \geq 0$ and every profile; for $m = 0$ this holds because $a_0 = 0$. For example, $a_1 = a_2 = 1$ and $a_3 = 3/2$, and the compositions (3), $(1, 2)$, $(2, 1)$ and $(1, 1, 1)$ of 3 have the sets of cut points $\emptyset$, $\{1\}$, $\{2\}$ and $\{1, 2\}$. Hence $V_1(3; f) = 3/2$, $V_2(3; f) = f(1) + f(2)$ and $V_3(3; f) = f(1)\, f(2)$, and

$$\Lambda_3[f] = \tfrac{3}{2} + f(1) + f(2) + f(1)\, f(2).$$

The following properties are immediate from the definition. Let f and g be cut profiles and let $m \geq 0$. The weight of a composition of m involves the values of the profile at the points $1, \ldots, m-1$ only. Hence, if $f(s) = g(s)$ for $0 < s < m$, then $V_q(m; f) = V_q(m; g)$ for every q, and $\Lambda_m[f] = \Lambda_m[g]$. If $f(s) \geq 0$ for $0 < s < m$, then $V_q(m; f) \geq 0$ for every q. If $f(s) > 0$ for $0 < s < m$, then $V_q(m; f) > 0$ for $1 \leq q \leq m$. If $0 \leq f(s) \leq g(s)$ for $0 < s < m$, then $\Lambda_m[f] \leq \Lambda_m[g]$.

When $m \geq 1$, f is nonnegative at the points $1, \ldots, m-1$ and $\Lambda_m[f] > 0$, the numbers $V_q(m; f)/\Lambda_m[f]$ with $1 \leq q \leq m$ are nonnegative and add up to one. They are the probabilities with which a composition of m that is drawn with probability proportional to its weight has q parts. Several statements of this section describe this law of the number of parts; we use the probabilistic language only to say what a sum means, and every such statement is an identity between finite sums.

The identities of this section rest on one observation: a composition can be split at each of its cut points. For a cut profile f and an integer $b \geq 0$ we write f^{+b} for the *shifted profile* $s \mapsto f(b + s)$. Let $\mathbf{p} = (p_1, \ldots, p_q)$ be a composition of m and let $b = s_j$ be one of its cut points. Then $\mathbf{p}' = (p_1, \ldots, p_j)$ is a composition of b, and $\mathbf{p}'' = (p_{j+1}, \ldots, p_q)$ is a composition of $m - b$. The cut points of $\mathbf{p}$ are the cut points of $\mathbf{p}'$, the point b, and the points $b + s$ for the cut points s of $\mathbf{p}''$. Conversely, for $0 < b < m$ every pair $(\mathbf{p}', \mathbf{p}'')$ of a composition of b and a composition of $m - b$ arises in this way from exactly one composition $\mathbf{p}$ of m that has a cut point at b, namely from the sequence $\mathbf{p}'$ followed by the sequence $\mathbf{p}''$. If $\mathbf{p}'$ has j parts and $\mathbf{p}''$ has r parts, then $\mathbf{p}$ has $j + r$ parts, and b is its j-th cut point.

Let $q \geq 1$. A composition of m with $q + 1$ parts has a last cut point $s = s_q$. Split at this point, it consists of a composition $\mathbf{p}'$ of s with q parts and of the composition $\mathbf{p}'' = (m - s)$ with one part, which has no cut point. The cut factors of the composition are therefore those of $\mathbf{p}'$ together with $f(s)$, and its last part contributes a_{m-s}. Sorting these compositions by their last cut point gives, for all $q \geq 1$ and $m \geq 0$,

$$V_{q+1}(m; f) = \sum_{0<s<m} V_q(s; f)\, f(s)\, a_{m-s}. \tag{11.1}$$

Together with the values $V_0(m; f) = 0$ and $V_1(m; f) = a_m$, the recursion (11.1) determines all the numbers $V_q(m; f)$. The following two lemmas are recursions for the total weights, by the last cut and by the first part.

Lemma 11.1. *For every function $f\colon \mathbb{N} \to \mathbb{R}$ and every integer $m \geq 0$,*

$$\Lambda_m[f] = a_m + \sum_{0<s<m} \Lambda_s[f]\, f(s)\, a_{m-s}.$$

Proof. For $m = 0$ both sides are zero. Let $m \geq 1$. Since $V_1(m; f) = a_m$, the recursion (11.1) gives

$$\begin{aligned} \Lambda_m[f] &= a_m + \sum_{q=1}^{m-1} V_{q+1}(m; f) \\ &= a_m + \sum_{0<s<m} \Big(\sum_{q=1}^{m-1} V_q(s; f) \Big) f(s)\, a_{m-s}. \end{aligned}$$

For $0 < s < m$ the inner sum is $\Lambda_s[f]$, because $s \leq m-1$ and $V_q(s; f) = 0$ for $q > s$. □

Lemma 11.2. *For every function $f\colon \mathbb{N} \to \mathbb{R}$ and every integer $m \geq 0$,*

$$\Lambda_m[f] = a_m + \sum_{0<p<m} a_p\, f(p)\, \Lambda_{m-p}[f^{+p}].$$

Proof. Let $\mathbf{p}$ be a composition of m, let b be one of its cut points, and let $\mathbf{p}'$ and $\mathbf{p}''$ be the compositions of b and of $m-b$ into which b splits it. The cut points of $\mathbf{p}$ are the cut points of $\mathbf{p}'$, the point b, and the points $b+s$ for the cut points s of $\mathbf{p}''$. Consequently

$$\mathrm{wt}_f(\mathbf{p}) = \mathrm{wt}_f(\mathbf{p}')\, f(b)\, \mathrm{wt}_{f^{+b}}(\mathbf{p}''). \tag{11.2}$$

By the correspondence between the pairs $(\mathbf{p}', \mathbf{p}'')$ and the compositions of m with a cut point at b described above, we may sum (11.2) over the pairs $(\mathbf{p}', \mathbf{p}'')$ with given numbers $j \geq 1$ and $r \geq 1$ of parts and obtain, for $0 < b < m$,

$$\sum_{\mathbf{p}} \mathrm{wt}_f(\mathbf{p}) = V_j(b; f)\, f(b)\, V_r(m-b; f^{+b}), \tag{11.3}$$

where $\mathbf{p}$ runs through the compositions of m with $j+r$ parts whose j-th cut point is b. Let $q \geq 1$. A composition of m with $q+1$ parts has a first cut point $p = s_1$. We sort these compositions by their first cut point, for which $j = 1$, $r = q$ and $V_1(p; f) = a_p$. This gives, for every $m \geq 0$,

$$V_{q+1}(m; f) = \sum_{0<p<m} a_p\, f(p)\, V_q(m-p; f^{+p}). \tag{11.4}$$

For $m = 0$ both sides of the assertion are zero. For $m \geq 1$ we argue as in the proof of Lemma 11.1, with (11.4) instead of (11.1):

$$\Lambda_m[f] = a_m + \sum_{0<p<m} a_p\, f(p) \sum_{q=1}^{m-1} V_q(m-p; f^{+p}),$$

and the inner sum is $\Lambda_{m-p}[f^{+p}]$, because $m - p \leq m-1$. □

The recursion by the last cut keeps the profile, whereas the recursion by the first part changes it: the cut points of the composition that remains lie at positions $p+s$ of the original segment.

The cut profile that the parking problem produces is the following. For real numbers y and N the *threshold profile* $\lambda^{y,N}$ is the cut profile

$$\lambda^{y,N}(s) = \frac{y}{N-s} \qquad (s \neq N).$$

If N is a nonnegative integer, the quotient is not defined at $s = N$, and we put $\lambda^{y,N}(N) = 0$, so that the profile is defined at every point of $\mathbb{N}$ for all real numbers y and N. The total weight for the threshold profile is written

$$\Lambda_m(y; N) = \Lambda_m[\lambda^{y,N}]$$

and is called a *chain sum*, and y is called its *threshold*. By the cut factors of the chain sum $\Lambda_m(y; N)$ we mean those of the compositions of m for the threshold profile, the numbers $\lambda^{y,N}(s)$ with $0 < s < m$.

A chain sum is thus defined for all real numbers y and N and every integer $m \geq 0$. Suppose first that $m \leq N$. Then every cut point of a composition of m is an integer s with $0 < s < m \leq N$. The cut factors of the chain sum are therefore the numbers $y/(N-s)$ with $0 < s < m$, their denominators are positive, and the value that we gave to the profile at the point N does not occur. For $y > 0$ the weight $y/(N-s)$ of a cut increases with $s < N$. For $m \geq 1$ the chain sum is then the sum that was displayed in the introduction. If $m > N$, on the other hand, a composition of m can have cut points $s \geq N$, at which the denominator $N - s$ is zero or negative. The identities below that are stated for all real N and every m include this case, and in them the profile has the value 0 at the point N.

Chain sums arise in Section 13, and the way in which they arise there is the reason for the form of the threshold profile. For an integer $k \geq 0$ and a threshold $j \geq 1$ that is one of the integers k and $k+1$, the linear relation of Proposition 4.9 is applied there repeatedly to the normalised counts $z_n^{(j)}$ with $0 \leq n \leq 2k+1$, and with $N = n+1$ this expresses the count, apart from one term, as a sum over the decreasing chains $N > t_1 > \cdots > t_i > k$ of integers. Such a chain corresponds to a composition of the integer $m = N - t_i$, in which the integers t_l of the chain other than the last give the cut points $s_l = N - t_l$, and the factor j/t_l with which such an integer occurs is a cut factor of the form $y/(N-s)$, the value of the threshold profile $\lambda^{j,N}$ at the cut point s_l. Since $t_i > 0$, the chain sums that arise in this way have $m < N$.

The properties listed above take the following form for the threshold profile. If $y \geq 0$, then $\lambda^{y,N}(s) \geq 0$ for every integer s with $0 \leq s \leq N$. Hence for $y \geq 0$ and an integer m with $1 \leq m \leq N$ the numbers $V_q(m; \lambda^{y,N})$ are nonnegative and $\Lambda_m(y; N) > 0$. A composition with q parts has $q-1$ cut points, and $\lambda^{y,N}(s) = y\,\lambda^{1,N}(s)$ at every point s of $\mathbb{N}$, the point $s = N$ included. Hence

$$V_q(m; \lambda^{y,N}) = y^{q-1}\, V_q(m; \lambda^{1,N}) \qquad (q \geq 1), \tag{11.5}$$

$$\Lambda_m(y; N) = \sum_{q=1}^{m} y^{q-1}\, V_q(m; \lambda^{1,N})$$

for all real y and N and every $m \geq 0$. The chain sum is thus a polynomial in the threshold, in which the compositions with q parts contribute to the coefficient of y^{q-1}; the coefficients depend on m and N only, and they are nonnegative when $m \leq N$. Finally, the shift of a threshold profile is a threshold profile, $(\lambda^{y,N})^{+p} =$

$\lambda^{y,N-p}$. Lemma 11.2 therefore reads

$$\Lambda_m(y;N) = a_m + \sum_{0<p<m} a_p \, \frac{y}{N-p} \, \Lambda_{m-p}(y;N-p) \qquad (m \le N).$$

Removing a first part of size p lowers the integer N to $N-p$, and in this form the recursion by the first part is one step of the iteration described above.

11.2. **Marked cut points.** In the estimates that lead to Proposition 16.2 a chain is separated at a cut point of its composition, that is, at one of its integers other than the last, and the two pieces are estimated separately. The sums that arise in this way are the following. For a cut profile f and an integer $m \ge 0$ we put

$$\Lambda^{(1)}_m[f] = \sum_{0<b<m} \Lambda_b[f] \, f(b) \, \Lambda_{m-b}[f^{+b}],$$
$$\Lambda^{(2)}_m[f] = \sum_{0<b<m} \Lambda^{(1)}_b[f] \, f(b) \, \Lambda_{m-b}[f^{+b}].$$

A composition together with one of its cut points is a composition with one *marked* cut point, and $\Lambda^{(1)}_m[f]$ is the total weight of the compositions of m with one marked cut point. The second definition repeats the first one level higher, and $\Lambda^{(2)}_m[f]$ is the total weight of the compositions of m with two marked cut points. Lemmas 11.5 and 11.6 below make these descriptions precise. The first sum is empty for $m \le 1$, so that $\Lambda^{(1)}_0[f] = \Lambda^{(1)}_1[f] = 0$, and therefore $\Lambda^{(2)}_m[f] = 0$ for $m \le 2$. For the threshold profile we write $\Lambda^{(1)}_m(y;N) = \Lambda^{(1)}_m[\lambda^{y,N}]$.

For the proof of Proposition 13.9 we need a recursion by the last cut for the sums with marked cut points as well. When the last part of a composition with two marked cut points is removed, its last cut point s was either the larger of the two marked points, and a composition of s with one marked cut point remains, or it was not marked, and a composition of s with two marked cut points remains.

Lemma 11.3. *For every function $f\colon \mathbb{N} \to \mathbb{R}$ and every integer $m \ge 0$,*

$$\Lambda^{(2)}_m[f] = \sum_{0<s<m} \big(\Lambda^{(1)}_s[f] + \Lambda^{(2)}_s[f]\big) \, f(s) \, a_{m-s}.$$

Proof. Let $0 < b < m$. Lemma 11.1, applied to the profile f^{+b} and to the integer $m-b$, gives, after a shift of the index of summation by b,

$$\Lambda_{m-b}[f^{+b}] = a_{m-b} + \sum_{b<s<m} \Lambda_{s-b}[f^{+b}] \, f(s) \, a_{m-s}.$$

We multiply by $\Lambda^{(1)}_b[f] \, f(b)$ and sum over $0 < b < m$. The left side becomes $\Lambda^{(2)}_m[f]$. On the right side the terms a_{m-b} contribute the terms of the asserted sum that contain $\Lambda^{(1)}$, and the other terms form a double sum over $0 < b < s < m$. Summed first over b, it is

$$\sum_{0<s<m} \Big(\sum_{0<b<s} \Lambda^{(1)}_b[f] \, f(b) \, \Lambda_{s-b}[f^{+b}] \Big) f(s) \, a_{m-s},$$

and the sum in brackets is $\Lambda^{(2)}_s[f]$ by definition. □

11.3. **Compositions without cut factors.** We now turn to the constant cut profile $\mathbf{1}$ with the value 1, for which the cuts carry no factor. For integers $m \geq 0$ and $q \geq 0$ let $T_{m,q} = V_q(m; \mathbf{1})$. Thus $T_{m,0} = 0$ and

$$T_{m,q} = \sum_{\substack{p_1+\cdots+p_q=m \\ p_i \geq 1}} a_{p_1} \cdots a_{p_q} \qquad (q \geq 1).$$

By Section 11.1 we have $T_{m,1} = a_m$, $T_{m,q} \geq 0$ for all m and q, $T_{m,q} > 0$ for $1 \leq q \leq m$, and $T_{m,q} = 0$ for $q > m$. For a real number h, regarded as a constant cut profile, and every integer $m \geq 0$, the total weight $\Lambda_m[h]$ is the sum of $T_{m,q}\, h^{q-1}$ over $1 \leq q \leq m$. The numbers $T_{m,q}$ are in this sense the weights of the model in which a composition is weighted by its parts alone, and Sections 20 and 21 compare the weights of the threshold profile with them.

The numbers $T_{m,q}$ can be evaluated in closed form by the Lagrange inversion formula. We state the result, which Sections 12, 20 and 21 use, with the probabilities of the birthday problem. For integers $m \geq 1$ and $j \geq 0$ let

$$B_{m,j} = \prod_{i=0}^{j-1} \Big(1 - \frac{i}{m}\Big) = \frac{m(m-1)\cdots(m-j+1)}{m^j}.$$

The number $B_{m,j}$, the *birthday tail*, is the probability that j independent draws, each uniformly distributed over m values, are pairwise distinct. Clearly $B_{m,0} = B_{m,1} = 1$ and

$$B_{m,j+1} = B_{m,j}\Big(1 - \frac{j}{m}\Big) \qquad (j \geq 0). \tag{11.6}$$

For $j \leq m$ we have $B_{m,j} = m!/((m-j)!\, m^j) > 0$, and for $j \geq m+1$ the product contains the factor with $i = m$, which is zero, so that $B_{m,j} = 0$; in particular $B_{m,m+1} = 0$, and $B_{m,j} \geq 0$ for all j.

Proposition 11.4. *Let $m \geq 0$ and $q \geq 1$ be integers. If $m \geq 1$, then*

$$T_{m,q} = a_m\, q\, B_{m,q}.$$

If $m = 0$, then $T_{0,q} = 0$.

Proof. The integer 0 has no composition, so that $T_{0,q} = 0$. Let $m \geq 1$. If $q > m$, then $T_{m,q} = 0$ and $B_{m,q} = 0$. Let $1 \leq q \leq m$. The tree series $T = \sum_{j \geq 1} a_j X^j$ of Section 4 (the letter T without indices denotes this series) has no constant term. The coefficient of X^m in its q-th power is therefore the sum of the products $a_{p_1} \cdots a_{p_q}$ over the sequences $(p_1, \ldots, p_q)$ of positive integers with sum m, which is $T_{m,q}$. Since the tree series satisfies the equation $T = Xe^T$, the Lagrange inversion formula [8, Theorem A.2] applies to it with the series $\phi(u) = e^u$, whose constant term is not zero. In its form for the powers of the solution of such an equation, the formula gives

$$[X^m]\, T^q = \frac{q}{m}\, [u^{m-q}]\, \phi(u)^m = \frac{q}{m}\, [u^{m-q}]\, e^{mu},$$

where $[u^{m-q}]$ denotes the coefficient of u^{m-q} in a power series in u. This coefficient of e^{mu} is $m^{m-q}/(m-q)!$. Since $a_m = m^{m-1}/m!$ and $B_{m,q} = m!/((m-q)!\, m^q)$, it follows that

$$T_{m,q} = \frac{q}{m} \cdot \frac{m^{m-q}}{(m-q)!} = \frac{q\, m^{m-q-1}}{(m-q)!} = a_m\, q\, B_{m,q}. \qquad \square$$

For instance, the proposition gives $T_{m,m} = 1$, the weight of the composition of m into m parts of size 1, and for $m = 3$ it gives 3/2, 2 and 1, the values of the example of Section 11.1 for the profile **1**.

Let $m \geq 1$. By (11.6), $(q/m)\, B_{m,q} = B_{m,q} - B_{m,q+1}$, and therefore

$$T_{m,q} = m\, a_m \left(B_{m,q} - B_{m,q+1}\right) \qquad (q \geq 1),$$

$$\sum_{q=1}^{m} T_{m,q} = m\, a_m \left(B_{m,1} - B_{m,m+1}\right) = m\, a_m, \tag{11.7}$$

the sum being a telescoping one. The *birthday law of* m is the law of a random variable Q with $\Pr(Q \geq j) = B_{m,j}$ for $j \geq 1$. It is the law of the number of draws that precede the first repetition when one draws independently and uniformly from m values, since that number is at least j exactly when the first j draws are pairwise distinct. It is carried by the integers $1, \ldots, m$, and its mean, which is the sum of $\Pr(Q \geq j)$ over $j \geq 1$, is $B_{m,1} + \cdots + B_{m,m}$. Section 12 studies this mean.

11.4. **The number of cut points.** We return to an arbitrary cut profile and evaluate the sums with marked cut points of Section 11.2 by the number of parts. A composition with q parts has $q - 1$ cut points, so that there are $q - 1$ ways to mark one of them.

Lemma 11.5. *For every function* $f\colon \mathbb{N} \to \mathbb{R}$ *and every integer* $m \geq 0$,

$$\Lambda_m^{(1)}[f] = \sum_{q=1}^{m} (q-1)\, V_q(m; f).$$

Proof. Let $0 < b < m$. We sum (11.3), which was shown in the proof of Lemma 11.2, over $j \geq 1$ and $r \geq 1$ and obtain

$$\sum_{\mathbf{p}} \mathrm{wt}_f(\mathbf{p}) = \Lambda_b[f]\, f(b)\, \Lambda_{m-b}[f^{+b}],$$

where $\mathbf{p}$ runs through all compositions of m that have a cut point at b. So the term of index b in the definition of $\Lambda_m^{(1)}[f]$ is the sum of the weights $\mathrm{wt}_f(\mathbf{p})$ of the compositions $\mathbf{p}$ of m that have a cut point at b. In the sum over b every composition of m therefore occurs once for each of its cut points, that is, $q - 1$ times if it has q parts. □

If $m \geq 1$, f is nonnegative at the points $1, \ldots, m-1$ and $\Lambda_m[f] > 0$, the lemma says that $\Lambda_m^{(1)}[f]/\Lambda_m[f]$ is the mean number of cut points of a composition of m that is drawn with probability proportional to its weight.

A composition with q parts has $(q-1)(q-2)/2$ pairs of cut points, and the sum with two marked cut points counts them.

Lemma 11.6. *For every function* $f\colon \mathbb{N} \to \mathbb{R}$ *and every integer* $m \geq 0$,

$$\Lambda_m^{(2)}[f] = \sum_{q=1}^{m} \frac{(q-1)(q-2)}{2}\, V_q(m; f).$$

Proof. Let $0 < b < m$. By Lemma 11.5, applied to the integer b, the number $\Lambda_b^{(1)}[f]$ is the sum of the weights $\mathrm{wt}_f(\mathbf{p}')$ of the compositions $\mathbf{p}'$ of b, each multiplied by the number of cut points of $\mathbf{p}'$. The number $\Lambda_{m-b}[f^{+b}]$ is the sum of the weights $\mathrm{wt}_{f^{+b}}(\mathbf{p}'')$ of the compositions $\mathbf{p}''$ of $m - b$. The pairs $(\mathbf{p}', \mathbf{p}'')$ correspond to the

compositions $\mathbf{p}$ of m that have a cut point at b, the weights are related by (11.2), which was shown in the proof of Lemma 11.2, and the cut points of $\mathbf{p}'$ are the cut points of $\mathbf{p}$ that are smaller than b. Hence the term of index b in the definition of $\Lambda_m^{(2)}[f]$ is the sum of the weights $\mathrm{wt}_f(\mathbf{p})$ of the compositions $\mathbf{p}$ of m that have a cut point at b, each multiplied by the number of its cut points that are smaller than b. In the sum over b every composition $\mathbf{p}$ of m therefore occurs with a multiplicity which is the number of pairs $b' < b$ of cut points of $\mathbf{p}$, and a composition with q parts has $(q-1)(q-2)/2$ such pairs. □

The two lemmas together give the second factorial moment of the number of parts. It is needed for the lower bound of the curvature term in Proposition 13.9: for the threshold profile the compositions with q parts contribute to the coefficient of y^q in $y\,\Lambda_m(y;N)$, and a second difference of the power y^q brings the factor $q(q-1)$.

Corollary 11.7. *For every function $f\colon \mathbb{N}\to\mathbb{R}$ and every integer $m\ge 0$,*

$$\sum_{q=1}^{m} q\,(q-1)\,V_q(m;f) = 2\,\Lambda_m^{(1)}[f] + 2\,\Lambda_m^{(2)}[f].$$

Proof. By Lemmas 11.5 and 11.6 the right side is the sum of $\big(2(q-1)+(q-1)(q-2)\big)\,V_q(m;f)$ over $1\le q\le m$, and $2(q-1)+(q-1)(q-2) = q\,(q-1)$. □

The identity $q(q-1)/2 = (q-1)+(q-1)(q-2)/2$ behind the corollary has a direct meaning. The $q-1$ cut points of a composition of m with q parts and the end point m of the segment are q points, and a pair of these points consists either of the end point and a cut point or of two cut points. So $\Lambda_m^{(1)}[f]+\Lambda_m^{(2)}[f]$ counts every composition with its weight times the number of pairs among its q points.

11.5. **The difference $\boldsymbol{\nu_m - \Lambda_m[f]}$.** The inequalities of Section 14 bound a total weight $\Lambda_m[f]$ from below. The proof of Proposition 16.2 also needs an upper bound, for the chain sum on the left side of the criterion of Proposition 13.10. The identity of this subsection makes such a bound possible. It expresses the amount by which $\Lambda_m[f]$ falls short of the number $\nu_m = m^m/m!$ as a combination of the totals $\Lambda_s[f]$ of the smaller integers $s<m$, and for a profile with values in the interval $[0,1]$ the coefficients of this combination are nonnegative. Lower bounds for the $\Lambda_s[f]$ then give an upper bound for $\Lambda_m[f]$, and this is how Proposition 14.8 is proved.

Lemma 11.8. *For every function $f\colon \mathbb{N}\to\mathbb{R}$ and every integer $m\ge 1$,*

$$\nu_m - \Lambda_m[f] = \sum_{0<s<m} \nu_{m-s}\,\big(1-f(s)\big)\,\Lambda_s[f].$$

Proof. Let $n\ge 1$. By Lemma 4.5,

$$\sum_{j=0}^{n} a_j\,\nu_{n-j} = \nu_n.$$

In this sum the term $j=0$ vanishes, the term $j=n$ is $a_n\,\nu_0 = a_n$, and with $s=n-j$ the other terms give

$$\nu_n = a_n + \sum_{0<s<n} a_{n-s}\,\nu_s \qquad (n\ge 1). \tag{11.8}$$

Put $\varphi_s = (1 - f(s))\,\Lambda_s[f]$ and $Y_m = \nu_m - \Lambda_m[f]$. We prove

$$Y_m = \sum_{0<s<m} \nu_{m-s}\,\varphi_s$$

by induction on $m \geq 1$, assuming it for all smaller positive integers. Subtracting the recursion of Lemma 11.1 from (11.8) with $n = m$, we get

$$Y_m = \sum_{0<s<m} a_{m-s}\big(\nu_s - f(s)\,\Lambda_s[f]\big) = \sum_{0<s<m} a_{m-s}\big(\varphi_s + Y_s\big).$$

By the induction hypothesis, Y_s is the sum of $\nu_{s-t}\,\varphi_t$ over $0 < t < s$, for every s with $0 < s < m$. We insert this and collect the coefficient of each φ_t:

$$Y_m = \sum_{0<t<m} \varphi_t \Big(a_{m-t} + \sum_{t<s<m} a_{m-s}\,\nu_{s-t}\Big).$$

With $u = s - t$ the inner sum is the sum of $a_{m-t-u}\,\nu_u$ over $0 < u < m - t$, so that the expression in brackets is ν_{m-t} by (11.8) with $n = m - t \geq 1$. This is the assertion. □

The identity of Lemma 11.8 can be understood as follows. For a composition with the cut points $s_1 < \cdots < s_j$,

$$1 - \prod_{i=1}^{j} f(s_i) = \sum_{i=1}^{j} \big(1 - f(s_i)\big) \prod_{l<i} f(s_l),$$

a telescoping sum. In the term of index i the cut points before s_i keep their factors, the point s_i carries the factor $1 - f(s_i)$, and the cut points after s_i carry the factor 1. When $0 \leq f \leq 1$, one may read $f(s)$ as the probability that the cut point s passes a test, independently of the other cut points. The identity then sorts the compositions with a failing cut point by the first such point.

For the profile $\mathbf{1}$ the lemma gives the total weight of all compositions of m when the cuts carry no factor.

Corollary 11.9. *For every integer $m \geq 1$,*

$$\Lambda_m[\mathbf{1}] = \nu_m = \frac{m^m}{m!}.$$

Proof. For the profile $f = \mathbf{1}$ every term on the right side of Lemma 11.8 vanishes. □

Since $\Lambda_m[\mathbf{1}]$ is the sum of $T_{m,q}$ over $1 \leq q \leq m$ and $\nu_m = m\,a_m$, the corollary says the same as (11.7). With the corollary the sums with marked cut points become convolutions for the profile $\mathbf{1}$. A shift does not change a constant profile, so that

$$\Lambda^{(1)}_m[\mathbf{1}] = \sum_{0<b<m} \nu_b\,\nu_{m-b}, \qquad \Lambda^{(2)}_m[\mathbf{1}] = \sum_{0<b<m} \Lambda^{(1)}_b[\mathbf{1}]\,\nu_{m-b}.$$

Lemmas 11.5 and 11.6 and Proposition 11.4 express the same two sums through the numbers $B_{m,q}$, and Section 12 draws the consequences.

12. Convolutions of the Poisson mass and the birthday law

The estimates that lead to Proposition 16.2 are expressed through two convolutions of the Poisson mass $P(m)$. We evaluate them here in closed form through the mean s_m of the birthday law of m (Lemmas 12.4 and 12.5), compute the mean positions in them, and bound s_m from both sides (Lemmas 12.11 and 12.13). The estimates for Proposition 16.2 use the lower bound, through Corollary 12.12; the upper bound is used only in Section 21, for Proposition 22.6. The two bounds are deduced from a theorem of Flajolet, Grabner, Kirschenhofer and Prodinger [7, Theorem 7].

For an integer $m \geq 0$ the *twofold convolution* and the *threefold convolution* of the Poisson mass are

$$P_2(m) = \sum_{0<g<m} P(g)\,P(m-g), \qquad P_3(m) = \sum_{0<g<m} P_2(g)\,P(m-g),$$

where g runs over the integers with $0 < g < m$. The proof of Proposition 16.2 rests on the criterion of Proposition 13.10, and Section 14 bounds the two sides of that criterion by expressions in P, P_2 and P_3. The reason is the following. By Corollary 11.9, for $m \geq 1$ the total weight $\Lambda_m[\mathbf{1}]$ of the compositions of m for the cut profile $\mathbf{1}$ is $\nu_m = e^m P(m)$. The chain sums on the two sides of the criterion have cut factors that are a little smaller than 1, and they are estimated by comparison with the profile $\mathbf{1}$. Where a chain of length m is divided at one marked point or at two, its pieces contribute a product of two or three of the numbers ν whose indices add up to m, which is e^m times a product of Poisson masses, and the sum over the positions of the marked points is $e^m P_2(m)$ or $e^m P_3(m)$.

The sum that defines $P_2(m)$ runs over the splittings $m = g + (m-g)$ of m into two positive parts, with the splitting point g; the terms $g = 0$ and $g = m$ of the full convolution, which are both equal to $P(m)$ because $P(0) = 1$, are left out. Clearly $P_2(0) = P_2(1) = 0$ and $P_3(m) = 0$ for $m \leq 2$, and $P_2(m) \geq 0$ and $P_3(m) \geq 0$ for every m. Inserting the definition of $P_2(g)$, as a sum over the integers h with $0 < h < g$, into that of $P_3(m)$, and writing $g_1 = h$, $g_2 = g - h$ and $g_3 = m - g$, we obtain

$$P_3(m) = \sum_{g_1+g_2+g_3=m} P(g_1)\,P(g_2)\,P(g_3). \tag{12.1}$$

Here and below, a sum over $g_1 + \cdots + g_q = m$ runs over the compositions $(g_1, \ldots, g_q)$ of m into q positive parts. The right-hand side of (12.1) does not change when the three parts are permuted. In particular, the threefold convolution can also be formed with the single factor first:

$$P_3(m) = \sum_{0<g<m} P(g)\,P_2(m-g).$$

12.1. **Moments of the birthday law.** For an integer $m \geq 1$ we put

$$s_m = \sum_{j=1}^{m} B_{m,j},$$

and we put $s_0 = 0$. Here $B_{m,j}$ is the birthday tail of Section 11. A random variable Q with the birthday law of m, which is defined there, takes the values $1, \ldots, m$, with $\Pr(Q = q) = (q/m)\,B_{m,q}$ for $1 \leq q \leq m$ and $\Pr(Q \geq j) = B_{m,j}$ for $j \geq 1$. The mean of a random variable with values in the positive integers is the sum of its tail

probabilities, so that s_m is the mean of Q, the *mean of the birthday law* of m. The symbol s_m is not related to the cut points $s_0 < s_1 < \cdots < s_q$ of a composition.

By Proposition 11.4 the compositions of m with q parts have the total weight $T_{m,q} = a_m\, q\, B_{m,q}$. Sums over compositions therefore lead to sums in which a function of q is multiplied by $q\, B_{m,q}$, which is m times a point mass of the birthday law, whereas s_m is a sum of the tails $B_{m,q}$. The following summation by parts passes from the first form to the second.

Lemma 12.1. *Let $m \geq 1$ be an integer and let ψ be a real-valued function on the nonnegative integers. Then the birthday tail $B_{m,q}$ satisfies*

$$\sum_{q=1}^{m} \psi(q)\, q\, B_{m,q} = m \Big(\psi(0) + \sum_{q=1}^{m} \big(\psi(q) - \psi(q-1) \big)\, B_{m,q} \Big).$$

Proof. For every integer $q \geq 0$ the recursion $B_{m,q+1} = B_{m,q}\,(1 - q/m)$ gives $q\, B_{m,q} = m\,(B_{m,q} - B_{m,q+1})$. Hence the left-hand side of the claim is m times

$$\begin{aligned} \sum_{q=1}^{m} \psi(q) \big(B_{m,q} - B_{m,q+1} \big) &= \sum_{q=1}^{m} \psi(q)\, B_{m,q} - \sum_{q=2}^{m+1} \psi(q-1)\, B_{m,q} \\ &= \sum_{q=1}^{m} \big(\psi(q) - \psi(q-1) \big)\, B_{m,q} \\ &\qquad + \psi(0)\, B_{m,1} - \psi(m)\, B_{m,m+1}, \end{aligned}$$

and we have $B_{m,1} = 1$ and $B_{m,m+1} = 0$. □

For the constant function $\psi = 1$, for $\psi(j) = j$ and for $\psi(j) = j - 1$ the lemma gives, for every integer $m \geq 1$,

$$\begin{aligned} \sum_{q=1}^{m} q\, B_{m,q} = m, \qquad \sum_{q=1}^{m} q^2\, B_{m,q} = m\, s_m, \\ \sum_{q=1}^{m} (q-1)\, q\, B_{m,q} = m\,(s_m - 1). \end{aligned} \tag{12.2}$$

The second and the third factorial moment of the birthday law, $\mathbb{E}\, Q(Q-1)$ and $\mathbb{E}\, Q(Q-1)(Q-2)$, are $1/m$ times the sums of the next two corollaries. The first of them enters the closed form of P_3 in Lemma 12.5, and both are used in Section 21.

Corollary 12.2. *For every integer $m \geq 1$,*

$$\sum_{q=1}^{m} q^2\, (q-1)\, B_{m,q} = 2m\,(m - s_m).$$

Proof. We apply Lemma 12.1 to $\psi(j) = j(j-1)$, for which $\psi(0) = 0$ and $\psi(q) - \psi(q-1) = 2(q-1)$. The sum of the claim is therefore

$$2m \sum_{q=1}^{m} (q-1)\, B_{m,q} = 2m\,(m - s_m),$$

by the first identity of (12.2) and the definition of s_m. □

Corollary 12.3. *For every integer $m \geq 1$,*

$$\sum_{q=1}^{m} q^2\,(q-1)(q-2)\,B_{m,q} = 3m\,\big((m+2)\,s_m - 3m\big).$$

Proof. We apply Lemma 12.1 to $\psi(j) = j(j-1)(j-2)$, for which $\psi(0) = 0$ and $\psi(q) - \psi(q-1) = 3(q-1)(q-2) = 3q^2 - 9q + 6$. The sum of the claim is therefore

$$m \sum_{q=1}^{m} \big(3q^2 - 9q + 6\big)\,B_{m,q} = m\,\big(3m\,s_m - 9m + 6\,s_m\big),$$

by the first two identities of (12.2) and the definition of s_m. The right-hand side is $3m\,((m+2)\,s_m - 3m)$. □

12.2. **Closed forms of the two convolutions.** We now evaluate the twofold convolution. The proof reads $e^m P_2(m)$ as the total weight, for the cut profile $\mathbf{1}$, of the compositions of m with one marked cut point, in which a composition with q parts is counted $q-1$ times, and it evaluates this weight by Proposition 11.4.

Lemma 12.4. *For every integer $m \geq 1$, the twofold convolution of the Poisson mass and the mean s_m of the birthday law of m satisfy*

$$P_2(m) = P(m)\,(s_m - 1).$$

Proof. Let g and m be integers with $0 < g < m$. By the definition of the Poisson mass, $P(g)\,P(m-g) = e^{-m}\,\nu_g\,\nu_{m-g}$, and by Corollary 11.9, $\nu_g = \Lambda_g[\mathbf{1}]$ and $\nu_{m-g} = \Lambda_{m-g}[\mathbf{1}]$. The constant profile $\mathbf{1}$ is equal to its shift $\mathbf{1}^{+g}$, so that the sum of these products over g is the sum that defines $\Lambda_m^{(1)}[\mathbf{1}]$ in Section 11:

$$e^m P_2(m) = \sum_{0<g<m} \Lambda_g[\mathbf{1}]\,\Lambda_{m-g}[\mathbf{1}] = \Lambda_m^{(1)}[\mathbf{1}]. \tag{12.3}$$

This holds for every integer $m \geq 0$; for $m \leq 1$ the sums are empty. Let now $m \geq 1$. By Lemma 11.5 for the profile $\mathbf{1}$, for which $V_q(m;\mathbf{1}) = T_{m,q}$, and by Proposition 11.4,

$$\Lambda_m^{(1)}[\mathbf{1}] = \sum_{q=1}^{m} (q-1)\,T_{m,q} = a_m \sum_{q=1}^{m} (q-1)\,q\,B_{m,q} = m\,a_m\,(s_m - 1),$$

where the last step is the third identity of (12.2). Since $m\,a_m = m^m/m! = \nu_m$ for $m \geq 1$, we conclude with (12.3) that $P_2(m) = e^{-m}\,\nu_m\,(s_m - 1) = P(m)\,(s_m - 1)$. □

The threefold convolution is treated in the same way, with two marked cut points instead of one. A composition with q parts has $(q-1)(q-2)/2$ pairs of cut points, which brings in the second factorial moment of the birthday law. The result expresses P_3 in terms of P and P_2, and Sections 15 and 16 use it to remove P_3 from their estimates.

Lemma 12.5. *For every integer $m \geq 1$, the threefold convolution of the Poisson mass satisfies*

$$P_3(m) = (m-1)\,P(m) - 2\,P_2(m).$$

Proof. Let g be an integer with $0 < g < m$. For $0 < h < g$ the definition of the Poisson mass and Corollary 11.9 give $P(h)\,P(g-h) = e^{-g}\,\Lambda_h[\mathbf{1}]\,\Lambda_{g-h}[\mathbf{1}]$, and

since $\mathbf{1}^{+h} = \mathbf{1}$, summing over h gives $e^g P_2(g) = \Lambda_g^{(1)}[\mathbf{1}]$. With this and with Corollary 11.9 at $m-g$,

$$P_2(g)\,P(m-g) = e^{-m}\,\Lambda_g^{(1)}[\mathbf{1}]\,\Lambda_{m-g}[\mathbf{1}],$$

and, as in the proof of Lemma 12.4, the sum of the right-hand sides over g is e^{-m} times the sum that defines $\Lambda_m^{(2)}[\mathbf{1}]$ in Section 11. Hence, by Lemma 11.6 and Proposition 11.4,

$$e^m P_3(m) = \Lambda_m^{(2)}[\mathbf{1}] = \sum_{q=1}^{m} \frac{(q-1)(q-2)}{2}\,T_{m,q} = a_m \sum_{q=1}^{m} \frac{(q-1)(q-2)}{2}\,q\,B_{m,q}.$$

Since $(q-1)(q-2)\,q/2 = q^2(q-1)/2 - (q-1)\,q$, Corollary 12.2 and the third identity of (12.2) give

$$\sum_{q=1}^{m} \frac{(q-1)(q-2)}{2}\,q\,B_{m,q} = m\,(m-s_m) - m\,(s_m-1) = m\,(m+1-2s_m).$$

With $m\,a_m = \nu_m$ we obtain

$$P_3(m) = P(m)\,(m+1-2s_m) = (m-1)\,P(m) - 2\,P(m)\,(s_m-1),$$

and $P(m)\,(s_m-1) = P_2(m)$ by Lemma 12.4. □

12.3. **Mean positions.** In the estimates of Section 14 the terms of the convolutions P_2 and P_3 are multiplied by a function of the splitting point that is linear, or that is bounded by a linear function. The sums that result are evaluated by the next three lemmas, which give the mean position of a splitting point. Their proofs use the symmetry of a product of Poisson masses in its factors.

Lemma 12.6. *For every integer $m \geq 0$,*

$$\sum_{0<g<m} g\,P(g)\,P(m-g) = \frac{m}{2}\,P_2(m).$$

Proof. With $g_1 = g$ and $g_2 = m-g$, the left-hand side of the claim is the sum of $g_1\,P(g_1)\,P(g_2)$ over the compositions (g_1, g_2) of m into two parts. The product $P(g_1)\,P(g_2)$ does not change when the two parts are exchanged, so that this sum equals the sum of $g_2\,P(g_1)\,P(g_2)$. The two sums add up to m times the sum of $P(g_1)\,P(g_2)$, which is $m\,P_2(m)$, and hence each of them is $(m/2)\,P_2(m)$. □

For the threefold convolution there are two sums of this kind, according to whether the splitting point separates the first two of the three factors from the third, or the first factor from the other two.

Lemma 12.7. *For every integer $m \geq 0$,*

$$\sum_{0<g<m} g\,P_2(g)\,P(m-g) = \frac{2m}{3}\,P_3(m).$$

Proof. As in the derivation of (12.1), in which $g = g_1 + g_2$, the left-hand side of the claim is

$$\sum_{g_1+g_2+g_3=m} (g_1+g_2)\,P(g_1)\,P(g_2)\,P(g_3).$$

The product $P(g_1)\,P(g_2)\,P(g_3)$ does not change when the three parts are permuted, so that the sums of g_1, of g_2 and of g_3 times this product over the compositions of m

into three parts are equal. They add up to m times the sum of the product, which is $m\,P_3(m)$ by (12.1), and hence each of them is $(m/3)\,P_3(m)$. The sum displayed above is the sum of the first two, that is, $2\cdot(m/3)\,P_3(m)$. □

Lemma 12.8. *For every integer $m \geq 0$,*

$$\sum_{0<g<m} g\,P(g)\,P_2(m-g) = \frac{m}{3}\,P_3(m).$$

Proof. Inserting the definition of $P_2(m-g)$, we see that the left-hand side of the claim is the sum of $g_1\,P(g_1)\,P(g_2)\,P(g_3)$ over the compositions of m into three parts, with $g_1 = g$. The product $P(g_1)\,P(g_2)\,P(g_3)$ does not change when the three parts are permuted, so that the sums of g_1, of g_2 and of g_3 times this product are equal. They add up to m times the sum of the product, which is $m\,P_3(m)$ by (12.1), and hence the first of them is $(m/3)\,P_3(m)$. □

For an integer $R \geq 0$ and the integers s with $0 < s < R$ let

$$\vartheta_s = P(R-s)\,(R-1-s)\,P_2(s),$$

where the dependence on R is not shown. The numbers ϑ_s occur in the proof of Proposition 14.8, in a sum over s in which ϑ_s is multiplied by the value at s of a concave function. That proof needs the total of the weights and a lower bound for their mean position, the sum of the numbers $s\,\vartheta_s$ divided by the sum of the weights. The total follows from Lemma 12.7.

Corollary 12.9. *For every integer $R \geq 0$, the weights ϑ_s satisfy*

$$\sum_{0<s<R} \vartheta_s = \frac{R-3}{3}\,P_3(R).$$

Proof. Writing $R-1-s = (R-1)-s$, we have

$$\sum_{0<s<R} \vartheta_s = (R-1)\sum_{0<s<R} P_2(s)\,P(R-s) - \sum_{0<s<R} s\,P_2(s)\,P(R-s).$$

The first sum on the right is $P_3(R)$, by the definition of the threefold convolution, and the second is $(2R/3)\,P_3(R)$, by Lemma 12.7. The claim follows. □

The lower bound for the mean position rests on Lemma 12.5, which turns the sums over s into sums over the compositions of R into five parts and into four parts.

Lemma 12.10. *For every integer $R \geq 0$, the weights ϑ_s satisfy*

$$\frac{2R}{5}\sum_{0<s<R} \vartheta_s \;\leq\; \sum_{0<s<R} s\,\vartheta_s.$$

Proof. Let s be an integer with $0 < s < R$. Lemma 12.5 at $R-s \geq 1$ gives $(R-s-1)\,P(R-s) = P_3(R-s) + 2\,P_2(R-s)$, and hence

$$\vartheta_s = P_2(s)\,P_3(R-s) + 2\,P_2(s)\,P_2(R-s).$$

For $q = 4$ and $q = 5$ let Σ_q denote the sum of the products $P(g_1)\cdots P(g_q)$ over the compositions $R = g_1 + \cdots + g_q$ of R into q parts. A composition of s into two parts, followed by a composition of $R-s$ into three parts, is a composition of R into five parts with $g_1 + g_2 = s$, and every composition of R into five parts arises

in this way for exactly one s. By the definition of P_2 and by (12.1), the sum of the products $P_2(s)\,P_3(R-s)$ over $0<s<R$ is therefore Σ_5, and

$$\sum_{0<s<R} s\,P_2(s)\,P_3(R-s) = \sum_{g_1+\cdots+g_5=R} (g_1+g_2)\,P(g_1)\cdots P(g_5) = \frac{2R}{5}\,\Sigma_5,$$

where the last step holds because the product $P(g_1)\cdots P(g_5)$ does not change when the five parts are permuted: the sums of $g_1,\dots,g_5$ times this product over the compositions of R into five parts are therefore equal, they add up to $R\,\Sigma_5$, and each of them is $(R/5)\,\Sigma_5$. In the same way, with compositions into four parts instead of five, the sum of the products $P_2(s)\,P_2(R-s)$ over $0<s<R$ is Σ_4, and

$$\sum_{0<s<R} s\,P_2(s)\,P_2(R-s) = \frac{2R}{4}\,\Sigma_4.$$

Consequently

$$\sum_{0<s<R} \vartheta_s = \Sigma_5 + 2\,\Sigma_4, \qquad \sum_{0<s<R} s\,\vartheta_s = \frac{2R}{5}\,\Sigma_5 + R\,\Sigma_4.$$

The right-hand side of the claim minus its left-hand side is thus $R\,\Sigma_4 - (4R/5)\,\Sigma_4 = (R/5)\,\Sigma_4$, which is nonnegative. □

12.4. **Bounds for the mean of the birthday law.** By Lemmas 12.4 and 12.5, bounds for $P_2(m)$ and $P_3(m)$ come down to bounds for the Poisson mass $P(m)$ and for s_m. For $P(m)$ these are the bounds (2.5). In this subsection we bound s_m from both sides.

The sum $s_m = 1 + (m-1)/m + (m-1)(m-2)/m^2 + \cdots$ is known as Ramanujan's Q-function [7, (1.3)], and we deduce the two bounds from a theorem of Flajolet, Grabner, Kirschenhofer and Prodinger, which is stated in terms of the partial sums of the exponential series. These partial sums are related to s_m as follows. For integers $m \ge 1$ and $0 \le j \le m$ we have $B_{m,j} = m!/((m-j)!\,m^j)$, so that $(m^m/m!)\,B_{m,j} = m^{m-j}/(m-j)!$. Summing over $0 \le j \le m$ and using $B_{m,0} = 1$, we get

$$\frac{m^m}{m!}\,(1+s_m) = \sum_{j=0}^{m} \frac{m^{m-j}}{(m-j)!} = \sum_{i=0}^{m} \frac{m^i}{i!}. \tag{12.4}$$

In the classical notation, a number θ_m is defined for every integer $m \ge 0$ by

$$\frac{1}{2}\,e^m = \sum_{i=0}^{m-1} \frac{m^i}{i!} + \theta_m\,\frac{m^m}{m!}.$$

For $m \ge 1$ the sum on the right-hand side is the last sum of (12.4) without its term $m^m/m!$ of index $i = m$, and hence equal to $(m^m/m!)\,s_m$. The definition of θ_m therefore reads $e^m/2 = (m^m/m!)\,(s_m + \theta_m)$, and since $P(m) = e^{-m}\,m^m/m!$, it gives

$$s_m = \frac{1}{2\,P(m)} - \theta_m \qquad (m \ge 1). \tag{12.5}$$

The theorem [7, Theorem 7] states that for every integer $m \ge 0$ there is a real number k_m with

$$\theta_m = \frac{1}{3} + \frac{4}{135\,(m+k_m)} \qquad \text{and} \qquad \frac{2}{21} \le k_m \le \frac{8}{45}.$$

Since $k_m > 0$, this gives

$$\frac{1}{3} < \theta_m < \frac{1}{3} + \frac{4}{135\,m} \qquad (m \geq 1). \tag{12.6}$$

The two lemmas of this subsection are stated with the constants that the later sections use.

Lemma 12.11. *For every integer $m \geq 2$, the mean s_m of the birthday law of m satisfies*

$$\frac{1}{2\,P(m)} - \frac{1}{3} - \frac{1}{15\,m} \;\leq\; s_m.$$

Proof. Since $4/135 < 1/15$, the relation (12.5) and the right-hand inequality of (12.6) give

$$s_m = \frac{1}{2\,P(m)} - \theta_m > \frac{1}{2\,P(m)} - \frac{1}{3} - \frac{4}{135\,m} > \frac{1}{2\,P(m)} - \frac{1}{3} - \frac{1}{15\,m}. \qquad \square$$

With the closed form of Lemma 12.4, the lower bound of s_m becomes a lower bound of the twofold convolution.

Corollary 12.12. *For every integer $m \geq 2$, the twofold convolution of the Poisson mass satisfies*

$$\frac{1}{2} - \Big(\frac{4}{3} + \frac{1}{15\,m}\Big)\,P(m) \;\leq\; P_2(m).$$

Proof. Multiplying the inequality of Lemma 12.11 by $P(m)$, we obtain $1/2 - P(m)/3 - P(m)/(15m) \leq P(m)\,s_m$, and $P(m)\,s_m = P_2(m) + P(m)$ by Lemma 12.4. $\square$

We turn to the upper bound. Here the left-hand inequality of (12.6) is combined with Robbins' bound for the factorial, which turns $1/(2P(m))$ into an explicit function of m.

Lemma 12.13. *For every integer $m \geq 1$, the mean s_m of the birthday law of m satisfies*

$$s_m \leq \sqrt{\frac{\pi m}{2}}\; e^{1/(12m)} - \frac{1}{3}.$$

Proof. By (12.5) and the left-hand inequality of (12.6), we have $s_m < 1/(2P(m)) - 1/3$. Moreover, $1/(2P(m)) = m!/(2\,(m/e)^m)$, and the right-hand inequality of (2.4) shows that

$$\frac{1}{2\,P(m)} < \frac{\sqrt{2\pi m}}{2}\; e^{1/(12m)} = \sqrt{\frac{\pi m}{2}}\; e^{1/(12m)}. \qquad \square$$

13. The frozen polynomial and a criterion for a positive step

By (3.4), the normalised step $\delta_n^{(k)}$ of a column $k \geq 2$ is a second difference of the normalised counts in the threshold. In this section we write this step as a second difference of a polynomial in a real variable y, the frozen polynomial, minus a multiple of a chain sum. Proposition 13.6 does this for the steps of two neighbouring columns in one row, with the same polynomial, and its two formulas are used separately. The first gives the expression of a step of the first layer in Corollary 13.7, from which Proposition 22.6 is proved in Section 22. The second gives a sufficient condition for a positive step in which the step itself no longer occurs (Proposition 13.10), from which Sections 14 to 16 derive Proposition 16.2.

13.1. **The kernel deficit.** For integers $k \geq 0$ and $t \geq 0$ the *kernel deficit* is the number

$$\varkappa_k(t) = a_t - \sum_{k<i<t} a_i\, a_{t-i}.$$

The sum is empty for $t \leq k+1$, so that $\varkappa_k(t) = a_t$ in this range. The kernel deficits arise as the terms without the threshold in a linear equation for the normalised counts of two consecutive thresholds, which is derived in the proof of Proposition 13.6, and they are the coefficients of the frozen polynomial. The next lemma shows that they are positive in a range that contains all their arguments in the first layer of column k.

Lemma 13.1. *Let $k \geq 0$ and t be integers with $1 \leq t \leq 2k+1$. Then $\varkappa_k(t) > 0$.*

Proof. The reflection $i \mapsto t - i$ maps the set of integers i with $k < i < t$ onto the set of integers j with $0 < j < t - k$, and it leaves the product $a_i\, a_{t-i}$ unchanged. An index $j < t - k$ satisfies $j \leq k$, because $t \leq 2k+1$; so the two sets are disjoint. Since the numbers a_i are nonnegative, it follows that

$$2 \sum_{k<i<t} a_i\, a_{t-i} \;\leq\; \sum_{i=0}^{t} a_i\, a_{t-i}.$$

By Lemma 2.6,

$$\sum_{i=0}^{t-1} i\, a_i\, a_{t-i} = (t-1)\, a_t.$$

Since $a_0 = 0$, the sum may be extended to $i = t$. Adding to the identity so obtained the same identity with i replaced by $t - i$, we get

$$t \sum_{i=0}^{t} a_i\, a_{t-i} = 2\,(t-1)\, a_t.$$

Therefore t times the sum over $k < i < t$ is at most $(t-1)\, a_t$, and $t\, \varkappa_k(t) \geq a_t > 0$. □

13.2. **The frozen polynomial and the curvature sums.** From here on we use the sums over compositions of Section 11: the weights $V_q(m; f)$ and $\Lambda_m[f]$ for a cut profile f, the weights $\Lambda_m^{(1)}[f]$ and $\Lambda_m^{(2)}[f]$ with one and with two marked cut points, the threshold profile $\lambda^{y,N}$ and the chain sums $\Lambda_m(y; N)$. The homogeneity (11.5) of the weights $V_q(m; \lambda^{y,N})$ in the threshold y, summed over the number q of parts, gives

$$\Lambda_m(y; N) = \sum_{q=1}^{m} y^{\,q-1}\, V_q\big(m; \lambda^{1,N}\big). \tag{13.1}$$

We also use two facts that are noted in Section 11: the weights $V_q(m; f)$ of a cut profile f with $f(s) \geq 0$ for $0 < s < m$ are nonnegative, and a threshold profile shifted by an integer $g \geq 0$ is again a threshold profile, namely $(\lambda^{y,N})^{+g} = \lambda^{y,N-g}$.

