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EXPLICIT LINEAR BOUNDS IN WARING'S PROBLEM

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Abstract. Let $G(k)$ denote the least number s such that every sufficiently large natural number is the sum of at most s positive integral k-th powers. Chow and Dietmann have recently shown that $G(k) \ll k$. We make their method numerically efficient, and prove that $G(k) \leqslant C_0\, k + O(\sqrt{k})$, where $C_0 = 6.3895\ldots$ is given by an explicit series; more precisely, $G(k) < C_0 k + 10\sqrt{k}$ for all $k \geqslant 37$ and $G(k) < C_0 k + 3\sqrt{k}$ for all $k \geqslant 1339$. In particular $G(k) < 7k$ for $k \geqslant 212$, $G(k) < 6.5\, k$ for $k \geqslant 927$, and $G(k) < 7.81\, k$ for every k. For the least number $G^+(k)$ of k-th powers that suffice for almost all natural numbers we obtain $G^+(k) \leqslant \frac{1}{2}C_0\, k + O(\sqrt{k})$ when k is not a power of 2, and $G^+(k) = 4k$ when $k \geqslant 8$ is a power of 2; the latter is new for $k \geqslant 64$. The k-th powers are those of integers $mp_1 \cdots p_J \leqslant P$, with primes $p_1 < \cdots < p_J$ from one interval and m unrestricted, which form a set of positive density, and the exponential sum F over these k-th powers is estimated on all arcs without the help of auxiliary smooth Weyl sums. We also show that $\int_0^1 |F(\alpha)|^s \,\mathrm{d}\alpha \ll P^{s-k}$ for all real $s \geqslant C_0 k + 10\sqrt{k}$ when $k \geqslant 37$, and we prove a large sieve estimate for F which gives the localised bound $\min_{1\leqslant n\leqslant N} \|\alpha n^k\| < N^{-1/(11k)}$, for every real α, when $k \geqslant 1339$ and N is large. In an appendix, a mean value argument with two coefficients, in the spirit of Heath-Brown's k-th derivative test, a finer treatment of rounding and a recursive form of the main induction replace C_0 by $C_0^\star = 6.2821\ldots$ and make the term of order $\sqrt{k}$ negative; for example, $G(k) < C_0^\star k$ for $k \geqslant 56$ and $G(k) < 6.43\, k$ for every k. For $k < 5000$ the proofs involve numerical computations.

1. Introduction

1.0. **Motivation.** The motivation for preparing this document is threefold. First, the remarkable progress on $G(k)$ obtained by Chow and Dietmann [10] through the use of OpenAI's GPT-6.2 Astra is somewhat difficult to apply, with summands apparently constructed in a tricky manner. The rather artificial nature of the summands precludes immediate application of the extensive toolbox of modern circle method practitioners. We address this issue by adapting the argument to make use of essentially conventional sets of smooth numbers in the form previously employed by Brüdern and Wooley in work on sums of eight cubes of smooth numbers [5, §5]. This permits the application of the modern toolbox of major and minor arcs estimates [7, 9], saving a number of variables in the final bounds which appreciably reduces the accessible bounds for $G(k)$. Second, beyond the significant improvements to be made by putting the ideas into the modern conventional framework, further sizeable improvements are made possible by more carefully employing the full force of modern versions of Vinogradov's mean value theorem [4, 44]. These ideas will be found in §4 of this manuscript. The significant improvements also include much more powerful control of intermediate arcs available via ideas of

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Brüdern and Wooley [6, §§4 and 5]. Third, and perhaps most importantly, progress on AI has radically altered the mathematical environment when it comes to the digestion and publication of significant work. In times past (indeed, the very recent past), one could reflect at length on new developments, confident that further progress would require the digestion of new ideas, so that a significant stride forward would be encompassed by the next publication of substance. It is now the case that another paper can be produced in hours or days that advances the subject. Obviously, these AI-assisted papers are not really worthy of formal publication. So we have worked with Claude to rapidly convert the initial work into a form that is readily employed, but also more easily accessible to improvement. My hope is that a community endeavour involving somewhere between 10 and 100 (or even more) individuals refines the new ideas into a form ultimately worthy of serious formal publication, ideally with a proof of the bound $G(k) \leqslant 2k+1$ when congruence constraints are absent. The more ambitious goal of showing under the same conditions that $G(k) = k+1$, meanwhile, would seem to require an entirely new array of ideas. The minor steps along the way, including this manuscript, will likely not be deemed worthy of formal publication. For this last reason, although I have guided Claude very carefully through the writing process, as well as through the research enhancements noted, and have edited the text further, this manuscript is not written to full "Wooley" standards. Nonetheless, it is to be hoped that it proves useful in the community endeavour described.

1.1. **Results.** Let k be a natural number, and let $G(k)$ denote the least integer s with the property that every sufficiently large natural number is the sum of at most s k-th powers of natural numbers. Ever since Hardy and Littlewood introduced their circle method a century ago [16], progress with the method has been reflected in the bounds for $G(k)$. We recall the main steps, following the account in [6, §1]. The first general bound,

$$G(k) \leqslant (k-2)2^{k-1} + 5,$$

was obtained by Hardy and Littlewood [17]. It was improved rapidly during the next four decades, a development that culminated in 1959 with Vinogradov's bound

$$G(k) \leqslant k\big(2\log k + 4\log\log k + 2\log\log\log k + 13\big) \qquad (k \geqslant 170\,000)$$

(see [36]). This was sharpened by Karatsuba [24] and, shortly afterwards, by Vaughan [31], whose iterative method introduced smooth numbers into the subject and gave

$$G(k) \leqslant 2k\Big(\log k + \log\log k + 1 + \log 2 + O\Big(\frac{\log\log k}{\log k}\Big)\Big).$$

A little more than thirty years after the work of Vinogradov, Wooley obtained a bound of roughly half this size, namely

$$G(k) \leqslant k\Big(\log k + \log\log k + 2 + O\Big(\frac{\log\log k}{\log k}\Big)\Big)$$

(see [38, 39] and [41, Theorem 1.4]). The secondary term of size $k\log\log k$ was removed by Brüdern and Wooley [6], who showed that

$$G(k) \leqslant \lceil k(\log k + 4.20032)\rceil \qquad (k \in \mathbb{N}), \tag{1.1}$$

and, in a slightly sharper form, that $G(k) < k(\log k + 4.200189\ldots) + 3.015478\ldots$ for all k, the constants being given explicitly there. For exponents of moderate size these bounds can be sharpened further; for example, $G(14) \leqslant 89$ and $G(20) \leqslant 137$ by [6, Theorem 1.3], and for smaller k we refer to [34, 42] and to the survey [35], which contains a detailed account of the history of the problem. In the other direction one has $G(k) \geqslant k+1$ for $k \geqslant 2$, and $G(k) \geqslant 4k$ when $k \geqslant 4$ is a power of 2 [19, §21.6].

All the upper bounds just described are of order $k \log k$ at least. In a remarkable recent paper, Chow and Dietmann [10] determined the order of magnitude of $G(k)$ by proving that

$$G(k) \ll k.$$

Their argument is short, and no attempt is made in [10] to control the implied constant: with the parameters chosen there it is 12 000 005 (see §12.3). It is stated in [10], without proof, that more elaborate reasoning gives $G(k) \leqslant 8.4\,k$ for all k.

The purpose of this paper is to show that the method of [10] can be made numerically efficient. Put

$$\begin{aligned} \Lambda &:= \tfrac{1}{3}\log 2 + \tfrac{1}{4}\log\tfrac{25}{9} + \tfrac{1}{5}\log 2 + \tfrac{1}{5}\log\tfrac{32}{21} + \tfrac{1}{6}\log\tfrac{9}{8} + \tfrac{1}{6}\log\tfrac{147}{103} \\ &\quad + \sum_{D\geqslant 6} \frac{1}{D+1}\log\Big(1+\frac{2}{D}\Big) \\ &= 1.096145\ldots \end{aligned} \tag{1.2}$$

and

$$C_0 := 2 + \log 9 + 2\Lambda = 6.389515\ldots. \tag{1.3}$$

Theorem 1.1. *For every $k \geqslant 37$ one has $G(k) < C_0\,k + 10\sqrt{k}$; for every $k \geqslant 697$ one has $G(k) < C_0\,k + 4\sqrt{k}$; and for every $k \geqslant 1339$ one has $G(k) < C_0\,k + 3\sqrt{k}$. Moreover*

$$G(k) \leqslant C_0\,k + 1.572\,\sqrt{k} + 5\log k + 55 \qquad (k \geqslant 5000).$$

In particular $\limsup_{k\to\infty} G(k)/k \leqslant C_0$.

Theorem 1.2. *One has $G(k) < 7.81\,k$ for every $k \in \mathbb{N}$. Moreover $G(k) < 7.38\,k$ for $k \geqslant 37$, $G(k) < 7k$ for $k \geqslant 212$, and $G(k) < 6.5\,k$ for $k \geqslant 927$.*

Both theorems are consequences of an inequality for $G(k)$ that is valid for all $k \geqslant 37$ and involves a free parameter (Theorem 2.3). For $k \leqslant 36$ the first assertion of Theorem 1.2 is contained in (1.1); for $k \geqslant 37$ our bounds are smaller than (1.1), by 17 when $k = 37$ and by more than k when $k \geqslant 49$. For example, we obtain $G(50) \leqslant 353$, $G(100) \leqslant 709$ and $G(1000) \leqslant 6492$, where (1.1) gives 406, 881 and 11109 (see Table 3). Since $G(k) \geqslant 4k$ when $k \geqslant 4$ is a power of 2, no bound of the shape $G(k) \leqslant Ck$ can hold for all k with $C < 4$.

The method applies equally to the representation of almost all natural numbers. Let $G^+(k)$ denote the least number s with the property that almost all natural numbers, in the sense of natural density, are sums of at most s k-th powers of natural numbers. By Bessel's inequality, a bound for the mean value of order $2s$ of the generating function over the minor arcs shows that almost all natural numbers are sums of s k-th powers, as far as the singular series permits. In this way an upper bound t for $G(k)$ that is obtained by the circle method is accompanied by the upper bound $\frac{1}{2}(t+1)$ for $G^+(k)$, unless the

congruence conditions intervene. For the bound (1.1) this is carried out by Brüdern and Wooley in [6, §9], where the bound $G^+(k) \leqslant \lceil \frac{1}{2}k(\log k + 4.20032)\rceil$ is recorded for all k other than 4, 8, 16 and 32. The congruence conditions do intervene when k is a power of 2: if $k \geqslant 8$ is a power of 2, then no integer $n \equiv 4k \pmod{8k}$ is the sum of fewer than $4k$ k-th powers (Lemma 9.13), so that $G^+(k) \geqslant 4k$. The equation $G^+(k) = 4k$ was proved by Vaughan [30] for $k = 8$, and by Wooley [39] for $k = 16$ and $k = 32$. For $k = 4$ one has $G^+(4) = 15$, by a theorem of Hardy and Littlewood [18]; see also the survey [35].

Theorem 1.3. (i) *Let $k \geqslant 8$ be a power of 2. Then $G^+(k) = 4k$.*

(ii) *Let $k \geqslant 37$, and suppose that k is not a power of 2. Then $G^+(k) < \frac{1}{2}C_0\, k + 5\sqrt{k}$. Moreover*

$$G^+(k) \leqslant \tfrac{1}{2}C_0\, k + 0.786\,\sqrt{k} + \tfrac{5}{2}\log k + 28 \qquad (k \geqslant 5000).$$

Part (i) is new for $k \geqslant 64$. In part (ii) the bounds are those for $G(k)$, halved; for example, we obtain $G^+(37) \leqslant 137$, $G^+(50) \leqslant 177$, $G^+(100) \leqslant 355$ and $G^+(1000) \leqslant 3246$, where the bound of [6] just quoted gives 145, 203, 441 and 5555. Together with the results for $k \leqslant 36$ that were recalled above, Theorem 1.3 shows that

$$G^+(k) \leqslant 4k \qquad (k \in \mathbb{N}),$$

with equality if and only if $k \geqslant 8$ is a power of 2 (§10.3). Thus $\limsup_{k\to\infty} G^+(k)/k = 4$, whereas the upper limit of $G^+(k)/k$ over the numbers k that are not powers of 2 is at most $\frac{1}{2}C_0 = 3.1947\ldots$.

Theorem 1.3 is a consequence of Theorem 2.4 below, in which the number of exceptional $n \leqslant N$ is $O(N^{1-\delta})$ for some $\delta > 0$. That theorem shows in addition that, when k is a power of 2, fewer than $4k$ k-th powers suffice for almost all integers in those residue classes modulo $4k$ which permit this; for instance, almost all natural numbers $n \equiv j \pmod{256}$ with $1 \leqslant j \leqslant 229$ are sums of 229 sixty-fourth powers.

Mean values. All these results rest on mean value estimates for an exponential sum over the k-th powers of the elements of a set of natural numbers of the following kind. Let J be a natural number and ω a positive real number. For large real numbers P put $R := P^{1/(J+1)}$ and $H := \lfloor \omega \log R/\log 2\rfloor$, and let

$$\mathcal{C}(P) := \big\{mp_1\cdots p_J \leqslant P:\ m \in \mathbb{N},\ p_j \text{ prime},\ R < p_1 < \cdots < p_J \leqslant 2^H R\big\}. \tag{1.4}$$

Thus $p_1, \ldots, p_J$ are distinct primes from the union of the H dyadic intervals $(2^hR, 2^{h+1}R]$ with $0 \leqslant h < H$, and the factor m is otherwise unrestricted. Since $p_1\cdots p_J > R^J = P/R$, one has $m < R$, so that $p_1, \ldots, p_J$ are the prime factors of $mp_1\cdots p_J$ that exceed R, and every element of $\mathcal{C}(P)$ has exactly one representation of the shape displayed. Put $F(\alpha) := \sum_{n\in\mathcal{C}(P)} e(\alpha n^k)$. Mean value estimates of this kind are the main tool in most applications of the circle method to problems that involve k-th powers, and we therefore record the outcome in explicit form.

Theorem 1.4. *Let $k \geqslant 37$ and $s_1 := C_0\, k + 10\sqrt{k}$, or let $k \geqslant 1339$ and $s_1 := C_0\, k + 3\sqrt{k}$. Let J be a natural number that is sufficiently large in terms of k, and let $\mathcal{C}(P)$ be the set (1.4) with $\omega := 1/\big(64\, k\, s_1(J+1)\big)$. Then, for all large P, one has $\#\mathcal{C}(P) \gg P$ and*

$$\int_0^1 \Big|\sum_{n\in\mathcal{C}(P)} e(\alpha n^k)\Big|^s \,\mathrm{d}\alpha \ll P^{s-k} \qquad \textit{for all real } s \geqslant s_1.$$

The implied constants depend on k, J and s.

This is best possible: the sum is at least $\frac{1}{2}\#\mathcal{C}(P)$ in modulus when $|\alpha| \leqslant (4\pi P^k)^{-1}$. The elements of $\mathcal{C}(P)$ have no prime factors exceeding $2^H R \leqslant P^{(1+\omega)/(J+1)}$. That $\mathcal{C}(P)$ has positive density is due to the number H of dyadic intervals, which is proportional to $\log P$; with $H = 1$ the density would be of order $(\log P)^{-J}$ (§3.2). Theorem 1.4 is a special case of Theorem 2.5, which shows in addition that the part of the integral away from the major arcs is smaller by a power of P, and of Corollary 2.6, in which the condition on s takes the form $s \geqslant A_1 k + A_2\sqrt{k}$ for various pairs (A_1, A_2), depending on the size of k; for instance, $s \geqslant 7k$ suffices when $k \geqslant 212$, and $s \geqslant C_0 k + 2\sqrt{k}$ when $k \geqslant 70000$. For comparison, for the sum over all $n \leqslant P$ a bound of this strength is known when s is of order k^2 [3, 4, 44], and for the sum over the smooth numbers $n \leqslant P$ when s is $k \log k + O(k)$ [6].

We expect Theorem 1.4 to be useful beyond Waring's problem. An application of the circle method typically requires a mean value estimate of this kind, a nontrivial bound on the minor arcs, and an asymptotic formula on the major arcs. For the sum F all three are available, with a saving of a power of P on the minor arcs and in the asymptotic formula (see (i)–(iv) in §1.2, Proposition 9.1 and Theorem 11.1), and the underlying set has positive density. Thus F can take the place of a classical or a smooth Weyl sum in other problems in which k-th powers occur, and one may hope that a number of k-th powers proportional to k will then suffice where the known methods need $k \log k$ of them, or more. Examples are representation problems in which k-th powers are accompanied by variables of another kind, such as squares [8]; questions about exceptional sets, of which Theorem 1.3 is an instance; and questions about the distribution of αn^k modulo 1. Our last theorem is an application of the last kind.

Fractional parts. Let $\|x\|$ denote the distance from the real number x to the nearest integer. A classical problem asks for upper bounds for $\min_{1\leqslant n\leqslant N} \|\alpha n^k\|$ that hold for all real α and all large N; these are known as localised bounds. Following work of Vinogradov and Heilbronn, Danicic proved that the minimum is smaller than $N^{\varepsilon - 2^{1-k}}$ (see [1]). For large k the sharpest result is due to Brüdern and Wooley [9, Theorem 1.2]: when $k \geqslant 6$ and N is sufficiently large, one has

$$\min_{1\leqslant n\leqslant N} \|\alpha n^k\| \leqslant N^{-1/(k(\log k + 8.02113))}.$$

This refines bounds of Wooley [40, Theorem 1.2], [41, Theorem 1.2], in which the exponents are $1/(2k(\log k + \log\log k + 2 + o(1)))$ and $1/(k(\log k + O(\log\log k)))$. In recent work on small fractional parts of polynomials, Baker [2, Theorem 2] showed that the resolution of the main conjecture in Vinogradov's mean value theorem [4, 44] leads to the bound $N^{\varepsilon - 1/(k(k-1))}$ for $k \geqslant 6$, which is the sharper one when k is small. If a bound is required for infinitely many n only, then more is known: Wooley [40, Theorem 1.1 and Corollary 2] proved that for $k \geqslant 4$, for every real α and every $\varepsilon > 0$, there are infinitely many natural numbers n with

$$\|\alpha n^k\| \leqslant n^{\varepsilon - 1/(9.028\, k)},$$

improving a bound of the same shape that is due to Heath-Brown [22]. Thus the exponent in this unlocalised bound is of order $1/k$, whereas the exponents in the localised bounds

are of order $1/(k\log k)$ at best. We show that there is a localised bound with an exponent of order $1/k$.

Theorem 1.5. *Let $k \geqslant 37$, and let α be a real number. Then, whenever N is sufficiently large in terms of k,*

$$\min_{1\leqslant n\leqslant N} \|\alpha n^k\| < N^{-1/(10.9035\,k+12\sqrt{k})}.$$

If $k \geqslant 900$, this holds with $4\sqrt{k}$ in place of $12\sqrt{k}$, and if $k \geqslant 5000$, with $2\sqrt{k}+5\log k+58$. Moreover, the minimum is smaller than $N^{-1/(12k)}$ when $k \geqslant 42$, and smaller than $N^{-1/(11k)}$ when $k \geqslant 1339$ (Proposition 11.10).

Here 10.9035 is a rounded-up value of the number $C_0 + \frac{1}{2}\cdot 9.027901 = 10.90346\ldots$, which is the constant that the method gives. The method gives a bound that is sharper than the one of [9] displayed above for every $k \geqslant 46$ (Proposition 11.10); for example, when $k = 100$ and $k = 1000$ it gives the exponents $1/1175$ and $1/11023$, in place of $1/1262.6\ldots$ and $1/14928.8\ldots$. For large k the exponent is about 83 per cent of the one in the unlocalised bound of [40]. Theorem 1.5 is deduced in §11.5 from a bound of Weyl type for F, in the manner of [40]: for large k one has $F(\beta) \ll P^{1-1/((10.9035+o(1))k)}$ for all real β that have no rational approximation a/q with $q \leqslant P^{k/(k+1)}$ and $|q\beta - a| \leqslant P^{k/(k+1)-k}$ (Corollary 11.2). This bound has two sources: the large sieve, applied with the moments of F of order about C_0k, which accounts for the term C_0; and the bound of Weyl type with the exponent $1/(9.027901\,k)$ that is known for smooth Weyl sums [6, Lemma 7.1], which accounts for the term $\frac{1}{2}\cdot 9.027901$. The general statement is Theorem 2.7.

A refinement. The estimates for complete exponential sums on which these results rest (§4) can be refined by a mean value argument that uses two coefficients. This is carried out in Appendix A, where the theorems of this section are sharpened: the constant C_0 is replaced by $C_0^\star = 6.2821\ldots$, the term of order $\sqrt{k}$ in Theorem 1.1 becomes negative, and one has, for example, $G(k) < 6.43\,k$ for every k and $G(k) < C_0^\star k$ for all $k \geqslant 56$ (Theorems A.1–A.5); for $13 \leqslant k \leqslant 36$ the bounds for $G(k)$ and for $G^+(k)$ known before are improved as well. For $k \leqslant 5000$ these bounds rest, in addition, on a recursive form of the potential of §8, with which the arcs of large height are pruned by means of large values of exponential sums instead of moments (§A.6). We have deferred these results to the appendix, since their proofs are sufficiently complicated that our main arguments would be obscured were they to be included in the main body of the text, and in any case they are of relatively modest impact. The main text does not depend on the appendix.

1.2. **The method.** We describe the argument of [10] and the points at which we depart from it. Let N be large and $P = N^{1/k}$. In [10] one works with a set of integers $n = p_1\cdots p_l\,u \leqslant P$, in which the p_i are primes from l disjoint intervals of increasing size, one from each, and u is a smooth number. We use the integers

$$n = m\,p_1\cdots p_J \leqslant P, \qquad R < p_1 < \cdots < p_J \leqslant \zeta R, \qquad \text{where} \quad R = P^{1/(J+1)}, \tag{1.5}$$

with a large number J, a natural number m that is not restricted otherwise (so that $m \leqslant R$), and $2 \leqslant \zeta \leqslant R^\omega$ for a small $\omega > 0$. Let $F(\alpha)$ be the exponential sum over the k-th powers of these integers. Grouping the terms according to a divisor d of $p_1\cdots p_J$

TABLE 1. The argument of [10] and the present one. Here $q = X^U$ is a modulus dividing d^k, $W = X^z$, and $d \leqslant X^{\theta/k}$.

