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The free von Neumann algebra zoo

Teo 2.0 and GPT 6.2

Abstract

We study the von Neumann algebras of free quantum groups, and of their basic homogeneous spaces. Let $F = L(F_N)$ be the free group factor. We have $L^\infty(H_4^{s+}) \simeq F$ for $s = 2, 3, \ldots, \infty$. For $N \geq 2$ each matrix entry generates an irreducible copy of F in $L^\infty(U_N^+)$. For $N \geq 4$ the signs of 3 entries in one row yield such a copy in $L^\infty(O_N^+)$. For $N \geq 2$ we have $L^\infty(S^{N-1}_{\mathbb{R},+}) \simeq L^\infty(P_+^{N-1}) \rtimes \mathbb{Z}_2$ and $L^\infty(S^{N-1}_{\mathbb{C},+}) \simeq L(\mathbb{Z}) * L^\infty(P_+^{N-1})$. These formulas extend to frames, with infinite index between distinct row lengths in the factorial ranges. For $M, N \geq 3$, let O_{MN}^{1+} be the space of real rank-one free partial isometries of size $M \times N$. The magnitude inclusion in $L^\infty(O_{MN}^{1+})$ determines both side lengths up to exchange. This prime nonsolid factor contains a tensor product of 2 nonamenable factors at index 2. In $L^\infty(U_{MN}^{1+})$ every such subfactor has infinite index for $M, N \geq 2$. For $N \geq 3$, the radial free ball algebra $L^\infty(B^N_{\mathbb{R},+}, \tau_{\rm rad})$ has diffuse center, and each projected free ball algebra $L^\infty(B^N_{\mathbb{R},+}, \tau_{N,j})$ is a strongly solid factor for $j \geq 1$.

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Introduction

Let $F = L(F_N)$ be the free group factor [42]. The free orthogonal and unitary groups O_N^+, U_N^+ give two different starting points for the comparison with F. The orthogonal factor is separated from F by strong 1-boundedness [12]. The unitary factor escapes this obstruction in every dimension and is known to be F in dimension 2 [2, 11]. For the quantum reflection groups we can identify the von Neumann algebra at size 4. Let H_N^{s+} be the free wreath product of $\widehat{\mathbb{Z}}_s$ by S_N^+, with $\mathbb{Z}_\infty = \mathbb{Z}$, in the sense of Bichon [8]. We have

$$L^\infty(H_4^{s+}) \simeq L(F_{2-1/s}) \simeq F$$

for every integer $s \geq 2$ and for $s = \infty$, where $1/\infty = 0$. In particular the free hyperoctahedral group H_4^+ gives F. The free dimension calculation gives the parameter $2 - 1/s$. The preprint

[42] identifies the resulting factor with F. We also use it to compare the even sign actions across dimensions, both for rows and for rectangles. The decompositions of the sphere and rectangle algebras hold independently of this identification. For O_N^+ and U_N^+ we can ask how much of the matrix one needs to generate a copy of F.

Write $u = (u_{ij})$ for the fundamental representation of O_N^+ and put $x_i = u_{1i}$. Projectivization removes the scalars, so the real and complex free spheres have the same projective row algebra [1], namely

$$L^\infty(P_+^{N-1}) = W^*(x_i x_j \mid 1 \leq i, j \leq N).$$

The distinction between this one-row algebra and the full projective group algebra $L^\infty(PO_N^+)$ will matter throughout.

Theorem A. *For $N \geq 3$ the projective row algebra $L^\infty(P_+^{N-1})$ has trivial relative commutant in $L^\infty(O_N^+)$. For $N \geq 4$ the 2 free Haar unitaries $s_1 s_2, s_1 s_3$, where $s_i = sgn(x_i)$, already generate an irreducible copy of F in $L^\infty(O_N^+)$. In the fundamental matrix v of U_N^+, any single entry generates an irreducible copy of F for $N \geq 2$. In each case every intermediate algebra is a nonamenable strongly solid II_1 factor.*

In every dimension $N \geq 3$ the unitaries $s_1 s_2, \ldots, s_1 s_N$ give a canonical copy of F inside $L^\infty(P_+^{N-1})$. They generate a proper subalgebra, already because x_1^2 is diffuse and its conditional expectation onto that copy is scalar. This leaves open whether other free Haar unitaries generate the entire projective algebra.

Theorem B. *For $N \geq 2$ we have trace-preserving isomorphisms*

$$L^\infty(S_{\mathbb{R},+}^{N-1}) \simeq L^\infty(P_+^{N-1}) \rtimes \mathbb{Z}_2, \qquad L^\infty(S_{\mathbb{C},+}^{N-1}) \simeq L(\mathbb{Z}) * L^\infty(P_+^{N-1}),$$

where the generator of $\mathbb{Z}_2$ acts by conjugation with $s_1 = sgn(x_1)$. The action is outer for $N \geq 3$. For a frame of L rows the same formulas hold with the projective row algebra replaced by the algebra generated by all products of 2 real frame entries. Increasing the row length gives irreducible inclusions of infinite index in the real tower for $N \geq 3$ and in the complex tower for $N \geq 2$.

At full frame length the second formula gives a free product presentation of $L^\infty(U_N^+)$ with $L^\infty(PO_N^+)$ as its other component. At frame length one it gives a test for the conjecture $L^\infty(P_+^{N-1}) \simeq F$, since this conjecture would imply $L^\infty(S_{\mathbb{C},+}^{N-1}) \simeq F$. The real formula imposes a different task, that of identifying an outer action on a factor.

Theorem C. *For $M, N \geq 3$ the algebra $L^\infty(O_{MN}^{1+})$ is a prime, nonsolid II_1 factor. It is the fixed algebra of $L^\infty(S_{\mathbb{R},+}^{M-1}) \bar{\otimes} L^\infty(S_{\mathbb{R},+}^{N-1})$ under simultaneous sign change, and contains its projective algebra $L^\infty(P_+^{M-1}) \bar{\otimes} L^\infty(P_+^{N-1})$ with index 2.*

Thus solidity separates the rectangular factor from F, from the free orthogonal and unitary factors, and from $L^\infty(S_N^+)$ for $N \geq 5$. Primeness separates it from its own projectivization, even if the 2 projective legs should both be F. When one side has length 2 and the other has length $N \geq 3$, the rectangle has diffuse center and constant central fiber $M_2(\mathbb{C}) \bar{\otimes} L^\infty(P_+^{N-1})$.

How much of N survives in the involution on the projective row algebra? Its restrictions to the even sign algebras are conjugate for all $N \geq 3$. To compare the full actions we retain one subalgebra. For real sphere coordinates x_i, let D be generated by all $u|x_i|u^*$, where u ranges over the words in the coordinate signs. We call D the magnitude algebra. For a minimal central projection p of D, write

$$W(p) = \mathcal{N}_{pL^\infty(P_+^{N-1})p}(pDp)/\mathcal{U}(pDp)$$

for the normalizing unitaries of pDp modulo its own unitary group.

Theorem D. *Let $N \geq 3$ and let D be the magnitude algebra in $L^\infty(P_+^{N-1})$. Its center has finite dimension k, and*

$$W(p) \simeq F_{1+k(N-2)}$$

for every minimal central projection p of D. For $N \geq 4$, D is a factor and $W(1) \simeq F_{N-1}$. In every dimension $N \geq 3$ the inclusion determines N.

For $1 \leq K \leq \min(M, N)$, let O_{MN}^{K+} denote the space of real rank-K free partial isometries of size $M \times N$ [1]. Its universal coordinate matrix $z = (z_{ia})$ has self-adjoint entries, and zz^t is a projection with unnormalized matrix trace $\sum_i (zz^t)_{ii} = K\,1$. In the orbit trace its coordinates are

$$z_{ia} = \sum_{r=1}^{K} u_{ri} \otimes v_{ra},$$

where u and v are the fundamental matrices of O_M^+ and O_N^+ on separate tensor legs. At $K = 1$ this gives the rectangular coordinates $z_{ia} = x_i \otimes y_a$, with $x_i = u_{1i}$ and $y_a = v_{1a}$. The absolute values $|z_{ia}|$ and their conjugates by words in the coordinate signs generate the rectangular magnitude algebra E. Thus we can ask the dimensional question inside the rectangle without specifying either of its 2 tensor legs.

Theorem E. *Let $M, N \geq 3$ and let $E \subset L^\infty(O_{MN}^{1+})$ be the rectangular magnitude algebra. For this inclusion we have*

$$\frac{\chi(W(p))}{\dim Z(E)} = \frac{(M-2)(N-2)}{2}$$

at every minimal central projection p of E. Here $W(p)$ is the unitary normalizer of pEp in $pL^\infty(O_{MN}^{1+})p$ modulo $\mathcal{U}(pEp)$, and χ is the rational Euler characteristic. The isomorphism class of the inclusion determines both side lengths up to exchange.

In a square of rank $K = N - 1$ with $N \geq 3$, summing products of rectangular coordinates recovers the projective algebra of each omitted row. The rectangular algebra is again a nonsolid factor. Both inclusions, from the product of the projective rows into the rectangle and from the rectangle into the product of the frames, now have infinite index.

For the real rectangles of sizes $(3, 3)$ and $(3, 4)$, the normalized Euler characteristic of the magnitude inclusion equals $1/2$ and 1, respectively. Thus no isomorphism between their von Neumann algebras can carry the magnitude algebra of one onto that of the other. For $(3, 6)$ and $(4, 4)$ the ratio is 2 in both cases, and the center and the normalizer quotient together distinguish the inclusions. For the factors themselves, the comparison involves the full involutions on the projective rows.

On the closed real free ball $B_{\mathbb{R},+}^N$, with boundary sphere $S_{\mathbb{R},+}^{N-1}$, we consider 2 choices of volume. The divergence prescription in [18] uses a scalar radius and the invariant trace of the boundary sphere. A second prescription keeps N coordinates of a sphere with $N + j$ coordinates. Classically the 2 volumes agree for $j = 2$.

Theorem F. *For $N \geq 3$ the center of the radial free ball algebra $L^\infty(B_{\mathbb{R},+}^N, \tau_{\mathrm{rad}})$ is the diffuse algebra generated by its radius. For every $j \geq 1$ the projected free ball algebra $L^\infty(B_{\mathbb{R},+}^N, \tau_{N,j})$ is an irreducible nonamenable strongly solid II_1 subfactor of $L^\infty(O_{N+j}^+)$.*

A rectangular block of a real free Stiefel frame also generates an irreducible subfactor as soon as 3 entries of a row are retained in ambient dimension at least 4. The coordinates of a higher Grassmannian are sums over rows. The first case where these sums intervene is Gr_{24}^+.

We now compare a square and a rectangle sharing one side. The normalized Euler characteristic uses the rectangular magnitude algebra as part of the inclusion being compared. We can also start from an arbitrary isomorphism of the rectangular factors and ask what it implies about a single row.

Theorem G. *For $M, N \geq 3$, let α_k be conjugation by $sgn(u_{11})$ on the projective row algebra $L^\infty(P_+^{k-1})$ of O_k^+. Then*

$$L^\infty(O_{MM}^{1+}) \simeq L^\infty(O_{MN}^{1+}) \quad \iff \quad \alpha_M \text{ and } \alpha_N \text{ are conjugate.}$$

Equivalently, the 2 inclusions of the projective row algebra in the real sphere algebra are isomorphic.

In particular the comparison of the unmarked factors for $(3,3)$ and $(3,4)$ is precisely the conjugacy question for the full involutions at 3 and 4. Their restrictions to the even sign algebras are conjugate. The full conjugacy question remains open.

For the unitary rectangle the 2 column phases can turn through opposite angles. In the product of the 2 complex sphere algebras the fixed Fourier degrees form a diagonal copy of $\mathbb{Z}$ in $\mathbb{Z}^2$. This gives a distinction visible in the unmarked von Neumann algebras.

Theorem H. *For $M, N \geq 2$ the orbit algebra $L^\infty(U_{MN}^{1+})$ is a full, prime, nonsolid II_1 factor. Every subfactor in it which is a tensor product of 2 nonamenable factors has infinite index. In particular it is nonisomorphic to $L^\infty(O_{KL}^{1+})$ for every $K, L \geq 3$.*

The smallest unitary square, of size $(2,2)$, already gives an explicit group factor with these properties. The group consists of pairs in $F_2 \times F_2$ having the same image under a fixed surjection $F_2 \to \mathbb{Z}$.

The paper is organized as follows. Section 1 compares the quantum group factors with F and computes $L^\infty(H_4^{s+})$. Sections 2–5 pass from the signs in an orthogonal row to the real and complex sphere algebras and the involution on the projective row. Sections 6 and 7 treat the 2 volumes on free balls and the algebras of frames, flags and Grassmannians. Section 8 treats real rectangles of rank one and squares of corank one. Section 9 recovers both side lengths from the rank-one magnitude inclusion. Sections 10 and 11 use amplifications of involutions to compare a square and a rectangle after forgetting that inclusion. Section 12 treats unitary rectangles and separates their factors from the real ones by the index of tensor product subfactors.

Acknowledgements. This is the first of a series of papers on von Neumann algebras and freeness, motivated by the high-energy physics program in [53] and its relativistic ramifications. We would like to thank Claude for his contributions to this new mathematical sub-program, and notably for his work on [17], heavily used in the present paper.

1 The free quantum group factors

Let $F = L(F_N)$ be the free group factor, with F_N being the free group on N generators. By [42, 51] the notation F also applies to the interpolated free group factors of Dykema and Rădulescu [25, 47] throughout the range $1 < r \leq \infty$.

We are interested in this paper in the von Neumann algebras of other free discrete quantum groups, best viewed as algebras of L^∞ functions on the corresponding compact quantum group duals. We will be interested as well in the basic homogeneous spaces over these latter compact quantum groups. To be more precise, we use the following formalism.

Definition 1.1. *For a discrete group Γ, $L(\Gamma)$ is the von Neumann algebra generated by the left translations $\lambda(g)$ on $l^2(\Gamma)$, with trace $\tau(T) = \langle T\delta_1, \delta_1 \rangle$. For a compact matrix quantum group G of Kac type, $L^\infty(G) = \pi_h(C(G))''$ in the GNS representation of its Haar trace h in Woronowicz's theory [63]. For an orbit map $\rho : C(X) \to C(G)$ we put*

$$L^\infty(X) = \pi_h(\rho(C(X)))'' \subset L^\infty(G)$$

and equip it with the restriction of the trace of $L^\infty(G)$.

In particular the integral of a coordinate polynomial p on X is $h(\rho(p))$. This is the measure-theoretic side of Connes' noncommutative geometry [20], with $C(X)$ playing the role of continuous functions and $L^\infty(X)$ that of essentially bounded measurable functions. We use normal trace-preserving isomorphisms, tracial free products $*$ and spatial tensor products $\bar{\otimes}$ throughout.

In the classical type classification [39, 40, 60], a II_1 factor is a finite factor without minimal projections. Murray and von Neumann showed in [41] that $L(F_2)$ is distinct from the hyperfinite II_1 factor. The classical free Haar unitary criterion [59] recognizes a free group factor directly from its generators.

Proposition 1.2. *Let (A, τ) be a tracial von Neumann algebra with faithful trace, generated by $d \geq 2$ unitaries $v_1, \ldots, v_d$. If every nonempty reduced word in these unitaries and their adjoints has trace zero, then $A \simeq L(F_d)$.*

Proof. For $g \in F_d$ let v_g be the corresponding product. Since $v_h^* v_g = v_{h^{-1}g}$, we have

$$\tau(v_h^* v_g) = \begin{cases} 1 & g = h, \\ 0 & g \neq h. \end{cases}$$

The vectors v_g span a dense subspace of $L^2(A, \tau)$ by the generation hypothesis. Sending δ_g to v_g thus gives a unitary from $l^2(F_d)$ onto $L^2(A, \tau)$. It intertwines left translation by each generator with left multiplication by the corresponding v_i. Taking von Neumann closures gives $L(F_d) \simeq A$, and $L(F_d) \simeq F$ by [42]. □

The word condition says that the v_i are free Haar unitaries in the sense of Voiculescu, Dykema and Nica [59]. The same proof for a system in a larger algebra identifies precisely $W^*(v_1, \ldots, v_d)$. Generation of the larger algebra is a separate issue. Shlyakhtenko constructs a free complement to the basic commutator in $L(F_2)$ in [50]. In [42] small perturbations of free Haar generators allow an additional generator to be approximated in L^2 by words in the limiting system. The semicircular argument of [51] gives an isomorphism even for the C^*-algebras generated by free semicircular systems. For a row of O_N^+ we shall have explicit free unitaries and an explicit element outside their algebra.

To prove that an algebra is different from F, one can bound the size of its matrix approximations. Fix a self-adjoint generating list $X = (x_1, \ldots, x_m)$ and $R > \max_i \|x_i\|$. Its microstates of size d, order l and tolerance η are lists $T = (T_1, \ldots, T_m)$ of self-adjoint $d \times d$ matrices of norm at most R such that

$$|tr_d(w(T)) - \tau(w(X))| < \eta$$

for every word w of length at most l, where $tr_d(1_d) = 1$. Denote this set by $\Gamma_R(X; l, d, \eta)$. At each size d we use the distance with square $\sum_i tr_d((T_i - S_i)^2)$. Normalizing the trace gives the same metric scale at every size when we compare covering numbers.

Definition 1.3. *Write K_ε for the least number of ε-balls covering a microstate set and put*

$$\mathcal{K}_\varepsilon(X) = \inf_{l,\eta} \limsup_{d \to \infty} \frac{1}{d^2} \log K_\varepsilon(\Gamma_R(X; l, d, \eta)).$$

The list is α-bounded if $\mathcal{K}_\varepsilon(X) \leq \alpha |\log \varepsilon| + C$ for all sufficiently small ε, with a constant C independent of ε. The algebra is strongly 1-bounded if it has a 1-bounded generating list containing an element of finite free entropy.

The normalization by d^2 reflects the real dimension of the space of self-adjoint $d \times d$ matrices. A relation among the generators can reduce the coefficient of $|\log \varepsilon|$. The following criterion explains which property of the derivative gives the bound needed here.