The frozen polynomial is modelled on the repeated substitution into itself of the linear equation of the counts that the proof of Proposition 13.6 derives. Let $k \geq 0$ be an integer, let $j \geq 1$ be one of the integers k and $k+1$, and let $N \geq 1$ be an integer. In this substitution for the count $z_{N-1}^{(j)}$, each count on the right-hand side is replaced by the right-hand side of the same equation at a smaller

argument; a term that arises in this way belongs to a decreasing chain of integers $N > t_1 > \cdots > t_i > k$, and every integer t that the chain visits contributes a factor j/t. Put $m = N - t_i$. The differences $N - t_1$, $t_1 - t_2$, …, $t_{i-1} - t_i$ of a chain are the parts of a composition of m, whose cut points are $s_l = N - t_l$ for $1 \le l < i$, and the factor j/t_l of the integer that the chain visits at the cut s_l is the value $\lambda^{j,N}(s_l)$ of the threshold profile. The last integer of the chain contributes $(j/t_i)\,\varkappa_k(t_i)$. This is the origin of the following definition, in which the threshold has become a real variable; the frozen polynomial that it introduces is not related to the frozen kernel of Section 9.

Definition 13.2. Let $k \ge 0$ and $n \ge 0$ be integers, and put $N = n + 1$. The *frozen polynomial* of column k and row n is the function of the real variable y given by

$$\Phi_{n,k}(y) = \varkappa_k(N) + \sum_{m=1}^{n-k} \frac{\varkappa_k(N-m)}{N-m}\, y\, \Lambda_m(y; N).$$

The sum is empty when $n \le k$.

By (13.1) the function $y\,\Lambda_m(y; N)$ is a polynomial in y of degree at most m without constant term. So $\Phi_{n,k}$ is a polynomial, of degree at most $n - k$ when $n \ge k$, with constant term $\varkappa_k(N)$; for $n \le k$ it is the constant a_{n+1}.

The proof of Proposition 13.6 identifies the counts of the thresholds k and $k+1$ with values of the frozen polynomial of column k. The name of the frozen polynomial refers to the weight of the last-car recursion that equals t for $t \le k$ and a free value for $t > k$: the point k at which the weight stops growing is fixed, while the value at which it stops is left free.

The coefficients of the frozen polynomial are built from kernel deficits and from composition weights whose signs are known in the first layer. Indeed, for $k \le n \le 2k - 1$ the arguments N and $N - m$ of the kernel deficits in Definition 13.2 lie between $k+1$ and $2k$, so that these kernel deficits are positive by Lemma 13.1, and the coefficients $V_q(m; \lambda^{1,N})$ in (13.1) are nonnegative.

The frozen polynomial depends on y only through the functions $y\,\Lambda_m(y; N)$. Their second differences with step one are the quantities in which the steps will be expressed.

Definition 13.3. For real numbers y and N and an integer $m \ge 0$, the *curvature sum* is

$$\mathsf{C}_m(y; N) = (y+1)\,\Lambda_m(y+1; N) - 2\,y\,\Lambda_m(y; N) + (y-1)\,\Lambda_m(y-1; N).$$

In this section the curvature sum is used with positive integers y and N and with $m < N$, so that the denominators $N - s$ of the threshold profile are positive at the cut points $0 < s < m$.

Lemma 13.4. *Let $k \ge 0$ and $n \ge 0$ be integers, let y be a real number, and put $N = n + 1$. Then*

$$\Phi_{n,k}(y+1) - 2\,\Phi_{n,k}(y) + \Phi_{n,k}(y-1) = \sum_{m=1}^{n-k} \frac{\varkappa_k(N-m)}{N-m}\, \mathsf{C}_m(y; N).$$

The sum is empty when $n \le k$.

Proof. This is immediate from Definitions 13.2 and 13.3, since the constant term $\varkappa_k(N)$ and the coefficients $\varkappa_k(N-m)/(N-m)$ do not depend on y. □

A curvature sum can in turn be taken power by power in the threshold. The later statements about curvature sums start from the resulting expansion by the number of parts.

Lemma 13.5. *For all real numbers y and N and every integer $m \geq 0$,*

$$\mathsf{C}_m(y;N) = \sum_{q=1}^{m} \big((y+1)^q - 2\,y^q + (y-1)^q\big)\, V_q\big(m;\lambda^{1,N}\big).$$

Proof. Multiplying (13.1) by the threshold, we have for every real number x

$$x\,\Lambda_m(x;N) = \sum_{q=1}^{m} x^q\, V_q\big(m;\lambda^{1,N}\big).$$

Inserting this with $x = y+1$, $x = y$ and $x = y-1$ into Definition 13.3 gives the statement. □

The weights $V_q(m;\lambda^{1,N})$ of this expansion do not contain y, and they are nonnegative when $N \geq m$. By the binomial theorem,

$$(y+1)^q - 2\,y^q + (y-1)^q = 2\sum_{i\geq 1} \binom{q}{2i}\, y^{\,q-2i} \tag{13.2}$$

for every real y and every integer $q \geq 0$. This is zero for $q = 1$, and for $y > 0$ and $q \geq 2$ all its terms are nonnegative and the first of them is $q(q-1)\,y^{q-2}$. The curvature sums satisfy $\mathsf{C}_1(y;N) = 0$ and $\mathsf{C}_m(y;N) > 0$ for $y > 0$, $m \geq 2$ and $N \geq m$.

13.3. **The step as a curvature term minus a boundary term.** By (3.4) the step of column k in row n is the second difference of the counts of thresholds $k+1$, k and $k-1$. The next proposition writes the steps of columns k and $k+1$ in row n as a second difference of the frozen polynomial $\Phi_{n,k}$ minus a multiple of a chain sum.

Proposition 13.6. *Let $k \geq 2$ and r be integers with $1 \leq r \leq k-1$, and put $n = k+r$ and $N = n+1$. Then the normalised steps of columns k and $k+1$ in row n are*

$$\delta_n^{(k)} = \big(\Phi_{n,k}(k+1) - 2\,\Phi_{n,k}(k) + \Phi_{n,k}(k-1)\big) - \frac{a_k}{k}\,\Lambda_{r+1}(k-1;N), \tag{13.3}$$

$$\delta_n^{(k+1)} = \big(\Phi_{n,k}(k+2) - 2\,\Phi_{n,k}(k+1) + \Phi_{n,k}(k)\big) - \frac{a_{k+1}}{k+1}\,\Lambda_{r}(k+2;N). \tag{13.4}$$

Proof. We first compare the frozen polynomials of two consecutive columns, for arbitrary integers k and n with $1 \leq k \leq n$: for every real number y, with $N = n+1$,

$$\Phi_{n,k-1}(y) - \Phi_{n,k}(y) = (y-k)\,\frac{a_k}{k}\,\Lambda_{N-k}(y;N). \tag{13.5}$$

To see this, we compare the two definitions term by term. Lowering k by one adds the term of index $i = k$ to the sum that defines the kernel deficit: for every integer $t > k$,

$$\varkappa_{k-1}(t) = \varkappa_k(t) - a_k\,a_{t-k}, \qquad \text{and} \qquad \varkappa_{k-1}(k) = a_k. \tag{13.6}$$

The sum of $\Phi_{n,k-1}$ has one term more than that of $\Phi_{n,k}$, the term $m = N-k$, which is $(a_k/k)\,y\,\Lambda_{N-k}(y;N)$ because $\varkappa_{k-1}(k) = a_k$. In the constant term and in the terms with $1 \leq m \leq n-k$ the arguments of the kernel deficits are larger than

k, so that by (13.6) the kernel deficits of the two polynomials differ by $-a_k\,a_{N-k}$ and by $-a_k\,a_{N-m-k}$. These terms therefore contribute

$$-a_k\Big(a_{N-k}+\sum_{m=1}^{n-k}\Lambda_m(y;N)\,\frac{y}{N-m}\,a_{N-k-m}\Big)=-a_k\,\Lambda_{N-k}(y;N)$$

to the difference, by the recursion of Lemma 11.1 for the profile $\lambda^{y,N}$ and the compositions of $N-k$. The two contributions add up to the right-hand side of (13.5).

Next we show that the counts of two consecutive thresholds are values of one frozen polynomial: if $k\ge 0$ is an integer and $j\ge 1$ is one of the integers k and $k+1$, then

$$z_n^{(j)}=\Phi_{n,k}(j)\qquad\text{for } 0\le n\le 2k+1. \tag{13.7}$$

By Corollary 3.3 the sequence $z^{(j)}$ is the solution of the weighted last-car recursion with weight $\rho(t)=\min(t,j)$. This weight satisfies $\rho(t)=t$ for $1\le t\le k$ and $\rho(t)=j$ for $t\ge k+1$. Proposition 4.9, applied with this k, therefore gives for every integer n with $0\le n\le 2k+1$

$$c_n-z_n^{(j)}=\sum_{t=k+1}^{n}c_{n-t}\Big(c_{t-1}-\frac{j}{t}\,z_{t-1}^{(j)}\Big).$$

We solve this for $z_n^{(j)}$. Since $c_i=a_{i+1}$, the terms that contain no count are a_{n+1} and the products $-a_{n+1-t}\,a_t$ with $k<t\le n$, and they add up to $\varkappa_k(n+1)$. With $N=n+1$ the equation becomes

$$z_{N-1}^{(j)}=\varkappa_k(N)+\sum_{k<t<N}a_{N-t}\,\frac{j}{t}\,z_{t-1}^{(j)}\qquad(1\le N\le 2k+2). \tag{13.8}$$

With $\tau=N-m$ as the index of summation, Definition 13.2 reads, for every integer $N\ge 1$ and every real number y,

$$\Phi_{N-1,k}(y)=\varkappa_k(N)+\sum_{k<\tau<N}\varkappa_k(\tau)\,\frac{y}{\tau}\,\Lambda_{N-\tau}(y;N). \tag{13.9}$$

Let k and j be as above. We show that $z_{N-1}^{(j)}=\Phi_{N-1,k}(j)$ for $1\le N\le 2k+2$ by induction on N. Let N be an integer of this range, and suppose that the identity holds for all smaller values of N. By this hypothesis we may replace every $z_{t-1}^{(j)}$, $k<t<N$, in (13.8) by the right-hand side of (13.9), taken with t instead of N and with $y=j$. We then collect the terms that contain $\varkappa_k(\tau)$ for a given τ. The result is

$$z_{N-1}^{(j)}=\varkappa_k(N)+\sum_{k<\tau<N}\varkappa_k(\tau)\,\frac{j}{\tau}\,\Big(a_{N-\tau}+\sum_{\tau<t<N}a_{N-t}\,\frac{j}{t}\,\Lambda_{t-\tau}(j;t)\Big),$$

where $a_{N-\tau}$ comes from the term $t=\tau$ of (13.8). The expression in parentheses is $\Lambda_{N-\tau}(j;N)$. Indeed, the profile $\lambda^{j,N}$ shifted by g is $\lambda^{j,N-g}$, so that the recursion of Lemma 11.2, for the profile $\lambda^{j,N}$ and the compositions of $N-\tau$, is

$$\Lambda_{N-\tau}(j;N)=a_{N-\tau}+\sum_{0<g<N-\tau}a_g\,\frac{j}{N-g}\,\Lambda_{N-\tau-g}(j;N-g),$$

and with $t = N - g$ the right-hand side is the expression in parentheses. Comparing with (13.9) for $y = j$, we obtain $z_{N-1}^{(j)} = \Phi_{N-1,k}(j)$. This completes the induction, and with $n = N - 1$ it proves (13.7).

Now let k, r, n and N be as in the statement. By (13.7) for column k we have $z_n^{(k)} = \Phi_{n,k}(k)$ and $z_n^{(k+1)} = \Phi_{n,k}(k+1)$, since $n \le 2k+1$. For column $k-1$ and its threshold $k-1 \ge 1$, (13.7) gives $z_n^{(k-1)} = \Phi_{n,k-1}(k-1)$; this requires $n \le 2(k-1)+1$, and it is here that the hypothesis $r \le k-1$ is used. For column $k+1$ and its threshold $k+2$ it gives $z_n^{(k+2)} = \Phi_{n,k+1}(k+2)$. We pass from the polynomials of columns $k-1$ and $k+1$ to $\Phi_{n,k}$ by (13.5). At $y = k-1$, where $N - k = r+1$, this identity gives

$$\Phi_{n,k-1}(k-1) = \Phi_{n,k}(k-1) - \frac{a_k}{k}\,\Lambda_{r+1}(k-1;N).$$

With k replaced by $k+1$, which is allowed because $k+1 \le n$, and at $y = k+2$, where $N - (k+1) = r$, it gives

$$\Phi_{n,k+1}(k+2) = \Phi_{n,k}(k+2) - \frac{a_{k+1}}{k+1}\,\Lambda_{r}(k+2;N).$$

Inserting the four values into (3.4), for column k and for column $k+1$, we obtain (13.3) and (13.4). □

The rows $n = k + r$ of Proposition 13.6 are the rows $k+1 \le n \le 2k-1$, which form the first layer of column k; so (13.3) holds at every position of this layer. In each of the formulas (13.3) and (13.4) the second difference of the frozen polynomial is the *curvature term* of the step, and the multiple of a chain sum is its *boundary term*.

Writing out the curvature term of the first formula, (13.3), by Lemma 13.4, we obtain the expression of a first-layer step from which Proposition 22.6 is proved in Section 22.

Corollary 13.7. *Let $k \ge 2$ and r be integers with $1 \le r \le k-1$, and put $N = k + r + 1$. Then*

$$\frac{d_{k+r,k}}{(k+r)!} = \sum_{m=1}^{r} \frac{\varkappa_k(N-m)}{N-m}\,\mathsf{C}_m(k;N) - \frac{a_k}{k}\,\Lambda_{r+1}(k-1;N).$$

Proof. The left-hand side is $\delta_{k+r}^{(k)}$. By Lemma 13.4 for column k, row $n = k+r$ and $y = k$, where $n+1 = N$ and $n - k = r$, the second difference in (13.3) is the sum on the right. □

The boundary term is positive: the chain sum $\Lambda_{r+1}(k-1;N)$ is a sum of nonnegative weights $V_q(r+1;\lambda^{k-1,N})$, of which the one with $q = 1$ is $a_{r+1} > 0$. In the curvature term the coefficients are positive by Lemma 13.1, since $k+1 \le N-m \le 2k-1$; the curvature sum $\mathsf{C}_1(k;N)$ vanishes, and those with $m \ge 2$ are positive, as noted after Lemma 13.5. The sign of the step is decided by the comparison of the two terms.

13.4. **The pair sum and the criterion.** For the criterion we read the second formula of Proposition 13.6 for column $k-1$. Let $k \ge 3$ and R be integers with $1 \le R \le k-2$, and put $n = k + R - 1$ and $N = n + 1 = k + R$. The proposition applies to column $k-1 \ge 2$ with $r = R \le (k-1) - 1$, for which its row is

$(k-1)+R=n$ and its frozen polynomial is $\Phi_{n,k-1}$. Formula (13.4) then expresses the step of column k:

$$\delta_n^{(k)} = \big(\Phi_{n,k-1}(k+1) - 2\,\Phi_{n,k-1}(k) + \Phi_{n,k-1}(k-1)\big) - \frac{a_k}{k}\,\Lambda_R(k+1;N). \tag{13.10}$$

The rows are $k \le n \le 2k-3$, and $n-k = R-1$. For $R \ge 2$, position (n,k) lies in the first layer of column k and has the deficit $R-1$, so that (13.3) holds at this position as well, with $r = R-1$. The number $\delta_n^{(k)}$ then has two expressions as a curvature term minus a boundary term, (13.3) and (13.10), and these are different decompositions of the step. The curvature term of (13.3) is a second difference of $\Phi_{n,k}$, the frozen polynomial of column k, whereas that of (13.10) is a second difference of $\Phi_{n,k-1}$, the frozen polynomial of column $k-1$ in the same row; both are taken at the points $k+1$, k and $k-1$. The chain sums of the two boundary terms run over the compositions of the same number R, with threshold $k-1$ in (13.3) and with threshold $k+1$ in (13.10).

We bound the curvature term of (13.10) from below by a quantity in which the frozen polynomial no longer occurs. The quantity is suggested by the chains on which Definition 13.2 is modelled, taken for column $k-1$: the decreasing chains of integers $N > t_1 > \cdots > t_i \ge k$, in which every integer t_l that a chain visits contributes one factor y/t_l, so that a chain that visits i integers belongs to the power y^i. In the proof of Proposition 13.9, k^2 times the second difference of a power y^i at $y=k$ is bounded from below by $i(i-1)\,k^i$, and $i(i-1)$ is twice the number of pairs of visited integers. This leads to the chains in which two of the visited integers are marked. We sort these chains by the lower of their two marked integers, which we write $N-b$. This integer contributes the factor $k/(N-b)$. The part of the chain from N down to $N-b$ is a composition of b, and the upper marked integer is one of the integers that the chain visits between N and $N-b$, that is, one of the cut points of this composition. In the following quantity the compositions of b with their marked cut point are collected in the weight $\Lambda_b^{(1)}$, and the part of the chain below $N-b$ is replaced by the count $z^{(k)}_{N-b-1}$.

Definition 13.8. Let $k \ge 1$ and $R \ge 0$ be integers, and put $N = k+R$. The *pair sum* is

$$\mathcal{P}(k,R) = \sum_{b=1}^{R} \Lambda_b^{(1)}(k;N) \cdot \frac{k}{N-b} \cdot z^{(k)}_{N-b-1}.$$

Here $\Lambda_b^{(1)}(k;N) = \Lambda_b^{(1)}[\lambda^{k,N}]$ is the weight of the compositions of b with one marked cut point for the threshold profile. We write a for the marked cut point, so that $0 < a < b$. From here to the end of the section the letters a and b without an index denote integers of this kind, the positions of the two marked points; the Cayley coefficients a_j and the numbers $b_{n,\ell}$ always carry their indices. By the definition of the weight $\Lambda_b^{(1)}$, and since the profile $\lambda^{k,N}$ shifted by a is $\lambda^{k,N-a}$,

$$\mathcal{P}(k,R) = \sum_{1 \le a < b \le R} \Lambda_a(k;N)\,\frac{k}{N-a}\,\Lambda_{b-a}(k;N-a)\,\frac{k}{N-b}\,z^{(k)}_{N-b-1},$$

a sum over the pairs $a<b$ of marked points. In the chains from which we started, the two marked integers are $N-a > N-b$. The factor $\Lambda_a(k;N)$ is the sum over

the part of the chain above $N-a$, which is a composition of a, and the factor $\Lambda_{b-a}(k;N-a)$ is the sum over the part between the two marked integers, which is a composition of $b-a$; the factors $k/(N-a)$ and $k/(N-b)$ belong to the marked integers themselves.

The next proposition bounds k^2 times the curvature term of (13.10) from below by twice the pair sum. In its proof, which does not expand into chains, the second differences of the powers produce a sum that is organised by the last point m of a composition, with the coefficient $\varkappa_{k-1}(N-m)$, whereas the pair sum is organised by the lower marked point b, with everything below it collected in a count. The proof identifies the two sums by a duality between the linear equation of the counts $z^{(k)}_{N-b-1}$, which expresses the count at a point b by the counts at larger points, and the recursion of Lemma 11.3 for the weights with one or two marked cut points, which expresses the weight at a point by the weights at smaller points; the two equations have the same kernel.

Proposition 13.9. *Let $k \geq 1$ and $R \geq 0$ be integers such that either $R = 0$, or $R \geq 1$ and $k \geq R+2$, and put $n = k+R-1$. Then*

$$2\,\mathcal{P}(k,R) \;\leq\; k^2\,\big(\Phi_{n,k-1}(k+1) - 2\,\Phi_{n,k-1}(k) + \Phi_{n,k-1}(k-1)\big).$$

Proof. For $R = 0$ the pair sum is an empty sum and the polynomial $\Phi_{n,k-1} = \Phi_{k-1,k-1}$ is constant, so that both sides are 0. Let $R \geq 1$; then $k \geq R+2 \geq 3$. We put $N = k+R = n+1$, and we write $\varkappa$ for $\varkappa_{k-1}$, λ for the threshold profile $\lambda^{k,N}$, so that $\lambda(s) = k/(N-s)$, $\Lambda^{(1)}_m$ and $\Lambda^{(2)}_m$ for $\Lambda^{(1)}_m[\lambda]$ and $\Lambda^{(2)}_m[\lambda]$, and σ_m for the sum $\Lambda^{(1)}_m + \Lambda^{(2)}_m$.

By Lemma 13.4 for column $k-1$, row n and $y = k$, where $n-(k-1) = R$,

$$\Phi_{n,k-1}(k+1) - 2\,\Phi_{n,k-1}(k) + \Phi_{n,k-1}(k-1) \qquad (13.11)$$
$$= \sum_{m=1}^{R} \frac{\varkappa(N-m)}{N-m}\;\mathsf{C}_m(k;N).$$

Fix m with $1 \leq m \leq R$, and write V_q for $V_q(m;\lambda^{1,N})$; these numbers are nonnegative, because $m < N$. By Lemma 13.5, $\mathsf{C}_m(k;N)$ is the sum over $1 \leq q \leq m$ of $\big((k+1)^q - 2k^q + (k-1)^q\big)V_q$. For $y = k$ all terms on the right-hand side of (13.2) are nonnegative, and the first of them is $q(q-1)\,k^{q-2}$; hence

$$k^2\big((k+1)^q - 2\,k^q + (k-1)^q\big) \;\geq\; q\,(q-1)\,k^q$$

for every $q \geq 1$. On the other hand, $V_q(m;\lambda) = k^{q-1}\,V_q$, and Corollary 11.7 for the profile λ states that the numbers $q(q-1)\,V_q(m;\lambda)$, $1 \leq q \leq m$, add up to $2\,\sigma_m$. Together, for every m with $1 \leq m \leq R$,

$$k^2\,\mathsf{C}_m(k;N) \;\geq\; \sum_{q=1}^{m} q\,(q-1)\,k^q\,V_q = 2\,k\,\sigma_m. \qquad (13.12)$$

For these m we have $1 \leq N-m \leq 2k-3$, so that $\varkappa(N-m) > 0$ by Lemma 13.1. We multiply (13.12) by $\varkappa(N-m)/(N-m)$ and sum over m. Since $k/(N-m) = \lambda(m)$, this gives

$$k^2 \sum_{m=1}^{R} \frac{\varkappa(N-m)}{N-m}\;\mathsf{C}_m(k;N) \;\geq\; 2\sum_{m=1}^{R} \varkappa(N-m)\,\lambda(m)\,\sigma_m, \qquad (13.13)$$

and by (13.11) the left-hand side is the right-hand side of the proposition.

It remains to show that the sum on the right is the pair sum:

$$\sum_{m=1}^{R} \varkappa(N-m)\,\lambda(m)\,\sigma_m = \mathcal{P}(k,R). \tag{13.14}$$

For $1 \le b \le R$ put $v_b = z^{(k)}_{N-b-1}$. Equation (13.8), which was shown in the proof of Proposition 13.6, holds for column $k-1$ and its threshold k when the N of that equation is replaced by any integer from 1 to $2k$. We use it for the integer $N-b$, which is allowed because $1 \le N-b \le 2k-3$, and with $m = N-t$ as the index of summation. Since $k/t = \lambda(m)$ and $z^{(k)}_{t-1} = v_m$, it reads

$$v_b = \varkappa(N-b) + \sum_{b<m\le R} a_{m-b}\,\lambda(m)\,v_m \qquad (1 \le b \le R). \tag{13.15}$$

On the other side, Lemma 11.3 for the profile λ states that $\Lambda^{(2)}_m$ is the sum of $\sigma_s\,\lambda(s)\,a_{m-s}$ over $0 < s < m$. Since $\sigma_m = \Lambda^{(1)}_m + \Lambda^{(2)}_m$, this gives

$$\sigma_m = \Lambda^{(1)}_m + \sum_{0<s<m} a_{m-s}\,\lambda(s)\,\sigma_s \qquad (1 \le m \le R). \tag{13.16}$$

We solve (13.15) for $\varkappa(N-s)$, multiply by $\lambda(s)\,\sigma_s$ and sum over $1 \le s \le R$:

$$\sum_{s=1}^{R} \varkappa(N-s)\,\lambda(s)\,\sigma_s = \sum_{s=1}^{R} v_s\,\lambda(s)\,\sigma_s - \sum_{1\le s<m\le R} \lambda(m)\,v_m\,a_{m-s}\,\lambda(s)\,\sigma_s.$$

Summing over s first in the double sum and using (13.16), we see that the right-hand side is

$$\sum_{m=1}^{R} v_m\,\lambda(m)\,\sigma_m - \sum_{m=1}^{R} \lambda(m)\,v_m\,\bigl(\sigma_m - \Lambda^{(1)}_m\bigr) = \sum_{m=1}^{R} \Lambda^{(1)}_m\,\lambda(m)\,v_m,$$

which is $\mathcal{P}(k,R)$ by Definition 13.8. This proves (13.14), and the proposition follows from (13.13). □

Remark. The one inequality in the proof keeps, for each q, the first term of the sum (13.2) at $y = k$, and the second term is smaller than the first by the factor $(q-2)(q-3)/(12k^2)$.

Inserted into (13.10), the proposition gives the criterion.

Proposition 13.10. *Let k and R be integers with $1 \le R \le k-2$, so that $k \ge 3$, and put $N = k+R$ and $n = k+R-1$. If*

$$k\,a_k\,\Lambda_R(k+1;N) < 2\,\mathcal{P}(k,R),$$

then the normalised step $\delta^{(k)}_n$ of column k in row n is positive.

Proof. We multiply (13.10) by k^2 and apply Proposition 13.9, whose hypothesis holds because $R \ge 1$ and $k \ge R+2$. The hypothesis of the criterion then gives

$$k^2\,\delta^{(k)}_n \;\ge\; 2\,\mathcal{P}(k,R) - k\,a_k\,\Lambda_R(k+1;N) \;>\; 0. \qquad \square$$

14. The two sides of the criterion

Proposition 13.10 reduces the positivity of the step of column k in row $k+R-1$, for integers k and R with $1 \le R \le k-2$, to an inequality in which the step no longer occurs: with $N = k+R$, the step is positive as soon as

$$k\, a_k\, \Lambda_R(k+1; N) < 2\, \mathcal{P}(k, R).$$

We now estimate the two sides. Proposition 14.7 bounds the pair sum $\mathcal{P}(k, R)$ from below, and Proposition 14.8 bounds the chain sum $\Lambda_R(k+1; N)$ of the boundary term from above. Both bounds are expressed in terms of the Poisson mass P and of its convolutions P_2 and P_3 of Section 12. Section 15 bounds the three sums of these convolutions that occur in the first bound, and Section 16 then inserts both bounds into the criterion.

14.1. **A tangent inequality for a Volterra equation.** The left-hand side of the criterion contains a chain sum over the compositions of R, and the pair sum is built from chain sums and from normalised counts $z_n^{(k)}$. In a chain sum $\Lambda_m(y; N)$ a composition of m carries, besides the Cayley coefficients of its parts, the value $y/(N-s)$ of the threshold profile at each of its cut points s; these values are its cut factors (Section 11). For the counts, the proof of Lemma 14.4 uses a linear equation in which an integer $t > k$ carries the factor k/t; in this section we call these factors cut factors as well.

If every cut factor is replaced by 1, the chain sum $\Lambda_R(k+1; N)$ becomes $\nu_R = e^R P(R)$ (Corollary 11.9). The two bounds of this section take the cut factors into account instead of replacing them by 1.

The common tool is a lower bound for sums of products of cut factors. Write a cut factor as e^{-x} with a real number x. A product of cut factors is then e^{-X}, where X is the sum of the exponents, and the exponential function lies above each of its tangents: for all real numbers X and μ,

$$e^{-X} \ge e^{-\mu}\,(1 + \mu - X). \tag{14.1}$$

The right-hand side is linear in X, so that its sum over the compositions is known as soon as the sum of the exponents is known. The sums in question are solutions of linear equations of one type, and Lemma 14.1 carries out this summation for all of them at once. The inequalities that follow from it apply directly to the pair sum, for which a lower bound is wanted. For the chain sum of the boundary term, which has to be bounded from above, they are applied to the amount by which the chain sum falls short of its value without cut factors, which Lemma 11.8 expresses through the chain sums over the compositions of the integers smaller than R.

We consider real-valued functions on the nonnegative integers and equations of the form

$$u(M) = h(M) + \sum_{t_0 < t < M} a_{M-t}\, w(t)\, u(t), \tag{14.2}$$

in which the value of a function u at M is expressed in terms of its values at smaller arguments, of a function h and of a weight w; here $t_0 \ge 0$ is an integer. Such an equation is a Volterra equation of the second kind in discrete form, whose kernel a_{M-t} is formed from the Cayley coefficients. Two sequences of this paper satisfy an equation of this form. The total weights $\Lambda_M[f]$ of the compositions of M for a cut profile f satisfy it with $h(M) = a_M$, $w = f$ and $t_0 = 0$; this is the last-cut

recursion of Lemma 11.1. The numbers ν_M satisfy it with the same h and t_0 and with the weight 1: in the identity of Lemma 4.5 the term $j = 0$ vanishes and the term $j = m$ is $a_m\,\nu_0 = a_m$, so that

$$\nu_m = a_m + \sum_{0<s<m} a_{m-s}\,\nu_s \qquad (m \geq 1). \tag{14.3}$$

The proof of Lemma 14.4 applies Lemma 14.1 also to the normalised counts $z^{(k)}_{M-1}$ of a threshold $k \geq 1$.

The following lemma compares the solution u of (14.2) for a weight of the form $w(t) = e^{-x(t)}$ with the solution v of the same equation without weights.

Lemma 14.1. *Let $t_0 \geq 0$ and $B \geq 0$ be integers, and let h, v, u and x be real-valued functions on the nonnegative integers such that for every integer M with $1 \leq M \leq B$ the following three conditions hold.*

(1) $h(M) \geq 0$.
(2) $v(M) = h(M) + \sum_{t_0<t<M} a_{M-t}\,v(t)$.
(3) $u(M) = h(M) + \sum_{t_0<t<M} a_{M-t}\,e^{-x(t)}\,u(t)$.

Then for every real number μ and every integer M with $1 \leq M \leq B$,

$$e^{-\mu}\Big((1+\mu)\,v(M) - \sum_{t_0<t<M} \nu_{M-t}\,v(t)\,x(t)\Big) \leq u(M).$$

Proof. For an integer $M \geq 0$ let $S(M) = \sum_{t_0<t<M} \nu_{M-t}\,v(t)\,x(t)$. We first show that for every $M \geq 0$

$$\sum_{t_0<t<M} a_{M-t}\,\big(v(t)\,x(t) + S(t)\big) = S(M). \tag{14.4}$$

When the definition of $S(t)$ is inserted on the left-hand side and the two summations are exchanged, a number $v(c)\,x(c)$ with $t_0 < c < M$ occurs with the total coefficient $a_{M-c} + \sum_{0<s<M-c} a_{M-c-s}\,\nu_s$, and this equals ν_{M-c} by (14.3) for $m = M - c \geq 1$.

We now prove the inequality of the lemma by strong induction on M, for all real numbers μ at once. Let $1 \leq M \leq B$, let μ be a real number, and suppose that the inequality holds at every integer t with $1 \leq t < M$ and for every real value of the parameter. Let t be an integer with $t_0 < t < M$. Then $1 \leq t \leq B$, and the inequality at t with the parameter $\mu - x(t)$ reads

$$e^{-(\mu-x(t))}\Big(\big(1+\mu-x(t)\big)\,v(t) - S(t)\Big) \leq u(t).$$

Multiplying it by $a_{M-t}\,e^{-x(t)} \geq 0$, we obtain

$$e^{-\mu}\,a_{M-t}\big((1+\mu)\,v(t) - v(t)\,x(t) - S(t)\big) \leq a_{M-t}\,e^{-x(t)}\,u(t). \tag{14.5}$$

The shift of the parameter by $x(t)$ is what turns the weight $e^{-x(t)}$ in the equation of u into the common factor $e^{-\mu}$. Moreover $1 + \mu \leq e^{\mu}$ and $h(M) \geq 0$, so that

$$e^{-\mu}\,(1+\mu)\,h(M) \leq h(M).$$

We add this inequality and the inequalities (14.5) for all t with $t_0 < t < M$. On the right we obtain $u(M)$, by the equation of u. On the left we obtain $e^{-\mu}$ times

$$(1+\mu)\Big(h(M) + \sum_{t_0<t<M} a_{M-t}\,v(t)\Big) - \sum_{t_0<t<M} a_{M-t}\,\big(v(t)\,x(t) + S(t)\big)$$
$$= (1+\mu)\,v(M) - S(M),$$

by the equation of v and by (14.4). This is the inequality at M. □

Lemma 14.1 can be read as follows, although its proof does not proceed in this way. Iterating the equation of u expresses $u(M)$ as a sum over the decreasing chains $M > t_1 > \cdots > t_j > t_0$ of integers, $j \geq 0$. Such a chain has the weight

$$a_{M-t_1}\, a_{t_1-t_2} \cdots a_{t_{j-1}-t_j}\, h(t_j),$$

the empty chain having the weight $h(M)$, and it carries in addition the factor e^{-X}, where $X = x(t_1) + \cdots + x(t_j)$ is the sum of the exponents at the points that the chain visits. The function v is the same sum without the factors e^{-X}. By the first condition the weights are nonnegative, and (14.1), summed over the chains, bounds $u(M)$ from below by $e^{-\mu}$ times the difference of $(1+\mu)\, v(M)$ and the sum of X over the chains, each taken with its weight. To this last sum a point t contributes $x(t)$ once for every chain that visits it. Such a chain consists of a chain from M down to t and of a chain that starts at t. The chains of the first kind are the compositions of $M - t$ without cut factors, whose total weight is ν_{M-t} (Corollary 11.9), and the total weight of the chains of the second kind is $v(t)$. So the sum of X over the chains is $\sum_{t_0<t<M} \nu_{M-t}\, v(t)\, x(t)$, which is the sum in the lemma.

If $v(M) > 0$, one can take $\mu = S(M)/v(M)$ in the lemma, in the notation of its proof. The expression in parentheses then equals $v(M)$, and the lemma reads $v(M)\, e^{-S(M)/v(M)} \leq u(M)$. In the reading by chains this is Jensen's inequality for the convex function $X \mapsto e^{-X}$: the mean of e^{-X} over the chains is at least $e^{-\bar{X}}$, where $\bar{X}$ is the mean of X. Corollary 14.3 and Lemma 14.4 are of this form. The proof of Proposition 14.8 uses the lemma for a general μ, with one and the same value for several chain sums.

For the total weight $\Lambda_m[f]$ of the compositions of m for a cut profile f (Section 11) the lemma takes the following form. For a real-valued function x on the nonnegative integers we write e^{-x} for the cut profile $s \mapsto e^{-x(s)}$; every positive cut profile is of this form.

Lemma 14.2. *Let x be a real-valued function on the nonnegative integers, let μ be a real number and let $m \geq 1$ be an integer. Then*

$$e^{-\mu}\Big((1+\mu)\,\nu_m - \sum_{0<c<m} \nu_{m-c}\,\nu_c\, x(c)\Big) \leq \Lambda_m[e^{-x}].$$

Proof. We apply Lemma 14.1 with $t_0 = 0$, $B = m$, $h(M) = a_M$, $v(M) = \nu_M$ and $u(M) = \Lambda_M[e^{-x}]$. Its first condition is clear, the equation of v is (14.3), and the equation of u is the last-cut recursion of Lemma 11.1 for the profile e^{-x}. The conclusion of Lemma 14.1 at $M = m$ is the claim. □

In the reading of Lemma 14.1 by chains, the chains are now the compositions of m, and the sum in the lemma, divided by ν_m, is the mean of the sum of the exponents at the cut points, for a composition that is drawn with a probability proportional to the product of the Cayley coefficients of its parts. Taking this mean for μ, we obtain Jensen's inequality.

Corollary 14.3. *Let x be a real-valued function on the nonnegative integers and let $m \geq 1$ be an integer. Then*

$$\nu_m\, \exp\Big(-\frac{1}{\nu_m} \sum_{0<c<m} \nu_{m-c}\,\nu_c\, x(c)\Big) \leq \Lambda_m[e^{-x}].$$

Proof. Let $S = \sum_{0<c<m} \nu_{m-c}\, \nu_c\, x(c)$. We apply Lemma 14.2 with $\mu = S/\nu_m$, for which $(1+\mu)\,\nu_m - S = \nu_m$. □

A term of the pair sum contains, besides two chain sums, a normalised count $z_n^{(k)}$ of threshold k (Section 3), and the next lemma is the inequality of the same kind for these numbers.

Lemma 14.4. *Let $k \geq 1$ and M be integers with $1 \leq M \leq 2k+1$. Then the normalised count of threshold k satisfies*

$$a_M \, \exp\Big(-\frac{1}{a_M} \sum_{k<t<M} \nu_{M-t}\, a_t \, \log\frac{t}{k}\Big) \leq z_{M-1}^{(k)}.$$

Proof. By (13.8), which was shown in the proof of Proposition 13.6, taken with $j = k$, which is admissible there because $k \geq 1$, and with M in the place of N, the normalised counts $z_{M-1}^{(k)}$ satisfy for every integer M with $1 \leq M \leq 2k+2$ an equation of the form (14.2), with $t_0 = k$, with the weight $w(t) = k/t$ and with the kernel deficit $\varkappa_k$ of Section 13 as the function h.

We now apply Lemma 14.1 with $t_0 = k$, $B = 2k+1$, $h = \varkappa_k$ and $v(M) = a_M$, with $u(M) = z_{M-1}^{(k)}$ for $M \geq 1$ and with $x(t) = \log(t/k)$ for $t \geq 1$; the values of u and x at 0 do not enter. Its three conditions hold for $1 \leq M \leq 2k+1$: in this range $\varkappa_k(M) > 0$ by Lemma 13.1; the equation of v is the definition of $\varkappa_k(M)$; and the equation of u is (13.8) with $j = k$, in which $k/t = e^{-x(t)}$. Hence, for every real number μ,

$$e^{-\mu}\big((1+\mu)\,a_M - S\big) \leq z_{M-1}^{(k)}, \qquad S = \sum_{k<t<M} \nu_{M-t}\, a_t \, \log\frac{t}{k}.$$

For $\mu = S/a_M$ we have $(1+\mu)\,a_M - S = a_M$, and the inequality becomes the claim. □

The number a_M on the left-hand side of Lemma 14.4 is c_{M-1}, the number of all parking functions of length $M-1$ divided by $(M-1)!$, so that the lemma bounds from below the proportion of the parking functions of length $M-1$ all of whose displacements are smaller than k.

14.2. **The Borel distribution.** The numbers $e^{-m}a_m$, $m \geq 1$, are the probabilities of the Borel distribution with parameter 1; their sum is 1 by Lemma 9.8. The lower bound of the pair sum contains quotients $e^{-i}a_i/(e^{-j}a_j)$ of two of these probabilities with $i \leq j$, which come from the Cayley coefficients in Lemma 14.4. The next lemma shows that $(j/i)^{3/2}$ is an upper bound for such a quotient.

Lemma 14.5. *For all integers i and j with $1 \leq i \leq j$, the Cayley coefficients satisfy*

$$\log a_i - i + \tfrac{3}{2}\log i \leq \log a_j - j + \tfrac{3}{2}\log j.$$

Proof. For an integer $m \geq 1$ let $\phi(m) = \log a_m - m + \frac{3}{2}\log m$. It suffices to show that $\phi(j) \leq \phi(j+1)$ for every $j \geq 1$. Since $a_{j+1} = (1+1/j)^{j-1}a_j$ by (2.1), we have

$$\begin{aligned}\phi(j+1) - \phi(j) &= (j-1)\log\Big(1+\frac{1}{j}\Big) - 1 + \frac{3}{2}\log\Big(1+\frac{1}{j}\Big)\\ &= \Big(j+\frac{1}{2}\Big)\log\Big(1+\frac{1}{j}\Big) - 1.\end{aligned}$$

For every real number $u \geq 0$ we have $\log(1+u) \geq 2u/(u+2)$, as one checks by differentiating the difference of the two sides, which vanishes at $u = 0$. At $u = 1/j$ this inequality reads $\log(1+1/j) \geq 2/(2j+1)$, so that $(j+\frac{1}{2})\log(1+1/j) \geq 1$ and $\phi(j+1) - \phi(j) \geq 0$. □

In the proof of Proposition 14.7 a bound for the quotient $e^{-i}a_i/(e^{-j}a_j)$ is multiplied by nonnegative weights and summed over one of the two indices, and the sums of those weights against the index and against its square are known or easily bounded. A power with the exponent $3/2$ cannot be summed in this way, whereas a polynomial can. We therefore replace the bound $(j/i)^{3/2} = (1+u)^{3/2}$, with $u = (j-i)/i$, by its Taylor polynomial of degree two at $u = 0$, which lies above it for $u \geq 0$.

Lemma 14.6. *For all integers i and j with $1 \leq i \leq j$, the Cayley coefficients satisfy*

$$\frac{e^{\,j-i}\,a_i}{a_j} \leq 1 + \frac{3}{2}\cdot\frac{j-i}{i} + \frac{3}{8}\Big(\frac{j-i}{i}\Big)^2.$$

Proof. Let $u = (j-i)/i$, so that $u \geq 0$ and $1+u = j/i$, and let $q = e^{\,j-i}a_i/a_j$. By Lemma 14.5,

$$\log q = (j-i) + \log a_i - \log a_j \leq \tfrac{3}{2}\,(\log j - \log i) = \tfrac{3}{2}\log(1+u),$$

and hence $q^2 \leq (1+u)^3$. On the other hand, a direct computation gives

$$\Big(1 + \frac{3}{2}\,u + \frac{3}{8}\,u^2\Big)^2 = (1+u)^3 + \frac{1}{8}\,u^3 + \frac{9}{64}\,u^4 \geq (1+u)^3.$$

Thus $q^2 \leq (1 + \frac{3}{2}u + \frac{3}{8}u^2)^2$, and the claim follows on taking square roots. □

14.3. **The lower bound of the pair sum.** Let $k \geq 1$ and $R \geq 0$ be integers and $N = k + R$. By Definition 13.8 and by the definition of the weight $\Lambda_b^{(1)}[f]$ of the compositions of b with one marked cut point (Section 11), the pair sum is a sum over the pairs of marked cut points $a < b$,

$$\begin{aligned} \mathcal{P}(k,R) &= \sum_{b=1}^{R} \sum_{0<a<b} \mathcal{P}_{ab}, \\ \mathcal{P}_{ab} &= \Lambda_a(k;N)\,\frac{k}{N-a}\,\Lambda_{b-a}(k;N-a)\,\frac{k}{N-b}\,z^{(k)}_{N-b-1}. \end{aligned} \tag{14.6}$$

Here we have used that the threshold profile $\lambda^{k,N}$, shifted by a, is the threshold profile $\lambda^{k,N-a}$. In this subsection a and b without an index denote cut points; the Cayley coefficients always carry an index.

The cut factor at a cut point $s \leq R$ is $k/(N-s)$, and $N - s = k + (R-s)$. We write it as

$$\frac{k}{N-s} = e^{-L(R-s)}, \qquad \text{where} \quad L(y) = \log\Big(1 + \frac{y}{k}\Big).$$

Each of the three sums in $\mathcal{P}_{ab}$, the two chain sums and the normalised count, is thus a sum of products of exponentials, and Corollary 14.3 and Lemma 14.4 bound it from below by a single exponential. For the pair $a < b$ this gives a lower bound of the form $a_k\,e^R\,P(a)\,P(b-a)\,e^{-X_{ab}}$. Here $P(a)\,P(b-a)$ is, up to the factor e^b, the value $\nu_a\,\nu_{b-a}$ of the two chain sums without cut factors, and the exponent X_{ab} consists of means of sums of values of L. Two tangents then make the sum over the pairs computable. The inequality (14.1) replaces $e^{-X_{ab}}$ by an expression that

is linear in X_{ab}. And the concave function L lies below each of its tangents: for a real number $\theta \geq 0$ the tangent of L at the point $y = k\theta$ is a line $\alpha + \beta y$, whose coefficients are written out in Proposition 14.7. When L is replaced by a line, the sum of the exponents over the pairs is evaluated by the mean positions in the convolutions of the Poisson mass that were computed in Section 12. For the sum of the logarithms themselves no such evaluation is available, and this is the reason for the second tangent.

The result is expressed in terms of three sums. For a real number N and an integer $c \neq N$ let

$$\varrho(N, c) = \frac{c}{2\,(N-c)} + \frac{c^2}{8\,(N-c)^2}.$$

Clearly $\varrho(k+R, c) \geq 0$ for integers $k \geq 0$, $R \geq 0$ and c with $0 \leq c \leq R$ and $c < k + R$. For integers $k \geq 0$ and $R \geq 0$ let

$$\mathcal{W}(R) = \sum_{b=1}^{R} P_2(b), \tag{14.7}$$

$$\begin{aligned} \mathcal{G}(k, R) = {} & \sum_{b=1}^{R} \Big(\frac{7}{2}\, P_2(b) + 2\, P_3(b) \Big) \\ & + \sum_{0<c<R} P_3(c) \, \big(1 + \varrho(k+R, c) \big), \end{aligned} \tag{14.8}$$

$$\begin{aligned} \mathcal{A}(k, R) = {} & \sum_{b=1}^{R} \Big(\big(R - \tfrac{b}{2} \big) \big(P_2(b) + 2\, P_3(b) \big) + \tfrac{5}{2}\, (R-b)\, P_2(b) \Big) \\ & + \sum_{0<c<R} (R-c)\, P_3(c) \, \big(1 + \varrho(k+R, c) \big), \end{aligned} \tag{14.9}$$

where P_2 and P_3 are the twofold and the threefold convolution of the Poisson mass (Section 12).

By the definition of P_2, the sum $\mathcal{W}(R)$ is the sum of the weights $P(a)\, P(b-a)$ over the pairs $0 < a < b \leq R$; this is the total weight of the pairs. When every value $L(y)$ in the exponents X_{ab} is replaced by the line $\alpha + \beta y$, the sum of the exponents over the pairs becomes α times the total weight of these values plus β times the total of the arguments y at which they are taken, and $\mathcal{G}(k, R)$ and $\mathcal{A}(k, R)$ are bounds for these two totals. The summand of the first sum of (14.8) collects, for a fixed larger marked point b, the two marked points with the weight $P_2(b)$ each, the cut points of the two chain sums with the weight $P_3(b)$ each, and a term with the weight $\frac{3}{2} P_2(b)$ that comes from the comparison of the Cayley coefficients a_k and a_{N-b}; the second sum belongs to the normalised counts. Column k enters $\mathcal{G}$ and $\mathcal{A}$ only through the correction $\varrho(k+R, c)$, which comes from Lemma 14.6. The proof of the proposition first proves the lower bound described above for the term $\mathcal{P}_{ab}$ of one pair. It then shows that the sum of the exponents X_{ab} over the pairs, taken with the weights $P(a)\, P(b-a)$, is at most $\alpha\, \mathcal{G}(k, R) + \beta\, \mathcal{A}(k, R)$, which is its longest part, and it concludes by (14.1).

Proposition 14.7. *Let $k \geq 1$ and $R \geq 0$ be integers with $R \leq k$, and let $\theta \geq 0$ and μ be real numbers. Put*

$$\alpha = \log(1+\theta) - \frac{\theta}{1+\theta}, \qquad \beta = \frac{1}{k\,(1+\theta)}.$$

Then the pair sum $\mathcal{P}(k,R)$ of Definition 13.8 satisfies

$$a_k\, e^R\, e^{-\mu}\Big((1+\mu)\,\mathcal{W}(R) - \big(\alpha\,\mathcal{G}(k,R) + \beta\,\mathcal{A}(k,R)\big)\Big) \le \mathcal{P}(k,R),$$

where $\mathcal{W}(R)$, $\mathcal{G}(k,R)$ and $\mathcal{A}(k,R)$ are the sums (14.7), (14.8) *and* (14.9).

Proof. Throughout the proof $N = k+R$, and $L(y) = \log(1+y/k)$ for real numbers $y \ge 0$. We use the following properties of L. It is nonnegative, and for all integers $j \ge k$ and $0 \le s \le R$,

$$L(j-k) = \log\frac{j}{k}, \qquad \frac{k}{N-s} = e^{-L(R-s)}. \tag{14.10}$$

Moreover, for every real number $y \ge 0$,

$$L(y) \le \alpha + \beta\, y. \tag{14.11}$$

Indeed, the inequality $\log t \le t-1$ at the point $t = (1+y/k)/(1+\theta)$ gives

$$L(y) - \log(1+\theta) \le \frac{1+y/k}{1+\theta} - 1 = -\frac{\theta}{1+\theta} + \frac{y}{k\,(1+\theta)}.$$

We first bound the term of one pair from below. Let a and b be integers with $0 < a < b \le R$. We define six numbers, whose indices show on which of the two marked points they depend:

$$X_a^{(1)} = \frac{1}{\nu_a}\sum_{0<c<a} \nu_{a-c}\,\nu_c\, L(R-c), \qquad X_a^{(2)} = L(R-a),$$

$$X_{ab}^{(3)} = \frac{1}{\nu_{b-a}}\sum_{0<c<b-a} \nu_{b-a-c}\,\nu_c\, L(R-a-c), \qquad X_b^{(4)} = L(R-b),$$

$$X_b^{(5)} = \log\frac{a_k\, e^{R-b}}{a_{N-b}}, \qquad X_b^{(6)} = \frac{1}{a_{N-b}}\sum_{k<t<N-b} \nu_{N-b-t}\, a_t\, \log\frac{t}{k},$$

and we let X_{ab} be their sum. We claim that

$$a_k\, e^R\, P(a)\, P(b-a)\, e^{-X_{ab}} \le \mathcal{P}_{ab}. \tag{14.12}$$

The five factors of $\mathcal{P}_{ab}$ are bounded from below one by one. By (14.10) the threshold profile $\lambda^{k,N}$ agrees at the cut points $0 < s < a$ with the profile e^{-x} for a function x with $x(s) = L(R-s)$ at these points, and the total weight $\Lambda_a[f]$ of the compositions of a involves the values of f at the cut points $0 < s < a$ only (Section 11). Corollary 14.3 therefore gives

$$\nu_a\, e^{-X_a^{(1)}} \le \Lambda_a(k;N).$$

In the same way the profile $\lambda^{k,N-a}$ agrees at the cut points $0 < s < b-a$ with e^{-x} for $x(s) = L(R-a-s)$, and

$$\nu_{b-a}\, e^{-X_{ab}^{(3)}} \le \Lambda_{b-a}(k;N-a).$$

The two cut factors are $k/(N-a) = e^{-X_a^{(2)}}$ and $k/(N-b) = e^{-X_b^{(4)}}$, again by (14.10). Lemma 14.4 applies with $M = N-b$, since $1 \le b \le R \le k$ gives $k \le N-b \le 2k-1$, and it states that

$$a_{N-b}\, e^{-X_b^{(6)}} \le z_{N-b-1}^{(k)}.$$

Finally $a_{N-b} = a_k\, e^{R-b}\, e^{-X_b^{(5)}}$ by the definition of $X_b^{(5)}$. Multiplying the bounds for the five factors, all of which are positive, we obtain

$$\nu_a\, \nu_{b-a}\, a_k\, e^{R-b}\, e^{-X_{ab}} \le \mathcal{P}_{ab},$$

and this is (14.12), because $\nu_a = e^a P(a)$ and $\nu_{b-a} = e^{b-a} P(b-a)$.

Next we bound the sum of the exponents over the pairs. We claim that

$$\sum_{b=1}^{R} \sum_{0<a<b} P(a)\, P(b-a)\, X_{ab} \le \alpha\, \mathcal{G}(k,R) + \beta\, \mathcal{A}(k,R). \tag{14.13}$$

To prove this we use that the Poisson mass is positive, that P_2 and P_3 are non-negative, and that $e^{-m}\, \nu_g\, \nu_{m-g} = P(g)\, P(m-g)$ for $0 \le g \le m$. We also use repeatedly the following consequence of (14.11): for an integer $m \ge 0$ and a real number $y \ge m$,

$$\sum_{0<g<m} P(g)\, P(m-g)\, L(y-g) \le \Big(\alpha + \beta\,\big(y - \tfrac{m}{2}\big)\Big)\, P_2(m). \tag{14.14}$$

Indeed, $y - g \ge 0$ and hence $L(y-g) \le \alpha + \beta y - \beta g$ by (14.11), the weights $P(g)\, P(m-g)$ add up to $P_2(m)$ by definition, and $\sum_{0<g<m} g\, P(g)\, P(m-g) = \frac{m}{2}\, P_2(m)$ by Lemma 12.6.

Fix b with $1 \le b \le R$. We sum the parts of X_{ab} over a one after the other. For the cut factor at a, (14.14) with $m = b$ and $y = R$ gives directly

$$\sum_{0<a<b} P(a)\, P(b-a)\, X_a^{(2)} \le \Big(\alpha + \beta\,\big(R - \tfrac{b}{2}\big)\Big)\, P_2(b).$$

For the first chain sum we have $P(a)\, X_a^{(1)} = \sum_{0<c<a} P(c)\, P(a-c)\, L(R-c)$, because $P(a)/\nu_a = e^{-a}$, and by (14.14) with $m = a$ and $y = R$ this is at most $(\alpha + \beta(R - a/2))\, P_2(a)$. We multiply by $P(b-a)$ and sum over a. Since $\sum_{0<a<b} P_2(a)\, P(b-a) = P_3(b)$ by definition and $\sum_{0<a<b} a\, P_2(a)\, P(b-a) = \frac{2b}{3}\, P_3(b)$ by Lemma 12.7, we obtain

$$\sum_{0<a<b} P(a)\, P(b-a)\, X_a^{(1)} \le \Big(\alpha + \beta\,\big(R - \tfrac{b}{3}\big)\Big)\, P_3(b).$$

For the second chain sum we have in the same way

$$P(b-a)\, X_{ab}^{(3)} = \sum_{0<c<b-a} P(c)\, P(b-a-c)\, L(R-a-c),$$

and (14.14) with $m = b-a$ and $y = R-a$ bounds this by $(\alpha + \beta(R - b/2 - a/2))\, P_2(b-a)$. We multiply by $P(a)$ and sum over a. The substitution $a \mapsto b-a$ shows that $\sum_{0<a<b} P(a)\, P_2(b-a) = P_3(b)$, and $\sum_{0<a<b} a\, P(a)\, P_2(b-a) = \frac{b}{3}\, P_3(b)$ by Lemma 12.8. Hence

$$\sum_{0<a<b} P(a)\, P(b-a)\, X_{ab}^{(3)} \le \Big(\alpha + \beta\,\big(R - \tfrac{2b}{3}\big)\Big)\, P_3(b).$$

The parts $X_b^{(4)}$ and $X_b^{(5)}$ do not depend on a. By Lemma 14.5 with $i = k$ and $j = N - b$, and by (14.10),

$$X_b^{(5)} = \log a_k + (R-b) - \log a_{N-b} \le \tfrac{3}{2}\,\big(\log(N-b) - \log k\big) = \tfrac{3}{2}\, L(R-b).$$

Hence $X_b^{(4)} + X_b^{(5)} \le \frac{5}{2} L(R-b) \le \frac{5}{2}(\alpha + \beta(R-b))$ by (14.11), and since the weights add up to $P_2(b)$,

$$\sum_{0<a<b} P(a)\, P(b-a)\, \big(X_b^{(4)} + X_b^{(5)}\big) \le \tfrac{5}{2}\, \big(\alpha + \beta\,(R-b)\big)\, P_2(b).$$

The sum of these four bounds is

$$\alpha\Big(\tfrac{7}{2}\, P_2(b) + 2\, P_3(b)\Big) + \beta\Big(\big(R - \tfrac{b}{2}\big)\big(P_2(b) + 2\, P_3(b)\big) + \tfrac{5}{2}\,(R-b)\, P_2(b)\Big),$$

where we have used that $(R - b/3) + (R - 2b/3) = 2(R - b/2)$. Summed over $1 \le b \le R$, this is α times the first sum of (14.8) plus β times the first sum of (14.9).