	[10]	this paper
variables	$p_1 \cdots p_l u \leqslant P$ with primes p_i from l disjoint intervals of increasing size, and u smooth	$mp_1 \cdots p_J \leqslant P$ with distinct primes from one interval $(R, \zeta R]$, and m unrestricted; positive density for suitable ζ (§3)
saving in complete sums, $q = X^U > X$	X^{-10^{-3}/θ^2} for all $U \leqslant \theta$, from a Weyl-type bound on each residue class	$X^{-\varsigma(U)}$ with $\varsigma(U) \asymp U^{-2}$: van der Corput differencing adapted to the modulus for small U, a mean value over residue classes for large U (§4)
loss in one step	W^3	W^3/V when the shorter sums are of relative size $1/V$; in effect W^2 (§6)
step size	$\theta \leqslant 3 \log k$, tied to z by $\theta \asymp z^{-1/2}$	$\theta \leqslant \Theta$ with $\Theta \asymp k^{1/2}$; the optimal $\theta = \theta(z)$ (§8)
large W	trivial bound for $z \geqslant 2.5 \cdot 10^{-7}$	mean values over smooth numbers for $z \geqslant \frac{1}{10}$ (§7)
minor arcs and pruning	$2k$ auxiliary smooth Weyl sums	no auxiliary sums: estimates for F on arcs of a given height (§9)
major arcs	theory of smooth Weyl sums	classical, with the density of the set as a factor (Lemma 9.10)
constant	12 000 005	6.3895; 6.2821 in Appendix A

with a given number i of prime factors gives

$$\binom{J}{i} F(\alpha) = \sum_d F_d(d^k \alpha), \tag{1.6}$$

where F_d is a sum of the same kind of length X/d, with $X = P$, in which $J - i$ primes remain and the primes dividing d are excluded. If $|F(\alpha)| \geqslant X/W$, then $|F_d(d^k\alpha)|$ is large for many d. The central observation of [10] is a large sieve inequality on the fibres of the map $\alpha \mapsto d^k\alpha$ of $\mathbb{T} = \mathbb{R}/\mathbb{Z}$: for a given β, the number of α with $d^k\alpha = \beta$ at which an exponential sum $\sum_{n \leqslant X} c_n e(\alpha n^k)$ with bounded coefficients exceeds X/W is $O(W^2)$, provided that the complete sums $\sum_{n \leqslant X,\,(n,d)=1} e(an^k/q)$ to moduli $q \mid d^k$ with $q > X$ are smaller than X/W^2; the proof is in the manner of Halász and Montgomery's treatment of the large values of Dirichlet polynomials (see §5). This inequality converts a bound for the measure of the set where F_d is large into a bound for the measure of the set where F is large, at the cost of a power of W, while the length of the sum is reduced from X to X/d. Iterating one obtains an essentially optimal estimate for $\int_{\mathbb{T}} |F|^r \, \mathrm{d}\alpha$ with $r \ll k$, and in [10] the circle method, with $2k$ further smooth variables to deal with the minor and major arcs, completes the proof.

The set (1.5) has less structure than that of [10]: all primes come from one interval, the size of a divisor d is governed by the number of its prime factors alone, and the complementary factor m is not required to be smooth. The price is that the primes have to be distinct, which is accounted for by the excluded primes in the sums F_d, and that every integer occurs $\binom{J}{i}$ times in (1.6); neither causes any difficulty. When $\zeta = 2$, the density of the set (1.5) is of order $(\log P)^{-J}$. This loss is recovered when $(R, \zeta R]$ is the union of the dyadic intervals $(2^h R, 2^{h+1}R]$ with $0 \leqslant h < H$ and H is proportional to $\log P$: the set is then a union of about H^J sets of the former kind and has positive density (§3.2); for $H = \lfloor \omega \log R/\log 2 \rfloor$ it is the set (1.4). This device is adapted from the work of Brüdern and Wooley on cubes of smooth numbers [5, §5]. All our estimates are uniform in ζ.

We keep the framework of [10] and change seven things, which are summarised in Table 1. The constant C_0 decomposes as $(2 + \log 9) + 2\Lambda$: the term $2 + \log 9 = 4.197\ldots$ comes from the range $W \geqslant X^{1/10}$ of large values, which is controlled by the known mean values of smooth Weyl sums, and $2\Lambda = 2.192\ldots$ is the cost of the fibre steps. This is explained in §2.2.

The two rows of the table that concern the minor and the major arcs require some comment. In [10] the sum F, whose moments are under control, is accompanied by $2k$ smooth Weyl sums, for which pointwise bounds are available on the minor arcs and near the major arcs. We show that F itself has all the necessary properties. Let $\mathfrak{N}(Q)$ denote the set of α having a rational approximation a/q with $q + P^k|q\alpha - a|$ of exact order Q, where $1 \leqslant Q \leqslant P^{k/2}$. Then the following hold, for exponents r that are admitted by the iteration and suitable small $b > 0$. In (i)–(iii), ε denotes a positive exponent that can be made as small as we please by the choice of the parameters, in particular by taking J large.

(i) *Mean values.* One has $\int |F|^r \,\mathrm{d}\alpha \ll P^{r-k+\varepsilon}Q^{2b/k}$, the integral being over the union of the sets $\mathfrak{N}(Q')$ with $Q' \leqslant Q$ (Corollary 9.3). This follows from (1.6) with $d \approx P\,Q^{-2/k}$: the substitution $\beta = d^k\alpha$ maps these arcs to the corresponding ones for sums of length P/d, on which the trivial extension to $\mathbb{T}$ loses little.

(ii) *A bound of Weyl type.* For $Q \geqslant P^{1/4}$ and $\alpha \in \mathfrak{N}(Q)$ one has $F(\alpha) \ll P^{1+\varepsilon}Q^{-2\tau_k/k}$, where $\tau_k := 1/(9.027901\,k)$; see Lemma 9.8. This rests on a bilinear argument of Wooley [41, Lemma 3.1], which applies to any bilinear sum once mean values for the two sets of variables are known; for F these are inherited from smooth numbers.

(iii) *Small heights.* For $Q \leqslant P^{1/4}$ and $\alpha \in \mathfrak{N}(Q)$ one has $F(\alpha) \ll P^{1+\varepsilon}Q^{-1/(2k)}$; see Lemma 9.9. Here the sum over one of two factors of size $P^{1/2}$ is completed to a classical Weyl sum.

(iv) *Major arcs.* For $q \leqslant R^{1/4}$ and $|\beta| \leqslant R^{1/4}P^{-k}$ one has

$$F(a/q + \beta) = \Pi\, q^{-1}S(q,a)v(\beta) + O(PR^{-1/2}),$$

where $q^{-1}S(q,a)v(\beta)$ is the classical approximation to a Weyl sum of length P and Π is the density of the set (1.5); see Lemma 9.10. This rests on the unrestricted factor m alone.

For smooth Weyl sums the estimates corresponding to (i) and (ii) are due to Brüdern and Wooley [6, §§4 and 5], and the exponent τ_k is that of [6, Lemma 7.1]. With (i)–(iv) the

PRINCIPAL NOTATION

symbol	meaning	defined in
$G(k)$, $G^+(k)$	the least s for all large n, and for almost all n	§1.1
Λ, C_0	$\Lambda = 1.0961\ldots$ and $C_0 = 2 + \log 9 + 2\Lambda = 6.3895\ldots$	(1.2), (1.3)
ς; σ, θ^*, I, T	the saving in complete sums; functions derived from it	Def. 2.1
κ, ϖ, η_k	rounding losses; admissibility of the largest step Θ	Def. 2.2
R_k, $C(h)$, h_k	further terms of the bound of Theorem 2.3	§2
τ_k, y_k, W_k	the Weyl exponent $1/(9.027901\,k)$, and related terms	(2.3)
$B(k,\Theta)$, $\Theta^\sharp(k)$, a_k	$G(k) \leqslant \lfloor kB \rfloor$; the choice of Θ; the root of $a(a+5)=k$	§10.2
J, η, R, ω, H, ζ	parameters: $\eta = 1/(J+1)$, $R = P^\eta$, $\zeta = 2^H \leqslant R^\omega$	§3.1
$\mathcal{P}$, $\mathcal{D}_j^{\mathfrak{e}}$	the primes in $(R, \zeta R]$; products of j of them, not in $\mathfrak{e}$	§3.1
$\mathcal{C}_j^{\mathfrak{e}}(X)$, $F_j^{\mathfrak{e}}(\alpha;X)$	integers $dm \leqslant X$ with $d \in \mathcal{D}_j^{\mathfrak{e}}$, $m \leqslant R$; their sum	(3.4)
$\mathcal{C}$, F; $\mathcal{C}(P)$	$\mathcal{C}_J(P)$ and $F_J(\,\cdot\,;P)$; the case of the largest H	§3.1, (1.4)
$D_j(X,W)$	X^k times the measure of the set where $\lvert F_j^{\mathfrak{e}}\rvert \geqslant X/W$	§3.1
Π	the density of $\mathcal{C}$	§3.2
$\mathcal{S}$, ϑ, U	complete sum to a modulus $q = X^U \mid m^k$, where $m = X^\vartheta$	§4
$\mathfrak{d}(t)$, Ψ	exponents for smooth Weyl sums; $\Psi(z) = 2+\log((1-z)/z)$	§7
$\mathcal{G}$, $\mathfrak{g}$, Φ	the potential; $\mathfrak{g}(z) = -z\mathcal{G}'(z)$; $\Phi(x,\lambda) \approx k\lambda\,\mathcal{G}(\lambda/x)$	§8.1
u_Θ, c_Θ	$\mathcal{G}(z) = \mathcal{G}(u_\Theta) + c_\Theta \log(u_\Theta/z)$ for $0 < z \leqslant u_\Theta$	(8.4)
a_*	moments of order $k(\mathcal{G}(u_\Theta)+a_*)$ have excess at most X^b	(8.8)
$\mathfrak{M}(Q;Y)$, $\mathfrak{N}(Q;Y)$	arcs of height at most Q; $\mathfrak{M}(Q;Y) \setminus \mathfrak{M}(\frac12 Q;Y)$	§9.1
$\Upsilon(\alpha)$	$(q + P^k\lvert q\alpha - a\rvert)^{-1}$	§9.1
r, b, t, s, ν	order r, excess P^b, t further variables, $s = r+t$, $\nu = t\tau_k - b$	(9.1)
δ_*	$\mathfrak{d}(2k) = 0.27846\ldots$	§9.3
$E^\sharp(k)$, $s_0^\sharp(k)$, χ_0	the exponents $\rho < 1/E^\sharp(k)$ of (11.5); ingredients of $E^\sharp(k)$	§11.5
$\Lambda^\star$, $C_0^\star$	$\Lambda^\star = 1.0424\ldots$ and $C_0^\star = 2 + \log 9 + 2\Lambda^\star = 6.2821\ldots$	(A.1)
$\varsigma^\star$; σ_k, θ_k	the saving with its actual rounding losses; functions derived from it	Def. A.11
(Ψ°, τ°)	the a priori bound and the exponent of Weyl type, in two cases	§A.4
z_k, Θ_k, $\mathcal{Y}_k(\Theta)$	$G(k) \leqslant \lfloor \mathcal{Y}_k(\Theta) \rfloor + 1$ for $\frac94 \leqslant \Theta \leqslant \Theta_k$	Thm. A.17
$\mathcal{Y}^\flat(k)$, $G^\flat(k)$, $\ell^\flat(k)$	the bounds $G(k) \leqslant G^\flat(k)$ and $G^+(k) \leqslant \ell^\flat(k)$ that are obtained from a recursion	§A.7
$\mathcal{Y}^{\mathrm{VW}}$, G^{VW}, ℓ^{VW}	the same for $13 \leqslant k \leqslant 20$, with the tables of [34]	§A.7
$B^\natural(k)$, $\Theta^\natural(k)$, $E^\natural(k)$	a bound for $\mathcal{Y}_k$ from chains; the step used; exponents for $\lVert \alpha n^k \rVert$	§A.8

circle method can be carried out with $s = r + t$ copies of F as soon as $t\tau_k > b$, and both t and the excess of r over $C_0 k$ turn out to be $O(\sqrt{k})$. It gives an asymptotic formula, with a power saving, for the number of representations by k-th powers of the integers (1.5); see (9.20).

The exponent τ_k in (ii) is the one known for smooth Weyl sums. The mean values of F of order about $C_0 k$ are much stronger than those of a smooth Weyl sum, and in §11 we use them, by way of the large sieve, to prove a second estimate of Weyl type (Theorem 11.1). It is stronger than (ii) on the sets $\mathfrak{N}(Q)$ with $Q < P^{0.41k}$, roughly, and it

is the estimate on which Theorem 1.5 rests. The number of variables, however, is decided by the arcs of larger height, on which (ii) is the stronger estimate, and the mean values of F do not improve the exponent τ_k in the argument behind (ii) (§11.4). Thus the large sieve estimate does not enter the proofs of Theorems 1.1–1.4.

For finite k the moduli occurring in §4 are only approximately of the size one would like, and steps longer than Θ, which in Theorem 2.3 must be smaller than $\sqrt{k}$, are not available (Appendix A admits somewhat larger steps). The second restriction is absorbed by the saving (ii), which is why a positive b can be tolerated in (i), and balancing the two effects gives the term of order $\sqrt{k}$ in Theorem 1.1 (see §2.2).

All proofs are given in full, including the part of the argument that is taken from [10] without change, the fibre proposition (§5). Thus the paper is self-contained apart from Weil's bound, Vinogradov's mean value theorem [4, 44], the mean value estimates for smooth Weyl sums of [9, 40] that are quoted in (7.1), the prime number theorem, the classical theory of Weyl sums, the singular series and the singular integral [32], and, for Theorem 1.5, a lemma of Harman [20]. In particular no pointwise estimate for smooth Weyl sums is used, on minor or on major arcs. Theorems 1.1–1.5 depend in addition on the evaluation of an explicit closed expression at 4964 points, one for each k with $37 \leqslant k \leqslant 5000$; this is discussed in §10.4.

1.3. **Organisation and notation.** The explicit bound is stated in §2. Sections 3–7 contain the tools: the set of variables and its density, exponential sums to moduli dividing m^k, the fibre proposition, the iterative inequality, and the bounds coming from smooth numbers. The iteration is carried out in §8 by means of a potential function. Section 9 contains the estimates for F on arcs and the application of the circle method, first to all large natural numbers and then, in §9.6, to almost all natural numbers; the mean value estimate is proved in §9.7. The resulting bounds are evaluated in §10, where Theorems 1.1, 1.2, 1.3 and 1.4 are proved. The large sieve estimate is the subject of §11. It is applied to the fractional parts of αn^k in §11.5, where Theorem 1.5 is proved, and it is not used elsewhere in the paper. Some remarks are collected in §12, and Appendix A contains the refinement that was mentioned at the end of §1.1.

We write $e(x) = \mathrm{e}^{2\pi i x}$, and e for the base of the natural logarithm; $\mathbb{T} = \mathbb{R}/\mathbb{Z}$; and $\|x\|$ is the distance from x to the nearest integer. The letter p denotes a prime, $v_p(q)$ is the exponent of p in q, (a, b) is the greatest common divisor, and φ is Euler's function. For real numbers $Y \geqslant Z \geqslant 1$ we write $\mathcal{A}(Y, Z)$ for the set of natural numbers $u \leqslant Y$ all of whose prime factors are at most Z. The parameter ε is fixed in §9; in §4, where the statements hold for every $\varepsilon > 0$, this is said explicitly. The dependence of implied constants is described at the end of §3. The principal notation is collected in the table on page 9.

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2. The explicit bound

2.1. **Statement.** The bound involves a function ς which measures the saving in certain complete exponential sums, and some quantities derived from it.

Definition 2.1 (saving function)**.** For $U \geqslant 1$ let

$$\varsigma(U) := \begin{cases} (2-U)/2, & 1 \leqslant U \leqslant \frac{3}{2}, \\ (3-U)/6, & \frac{3}{2} \leqslant U \leqslant \frac{9}{4}, \\ (4-U)/14, & \frac{9}{4} \leqslant U \leqslant \frac{25}{8}, \\ (5-U)/30, & \frac{25}{8} \leqslant U \leqslant \frac{50}{13}, \\ U/100, & \frac{50}{13} \leqslant U \leqslant 4, \\ (5-U)/25, & 4 \leqslant U \leqslant \frac{160}{37}, \\ (6-U)/62, & \frac{160}{37} \leqslant U \leqslant \frac{540}{121}, \\ U/180, & \frac{540}{121} \leqslant U \leqslant 5, \\ \max\Big\{\dfrac{D+1-U}{(D+1)^2},\ \dfrac{U}{(D+1)(D+2)^2}\Big\}, & U \geqslant 5, \end{cases}$$

where in the last line $D \geqslant 5$ is an integer with $D \leqslant U \leqslant D+1$. Further let

$$\sigma(\theta) := \min_{1 \leqslant U \leqslant \theta} \varsigma(U) \quad (\theta \geqslant 1), \qquad \theta^*(s) := \max\{\theta \geqslant 1 :\ \sigma(\theta) \geqslant s\} \quad (0 < s \leqslant \tfrac{1}{2}),$$

$$I(s) := \int_0^s \frac{\mathrm{d}t}{t\,\theta^*(t)} \quad (0 < s \leqslant \tfrac{1}{2}), \qquad T(\Theta) := I\big(\sigma(\Theta)\big) \quad (\Theta \geqslant 1).$$

The definitions at common end points agree, so that ς is continuous, and $\varsigma(U) \to 0$ as $U \to \infty$. Hence σ is continuous and nonincreasing with $\sigma(1) = \frac{1}{2}$ and $\sigma(\theta) \to 0$, and θ^* is well defined and nonincreasing. These functions are made explicit in §10; in particular $\theta^*(t) \asymp t^{-1/2}$, so that the integral converges, and (Lemma 10.2)

$$I\big(\tfrac{1}{5}\big) = \Lambda. \tag{2.1}$$

The meaning of ς is as follows (Lemma 4.5): if m is a squarefree number composed of a bounded number of large primes and $q = X^U$ is a divisor of m^k with $U > 1$, then $\sum_{n \leqslant X,\,(n,m)=1} e(an^k/q) \ll X^{1-\varsigma(U)+\varepsilon}$ whenever $(a,q) = 1$, up to a small loss depending on the size of m. Thus $X^{-\sigma(\theta)}$ is the saving that is available for all such moduli when $m \leqslant X^{\theta/k}$, and $\theta^*(s)$ is the largest θ for which a saving X^{-s} is guaranteed.

Definition 2.2 (admissible parameters)**.** For $\theta \geqslant 1$ let

$$\kappa(\theta) := \begin{cases} 4, & \theta \leqslant \frac{9}{4}, \\ 8, & \frac{9}{4} < \theta \leqslant \frac{25}{8}, \\ 13, & \frac{25}{8} < \theta \leqslant \frac{160}{37}, \\ \max\{21, \theta+5\}, & \theta > \frac{160}{37}, \end{cases} \qquad \varpi(\theta) := \begin{cases} \frac{1}{16}, & \theta \leqslant \frac{25}{8}, \\ \frac{1}{26}, & \frac{25}{8} < \theta \leqslant \frac{160}{37}, \\ \frac{3}{121}, & \theta > \frac{160}{37}, \end{cases}$$

and $\eta_k(\theta) := \theta\,\kappa(\theta)/k$. A real number Θ is called *admissible for* k if

$$\Theta \geqslant \tfrac{9}{4}, \qquad \Theta \leqslant \varpi(\Theta)\,k, \qquad \Theta(\Theta+5) < k.$$

For $k \geqslant 37$ the admissible numbers form an interval with left end point $\frac{9}{4}$. Since κ is nondecreasing and ϖ is nonincreasing, an admissible Θ satisfies

$$\theta \leqslant \varpi(\theta)\,k \qquad \text{and} \qquad \eta_k(\theta) \leqslant \eta_k(\Theta) < 1 \qquad (1 \leqslant \theta \leqslant \Theta); \tag{2.2}$$

indeed $\eta_k(\Theta) \leqslant \frac{63}{121}$ if $\Theta \leqslant 16$, and $\eta_k(\Theta) = \Theta(\Theta+5)/k$ if $\Theta > 16$. The number Θ will be the largest step size used in the iteration, and $\eta_k(\theta)$ bounds the relative loss in the saving

$\sigma(\theta)$ that is caused by rounding. The requirement $\Theta \geqslant \frac{9}{4}$ is a normalisation which keeps the two regimes of §2.2 apart (see (8.3)). Next let

$$R_k(\Theta) := \int_{9/5}^{\Theta} \frac{\mathrm{d}\big(-\log(1-\eta_k(\theta))\big)}{\theta},$$

$$C(h) := 2 + 2\Lambda + \log\frac{45+6h-h^2}{(1-h)(5-h)} - \frac{2}{3}\log\frac{15+2h}{15-3h} \qquad (0 \leqslant h < 1).$$

The first of these is a Riemann–Stieltjes integral with respect to a nondecreasing function which has jumps at $\frac{9}{4}$, $\frac{25}{8}$ and $\frac{160}{37}$; its closed form is given in Lemma 10.4. Note that $C(0) = C_0$. Finally let

$$\tau_k := \frac{1}{9.027901\,k}, \qquad y_k(\Theta) := \frac{(1-\eta_k(\Theta))\,\sigma(\Theta)}{2\,\tau_k}, \qquad W_k(\Theta) := \frac{2}{\Theta}\log y_k(\Theta). \tag{2.3}$$

The number τ_k is the exponent in a bound of Weyl type for the exponential sum F of §1.2 on the extreme minor arcs (Lemmas 9.7 and 9.8); it is the exponent obtained for smooth Weyl sums by Brüdern and Wooley [6, Lemma 7.1]. Note that $W_k(\Theta)$ is negative when $y_k(\Theta) < 1$.