Theorem 1.4. *Let P be a list of polynomial relations $P(X) = 0$ and put $D = \partial P(X)$, the matrix of free difference quotients over $A \bar{\otimes} A^{opp}$, where $A = W^*(X)$. If D is of determinant class, then X is $(m - \operatorname{rank} D)$-bounded. If $\operatorname{rank} D = m - 1$ and one x_i has finite free entropy, then A is strongly 1-bounded and $A \not\cong F$.*

Proof. The column of D indexed by i records the effect of varying x_i. On a monomial, an occurrence ax_ib contributes $a \otimes b^{opp}$ to its derivative, representing the variation $a\,\Delta x_i\,b$. The rank is the trace of the range projection, using the product trace on $A \bar{\otimes} A^{opp}$ and the unnormalized trace on matrix indices. The determinant-class hypothesis controls the small positive singular values by requiring

$$\int_{(0,1)} |\log t|\, d\mu_{|D|}(t) < \infty.$$

The kernel is accounted for by $m - \operatorname{rank} D$. Under this integrability hypothesis the rank theorem of Jung and Shlyakhtenko [37, 49], in the form given in [12], gives the asserted covering bound.

At rank $m - 1$ we obtain a 1-bounded list. The finite free entropy assumption gives strong 1-boundedness by definition. Jung's generator theorem [36] now gives $\delta_0(Y) \leq 1$ for every finite self-adjoint generating list Y of A, where δ_0 is the microstates free entropy dimension. A free semicircular pair generates F and has $\delta_0 = 2$ by Voiculescu's formula in [58]. A trace-preserving isomorphism would transport this pair to a generating pair of A with the same joint law, a contradiction. □

For O_N^+ the matrix u is orthogonal with self-adjoint entries. For U_N^+ the fundamental matrix and its entrywise conjugate are unitary. For Wang's S_N^+ the fundamental matrix is magic: its entries are projections with each row and column summing to 1 [61]. The orthogonality relations give the following application of the criterion.

Theorem 1.5. *For $N \geq 3$ both $L^\infty(O_N^+)$ and $L^\infty(S_{N^2}^+)$ are strongly 1-bounded factors. In particular neither is isomorphic to F.*

Proof. Take the N^2 entries of u and the 2 matrix relations $u^t u - 1_N = uu^t - 1_N = 0$. Brannan and Vergnioux [12] compute the rank of their differential as $N^2 - 1$. They prove determinant class by establishing logarithmic integrability of $1 + Re\,\Theta$, where Θ reverses the edges of the quantum Cayley tree [56]. The law of u_{11} has bounded compactly supported density [4, 12], so Voiculescu's one-variable formula [57] gives finite free entropy. Theorem 1.4 now gives strong 1-boundedness for $L^\infty(O_N^+)$.

For the permutation case the relevant dimension is $\dim M_N(\mathbb{C}) = N^2$. The trace-preserving quantum automorphism groups of $M_N(\mathbb{C})$ and $\mathbb{C}^{N^2}$ are PO_N^+ and $S_{N^2}^+$, respectively. The inclusion $L^\infty(PO_N^+) \subset L^\infty(O_N^+)$ has index 2. Brannan, Elzinga, Harris and Yamashita realize $L^\infty(S_{N^2}^+)$ as a finite-index subfactor of a matrix amplification of $L^\infty(PO_N^+)$. Strong 1-boundedness passes in both directions through a finite-index inclusion of II_1 factors and is stable under amplification [11]. Applying these facts successively gives the assertion for $S_{N^2}^+$. □

The same obstruction fails for $L^\infty(U_N^+)$ [11]. At $N = 2$ one has the stronger conclusion $L^\infty(U_2^+) \simeq F$ [1]. We collect these comparisons, including the reflection case proved below.

Algebra	Parameters	Comparison with F
$L^\infty(O_N^+)$	$N \geq 3$	Strongly 1-bounded, hence distinct [12]
$L^\infty(U_2^+)$	$N = 2$	Isomorphic [2]
$L^\infty(H_4^{s+})$	$s = 2, 3, \ldots, \infty$	Isomorphic (Theorem 1.8)
$L^\infty(U_N^+)$	$N \geq 3$	Failure of strong 1-boundedness [11]
$L^\infty(S_{N^2}^+)$	$N \geq 3$	Strongly 1-bounded, hence distinct [11]
$L^\infty(S_N^+)$	$N \geq 5$	Strongly solid [22, 33]

The permutation series is classical for $N \leq 3$, and $S_4^+ \simeq SO_{-1}(3)$ is a twist of the classical rotation group [1]. For $N \geq 5$ we can use solidity even when the entropy comparison is open.

Definition 1.6. *A finite von Neumann algebra is* solid *if the relative commutant of every diffuse von Neumann subalgebra is amenable. It is* strongly solid *if the unitaries normalizing any diffuse amenable subalgebra generate an amenable algebra.*

These properties are understood for unital inclusions. For the free orthogonal and unitary factors, strong solidity follows from Isono's theorem [33]. See also Caspers [14, 15]. For S_N^+ the approximation theorem of De Commer, Freslon and Yamashita [22] and the solidity theorem of Isono [32, 33] give strong solidity for $N \geq 5$. Brannan proves factoriality and fullness for $N \geq 8$ in [10]. For $5 \leq N \leq 7$, factoriality is also stated in [11], which cites work of Gao and Weeks then in preparation. Only strong solidity is needed in our comparison with rectangles. For F we use Ozawa and Popa [43, 45]. For the single-row examples we still need the distinction between finding free unitaries and generating the entire algebra. Indeed, we shall exhibit F inside the strongly 1-bounded orthogonal factor. The finite-index hypothesis in the permanence theorem is essential.

Conjecture 1.7. *We have $L^\infty(U_N^+) \simeq F$ for $N \geq 3$. For all $N \geq 5$, the algebra $L^\infty(S_N^+)$ is strongly 1-bounded.*

The unitary assertion has a concrete free product formulation in Section 4. The permutation assertion extends the square-dimensional result in the table. Even when $M = N^2$ one can ask separately whether $L^\infty(S_M^+)$ and $L^\infty(O_N^+)$ are isomorphic. Another independent question is whether $L^\infty(O_N^+)$ varies with $N \geq 3$. The entropy obstruction alone gives no count of the isomorphism classes in either series.

For the real and complex free bistochastic groups the fundamental matrix fixes the constant vector. Choose a real orthogonal basis whose first vector is the normalized constant vector. In this basis the matrix has the form $diag(1, w)$. The universal relations on w are precisely those of O_{N-1}^+ or U_{N-1}^+, respectively, since inserting any such w gives a bistochastic matrix upon changing back to the original basis. Thus

$$B_N^+ \simeq O_{N-1}^+, \qquad C_N^+ \simeq U_{N-1}^+$$

for $N \geq 2$ [1, 48]. In particular $L^\infty(C_3^+) \simeq F$.

For the free hyperoctahedral group H_N^+ the squares of its matrix entries form a magic unitary. Sending the magic generators to these squares embeds $C(S_N^+)$, since the map back sending each cubic generator to the corresponding magic generator is a left inverse. The embedding respects coproducts and hence preserves the invariant traces. We can identify the entire ambient algebra at $N = 4$. The same calculation works for all the quantum reflection groups

$$H_N^{s+} = \widehat{\mathbb{Z}}_s \wr_* S_N^+,$$

where $\mathbb{Z}_\infty = \mathbb{Z}$ and $H_N^{2+} = H_N^+$ [7, 8]. We first determine the center, since the formula for the free dimension also applies to direct sums.

Theorem 1.8. *Let s be an integer at least 2 or let $s = \infty$. We have*

$$L^\infty(H_4^{s+}) \simeq L(F_{2-1/s}) \simeq F,$$

where $1/\infty = 0$ and the middle term denotes the interpolated free group factor.

Proof. Write $A = L^\infty(S_4^+)$ and $C = L^\infty(H_4^{s+})$. If $p = (p_{ij})$ is the fundamental magic unitary, put $D_i = W^*(p_{i1}, \ldots, p_{i4})$. Each D_i is a copy of $\mathbb{C}^4$ with its uniform trace. The tracial free

wreath product construction of Fima and Troupel [27] gives the following 4 successive tracial amalgamated free products

$$A_0 = A, \qquad A_i = A_{i-1} *_{D_i} (D_i \bar{\otimes} L(\mathbb{Z}_s)) \quad (1 \le i \le 4), \qquad A_4 = C.$$

The 2 embeddings of D_i here are the ith row in $A \subset A_{i-1}$ and the first tensor leg in the new algebra. Let v_i be the canonical cyclic generator in its second leg. We have $E_{D_i}(v_i) = 0$ for every allowed s.

The algebra A is diffuse and amenable, and is maximal amenable in C by [27], since $\widehat{\mathbb{Z}}_s$ is of Kac type. If $z \in Z(C)$, adjoining the central abelian algebra $W^*(z)$ to A gives an amenable algebra. Indeed, approximate $W^*(z)$ by increasing finite-dimensional abelian algebras. For each such algebra with atoms e_j, the join with A is $\bigoplus_j Ae_j$, a direct sum of amenable quotients of A. Their increasing union generates $A \vee W^*(z)$. Maximal amenability gives $z \in A$. At the ith step amalgamated freeness gives

$$E_{A_{i-1}}(v_i z v_i^*) = E_{D_i}(z).$$

To see this, subtract $E_{D_i}(z)$ from z. The resulting product is a reduced word beginning and ending in the new leg, and its expectation onto A_{i-1} vanishes. As z commutes with v_i, it follows that $z \in D_i$ for every i.

Already $D_1 \cap D_2 = \mathbb{C}1$. An element of the intersection has 2 expressions $\sum_j a_j p_{1j} = \sum_k b_k p_{2k}$. This is an identity in the polynomial algebra, on which the invariant trace is faithful. Evaluation on classical permutation matrices gives $a_j = b_k$ whenever $j \neq k$. A column distinct from 2 prescribed columns therefore shows that all the coefficients agree. This proves factoriality. Also v_1 is a unitary with zero expectation onto A, so $A \subsetneq C$. Maximal amenability of A now implies that C is nonamenable.

It remains to compute the parameter. Dykema and Redelmeier [26] show that finite direct sums of interpolated free group factors and a hyperfinite summand are closed under these amalgamations. The construction parameter is obtained by adding the free dimensions of the 2 legs and subtracting that of the amalgam. The diffuse hyperfinite algebra A starts at 1. A uniform d-point algebra contributes $1 - 1/d$, so $\mathbb{C}^4$ contributes $1 - 1/4$. For finite s the new leg is a uniform $4s$-point algebra. For $s = \infty$ it is diffuse abelian and contributes 1. After 4 steps the parameter is

$$1 + 4\left[\left(1 - \frac{1}{4s}\right) - \left(1 - \frac{1}{4}\right)\right] = 2 - \frac{1}{s}.$$

Factoriality and nonamenability leave a single interpolated free group factor in the decomposition. The identification of the interpolated factors in [42, 25, 47] gives the second isomorphism. □

At $s = 2$ the construction gives the interpolated factor at parameter $3/2$. At $s = \infty$ it gives $L(F_2)$ directly. The distinction between the cyclic orders survives in the K-theory of the reduced C^*-algebras [27]. Their von Neumann algebras at size 4 all have the same isomorphism type. At $N \ge 5$ the starting algebra $L^\infty(S_N^+)$ is already nonamenable, and the above computation from a diffuse hyperfinite starting algebra no longer applies.

2 Row signs and the sphere relation

The free real sphere has self-adjoint coordinates $x_1, \ldots, x_N$ satisfying $\sum_i x_i^2 = 1$. Its invariant trace is the restriction of the trace of $L^\infty(O_N^+)$ under $x_i = u_{1i}$ [5, 18]. For its complex counterpart the relations are $\sum_i y_i^* y_i = \sum_i y_i y_i^* = 1$, and the orbit realization is $y_i = v_{1i}$. With $\gamma(x_i) = -x_i$, the fixed algebra of γ in $L^\infty(S_{\mathbb{R},+}^{N-1})$ is precisely $L^\infty(P_+^{N-1})$. Indeed, averaging a polynomial over $\{1, \gamma\}$ selects its even words, and every even word is a product of adjacent pairs. Polynomial density and normality of the averaging map give the assertion in the von Neumann closure. The matrix $(x_i x_j)$ is a projection of trace 1, the coordinates of the projective orbit in [17].

Consider the discrete group $\Gamma_N = \langle g_1, \ldots, g_N \mid g_i^2 = 1 \rangle$. Its dual is the diagonal quantum subgroup of O_N^+, obtained on polynomial functions by $u_{ij} \to \delta_{ij} g_i$. Right translation by this subgroup gives a grading on the entire orthogonal algebra. Each entry in column j has degree g_j. Degrees multiply in the order of the entries. A monomial of degree $g \neq 1$ has trace zero, by right invariance. Thus monomials of distinct degrees are orthogonal in L^2. Writing K_g for the closed span of the monomials of degree g gives

$$L^2(O_N^+) = \bigoplus_{g \in \Gamma_N} K_g.$$

This construction on polynomials also describes the restricted right regular unitary representation, with K_1 its fixed vector space. The grading thus extends to $L^2(O_N^+)$. We write z_g for the component in K_g of the vector determined by z.

For $N \geq 2$ the law of u_{ri} is symmetric and atomless [4]. Its support is the interval $[-c_N, c_N]$, where $c_N = 2/\sqrt{N+2}$. In particular $s_i = sgn(x_i)$ is a self-adjoint unitary. It has degree g_i. To see this, approximate the sign in L^2 of the coordinate law by uniformly bounded odd functions obtained by polynomial approximation of continuous odd cutoffs. Every odd polynomial in x_i has degree g_i. The same approximations converge strongly as bounded operators on the tracial representation. Left or right multiplication by s_i therefore transports each homogeneous subspace to the subspace of the corresponding product degree.

Proposition 2.1. *The assignment $g_i \to s_i$ identifies $L(\Gamma_N)$ with the algebra of the signs in $L^\infty(S^{N-1}_{\mathbb{R},+})$ for $N \geq 2$. The elements $s_1 s_i$, $2 \leq i \leq N$, are free Haar unitaries and generate a copy of $L(F_{N-1})$ in $L^\infty(P_+^{N-1})$.*

Proof. For a reduced word $g = g_{i_1} \ldots g_{i_k}$ put $s_g = s_{i_1} \ldots s_{i_k}$. The symmetries satisfy the defining relations, and s_g is homogeneous of degree g. Consequently $\tau(s_g) = \delta_{g,1}$. This identifies the 2 tracial group representations by their GNS constructions.

The homomorphism $\Gamma_N \to \mathbb{Z}_2$ taking every g_i to $1 \in \mathbb{Z}_2$ has kernel Γ_N^+. With $a_1 = 1$ and $a_i = g_1 g_i$, every pair $g_i g_j$ is $a_i^{-1} a_j$. More explicitly, the semidirect product of the free group on $a_2, \ldots, a_N$ by the involution $a_i \to a_i^{-1}$ has presentation Γ_N, with the involution implemented by g_1 and $g_i = g_1 a_i$. This proves the free basis assertion. Finally $\gamma(s_i) = -s_i$, so the even sign words belong to $L^\infty(P_+^{N-1})$. □

This also identifies the von Neumann algebras of the free diagonal tori. For the complex torus the group is F_N, and its group algebra is F for $N \geq 2$. For the real torus the tracial free product of N copies of $L(\mathbb{Z}_2)$ is an interpolated free group factor for $N \geq 3$, with construction parameter $N/2$ by Dykema's formula [24]. Each 2-point algebra of equal weights contributes $1 - 2(1/2)^2 = 1/2$, and the group is ICC by Lemma 3.1, so no finite-dimensional summand occurs. The resulting algebra is again F. The even sign algebra and the full sign algebra thus already have the same abstract type, despite their index-2 inclusion.

The distinction between embedding and generation has a direct illustration. Let E be the trace-preserving expectation of $L^\infty(P_+^{N-1})$ onto the even sign algebra. Its orthonormal group basis is $\{s_g \mid g \in \Gamma_N^+\}$. Since x_1^2 has degree 1, it is orthogonal to every nonidentity basis element. Column symmetry and the sphere relation give

$$E(x_1^2) = \tau(x_1^2)1 = \frac{1}{N}\,1.$$

The square has diffuse law. Thus the inclusion in Proposition 2.1 is proper, including at $N = 3$ where its domain is already F.

The signs above give free generators for a subalgebra of the row. To bound the free entropy dimension of the row coordinates themselves, we apply Theorem 1.4 to their sphere relation.

Theorem 2.2. *Let $N \geq 3$ and $x = (x_1, \ldots, x_N)$ be the canonical real sphere generators in their invariant trace. The free differential of $\sum_i x_i^2 - 1$ has rank 1 and is of determinant class. Consequently x is $(N-1)$-bounded and $\delta_0(x) \leq N - 1$.*

Proof. Put $A = L^\infty(S^{N-1}_{\mathbb{R},+})$ and work over $B = A \bar{\otimes} A^{opp}$ with trace $\sigma = \tau \otimes \tau$. The differential is the one-row matrix $D = (d_1, \ldots, d_N)$ with self-adjoint entries

$$d_i = x_i \otimes 1 + 1 \otimes x_i^{opp}, \qquad DD^* = \sum_i d_i^2 = 2 + 2\sum_i x_i \otimes x_i^{opp}.$$

The 2 commuting terms of d_1 have their classical product law under σ. With μ the law of x_1, the law of d_1 is $\mu * \mu$ in the classical convolution sense. By [4, 12], μ has a bounded, compactly supported density. The same holds for $\mu * \mu$. In particular

$$\ker d_1 = 0, \qquad \int_{|t|<1} |\log|t|| \, d(\mu * \mu)(t) < \infty.$$

Faithfulness and $DD^* \geq d_1^2$ show that DD^* has support 1. Its range projection has trace 1, as asserted. For every $\varepsilon > 0$ operator monotonicity of the logarithm gives

$$\sigma\big(\log(DD^* + \varepsilon)\big) \;\geq\; \sigma\big(\log(d_1^2 + \varepsilon)\big).$$

The right side has a finite limit as ε decreases to zero. Since DD^* is bounded, the positive part of its logarithm is bounded as well. Thus $\log(DD^*)$ is integrable. The polar decomposition of D identifies the positive spectral parts of DD^* and D^*D with their respective unnormalized matrix traces. Thus the free differential of the sphere relation has rank 1 and is of determinant class. Applying Theorem 1.4 to the N coordinates gives $(N-1)$-boundedness. □

At $N = 3$ this gives $\delta_0(x_1, x_2, x_3) \leq 2$ for the coordinates of the real free sphere. The corresponding lower bound problem is discussed in Section 4.