It remains to treat the parts $X_b^{(6)}$. They do not depend on a either, so that their contribution to the left-hand side of (14.13) is $\sum_{b=1}^{R} P_2(b)\, X_b^{(6)}$. We show that

$$\sum_{b=1}^{R} P_2(b)\, X_b^{(6)} \le \sum_{0<c<R} P_3(c)\, \big(1 + \varrho(N,c)\big)\, \big(\alpha + \beta\,(R-c)\big). \tag{14.15}$$

The right-hand side is α times the second sum of (14.8) plus β times the second sum of (14.9), so that (14.13) follows from (14.15). In the sum that defines $X_b^{(6)}$ we substitute $t = N - c$. The range $k < t < N - b$ becomes $b < c < R$, and $\log(t/k) = L(R-c)$ by (14.10), so that

$$X_b^{(6)} = \sum_{b<c<R} \frac{\nu_{c-b}\, a_{N-c}}{a_{N-b}}\, L(R-c).$$

By Lemma 14.6 with $i = N - c$ and $j = N - b$, and since $\nu_{c-b} = e^{c-b} P(c-b)$,

$$\frac{\nu_{c-b}\, a_{N-c}}{a_{N-b}} \le P(c-b)\, \Big(1 + \frac{3}{2} \cdot \frac{c-b}{N-c} + \frac{3}{8}\Big(\frac{c-b}{N-c}\Big)^2\Big).$$

We insert this bound, using that $L(R-c) \ge 0$ and $P_2(b) \ge 0$, and exchange the two summations, so that c runs over $0 < c < R$ and b over $0 < b < c$. For a fixed c let $D = N - c > 0$ and $u_b = (c-b)/D$ for $0 < b < c$. Since $0 \le c - b \le c$, we have $u_b^2 \le (c/D)\, u_b$, and therefore

$$1 + \frac{3}{2}\, u_b + \frac{3}{8}\, u_b^2 \le 1 + (c-b)\Big(\frac{3}{2D} + \frac{3c}{8D^2}\Big).$$

We multiply by $P_2(b)\, P(c-b) \ge 0$ and sum over $0 < b < c$. By the definition of P_3 and by Lemma 12.7, these weights add up to $P_3(c)$ and $\sum_{0<b<c} (c-b)\, P_2(b)\, P(c-b) = \frac{c}{3}\, P_3(c)$. Hence

$$\begin{aligned}\sum_{0<b<c} P_2(b)\, P(c-b)\, \Big(1 + \frac{3}{2}\, u_b + \frac{3}{8}\, u_b^2\Big) &\le P_3(c)\, \Big(1 + \frac{c}{3}\Big(\frac{3}{2D} + \frac{3c}{8D^2}\Big)\Big) \\ &= P_3(c)\, \big(1 + \varrho(N,c)\big).\end{aligned}$$

Finally $L(R-c) \le \alpha + \beta(R-c)$ by (14.11), and the factor $P_3(c)\,(1 + \varrho(N,c))$ is nonnegative. This proves (14.15).

By (14.1) with $X = X_{ab}$ and by (14.12), for all integers a and b with $0 < a < b \le R$,

$$a_k\, e^R\, e^{-\mu}\, P(a)\, P(b-a)\, \big(1 + \mu - X_{ab}\big) \le \mathcal{P}_{ab}.$$

We sum over the pairs. The right-hand sides add up to $\mathcal{P}(k,R)$, by (14.6). On the left the weights $P(a)\,P(b-a)$ add up to $\sum_{b=1}^{R} P_2(b) = \mathcal{W}(R)$, so that the sum of the left-hand sides is

$$a_k\, e^R\, e^{-\mu}\Big((1+\mu)\,\mathcal{W}(R) - \sum_{b=1}^{R}\ \sum_{0<a<b} P(a)\,P(b-a)\,X_{ab}\Big).$$

By (14.13) the double sum is at most $\alpha\,\mathcal{G}(k,R) + \beta\,\mathcal{A}(k,R)$, which gives the claim. □

The two parameters are left free because the proposition is used with bounds for the three sums instead of the sums themselves. Section 15 proves a lower bound for $\mathcal{W}(R)$ and upper bounds for $\mathcal{G}(k,R)$ and $\mathcal{A}(k,R)$, and Section 16 inserts them and chooses θ and μ in terms of them.

To see what the proposition gives, let $R \geq 2$, write $\mathcal{W}$, $\mathcal{G}$ and $\mathcal{A}$ for the three sums, suppose that they are positive, and take $\theta = \mathcal{A}/(k\,\mathcal{G})$ and $\mu = (\mathcal{G}/\mathcal{W})\log(1+\theta)$. For this θ we have $\beta\,\mathcal{A} = \theta\,\mathcal{G}/(1+\theta)$, so that $\alpha\,\mathcal{G} + \beta\,\mathcal{A} = \mathcal{G}\log(1+\theta) = \mu\,\mathcal{W}$, and the proposition reads

$$\mathcal{P}(k,R) \geq a_k\, e^R\, \mathcal{W}\left(1 + \frac{\mathcal{A}}{k\,\mathcal{G}}\right)^{-\mathcal{G}/\mathcal{W}}.$$

In this form the bound can be read as follows: there are $\mathcal{W}$ pairs, counted with their weights, and each of them carries in the mean $\mathcal{G}/\mathcal{W}$ cut factors $k/(k+y)$, with $y = \mathcal{A}/\mathcal{G}$ in the mean.

14.4. **The upper bound of the boundary chain sum.** The left-hand side of the criterion of Proposition 13.10 contains the chain sum $\Lambda_R(k+1;k+R)$, whose cut factors are $(k+1)/(k+R-s)$ for $0<s<R$. The following proposition bounds e^{-R} times this chain sum from above by $P(R)$ minus an explicit expression. It concerns the chain sum $\Lambda_R(y+1;y+R)$, in which column k is replaced by a real number $y \geq 0$, and it contains two parameters. The number K is an upper bound for y: the bound holds for every $y \leq K$, and Section 16 uses this to apply the proposition with a number K that depends on R alone. The number μ is the point at which the tangent (14.1) is taken; Section 16 uses one fixed numerical value.

Proposition 14.8. *Let $R \geq 1$ be an integer, let y and K be real numbers with $0 \leq y \leq K$ and $R \leq K$, and let μ be a real number with $1+\mu \geq 0$. Then the chain sum $\Lambda_R(y+1;y+R)$ satisfies*

$$e^{-R}\,\Lambda_R(y+1;y+R) \leq P(R) - e^{-\mu}\Big((1+\mu)\,\frac{R-2}{2\,(K+R-1)}\,P_2(R) - \frac{1}{K+1}\cdot\frac{R-3}{3}\,P_3(R)\cdot\frac{\frac{4}{5}R-1}{K+\frac{3}{5}R}\Big).$$

Proof. For an integer s with $0<s<R$ let

$$\varphi(s) = 1 - \frac{K+1}{K+R-s} = \frac{R-1-s}{K+R-s}.$$

Since $K \geq R \geq 1$, the denominator is positive and $0 \leq \varphi(s) < 1$. For $1 \leq s \leq R$ we abbreviate $\widetilde{\Lambda}_s = e^{-s}\,\Lambda_s(K+1;K+R)$.

We first replace y by K and show that

$$e^{-R}\,\Lambda_R(y+1;y+R) \leq \widetilde{\Lambda}_R. \tag{14.16}$$

At a cut point $0 < s < R$ the cut factor $(y+1)/(y+R-s)$ is positive, and it is at most the cut factor $(K+1)/(K+R-s)$, since cross-multiplication turns this inequality into $(K-y)(R-s-1) \geq 0$. The total weight $\Lambda_R[f]$ of the compositions of R does not decrease when a nonnegative cut profile f is increased at the cut points, because the weight of every composition is a product of nonnegative Cayley coefficients and of values of the profile at its cut points (Section 11). This gives (14.16).

Next we show that, for every real number μ,

$$\widetilde{\Lambda}_R \leq P(R) - e^{-\mu}\Big((1+\mu)\, I_1 - \frac{1}{K+1}\, I_2\Big), \tag{14.17}$$

where

$$I_1 = \sum_{0<s<R} \varphi(s)\, P(s)\, P(R-s),$$
$$I_2 = \sum_{0<s<R} \varphi(s)\, \big(R-1-\tfrac{s}{2}\big)\, P(R-s)\, P_2(s).$$

By Lemma 11.8 for the threshold profile $f = \lambda^{K+1,K+R}$, for which $1 - f(s) = \varphi(s)$,

$$\nu_R - \Lambda_R(K+1;K+R) = \sum_{0<s<R} \nu_{R-s}\, \varphi(s)\, \Lambda_s(K+1;K+R).$$

We multiply by $e^{-R} = e^{-s}\, e^{-(R-s)}$ and obtain

$$P(R) - \widetilde{\Lambda}_R = \sum_{0<s<R} \varphi(s)\, P(R-s)\, \widetilde{\Lambda}_s. \tag{14.18}$$

The chain sums on the right are bounded from below by Lemma 14.2. Let $0 < s < R$. At a cut point $0 < g < s$ the cut factor is $(K+1)/(K+R-g) = e^{-x(g)}$ with $x(g) = \log\big((K+R-g)/(K+1)\big)$. The lemma, with $m = s$ and the same μ and multiplied by e^{-s}, therefore gives

$$e^{-\mu}\Big((1+\mu)\, P(s) - \sum_{0<g<s} P(g)\, P(s-g)\, x(g)\Big) \leq \widetilde{\Lambda}_s;$$

here we have used that $e^{-s}\, \nu_g\, \nu_{s-g} = P(g)\, P(s-g)$. Since $\log t \leq t-1$, we have $x(g) \leq (K+R-g)/(K+1) - 1 = (R-1-g)/(K+1)$. The weights $P(g)\, P(s-g)$ are positive, their sum is $P_2(s)$, and $\sum_{0<g<s} g\, P(g)\, P(s-g) = \frac{s}{2}\, P_2(s)$ by Lemma 12.6. Hence

$$e^{-\mu}\Big((1+\mu)\, P(s) - \frac{1}{K+1}\, \big(R-1-\tfrac{s}{2}\big)\, P_2(s)\Big) \leq \widetilde{\Lambda}_s. \tag{14.19}$$

In (14.18) every term has the nonnegative factors $\varphi(s)$ and $P(R-s)$, so that (14.19) can be inserted, and the result is (14.17).

It remains to bound I_1 from below and I_2 from above. For $0 < s < R$ we have $R-1-s \geq 0$ and $K+R-s \leq K+R-1$, hence $\varphi(s) \geq (R-1-s)/(K+R-1)$. By the definition of P_2 and by Lemma 12.6, $\sum_{0<s<R} (R-1-s)\, P(s)\, P(R-s) = (R-1-R/2)\, P_2(R)$, and therefore

$$I_1 \geq \frac{R-2}{2\,(K+R-1)}\, P_2(R). \tag{14.20}$$

For the second sum we write $I_2 = \sum_{0<s<R} \vartheta_s\, \chi(s)$ with the weights $\vartheta_s = P(R-s)\,(R-1-s)\,P_2(s)$ of Section 12, which are nonnegative, and with

$$\chi(s) = \frac{R-1-\frac{s}{2}}{K+R-s} = \frac{1}{2} - \frac{K-R+2}{2\,(K+R-s)}.$$

Since $K-R+2>0$, the second expression shows that χ is a concave function of the real variable $s < K+R$. We know the sum of the weights and a bound for their mean position: $\sum_{0<s<R} \vartheta_s = \frac{R-3}{3}\, P_3(R)$ by Corollary 12.9, and $\frac{2R}{5} \sum_{0<s<R} \vartheta_s \le \sum_{0<s<R} s\,\vartheta_s$ by Lemma 12.10. Accordingly we compare χ with its tangent at the point $s^* = 2R/5$. Let $D = K+R-s^* = K+\frac{3}{5}R$. The derivative of χ at s^* is $-(K-R+2)/(2D^2)$, and a concave function lies below its tangents, so that

$$\chi(s) \le \chi(s^*) - \frac{K-R+2}{2D^2}\,(s-s^*) \qquad (0<s<R).$$

We multiply by $\vartheta_s \ge 0$ and sum over $0<s<R$:

$$\begin{aligned} I_2 &\le \chi(s^*) \sum_{0<s<R} \vartheta_s - \frac{K-R+2}{2D^2}\Big(\sum_{0<s<R} s\,\vartheta_s - \frac{2R}{5} \sum_{0<s<R} \vartheta_s \Big) \\ &\le \chi(s^*) \sum_{0<s<R} \vartheta_s, \end{aligned}$$

because $K-R+2>0$ and the expression in parentheses is nonnegative. Since $\chi(s^*) = (\frac{4}{5}R-1)/(K+\frac{3}{5}R)$, this is

$$I_2 \le \frac{R-3}{3}\, P_3(R) \cdot \frac{\frac{4}{5}R-1}{K+\frac{3}{5}R}. \tag{14.21}$$

The proposition now follows from (14.16) and (14.17), when I_1 and I_2 are replaced by the bounds (14.20) and (14.21); for I_1 we use that $1+\mu \ge 0$. □

The expression in parentheses on the right-hand side of Proposition 14.8 has a main term, with $P_2(R)$, and a correction of the opposite sign, with $P_3(R)$. The main term is the effect of the cut factors to first order: in the chain sum $\Lambda_R(K+1;K+R)$ the cut factor at s is $(K+1)/(K+R-s)$, so that a cut at s loses the proportion $(R-1-s)/(K+R-s)$ of the weight, and summing this over the position of one cut gives a multiple of $P_2(R)$. The correction accounts for the cut factors at the cut points before the one considered, which have reduced the weight already.

15. Three sums of convolutions

Proposition 14.7 bounds the pair sum $\mathcal{P}(k,R)$ from below by an expression that contains three sums of the convolutions P_2 and P_3 of the Poisson mass: the total weight of the pairs, $\mathcal{W}(R)$, and the sums $\mathcal{G}(k,R)$ and $\mathcal{A}(k,R)$ of Section 14. In that bound $\mathcal{W}$ carries a factor that is positive for the parameters that will be chosen, and $\mathcal{G}$ and $\mathcal{A}$ are subtracted, each with a nonnegative factor. We therefore bound $\mathcal{W}$ from below and $\mathcal{G}$ and $\mathcal{A}$ from above, by explicit polynomials in $s=\sqrt{R}$ (Lemmas 15.1, 15.2 and 15.3), in the range in which the proof of Proposition 16.1 needs them: for $R \ge 42$ and, where the sum depends on k, for $R^{3/2} \le 1.07593\,k$.

15.1. **The three sums and the scheme of the estimates.** The three sums are defined in Section 14, for integers $k \ge 0$ and $R \ge 0$: the total weight of the pairs $\mathcal{W}(R)$ in (14.7), the sum $\mathcal{G}(k,R)$ in (14.8) and the sum $\mathcal{A}(k,R)$ in (14.9). The definitions of $\mathcal{G}$ and $\mathcal{A}$ contain the numbers $\varrho(k+R,c)$, which we call corrections; the function ϱ is defined there as well, immediately before (14.7).

The three proofs follow one scheme. Each of the sums $\mathcal{G}$ and $\mathcal{A}$ consists of a *main part*, a sum of multiples of $P_2(b)$ and $P_3(b)$ over $1 \le b \le R$, and of a *correction part*, in which the numbers $P_3(c)$ with $0 < c < R$ are multiplied by the corrections $\varrho(k+R,c)$. In the main part the closed form $P_3(b) = (b-1)\,P(b) - 2\,P_2(b)$ of Lemma 12.5 removes P_3. What is left of the term of index b is a nonnegative multiple of the Poisson mass $P(b)$ minus a positive multiple of $P_2(b)$. An upper bound of the term therefore needs an upper bound of $P(b)$, which is that of (2.5), and a lower bound of $P_2(b)$, which is Corollary 12.12; the same lower bound of P_2 gives the lower bound of $\mathcal{W}$ directly. With these inserted, the term is a combination of the powers $b^{3/2}$, $b^{1/2}$, $b^{-1/2}$ and $b^{-3/2}$ and of a polynomial in b, and the sum over b is bounded by an explicit function of s. The correction part is bounded more roughly.

For $\mathcal{W}$ and for $\mathcal{A}$ we state the bounds that the estimates themselves produce: the two polynomials consist of the leading terms that result from the scheme above, with decimal coefficients that are rounded in the direction that keeps the inequality.

In the numerical comparisons we use that, for every integer $R \ge 42$,

$$s = \sqrt{R} \ge 6.48, \tag{15.1}$$

because $6.48^2 \le 42$.

15.2. **Sums of half-integer powers.** For an integer $R \ge 2$ we write

$$\Sigma_j = \sum_{b=2}^{R} b^{j/2} \qquad \text{for } j \in \{-3,-1,1,3\},$$

without showing the dependence on R. These are the sums to which $\mathcal{W}$ and the main parts of $\mathcal{G}$ and $\mathcal{A}$ lead. They begin at $b = 2$ because the terms of index 1 will be seen to vanish, and because the lower bound of $P_2(b)$ in Corollary 12.12 holds from $b = 2$ on. The correction parts lead to two sums of the powers $c^{3/2}$ over the integers c with $0 < c < R$. We use the following bounds. For every integer $R \ge 2$,

$$\Sigma_{-1} = \sum_{b=2}^{R} \frac{1}{\sqrt{b}} \le 2\sqrt{R} + \frac{1}{2\sqrt{R}} - 2.44948, \tag{15.2}$$

$$\Sigma_{-3} = \sum_{b=2}^{R} \frac{1}{b\sqrt{b}} \le 1.63300, \tag{15.3}$$

$$\Sigma_{1} = \sum_{b=2}^{R} \sqrt{b} \le \frac{2}{3}\,R\sqrt{R} + \frac{1}{2}\,\sqrt{R} - 1.17851, \tag{15.4}$$

$$\Sigma_{3} = \sum_{b=2}^{R} b\sqrt{b} \ge \frac{2}{5}\,R^2\sqrt{R} + \frac{1}{2}\,R\sqrt{R} - 0.84853, \tag{15.5}$$

and for every integer $R \geq 0$,

$$\sum_{0<c<R} c\sqrt{c} \leq \frac{2}{5}\, R^2\sqrt{R}, \tag{15.6}$$

$$\sum_{0<c<R} c\sqrt{c}\,(R-c) \leq \frac{4}{35}\,(R+1)^3\sqrt{R+1}. \tag{15.7}$$

The bound (15.5) is a lower bound because Σ_3 will occur with a negative coefficient.

Each of the six bounds compares a sum with an integral. For the first four we use the midpoint rule and the trapezoid rule, which for a convex or a concave function err in a known direction. Let φ be a differentiable convex function on an interval $[x, x+1]$. Then

$$\varphi\Big(x+\frac{1}{2}\Big) \leq \int_x^{x+1} \varphi(t)\,dt \leq \frac{\varphi(x)+\varphi(x+1)}{2}. \tag{15.8}$$

Indeed, φ lies above its tangent at the point $x+1/2$ and below its chord. For a concave function both inequalities are reversed.

The functions $t^{-1/2}$ and $t^{-3/2}$ are convex on the positive half-line. We apply the first inequality of (15.8) to them on the intervals $[b-1/2,\, b+1/2]$ with $2 \leq b \leq R$, which together make up the interval $[3/2,\, R+1/2]$, and obtain

$$\Sigma_{-1} \leq \int_{3/2}^{R+1/2} \frac{dt}{\sqrt{t}} = 2\sqrt{R+\frac{1}{2}} - \sqrt{6},$$

$$\Sigma_{-3} \leq \int_{3/2}^{R+1/2} \frac{dt}{t\sqrt{t}} = \sqrt{\frac{8}{3}} - \frac{2}{\sqrt{R+1/2}}.$$

Since $\sqrt{R+1/2} \leq \sqrt{R}+1/(4\sqrt{R})$ and $\sqrt{6} \geq 2.44948$, the first of these gives (15.2). In the second we leave out the negative term and use $\sqrt{8/3} \leq 1.63300$; this gives (15.3).

The function $\sqrt{t}$ is concave and the function $t\sqrt{t}$ is convex on the half-line $t \geq 0$. We apply the second inequality of (15.8), reversed for $\sqrt{t}$, on the intervals $[b, b+1]$ with $2 \leq b \leq R-1$, and add. Since

$$\sum_{b=2}^{R-1} \frac{\varphi(b)+\varphi(b+1)}{2} = \sum_{b=2}^{R} \varphi(b) - \frac{\varphi(2)+\varphi(R)}{2}$$

for every function φ and every integer $R \geq 2$, this gives

$$\Sigma_1 - \frac{\sqrt{2}+\sqrt{R}}{2} \leq \int_2^R \sqrt{t}\,dt = \frac{2}{3}\big(R\sqrt{R}-2\sqrt{2}\big),$$

$$\Sigma_3 - \frac{2\sqrt{2}+R\sqrt{R}}{2} \geq \int_2^R t\sqrt{t}\,dt = \frac{2}{5}\big(R^2\sqrt{R}-4\sqrt{2}\big),$$

and therefore

$$\Sigma_1 \leq \frac{2}{3}\,R\sqrt{R} + \frac{1}{2}\,\sqrt{R} - \frac{5}{6}\,\sqrt{2}, \qquad \Sigma_3 \geq \frac{2}{5}\,R^2\sqrt{R} + \frac{1}{2}\,R\sqrt{R} - \frac{3}{5}\,\sqrt{2}.$$

Since $(5/6)\sqrt{2} \geq 1.17851$ and $(3/5)\sqrt{2} \leq 0.84853$, the bounds (15.4) and (15.5) follow.

For the last two bounds monotonicity is enough, since they serve for the correction parts only. The functions $t\sqrt{t}$ and $t^2\sqrt{t}$ increase on the half-line $t \geq 0$, so that

the value of each at an integer $c \geq 0$ is at most its integral over $[c, c+1]$. For $t\sqrt{t}$ this gives (15.6):

$$\sum_{0<c<R} c\sqrt{c} \leq \int_0^R t\sqrt{t}\,dt = \frac{2}{5}\,R^2\sqrt{R}.$$

In (15.7) the summand $c\sqrt{c}\,(R-c)$ is not monotone in c. For $R \leq 1$ the sum is empty, so let $R \geq 2$. We write the factor $R-c$ of the summand as the number of integers n with $c \leq n < R$, which turns the sum into a double sum of an increasing function:

$$\begin{aligned}\sum_{0<c<R} c\sqrt{c}\,(R-c) = \sum_{n=1}^{R-1}\sum_{c=1}^{n} c\sqrt{c} &\leq \frac{2}{5}\sum_{n=1}^{R-1}(n+1)^2\sqrt{n+1} \\ &\leq \frac{2}{5}\int_2^{R+1} t^2\sqrt{t}\,dt \leq \frac{2}{5}\cdot\frac{2}{7}\,(R+1)^3\sqrt{R+1}.\end{aligned}$$

Here the first inequality is (15.6) with R replaced by $n+1$, and the second is the same comparison for $t^2\sqrt{t}$ at the integers $m = n+1$ from 2 to R, whose intervals $[m, m+1]$ make up the interval $[2, R+1]$.

15.3. **The single terms.** The terms of the three sums are bounded through the upper bound $P(b) \leq c_{\mathrm{S}}/\sqrt{b}$ of (2.5), through two inequalities for the convolutions P_2 and P_3 at one point, and through a bound of the correction. For the constant c_{S} we use the bound $c_{\mathrm{S}} \leq 0.39894229$ of Section 2.

The proofs below derive the two inequalities for P_2 and P_3 from Corollary 12.12 and Lemma 12.5. The numbers $P_2(m)$ and $P_3(m)$ are sums of products of values of the Poisson mass and hence nonnegative for every integer $m \geq 0$. We also use that $P_2(1) = 0$.

The correction is bounded as follows. Let $k \geq 1$ and R be integers such that $s = \sqrt{R}$ satisfies $s \geq 6.48$, and let $R^{3/2} \leq 1.07593\,k$. Then for every integer c with $0 < c < R$,

$$0 \leq \varrho(k+R, c) \leq 0.560302\,\frac{c}{s^3}. \tag{15.9}$$

To see this, let $N = k + R$, so that $N - c > k$ and

$$\varrho(N,c) = \frac{c}{2\,(N-c)} + \frac{c^2}{8\,(N-c)^2} = \frac{c}{2\,(N-c)}\Big(1 + \frac{c}{4\,(N-c)}\Big),$$

which is nonnegative. The hypothesis on k gives $1/(N-c) \leq 1/k \leq 1.07593/s^3$. With $c \leq R = s^2$ and $s \geq 6.48$ it follows that $c/(N-c) \leq 1.07593/s \leq 1.07593/6.48$. Hence

$$\varrho(N,c) \leq \frac{1.07593\,c}{2s^3}\Big(1 + \frac{1.07593}{4\cdot 6.48}\Big) \leq 0.560302\,\frac{c}{s^3}.$$

15.4. **The total weight of the pairs.** We begin with the sum $\mathcal{W}$, which has no correction part and does not contain P_3.

Lemma 15.1. *Let $R \geq 42$ be an integer. Then the total weight of the pairs satisfies*

$$\frac{R}{2} - 1.06385\,\sqrt{R} + 0.7184 \leq \mathcal{W}(R).$$

Proof. Let $s = \sqrt{R}$. For every integer $b \geq 2$,

$$P_2(b) \geq \frac{1}{2} - \frac{4}{3} \cdot \frac{c_{\mathrm{S}}}{\sqrt{b}} - \frac{1}{15} \cdot \frac{c_{\mathrm{S}}}{b\sqrt{b}},$$

by Corollary 12.12 and the upper bound of (2.5). The term $b = 1$ of $\mathcal{W}(R)$ is $P_2(1) = 0$, and the sum of this bound over the $R - 1$ integers b with $2 \leq b \leq R$ gives

$$\mathcal{W}(R) \geq \frac{R-1}{2} - \frac{4}{3}\, c_{\mathrm{S}}\, \Sigma_{-1} - \frac{1}{15}\, c_{\mathrm{S}}\, \Sigma_{-3}.$$

Here c_{S} may be replaced by its upper bound 0.39894229. By (15.3) we have $\Sigma_{-3} \leq 1.633$. In (15.2) we have $\sqrt{R} = s$ and, by (15.1), $1/(2s) \leq 1/12.96 < 0.07717$, so that $\Sigma_{-1} \leq 2s - 2.37231$. With $R = s^2$ we obtain

$$\begin{aligned} \mathcal{W}(R) \geq \frac{s^2}{2} - \frac{1}{2} - \frac{4}{3} \cdot 0.39894229 \cdot (2s - 2.37231) \\ - \frac{1}{15} \cdot 0.39894229 \cdot 1.633. \end{aligned}$$

On the right-hand side the coefficient of s is $-(8/3) \cdot 0.39894229 \geq -1.06385$, and the constant term is

$$\begin{aligned} -\frac{1}{2} + \frac{4}{3} \cdot 0.39894229 \cdot 2.37231 \\ - \frac{1}{15} \cdot 0.39894229 \cdot 1.633 \geq 0.7184. \end{aligned}$$

Hence the right-hand side is at least $s^2/2 - 1.06385\, s + 0.7184$, which is the claim. □

15.5. **The sum $\mathcal{G}$.** We turn to the two sums that are subtracted in Proposition 14.7, and begin with $\mathcal{G}$. The comparison of Section 16 goes through with a weaker bound of $\mathcal{G}$ than the estimate produces, and the following lemma states such a bound: its polynomial is the polynomial that bounds $\mathcal{A}(k, R)$ in Lemma 15.3, multiplied by $2/s^2$. With this ratio, one of the two free parameters of Proposition 14.7 can be taken there so that, after the sums have been replaced by their bounds, the term with $\mathcal{A}$ cancels against a part of the term with $\mathcal{G}$.

Lemma 15.2. *Let $k \geq 1$ and $R \geq 42$ be integers with $R^{3/2} \leq 1.07593\, k$, and let $s = \sqrt{R}$. Then*

$$\mathcal{G}(k, R) \leq 2\,\big(0.478731\, s^3 - 0.974453\, s^2 + 0.642741\, s\big).$$

Proof. We split the factor $1 + \varrho(k + R, c)$ of the second sum in the definition (14.8) of $\mathcal{G}(k, R)$ into its two terms. The sum of $P_3(c)$ over $0 < c < R$ is at most the sum over $1 \leq c \leq R$, because $P_3(R) \geq 0$. Hence $\mathcal{G}(k, R) \leq \mathcal{G}' + \mathcal{G}''$, where

$$\mathcal{G}' = \sum_{b=1}^{R} \Big(\frac{7}{2}\, P_2(b) + 3\, P_3(b) \Big), \qquad \mathcal{G}'' = \sum_{0<c<R} P_3(c)\, \varrho(k + R, c).$$

By Lemma 12.5 we have $P_3(b) = (b-1)\, P(b) - 2\, P_2(b)$ for every $b \geq 1$, so that the term of index b of $\mathcal{G}'$ is

$$\frac{7}{2}\, P_2(b) + 3\, P_3(b) = 3\,(b-1)\, P(b) - \frac{5}{2}\, P_2(b).$$

For $b = 1$ it is zero, because $P_2(1) = 0$. For $2 \leq b \leq R$,

$$P_2(b) \geq \frac{1}{2} - \frac{4}{3} \cdot \frac{c_{\mathrm{S}}}{\sqrt{b}} - \frac{1}{15} \cdot \frac{c_{\mathrm{S}}}{b\sqrt{b}}$$

by Corollary 12.12 and the upper bound of (2.5), and we insert this lower bound and the upper bound of (2.5):

$$\begin{aligned} 3\,(b-1)\,P(b) - \frac{5}{2}\,P_2(b) &\le 3\,c_{\mathrm{S}}\Big(\sqrt{b} - \frac{1}{\sqrt{b}}\Big) - \frac{5}{4} + \frac{10}{3}\cdot\frac{c_{\mathrm{S}}}{\sqrt{b}} + \frac{1}{6}\cdot\frac{c_{\mathrm{S}}}{b\sqrt{b}} \\ &= c_{\mathrm{S}}\Big(3\sqrt{b} + \frac{1}{3}\cdot\frac{1}{\sqrt{b}} + \frac{1}{6}\cdot\frac{1}{b\sqrt{b}}\Big) - \frac{5}{4}. \end{aligned}$$

Summing over the $R-1$ integers b with $2 \le b \le R$, we obtain

$$\mathcal{G}' \le c_{\mathrm{S}}\Big(3\,\Sigma_1 + \frac{1}{3}\,\Sigma_{-1} + \frac{1}{6}\,\Sigma_{-3}\Big) - \frac{5}{4}\,(R-1).$$

For every integer $c \ge 1$, Lemma 12.5 gives $P_3(c) = (c-1)\,P(c) - 2\,P_2(c) \le c\,P(c)$, and $c\,P(c) \le c_{\mathrm{S}}\sqrt{c}$ by the upper bound of (2.5), so that

$$P_3(c) \le c_{\mathrm{S}}\,\sqrt{c}.$$

The two factors of a term of $\mathcal{G}''$ are bounded by this inequality and by (15.9), which applies because $s \ge 6.48$ by (15.1); with (15.6) and $R^2\sqrt{R} = s^5$ this gives

$$\mathcal{G}'' \le \frac{0.560302\,c_{\mathrm{S}}}{s^3} \sum_{0<c<R} c\sqrt{c} \le 0.560302 \cdot \frac{2}{5}\,c_{\mathrm{S}}\,s^2.$$

It remains to insert the bounds of the power sums and of c_{S}. By (15.4) we have $3\,\Sigma_1 \le 2s^3 + 3s/2 - 3.53553$. By (15.2), in which $1/(2s) \le 1/12.96 < 0.07717$, we have $\Sigma_{-1} \le 2s - 2.37231$ and hence $\Sigma_{-1}/3 \le 2s/3 - 0.79077$. By (15.3) we have $\Sigma_{-3}/6 \le 0.27217$. Adding these three bounds, we find

$$3\,\Sigma_1 + \frac{1}{3}\,\Sigma_{-1} + \frac{1}{6}\,\Sigma_{-3} \le 2s^3 + \frac{13}{6}\,s - 4.0541.$$

Since $c_{\mathrm{S}} \le 0.39894229$, it follows that

$$\begin{aligned} c_{\mathrm{S}}\Big(3\,\Sigma_1 + \frac{1}{3}\,\Sigma_{-1} + \frac{1}{6}\,\Sigma_{-3}\Big) &\le 0.39894229\,\Big(2s^3 + \frac{13}{6}\,s - 4.0541\Big) \\ &\le 0.79788458\,s^3 + 0.86438\,s - 1.61735. \end{aligned}$$

Likewise $0.560302 \cdot (2/5)\,c_{\mathrm{S}} \le 0.08942$. With $R = s^2$ we arrive at

$$\begin{aligned} \mathcal{G}(k,R) &\le 0.79788458\,s^3 + 0.86438\,s - 1.61735 - \frac{5}{4}\,(s^2-1) + 0.08942\,s^2 \\ &\le 0.79788458\,s^3 - 1.16058\,s^2 + 0.86438\,s, \end{aligned}$$

where the constant $5/4 - 1.61735$, which is negative, has been left out. The right-hand side of the claim minus this bound is

$$0.15957742\,s^3 - 0.788326\,s^2 + 0.421102\,s,$$

and this is positive, because by (15.1)

$$0.15957742\,s \ge 0.15957742 \cdot 6.48 > 0.788326. \qquad \square$$

15.6. **The sum $\mathcal{A}$.** The bound of $\mathcal{A}$ is obtained in the same way as that of $\mathcal{G}$, with the weights $R - b/2$, $R - b$ and $R - c$ in the sums and with a longer computation.

Lemma 15.3. *Let $k \ge 1$ and $R \ge 42$ be integers with $R^{3/2} \le 1.07593\,k$, and let $s = \sqrt{R}$. Then*

$$\mathcal{A}(k,R) \le 0.478731\,s^5 - 0.974453\,s^4 + 0.642741\,s^3.$$

Proof. We split the factor $1+\varrho(k+R,c)$ of the second sum in the definition (14.9) of $\mathcal{A}(k,R)$ into its two terms. The sum of $(R-c)\,P_3(c)$ over $0<c<R$ does not change when the term $c=R$, which is zero, is added to it. Hence $\mathcal{A}(k,R)=\mathcal{A}'+\mathcal{A}''$, where the correction part is

$$\mathcal{A}''=\sum_{0<c<R}(R-c)\,P_3(c)\,\varrho(k+R,c)$$

and the main part $\mathcal{A}'$ is the sum over $1\le b\le R$ of the terms

$$\Big(R-\frac{b}{2}\Big)\big(P_2(b)+2\,P_3(b)\big)+\frac{5}{2}\,(R-b)\,P_2(b)+(R-b)\,P_3(b).$$

Step 1: the terms of the main part. Inserting $P_3(b)=(b-1)\,P(b)-2\,P_2(b)$, which holds for $b\ge 1$ by Lemma 12.5, and collecting terms, we find that the term of index b of $\mathcal{A}'$ is

$$(3R-2b)\,(b-1)\,P(b)-\Big(\frac{5}{2}\,R-b\Big)P_2(b).$$

For $b=1$ it is zero, because $P_2(1)=0$. Let $2\le b\le R$. Then $(3R-2b)(b-1)\ge 0$ and $5R/2-b>0$, and

$$P_2(b)\ge\frac{1}{2}-\frac{4}{3}\cdot\frac{c_{\mathrm{S}}}{\sqrt{b}}-\frac{1}{15}\cdot\frac{c_{\mathrm{S}}}{b\sqrt{b}}$$

by Corollary 12.12 and the upper bound of (2.5), so that the upper bound of (2.5) and this lower bound may be inserted:

$$\begin{aligned}(3R-2b)\,(b-1)\,P(b)&\le c_{\mathrm{S}}\,(3R-2b)\Big(\sqrt{b}-\frac{1}{\sqrt{b}}\Big),\\ -\Big(\frac{5}{2}\,R-b\Big)P_2(b)&\le-\frac{1}{2}\Big(\frac{5}{2}\,R-b\Big)+c_{\mathrm{S}}\Big(\frac{5}{2}\,R-b\Big)\Big(\frac{4}{3\sqrt{b}}+\frac{1}{15\,b\sqrt{b}}\Big).\end{aligned}$$

Multiplying out the two products on the right, we see that the term of index b of $\mathcal{A}'$ is at most

$$c_{\mathrm{S}}\Big(-2\,b\sqrt{b}+\Big(3R+\frac{2}{3}\Big)\sqrt{b}+\Big(\frac{R}{3}-\frac{1}{15}\Big)\frac{1}{\sqrt{b}}+\frac{R}{6}\cdot\frac{1}{b\sqrt{b}}\Big)\\ -\frac{1}{2}\Big(\frac{5}{2}\,R-b\Big).$$

Step 2: the sum of the main part. We sum the bound of Step 1 over $2\le b\le R$. Its last terms add up to

$$\sum_{b=2}^{R}\frac{1}{2}\Big(\frac{5}{2}\,R-b\Big)=R^2-\frac{3}{2}\,R+\frac{1}{2}.$$

The factors of c_{S} add up to the number

$$Y=-2\,\Sigma_3+\Big(3R+\frac{2}{3}\Big)\Sigma_1+\Big(\frac{R}{3}-\frac{1}{15}\Big)\Sigma_{-1}+\frac{R}{6}\,\Sigma_{-3},$$

and therefore

$$\mathcal{A}'\le c_{\mathrm{S}}\,Y-\Big(R^2-\frac{3}{2}\,R+\frac{1}{2}\Big).$$

In Y only Σ_3 has a negative coefficient. We replace Σ_1, Σ_{-1} and Σ_{-3} by their upper bounds (15.4), (15.2) and (15.3) and Σ_3 by its lower bound (15.5). With $\sqrt{R} = s$ and $R = s^2$ this gives

$$Y \le -2\Big(\frac{2}{5}\,s^5 + \frac{1}{2}\,s^3 - 0.84853\Big) + \Big(3s^2 + \frac{2}{3}\Big)\Big(\frac{2}{3}\,s^3 + \frac{1}{2}\,s - 1.17851\Big) + \Big(\frac{s^2}{3} - \frac{1}{15}\Big)\Big(2s + \frac{1}{2s} - 2.44948\Big) + \frac{s^2}{6}\cdot 1.633.$$

Expanding the products, we find that the right-hand side is a polynomial in s plus the term $-1/(30s)$, which is negative. The polynomial has the coefficients $-4/5{+}2 = 6/5$ of s^5, $-1{+}3/2{+}4/9{+}2/3 = 29/18$ of s^3 and $1/3{+}1/6{-}2/15 = 11/30$ of s, and for its coefficient of s^2 and its constant term we have

$$-3\cdot 1.17851 - \frac{2.44948}{3} + \frac{1.633}{6} \le -4.07985,$$
$$2\cdot 0.84853 - \frac{2}{3}\cdot 1.17851 + \frac{2.44948}{15} \le 1.07469.$$

Hence $Y \le Y_0$, where

$$Y_0 = \frac{6}{5}\,s^5 + \frac{29}{18}\,s^3 - 4.07985\,s^2 + \frac{11}{30}\,s + 1.07469.$$

The number Y_0 is positive, because $(29/18)\,s^3 \ge (29/18)\cdot 6.48\,s^2 > 4.07985\,s^2$ by (15.1) and the other terms of Y_0 are positive. Since $c_{\mathrm{S}} > 0$, we have $c_{\mathrm{S}}\,Y \le c_{\mathrm{S}}\,Y_0$, and since $Y_0 > 0$ and $c_{\mathrm{S}} \le 0.39894229$, we have $c_{\mathrm{S}}\,Y_0 \le 0.39894229\,Y_0$. Multiplying the coefficients of Y_0 by 0.39894229 and rounding upwards, we obtain

$$c_{\mathrm{S}}\,Y \le 0.478731\,s^5 + 0.642741\,s^3 - 1.62762\,s^2 + 0.14628\,s + 0.42874.$$

Step 3: the correction part. For every integer $c \ge 1$, Lemma 12.5 gives $P_3(c) = (c-1)\,P(c) - 2\,P_2(c) \le c\,P(c)$, and $c\,P(c) \le c_{\mathrm{S}}\sqrt{c}$ by the upper bound of (2.5), so that $P_3(c) \le c_{\mathrm{S}}\sqrt{c}$. By this inequality and by (15.9), which applies because $s \ge 6.48$ by (15.1),

$$\mathcal{A}'' \le \frac{0.560302\,c_{\mathrm{S}}}{s^3} \sum_{0<c<R} c\sqrt{c}\,(R-c),$$

and by (15.7), in which $R+1 = s^2+1$, the sum on the right is at most $4/35$ times $(s^2+1)^3\sqrt{s^2+1}$. We have

$$\sqrt{s^2+1} \le s + \frac{1}{2s} = \frac{2s^2+1}{2s}$$

and

$$(s^2+1)^3\,(2s^2+1) = 2s^8 + 7s^6 + 9s^4 + 5s^2 + 1 \le 2s^8 + 7.22\,s^6.$$

For the last inequality we use twice that $s^2 \ge 6.48^2 \ge 41.99$ by (15.1): this gives $0.22\,s^6 \ge 9.2378\,s^4$, and then $0.2378\,s^4 \ge 9.98\,s^2 \ge 5s^2 + 1$. Therefore

$$\sum_{0<c<R} c\sqrt{c}\,(R-c) \le \frac{4}{35}\cdot\frac{2s^8 + 7.22\,s^6}{2s} = \frac{4}{35}\,\big(s^7 + 3.61\,s^5\big).$$

Multiplying by $0.560302\,c_{\mathrm{S}}/s^3$ and using $c_{\mathrm{S}} \le 0.39894229$, we obtain

$$\mathcal{A}'' \le 0.560302\cdot\frac{4}{35}\cdot 0.39894229\,\big(s^4 + 3.61\,s^2\big).$$

Since $0.560302 \cdot (4/35) \cdot 0.39894229 \le 0.025547$ and $0.025547 \cdot 3.61 \le 0.09223$, it follows that

$$\mathcal{A}'' \le 0.025547\, s^4 + 0.09223\, s^2.$$

Step 4: the comparison. By the bounds of $\mathcal{A}'$ and of $c_{\mathrm{S}}\, Y$ in Step 2, in which $R = s^2$, and by the bound of $\mathcal{A}''$ in Step 3, the sum $\mathcal{A}(k,R) = \mathcal{A}' + \mathcal{A}''$ is at most

$$0.478731\, s^5 + 0.642741\, s^3 - 1.62762\, s^2 + 0.14628\, s + 0.42874 \\ - \Big(s^4 - \frac{3}{2}\, s^2 + \frac{1}{2}\Big) + 0.025547\, s^4 + 0.09223\, s^2.$$

Collecting terms, we see that this polynomial is the right-hand side of the claim plus

$$-0.03539\, s^2 + 0.14628\, s - 0.07126,$$

and this is negative, because by (15.1)

$$0.03539\, s \ge 0.03539 \cdot 6.48 > 0.14628. \qquad \square$$

16. A positive step next to the curve and the change of sign in the first layer

In this section we prove that every column $k \ge 265$ has a positive step at most one row beyond the curve $k^2 = (n-k)^3$: there is a row $j > k$ with $(j-k-1)^3 \le k^2$ and $d_{j,k} > 0$ (Proposition 16.2). For its proof it remains to insert the bounds of Sections 14 and 15 into the criterion of Proposition 13.10. We do this at the positions $(k+r, k)$ with $r \ge 41$ and $(r-1)^3 \le k^2 \le r^3$, and prove that the step is positive there (Proposition 16.1).

Combined with Lemma 6.6 and with Proposition 10.5, Proposition 16.2 gives the single change of sign in the first layer of every column $k \ge 82$ (Proposition 16.4). From this and Lemma 6.4 we obtain the column rule in the first layer of every column $k \ge 2$: a positive step in a row of the layer is followed by a positive step in the next row of the layer (Proposition 16.5).

16.1. **A positive step next to the curve.** As in Sections 13 to 15, the position under study is written $(k+R-1, k)$, so that R is one more than its deficit, and $s = \sqrt{R}$. By Proposition 13.10, the step at such a position with $1 \le R \le k-2$ is positive as soon as

$$k\, a_k\, \Lambda_R(k+1; k+R) < 2\, \mathcal{P}(k,R), \tag{16.1}$$

and Sections 14 and 15 bound the two sides of this inequality. We prove (16.1) at the positions $(k+r, k)$ with $r \ge 41$ and $(r-1)^3 \le k^2 \le r^3$, for which $R = r+1$. The first of the two inequalities says that the position one row above, $(k+r-1, k)$, lies behind the curve, and the second that position $(k+r, k)$ itself lies on the curve, when $k^2 = r^3$, or beyond it, when $k^2 < r^3$.

The proof divides (16.1) by $k\, a_k\, e^R$ and separates its two sides by one explicit function of s. For the right side it inserts the bounds of Section 15 into Proposition 14.7 and chooses the two parameters of that proposition; here the hypothesis on k enters in the form $k \le s^3 \le 1.07593\, k$. For the left side it applies Proposition 14.8 with a number K that depends on R alone, which the inequality $k^2 \le r^3$ permits, and the comparison of the resulting bound with the function of s comes down to the positivity of one polynomial of degree 15 on the half-line $s \ge 6.48$.

Proposition 16.1. *Let k and r be nonnegative integers with*

$$r \geq 41 \qquad \text{and} \qquad (r-1)^3 \leq k^2 \leq r^3.$$

Then $d_{k+r,k} > 0$.

Proof. Step 1: reduction to two inequalities. Let $R = r+1$ and $s = \sqrt{R}$. Then $R \geq 42$, row $k+r$ is row $k+R-1$, and the hypothesis reads

$$(R-2)^3 \leq k^2 \leq (R-1)^3. \tag{16.2}$$

The normalised step $\delta_n^{(k)} = d_{n,k}/n!$ has the sign of $d_{n,k}$, so that we have to show that $\delta_{k+R-1}^{(k)} > 0$. Since $R \geq 42$, we have $s \geq 6.48$ by (15.1).

We first show that

$$k \leq s^3 \leq 1.07593\, k. \tag{16.3}$$

By (16.2), $k^2 \leq (R-1)^3 \leq R^3 = (s^3)^2$, and hence $k \leq s^3$. For the other inequality we use that $R \leq \frac{21}{20}(R-2)$, which is equivalent to $R \geq 42$. Taking third powers and using (16.2) again, we get

$$(s^3)^2 = R^3 \leq \Big(\frac{21}{20}\Big)^3 (R-2)^3 \leq \Big(\frac{21}{20}\Big)^3 k^2 \leq (1.07593\, k)^2,$$

where the last step holds because $(21/20)^3 \leq 1.07593^2$. This proves (16.3).

From (16.3) we get $k \geq R+2$: since $1.07593\, k \geq s^3 = R\, s \geq 6.48\, R$ and $6.48 \geq 6 \cdot 1.07593$, we have $k \geq 6R \geq R+2$. Thus $1 \leq R \leq k-2$, which is the hypothesis of Proposition 13.10, so that it suffices to prove (16.1). Moreover $k \geq 1$, so that $a_k > 0$.

For a real number t let

$$\begin{aligned} \mathcal{W}_0(t) &= \tfrac{1}{2}\, t^2 - 1.06385\, t + 0.7184, \\ \mathcal{Q}_0(t) &= 0.478731\, t^2 - 0.974453\, t + 0.642741. \end{aligned} \tag{16.4}$$

We shall prove the two inequalities

$$e^{-R}\, \Lambda_R(k+1; k+R) < \frac{2 \cdot 0.38385\, \mathcal{W}_0(s)\, \big(1 + \frac{1}{10s}\big)}{s^3}, \tag{16.5}$$

$$\mathcal{P}(k,R) \geq a_k\, e^R \cdot 0.38385 \Big(1 + \frac{1}{10s}\Big) \frac{k}{s^3}\, \mathcal{W}_0(s). \tag{16.6}$$

They give (16.1): multiplying (16.5) by the positive number $k\, a_k\, e^R$ and using (16.6), we get

$$\begin{aligned} k\, a_k\, \Lambda_R(k+1; k+R) &< 2\, a_k\, e^R \cdot 0.38385 \Big(1 + \frac{1}{10s}\Big) \frac{k}{s^3}\, \mathcal{W}_0(s) \\ &\leq 2\, \mathcal{P}(k,R). \end{aligned}$$

Step 2: the lower bound of the pair sum. We prove (16.6). Since $R \geq 42$ and $s^3 \leq 1.07593\, k$, Lemmas 15.1, 15.2 and 15.3 give

$$\mathcal{W}(R) \geq \mathcal{W}_0(s), \qquad \mathcal{G}(k,R) \leq 2s\, \mathcal{Q}_0(s), \qquad \mathcal{A}(k,R) \leq s^3\, \mathcal{Q}_0(s).$$

Both $\mathcal{W}_0(s)$ and $\mathcal{Q}_0(s)$ are positive, since $s/2 > 1.06385$ and $0.478731\, s > 0.974453$. Let

$$\theta = \frac{s^2}{2k}, \qquad x = \frac{s^3}{k}, \qquad \mu = \frac{2s\, \mathcal{Q}_0(s)\, \log(1+\theta)}{\mathcal{W}_0(s)}, \qquad h = 0.9575 - \frac{1}{10s},$$

and, as in Proposition 14.7,

$$\alpha = \log(1+\theta) - \frac{\theta}{1+\theta}, \qquad \beta = \frac{1}{k\,(1+\theta)}.$$

Then $\theta > 0$ and $\mu > 0$, we have $2s\,\theta = x$, and (16.3) says that $1 \le x \le 1.07593$. Since $k \ge 1$ and $R \le k$, Proposition 14.7 applies to these values of θ and μ, and it gives

$$a_k\, e^R\, e^{-\mu} \big((1+\mu)\,\mathcal{W}(R) - \alpha\,\mathcal{G}(k,R) - \beta\,\mathcal{A}(k,R)\big) \le \mathcal{P}(k,R). \tag{16.7}$$

We first show that

$$(1+\mu)\,\mathcal{W}(R) - \alpha\,\mathcal{G}(k,R) - \beta\,\mathcal{A}(k,R) \ge \mathcal{W}_0(s). \tag{16.8}$$

The three coefficients $1+\mu$, α and β are nonnegative; for α this is the inequality $\log u \ge 1 - 1/u$ at $u = 1+\theta$. By the bounds of the three sums, the left side of (16.8) is therefore at least

$$(1+\mu)\,\mathcal{W}_0(s) - \Big(\log(1+\theta) - \frac{\theta}{1+\theta}\Big)\,2s\,\mathcal{Q}_0(s) - \frac{s^3\mathcal{Q}_0(s)}{k\,(1+\theta)}.$$

Here $2s\,\mathcal{Q}_0(s)\,\theta/(1+\theta) = s^3\mathcal{Q}_0(s)/(k\,(1+\theta))$, because $2s\,\theta = s^3/k$, so that these two terms cancel; and $\mu\,\mathcal{W}_0(s) = 2s\,\mathcal{Q}_0(s)\log(1+\theta)$ by the choice of μ, so that these two terms cancel as well. What remains is $\mathcal{W}_0(s)$.

Next we show that $\mu \le x\,h$. By the Taylor bound $\log(1+u) \le u - u^2/2 + u^3/3$ for $u \ge 0$, which is recalled in Section 2, and by $\theta = x/(2s)$,

$$\begin{aligned}\mu\,\mathcal{W}_0(s) = 2s\,\mathcal{Q}_0(s)\log(1+\theta) &\le 2s\,\mathcal{Q}_0(s)\Big(\frac{x}{2s} - \frac{x^2}{8s^2} + \frac{x^3}{24s^3}\Big)\\ &= \frac{x\,\mathcal{Q}_0(s)}{s^2}\Big(s^2 - \frac{x\,s}{4} + \frac{x^2}{12}\Big).\end{aligned}$$

The last bracket is largest at the smallest value $x = 1$, since

$$\Big(s^2 - \frac{s}{4} + \frac{1}{12}\Big) - \Big(s^2 - \frac{x\,s}{4} + \frac{x^2}{12}\Big) = (x-1)\Big(\frac{s}{4} - \frac{x+1}{12}\Big) \ge 0;$$

here $x \ge 1$ and $s/4 \ge 1.62 > (x+1)/12$. Hence

$$\mu\,\mathcal{W}_0(s)\,s^2 \le x\,\mathcal{Q}_0(s)\,\Big(s^2 - \frac{s}{4} + \frac{1}{12}\Big). \tag{16.9}$$

We compare the right side with $x\,h\,\mathcal{W}_0(s)\,s^2$. Since $h\,s^2 = 0.9575\,s^2 - s/10$, multiplying out the polynomials (16.4) gives

$$\begin{aligned}&\mathcal{W}_0(s)\,\Big(0.9575\,s^2 - \frac{s}{10}\Big) - \mathcal{Q}_0(s)\,\Big(s^2 - \frac{s}{4} + \frac{1}{12}\Big)\\ &\qquad = \frac{19}{10^6}\,s^4 + s^2\,\Big(\frac{40799}{1600000}\,s - \frac{263991}{2000000}\Big) + \Big(\frac{510149}{3000000}\,s - \frac{214247}{4000000}\Big).\end{aligned}$$

For $s \ge 6.48$ the three terms on the right are positive, since the two brackets increase with s and are positive at $s = 6.48$. Together with (16.9) this gives $\mu\,\mathcal{W}_0(s)\,s^2 \le x\,h\,\mathcal{W}_0(s)\,s^2$, and hence $\mu \le x\,h$.