Theorem 2.3. *Let $k \geqslant 37$, let Θ be admissible for k, and put $h_k := \frac{36}{5(k-1)}$. Then*

$$G(k) \leqslant \Big\lfloor k\Big(C(h_k) - 2\,T(\Theta) + 2R_k(\Theta) + W_k(\Theta)\Big)\Big\rfloor + 1.$$

Moreover $k\,C(h_k) \leqslant C_0\,k + 11$, so that $G(k) \leqslant k\big(C_0 - 2\,T(\Theta) + 2R_k(\Theta) + W_k(\Theta)\big) + 12$.

The three terms depending on Θ are effects of finite k: steps larger than Θ are not used ($-2T$), moduli have to be rounded ($2R_k$), and the potential of §8 is needed down to the level τ_k, while the first two terms account for it only down to $\frac{1}{2}(1-\eta_k(\Theta))\sigma(\Theta) = y_k(\Theta)\,\tau_k$ (W_k). The term $W_k(\Theta)$ vanishes when $y_k(\Theta) = 1$, that is, when the saving available for the largest step is twice the Weyl exponent, and it is near this value of Θ that the bound is smallest. An admissible Θ is smaller than $\sqrt{k}$. For $\Theta = \lambda\sqrt{k}$ with a fixed $\lambda \in (0,1)$ one has $T(\Theta) \sim 2/\Theta$, $R_k(\Theta) \sim k^{-1/2}\log\frac{1+\lambda}{1-\lambda}$ and $W_k(\Theta) \ll 1/\Theta$, so that all three terms are of order $k^{-1/2}$; this leads to Theorem 1.1 (see §10.3). For $37 \leqslant k \leqslant 5000$ we evaluate the bound at an explicit $\Theta = \Theta^\sharp(k)$; see §10.2.

For the representation of almost all natural numbers half as many k-th powers suffice, as far as the congruence conditions permit. Recall the definition of $G^+(k)$ from §1.

Theorem 2.4. *Let $k \geqslant 37$, let Θ be admissible for k, and let ℓ be an integer with*

$$\ell > \tfrac{1}{2}\,k\Big(C(h_k) - 2\,T(\Theta) + 2R_k(\Theta) + W_k(\Theta)\Big).$$

Then there is a number $\delta > 0$ with the following properties.

(i) *Suppose that k is not a power of 2. Then all natural numbers $n \leqslant N$, with at most $O(N^{1-\delta})$ exceptions, are sums of ℓ k-th powers of natural numbers. In particular*

$$G^+(k) \leqslant \Big\lfloor \tfrac{1}{2}\,k\Big(C(h_k) - 2\,T(\Theta) + 2R_k(\Theta) + W_k(\Theta)\Big)\Big\rfloor + 1.$$

(ii) *Suppose that k is a power of 2, and let j be an integer with $1 \leqslant j \leqslant \min\{\ell, 4k\}$. Then all natural numbers $n \leqslant N$ with $n \equiv j \pmod{4k}$, with at most $O(N^{1-\delta})$ exceptions, are sums of ℓ k-th powers of natural numbers. In particular $G^+(k) \leqslant \max\{\ell, 4k\}$.*

The proof is in §9.6. For $k = 16$ and $k = 32$ a result of the same kind as part (ii) is [6, Theorem 9.2]. When k is a power of 2 and $\ell < 4k$, the condition $j \leqslant \ell$ in part (ii) cannot be omitted: an integer $n \equiv j \pmod{4k}$ with $1 \leqslant j < 4k$ is not the sum of fewer than j k-th powers (Lemma 9.13).

The mean value estimate behind Theorems 2.3 and 2.4 is of independent interest, and we state it separately. Let $\mathcal{C}$ be the set of the integers (1.5). It is defined in §3; it depends on P, on a natural number H (the interval $(R, \zeta R]$ is the union of H dyadic intervals), and on a number of further parameters, among them the numbers J and $\eta = 1/(J+1)$ with $R = P^\eta$. When H is the largest integer that is permitted in §3, $\mathcal{C}$ is the set $\mathcal{C}(P)$ of (1.4). As before, $F(\alpha) := \sum_{n\in\mathcal{C}} e(\alpha n^k)$. For $Q \geqslant 1$ let $\mathfrak{M}(Q;P)$ be the set of all $\alpha \in \mathbb{T}$ for which there are coprime integers a and q with $1 \leqslant q \leqslant Q$ and $|q\alpha - a| \leqslant QP^{-k}$, as in §9.1.

Theorem 2.5. *Let $k \geqslant 37$, let Θ be admissible for k, and let s_1 be a real number with*

$$s_1 > k\Big(C(h_k) - 2\,T(\Theta) + 2R_k(\Theta) + W_k(\Theta)\Big).$$

Then the parameters of §3 other than P and H can be chosen, in terms of k, Θ and s_1, in such a way that for a certain number $\nu_1 > 0$ the following holds for all real $s \geqslant s_1$, all large P and all H that are permitted in §3.

(i) *One has* $\displaystyle\int_{\mathbb{T}} |F(\alpha)|^s \,\mathrm{d}\alpha \asymp (\#\mathcal{C})^s\, P^{-k}$.

(ii) *For $1 \leqslant Q \leqslant P^{\nu_1}$ one has*

$$\int_{\mathbb{T}\setminus\mathfrak{M}(Q;P)} |F(\alpha)|^s \,\mathrm{d}\alpha \ll (\#\mathcal{C})^s\, P^{-k}\big(Q^{-2} + P^{-\nu_1}\big).$$

(iii) *The elements of $\mathcal{C}$ are natural numbers $n \leqslant P$ without prime factors exceeding $P^{2\eta}$, and $P(H/\log P)^J \ll \#\mathcal{C} \leqslant P$. In particular $\#\mathcal{C} \gg P$ when H is the largest integer that is permitted in §3.*

The implied constants depend on k, Θ, s_1, s and the parameters, but not on H.

Thus $\int_{\mathbb{T}} |F|^s \,\mathrm{d}\alpha \ll P^{s-k}$ in all cases, and $\int_{\mathbb{T}} |F|^s \,\mathrm{d}\alpha \asymp P^{s-k}$ for the largest H. The proof is in §9.7. For smaller orders s, Corollary 8.3 provides bounds of the shape $P^{s-k+\Delta+\varepsilon'}$, in which Δ is determined by the function $\mathcal{G}$ of §8; such a bound is used in the proof of Theorem 2.7. The following corollary makes the condition on s_1 explicit; here $\Theta = \Theta^\sharp(k)$ is the number defined in §10.2.

Corollary 2.6. *Let $k \geqslant 37$. The conclusions of Theorem 2.5 hold with $\Theta = \Theta^\sharp(k)$ and $s_1 = A_1 k + A_2\sqrt{k}$ whenever $k \geqslant k_1$ and (k_1, A_1, A_2) is one of the columns of the following table.*

k_1	37	37	212	697	927	1339	70000
A_1	7.39	C_0	7	C_0	6.5	C_0	C_0
A_2	0	10	0	4	0	3	2

For $k \geqslant 5000$ they hold with $s_1 = C_0 k + 1.572\sqrt{k} + 5\log k + 54$. Moreover, for every $\varepsilon_0 > 0$ there is a number $k_0(\varepsilon_0)$ such that for $k \geqslant k_0(\varepsilon_0)$ they hold with $\Theta = 0.90478\sqrt{k}$ and $s_1 = C_0 k + (1.5712 + \varepsilon_0)\sqrt{k}$.

The proof is in §10.3.

For the fractional parts of αn^k the general statement is as follows. Recall that $\|x\|$ denotes the distance from x to the nearest integer, and that $\tau_k = 1/(9.027901\,k)$, so that $4.5139505\,k = 1/(2\tau_k)$.

Theorem 2.7. *Let $k \geqslant 37$, let Θ be admissible for k, let $0 < b < 1$, and let $s_0 \geqslant k$ be a natural number with*

$$2s_0 > k\Big(C(h_k) - 2\,T(\Theta) + 2R_k(\Theta) + W_k(\Theta)\Big) - \frac{2k}{\Theta}\Big(1 + \log\frac{9.027901\,\Theta\,b}{2}\Big).$$

Let ρ be a real number with

$$\rho < \frac{1-b}{2s_0 + 4.5139505\,k}.$$

Then there is a number N_0 such that $\min_{1\leqslant n\leqslant N}\|\alpha n^k\| < N^{-\rho}$ for all real numbers α and all $N \geqslant N_0$.

The proof is in §11.5. The number b is the exponent by which the moment of order $2s_0$ of F is allowed to exceed its expected size; the best choice is $b \approx 0.18/\Theta$, and then the second term on the right of the condition on s_0 is about $1.6\,k/\Theta$. With Θ of order $\sqrt{k}$ this gives exponents ρ with $1/\rho = (C_0 + 4.5139505)k + O(\sqrt{k})$; they are evaluated in Proposition 11.10.

2.2. **Where the constant comes from.** The following heuristic computation explains the shape of Theorem 2.3; it is not used in the proofs. In the notation of §1.2, let $D(X,W)$ be X^k times the measure of the set of $\alpha \in \mathbb{T}$ at which a sum of the type F, of length $X = P^x$, has modulus at least X/W, where $W = P^\lambda$. The trivial bound is $D(X,W) \leqslant X^k$, and the truth should be $D(X,W) \approx W^{2k}$ for small W. Write $z := \lambda/x$, so that $W = X^z$, and suppose that $D(X,W) \leqslant P^{k\lambda\,\mathcal{G}(z)}$ for some function $\mathcal{G}$. Two kinds of information are available.

(i) Chebyshev's inequality and the sharpest known mean values for smooth Weyl sums give $\mathcal{G}(z) \leqslant \Psi(z) := 2 + \log\frac{1-z}{z}$ for $0 < z \leqslant \frac12$ (Lemma 7.1).

(ii) A fibre step with a divisor $d \approx X^{\theta/k}$ bounds the quantity $D(X,W)$ by $W^2 D(X/d,W)$ (Lemma 6.1). It is permitted when W^2 is smaller than the saving $X^{\sigma(\theta)}$ in the complete sums (Proposition 5.1), that is, when $\sigma(\theta) \geqslant 2z$. The step replaces (x,λ) by $(x(1-\theta/k),\lambda)$, and so it is consistent with the bound $P^{k\lambda\mathcal{G}(z)}$ if $2\lambda + k\lambda\,\mathcal{G}\big(z/(1-\theta/k)\big) \leqslant k\lambda\,\mathcal{G}(z)$; for large k this means $-z\mathcal{G}'(z) \geqslant 2/\theta$. The best choice is the largest permitted step, $\theta = \theta^*(2z)$.

The corresponding quantity for Ψ is $-z\Psi'(z) = 1/(1-z)$, and one checks from Lemma 10.1 that $2/\theta^*(2z) < 1/(1-z)$ exactly when $z < \frac{1}{10}$. The best strategy is therefore to use (i) for $z \geqslant \frac{1}{10}$ and (ii) for $z < \frac{1}{10}$, which leads to

$$\mathcal{G}(z) = \Psi\big(\tfrac{1}{10}\big) + 2\int_z^{1/10} \frac{\mathrm{d}u}{u\,\theta^*(2u)} \longrightarrow 2 + \log 9 + 2\,I\big(\tfrac15\big) = 2 + \log 9 + 2\Lambda \qquad (z \to 0).$$

Consequently $\int_{\mathbb{T}} |F|^r\,\mathrm{d}\alpha \ll P^{r-k+\varepsilon}$ should hold as soon as $r > k(2 + \log 9 + 2\Lambda)$. As the sum F can be controlled on all arcs without the help of other generating functions (§9), only $o(k)$ further variables of the same kind are needed for the circle method, and this gives $G(k) \leqslant (C_0 + o(1))k$ with $C_0 = (2 + \log 9) + 2\Lambda$.

In reality only steps with $\theta \leqslant \Theta$ can be used, for some Θ which in Theorem 2.3 must be smaller than $k^{1/2}$ because of the rounding of moduli. Then $-z\mathcal{G}'(z) = 2/\Theta$ for small z, so that $\mathcal{G}(z)$ grows like $(2/\Theta)\log(1/z)$ and is unbounded. However, on the extreme minor arcs, to which the other arcs are reduced by rescaling, t factors F save a factor $P^{t\tau_k}$, by the bound of Weyl type, and to exploit this one only needs $k\lambda\,\mathcal{G}(\lambda) \leqslant r\lambda + b$ for $0 < \lambda \leqslant 1$, with some $b < t\tau_k$. This is the origin of the terms $-2T(\Theta)$ and $W_k(\Theta)$: the former is the part of the integral above that is saved by stopping at $\sigma(\Theta)$, and the latter is what the tail of $\mathcal{G}$ costs, in units of k variables, after an optimal choice of b and t. The outcome is simply that $r + t$ must exceed $k\,\mathcal{G}(\tau_k)$, when τ_k lies in the range $z \leqslant \frac{1}{2}\sigma(\Theta)$ in which $-z\mathcal{G}'(z) = 2/\Theta$: levels $|F| > P^{1-\tau_k}$ do not occur on the extreme minor arcs, so the potential is needed only for $z \geqslant \tau_k$. (When τ_k is larger than $\frac{1}{2}\sigma(\Theta)$, the bound of Theorem 2.3 is k times the value at τ_k of the linear function of $\log z$ that coincides with $\mathcal{G}$ for $z \leqslant \frac{1}{2}\sigma(\Theta)$; this is larger than $k\,\mathcal{G}(\tau_k)$.) Thus the exponent of the bound of Weyl type enters the number of variables only through the value of $\mathcal{G}$ at that exponent; see §11.4.

3. Setup

In [10] the variables are products of primes taken from a large number of disjoint intervals of increasing size, one from each, multiplied by a smooth number. We use a set with less structure: the integers $mp_1 \cdots p_J \leqslant P$ in which $p_1 < \cdots < p_J$ are primes from a single interval $(R, \zeta R]$, where $R^{J+1} = P$, and $m \leqslant R$ is an arbitrary natural number. In the simplest case $\zeta = 2$. In general $(R, \zeta R]$ is a union of dyadic intervals, and when their number is proportional to $\log P$ the set has positive density; see §3.2.

3.1. **Parameters and sets.** Throughout §§3–9 we fix an integer $k \geqslant 37$ and a number Θ that is admissible for k. The parameters are chosen in the following order. First

$$\begin{gathered} 0 < \delta_0 < \min\{\tfrac{1}{40}, \tfrac{1}{2}(1 - \eta_k(\Theta))\}, \qquad 0 < \varepsilon_2 \leqslant \min\{\tfrac{1}{100}, \delta_0\}, \\ 0 < \gamma \leqslant \frac{1}{100\,k}, \qquad 0 < \varepsilon \leqslant \tfrac{1}{2}\delta_0\,\sigma(\Theta). \end{gathered} \tag{3.1}$$

The numbers γ and ε will be specified in §9, in terms of k, Θ, δ_0, ε_2 and the number of variables; see (9.2). In §§3–8 they are arbitrary numbers satisfying (3.1). Next, J is a natural number that is sufficiently large in terms of k, Θ, δ_0, ε_2, γ and ε, and

$$\eta := \frac{1}{J+1}, \qquad \text{where in particular} \quad \eta \leqslant \frac{3\gamma\,\varepsilon_2}{k}; \qquad 0 < \omega \leqslant \frac{\eta}{8}. \tag{3.2}$$

The number ω will be specified in (9.3); in §§3–8 it is arbitrary. Finally, P is a real number that is sufficiently large in terms of all the preceding parameters, and

$$R := P^{\eta}, \qquad H \in \mathbb{N} \ \text{ with } \ 1 \leqslant H \leqslant \frac{\omega \log R}{\log 2}, \qquad \zeta := 2^H.$$

The integer H may depend on P. We put

$$\mathcal{P}_h := \{p :\ 2^h R < p \leqslant 2^{h+1}R\} \quad (0 \leqslant h < H), \qquad \mathcal{P} := \bigcup_{0 \leqslant h < H} \mathcal{P}_h = \{p :\ R < p \leqslant \zeta R\}.$$

Then

$$R^{J+1} = P, \qquad 2 \leqslant \zeta \leqslant R^{\omega}, \qquad \zeta^J \leqslant \min\{R^{1/8}, P^{\omega}\}, \qquad \sum_{p \in \mathcal{P}} \frac{1}{p} \leqslant 1, \tag{3.3}$$

and every prime in $\mathcal{P}$ exceeds k. Indeed $\zeta^J \leqslant R^{J\omega}$, where $J\omega < \frac{1}{8}$ and $R^J < P$; and by Mertens' theorem

$$\sum_{p\in\mathcal{P}} \frac{1}{p} = \log\frac{\log \zeta R}{\log R} + O\Big(\frac{1}{\log R}\Big) \leqslant \log(1+\omega) + O\Big(\frac{1}{\log R}\Big) < 1$$

when P is large.

For a natural number d let $[d]$ denote the set of its prime divisors. Let $\mathfrak{e}$ be a finite set of primes (the *excluded* primes), and let $0 \leqslant j \leqslant J$. We write $\mathcal{D}_j^{\mathfrak{e}}$ for the set of all products of j distinct primes from $\mathcal{P} \setminus \mathfrak{e}$, so that $\mathcal{D}_0^{\mathfrak{e}} = \{1\}$, and for real $X \geqslant 1$ we put

$$\begin{aligned} \mathcal{C}_j^{\mathfrak{e}}(X) &:= \big\{dm :\ d \in \mathcal{D}_j^{\mathfrak{e}},\ m \in \mathbb{N},\ m \leqslant R,\ dm \leqslant X\big\}, \\ F_j^{\mathfrak{e}}(\alpha; X) &:= \sum_{n\in\mathcal{C}_j^{\mathfrak{e}}(X)} e(\alpha n^k). \end{aligned} \tag{3.4}$$

When $\mathfrak{e} = \emptyset$ we omit it from the notation. In particular $\mathcal{D}_j := \mathcal{D}_j^{\emptyset}$, and

$$\mathcal{C} := \mathcal{C}_J(P), \qquad F(\alpha) := F_J(\alpha; P) = \sum_{n\in\mathcal{C}} e(\alpha n^k).$$

Thus $\mathcal{C}$ is the set of the integers $n = mp_1\cdots p_J \leqslant P$ with primes $R < p_1 < \cdots < p_J \leqslant \zeta R$ and a natural number m; the condition $m \leqslant R$ is automatic here, as $p_1\cdots p_J > R^J = P/R$. We say that X is *attached to* j if

$$\zeta^{j-J} R^{j+1} \leqslant X \leqslant R^{j+1};$$

thus P is attached to J. For $W > 0$ let

$$D_j(X, W) := \max_{\mathfrak{e}} X^k \operatorname{meas}\{\alpha \in \mathbb{T} :\ |F_j^{\mathfrak{e}}(\alpha; X)| \geqslant X/W\}, \qquad K_0 := 2^{J+1},$$

where the maximum is over all finite sets $\mathfrak{e}$ of primes; it exists because $F_j^{\mathfrak{e}}$ depends only on $\mathfrak{e} \cap \mathcal{P}$. The excluded primes are needed because the primes $p_1, \dots, p_J$ are distinct: when a divisor d of n is split off, the cofactor has to avoid the primes dividing d.

Lemma 3.1. *Let $0 \leqslant j \leqslant J$, let $\mathfrak{e}$ be a finite set of primes, and let $X = P^x$ be attached to j.*

(a) *One has $(j + \frac{7}{8})\eta \leqslant x \leqslant (j+1)\eta$.*

(b) *Every $n \in \mathcal{C}_j^{\mathfrak{e}}(X)$ has exactly one representation $n = dm$ with $d \in \mathcal{D}_j^{\mathfrak{e}}$ and $m \leqslant R$. For $d \in \mathcal{D}_j^{\mathfrak{e}}$ one has*

$$R^j \leqslant d \leqslant (\zeta R)^j \qquad \text{and} \qquad \zeta^{-J} R \leqslant \frac{X}{d} \leqslant R.$$

Consequently

$$F_j^{\mathfrak{e}}(\alpha; X) = \sum_{d\in\mathcal{D}_j^{\mathfrak{e}}} \sum_{m\leqslant X/d} e\big(\alpha(dm)^k\big), \qquad \mathcal{C}_j^{\mathfrak{e}}(X) \subseteq \mathcal{A}(X, \zeta R), \qquad |F_j^{\mathfrak{e}}(\alpha; X)| \leqslant X,$$

and $D_j(X, W) = 0$ for $W < 1$.

(c) *Let $1 \leqslant i \leqslant j$. For $d \in \mathcal{D}_i^{\mathfrak{e}}$ the number X/d is attached to $j - i$, and*

$$\binom{j}{i} F_j^{\mathfrak{e}}(\alpha; X) = \sum_{d\in\mathcal{D}_i^{\mathfrak{e}}} F_{j-i}^{\mathfrak{e}\cup[d]}\big(d^k\alpha; X/d\big), \qquad \sum_{d\in\mathcal{D}_i^{\mathfrak{e}}} \frac{1}{d} \leqslant 1.$$

(d) *Let $0 \leqslant i \leqslant j$ and $d \in \mathcal{D}_i^{\mathfrak{e}}$. Then*

$$\sum_{\substack{n \in \mathcal{C}_j^{\mathfrak{e}}(X) \\ d \mid n}} e(\alpha n^k) = F_{j-i}^{\mathfrak{e}\cup[d]}\big(d^k\alpha; X/d\big) \qquad \text{and} \qquad \sum_{\substack{n \in \mathcal{C}_j^{\mathfrak{e}}(X) \\ (n,d)=1}} e(\alpha n^k) = F_j^{\mathfrak{e}\cup[d]}(\alpha; X).$$

Proof. (a) This is the definition, since $R = P^\eta$ and $1 \leqslant \zeta^{J-j} \leqslant \zeta^J \leqslant R^{1/8}$ by (3.3).