3 An irreducible projective row

We first compute the commutant of a full projective row. We then pass to 3 entries of that row. In both calculations the Fourier components of an element of the ambient algebra can have odd degree.

Lemma 3.1. *For $N \geq 3$ every nonidentity element of Γ_N has infinite orbit under conjugation by Γ_N^+.*

Proof. Let $g \neq 1$ be a reduced word. Choose k different from both its first and its last letter index, and choose $j \neq k$. The words $h_M = (g_j g_k)^M$ belong to Γ_N^+. There is no cancellation at either end of g in $h_M g h_M^{-1}$. Their lengths are $4M + |g|$, for $M \geq 1$. □

Theorem 3.2. *For $N \geq 3$ we have*

$$L^\infty(P_+^{N-1})' \cap L^\infty(O_N^+) = \mathbb{C}1.$$

The same conclusion holds for the projective algebra of any single row.

Proof. Denote the relative commutant by D. If $z \in D$, Proposition 2.1 gives $s_h z s_h^* = z$ for all $h \in \Gamma_N^+$. Conjugation by s_h takes K_g unitarily onto $K_{hgh^{-1}}$. Comparing the components of this identity gives

$$\|z_{hgh^{-1}}\|_2 = \|z_g\|_2.$$

The squares of these norms are summable. By Lemma 3.1, all components with $g \neq 1$ vanish. Thus $D \subset K_1$. Each classical diagonal sign matrix acts on homogeneous polynomials through a

character of Γ_N. This character takes the value 1 at the group unit, so by continuity the sign matrix acts trivially on K_1.

Right translation by a classical $Q \in O_N$ rotates the coordinates of the first row by Q. It preserves $L^\infty(P_+^{N-1})$, hence preserves D. Choose the diagonal reflection $d = diag(-1, 1, \dots, 1)$. Since $D \subset K_1$, this reflection fixes D pointwise. Therefore every QdQ^{-1} fixes D pointwise as well: first apply Q^{-1} to an element of D, use that d fixes the result, and apply Q. Hyperplane reflections generate O_N. Every element of D is now fixed by the diagonal quantum subgroup and by the full classical orthogonal group.

By Chirvasitu's topological generation theorem [16], these 2 subgroups generate O_N^+. To apply this on L^2, decompose the right regular representation into finite-dimensional irreducibles. In each coefficient block the common fixed vectors for the 2 subgroups are exactly the fixed vectors for the generated group, by the intertwiner characterization of topological generation coming from Woronowicz's Tannaka–Krein duality [62]. The projections onto the blocks commute with restriction, so the conclusion passes to their Hilbert direct sum. The trivial block of the right regular representation consists of the constants. This proves $D = \mathbb{C}1$. Classical row permutations on the left transport this calculation to any chosen row. □

For a shortened row the rotations mixing retained and discarded columns need no longer preserve the algebra. We recover the required rotations by examining the commutant of each sign word separately.

Theorem 3.3. *Fix a row of O_N^+, with $N \geq 4$, and write s_1, s_2, s_3 for the signs in any 3 distinct columns. For arbitrary positive integers a, b, c, the 3 unitaries*

$$(s_1 s_2)^a, \qquad (s_1 s_3)^b, \qquad (s_2 s_3)^c$$

have scalar common commutant in $L^\infty(O_N^+)$. In particular $W^*(s_1 s_2, s_1 s_3) \simeq F$ is irreducible there.

Proof. Permute the columns so that the chosen indices are $1, 2, 3$. Fix a pair i, j and let C_{ij} be the commutant of the indicated power for that pair. Put $t = g_i g_j$ and $H_{ij} = \langle g_i, g_j \rangle$. For $z \in C_{ij}$ the norms of its Fourier components are constant on conjugacy orbits under the corresponding power of t. Square summability forces each g with $z_g \neq 0$ to have finite orbit, so g commutes with some positive power of t.

We claim that such a g belongs to H_{ij}. Indeed Γ_N is the free product of H_{ij} and the subgroup generated by the other letters. If $g \notin H_{ij}$, write $g = hwh'$ with $h, h' \in H_{ij}$ and the reduced word w beginning and ending in the other free factor. For any $v \neq 1$ in H_{ij}, the middle syllable $h'v(h')^{-1}$ survives in gvg^{-1}, as do the adjacent syllables of w and w^{-1}. Thus $gvg^{-1} \notin H_{ij}$. Taking v to be the power of t proves the claim. The 3 support restrictions now force a common commuting element to lie in K_1, since $H_{12} \cap H_{13} \cap H_{23} = \{1\}$.

There is also a classical consequence for each pair. The diagonal reflection changing the sign of one coordinate outside i, j acts by a character equal to 1 on H_{ij}, so it fixes C_{ij} pointwise. The classical orthogonal group on the complementary coordinate space fixes s_i and s_j and preserves C_{ij}. Inside that group the conjugates of the given reflection are all hyperplane reflections. The entire complementary orthogonal group therefore fixes C_{ij} pointwise.

Choose a fourth axis. For each $i = 1, 2, 3$ the complement of the other 2 chosen axes contains the plane of i and the fourth axis. Each further axis and the fourth axis also lie in one of the 3 complements. Thus these groups contain all plane rotations through the fourth axis and another coordinate axis. Conjugating by quarter turns gives rotations in every coordinate plane. These generate SO_N, and the groups contain a reflection as well. A common commuting element is therefore fixed by O_N and, as it lies in K_1, by the diagonal quantum subgroup. The topological generation argument of Theorem 3.2 makes it scalar. Finally $s_2 s_3 = (s_1 s_2)^*(s_1 s_3)$, and Proposition 2.1 gives freeness. □

There is no invariance assumption on the intermediate algebra in the following consequence. In particular this is a statement about the inclusion, beyond factoriality of its 2 endpoints.

Corollary 3.4. *Every algebra between $L^\infty(P_+^{N-1})$ and $L^\infty(O_N^+)$ is a nonamenable strongly solid II_1 factor for $N \geq 3$. For $N \geq 4$ the same holds for every algebra between $W^*(s_1s_2, s_1s_3)$ and $L^\infty(O_N^+)$. Each intermediate algebra has scalar commutant in the ambient algebra.*

Proof. Write B for an intermediate algebra. Its relative commutant, and hence its center, is scalar by Theorem 3.2 or 3.3. It contains a nonamenable free group factor with a trace-preserving expectation, so it is nonamenable. A finite nonamenable factor has type II_1. For strong solidity, take a diffuse amenable subalgebra of B. Its normalizing unitaries in B are among its normalizing unitaries in $L^\infty(O_N^+)$. The latter generate an amenable algebra by [14], and every von Neumann subalgebra of a finite amenable algebra is expected and amenable. □

For a full row the dimension restriction is sharp. When $N = 2$, the even sign group is $\mathbb{Z}$, and even degrees can have singleton conjugacy orbits under that group. In this dimension $P_+^1 = P_{\mathbb{C}}^1$ [17]. Its orbit algebra is diffuse abelian: in a commutative algebra an atom would force a point mass in the law of x_1^2. Separability identifies $L^\infty(P_+^1)$ as a tracial algebra with $L^\infty([0,1])$. At $N = 1$ the projective space is a point.

4 Free spheres and their projective algebra

For $1 \leq L \leq N$, a real Stiefel frame is an ordered list of L orthonormal vectors in $\mathbb{R}^N$. We write St_{LN}^+ for the real free Stiefel manifold [17]. Its orbit coordinates are the entries of the first L rows of the fundamental matrix of O_N^+. The complex free Stiefel manifold $St_{LN,\mathbb{C}}^+$ is obtained in the same way from U_N^+. At $L = 1$ we recover the real and complex free spheres, and at $L = N$ we recover O_N^+ and U_N^+.

Let γ act on the real frame by simultaneous sign change. Its fixed algebra is

$$L^\infty(St_{LN}^+)^\gamma = W^*(u_{ri}u_{sj} \mid r, s \leq L,\ i, j \leq N).$$

Indeed the even polynomial words are precisely the products of adjacent pairs. At $L = 1$ this gives $L^\infty(P_+^{N-1})$, and at $L = N$ it gives $L^\infty(PO_N^+)$. The latter contains mixed-row products. For example a change of sign of row 2 fixes $L^\infty(P_+^{N-1})$ pointwise and reverses $u_{11}u_{21}$. This product is nonzero, since both factors have trivial kernel. Thus even the concrete inclusion $L^\infty(P_+^{N-1}) \subset L^\infty(PO_N^+)$ is proper for $N \geq 2$.

The polar-unitary argument for U_2^+ in [2] starts with the following free Haar unitary calculation, there stated for a centered unitary. De Commer used the same method for the size-2 groups $A_u(Q)$ in [21]. We give the calculation for an arbitrary unitary and then apply it to all frame lengths.

Lemma 4.1. *Let z be a Haar unitary free from a tracial algebra B, and let $s \in B$ be unitary. Then $w = zs$ is a Haar unitary free from B.*

Proof. Positive powers of zs have trace zero by expanding the intervening s's into scalar and centered parts. Every surviving power of z has positive exponent. Adjoints give the negative powers. For freeness, take the trace of a cyclic word $w^{k_1}b_1 \dots w^{k_l}b_l$ with $k_i \neq 0$ and each b_i centered in B. Expand the powers into letters z, s, s^*, z^*. Between a positive and a negative z letter the intervening coefficient is sb_is^*, and between a negative and a positive letter it is b_i. Both are centered. At each other position expand the coefficient in B into its scalar and centered parts. Removing a scalar only merges powers of z of the same sign. The remaining cyclic words are either nonzero powers of z up to a scalar, or alternating products of centered B coefficients and nonzero powers of z. All have trace zero. These cyclic tests imply freeness by traciality and centering successive end factors. □

For the real frame choose $s = sgn(u_{11})$. Changing the sign of the frame sends s to $-s$. For the complex frame use the free complexification model [1], in which $v_{ri} = zu_{ri}$ inside $L(\mathbb{Z}) * L^\infty(O_N^+)$, where z is the canonical unitary of $L(\mathbb{Z})$. The product trace restricts to the invariant trace of U_N^+ and hence gives the orbit trace on any chosen rows.

Theorem 4.2. *For $N \geq 2$ and $1 \leq L \leq N$ we have trace-preserving isomorphisms*

$$L^\infty(St_{LN}^+) \simeq L^\infty(St_{LN}^+)^\gamma \rtimes \mathbb{Z}_2,$$
$$L^\infty(St_{LN,\mathbb{C}}^+) \simeq L(\mathbb{Z}) * L^\infty(St_{LN}^+)^\gamma.$$

The generator of $\mathbb{Z}_2$ acts on the even algebra by conjugation with $s = sgn(u_{11})$. Its action is outer for $N \geq 3$.

Proof. Put $D = L^\infty(St_{LN}^+)^\gamma$. The symmetry s normalizes D. Every odd element b is $(bs)s$ with $bs \in D$. Decomposing an element into its even and odd parts gives $L^\infty(St_{LN}^+) = D + Ds$. The expectation $(1+\gamma)/2$ kills Ds, and the trace is the crossed product trace. The representation of the crossed product defined by its generators is therefore faithful. If $Ad\, s|_D$ were inner, a unitary $t \in D$ implementing it would make t^*s commute with D. Since $L^\infty(P_+^{N-1}) \subset D$, Theorem 3.2 would make t^*s scalar. It is an odd unitary, a contradiction.

In the complex model, $v_{ri}^* v_{sj} = u_{ri} u_{sj}$ puts D inside $L^\infty(St_{LN,\mathbb{C}}^+)$. The polar unitary of v_{11} is $w = zs$, since u_{11} has zero kernel and dense range. Thus $w \in L^\infty(St_{LN,\mathbb{C}}^+)$, and Lemma 4.1 makes it free from D. Conversely su_{ri} is even, and $v_{ri} = w(su_{ri})$. This recovers every frame coordinate from D and w. Faithfulness of the free product trace gives the second isomorphism. □

At $L = 1$ the action on the projective algebra is $a \to sas$, with $s = sgn(x_1)$. The 2 sphere formulas become

$$L^\infty(S_{\mathbb{R},+}^{N-1}) = L^\infty(P_+^{N-1}) \rtimes \mathbb{Z}_2,$$
$$L^\infty(S_{\mathbb{C},+}^{N-1}) \simeq L(\mathbb{Z}) * L^\infty(P_+^{N-1}).$$

For $N \geq 3$ the first inclusion $L^\infty(P_+^{N-1}) \subset L^\infty(S_{\mathbb{R},+}^{N-1})$ has index 2. It follows from [11] that $L^\infty(P_+^{N-1})$ is strongly 1-bounded exactly when $L^\infty(S_{\mathbb{R},+}^{N-1})$ is. In dimension 2 the second formula yields $L^\infty(S_{\mathbb{C},+}^1) \simeq L(\mathbb{Z}) * L(\mathbb{Z}) = F$. The first formula is compatible with the central radial variable of the real free circle in [18]. There is a simple mixed moment distinguishing the 2 phase constructions. If $a \in L^\infty(P_+^{N-1})$ is centered and nonzero, then $b = sa^*s \in L^\infty(P_+^{N-1})$ is centered, while $\tau(sasb) = \tau(aa^*) > 0$. Thus the real sign and the fixed algebra fail the alternating centered moment test for freeness. At $N = 1$ the real and complex spheres have von Neumann algebras $L(\mathbb{Z}_2)$ and $L(\mathbb{Z})$.

Conjecture 4.3. *For $N \geq 3$ we have $L^\infty(P_+^{N-1}) \simeq F$.*

The polar decompositions of the row coordinates give a concrete formulation of this conjecture. Put $b_i = s_1 x_i$ in $L^\infty(P_+^{N-1})$. These variables generate $L^\infty(P_+^{N-1})$ because $b_i^* b_j = x_i x_j$. Their polar decomposition is

$$b_1 = |x_1|, \qquad b_i = (s_1 s_i)|x_i|, \qquad \sum_i |x_i|^2 = 1.$$

All the absolute values belong to $L^\infty(P_+^{N-1})$. Thus adjoining them to the free sign algebra gives the entire projective algebra. Their correlations with the signs enter the generation problem even in the smallest dimension. In the case $N = 3$ the concrete task is to find 2 free Haar unitaries whose word algebra approximates b_1, b_2, b_3 in L^2. Conversely, if $L^\infty(P_+^2) \simeq F$ such a pair exists by transport of generators. For the expectation E onto the even sign algebra, Section 2 gives $E(x_1^2) = 1/N$. The fourth moment in Proposition 7.3 at $L = 1$ therefore gives

$$\|x_1^2 - E(x_1^2)\|_2^2 = \frac{2}{N(N+1)} - \frac{1}{N^2} = \frac{N-1}{N^2(N+1)}.$$

It equals $1/18$ for the free projective plane. Thus x_1^2 cannot be approximated in L^2 by words in the original even signs. For the first higher Grassmannian even the existence of the initial system of homogeneous unitaries remains a separate problem, as in Question 7.4.

Theorem 2.2 gives $\delta_0(x_1, x_2, x_3) \leq 2$. Establishing a strict lower bound greater than 1 for any generating system of $L^\infty(S^2_{\mathbb{R},+})$ would already separate $L^\infty(S^2_{\mathbb{R},+})$ and, by index 2, $L^\infty(P^2_+)$ from the strongly 1-bounded factors. Such a lower bound would still leave the free generation conjecture open.

Conjecture 4.3 implies $L^\infty(S^{N-1}_{\mathbb{C},+}) \simeq F$, because $L(\mathbb{Z}) * F \simeq L(F_3) \simeq F$. It also implies that $L^\infty(S^{N-1}_{\mathbb{R},+})$ fails strong 1-boundedness, by finite index. To identify $L^\infty(S^{N-1}_{\mathbb{R},+})$ itself one would still have to understand the action in Theorem 4.2. At the opposite endpoint of the frame tower one gets

$$L^\infty(U_N^+) \simeq L(\mathbb{Z}) * L^\infty(PO_N^+).$$

Here the second component is strongly 1-bounded for $N \geq 3$ by [11]. Thus the one-row conjecture and the full unitary group conjecture ask genuinely different free product questions. The common phase of the row and the matrix has another consequence.

Corollary 4.4. *For $N \geq 2$, $L^\infty(S^{N-1}_{\mathbb{C},+})' \cap L^\infty(U_N^+) = \mathbb{C}1$. Each complex Stiefel row algebra $L^\infty(St^+_{LN,\mathbb{C}})$ is a nonamenable strongly solid II_1 factor and fails strong 1-boundedness.*

Proof. Write $L^\infty(U_N^+) = A * L^\infty(PO_N^+)$ as in Theorem 4.2, where $A = W^*(w)$ and w is the polar unitary of v_{11}. In this free product the commutant of w is A. Indeed the reduced words containing a centered letter of $L^\infty(PO_N^+)$ span the orthogonal complement of $L^2(A)$. Grouping them by the alternating word from the first to the last such letter identifies this complement, as an A-bimodule, with copies of $L^2(A) \otimes L^2(A)$. Conjugation by w acts there by the function $(z_1, z_2) \to z_1 \bar{z}_2$ on the product of 2 circles with normalized arc length. Its level set at 1 has product measure zero.

If a commutes with $L^\infty(S^{N-1}_{\mathbb{C},+})$, then $a \in A$ and a also commutes with the centered unitary $b = s_1 s_2 \in L^\infty(P_+^{N-1})$. For $a_0 = a - \tau(a)1$, the reduced words $a_0 b$ and $b a_0$ are orthogonal, so equality forces $a_0 = 0$. This gives the asserted relative commutant. The row contains $W^*(w, b) \simeq L(F_2)$, since $b = s_1 s_2$ has infinite order and all its nonzero powers have trace zero by Proposition 2.1. Thus every intermediate frame algebra is nonamenable, also at $N = 2$. Strong solidity passes from the ambient unitary factor as in Corollary 3.4.