For decimal values of the exponential function we use that, for every real number y with $0 \le y \le 1$,

$$\sum_{j=0}^{7} \frac{y^j}{j!} \le e^y \le \sum_{j=0}^{7} \frac{y^j}{j!} + \frac{9\,y^8}{8\cdot 8!}. \tag{16.10}$$

The lower bound is clear. For the upper bound, $(8+i)! \geq 8! \cdot 9^i$ for every $i \geq 0$, so that the terms of the exponential series from index 8 on add up to at most $(y^8/8!)\sum_{i\geq 0} 9^{-i} = 9y^8/(8\cdot 8!)$.

We now show that

$$e^{-xh} \geq 0.38385 \Big(1 + \frac{1}{10s}\Big)\frac{1}{x}. \tag{16.11}$$

Since $-x\,h = -0.9575 + 1/(10s) - (x-1)\,h$, the left side is the product of the three positive numbers $e^{-0.9575}$, $e^{1/(10s)}$ and $e^{-(x-1)h}$, and we bound each of them from below. By (16.10), applied to the number 0.9575, we have $e^{0.9575} \leq 2.60518$, and $0.38385 \cdot 2.60518 \leq 1$; hence $e^{-0.9575} \geq 0.38385$. Next, $e^{1/(10s)} \geq 1 + 1/(10s)$. For the third factor, the Taylor bound $\log(1+u) \geq u - u^2/2$ for $u \geq 0$, also recalled in Section 2, gives at $u = x - 1 \leq 0.07593$

$$\log x \geq (x-1)\Big(1 - \frac{x-1}{2}\Big) \geq 0.962\,(x-1) \geq (x-1)\,h,$$

because $h \leq 0.9575$; hence $e^{-(x-1)h} \geq 1/x$. The product of the three lower bounds is the right side of (16.11).

Finally, since $\mu \leq x\,h$, we have $e^{-\mu} \geq e^{-xh}$. With (16.11) and $1/x = k/s^3$, and since $a_k\,e^R > 0$ and $\mathcal{W}_0(s) > 0$,

$$a_k\,e^R \cdot 0.38385 \Big(1 + \frac{1}{10s}\Big)\frac{k}{s^3}\,\mathcal{W}_0(s) \leq a_k\,e^R\,e^{-\mu}\,\mathcal{W}_0(s).$$

By (16.8), the right side is at most the left side of (16.7), and hence at most $\mathcal{P}(k,R)$. This proves (16.6).

Step 3: the upper bound of the chain sum. It remains to prove (16.5). Proposition 14.8 bounds the chain sum $\Lambda_R(k+1; k+R)$ for every real number $K \geq \max(k,R)$, and its bound contains a second real parameter, which may take any value not smaller than -1. We take $K = (R-1)(2R-1)/(2s)$, and for the second parameter the value 0.35. The number K depends on R alone and is a rational function of s; in this way the bound of the chain sum no longer depends on k. We have $R \leq K$, because

$$2s\,(K-R) = (s^2-1)(2s^2-1) - 2s^3 = s^2\,(2s^2 - 2s - 3) + 1 > 0,$$

the bracket being positive for $s \geq 6.48$. Also $k \leq K$: we have $K = (R-1)\big(s - 1/(2s)\big)$ and $\big(s - 1/(2s)\big)^2 = R - 1 + 1/(4R) \geq R-1$, so that $K^2 \geq (R-1)^3 \geq k^2$ by (16.2). Proposition 14.8 therefore applies to the integer $R \geq 1$, to the real numbers k and K, which satisfy $0 \leq k \leq K$ and $R \leq K$, and to the value 0.35 of its second parameter. With the abbreviations

$$\chi_1 = \frac{R-2}{2\,(K+R-1)}, \qquad \chi_2 = \frac{1}{K+1}\cdot\frac{R-3}{3}\cdot\frac{\frac{4}{5}R-1}{K+\frac{3}{5}R},$$

which are positive numbers, it gives

$$e^{-R}\,\Lambda_R(k+1; k+R) \leq P(R) - e^{-0.35}\,\big(1.35\,\chi_1\,P_2(R) - \chi_2\,P_3(R)\big). \tag{16.12}$$

Let

$$\phi = \frac{4}{3} + \frac{1}{15R}, \qquad \bar{v} = \frac{c_{\mathrm{S}}}{s}.$$

By the upper bound of (2.5), $P(R) \le \bar v$, and $0 < c_{\mathrm S} \le 0.39894229$ by the enclosure of $c_{\mathrm S}$ in Section 2. By Corollary 12.12 and Lemma 12.5,

$$P_2(R) \ge \frac{1}{2} - \phi\, P(R), \qquad P_3(R) = (R-1)\, P(R) - 2\, P_2(R),$$

and hence

$$P_3(R) \le (R - 1 + 2\phi)\, P(R) - 1.$$

Finally, (16.10), applied to the number 0.35, shows that $e^{0.35}$ lies between 1.4190675 and 1.4190676. Since

$$0.704688 \cdot 1.4190676 \le 1 \le 0.704689 \cdot 1.4190675,$$

it follows that

$$0.704688 \le e^{-0.35} \le 0.704689.$$

Since $e^{-0.35}$, χ_1 and χ_2 are positive, the bounds for $P_2(R)$ and $P_3(R)$ can be inserted into (16.12): its right side is at most

$$P(R) - e^{-0.35}\Big(1.35\,\chi_1\big(\tfrac12 - \phi\, P(R)\big) - \chi_2\big((R-1+2\phi)\, P(R) - 1\big)\Big).$$

In this expression $P(R)$ has the coefficient

$$1 + e^{-0.35}\big(1.35\,\chi_1\phi + \chi_2\,(R - 1 + 2\phi)\big),$$

which is positive, so that the expression does not decrease when $P(R)$ is replaced by the larger number $\bar v$:

$$\begin{aligned} e^{-R}\,\Lambda_R(k+1; k+R) \le \bar v - e^{-0.35}\Big(&1.35\,\chi_1\big(\tfrac12 - \phi\,\bar v\big) \\ &- \chi_2\big((R-1+2\phi)\,\bar v - 1\big)\Big). \end{aligned} \tag{16.13}$$

Next we bound the right side of (16.13) by a multiple of $\bar v$. Let

$$J_1 = 1.253314\, s - \phi, \qquad J_2 = s^2 - 2.5066\, s + 1.67.$$

Both numbers are positive: we have $\phi \le 4/3 + 1/600 = 1.335$, because $15R \ge 600$, so that $J_1 \ge 1.253314 \cdot 6.48 - 1.335 > 0$, and $J_2 = s\,(s - 2.5066) + 1.67$. We compare the two inner brackets of (16.13) with $\bar v\, J_1$ and $\bar v\, J_2$, using that $s\,\bar v = c_{\mathrm S} \le 0.39894229$. First,

$$\bar v\, J_1 = 1.253314\, c_{\mathrm S} - \phi\,\bar v \le \tfrac12 - \phi\,\bar v,$$

because $1.253314 \cdot 0.39894229 \le 1/2$. Secondly,

$$\bar v\, J_2 - \big((R-1+2\phi)\,\bar v - 1\big) = 1 - 2.5066\, c_{\mathrm S} + (2.67 - 2\phi)\,\bar v \ge 0,$$

because $2.5066 \cdot 0.39894229 \le 1$ and $2\phi \le 2.67$. Inserting the two comparisons into (16.13), we see that its right side is at most

$$\bar v - e^{-0.35}\big(1.35\,\chi_1 J_1 - \chi_2 J_2\big)\,\bar v.$$

Here the term $\chi_1 J_1 \bar v$, which is subtracted, and the term $\chi_2 J_2 \bar v$, which is added, are both positive. We use the lower bound of $e^{-0.35}$ in the first and its upper bound in the second, and arrive at

$$e^{-R}\,\Lambda_R(k+1; k+R) \le \bar v\,\Omega_1, \tag{16.14}$$

where

$$\Omega_1 = 1 - 1.35 \cdot 0.704688\,\chi_1 J_1 + 0.704689\,\chi_2 J_2.$$

Step 4: the comparison of the two bounds. Let

$$\Omega_2 = \frac{1.9243\,\mathcal{W}_0(s)\,(10s+1)}{10\,s^3}.$$

We claim that $\Omega_1 < \Omega_2$. To see this we write Ω_1 as a rational function of s. By the choice of K, $2s\,K = (s^2-1)(2s^2-1)$. Let

$$\begin{aligned} M_1 &= (s^2-1)(2s^2+2s-1),\\ M_2 &= (s^2-1)(2s^2-1) + \tfrac{6}{5}\,s^3,\\ M_3 &= (s^2-1)(2s^2-1) + 2s. \end{aligned}$$

Then $2s\,(K+R-1) = M_1$, $2s\,(K+\frac{3}{5}R) = M_2$ and $2s\,(K+1) = M_3$, and the three numbers are positive. Consequently

$$\chi_1 = \frac{s\,(s^2-2)}{M_1}, \qquad \chi_2 = \frac{4s^2\,(s^2-3)\big(\frac{4}{5}s^2-1\big)}{3\,M_2M_3},$$
$$J_1 = \frac{18.79971\,s^3 - 20\,s^2 - 1}{15\,s^2},$$

the last because $15s^2\phi = 20s^2+1$ and $15\cdot 1.253314 = 18.79971$. We multiply $\Omega_2-\Omega_1$ by the positive number $30\,s^3M_1M_2M_3$. The result is a polynomial in s:

$$30\,s^3\,M_1M_2M_3\,(\Omega_2-\Omega_1) = \mathcal{X}(s),$$

where

$$\begin{aligned} \mathcal{X}(s) = {}& 5.7729\,\mathcal{W}_0(s)\,(10s+1)\,M_1M_2M_3 - 30\,s^3\,M_1M_2M_3\\ &+ 1.35\cdot 0.704688\cdot 2s^2\,(s^2-2)\,\big(18.79971\,s^3 - 20\,s^2 - 1\big)\,M_2M_3\\ &- 0.704689\cdot 40\,s^5\,(s^2-3)\big(\tfrac{4}{5}s^2-1\big)\big(s^2-2.5066\,s+1.67\big)\,M_1. \end{aligned}$$

The four terms come from Ω_2, from the term 1 of Ω_1, and from its terms with χ_1J_1 and with χ_2J_2; the factor 5.7729 is $3\cdot 1.9243$. Each of them is a polynomial of degree 15 in s whose coefficients are finite decimal fractions. Multiplying out the four products and collecting the powers of s, we obtain

$$\begin{aligned} \mathcal{X}(s) = {}& 88.893548437184\,s^{15} - 481.1809899040896\,s^{14}\\ &- 940.74918031232\,s^{13} + 2462.6087604838976\,s^{12}\\ &+ 2054.9679357758624\,s^{11} - 5874.3007648560576\,s^{10}\\ &- 1064.5268494392928\,s^{9} + 7091.8315979794336\,s^{8}\\ &- 1196.745291572152\,s^{7} - 4127.091835775184\,s^{6}\\ &+ 1376.539082916408\,s^{5} + 1128.517422882\,s^{4}\\ &- 439.556846403\,s^{3} - 108.63749913\,s^{2}\\ &+ 35.331013935\,s + 4.14725136. \end{aligned}$$

These decimal coefficients are exact; none of them is rounded.

The coefficients are of both signs, so that the sign of $\mathcal{X}(s)$ cannot be read off from them. We separate the four terms of highest degree from the others. For a real number t let

$$\begin{aligned} \mathcal{X}_1(t) = {}& 88.893548437184\,t^3 - 481.1809899040896\,t^2\\ &- 940.74918031232\,t + 2462.6087604838976, \end{aligned}$$

and let $\mathcal{X}_0(s)$ be the sum of the twelve terms of $\mathcal{X}(s)$ of degree at most 11, so that

$$\mathcal{X}(s) = s^{12}\,\mathcal{X}_1(s) + \mathcal{X}_0(s).$$

Four terms are taken together because the sum of the three terms of highest degree is negative at $s = 6.48$. It suffices to show that $\mathcal{X}_1(s)$ and $\mathcal{X}_0(s)$ are positive.

The polynomial $\mathcal{X}_1$ increases on the half-line $t \geq 6.48$. Indeed, for $t \geq 0$ its derivative satisfies

$$\mathcal{X}_1'(t) \geq t\,(3 \cdot 88.89\,t - 2 \cdot 481.19) - 940.75,$$

because 88.89 is smaller than the coefficient of t^3 in $\mathcal{X}_1(t)$, and 481.19 and 940.75 are larger than the absolute values of its coefficients of t^2 and of t. For $t \geq 6.48$ the bracket is at least 765, because $3 \cdot 88.89 \cdot 6.48 > 1728$ and $2 \cdot 481.19 < 963$, and therefore

$$\mathcal{X}_1'(t) \geq 765 \cdot 6.48 - 940.75 > 0.$$

Moreover $\mathcal{X}_1(6.48) > 0$. We have $6.48^2 = 41.9904$ and $6.48^3 = 272.097792$. Replacing the first of these powers by 42 and the second by 272.09, and rounding the coefficients in the same way, the constant term to 2462.6, we only make the value smaller, so that

$$\begin{aligned}\mathcal{X}_1(6.48) &\geq 88.89 \cdot 272.09 - 481.19 \cdot 42 - 940.75 \cdot 6.48 + 2462.6\\ &\geq 24186.08 - 20209.98 - 6096.06 + 2462.6 = 342.64.\end{aligned}$$

Hence $\mathcal{X}_1(s) \geq \mathcal{X}_1(6.48) > 0$, because $s \geq 6.48$.

We turn to $\mathcal{X}_0(s)$. Its two terms of highest degree are $2054.96\ldots\, s^{11}$ and $-5874.30\ldots\, s^{10}$, and since $s \geq 6.48$ their sum is at least

$$(2054 \cdot 6.48 - 5875)\, s^{10} \geq 7000\, s^{10}.$$

Of the coefficients of $s^9, s^8, \ldots, s^0$ none is smaller than -4128. We add the terms of $\mathcal{X}_0(s)$ one at a time, in the order of decreasing degree, and show for $j = 10, 9, \ldots, 0$ that the sum of the terms of degree at least j is at least $7000\, s^j$. For $j = 10$ this has just been shown. If it holds for $j+1$, where $0 \leq j \leq 9$, then the sum of the terms of degree at least j is at least

$$7000\, s^{j+1} - 4128\, s^j = (7000\, s - 4128)\, s^j \geq 7000\, s^j,$$

because $7000 \cdot 6.48 - 4128 > 7000$. For $j = 0$ this says that $\mathcal{X}_0(s) \geq 7000$.

Hence $\mathcal{X}(s) > 0$, and therefore $\Omega_1 < \Omega_2$.

Since $\bar{v}$ is positive, (16.14) and the inequality $\Omega_1 < \Omega_2$ show that the left side of (16.5) is smaller than $\bar{v}\,\Omega_2$. By the definitions of $\bar{v}$ and of Ω_2,

$$\bar{v}\,\Omega_2 = 1.9243\, c_{\mathrm{S}} \cdot \frac{\mathcal{W}_0(s)\left(1 + \frac{1}{10s}\right)}{s^3},$$

and the fraction on the right is positive, because $\mathcal{W}_0(s) > 0$. Moreover

$$1.9243\, c_{\mathrm{S}} \leq 1.9243 \cdot 0.39894229 \leq 2 \cdot 0.38385.$$

This proves (16.5). □

We apply Proposition 16.1 in the columns $k \geq 265$, for which Lemma 6.6 supplies no positive row, in the first row of the column beyond the curve. This gives the following bound for the first positive row of the column. The bound is used in Section 23, together with Corollary 16.6 and Proposition 22.6, for the diagonal rule in the first layer of such a column.

Proposition 16.2. *For every integer $k \geq 265$ there is an integer j with*

$$k < j, \qquad (j-k-1)^3 \leq k^2, \qquad d_{j,k} > 0.$$

Proof. Let r be the least nonnegative integer with $k^2 < r^3$. Then $(r-1)^3 \leq k^2 < r^3$, and $r \geq 42$: for $r \leq 41$ we would have $r^3 \leq 41^3 < 265^2 \leq k^2$. Proposition 16.1 therefore gives $d_{k+r,k} > 0$, and the integer $j = k + r$ has the required properties, since $(j-k-1)^3 = (r-1)^3 \leq k^2$. □

16.2. **The change of sign in the first layer.** Proposition 10.5 shows that a column $k \geq 82$ which is not positive in the rows k to $q-1$ and is positive in a row $q \geq k+1$ stays positive from row q to the end of the first layer, under the assumption that q lies in the first third of the layer, $3\,(q-k) \leq k$, and the argument of Section 19 for the rows beyond the first layer needs the bound $100\,(q-k) \leq 27\,k$. The following corollary supplies a positive row within this distance of row k.

Corollary 16.3. *For every integer $k \geq 82$ there is an integer r with*

$$r \geq 1, \qquad 100\,r \leq 27\,k, \qquad d_{k+r,k} > 0.$$

Proof. First let $82 \leq k \leq 264$, and let $\zeta(k)$ be the number that Definition 6.5 lists for column k. By Lemma 6.6, which holds for $9 \leq k \leq 264$, column k is positive in row $k + \zeta(k) + 1$. We take $r = \zeta(k) + 1$, so that $r \geq 1$, and have to verify that $100\,(\zeta(k)+1) \leq 27\,k$. This is read from the table of Definition 6.5: we have $\zeta(k) \leq 20$ for $k \leq 101$, $\zeta(k) \leq 24$ for $k \leq 141$, and $\zeta(k) \leq 35$ for every $k \leq 264$. Hence $100\,(\zeta(k)+1)$ is at most $2100 \leq 27 \cdot 82$ for $82 \leq k \leq 101$, at most $2500 \leq 27 \cdot 102$ for $102 \leq k \leq 141$, and at most $3600 \leq 27 \cdot 142$ for $142 \leq k \leq 264$.

Now let $k \geq 265$. By Proposition 16.2 there is an integer $j > k$ with $(j-k-1)^3 \leq k^2$ and $d_{j,k} > 0$. Let $t = j-k-1$ and $r = t+1$. Then $t \geq 0$ and $t^3 \leq k^2$, we have $r \geq 1$, and $d_{k+r,k} = d_{j,k} > 0$. It remains to show that $100\,(t+1) \leq 27\,k$. Suppose that $100\,(t+1) > 27\,k$. Then $400\,t > 108\,k - 400 \geq 100\,k$, because $8k \geq 400$, and so $k < 4t$. Hence $k^3 < 64\,t^3 \leq 64\,k^2$, so that $k < 64$, which contradicts $k \geq 265$. □

The proof of the next proposition applies Proposition 10.5 to the first positive row q of a column $k \geq 82$ instead of the row of the corollary. Proposition 16.4 describes the signs of the column from row k to the end of the first layer: read down the column, the step changes its sign exactly once. The signs in the rows from k to $2k-1$ are also the start of the induction on the row in Section 19.

Proposition 16.4. *For every integer $k \geq 82$ there is an integer q with the following three properties.*

(1) *One has $k+1 \leq q$ and $100\,(q-k) \leq 27\,k$.*
(2) *One has $d_{j,k} \leq 0$ for every integer j with $k \leq j < q$.*
(3) *One has $d_{n,k} > 0$ for every integer n with $q \leq n \leq 2k-1$.*

Proof. By Corollary 16.3 there is an integer $r \geq 1$ with $100\,r \leq 27\,k$ and $d_{k+r,k} > 0$. Let q be the first positive row of column k, that is, the least row $j \geq k+1$ with $d_{j,k} > 0$. Then $k+1 \leq q \leq k+r$, hence $100\,(q-k) \leq 100\,r \leq 27\,k$, and $d_{q,k} > 0$.

For the second property let $k \leq j < q$. If $j = k$, then $d_{k,k}/k! = \delta_k^{(k)} = -a_k/k < 0$ by Lemma 4.11, and if $k+1 \leq j < q$, then $d_{j,k} \leq 0$ by the choice of q.

For the third property let $r_q = q-k$. Then $r_q \geq 1$, and $3\,r_q \leq k$, because $100\,r_q \leq 27\,k$. We have just seen that $d_{j,k} \leq 0$ for every j with $k \leq j < k + r_q$, and that $d_{k+r_q,k} > 0$. These are the hypotheses of Proposition 10.5 for column $k \geq 82$, and its conclusion is that $d_{n,k} > 0$ for every n with $k + r_q \leq n \leq 2k-1$. □

We can now prove the column rule in the case in which its conclusion lies in the first layer, for all columns $k \geq 2$. The case in which the conclusion lies beyond the first layer is treated in Section 19 (Corollary 19.7).

Proposition 16.5. *Let $k \geq 2$ and n be integers with $k+1 \leq n \leq 2k-2$. If $d_{n,k} > 0$, then $d_{n+1,k} > 0$.*

Proof. For $2 \leq k \leq 81$ the claim is part 2 of Lemma 6.4.

Now let $k \geq 82$, and let q be a row with the properties of Proposition 16.4. Since $d_{n,k} > 0$ and $n \geq k$, the second property of q gives $q \leq n$. Then $q \leq n+1 \leq 2k-1$, and the third property gives $d_{n+1,k} > 0$. □

Applied repeatedly, Proposition 16.5 carries a positive step to every later row of the first layer. Section 23 uses this for the diagonal rule, where a positive row of the first layer is known and a step of the first layer that is not positive must then lie in an earlier row.

Corollary 16.6. *Let $k \geq 2$, m and n be integers with $k+1 \leq m \leq n \leq 2k-1$. If $d_{m,k} > 0$, then $d_{n,k} > 0$.*

Proof. This follows by induction on n from Proposition 16.5, which applies to every row n with $m \leq n \leq 2k-2$. □

17. The prime kernel at the critical tilt

Beyond the first layer the column equation of Proposition 4.12 is no longer available. What remains is the inequality of Proposition 5.3, in which the feedback of the earlier rows is formed with the prime kernel $\kappa^{(k)}$ of threshold k. From this inequality Section 19 derives that the step of a column $k \geq 82$ is positive in every row $n \geq 2k$ (Proposition 19.5), by an argument at the critical tilt x_k of Section 5. Here we prove what that argument uses about the tilted kernel $\bar{\kappa}_m = \kappa_m^{(k)} x_k^m$: its monotonicity and decay, its size for $k \geq 82$, and the order of the weights that it gives to the rows before and after the first positive row of the column (Lemmas 17.8 and 17.9).

17.1. **Monotonicity and harmonic decay.** Recall from Section 5 that the critical tilt of a column $k \geq 1$ is $x_k = (k/(k+1))^{k-1}$, that $0 < x_k \leq 1$, and that at this tilt the tilted Cayley masses $\omega_n = c_n x_k^n$ do not increase up to index k (Lemma 5.6). We first show that at the same tilt the tilted free prime kernel $\varepsilon_m x_k^m$ does not increase up to index k either; here $\varepsilon_m = (m-1)^{m-1}/m!$, with $\varepsilon_1 = 1$, is the free prime kernel of Section 4. The proof derives this monotonicity from that of the tilted Cayley masses.

Lemma 17.1. *Let k and m be integers with $1 \leq m < k$. Then the free prime kernel ε and the critical tilt x_k of column k satisfy*

$$\varepsilon_{m+1} x_k^{m+1} \leq \varepsilon_m x_k^m.$$

Proof. For an integer $i \geq 1$ let $\varpi_i = (1+1/i)^i$, and let $\varpi_0 = 1$. Then

$$a_{i+1} = \varpi_i \varepsilon_{i+1} \qquad \text{for every integer } i \geq 0, \tag{17.1}$$

as one checks directly from the definitions. The factor ϖ_i does not decrease with i: we have $\varpi_0 = 1 \leq 2 = \varpi_1$, and $\varpi_{i-1} \leq \varpi_i$ for $i \geq 2$ by the monotonicity of $(1+1/i)^i$ that is recalled in Section 2.

Write $x = x_k$. Since $a_i = c_{i-1}$, we have $a_i\, x^i = x\, \omega_{i-1}$ for every integer $i \geq 1$. Since $m \leq k$, Lemma 5.6 gives $\omega_m \leq \omega_{m-1}$, and hence $a_{m+1}\, x^{m+1} \leq a_m\, x^m$. With (17.1) for indices m and $m-1$ it follows that

$$\varpi_m\, \varepsilon_{m+1}\, x^{m+1} = a_{m+1}\, x^{m+1} \leq a_m\, x^m = \varpi_{m-1}\, \varepsilon_m\, x^m \leq \varpi_m\, \varepsilon_m\, x^m,$$

where the last step uses $\varpi_{m-1} \leq \varpi_m$. Dividing by ϖ_m gives the claim. □

We now consider an arbitrary threshold $y \geq 2$ and an arbitrary tilt $x \geq 0$. Recall from Section 4 that $(\kappa_m^{(y)}\, x^m)_{m \geq 0}$ is the tilted prime kernel of threshold y, and that in the free range $1 \leq m \leq y$ it agrees with the tilted free prime kernel, $\kappa_m^{(y)}\, x^m = \varepsilon_m\, x^m$ (Lemma 4.6). Beyond the free range the tilted prime kernel is given by the recursion (4.10), which holds for every integer $N \geq y$ and in which the tilted weights $(y-\gamma)\, a_\gamma\, x^\gamma$ are nonnegative and do not depend on N.

The next two lemmas concern a tilt at which the tilted prime kernel does not increase on the free range. They show that the numbers $m\, \kappa_m^{(y)}\, x^m$ then do not increase beyond the free range, so that the tilted prime kernel decays at least like $1/m$, which is its *harmonic decay*, and that the tilted prime kernel itself does not increase from index 1 on. The two lemmas are proved by one induction.

Lemma 17.2. *Let $y \geq 2$ be an integer and $x \geq 0$ a real number, and suppose that the free prime kernel ε satisfies*

$$\varepsilon_{m+1}\, x^{m+1} \leq \varepsilon_m\, x^m \qquad \text{for every integer } m \text{ with } 1 \leq m < y.$$

Then the prime kernel $\kappa^{(y)}$ of threshold y satisfies

$$(m+1)\, \kappa_{m+1}^{(y)}\, x^{m+1} \leq m\, \kappa_m^{(y)}\, x^m \qquad \text{for every integer } m \geq y.$$

Proof. We prove by induction on $m \geq 1$ that

$$\kappa_{m+1}^{(y)}\, x^{m+1} \leq \kappa_m^{(y)}\, x^m, \tag{17.2}$$

and that the inequality of the lemma holds at m if $m \geq y$. Let $m \geq 1$, and suppose that (17.2) holds with m replaced by any index i with $1 \leq i < m$.

If $m+1 \leq y$, the indices m and $m+1$ lie in the free range, where $\kappa_m^{(y)} = \varepsilon_m$ and $\kappa_{m+1}^{(y)} = \varepsilon_{m+1}$ by Lemma 4.6, and (17.2) is the hypothesis of the lemma.

Let $m \geq y$. By (4.10) for $N = m+1$ and for $N = m$,

$$(m+1)\, \kappa_{m+1}^{(y)}\, x^{m+1} = \sum_{\gamma=1}^{y-1} (y-\gamma)\, a_\gamma\, x^\gamma \cdot \kappa_{m+1-\gamma}^{(y)}\, x^{m+1-\gamma},$$

$$m\, \kappa_m^{(y)}\, x^m = \sum_{\gamma=1}^{y-1} (y-\gamma)\, a_\gamma\, x^\gamma \cdot \kappa_{m-\gamma}^{(y)}\, x^{m-\gamma}.$$

For $1 \leq \gamma \leq y-1$ the index $i = m-\gamma$ satisfies $1 \leq i < m$, because $m \geq y$, so that the induction hypothesis gives

$$\kappa_{m+1-\gamma}^{(y)}\, x^{m+1-\gamma} \leq \kappa_{m-\gamma}^{(y)}\, x^{m-\gamma}.$$

Multiplying by the weight $(y-\gamma)\, a_\gamma\, x^\gamma \geq 0$ and summing over γ, we obtain the inequality of the lemma at m. Moreover $\kappa_{m+1}^{(y)}\, x^{m+1} \geq 0$ by Lemma 4.8, and hence

$$m\, \kappa_{m+1}^{(y)}\, x^{m+1} \leq (m+1)\, \kappa_{m+1}^{(y)}\, x^{m+1} \leq m\, \kappa_m^{(y)}\, x^m.$$

Dividing by m, we obtain (17.2) at m. This completes the induction. □

Lemma 17.3. *Let $y \geq 2$ be an integer and $x \geq 0$ a real number, and suppose that the free prime kernel ε satisfies*

$$\varepsilon_{m+1}\, x^{m+1} \leq \varepsilon_m\, x^m \qquad \text{for every integer } m \text{ with } 1 \leq m < y.$$

Then the prime kernel $\kappa^{(y)}$ of threshold y satisfies

$$\kappa^{(y)}_{m+1}\, x^{m+1} \leq \kappa^{(y)}_m\, x^m \qquad \text{for every integer } m \geq 1.$$

Proof. The hypotheses are those of Lemma 17.2, and the claim is the inequality (17.2), which the proof of that lemma establishes for every integer $m \geq 1$. □

We return to the critical tilt. Let $k \geq 2$ be an integer. By Lemma 17.1, the hypothesis of Lemmas 17.2 and 17.3 holds for threshold $y = k$ and tilt $x = x_k$. The tilted prime kernel of threshold k at this tilt, $\bar\kappa_m = \kappa^{(k)}_m\, x_k^m$, therefore has the following five properties. The first is Lemma 4.8, the second is Lemma 4.6, the third and the fourth are the conclusions of Lemmas 17.3 and 17.2, and the last is (4.10) for threshold $y = k$ and tilt $x = x_k$:

$$\bar\kappa_m \geq 0 \qquad (m \geq 0), \tag{17.3}$$

$$\bar\kappa_m = \varepsilon_m\, x_k^m \qquad (1 \leq m \leq k), \tag{17.4}$$

$$\bar\kappa_{m+1} \leq \bar\kappa_m \qquad (m \geq 1), \tag{17.5}$$

$$(m+1)\,\bar\kappa_{m+1} \leq m\,\bar\kappa_m \qquad (m \geq k), \tag{17.6}$$

$$N\,\bar\kappa_N = \sum_{\gamma=1}^{k-1} (k-\gamma)\, a_\gamma\, x_k^\gamma\, \bar\kappa_{N-\gamma} \qquad (N \geq k). \tag{17.7}$$

By (17.5) we have $\bar\kappa_i \leq \bar\kappa_m$ whenever $1 \leq m \leq i$. These properties are used throughout the rest of the section, in which $k \geq 2$ is a column and $\bar\kappa$ always denotes the prime kernel of threshold k at the critical tilt x_k; the notation does not show the column, and we refer to $\bar\kappa$ as the kernel. Section 19 uses the harmonic decay (17.6) in the following iterated form.

Corollary 17.4. *Let $k \geq 2$ and $L \geq k$ be integers. Then the prime kernel $\bar\kappa$ of threshold k at the critical tilt satisfies*

$$L\,\bar\kappa_L \leq k\,\bar\kappa_k.$$

Proof. This follows from (17.6) by induction on L, starting from the equality at $L = k$. □

17.2. **Faster decay beyond the free range.** The next lemma bounds the kernel from index $2k-1$ on in terms of its value $\bar\kappa_k$. Its proof carries a bound for the ratio of consecutive values of the kernel from a window of $k-1$ indices to the next index by the recursion (17.7). Section 19 uses the lemma for the rows that are far from the first positive row of the column.

Lemma 17.5. *Let $k \geq 2$ and $L \geq 2k-1$ be integers. Then the prime kernel $\bar\kappa$ of threshold k at the critical tilt satisfies*

$$(L-1)\,L\,\bar\kappa_L \leq 2k\,(k-1)\,\bar\kappa_k.$$

Proof. We claim that for all integers $t \geq 1$ and m with $m \geq k + (t-1)(k-1)$,

$$(m+1)\,\bar\kappa_{m+1} \leq (m+1-t)\,\bar\kappa_m. \tag{17.8}$$

For $t = 1$ this is (17.6). Let $m \geq k$ be an integer and ρ a real number such that $\bar{\kappa}_{i+1} \leq \rho\, \bar{\kappa}_i$ for every integer i with $m - k + 1 \leq i \leq m - 1$. Then

$$(m+1)\, \bar{\kappa}_{m+1} \leq \rho\, m\, \bar{\kappa}_m. \tag{17.9}$$

Indeed, by (17.7) for $N = m + 1$ and for $N = m$,

$$\begin{aligned}(m+1)\, \bar{\kappa}_{m+1} &= \sum_{\gamma=1}^{k-1} (k-\gamma)\, a_\gamma\, x_k^\gamma\, \bar{\kappa}_{m+1-\gamma} \\ &\leq \rho \sum_{\gamma=1}^{k-1} (k-\gamma)\, a_\gamma\, x_k^\gamma\, \bar{\kappa}_{m-\gamma} = \rho\, m\, \bar{\kappa}_m,\end{aligned}$$

because the weights $(k-\gamma)\, a_\gamma\, x_k^\gamma$ are nonnegative and the indices $i = m - \gamma$ with $1 \leq \gamma \leq k - 1$ are those of the window.

We prove (17.8) by induction on t; the case $t = 1$ has just been noted. Suppose that (17.8) holds for some $t \geq 1$, and let $m \geq k + t\,(k-1)$. Every index i with $m-k+1 \leq i \leq m-1$ satisfies $i \geq k+(t-1)(k-1)$, so that $(i+1)\, \bar{\kappa}_{i+1} \leq (i+1-t)\, \bar{\kappa}_i$ by the induction hypothesis, and $(m-i-1)\, \bar{\kappa}_{i+1} \leq (m-i-1)\, \bar{\kappa}_i$ by (17.5). Adding these two inequalities, we get $m\, \bar{\kappa}_{i+1} \leq (m-t)\, \bar{\kappa}_i$. The hypothesis of (17.9) thus holds with $\rho = (m-t)/m$, and (17.9) gives $(m+1)\, \bar{\kappa}_{m+1} \leq (m-t)\, \bar{\kappa}_m$, which is (17.8) with t replaced by $t+1$.

We now prove the claim of the lemma by induction on L. For $L = 2k-1$, Corollary 17.4 gives $(2k-1)\, \bar{\kappa}_{2k-1} \leq k\, \bar{\kappa}_k$, and multiplying by $2k-2$ we obtain the claim. Let $L \geq 2k-1$, and suppose that the claim holds for L. Since $L \geq k+(k-1)$, the inequality (17.8) with $t = 2$ and $m = L$ gives $(L+1)\, \bar{\kappa}_{L+1} \leq (L-1)\, \bar{\kappa}_L$. Multiplying by L and using the induction hypothesis, we get

$$L\,(L+1)\, \bar{\kappa}_{L+1} \leq (L-1)\, L\, \bar{\kappa}_L \leq 2k\,(k-1)\, \bar{\kappa}_k,$$

which is the claim for $L + 1$. □

17.3. **The size of the kernel in the free range.** Corollary 17.4 and Lemma 17.5 bound the kernel beyond the free range in terms of its value $\bar{\kappa}_k$. We now bound the kernel in the free range, where it is explicit, in absolute terms. The proof of the next lemma writes the kernel in the free range with the Poisson mass P of Section 2, and the lemma is the first of the two bounds that Section 19 uses.

Lemma 17.6. *For every integer $k \geq 82$, the prime kernel $\bar{\kappa}$ of threshold k at the critical tilt satisfies*

$$\bar{\kappa}_k \leq \frac{0.662}{k\, \sqrt{k}}.$$

Proof. Let $1 \leq m \leq k$ be an integer. By (17.4) we have $\bar{\kappa}_m = \varepsilon_m\, x_k^m$, and

$$m\, \varepsilon_m = \frac{(m-1)^{m-1}}{(m-1)!} = \nu_{m-1} = e^{m-1}\, P(m-1),$$

where ν and the Poisson mass P are as in Section 2. Hence

$$\bar{\kappa}_m = \frac{P(m-1)\, (e\, x_k)^m}{e\, m} \qquad (1 \leq m \leq k).$$

For $m \geq 2$ we have $P(m-1) \leq 1/\sqrt{2\pi(m-1)}$ by the upper bound of (2.5), and $(e\,x_k)^m \leq e^{3m/(2k)}$ by Lemma 5.7. Therefore

$$\bar{\kappa}_m \leq \frac{1}{m\,\sqrt{2\pi(m-1)}}\,\exp\Big(\frac{3m}{2k}-1\Big) \qquad (2 \leq m \leq k). \tag{17.10}$$

By (17.10) with $m = k$ we have $\bar{\kappa}_k \leq e^{1/2}/\big(k\sqrt{2\pi(k-1)}\big)$, so that it suffices to show that $e^{1/2}\sqrt{k} \leq 0.662\,\sqrt{2\pi(k-1)}$, or, after squaring, that $e\,k \leq 0.662^2 \cdot 2\pi\,(k-1)$. Since $e < 2.7183$ and $0.662^2 \cdot 2\pi > 2.7534$, this follows from the inequality $2.7183\,k \leq 2.7534\,(k-1)$, which holds because $k \geq 82$. □

The second bound is for the number $\bar{\kappa}_{n-q+1}$, where $n \geq 2k$ is a row beyond the first layer and q is the first positive row of the column; this is the number with which Section 17.4 compares the weights that the kernel gives to the rows. There $q \leq k + 27k/100$, so that the index $L = n - q + 1$ satisfies $L - 1 = n - q \geq 73k/100$; this index may lie in the free range or beyond it.

Lemma 17.7. *Let $k \geq 82$ and L be integers with $100\,(L-1) \geq 73\,k$. Then the prime kernel $\bar{\kappa}$ of threshold k at the critical tilt satisfies*

$$\bar{\kappa}_L \leq \frac{0.73}{k\,\sqrt{k}}.$$

Proof. Let m_0 be the least integer with $100\,(m_0-1) \geq 73\,k$, so that $73\,k \leq 100\,(m_0-1) \leq 73\,k + 99$. Then $m_0 \leq L$ and $m_0 \geq 2$, and $m_0 \leq k$ because $100\,m_0 \leq 73\,k + 199 \leq 100\,k$ for $k \geq 82$. By (17.5) and by (17.10) with $m = m_0$, which was shown in the proof of Lemma 17.6,

$$\bar{\kappa}_L \leq \bar{\kappa}_{m_0} \leq \frac{1}{m_0\,\sqrt{2\pi(m_0-1)}}\,\exp\Big(\frac{3}{2}\,\sigma - 1\Big), \qquad \text{where } \sigma = \frac{m_0}{k}.$$

The bounds for m_0 give $0.73 \leq \sigma \leq 0.73 + 1.99/k \leq 0.7543$, so that the exponent $(3/2)\,\sigma - 1$ lies between 0 and 1.7. By the inequality $e^v \leq 1 + v + v^2$ for $0 \leq v \leq 1.7$, which is recalled in Section 2,

$$\exp\Big(\frac{3}{2}\,\sigma - 1\Big) \leq 1 + \Big(\frac{3}{2}\,\sigma - 1\Big) + \Big(\frac{3}{2}\,\sigma - 1\Big)^2 = 1 - \frac{3}{2}\,\sigma + \frac{9}{4}\,\sigma^2.$$

The right-hand side is at most $1.563368\,\sigma$. Indeed, the difference

$$\frac{9}{4}\,\sigma^2 - 3.063368\,\sigma + 1$$

is a convex function of σ, and it is negative at $\sigma = 0.73$ and at $\sigma = 0.7543$; hence it is negative on the interval between these two points. Finally

$$2\pi\,(m_0 - 1) \geq 2\pi \cdot 0.73\,k \geq 2.1416^2\,k,$$

where the second inequality is checked by direct computation, with $2\pi > 6.283$. Altogether,

$$\bar{\kappa}_L \leq \frac{1.563368\,\sigma}{m_0 \cdot 2.1416\,\sqrt{k}} = \frac{1.563368}{2.1416\,k\,\sqrt{k}} = \frac{0.73}{k\,\sqrt{k}}.$$ □

17.4. **The weight Ψ_k and the order of the kernel weights.** For integers $k \geq 1$ and $N \geq 0$ let

$$\Psi_k(N) = \max(N,\, 2k-1).$$

The number $\Psi_k(N)$ is the *weight* of row N in the argument for column k. It is equal to $2k-1$ up to row $2k-1$, the last row of the first layer, and to the row number from there on. Clearly $\Psi_k(N) \geq N$ for every N, and $\Psi_k(N) \leq \Psi_k(N')$ whenever $N \leq N'$.

The weight serves the following step of Section 19. Let $k \geq 2$, and let η_j and U_j be the tilted step of column k and the tilted column count that are defined in Section 7, with (7.13): they are the normalised step of column k and the normalised count of column $k-1$, both tilted by the critical tilt x_k. Multiplied by x_k^n, the inequality of Proposition 5.3 for a row $n \geq k+1$ reads

$$n\,\eta_n \;\geq\; (k+1)\,x_k\,\eta_{n-1} + 2\,x_k\,U_{n-1} + n\sum_{j=k}^{n-1} \bar\kappa_{n-j}\,\eta_j.$$

The sum is the feedback of rows $k \leq j \leq n-1$, in which row j has the *kernel weight* $\bar\kappa_{n-j}$. In the situation of Section 19 the column has a first positive row q, and row n is reached by an induction on the row: the tilted step η_j is not positive for $k \leq j < q$ and positive for $q \leq j \leq n-1$. The feedback thus contains terms that are not positive, and nothing is known about the size of the single steps in the rows before q. What is known about these rows is a lower bound for one weighted sum: Lemma 19.4 bounds the sum $\sum_{j=k}^{J} \Psi_k(j)\,\eta_j$ from below by terms of rows J and $J-1$. To use this bound for the feedback, the kernel weights $\bar\kappa_{n-j}$ have to be exchanged for the weights $\Psi_k(j)$. This is possible term by term, without any knowledge of the sizes of the η_j, as soon as one number $\theta \geq 0$ separates the ratios of the two weights at row q:

$$\frac{\bar\kappa_{n-j}}{\Psi_k(j)} \leq \theta \quad (k \leq j < q), \qquad \frac{\bar\kappa_{n-j}}{\Psi_k(j)} \geq \theta \quad (q \leq j \leq n-1). \tag{17.11}$$

For then $\bar\kappa_{n-j}\,\eta_j \geq \theta\,\Psi_k(j)\,\eta_j$ in every row $k \leq j \leq n-1$, because η_j is not positive in the rows of the first kind and positive in those of the second, and the feedback is at least θ times the sum $\sum_{j=k}^{n-1} \Psi_k(j)\,\eta_j$. The two lemmas of this subsection show that (17.11) holds with

$$\theta = \frac{\bar\kappa_{n-q+1}}{2k-1},$$

the value of the ratio at row $j = q-1$, for every row $n \geq 2k$, provided that $k+1 \leq q \leq k + 27k/100$. This is the range in which the first positive row of a column $k \geq 82$ lies (Proposition 16.4).

The weight has its form because it has to meet two requirements. One is the separation (17.11). The other comes from Lemma 19.4. That lemma is proved for columns $k \geq 7$, and for such a column it rests on a convexity along the diagonal of the tilted normalised counts of three consecutive rows $N-1$, N and $N+1$, in which each count is multiplied by the weight of its row (Corollary 19.3). With a constant weight this convexity is proved for $k \leq N \leq 2k-2$ (Corollary 18.13), and with each count multiplied by its row number it is proved for $N \geq 2k-1$ (Proposition 19.2). The weight Ψ_k is chosen so that the two statements together give the convexity for every $N \geq k$.

For the separation a constant weight is the convenient one, since (17.11) then asks only that the kernel do not increase. This is the situation on the rows before q.

Lemma 17.8. *Let $k \geq 2$, q, n and j be nonnegative integers with $100\,(q-k) \leq 27\,k$, $n \geq 2k$ and $j < q$. Then the prime kernel $\bar{\kappa}$ of threshold k at the critical tilt and the weight Ψ_k satisfy*

$$\bar{\kappa}_{n-j}\,(2k-1) \leq \bar{\kappa}_{n-q+1}\,\Psi_k(j).$$

Proof. Since $27\,k \leq 100\,(k-1)$ for $k \geq 2$, the hypothesis on q gives $q \leq 2k-1$. Hence $j < 2k-1$, so that $\Psi_k(j) = 2k-1$, and $q < n$, so that $1 \leq n-q+1 \leq n-j$. By (17.5) we have $\bar{\kappa}_{n-j} \leq \bar{\kappa}_{n-q+1}$, and multiplying by $2k-1$ gives the claim. □

On the rows from q on the weight is constant only up to row $2k-1$, and on these rows the comparison is once more the monotonicity of the kernel. For a row $j \geq 2k$ the weight is j, and the second half of (17.11) says that, as j increases and the index $n-j$ of the kernel decreases, the kernel grows at least in proportion to j. The proof of this half depends on where the indices $n-j$ and $n-q+1$ lie relative to the free range, and it rests on two further properties of the kernel, one beyond the free range and one inside it.

We can now prove the second half of (17.11).

Lemma 17.9. *Let $k \geq 2$, q, n and j be integers with $k+1 \leq q$, $100\,(q-k) \leq 27\,k$, $n \geq 2k$ and $q \leq j \leq n-1$. Then the prime kernel $\bar{\kappa}$ of threshold k at the critical tilt and the weight Ψ_k satisfy*

$$\bar{\kappa}_{n-q+1}\,\Psi_k(j) \leq \bar{\kappa}_{n-j}\,(2k-1).$$

Proof. Put $L = n-q+1$ and $\ell = n-j$. As in the proof of Lemma 17.8 we have $q \leq 2k-1 < n$, and $q \leq j \leq n-1$ gives $1 \leq \ell < L$.

If $j \leq 2k-1$, then $\Psi_k(j) = 2k-1$, and the claim is $\bar{\kappa}_L \leq \bar{\kappa}_\ell$, which holds by (17.5).

Let $j \geq 2k$. Then $\Psi_k(j) = j$, and we have to show that

$$j\,\bar{\kappa}_L \leq (2k-1)\,\bar{\kappa}_\ell. \tag{17.12}$$

We distinguish three cases, according to the position of indices ℓ and L relative to the free range.

Suppose first that $\ell \geq k$. The kernel decays by a fixed factor beyond the free range: for every integer $m \geq k$,

$$(2k-1)\,\bar{\kappa}_{m+1} \leq (2k-2)\,\bar{\kappa}_m. \tag{17.13}$$

To prove it, let $t \geq 1$ be the integer with $k+(t-1)(k-1) \leq m < k+t\,(k-1)$. By (17.8), which was shown in the proof of Lemma 17.5, $(m+1)\,\bar{\kappa}_{m+1} \leq (m+1-t)\,\bar{\kappa}_m$. Moreover $m+1 \leq k+t\,(k-1) \leq t\,(2k-1)$, and therefore

$$(2k-1)\,(m+1-t) - (2k-2)\,(m+1) = m+1-t\,(2k-1) \leq 0.$$

Since $\bar{\kappa}_m \geq 0$ by (17.3), the two inequalities give

$$(2k-1)\,(m+1)\,\bar{\kappa}_{m+1} \leq (2k-1)\,(m+1-t)\,\bar{\kappa}_m \leq (2k-2)\,(m+1)\,\bar{\kappa}_m,$$

and division by $m+1$ gives (17.13).

We show by induction on i that, for every integer i with $2k-1 \leq i \leq n-k$,

$$i\,\bar{\kappa}_{n-2k+1} \leq (2k-1)\,\bar{\kappa}_{n-i}. \tag{17.14}$$

For $i = 2k-1$ the two sides are equal. Suppose that (17.14) holds for i, and that $i+1 \le n-k$. By (17.13) with $m = n-i-1 \ge k$ we have $(2k-1)\,\bar\kappa_{n-i} \le (2k-2)\,\bar\kappa_{n-i-1}$, so that $i\,\bar\kappa_{n-2k+1} \le (2k-2)\,\bar\kappa_{n-i-1}$. Since $(2k-2)\,(i+1) \le (2k-1)\,i$ for $i \ge 2k-2$, it follows that $(i+1)\,\bar\kappa_{n-2k+1} \le (2k-1)\,\bar\kappa_{n-i-1}$, which is (17.14) for $i+1$. Now $2k \le j = n-\ell \le n-k$, so that (17.14) with $i = j$ gives $j\,\bar\kappa_{n-2k+1} \le (2k-1)\,\bar\kappa_\ell$. Since $q \le 2k$, we have $L \ge n-2k+1 \ge 1$, hence $\bar\kappa_L \le \bar\kappa_{n-2k+1}$ by (17.5), and (17.12) follows.

Suppose next that $\ell < k$ and $L \le k$. Let $1 \le m \le k$ be an integer. By (17.4) we have $\bar\kappa_m = \varepsilon_m\, x_k^m$, and

$$m\,\varepsilon_m = \frac{(m-1)^{m-1}}{(m-1)!} = \nu_{m-1} = e^{m-1}\,P(m-1),$$

where ν and the Poisson mass P are as in Section 2. Hence

$$\bar\kappa_m = \frac{P(m-1)\,(e\,x_k)^m}{e\,m} \qquad (1 \le m \le k). \tag{17.15}$$

We show that the kernel drops across a long gap in the free range: for all integers i and m with $1 \le i < m \le k$ and $100\,(m-i) \ge 73\,k$,

$$1.22\,\bar\kappa_m \le \bar\kappa_i. \tag{17.16}$$

For $i = 1$ we have $\bar\kappa_1 = x_k$ and $\bar\kappa_2 = x_k^2/2$ by (17.4), and since $m \ge 2$ and $x_k \le 1$, the monotonicity (17.5) gives $\bar\kappa_m \le \bar\kappa_2 \le \bar\kappa_1/2$. Let $i \ge 2$. By (17.15) for indices m and i,

$$\bar\kappa_m = \frac{P(m-1)}{P(i-1)}\,(e\,x_k)^{m-i}\,\frac{i}{m}\,\bar\kappa_i.$$

Here $P(m-1) \le 1/\sqrt{2\pi(m-1)}$ by the upper bound of (2.5), and $P(i-1) \ge (11/12)/\sqrt{2\pi(i-1)}$ by its lower bound, because $1 - 1/(12\,(i-1)) \ge 11/12$. By Lemma 5.7, $(e\,x_k)^{m-i} \le e^{3(m-i)/(2k)} \le e^{3/2}$, because $m-i \le k$. The gap gives $m-i \ge 0.73\,k \ge 0.73\,m$, hence $i \le 0.27\,m$ and $i-1 \le 0.27\,(m-1)$. Therefore

$$\bar\kappa_m \le \frac{12}{11}\sqrt{\frac{i-1}{m-1}}\;e^{3/2}\,\frac{i}{m}\,\bar\kappa_i \le \frac{12}{11}\,e^{3/2}\,0.27^{3/2}\,\bar\kappa_i.$$

The number $1.22 \cdot (12/11)\,e^{3/2}\,0.27^{3/2}$ is smaller than 1, because its square $1.22^2 \cdot (144/121)\,e^3 \cdot 0.27^3$ is smaller than 0.71, as a direct computation shows. This proves (17.16).

We have $L - \ell = j - q + 1$, and therefore

$$100\,(L-\ell) \ge 100\,(2k-q+1) = 100\,(k+1) - 100\,(q-k) \ge 73\,k.$$

Thus (17.16) applies to indices $\ell < L$ and gives $1.22\,\bar\kappa_L \le \bar\kappa_\ell$. On the other hand, $L \le k$ says that $n \le k+q-1$, so that $j \le n-1 \le k+q-2$ and

$$100\,j \le 100\,(2k-2) + 100\,(q-k) \le 227\,k - 200 \le 122\,(2k-1).$$

Hence

$$j\,\bar\kappa_L \le 1.22\,(2k-1)\,\bar\kappa_L \le (2k-1)\,\bar\kappa_\ell.$$

Suppose finally that $\ell < k < L$. By Corollary 17.4 we have $L\,\bar\kappa_L \le k\,\bar\kappa_k$, and it remains to compare $\bar\kappa_k$ with $\bar\kappa_\ell$; how this is done depends on the size of ℓ. Note that $j = n-\ell = L+q-1-\ell$.

If $100\,\ell \le 27\,k$, then (17.16) applies to indices $\ell < k$ and gives $1.22\,\bar\kappa_k \le \bar\kappa_\ell$. Since $\ell \ge 1$ and $100\,(q-k) \le 27\,k$, we have $100\,j \le 100\,L + 127\,k - 200$. Moreover, since $L \ge k+1$,

$$127\,k^2 - 200\,k \le (144\,k - 122)\,(k+1) \le (144\,k - 122)\,L.$$

Adding $100\,k\,L$, we obtain

$$100\,k\,j \le 100\,k\,L + 127\,k^2 - 200\,k \le 122\,(2k-1)\,L,$$

so that $k\,j \le 1.22\,(2k-1)\,L$. Hence

$$j\,L\,\bar\kappa_L \le k\,j\,\bar\kappa_k \le 1.22\,(2k-1)\,L\,\bar\kappa_k \le (2k-1)\,L\,\bar\kappa_\ell.$$

If $100\,\ell > 27\,k$, then $\bar\kappa_k \le \bar\kappa_\ell$ by (17.5), because $\ell < k$. Moreover

$$100\,(q-\ell) = 100\,(q-k) + 100\,(k-\ell) < 100\,k,$$

so that $q - \ell \le k-1$ and $j \le L + k - 2$. Since $L \ge k+1$, this gives

$$k\,j \le k\,L + k^2 - 2k \le k\,L + (k-1)\,(k+1) \le (2k-1)\,L,$$

and hence

$$j\,L\,\bar\kappa_L \le k\,j\,\bar\kappa_k \le (2k-1)\,L\,\bar\kappa_k \le (2k-1)\,L\,\bar\kappa_\ell.$$

For both sizes of ℓ, division by L gives (17.12). □

18. Property K and diagonal convexity in the first layer

In this section we prove that, for a column $k \ge 7$ and at the critical tilt of that column, the tilted normalised counts of thresholds $k-1$, k and $k+1$ are convex along the diagonal in the first layer (Corollary 18.13), and that they remain so when they are multiplied by the weights $c-1$, c and $c+1$ for a real number $c \ge k$ (Corollary 18.12). These two inequalities are what the argument beyond the first layer, in Section 19, takes from the first layer. We prove them by an induction along the last-car recursion, which is carried out for a stronger property, property K (Definition 18.1).

18.1. **Diagonal triples.** Throughout the section the tilt is the critical tilt $x_k = (k/(k+1))^{k-1}$ of a column $k \ge 1$ (Section 5). As before, for integers $n \ge 0$, $N \ge 0$ and $y \ge 1$,

$$\omega_n = c_n\,x_k^n \qquad \text{and} \qquad u_N^{(y)} = z_N^{(y)}\,x_k^N$$

denote the tilted Cayley masses and the tilted normalised count of threshold y at this tilt; we say masses and tilted counts for short, and column k is not shown in the notation. The index N of a tilted count is its *size*, the length of the parking functions that the count refers to. The tilt is positive, so that $\omega_n > 0$ and $u_N^{(y)} \ge 0$. By Proposition 3.4 a tilted count in the free range of its threshold is a mass:

$$u_N^{(y)} = \omega_N \qquad (N \le y). \tag{18.1}$$

The *offset* of a tilted count $u_N^{(y)}$ is the integer $N - y$, so that the counts of (18.1) are those whose offset is at most 0.