(b) Let $n = dm$ with $d \in \mathcal{D}_j^{\mathfrak{e}}$ and $m \leqslant R$. Every prime factor of d exceeds R, and no prime factor of m does. So d is the product of the prime factors of n that exceed R, and $m = n/d$. The bounds for d are clear, as d is a product of j primes from $(R, \zeta R]$, and they give $X/d \leqslant R^{j+1}/R^j$ and $X/d \geqslant \zeta^{j-J}R^{j+1}/(\zeta R)^j$. Hence the condition $m \leqslant R$ in (3.4) follows from $dm \leqslant X$, which gives the formula for $F_j^{\mathfrak{e}}$. The remaining assertions are clear, as $\mathcal{C}_j^{\mathfrak{e}}(X) \subseteq [1, X]$.

(c) We have $X/d \leqslant R^{j+1}/R^i$ and $X/d \geqslant \zeta^{j-J}R^{j+1}/(\zeta R)^i$, so that X/d is attached to $j - i$. Let $n = d'm \in \mathcal{C}_j^{\mathfrak{e}}(X)$, with $d' \in \mathcal{D}_j^{\mathfrak{e}}$ and $m \leqslant R$. Exactly $\binom{j}{i}$ divisors d of d' belong to $\mathcal{D}_i^{\mathfrak{e}}$, namely the products of i of the j prime factors of d', and for each of them $n/d = (d'/d)\, m$ belongs to $\mathcal{C}_{j-i}^{\mathfrak{e}\cup[d]}(X/d)$. Conversely, if $d \in \mathcal{D}_i^{\mathfrak{e}}$ and $n' = d''m \in \mathcal{C}_{j-i}^{\mathfrak{e}\cup[d]}(X/d)$, with $d'' \in \mathcal{D}_{j-i}^{\mathfrak{e}\cup[d]}$ and $m \leqslant R$, then $dd'' \in \mathcal{D}_j^{\mathfrak{e}}$, because d'' is coprime to d, and so $dn' \in \mathcal{C}_j^{\mathfrak{e}}(X)$. Thus every $n \in \mathcal{C}_j^{\mathfrak{e}}(X)$ arises from exactly $\binom{j}{i}$ pairs (d, n'), which is the identity. Finally $\sum_{d \in \mathcal{D}_i^{\mathfrak{e}}} d^{-1} \leqslant \frac{1}{i!}\big(\sum_{p \in \mathcal{P}} p^{-1}\big)^i \leqslant 1$ by (3.3).

(d) Let $n = d'm \in \mathcal{C}_j^{\mathfrak{e}}(X)$ be as before. Since $m \leqslant R$ has no prime factor in $\mathcal{P}$, we have $d \mid n$ if and only if $d \mid d'$, and $(n, d) = 1$ if and only if $(d', d) = 1$. The first condition holds if and only if $n = dn'$ with $n' = (d'/d)\, m \in \mathcal{C}_{j-i}^{\mathfrak{e}\cup[d]}(X/d)$, and the second if and only if $d' \in \mathcal{D}_j^{\mathfrak{e}\cup[d]}$. □

Convention. From now on, implied constants, and the constants $C_1, C_2, \dots$ and C_*, may depend on k, Θ, δ_0, ε_2, γ, ε, J and ω, but never on P, on H, on j and the excluded primes, or on X, W and α. In §4 the dependence is stated in each lemma. In §§4 and 5, which do not depend on the present section, the letter m denotes a squarefree modulus. Everywhere else m is the unrestricted factor of the integers in $\mathcal{C}_j^{\mathfrak{e}}(X)$, and the moduli to which §§4 and 5 are applied are the numbers $d \in \mathcal{D}_i$.

3.2. **The density of the set, and unions of dyadic intervals.** Let

$$\Pi := \sum_{d \in \mathcal{D}_J} \frac{1}{d}.$$

By Lemma 3.1(c), $\Pi \leqslant 1$. This number is the density of $\mathcal{C}$ in $[1, P]$, and it is the factor by which the major arc approximation to F differs from that of a classical Weyl sum (Lemma 9.10).

Lemma 3.2. *One has $\#\mathcal{C} = \Pi\, P + O\big(PR^{-7/8}\big)$, and*

$$\frac{1}{2\,J!}\Big(\frac{H}{3\log R}\Big)^J \leqslant \Pi \leqslant \frac{1}{J!}\Big(\frac{2H}{\log R}\Big)^J.$$

Proof. By Lemma 3.1(b),

$$\#\mathcal{C} = \sum_{d \in \mathcal{D}_J} \lfloor P/d \rfloor = \Pi\, P + O(\#\mathcal{D}_J),$$

and $\#\mathcal{D}_J \leqslant (\zeta R)^J \leqslant PR^{-7/8}$ by (3.3). Let $S_1 := \sum_{p\in\mathcal{P}} p^{-1}$ and $S_2 := \sum_{p\in\mathcal{P}} p^{-2} \leqslant 2/R$. The number $J!\,\Pi$ is the sum of $(p_1\cdots p_J)^{-1}$ over all J-tuples of distinct primes from $\mathcal{P}$. The sum over all J-tuples is S_1^J, and those with $p_a = p_b$ for a given pair $a < b$ contribute at most $S_2 S_1^{J-2}$. Hence

$$S_1^J - \binom{J}{2} S_2 S_1^{J-2} \leqslant J!\,\Pi \leqslant S_1^J.$$

By the prime number theorem, the number of primes in $(y, 2y]$ is $(1 + o(1))\, y/\log y$ as $y \to \infty$, so that the sum of p^{-1} over these primes lies between $(\frac{1}{2} + o(1))/\log y$ and $(1 + o(1))/\log y$. We apply this with $y = 2^h R$ for $0 \leqslant h < H$, noting that $\log R \leqslant \log(2^h R) \leqslant (1 + \omega)\log R$. For large P it follows that

$$\frac{H}{3\log R} \leqslant S_1 \leqslant \frac{2H}{\log R},$$

and then $\binom{J}{2} S_2 S_1^{-2} \leqslant 9J^2(\log R)^2/R \leqslant \frac{1}{2}$. This gives the lemma. □

When $H = 1$ the primes are taken from the single dyadic interval $(R, 2R]$. This is the simplest set of variables to which the method applies. Its density Π is of order $(\log P)^{-J}$: each of the J prime variables costs a factor $\log P$, and the number of representations counted in §9.5 is then of order $P^{s-k}(\log P)^{-Js}$. For the bound on $G(k)$ this is of no importance.

The loss is recovered by taking a union over dyadic ranges whose number is proportional to $\log P$. For $0 \leqslant h_1 \leqslant \ldots \leqslant h_J < H$ let $\mathcal{C}(h_1, \ldots, h_J)$ be the set of the integers $mp_1\cdots p_J \leqslant P$ with $p_1 < \cdots < p_J$ and $p_l \in \mathcal{P}_{h_l}$ for $1 \leqslant l \leqslant J$. Each of these sets is of the same nature as the set $\mathcal{C}$ that belongs to $H = 1$, with the interval $(R, 2R]$ replaced by intervals $(2^{h_l}R, 2^{h_l+1}R]$, and $\mathcal{C}$ is their disjoint union. There are $\binom{H+J-1}{J} \asymp H^J$ of them, and Lemma 3.2 shows that $\Pi \asymp (H/\log P)^J$. In particular

$$\Pi \asymp 1 \qquad \text{when} \quad H = \Big\lfloor \frac{\omega \log R}{\log 2} \Big\rfloor,$$

so that $\mathcal{C}$ then has positive density, and the number of representations counted in §9.5 has the order of magnitude P^{s-k} that is expected for unrestricted variables. For this the primes $p_1, \ldots, p_J$ must be allowed to come from different dyadic intervals. The sets $\mathcal{C}(h, \ldots, h)$ with $0 \leqslant h < H$, in which all J primes lie in the same interval $(2^h R, 2^{h+1}R]$, are the sets of the case $H = 1$ with $2^h R$ in place of R, as far as the primes are concerned; their union has $\asymp HP(\log P)^{-J}$ elements only, which recovers one of the J logarithms.

The device of letting the prime factors range over a union of dyadic intervals, in a set of integers that have in addition an unrestricted factor, is adapted from the work of Brüdern and Wooley on sums of cubes of smooth numbers [5, §5]. There the variables are the integers lm in which $1 \leqslant l \leqslant \sqrt{R}$ and all prime factors of m lie in $(\sqrt{R}, R]$; the factor l, like our factor m, makes the major arcs accessible to the classical theory of Weyl sums. Everything that follows is uniform in H, because the argument uses only that the primes lie in $(R, \zeta R]$ and that ζ^J is a small power of R; the case $H = 1$ is in no way simpler.

4. Exponential sums to moduli dividing m^k

The results of this section do not depend on the choices made in §3: here $k \geqslant 3$ and $J \geqslant 1$ are arbitrary integers, $X \geqslant 2$ is real, and m is a squarefree natural number with at

most J prime factors, all exceeding k. We write

$$\vartheta := \frac{\log m}{\log X}.$$

Further q is a divisor of m^k with $q > X$, $U := \log q/\log X$ (so that $1 < U \leqslant k\vartheta$), $e_p := v_p(q)$ for $p \mid m$, a is an integer coprime to q, and

$$\mathcal{S} := \sum_{\substack{n\leqslant X\\(n,m)=1}} e\Big(\frac{an^k}{q}\Big).$$

Implied constants in this section depend at most on k, J and, where it occurs, ε. In [10, Lemma 3.1], the sum $\mathcal{S}$ is bounded by splitting it into residue classes modulo a suitable divisor A of a power of m and applying a Weyl-type estimate [3, Theorem 5] to each class. We use two other devices. The first is van der Corput differencing with shifts that are multiples of divisors of q, followed by completion; this is in the spirit of the q-analogue of van der Corput's method [21, 14] and of its p-adic versions [25]. The second also splits into residue classes modulo A, as in the work of Postnikov and Gallagher [13] on character sums to powerful moduli, but applies Vinogradov's mean value theorem to all classes simultaneously. The second device is the better one for large U; see §12.1 for a comparison with the method of [10].

Lemma 4.1 (local sums)**.** *Let $p > k$ be prime, $e \geqslant 1$, $2 \leqslant d \leqslant k$ and $\ell \in \mathbb{Z}$, and let $f \in \mathbb{Z}[t]$ be a polynomial all of whose coefficients agree modulo p with those of ct^d, where $p \nmid c$. Then*

$$\Big|\sum_{\substack{n \bmod p^e\\ p\nmid n}} e\Big(\frac{f(n)+\ell n}{p^e}\Big)\Big| \leqslant d\,p^{e/2}.$$

Proof. Put $f_\ell := f + \ell t$. Then $f_\ell' \equiv cd\,t^{d-1} + \ell$ and $f_\ell'' \equiv cd(d-1)t^{d-2} \pmod p$, and $p \nmid cd(d-1)$ because $2 \leqslant d \leqslant k < p$. Hence $f_\ell''(y) \not\equiv 0 \pmod p$ whenever $p \nmid y$, and the congruence $f_\ell'(y) \equiv 0 \pmod p$ has at most $d-1$ solutions with $p \nmid y$, all of them simple.

For $e = 1$ the reduction of f_ℓ modulo p has degree d with $2 \leqslant d < p$, so the complete sum $\sum_{n \bmod p} e(f_\ell(n)/p)$ is at most $(d-1)\sqrt{p}$ in modulus by Weil's bound [37]; removing the term $n = 0$ costs at most 1.

Let $e \geqslant 2$ and $u := \lfloor e/2 \rfloor$. Write $n = y + p^u t$ with y modulo p^u, $p \nmid y$, and t modulo p^{e-u}. As $3u \geqslant e$, Taylor's formula gives $f_\ell(n) \equiv f_\ell(y) + f_\ell'(y)p^u t + \frac12 f_\ell''(y)p^{2u}t^2 \pmod{p^e}$. If $e = 2u$, the sum over t is p^u when $p^u \mid f_\ell'(y)$ and 0 otherwise. If $e = 2u+1$, write $t = t_0 + pt_1$ with t_0 modulo p and t_1 modulo p^u. The sum over t_1 vanishes unless $p^u \mid f_\ell'(y)$, and if $f_\ell'(y) = p^u w$ then the sum over t equals p^u times the quadratic Gauss sum $\sum_{t_0 \bmod p} e\big((wt_0 + \frac12 f_\ell''(y)t_0^2)/p\big)$, which has modulus $\sqrt{p}$ since p is odd and $p \nmid f_\ell''(y)$. By Hensel's lemma there are at most $d-1$ residues y modulo p^u with $p \nmid y$ and $p^u \mid f_\ell'(y)$. In both cases the sum in question is at most $(d-1)p^{e/2}$ in modulus. □

Lemma 4.2 (completion)**.** *Let $q_1 \mid m^k$, $(a_1, q_1) = 1$ and $2 \leqslant d \leqslant k$, and let $f \in \mathbb{Z}[t]$ be such that for every prime $p \mid q_1$ the coefficients of f agree modulo p with those of $c_p t^d$,*

where $p \nmid c_p$. Then for every interval $\mathcal{I}$ of length at most X,

$$\sum_{\substack{n\in\mathcal{I}\\(n,m)=1}} e\Big(\frac{a_1 f(n)}{q_1}\Big) \ll Xq_1^{-1/2} + q_1^{1/2}\log(2q_1m).$$

Proof. Let m_0 be the product of the primes dividing m but not q_1, and $q_2 := q_1m_0$. The function $\psi(n) := e(a_1f(n)/q_1)\mathbf{1}_{(n,m)=1}$ has period q_2, so that $\psi(n) = \sum_{\ell \bmod q_2} \widehat{\psi}(\ell)e(\ell n/q_2)$ with $\widehat{\psi}(\ell) = q_2^{-1}\sum_{n \bmod q_2}\psi(n)e(-\ell n/q_2)$. By the Chinese remainder theorem, $q_2\widehat{\psi}(\ell)$ is a product of local sums, one for each prime $p \mid m$. For $p \mid q_1$ the local sum is of the type treated in Lemma 4.1, with modulus p^e, $e = v_p(q_1)$, and with a multiple of f by an integer prime to p in place of f; it is at most $kp^{e/2}$ in modulus. For $p \mid m_0$ it is the Ramanujan sum $\sum_{n \bmod p,\, p\nmid n} e(\ell' n/p)$ with $(\ell', p) = (\ell, p)$, which is at most (ℓ, p) in modulus. Hence $|\widehat{\psi}(\ell)| \leqslant k^J q_1^{1/2}(\ell, m_0)/q_2$. Since $|\sum_{n\in\mathcal{I}} e(\ell n/q_2)| \leqslant \min\{X+1, q_2/(2|\ell|)\}$ for $|\ell| \leqslant q_2/2$, the sum in question is at most

$$k^J q_1^{1/2}\Big(\frac{X+1}{q_1} + \sum_{1\leqslant\ell\leqslant q_2/2}\frac{(\ell,m_0)}{\ell}\Big) \leqslant k^J\big((X+1)q_1^{-1/2} + 2^J q_1^{1/2}(1+\log q_2)\big),$$

because $\sum_{\ell\leqslant q_2/2}(\ell,m_0)/\ell \leqslant \sum_{g\mid m_0}\sum_{\ell'\leqslant q_2/(2g)} 1/\ell'$ and m_0 has at most 2^J divisors. This gives the lemma, since $X+1 \leqslant 2X$ and $q_2 \leqslant q_1m$. □

We shall use the following form of the Weyl–van der Corput inequality. Let $(\xi_n)_{n\in\mathbb{Z}}$ be complex numbers with $\xi_n = 0$ for $n \notin [1, X]$, and let $A, Y \in \mathbb{N}$ with $AY \leqslant X$. Then

$$\Big|\sum_n \xi_n\Big|^2 \leqslant \frac{2X}{Y}\sum_{|y|<Y}\Big|\sum_n \xi_{n+yA}\overline{\xi_n}\Big|. \tag{4.1}$$

Indeed, $Y\sum_n\xi_n = \sum_n\sum_{0\leqslant y<Y}\xi_{n+yA}$, where n runs over the integers in $(-AY, X]$, of which there are at most $2X$. By Cauchy's inequality

$$Y^2\Big|\sum_n\xi_n\Big|^2 \leqslant 2X\sum_n\Big|\sum_{0\leqslant y<Y}\xi_{n+yA}\Big|^2 = 2X\sum_{|y|<Y}(Y-|y|)\sum_n\xi_{n+yA}\overline{\xi_n},$$

which gives (4.1).

Lemma 4.3 (differencing)**.** (i) *One has $\mathcal{S} \ll q^{1/2}\log(2qm)$.*

(ii) *Let $1 \leqslant j \leqslant k-2$ and $U \geqslant j + 2^{-j}$, and put $\varrho := (j+2-U)/(2^{j+2}-2)$. If $\varrho \geqslant \vartheta$, then*

$$\mathcal{S} \ll X^{1-\varrho+\vartheta/2}\log(2qm).$$

Proof. (i) is Lemma 4.2 with $q_1 = q$, $a_1 = a$ and $f = t^k$, since $X < q$.

(ii) Note that $\varrho \geqslant \vartheta > 0$. The hypothesis $U \geqslant j + 2^{-j}$ is equivalent to $2^{j+1}\varrho \leqslant 1$, so the numbers $\mu_i := (1-2^i\varrho)/U$ $(1 \leqslant i \leqslant j)$ are positive. Put

$$s_{i,p} := \max\{1, \lceil \mu_i e_p\rceil\}, \qquad A_i := \prod_{p\mid m} p^{s_{i,p}}, \qquad Y_i := \lfloor X/A_i\rfloor.$$

Then $A_i \leqslant q^{\mu_i}m = X^{1-2^i\varrho+\vartheta} \leqslant X^{1-\varrho}$, so that $Y_i \geqslant 1$ and $Y_i \geqslant \frac{1}{2}X^{2^i\varrho-\vartheta}$. Every prime dividing m divides every A_i; hence a shift $n \mapsto n + yA_i$ preserves the condition $(n, m) = 1$.

For $0 \leqslant i \leqslant j$ and a tuple $\mathbf{y} = (y_1, \dots, y_i)$ of integers let $\Delta_{\mathbf{y}}$ be the i-fold difference of an^k/q with the steps $y_1A_1, \dots, y_iA_i$, that is,

$$\Delta_{\mathbf{y}}(n) := \frac{a}{q} \sum_{\boldsymbol{\epsilon} \in \{0,1\}^i} (-1)^{i-\epsilon_1-\dots-\epsilon_i} \big(n + \epsilon_1 y_1 A_1 + \dots + \epsilon_i y_i A_i\big)^k,$$

let $\mathcal{I}_{\mathbf{y}}$ be the set of real numbers n such that $n + \epsilon_1 y_1 A_1 + \dots + \epsilon_i y_i A_i \in [1, X]$ for every $\boldsymbol{\epsilon} \in \{0,1\}^i$, which is an interval contained in $[1, X]$, and put

$$\mathcal{S}_{\mathbf{y}} := \sum_{\substack{n \in \mathcal{I}_{\mathbf{y}} \\ (n,m)=1}} e\big(\Delta_{\mathbf{y}}(n)\big);$$

thus $\mathcal{S}_{\mathbf{y}} = \mathcal{S}$ when $i = 0$. We show by induction on i that

$$|\mathcal{S}|^{2^i} \ll X^{2^i} \sum_{l=1}^{i} Y_l^{-2^{i-l}} + \frac{X^{2^i-1}}{Y_1 \cdots Y_i} \sum_{\substack{\mathbf{y}=(y_1,\dots,y_i) \\ 0<|y_l|<Y_l}} |\mathcal{S}_{\mathbf{y}}| \qquad (1 \leqslant i \leqslant j). \tag{4.2}$$

Let $1 \leqslant i \leqslant j$, and let $\mathbf{y}' = (y_1, \dots, y_{i-1})$. We apply (4.1), with $A = A_i$ and $Y = Y_i$, to the numbers $\xi_n := e(\Delta_{\mathbf{y}'}(n))$ for $n \in \mathcal{I}_{\mathbf{y}'}$ with $(n, m) = 1$, and $\xi_n := 0$ for all other n. Since $(n + y_iA_i, m) = (n, m)$, and since $\Delta_{\mathbf{y}'}(n + y_iA_i) - \Delta_{\mathbf{y}'}(n) = \Delta_{\mathbf{y}}(n)$ with $\mathbf{y} = (y_1, \dots, y_i)$, the inner sum on the right of (4.1) is $\mathcal{S}_{\mathbf{y}}$ when $y_i \neq 0$, and it is $\sum_n |\xi_n|^2 \leqslant X$ when $y_i = 0$. Hence

$$|\mathcal{S}_{\mathbf{y}'}|^2 \leqslant \frac{2X^2}{Y_i} + \frac{2X}{Y_i} \sum_{0<|y_i|<Y_i} |\mathcal{S}_{(y_1,\dots,y_i)}|.$$

For $i = 1$ this is (4.2). For $2 \leqslant i \leqslant j$ we square (4.2) for $i - 1$, apply Cauchy's inequality to the sum over the at most $2^{i-1}Y_1 \cdots Y_{i-1}$ tuples $\mathbf{y}'$, and insert the last display; this gives (4.2) for i. From now on let $i = j$ in (4.2), and let $\mathbf{y} = (y_1, \dots, y_j)$. Expanding $\Delta_{\mathbf{y}}$ by the multinomial theorem, only the monomials containing every step y_iA_i survive, so that

$$\Delta_{\mathbf{y}}(n) = \frac{a\, y_1 \cdots y_j\, A_1 \cdots A_j}{q}\, f_{\mathbf{y}}(n), \qquad f_{\mathbf{y}} \in \mathbb{Z}[t],$$

and the coefficients of $f_{\mathbf{y}}$ agree with those of $\frac{k!}{(k-j)!}\, t^{k-j}$ modulo every prime $p \mid m$, because every other term carries a further factor y_iA_i. Note that $p \nmid k!$ and $k - j \geqslant 2$. Let $q_A := q/(q, A_1 \cdots A_j)$. A prime dividing q_A does not divide $A_1 \cdots A_j/(q, A_1 \cdots A_j)$, so the reduced denominator of $ay_1 \cdots y_jA_1 \cdots A_j/q$ is $q_A/(q_A, y_1 \cdots y_j)$. Lemma 4.2 with $d = k - j$ gives

$$\mathcal{S}_{\mathbf{y}} \ll X(q_A, y_1 \cdots y_j)^{1/2} q_A^{-1/2} + q_A^{1/2} \log(2qm).$$