Finally $L^\infty(St^+_{LN})^\gamma$ is diffuse and embeds into a tracial matrix ultraproduct since $L^\infty(O_N^+)$ does [16]. By the free product obstruction of Jung [36], applied as in [12], a free product of 2 diffuse finite algebras with matricial microstates cannot be strongly 1-bounded. Apply this to $A * L^\infty(St^+_{LN})^\gamma$. At $L = N$ this gives another proof of the failure of strong 1-boundedness for $L^\infty(U_N^+)$ established in [11]. □

In the unitary group a single entry has both a phase and a diffuse modulus. This already gives the analogue of the irreducible copy of F in Theorem 3.3.

Theorem 4.5. *For $N \geq 2$ and any entry v_{ij} of the fundamental matrix of U_N^+, we have*

$$W^*(v_{ij}) \simeq F, \qquad W^*(v_{ij})' \cap L^\infty(U_N^+) = \mathbb{C}1.$$

The inclusion is proper, even when $N = 2$.

Proof. By permuting the rows and columns take $i = j = 1$. In the free complexification $v_{11} = z u_{11} = w|u_{11}|$, where $w = z\, sgn(u_{11})$ is its polar unitary. Both polar factors belong to $W^*(v_{11})$, since u_{11} has zero kernel. The modulus has atomless law, and Lemma 4.1 gives freeness of w from the entire orthogonal algebra. The modulus therefore generates a diffuse abelian tracial algebra free from $W^*(w)$. Their free product is $L(F_2)$.

In the full frame decomposition $L^\infty(U_N^+) = W^*(w) * L^\infty(PO_N^+)$, the commutant of w is $W^*(w)$ by the reduced-word argument in Corollary 4.4. The centered modulus is a nonzero element of the other free component. A centered element of $W^*(w)$ times that element in the 2 possible orders gives orthogonal vectors, so their commutation forces the first element to vanish. Finally changing the sign of row 2 fixes v_{11} and moves $v_{21} \neq 0$, giving properness. □

The same polar decomposition works for any nonempty set I of positions in a unitary matrix. Choose a distinguished position $e \in I$ and put $w = z\, sgn(u_e)$. We have

$$W^*(v_f \mid f \in I) = W^*(w) * W^*(u_f u_g \mid f, g \in I).$$

Indeed $u_f u_g = v_f^* v_g$ belongs to the left side. Within the algebra of products the elements $(|u_e|+\varepsilon)^{-1} u_e u_f$ converge strongly to $sgn(u_e)u_f$, since the first factor multiplied by u_e is uniformly bounded. Multiplying the limit by w recovers v_f. Freeness follows from Lemma 4.1. In particular every nonempty projected unitary row or frame block generates an irreducible nonamenable strongly solid factor in the ambient $L^\infty(U_N^+)$. For $N = 2$ the ambient algebra is F and for $N \geq 3$ use [14]. The 2 free components are diffuse and have matricial microstates, giving the same obstruction to strong 1-boundedness as in Corollary 4.4. Keeping one complex coordinate gives F in every ambient dimension $N \geq 2$.

The unitary conjecture consequently implies a constraint on the projective group which can be tested independently. Write $h(D)$ for Hayes' 1-bounded entropy [28] and $\mathcal{F}(D)$ for the fundamental group of a factor, the set of $t > 0$ for which its trace-normalized amplification D^t is isomorphic to D.

Proposition 4.6. *Let $N \geq 3$ and $D = L^\infty(PO_N^+)$. If $L^\infty(U_N^+) \simeq F$, then $0 < h(D) < \infty$ and $\mathcal{F}(D) = \{1\}$. In particular, either $h(D) = 0$ or a single nontrivial self-amplification of D would disprove the unitary conjecture at that N.*

Proof. The free product formula embeds D into F under the hypothesis. By Corollary 3.4 it is nonamenable. Hayes, Jekel and Kunnawalkam Elayavalli [29] give $h(D : F) > 0$ for a nonamenable subalgebra of F. The entropy in the presence of F is bounded above by $h(D)$, and $h(D) < \infty$ follows from strong 1-boundedness [11, 29]. If $0 < t < 1$ belongs to $\mathcal{F}(D)$, the compression formula of Hayes, Jekel and Kunnawalkam Elayavalli [30] yields $h(D) = t^{-2}h(D)$, which is impossible for a finite strictly positive number. Inversion excludes $t > 1$ as well. □

5 The projective involution and the magnitudes

We first compare the part of the real sphere extension generated by coordinate signs. Write $\alpha_N = Ad\, s_1$ on $L^\infty(P_+^{N-1})$ and let $B = W^*(s_1 s_2, \ldots, s_1 s_N)$. In the free generating system of Proposition 2.1, α_N takes each generator to its adjoint. An isomorphism of the abstract factors B need not intertwine these actions. The following conjugacy uses the isomorphism of free group factors in [42].

Theorem 5.1. *For all $M, N \geq 3$ the restrictions of α_M, α_N to the even sign algebras are trace-preservingly conjugate. The conjugacy extends to an isomorphism of the full sign algebras carrying the first coordinate sign to the first coordinate sign.*

Proof. Put $d = N - 1$. On the circle with normalized arc length, complex conjugation and the half-turn are measurably conjugate. Both exchange 2 sets of measure $1/2$ with diffuse normalized measure. A measure-preserving identification on one half extends equivariantly to the other. We can therefore choose free Haar generators $u_1, \ldots, u_d$ for which $\alpha_N(u_j) = -u_j$. The free group change of basis $u = u_1$, $v_j = u_1^* u_j$ for $2 \leq j \leq d$ gives

$$B = W^*(u) * Q, \qquad Q = W^*(v_2, \ldots, v_d).$$

The involution changes the sign of u and fixes Q pointwise. Denote this action on the displayed free product by δ_d. For $d \geq 3$ all the algebras $Q \simeq L(F_{d-1})$ are isomorphic by [42]. Extending such an isomorphism by the identity on $W^*(u)$ conjugates any 2 of these actions.

For $d = 2$, Q is abelian, so we use the order-2 free shift of Hiai and Ueda [31]. Here are the coordinates for this step. Put $P = W^*(u^2)$ and choose a symmetry t equal to 1 on one semicircle and -1 on the opposite one. Then $\delta_d(t) = -t$, and $W^*(u) = P \bar{\otimes} W^*(t)$ in the product trace. The free shift formula identifies the fixed algebra and the action of its odd symmetry as

$$K = B^{\delta_d} = P * Q * (tQt), \qquad B = K \rtimes \mathbb{Z}_2.$$

Conjugation by t fixes P and exchanges the 2 copies of Q. The dual involution fixes K pointwise and sends t to $-t$, so it is exactly δ_d.

Choose one semicircular generator of P and $d-1$ free semicircular generators a_j of Q, using the diffuse abelian algebra of each generator. In K the variables a_j, ta_jt are part of a free semicircular system. Replacing each pair by $(a_j + ta_jt)/\sqrt{2}$ and $(a_j - ta_jt)/\sqrt{2}$ preserves its law and the generated algebra. The action on K now fixes d semicirculars and changes the signs of $d-1$. Each odd semicircular admits a Haar generator of its abelian algebra on which the action is the half-turn, by the same measurable identification on 2 halves. Make the fixed semicirculars Haar as well. Multiplying each other odd generator on the left by the adjoint of the first odd one is again a free group change of basis. This identifies the action on K with δ_{2d-1}.

For $d = 2$ and $d = 3$ the 2 actions on K are consequently δ_3 and δ_5, already conjugate by the first paragraph. Extend this conjugacy to their crossed products by sending the implementing symmetry to the implementing symmetry. It intertwines the dual involutions δ_2 and δ_3. This proves conjugacy for every $d \geq 2$. Finally, the full sign algebra is the crossed product of the even sign algebra by $Ad\, s_1$, by Proposition 2.1. Extending the conjugacy once more gives the assertion about s_1. □

We now put the absolute values back into the row. For this section put $T = L^\infty(S^{N-1}_{\mathbb{R},+})$, $A = L^\infty(P^{N-1}_+)$, and write s_g for the word in the signs associated to $g \in \Gamma_N$. Define

$$D = W^*(s_g|x_i|s_g^* \mid g \in \Gamma_N,\ 1 \leq i \leq N).$$

All these operators have degree 1. The algebra D contains the diffuse algebra generated by $|x_1|$, and conjugation by each s_g preserves D.

Proposition 5.2. *We have*

$$T = D \rtimes \Gamma_N, \qquad A = D \rtimes \Gamma_N^+,$$

where both actions are conjugation by the sign words. In particular D is the degree-1 algebra of the diagonal grading on the row.

Proof. The s_g form a unitary representation by Proposition 2.1. Since $x_i = s_i|x_i|$, the algebra generated by D and the s_g is T. For $d \in D$ and $g \neq 1$, the vector ds_g has degree g and hence trace zero. The trace on finite sums $\sum_g d_g s_g$ is therefore the crossed product trace. Its faithfulness gives the first identification, with gth Fourier space $L^2(D)s_g$. The antipodal action fixes D and acts on s_g by word parity. Its fixed algebra is the indicated even crossed product. □

The next computation uses the full sphere as well as its projective part. This will allow us to use odd sign words on both sides of a rectangle.

Lemma 5.3. *For $N \geq 3$ the algebra D is nonamenable, $D' \cap T = Z(D)$ is finite dimensional, and Γ_N^+ acts transitively on the minimal projections of $Z(D)$.*

Proof. The factors A and T are strongly solid by Corollary 3.4. The algebra D is diffuse since it contains $W^*(|x_1|)$ unitally. If D were amenable, its unitary normalizer in A would generate an amenable algebra. This normalizer contains the even signs, which generate $L(F_{N-1})$.

Put $C = D' \cap T$. Conjugation by Γ_N preserves C and acts ergodically on $Z(C)$, since an invariant element there commutes with D and every s_g and hence is in $Z(T) = \mathbb{C}$. The central projection of C supporting its atomic part is invariant. If it were zero, C would be diffuse. Strong solidity of T implies solidity, since an arbitrary diffuse subalgebra contains a diffuse abelian algebra, whose commutant is contained in its amenable normalizing algebra. Solidity of T would then make $C' \cap T$ amenable. This is impossible because $C' \cap T$ contains D with trace-preserving expectation. Thus C is atomic. Its minimal central projections form one orbit by ergodicity. They have equal positive trace, so there are finitely many; each atomic summand of a finite algebra is a matrix algebra.

Each Fourier projection of $T = D \rtimes \Gamma_N$ preserves C. Indeed, commutation with $a \in D$ for a coefficient $d_g s_g$ is the equation $ad_g = d_g(s_g a s_g^*)$. Distinct nonzero Fourier spaces in C are orthogonal, so finite dimensionality permits only finitely many degrees. Their set is invariant under conjugation by Γ_N, and Lemma 3.1 leaves only the unit. Consequently $C \subset D$, giving $C = Z(D)$. Finally a Γ_N^+-invariant central projection of D belongs to the center of $A = D \rtimes \Gamma_N^+$. Factoriality of A proves transitivity. □

We record how to extract a discrete group from a corner. For a unital inclusion $D \subset A$, write $\mathcal{N}_A(D)$ for the group of unitaries $u \in A$ with $uDu^* = D$.

Lemma 5.4. *Suppose $A = D \rtimes G$ is a tracial crossed product with $D' \cap A = Z(D)$, and p is a minimal projection of $Z(D)$. Then*

$$\mathcal{N}_{pAp}(pDp)/\mathcal{U}(pDp) \simeq G_p,$$

where G_p is the stabilizer of p for the given action.

Proof. Write u_g for the implementing unitaries. If g moves p, then $pu_g p = 0$. Compression of the Fourier expansion gives $pAp = (pDp) \rtimes G_p$. This action on the factor pDp is outer. Indeed if $g \neq 1$ acted there by $Ad\, v$, then $v^* p u_g$ would commute with D (it is supported on p) and have nonidentity degree, contradicting $D' \cap A = Z(D)$.

For a unitary w normalizing pDp, each Fourier coefficient of w intertwines the corresponding group automorphism with $Ad\, w$ on pDp. Both absolute squares of a nonzero intertwiner are scalar in this factor, so it is a scalar multiple of a unitary. Two distinct nonzero coefficients would therefore identify 2 group automorphisms modulo inner automorphisms. Outerness leaves a single degree and $w = vpu_g$ for some $v \in \mathcal{U}(pDp)$. Uniqueness of the degree and multiplication of the pu_g give the claimed quotient. □

We can now read the dimension from 2 quantities attached to the inclusion. The group rank below is the number of generators of a free basis.

Theorem 5.5. *Let $N \geq 3$, $A = L^\infty(P_+^{N-1})$ and $D \subset A$ be the magnitude algebra. Put $k = \dim Z(D)$. For every minimal central projection p of D we have*

$$W(p) = \mathcal{N}_{pAp}(pDp)/\mathcal{U}(pDp) \simeq F_{1+k(N-2)}.$$

Thus the inclusion recovers $N = 2 + (rank\, W(p) - 1)/k$. For $N \geq 4$ we have $D' \cap T = \mathbb{C}1$ and $W(1) \simeq F_{N-1}$.

Proof. Lemma 5.3 gives $D' \cap A = Z(D)$ and an orbit of size k. By Lemma 5.4, $W(p)$ is an index-k subgroup of $\Gamma_N^+ \simeq F_{N-1}$. Its graph covering of the bouquet of $N-1$ circles has k vertices and $k(N-1)$ edges. Contracting a spanning tree leaves $1 + k(N-2)$ free generators. Both k and the corner normalizer quotient are preserved by isomorphism of inclusions.

Suppose now that $N \geq 4$. Conjugation by each of s_1s_2, s_1s_3, s_2s_3 permutes the finitely many minimal projections of $Z(D)$. A positive power of each word fixes p. Theorem 3.3 applied to these 3 powers gives $p = 1$. Thus $k = 1$, and Lemma 5.3 gives the assertion in the full sphere. □

At $N = 4$ the magnitude algebra is a factor with discrete normalizer quotient F_3. At $N = 3$ either its center is nontrivial or it is a factor with quotient F_2. Thus an isomorphism of the projective factors carrying one magnitude algebra onto a unitary conjugate of the other is impossible. In particular any extension of the sign conjugacy in Theorem 5.1 to the full actions would have to move the magnitude algebra outside that conjugacy class.

There is a second presentation in dimension 4 which explains the scope of this obstruction. The retraction $\Gamma_4 \to \Gamma_3$ fixing the first 3 generators and sending g_4 to g_1 preserves parity. Its kernel H consists of even words, contains g_4g_1, and gives $\Gamma_4 = H \rtimes \Gamma_3$ by the indicated splitting. Enlarge the dimension-4 magnitude algebra to $D_* = D \rtimes H$. Fourier degrees show that $D \subsetneq D_* \subset A$, and normality of H makes D_* invariant under $Ad\, s_1$. We obtain

$$A = D_* \rtimes \Gamma_3^+, \qquad T = D_* \rtimes \Gamma_3$$

with $\Gamma_3^+ \simeq F_2$. Thus dimension 4 already admits a presentation with the same quotient group and the same induced involution on that group as dimension 3. The conjugacy question for α_3, α_4 on the full factors is open.

Even within a single dimension the choice of axes matters. The following rotation gives an automorphism commuting with the full projective involution and makes the issue of identifying the magnitude algebra quite explicit.

Proposition 5.6. *For every $N \geq 3$ there is a trace-preserving automorphism θ of $L^\infty(P_+^{N-1})$ such that $\theta\alpha_N = \alpha_N\theta$ and $\theta(D) \neq D$.*

Proof. Rotate the second and third columns of O_N^+ through $\pi/4$, choosing the orientation such that x_2 goes to $(x_2 + x_3)/\sqrt{2}$. Right translation preserves the trace and both the row and projective algebras. It fixes x_1 and hence s_1. On $x_2^2 \in D$ this gives

$$\theta(x_2^2) = \frac{1}{2}(x_2^2 + x_3^2) + \frac{1}{2}x_2x_3 + \frac{1}{2}x_3x_2.$$

The last 2 terms have distinct degrees g_2g_3 and g_3g_2 and are both nonzero, since x_2 and x_3 have trivial kernel. Proposition 5.2 identifies D with the degree-1 algebra, so the displayed element is outside D. □

The rotation moves D as a subalgebra. Whether it moves D outside its unitary conjugacy class would require a further argument. For different dimensions Theorem 5.5 already gives this stronger obstruction.

6 Volumes on the free ball

Write $C(B_{\mathbb{R},+}^N)$ for the universal unital C^*-algebra on self-adjoint $x_1, \ldots, x_N$ with $\sum_i x_i^2 \leq 1$. Imposing equality gives the boundary quotient $C(B_{\mathbb{R},+}^N) \to C(S_{\mathbb{R},+}^{N-1})$. Each ball point also extends to a point of $S_{\mathbb{R},+}^N$ by adjoining $x_{N+1} = (1 - \sum_i x_i^2)^{1/2}$. Passing from this relation to a tracial von Neumann algebra requires a choice of measure. We use the radial and projected integrals of [18], and write $L^\infty(B_{\mathbb{R},+}^N, \tau) = \pi_\tau(C(B_{\mathbb{R},+}^N))''$. Put $R = (\sum_i x_i^2)^{1/2}$.

For the radial volume let $u_1, \ldots, u_N$ be the coordinates of the real free sphere with invariant trace h_N, and put $d\mu_N(r) = Nr^{N-1}\, dr$ on $[0, 1]$. The divergence prescription multiplies the integral of a homogeneous boundary polynomial of degree d by $N/(N + d)$. For the projected

volume keep the first N coordinates of a row of O^+_{N+j}. Thus the 2 traces are specified on polynomials by

$$\tau_{\rm rad}(p) = \int_0^1 h_N\big(p(ru_1,\dots,ru_N)\big)\,d\mu_N(r), \qquad \tau_{N,j}(p) = h_{N+j}\big(p(u_1,\dots,u_N)\big).$$

In the second integral the u_i are coordinates of the larger sphere. For $j=2$ the 2 traces give x_1^2 the same integral $1/(N+2)$, while their fourth moments already differ [18]. To distinguish the abstract von Neumann algebras we compute their centers.