For integers $k \ge 2$ and $\rho \ge 0$ the *diagonal triple of offset* ρ of column k is

$$\big(u_{k-1+\rho}^{(k-1)},\ u_{k+\rho}^{(k)},\ u_{k+1+\rho}^{(k+1)}\big).$$

Its members are the tilted counts of the three thresholds $k-1$, k and $k+1$, from which the quantities of column k are formed (Section 3), each at offset ρ and all

tilted by the same number x_k. For $0 \le \rho \le k-2$ the largest of the three sizes, $k+1+\rho$, runs through rows $k+1, \dots, 2k-1$ of the first layer of column k.

The induction on the offset uses the last-car recursion in a form that the proof of Proposition 18.11 derives from Lemma 18.10. For integers ρ and y with $1 \le \rho \le y-1$ the *direct term* is

$$S_\rho(y) = x_k\, \omega_{\rho-1}\, \omega_y + \frac{x_k}{2} \cdot \frac{y+\rho+1}{y+\rho}\, Q_\rho(y-\rho-1),$$

where, for integers $\rho \ge 0$ and $j \ge 0$,

$$Q_\rho(j) = \sum_{i=0}^{j} \omega_{\rho+i}\, \omega_{\rho+j-i}. \tag{18.2}$$

That proof writes the tilted count $u^{(y)}_{y+\rho}$ as the sum of the direct term and of the *chain part*, which comes from the sum in Lemma 18.10 and contains the counts $u^{(y)}_{y+\rho-1-i}$ of the same threshold at offsets $1, \dots, \rho-1$, each multiplied by a number $x_k\, \omega_i$ and by a factor $(y+i+1)/(y+\rho)$, for $0 \le i \le \rho-2$. Taken for the three thresholds $y = k-1, k, k+1$, the proof thus builds the diagonal triple of offset ρ from the diagonal triples of smaller offset and from the triple of the direct terms by three operations: sums; multiplication by the numbers $x_k\, \omega_i \ge 0$, which are the same for the three thresholds; and multiplication by the factors $(y+i+1)/(y+\rho)$, which depend on the threshold and increase with it.

18.2. **Property K and its stability under factors.** Property K concerns a triple of real numbers, whose members we write with indices $-$, 0 and $+$ in their order, and one further real number k, which in the applications is the column.

Definition 18.1. Let $k \ne 0$ be a real number. A triple (t_-, t_0, t_+) of real numbers *has property K at* k if

$$2\, e^{\mu/k}\, t_0 \le e^{\mu}\, t_+ + e^{-\mu}\, t_- \qquad \text{for every real number } \mu \ge 0.$$

In the applications t_-, t_0, t_+ are the values at the three consecutive thresholds $k-1$, k, $k+1$ of one quantity that depends on a threshold. The order of the triple is the order of the thresholds, so that the value t_- at the lower threshold carries the factor $e^{-\mu}$ and the value t_+ at the upper threshold the factor e^{μ}. For $\mu = 0$ the inequality is the convexity $2\, t_0 \le t_- + t_+$ of the three values. For $\mu > 0$ it says that the triple stays convex after the lower value has been multiplied by $e^{-\mu}$ and the upper value by e^{μ}, and that it does so with the additional factor $e^{\mu/k}$ on the middle value. For each fixed μ the inequality is linear in the triple. Hence the sum of two triples with property K at k has property K at k, and so has the multiple of such a triple by a nonnegative real number.

Convexity alone is not preserved when the three values are multiplied by positive factors that increase with the threshold, as the factors of the chain part do. The triple $(2, 1, 0)$ is convex. Multiplied by factors $\theta_- < \theta_0 < \theta_+$ it becomes $(2\theta_-, \theta_0, 0)$, which is not convex, because $2\theta_0 > 2\theta_-$. For this reason the induction on the offset cannot be carried out for convexity itself, and we carry it out for property K. Its stability under factors rests on the following computation. Let $k > 0$ be a real number and let (t_-, t_0, t_+) be a triple with property K at k. Then for all real numbers L, M and μ with $\mu + (L-M)/2 \ge 0$,

$$2\, e^{\mu/k} \exp\Big(\frac{(k+1)L + (k-1)M}{2k}\Big)\, t_0 \le e^{\mu}\, e^{L}\, t_+ + e^{-\mu}\, e^{M}\, t_-. \tag{18.3}$$

Indeed, the number $\mu' = \mu + (L-M)/2$ is nonnegative, so that $2e^{\mu'/k}\,t_0 \le e^{\mu'}\,t_+ + e^{-\mu'}\,t_-$. Multiplying this inequality by $e^{(L+M)/2}$ gives (18.3): the exponents on the right become $\mu + L$ and $-\mu + M$, and the exponent on the left becomes

$$\frac{L+M}{2} + \frac{\mu'}{k} = \frac{\mu}{k} + \frac{(k+1)L + (k-1)M}{2k}.$$

In words, to multiply the upper value by e^L and the lower value by e^M has the same effect on the inequality of property K as to move the parameter from μ to $\mu + (L-M)/2$ and to multiply the middle value by the exponential factor on the left-hand side of (18.3). If $L \ge M$, the parameter moves upward, and (18.3) holds for every $\mu \ge 0$. This is why the property is required for all $\mu \ge 0$: a hypothesis that held for $\mu = 0$ only could not be used after a single increasing factor. If $L < M$, the parameter moves downward, and (18.3) is available only for $\mu \ge (M-L)/2$, because property K says nothing for negative parameters. We meet this case in the proof of Lemma 18.9.

The proofs of the two stability lemmas below use (18.3) for positive factors θ_-, θ_0, θ_+ that do not decrease from the lower to the upper threshold, and they verify that the number

$$(k+1)\log\frac{\theta_+}{\theta_0} + (k-1)\log\frac{\theta_-}{\theta_0}$$

is nonnegative. This condition on the factors explains the factor $e^{\mu/k}$ in Definition 18.1. When θ_-, θ_0, θ_+ are the values at $k-1$, k, $k+1$ of a positive function θ of the threshold, the number above is the second difference of the function $y \mapsto y\log\theta(y)$ at these three points. The weights $k+1$ and $k-1$ come from the exponent μ/k. Without the factor $e^{\mu/k}$, the same computation would lead to the condition $\log\theta_+ + \log\theta_- \ge 2\log\theta_0$, the log-convexity of θ at the three points. With it, the condition is the one that the proof of Lemma 18.2 verifies for the factors of the chain part, through the relative increments of the factors.

The first of the two stability lemmas treats the factors $(y+i+1)/(y+\rho)$ that thresholds $y = k-1$, k and $k+1$ carry in the chain part.

Lemma 18.2. *Let $k > 1$ be a real number, let (t_-, t_0, t_+) be a triple of real numbers that has property K at k and satisfies $t_0 \ge 0$, and let α and β be real numbers with $0 \le \alpha \le \beta$. Then the triple*

$$\Big(\frac{k-1+\alpha}{k-1+\beta}\,t_-,\ \frac{k+\alpha}{k+\beta}\,t_0,\ \frac{k+1+\alpha}{k+1+\beta}\,t_+\Big)$$

has property K at k.

Proof. Let θ_-, θ_0, θ_+ be any positive real numbers such that

$$\theta_- \le \theta_+ \qquad \text{and} \qquad (k+1)\log\frac{\theta_+}{\theta_0} + (k-1)\log\frac{\theta_-}{\theta_0} \ge 0. \tag{18.4}$$

Then the triple $(\theta_- t_-, \theta_0 t_0, \theta_+ t_+)$ has property K at k. To see this, put $L = \log(\theta_+/\theta_0)$ and $M = \log(\theta_-/\theta_0)$, and let $E = (k+1)L + (k-1)M$ be the left-hand side of the second condition. Since $L \ge M$, every $\mu \ge 0$ satisfies $\mu + (L-M)/2 \ge 0$, and (18.3) gives

$$2\,e^{\mu/k}\,e^{E/(2k)}\,t_0 \le e^{\mu}\,\frac{\theta_+}{\theta_0}\,t_+ + e^{-\mu}\,\frac{\theta_-}{\theta_0}\,t_-.$$

As $e^{E/(2k)} \ge 1$ and $t_0 \ge 0$, the left-hand side is at least $2e^{\mu/k}\,t_0$. Multiplying by θ_0, we obtain the inequality of property K for the triple $(\theta_- t_-, \theta_0 t_0, \theta_+ t_+)$.

For such positive numbers θ_-, θ_0, θ_+ put

$$\sigma_+ = \frac{\theta_+}{\theta_0} - 1, \qquad \sigma_- = \frac{\theta_0}{\theta_-} - 1.$$

If

$$\sigma_+ \geq 0 \qquad \text{and} \qquad (k-1)\,\sigma_- \leq (k+1)\,\frac{2\sigma_+}{\sigma_+ + 2}, \tag{18.5}$$

then the second condition of (18.4) holds. This follows from the elementary inequalities $\log(1+u) \geq 2u/(u+2)$ for $u \geq 0$, which is easily verified by differentiation, and $\log r \geq 1 - 1/r$ for $r > 0$. The first, with $u = \sigma_+$, gives $\log(\theta_+/\theta_0) \geq 2\sigma_+/(\sigma_+ + 2)$, and the second, with $r = \theta_-/\theta_0$, gives $\log(\theta_-/\theta_0) \geq -\sigma_-$. Hence

$$(k+1)\log\frac{\theta_+}{\theta_0} + (k-1)\log\frac{\theta_-}{\theta_0} \geq (k+1)\,\frac{2\sigma_+}{\sigma_+ + 2} - (k-1)\,\sigma_- \geq 0.$$

Now let $\theta(y) = (y+\alpha)/(y+\beta)$, and put $\theta_- = \theta(k-1)$, $\theta_0 = \theta(k)$ and $\theta_+ = \theta(k+1)$. These numbers are positive, and the triple of the lemma is $(\theta_- t_-, \theta_0 t_0, \theta_+ t_+)$. So it suffices to verify the two conditions of (18.4), and for the second of them we verify (18.5).

With the denominators cleared, the inequality $\theta_- \leq \theta_+$ reduces to $2(\beta - \alpha) \geq 0$. A direct computation of the two quotients θ_+/θ_0 and θ_0/θ_- gives

$$\sigma_+ = \frac{\beta - \alpha}{(k+\alpha)(k+1+\beta)}, \qquad \sigma_- = \frac{\beta - \alpha}{(k+\beta)(k-1+\alpha)}.$$

Thus $\sigma_+ \geq 0$, and with the denominators cleared, the second inequality of (18.5) reads

$$(k-1)(\beta-\alpha)\big[2(k+\alpha)(k+1+\beta) + (\beta-\alpha)\big] \leq 2(k+1)(\beta-\alpha)(k+\beta)(k-1+\alpha),$$

and when both sides are expanded, the right-hand side minus the left-hand side is found to be

$$(\beta-\alpha)\big[\alpha\,(4\beta + 3k + 1) + \beta\,(k-1)\big],$$

which is nonnegative. □

The second stability lemma treats the weights $c-1$, c and $c+1$ of Corollary 18.12, with a real number $c \geq k$. These weights are the values at $y = k-1$, k and $k+1$ of the factor $y + \alpha$ with $\alpha = c - k$. This factor increases with the threshold, but it is not a quotient of the kind treated in Lemma 18.2. We state the lemma for the more general factor $(y+\alpha)(y+\beta)/y$ with $\alpha \geq 0$ and $\beta \geq 0$, of which the factor $y + \alpha$ is the case $\beta = 0$.

Lemma 18.3. *Let $k > 1$ be a real number, let (t_-, t_0, t_+) be a triple of real numbers that has property K at k and satisfies $t_0 \geq 0$, and let $\alpha \geq 0$ and $\beta \geq 0$ be real numbers with $\alpha\beta \leq k^2 - 1$. Then the triple*

$$\Big(\frac{(k-1+\alpha)(k-1+\beta)}{k-1}\,t_-,\ \frac{(k+\alpha)(k+\beta)}{k}\,t_0,\ \frac{(k+1+\alpha)(k+1+\beta)}{k+1}\,t_+\Big)$$

has property K at k.

Proof. Let $\theta(y) = (y+\alpha)(y+\beta)/y$ for $y > 0$, and put $\theta_- = \theta(k-1)$, $\theta_0 = \theta(k)$ and $\theta_+ = \theta(k+1)$. These numbers are positive, and the triple of the lemma is $(\theta_- t_-, \theta_0 t_0, \theta_+ t_+)$. By what was shown in the proof of Lemma 18.2 for any positive numbers θ_-, θ_0, θ_+, it suffices to verify the first condition of (18.4) and the two conditions of (18.5).

Since $\theta(y) = y + \alpha + \beta + \alpha\beta/y$, we have

$$\theta(k+1) - \theta(k-1) = 2 - \frac{2\,\alpha\beta}{k^2-1} \geq 0,$$

because $\alpha\beta \leq k^2 - 1$; this is the first condition. In the same way $\theta(k+1) - \theta(k) = 1 - \alpha\beta/(k(k+1))$ and $\theta(k) - \theta(k-1) = 1 - \alpha\beta/(k(k-1))$, and dividing by $\theta(k)$ and by $\theta(k-1)$ we obtain

$$\sigma_+ = \frac{k(k+1) - \alpha\beta}{(k+1)(k+\alpha)(k+\beta)}, \qquad \sigma_- = \frac{k(k-1) - \alpha\beta}{k\,(k-1+\alpha)(k-1+\beta)}.$$

Here $\sigma_+ \geq 0$, because $\alpha\beta \leq k^2 - 1 \leq k(k+1)$; the number σ_- may be negative. With the denominators cleared, the second inequality of (18.5) reads

$$\begin{aligned}(k-1)\big(k(k-1) - \alpha\beta\big)\big[2(k+1)(k+\alpha)(k+\beta) + k(k+1) - \alpha\beta\big] \\ \leq 2k\,(k+1)\big(k(k+1) - \alpha\beta\big)(k-1+\alpha)(k-1+\beta).\end{aligned}$$

To compare the two sides we write them in terms of $s = \alpha + \beta$ and $p = \alpha\beta$. We have

$$\begin{aligned}(k+\alpha)(k+\beta) &= k^2 + ks + p, \\ (k-1+\alpha)(k-1+\beta) &= (k-1)^2 + (k-1)\,s + p.\end{aligned}$$

The left-hand side of the inequality is therefore $k-1$ times

$$\begin{aligned}&\big(k(k-1) - p\big)\big[2(k+1)(k^2+ks+p) + k(k+1) - p\big] \\ &\quad = \big(k(k-1) - p\big)\big[k(k+1)(2k+1) + 2k(k+1)\,s + (2k+1)\,p\big] \\ &\quad = k^2(k^2-1)(2k+1) + 2k^2(k^2-1)\,s - 2k(k+1)\,sp \\ &\qquad - 2k(2k+1)\,p - (2k+1)\,p^2,\end{aligned}$$

and the right-hand side is $2k(k+1)$ times

$$\begin{aligned}&\big(k(k+1) - p\big)\big[(k-1)^2 + (k-1)\,s + p\big] \\ &\quad = k(k+1)(k-1)^2 + k(k^2-1)\,s - (k-1)\,sp + (3k-1)\,p - p^2.\end{aligned}$$

In the first expansion the coefficient of p is $2k+1$ times $k(k-1) - k(k+1) = -2k$, and in the second it is $k(k+1) - (k-1)^2 = 3k - 1$. The terms with sp are $-2k(k-1)(k+1)\,sp$ on both sides of the inequality, so that they cancel in the difference of the two sides. For the monomials 1, s, p and p^2, in this order, the coefficient on the right-hand side minus the coefficient on the left-hand side is

$$\begin{aligned}2k^2(k+1)^2(k-1)^2 - k^2(k-1)^2(k+1)(2k+1) &= k^2(k-1)^2(k+1), \\ 2k^2(k+1)(k^2-1) - 2k^2(k-1)(k^2-1) &= 4k^2(k^2-1), \\ 2k(k+1)(3k-1) + 2k(k-1)(2k+1) &= 10k^3 + 2k^2 - 4k, \\ -2k(k+1) + (k-1)(2k+1) &= -(3k+1).\end{aligned}$$

With $s = \alpha + \beta$ and $p = \alpha\beta$ inserted again, the right-hand side minus the left-hand side is therefore

$$k^2(k-1)^2(k+1) + 4k^2(k^2-1)(\alpha+\beta) \\ + \alpha\beta\,\big(10k^3 + 2k^2 - 4k - (3k+1)\,\alpha\beta\big).$$

The first two terms are nonnegative. So is the third, because $\alpha\beta \geq 0$ and $\alpha\beta \leq k^2-1$ gives

$$10k^3 + 2k^2 - 4k - (3k+1)\,\alpha\beta \geq 10k^3 + 2k^2 - 4k - (3k+1)(k^2-1) \\ = 7k^3 + k^2 - k + 1 > 0. \qquad \square$$

18.3. **The tilted Cayley masses.** In this subsection we prove that three consecutive tilted Cayley masses with indices up to $k+1$ satisfy the inequality of property K at k. For the last triple, $(\omega_{k-1}, \omega_k, \omega_{k+1})$, this is the base of the induction on the offset. For the other triples it is the hypothesis under which Section 18.4 treats the sums of products of two masses that occur in the direct term.

Everything is read off from the ratio of consecutive masses. For a real number $y > 0$ let

$$\varphi(y) = (y-1)\log\Big(1 + \frac{1}{y}\Big).$$

For an integer $n \geq 1$ we have $e^{\varphi(n)} = (1 + 1/n)^{n-1}$. Hence $x_n = e^{-\varphi(n)}$, and (2.1), taken at $j = n+1$ and written for the numbers $c_n = a_{n+1}$, says that $c_{n+1} = e^{\varphi(n+1)}\,c_n$ for every integer $n \geq 0$. Multiplying by x_k^{n+1} and using $x_k = e^{-\varphi(k)}$, we obtain for every column $k \geq 1$

$$\omega_{n+1} = \omega_n\, e^{\varphi(n+1) - \varphi(k)} \qquad (n \geq 0). \tag{18.6}$$

By Lemma 2.1 the function φ does not decrease on the positive integers. So the ratio ω_{n+1}/ω_n does not decrease in n, and it equals 1 for $n + 1 = k$:

$$\omega_k = \omega_{k-1} \qquad (k \geq 1). \tag{18.7}$$

The sequence ω is also log-convex, in the following form: for all integers $\rho \geq 0$, $m \geq 0$ and i with $0 \leq i \leq m$,

$$\omega_{\rho+i}\,\omega_{\rho+m-i} \leq \omega_\rho\,\omega_{\rho+m}. \tag{18.8}$$

Indeed, since $c_n = a_{n+1}$, Lemma 2.2, applied to indices $\rho + 1 \leq \rho + m - i + 1$ with the shift i, gives $c_{\rho+i}\,c_{\rho+m-i} \leq c_\rho\,c_{\rho+m}$, and the two sides of (18.8) carry the same power $x_k^{2\rho+m}$ of the tilt. The inequality is used in the proof of Lemma 18.9.

By (18.6), the inequality of property K for three consecutive masses, divided by the middle one, is an inequality between exponentials of increments of φ. The proofs below bound these increments with the help of the formula

$$\log\Big(1 + \frac{1}{y}\Big) = \int_0^1 \frac{dt}{y+t}, \tag{18.9}$$

which holds for every real number $y > 0$.

They also use the inequality between the arithmetic and the geometric mean, with the weights that the exponent μ/k of property K dictates. For a real number $k > 1$ let

$$\chi_k = (k+1)\log\frac{k+1}{k} + (k-1)\log\frac{k-1}{k};$$

this is the second difference of the function $t \mapsto t \log t$ at the points $k-1$, k and $k+1$.

We now prove the inequality of property K for the triples of masses other than the last one.

Lemma 18.4. *Let $k \geq 3$ be an integer, and let ω_n be the tilted Cayley masses at the critical tilt x_k of column k. Then for every integer $j \geq 0$ with $j+2 \leq k$ and every real number μ,*

$$2\, e^{\mu/k}\, \omega_{j+1} \leq e^{\mu}\, \omega_{j+2} + e^{-\mu}\, \omega_j.$$

Proof. For all real numbers L, M and μ,

$$e^{\mu+L} + e^{-\mu+M} \geq 2\, e^{\mu/k} \exp\Big(\frac{(k+1)L + (k-1)M - \chi_k}{2k} \Big). \tag{18.10}$$

To prove this, we write

$$e^{\mu+L} = \frac{k+1}{k}\, e^{p_+}, \qquad e^{-\mu+M} = \frac{k-1}{k}\, e^{p_-},$$

with $p_+ = \mu + L - \log \frac{k+1}{k}$ and $p_- = -\mu + M - \log \frac{k-1}{k}$. The two coefficients are positive and add up to 2. Let

$$\bar{p} = \frac{k+1}{2k}\, p_+ + \frac{k-1}{2k}\, p_- = \frac{\mu}{k} + \frac{(k+1)L + (k-1)M - \chi_k}{2k},$$

so that the right-hand side of (18.10) is $2e^{\bar{p}}$. As $\bar{p}$ is the mean of p_+ and p_- with these coefficients as weights, Jensen's inequality for the exponential function, in the form of Section 2, shows that the left-hand side is at least $2e^{\bar{p}}$.

Now let $n = j+1$, so that $1 \leq n \leq k-1$, and put $L = \varphi(n+1) - \varphi(k)$ and $M = \varphi(k) - \varphi(n)$. By (18.6) we have $\omega_{j+2} = \omega_{j+1}\, e^L$ and $\omega_j = \omega_{j+1}\, e^M$. Since $\omega_{j+1} > 0$, the claim is that $e^{\mu+L} + e^{-\mu+M} \geq 2e^{\mu/k}$, and by (18.10) it suffices to show that $(k+1)L + (k-1)M \geq \chi_k$. For $1 \leq m \leq k-1$ let

$$h_m = (k+1)\, \varphi(m+1) - (k-1)\, \varphi(m).$$

Then $(k+1)L + (k-1)M = h_n - 2\, \varphi(k)$, and what has to be shown is that

$$h_n - 2\, \varphi(k) \geq \chi_k. \tag{18.11}$$

For every real number $y \geq 2$,

$$\varphi(y) - \varphi(y-1) \geq \frac{3}{2\, y\, (y+1)}, \tag{18.12}$$

$$(y+2)\big(\varphi(y+1) - \varphi(y)\big) \leq y\, \big(\varphi(y) - \varphi(y-1)\big). \tag{18.13}$$

For (18.12) we write the left-hand side as $(y-1)\log(1+1/y) - (y-2)\log(1+1/(y-1))$. By (18.9) it is the integral over $[0,1]$ of

$$\frac{y-1}{y+t} - \frac{y-2}{y-1+t} = \frac{1+t}{(y+t)(y-1+t)} \geq \frac{1+t}{y\,(y+1)},$$

where the inequality holds because $0 < (y+t)(y-1+t) \leq (y+1)\, y$ for $0 \leq t \leq 1$; and the integral of $1+t$ over $[0,1]$ is $3/2$. For (18.13), the right-hand side minus

the left-hand side is $(2y+2)\,\varphi(y) - y\,\varphi(y-1) - (y+2)\,\varphi(y+1)$. By the definition of φ and by (18.9) this is the integral over $[0,1]$ of

$$\frac{2\,(y^2-1)}{y+t} - \frac{y\,(y-2)}{y-1+t} - \frac{y\,(y+2)}{y+1+t} = \frac{2\,(1-t^2)}{(y-1+t)(y+t)(y+1+t)},$$

which is nonnegative for $0 \le t \le 1$.

We now show that $h_{m+1} \le h_m$ for $1 \le m \le k-2$. By (18.13) with $y = m+1$,

$$(m+3)\big(\varphi(m+2) - \varphi(m+1)\big) \le (m+1)\big(\varphi(m+1) - \varphi(m)\big).$$

Moreover $\varphi(m+1) - \varphi(m) \ge 0$ by (18.12), and $(k+1)(m+1) \le (k-1)(m+3)$, because $m \le k-2$. Hence

$$(k+1)\big(\varphi(m+2) - \varphi(m+1)\big) \le (k-1)\big(\varphi(m+1) - \varphi(m)\big),$$

which is the inequality $h_{m+1} \le h_m$. Consequently $h_n \ge h_{k-1}$.

It remains to compare $h_{k-1} - 2\,\varphi(k)$ with χ_k. We have $h_{k-1} - 2\,\varphi(k) = (k-1)\big(\varphi(k) - \varphi(k-1)\big)$ and $\chi_k = (k+1)\log(1+1/k) - (k-1)\log(1+1/(k-1))$, and inserting the definition of φ we find

$$h_{k-1} - 2\,\varphi(k) - \chi_k = (k-3)\Big(k\log\Big(1+\frac{1}{k}\Big) - (k-1)\log\Big(1+\frac{1}{k-1}\Big)\Big).$$

This is nonnegative, because $k \ge 3$ and because $(1+1/i)^i$ does not decrease in $i \ge 1$, as is recalled in Section 2. This proves (18.11). □

For the last triple, $(\omega_{k-1}, \omega_k, \omega_{k+1})$, the next lemma bounds from below the amount by which the inequality of property K holds; the bound is needed in the proof of Lemma 18.9.

Lemma 18.5. *Let $k \ge 3$ be an integer, and let ω_n be the tilted Cayley masses at the critical tilt x_k of column k. Then for every real number μ,*

$$\frac{k^2-2k-1}{2k^2(k+1)(k+2)}\; e^{\mu/k}\,\omega_k \le e^{\mu}\,\omega_{k+1} + e^{-\mu}\,\omega_{k-1} - 2\,e^{\mu/k}\,\omega_k.$$

Proof. Let $L = \varphi(k+1) - \varphi(k)$. By (18.6) and (18.7) we have $\omega_{k+1} = \omega_k\,e^L$ and $\omega_{k-1} = \omega_k$, so that

$$e^{\mu}\,\omega_{k+1} + e^{-\mu}\,\omega_{k-1} - 2\,e^{\mu/k}\,\omega_k = \omega_k\,\big(e^{\mu+L} + e^{-\mu} - 2\,e^{\mu/k}\big). \tag{18.14}$$

The inequality (18.10), which was shown in the proof of Lemma 18.4, with $M = 0$ gives

$$e^{\mu+L} + e^{-\mu} \ge 2\,e^{\mu/k}\exp\Big(\frac{(k+1)L - \chi_k}{2k}\Big),$$

and we bound the number $(k+1)L - \chi_k$ from below. By the definitions of φ and of χ_k,

$$(k+1)L - \chi_k = (k-1)\log\Big(1+\frac{1}{k-1}\Big) - k\,(k+1)\log\Big(1+\frac{1}{k}\Big) + k\,(k+1)\log\Big(1+\frac{1}{k+1}\Big).$$

By (18.9) this is the integral over $[0,1]$ of

$$\frac{k-1}{k-1+t} - \frac{k\,(k+1)}{k+t} + \frac{k\,(k+1)}{k+1+t} = \frac{t\,\big(k^2-2k-1+(k-1)\,t\big)}{(k-1+t)(k+t)(k+1+t)}.$$

For $0 \le t \le 1$ the numerator is at least $(k^2-2k-1)\,t$, where $k^2-2k-1 > 0$ because $k \ge 3$, and the denominator is positive and at most $k\,(k+1)(k+2)$. Since the integral of t over $[0,1]$ is $1/2$,

$$(k+1)L - \chi_k \ge \frac{k^2-2k-1}{2k\,(k+1)(k+2)}.$$

With $e^b \ge 1+b$ for $b = ((k+1)L-\chi_k)/(2k)$ we conclude that

$$\begin{aligned} e^{\mu+L} + e^{-\mu} - 2\,e^{\mu/k} &\ge e^{\mu/k}\,\frac{(k+1)L-\chi_k}{k} \\ &\ge \frac{k^2-2k-1}{2k^2(k+1)(k+2)}\;e^{\mu/k}, \end{aligned}$$

and the lemma follows from (18.14), because $\omega_k > 0$. □

For the base of the induction on the offset only the sign of this amount is needed.

Corollary 18.6. *Let $k \ge 3$ be an integer, and let ω_n be the tilted Cayley masses at the critical tilt x_k of column k. Then the triple $(\omega_{k-1}, \omega_k, \omega_{k+1})$ has property K at k.*

Proof. This is immediate from Lemma 18.5, since the left-hand side of its inequality is nonnegative for $k \ge 3$. □

18.4. **Self-convolutions and the pair sums of the masses.** The direct term $S_\rho(y)$ of Section 18.1 contains the number $Q_\rho(y-\rho-1)$, one of the numbers $Q_\rho(j)$ that (18.2) defines for integers $\rho \ge 0$ and $j \ge 0$. We call $Q_\rho(j)$ a *pair sum of the masses*, of length j. In this section a pair sum is always one of the numbers $Q_\rho(j)$, and not the pair sum $\mathcal{P}(k,R)$ of Definition 13.8. When y runs through $k-1$, k, $k+1$ and ρ is fixed, the length $j = y-\rho-1$ runs through three consecutive values n, $n+1$, $n+2$ with $\rho+n+2 = k$, and the largest index of a mass that occurs in these three pair sums of the masses is k. The aim of this subsection is to show that the three pair sums of the masses form a triple with property K at k (Corollary 18.8).

The pair sum of the masses $Q_\rho(j)$ is a self-convolution. For a sequence $(v_i)_{i\ge0}$ of real numbers and an integer $j \ge 0$, the *self-convolution* of v at length j is the sum of the products $v_i\,v_{j-i}$ over $0 \le i \le j$, and $Q_\rho(j)$ is the self-convolution at length j of the shifted sequence $v_i = \omega_{\rho+i}$, whose last two terms that occur are $v_{n+1} = \omega_{k-1}$ and $v_{n+2} = \omega_k$. The argument uses of this sequence only what

Section 18.3 has provided, and the next lemma is stated for an arbitrary sequence with these properties.

The proof of the lemma sorts the combination of the three self-convolutions that property K asks about by one of the two indices of the products: the inner terms are then nonnegative, because three consecutive terms of v satisfy the inequality of property K at k, but end terms remain, which may be negative. For $\cosh\mu \geq e^{\mu/k}$ the proof replaces each value of v in the inner terms by the smallest one, v_{n+1}; for the remaining parameters it compares a positive multiple of the sequence $(e^{\mu i}\, v_i)$ with the solution of the three-term recurrence $\psi(t+1)+\psi(t-1) = 2\, e^{\mu/k}\, \psi(t)$ that agrees with this multiple at indices $n+1$ and $n+2$. This solution cancels the end terms, and what is left is an explicit expression in the two values v_{n+1} and v_{n+2}, whose sign is decided by the hypothesis $n+2 \leq k$.

Lemma 18.7. *Let k and n be integers with $n \geq 0$ and $n+2 \leq k$, and let $(v_i)_{i\geq 0}$ be a sequence of real numbers with the following four properties.*

(1) *One has $v_i > 0$ for every i with $0 \leq i \leq n+2$.*
(2) *One has $v_{n+1} \leq v_i$ for every i with $0 \leq i \leq n$.*
(3) *One has $v_{n+2} = v_{n+1}$.*
(4) *For every real number $\mu \geq 0$ and every j with $0 \leq j \leq n$,*

$$2\, e^{\mu/k}\, v_{j+1} \leq e^{\mu}\, v_{j+2} + e^{-\mu}\, v_j.$$

Then the self-convolutions

$$\sum_{i=0}^{j} v_i\, v_{j-i} \qquad (j = n,\ n+1,\ n+2)$$

of v form, in this order, a triple with property K at k.

Proof. Fix a real number $\mu \geq 0$, and for an integer $j \geq 0$ let Σ_j denote the self-convolution of v at length j. We have to show that the number

$$I = e^{\mu}\, \Sigma_{n+2} + e^{-\mu}\, \Sigma_n - 2\, e^{\mu/k}\, \Sigma_{n+1}$$

is nonnegative. In the three sums we collect the terms according to the factor v_{n-j} with $0 \leq j \leq n$:

$$\begin{aligned} \Sigma_n &= \sum_{j=0}^{n} v_{n-j}\, v_j, \\ \Sigma_{n+1} &= \sum_{j=0}^{n} v_{n-j}\, v_{j+1} + v_{n+1}\, v_0, \\ \Sigma_{n+2} &= \sum_{j=0}^{n} v_{n-j}\, v_{j+2} + v_{n+1}\, v_1 + v_{n+2}\, v_0. \end{aligned} \tag{18.15}$$

These identities hold for every sequence of real numbers. We distinguish two cases, according to whether $\cosh\mu \geq e^{\mu/k}$ or $\cosh\mu < e^{\mu/k}$.

Suppose first that $\cosh\mu \geq e^{\mu/k}$. By (18.15),

$$\begin{aligned} I = \sum_{j=0}^{n} v_{n-j} \left(e^{\mu}\, v_{j+2} + e^{-\mu}\, v_j - 2\, e^{\mu/k}\, v_{j+1}\right) \\ + v_{n+1} \left(e^{\mu}\, v_1 - 2\, e^{\mu/k}\, v_0\right) + e^{\mu}\, v_{n+2}\, v_0. \end{aligned}$$

By hypothesis (4) each expression in parentheses under the sum is nonnegative, and by hypothesis (2) we have $v_{n-j} \geq v_{n+1}$. So the sum does not increase when each factor v_{n-j} is replaced by v_{n+1}. Let $\sigma = v_0 + v_1 + \cdots + v_{n+1}$. Then

$$\sum_{j=0}^{n} v_j = \sigma - v_{n+1}, \qquad \sum_{j=0}^{n} v_{j+1} = \sigma - v_0,$$

$$\sum_{j=0}^{n} v_{j+2} = \sigma - v_0 - v_1 + v_{n+2}.$$

Inserting these three sums and using $v_{n+2} = v_{n+1}$, which is hypothesis (3), we obtain after cancellation

$$I \geq 2\, v_{n+1} \Big(\big(\cosh \mu - e^{\mu/k}\big)\, \sigma + \sinh \mu \cdot v_{n+1} \Big).$$

The right-hand side is nonnegative, because v_{n+1} and σ are positive by hypothesis (1) and $\cosh \mu \geq e^{\mu/k}$ in the present case.

Suppose now that $\cosh \mu < e^{\mu/k}$. Then $\mu > 0$, since both sides are equal to 1 for $\mu = 0$. As $e^{\mu/k} > 1$, there is a real number $\tau > 0$ with $\cosh \tau = e^{\mu/k}$, and $\cosh \mu < \cosh \tau$ gives $\mu < \tau$. We pass to the sequence $\tilde{v}_i = e^{\mu i}\, v_i$ and write $\widetilde{\Sigma}_j$ for its self-convolution at length j, so that $\widetilde{\Sigma}_j = e^{\mu j}\, \Sigma_j$. Hypothesis (4) for the parameter μ, multiplied by $e^{\mu(j+1)}$, says that the numbers

$$w_j = \tilde{v}_{j+2} + \tilde{v}_j - 2 \cosh \tau \cdot \tilde{v}_{j+1} \qquad (0 \leq j \leq n)$$

are nonnegative, and the number

$$J = \widetilde{\Sigma}_{n+2} + \widetilde{\Sigma}_n - 2 \cosh \tau \cdot \widetilde{\Sigma}_{n+1}$$

is $e^{\mu(n+1)}\, I$, so that we have to show that $J \geq 0$. By (18.15) for the sequence $\tilde{v}$,

$$\begin{aligned} J = & \sum_{j=0}^{n} \tilde{v}_{n-j}\, w_j \\ & + \tilde{v}_{n+1} \big(\tilde{v}_1 - 2 \cosh \tau \cdot \tilde{v}_0 \big) + \tilde{v}_{n+2}\, \tilde{v}_0. \end{aligned} \tag{18.16}$$

For a real number t let

$$\psi(t) = \sinh\big((n+2-t)\,\tau\big)\, \tilde{v}_{n+1} - \sinh\big((n+1-t)\,\tau\big)\, \tilde{v}_{n+2}.$$

By the addition formula $\sinh(y+\tau) + \sinh(y-\tau) = 2 \cosh \tau \cdot \sinh y$,

$$\psi(t+1) + \psi(t-1) = 2 \cosh \tau \cdot \psi(t) \tag{18.17}$$

for every real number t, and from the definition, $\psi(n+1) = \sinh \tau \cdot \tilde{v}_{n+1}$ and $\psi(n+2) = \sinh \tau \cdot \tilde{v}_{n+2}$. For $0 \leq j \leq n+2$ let

$$r_j = \sinh \tau \cdot \tilde{v}_j - \psi(j).$$

Then $r_{n+1} = r_{n+2} = 0$, and by (18.17),

$$r_{j+2} + r_j - 2 \cosh \tau \cdot r_{j+1} = \sinh \tau \cdot w_j \geq 0 \qquad (0 \leq j \leq n).$$

We claim that $0 \leq r_{j+1} \leq r_j$ for $0 \leq j \leq n+1$, and prove this by descending induction on j. For $j = n+1$ both numbers are 0. If $j \leq n$ and $0 \leq r_{j+2} \leq r_{j+1}$, then

$$r_j \geq 2 \cosh \tau \cdot r_{j+1} - r_{j+2} \geq 2\, r_{j+1} - r_{j+2} \geq r_{j+1},$$

where we used $\cosh \tau \geq 1$ and $r_{j+1} \geq 0$ in the second step and $r_{j+2} \leq r_{j+1}$ in the third. In particular $r_j \geq 0$ for $0 \leq j \leq n$.

We multiply (18.16) by $\sinh\tau > 0$ and write $\sinh\tau\cdot\tilde v_{n-j} = r_{n-j} + \psi(n-j)$ in the sum. The products $r_{n-j}\,w_j$ are nonnegative. For the other part let

$$Y_j = \psi(n+1-j)\,\tilde v_{j+1} - \psi(n-j)\,\tilde v_j \qquad (0\le j\le n+1).$$

By (18.17) at $t = n-j$, we have for $0\le j\le n$

$$\begin{aligned} Y_{j+1} - Y_j &= \psi(n-j)\,\big(\tilde v_{j+2} + \tilde v_j\big) \\ &\qquad - \big(\psi(n-j-1) + \psi(n-j+1)\big)\,\tilde v_{j+1} \\ &= \psi(n-j)\,w_j, \end{aligned}$$

so that the sum of $\psi(n-j)\,w_j$ over $0\le j\le n$ is $Y_{n+1} - Y_0$. Now (18.17) at $t = n+1$ gives

$$\psi(n) = 2\cosh\tau\cdot\psi(n+1) - \psi(n+2) = \sinh\tau\,\big(2\cosh\tau\cdot\tilde v_{n+1} - \tilde v_{n+2}\big),$$

and hence

$$\begin{aligned} Y_0 &= \psi(n+1)\,\tilde v_1 - \psi(n)\,\tilde v_0 \\ &= \sinh\tau\,\Big(\tilde v_{n+1}\,\big(\tilde v_1 - 2\cosh\tau\cdot\tilde v_0\big) + \tilde v_{n+2}\,\tilde v_0\Big), \end{aligned}$$

which is $\sinh\tau$ times the two end terms of (18.16). These terms therefore cancel against $-Y_0$, and

$$\sinh\tau\cdot J = \sum_{j=0}^{n} r_{n-j}\,w_j + Y_{n+1} \ge Y_{n+1}.$$

It remains to show that $Y_{n+1}\ge 0$. Reading $\psi(0)$ and $\psi(-1)$ from the definition of ψ, we find

$$\begin{aligned} Y_{n+1} &= \psi(0)\,\tilde v_{n+2} - \psi(-1)\,\tilde v_{n+1} \\ &= 2\,\tilde v_{n+1}\,\tilde v_{n+2}\,\sinh\big((n+2)\,\tau\big) - \tilde v_{n+2}^2\,\sinh\big((n+1)\,\tau\big) \\ &\quad - \tilde v_{n+1}^2\,\sinh\big((n+3)\,\tau\big). \end{aligned}$$

By hypothesis (3) we have $\tilde v_{n+2} = e^{\mu}\,\tilde v_{n+1}$. With $m = n+1$ this gives $Y_{n+1} = \tilde v_{n+1}^2\,A$, where

$$A = 2\,e^{\mu}\sinh\big((m+1)\,\tau\big) - e^{2\mu}\sinh(m\tau) - \sinh\big((m+2)\,\tau\big).$$

Put $h = m\tau/2$. Then

$$A = 2\,\big(\sinh(h+\tau) - e^{\mu}\sinh h\big)\,\big(e^{\mu}\cosh h - \cosh(h+\tau)\big);$$

indeed, by the addition formula for sinh and the formula $2\sinh y\cosh y = \sinh(2y)$ the product on the right is $2\,e^{\mu}\sinh(2h+\tau) - \sinh(2h+2\tau) - e^{2\mu}\sinh(2h)$. Both factors are nonnegative. For the first, we have $e^{\mu}\le e^{\tau} = \cosh\tau + \sinh\tau$, because $\mu\le\tau$, and $0\le\sinh h\le\cosh h$, so that

$$e^{\mu}\sinh h \le \sinh h\cosh\tau + \cosh h\sinh\tau = \sinh(h+\tau).$$

For the second,

$$\begin{aligned}\cosh(h+\tau) &= \cosh h\cosh\tau + \sinh h\sinh\tau \\ &\le \cosh h\cosh\tau\left(1+\frac{m\,\tau^2}{2}\right) \\ &\le \cosh h\cosh\tau\left(1+\frac{\tau^2}{2}\right)^m \le \cosh h\,(\cosh\tau)^{m+1} \\ &\le \cosh h\,(\cosh\tau)^k = e^{\mu}\cosh h.\end{aligned}$$

Here the first inequality uses the elementary inequality $\sinh y \le y\cosh y$ for $y \ge 0$, applied to $y = h$ and to $y = \tau$, together with $h\,\tau = m\,\tau^2/2$. The second is Bernoulli's inequality, and the third is $1+\tau^2/2 \le \cosh\tau$. The fourth holds because $m+1 = n+2 \le k$ and $\cosh\tau \ge 1$, and the last equality is the choice of τ. Hence $A \ge 0$ and $Y_{n+1} \ge 0$, which completes the proof. □

Applied to the shifted sequence of the masses, the lemma gives the result on the pair sums of the masses.

Corollary 18.8. *Let $k \ge 3$, $\rho \ge 0$ and $n \ge 0$ be integers with $\rho + n + 2 = k$. Then the pair sums $Q_\rho(n)$, $Q_\rho(n+1)$ and $Q_\rho(n+2)$ of the tilted Cayley masses at the critical tilt x_k of column k form, in this order, a triple with property K at k.*

Proof. We apply Lemma 18.7 to the integers k and n and to the sequence $v_i = \omega_{\rho+i}$, whose self-convolution at length j is $Q_\rho(j)$. The masses are positive, which is hypothesis (1). Hypothesis (2) follows from Lemma 5.6, since the indices $\rho+i$ with $0 \le i \le n+1$ are at most $k-1$, and hypothesis (3) is (18.7). Finally, for $0 \le j \le n$ the inequality of (4) is that of Lemma 18.4 at index $\rho + j$, which is admissible because $\rho + j + 2 \le k$. □

18.5. **The direct term.** Recall that for integers ρ and y with $1 \le \rho \le y-1$ the direct term is

$$S_\rho(y) = x_k\,\omega_{\rho-1}\,\omega_y + \frac{x_k}{2}\cdot\frac{y+\rho+1}{y+\rho}\,Q_\rho(y-\rho-1).$$

In the induction on the offset it is the part of the recursion that the induction hypothesis does not cover, and its property K has to be proved directly.

Both ingredients of $S_\rho(y)$ have property K by themselves when the threshold runs through $k-1$, k, $k+1$: the masses ω_y by Corollary 18.6, and the pair sums of the masses $Q_\rho(y-\rho-1)$ by Corollary 18.8. But the pair sum of the masses is multiplied by the factor $\theta(y) = (y+\rho+1)/(y+\rho)$, which decreases as the threshold grows, whereas the stability statements of Section 18.2 require factors that do not decrease. A decreasing factor is the case $L < M$ of (18.3), which is available only for parameters $\mu \ge (M-L)/2$. The proof of the next lemma therefore compares two amounts: the amount by which the triple $(\omega_{k-1}, \omega_k, \omega_{k+1})$ exceeds the inequality of property K, which it bounds from below, and the amount by which the triple of the terms $\theta(y)\,Q_\rho(y-\rho-1)$ may fail it, which it bounds from above. It divides the parameters at $\mu = 1/(k(k+2))$: above this value the downward shift caused by the factor stays within the nonnegative parameters, so that (18.3) applies, and below it the second amount is bounded through the two end terms of the longest of the three pair sums of the masses. In both ranges the two bounds combine to a nonnegative multiple of $\omega_{\rho-1} - \omega_\rho$.

Lemma 18.9. *Let $k \geq 7$ and ρ be integers with $1 \leq \rho \leq k-2$. Then the direct terms $S_\rho(k-1)$, $S_\rho(k)$ and $S_\rho(k+1)$ at the critical tilt x_k of column k form, in this order, a triple with property K at k.*

Proof. Fix a real number $\mu \geq 0$, and put $N = k + \rho$, so that $k+1 \leq N \leq 2k-2$. We write

$$\theta_- = \frac{N}{N-1}, \qquad \theta_0 = \frac{N+1}{N}, \qquad \theta_+ = \frac{N+2}{N+1}$$

for the values of the factor $\theta(y) = (y+\rho+1)/(y+\rho)$ at $y = k-1, k, k+1$, and

$$Q^- = Q_\rho(k-\rho-2), \qquad Q^0 = Q_\rho(k-\rho-1), \qquad Q^+ = Q_\rho(k-\rho)$$

for the three pair sums of the masses that occur in the direct terms. These are positive, as sums of products of positive masses, and by Corollary 18.8, applied with $n = k-\rho-2$, the triple (Q^-, Q^0, Q^+) has property K at k. Inserting the definition of the direct terms $S_\rho(y)$ at $y = k-1$, k and $k+1$, we find

$$\begin{aligned} e^\mu\, S_\rho(k+1) + e^{-\mu}\, S_\rho(k-1) - 2\, e^{\mu/k}\, S_\rho(k) \\ = x_k\, \omega_{\rho-1}\, \mathrm{Sur} + \frac{x_k}{2}\, \mathrm{Def}, \end{aligned} \tag{18.18}$$

where

$$\begin{aligned} \mathrm{Sur} &= e^\mu\, \omega_{k+1} + e^{-\mu}\, \omega_{k-1} - 2\, e^{\mu/k}\, \omega_k, \\ \mathrm{Def} &= e^\mu\, \theta_+\, Q^+ + e^{-\mu}\, \theta_-\, Q^- - 2\, e^{\mu/k}\, \theta_0\, Q^0. \end{aligned}$$

The number Sur is the amount by which the triple $(\omega_{k-1}, \omega_k, \omega_{k+1})$ exceeds the inequality of property K at the parameter μ, and Def, which may be negative, is the corresponding amount for the pair sums of the masses with their factors. The left-hand side of (18.18) is this amount for the triple of the lemma, and we have to show that it is nonnegative. We show that there is a real number $\lambda \geq 0$ such that

$$\mathrm{Sur} \geq \lambda\, \omega_k \qquad \text{and} \qquad \mathrm{Def} \geq -2\, \lambda\, \omega_\rho\, \omega_k. \tag{18.19}$$

This proves the lemma. Indeed, by (18.19) the right-hand side of (18.18) is at least $x_k\, \lambda\, \omega_k\, (\omega_{\rho-1} - \omega_\rho)$, and this is nonnegative, because $\omega_\rho \leq \omega_{\rho-1}$ by Lemma 5.6, as $\rho \leq k$. We distinguish two cases, according to whether $\mu \leq 1/(k(k+2))$ or $\mu \geq 1/(k(k+2))$.

Suppose first that $\mu \leq 1/(k(k+2))$. We take $\lambda = e^\mu/(N(N+1))$. The differences of the factors are

$$\theta_0 - \theta_+ = \frac{1}{N(N+1)}, \qquad \theta_- - \theta_0 = \frac{1}{N(N-1)} \geq \frac{1}{N(N+1)},$$

and we write

$$\begin{aligned} \mathrm{Def} = \theta_0 \left(e^\mu\, Q^+ + e^{-\mu}\, Q^- - 2\, e^{\mu/k}\, Q^0\right) \\ - (\theta_0 - \theta_+)\, e^\mu\, Q^+ + (\theta_- - \theta_0)\, e^{-\mu}\, Q^-. \end{aligned}$$

The first term is nonnegative by property K of the pair sums of the masses, and in the last term $\theta_- - \theta_0$ can be replaced by the smaller number $\theta_0 - \theta_+$. Hence

$$\mathrm{Def} \geq -\frac{1}{N(N+1)} \left(e^\mu\, Q^+ - e^{-\mu}\, Q^-\right). \tag{18.20}$$

We bound the difference in parentheses by the two end terms of Q^+. Let $n = k-\rho-2$, so that $Q^- = Q_\rho(n)$ and $Q^+ = Q_\rho(n+2)$. The sum Q^+ consists of the products $\omega_{\rho+i}\, \omega_{\rho+n+2-i}$ with $0 \leq i \leq n+2$. Its terms $i = 0$ and $i = n+2$ are

both equal to $\omega_\rho\,\omega_k$, because $\rho + n + 2 = k$, and its other terms are the products $\omega_{\rho+i+1}\,\omega_{\rho+n-i+1}$ with $0 \le i \le n$. Since $\mu \le 1/(k(k+2))$, we have for every integer $j \ge 0$ with $j + 2 \le k$

$$e^{\mu}\,\omega_{j+1} \le \omega_j. \tag{18.21}$$

Indeed, by (18.6) we have $\omega_j = \omega_{j+1}\,e^{\varphi(k)-\varphi(j+1)}$, so that it suffices to show that $\mu \le \varphi(k) - \varphi(j+1)$. Since $j + 1 \le k - 1$ and φ does not decrease on the positive integers, $\varphi(k) - \varphi(j+1) \ge \varphi(k) - \varphi(k-1)$. By (18.12), which was shown in the proof of Lemma 18.4, with $y = k$ this is at least $3/(2k(k+1)) \ge 1/(k(k+2))$. For such i the indices $\rho + i$ and $\rho + n - i$ are at most $k - 2$, so that (18.21) applies to both of them: $e^{\mu}\,\omega_{\rho+i+1} \le \omega_{\rho+i}$ and $e^{\mu}\,\omega_{\rho+n-i+1} \le \omega_{\rho+n-i}$. Multiplying these two inequalities and then multiplying by $e^{-\mu}$, we get

$$e^{\mu}\,\omega_{\rho+i+1}\,\omega_{\rho+n-i+1} \le e^{-\mu}\,\omega_{\rho+i}\,\omega_{\rho+n-i}.$$

The right-hand side is $e^{-\mu}$ times the term of index i of Q^-. Summing over $0 \le i \le n$, we obtain $e^{\mu}\,(Q^+ - 2\,\omega_\rho\,\omega_k) \le e^{-\mu}\,Q^-$, and hence

$$e^{\mu}\,Q^+ - e^{-\mu}\,Q^- \le 2\,e^{\mu}\,\omega_\rho\,\omega_k.$$

With (18.20) this is the second inequality of (18.19).

For the first inequality, (18.6) and (18.7) give $\omega_{k+1} = \omega_k\,e^{\varphi(k+1)-\varphi(k)}$ and $\omega_{k-1} = \omega_k$, so that

$$\mathrm{Sur} = \omega_k\,\bigl(e^{\mu+\varphi(k+1)-\varphi(k)} + e^{-\mu} - 2\,e^{\mu/k}\bigr).$$

Since $e^t \ge 1 + t$ for every real number t, the sum of the first two exponentials is at least $2 + \varphi(k+1) - \varphi(k)$, and $\varphi(k+1) - \varphi(k) \ge 3/(2(k+1)(k+2))$ by (18.12) at the point $k + 1$. Hence

$$\mathrm{Sur} \ge \omega_k\,\Bigl(\frac{3}{2\,(k+1)(k+2)} - 2\,\bigl(e^{\mu/k} - 1\bigr)\Bigr),$$

and it remains to see that the number in parentheses is at least λ. Since $N \ge k+1$, we have $\lambda \le e^{\mu}/((k+1)(k+2))$. As $k \ge 7$, the parameter satisfies $\mu \le 1/63$ and $\mu/k \le 1/(k^2(k+2)) \le 1/441$. By the inequality $e^t \le 1/(1-t)$ for $0 \le t < 1$ this gives $e^{\mu} \le 63/62$ and

$$\begin{aligned} e^{\mu/k} - 1 \le \frac{\mu/k}{1 - \mu/k} &\le \frac{441}{440}\cdot\frac{1}{k^2(k+2)} \\ &\le \frac{441}{440}\cdot\frac{8}{49}\cdot\frac{1}{(k+1)(k+2)}, \end{aligned}$$

where the last step holds because $8k^2 - 49\,(k+1) = (k-7)(8k+7) \ge 0$. Therefore the number in parentheses is at least

$$\Bigl(\frac{3}{2} - 2\cdot\frac{441}{440}\cdot\frac{8}{49}\Bigr)\frac{1}{(k+1)(k+2)},$$

and this is at least λ, because $3/2 - 2\cdot(441/440)\cdot(8/49) \ge 63/62 \ge e^{\mu}$.