The number of $\mathbf{y}$ with $g \mid y_1 \cdots y_j$ is at most $d_j(g)2^jY_1 \cdots Y_j/g$, where $d_j(g)$ is the number of factorisations of g into j factors, and q_A has at most $(k + 1)^j$ divisors; hence $\sum_{\mathbf{y}}(q_A, y_1 \cdots y_j)^{1/2} \ll Y_1 \cdots Y_j$. Thus

$$|\mathcal{S}|^{2^j} \ll X^{2^j} \sum_{i=1}^{j} Y_i^{-2^j-i} + X^{2^j-1}\big(Xq_A^{-1/2} + q_A^{1/2} \log(2qm)\big).$$

Now $v_p(q_A) = \max\{e_p - \sum_i s_{i,p}, 0\}$ lies between $e_p(1-\sum_i \mu_i) - j$ and $e_p(1-\sum_i \mu_i)$, and

$$E := U\Big(1 - \sum_{i=1}^{j} \mu_i\Big) = U - j + (2^{j+1}-2)\varrho = 2 - 2^{j+1}\varrho \geqslant 1,$$

the last equality being the definition of ϱ. Consequently $X^{E-j\vartheta} \leqslant q_A \leqslant X^E$, and the three terms above are at most

$$2^{2^j} j X^{2^j(1-\varrho)+2^{j-1}\vartheta}, \qquad X^{2^j-1+2^j\varrho+j\vartheta/2}, \qquad X^{2^j(1-\varrho)}\log(2qm),$$

respectively. The middle one does not exceed $X^{2^j(1-\varrho)+2^{j-1}\vartheta}$, because $2^{j+1}\varrho \leqslant 1$ and $j \leqslant 2^j$. □

Lemma 4.4 (mean value over residue classes)**.** *Let D be an integer with $4 \leqslant D \leqslant k-1$ and $U/(D+1) + \vartheta < 1$. Then, for every $\varepsilon > 0$,*

$$\mathcal{S} \ll X^{1+\varepsilon-\varrho'}, \qquad \varrho' := \frac{1}{D(D+1)} \min\Big\{\frac{U}{D+1} - D\vartheta,\ D\Big(1 - \frac{U}{D+1} - \vartheta\Big)\Big\}.$$

Proof. Let $s_p := \max\{1, \lceil e_p/(D+1)\rceil\}$ and $A := \prod_{p|m} p^{s_p}$, so that $A \leqslant q^{1/(D+1)}m < X$. Every $n \leqslant X$ with $(n,m) = 1$ can be written uniquely as $n = n_0 + Ay$ with $1 \leqslant n_0 \leqslant A$, $(n_0, m) = 1$ and $0 \leqslant y < Y_{n_0}$, where $Y_{n_0} \leqslant Y := \lfloor X/A\rfloor + 1$. Since $(D+1)s_p \geqslant e_p$, the coefficient of y^i in $a(n_0+Ay)^k/q$ is an integer for $i > D$, so that

$$e\Big(\frac{a(n_0+Ay)^k}{q}\Big) = e\Big(\sum_{i=0}^{D} \alpha_i(n_0) y^i\Big), \qquad \alpha_i(n_0) := \frac{a}{q}\binom{k}{i} A^i n_0^{k-i}.$$

For $\boldsymbol{\beta} \in \mathbb{R}^D$ and $t \in \mathbb{N}$ let $g_t(\boldsymbol{\beta}) := \sum_{0\leqslant y<t} e(\beta_1 y + \cdots + \beta_D y^D)$ and $g^*(\boldsymbol{\beta}) := \max_{t\leqslant Y} |g_t(\boldsymbol{\beta})|$, and write $\boldsymbol{\alpha}(n_0) := (\alpha_1(n_0), \dots, \alpha_D(n_0))$. With $2w := D(D+1)$, Hölder's inequality gives

$$|\mathcal{S}|^{2w} \leqslant \Big(\sum_{n_0} g^*(\boldsymbol{\alpha}(n_0))\Big)^{2w} \leqslant A^{2w-1} \sum_{n_0} g^*(\boldsymbol{\alpha}(n_0))^{2w}. \tag{4.3}$$

Let $\mathcal{Q} := \prod_{i\leqslant D}[-Y^{-i}, Y^{-i}]$. If $\boldsymbol{\beta} \in \boldsymbol{\alpha} + \mathcal{Q}$, then $y \mapsto e\big(\sum_i (\alpha_i - \beta_i) y^i\big)$ has total variation at most $2\pi D$ on $[0, Y]$, since each y^i is monotone there, so that the polynomial $\sum_i (\alpha_i - \beta_i) y^i$ has total variation at most $\sum_i |\alpha_i - \beta_i| Y^i \leqslant D$ on $[0, Y]$; and partial summation gives $g^*(\boldsymbol{\alpha}) \leqslant (1 + 2\pi D) g^*(\boldsymbol{\beta})$. Averaging this over $\boldsymbol{\beta} \in \boldsymbol{\alpha}(n_0) + \mathcal{Q}$, summing over n_0, and using that g^* has period 1 in each variable, we find that (for given n_0, a point of $\mathbb{R}^D$ lies in at most 3^D of the sets $\boldsymbol{\alpha}(n_0) + \mathcal{Q} + \mathbf{m}$ with $\mathbf{m} \in \mathbb{Z}^D$, because the sides of $\mathcal{Q}$ have length at most 2)

$$\sum_{n_0} g^*(\boldsymbol{\alpha}(n_0))^{2w} \ll Y^{D(D+1)/2} \mathcal{N} \int_{\mathbb{T}^D} g^*(\boldsymbol{\beta})^{2w}\, \mathrm{d}\boldsymbol{\beta}, \tag{4.4}$$

where

$$\mathcal{N} := \sup_{\boldsymbol{\beta}\in\mathbb{R}^D} \#\{n_0 :\ \boldsymbol{\beta} \in \boldsymbol{\alpha}(n_0) + \mathcal{Q} + \mathbb{Z}^D\}.$$

Next, with $\mathbf{e}_1 := (1, 0, \dots, 0)$, the identity

$$g_t(\boldsymbol{\beta}) = \int_0^1 g_Y(\boldsymbol{\beta} + u\,\mathbf{e}_1) \sum_{0\leqslant y<t} e(-uy)\, \mathrm{d}u \qquad (t \leqslant Y),$$

the bound $|\sum_{0\leqslant y<t} e(-uy)| \leqslant \min\{Y, 1/(2\|u\|)\}$ and Hölder's inequality yield

$$\int_{\mathbb{T}^D} g^*(\boldsymbol{\beta})^{2w}\,\mathrm{d}\boldsymbol{\beta} \ll (\log 2Y)^{2w} \int_{\mathbb{T}^D} |g_Y(\boldsymbol{\beta})|^{2w}\,\mathrm{d}\boldsymbol{\beta} \ll Y^{w+\varepsilon},$$

the last bound being Vinogradov's mean value theorem in the form proved by Bourgain, Demeter and Guth [4] for degrees $D \geqslant 4$, and by Wooley [44], by means of nested efficient congruencing, for all degrees. As $D(D+1)/2 = w$, we infer from (4.3) and (4.4) that

$$|\mathcal{S}|^{2w} \ll A^{2w-1}\,\mathcal{N}\,Y^{2w+\varepsilon}. \tag{4.5}$$

We bound $\mathcal{N}$ using the top coefficient only. In lowest terms $\alpha_D(n_0) \equiv a_D\, n_0^{k-D}/q_D \pmod 1$ with $(a_D, q_D) = 1$ and $q_D := \prod_p p^{\max\{e_p - Ds_p, 0\}}$, because $p \nmid a\binom{k}{D}$. Moreover $q_D \mid A$, since $e_p - Ds_p \leqslant s_p$. As n_0 runs over $[1, A]$, each residue class modulo q_D is hit A/q_D times; the map $r \mapsto r^{k-D}$ on the reduced residues modulo q_D is at most k^J-to-one (note that $k - D \geqslant 1$ and that q_D has at most J prime factors, all odd); and at most $2q_DY^{-D} + 1$ residues x modulo q_D satisfy $\|x/q_D - \beta_D\| \leqslant Y^{-D}$. Therefore $\mathcal{N} \leqslant k^J(A/q_D)(2q_DY^{-D} + 1)$, and by (4.5)

$$\mathcal{S} \ll AY^{1+\varepsilon}(\mathcal{N}/A)^{1/(2w)} \ll X^{1+\varepsilon}\big(Y^{-D} + q_D^{-1}\big)^{1/(D(D+1))}.$$

Finally $Y \geqslant X/A \geqslant X^{1-U/(D+1)-\vartheta}$, and the inequality $e_p - Ds_p \geqslant e_p/(D+1) - D$ gives

$$q_D \geqslant q^{1/(D+1)} m^{-D} = X^{U/(D+1)-D\vartheta}. \qquad \square$$

The next lemma combines the preceding ones. Recall Definitions 2.1 and 2.2, and put

$$v_D := \frac{1}{D^2 + 3D + 1}, \qquad U_D := D(D+1)^2\, v_D \qquad (D \geqslant 6). \tag{4.6}$$

For $D \geqslant 5$ the two terms in the maximum defining ς on $[D, D+1]$ coincide at $U = U_{D+1}$, where their common value is v_{D+1}; the first is the larger one for $U \leqslant U_{D+1}$.

Lemma 4.5. *Let $k \geqslant 37$, let Θ be admissible for k, let $1 < \theta \leqslant \Theta$ and suppose that $m \leqslant X^{\theta/k}$. Then, for every $\varepsilon > 0$,*

$$\mathcal{S} \ll X^{1-(1-\eta_k(\theta))\varsigma(U)+\varepsilon} \leqslant X^{1-(1-\eta_k(\theta))\sigma(\theta)+\varepsilon}.$$

Proof. Here $\vartheta \leqslant \theta/k \leqslant \varpi(\theta)$ and $1 < U \leqslant k\vartheta \leqslant \theta$, which gives the second inequality. We claim that in each of the ranges of U listed in Table 2, the lemma indicated there gives $\mathcal{S} \ll X^{1-\varsigma(U)+\Delta+\varepsilon}$ with the value of Δ shown, and that $\Delta \leqslant \vartheta\,\kappa(\theta)\,\varsigma(U)$. This suffices, because $\vartheta\,\kappa(\theta) \leqslant \eta_k(\theta)$.

The exponents. In row 1, Lemma 4.3(i) gives $\mathcal{S} \ll X^{U/2}\log X$. In rows 2, 3, 4 and 6, the number ϱ of Lemma 4.3(ii) equals $\varsigma(U)$, the hypothesis $U \geqslant j + 2^{-j}$ holds, and $\Delta = \vartheta/2$. In Lemma 4.4 the minimum defining ϱ' is attained by its first term when $U \leqslant D$ and by its second term when $U \geqslant D$. Hence $\varrho' = \frac{U}{100} - \frac{\vartheta}{5}$ for $U \leqslant 4$ and $\varrho' = \frac{5-U}{25} - \frac{\vartheta}{5}$ for $U \geqslant 4$ when $D = 4$; $\varrho' = \frac{U}{180} - \frac{\vartheta}{6}$ when $D = 5$ and $U \leqslant 5$; and, for $D < U \leqslant D+1$, the lemma with D gives $\varrho' = (D+1-U)/(D+1)^2 - \vartheta/(D+1)$, while with $D+1$ in place of D it gives $\varrho' = U/((D+1)(D+2)^2) - \vartheta/(D+2)$. The factors $\log X$ and $\log(2qm) \ll \log X$ are absorbed in X^{ε}.

The lower bounds for ς. On each range ς is monotone or, in row 5, increasing and then decreasing; so the bounds are obtained by evaluating at the end points. In rows 8 and 9

TABLE 2. The cases in the proof of Lemma 4.5. In the last two rows $D \geqslant 5$ is the integer with $D < U \leqslant D+1$.

	range of U	lemma	$\varsigma(U)$	$\varsigma(U) \geqslant$	Δ	$\Delta/\varsigma(U) \leqslant$
1	$1 < U \leqslant \frac{3}{2}$	4.3(i)	$(2-U)/2$	$\frac{1}{4}$	0	0
2	$\frac{3}{2} < U \leqslant \frac{9}{4}$	4.3(ii), $j=1$	$(3-U)/6$	$\frac{1}{8}$	$\vartheta/2$	4ϑ
3	$\frac{9}{4} < U \leqslant \frac{25}{8}$	4.3(ii), $j=2$	$(4-U)/14$	$\frac{1}{16}$	$\vartheta/2$	8ϑ
4	$\frac{25}{8} < U \leqslant \frac{50}{13}$	4.3(ii), $j=3$	$(5-U)/30$	$\frac{1}{26}$	$\vartheta/2$	13ϑ
5	$\frac{50}{13} < U \leqslant \frac{160}{37}$	4.4, $D=4$	$\min\{\frac{U}{100}, \frac{5-U}{25}\}$	$\frac{1}{37}$	$\vartheta/5$	7.4ϑ
6	$\frac{160}{37} < U \leqslant \frac{540}{121}$	4.3(ii), $j=4$	$(6-U)/62$	$\frac{3}{121}$	$\vartheta/2$	20.2ϑ
7	$\frac{540}{121} < U \leqslant 5$	4.4, $D=5$	$U/180$	$\frac{3}{121}$	$\vartheta/6$	6.8ϑ
8	$D < U \leqslant U_{D+1}$	4.4 with D	$(D+1-U)/(D+1)^2$	v_{D+1}	$\vartheta/(D+1)$	$(U+4.2)\vartheta$
9	$U_{D+1} < U \leqslant D+1$	4.4 with $D+1$	$U/((D+1)(D+2)^2)$	v_{D+1}	$\vartheta/(D+2)$	$(U+4.2)\vartheta$

we use the remark following (4.6). The last column follows; in rows 8 and 9 note that $\Delta/\varsigma(U) \leqslant \vartheta(D^2+5D+5)/(D+1) \leqslant \vartheta(D+4.2)$.

Comparison with κ. A range contains a point $U \leqslant \theta$ only if θ exceeds its left end point. Thus rows 3 and 4 occur only if $\theta > \frac{9}{4}$ and $\theta > \frac{25}{8}$, when $\kappa(\theta) \geqslant 8$ and $\kappa(\theta) \geqslant 13$ respectively; row 5 only if $\theta > \frac{25}{8}$; and rows 6 to 9 only if $\theta > \frac{160}{37}$, when $\kappa(\theta) = \max\{21, \theta+5\}$ exceeds 20.2, 6.8 and $U+4.2$. In rows 1 and 2 we use $\kappa(\theta) \geqslant 4$. This proves that $\Delta \leqslant \vartheta\,\kappa(\theta)\,\varsigma(U)$ in all cases.

The side conditions. In Lemma 4.3(ii) we need $j \leqslant k-2$, which is clear, and $\varrho \geqslant \vartheta$. For $j=1$ this holds as $\varrho \geqslant \frac{1}{8}$ and $\vartheta \leqslant \frac{1}{16}$. For $j = 2, 3, 4$ we have $\varrho \geqslant \frac{1}{16}, \frac{1}{26}, \frac{3}{121}$, and the range contains a point $U \leqslant \theta$ only if $\theta > \frac{9}{4}, \frac{25}{8}, \frac{160}{37}$, in which case $\vartheta \leqslant \varpi(\theta)$ is at most $\frac{1}{16}, \frac{1}{26}, \frac{3}{121}$ respectively. In Lemma 4.4 the degree is at most $\theta+1 \leqslant k-1$, and we need $U/(D+1)+\vartheta < 1$. For $D=4$ and $D=5$ in rows 5 and 7 one has $1 - \frac{U}{D+1} \geqslant \frac{5}{37} > \frac{1}{26} \geqslant \vartheta$. In rows 8 and 9, $1 - \frac{U}{D+1} \geqslant (D+1)v_{D+1} \geqslant \frac{1}{\theta+5}$ and $1 - \frac{U}{D+2} \geqslant \frac{1}{D+2} \geqslant \frac{1}{\theta+5}$ respectively, while $\vartheta \leqslant \frac{\theta}{k} < \frac{1}{\theta+5}$ because $\theta(\theta+5) \leqslant \Theta(\Theta+5) < k$. □

5. Large values on a fibre

The following proposition is [10, Proposition 3.2], with the hypothesis on complete exponential sums isolated and with explicit constants. We include the proof.

The argument is of the type introduced by Halász and Montgomery in the theory of Dirichlet polynomials [15, 26]; see [27, Chapter 8]. In their method the number of points at which a Dirichlet polynomial $\sum_{n \leqslant N} a_n n^{-\mathrm{i}t}$ is large is estimated by duality, in terms of the bilinear form whose coefficients are the sums $\sum_{n \leqslant N} n^{-\mathrm{i}(t_r - t_{r'})}$ at the differences of the points, and the coefficients with $r \neq r'$ are estimated individually. Here the points lie on a fibre of the map $\alpha \mapsto m^k\alpha$, the coefficients are the entries of the Gram matrix K in (5.3) below, and these are complete exponential sums at the differences of the points, which are rational numbers with denominators dividing m^k. The entries with a denominator $q > X$ are small by hypothesis and are estimated individually, as in the classical case. Those with $q \leqslant X$ are not small, but they form a principal submatrix of a circulant matrix, the norm of which can be computed. As in the classical case, the outcome is a bound of the

quality of the large sieve for the number of large values, here $O(W^2)$, in a range of W that is limited by the individual estimates.

Proposition 5.1. *Let k, J, X and m be as in §4, and let $W \geqslant 1$. Suppose that there are numbers $\delta_F > 0$ and $C_F \geqslant 1$ such that*

$$\Big| \sum_{\substack{n \leqslant X \\ (n,m)=1}} e\Big(\frac{an^k}{q}\Big)\Big| \leqslant C_F X^{1-\delta_F} \quad \text{whenever } q \mid m^k,\ q > X \text{ and } (a,q)=1, \tag{5.1}$$

and that

$$8\, C_F W^2 \leqslant X^{\delta_F}. \tag{5.2}$$

Let $(c_n)_{n\leqslant X}$ be complex numbers with $\sum_n |c_n|^2 \leqslant X$. Then for every $\beta \in \mathbb{T}$ the number of $\alpha \in \mathbb{T}$ with

$$m^k\alpha = \beta \qquad \text{and} \qquad \Big| \sum_{\substack{n \leqslant X \\ (n,m)=1}} c_n e(\alpha n^k)\Big| \geqslant \frac{X}{2W}$$

is at most $C_1 W^2$, where $C_1 := 16\, k^J (2k+1)^J$.

Proof. Let $\beta_0 \in [0,1)$ represent β. The solutions of $m^k\alpha = \beta$ are $\alpha_j := (\beta_0 + j)/m^k$ with $j \in \mathbb{Z}/m^k\mathbb{Z}$. Let $\mathcal{H}$ be the set of j for which the sum in question is at least $X/(2W)$ in modulus. Let $\mathcal{U} := \{n \leqslant X : (n,m)=1\}$, and consider the matrix $V := \big(e(\alpha_j n^k)\big)_{j\in\mathcal{H},\, n\in\mathcal{U}}$ and the vector $\mathbf{c} := (c_n)_{n\in\mathcal{U}}$. Then

$$\frac{\#\mathcal{H}\cdot X^2}{4W^2} \leqslant \|V\mathbf{c}\|_2^2 \leqslant \|V\|_{\mathrm{op}}^2\, \|\mathbf{c}\|_2^2 \leqslant X\, \|VV^*\|_{\mathrm{op}}. \tag{5.3}$$

The Gram matrix $K := VV^*$ has the entries

$$K_{ij} = \sum_{n\in\mathcal{U}} e\big((\alpha_i - \alpha_j)n^k\big) = w(i-j), \qquad \text{where} \quad w(h) := \sum_{n\in\mathcal{U}} e\Big(\frac{hn^k}{m^k}\Big) \quad (h \in \mathbb{Z}/m^k\mathbb{Z}).$$

For $h \in \mathbb{Z}/m^k\mathbb{Z}$ let $q(h) := m^k/(h, m^k)$ be the denominator of h/m^k in lowest terms. We write $w = w_\leqslant + w_>$, where $w_\leqslant(h) := w(h)$ if $q(h) \leqslant X$ and $w_\leqslant(h) := 0$ otherwise, and correspondingly $K = K_\leqslant + K_>$. Since $w(-h) = \overline{w(h)}$ and $q(-h) = q(h)$, the matrices $K_\leqslant$ and $K_>$ are Hermitian.