Theorem 6.1. *For $N\geq 3$ the radial completion is*

$$L^\infty(B^N_{\mathbb{R},+},\tau_{\rm rad}) \simeq L^\infty([0,1],\mu_N)\,\bar\otimes\, L^\infty(S^{N-1}_{\mathbb{R},+}),$$

with center $W^*(R)$. For every $j\geq 1$, $L^\infty(B^N_{\mathbb{R},+},\tau_{N,j})$ is an irreducible nonamenable strongly solid II_1 subfactor of $L^\infty(O^+_{N+j})$. In particular the radial and projected completions are nonisomorphic.

Proof. In the tensor product the generators are $x_i = r\otimes u_i$, so that $R = r\otimes 1$ belongs to their algebra. For $\varepsilon>0$ the elements $(R+\varepsilon)^{-1}x_i$ are uniformly bounded and converge strongly to $1\otimes u_i$, since $\mu_N(\{0\})=0$. We recover both tensor legs, and the product trace is faithful and agrees with the defining radial integral. The sphere leg is a factor by Corollary 3.4, proving the assertion about the center.

For the projected trace take the first N entries of a row of O^+_{N+j}. Their squares sum to at most one by the sphere relation. The restricted trace is faithful, so the GNS algebra is $W^*(u_1,\dots,u_N)$. Borel functional calculus gives the signs of its first 3 coordinates. As $N+j\geq 4$, Theorem 3.3 and Corollary 3.4 apply in this ambient algebra. Its projected subfactor has scalar center, and the radial center is diffuse. □

For $N=2$ the projected trace already has noncentral squared radius [18]. The first unsettled projected disk is $W^*(u_{11},u_{12})\subset L^\infty(O_3^+)$. Is its center trivial? The 2 coordinate signs generate the amenable infinite dihedral group algebra, and the 3-sign argument begins only with a third retained coordinate. For $N\geq 3$ the factor conclusion holds at every codimension, leaving open whether its isomorphism class depends on j.

There is a direct extension from one projected row to several. The first r rows of O_M^+ form a Stiefel frame. Retaining the first $N<M$ columns gives an $r\times N$ matrix X. If Y consists of the discarded columns, the full frame relation gives $1_r - XX^t = YY^t \geq 0$.

Corollary 6.2. *For $3\leq N<M$ and $1\leq r\leq M$, the algebra generated by the entries of this projected frame is an irreducible nonamenable strongly solid II_1 subfactor of $L^\infty(O_M^+)$.*

Proof. The first retained row gives the ball algebra for $\tau_{N,M-N}$. Its first 3 coordinate signs give the 2 unitaries of Theorem 3.3. The block algebra lies between their algebra and $L^\infty(O_M^+)$, so Corollary 3.4 applies. □

For example, take the upper left 2×3 block of O_6^+. On the same 6 self-adjoint generators with $XX^t\leq 1_2$, a second trace comes from $X=tV$, where V consists of 2 rows of O_3^+ and t is independent and uniform on $[0,1]$. Here $XX^t = t^2 1_2$. The first diagonal entry recovers t, and division by $t+\varepsilon$ recovers V in the strong limit. This model has center $L^\infty([0,1])$, since the 2-row frame algebra is a factor by Corollary 3.4. The projected block in O_6^+ has scalar center by Corollary 6.2.

For the closed complex free ball $B^N_{\mathbb{C},+}$ the coordinate algebra is universal on $x_1,\dots,x_N$ with both $\sum_i x_i^*x_i\leq 1$ and $\sum_i x_ix_i^*\leq 1$. Imposing equality in both relations gives $C(S^{N-1}_{\mathbb{C},+})$. Imposing self-adjointness of the coordinates gives $C(B^N_{\mathbb{R},+})$.

A complex ball point need not extend to a sphere point by adjoining a single coordinate. For example, take $x_1 = e_{11}/\sqrt{2}$ and $x_2 = e_{12}/\sqrt{2}$ in $M_2(\mathbb{C})$, with e_{ij} the matrix units. A third coordinate w completing this point to $S^2_{\mathbb{C},+}$ would satisfy

$$w^*w = 1_2 - \sum_{i=1}^{2} x_i^* x_i = \frac{1}{2}\,1_2, \qquad ww^* = 1_2 - \sum_{i=1}^{2} x_i x_i^* = e_{22}.$$

The 2 required operators have norms $1/2$ and 1, whereas $\|w^*w\| = \|ww^*\|$.

In the projected trace the retained generators are entries of a row of U^+_{N+j}. By Theorem 4.5 every projection retaining at least one coordinate generates an irreducible factor for $N, j \geq 1$. In the scalar radial model put $x_i = ru_i$ with u_i the boundary complex sphere coordinates. For instance take the classical probability density $2Nr^{2N-1}\,dr$ on $[0,1]$. Since $r = (\sum_i x_i^* x_i)^{1/2}$, the proof of Theorem 6.1 gives the tensor product of the radial and boundary algebras. At $N = 1$ this model is the abelian algebra of the ordinary disk with normalized area, whereas retaining one coordinate of any U^+_{1+j} gives F.

7 Frames, flags and Grassmannians

There are 2 operations on an orthogonal frame that retain different amounts of information. One can remember every entry of the first L rows. One can instead remember only the projection onto their span. The book [1] treats the first construction among the affine row spaces, and [17] realizes the second inside the same quantum group. At the operator algebra level the first construction is $L^\infty(St^+_{LN})$. The second is

$$L^\infty(Gr^+_{LN}) = W^*(p_{ij} \mid 1 \leq i, j \leq N), \qquad p_{ij} = \sum_{r=1}^{L} u_{ri} u_{rj}.$$

At $L = 1$ this is $L^\infty(P^{N-1}_+)$. For general L the Grassmannian algebra lies in the even frame algebra. At $L = N$ the projection is the identity, so $L^\infty(Gr^+_{NN}) = \mathbb{C}$ while $L^\infty(PO^+_N)$ is an entire projective quantum group algebra.

For the rectangular presentation of the real Stiefel manifold, $uu^t = 1_L$. Its orbit coordinates have the joint law of the first L rows of O^+_N. Indeed, left translation by the block subgroup $diag(O^+_L, 1)$ rotates these rows and preserves their joint law. More explicitly, the 2-group formula of [1] uses $\sum_{a \leq L} v_{ar} \otimes w^*_{ai}$ for row r, column i. In the unitary case transposition of the first fundamental matrix preserves its joint law (it interchanges coproduct and opposite coproduct), and entrywise adjoint on the second is a trace-preserving quantum group automorphism. The resulting coordinates $\sum_{a \leq L} v_{ra} \otimes w_{ai}$ are exactly the block translation of the first L rows. In the orthogonal case the entries are already self-adjoint. This verifies the row convention of [6, 17] for both traces.

The index measures what happens when one more row is retained. In the real case the left diagonal torus records its row index. In the complex case a scalar phase on the new row suffices.

Theorem 7.1. *In the real Stiefel tower for $N \geq 3$ and in the complex Stiefel tower for $N \geq 2$, every inclusion between distinct row lengths is irreducible and has infinite Jones index. Each tower runs from the sphere at length 1 to the full quantum group at length N, and all its members are nonamenable strongly solid II_1 factors.*

Proof. Let $1 \leq L < J \leq N$ and denote the 2 frame algebras by $A \subset B$. The projective first row in the real case and any one entry of that row in the complex case have scalar commutant in the ambient group factor, by Theorems 3.2 and 4.5. The intermediate factor conclusions follow from Corollary 3.4 and the corresponding complex argument following Theorem 4.5.

We construct a unitary $w \in B$ such that $E_A(w^k) = 0$ for every nonzero integer k. The subspaces $w^j L^2(A)$, for $j \geq 0$, are then pairwise orthogonal right A-modules in $L^2(B)$. Each has right dimension 1, whence $[B : A] = \dim_A L^2(B) = \infty$ [35].

For the real tower apply the diagonal grading of Section 2 on the left. The entries of row r have degree g_r, and so do their signs by the same odd polynomial approximation. All words in A have degrees in $\langle g_1, \ldots, g_L \rangle$. Take

$$w = sgn(u_{L+1,1}) sgn(u_{11}).$$

Every nonzero power has degree $(g_{L+1} g_1)^k$, whose reduced word contains g_{L+1}. Orthogonality of the homogeneous spaces gives the required expectation.

For the complex tower take w to be the polar unitary of $v_{L+1,1}$. Left translation by the diagonal matrix with $\lambda \in \mathbb{T}$ at position $L+1$ and 1 elsewhere fixes A pointwise and takes w to λw. The expectation onto A is equivariant by uniqueness of the trace-preserving expectation. Thus $E_A(w^k) = \lambda^k E_A(w^k)$ for every λ, giving the conclusion also in this case. □

The same circle argument computes the index of the single-entry inclusion in Theorem 4.5. For $A = W^*(v_{11})$, use the polar unitary of v_{21} and rotate the second row. In particular at $N = 2$ the entry gives an irreducible inclusion of infinite index between 2 copies of F. On the real side, the unitary used in the proof is a product of 2 signs and thus belongs even to the projective algebra of the larger frame. The inclusions between distinct even frame algebras have infinite index as well for $N \geq 3$.

For a composition $d = (d_1, \ldots, d_k)$ of N, let I_s be the successive blocks. The flag orbit has algebra

$$L^\infty(Fl_d^+) = W^* \left(q_{ij}^s \mid 1 \leq s \leq k,\ i, j \leq N \right), \qquad q_{ij}^s = \sum_{r \in I_s} u_{ri} u_{rj}.$$

This is the realization of the free flags in [17]. The 2-block flag $(L, N-L)$ yields $L^\infty(Gr_{LN}^+)$, since the second matrix is $1 - p$. The full flag is obtained by taking all block sizes equal to one. A sum of adjacent block projections implements coarsening on the level of orbit coordinates. All these assertions concern the algebra generated by the indicated coordinates. The full stabilizer coinvariant algebra can be larger [6, 17].

Theorem 7.2. *If $N \geq 3$ and some $d_s = 1$, then $L^\infty(Fl_d^+)' \cap L^\infty(O_N^+) = \mathbb{C}1$ and $L^\infty(Fl_d^+)$ is a nonamenable strongly solid II_1 factor. The orthogonal and unitary orbit realizations give the same tracial algebra.*

Proof. If $I_s = \{r\}$, the entries of q^s contain the whole projective row $W^*(u_{ri} u_{rj} \mid i, j)$. Theorem 3.2 applied to row r gives the commutant assertion, and Corollary 3.4 applies after a row permutation. In the free complexification model the unitary orbit coordinates are literally the same elements, because

$$\sum_{r \in I_s} v_{ri}^* v_{rj} = \sum_{r \in I_s} u_{ri} z^* z u_{rj} = q_{ij}^s.$$

This also gives the equality of orbit traces from [17]. □

For instance the full flag at $N = 3$ is generated by the 3 projective rows. The sum of their projection matrices is 1, so 2 suffice. It contains the projective plane algebra irreducibly. No freeness of these projective rows is asserted: the sum relation couples their coordinates. In all dimensions a full flag contains copies of $L^\infty(P_+^{N-1})$ coming from each row.

Taking the orthogonal complement of the Grassmannian projection, followed by a row permutation, gives

$$L^\infty(Gr_{LN}^+) \simeq L^\infty(Gr_{N-L,N}^+).$$

The theorem therefore settles both extremes $L = 1, N-1$. For $2 \leq L \leq N-2$ the visible homogeneous elements are sums of several row products. Their degrees are $g_i g_j$, but a homogeneous

operator can have a proper support projection. Its polar part cannot be used as a group unitary until both supports are controlled.

Proposition 7.3. *At $1 \le L < N$, the diagonal Grassmannian coordinate satisfies*

$$\tau(p_{11}) = \frac{L}{N}, \qquad \tau(p_{11}^2) = \frac{L(L+1)}{N(N+1)}, \qquad V(p_{11}) = \frac{L(N-L)}{N^2(N+1)}.$$

Here $V(p_{11}) = \tau(p_{11}^2) - \tau(p_{11})^2$ is the variance. For fixed L, Np_{11} converges in moments to the free Poisson law of parameter L.

Proof. These are the orbit integration formulas of [17]. For the first 2, the column coordinates $t_r = u_{r1}$ have $\tau(t_r^2) = 1/N$, $\tau(t_r^4) = 2/[N(N+1)]$, and $\tau(t_r^2 t_s^2) = 1/[N(N+1)]$ for $r \neq s$, by the 2 noncrossing pairings in the integration formula of [3] in degree 4. Expanding $(\sum_{r \le L} t_r^2)^2$ gives the stated second moment. More generally the scaled column converges on each fixed set of entries to a free semicircular system [4, 17]. Its first L squares sum to a free Poisson variable of parameter L. □

The Peter–Weyl spectrum of this orbit algebra is also known from [17]. Each even irreducible r_{2k} occurs once and no odd irreducible occurs. It is the same list for all $1 \le L < N$. Multiplicities describe L^2 as a representation. Multiplication and the position of the corresponding coefficient spaces in $L^2(O_N^+)$ retain information about L.

Question 7.4. *Does $L^\infty(Gr_{24}^+)$ have trivial center? More strongly, is $L^\infty(Gr_{24}^+)' \cap L^\infty(O_4^+) = \mathbb{C}1$? If it is a factor, is it isomorphic to the projective algebra $L^\infty(P_+^3)$?*

In this example $\tau(p_{11}) = 1/2$ and $\tau(p_{11}^2) = 3/10$. A possible way to address the relative commutant is to find homogeneous unitaries in $L^\infty(Gr_{24}^+)$ whose degrees generate enough conjugations to force every commuting vector into K_1. The reflection argument would then apply, because right orthogonal rotations preserve the whole projection algebra. This isolates the step for which the rank-one proof presently has no substitute.

8 Rectangles and their projectivizations

Let x_i and y_a be the coordinates on the 2 tensor legs of the product of the real free spheres $S_{\mathbb{R},+}^{M-1}$ and $S_{\mathbb{R},+}^{N-1}$. For the space of real rectangular partial isometries of rank one, the orbit integration in [1] realizes the coordinates as

$$z_{ia} = x_i \otimes y_a, \qquad L^\infty(O_{MN}^{1+}) = W^*(z_{ia} \mid 1 \le i \le M,\ 1 \le a \le N).$$

This is also the rank-one sphere product model inside the determinantal incidence space of [17]. The full universal incidence space includes its auxiliary projection; the assertion here concerns the specified orbit trace.

Theorem 8.1. *Put $\gamma_M(x_i) = -x_i$ and $\gamma_N(y_a) = -y_a$. For $M, N \ge 2$ we have*

$$L^\infty(O_{MN}^{1+}) = (L^\infty(S_{\mathbb{R},+}^{M-1}) \bar{\otimes} L^\infty(S_{\mathbb{R},+}^{N-1}))^{\gamma_M \otimes \gamma_N}.$$

The von Neumann algebra generated by all $z_{ia}z_{jb}$ is $L^\infty(P_+^{M-1}) \bar{\otimes} L^\infty(P_+^{N-1})$.

Proof. Each z_{ia} is fixed under simultaneous sign change. On the polynomial tensor product the averaging projection for this action retains the pairs of words whose lengths have the same parity. In such a pair, insert $\sum_i x_i^2 = 1$ or $\sum_a y_a^2 = 1$ into the shorter word until both lengths agree. The resulting sum is a sum of words in the z_{ia}. Normality of the averaging projection and weak density give the reverse inclusion.

A product of 2 rectangular coordinates is $x_i x_j \otimes y_a y_b$. Within the algebra they generate the 2 partial sums

$$\sum_a z_{ia} z_{ja} = x_i x_j \otimes 1, \qquad \sum_i z_{ia} z_{ib} = 1 \otimes y_a y_b$$

recover both full projective legs. Conversely each product lies in the tensor product of these legs. These are the von Neumann versions of the 2 partial traces of the Segre embedding in [17]. □

The 2 inclusions around $L^\infty(O_{MN}^{1+})$ have the same finite index. We give the actions explicitly since the algebras on either side of the rectangular algebra each have a geometric meaning.

Corollary 8.2. *For $M, N \geq 3$ the inclusions*

$$L^\infty(P_+^{M-1} \times P_+^{N-1}) \subset L^\infty(O_{MN}^{1+}) \subset L^\infty(S_{\mathbb{R},+}^{M-1} \times S_{\mathbb{R},+}^{N-1})$$

are inclusions of II_1 factors, both of index 2. The middle factor is nonsolid. In particular it is distinct from F, from $L^\infty(O_k^+), L^\infty(U_k^+)$ for $k \geq 3$, and from $L^\infty(S_k^+)$ for $k \geq 5$.

Proof. Put $s = sgn(x_1)$, $t = sgn(y_1)$, and $A = L^\infty(P_+^{M-1}) \bar{\otimes} L^\infty(P_+^{N-1})$. The parity decomposition of Theorem 4.2 gives the even-even and odd-odd summands as

$$L^\infty(O_{MN}^{1+}) = A + A(s \otimes t).$$

The symmetry $s \otimes t$ implements on the tensor product the tensor product of the outer actions from Theorem 4.2. This tensor product action is outer. Indeed if a unitary v implemented it, then $v(a \otimes 1) = (\alpha(a) \otimes 1)v$ on the first leg. A nonzero normal slice on the second leg gives an intertwiner in the first factor from the identity to α. Its 2 absolute squares are scalar by factoriality, so a scalar multiple implements α, a contradiction. The first inclusion is therefore an outer crossed product inclusion of index 2, with the trace given by its parity expectation.

The antipodal automorphism of $L^\infty(S_{\mathbb{R},+}^{M-1})$ is outer as well. A unitary implementing it would commute with its fixed algebra $L^\infty(P_+^{M-1})$, so Theorem 3.2 would make it scalar. The same slice argument makes $\gamma_M \otimes \gamma_N$ outer. The fixed algebra inclusion of an outer action of a group of order 2 on a factor has index 2 [35]. Equivalently the odd unitary $s \otimes 1$ gives the other summand of $L^2(S_{\mathbb{R},+}^{M-1} \times S_{\mathbb{R},+}^{N-1})$ over the fixed algebra.