Suppose now that $\mu \ge 1/(k(k+2))$. We take

$$\lambda = e^{\mu/k}\,\theta_0\,\frac{(2k-N)(N-k+1)}{N^2\,(N+2)\,k},$$

which is positive, because $k+1 \le N \le 2k-2$. Let $L = \log(\theta_+/\theta_0)$ and $M = \log(\theta_-/\theta_0)$. We have $\theta_+/\theta_0 = N(N+2)/(N+1)^2$ and $\theta_-/\theta_0 = N^2/(N^2-1)$, and the inequalities $1 - 1/r \le \log r \le r - 1$ for $r > 0$ give

$$L \ge -\frac{1}{N(N+2)}, \qquad \frac{1}{N^2} \le M \le \frac{1}{(N-1)(N+1)}.$$

Hence

$$\begin{aligned}\frac{M-L}{2} &\le \frac{1}{2}\Big(\frac{1}{(N-1)(N+1)} + \frac{1}{N(N+2)}\Big) \\ &\le \frac{1}{(N-1)(N+1)} \le \frac{1}{k\,(k+2)} \le \mu,\end{aligned}$$

where the third step uses $N \ge k+1$. So $\mu + (L-M)/2 \ge 0$, and (18.3), applied to the triple (Q^-, Q^0, Q^+), gives

$$2\,e^{\mu/k}\,e^{E/(2k)}\,Q^0 \le e^{\mu}\,\frac{\theta_+}{\theta_0}\,Q^+ + e^{-\mu}\,\frac{\theta_-}{\theta_0}\,Q^-$$

with $E = (k+1)L + (k-1)M$. Multiplying by θ_0 and subtracting $2\,e^{\mu/k}\,\theta_0\,Q^0$ on both sides, we get $\mathrm{Def} \ge 2\,e^{\mu/k}\,\theta_0\,Q^0\,(e^{E/(2k)} - 1)$. By the bounds for L and M,

$$E \ge -\frac{k+1}{N(N+2)} + \frac{k-1}{N^2} = -\frac{2\,(N-k+1)}{N^2\,(N+2)},$$

so that $e^{E/(2k)} - 1 \ge E/(2k) \ge -(N-k+1)/(N^2(N+2)\,k)$. As $Q^0 > 0$, this gives

$$\mathrm{Def} \ge -2\,e^{\mu/k}\,\theta_0\,\frac{N-k+1}{N^2\,(N+2)\,k}\,Q^0.$$

Finally, the sum Q^0 consists of the $k-\rho$ products $\omega_{\rho+i}\,\omega_{k-1-i}$ with $0 \le i \le k-\rho-1$, and by (18.8) with $m = k-\rho-1$ each of them is at most $\omega_\rho\,\omega_{k-1}$, which is equal to $\omega_\rho\,\omega_k$ because $\omega_{k-1} = \omega_k$ by (18.7). So $Q^0 \le (k-\rho)\,\omega_\rho\,\omega_k = (2k-N)\,\omega_\rho\,\omega_k$, and the second inequality of (18.19) follows.

For the first inequality, Lemma 18.5 gives

$$\mathrm{Sur} \ge \frac{k^2-2k-1}{2k^2(k+1)(k+2)}\,e^{\mu/k}\,\omega_k.$$

So it suffices to show that λ is at most the coefficient of ω_k on the right. After division by $e^{\mu/k}$ and multiplication by k, and with $\theta_0 = (N+1)/N$, this is the inequality

$$\frac{(N+1)(2k-N)(N-k+1)}{N^3\,(N+2)} \le \frac{k^2-2k-1}{2k\,(k+1)(k+2)}.$$

The numbers $2k-N$ and $N-k+1$ are positive and add up to $k+1$, so that their product is at most $(k+1)^2/4$; and $(N+1)(k+1)^3 \le (N+2)\,N^3$, because $k+1 \le N$. Hence the left-hand side is at most $1/(4(k+1))$. And the inequality $1/(4(k+1)) \le (k^2-2k-1)/(2k(k+1)(k+2))$ is equivalent to $k\,(k+2) \le 2\,(k^2-2k-1)$, or $k^2-6k-2 \ge 0$, which holds because $k \ge 7$ (and fails for $k = 6$). □

18.6. **The recursion in the first layer and the diagonal triples.** We now sort the terms of the tilted last-car recursion (3.6) by the sizes of the two tilted counts in each term. For integers $y \geq 1$ and p with $0 \leq p \leq y$ let

$$\mathsf{D}(y,p) = \sum_{i=0}^{y-p} \min(p+i+1,\, y)\, \omega_{p+i}\, \omega_{y-i},$$

the *direct part* of the recursion at size $y+p+1$ and threshold y. The next lemma separates it from the terms of the recursion that contain a tilted count of a size larger than the threshold.

Lemma 18.10. *Let $k \geq 1$, $y \geq 1$ and $p \geq 0$ be integers with $p \leq y$. Then the tilted normalised counts $u_N^{(y)}$ and the tilted Cayley masses ω_n at the critical tilt x_k of column k satisfy*

$$(y+p+1)\, u_{y+p+1}^{(y)} = x_k \Big(\sum_{i=0}^{p-1} (y+i+1)\, \omega_i\, u_{y+p-i}^{(y)} + \mathsf{D}(y,p) \Big),$$

where $\mathsf{D}(y,p)$ is the direct part of the recursion at size $y+p+1$ and threshold y.

Proof. The recursion (3.6), for threshold y, the tilt x_k and $N = y+p$, reads

$$(y+p+1)\, u_{y+p+1}^{(y)} = x_k \sum_{j=0}^{y+p} \min(j+1, y)\, u_j^{(y)}\, u_{y+p-j}^{(y)}.$$

We split the range of j into the three parts $0 \leq j < p$, $p \leq j \leq y$ and $y < j \leq y+p$. For $0 \leq i < p$ we have $i < y$, because $p \leq y$. By (18.1), the term of the first part with $j = i$ is therefore $(i+1)\, \omega_i\, u_{y+p-i}^{(y)}$, and the term of the third part with $j = y+p-i$ is $y\, \omega_i\, u_{y+p-i}^{(y)}$. The two parts together give the first sum of the lemma. In the second part we write $j = p+i$ with $0 \leq i \leq y-p$. Both $p+i$ and $y+p-j = y-i$ are at most y, so that both factors are Cayley masses by (18.1), and the terms are those of $\mathsf{D}(y,p)$. □

The next proposition uses the lemma for an induction on the offset, which is carried out for the three thresholds $k-1$, k and $k+1$ at once.

Proposition 18.11. *Let $k \geq 7$ and $\rho \geq 0$ be integers with $\rho + 2 \leq k$. Then the diagonal triple of offset ρ of column k,*

$$\big(u_{k-1+\rho}^{(k-1)},\ u_{k+\rho}^{(k)},\ u_{k+1+\rho}^{(k+1)} \big),$$

has property K at k.

Proof. We first derive the recursion (18.23). For integers $p \geq 0$ and y with $p+2 \leq y$ the direct part is

$$\mathsf{D}(y,p) = (y+p+1)\, \omega_p\, \omega_y + \frac{y+p+2}{2}\, Q_{p+1}(y-p-2). \tag{18.22}$$

Indeed, the terms $i = 0$ and $i = y-p$ of the sum that defines $\mathsf{D}(y,p)$ are $(p+1)\, \omega_p\, \omega_y$ and $y\, \omega_y\, \omega_p$, and their sum is the first term of (18.22). Let $m = y-p-2 \geq 0$. For $0 \leq j \leq m$ the term $i = j+1$ is $(p+2+j)\, \pi_j$, where $\pi_j = \omega_{p+1+j}\, \omega_{p+1+m-j}$. Since $\pi_{m-j} = \pi_j$, the sum of $(p+2+j)\, \pi_j$ over $0 \leq j \leq m$ is equal to the sum of $(p+2+m-j)\, \pi_j$, and hence to $(2p+4+m)/2$ times the sum of the π_j. Here $2p+4+m = y+p+2$, and the sum of the π_j is $Q_{p+1}(m)$, which gives (18.22). Now

let y and ρ be integers with $1 \le \rho \le y-1$. We apply Lemma 18.10 and (18.22) with $p = \rho - 1$, for which $p+2 \le y$, and divide by $y+\rho$. Since

$$\begin{aligned}\frac{x_k}{y+\rho}\,\mathsf{D}(y,\rho-1) &= x_k\,\omega_{\rho-1}\,\omega_y + \frac{x_k}{2}\cdot\frac{y+\rho+1}{y+\rho}\,Q_\rho(y-\rho-1)\\ &= S_\rho(y),\end{aligned}$$

the result is

$$u^{(y)}_{y+\rho} = \sum_{i=0}^{\rho-2} x_k\,\omega_i\,\frac{y+i+1}{y+\rho}\,u^{(y)}_{y+\rho-1-i} + S_\rho(y). \tag{18.23}$$

We now prove the proposition by induction on ρ, assuming it for all smaller offsets. For $\rho = 0$ the sizes $k-1$, k, $k+1$ do not exceed thresholds $k-1$, k, $k+1$, so that by (18.1) the triple is $(\omega_{k-1}, \omega_k, \omega_{k+1})$, which has property K at k by Corollary 18.6.

Let $1 \le \rho \le k-2$. Then (18.23) holds for each of the thresholds $y = k-1$, k and $k+1$, because $\rho \le (k-1)-1$. The range of i and the numbers $x_k\,\omega_i$ in it are the same for the three thresholds, so that the triple of the proposition is the sum of the following triples. For each i with $0 \le i \le \rho-2$ there is the triple whose members are the numbers

$$x_k\,\omega_i\,\frac{y+i+1}{y+\rho}\,u^{(y)}_{y+\rho'} \qquad (y = k-1,\ k,\ k+1),$$

where $\rho' = \rho - 1 - i$ satisfies $1 \le \rho' \le \rho - 1$. By the induction hypothesis the diagonal triple of offset ρ' has property K at k, and its middle member $u^{(k)}_{k+\rho'}$ is nonnegative. By Lemma 18.2 with $\alpha = i+1$ and $\beta = \rho$ it keeps the property when its members are multiplied by the values of $(y+i+1)/(y+\rho)$ at $y = k-1$, k, $k+1$, and it keeps it again when all three members are multiplied by the nonnegative number $x_k\,\omega_i$. The remaining triple consists of the direct terms $S_\rho(k-1)$, $S_\rho(k)$ and $S_\rho(k+1)$, and it has property K at k by Lemma 18.9. Since a sum of triples with property K at k has property K at k, the proposition follows. □

The two inequalities that Section 19 uses are now read off. In the first the three counts of a diagonal triple carry the weights $c-1$, c and $c+1$, where c is a real number, not to be confused with the numbers c_n.

Corollary 18.12. *Let $k \ge 7$ and N be integers with $k \le N \le 2k-2$, and let $c \ge k$ be a real number. Then the tilted normalised counts $u^{(y)}_n$ at the critical tilt x_k of column k satisfy*

$$(c+1)\,u^{(k+1)}_{N+1} - 2\,c\,u^{(k)}_N + (c-1)\,u^{(k-1)}_{N-1} \ge 0.$$

Proof. Let $\rho = N - k$, so that $0 \le \rho \le k-2$. By Proposition 18.11 the triple with the members $u^{(k-1)}_{N-1}$, $u^{(k)}_N$ and $u^{(k+1)}_{N+1}$ has property K at k, and its middle member is nonnegative. We apply Lemma 18.3 with $\alpha = c-k \ge 0$ and $\beta = 0$; its hypothesis $\alpha\beta \le k^2 - 1$ holds because $\alpha\beta = 0$. For $\beta = 0$ the three factors of that lemma are $k-1+\alpha$, $k+\alpha$ and $k+1+\alpha$, that is, $c-1$, c and $c+1$, so that the triple

$$\bigl((c-1)\,u^{(k-1)}_{N-1},\ c\,u^{(k)}_N,\ (c+1)\,u^{(k+1)}_{N+1}\bigr)$$

has property K at k. For $\mu = 0$ the inequality of property K for this triple is the claim. □

The second inequality is the convexity of the diagonal triple itself.

Corollary 18.13. *Let $k \geq 7$ and N be integers with $k \leq N \leq 2k-2$. Then the tilted normalised counts $u_n^{(y)}$ at the critical tilt x_k of column k satisfy*

$$u_{N+1}^{(k+1)} - 2\,u_N^{(k)} + u_{N-1}^{(k-1)} \geq 0.$$

Proof. This is the inequality of property K at the parameter $\mu = 0$ for the triple of Proposition 18.11 with $\rho = N - k$, for which $0 \leq \rho \leq k-2$. □

19. The signs beyond the first layer

In this section we determine the signs of the steps beyond the first layer: in every column $k \geq 1$ and every row $n \geq 2k$ the step $d_{n,k}$ is positive, unless (n,k) is one of the eleven exceptional positions, which are the positions $(2k,k)$ with $1 \leq k \leq 7$ and $(2k+1,k)$ with $2 \leq k \leq 5$ (Proposition 19.6). From this statement and from the computed negative signs of Lemma 6.1 we derive the column rule and the diagonal rule beyond the first layer (Corollaries 19.7 and 19.8); in Section 23 these are joined with the corresponding statements for the first layer. Columns $k \leq 81$ need no new estimate. Most of the section treats columns $k \geq 82$, by an induction on the row (Proposition 19.5) in which the rows before the first positive row of the column contribute terms that are not positive and whose size is not known; we bound a weighted sum of these terms by a summation by parts (Lemma 19.4), which rests on a convexity of the tilted normalised counts along the diagonal (Corollary 19.3).

19.1. **Convexity along the diagonal.** The arguments for columns $k \geq 82$ are carried out at the critical tilt of Section 5. For an integer $k \geq 1$ this is the number $x_k = (k/(k+1))^{k-1}$; it is positive and satisfies (5.10). As in that section, the tilted Cayley masses at this tilt are written $\omega_n = c_n\,x_k^n$. They are positive, and (5.9), taken at the tilt $x = x_k$, gives the ratio of two consecutive masses: for every integer $j \geq 0$ the mass ω_{j+1} is the mass ω_j multiplied by $x_k\,(1+1/(j+1))^j$. By Lemma 5.6 the masses do not increase up to index k. We begin with the counterpart of that lemma: from index $k-1$ on they do not decrease.

Lemma 19.1. *Let $k \geq 1$ and $n \geq 0$ be integers with $k \leq n+1$. Then, at the critical tilt x_k of column k,*

$$c_n\,x_k^n \leq c_{n+1}\,x_k^{n+1}.$$

Proof. By (5.9) for $x = x_k$ and $j = n$ the right-hand side is the left-hand side multiplied by $x_k\,(1+1/(n+1))^n$. This factor is at least $x_k\,(1+1/k)^{k-1}$ by Lemma 2.1, applied to the integers $k \leq n+1$, and $x_k\,(1+1/k)^{k-1} = 1$ by (5.10). □

Now let $k \geq 2$ be an integer. In the rest of this section we use the notation of the earlier sections at the critical tilt of column k; the column is not shown in it. For a threshold $y \geq 1$ and an integer $N \geq 0$, $u_N^{(y)}$ is the tilted normalised count of threshold y (Section 5), and U_N, U_N^+ and η_N are the tilted column counts and the tilted step of column k (Section 7). As in Section 18, the index N of $u_N^{(y)}$ is called its size, and a tilted normalised count is also called a tilted count for short. The tilted normalised counts are nonnegative and do not decrease in the threshold,

$$0 \leq u_N^{(y)} \leq u_N^{(y')} \qquad (1 \leq y \leq y'), \tag{19.1}$$

and in the free range $N \leq y$ of a threshold y they are the tilted Cayley masses ω_N, by (18.1). By (7.13) the tilted column counts are the normalised counts of columns $k-1$ and k, tilted by x_k. Both are nonnegative. By their definition, U_N and U_N^+

are the differences $u_N^{(k)} - u_N^{(k-1)}$ and $u_N^{(k+1)} - u_N^{(k)}$ of the tilted counts of consecutive thresholds, and the tilted step is the difference of the two tilted column counts:

$$\eta_N = U_N^+ - U_N = u_N^{(k+1)} - 2\,u_N^{(k)} + u_N^{(k-1)}. \tag{19.2}$$

The tilted step η_N has the sign of the step $d_{N,k}$. Finally, at the tilt x_k the tilted last-car recursion (3.6) reads

$$(N+1)\,u_{N+1}^{(y)} = x_k \sum_{i=0}^{N} \min(i+1, y)\,u_i^{(y)}\,u_{N-i}^{(y)} \qquad (y \geq 1,\ N \geq 0). \tag{19.3}$$

The convexity that we need concerns three consecutive points of a diagonal. For an integer $M \geq 1$ and a real number c let

$$\nabla_c(M) = (c+1)\,u_{M+1}^{(k+1)} - 2c\,u_M^{(k)} + (c-1)\,u_{M-1}^{(k-1)}. \tag{19.4}$$

This is a second difference along the diagonal with weights, and for $c = M$ each tilted count is multiplied by its own size. Section 18 treats the sizes of the first layer. For $k \geq 7$ and $k \leq M \leq 2k-2$ the second difference without weights, $u_{M+1}^{(k+1)} - 2\,u_M^{(k)} + u_{M-1}^{(k-1)}$, is nonnegative (Corollary 18.13), and so is $\nabla_c(M)$ for every real number $c \geq k$ (Corollary 18.12). The following proposition treats all larger sizes, for the same columns $k \geq 7$ and with the weight equal to the size. This is the weight that the last-car recursion produces, since the left-hand side of (19.3) is a tilted count multiplied by its size.

The proof of the proposition is an induction on the size. It writes the three terms of $\nabla_N(N)$ as convolutions by (19.3) and splits each convolution according to whether one of its two indices is at most $k-2$. The terms with such an index combine to positive multiples of numbers $\nabla_c(M)$ at smaller sizes M, which are nonnegative by Corollary 18.12 or by the induction hypothesis, and the remaining terms are controlled by Corollary 7.3.

Proposition 19.2. *Let $k \geq 7$ and $N \geq 2k-1$ be integers. Then the tilted normalised counts $u_M^{(y)}$ at the critical tilt x_k of column k satisfy*

$$(N+1)\,u_{N+1}^{(k+1)} - 2N\,u_N^{(k)} + (N-1)\,u_{N-1}^{(k-1)} \geq 0.$$

Proof. In the notation (19.4) the claim is $\nabla_N(N) \geq 0$. We prove it by induction on N. Let $N \geq 2k-1$, suppose that $\nabla_M(M) \geq 0$ for every integer M with $2k-1 \leq M < N$, and put $r = N - (2k-1)$, so that $r \geq 0$.

By Proposition 7.1 we have $z_i^{(y)}/c_i \leq z_{i+1}^{(y+1)}/c_{i+1}$ for all integers $y \geq 1$ and $i \geq 0$. Multiplying this inequality by ω_i and using that $\omega_i \leq \omega_{i+1}$ for $i \geq k-1$, by Lemma 19.1, we see that at the critical tilt the tilted normalised counts do not decrease along a diagonal step, on which the threshold and the size are both raised by one, as soon as the size is at least $k-1$:

$$u_i^{(y)} \leq u_{i+1}^{(y+1)} \qquad (y \geq 1,\ i \geq k-1). \tag{19.5}$$

We now show that the numbers $\nabla_c(M)$ at smaller sizes are nonnegative: for every integer M and every real number c in the ranges indicated,

$$\nabla_c(M) \geq 0 \qquad (k \leq M \leq N-1,\ \ k \leq c \leq 2k-1). \tag{19.6}$$

If $M \leq 2k-2$, this is Corollary 18.12, and it is here that the hypothesis $k \geq 7$ is used. If $2k-1 \leq M \leq N-1$, then $\nabla_M(M) \geq 0$ by the induction hypothesis, and

a comparison of the coefficients of the three tilted counts shows that

$$M\,\nabla_c(M) = c\,\nabla_M(M) + (M-c)\,\bigl(u_{M+1}^{(k+1)} - u_{M-1}^{(k-1)}\bigr).$$

Here $c > 0$ and $M - c \geq 0$, and $u_{M-1}^{(k-1)} \leq u_M^{(k)} \leq u_{M+1}^{(k+1)}$ by (19.5); hence $M\,\nabla_c(M) \geq 0$. We also need the weight k at size $k-1$, which lies below the range of (19.6):

$$\nabla_k(k-1) = (k+1)\,\omega_k - 2k\,\omega_{k-1} + (k-1)\,\omega_{k-2} \geq 0. \tag{19.7}$$

Here the equality holds by (18.1), and the inequality because $\omega_k \geq \omega_{k-1}$ by Lemma 19.1 and $\omega_{k-2} \geq \omega_{k-1}$ by Lemma 5.6.

Next we apply (19.3) to threshold $k+1$ at size $N+1$, to threshold k at size N and to threshold $k-1$ at size $N-1$. This gives $(N+1)\,u_{N+1}^{(k+1)} = x_k\,\Sigma^{(k+1)}$, $N\,u_N^{(k)} = x_k\,\Sigma^{(k)}$ and $(N-1)\,u_{N-1}^{(k-1)} = x_k\,\Sigma^{(k-1)}$, where

$$\begin{aligned}
\Sigma^{(k+1)} &= \sum_{i=0}^{N} \min(i+1, k+1)\,u_i^{(k+1)}\,u_{N-i}^{(k+1)},\\
\Sigma^{(k)} &= \sum_{i=0}^{N-1} \min(i+1, k)\,u_i^{(k)}\,u_{N-1-i}^{(k)},\\
\Sigma^{(k-1)} &= \sum_{i=0}^{N-2} \min(i+1, k-1)\,u_i^{(k-1)}\,u_{N-2-i}^{(k-1)}.
\end{aligned}$$

Hence $\nabla_N(N)$ is x_k times $\Sigma^{(k+1)} - 2\,\Sigma^{(k)} + \Sigma^{(k-1)}$, and it suffices to show that

$$\Sigma^{(k+1)} - 2\,\Sigma^{(k)} + \Sigma^{(k-1)} \geq 0.$$

To this end we sort the terms of the three sums. Every term is a weight times a product of two tilted counts. The indices of the two counts, the first and the second index of the term, add up to N, to $N-1$ and to $N-2$ in the three sums. In each sum we separate the terms whose first index is at most $k-2$, the terms whose second index is at most $k-2$, and the terms in which both indices are at least $k-1$. No term has both indices at most $k-2$, because the two indices add up to at least $N-2 \geq 2k-3$.

Let $0 \leq \ell \leq k-2$. By (18.1) a tilted count with index ℓ is equal to ω_ℓ for each of the three thresholds. In the term with the first index ℓ the weight $\min(\ell+1, y)$ is $\ell+1$ for each of the thresholds $y = k+1$, k, $k-1$. In the term with the second index ℓ the first index is $N-\ell$, $N-1-\ell$ or $N-2-\ell$, and since $N-2-\ell \geq k-1$, the weight is the threshold itself. The two terms with index ℓ therefore contribute to $\Sigma^{(k+1)}$, to $\Sigma^{(k)}$ and to $\Sigma^{(k-1)}$ the amounts

$$(\ell+k+2)\,\omega_\ell\,u_{N-\ell}^{(k+1)}, \qquad (\ell+k+1)\,\omega_\ell\,u_{N-1-\ell}^{(k)}, \qquad (\ell+k)\,\omega_\ell\,u_{N-2-\ell}^{(k-1)},$$

and to $\Sigma^{(k+1)} - 2\,\Sigma^{(k)} + \Sigma^{(k-1)}$ the amount $\omega_\ell\,\nabla_{\ell+k+1}(N-1-\ell)$.

The terms in which both indices are at least $k-1$ are sorted by their second index, which we write as $m_t = N-k-t$ with an integer t. In $\Sigma^{(k)}$ these are the terms with $0 \leq t \leq r$; their first index is $k-1+t$, and their weight is k. In $\Sigma^{(k+1)}$ the terms with $0 \leq t \leq r$ have the first index $k+t$ and the weight $k+1$, and there is one further term, with the first index $k-1$ and the second index $N-k+1$; its weight is k, and by (18.1) it is equal to $k\,\omega_{k-1}\,u_{N-k+1}^{(k+1)}$. In $\Sigma^{(k-1)}$ the first index is $k-2+t$ and the weight is $k-1$, and the terms in question are those with $1 \leq t \leq r$,

since for $t = 0$ the first index is $k-2$. In order that the three sums run over the same values of t, we add to the terms of $\Sigma^{(k-1)}$ the product for $t = 0$, which is

$$(k-1)\,u_{k-2}^{(k-1)}\,u_{m_0}^{(k-1)} = (k-1)\,\omega_{k-2}\,u_{N-k}^{(k-1)},$$

and subtract it again. For $0 \le t \le r$ let

$$\Upsilon_t = (k+1)\,u_{k+t}^{(k+1)}\,u_{m_t}^{(k+1)} - 2k\,u_{k-1+t}^{(k)}\,u_{m_t}^{(k)} + (k-1)\,u_{k-2+t}^{(k-1)}\,u_{m_t}^{(k-1)}.$$

Then we have found that

$$\begin{aligned}\Sigma^{(k+1)} - 2\,\Sigma^{(k)} + \Sigma^{(k-1)} = {} & \sum_{\ell=0}^{k-2} \omega_\ell\,\nabla_{\ell+k+1}(N-1-\ell) + \sum_{t=0}^{r} \Upsilon_t \\ & + k\,\omega_{k-1}\,u_{N-k+1}^{(k+1)} - (k-1)\,\omega_{k-2}\,u_{N-k}^{(k-1)}.\end{aligned} \tag{19.8}$$

It remains to show that the right-hand side of (19.8) is nonnegative. For $0 \le \ell \le k-2$ the weight $c = \ell+k+1$ satisfies $k+1 \le c \le 2k-1$, and the size $M = N-1-\ell$ satisfies $k \le N-k+1 \le M \le N-1$. Hence $\nabla_{\ell+k+1}(N-1-\ell) \ge 0$ by (19.6), and the first sum on the right of (19.8) is nonnegative.

We turn to the sum of the Υ_t. Let $0 \le t \le r$. We insert $u_{m_t}^{(k+1)} = u_{m_t}^{(k)} + U_{m_t}^{+}$ and $u_{m_t}^{(k-1)} = u_{m_t}^{(k)} - U_{m_t}$ into Υ_t and obtain

$$\Upsilon_t = u_{m_t}^{(k)}\,\nabla_k(k-1+t) + (k+1)\,u_{k+t}^{(k+1)}\,U_{m_t}^{+} - (k-1)\,u_{k-2+t}^{(k-1)}\,U_{m_t}.$$

Here $\nabla_k(k-1+t) \ge 0$: for $t = 0$ this is (19.7), and for $t \ge 1$ it is (19.6) with the weight k and the size $k-1+t$, which lies between k and $k-1+r = N-k$. Moreover $(k+1)\,U_{m_t}^{+} \ge k\,U_{m_t-1}$ by Corollary 7.3, applied at row $m_t \ge 1$. Since $u_{m_t}^{(k)}$ and $u_{k+t}^{(k+1)}$ are nonnegative by (19.1), it follows that

$$\Upsilon_t \ge k\,u_{k+t}^{(k+1)}\,U_{m_t-1} - (k-1)\,u_{k-2+t}^{(k-1)}\,U_{m_t}.$$

We sum over $0 \le t \le r$ and combine the positive term of t with the negative term of $t+1$, which is $(k-1)\,u_{k-1+t}^{(k-1)}\,U_{m_t-1}$ because $m_{t+1} = m_t - 1$. The positive term of $t = r$, in which $m_r - 1 = k-2$, and the negative term of $t = 0$, in which $m_0 = N-k$, are left over, so that

$$\begin{aligned}\sum_{t=0}^{r} \Upsilon_t \ge \sum_{t=0}^{r-1} U_{m_t-1}\left(k\,u_{k+t}^{(k+1)} - (k-1)\,u_{k-1+t}^{(k-1)}\right) \\ + k\,u_{k+r}^{(k+1)}\,U_{k-2} - (k-1)\,\omega_{k-2}\,U_{N-k}.\end{aligned}$$

Every term of the sum on the right is nonnegative, because $U_{m_t-1} \ge 0$ and $u_{k-1+t}^{(k-1)} \le u_{k+t}^{(k)} \le u_{k+t}^{(k+1)}$ by (19.5) and (19.1). Further, $U_{k-2} = 0$ by (18.1). Hence the sum of the Υ_t is at least $-(k-1)\,\omega_{k-2}\,U_{N-k}$, and the last three terms on the right of (19.8) together are at least

$$\begin{aligned}& k\,\omega_{k-1}\,u_{N-k+1}^{(k+1)} - (k-1)\,\omega_{k-2}\left(U_{N-k} + u_{N-k}^{(k-1)}\right) \\ & \qquad = k\,\omega_{k-1}\,u_{N-k+1}^{(k+1)} - (k-1)\,\omega_{k-2}\,u_{N-k}^{(k)}.\end{aligned}$$

This number is nonnegative. Indeed, $u_{N-k}^{(k)} \le u_{N-k+1}^{(k+1)}$ by (19.5), which applies because $N-k \ge k-1$, and $(k-1)\,\omega_{k-2} \le k\,\omega_{k-1}$: by (5.9) for $x = x_k$ and

$j = k-2$,

$$k\,\omega_{k-1} = k\,x_k\left(1+\frac{1}{k-1}\right)^{k-2}\omega_{k-2} = (k-1)\,x_k\left(1+\frac{1}{k-1}\right)^{k-1}\omega_{k-2},$$

and the factor $x_k\,(1+1/(k-1))^{k-1}$ is at least $x_k\,(1+1/k)^{k-1}$, which is 1 by (5.10). Since the first sum on the right of (19.8) is nonnegative as well, the right-hand side of (19.8) is nonnegative, which completes the induction. □

The proposition and the convexity of the first layer are joined into one statement by the weight $\Psi_k(N) = \max(N, 2k-1)$ of Section 17. It is constant on the sizes $N \le 2k-1$, where the convexity without weights is available, and equal to the size from there on, where only the weighted form of Proposition 19.2 is. One weight for all sizes is needed because the sum of the next subsection runs through the first layer and beyond it.

Corollary 19.3. *Let $k \ge 7$ and $N \ge k$ be integers. Then the tilted normalised counts $u_M^{(y)}$ at the critical tilt x_k of column k and the weight Ψ_k satisfy*

$$\Psi_k(N+1)\,u_{N+1}^{(k+1)} - 2\,\Psi_k(N)\,u_N^{(k)} + \Psi_k(N-1)\,u_{N-1}^{(k-1)} \ge 0.$$

Proof. If $k \le N \le 2k-2$, the three weights are equal to $2k-1$, and the left-hand side is $2k-1$ times $u_{N+1}^{(k+1)} - 2\,u_N^{(k)} + u_{N-1}^{(k-1)}$, which is nonnegative by Corollary 18.13. If $N = 2k-1$, the weights are $\Psi_k(N+1) = N+1$, $\Psi_k(N) = N$ and $\Psi_k(N-1) = 2k-1 = N$, and the left-hand side is

$$\big((N+1)\,u_{N+1}^{(k+1)} - 2N\,u_N^{(k)} + (N-1)\,u_{N-1}^{(k-1)}\big) + u_{N-1}^{(k-1)};$$

this is nonnegative by Proposition 19.2 and because $u_{N-1}^{(k-1)} \ge 0$. If $N \ge 2k$, the weights are $N+1$, N and $N-1$, and the claim is Proposition 19.2. □

19.2. **The weighted debt of a column.** Let $k \ge 7$, and let $J \ge k$ be an integer. We now sum the tilted steps η_n of column k over rows $k \le n \le J$, with the weights $\Psi_k(n)$. In the rows before the first positive row of the column the steps are not positive, so that such a sum can be negative, and the following lemma says how negative it can be: the sum is at least minus a number that consists of quantities of the last two rows $J-1$ and J alone. We call this number, which stands in the large parentheses on the right-hand side of the lemma, the *weighted debt* of column k at row J. Nothing about the single rows enters it, and nothing about the row in which the sign changes.

The reason is a summation by parts. By (19.2) the tilted step η_n is a second difference in the threshold, at the fixed size n. When the terms $\Psi_k(n)\,\eta_n$ of rows $k, \dots, J$ are added and regrouped along the diagonals, they become, up to terms of rows $J-1$ and J, the left-hand sides of Corollary 19.3 at sizes $k, \dots, J-1$ and one term of rows $k-1$ and k, the number $(2k-1)\,(\omega_k - \omega_{k-1})$. Each of these left-hand sides is nonnegative, and so is this number, by Lemma 19.1.

Lemma 19.4. *Let $k \ge 7$ and $J \ge k$ be integers. Then, at the critical tilt x_k of column k, the tilted steps η_n of the column, its tilted column counts U_N, the tilted normalised counts $u_N^{(k)}$ of threshold k and the weight Ψ_k satisfy*

$$\sum_{n=k}^{J} \Psi_k(n)\,\eta_n \ge -\Big(\Psi_k(J)\,U_J + \Psi_k(J-1)\,U_{J-1} + \big(\Psi_k(J) - \Psi_k(J-1)\big)\,u_J^{(k)}\Big).$$

Proof. For an integer $J \geq k$ let

$$Y_J = \sum_{n=k}^{J} \Psi_k(n)\,\eta_n + \Psi_k(J)\,U_J + \Psi_k(J-1)\,U_{J-1} - \Psi_k(J-1)\,u_{J-1}^{(k)} + \Psi_k(J)\,u_J^{(k)}.$$

We show by induction on J that $Y_J \geq 0$.

For $J = k$ the weights $\Psi_k(k)$ and $\Psi_k(k-1)$ are both equal to $2k-1$. Inserting (19.2) for η_k and the definitions of U_k and U_{k-1}, we find by a direct computation that all terms cancel except two, and by (18.1),

$$Y_k = (2k-1)\left(u_k^{(k+1)} - u_{k-1}^{(k-1)}\right) = (2k-1)\,(\omega_k - \omega_{k-1}).$$

This is nonnegative by Lemma 19.1.

Let $J \geq k$. In the difference $Y_{J+1} - Y_J$ the terms $\Psi_k(J)\,U_J$ cancel, and

$$Y_{J+1} - Y_J = \Psi_k(J+1)\left(\eta_{J+1} + U_{J+1} + u_{J+1}^{(k)}\right) - 2\,\Psi_k(J)\,u_J^{(k)} + \Psi_k(J-1)\left(u_{J-1}^{(k)} - U_{J-1}\right).$$

By (19.2) and the definition of U_N, the first parenthesis is $u_{J+1}^{(k+1)}$ and the second is $u_{J-1}^{(k-1)}$. Hence

$$Y_{J+1} - Y_J = \Psi_k(J+1)\,u_{J+1}^{(k+1)} - 2\,\Psi_k(J)\,u_J^{(k)} + \Psi_k(J-1)\,u_{J-1}^{(k-1)},$$

which is nonnegative by Corollary 19.3, applied with $N = J$. This completes the induction.

From $Y_J \geq 0$ we get

$$\sum_{n=k}^{J} \Psi_k(n)\,\eta_n \geq -\Psi_k(J)\,U_J - \Psi_k(J-1)\,U_{J-1} + \Psi_k(J-1)\,u_{J-1}^{(k)} - \Psi_k(J)\,u_J^{(k)}.$$

By Proposition 5.9 we have $u_J^{(k)} \leq u_{J-1}^{(k)}$. Therefore

$$\Psi_k(J-1)\,u_{J-1}^{(k)} - \Psi_k(J)\,u_J^{(k)} \geq -\big(\Psi_k(J) - \Psi_k(J-1)\big)\,u_J^{(k)},$$

and the lemma follows. □

19.3. **Columns $k \geq 82$.** Let $k \geq 82$. By Proposition 16.4 column k changes its sign once between rows k and $2k-1$: there is a row q with $k+1 \leq q \leq k+27k/100$ such that the step of the column is not positive in rows $k \leq j < q$ and positive in rows $q \leq j \leq 2k-1$. Row q is the first positive row of the column. We now show that the step stays positive in every later row.

The proof is an induction on the row: for a row $n \geq 2k$ it starts from the inequality of Proposition 5.3 at the critical tilt, to which the rows before row q contribute terms that are not positive and whose size is not known. Lemmas 17.8 and 17.9 compare the weights that the prime kernel gives to the rows before row n in this inequality with the weights Ψ_k, so that by Lemma 19.4 the contribution of all these rows is at least minus a nonnegative multiple of the weighted debt of the column at row $n-1$. The proof then bounds the debt by the quantities η_{n-1} and U_{n-1} of the other two terms of the inequality; the debt enters with coefficients that are at most a fixed multiple of $1/\sqrt{k}$, whereas the coefficients of those two terms are bounded below by a positive constant.

Proposition 19.5. *Let $k \geq 82$ and $n \geq 2k$ be integers. Then $d_{n,k} > 0$.*

Proof. We use the notation of Section 19.1 at the critical tilt x_k of column k, and the prime kernel $\bar{\kappa}_m = \kappa_m^{(k)} x_k^m$ of threshold k at this tilt; the numbers $\bar{\kappa}_m$ are nonnegative by Lemma 4.8. Since η_j has the sign of $d_{j,k}$, the claim is $\eta_n > 0$. Let q be a row as in Proposition 16.4, so that $k+1 \leq q$ and $100\,(q-k) \leq 27\,k$, that $\eta_j \leq 0$ for $k \leq j < q$, and that $\eta_j > 0$ for $q \leq j \leq 2k-1$. It suffices to prove the following claim: if $n \geq 2k$ is an integer and $\eta_j > 0$ for every j with $q \leq j \leq n-1$, then $\eta_n > 0$. Indeed, since $q < 2k$ and $\eta_m > 0$ for $q \leq m \leq 2k-1$ by the choice of q, the claim gives $\eta_m > 0$ for every integer $m \geq q$ by induction on m.

To prove the claim, let $n \geq 2k$, and suppose that $\eta_j > 0$ for $q \leq j \leq n-1$. Since $q < 2k \leq n$, row $n-1$ is one of these rows, so that $\eta_{n-1} > 0$. We multiply the inequality of Proposition 5.3, which applies because $n \geq k+1$, by x_k^n and index its sum by row $j = n-m$. Since $\kappa_m^{(k)}\,\delta_{n-m}^{(k)}\,x_k^n = \bar{\kappa}_m\,\eta_{n-m}$ and $\beta_{n-1}^{(k-1)}\,x_k^n = x_k\,U_{n-1}$, it becomes

$$n\,\eta_n \geq (k+1)\,x_k\,\eta_{n-1} + 2\,x_k\,U_{n-1} + n\sum_{j=k}^{n-1} \bar{\kappa}_{n-j}\,\eta_j. \tag{19.9}$$

The first term on the right is a multiple of the tilted step η_{n-1} of the row before, which is positive; the second is the source, which comes from column $k-1$ and is nonnegative; and the sum is the feedback of rows $k \leq j \leq n-1$, in which row j has the kernel weight $\bar{\kappa}_{n-j}$. Let

$$L = n - q + 1, \qquad \theta = \frac{\bar{\kappa}_L}{2k-1}, \qquad \mathfrak{K} = n(n-1)\,\bar{\kappa}_L.$$

Then $\theta \geq 0$ and $n(n-1)\,\theta = \mathfrak{K}/(2k-1)$.

We first bound the feedback from below in terms of the weighted debt of the column. Let $\mathfrak{S}$ be the feedback in (19.9), the sum of $\bar{\kappa}_{n-j}\,\eta_j$ over $k \leq j \leq n-1$. Lemmas 17.8 and 17.9 apply to the rows q and n, because $k+1 \leq q$, $100\,(q-k) \leq 27\,k$ and $n \geq 2k$. For $k \leq j < q$ we have $\eta_j \leq 0$, and Lemma 17.8 gives $\bar{\kappa}_{n-j}\,(2k-1) \leq \bar{\kappa}_L\,\Psi_k(j)$, so that $\bar{\kappa}_{n-j} \leq \theta\,\Psi_k(j)$. For $q \leq j \leq n-1$ we have $\eta_j > 0$, and Lemma 17.9 gives $\bar{\kappa}_L\,\Psi_k(j) \leq \bar{\kappa}_{n-j}\,(2k-1)$, so that $\bar{\kappa}_{n-j} \geq \theta\,\Psi_k(j)$. In both cases $\bar{\kappa}_{n-j}\,\eta_j \geq \theta\,\Psi_k(j)\,\eta_j$. We sum over j and apply Lemma 19.4 with $J = n-1$:

$$\mathfrak{S} \geq \theta \sum_{j=k}^{n-1} \Psi_k(j)\,\eta_j \geq -\theta\,\mathfrak{D}, \tag{19.10}$$

where

$$\mathfrak{D} = \Psi_k(n-1)\,U_{n-1} + \Psi_k(n-2)\,U_{n-2} + \big(\Psi_k(n-1) - \Psi_k(n-2)\big)\,u_{n-1}^{(k)}$$

is the weighted debt of the column at row $n-1$. Since $n-1 \geq 2k-1$, we have $\Psi_k(n-1) = n-1$ and $n-2 \leq \Psi_k(n-2) \leq n-1$. As $U_{n-2} \geq 0$ and $u_{n-1}^{(k)} \geq 0$, it follows that

$$\mathfrak{D} \leq (n-1)\,(U_{n-1} + U_{n-2}) + u_{n-1}^{(k)}. \tag{19.11}$$

Next we bound U_{n-2} and $u_{n-1}^{(k)}$ by the quantities η_{n-1} and U_{n-1} of the first two terms on the right of (19.9). Corollary 7.3, applied at row $n-1$, states that $k\,U_{n-2} \leq (k+1)\,U_{n-1}^+$, and $U_{n-1}^+ = U_{n-1} + \eta_{n-1}$ by (19.2). Hence

$$U_{n-2} \leq \Big(1 + \frac{1}{k}\Big)\,(U_{n-1} + \eta_{n-1}). \tag{19.12}$$

For $u_{n-1}^{(k)}$ we show that

$$x_k\,\Omega\,u_{n-1}^{(k)} \le (n-1)\,U_{n-1}, \qquad \text{where } \Omega = \sum_{j=0}^{8}\omega_j. \tag{19.13}$$

We subtract the recursion (19.3) of threshold $k-1$ at size $n-1$ from that of threshold k at the same size, and obtain

$$(n-1)\,U_{n-1} = x_k \sum_{i=0}^{n-2} \mathfrak{T}_i,$$

where

$$\mathfrak{T}_i = \min(i+1,k)\,u_i^{(k)}\,u_{n-2-i}^{(k)} - \min(i+1,k-1)\,u_i^{(k-1)}\,u_{n-2-i}^{(k-1)}.$$

Every $\mathfrak{T}_i$ is nonnegative, by (19.1) and because $\min(i+1,k-1) \le \min(i+1,k)$. Let $0 \le j \le 8$ and $i = n-2-j$. Then $i+1 \ge 2k-9 \ge k$, so that the two minima are k and $k-1$, and

$$\mathfrak{T}_{n-2-j} = k\,u_{n-2-j}^{(k)}\,u_j^{(k)} - (k-1)\,u_{n-2-j}^{(k-1)}\,u_j^{(k-1)} \ge u_{n-2-j}^{(k)}\,u_j^{(k)} \ge u_{n-1}^{(k)}\,\omega_j.$$

Here the first inequality is (19.1). The second holds because $u_j^{(k)} = \omega_j$ by (18.1), as $j \le 8 \le k$, and because $u_{n-2-j}^{(k)} \ge u_{n-1}^{(k)}$ by Proposition 5.9. Keeping only these nine terms of the sum, we obtain (19.13). Now we insert (19.12) and (19.13) into (19.11). With the number

$$\sigma = 2 + \frac{1}{k} + \frac{1}{x_k\,\Omega}$$

we find

$$\mathfrak{D} \le (n-1)\left(\sigma\,U_{n-1} + \left(1+\frac{1}{k}\right)\eta_{n-1}\right). \tag{19.14}$$

By (19.9), (19.10) and (19.14), and since $n(n-1)\,\theta = \mathfrak{K}/(2k-1)$,

$$\begin{aligned} n\,\eta_n &\ge (k+1)\,x_k\,\eta_{n-1} + 2\,x_k\,U_{n-1} - n\,\theta\,\mathfrak{D} \\ &\ge U_{n-1}\left(2\,x_k - \frac{\mathfrak{K}\,\sigma}{2k-1}\right) + (k+1)\,\eta_{n-1}\left(x_k - \frac{\mathfrak{K}}{(2k-1)\,k}\right). \end{aligned}$$

Since $U_{n-1} \ge 0$ and $\eta_{n-1} > 0$, we obtain $\eta_n > 0$ as soon as

$$\mathfrak{K}\,\sigma \le 2\,x_k\,(2k-1) \qquad \text{and} \qquad \mathfrak{K} < x_k\,k\,(2k-1). \tag{19.15}$$

It remains to verify the two conditions (19.15). To this end we first show that

$$\mathfrak{K} = n(n-1)\,\bar\kappa_L \le 3.762\,\sqrt{k}. \tag{19.16}$$

Since $q \le 1.27\,k$, we have $n = L + q - 1 \le L + 1.27\,k - 1$. We distinguish three cases according to the size of L.

If $L \le k$, then $n \le 2.27\,k$. Moreover $100\,(L-1) = 100\,(n-q) \ge 73\,k$, because $n \ge 2k$ and $100\,q \le 127\,k$, so that $\bar\kappa_L \le 0.73/(k\sqrt{k})$ by Lemma 17.7. Hence

$$\mathfrak{K} \le n^2\,\bar\kappa_L \le 2.27^2 \cdot 0.73\,\sqrt{k} \le 3.762\,\sqrt{k}.$$

If $k < L \le 2k$, Corollary 17.4 gives $L\,\bar\kappa_L \le k\,\bar\kappa_k$, and $\bar\kappa_k \le 0.662/(k\sqrt{k})$ by Lemma 17.6; hence $L\,\bar\kappa_L \le 0.662/\sqrt{k}$. Furthermore $n^2 \le (L + 1.27\,k)^2 \le 5.35\,k\,L$. With $t = L/k$ the second of these inequalities reads $t^2 - 2.81\,t + 1.6129 \le 0$; the

polynomial on the left is convex and negative at $t = 1$ and at $t = 2$, so that it is negative on the interval $1 \le t \le 2$. Hence

$$\mathfrak{K} \le n^2\,\bar\kappa_L \le 5.35\,k\,L\,\bar\kappa_L \le 5.35 \cdot 0.662\,\sqrt{k} \le 3.762\,\sqrt{k}.$$

If $L > 2k$, Lemma 17.5 gives $(L-1)\,L\,\bar\kappa_L \le 2k(k-1)\,\bar\kappa_k$, and with Lemma 17.6 we get $(L-1)\,L\,\bar\kappa_L \le 2 \cdot 0.662\,\sqrt{k}$. In this case $k \le (L-1)/2$, so that $n \le (L-1) + 1.27\,k \le 1.635\,(L-1)$, and therefore

$$n(n-1) \le 1.635^2\,(L-1)\,L \le 2.675\,(L-1)\,L.$$

Hence $\mathfrak{K} \le 2.675 \cdot 2 \cdot 0.662\,\sqrt{k} \le 3.762\,\sqrt{k}$.

To derive (19.15) from (19.16), we bound x_k and Ω from below. By (5.10), $1/x_k = (1+1/k)^{k-1} \le e^{(k-1)/k} \le e$, so that $x_k \ge 1/e \ge 0.3678$. Hence Ω, which is the sum of $c_j\,x_k^j$ over $0 \le j \le 8$, is at least the value of this sum with x_k replaced by 0.3678, and a direct computation with the numbers $c_j = (j+1)^{j-1}/j!$ shows that this value is at least 2.01. Therefore $\Omega \ge 2.01$. It follows that

$$\sigma \le 2 + \frac{1}{82} + \frac{1}{0.3678 \cdot 2.01} \le 3.3652.$$

By (19.16), the first condition of (19.15) follows from

$$3.762 \cdot 3.3652\,\sqrt{k} \le 12.66\,\sqrt{k} \le 2 \cdot 0.3678\,(2k-1).$$

With $s = \sqrt{k}$ the second of these inequalities reads

$$1.4712\,s^2 - 12.66\,s - 0.7356 \ge 0.$$

The left-hand side increases for $s \ge 4.31$ and is positive at $s = 9.05$, and $\sqrt{k} \ge 9.05$, because $9.05^2 \le 82$. The second condition of (19.15) holds as well: since $2k - 1 \ge 163$,

$$\mathfrak{K} \le 3.762\,k < 0.3678\,k\,(2k-1) \le x_k\,k\,(2k-1).$$

This proves the claim, and with it the proposition. □

19.4. **All columns, and the two rules beyond the first layer.** We now assemble the statement for all columns. The positions that the hypothesis of the following proposition excludes are the eleven exceptional positions.

Proposition 19.6. *Let $k \ge 1$ and $n \ge 2k$ be integers, and suppose that neither $n = 2k$ and $k \le 7$, nor $n = 2k+1$ and $2 \le k \le 5$. Then $d_{n,k} > 0$.*

Proof. We distinguish column $k = 1$, columns $2 \le k \le 81$ and columns $k \ge 82$.

Let first $k = 1$. Then the supposition excludes $n = 2$, so that $n \ge 3$. Since $b_{n,0} = A_{n,0}$ and $b_{n,1} = A_{n,1} - A_{n,0}$ (Section 3), the claim $d_{n,1} > 0$ is the inequality

$$2\,A_{n,0} < A_{n,1}, \tag{19.17}$$

which we prove by induction on $n \ge 3$. Starting from $A_{0,\ell} = 1$, the last-car recursion (Proposition 3.1) gives the values 1, 2, 6 of $A_{n,0}$ and the values 1, 3, 13 of $A_{n,1}$ for $n = 1, 2, 3$. Thus $2\,A_{3,0} = 12 < 13 = A_{3,1}$. Now let $n \ge 3$, and suppose that (19.17) holds for n. For $\ell = 0$ and for $\ell = 1$ the last-car recursion reads

$$A_{n+1,0} = \sum_{j=0}^{n} \binom{n}{j} A_{j,0}\,A_{n-j,0}, \qquad A_{n+1,1} = \sum_{j=0}^{n} \min(j+1,2) \binom{n}{j} A_{j,1}\,A_{n-j,1}.$$

We compare twice the first sum with the second sum term by term. For $j \ge 1$ the term of the second sum carries the factor $\min(j+1,2) = 2$, and $A_{j,0}\,A_{n-j,0} \le$

$A_{j,1}\,A_{n-j,1}$, because the cumulative counts do not decrease in the bound on the displacement. For $j = 0$ the two terms are $A_{n,0}$ and $A_{n,1}$, and $2\,A_{n,0} < A_{n,1}$ by the induction hypothesis. Adding up, we obtain $2\,A_{n+1,0} < A_{n+1,1}$.

Now let $2 \le k \le 81$, and let $n_0 = n_0(k)$ be the row of Definition 6.2. By Lemma 6.3 we have $2k+3 \le n_0$, and $d_{m,k} > 0$ for every row m with $2k \le m < n_0$ other than $m = 2k$ when $k \le 7$ and $m = 2k+1$ when $k \le 5$. If $n < n_0$, row n is one of these rows. Indeed, for $k \ge 2$ the supposition of the proposition says that $n \ne 2k$ when $k \le 7$ and that $n \ne 2k+1$ when $k \le 5$, so that the rows which it excludes are the rows which the lemma excepts. Hence the claim is part of the lemma.

Let $n \ge n_0$, and put $m = n_0 - 1$. Then $2k+2 \le m < n_0$, so that m is one of the rows of Lemma 6.3 and is not excepted. Hence $d_{m,k} > 0$. We apply Proposition 5.5 to column k, to row $m \ge k$ and to the tilt $x = x_k$, which satisfies $0 < x_k < 1$ because $k \ge 2$. Its hypotheses hold. The tilted normalised counts $z_N^{(k)}\,x_k^N$ of threshold k do not increase in N, by Proposition 5.9. The sums of $\xi_t^{(k)}\,x_k^t$ over $s \le t \le m+1$, where $\xi^{(k)}$ is the tree curvature of column k (Definition 5.1), are nonnegative for every s with $k \le s \le m+1$; as $m+1 = n_0$, this is part of Lemma 6.4. And $\delta_m^{(k)} > 0$, because $d_{m,k} > 0$. The proposition gives $\delta_j^{(k)} > 0$ for every $j \ge m$, and in particular $d_{n,k} > 0$.

Finally, for $k \ge 82$ the claim is Proposition 19.5. □

At the ten exceptional positions with $k \ge 2$ the step is negative, by Lemma 6.1. The eleventh is $(2,1)$, the one position of row 2.

We turn to the two rules. Beyond the first layer both follow from Proposition 19.6 and Lemma 6.1. The first corollary is the column rule in the case in which the lower position $(n+1,k)$ lies beyond the first layer.

Corollary 19.7. *Let $k \ge 1$ and n be integers with $n \ge k+1$ and $n+1 \ge 2k$. If $d_{n,k} > 0$, then $d_{n+1,k} > 0$.*

Proof. Suppose first that $(n+1,k)$ is not an exceptional position. Then Proposition 19.6, applied to column k and to row $n+1 \ge 2k$, gives $d_{n+1,k} > 0$.

Suppose now that $(n+1,k)$ is exceptional. Then $d_{n,k} < 0$ by Lemma 6.1, so that the hypothesis $d_{n,k} > 0$ fails and there is nothing to prove. Indeed, if $n+1 = 2k$ and $k \le 7$, then $n = 2k-1$, and $n \ge k+1$ gives $k \ge 2$, so that the lemma applies with its condition that $n \le 2k-1$ and $k \le 8$. If $n+1 = 2k+1$ and $2 \le k \le 5$, then $n = 2k$, and the lemma applies with its condition that $n \le 2k$ and $k \le 7$. □

The second corollary is the diagonal rule in the case in which the upper position (n,k) lies beyond the first layer.

Corollary 19.8. *Let $k \ge 1$ and $n \ge 2k$ be integers. If $d_{n,k} \le 0$, then $d_{n+1,k+1} < 0$.*

Proof. If neither $n = 2k$ and $k \le 7$, nor $n = 2k+1$ and $2 \le k \le 5$, then Proposition 19.6 gives $d_{n,k} > 0$, against the hypothesis. Hence $n = 2k$ and $k \le 7$, or $n = 2k+1$ and $2 \le k \le 5$.

Put $k' = k+1$ and $n' = n+1$. Then $k' \ge 2$, and $n' \ge k'+1$ because $n \ge 2k \ge k+1$. If $n = 2k$ and $k \le 7$, then $n' = 2k'-1$ and $k' \le 8$, and Lemma 6.1 applies with its condition that $n' \le 2k'-1$ and $k' \le 8$. If $n = 2k+1$ and $2 \le k \le 5$, then $n' = 2k'$ and $k' \le 6$, and the lemma applies with its condition that $n' \le 2k'$ and $k' \le 7$. In both cases $d_{n+1,k+1} = d_{n',k'} < 0$. □

20. The gain of the boundary term along the diagonal

Proposition 22.6, the diagonal rule at the positions behind the curve in columns $k \geq 265$, is proved in Section 22 from three comparisons along the diagonal, and we begin with the one for the boundary term. At a position $(k+r, k)$ of the first layer the normalised step is a curvature term minus the boundary term $(a_k/k)\,\Lambda_{r+1}(k-1; k+r+1)$ (Corollary 13.7), and at the diagonal neighbour $(k+r+1, k+1)$ the boundary term is $(a_{k+1}/(k+1))\,\Lambda_{r+1}(k; k+r+2)$. For Proposition 22.6 the boundary term has to grow along the diagonal by more than the terms of the curvature term do, and Proposition 20.2 bounds the growth of its chain sum from below by an explicit factor.