Every entry of $K_>$ is either 0 or a sum as in (5.1), hence at most $C_F X^{1-\delta_F}$ in modulus. As $K_>$ is Hermitian, its operator norm is at most the largest ℓ^1-norm of a row, so

$$\|K_>\|_{\mathrm{op}} \leqslant \#\mathcal{H}\cdot C_F X^{1-\delta_F}. \tag{5.4}$$

The matrix $K_\leqslant$ is a principal submatrix of the circulant matrix $\widetilde{K} := \big(w_\leqslant(i-j)\big)_{i,j}$, with rows and columns indexed by $\mathbb{Z}/m^k\mathbb{Z}$, which is Hermitian for the same reason. Hence $\|K_\leqslant\|_{\mathrm{op}} \leqslant \|\widetilde{K}\|_{\mathrm{op}} = \max_t |\lambda_t|$, where the λ_t are the eigenvalues of $\widetilde{K}$. The vectors $(e(jt/m^k))_j$, $t \in \mathbb{Z}/m^k\mathbb{Z}$, form an orthogonal basis of eigenvectors, and

$$\lambda_t = \sum_{h \bmod m^k} w_\leqslant(h)\, e\Big(-\frac{ht}{m^k}\Big) = \sum_{n\in\mathcal{U}} \sum_{\substack{q \mid m^k \\ q\leqslant X}} \sum_{\substack{b \bmod q \\ (b,q)=1}} e\Big(\frac{b(n^k - t)}{q}\Big),$$

on sorting the h according to $q = q(h)$ and writing $h/m^k = b/q$. The innermost sum is Ramanujan's sum, which equals $\sum_{d|(q,\, n^k - t)} d\,\mu(q/d)$. Therefore

$$\lambda_t = \sum_{\substack{q|m^k\\ q\leqslant X}} \sum_{d|q} \mu(q/d)\, d\, \#\{n \in \mathcal{U} :\ n^k \equiv t \pmod d\}.$$

Let $d \mid m^k$ with $d \leqslant X$. Every prime factor of d divides m and is odd, and every $n \in \mathcal{U}$ is coprime to d. Since the group of reduced residues modulo an odd prime power is cyclic, the congruence $y^k \equiv t \pmod d$ has at most k^J solutions y modulo d with $(y,d) = 1$, and each residue class modulo d contains at most $X/d + 1$ integers $n \leqslant X$. Thus $d\,\#\{n \in \mathcal{U} : n^k \equiv t \pmod d\} \leqslant k^J(X + d) \leqslant 2k^J X$. The number of pairs (d, q) with $d \mid q \mid m^k$ and q/d squarefree is at most $(2k+1)^J$, since for each prime $p \mid m$ there are $2k + 1$ possibilities for $(v_p(d), v_p(q))$. It follows that

$$\|K_{\leqslant}\|_{\mathrm{op}} \leqslant \max_t |\lambda_t| \leqslant 2k^J(2k+1)^J X = \tfrac{1}{8} C_1 X. \tag{5.5}$$

Combining (5.3), (5.4) and (5.5) with (5.2), we obtain

$$\frac{\#\mathcal{H}}{4W^2} \leqslant \frac{C_1}{8} + \#\mathcal{H} \cdot C_F X^{-\delta_F} \leqslant \frac{C_1}{8} + \frac{\#\mathcal{H}}{8W^2},$$

that is, $\#\mathcal{H} \leqslant C_1 W^2$. □

If $m \leqslant X^{1/k}$ then no divisor of m^k exceeds X, the hypothesis (5.1) is void, and Proposition 5.1 is a large sieve inequality on the fibre. The point of [10] is that m may be taken much larger when W is small, because then a small saving δ_F suffices. With Lemma 4.5 we obtain the following.

Corollary 5.2. *Let $k \geqslant 37$, let Θ be admissible for k, and let $\varepsilon > 0$. There is a number $X_0 = X_0(k, J, \varepsilon)$ with the following property. Let $X \geqslant X_0$, let m be as in §4 with $m \leqslant X^{\theta/k}$ for some $\theta \in (1, \Theta]$, and let $W \geqslant 1$ with*

$$W^2 \leqslant X^{(1-\eta_k(\theta))\,\sigma(\theta) - 2\varepsilon}.$$

Then the conclusion of Proposition 5.1 holds for X, m and W.

Proof. By Lemma 4.5, the hypothesis (5.1) holds with $\delta_F := (1 - \eta_k(\theta))\sigma(\theta) - \varepsilon$ and with a constant $C_F \geqslant 1$ that depends only on k, J and ε; here $\delta_F \geqslant \varepsilon > 0$, since $W \geqslant 1$ and the hypothesis on W give $(1 - \eta_k(\theta))\sigma(\theta) \geqslant 2\varepsilon$. Then $8C_F W^2 \leqslant 8C_F X^{-\varepsilon} X^{\delta_F} \leqslant X^{\delta_F}$ as soon as $X^{\varepsilon} \geqslant 8C_F$, which is (5.2). □

6. The iterative inequality

We return to the setting of §3. The next lemma is a refinement of [10, Lemma 4.1], for the sums $F_j^{\mathfrak{e}}$.

Lemma 6.1. *Let $1 \leqslant i \leqslant j \leqslant J$, let X be attached to j and $W \geqslant 1$, and suppose that the conclusion of Proposition 5.1 holds for X, W and $m = d$, for every $d \in \mathcal{D}_i$. Put*

$N_W := \lceil \log_2(2W) \rceil$. *Then*

$$\begin{aligned} D_j(X,W) \leqslant 2C_1 N_W \sum_{l=0}^{N_W-1} \frac{W^3}{2^l} \sum_{d\in\mathcal{D}_i} \frac{D_{j-i}(X/d, 2^{l+1})}{d} \\ + \sum_{i_1=1}^{i} \sum_{d_1\in\mathcal{D}_{i_1}} d_1^k\, D_{j-i_1}\Big(\frac{X}{d_1}, \frac{K_0 W}{d_1}\Big). \end{aligned} \tag{6.1}$$

Proof. Fix a finite set $\mathfrak{e}$ of primes, and let $E_* := \{\alpha \in \mathbb{T} : |F_j^{\mathfrak{e}}(\alpha; X)| \geqslant X/W\}$; we have to bound $X^k \operatorname{meas}(E_*)$ by the right hand side of (6.1). For $d \in \mathcal{D}_i^{\mathfrak{e}}$ we have, by Lemma 3.1(d),

$$F_j^{\mathfrak{e}\cup[d]}(\alpha; X) = \sum_{\substack{n\in\mathcal{C}_j^{\mathfrak{e}}(X)\\(n,d)=1}} e(\alpha n^k).$$

We split E_* into the set E' of those α for which $|F_j^{\mathfrak{e}\cup[d]}(\alpha; X)| < X/(2W)$ for some $d \in \mathcal{D}_i^{\mathfrak{e}}$, and the complement $E := E_* \setminus E'$.

The set E'. Let $\alpha \in E'$, and let $d \in \mathcal{D}_i^{\mathfrak{e}}$ be such that $|F_j^{\mathfrak{e}\cup[d]}(\alpha; X)| < X/(2W)$. Then $|F_j^{\mathfrak{e}}(\alpha; X) - F_j^{\mathfrak{e}\cup[d]}(\alpha; X)| > X/(2W)$. An element of $\mathcal{C}_j^{\mathfrak{e}}(X)$ has a common factor with d if and only if it is divisible by one of the i primes dividing d. For a divisor d_1 of d let i_1 denote the number of its prime factors, so that $d_1 \in \mathcal{D}_{i_1}^{\mathfrak{e}}$. By inclusion and exclusion and Lemma 3.1(d),

$$F_j^{\mathfrak{e}}(\alpha; X) - F_j^{\mathfrak{e}\cup[d]}(\alpha; X) = \sum_{\substack{d_1\mid d\\ d_1>1}} (-1)^{i_1+1} \sum_{\substack{n\in\mathcal{C}_j^{\mathfrak{e}}(X)\\ d_1\mid n}} e(\alpha n^k) = \sum_{\substack{d_1\mid d\\ d_1>1}} (-1)^{i_1+1} F_{j-i_1}^{\mathfrak{e}\cup[d_1]}\big(d_1^k\alpha; X/d_1\big).$$

There are fewer than 2^J terms, so for some divisor $d_1 > 1$ of d we have

$$\big|F_{j-i_1}^{\mathfrak{e}\cup[d_1]}(d_1^k\alpha; X/d_1)\big| \geqslant \frac{X}{2^{J+1}W} = \frac{X/d_1}{K_0W/d_1}.$$

Here X/d_1 is attached to $j-i_1$, by Lemma 3.1(c), and the set of excluded primes depends on d_1 only. Multiplication by d_1^k preserves the measure on $\mathbb{T}$. Hence, by the union bound over all i_1 with $1 \leqslant i_1 \leqslant i$ and all $d_1 \in \mathcal{D}_{i_1}^{\mathfrak{e}}$,

$$X^k \operatorname{meas}(E') \leqslant \sum_{i_1=1}^{i} \sum_{d_1\in\mathcal{D}_{i_1}^{\mathfrak{e}}} X^k\Big(\frac{X}{d_1}\Big)^{-k} D_{j-i_1}\Big(\frac{X}{d_1}, \frac{K_0W}{d_1}\Big),$$

which is at most the second term in (6.1).

The set E. For $d \in \mathcal{D}_i^{\mathfrak{e}}$ let $u_d(\alpha) := \frac{d}{X}\big|F_{j-i}^{\mathfrak{e}\cup[d]}(d^k\alpha; X/d)\big|$, so that $0 \leqslant u_d(\alpha) \leqslant 1$ by Lemma 3.1(b) and (c). By Lemma 3.1(c), $\sum_d u_d(\alpha)/d \geqslant \binom{j}{i}/W \geqslant 1/W$ for $\alpha \in E_*$, and the d with $u_d(\alpha) \leqslant 2^{-N_W} \leqslant 1/(2W)$ contribute at most $1/(2W)$ to this sum. Sorting the other d according to $2^{-l-1} < u_d(\alpha) \leqslant 2^{-l}$ $(0 \leqslant l < N_W)$, we find for each $\alpha \in E$ an index $l < N_W$ with

$$\sum_{\substack{d\in\mathcal{D}_i^{\mathfrak{e}}\\ u_d(\alpha)>2^{-l-1}}} \frac{1}{d} \geqslant \frac{2^l}{2WN_W}. \tag{6.2}$$

Let E_l be the set of $\alpha \in E$ for which (6.2) holds. Integrating (6.2) over E_l gives

$$\frac{2^l}{2WN_W}\operatorname{meas}(E_l) \leqslant \sum_{d\in\mathcal{D}_i^{\mathfrak{e}}}\frac{1}{d}\operatorname{meas}\{\alpha\in E:\ u_d(\alpha)>2^{-l-1}\}. \tag{6.3}$$

Fix $d \in \mathcal{D}_i^{\mathfrak{e}}$. The condition $u_d(\alpha) > 2^{-l-1}$ depends only on $\beta := d^k\alpha$, and it implies that β lies in the set $\mathcal{L} := \{\beta \in \mathbb{T} : |F_{j-i}^{\mathfrak{e}\cup[d]}(\beta; X/d)| \geqslant (X/d)/2^{l+1}\}$, of measure at most $(X/d)^{-k}D_{j-i}(X/d, 2^{l+1})$. On the other hand, $\alpha \in E$ implies $|F_j^{\mathfrak{e}\cup[d]}(\alpha; X)| \geqslant X/(2W)$, and

$$F_j^{\mathfrak{e}\cup[d]}(\alpha;X) = \sum_{\substack{n\leqslant X\\(n,d)=1}} c_n e(\alpha n^k) \qquad \text{with } c_n \in \{0,1\},$$

where c_n is the indicator function of the set $\mathcal{C}_j^{\mathfrak{e}\cup[d]}(X)$, all of whose elements are coprime to d. By hypothesis, each $\beta \in \mathbb{T}$ therefore has at most C_1W^2 preimages in E under the map $\alpha \mapsto d^k\alpha$. This map multiplies lengths by d^k locally, so that the integral over $\beta \in \mathbb{T}$ of the number of preimages of β in a measurable set $\mathcal{M} \subseteq \mathbb{T}$ is $d^k \operatorname{meas}(\mathcal{M})$. For $\mathcal{M} = \{\alpha \in E :\ u_d(\alpha) > 2^{-l-1}\}$ this number is at most C_1W^2, and it vanishes when $\beta \notin \mathcal{L}$. Hence

$$\operatorname{meas}\{\alpha\in E:\ u_d(\alpha)>2^{-l-1}\} \leqslant \frac{C_1W^2}{d^k}\operatorname{meas}(\mathcal{L}) \leqslant C_1W^2X^{-k}D_{j-i}\Big(\frac{X}{d},2^{l+1}\Big).$$

Inserting this in (6.3) and summing $X^k \operatorname{meas}(E_l)$ over l gives the first term of (6.1). □

In [10] only the level $u_d \geqslant 1/(2W)$ is used, and one step costs a factor W^3. In (6.1) the cost is W^3/V when the shorter sums are of relative size $1/V$; since the bounds for $D_{j-i}(\cdot, V)$ that we shall use grow at least linearly in V (Lemma 8.1(ii)), the worst case is $V \approx W$, with cost W^2. The second term in (6.1) will turn out to be harmless, because of the factor d_1^{-k} gained from W/d_1. In the identity of Lemma 3.1(c) each integer is counted $\binom{j}{i}$ times; this helps, and we have not used it.

7. Large values and mean values over smooth numbers

For $t \geqslant 0$ let $\mathfrak{d}(t) \in (0, 1]$ denote the solution of the equation

$$\mathfrak{d}(t) + \log \mathfrak{d}(t) = 1 - t/k,$$

and let $\Psi(z) := 2 + \log \frac{1-z}{z}$ for $0 < z \leqslant \frac{1}{2}$. We need the following fact about mean values over sets of smooth numbers: for every real number $v \geqslant 0$ and every $\varepsilon' > 0$ there is $\eta' > 0$ such that

$$\int_{\mathbb{T}}\Big|\sum_{n\in\mathcal{K}} e(\alpha n^k)\Big|^v \,\mathrm{d}\alpha \ll Y^{v-k+k\mathfrak{d}(v)+\varepsilon'} \qquad \big(\mathcal{K}\subseteq\mathcal{A}(Y,Z),\ 2\leqslant Z\leqslant Y^{\eta'}\big), \tag{7.1}$$

where the implied constant depends on k, v and ε' only. For $v \geqslant 4$ this is contained in [9, Theorem 2.2]. That theorem concerns the sum over $\mathcal{A}(Y, Z)$ with arbitrary complex weights $w(n)$, and bounds its mean value of order v by $\max_{n\leqslant Y}|w(n)|^v$ times the right hand side of (7.1), uniformly in the weights; we take for w the indicator function of $\mathcal{K}$. (For even v and $\mathcal{K} = \mathcal{A}(Y, Z)$ the estimate goes back to [40]; see [6, §7].) For $0 \leqslant v \leqslant 4$ we appeal to Hua's lemma. The mean value of order 4 is the number of solutions of $n_1^k + n_2^k = n_3^k + n_4^k$ with $n_i \in \mathcal{K}$; this is at most the number of solutions in natural numbers

$n_i \leqslant Y$, which is $\ll Y^{2+\varepsilon'}$ by [32, Lemma 2.5]. By Hölder's inequality the left hand side of (7.1) is therefore $\ll Y^{v/2+\varepsilon'}$. This suffices, since $\frac{v}{2} \leqslant v-k+k\mathfrak{d}(v)$: the number $y := 1-\frac{v}{2k}$ lies in $(0,1]$ and satisfies $y + \log y \leqslant 2y - 1 = 1 - \frac{v}{k}$, whence $y \leqslant \mathfrak{d}(v)$.

Lemma 7.1. *Let $0 < z_* \leqslant \frac{1}{2}$. Suppose that J is sufficiently large in terms of k, γ, ε and z_*. Let $0 \leqslant j \leqslant J$, let $X = P^x$ be attached to j with $x \geqslant \gamma$, let $W = P^\lambda \geqslant 1$ and $z := \lambda/x$. Then $D_j(X,W) \leqslant X^k$, and if $z_* \leqslant z \leqslant \frac{1}{2}$ then*

$$D_j(X,W) \ll P^{\,k\lambda\Psi(z)+\varepsilon}.$$

Proof. The first bound is trivial. Let $\mathfrak{e}$ be a finite set of primes. By Lemma 3.1(b) the set $\mathcal{C}_j^{\mathfrak{e}}(X)$ is contained in $\mathcal{A}(X,\zeta R)$, and $\zeta R \leqslant P^{2\eta} \leqslant X^{\eta_1}$ with $\eta_1 := 2\eta/\gamma$, by (3.3) and because $x \geqslant \gamma$. Let $v \geqslant 0$ be a real number. By Chebyshev's inequality, and by (7.1) with $\mathcal{K} = \mathcal{C}_j^{\mathfrak{e}}(X)$, $Y = X$, $Z = \zeta R$ and $\varepsilon' = \frac{1}{2}\varepsilon$, if $\eta_1 \leqslant \eta'(v, \frac{1}{2}\varepsilon)$ then

$$D_j(X,W) \leqslant W^v X^{k-v} \max_{\mathfrak{e}} \int_{\mathbb{T}} |F_j^{\mathfrak{e}}(\alpha;X)|^v \,\mathrm{d}\alpha \ll W^v X^{k\mathfrak{d}(v)} P^{\varepsilon/2}.$$

Now let $z_* \leqslant z \leqslant \frac{1}{2}$. Put $\delta := z/(1-z) \leqslant 1$ and $t := k(1-\delta-\log\delta) \geqslant 0$, so that $\mathfrak{d}(t) = \delta$, and let v be the least integral multiple of $\frac{1}{2}\varepsilon$ with $v \geqslant t$. Then $\mathfrak{d}(v) \leqslant \delta$ and $v \leqslant t + \frac{1}{2}\varepsilon$, and $\lambda \leqslant x \leqslant 1$ by Lemma 3.1(a), so that

$$W^v X^{k\mathfrak{d}(v)} \leqslant P^{(t+\varepsilon/2)\lambda + k\delta x} = P^{k\lambda(1-\delta-\log\delta+\delta/z)+\varepsilon\lambda/2} \leqslant P^{k\lambda\Psi(z)+\varepsilon/2},$$

since $1-\delta+\delta/z = 2$ and $-\log\delta = \log\frac{1-z}{z}$. As $t \leqslant k(1+\log(1/z_*))$, the number v belongs to a finite set that depends on k, ε and z_* only, and $\eta_1 = 2\eta/\gamma$ is small enough for all of its elements when J is large. □

The choice of t in this proof is the optimal one: for fixed x and λ the function $v \mapsto v\lambda + k\mathfrak{d}(v)x$ is convex, and its derivative $\lambda - x\,\mathfrak{d}(v)/(1+\mathfrak{d}(v))$ vanishes at $v = t$. This is the place where it matters that (7.1) holds for all real orders v, and not only for even integers. Note that $k\lambda\Psi(z) = kx$ for $z = \frac{1}{2}$, so that the bound of the lemma then agrees with the trivial one up to the factor P^ε.

8. The potential and the main induction

We shall bound $D_j(P^x, P^\lambda)$ by $P^{\Phi(x,\lambda)}$, up to harmless factors, for a function Φ of the shape $k\lambda\,\mathcal{G}(\lambda/x)$ which is constructed along the lines of §2.2. The parameters k, Θ, γ, δ_0, ε_2, ε are those of §3.

8.1. **Definitions.** For $1 \leqslant \theta \leqslant \Theta$ let

$$M(\theta) := \tfrac{1}{2}\big(1 - \delta_0 - \eta_k(\theta)\big)\,\sigma(\theta).$$

By (2.2) and (3.1), $1 - \delta_0 - \eta_k(\theta) \geqslant 1 - \delta_0 - \eta_k(\Theta) > 0$. As η_k is increasing and left-continuous and σ is continuous and nonincreasing, the function M is positive, strictly decreasing and left-continuous; its only discontinuities are downward jumps at those of the points $\frac{9}{4}$, $\frac{25}{8}$, $\frac{160}{37}$ that lie in $[1,\Theta)$. Hence

$$\theta(z) := \max\{\theta \in [1,\Theta] :\ M(\theta) \geqslant z\} \quad (0 \leqslant z \leqslant M(1)), \qquad u_\Theta := M(\Theta),$$

is well defined and nonincreasing, $M(\theta(z)) \geqslant z$, and $\theta(z) = \Theta$ exactly when $z \leqslant u_\Theta$. Since $\sigma(\theta(z)) \geqslant \sigma(\Theta)$, we have

$$\big(1 - \eta_k(\theta)\big)\sigma(\theta) \geqslant 2z + \delta_0\,\sigma(\Theta) \qquad \text{for } \theta = \theta(z). \tag{8.1}$$

Next let

$$\eta_c := 2\delta_0 + \frac{36}{5(k-1)}, \qquad w_c := \frac{1}{10} - \frac{\eta_c}{50} - \frac{\varepsilon_2}{4}, \qquad z_c := (1-\eta_c)\,w_c,$$
$$\theta_0 := 3 - 12w_c = \frac{9}{5} + \frac{6\eta_c}{25} + 3\varepsilon_2.$$

Then $\eta_c < \frac14$, because $\delta_0 < \frac{1}{40}$ and $k \geqslant 37$, and $\frac95 < \theta_0 < 2$. So $\sigma(\theta_0) = (3-\theta_0)/6 = 2w_c$ and $\eta_k(\theta_0) = 4\theta_0/k$. Now

$$\frac{4\theta_0}{k} = \frac{36}{5k} + \frac{24\,\eta_c}{25k} + \frac{12\,\varepsilon_2}{k} \leqslant \delta_0 + \frac{36}{5(k-1)} = \eta_c - \delta_0,$$

because $\frac{36}{5(k-1)} - \frac{36}{5k} = \frac{36}{5k(k-1)} \geqslant \frac{24}{25k} \cdot \frac{36}{5(k-1)}$ and $\frac{48\delta_0}{25k} + \frac{12\varepsilon_2}{k} \leqslant \delta_0$. Hence $M(\theta_0)$ is at least $\frac12(1-\eta_c)\cdot 2w_c = z_c$, that is, $\theta(z_c) \geqslant \theta_0$. Moreover $1 - z_c \leqslant 1 - w_c + \frac{\eta_c}{10} = \frac{9}{10} + \frac{3\eta_c}{25} + \frac{\varepsilon_2}{4}$, so that

$$2(1+\varepsilon_2)(1-z_c) \leqslant \frac95 + \frac{6\eta_c}{25} + \varepsilon_2\Big(\frac12 + \frac95 + \frac{6\eta_c}{25} + \frac{\varepsilon_2}{2}\Big) \leqslant \theta_0.$$

Thus

$$\theta(z) \geqslant \theta(z_c) \geqslant \theta_0 \geqslant 2(1+\varepsilon_2)(1-z_c) \qquad (0 \leqslant z \leqslant z_c). \tag{8.2}$$

Note also that

$$u_\Theta \leqslant M\big(\tfrac94\big) < \tfrac{1}{16} < z_c < \tfrac{1}{10}, \tag{8.3}$$

since $\sigma(\frac94) = \frac18$ and $z_c \geqslant \frac34\big(\frac{1}{10} - \frac{1}{200} - \frac{1}{400}\big)$. With Ψ as in §7, define

$$\mathcal{G}(z) := \begin{cases} 1/z, & z \geqslant \frac12, \\ \Psi(z), & z_c \leqslant z \leqslant \frac12, \\ \mathcal{G}(z_c) + 2(1+\varepsilon_2)\displaystyle\int_z^{z_c} \frac{\mathrm{d}u}{u\,\theta(u)}, & 0 < z \leqslant z_c. \end{cases}$$

Since $\theta(u) = \Theta$ for $u \leqslant u_\Theta$, we have

$$\mathcal{G}(z) = \mathcal{G}(u_\Theta) + c_\Theta \log\frac{u_\Theta}{z} \quad (0 < z \leqslant u_\Theta), \qquad \text{where} \quad c_\Theta := \frac{2(1+\varepsilon_2)}{\Theta}. \tag{8.4}$$

For $\gamma \leqslant x \leqslant 1$ and $\lambda > 0$ let

$$\Phi(x,\lambda) := k\lambda\,\mathcal{G}\Big(\frac{\max\{\lambda, \gamma u_\Theta\}}{x}\Big), \qquad \Phi(x,0) := 0,$$
$$\mathcal{G}_\gamma := \mathcal{G}(\gamma u_\Theta) = \mathcal{G}(u_\Theta) + c_\Theta \log(1/\gamma).$$

Thus $\Phi(x,\lambda) = k\lambda\,\mathcal{G}(\lambda/x)$ unless $\lambda < \gamma u_\Theta$; the floor γu_Θ only serves to keep Φ Lipschitz continuous. Observe that $\Phi(x,\lambda) = kx$ when $\lambda/x \geqslant \frac12$, and $\Phi(x,\lambda) = k\lambda\Psi(\lambda/x)$ when $z_c \leqslant \lambda/x \leqslant \frac12$; these are the trivial bound and the bound of Lemma 7.1.