Finally choose a diffuse abelian subalgebra of $L^\infty(P_+^{M-1}) \otimes 1$, for instance the one generated by $x_1^2 \otimes 1$. Its commutant in $L^\infty(O_{MN}^{1+})$ contains the nonamenable factor $1 \otimes L^\infty(P_+^{N-1})$ with expectation. Thus that commutant is nonamenable. The comparison factors in the statement are solid by [43, 14, 11]. □

The rectangular factor is also strongly 1-bounded. The tensor product of 2 diffuse factors has 1-bounded entropy at most zero [29], and finite-index permanence [11] applies to the first inclusion. The smallest pair in the corollary is $(M, N) = (3, 3)$. It produces a factor distinct from the quantum group examples before the isomorphism type of even one projective plane is known. For $M = 1$ the orbit coordinates reduce, up to a common scalar sign in the separate leg, to one sphere with its symmetric trace. This recovers $L^\infty(O_{1N}^{1+}) \simeq L^\infty(S_{\mathbb{R},+}^{N-1})$ and explains the role of having 2 nonamenable projective legs in the separation argument.

The remaining side length is 2. Here the real circle has a central coordinate square, and the projective involution can be computed on its whole algebra.

Theorem 8.3. *For $N \geq 2$ the real rank-one rectangle with side lengths $2, N$ has orbit algebra*

$$L^\infty(O_{2N}^{1+}) \simeq L^\infty([0,1]) \bar{\otimes} M_2(\mathbb{C}) \bar{\otimes} L^\infty(P_+^{N-1}),$$

with Lebesgue measure and the normalized matrix trace. For $N \geq 3$ the first tensor leg is the center. At $N = 2$ the algebra is of type I_2 with diffuse center.

Proof. Let x_1, x_2 be the 2 real circle coordinates and put $t = x_1^2$, $s_i = sgn(x_i)$ and $h = s_1 s_2$. Since $t = 1 - x_2^2$, it commutes with both coordinates. The circle algebra is generated by t, s_1, s_2, as $x_1 = s_1\sqrt{t}$ and $x_2 = s_2\sqrt{1-t}$. Its even subalgebra is therefore generated by t, h, since every even word in 2 symmetries is a power of their product.

The unitary h is Haar by Proposition 2.1. For $k \neq 0$, $f(t)h^k$ has nonidentity degree $(g_1 g_2)^k$ and hence trace zero for every bounded Borel f. This gives independence of t and h. If ν is the diffuse law of t and m denotes normalized arc length on $\mathbb{T}$, the projective leg is

$$A = L^\infty(P_+^1) = W^*(t) \bar{\otimes} W^*(h) \simeq L^\infty([0,1], \nu) \bar{\otimes} L^\infty(\mathbb{T}, m).$$

Conjugation by s_1 fixes t and sends h to h^*.

Put $B = L^\infty(P_+^{N-1})$ and let d be the sign of the first rectangular coordinate. In the tensor model $d = s_1 \otimes sgn(y_1)$. The even-even and odd-odd decomposition of Theorem 8.1 gives the rectangular algebra as $C = (A \bar{\otimes} B) + (A \bar{\otimes} B)d$ for every $N \geq 2$. Let e be the spectral projection of $h \otimes 1$ on the open upper semicircle. Then $ded = 1 - e$ and $\tau(e) = 1/2$. Compression kills the odd summand, giving

$$eCe = e(A \bar{\otimes} B)e, \qquad C \simeq M_2(\mathbb{C}) \bar{\otimes} eCe.$$

Indeed de has initial projection e and final projection $1 - e$ and hence supplies the off-diagonal matrix unit. The normalized corner is B tensored with the function algebra on the product of the t-space and the upper semicircle with normalized measure. This product is a standard diffuse probability space, hence isomorphic in measure to the unit interval.

For $N \geq 3$ the algebra B is a factor. Concretely the center in C is generated by $t \otimes 1$ and $(h + h^*) \otimes 1$. On the upper semicircle the real part determines h, and both generators commute with d. For $N = 2$ the additional projective leg B is diffuse abelian. Combining the abelian tensor legs gives $L^\infty([0,1]) \bar{\otimes} M_2(\mathbb{C})$. □

In particular the real $(2,3)$ rectangle has constant central fiber $M_2(\mathbb{C}) \bar{\otimes} L^\infty(P_+^2)$. Its central decomposition is explicit even though the isomorphism type of the projective plane factor is still open. In Section 12 the unitary rectangle is a factor already at $(2,2)$.

We return to $M, N \geq 3$. There is also an index-2 inclusion using just the signs of the rectangle. Let D_0 and E_0 denote the 2 even sign factors on the individual rows, and let $C_0 = W^*(sgn(z_{ia}) \mid i \leq M,\ a \leq N)$. The even-even subgroup of the rectangular sign group yields $D_0 \bar{\otimes} E_0 \subset C_0$. The conjugacy on even signs removes both parameters from this inclusion.

Corollary 8.4. *For all $M, N \geq 3$ the inclusions $D_0 \bar{\otimes} E_0 \subset C_0$ are mutually isomorphic, with the odd symmetry $sgn(z_{11})$ preserved.*

Proof. Write $sgn(z_{ia}) = s_i \otimes t_a$, where $t_a = sgn(y_a)$. Products using the same second index generate $D_0 \otimes 1$, and products using the same first index generate $1 \otimes E_0$. Each $s_i \otimes t_a$ differs from $s_1 \otimes t_1$ by an even sign word on each leg. Thus C_0 is generated by $D_0 \bar{\otimes} E_0$ and $s_1 \otimes t_1$. The odd summand has trace zero by parity, so this is the crossed product by the tensor product of the 2 sign involutions. Applying Theorem 5.1 on each leg gives a conjugacy of these tensor product actions. It extends to the crossed products by identifying their specified implementing symmetries. □

At general rank $L \leq \min(M, N)$ there is still an exact computation. In the orbit coordinates $z_{ia} = \sum_{r \leq L} u_{ri} \otimes w_{ra}$, let u, w denote the 2 independent orthogonal matrices. Row orthogonality gives

$$\sum_a z_{ia} z_{ja} = \sum_{r \leq L} u_{ri} u_{rj} \otimes 1, \qquad \sum_i z_{ia} z_{ib} = 1 \otimes \sum_{r \leq L} w_{ra} w_{rb}.$$

Consequently the rectangular algebra contains $L^\infty(Gr_{LM}^+) \bar{\otimes} L^\infty(Gr_{LN}^+)$. Whenever one of these legs contains a diffuse abelian algebra and the other is nonamenable, the same computation obstructs solidity. In a square of corank one, both Grassmannians are projective row spaces. This gives a higher-rank case in which both indices can be computed.

Proposition 8.5. *Let $N \geq 3$, put $K = N-1$, and take the first K rows of each of 2 independent copies of O_N^+. Let A be the tensor product of the projective algebras of their omitted rows, let $B = L^\infty(O_{NN}^{K+})$ in the orbit trace, and let $C = L^\infty(St_{KN}^+) \bar{\otimes} L^\infty(St_{KN}^+)$ be the product of the frame algebras. Then $A \subset B \subset C$ are irreducible inclusions of II_1 factors, with*

$$[B : A] = [C : B] = \infty,$$

and B is nonsolid.

Proof. Column orthogonality completes the preceding row sums. In the first tensor leg it gives

$$\sum_a z_{ia} z_{ja} = (\delta_{ij} 1 - u_{Ni} u_{Nj}) \otimes 1.$$

The other sum gives $1 \otimes (\delta_{ab} 1 - w_{Na} w_{Nb})$. Hence $A \subset B \subset C$. By Theorem 3.2 each omitted projective row has scalar commutant in its full orthogonal algebra. Taking commutants on the tensor legs gives $A' \cap C = \mathbb{C}1$. This proves factoriality and irreducibility. A diffuse abelian algebra in the first projective leg commutes with the nonamenable second leg by Corollary 3.4, so B is nonsolid.

Rotate rows $1, 2$ through the same angle in both frames. These trace-preserving automorphisms of C fix every z_{ia}. The action is faithful, as can already be seen on the first column of the first frame by evaluating its polynomial entries at the identity matrix. An inner rotation would be implemented by a unitary in $B' \cap C = \mathbb{C}1$. Thus the cyclic subgroup of order m acts outerly, and its fixed algebra in C has index m by Jones' index formula [35]. It contains B, so $[C : B] \geq m$ for all m.

For the other index, rotate rows $1, 2$ in the first frame alone, fixing both omitted rows. On $L^2(C)$ write P_m for the orthogonal projection onto the circle character $e^{im\theta}$, and let e_B be the orthogonal projection onto $L^2(B)$. Since the rotations fix A pointwise, P_m commutes with the left and right actions of A, as does e_B. Thus $T_m = e_B P_m|_{L^2(B)}$ is an A-bimodule endomorphism.

For $x = z_{11}$ the rotated polynomial x^l has Fourier degree at most l. Its coefficient at $m = l$ is nonzero. Indeed, evaluating both polynomial matrices at the identity gives $\cos^l \theta$, with top coefficient 2^{-l} for $l \geq 1$. This evaluation takes place on the polynomial algebras, where the product trace is faithful. Consequently

$$\langle T_m x^l, x^l \rangle = \| P_m x^l \|_2^2$$

vanishes for $m > l$ and is positive for $m = l$. Testing successively at $l = 1, 2, \ldots$ proves linear independence of the T_m for $m \geq 1$. If $A \subset B$ had finite index, the algebra of A-bimodule endomorphisms of $L^2(B)$ would be finite dimensional. Indeed it is $A' \cap B_1$ in the Jones basic construction, where $A \subset B_1$ would have finite index [35]. This proves $[B : A] = \infty$. □

At size 3×3 the inclusions from the projective legs to the rectangle and from the rectangle to the frame product have indices $(2, 2)$ at rank one and (∞, ∞) at rank 2. This distinguishes the inclusions. At size 4×4 and rank 2, the complementary projections have rank 2 again. A scalar relative commutant for $L^\infty(Gr_{24}^+)$ in $L^\infty(O_4^+)$ would give the same factoriality and nonsolidity argument there.

9 Primeness and rectangular magnitudes

We return to rank one and assume $M, N \geq 3$. Write $A = L^\infty(P_+^{M-1})$ and $B = L^\infty(P_+^{N-1})$ for the projective legs. By Corollary 8.2, the rectangular algebra $C = L^\infty(O_{MN}^{1+})$ is $(A \bar{\otimes} B) \rtimes \mathbb{Z}_2$, where the generator acts by the outer involution on each leg simultaneously. We first compare C to the tensor product it contains.

Theorem 9.1. *For $M, N \geq 3$ the factor $L^\infty(O_{MN}^{1+})$ is prime. In particular it is distinct from $F \bar{\otimes} F$ and from its own projective subfactor.*

Proof. Inside C the projective legs satisfy

$$A' \cap C = B, \qquad B' \cap C = A.$$

Indeed, in the 2-term Fourier expansion an odd coefficient commuting across the first leg intertwines its identity representation with its outer involution. Slicing on the second leg gives zero, by the intertwiner argument in Corollary 8.2. The even coefficients commuting with A give exactly B. Interchanging the legs gives the other equality. Their join has index 2 in C.

We use the following consequence of the tensor product theorem of Ding and Peterson [23]. In a tensor product of 2 finite biexact factors, a subfactor with nonamenable relative commutant intertwines into one leg. Here intertwining means the existence of a nonzero bimodule of finite right dimension inside the ambient L^2 space. In particular a corner of the subfactor embeds in a matrix amplification of that leg. Biexactness passes to amplifications and to expected subalgebras by Propositions 6.5 and 6.10 of [23], respectively. For tensor products of free quantum group factors see also Isono [34]. If the relative commutant is a factor, the unitary conjugacy argument of Ozawa and Popa [44] places the whole subfactor inside one leg after rescaling the 2 legs by reciprocal amplifications.

The 2 ambient factors $L^\infty(O_M^+), L^\infty(O_N^+)$ are biexact. Indeed, exactness of their reduced algebras [55] gives weak exactness of the von Neumann algebras. Together with Isono's condition $(\mathrm{AO})^+$ [32], this gives biexactness by Theorem 7.17 of [23]. In particular the projective subfactors are nonamenable biexact factors and hence full [23]. Their tensor product is full, by Connes' spectral gap characterization of fullness for tracial factors [19]. Its outer crossed product by the finite group $\mathbb{Z}_2$ is again full [38]. If $C = U \bar{\otimes} V$ were a tensor product of diffuse factors, both would be nonamenable, since a diffuse amenable factor with separable predual is hyperfinite and supplies nontrivial central sequences.

Apply the intertwining criterion first in the ambient product $L^\infty(O_M^+) \bar{\otimes} L^\infty(O_N^+)$. The commutant of U contains V with trace-preserving expectation. Thus a corner of U embeds in a matrix amplification of one ambient leg, so U is biexact. The same holds for V. Now apply the criterion inside $C = U \bar{\otimes} V$ to A, whose relative commutant is the nonamenable factor B. After interchanging, amplifying and unitarily conjugating the proposed tensor legs, we may write $C = U \bar{\otimes} V$ with $A \subset U$. Put $J = A' \cap U$. We have

$$B = J \bar{\otimes} V, \qquad A = B' \cap C = J' \cap U.$$

The algebra J is a factor because B is a factor. If J were diffuse, its commutant in B would contain the nonamenable factor V, contradicting solidity of B. Hence J is a matrix algebra. Matrix units give $U = J \vee (J' \cap U) = J \vee A$. It follows that $A \vee B = C$, contradicting the index 2 computed above. □

To ask a dimensional question inside this prime factor we use the rectangular coordinates themselves. Let $r_{ia} = sgn(z_{ia})$ and define

$$E = W^*(v|z_{ia}|v^* \mid i \leq M,\ a \leq N,\ v \text{ a word in the } r_{jb}).$$

Write $\epsilon_N : \Gamma_N \to \mathbb{Z}_2$ for parity and $D(N)$ for the one-row magnitude algebra in dimension N. Consider the subgroup

$$G = \{(g, h) \in \Gamma_M \times \Gamma_N \mid \epsilon_M(g) = \epsilon_N(h)\}.$$

It contains $\Gamma_M^+ \times \Gamma_N^+$ with index 2.

Proposition 9.2. *The rectangular magnitude algebra and its crossed product satisfy*

$$E = D(M) \bar{\otimes} D(N), \qquad C = E \rtimes G,$$

where (g, h) acts by conjugation by $s_g \otimes t_h$, the sign words on the 2 legs.

Proof. The coordinates on either leg have zero kernel, so $r_{ia} = s_i \otimes t_a$ and $|z_{ia}| = |x_i| \otimes |y_a|$. The pairs (g_i, g_a) of sign generators generate G. Indeed a word of a given parity on either side can be padded with adjacent equal letters until the 2 word lengths agree. Thus $E \subset D(M) \bar{\otimes} D(N)$. For the reverse inclusion use

$$\sum_a |z_{ia}|^2 = x_i^2 \otimes 1, \qquad \sum_i |z_{ia}|^2 = 1 \otimes y_a^2.$$

Taking positive square roots puts the 2 sets of magnitudes in E. The projections of G onto both sign groups are surjective. Conjugation by the rectangular signs therefore gives all the generators of each magnitude leg. Finally, Proposition 5.2 on the 2 spheres gives the full crossed product by $\Gamma_M \times \Gamma_N$. Simultaneous sign change fixes exactly the Fourier degrees in G, giving the formula for C by Theorem 8.1. □

We can use the rational Euler characteristic to count the remaining group degrees. For a group with a finite-index subgroup H admitting a finite classifying space, $\chi(G) = \chi(H)/[G : H]$. Passing to a common finite cover gives independence of H and multiplicativity under finite index [13]. The groups here have finite-index subgroups which are products of 2 free groups.

Theorem 9.3. *For $M, N \geq 3$, let $E \subset C = L^\infty(O_{MN}^{1+})$ be the rectangular magnitude inclusion and put $k = \dim Z(E)$. We have*

$$\frac{\chi(\mathcal{N}_{pCp}(pEp)/\mathcal{U}(pEp))}{k} = \frac{(M-2)(N-2)}{2}$$

for every minimal central projection p of E. Thus the inclusion determines $(M-2)(N-2)$.

Proof. By Lemma 5.3, each magnitude algebra has finite-dimensional center and its commutant in the corresponding sphere algebra equals its center. The tensor product commutant formula therefore gives

$$E' \cap (L^\infty(S_{\mathbb{R},+}^{M-1}) \bar{\otimes} L^\infty(S_{\mathbb{R},+}^{N-1})) = Z(D(M)) \bar{\otimes} Z(D(N)) = Z(E).$$

Intersecting with C gives $E' \cap C = Z(E)$. Each even sign group acts transitively on its own minimal central projections. Their product, and therefore G, acts transitively on the k minimal central projections of E. Lemma 5.4 identifies the displayed quotient with the index-k stabilizer G_p.

The even-even subgroup is $F_{M-1} \times F_{N-1}$. Its classifying space is a product of 2 bouquets, so its Euler characteristic is $(2-M)(2-N)$. Since it has index 2 in G, we obtain

$$\chi(G_p) = k\chi(G) = \frac{k}{2}(2-M)(2-N).$$

Division by k proves the assertion for the inclusion alone. □

The Euler characteristic counts the product of the 2 shifted dimensions. The discrete normalizer group contains more information. For example, the rectangles of sizes $(3, 6)$ and $(4, 4)$ both give 2 in Theorem 9.3.

Theorem 9.4. *For $M, N \geq 3$ the isomorphism class of the inclusion $E \subset L^\infty(O_{MN}^{1+})$ determines both side lengths up to exchange.*

Proof. Read $k = \dim Z(E)$ from the inclusion. If $k > 1$, Proposition 9.2 gives a magnitude leg with nontrivial center. By Theorem 5.5 its side length must be 3. The product $(M-2)(N-2)$ from Theorem 9.3 then gives the other side length by adding 2.

If $k = 1$, the normalizer quotient at $p = 1$ is the whole equal-parity group G. Inside G consider the 2 subgroups

$$H = \Gamma_M^+ \times \{1\}, \qquad K = \{1\} \times \Gamma_N^+.$$

We can identify their unordered pair from the abstract group. In Γ_d the centralizer of each nonidentity element is virtually cyclic. Indeed the Cayley graph in the involutive generators is a tree with free action on vertices. An element of finite order other than the identity exchanges the endpoints of an edge and fixes its midpoint only. Its centralizer preserves this edge and has order at most 2. An element of infinite order translates along its axis. Its centralizer acts on that axis by translations, faithfully since a vertex stabilizer is trivial.