The two chain sums run over the compositions of the same integer $q = r+1$, with the same weights of the parts, and they differ only in the weights of the cuts. The proof of Proposition 20.2 compares the weights of the compositions in the first chain sum with their weights when all cuts have the same weight, by an instance of the correlation inequality of Fortuin, Kasteleyn and Ginibre [10, Proposition 1] on the sets of cut points. Section 20.1 treats the compositions whose cuts all have the same weight and bounds their mean number of cuts from below (Lemma 20.1), and Section 20.2 writes the weighted sums over compositions as sums over sets of cut points and recalls the correlation inequality, whose hypothesis the proof derives from the log-convexity of the Cayley coefficients (Lemma 2.2).

Throughout the section q denotes the integer of Proposition 20.2, whose compositions the two chain sums run over. The statements of Sections 20.1 and 20.2 are about the compositions of an arbitrary integer m, and we write Q for the number of parts of a composition.

20.1. **The homogeneous law and its mean number of cuts.** We use the weighted sums over compositions of Section 11: the weight $V_Q(m; f)$ of the compositions of m with exactly Q parts for a cut profile f, the weight $\Lambda_m[f] = \sum_{Q=1}^{m} V_Q(m; f)$ of all compositions of m, and the numbers $T_{m,Q} = V_Q(m; \mathbf{1})$, which are nonnegative and satisfy $T_{m,1} = a_m$.

For an integer $m \geq 0$ and a real number h let

$$\tau_m(h) = \Lambda_m[h]$$

be the total weight of the compositions of m for the constant cut profile with the value h. Then

$$\tau_m(h) = \sum_{Q=1}^{m} T_{m,Q}\, h^{Q-1}. \tag{20.1}$$

For $m \geq 1$ and $h \geq 0$ we have $\tau_m(h) > 0$.

Let $m \geq 1$ and $h \geq 0$. The *homogeneous law with the parameter* h on the compositions of m gives every composition a probability proportional to its weight for the constant cut profile h. Under this law the number of parts takes the value Q with probability $T_{m,Q}\, h^{Q-1}/\tau_m(h)$. Accordingly, the mean number of cuts of a composition of m under the homogeneous law with the parameter h is

$$\mu_m(h) = \frac{\sum_{Q=1}^{m} (Q-1)\, T_{m,Q}\, h^{Q-1}}{\tau_m(h)}.$$

Clearly $\mu_m(h) \geq 0$. We use the definition in the form

$$\mu_m(h)\,\tau_m(h) = \sum_{Q=1}^{m} (Q-1)\,T_{m,Q}\,h^{Q-1}. \tag{20.2}$$

The mean $\mu_m(h)$ is a quotient of two sums of m terms, and Proposition 20.2 needs a lower bound for it that is explicit in m and h. The bound is expressed by the function

$$\mu_m^0(X) = \frac{\sqrt{(4+X)^2 + 12\,(m-1)} - (4+X)}{3},$$

defined for an integer $m \geq 1$ and a real number X. For $X \geq 0$ the square root is at least $4 + X$, so that $\mu_m^0(X) \geq 0$.

By Proposition 11.4 the number J of cuts under the homogeneous law takes the value j with a probability proportional to $(j+1)\,h^j B_{m,j+1}$, where $B_{m,j+1}$ is the birthday tail of Section 11, and the proof of Lemma 20.1 shows that the sequence of the numbers $h^j B_{m,j+1}$ is log-concave. The proof sums the relation between consecutive terms of this sequence over j, which gives an exact relation between the first two moments of J, and combines it with the inequality $\mathbb{E}[J(J-1)] \leq (3/2)\,\mathbb{E}[J]^2$; the result is the inequality $2\,(m-1) \leq \frac{3}{2}\,\mu^2 + (4+X)\,\mu$ for $\mu = \mu_m(h) = \mathbb{E}[J]$ and $X = m\,(1/h - 1)$, and the bound of the lemma is its solution for μ. For the inequality between the moments the factor $j+1$ is read as the number of pairs (i, ℓ) of nonnegative integers with $i + \ell = j$: then J has the law of the sum of the two coordinates of a random pair, and the proof derives from the log-concavity that the two coordinates are negatively correlated.

Lemma 20.1. *For every integer $m \geq 1$ and every real number h with $0 < h \leq 1$, the mean number of cuts $\mu_m(h)$ of a composition of m under the homogeneous law with the parameter h satisfies*

$$\mu_m^0\big(m\,(1/h - 1)\big) \leq \mu_m(h).$$

Proof. Fix m and h, and write $\mu = \mu_m(h)$ and $X = m\,(1/h - 1)$. For an integer $j \geq 0$ let

$$v_j = h^j\,B_{m,j+1} = h^j \prod_{i=0}^{j} \Big(1 - \frac{i}{m}\Big),$$

where $B_{m,j}$ is the birthday tail of Section 11. Since $h > 0$, we have $v_j > 0$ for $0 \leq j \leq m-1$, while $v_j = 0$ for $j \geq m$. Moreover,

$$m\,v_{j+1} = h\,(m-1-j)\,v_j \qquad \text{for every integer } j \geq 0. \tag{20.3}$$

We put

$$M_0 = \sum_{j=0}^{m-1} (j+1)\,v_j, \quad M_1 = \sum_{j=0}^{m-1} j\,(j+1)\,v_j, \quad M_2 = \sum_{j=0}^{m-1} j^2\,(j+1)\,v_j.$$

By Proposition 11.4 we have $T_{m,Q} = a_m\,Q\,B_{m,Q}$ for $1 \leq Q \leq m$. With $Q = j+1$, the formulas (20.1) and (20.2) become $\tau_m(h) = a_m\,M_0$ and $\mu\,\tau_m(h) = a_m\,M_1$. Since $a_m > 0$ and $\tau_m(h) > 0$, it follows that

$$M_0 > 0 \qquad \text{and} \qquad M_1 = \mu\,M_0. \tag{20.4}$$

We first show that

$$m\,M_1 = h\,\big(2\,(m-1)\,M_0 + (m-3)\,M_1 - M_2\big). \tag{20.5}$$

The term $j = 0$ of M_1 vanishes, and $v_m = 0$. Shifting the index of summation by one, we therefore have

$$M_1 = \sum_{j=0}^{m-1} (j+1)(j+2)\, v_{j+1}.$$

By (20.3), for every j,

$$\begin{aligned} m\,(j+1)(j+2)\, v_{j+1} &= h\,(j+1)(j+2)(m-1-j)\, v_j \\ &= h\,(j+1)\, v_j \,\big(2\,(m-1) + (m-3)\, j - j^2\big), \end{aligned}$$

and the sum of these equations over $0 \le j \le m-1$ is (20.5).

Next we show that

$$\tfrac{1}{6}\,(M_2 - M_1)\, M_0 \le \tfrac{1}{4}\, M_1^2. \tag{20.6}$$

The sequence (v_j) is log-concave in the following form: for all integers s, t and d with $0 \le s \le t$ and $d \ge 0$,

$$v_s\, v_{t+d} \le v_{s+d}\, v_t. \tag{20.7}$$

Indeed, if $t + d \ge m$, the left-hand side is 0 and the right-hand side is nonnegative. If $t + d \le m - 1$, the four numbers are positive, and

$$\frac{v_{t+d}}{v_t} = h^d \prod_{i=1}^{d} \Big(1 - \frac{t+i}{m}\Big) \le h^d \prod_{i=1}^{d} \Big(1 - \frac{s+i}{m}\Big) = \frac{v_{s+d}}{v_s},$$

because the factors of both products are positive and $s \le t$.

We use a weighted form of Chebyshev's sum inequality. Let χ, ψ and φ be real functions on the set $\{0, \dots, m-1\}$ such that

$$\chi(j')\,\psi(j) \le \chi(j)\,\psi(j') \quad \text{and} \quad \varphi(j) \le \varphi(j') \qquad \text{whenever } j \le j'.$$

Then

$$\Big(\sum_j \chi(j)\,\varphi(j)\Big)\Big(\sum_j \psi(j)\Big) \le \Big(\sum_j \psi(j)\,\varphi(j)\Big)\Big(\sum_j \chi(j)\Big), \tag{20.8}$$

all sums running over $0 \le j \le m-1$. When χ and ψ are positive, the hypothesis says that the quotient χ/ψ does not increase, and the conclusion says that the mean of the nondecreasing function φ for the weights χ is at most its mean for the weights ψ; but no assumption on the signs is needed. The inequality (20.8) follows by the usual symmetrisation: if D denotes its left-hand side minus its right-hand side, then

$$2D = \sum_j \sum_{j'} \big(\chi(j)\,\psi(j') - \chi(j')\,\psi(j)\big)\big(\varphi(j) - \varphi(j')\big),$$

and by the two hypotheses no term of this sum is positive.

We apply (20.8) twice. First let i and i' be integers with $0 \le i \le i' \le m-1$, and take $\chi(\ell) = v_{i'+\ell}$, $\psi(\ell) = v_{i+\ell}$ and $\varphi(\ell) = \ell$. For $\ell \le \ell'$ the inequality (20.7) with $s = i + \ell$, $t = i' + \ell$ and $d = \ell' - \ell$ reads $v_{i+\ell}\, v_{i'+\ell'} \le v_{i+\ell'}\, v_{i'+\ell}$, and this is the hypothesis on χ and ψ. Hence

$$\Big(\sum_\ell \ell\, v_{i'+\ell}\Big)\Big(\sum_\ell v_{i+\ell}\Big) \le \Big(\sum_\ell \ell\, v_{i+\ell}\Big)\Big(\sum_\ell v_{i'+\ell}\Big) \qquad (0 \le i \le i' \le m-1),$$

the sums running over $0 \le \ell \le m-1$. This in turn is the hypothesis of (20.8) for the functions $\chi(i) = \sum_\ell \ell\, v_{i+\ell}$ and $\psi(i) = \sum_\ell v_{i+\ell}$, and with $\varphi(i) = i$ we obtain

$$\Big(\sum_{i,\ell} i\,\ell\, v_{i+\ell}\Big)\Big(\sum_{i,\ell} v_{i+\ell}\Big) \le \Big(\sum_{i,\ell} i\, v_{i+\ell}\Big)\Big(\sum_{i,\ell} \ell\, v_{i+\ell}\Big), \tag{20.9}$$

where the double sums run over $0 \le i \le m-1$ and $0 \le \ell \le m-1$.

It remains to express the four double sums by M_0, M_1 and M_2. Since $v_j = 0$ for $j \ge m$, only the pairs (i,ℓ) with $i + \ell \le m-1$ contribute, and for every $j \le m-1$ the pairs with $i + \ell = j$ are the $j+1$ pairs $(i, j-i)$ with $0 \le i \le j$. Since $\sum_{i=0}^{j} i = j(j+1)/2$ and $\sum_{i=0}^{j} i\,(j-i) = (j-1)j(j+1)/6$, we find

$$\sum_{i,\ell} v_{i+\ell} = M_0, \qquad \sum_{i,\ell} i\, v_{i+\ell} = \sum_{i,\ell} \ell\, v_{i+\ell} = \tfrac{1}{2}\, M_1,$$
$$\sum_{i,\ell} i\,\ell\, v_{i+\ell} = \tfrac{1}{6}\,(M_2 - M_1).$$

With these values (20.9) is (20.6).

We now multiply (20.6) by $6h$ and eliminate $h\,M_2$ by means of (20.5). The result is

$$0 \le \tfrac{3}{2}\, h\, M_1^2 + (4h + m - mh)\, M_1\, M_0 - 2\,(m-1)\, h\, M_0^2.$$

We divide by $h\,M_0^2$, which is positive by (20.4) and by the hypothesis $h > 0$, and use $M_1 = \mu\, M_0$ and $(4h + m - mh)/h = 4 + X$. This gives

$$2\,(m-1) \le \tfrac{3}{2}\,\mu^2 + (4 + X)\,\mu.$$

Completing the square, we obtain $(4+X)^2 + 12\,(m-1) \le (3\mu + 4 + X)^2$. Since $X \ge 0$ by the hypothesis $h \le 1$, and $\mu \ge 0$, taking square roots gives

$$\sqrt{(4+X)^2 + 12\,(m-1)} \le 3\mu + 4 + X,$$

which is the inequality $\mu_m^0(X) \le \mu$. □

20.2. **Sets of cut points and the correlation inequality.** Let $m \ge 1$ be an integer. The cut points of a composition of m are the partial sums of its parts that lie strictly between 0 and m. Every set S of integers between 1 and $m-1$ is the set of cut points of exactly one composition of m: if $0 = s_0 < s_1 < \cdots < s_Q = m$ are the elements of $S \cup \{0, m\}$, this composition has the Q parts $s_i - s_{i-1}$ with $1 \le i \le Q$. Let

$$\Pi_m(S) = \prod_{i=1}^{Q} a_{s_i - s_{i-1}}$$

be the product of the Cayley coefficients of its parts, which is positive. The empty set belongs to the composition with the single part m, and $\Pi_m(\emptyset) = a_m$.

For a cut profile f, the weight of the composition with the set S of cut points is $\Pi_m(S)$ times the product of $f(s)$ over $s \in S$. So $V_Q(m; f)$ is the contribution of the sets with $Q-1$ elements to the sum over all sets, and for every cut profile f

$$\Lambda_m[f] = \sum_{S \subseteq \{1,\dots,m-1\}} \Pi_m(S) \prod_{s \in S} f(s). \tag{20.10}$$

In particular $T_{m,Q}$ is the sum of $\Pi_m(S)$ over the sets S with $Q-1$ elements, and (20.1) and (20.2) take the form

$$\tau_m(h) = \sum_S \Pi_m(S)\, h^{|S|}, \qquad \mu_m(h)\,\tau_m(h) = \sum_S |S|\,\Pi_m(S)\, h^{|S|} \tag{20.11}$$

for every real number $h \ge 0$, where S runs over the subsets of $\{1, \dots, m-1\}$ and $|S|$ is the number of elements of S. Thus the homogeneous law with the parameter h gives the set S the probability $\Pi_m(S)\, h^{|S|}/\tau_m(h)$, and $\mu_m(h)$ is the mean of $|S|$.

The proof of Proposition 20.2 applies the correlation inequality to weights of this kind. A function θ with positive values on the set of all subsets of a finite set Ω is called *log-supermodular* if $\theta(S)\,\theta(S') \le \theta(S \cap S')\,\theta(S \cup S')$ for all subsets S and S' of Ω.

For log-supermodular functions we use the correlation inequality of Fortuin, Kasteleyn and Ginibre [10, Proposition 1]. Their result concerns a positive function on a finite distributive lattice that satisfies the inequality above with the intersection and the union replaced by the meet and the join of two elements. It states that two nondecreasing real functions on the lattice are nonnegatively correlated when this function is taken as the weight: the mean of their product is at least the product of their means. The subsets of a finite set Ω, ordered by inclusion, form a distributive lattice whose meet and join are the intersection and the union. For this lattice, and after multiplication by the square of the total weight, the result reads as follows. Let θ be a log-supermodular function with positive values on the subsets of Ω, and let φ_1 and φ_2 be real functions on the subsets of Ω that do not decrease under inclusion. Then

$$\Big(\sum_S \theta(S)\,\varphi_1(S)\,\varphi_2(S)\Big)\Big(\sum_S \theta(S)\Big) \ge \Big(\sum_S \theta(S)\,\varphi_1(S)\Big)\Big(\sum_S \theta(S)\,\varphi_2(S)\Big), \tag{20.12}$$

where all sums run over the subsets S of Ω.

20.3. **The gain of the chain sum.** The proof of Proposition 20.2 first shows that the weight of the composition with the set S of cut points grows, from the first chain sum to the second, by at least the factor $(1+x)^{\sigma(S)/q}$, where x is the number of the proposition and $\sigma(S)$ is the sum of $q+1-s$ over the cut points s, so that by Jensen's inequality the chain sum grows by at least $1+x$ raised to the mean of σ/q for the weights of the first chain sum. It then bounds the weight of every cut of the first chain sum from below by the weight h of the cut at position 1, and it uses the correlation inequality to show that this mean of σ is at least the mean of σ under the homogeneous law with the parameter h. It computes the mean of σ under the homogeneous law as $(q+2)/2$ times the mean number of cuts, by reading the compositions backwards, and Lemma 20.1 bounds the mean number of cuts from below.

Proposition 20.2. *Let $k \ge 2$ and $q \ge 1$ be integers, and put*

$$x = \frac{q}{(k-1)(k+q)}, \qquad E_0 = \frac{q+2}{2q}\,\mu_q^0\Big(\frac{q^2}{k-1}\Big),$$

with the function μ_q^0 of Section 20.1. Then the chain sums satisfy

$$(1+x)^{E_0}\,\Lambda_q(k-1;k+q) \le \Lambda_q(k;k+q+1).$$

Proof. For $q = 1$ both chain sums are equal to $a_1 = 1$, and $E_0 = 0$ because $\mu_1^0(X) = 0$ for $X \geq 0$; the two sides are then equal. Let $q \geq 2$. For an integer s with $0 < s < q$ we write

$$\lambda_s = \frac{k-1}{k+q-s}, \qquad \lambda'_s = \frac{k}{k+q+1-s}$$

for the values at s of the threshold profiles $\lambda^{k-1,k+q}$ and $\lambda^{k,k+q+1}$ of Section 11; these numbers are positive, since $k \geq 2$. In this proof S denotes a subset of $\{1, \ldots, q-1\}$, a sum over S runs over all these subsets, and

$$\lambda_S = \prod_{s \in S} \lambda_s, \qquad \lambda'_S = \prod_{s \in S} \lambda'_s, \qquad \sigma(S) = \sum_{s \in S} (q+1-s).$$

We write L and L' for the two chain sums. By (20.10) with $m = q$,

$$\begin{aligned} L &= \Lambda_q(k-1; k+q) = \sum_S \Pi_q(S)\, \lambda_S, \\ L' &= \Lambda_q(k; k+q+1) = \sum_S \Pi_q(S)\, \lambda'_S, \end{aligned} \tag{20.13}$$

and all terms of both sums are positive.

We first compare the weights of one cut in the two sums. For $0 < s < q$ we have

$$\frac{\lambda'_s}{\lambda_s} = \frac{k\,(k+q-s)}{(k-1)(k+q+1-s)} = 1 + \frac{q+1-s}{(k-1)(k+q+1-s)}.$$

Because $s \geq 1$, the denominator of the last fraction is at most $(k-1)(k+q)$, so that $\lambda'_s/\lambda_s \geq 1 + x\,(q+1-s)/q$. The number $(q+1-s)/q$ lies between 0 and 1, and for an exponent in this range Bernoulli's inequality gives $(1+x)^{(q+1-s)/q} \leq 1 + x\,(q+1-s)/q$. Hence

$$\lambda_s\,(1+x)^{(q+1-s)/q} \leq \lambda'_s \qquad (0 < s < q).$$

We take the product of these inequalities over the elements s of a set S and multiply by $\Pi_q(S)$. This gives

$$\Pi_q(S)\, \lambda_S\, (1+x)^{\sigma(S)/q} \leq \Pi_q(S)\, \lambda'_S,$$

and by (20.13) the sum over S is

$$\sum_S \Pi_q(S)\, \lambda_S\, (1+x)^{\sigma(S)/q} \leq L'. \tag{20.14}$$

Now let $\bar\sigma$ be the mean of σ for the weights of the first chain sum, so that

$$\bar\sigma\, L = \sum_S \Pi_q(S)\, \lambda_S\, \sigma(S).$$

The function $t \mapsto (1+x)^t$ is convex. We apply Jensen's inequality for it, in the form recalled in Section 2 in which the weights need not be normalised, to the weights $\Pi_q(S)\,\lambda_S$ and the exponents $\sigma(S)/q$, whose mean is $\bar\sigma/q$, and obtain

$$(1+x)^{\bar\sigma/q}\, L \leq \sum_S \Pi_q(S)\, \lambda_S\, (1+x)^{\sigma(S)/q}. \tag{20.15}$$

We next compute the mean of σ when all cuts have the same weight. For every real number $h \geq 0$,

$$\sum_S \Pi_q(S)\, h^{|S|}\, \sigma(S) = \frac{q+2}{2}\, \mu_q(h)\, \tau_q(h). \tag{20.16}$$

To see this, let $S^* = \{q - s : s \in S\}$, so that $S \mapsto S^*$ is an involution of the set of subsets of $\{1, \dots, q-1\}$. The composition with the cut points S^* has the parts of the composition with the cut points S in the reverse order, so that $\Pi_q(S^*) = \Pi_q(S)$; moreover $|S^*| = |S|$ and $\sigma(S^*) = \sum_{s\in S}(s+1)$. Replacing S by S^* in the sum on the left of (20.16), we see that this sum is equal to the sum of $\Pi_q(S)\, h^{|S|} \sum_{s\in S}(s+1)$ over S. Since $(q+1-s)+(s+1) = q+2$, half the sum of the two expressions is

$$\frac{q+2}{2} \sum_S |S|\, \Pi_q(S)\, h^{|S|},$$

and this is the right-hand side of (20.16) by (20.11).

Next we show that, for all subsets S and S' of $\{1, \dots, q-1\}$,

$$\Pi_q(S)\, \Pi_q(S') \le \Pi_q(S \cap S')\, \Pi_q(S \cup S'). \tag{20.17}$$

To prove it, we first consider the effect of one additional cut point. Let $S \subseteq \{1, \dots, q-1\}$, and let u be an integer with $0 < u < q$ that does not belong to S. Among the elements of $S \cup \{0, q\}$ let $u - p$ be the largest one below u and $u + p'$ the smallest one above u, so that $p \ge 1$ and $p' \ge 1$. The composition with the cut points $S \cup \{u\}$ arises from the composition with the cut points S when the part $p + p'$ is cut into the two parts p and p', and therefore

$$\Pi_q(S \cup \{u\}) = \frac{a_p\, a_{p'}}{a_{p+p'}}\, \Pi_q(S). \tag{20.18}$$

The factor depends on p and p' only, and it does not increase when one of them grows: for all integers $p \ge 1$, $p' \ge 1$ and $d \ge 0$,

$$\frac{a_{p+d}\, a_{p'}}{a_{p+d+p'}} \le \frac{a_p\, a_{p'}}{a_{p+p'}}. \tag{20.19}$$

Indeed, after clearing denominators and cancelling $a_{p'}$ this is the inequality

$$a_{p+d}\, a_{p+p'} \le a_p\, a_{p+p'+d},$$

which is Lemma 2.2 for indices $p \le p + p'$ and the shift d. The factor is symmetric in p and p', so that the same holds when p' grows.

Now let $S_0 \subseteq S_1 \subseteq \{1, \dots, q-1\}$, and let u be an integer with $0 < u < q$ that does not belong to S_1. Let $u - p_0$ and $u + p_0'$ be the nearest elements of $S_0 \cup \{0, q\}$ below and above u, and let $u - p_1$ and $u + p_1'$ be the nearest elements of $S_1 \cup \{0, q\}$ below and above u. Since $S_0 \cup \{0, q\}$ is contained in $S_1 \cup \{0, q\}$, we have $p_1 \le p_0$ and $p_1' \le p_0'$. By (20.19), applied in the first argument with the shift $p_0 - p_1$ and then in the second with the shift $p_0' - p_1'$,

$$\frac{a_{p_0}\, a_{p_0'}}{a_{p_0+p_0'}} \le \frac{a_{p_1}\, a_{p_0'}}{a_{p_1+p_0'}} \le \frac{a_{p_1}\, a_{p_1'}}{a_{p_1+p_1'}},$$

and by (20.18) this says that

$$\frac{\Pi_q(S_0 \cup \{u\})}{\Pi_q(S_0)} \le \frac{\Pi_q(S_1 \cup \{u\})}{\Pi_q(S_1)} \qquad (S_0 \subseteq S_1,\ u \notin S_1). \tag{20.20}$$

In words, the factor by which a new cut point multiplies Π_q does not decrease when further cut points are present.

We prove (20.17) by induction on the number of elements of $S \setminus S'$. If $S \subseteq S'$, the two sides are equal. Otherwise choose $u \in S \setminus S'$ and put $S_u = S \setminus \{u\}$. Then $S_u \cap S' = S \cap S'$, and the induction hypothesis for the sets S_u and S' reads

$$\Pi_q(S_u)\, \Pi_q(S') \le \Pi_q(S \cap S')\, \Pi_q(S_u \cup S').$$

Since u does not belong to $S_u \cup S'$, we can apply (20.20) with $S_0 = S_u$ and $S_1 = S_u \cup S'$, and we obtain

$$\begin{aligned}\Pi_q(S)\,\Pi_q(S') &= \frac{\Pi_q(S_u \cup \{u\})}{\Pi_q(S_u)} \cdot \Pi_q(S_u)\,\Pi_q(S') \\ &\le \frac{\Pi_q(S \cup S')}{\Pi_q(S_u \cup S')} \cdot \Pi_q(S \cap S')\,\Pi_q(S_u \cup S') \\ &= \Pi_q(S \cap S')\,\Pi_q(S \cup S').\end{aligned}$$

This proves (20.17).

We now compare $\bar\sigma$ with the mean of σ for a constant weight. From now on let $h = (k-1)/(k-1+q)$, so that $0 < h < 1$. For $0 < s < q$ we have $\lambda_s \ge h$, because $k + q - s \le k - 1 + q$. We apply the correlation inequality (20.12) on the set $\Omega = \{1, \dots, q-1\}$ to

$$\theta(S) = \Pi_q(S)\,h^{|S|}, \qquad \varphi_1(S) = \sigma(S), \qquad \varphi_2(S) = \prod_{s \in S} \frac{\lambda_s}{h}.$$

The function θ has positive values, and it is log-supermodular by (20.17), because $|S| + |S'| = |S \cap S'| + |S \cup S'|$. Neither σ nor φ_2 decreases under inclusion, because the summands $q + 1 - s$ are positive and the factors λ_s/h are at least 1. Since $\theta(S)\,\varphi_2(S) = \Pi_q(S)\,\lambda_S$, the inequality (20.12) reads

$$\Big(\sum_S \Pi_q(S)\,\lambda_S\,\sigma(S)\Big)\,\tau_q(h) \ge \Big(\sum_S \Pi_q(S)\,h^{|S|}\,\sigma(S)\Big)\,L,$$

where the two sums without the factor σ have been evaluated by (20.11) and (20.13). The sum on the left is $\bar\sigma\,L$ by the definition of $\bar\sigma$, and the sum on the right is given by (20.16). Dividing by $\tau_q(h)\,L > 0$, we obtain

$$\bar\sigma \ge \frac{q+2}{2}\,\mu_q(h). \tag{20.21}$$

Finally, we apply Lemma 20.1 with $m = q$ and with the number h just chosen. Since $1/h - 1 = q/(k-1)$, it gives $\mu_q^0\big(q^2/(k-1)\big) \le \mu_q(h)$. Together with (20.21) and the definition of E_0 this shows that $E_0 \le \bar\sigma/q$, and hence $(1+x)^{E_0} \le (1+x)^{\bar\sigma/q}$, because $1 + x \ge 1$. By (20.15) and (20.14) we conclude that

$$(1+x)^{E_0}\,L \le (1+x)^{\bar\sigma/q}\,L \le \sum_S \Pi_q(S)\,\lambda_S\,(1+x)^{\sigma(S)/q} \le L'. \qquad \square$$

21. The growth of the curvature sum along the diagonal

We turn to the second of the three comparisons along the diagonal from which Proposition 22.6 is obtained in Section 22. By Corollary 13.7, the curvature term of a position $(k + r, k)$ of the first layer is a combination of the curvature sums $\mathsf{C}_m(k; N)$ with $1 \le m \le r$, where $N = k + r + 1$, and the curvature term of the diagonal neighbour $(k + r + 1, k + 1)$ is the corresponding combination of the sums $\mathsf{C}_m(k + 1; N + 1)$. Here we compare the two curvature sums for one $m \ge 2$ with $401\,m^2 \le 12\,k^2$: by Proposition 21.5, a diagonal step, from a position to its diagonal neighbour, multiplies C_m by at most $k/(k+1)$ times an exponential factor. The curvature sum is a sum over the compositions of m in which every cut carries a factor of the threshold profile, and the exponent of the exponential factor contains

a mean number of cuts of these compositions; we bound this mean through the birthday law of m (Lemma 21.4).

21.1. **Reflection and log-convexity in the cut profile.** We work with the weights $V_Q(m; f)$ of Section 11, the total weight of the compositions of m with exactly Q parts for a cut profile $f\colon \mathbb{N} \to \mathbb{R}$; we write Q for the number of parts. Recall that $V_0(m; f) = 0$ and that for integers $Q \geq 1$ and $m \geq 0$

$$V_Q(m; f) = \sum_{0=s_0<\cdots<s_Q=m} \prod_{i=1}^{Q} a_{s_i - s_{i-1}} \prod_{i=1}^{Q-1} f(s_i), \tag{21.1}$$

where the sum runs over the sequences of integers $0 = s_0 < s_1 < \cdots < s_Q = m$. Such a sequence is a composition of m with the parts $s_i - s_{i-1}$ and the cut points $s_1, \ldots, s_{Q-1}$. The cut points of a composition of m lie strictly between 0 and m, and the integers s with $0 < s < m$ are the *interior points* of m.

We use throughout the section the following elementary properties of these weights, which Section 11 records. The weight $V_Q(m; f)$ depends only on the values of f at the interior points of m. It is nonnegative for a nonnegative profile: if $f(s) \geq 0$ at every interior point s of m, then $V_Q(m; f) \geq 0$. It is homogeneous in the threshold of a threshold profile: for all real numbers y and N and every $Q \geq 1$

$$V_Q(m; \lambda^{y,N}) = y^{\,Q-1}\, V_Q(m; \lambda^{1,N}). \tag{21.2}$$

For the constant profile $\mathbf{1}$ the weight is $T_{m,Q} = V_Q(m; \mathbf{1})$, so that $T_{m,Q} \geq 0$, and we write $\rho\,\mathbf{1}$ for the constant profile with the value ρ.

The comparison of two profiles in Section 21.2 rests on two properties of the weights as functions of the profile. The first is a symmetry. Read from its right end, a composition of m has the same parts in the reverse order, and a cut point s has become the cut point $m - s$; the weights therefore do not change when the cut profile is reflected in the same way.

Lemma 21.1. *Let f be a cut profile, let $Q \geq 0$ and $m \geq 0$ be integers, and let $\tilde{f}$ be a cut profile with $\tilde{f}(s) = f(m - s)$ for every integer s with $0 < s < m$. Then*

$$V_Q(m; \tilde{f}) = V_Q(m; f).$$

Proof. For $Q = 0$ both sides are 0. Let $Q \geq 1$. The map that sends a sequence $0 = s_0 < s_1 < \cdots < s_Q = m$ to the sequence $s'_i = m - s_{Q-i}$, $0 \leq i \leq Q$, is an involution of the set of these sequences, and it reverses the order of the parts. The cut points of the image are the integers s'_i with $1 \leq i \leq Q - 1$. They satisfy $0 < s'_i < m$, so that $\tilde{f}(s'_i) = f(m - s'_i) = f(s_{Q-i})$ by the assumption on $\tilde{f}$. Hence the term of the image in the sum (21.1) for $\tilde{f}$ is equal to the term of the original sequence in the sum for f, and the claim follows. □

The second property is the log-convexity of the weight in the profile. By (21.1), $V_Q(m; f)$ is a polynomial in the values of f at the interior points of m, and the coefficients of this polynomial are nonnegative. For such a function Hölder's inequality gives log-convexity along geometric interpolation. We use Hölder's inequality for finite sums in the following form: if $\theta_1 > 0$ and $\theta_2 > 0$ are real numbers with $\theta_1 + \theta_2 = 1$, and $x_i \geq 0$ and $y_i \geq 0$ are real numbers indexed by a finite set I, then

$$\sum_{i \in I} x_i^{\theta_1}\, y_i^{\theta_2} \leq \Big(\sum_{i \in I} x_i\Big)^{\theta_1} \Big(\sum_{i \in I} y_i\Big)^{\theta_2}. \tag{21.3}$$

This is the usual inequality with the conjugate exponents $1/\theta_1$ and $1/\theta_2$, applied to the numbers $x_i^{\theta_1}$ and $y_i^{\theta_2}$.

Lemma 21.2. *Let $\theta_1 > 0$ and $\theta_2 > 0$ be real numbers with $\theta_1 + \theta_2 = 1$, let $Q \geq 0$ and $m \geq 0$ be integers, and let f and g be cut profiles such that $f(s) \geq 0$ and $g(s) \geq 0$ for every integer s with $0 < s < m$. Then*

$$V_Q\big(m; f^{\theta_1} g^{\theta_2}\big) \leq V_Q(m; f)^{\theta_1}\, V_Q(m; g)^{\theta_2},$$

where $f^{\theta_1} g^{\theta_2}$ denotes a cut profile that takes the value $f(s)^{\theta_1}\, g(s)^{\theta_2}$ at every integer s with $0 < s < m$.

Proof. For $Q = 0$ both sides are 0. Let $Q \geq 1$, and for a sequence $c = (s_0, \dots, s_Q)$ as in (21.1) let x_c and y_c be its terms in the sums for the profiles f and g. They are nonnegative, because f and g are nonnegative at the cut points of c, and since $\theta_1 + \theta_2 = 1$, the term of c in the sum for the profile $f^{\theta_1} g^{\theta_2}$ is $x_c^{\theta_1}\, y_c^{\theta_2}$. The claim is therefore (21.3) for the numbers x_c and y_c, indexed by the sequences c. □

21.2. **The comparison of two profiles.** We now compare one weighted sum over the compositions of m for two cut profiles. Let a weight $v_Q \geq 0$ be given for each number Q of parts with $1 \leq Q \leq m$, and for a cut profile ϕ that is nonnegative at the interior points of m put

$$S(\phi) = \sum_{Q=1}^{m} v_Q\, V_Q(m; \phi). \tag{21.4}$$

Since every $V_Q(m; \phi)$ is nonnegative, this is a nonnegative number. Lemma 21.3 bounds $S(f)$ by a multiple of $S(g)$ for two profiles f and g, without assuming that $f \leq g$. In Section 21.4 it is applied with f the profile of the diagonal neighbour of a position and g the profile of the position. The proof of the lemma needs a lower bound for the profile f; hypothesis (3) asks for one only for the products $f(s)\, f(m - s)$. A lower bound of these products by the square of a constant serves in the proof, by Lemmas 21.1 and 21.2, as a constant lower bound of f would; the proof of Proposition 21.5 verifies it for its profiles.

Lemma 21.3. *Let $k \geq 1$ and $m \geq 0$ be integers, let v_Q, $1 \leq Q \leq m$, be real numbers, let f and g be cut profiles, and let ρ and μ be real numbers. Assume the following.*

1. *One has $v_Q \geq 0$ for $1 \leq Q \leq m$.*
2. *For every integer s with $0 < s < m$ one has $f(s) \geq 0$, $g(s) \geq 0$ and $f(s)^{k+1} \leq g(s)^k$.*
3. *One has $0 < \rho \leq 1$, and $\rho^2 \leq f(s)\, f(m - s)$ for every integer s with $0 < s < m$.*
4. *The sum of $v_Q\, T_{m,Q}$ over $1 \leq Q \leq m$ is positive, and*

$$\sum_{Q=1}^{m} v_Q\, T_{m,Q}\, (Q - 1) \leq \mu \sum_{Q=1}^{m} v_Q\, T_{m,Q}.$$

Then

$$\Big(\sum_{Q=1}^{m} v_Q\, V_Q(m; f)\Big)^k \leq \rho^{-\mu} \Big(\sum_{Q=1}^{m} v_Q\, V_Q(m; g)\Big)^k.$$

Proof. By hypotheses (1) and (2) the numbers $S(f)$, $S(g)$ and $S(\mathbf{1})$ of (21.4) are defined and nonnegative, and $S(\mathbf{1}) > 0$ by hypothesis (4). We first show that

$$S(f)^{k+1} \le S(\mathbf{1})\, S(g)^k. \tag{21.5}$$

Let $\theta = k/(k+1)$, so that $0 < \theta < 1$. At an interior point s of m, hypothesis (2) gives $f(s) \le g(s)^\theta = 1^{1-\theta}\, g(s)^\theta$. By (21.1), $V_Q(m;\phi)$ is a polynomial with nonnegative coefficients in the values of a cut profile ϕ at the interior points of m. Hence, for cut profiles ϕ and ψ with $0 \le \phi(s) \le \psi(s)$ at every interior point s of m,

$$0 \le V_Q(m;\phi) \le V_Q(m;\psi). \tag{21.6}$$

By (21.6), and by Lemma 21.2 for the profiles $\mathbf{1}$ and g with the exponents $1-\theta$ and θ,

$$V_Q(m;f) \le V_Q\big(m;\mathbf{1}^{1-\theta} g^\theta\big) \le T_{m,Q}^{1-\theta}\, V_Q(m;g)^\theta \qquad (1 \le Q \le m).$$

We multiply by $v_Q = v_Q^{1-\theta}\, v_Q^\theta$, sum over Q, and apply (21.3) to the numbers $x_Q = v_Q\, T_{m,Q}$ and $y_Q = v_Q\, V_Q(m;g)$:

$$S(f) \le \sum_{Q=1}^{m} x_Q^{1-\theta}\, y_Q^\theta \le S(\mathbf{1})^{1-\theta}\, S(g)^\theta.$$

Raising this to the power $k+1$ gives (21.5).

Next we show that

$$\rho^\mu\, S(\mathbf{1}) \le S(f). \tag{21.7}$$

Let $\tilde f$ be a cut profile with $\tilde f(s) = f(m-s)$ at every interior point s of m. Then $\tilde f$ is nonnegative at the interior points of m by hypothesis (2), and hypothesis (3) gives $\rho \le f(s)^{1/2}\, \tilde f(s)^{1/2}$ there. By (21.6), by Lemma 21.2 for the profiles f and $\tilde f$ with the exponents $1/2$ and $1/2$, and by Lemma 21.1,

$$V_Q(m;\rho\,\mathbf{1}) \le V_Q\big(m; f^{1/2} \tilde f^{1/2}\big) \le V_Q(m;f)^{1/2}\, V_Q(m;\tilde f)^{1/2} = V_Q(m;f).$$

The left side is $\rho^{\,Q-1}\, T_{m,Q}$, because by (21.1) each of the $Q-1$ cut points of a composition with Q parts contributes the factor ρ to its term. Multiplying by $v_Q \ge 0$ and summing over Q, we get

$$\sum_{Q=1}^{m} v_Q\, T_{m,Q}\, \rho^{\,Q-1} \le S(f). \tag{21.8}$$

Since $S(\mathbf{1}) > 0$, there is a real number $\bar\mu$ with

$$\bar\mu\, S(\mathbf{1}) = \sum_{Q=1}^{m} v_Q\, T_{m,Q}\, (Q-1),$$

the mean of the number $Q-1$ of cuts with respect to the weights $v_Q\, T_{m,Q} \ge 0$. Jensen's inequality for the exponential function $t \mapsto \rho^t$, in the form given in Section 2 and applied with these weights and with the exponents $Q-1$, shows that the left side of (21.8) is at least $\rho^{\bar\mu}\, S(\mathbf{1})$. By hypothesis (4) we have $\bar\mu \le \mu$, and since $0 < \rho \le 1$ this implies $\rho^\mu \le \rho^{\bar\mu}$. Together with (21.8) this proves (21.7). In particular $S(f) > 0$.

By (21.5) and (21.7),

$$\rho^\mu\, S(f)^{k+1} \le \rho^\mu\, S(\mathbf{1})\, S(g)^k \le S(f)\, S(g)^k,$$

and dividing by $\rho^\mu\, S(f) > 0$ we obtain $S(f)^k \le \rho^{-\mu}\, S(g)^k$. □

21.3. **The moment ratio of the birthday law.** The mean number of cuts that the comparison needs in Section 21.4 is expressed by a number formed from the mean of the birthday law. Recall the birthday tail $B_{m,j}$, the product of $1 - i/m$ over $0 \le i < j$ (Section 11), and the mean

$$s_m = \sum_{j=1}^{m} B_{m,j}$$

of the birthday law of m (Section 12); the symbol s_m is not related to the cut points $s_0 < s_1 < \cdots < s_Q$ of a composition. Since $B_{m,j} \le 1$ for every j and $B_{m,2} < 1$, we have $s_m < m$ for $m \ge 2$. For an integer $m \ge 2$ the *moment ratio* of the birthday law of m is therefore well defined by

$$D_m = \frac{3\big((m+2)\,s_m - 3m\big)}{2\,(m - s_m)}. \tag{21.9}$$

By Proposition 11.4 the weights $T_{m,Q} = a_m\, Q\, B_{m,Q}$ are proportional to the point masses $(Q/m)\, B_{m,Q}$ of the birthday law of m, and D_m is the ratio of the third to the second factorial moment of this law: the sum of $Q(Q-1)(Q-2)\,T_{m,Q}$ over $1 \le Q \le m$ is D_m times the sum of $Q(Q-1)\,T_{m,Q}$, as is verified in the proof of Proposition 21.5. A composition with Q parts has $Q - 1 = (Q-2) + 1$ cuts. Hence $D_m + 1$ is the mean number of cuts of a composition of m when the weight of every composition with Q parts is multiplied by $Q(Q-1)$. We define D_m by the closed formula in m and s_m, and not as a quotient of two sums, because a bound for s_m then turns directly into a bound for D_m. Lemma 21.4 is obtained in this way from the upper bound of Lemma 12.13.

Lemma 21.4. *For every integer $m \ge 2$, the moment ratio D_m of the birthday law of m satisfies*

$$D_m + 1 \le \frac{3}{2}\sqrt{\frac{\pi m}{2}} - \frac{8}{5}.$$

Proof. Let $p = \sqrt{\pi m/2}$. The claim is $D_m \le 3p/2 - 13/5$. Multiplying by the denominator $2\,(m - s_m)$ of D_m, which is positive, and collecting the terms with s_m on the left side, we see that this is equivalent to

$$s_m \left(3m + 3p + \frac{4}{5}\right) \le m \left(3p + \frac{19}{5}\right). \tag{21.10}$$

For $m = 2$ and $m = 3$ we check (21.10) directly. By the definition of the birthday tails, $B_{2,1} = B_{3,1} = 1$, $B_{2,2} = 1/2$, $B_{3,2} = 2/3$ and $B_{3,3} = (2/3)(1/3) = 2/9$, so that the definition of s_m gives

$$s_2 = 1 + \frac{1}{2} = \frac{3}{2}, \qquad s_3 = 1 + \frac{2}{3} + \frac{2}{9} = \frac{17}{9}.$$

For $m = 2$ the inequality (21.10) rearranges to $p \ge 26/15$, which holds because $p^2 = \pi > 3.14$ and $(26/15)^2 < 3.01$. For $m = 3$ it is equivalent to $p \ge 32/15$, which holds because $p^2 = 3\pi/2 > 4.71$ and $(32/15)^2 < 4.56$.

Let $m \ge 4$. By Lemma 12.13, $s_m \le p\, e^{1/(12m)} - 1/3$. The number $y = 1/(12m)$ satisfies $0 \le y \le 1/48$, so that the inequality $e^y \le 1 + y + y^2$ of Section 2 gives

$$e^{1/(12m)} \le 1 + \frac{49}{48}\, y \le 1 + \frac{48}{47}\, y = 1 + \frac{4}{47m},$$

and hence

$$s_m \le p\Big(1+\frac{4}{47m}\Big)-\frac{1}{3}.$$

Since the factor of s_m in (21.10) is positive, it suffices to prove (21.10) with s_m replaced by this upper bound, and after multiplication by m this is the inequality

$$\Big(pm+\frac{4p}{47}-\frac{m}{3}\Big)\Big(3m+3p+\frac{4}{5}\Big) \le 3pm^2+\frac{19}{5}\,m^2.$$

We expand the product and replace p^2 by $\pi m/2$. The right side minus the left side is then

$$\Big(\frac{24}{5}-\frac{3\pi}{2}\Big)m^2-\frac{13}{235}\,pm-\frac{6\pi}{47}\,m-\frac{16}{235}\,p+\frac{4}{15}\,m, \tag{21.11}$$

and we have to show that this number is nonnegative. In (21.11) the numbers π and p occur with negative coefficients, so that they may be replaced by larger numbers. We use $\pi < 3.1416$ and

$$p \le \frac{p^2}{5}+\frac{5}{4} \le 0.31416\,m+1.25,$$

where the first inequality is $(p-5/2)^2 \ge 0$ and the second holds because $p^2=\pi m/2$. A direct computation with these two bounds shows that the number (21.11) is at least

$$0.0702\,m^2-0.225\,m-0.0852 = m\,(0.0702\,m-0.225)-0.0852.$$

For $m \ge 4$ we have $0.0702\,m-0.225 \ge 0.0558$, so that this is at least $4\cdot 0.0558-0.0852>0$. □

21.4. **The comparison of the curvature sums along the diagonal.** We turn to the curvature sum $\mathsf{C}_m(y;N)$ of Definition 13.3, the second difference in the threshold y of $y\,\Lambda_m(y;N)$. By Lemma 13.5 it is a sum over the number Q of parts, in which the weight of the compositions of m with Q parts is multiplied by the second difference $(y+1)^Q-2\,y^Q+(y-1)^Q$ of the power y^Q. Formula (13.2) expands this second difference in powers of y, for every real number y and every integer exponent $q \ge 0$; we use it with the letters Q and j in the place of q and i, and we write its right side as a power of y times a polynomial in $1/y$. For an integer $Q \ge 0$ and a real number x let

$$\sigma_Q(x) = 2\sum_{j\ge 1}\binom{Q}{2j}\,x^{2j-2}. \tag{21.12}$$

The sum is finite, because the binomial coefficient vanishes for $2j>Q$. The proof of Proposition 21.5 bounds σ_Q from both sides.

Now let $k \ge 1$, $m \ge 2$ and $q \ge m+1$ be integers. We compare the curvature sums $\mathsf{C}_m(k;k+q)$ and $\mathsf{C}_m(k+1;k+q+1)$. For $q=r+1$ these are the curvature sums of the positions $(k+r,k)$ and $(k+r+1,k+1)$ that were described at the beginning of the section: on a diagonal step the threshold y and the number N of the curvature sum $\mathsf{C}_m(y;N)$ are both raised by one, while the integer m whose compositions are summed stays the same. The threshold profiles $\lambda^{k,k+q}$ and $\lambda^{k+1,k+q+1}$ of the two sums, for which we write λ and λ', are

$$\lambda(s)=\frac{k}{k+q-s} \qquad \text{and} \qquad \lambda'(s)=\frac{k+1}{k+q+1-s}.$$

For an interior point s of m the number $h=q-s$ is positive, and the factors that the two profiles give to a cut at s are $\lambda(s)=1/(1+h/k)$ and $\lambda'(s)=1/(1+h/(k+1))$.

They are the values at $u = 1/k$ and at $u = 1/(k+1)$ of the function $u \mapsto 1/(1+hu)$, whose value at $u = 0$ is 1. A diagonal step thus moves u from $1/k$ to $1/(k+1)$, towards 0. The function $u \mapsto 1/(1+hu)$ is log-convex, and $1/(k+1)$ is the convex combination of 0 and $1/k$ with the coefficients $1/(k+1)$ and $k/(k+1)$. This log-convexity is the reason behind hypothesis (2) of Lemma 21.3 for $f = \lambda'$ and $g = \lambda$, which the proof of Proposition 21.5 verifies; the constant profile $\mathbf{1}$ of that lemma belongs to $u = 0$.

Proposition 21.5. *Let k, q and m be integers with $k \geq 1$, $m \geq 2$, $m+1 \leq q$ and $401\,m^2 \leq 12\,k^2$. Then the curvature sum of Definition 13.3 and the moment ratio D_m of the birthday law of m satisfy*

$$\mathsf{C}_m(k+1;\,k+q+1) \leq \frac{k}{k+1}\,\exp\!\left(\frac{\bigl(1+\frac{1}{400}\bigr)(D_m+1)\bigl(q-\frac{m}{2}\bigr)}{k\,(k+1)}\right)\mathsf{C}_m(k;\,k+q).$$

Proof. Let $\lambda = \lambda^{k,k+q}$ and $\lambda' = \lambda^{k+1,k+q+1}$ be the threshold profiles of the two curvature sums, so that $\lambda(s) = k/(k+q-s)$ and $\lambda'(s) = (k+1)/(k+q+1-s)$, and put

$$v_Q = \sigma_Q(1/k) \quad (1 \leq Q \leq m), \qquad \bar{h} = q - \frac{m}{2},$$
$$\rho = \frac{k+1}{k+1+\bar{h}}, \qquad \mu = \Bigl(1+\frac{1}{400}\Bigr)(D_m+1).$$

All sums over Q run over $1 \leq Q \leq m$. We first express the two curvature sums by the weights of their own profiles. Let (y, N) be one of the pairs $(k, k+q)$ and $(k+1, k+q+1)$. By Lemma 13.5, $\mathsf{C}_m(y;N)$ is the sum over Q of

$$\bigl((y+1)^Q - 2\,y^Q + (y-1)^Q\bigr)\,V_Q\bigl(m;\lambda^{1,N}\bigr).$$

Here $y \neq 0$, and the term j of the sum (21.12) at $x = 1/y$, multiplied by y^{Q-2}, is $\binom{Q}{2j}\,y^{Q-2j}$, the term $i = j$ of the sum in (13.2) for $q = Q$. Hence (13.2) says that the first factor is

$$(y+1)^Q - 2\,y^Q + (y-1)^Q = y^{Q-2}\,\sigma_Q(1/y).$$

The threshold profile $\lambda^{y,N}$ is y times the profile $\lambda^{1,N}$, so that the second factor is $y^{-(Q-1)}\,V_Q(m;\lambda^{y,N})$ by (21.2). Hence

$$\mathsf{C}_m(y;N) = \frac{1}{y}\sum_Q \sigma_Q(1/y)\,V_Q\bigl(m;\lambda^{y,N}\bigr).$$

We write X and Y for the sums of $v_Q\,V_Q(m;\lambda')$ and $v_Q\,V_Q(m;\lambda)$ over Q, so that $\mathsf{C}_m(k;k+q) = Y/k$. At an interior point s of m we have $q-s > q-m > 0$, so that both profiles are positive at the interior points of m, and the weights $V_Q(m;\lambda)$ and $V_Q(m;\lambda')$ are nonnegative, as recalled in Section 21.1. The polynomial σ_Q contains only even powers of x, its coefficients are nonnegative, and its constant term, the term $j = 1$ of (21.12), is $Q(Q-1)$. Consequently, for all real numbers x and x' with $0 \leq x \leq x'$,

$$Q(Q-1) \leq \sigma_Q(x) \leq \sigma_Q(x'). \tag{21.13}$$

Since $0 \leq \sigma_Q(1/(k+1)) \leq v_Q$ by (21.13), the formula for the pair $(k+1, k+q+1)$ gives $\mathsf{C}_m(k+1;k+q+1) \leq X/(k+1)$. As $\mu\,\bar{h}/(k\,(k+1))$ is the exponent in the

statement, it therefore suffices to show that

$$X \le \exp\Big(\frac{\mu\,\bar h}{k\,(k+1)}\Big)\,Y. \tag{21.14}$$

We derive (21.14) from Lemma 21.3, applied with the exponent k, the weights v_Q, the profiles $f = \lambda'$ and $g = \lambda$, and the numbers ρ and μ.

Hypothesis (1) holds by (21.13). For hypothesis (2), let s be an interior point of m and $h = q - s$, so that $\lambda(s) = 1/(1 + h/k)$ and $\lambda'(s) = 1/(1 + h/(k+1))$. The inequality between the geometric and the arithmetic mean of the $k+1$ numbers that consist of k copies of $1 + h/k$ and of the number 1 gives

$$\Big(1 + \frac{h}{k}\Big)^k \le \Big(1 + \frac{h}{k+1}\Big)^{k+1},$$

and taking reciprocals we obtain $\lambda'(s)^{k+1} \le \lambda(s)^k$. For hypothesis (3), we have $\bar h > 0$ because $q > m$, and hence $0 < \rho < 1$. For an interior point s of m,

$$\lambda'(s)\,\lambda'(m-s) = \frac{(k+1)^2}{(k+1+q-s)\,(k+1+q-m+s)}.$$

The two factors of the denominator are positive and their sum is $2\,(k+1+\bar h)$, so that their product is at most $(k+1+\bar h)^2$. Hence $\lambda'(s)\,\lambda'(m-s) \ge \rho^2$.

For hypothesis (4), let M_2 and M_3 be the sums over Q of $Q(Q-1)\,T_{m,Q}$ and $Q(Q-1)(Q-2)\,T_{m,Q}$. By Proposition 11.4 we have $T_{m,Q} = a_m\,Q\,B_{m,Q}$, so that M_2 and M_3 are a_m times the sums of $Q^2(Q-1)\,B_{m,Q}$ and $Q^2(Q-1)(Q-2)\,B_{m,Q}$. Corollaries 12.2 and 12.3 evaluate these two sums, and we obtain

$$M_2 = 2m\,a_m\,(m - s_m), \qquad M_3 = 3m\,a_m\,\big((m+2)\,s_m - 3m\big).$$

Since $a_m > 0$, and since $s_m < m$ as was noted before (21.9), the number M_2 is positive, and by the definition (21.9) of D_m,

$$M_3 = D_m\,M_2. \tag{21.15}$$

Clearly $M_3 \ge 0$, so that (21.15) shows that $D_m \ge 0$; in particular $\mu > 0$. By (21.13) we have $v_Q \ge Q(Q-1)$, and therefore

$$\sum_Q v_Q\,T_{m,Q} \ge M_2 > 0.$$

In the other direction, if $1 \le Q \le m$ and x is a real number with $m^2x^2 < 12$, then

$$\sigma_Q(x) \le \frac{Q(Q-1)}{1 - m^2x^2/12}. \tag{21.16}$$

To see this, we compare consecutive terms of (21.12). For $j \ge 1$ with $2j \le Q$,

$$\binom{Q}{2j+2} x^{2j} = \frac{(Q-2j)(Q-2j-1)}{(2j+1)(2j+2)}\,x^2 \cdot \binom{Q}{2j} x^{2j-2} \le \frac{m^2x^2}{12}\binom{Q}{2j} x^{2j-2},$$

since $0 \le (Q-2j)(Q-2j-1) \le Q^2 \le m^2$ and $(2j+1)(2j+2) \ge 12$; for $2j > Q$ both terms vanish. The terms of (21.12) are thus at most those of the geometric series with first term $Q(Q-1)$ and ratio $m^2x^2/12$, whose sum is the right side of (21.16). The hypothesis $401\,m^2 \le 12\,k^2$ says that $m^2/(12k^2) \le 1/401$, so that

(21.16) with $x = 1/k$ gives $v_Q \le (1 + 1/400)\, Q(Q-1)$ for $1 \le Q \le m$. Writing $Q - 1 = (Q-2) + 1$ and using (21.15), we obtain

$$\sum_Q v_Q\, T_{m,Q}\,(Q-1) \le \Big(1 + \frac{1}{400}\Big) \sum_Q Q(Q-1)\big((Q-2)+1\big)\, T_{m,Q}$$
$$= \Big(1 + \frac{1}{400}\Big)(M_3 + M_2) = \mu\, M_2 \le \mu \sum_Q v_Q\, T_{m,Q},$$

where the last step uses $\mu \ge 0$. This is hypothesis (4).