Lemma 8.1. (i) *The function $\mathcal{G}$ is continuous and nonincreasing, and*

$$\mathcal{G}(u_\Theta) \leqslant \Psi(z_c) + 2(1+\varepsilon_2)\Big(I(2w_c) - T(\Theta) + \int_{9/5}^{\Theta} \frac{\mathrm{d}\big(-\log(1-\delta_0-\eta_k(\theta))\big)}{\theta}\Big).$$

(ii) *The function Φ is locally Lipschitz continuous on $[\gamma,1]\times[0,\infty)$. Wherever it is differentiable, $0 \leqslant \partial_\lambda\Phi \leqslant \mathcal{G}_\gamma k$ and $\partial_x\Phi \geqslant 0$, and $\partial_\lambda\Phi \geqslant 2k$ if $\lambda/x \leqslant \frac15$.*

(iii) *If $\lambda/x \leqslant \frac15$ and $0 \leqslant t \leqslant \min\{\lambda, x-\gamma\}$, then $\Phi(x-t,\lambda-t) \leqslant \Phi(x,\lambda) - 2kt$.*

(iv) *Let $z := \lambda/x < z_c$ and $\theta := \theta(z)$, and let $(1+\varepsilon_2)^{-1}\theta x/k \leqslant y \leqslant \min\{\theta x/k,\, x-\gamma\}$. Then $\Phi(x,\lambda) - \Phi(x-y,\lambda) \geqslant 2\lambda$.*

(v) *For $0 < z \leqslant 1$ one has $\mathcal{G}(z) \leqslant \mathcal{G}(u_\Theta) + c_\Theta \log(u_\Theta/z)$, with c_Θ as in* (8.4)*. Moreover the function $z \mapsto z\,\mathcal{G}(z)$ is concave on $(0,1]$.*

Proof. (i) Continuity and monotonicity are clear from the definition, as Ψ is decreasing and $\Psi(\frac12) = 2$. Let $\theta_c := \theta(z_c)$, and let $\theta_c = t_0 < t_1 < \cdots < t_n = \Theta$ be a partition of $[\theta_c,\Theta]$. If $M(t_{i+1}) < u \leqslant M(t_i)$ then $\theta(u) \geqslant t_i$, and these intervals cover $(u_\Theta, z_c]$ because $M(\theta_c) \geqslant z_c$. Hence, with the nondecreasing functions $\psi_1 := -\log\sigma$ and $\psi_2 := -\log(1-\delta_0-\eta_k)$,

$$\int_{u_\Theta}^{z_c} \frac{\mathrm{d}u}{u\,\theta(u)} \leqslant \sum_{i<n} \frac{1}{t_i}\log\frac{M(t_i)}{M(t_{i+1})} = \sum_{i<n}\frac{\psi_1(t_{i+1})-\psi_1(t_i)}{t_i} + \sum_{i<n}\frac{\psi_2(t_{i+1})-\psi_2(t_i)}{t_i}.$$

As the mesh of the partition tends to 0, the two sums tend to the Riemann–Stieltjes integrals of $1/\theta$ with respect to ψ_1 and ψ_2 over $[\theta_c,\Theta]$. The second of these is at most the integral in the statement, because $\theta_c \geqslant \frac95$ by (8.2). For the first, note that $\sigma(t_{i+1}) < s \leqslant \sigma(t_i)$ implies $\theta^*(s) < t_{i+1}$, so that

$$\frac{\psi_1(t_{i+1})-\psi_1(t_i)}{t_{i+1}} \leqslant \int_{\sigma(t_{i+1})}^{\sigma(t_i)} \frac{\mathrm{d}s}{s\,\theta^*(s)}.$$

Replacing $1/t_i$ by $1/t_{i+1}$ in the first sum does not change its limit, and therefore

$$\int_{\theta_c}^{\Theta}\frac{\mathrm{d}\psi_1(\theta)}{\theta} \leqslant \int_{\sigma(\Theta)}^{\sigma(\theta_c)}\frac{\mathrm{d}s}{s\,\theta^*(s)} = I\big(\sigma(\theta_c)\big) - T(\Theta) \leqslant I(2w_c) - T(\Theta),$$

since $\sigma(\theta_c) \leqslant \sigma(\theta_0) = 2w_c$.

(ii) The function $\mathcal{G}$ is locally Lipschitz continuous on $(0,\infty)$, and so is Φ on the region described, as the argument of $\mathcal{G}$ is at least γu_Θ. Let $\mathfrak{g}(z) := -z\mathcal{G}'(z)$; thus

$$\mathfrak{g}(z) = \begin{cases} 2(1+\varepsilon_2)/\theta(z), & z < z_c, \\ 1/(1-z), & z_c < z < \frac12, \\ 1/z, & z > \frac12, \end{cases} \tag{8.5}$$

at the points of differentiability. As M is strictly decreasing, $\theta(\cdot)$ is continuous; hence $\mathcal{G}$ is continuously differentiable on $(0,\infty)\setminus\{z_c\}$ (at $z = \frac12$ both one-sided derivatives equal -4), and Φ is continuously differentiable outside the line $\lambda = \gamma u_\Theta$ and the ray $\lambda = z_c x$. The segments in (iii), and the lines on which x is constant, meet each of these two sets in at most one point, so that Φ is the integral of its derivative along them. Put $z^\flat := \max\{\lambda, \gamma u_\Theta\}/x$. If $\lambda > \gamma u_\Theta$ then $z^\flat = \lambda/x$, $\partial_\lambda\Phi = k(\mathcal{G}(z^\flat) - \mathfrak{g}(z^\flat))$ and $\partial_x\Phi = k\lambda\,\mathfrak{g}(z^\flat)/x$; if $\lambda < \gamma u_\Theta$ then $\partial_\lambda\Phi = k\,\mathcal{G}(z^\flat)$ and $\partial_x\Phi = k\lambda\,\mathfrak{g}(z^\flat)/x$. So $\partial_x\Phi \geqslant 0$, and $\partial_\lambda\Phi \leqslant k\,\mathcal{G}(z^\flat) \leqslant k\,\mathcal{G}(\gamma u_\Theta)$ because $z^\flat \geqslant \gamma u_\Theta$. Next, $\mathcal{G}(z) - \mathfrak{g}(z)$ vanishes for $z > \frac12$.

For $z_c < z < \frac{1}{2}$ it equals $\Psi(z) - \frac{1}{1-z}$, which is decreasing in z and vanishes at $z = \frac{1}{2}$, and it is at least $2 + \log 4 - \frac{5}{4} > 2$ for $z \leqslant \frac{1}{5}$. For $z < z_c$ we have $\mathfrak{g}(z) \leqslant 2(1+\varepsilon_2)/\theta(z_c) \leqslant \frac{2.02}{1.8}$ by (8.2), and $\mathcal{G}(z) \geqslant \mathcal{G}(z_c) \geqslant \Psi(\frac{1}{10}) > 4.19$, so $\mathcal{G} - \mathfrak{g} > 3$. Finally, if $\lambda < \gamma u_\Theta$ then $z^\flat \leqslant u_\Theta < z_c$ and $\mathcal{G}(z^\flat) > 4.19$.

(iii) Along the path $t \mapsto (x - t, \lambda - t)$ the ratio of the second coordinate to the first does not increase, since $\lambda \leqslant x$. So by (ii) the derivative of $t \mapsto \Phi(x - t, \lambda - t)$, which is $-\partial_x\Phi - \partial_\lambda\Phi$, is at most $-2k$ almost everywhere.

(iv) Let $z_1 := \max\{\lambda, \gamma u_\Theta\}/x$ and $z_1' := \max\{\lambda, \gamma u_\Theta\}/(x-y)$. Then $z \leqslant z_1 < z_c$ by (8.3), $\log(z_1'/z_1) = \log \frac{x}{x-y} \geqslant y/x \geqslant (1+\varepsilon_2)^{-1}\theta/k$, and $z_1' \leqslant 2z_1 < \frac{1}{5}$ since $\theta \leqslant k/2$. With $\mathfrak{g}$ as in (8.5),

$$\Phi(x,\lambda) - \Phi(x-y,\lambda) = k\lambda\big(\mathcal{G}(z_1) - \mathcal{G}(z_1')\big) = k\lambda \int_{z_1}^{z_1'} \mathfrak{g}(u)\,\frac{\mathrm{d}u}{u}.$$

Let $z_1 < u < z_1'$. If $u < z_c$ then $\mathfrak{g}(u) = 2(1+\varepsilon_2)/\theta(u) \geqslant 2(1+\varepsilon_2)/\theta$, because $u \geqslant z$ and $\theta(\cdot)$ is nonincreasing. If $z_c < u < \frac{1}{5}$ then $\mathfrak{g}(u) = \frac{1}{1-u} \geqslant \frac{1}{1-z_c} \geqslant 2(1+\varepsilon_2)/\theta(z_c) \geqslant 2(1+\varepsilon_2)/\theta$ by (8.2). Hence the integral is at least $\frac{2(1+\varepsilon_2)}{\theta} \cdot \frac{\theta}{(1+\varepsilon_2)k} = \frac{2}{k}$.

(v) For $z \leqslant u_\Theta$ there is equality, by (8.4). Let $\mathfrak{g}$ be as in (8.5). For almost all $u \in (u_\Theta, 1)$ we have $\mathfrak{g}(u) \geqslant c_\Theta$: if $u < z_c$, then $\mathfrak{g}(u) = 2(1+\varepsilon_2)/\theta(u)$ and $\theta(u) \leqslant \Theta$; if $z_c < u < \frac{1}{2}$, then $\mathfrak{g}(u) = 1/(1-u) \geqslant 1/(1-z_c) \geqslant 2(1+\varepsilon_2)/\theta(z_c) \geqslant c_\Theta$ by (8.2); and if $u > \frac{1}{2}$, then $\mathfrak{g}(u) = 1/u \geqslant 1 > c_\Theta$, since $\Theta \geqslant \frac{9}{4}$ and $\varepsilon_2 \leqslant \frac{1}{100}$. As $\mathcal{G}$ is locally Lipschitz continuous, it follows that $\mathcal{G}(u_\Theta) - \mathcal{G}(z) = \int_{u_\Theta}^{z} \mathfrak{g}(u)\,\mathrm{d}u/u \geqslant c_\Theta \log(z/u_\Theta)$ for $u_\Theta \leqslant z \leqslant 1$. This is the first assertion. The derivative of $z\,\mathcal{G}(z)$ is $\mathcal{G}(z) - \mathfrak{g}(z)$. The function $\mathfrak{g}$ is nondecreasing on $(0, \frac{1}{2})$, because $\theta(\cdot)$ is nonincreasing, $1/(1-z)$ is increasing, and $\mathfrak{g}$ jumps upwards at z_c, by (8.2). Hence $\mathcal{G} - \mathfrak{g}$ is nonincreasing on $(0, \frac{1}{2})$, where it is nonnegative (see the proof of (ii)), and it vanishes on $(\frac{1}{2}, 1)$. So $z\,\mathcal{G}(z)$ is concave. □

8.2. **The induction.** The following proposition is the main result of this section. It bounds the quantities $D_j(X, W)$ of §3 by means of the potential Φ.

Proposition 8.2. *There is a constant C_* such that, uniformly for $0 \leqslant j \leqslant J$, for $X = P^x$ attached to j with $x \geqslant \gamma$, and for $W = P^\lambda \geqslant 1$,*

$$D_j(X, W) \leqslant C_*^{j+1} (\log P)^{2j} P^{4k\gamma + \varepsilon + \Phi(x,\lambda)}.$$

Proof. We argue by induction on j. Put $z := \lambda/x$.

Case 1: $x \leqslant 4\gamma$. Then $D_j(X, W) \leqslant X^k \leqslant P^{4k\gamma}$. For $j = 0$ there is nothing to prove, because an X attached to 0 has $x \leqslant \eta < \gamma$, by Lemma 3.1(a) and (3.2).

Case 2: $z \geqslant z_c$. Then $\lambda \geqslant z_c\gamma > \gamma u_\Theta$ by (8.3), so $\Phi(x, \lambda) = k\lambda\,\mathcal{G}(z)$, which is kx if $z \geqslant \frac{1}{2}$ and $k\lambda\Psi(z)$ otherwise. The assertion follows from Lemma 7.1 with $z_* = z_c$, if C_* is large enough.

Case 3: $x > 4\gamma$ and $z < z_c$. Then $j \geqslant 1$. Let $\theta := \theta(z) \in [\theta_0, \Theta]$, and let

$$i := \Big\lceil \frac{\theta x}{(1+\varepsilon_2)\,k\,\eta} \Big\rceil.$$

For $d \in \mathcal{D}_i$ put $y := \log d/\log P$; then $i\eta < y \leqslant (1+\omega)\, i\eta$, since $R^i < d \leqslant (\zeta R)^i \leqslant R^{(1+\omega)i}$. We claim that

$$\frac{\theta x}{(1+\varepsilon_2)k} \leqslant y \leqslant \frac{\theta x}{k} \qquad (d \in \mathcal{D}_i), \qquad \text{and} \qquad 1 \leqslant i \leqslant j. \tag{8.6}$$

The lower bound for y holds by the choice of i. For the upper bound we note that $i\eta < \theta x/((1+\varepsilon_2)k) + \eta$, so that it suffices to show that

$$(1+\omega)\,\eta \leqslant \frac{\theta x}{k}\Big(1 - \frac{1+\omega}{1+\varepsilon_2}\Big) = \frac{\theta x}{k} \cdot \frac{\varepsilon_2 - \omega}{1+\varepsilon_2}.$$

Now $\theta x > \frac{9}{5} \cdot 4\gamma$, and $\omega \leqslant \frac{1}{8}\eta \leqslant \frac{1}{8}\varepsilon_2$ by (3.2), so that the right hand side exceeds $6\gamma\varepsilon_2/k$, while the left hand side is at most $\frac{9}{8}\eta \leqslant 4\gamma\varepsilon_2/k$. Finally $i \geqslant 1$ because $\theta x > 0$, and $i\eta < y \leqslant \theta x/k \leqslant x/2 \leqslant \frac{1}{2}(j+1)\eta$ by Lemma 3.1(a), since $\theta \leqslant k/2$; hence $i < \frac{1}{2}(j+1) \leqslant j$. This proves (8.6).

Thus $d \leqslant X^{\theta/k}$ for every $d \in \mathcal{D}_i$. By (8.1), and since $2\varepsilon \leqslant \delta_0\sigma(\Theta)$,

$$W^2 = X^{2z} \leqslant X^{(1-\eta_k(\theta))\sigma(\theta)-2\varepsilon}.$$

Every $d \in \mathcal{D}_i$ is squarefree with at most J prime factors, all exceeding k, and $X \geqslant P^{4\gamma}$ is large. So Corollary 5.2 applies with $m = d$ for every $d \in \mathcal{D}_i$, and Lemma 6.1 gives (6.1); call the two terms on the right Σ_1 and Σ_2.

The term Σ_1. Fix $l < N_W$ and $d \in \mathcal{D}_i$, and let $\mu := \log 2^{l+1}/\log P$, so that $0 < \mu \leqslant \lambda^+$, where $\lambda^+ := \lambda + \log 4/\log P$. By Lemma 3.1(c), $X/d = P^{x-y}$ is attached to $j-i$, and $x - y \geqslant x/2 \geqslant \gamma$ by (8.6), so the inductive hypothesis bounds $D_{j-i}(X/d, 2^{l+1})$ in terms of $\Phi(x-y,\mu)$. Since $y \leqslant x/2$ and $x > 4\gamma$, we have $\lambda^+/(x-y) \leqslant 2z + \log 4/(2\gamma\log P)$, and $2z < 2z_c \leqslant \frac{1}{5} - \frac{1}{2}\varepsilon_2$; so $\lambda^+/(x-y) \leqslant \frac{1}{5}$ for large P. By Lemma 8.1(ii) the function $\mu \mapsto \Phi(x-y,\mu) - \mu$ is therefore nondecreasing on $[0,\lambda^+]$, and

$$\begin{aligned} 3\lambda - \mu + \Phi(x-y,\mu) &\leqslant 2\lambda + \Phi(x-y,\lambda^+) \leqslant 2\lambda + \Phi(x-y,\lambda) + \frac{\mathcal{G}_\gamma k \log 4}{\log P} \\ &\leqslant \Phi(x,\lambda) + \frac{\mathcal{G}_\gamma k \log 4}{\log P}, \end{aligned}$$

the last step by Lemma 8.1(iv), which is applicable by (8.6). Now $W^3 2^{-l} = 2P^{3\lambda-\mu}$ and $N_W \leqslant 2 + \lambda\log P/\log 2 \leqslant 2\log P$, as $\lambda < z_c < \frac{1}{10}$, and the sum of d^{-1} over $d \in \mathcal{D}_i$ is at most 1. Hence

$$\Sigma_1 \leqslant 16\, C_1\, 4^{\mathcal{G}_\gamma k}\, C_*^j (\log P)^{2j} P^{4k\gamma+\varepsilon+\Phi(x,\lambda)}.$$

The term Σ_2. Let $\lambda^\dagger := \lambda + \log K_0/\log P$, and for $d_1 \in \mathcal{D}_{i_1}$ with $1 \leqslant i_1 \leqslant i$ let $y_1 := \log d_1/\log P$; then $y_1 \leqslant (1+\omega)\, i\eta \leqslant \theta x/k \leqslant x/2$, as in the proof of (8.6). The corresponding summand vanishes unless $K_0 W/d_1 \geqslant 1$, that is, $y_1 \leqslant \lambda^\dagger$. In that case $X/d_1 = P^{x-y_1}$ is attached to $j - i_1$, $x - y_1 \geqslant \gamma$, and $\lambda^\dagger/x \leqslant z + \log K_0/(4\gamma\log P) \leqslant \frac{1}{5}$ for large P; so by the inductive hypothesis and Lemma 8.1(iii),

$$\begin{aligned} d_1^k D_{j-i_1}\Big(\frac{X}{d_1}, \frac{K_0 W}{d_1}\Big) &\leqslant C_*^j (\log P)^{2j-2} P^{4k\gamma+\varepsilon+\Phi(x-y_1,\lambda^\dagger-y_1)} d_1^k \\ &\leqslant C_*^j (\log P)^{2j-2} P^{4k\gamma+\varepsilon+\Phi(x,\lambda^\dagger)} d_1^{-k}, \end{aligned}$$

and $\Phi(x,\lambda^\dagger) \leqslant \Phi(x,\lambda) + \mathcal{G}_\gamma k \log K_0/\log P$ by Lemma 8.1(ii). Each integer d_1 occurs for one value of i_1 only, so summing over $d_1 \geqslant 2$ gives

$$\Sigma_2 \leqslant K_0^{\mathcal{G}_\gamma k} C_*^j (\log P)^{2j} P^{4k\gamma+\varepsilon+\Phi(x,\lambda)}.$$

Taking $C_* \geqslant 16C_1 4^{\mathcal{G}_\gamma k} + K_0^{\mathcal{G}_\gamma k}$ closes the induction. □

Corollary 8.3. *Let $r \geqslant k+1$ and $\Delta \geqslant 0$ be real numbers such that*

$$k\, z\, \mathcal{G}(z) \leqslant r\, z + \Delta \qquad (0 < z \leqslant 1). \tag{8.7}$$

Then, uniformly for $0 \leqslant j \leqslant J$, for finite sets $\mathfrak{e}$ of primes, and for X attached to j,

$$\int_{\mathbb{T}} |F_j^{\mathfrak{e}}(\alpha;X)|^r \,\mathrm{d}\alpha \ll X^{r-k+\Delta} P^{4k\gamma+2\varepsilon}.$$

The hypothesis (8.7) *holds with $\Delta = b$ whenever $b > 0$ and $r \geqslant k\big(\mathcal{G}(u_\Theta) + a_*\big)$, where*

$$a_* := c_\Theta \log \frac{k\, c_\Theta\, u_\Theta}{\mathrm{e}\, b} \tag{8.8}$$

and c_Θ is as in (8.4). *The implied constants depend also on r and Δ.*

Proof. Let $X = P^x$. If $x < \gamma$, by Lemma 3.1(b) the integral is at most $X^r \leqslant X^{r-k}P^{k\gamma}$. Let $x \geqslant \gamma$, let $0 < \lambda \leqslant x$ and $z := \lambda/x \leqslant 1$. Since $\mathcal{G}$ is nonincreasing, $\Phi(x,\lambda) \leqslant k\lambda\,\mathcal{G}(z)$, and so $\Phi(x,\lambda) \leqslant r\lambda + \Delta x$ by (8.7). By Proposition 8.2 it follows that

$$\mathrm{meas}\{\alpha : |F_j^{\mathfrak{e}}(\alpha;X)| \geqslant XP^{-\lambda}\} \ll X^{-k} P^{4k\gamma+\frac{3}{2}\varepsilon+\Delta x+r\lambda} \qquad (0 \leqslant \lambda \leqslant x);$$

for $\lambda > x$ we use that the measure is at most 1, and for $\lambda < 0$ it is 0. Since $r > k$ and $X^{r-k} \geqslant 1$, we conclude that

$$\int_{\mathbb{T}} |F_j^{\mathfrak{e}}|^r \,\mathrm{d}\alpha = r\log P \int_{-\infty}^{\infty} (XP^{-\lambda})^r \,\mathrm{meas}\{\alpha : |F_j^{\mathfrak{e}}(\alpha;X)| \geqslant XP^{-\lambda}\}\,\mathrm{d}\lambda \ll X^{r-k}P^{4k\gamma+2\varepsilon+\Delta x}.$$

For the second assertion let $b > 0$ and $r \geqslant k(\mathcal{G}(u_\Theta)+a_*)$. By Lemma 8.1(v), for $0 < z \leqslant 1$,

$$k\, z\, \mathcal{G}(z) - r\, z \leqslant k\, z\big(\mathcal{G}(z) - \mathcal{G}(u_\Theta) - a_*\big) \leqslant k\, z\Big(c_\Theta \log \frac{u_\Theta}{z} - a_*\Big) \leqslant \frac{k\, c_\Theta\, u_\Theta}{\mathrm{e}}\, \mathrm{e}^{-a_*/c_\Theta} = b,$$

the maximum over $z > 0$ of the third expression being attained at $z = u_\Theta \mathrm{e}^{-1-a_*/c_\Theta}$. □

The point of Corollary 8.3 is that r need not exceed $k \sup \mathcal{G}$, which is infinite: the excess of $\Phi(x,\lambda)$ over $r\lambda$ is at most bx, and a factor P^b will be recovered on the minor arcs from a bound of Weyl type for F (§9.3). The number a_* is negative when $b > k\, c_\Theta\, u_\Theta/\mathrm{e}$, and r is then allowed to be smaller than $k\,\mathcal{G}(u_\Theta)$; this is possible because, by Lemma 8.1(v), the function $\mathcal{G}$ lies below its tail (8.4) also for $z > u_\Theta$. The uniformity in j, in the excluded primes and in X is needed in §9.2, where F is compared with the shorter sums $F_{J-i}^{[d]}$.