The elements of G with nonamenable centralizer are exactly $H \cup K$. On either axis the centralizer contains the opposite free group of rank at least 2. Off the 2 axes it is a subgroup of a product of 2 virtually cyclic groups and is amenable. The maximal subgroups contained in this union are exactly H and K, since the product of a nonidentity element on each axis lies outside the union. Their free ranks are $M-1$ and $N-1$. Adding one to each recovers the 2 side lengths, including the case of equal ranks. □

The tensor flip exchanges the 2 magnitude legs and the corresponding rectangular coordinates. Thus exchanging the side lengths always gives isomorphic inclusions. If we forget E, an isomorphism of the rectangular factors can carry the projective tensor product to a different subfactor. For a square there are 2 possibilities for its image. We first collect the facts about involutions and tensor products needed to describe both.

10 Amplifications of involutions

Write $X = (A, \alpha)$ for a II_1 factor equipped with an outer involution, and $X \simeq Y$ for conjugacy of the actions by a trace-preserving isomorphism of the factors. A tensor product of such pairs carries the simultaneous involution. The dual of X is the pair

$$DX = (A \rtimes \mathbb{Z}_2, \widehat{\alpha}),$$

where the generator acts on A by α, and $\widehat{\alpha}$ fixes A and changes the sign of the implementing symmetry. For the row involution of Theorem 4.2 the dual pair consists of the real sphere algebra with its antipodal involution.

To recognize an action on an index-2 subfactor, we are free to change the implementing symmetry by a cocycle. The following observation explains why ordinary conjugacy suffices here.

Lemma 10.1. *Let α be an outer involution on a II_1 factor A. If $Ad\, w \circ \alpha$ is an involution for a unitary $w \in A$, it is conjugate to α by an inner automorphism of A.*

Proof. We have $w\alpha(w) \in \mathbb{T}1$. Multiply w by a scalar of modulus one so that $w\alpha(w) = 1$. The involution $Ad\, diag(w, 1) \circ (id \otimes \alpha)$ on $M_2(A)$ is outer. Its fixed algebra is a factor and the 2 diagonal matrix projections have the same trace there. A fixed partial isometry between them gives a unitary $v \in A$ with $w\alpha(v) = v$. Thus $w = v\alpha(v)^*$ and $Ad\, w \circ \alpha = Ad\, v \circ \alpha \circ Ad\, v^*$. □

For $t > 0$ we form the amplification X^t by cutting $\alpha \otimes id$ on $A \bar{\otimes} B(l^2)$ by a projection of trace t in $A^\alpha \bar{\otimes} B(l^2)$, using the semifinite trace. Such projections of the same trace are equivalent in the stabilized fixed algebra. These amplifications multiply and leave the action outer, as matrix units in the stabilized fixed algebra recover the original action from its corner. We have

$$D(X^t) \simeq (DX)^t, \qquad D^2 X \simeq X^2.$$

For the first formula the cutting projection commutes with the implementing symmetry. Crossing the corner therefore gives the same corner of the crossed product with its dual action. The second formula is Takesaki duality for the group of order 2 [9, 52]. The bidual involution on $M_2(A)$ is α tensored with conjugation by the regular symmetry. That matrix conjugation is a cocycle perturbation and can be removed by Lemma 10.1. The exponent 2 denotes amplification of the normalized finite factor.

If $R = B_1 \bar{\otimes} B_2$ carries $\sigma = \beta_1 \otimes \beta_2$, choose each reciprocal amplification $R = B_1^t \bar{\otimes} B_2^{1/t}$ so that σ is the product of the amplified involutions. Indeed, choose projections of traces t and $1/t$ in the 2 stabilized fixed algebras. The product projection is equivalent to the original unit corner by a partial isometry in the stabilization of $B_1^{\beta_1} \bar{\otimes} B_2^{\beta_2}$. This identifies the product of the new fixed algebras with the original product of fixed algebras. Any 2 choices of the rescaled tensor legs are unitarily conjugate in R [44].

The next lemma places 2 commuting factors in separate tensor legs. It uses the same intertwining criterion as Theorem 9.1, in the setting of the tensor factorization results of [44, 34].

Lemma 10.2. *Let $R = B_1 \bar{\otimes} B_2$ for nonamenable biexact II_1 factors. Suppose $Q = U \bar{\otimes} V \subset R$ has $Q' \cap R = \mathbb{C}$, with U, V nonamenable factors. After an exchange of ambient legs if needed, there are $t > 0$ and a unitary $u \in R$ such that $uUu^* \subset B_1^t$ and $uVu^* \subset B_2^{1/t}$ are both irreducible.*

Proof. The relative commutant $U' \cap R$ contains V with expectation and is nonamenable. Its center commutes with U and V, so it is a factor. By the tensor product intertwining criterion of Ding and Peterson [23], U intertwines into one ambient leg. The unitary placement argument of Ozawa and Popa [44] places U in that leg after reciprocal amplification and conjugation. Call the resulting legs B, H. We have $U' \cap R = J \bar{\otimes} H$ with $J = U' \cap B$ a factor. If J were diffuse, solidity of B would make $J' \cap B$ amenable, although it contains U with expectation. Hence J is a matrix factor. Move it from $B = (J' \cap B) \bar{\otimes} J$ to the other leg, changing the reciprocal amplification. Now $U \subset J' \cap B$ and $V \subset J \bar{\otimes} H$. Irreducibility of Q makes both inclusions irreducible. □

When the commuting factors generate the whole product, we can also identify their involutions.

Proposition 10.3. *Let X_1, X_2, Y_1, Y_2 be outer involutions paired with their nonamenable biexact II_1 factors. If $X_1 \bar{\otimes} X_2 \simeq Y_1 \bar{\otimes} Y_2$, then, up to exchanging Y_1 and Y_2, we have*

$$X_1 \simeq Y_1^t, \qquad X_2 \simeq Y_2^{1/t}$$

for some $t > 0$.

Proof. Lemma 10.2 identifies the 2 source legs with the target legs amplified by $t, 1/t$, up to order and a unitary. Indeed the 2 images generate the target, so both inclusions are equalities. Choose the amplified target actions as above. The unitary conjugation changes the action on the product by an inner perturbation. If a product of 2 automorphisms is inner, each is inner. Indeed a nonzero normal slice of an implementing unitary gives an intertwiner in each factor and its polar part implements the automorphism. The corresponding involutions agree up to inner perturbations, which can be removed by Lemma 10.1. This proves $X_1 \simeq Y_1^t$ and $X_2 \simeq Y_2^{1/t}$ up to order. □

11 Cancellation for the unmarked rectangle

Fix $M \geq 3$. For the square of side M put $A = L^\infty(P_+^{M-1})$, $\alpha = \alpha_M$ and $s = sgn(u_{11})$. The dual of (A, α) consists of $T = L^\infty(S_{\mathbb{R},+}^{M-1})$ with the antipodal involution γ. We use both realizations of the square from Theorem 8.1 and Corollary 8.2. With $R = T \bar{\otimes} T$ and $\sigma = \gamma \otimes \gamma$ they read

$$P = A \bar{\otimes} A \ \subset \ C = L^\infty(O_{MM}^{1+}) = R^\sigma \ \subset \ R, \qquad C = P \rtimes \mathbb{Z}_2.$$

The generator acts on P by $\alpha\otimes\alpha$. Both inclusions have index 2. Outerness gives $C'\cap R=\mathbb{C}$, and $A'\cap T=\mathbb{C}$ in each leg gives $P'\cap R=\mathbb{C}$. The factors A and T are nonamenable by Corollary 3.4 and biexact as expected subalgebras of $L^\infty(O_M^+)$, as in the proof of Theorem 9.1.

Lemma 11.1. *Suppose $Q=U\bar{\otimes}V\subset C$ with U,V nonamenable factors, and $C=Q\rtimes\mathbb{Z}_2$ for an outer involution η implemented by a symmetry v. Put $X=(A,\alpha)$ and $Y=DX=(T,\gamma)$. Then (Q,η) is conjugate to $X\bar{\otimes}X$ if $Q'\cap R=\mathbb{C}$, and to $(Y\bar{\otimes}Y)^{1/2}$ otherwise.*

Proof. First suppose $Q'\cap R=\mathbb{C}$. Apply Lemma 10.2, choosing the reciprocal amplification compatibly with σ. For a unitary $u\in R$ we have $uQu^*=U_1\bar{\otimes}U_2$, with each U_i irreducible in its respective ambient leg B_i. Write $\sigma=\beta_1\otimes\beta_2$ in these legs. The involution $Ad\,u\circ\sigma\circ Ad\,u^*$ fixes uQu^* pointwise. Thus $w=u\sigma(u)^*$ intertwines $\beta_1\otimes\beta_2$ with the identity on $U_1\bar{\otimes}U_2$.

A nonzero slice of w on the opposite leg gives an intertwiner w_i from $\beta_i|_{U_i}$ to the identity on U_i. Both absolute squares are scalar by irreducibility. Normalize w_i to be unitary. Then $(w_1\otimes w_2)^*w$ commutes with $\sigma(uQu^*)$ and is scalar. Each $Ad\,w_i\circ\beta_i$ squares to the identity, since their product does, and it fixes U_i pointwise. Lemma 10.1 removes both inner perturbations and gives a unitary carrying Q into $B_1^{\beta_1}\bar{\otimes}B_2^{\beta_2}=P$. Both inclusions in R have index 4, so the containment is equality.

If $zQz^*=P$ in R, then $z^*\sigma(z)\in Q'\cap R$ since σ fixes both P and Q pointwise. Hence $\sigma(z)=\pm z$. In the odd case multiply z on the left by $s\otimes 1$, an odd unitary normalizing P. This conjugates Q to P inside C. In an outer crossed product by $\mathbb{Z}_2$ the normalizer quotient of its base factor by its own unitary group is $\mathbb{Z}_2$. Indeed the 2 Fourier coefficients of a normalizer intertwine the identity or the outer involution with the same automorphism, so at most one is nonzero. Consequently the transported generator induces the same outer class on P as $\alpha\otimes\alpha$. This gives the first alternative.

Now suppose $Q'\cap R\neq\mathbb{C}$. Its fixed algebra under σ is $Q'\cap C=\mathbb{C}$. A nonzero self-adjoint odd element has scalar square and can be normalized to a symmetry d. Multiplying by d sends every odd element to a scalar, so $Q'\cap R=\mathbb{C}1+\mathbb{C}d$. Conjugation by d preserves C and is outer there, since $C'\cap R=\mathbb{C}$. Its fixed algebra contains Q and has the same index 2. Hence $Q=C^{Ad\,d}$. Put $e=(1+d)/2$. We have $\sigma(e)=1-e$, and $q\to qe$ identifies Q with eRe. For $a\in eRe$ the element $a+\sigma(a)$ lies in Q and compresses to a.

The symmetry v normalizes $Q'\cap R$ and must exchange e and $1-e$. Otherwise it would commute with d and belong to Q. Thus $Ad\,v\circ\sigma$ preserves e and induces η under this identification. Move e by a unitary $b\in R$ to a σ-fixed projection p of the same trace $1/2$. The transported action is $Ad\,(bv\sigma(b)^*)\circ\sigma$ and its cocycle commutes with p, since both actions fix p. On pRp this is an inner perturbation of the amplified involution. Lemma 10.1 gives the second alternative. □

The second possibility occurs concretely. Take $d=s\otimes 1$ and write $w=s\otimes s$ for the implementing symmetry of $P\subset C$. The fixed algebra of $Ad\,d$ on C is generated by $A^\alpha\otimes 1$, $1\otimes A$ and w, as one sees on the 2 Fourier coefficients. The last 2 generate a copy of T commuting with the first, so this fixed algebra is $A^\alpha\bar{\otimes}T$. For 2 coordinate signs $s_1=s,s_2$ on the first sphere, s_1s_2 is Haar and α takes it to its adjoint. Thus $sgn(Im\,(s_1s_2))\otimes 1$ is an odd symmetry for $Ad\,d$ on C. It generates the other Fourier summand over the fixed algebra. The commutant of the fixed algebra in R contains d, while $P'\cap R=\mathbb{C}$.

All the subfactors Q in the second case are unitarily conjugate inside C. If e,f are the 2 projections obtained from 2 such subfactors as in the proof, choose $a\in R$ with $a^*a=e$ and $aa^*=f$. The partial isometry $\sigma(a)$ has initial and final projections $1-e,1-f$. Thus $a+\sigma(a)$ is a unitary of C carrying e to f and conjugating the 2 fixed algebras. The example and P cannot be conjugate inside C, since their commutants in R are different in dimension.

We now transport the projective tensor product of the rectangle into the square. The 2 possibilities give different amplifications, and both can be cancelled.

Theorem 11.2. *For $M, N \geq 3$ the factors $L^\infty(O_{MM}^{1+})$ and $L^\infty(O_{MN}^{1+})$ are isomorphic if and only if the full projective row involutions α_M and α_N are conjugate. Equivalently, the single-row inclusions*

$$L^\infty(P_+^{k-1}) \subset L^\infty(S_{\mathbb{R},+}^{k-1})$$

are isomorphic for $k = M$ and $k = N$.

Proof. Write $X_k = (L^\infty(P_+^{k-1}), \alpha_k)$ and put $X = X_M$, $Y = DX$. Transport the canonical projective tensor product and implementing symmetry of $L^\infty(O_{MN}^{1+})$ into $L^\infty(O_{MM}^{1+})$ by the proposed isomorphism. Lemma 11.1 gives

$$X \mathbin{\bar{\otimes}} X_N \simeq X \mathbin{\bar{\otimes}} X \qquad \text{or} \qquad X \mathbin{\bar{\otimes}} X_N \simeq (Y \mathbin{\bar{\otimes}} Y)^{1/2}.$$

In the first case Proposition 10.3 gives $X \simeq X^t$ and $X_N \simeq X^{1/t}$ for some $t > 0$. Amplifying the first conjugacy by $1/t$ gives $X_N \simeq X$.

In the second case choose the corner of trace $1/2$ on one leg. The same argument, allowing an exchange of legs, gives

$$X \simeq Y^t, \qquad X_N \simeq Y^{1/(2t)}.$$

Apply D to the first conjugacy. Since $Y = DX$ and $D^2X \simeq X^2$, we get $Y \simeq X^{2t}$. Amplifying by $1/(2t)$ gives $Y^{1/(2t)} \simeq X$. Thus $X_N \simeq X$ also in this case.

Conversely a conjugacy $X \simeq X_N$ tensored with the identity intertwines the diagonal actions and extends to their crossed products. The normalizer computation in Lemma 11.1 also shows that an isomorphism of the single-row inclusions identifies their actions up to an inner perturbation, hence up to conjugacy. □

Theorem 11.2 turns the first comparison of different side lengths into a question about one row.

Question 11.3. *Is there an isomorphism $\theta : L^\infty(P_+^2) \to L^\infty(P_+^3)$ such that $\theta\alpha_3 = \alpha_4\theta$? Equivalently, are $L^\infty(O_{33}^{1+})$ and $L^\infty(O_{34}^{1+})$ isomorphic?*

Theorem 5.1 gives a conjugacy on the even sign algebras. Theorem 5.5 shows that any conjugacy on the full factors would have to carry the dimension-3 magnitude algebra outside the unitary conjugacy class of its dimension-4 counterpart. The same obstruction holds for an arbitrary isomorphism of the rectangles by Theorem 9.4. Whether the full row involutions are conjugate remains open.

12 Unitary rank-one rectangles

For $M, N \geq 2$ take one column in each of U_M^+, U_N^+. The rank-one orbit map sends z_{ia} to the product of the first column entry on the left and the adjoint of the second column entry on the right, in separate tensor legs [1]. Entrywise conjugation of the fundamental matrix of U_N^+ is a trace-preserving quantum group automorphism. We may therefore name the adjointed column on the second side y_a and write $z_{ia} = x_i \otimes y_a$, with both columns having the usual complex sphere law. In these coordinates the scalar action fixing the rectangle is $(x_i, y_a) \to (\lambda x_i, \bar{\lambda} y_a)$.

Write S and T for the 2 complex sphere algebras. Theorem 4.2 gives $S = W^*(w) * A$ and $T = W^*(v) * B$, where w, v are the polar unitaries of x_1, y_1, and

$$A = W^*(x_i^* x_j \mid i, j) \simeq L^\infty(P_+^{M-1}), \qquad B = W^*(y_a^* y_b \mid a, b) \simeq L^\infty(P_+^{N-1}).$$

The same row law holds for a column by transposition. The scalar action on a single column multiplies its polar unitary by λ and fixes its projective algebra. Define

$$K = W^*(w^k A w^{-k} \mid k \in \mathbb{Z}), \qquad H = W^*(v^k B v^{-k} \mid k \in \mathbb{Z}).$$

The translates of A are freely independent: between centered letters from different copies a nonzero power of w remains in the free product expansion. Reduced words also show $\tau(aw^k) = 0$ for $a \in K$, $k \neq 0$, or one can average the scalar action. Thus K is the tracial free product of the indicated copies, $S = K \rtimes \mathbb{Z}$ for the shift $Ad\, w$, and K is the scalar fixed algebra. The same holds on the other side. In particular H and K are nonamenable factors: these are infinite free products of diffuse algebras. For factoriality one can take a Haar unitary in each of 2 free components. The commutant of the first unitary lies in its component by the reduced-word decomposition over that component, and intersecting with the commutant of the second gives scalars. The 2 unitaries generate F with expectation.

Theorem 12.1. *The orbit algebra of the unitary rank-one rectangle satisfies*

$$C = L^\infty(U_{MN}^{1+}) = (S \bar{\otimes} T)^{\mathbb{T}} = (K \bar{\otimes} H) \rtimes \mathbb{Z}.$$

The circle acts with opposite phases in the 2 *chosen coordinates. The generator of* $\mathbb{Z}$ *shifts both free products by one place. Both inclusions* $K \bar{\otimes} H \subset C \subset S \bar{\otimes} T$ *are irreducible inclusions of factors of infinite index.*

Proof. The polar unitary $d = w \otimes v$ of z_{11} belongs to C. The products $z_{ia}^* z_{jb} = x_i^* x_j \otimes y_a^* y_b$ generate $A \bar{\otimes} B$. Indeed summing over $a = b$ gives $x_i^* x_j \otimes 1$, and summing over $i = j$ gives $1 \otimes y_a^* y_b$. Conversely $d^* z_{ia} = w^* x_i \otimes v^* y_a \in A \bar{\otimes} B$ by the polar construction above. We obtain

$$C = W^*(A \bar{\otimes} B, d) = W^*(K \bar{\otimes} H, d).$$

For the last equality conjugate the 2 separate projective legs by every power of d. Inside $S \bar{\otimes} T = (K \bar{\otimes} H) \rtimes \mathbb{Z}^2$ we have retained exactly the diagonal degrees (k, k). These are exactly the degrees fixed by the specified circle action.