Lemma 21.3 now gives $X^k \le \rho^{-\mu}\, Y^k$. Since $\rho^{-1} = 1 + \bar{h}/(k+1) \le \exp\big(\bar{h}/(k+1)\big)$ and $\mu \ge 0$,

$$X^k \le \exp\Big(\frac{\mu\,\bar{h}}{k+1}\Big)\, Y^k = \left(\exp\Big(\frac{\mu\,\bar{h}}{k\,(k+1)}\Big)\, Y\right)^k.$$

Since X and Y are nonnegative, (21.14) follows by taking k-th roots. □

22. The diagonal step behind the curve

We can now prove Proposition 22.6: for $k \ge 265$, $r \ge 1$ and $r^3 \le k^2$, a step $d_{k+r,k}$ that is not positive is followed by a negative step $d_{k+r+1,k+1}$ at the diagonal neighbour. Together with Proposition 16.2 and Corollary 16.6, this gives the diagonal rule in the first layer of columns $k \ge 265$ in Section 23. The proof compares the two positions term by term, through the comparisons of Propositions 20.2 and 21.5 and a third one, for the kernel deficit (Proposition 22.2).

22.1. **The two positions.** The two positions are compared through Corollary 13.7, which writes the step at a position of the first layer as a curvature term minus a boundary term. Let $k \ge 2$ and r be integers with $1 \le r \le k-1$, and put $N = k + r + 1$. For $1 \le m \le r$ let

$$\chi_m = \frac{\varkappa_k(N-m)}{N-m}\, \mathsf{C}_m(k;N), \qquad B = \frac{a_k}{k}\, \Lambda_{r+1}(k-1;N), \tag{22.1}$$

and let χ'_m and B' be the same expressions for the diagonal neighbour $(k+r+1, k+1)$, that is, with k and N replaced by $k+1$ and $N+1$:

$$\chi'_m = \frac{\varkappa_{k+1}(N+1-m)}{N+1-m}\, \mathsf{C}_m(k+1;N+1),$$
$$B' = \frac{a_{k+1}}{k+1}\, \Lambda_{r+1}(k;N+1). \tag{22.2}$$

Here $\varkappa$ is the kernel deficit of Section 13, $\mathsf{C}_n(y;L)$ is the curvature sum of Definition 13.3, and $\Lambda_n(y;L)$ is the chain sum of Section 11. Let $C = \chi_1 + \cdots + \chi_r$ and $C' = \chi'_1 + \cdots + \chi'_r$. Corollary 13.7, applied to column k and to column $k+1$ with the same r, states that

$$\frac{d_{k+r,k}}{(k+r)!} = C - B, \qquad \frac{d_{k+r+1,k+1}}{(k+r+1)!} = C' - B'. \tag{22.3}$$

The numbers C and B are the curvature term and the boundary term of position $(k+r, k)$, and C' and B' are those of its diagonal neighbour. The two positions have the same deficit r. Their curvature terms are therefore sums over the same indices m, and the chain sums of their boundary terms run over the compositions of the same number $r+1$.

The chain sums of the boundary terms are positive. Indeed, by the properties of the threshold profile noted in Section 11, for a real number y and integers n and L we have

$$\Lambda_n(y;L) > 0 \qquad (y > 0,\ 1 \le n < L). \tag{22.4}$$

By (22.3), the hypothesis of Proposition 22.6 is $C \le B$, and its claim is $C' < B'$. The terms with $m = 1$ do not contribute to the two curvature terms: a curvature sum $\mathsf{C}_1(y;L)$ is zero, as noted after Lemma 13.5 in Section 13. Hence

$$\chi_1 = \chi_1' = 0. \tag{22.5}$$

For $r \ge 2$, and for $k \ge 265$ and $r^3 \le k^2$ as in the proposition, we shall derive the claim from the *cross inequality*

$$C'\,B < C\,B', \tag{22.6}$$

which says that the quotient of the curvature term by the boundary term decreases strictly from the position to its diagonal neighbour. The cross inequality does not involve the hypothesis $C \le B$, and it is the reason why the conclusion is strict although the hypothesis admits a step equal to zero. It is restricted to $r \ge 2$, because for $r = 1$ both of its sides are zero.

The cross inequality is proved term by term: we show that $\chi_m'\,B < \chi_m\,B'$ for $2 \le m \le r$ (Proposition 22.5). Each side of the inequality for one term is a product of three ingredients, namely a coefficient formed from a kernel deficit, a curvature sum, and the chain sum of a boundary term, and each ingredient can be followed from the position to its diagonal neighbour by itself. For the chain sum of the boundary term this was done in Proposition 20.2, by which it grows by at least an explicit factor, and for the curvature sum in Proposition 21.5, by which it grows by at most an explicit factor; the comparison of the coefficients is Proposition 22.2 below. What the three comparisons leave to be shown is an inequality between their factors (Lemma 22.4). After taking logarithms and inserting closed bounds for the factors, it becomes an inequality between explicit functions of k, r and m (Lemma 22.3), and this is where the hypothesis $r^3 \le k^2$ is needed.

22.2. **The kernel deficit along the diagonal.** The coefficient of the term χ_m in (22.1) is $\varkappa_k(t)/t$ with $t = N - m$. We write $t = k + a$, so that $a = r + 1 - m$; here and below a without an index denotes a nonnegative integer, while a_j continues to denote the Cayley coefficients. In χ_m' the coefficient is $\varkappa_{k+1}(t+1)/(t+1)$: the index of the kernel deficit and its argument have grown by one each, and their difference a is the same. We compare the two coefficients after dividing each kernel deficit by the Cayley coefficient of its argument. When the proof of Proposition 22.5 returns from these quotients to the coefficients, a diagonal step brings the factor γ_{k+a} into a term of the curvature term and the factor γ_k into the boundary term, where $\gamma_t = (1 + 1/t)^{t-2}$ is the diagonal factor of Section 7. We need to know by how much the first exceeds the second, and the following lemma bounds the difference $\log\gamma_i - \log\gamma_k$ from above by $(5/2)(1/k - 1/i)$.

Lemma 22.1. *For all integers k and i with $2 \le k \le i$, the diagonal factors satisfy*

$$\log\gamma_i - \log\gamma_k \le \frac{5}{2}\Big(\frac{1}{k} - \frac{1}{i}\Big).$$

Proof. It suffices to show that

$$\log \gamma_{m+1} - \log \gamma_m \le \frac{5}{2\,m\,(m+1)} = \frac{5}{2}\Big(\frac{1}{m} - \frac{1}{m+1}\Big) \tag{22.7}$$

for every integer $m \ge 2$; the lemma follows by adding these inequalities for $m = k, \dots, i-1$. By the definition of the diagonal factors,

$$\log \gamma_{m+1} = (m-1)\,\log\Big(1 + \frac{1}{m+1}\Big), \qquad \log \gamma_m = (m-2)\,\log\Big(1 + \frac{1}{m}\Big),$$

and the factors $m-1$ and $m-2$ are nonnegative. We bound the first logarithm from above and the second from below by the Taylor bounds of Section 2, namely $\log(1+u) \le u - u^2/2 + u^3/3$ at $u = 1/(m+1)$ and $\log(1+u) \ge u - u^2/2 + u^3/3 - u^4/4$ at $u = 1/m$. This gives

$$\begin{aligned}\log \gamma_{m+1} - \log \gamma_m \le{}& (m-1)\Big(\frac{1}{m+1} - \frac{1}{2\,(m+1)^2} + \frac{1}{3\,(m+1)^3}\Big)\\ &- (m-2)\Big(\frac{1}{m} - \frac{1}{2\,m^2} + \frac{1}{3\,m^3} - \frac{1}{4\,m^4}\Big).\end{aligned}$$

A straightforward computation shows that $5/(2m(m+1))$ minus the right-hand side equals

$$\frac{29\,(m-2)^4 + 253\,(m-2)^3 + 823\,(m-2)^2 + 1191\,(m-2) + 656}{12\,m^4\,(m+1)^3},$$

which is positive for $m \ge 2$. This proves (22.7). □

We turn to the kernel deficits $\varkappa_k(t)$ of Section 13. Let $k \ge 0$ and $a \ge 0$ be integers such that $t = k + a \ge 1$. The quotient $\varkappa_k(t)/a_t$ is the *normalised kernel deficit*. On a diagonal step it is replaced by $\varkappa_{k+1}(t+1)/a_{t+1}$, and we bound the increase by a multiple of the normalised kernel deficit itself.

The exact bound is expressed by the following closed form. For integers $k \ge 1$ and $a \ge 0$ with $2a \le k$ we put $\xi = a/k$ and

$$\Theta(k,a) = \frac{3}{2}\,\xi\,\Big(1 + \frac{2}{3\,(k+2)}\Big)\cdot \frac{1 + \frac{3}{2}\,\xi + \frac{3}{8}\,\xi^2}{\frac{23}{24} - \frac{3}{2}\,\xi\,\big(1 + \frac{\xi}{4}\big)}. \tag{22.8}$$

The ratio ξ carries no index and is unrelated to the tree curvature of Definition 5.1. Since $0 \le \xi \le 1/2$, we have $(3/2)\,\xi\,(1 + \xi/4) \le 27/32 < 23/24$. The denominator in (22.8) is therefore positive, and $\Theta(k,a) \ge 0$.

The following proposition is the relative bound in its exact form, and the form of Θ is the one that its proof produces. The proof bounds the normalised kernel deficit from below and its increase from above, both in terms of the sum of the Poisson masses $P(j)$ over $0 < j < a$. It then compares this sum with the tail of the Borel distribution by the two bounds of (2.5), and this comparison turns the bound for the increase into a multiple of the normalised kernel deficit.

Proposition 22.2. *Let $k \ge 1$ and $a \ge 0$ be integers with $2a \le k$. Then the normalised kernel deficits satisfy*

$$\frac{\varkappa_{k+1}(k+a+1)}{a_{k+a+1}} \le \Big(1 + \frac{\Theta(k,a)}{k}\Big)\,\frac{\varkappa_k(k+a)}{a_{k+a}},$$

where $\Theta(k,a)$ is the closed form (22.8).

Proof. Let $t = k + a$ and $\xi = a/k$, so that $0 \le \xi \le 1/2$, and write

$$\sigma = \frac{\varkappa_k(t)}{a_t}, \qquad \sigma' = \frac{\varkappa_{k+1}(t+1)}{a_{t+1}},$$
$$\Sigma = \sum_{0<j<a} P(j), \qquad \phi = 1 - \sum_{0<j<a} \frac{P(j)}{j}.$$

Further, let

$$\theta = \frac{3}{2}\Big(1 + \frac{2}{3\,(k+2)}\Big)\Big(1 + \frac{3}{2}\,\xi + \frac{3}{8}\,\xi^2\Big), \qquad \rho = \frac{23}{24} - \frac{3}{2}\,\xi\,\Big(1 + \frac{\xi}{4}\Big),$$

so that $\rho > 0$ and $\Theta(k,a) = \theta\,\xi/\rho$. The assertion is $\sigma' \le (1 + \Theta(k,a)/k)\,\sigma$.

We begin with two comparisons of Cayley coefficients. First, for all integers $i \ge 1$ and $j \ge 1$,

$$\frac{a_j\,a_i}{a_{i+j}} \le \frac{P(j)}{j}\,\Big(1 + \frac{3}{2}\cdot\frac{j}{i} + \frac{3}{8}\cdot\frac{j^2}{i^2}\Big). \tag{22.9}$$

Indeed, $a_j = e^j P(j)/j$, so that the left-hand side is $P(j)/j$ times $e^j a_i/a_{i+j}$, and Lemma 14.6, applied to indices $i \le i + j$, bounds $e^j a_i/a_{i+j}$ by the polynomial in j/i on the right. Secondly, for all integers $i \ge 2$ and $j \ge 0$,

$$\frac{a_i}{a_{i+j}} - \frac{a_{i+1}}{a_{i+j+1}} \le \frac{a_i}{a_{i+j}}\cdot\frac{j}{i+j}\,\Big(\frac{3}{2\,i} + \frac{1}{i\,(i+1)}\Big). \tag{22.10}$$

To see this, recall that $a_{n+1} = \gamma_n\,(1 + 1/n)\,a_n$ for $n \ge 2$, by (2.1) and (7.8). The left-hand side of (22.10) is therefore $(a_i/a_{i+j})\,(1 - 1/u)$ with

$$u = \frac{\gamma_{i+j}}{\gamma_i}\cdot\frac{1 + 1/(i+j)}{1 + 1/i} > 0.$$

Since $\log y \le y - 1$ for every $y > 0$, we have $1 - 1/u \le \log u$, and $\log u$ is the sum of $\log\gamma_{i+j} - \log\gamma_i$ and of the logarithm of the second factor of u. The first of these is at most $(5/2)(1/i - 1/(i+j)) = 5j/(2i(i+j))$ by Lemma 22.1, and the second, by the same inequality for the logarithm, is at most

$$\frac{1 + 1/(i+j)}{1 + 1/i} - 1 = -\frac{j}{(i+j)\,(i+1)}.$$

The two bounds add up to

$$\frac{j}{i+j}\,\Big(\frac{5}{2\,i} - \frac{1}{i+1}\Big) = \frac{j}{i+j}\,\Big(\frac{3}{2\,i} + \frac{1}{i\,(i+1)}\Big),$$

and (22.10) follows, because $a_i/a_{i+j} > 0$.

Next we bound the normalised kernel deficit σ from below: we show that

$$\sigma \ge \phi - \frac{3}{2}\,\Big(1 + \frac{\xi}{4}\Big)\,\frac{\Sigma}{k+1}. \tag{22.11}$$

On replacing the index of summation i in the definition of $\varkappa_k(t)$ by $j = t - i$, which runs over $0 < j < a$, and dividing by a_t, we obtain

$$\frac{\varkappa_k(t)}{a_t} = 1 - \sum_{0<j<a} \frac{a_j\,a_{t-j}}{a_t}. \tag{22.12}$$

Let $0 < j < a$ and $i = t - j$, so that $i \geq k+1$. Then $j/i \leq j/(k+1)$ and $j/i \leq a/k = \xi$, and (22.9) gives

$$\begin{aligned}\frac{a_j\, a_{t-j}}{a_t} &\leq \frac{P(j)}{j}\Big(1 + \frac{3}{2}\Big(1 + \frac{j}{4\,i}\Big)\frac{j}{i}\Big)\\ &\leq \frac{P(j)}{j} + \frac{3}{2}\Big(1 + \frac{\xi}{4}\Big)\frac{P(j)}{k+1}.\end{aligned}$$

Summing over j and inserting the result into (22.12), we obtain (22.11).

For the increase $\sigma' - \sigma$ we show that

$$\sigma' - \sigma \leq \theta \cdot \frac{\Sigma}{(k+a)\,(k+1)}. \tag{22.13}$$

The identity (22.12) holds for the pair (k,t) and for the pair $(k+1, t+1)$, with the same range $0 < j < a$ of summation. Subtracting the two identities, we get

$$\sigma' - \sigma = \sum_{0<j<a} a_j \Big(\frac{a_{t-j}}{a_t} - \frac{a_{t+1-j}}{a_{t+1}}\Big).$$

Let $0 < j < a$ and $i = t - j$, so that $i \geq k+1 \geq 2$ and $i + j = k + a$. By (22.10) and then (22.9), the term with index j is at most

$$\frac{P(j)}{j}\Big(1 + \frac{3}{2}\cdot\frac{j}{i} + \frac{3}{8}\cdot\frac{j^2}{i^2}\Big)\cdot\frac{j}{k+a}\Big(\frac{3}{2\,i} + \frac{1}{i\,(i+1)}\Big).$$

Here $j/i \leq \xi$, and since $i \geq k+1$,

$$\frac{3}{2\,i} + \frac{1}{i\,(i+1)} \leq \frac{3}{2\,(k+1)} + \frac{1}{(k+1)(k+2)} = \frac{3}{2\,(k+1)}\Big(1 + \frac{2}{3\,(k+2)}\Big).$$

The factors j cancel, so that this term is at most $\theta\, P(j)/((k+a)(k+1))$, and the sum over j gives (22.13).

We now compare the sum Σ of the Poisson masses with ϕ, and show that

$$\frac{23}{24}\,\Sigma \leq a\,\phi. \tag{22.14}$$

For $a \leq 1$ the two sums over j are empty, so that $\Sigma = 0$ and $\phi = 1$, and (22.14) holds. Let $a \geq 2$. By the upper bound of (2.5) we have $P(j) \leq c_{\mathrm{S}}/\sqrt{j}$ for $j \geq 1$, and the sum of $1/\sqrt{j}$ over $1 \leq j \leq a-1$ is at most $2\sqrt{a-1} - 1$ (Section 2). Hence $\Sigma \leq c_{\mathrm{S}}\,(2\sqrt{a-1} - 1)$. For the tail, let $n \geq a$. The lower bound of (2.5) gives

$$P(n) \geq \frac{c_{\mathrm{S}}}{\sqrt{n}}\Big(1 - \frac{1}{12\,n}\Big) \geq \frac{c_{\mathrm{S}}}{\sqrt{n}}\Big(1 - \frac{1}{12\,a}\Big),$$

and $1/(n\sqrt{n}) \geq 2\,(1/\sqrt{n} - 1/\sqrt{n+1})$ (Section 2). Hence

$$\frac{P(n)}{n} \geq 2\,c_{\mathrm{S}}\Big(1 - \frac{1}{12\,a}\Big)\Big(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\Big) \qquad (n \geq a).$$

The numbers $P(j)/j = e^{-j}a_j$, $j \geq 1$, are nonnegative, and their series has the sum 1 by Lemma 9.8; so ϕ is the sum of $P(n)/n$ over all $n \geq a$. The differences on the right of the last display telescope to $1/\sqrt{a}$, and we obtain

$$\phi \geq \frac{2\,c_{\mathrm{S}}}{\sqrt{a}}\Big(1 - \frac{1}{12\,a}\Big) = \frac{c_{\mathrm{S}}\,(2a - \frac{1}{6})}{a\,\sqrt{a}}.$$

Now let $u = \sqrt{a-1}$ and $v = \sqrt{a}$. Then $2uv \le u^2 + v^2 = 2a - 1$, hence $(2u-1)\,v \le 2a-1$, and

$$\frac{23}{24}\,\Sigma \le \frac{23}{24}\,c_{\mathrm{S}}\,(2u-1) \le \frac{23\,c_{\mathrm{S}}\,(2a-1)}{24\,v} \le \frac{c_{\mathrm{S}}\,(2a-\frac{1}{6})}{v} \le a\,\phi,$$

where the third inequality holds because $23\,(2a-1) \le 24\,(2a-1/6)$. This is (22.14).

It remains to combine the three bounds. By (22.14),

$$\frac{23}{24}\cdot\frac{\Sigma}{k+1} \le \frac{23}{24}\cdot\frac{\Sigma}{k} \le \frac{a\,\phi}{k} = \xi\,\phi.$$

We use this and (22.11), multiplied by $\xi \ge 0$, and get

$$\begin{aligned}\rho\cdot\frac{\Sigma}{k+1} &= \frac{23}{24}\cdot\frac{\Sigma}{k+1} - \frac{3}{2}\,\xi\Big(1+\frac{\xi}{4}\Big)\,\frac{\Sigma}{k+1}\\ &\le \xi\,\phi - \frac{3}{2}\,\xi\Big(1+\frac{\xi}{4}\Big)\,\frac{\Sigma}{k+1} \le \xi\,\sigma,\end{aligned}$$

and hence $\Sigma/(k+1) \le \xi\,\sigma/\rho$, because $\rho > 0$. In (22.13) we bound $1/(k+a)$ by $1/k$ and insert this inequality:

$$\sigma' - \sigma \le \frac{\theta}{k}\cdot\frac{\Sigma}{k+1} \le \frac{\theta}{k}\cdot\frac{\xi\,\sigma}{\rho} = \frac{\Theta(k,a)}{k}\,\sigma. \qquad \square$$

22.3. **The inequality between closed forms.** We now collect what the three comparisons along the diagonal give for one term. Consider a position $(k+r,k)$ as in (22.1) and an index m with $2 \le m \le r$, and put

$$q = r+1, \qquad a = q-m,$$

so that $1 \le a \le q-2$ and $k+q = N$. The proof of Proposition 22.5 bounds the growth of the coefficient of the term χ_m on a diagonal step by the factor $\gamma_{k+a}\,(1+\Theta(k,a)/k)$, by means of Proposition 22.2. By Proposition 21.5 its curvature sum grows by at most the factor $(k/(k+1))\,e^{\Upsilon_m}$, where we write

$$\Upsilon_m = \frac{\big(1+\frac{1}{400}\big)\,(D_m+1)\,\big(q-\frac{m}{2}\big)}{k\,(k+1)} \tag{22.15}$$

for the exponent of that proposition; here D_m is the moment ratio of the birthday law (Section 21). The same proof bounds the growth of the boundary term from below by the factor $\gamma_k\,(1+x)^{E_0}$, by means of Proposition 20.2, where x and E_0 are the two numbers of that proposition for the integers k and q,

$$x = \frac{q}{(k-1)(k+q)}, \qquad E_0 = \frac{q+2}{2\,q}\,\mu_q^0\Big(\frac{q^2}{k-1}\Big), \tag{22.16}$$

with the explicit function

$$\mu_q^0(X) = \frac{\sqrt{(4+X)^2 + 12\,(q-1)} - (4+X)}{3}$$

of Section 20. The hypotheses of the three statements are verified in the proof of Proposition 22.5. That proof multiplies the three comparisons and so reduces $\chi_m'\,B < \chi_m\,B'$ to the inequality

$$\gamma_{k+a}\Big(1+\frac{\Theta(k,a)}{k}\Big)\cdot\frac{k}{k+1}\cdot e^{\Upsilon_m} < \gamma_k\,(1+x)^{E_0}. \tag{22.17}$$

This is Lemma 22.4 below, and it is the source of the strict sign in the cross inequality.

We prove (22.17) in its logarithmic form, with each factor replaced by a closed bound, and this form is the following lemma. Its proof first bounds k^2 times the left side from above by an explicit expression in a, q and k, and k^2 times the right side from below by $k - 1/2$ plus a nonnegative term that comes from the gain. It then compares the two bounds, for $q \le 36$ without the gain and for $q > 36$ with it.

Lemma 22.3. *Let k, q and a be integers with $k \ge 265$, $(q-1)^3 \le k^2$, $a \ge 1$ and $a + 2 \le q$, and let $\Theta(k,a)$, x and E_0 be given by* (22.8) *and* (22.16)*. Then*

$$\frac{5}{2}\Big(\frac{1}{k} - \frac{1}{k+a}\Big) + \frac{\Theta(k,a)}{k} + \frac{401}{400}\Big(\frac{3}{2}\sqrt{\frac{\pi\,(q-a)}{2}} - \frac{8}{5}\Big) \cdot \frac{(q+a)/2}{k\,(k+1)} < \log\Big(1 + \frac{1}{k}\Big) + E_0\,\log(1+x).$$

Proof. Let $m = q - a$ and $\xi = a/k$, so that $m \ge 2$ and $q \ge 3$, and put

$$\Omega = 4.0705\,a + 6.675\,\frac{a^2}{k} + (q+a)\,\big(0.94235\,\sqrt{m} - 0.8\big),$$

$$J = k^2\,E_0\,\Big(x - \frac{x^2}{2}\Big).$$

We first show that

$$q^3 \le 1.036^2\,k^2 \qquad \text{and} \qquad q - 2 \le 0.151\,k. \tag{22.18}$$

If $q \le 42$, then $q^3 \le 42^3 < 1.036^2 \cdot 265^2$ and $q - 2 \le 40 < 0.151 \cdot 265$, so that both inequalities follow from $k \ge 265$. Let $q \ge 43$. Then $q/(q-1) \le 43/42$, hence $q^3 \le (43/42)^3\,(q-1)^3 \le 1.036^2\,k^2$, because $(43/42)^3 < 1.036^2$. Moreover the quotient $(q-2)^2/(q-1)^3$ decreases for $q \ge 4$, and for $q = 43$ it equals $41^2/42^3 < 0.151^2$. Hence $(q-2)^2 \le 0.151^2\,(q-1)^3 \le (0.151\,k)^2$. This proves (22.18). Since $a \le q - 2$, the second inequality gives $\xi \le 0.151$; in particular $2a \le k$, so that $\Theta(k,a)$ is defined.

Next we show that the left side of the asserted inequality is at most Ω/k^2, bounding its three terms in turn. The first is

$$\frac{5}{2}\Big(\frac{1}{k} - \frac{1}{k+a}\Big) = \frac{5\,a}{2\,k\,(k+a)} \le \frac{5}{2} \cdot \frac{a}{k^2}.$$

For the second, let $\rho = 23/24 - (3/2)\,\xi\,(1 + \xi/4)$ be the denominator in (22.8), which is positive. Since $k \ge 265$, we have $1 + 2/(3(k+2)) \le 1.0025$. We claim that

$$1.0025\,\Big(1 + \frac{3}{2}\,\xi + \frac{3}{8}\,\xi^2\Big) \le (1.047 + 4.45\,\xi)\,\rho. \tag{22.19}$$

Indeed, the right-hand side minus the left-hand side is

$$\frac{7}{8000} + \frac{3571}{3000}\,\xi - \frac{119097}{16000}\,\xi^2 - \frac{267}{160}\,\xi^3.$$

Since $0 \le \xi \le 0.151$, we have $\xi^2 \le 0.151\,\xi$ and $\xi^3 \le 0.151^2\,\xi$, so that this number is at least

$$\frac{7}{8000} + (1.1903 - 1.1240 - 0.0381)\,\xi \ge 0.$$

Dividing (22.19) by ρ and multiplying by $(3/2)\,\xi$, we get $\Theta(k,a) \le (3/2)\,\xi\,(1.047 + 4.45\,\xi)$, and with $\xi = a/k$

$$\frac{\Theta(k,a)}{k} \le \frac{1}{k^2}\,\Big(1.5705\,a + 6.675\,\frac{a^2}{k}\Big).$$

For the third term we use $\pi < 3.1416$, so that $\pi/2 < 1.5708 < 1.25332^2$ and $\sqrt{\pi m/2} \le 1.25332\,\sqrt{m}$. Since $(401/400)\cdot(3/2)\cdot 1.25332 < 1.8847$ and $(401/400)\cdot(8/5) \ge 1.6$,

$$\frac{401}{400}\Big(\frac{3}{2}\sqrt{\frac{\pi\, m}{2}} - \frac{8}{5}\Big) \le 1.8847\,\sqrt{m} - 1.6 = 2\,\big(0.94235\,\sqrt{m} - 0.8\big).$$

The right-hand side is positive, because $\sqrt{m} \ge 1.4$. As $q + a \ge 0$ and $k\,(k+1) \ge k^2$, the third term is at most $(q+a)\,(0.94235\,\sqrt{m} - 0.8)/k^2$. The three bounds add up to Ω/k^2.

We turn to the right side of the asserted inequality. Let $X = q^2/(k-1)$ be the number at which the function μ_q^0 is taken in (22.16), so that E_0 is $(q+2)/(2q)$ times $\mu_q^0(X)$. Since $q \ge 1$ and $X \ge 0$, the square root in $\mu_q^0(X)$ is at least $4 + X$, so that $\mu_q^0(X) \ge 0$ and $E_0 \ge 0$. Moreover $0 \le x \le 1/(k-1)$. By the inequality $\log(1+u) \ge u - u^2/2$ for $u \ge 0$ (Section 2), applied at $u = 1/k$ and at $u = x$, the right side of the asserted inequality is at least

$$\frac{1}{k} - \frac{1}{2\,k^2} + E_0\,\Big(x - \frac{x^2}{2}\Big) = \frac{k - \frac{1}{2} + J}{k^2},$$

and $J \ge 0$. It remains to show that $\Omega < k - 1/2 + J$, and for this we distinguish the cases $q \le 36$ and $q \ge 37$.

Suppose first that $q \le 36$. Here we show that $\Omega < k - 1/2$, which suffices because $J \ge 0$. The inequality $(\sqrt{m} - 7/2)^2 \ge 0$ gives $\sqrt{m} \le (m + 12.25)/7$, and hence

$$0.94235\,\sqrt{m} - 0.8 \le 0.134622\,m + 0.8491125.$$

We insert this bound, with $m = q - a$, and the bound $a^2/k \le a^2/265$ into the definition of Ω. Since $6.675/265 < 0.025189$, expanding gives

$$\Omega \le 0.134622\,q^2 + 0.8491125\,q + 4.9196125\,a - 0.109433\,a^2.$$

The first two terms increase with q, and for $q = 36$ their sum is less than 205.1. The last two terms are together at most $4.9196125^2/(4\cdot 0.109433) < 55.3$. Hence $\Omega < 260.4 < 264.5 \le k - 1/2$.

Now suppose that $q \ge 37$. Then $\sqrt{q} > 6$, and $q^{3/2} \le 1.036\,k$ by (22.18). We first bound J from below. Let $\varsigma = 4.657\,\sqrt{q} + 5.25$ and $u = 4\,(q-1)/\varsigma$. We have

$$(2.577\,\sqrt{q} - 2.75)\,\varsigma - 12\,(q-1) = 0.001089\,q + 0.7225\,\sqrt{q} - 2.4375,$$

and this is positive because $\sqrt{q} > 6$. As $3u\,\varsigma = 12\,(q-1)$ and $\varsigma > 0$, it follows that $3u < 2.577\,\sqrt{q} - 2.75$. Since $1.036\,k \le 1.04\,(k-1)$ for $k \ge 260$, we have $q^{3/2} \le 1.04\,(k-1)$. As $q^2 = \sqrt{q}\cdot q^{3/2}$, this gives $X \le 1.04\,\sqrt{q}$. Hence

$$2\,(4 + X) + 3u \le 8 + 2.08\,\sqrt{q} + 2.577\,\sqrt{q} - 2.75 = \varsigma,$$

and therefore

$$\begin{aligned}(4 + X + 3u)^2 &= (4+X)^2 + 3u\,\big(2\,(4+X) + 3u\big)\\ &\le (4+X)^2 + 3u\,\varsigma = (4+X)^2 + 12\,(q-1).\end{aligned}$$

Taking square roots, we get $u \le \mu_q^0(X)$, and so

$$E_0 \ge \frac{q+2}{2\,q}\cdot\frac{4\,(q-1)}{\varsigma} \ge \frac{2\,q}{\varsigma},$$

because $(q+2)(q-1) \geq q^2$. Next, $x - x^2/2 \geq 0.998\,x$, because $x \leq 1/(k-1) \leq 1/264$. Finally, since $q^{3/2} \leq 1.036\,k$,

$$\sqrt{q}\,(k-1)(k+q) \leq \sqrt{q}\,k^2 + k\,q^{3/2} \leq (\sqrt{q} + 1.036)\,k^2,$$

and hence

$$k^2\,x = \frac{k^2\,q}{(k-1)(k+q)} \geq \frac{q^{3/2}}{\sqrt{q} + 1.036}.$$

The product of the three bounds is

$$J \geq 0.998 \cdot \frac{q^{3/2}}{\sqrt{q} + 1.036} \cdot \frac{2\,q}{\varsigma} = \frac{1.996\,q^{5/2}}{(\sqrt{q} + 1.036)\,\varsigma} \geq 0.305\,q^{3/2}.$$

The last inequality holds because

$$1.996\,q - 0.305\,(\sqrt{q} + 1.036)\,\varsigma = 0.575615\,q - 3.07276886\,\sqrt{q} - 1.658895,$$

and the right-hand side, which is a polynomial of degree two in $\sqrt{q}$, is positive for $\sqrt{q} = 6$ and increases for $\sqrt{q} \geq 6$.

We now bound Ω from above. The inequality $(\sqrt{m} - 3\sqrt{q}/5)^2 \geq 0$ gives $\sqrt{m} \leq (m + 0.36\,q)/(1.2\,\sqrt{q})$, and $1/k \leq 1.036/q^{3/2}$. Let $h = 1/\sqrt{q}$ and $v = m/q$, so that $0 < h < 1/6$ and $0 < v < 1$, and

$$a = q\,(1-v), \qquad q + a = q\,(2-v), \qquad m = q\,v.$$

We insert these expressions and the two bounds into the definition of Ω. Since $6.675 \cdot 1.036 = 6.9153$ and $0.94235/1.2 = 18847/24000$, the result is $\Omega + 1/2 \leq q^{3/2}\,\mathcal{M}(h,v)$, where

$$\mathcal{M}(h,v) = 4.0705\,(1-v)\,h + 6.9153\,(1-v)^2\,h^2 + (2-v)\,\Big(\frac{18847}{24000}\,(v + 0.36) - 0.8\,h\Big) + \frac{h^3}{2};$$

the term $h^3/2$ accounts for $1/2 = q^{3/2}h^3/2$. We compare $\mathcal{M}(h,v)$ with its values for $h = 0$ and for $h = 1/6$. Replacing h^2 by $h/6$ and h^3 by $h/36$ in $\mathcal{M}(h,v)$ gives an expression that is affine in h and that agrees with $\mathcal{M}(h,v)$ for $h = 0$ and for $h = 1/6$. Since the coefficients of h^2 and of h^3 in $\mathcal{M}(h,v)$ are not negative, and since $h^2 \leq h/6$ and $h^3 \leq h/36$, the number $\mathcal{M}(h,v)$ is at most the value of this expression, that is,

$$\mathcal{M}(h,v) \leq (1 - 6h)\,\mathcal{M}(0,v) + 6h\,\mathcal{M}(1/6,v),$$

where

$$\mathcal{M}(0,v) = \frac{18847}{24000}\,(2-v)\,(v + 0.36) = \frac{18847}{24000}\,\big(0.72 + 1.64\,v - v^2\big),$$

$$\mathcal{M}(1/6,v) = \mathcal{M}(0,v) + \frac{1}{6}\,\Big(4.0705\,(1-v) + \frac{6.9153}{6}\,(1-v)^2 - 0.8\,(2-v) + \frac{1}{72}\Big).$$

Both are polynomials of degree two in v. The first is largest at $v = 0.82$, where its value is less than 1.094. Expanding the second, one finds that, for $v \geq 0$,

$$\mathcal{M}(1/6,v) \leq 1.1716 + 0.3587\,v - 0.5932\,v^2,$$

and the right-hand side of this inequality is at most $1.1716 + 0.3587^2/(4 \cdot 0.5932) < 1.226$. As $0 \le 6h \le 1$, it follows that $\mathcal{M}(h, v) \le 1.226$, and therefore

$$\Omega + \frac{1}{2} \le 1.226\, q^{3/2} < \Big(\frac{250}{259} + 0.305\Big)\, q^{3/2} \le k + J.$$

Here $250/259 = 1/1.036$, so that $(250/259)\, q^{3/2} \le k$, and $0.305\, q^{3/2} \le J$. Thus $\Omega < k - 1/2 + J$ in this case as well. □

With the bounds for the single factors, the lemma gives the inequality (22.17) between the factors of the three comparisons.

Lemma 22.4. *Let k, q and a be integers with $k \ge 265$, $(q-1)^3 \le k^2$, $a \ge 1$ and $a+2 \le q$, and put $m = q - a$. Let $\Theta(k, a)$ be given by (22.8), the exponent Υ_m of the curvature sum by (22.15), and the numbers x and E_0 of the gain of the boundary term by (22.16). Then the diagonal factors γ_{k+a} and γ_k satisfy*

$$\gamma_{k+a} \Big(1 + \frac{\Theta(k,a)}{k}\Big) \cdot \frac{k}{k+1} \cdot e^{\Upsilon_m} < \gamma_k\, (1+x)^{E_0}.$$

Proof. Since $(2\,(q-1))^3 \le 8\,k^2 \le k^3$, we have $2a \le 2\,(q-1) \le k$, so that $\Theta(k, a)$ is defined and nonnegative. Both sides of the asserted inequality are positive, and it suffices to compare their logarithms, that is, to show that

$$\log \gamma_{k+a} - \log \gamma_k + \log\Big(1 + \frac{\Theta(k,a)}{k}\Big) + \Upsilon_m \\ < \log\Big(1 + \frac{1}{k}\Big) + E_0\, \log(1+x).$$

By Lemma 22.1, applied to indices $2 \le k \le k+a$, the difference $\log \gamma_{k+a} - \log \gamma_k$ is at most $(5/2)(1/k - 1/(k+a))$. The logarithm of $1 + \Theta(k,a)/k$ is at most $\Theta(k,a)/k$, because $\log y \le y - 1$ for $y > 0$. By Lemma 21.4, which applies because $m \ge 2$, we have $D_m + 1 \le (3/2)\sqrt{\pi m/2} - 8/5$, and $q - m/2 = (q+a)/2 \ge 0$, so that

$$\Upsilon_m \le \frac{401}{400}\Big(\frac{3}{2}\sqrt{\frac{\pi\, m}{2}} - \frac{8}{5}\Big) \cdot \frac{(q+a)/2}{k\,(k+1)}.$$

Since $m = q - a$, the sum of the three bounds is the left side of the inequality of Lemma 22.3, which has the same hypotheses, and the right side of that inequality is the right side of the inequality to be shown. The assertion follows. □

22.4. **The cross inequality.** We can now compare one term of the curvature term with the boundary term, at a position behind the curve and at its diagonal neighbour. In the notation of (22.1) and (22.2), the following proposition states that $\chi'_m\, B < \chi_m\, B'$ for $2 \le m \le r$.

Proposition 22.5. *Let k, r and m be integers with $k \ge 265$, $r^3 \le k^2$ and $2 \le m \le r$, and put $N = k + r + 1$. Then*

$$\frac{\varkappa_{k+1}(N+1-m)}{N+1-m}\, \mathsf{C}_m(k+1; N+1) \cdot \frac{a_k}{k}\, \Lambda_{r+1}(k-1; N) \\ < \frac{\varkappa_k(N-m)}{N-m}\, \mathsf{C}_m(k; N) \cdot \frac{a_{k+1}}{k+1}\, \Lambda_{r+1}(k; N+1).$$

Proof. Let $q = r + 1$, $a = q - m$ and $t = k + a$. Then $1 \le a \le r - 1$, $a + 2 \le q$, $t = N - m$ and $t + 1 = N + 1 - m$, and $N = k + q$. Let Υ_m be the exponent (22.15), and let x and E_0 be the numbers (22.16).

We have $2r \le k$, because $(2r)^3 \le 8\,k^2 \le k^3$. Hence $r \le k-1$, so that the notation of (22.1) and (22.2) is available, and $2a \le 2\,(r-1) \le k$. Moreover $401\,m^2 \le 12\,k^2$. Indeed, if $r \le 41$, then $m \le 41$ and $401 \cdot 41^2 \le 12 \cdot 265^2$; and if $r \ge 42$, then $m^2 \le r^2 \le r^3/42 \le k^2/42$ and $401 \le 12 \cdot 42$.

We first compare the coefficients $\varkappa_k(t)/t$ and $\varkappa_{k+1}(t+1)/(t+1)$ of the terms χ_m and χ'_m. Proposition 22.2 applies because $2a \le k$. By (2.1) and (7.8) we have $a_{n+1} = \gamma_n\,(1+1/n)\,a_n$ for $n \ge 2$, and on dividing by $n+1$ we obtain

$$\frac{a_{n+1}}{n+1} = \gamma_n\,\frac{a_n}{n} \qquad (n \ge 2). \tag{22.20}$$

We multiply the inequality of Proposition 22.2 by $a_{t+1}/(t+1)$, which equals $\gamma_t\,a_t/t$ by (22.20), and obtain

$$\frac{\varkappa_{k+1}(t+1)}{t+1} \le \gamma_t\,\Big(1 + \frac{\Theta(k,a)}{k}\Big)\,\frac{\varkappa_k(t)}{t}. \tag{22.21}$$

Next, Proposition 21.5 compares the curvature sums $\mathsf{C}_m(k+1;k+q+1)$ and $\mathsf{C}_m(k;k+q)$ under the hypotheses $k \ge 1$, $m \ge 2$, $m+1 \le q$ and $401\,m^2 \le 12\,k^2$. These hold for $q = r+1$, and since $k+q = N$ the proposition gives

$$\mathsf{C}_m(k+1;N+1) \le \frac{k}{k+1}\,e^{\Upsilon_m}\,\mathsf{C}_m(k;N). \tag{22.22}$$

For the boundary terms, Proposition 20.2, for the integers $k \ge 2$ and $q = r+1 \ge 1$, states that

$$(1+x)^{E_0}\,\Lambda_{r+1}(k-1;N) \le \Lambda_{r+1}(k;N+1).$$

We multiply by $\gamma_k\,a_k/k$, which equals $a_{k+1}/(k+1)$ by (22.20), and obtain

$$\gamma_k\,(1+x)^{E_0}\,B \le B'. \tag{22.23}$$

Before we multiply these inequalities, we check the signs of their terms. The kernel deficits $\varkappa_k(t)$ and $\varkappa_{k+1}(t+1)$ are positive by Lemma 13.1, since $1 \le t \le 2k+1$ and $1 \le t+1 \le 2\,(k+1)+1$. The boundary term B is positive by (22.4). The curvature sum $\mathsf{C}_m(k;N)$ is positive as well. Indeed, by Lemma 13.5,

$$\mathsf{C}_m(k;N) = \sum_{Q=1}^{m}\big((k+1)^Q - 2\,k^Q + (k-1)^Q\big)\,V_Q\big(m;\lambda^{1,N}\big).$$

Here Q is the number of parts, which Lemma 13.5 writes q; in this proof q denotes $r+1$. No term of this sum is negative: the second differences $(k+1)^Q - 2\,k^Q + (k-1)^Q$ are nonnegative by convexity, and so are the weights $V_Q(m;\lambda^{1,N})$, whose cut factors $1/(N-s)$ are positive for $0 < s < m$. The term with $Q = 2$ is $2\,V_2(m;\lambda^{1,N})$, twice the sum of $a_s\,a_{m-s}/(N-s)$ over $0 < s < m$, which is positive because $m \ge 2$. Hence $\mathsf{C}_m(k;N) > 0$, and $\chi_m > 0$.

Multiplying (22.22) by $\varkappa_{k+1}(t+1)/(t+1)$ and using (22.21), we now get

$$\chi'_m \le \gamma_t\,\Big(1 + \frac{\Theta(k,a)}{k}\Big)\,\frac{k}{k+1}\,e^{\Upsilon_m}\cdot\chi_m. \tag{22.24}$$

By Lemma 22.4 for k, q and a, whose hypothesis $(q-1)^3 \le k^2$ is $r^3 \le k^2$ and in which $q - a = m$ and $k + a = t$, the factor of χ_m in (22.24) is smaller than $\gamma_k\,(1+x)^{E_0}$. As $\chi_m\,B > 0$, we conclude with (22.23) that

$$\begin{aligned}\chi'_m\,B &\le \gamma_t\,\Big(1 + \frac{\Theta(k,a)}{k}\Big)\,\frac{k}{k+1}\,e^{\Upsilon_m}\cdot\chi_m\,B\\ &< \gamma_k\,(1+x)^{E_0}\cdot\chi_m\,B \le \chi_m\,B',\end{aligned}$$

which is the assertion. □

The proof of the following proposition sums the inequalities of Proposition 22.5 over m and obtains the cross inequality (22.6) for $r \geq 2$.

Proposition 22.6. *Let k and r be integers with $k \geq 265$, $r \geq 1$ and $r^3 \leq k^2$. If $d_{k+r,k} \leq 0$, then $d_{k+r+1,k+1} < 0$.*

Proof. Since $(2r)^3 \leq 8\,k^2 \leq k^3$, we have $2r \leq k$, and hence $1 \leq r \leq k-1$. Both positions therefore lie in the first layer of their columns, and (22.3) holds with the notation of (22.1) and (22.2). The hypothesis $d_{k+r,k} \leq 0$ is $C \leq B$, and the claim $d_{k+r+1,k+1} < 0$ is $C' < B'$. The boundary terms B and B' are positive by (22.4), and the terms with $m = 1$ vanish by (22.5).

If $r = 1$, then $C' = \chi'_1 = 0$, and $C' < B'$ holds because $B' > 0$. Let $r \geq 2$. For $2 \leq m \leq r$, Proposition 22.5 gives $\chi'_m\, B < \chi_m\, B'$, and $\chi'_1\, B = \chi_1\, B'$. The sum of these relations over $1 \leq m \leq r$, in which the one for $m = 2$ is strict, is the cross inequality $C'\, B < C\, B'$. Since $C \leq B$ and $B' > 0$, it follows that $C'\, B < C\, B' \leq B\, B'$, and hence $C' < B'$. □

The restriction of Proposition 22.6 to the positions behind the curve comes from Lemma 22.3, whose inequality is proved in the range $(q-1)^3 \leq k^2$. For columns $k \leq 264$ the diagonal rule in the first layer is obtained with the help of the computed lemmas of Section 6, in Proposition 23.2.

23. The two rules and the proofs of the theorems

We now prove Theorems 1.2 and 1.1. The column rule and the diagonal rule for the array $(b_{n,k})$ are assembled from the statements of the earlier sections, region by region (Propositions 23.1 and 23.3), and an induction on the row, for an arbitrary array of natural numbers, derives from them the conclusion of Theorem 1.2 (Proposition 23.4). We pass between an inequality for a step $d_{n,k}$ and the corresponding inequality between two entries of the array without comment.

23.1. **The two rules.** We begin with the column rule, which has been proved in two parts, according to the region in which the lower position $(n+1, k)$ lies.

Proposition 23.1. *Let $n \geq 2$ and k be integers with $1 \leq k \leq n-1$. If $d_{n,k} > 0$, then $d_{n+1,k} > 0$.*

Proof. Suppose that $d_{n,k} > 0$. If $n+1 \leq 2k-1$, then $k+1 \leq n \leq 2k-2$, and Proposition 16.5 gives $d_{n+1,k} > 0$. If $n+1 \geq 2k$, the claim is Corollary 19.7. □

We turn to the diagonal rule, and begin with the first layer of columns $2 \leq k \leq 264$, which is as far as the computations of Section 6 reach.

Proposition 23.2. *Let k and n be integers with $2 \leq k \leq 264$ and $k+1 \leq n \leq 2k-1$. If $d_{n,k} \leq 0$, then $d_{n+1,k+1} < 0$.*

Proof. Let first $2 \leq k \leq 8$. Then $3 \leq k+1 \leq 9$ and $(k+1)+1 \leq n+1 \leq 2(k+1)-2$, so that Lemma 6.1, applied to column $k+1$ and to row $n+1$ with its fourth condition, gives $d_{n+1,k+1} < 0$.

Now let $9 \leq k \leq 264$, and put $j = k + \zeta(k) + 1$, where $\zeta(k)$ is the number of Definition 6.5. By Lemma 6.6 we have $d_{j,k} > 0$. If $n \geq j$, then $k+1 \leq j \leq n \leq 2k-1$, and Corollary 16.6 gives $d_{n,k} > 0$, contrary to the hypothesis. Hence $n \leq j-1$, so that $k+2 \leq n+1 \leq k+\zeta(k)+1$, and Lemma 6.7 gives $d_{n+1,k+1} < 0$. □

We can now prove the diagonal rule at all positions.

Proposition 23.3. *Let $n \geq 2$ and k be integers with $1 \leq k \leq n-1$. If $d_{n,k} \leq 0$, then $d_{n+1,k+1} < 0$.*

Proof. Suppose that $d_{n,k} \leq 0$. If $n \geq 2k$, the claim is Corollary 19.8. Let $n \leq 2k-1$, so that $k \geq 2$ and $k+1 \leq n \leq 2k-1$. For $k \leq 264$ the claim is Proposition 23.2.

It remains to consider $k \geq 265$, with $k+1 \leq n \leq 2k-1$ as before. By Proposition 16.2 there is an integer $j > k$ with $(j-k-1)^3 \leq k^2$ and $d_{j,k} > 0$. If $n \geq j$, then $k+1 \leq j \leq n \leq 2k-1$, and Corollary 16.6 gives $d_{n,k} > 0$, contrary to the hypothesis. Hence $n \leq j-1$, so that the deficit $r = n-k$ satisfies $1 \leq r \leq j-k-1$ and

$$r^3 \leq (j-k-1)^3 \leq k^2.$$

Since $d_{k+r,k} = d_{n,k} \leq 0$, Proposition 22.6 gives $d_{n+1,k+1} < 0$. □

23.2. **The induction on the row.** It remains to pass from the two rules to the shape of the rows. We do this for an arbitrary array of natural numbers, with the notions of Section 1.

Proposition 23.4. *Let $f(n,k)$, for integers $n \geq 0$ and $k \geq 0$, be an array of natural numbers. Suppose that for all integers $n \geq 2$ and k with $1 \leq k \leq n-1$ the following two implications hold: the column rule,*

$$f(n,k-1) < f(n,k) \quad \textit{implies} \quad f(n+1,k-1) < f(n+1,k),$$

and the diagonal rule,

$$f(n,k) \leq f(n,k-1) \quad \textit{implies} \quad f(n+1,k+1) < f(n+1,k).$$

Then for every integer $n \geq 2$ there is an integer $p \geq 0$ such that row n of f has its peak at position p, and row $n+1$ of f has its peak at position p or a strict peak at position $p+1$.

Proof. Let $n \geq 2$, and suppose that row n of f has its peak at position p. Then $p \leq n-1$, and we have $f(n,k-1) < f(n,k)$ for $1 \leq k \leq p$ and $f(n,k) \leq f(n,k-1)$ for $p+1 \leq k \leq n-1$. The column rule at the positions (n,k) with $k \leq p$ and the diagonal rule at those with $k \geq p+1$ therefore give

$$f(n+1,0) < \cdots < f(n+1,p) \quad \text{and} \quad f(n+1,p+1) > \cdots > f(n+1,n).$$

Hence row $n+1$ has a strict peak at position $p+1$ if $f(n+1,p) < f(n+1,p+1)$, and its peak at position p otherwise; in either case it has its peak at some position. Since row 2 consists of two entries, it has its peak at position 0 or 1, and it follows by induction on n that every row $n \geq 2$ of f has its peak at some position p. By what we have shown, row $n+1$ then has its peak at position p or a strict peak at position $p+1$. □

23.3. **Proofs of the two theorems.**

Proof of Theorem 1.2. By Propositions 23.1 and 23.3, the array $(b_{n,k})$ satisfies the column rule and the diagonal rule. The theorem therefore follows from Proposition 23.4, applied to this array. □

Proof of Theorem 1.1. For $n = 1$ the sequence has a single term and is unimodal. For $n \geq 2$, Theorem 1.2 gives a position p at which row n of the array $(b_{n,k})$ has its peak, and the definition of a unimodal sequence is satisfied with $j^* = p$. □

References

1. N. H. Abel, *Beweis eines Ausdruckes, von welchem die Binomial-Formel ein einzelner Fall ist*, Journal für die reine und angewandte Mathematik **1** (1826), 159–160.
2. Bjorn Andreas Ager-Hart, Melissa Beerbower, Pamela E. Harris, Joakim Jakovleski, and Matt McClinton, *Counting ℓ-interval Fubini rankings through their parking outcome*, 2026, Preprint, arXiv:2603.00693.
3. Tomás Aguilar-Fraga, Jennifer Elder, Rebecca E. Garcia, Kimberly P. Hadaway, Pamela E. Harris, Kimberly J. Harry, Imhotep B. Hogan, Jakeyl Johnson, Jan Kretschmann, Kobe Lawson-Chavanu, J. Carlos Martínez Mori, Casandra D. Monroe, Daniel Quiñonez, Dirk Tolson, and Dwight Anderson Williams, *Interval and ℓ-interval rational parking functions*, Discrete Mathematics & Theoretical Computer Science **26** (2024), no. 1, Paper No. 10, 29 pp.
4. David Aldous and Jim Pitman, *Tree-valued Markov chains derived from Galton-Watson processes*, Annales de l'Institut Henri Poincaré (B) Probability and Statistics **34** (1998), no. 5, 637–686.
5. Kyle Celano, Jennifer Elder, Kimberly P. Hadaway, Pamela E. Harris, Jeremy L. Martin, Amanda Priestley, and Gabe Udell, *ℓ-Interval parking functions: Foata invariance, statistic generating functions, and ciphers*, 2026, Preprint, arXiv:2507.07243v2; listed on arXiv under the title of its first version, "Statistics on ℓ-interval parking functions".
6. N. G. de Bruijn and P. Erdős, *On a recursion formula and on some Tauberian theorems*, Journal of Research of the National Bureau of Standards **50** (1953), no. 3, 161–164, Research Paper 2404.
7. Philippe Flajolet, Peter J. Grabner, Peter Kirschenhofer, and Helmut Prodinger, *On Ramanujan's Q-function*, Journal of Computational and Applied Mathematics **58** (1995), no. 1, 103–116.
8. Philippe Flajolet and Robert Sedgewick, *Analytic combinatorics*, Cambridge University Press, Cambridge, 2009.
9. Dominique Foata and John Riordan, *Mappings of acyclic and parking functions*, Aequationes Mathematicae **10** (1974), no. 1, 10–22.
10. C. M. Fortuin, P. W. Kasteleyn, and J. Ginibre, *Correlation inequalities on some partially ordered sets*, Communications in Mathematical Physics **22** (1971), no. 2, 89–103.
11. Svante Janson and Niclas Petersson, *The integral of the supremum process of Brownian motion*, Journal of Applied Probability **46** (2009), no. 2, 593–600.
12. Alan G. Konheim and Benjamin Weiss, *An occupancy discipline and applications*, SIAM Journal on Applied Mathematics **14** (1966), no. 6, 1266–1274.
13. Joseph Pappe, *A note on the log-concavity of parking functions*, 2024, Preprint, arXiv:2412.19783.
14. Niclas Petersson, *The maximum displacement for linear probing hashing*, Ph.D. thesis, Uppsala University, Uppsala, 2009, Uppsala Dissertations in Mathematics 63.
15. Herbert Robbins, *A remark on Stirling's formula*, The American Mathematical Monthly **62** (1955), no. 1, 26–29.
16. Thomas Selig and Haoyue Zhu, *Parking functions and Łukasiewicz paths*, Discrete Mathematics Letters **14** (2024), 77–84.
17. Catherine H. Yan, *Parking functions*, Handbook of Enumerative Combinatorics (Miklós Bóna, ed.), Discrete Mathematics and its Applications, CRC Press, Boca Raton, FL, 2015, pp. 835–893.