The shift on K is mixing: 2 finite words with centered parts have zero correlation once their sets of free indices are disjoint. Density gives mixing on $L^2(K) \ominus \mathbb{C}1$. Hence no nonzero power is inner. An implementing unitary for such a power would be fixed by that power, therefore scalar by mixing, while the power moves free components. The same holds for H. A nonzero normal slice of an implementing unitary for a tensor product of automorphisms would implement each automorphism up to scalar after polar decomposition. It follows that the action of $\mathbb{Z}^2$ on $K \bar{\otimes} H$ is outer. Fourier coefficients now give $(K \bar{\otimes} H)' \cap (S \bar{\otimes} T) = \mathbb{C}1$. The first index is infinite by the mutually orthogonal Fourier modules for the diagonal group, and the second by the infinitely many cosets of that group in $\mathbb{Z}^2$. □

Already the first square has an explicit group model. At $M = N = 2$ the projective algebra in each leg is diffuse abelian. Choose a Haar generator b there. Each sphere is $L(F_2)$ on the 2 free generators w, b. If $\epsilon : F_2 \to \mathbb{Z}$ takes w to 1 and b to 0, the diagonal degree computation gives

$$L^\infty(U_{22}^{1+}) \simeq L(\{(g, h) \in F_2 \times F_2 \mid \epsilon(g) = \epsilon(h)\}).$$

For larger sizes the formula in Theorem 12.1 depends only on the tracial isomorphism types of the 2 projective algebras. An isomorphism on either base extends copy by copy and commutes with its shift. In particular Conjecture 4.3 implies a common isomorphism type for all unitary rectangles with $M, N \geq 3$.

There are 2 different subalgebras associated with degree zero. The Fourier coefficient algebra $K \bar{\otimes} H$ is regular in C since the shift normalizes it. The smaller algebra $P = A \bar{\otimes} B$ comes directly from the projective coordinates $z_{ia}^* z_{jb}$. Its normalizer has the following description.

Proposition 12.2. *For* $M, N \geq 2$ *the projective algebra* $P = W^*(z_{ia}^* z_{jb} \mid i, j, a, b)$ *is singular in the complex rectangle. In the ambient product of complex spheres it satisfies*

$$\mathcal{N}_{S \bar{\otimes} T}(P) = \mathcal{U}(P).$$

At $(M, N) = (2, 2)$ *it is a singular maximal abelian subalgebra of the rectangular factor.*

Proof. Choose Haar unitaries $a \in A$ and $b \in B$ and put $h_j = a^j \otimes b^j$. For x, y orthogonal to P in $S \mathbin{\bar{\otimes}} T$, we claim that

$$\|E_P(xh_jy)\|_2 \longrightarrow 0.$$

In $S = A * W^*(w)$, first take reduced words x_1, y_1 each containing a centered letter from $W^*(w)$. The expectation $E_A(x_1a^jy_1)$ can survive only from the scalar part of the intervening A-letter at the junction. That scalar has the form $\tau(ca^jc')$ for fixed $c, c' \in A$ and tends to zero because a^j tends weakly to zero. If just one word is outside A and the other belongs to A, the expectation is zero by bimodularity. The same computations apply in the second leg. Taking tensor products shows the claim on finite sums of elementary reduced words, and bounded L^2 approximation gives the general case.

For a unitary u normalizing P, write $q = E_P(u)$. The claim with $x = u - q$ and $y = x^*$ shows $\|uh_ju^* - qh_jq^*\|_2 \to 0$. The first term is unitary, whereas $\|qh_jq^*\|_2 \le \|q\|_2$ because q is a contraction. Hence $\|q\|_2 = 1$ and $u = q \in P$. At size $(2, 2)$ both projective legs are abelian. All unitaries commuting with P normalize it, giving maximal abelianness as well. □

The real rectangle has a tensor product of 2 nonamenable factors at index 2. To test for such a subfactor anywhere in the unitary rectangle, we use its inclusion in the ambient $\mathbb{Z}^2$ crossed product. A translate of a coordinate axis meets the diagonal in just one point.

Theorem 12.3. *For $M, N \ge 2$, every unital subfactor $Q = U \mathbin{\bar{\otimes}} V$ of $C = L^\infty(U_{MN}^{1+})$ with U, V nonamenable has infinite Jones index. The factor C is full, prime and nonsolid. In particular it is nonisomorphic to every real rectangular factor $L^\infty(O_{KL}^{1+})$ with $K, L \ge 3$.*

Proof. Put $D = K \mathbin{\bar{\otimes}} H$ and $R = S \mathbin{\bar{\otimes}} T = D \rtimes \mathbb{Z}^2$. The sphere factors S, T are biexact. For a side of length at least 3 this follows by expectation from the free unitary group factor [23]; for a side of length 2 use $L(F_2)$. We first show that on the unit ball of U the diagonal Fourier degrees can be cut down to a common finite set with arbitrarily small L^2 error.

The commutant $U' \cap R$ has no amenable direct summand, since V embeds with expectation in every nonzero central summand. Thus for every nonzero projection $p \in U' \cap R$, the algebra $Up \subset pRp$ has nonamenable relative commutant there. By the tensor product intertwining criterion of Ding and Peterson [23], Up intertwines in R into one of the 2 sphere legs. To see the corner formulation, unitarily move p to $p_1 \otimes 1$ and use the criterion in $(p_1Sp_1) \mathbin{\bar{\otimes}} T$. Its first leg is a corner of S and its second is a compression of T by a commuting projection, so the resulting intertwiner also intertwines into the corresponding original leg in R. Each leg is contained in $D \rtimes \Sigma$ for its coordinate axis $\Sigma \subset \mathbb{Z}^2$.

Apply the equivalence (a)–(b) of Vaes' Lemma 2.5 in [54] to the 2 coordinate axes. The condition on every p just verified says that for each $\varepsilon > 0$ a finite union of translates of the axes captures the Fourier support of every element of the unit ball of U up to L^2 error ε. All these elements are in C, so their degrees lie on the diagonal. The intersection with the chosen union is finite. Interchanging U, V gives the same conclusion for the unit ball of V.

For clarity, use the Fejér multipliers for the diagonal $\mathbb{Z}$ to make these finite Fourier approximations contractive in operator norm. Their weights $(1 - |k|/l)_+$ converge to one on each fixed finite set, so the L^2 approximations are uniform on both balls. The product of 2 such contractions approximates uv uniformly for $u \in \mathcal{U}(U)$ and $v \in \mathcal{U}(V)$, with error at most the sum of the 2 errors. Its support lies in the sum of the 2 finite intervals. Thus some finite $J \subset \mathbb{Z}$ satisfies

$$\sum_{k \in J} \|E_D(uvd^{-k})\|_2^2 \ge \frac{1}{2} \qquad (u \in \mathcal{U}(U),\ v \in \mathcal{U}(V)).$$

The products uv form a group generating Q. Popa's intertwining criterion [46], in its generating-group form in [54], gives $Q \prec_C D$.

If $[C : Q]$ were finite, the resulting nonzero Q–D submodule of $L^2(C)$ of finite right D-dimension would generate a C–D submodule of finite right dimension. Indeed left multiplication

by a finite right Q-basis of C gives a finite sum of images of the original module, whose closure has finite right dimension. This would give $C \prec_C D$. The unitaries d^k exclude that intertwining: $\|E_D(xd^k y)\|_2 \to 0$ for $x, y \in C$, first by disjoint degrees for finite Fourier sums and then by bounded approximation. This proves the infinite-index assertion.

Finally R is a tensor product of 2 full factors and hence full. The unitary $w \otimes 1$ normalizes C and supplies a complement to the diagonal subgroup of $\mathbb{Z}^2$. Consequently $R = C \rtimes \mathbb{Z}$, for conjugation by $w \otimes 1$. The amenable-group direction of Marrakchi's fullness theorem [38] gives fullness of C. A diffuse amenable tensor factor would give nontrivial central sequences, and 2 nonamenable tensor factors would contradict the index assertion applied at index one. Thus C is prime. It is nonsolid because the commutant of a diffuse abelian subalgebra of $K \otimes 1$ contains the nonamenable factor $1 \otimes H$. The real rectangles in the statement have a product of nonamenable projective factors at index 2 by Corollary 8.2. □

The unitary rectangle is also strongly 1-bounded. Indeed $K \bar{\otimes} H$ is a tensor product of diffuse factors, so its 1-bounded entropy is at most zero [29]. Its normalizer generates C, and the normalizer inequality of Hayes [28] gives $h(C) \leq h(K \bar{\otimes} H) \leq 0$. Thus the entropy obstruction is present for both types of rectangles. The finite-index distinction comes from the diagonal Fourier degrees.

References

[1] T. Banica, Affine noncommutative geometry, arXiv:2012.10973 (2026).

[2] T. Banica, Le groupe quantique compact libre U(n), *Comm. Math. Phys.* **190** (1997), 143–172.

[3] T. Banica and B. Collins, Integration over compact quantum groups, *Publ. Res. Inst. Math. Sci.* **43** (2007), 277–302.

[4] T. Banica, B. Collins and P. Zinn-Justin, Spectral analysis of the free orthogonal matrix, *Int. Math. Res. Not. IMRN* **2009** (2009), 3286–3309.

[5] T. Banica and D. Goswami, Quantum isometries and noncommutative spheres, *Comm. Math. Phys.* **298** (2010), 343–356.

[6] T. Banica, A. Skalski and P.M. Soltan, Noncommutative homogeneous spaces: the matrix case, *J. Geom. Phys.* **62** (2012), 1451–1466.

[7] T. Banica and R. Vergnioux, Fusion rules for quantum reflection groups, *J. Noncommut. Geom.* **3** (2009), 327–359.

[8] J. Bichon, Free wreath product by the quantum permutation group, *Algebr. Represent. Theory* **7** (2004), 343–362.

[9] B. Blackadar, Operator algebras: Theory of C^*-algebras and von Neumann algebras, Encyclopaedia of Mathematical Sciences **122**, Springer (2006).

[10] M. Brannan, Reduced operator algebras of trace-preserving quantum automorphism groups, *Doc. Math.* **18** (2013), 1349–1402.

[11] M. Brannan, F. Elzinga, S.J. Harris and M. Yamashita, Crossed product equivalence of quantum automorphism groups of finite dimensional C^*-algebras, *Int. Math. Res. Not. IMRN* **2023** (2023), 17749–17787.

[12] M. Brannan and R. Vergnioux, Orthogonal free quantum group factors are strongly 1-bounded, *Adv. Math.* **329** (2018), 133–156.

[13] K.S. Brown, Cohomology of groups, Graduate Texts in Mathematics **87**, Springer (1982).

[14] M. Caspers, Gradient forms and strong solidity of free quantum groups, *Math. Ann.* **379** (2021), 271–324.

[15] M. Caspers, Riesz transforms on compact quantum groups and strong solidity, *J. Inst. Math. Jussieu* **21** (2022), 2135–2171.

[16] A. Chirvasitu, Topological generation results for free unitary and orthogonal groups, *Internat. J. Math.* **31** (2020), 2050003.

[17] Claude O and Teo B, Italian projective geometry goes free, zenodo:24448353 (2026).

[18] Codex and Teo, The magic of free spheres, zenodo:24434373 (2026).

[19] A. Connes, Classification of injective factors. Cases II_1, II_∞, III_λ, $\lambda \neq 1$, *Ann. of Math.* **104** (1976), 73–115.

[20] A. Connes, Noncommutative geometry, Academic Press (1994).

[21] K. De Commer, A note on the von Neumann algebra underlying some universal compact quantum groups, *Banach J. Math. Anal.* **3** (2009), 103–108.

[22] K. De Commer, A. Freslon and M. Yamashita, CCAP for universal discrete quantum groups, *Comm. Math. Phys.* **331** (2014), 677–701.

[23] C. Ding and J. Peterson, Biexact von Neumann algebras, arXiv:2309.10161 (2026).

[24] K. Dykema, Free products of hyperfinite von Neumann algebras and free dimension, *Duke Math. J.* **69** (1993), 97–119.

[25] K. Dykema, Interpolated free group factors, *Pacific J. Math.* **163** (1994), 123–135.

[26] K. Dykema and D. Redelmeier, The amalgamated free product of hyperfinite von Neumann algebras over finite dimensional subalgebras, *Houston J. Math.* **39** (2013), 1313–1331.

[27] P. Fima and A. Troupel, Operator algebras of free wreath products, *Adv. Math.* **441** (2024), 109546.

[28] B. Hayes, 1-bounded entropy and regularity problems in von Neumann algebras, *Int. Math. Res. Not. IMRN* **2018** (2018), 57–137.

[29] B. Hayes, D. Jekel and S. Kunnawalkam Elayavalli, Consequences of the random matrix solution to the Peterson–Thom conjecture, *Anal. PDE* **18** (2025), 1805–1834.

[30] B. Hayes, D. Jekel and S. Kunnawalkam Elayavalli, Property (T) and strong 1-boundedness for von Neumann algebras, *J. Inst. Math. Jussieu* **25** (2026), 2999–3032.

[31] F. Hiai and Y. Ueda, Automorphisms of free product-type and their crossed-products, *J. Operator Theory* **50** (2003), 119–130.

[32] Y. Isono, Examples of factors which have no Cartan subalgebras, *Trans. Amer. Math. Soc.* **367** (2015), 7917–7937.

[33] Y. Isono, On bi-exactness of discrete quantum groups, *Int. Math. Res. Not. IMRN* **2015** (2015), 3619–3650.

[34] Y. Isono, Some prime factorization results for free quantum group factors, *J. Reine Angew. Math.* **722** (2017), 215–250.

[35] V.F.R. Jones, Index for subfactors, *Invent. Math.* **72** (1983), 1–25.

[36] K. Jung, Strongly 1-bounded von Neumann algebras, *Geom. Funct. Anal.* **17** (2007), 1180–1200.

[37] K. Jung, The rank theorem and L^2-invariants in free entropy: global upper bounds, arXiv:1602.04726 (2016).

[38] A. Marrakchi, Fullness of crossed products of factors by discrete groups, *Proc. Roy. Soc. Edinburgh Sect. A* **150** (2020), 2368–2378.

[39] F.J. Murray and J. von Neumann, On rings of operators, *Ann. of Math.* **37** (1936), 116–229.

[40] F.J. Murray and J. von Neumann, On rings of operators, II, *Trans. Amer. Math. Soc.* **41** (1937), 208–248.

[41] F.J. Murray and J. von Neumann, On rings of operators, IV, *Ann. of Math.* **44** (1943), 716–808.

[42] OpenAI, An isomorphism of the free group factors, github (2026).

[43] N. Ozawa, Solid von Neumann algebras, *Acta Math.* **192** (2004), 111–117.

[44] N. Ozawa and S. Popa, Some prime factorization results for type II_1 factors, *Invent. Math.* **156** (2004), 223–234.

[45] N. Ozawa and S. Popa, On a class of II_1 factors with at most one Cartan subalgebra, *Ann. of Math.* **172** (2010), 713–749.

[46] S. Popa, Strong rigidity of II_1 factors arising from malleable actions of w-rigid groups, I, *Invent. Math.* **165** (2006), 369–408.

[47] F. Rădulescu, Random matrices, amalgamated free products and subfactors of the von Neumann algebra of a free group, of noninteger index, *Invent. Math.* **115** (1994), 347–389.

[48] S. Raum, Isomorphisms and fusion rules of orthogonal free quantum groups and their free complexifications, *Proc. Amer. Math. Soc.* **140** (2012), 3207–3218.

[49] D. Shlyakhtenko, Von Neumann algebras of sofic groups with $\beta_1^{(2)} = 0$ are strongly 1-bounded, *J. Operator Theory* **85** (2021), 217–228.

[50] D. Shlyakhtenko, On the II_1 factors of Fuchsian groups, arXiv:2609.11074 (2026).

[51] D. Shlyakhtenko, On the isomorphism of semicircular algebras, arXiv:2610.19590 (2026).

[52] M. Takesaki, Duality for crossed products and the structure of von Neumann algebras of type III, *Acta Math.* **131** (1973), 249–310.

[53] Teo Banica and Claude Opus, Quantum permutations and the Standard Model algebra, zenodo:24403549 (2026).

[54] S. Vaes, One-cohomology and the uniqueness of the group measure space decomposition of a II_1 factor, *Math. Ann.* **355** (2013), 661–696.

[55] S. Vaes and R. Vergnioux, The boundary of universal discrete quantum groups, exactness, and factoriality, *Duke Math. J.* **140** (2007), 35–84.

[56] R. Vergnioux, Orientation of quantum Cayley trees and applications, *J. Reine Angew. Math.* **580** (2005), 101–138.

[57] D. Voiculescu, The analogues of entropy and of Fisher's information measure in free probability theory, I, *Comm. Math. Phys.* **155** (1993), 71–92.

[58] D. Voiculescu, The analogues of entropy and of Fisher's information measure in free probability theory, II, *Invent. Math.* **118** (1994), 411–440.

[59] D.V. Voiculescu, K.J. Dykema and A. Nica, Free random variables, CRM Monograph Series **1**, Amer. Math. Soc. (1992).

[60] J. von Neumann, On rings of operators, III, *Ann. of Math.* **41** (1940), 94–161.

[61] S. Wang, Quantum symmetry groups of finite spaces, *Comm. Math. Phys.* **195** (1998), 195–211.

[62] S.L. Woronowicz, Tannaka–Krein duality for compact matrix pseudogroups. Twisted $SU(N)$ groups, *Invent. Math.* **93** (1988), 35–76.

[63] S.L. Woronowicz, Compact quantum groups, in Symétries quantiques (Les Houches, 1995), North-Holland (1998), 845–884.