9. The circle method

In this section we prove Theorems 2.3 and 2.4. In [10] the sum F is accompanied by $2k$ smooth Weyl sums, which supply a bound of Weyl type on the minor arcs and make it possible to prune back to the major arcs. Here all s variables are of the kind counted by F. This requires three pieces of information about F on arcs of intermediate height: mean values restricted to such arcs (§9.2), a bound of Weyl type (§9.3), and pointwise bounds on arcs of small height (§9.4). For smooth Weyl sums the corresponding results are due to Brüdern and Wooley [6, §§2–6]. They can be proved for F because the integers counted by F have a divisor of any prescribed size, up to factors $P^{O(\eta)}$. Close to the major arcs F is approximated by the main term of a classical Weyl sum, multiplied by the density Π of §3.2; this is a consequence of the unrestricted factor m.

9.1. **Arcs, and the pruning estimate.** We identify $\mathbb{T}$ with $[0,1)$. For real numbers $Y \geqslant 1$ and $Q > 0$ and for $q \in \mathbb{N}$ let $\mathfrak{M}_q(Q;Y)$ be the set of $\alpha \in \mathbb{T}$ for which there is an integer a with $(a,q) = 1$ and $|q\alpha - a| \leqslant QY^{-k}$, and put

$$\mathfrak{M}(Q;Y) := \bigcup_{1 \leqslant q \leqslant Q} \mathfrak{M}_q(Q;Y), \qquad \mathfrak{N}(Q;Y) := \mathfrak{M}(Q;Y) \setminus \mathfrak{M}(Q/2;Y).$$

We shall use the following simple facts.

(A1) One has $\mathfrak{M}(Y^{k/2};Y) = \mathbb{T}$, by Dirichlet's theorem on Diophantine approximation, and $\operatorname{meas}\mathfrak{M}(Q;Y) \leqslant \sum_{q \leqslant Q} \varphi(q) \cdot 2Q/(qY^k) \leqslant 2Q^2Y^{-k}$.

(A2) If $Q \leqslant \frac{1}{2}Y^{k/2}$, then the arcs $\{\alpha \in \mathbb{T} : \|\alpha - a/q\| \leqslant Q/(qY^k)\}$ with $1 \leqslant a \leqslant q \leqslant Q$ and $(a,q) = 1$ are pairwise disjoint: two of them with centres $a/q \neq a'/q'$ can meet only if $1/(qq') \leqslant Q(q+q')/(qq'Y^k)$, which implies that $Y^k \leqslant 2Q^2$. Thus $\mathfrak{M}_q(Q;Y)$ consists of $\varphi(q)$ arcs of length $2Q/(qY^k)$, and $\mathfrak{M}(Q;Y)$ is the disjoint union of the sets $\mathfrak{M}_q(Q;Y)$ with $q \leqslant Q$. In particular, for $\alpha \in \mathfrak{M}(\frac{1}{2}Y^{k/2};Y)$ there is exactly one arc of this family with $Q = \frac{1}{2}Y^{k/2}$ that contains α, and with the corresponding a and q we put

$$\Upsilon_Y(\alpha) := \big(q + Y^k|q\alpha - a|\big)^{-1}, \qquad \Upsilon := \Upsilon_P.$$

(A3) Let $\alpha \in \mathfrak{N}(Q;Y)$, and let a and q be coprime with $q \leqslant Q$ and $|q\alpha - a| \leqslant QY^{-k}$. Then $\frac{1}{2}Q < q + Y^k|q\alpha - a| \leqslant 2Q$, since otherwise $\alpha \in \mathfrak{M}(Q/2;Y)$.

Throughout this section b is a positive number, a_* is given by (8.8), and r, t are real numbers with

$$r \geqslant \max\big\{k\big(\mathcal{G}(u_\Theta) + a_*\big),\, 2k\big\}, \qquad t > 0, \qquad \nu := t\,\tau_k - b > 0; \qquad s := r + t. \tag{9.1}$$

Here $\tau_k = 1/(9.027901\,k)$, as in §2. By Corollary 8.3, the moments of order r of the sums $F_j^{\mathfrak{e}}(\,\cdot\,;X)$ are $\ll X^{r-k+b}P^{4k\gamma+2\varepsilon}$. Moreover

$$s > k\Big(\mathcal{G}(u_\Theta) + c_\Theta\,\log\frac{u_\Theta}{\tau_k}\Big) \geqslant k\,\mathcal{G}(\tau_k) > 4.19\,k \geqslant 4k+1.$$

Indeed, with $v := b/(k\,c_\Theta\,\tau_k)$ we have $k\,a_* + b/\tau_k = k\,c_\Theta\,\big(\log\frac{u_\Theta}{\tau_k} + v - 1 - \log v\big) \geqslant k\,c_\Theta\,\log\frac{u_\Theta}{\tau_k}$, and $t > b/\tau_k$; this gives the first inequality. The second is Lemma 8.1(v), and the third

holds because $\tau_k < \frac{1}{16} < z_c < \frac{1}{10}$, by (8.3), so that $\mathcal{G}(\tau_k) \geqslant \mathcal{G}(z_c) \geqslant \Psi(\frac{1}{10}) > 4.19$. We now fix the parameters γ and ε of (3.1) by

$$\gamma := \min\Big\{\frac{1}{100k},\ \frac{\nu}{40k^2(1+b)}\Big\}, \qquad \varepsilon := \min\Big\{\frac{\delta_0\,\sigma(\Theta)}{2},\ \frac{\gamma}{16k^2s}\Big\}, \tag{9.2}$$

and we suppose, as we may, that

$$\eta \leqslant \frac{\nu}{100\,k\,(t+b)}, \qquad \text{and we put} \qquad \omega := \frac{\eta}{64\,k\,s}; \tag{9.3}$$

moreover J is to be so large that the mean value estimates for smooth numbers quoted in the proof of Lemma 9.7 are applicable. In this section implied constants may depend on b, r and t as well. The main result of §§9.1–9.4 is as follows.

Proposition 9.1. *Suppose that* (9.1), (9.2) *and* (9.3) *hold. Then*

$$\int_{\mathfrak{N}(Q;P)} |F(\alpha)|^s\,\mathrm{d}\alpha \ll P^{s-k}Q^{-\nu/k} \qquad (P^{1/4} \leqslant Q \leqslant P^{k/2}), \tag{9.4}$$

$$\int_{\mathfrak{N}(Q;P)} |F(\alpha)|^s\,\mathrm{d}\alpha \ll P^{s-k}Q^{-1/(4k)} \qquad (R^{1/4} \leqslant Q \leqslant P^{1/4}), \tag{9.5}$$

$$\int_{\mathfrak{M}(R^{1/4};P)\setminus\mathfrak{M}(Q;P)} |F(\alpha)|^s\,\mathrm{d}\alpha \ll \Pi^s P^{s-k}Q^{-2} + P^{s-k}R^{-1} \qquad (1 \leqslant Q \leqslant R^{1/4}). \tag{9.6}$$

Consequently, with $\nu_0 := \min\{\nu/(4k), 1/(16k)\}$,

$$\int_{\mathbb{T}\setminus\mathfrak{M}(Q;P)} |F(\alpha)|^s\,\mathrm{d}\alpha \ll \Pi^s P^{s-k}Q^{-2} + P^{s-k}R^{-\nu_0} \qquad (1 \leqslant Q \leqslant R^{1/4}). \tag{9.7}$$

The estimate (9.4) is of the type of [6, Theorems 5.2 and 5.3], and (9.7) is of the type of [6, Theorem 6.1]; the number b plays the part of an admissible exponent for the r-th moment of F, and $b - t\tau_k$ that of the exponent Δ_s^* of [6]. In (9.6) and (9.7) the factor Π^s is the s-th power of the density of the set $\mathcal{C}$. When $H = 1$ it is of order $(\log P)^{-Js}$, and it is for this reason that on the arcs of height at most $R^{1/4}$ no factor P^{ε}, and not even a power of $\log P$, is lost. To deduce (9.7) from (9.4)–(9.6), let $Q_i := 2^{-i}P^{k/2}$. By (A1), the set $\mathbb{T} \setminus \mathfrak{M}(Q;P)$ is covered by the sets $\mathfrak{N}(Q_i;P)$ with $Q_i > Q$. Summing (9.4) over the i with $Q_i \geqslant P^{1/4}$ gives the bound $P^{s-k}P^{-\nu/(4k)}$, and summing (9.5) over those with $R^{1/4} \leqslant Q_i < P^{1/4}$ gives $P^{s-k}R^{-1/(16k)}$. The part of $\mathbb{T} \setminus \mathfrak{M}(Q;P)$ that is not covered by these sets is contained in $\mathfrak{M}(R^{1/4};P) \setminus \mathfrak{M}(Q;P)$, and there we use (9.6).

9.2. **Mean values over arcs of height Q.** For $q \in \mathbb{N}$ and squarefree d let $d_q(d)$ denote the product of the primes that divide both d and q. If $q \leqslant P^{k/2}$, then, for $0 \leqslant i \leqslant J$ and every finite set $\mathfrak{e}$ of primes,

$$\sum_{d\in\mathcal{D}_i^{\mathfrak{e}}} \frac{d_q(d)}{d} \leqslant 2^{kJ}. \tag{9.8}$$

Indeed, every prime in $\mathcal{P}$ exceeds $R = P^{\eta}$, so the number n_q of primes in $\mathcal{P}$ that divide q satisfies $n_q \leqslant k/(2\eta) \leqslant kJ$. We sort the $d \in \mathcal{D}_i^{\mathfrak{e}}$ according to $c := d_q(d)$, which is a product of some number $l \leqslant i$ of these n_q primes. Then $d/c \in \mathcal{D}_{i-l}^{\mathfrak{e}}$, so that the sum of c/d over the d that belong to a given c is at most 1, by Lemma 3.1(c). Hence the sum in (9.8) is at most $\sum_l \binom{n_q}{l} \leqslant 2^{n_q}$.

The next lemma is the counterpart for F of the scaling principle [6, Lemma 2.3].

Lemma 9.2. *Let $r \geqslant k+1$ be real, let $1 \leqslant Q \leqslant \frac{1}{2}P^{k/2}$, and let i be an integer with $0 \leqslant i \leqslant J$ and $(\zeta R)^i \leqslant P(2Q)^{-2/k}$. Then*

$$\int_{\mathfrak{M}(Q;P)} |F(\alpha)|^r \,\mathrm{d}\alpha \leqslant C_2 \sum_{d\in\mathcal{D}_i} d^{r-1-k} \int_{\mathfrak{M}(Q;P/d)} \bigl|F_{J-i}^{[d]}(\beta;P/d)\bigr|^r \,\mathrm{d}\beta,$$

where $C_2 := (k+1)^J 2^{kJr}$.

Proof. Let $q \leqslant Q$, let $d \in \mathcal{D}_i$, and put $Y := P/d$, $g := (q,d^k)$, $q'' := q/g$ and $d' := d/d_q(d)$. Then $d \leqslant (\zeta R)^i$ and therefore $Q \leqslant \frac{1}{2}Y^{k/2}$, so that (A2) is applicable to Y as well as to P. We claim that for every 1-periodic integrable function $f : \mathbb{R} \to [0,\infty)$

$$\int_{\mathfrak{M}_q(Q;P)} f(d^k\alpha)\,\mathrm{d}\alpha = \frac{\varphi(q)}{d^k\varphi(q'')} \int_{\mathfrak{M}_{q''}(Q/g;Y)} f(\beta)\,\mathrm{d}\beta \leqslant \frac{1}{d'^k} \int_{\mathfrak{M}_{q''}(Q;Y)} f(\beta)\,\mathrm{d}\beta. \tag{9.9}$$

To see this, consider one of the arcs $|\alpha - a/q| \leqslant Q/(qP^k)$ of $\mathfrak{M}_q(Q;P)$. The substitution $\beta = d^k\alpha$ maps it onto the interval $|\beta - a''/q''| \leqslant (Q/g)/(q''Y^k)$, where $a'' := ad^k/g$ is coprime to q''; this interval has length at most 1, and its image in $\mathbb{T}$ is one of the arcs of $\mathfrak{M}_{q''}(Q/g;Y)$. As a runs over the reduced residues modulo q, the number a'' runs over the reduced residues modulo q'', each of which occurs $\varphi(q)/\varphi(q'')$ times, because d^k/g is coprime to q''. This proves the equation in (9.9). The inequality holds because $\varphi(q) \leqslant g\,\varphi(q'')$ and g divides $d_q(d)^k$, so that $g\,d^{-k} \leqslant d'^{-k}$, and because the arcs $\mathfrak{M}_{q''}(Q/g;Y)$ are contained in $\mathfrak{M}_{q''}(Q;Y)$.

Now let $\alpha \in \mathfrak{M}_q(Q;P)$. By Lemma 3.1(c), in which the factor $\binom{J}{i} \geqslant 1$ is discarded, by Hölder's inequality and by (9.8),

$$\begin{aligned} |F(\alpha)|^r &\leqslant \Bigl(\sum_{d\in\mathcal{D}_i} \frac{1}{d'}\Bigr)^{r-1} \sum_{d\in\mathcal{D}_i} d'^{\,r-1} \bigl|F_{J-i}^{[d]}(d^k\alpha;P/d)\bigr|^r \\ &\leqslant 2^{kJ(r-1)} \sum_{d\in\mathcal{D}_i} d'^{\,r-1} \bigl|F_{J-i}^{[d]}(d^k\alpha;P/d)\bigr|^r; \end{aligned}$$

for $i = 0$ this is trivial. We integrate over $\mathfrak{M}_q(Q;P)$, apply (9.9) with $f = |F_{J-i}^{[d]}(\,\cdot\,;P/d)|^r$ to each term, and use that $d'^{\,r-1-k} \leqslant d^{r-1-k}$. This gives

$$\int_{\mathfrak{M}_q(Q;P)} |F(\alpha)|^r \,\mathrm{d}\alpha \leqslant 2^{kJr} \sum_{d\in\mathcal{D}_i} d^{r-1-k} \int_{\mathfrak{M}_{q/(q,d^k)}(Q;P/d)} \bigl|F_{J-i}^{[d]}(\beta;P/d)\bigr|^r \,\mathrm{d}\beta.$$

Finally we sum over $q \leqslant Q$. For given d and q'' there are at most $(k+1)^J$ numbers q with $q/(q,d^k) = q''$, since (q,d^k) is a divisor of d^k and d has at most J prime factors. By (A2) for $Y = P/d$, the lemma follows. □

Corollary 9.3. *Let b and r be as in* (9.1)*. Then, uniformly for $1 \leqslant Q \leqslant P^{k/2}$,*

$$\int_{\mathfrak{M}(Q;P)} |F(\alpha)|^r \,\mathrm{d}\alpha \ll P^{r-k+4k\gamma+2\varepsilon}\bigl(Q^{2/k}P^{2\eta}\bigr)^b.$$

Proof. If $Q > \frac{1}{2}P^{k/2}$ this is contained in Corollary 8.3. Let $Q \leqslant \frac{1}{2}P^{k/2}$, define $\mu \in [0,1]$ by $P^\mu = (2Q)^{2/k}$, and let i be the largest integer in $[0,J]$ with $(\zeta R)^i \leqslant P^{1-\mu}$. Then

Lemma 9.2 applies. For $d \in \mathcal{D}_i$ the number P/d is attached to $J-i$, by Lemma 3.1(c). So Corollary 8.3, applied to $F_{J-i}^{[d]}(\,\cdot\,;P/d)$, and Lemma 3.1(c) give

$$\int_{\mathfrak{M}(Q;P)} |F(\alpha)|^r \,\mathrm{d}\alpha \ll P^{4k\gamma+2\varepsilon} \sum_{d\in\mathcal{D}_i} d^{r-1-k}\Big(\frac{P}{d}\Big)^{r-k+b} \leqslant P^{r-k+4k\gamma+2\varepsilon} \max_{d\in\mathcal{D}_i}\Big(\frac{P}{d}\Big)^b.$$

If $i=J$, then $P/d \leqslant R$ for $d\in\mathcal{D}_J$, by Lemma 3.1(b). If $i<J$, then $(\zeta R)^{i+1} > P^{1-\mu}$, and $d \geqslant R^i \geqslant (\zeta R)^{i+1}/(\zeta^J R)$ for $d\in\mathcal{D}_i$, whence by (3.3)

$$P^\mu \leqslant \frac{P}{d} < P^\mu \zeta^J R \leqslant P^\mu R^{9/8} \qquad (d\in\mathcal{D}_i). \tag{9.10}$$

In both cases $P/d \leqslant (2Q)^{2/k} R^{9/8} \leqslant Q^{2/k} P^{2\eta}$. □

Corollary 9.3 corresponds to [6, Theorem 4.2]. For F the proof is short, because the decomposition of Lemma 3.1(c), which for smooth Weyl sums has to be constructed [6, §3], is part of the definition, and because the moment bound of Corollary 8.3 is uniform in the length of the sum.

9.3. **A bound of Weyl type.** The integers counted by $F_j^{\mathfrak{e}}(\,\cdot\,;X)$ are products dn in which d runs over a set of numbers of size about $X^{1/2}$. For exponential sums over such products we use the following estimate. It is the argument by which Wooley proves [41, Lemma 3.1] for smooth Weyl sums, and it has the feature that nothing is required of the two sets of integers beyond mean value estimates. For a finite set $\mathcal{A}\subset\mathbb{N}$ and $u\in\mathbb{N}$ let $S_u(\mathcal{A})$ denote the number of solutions of $m_1^k+\cdots+m_u^k = m_{u+1}^k+\cdots+m_{2u}^k$ with $m_i\in\mathcal{A}$. Thus

$$S_u(\mathcal{A}) = \int_{\mathbb{T}} \Big|\sum_{m\in\mathcal{A}} e(\alpha m^k)\Big|^{2u} \mathrm{d}\alpha.$$

Lemma 9.4. *Let $u,w\in\mathbb{N}$ and $A,B\geqslant 1$, let $\mathcal{A}\subseteq\mathbb{N}\cap[1,A]$ and $\mathcal{B}\subseteq\mathbb{N}\cap[1,B]$, and let $(\xi_m)_{m\in\mathcal{A}}$ be complex numbers with $|\xi_m|\leqslant 1$. Let $\alpha\in\mathbb{R}$, and suppose that $a_1\in\mathbb{Z}$ and $q_1\in\mathbb{N}$ are coprime and $|\alpha-a_1/q_1|\leqslant q_1^{-2}$. Then*

$$\begin{aligned}&\Big(\sum_{n\in\mathcal{B}}\Big|\sum_{m\in\mathcal{A}}\xi_m\, e\big(\alpha(mn)^k\big)\Big|\Big)^{2uw}\\ &\ll (\#\mathcal{A})^{2uw-2u}(\#\mathcal{B})^{2uw-2w} S_u(\mathcal{A})\, S_w(\mathcal{B})\,(AB)^k\Big(\frac{1}{q_1}+\frac{1}{A^k}+\frac{1}{B^k}+\frac{q_1}{(AB)^k}\Big)\log(2q_1B),\end{aligned}$$

where the implied constant depends only on u, w and k.

Proof. For $n\in\mathcal{B}$ let $v_n := \sum_{m\in\mathcal{A}}\xi_m e(\alpha(mn)^k)$, and choose a complex number ϵ_n of modulus 1 with $|v_n|^u = \epsilon_n v_n^u$. Expanding the power we get

$$v_n^u = \sum_{1\leqslant c\leqslant uA^k} \xi_c^{(u)}\, e(\alpha c n^k), \qquad \xi_c^{(u)} := \sum_{\substack{m_1,\dots,m_u\in\mathcal{A}\\ m_1^k+\cdots+m_u^k=c}} \xi_{m_1}\cdots\xi_{m_u}.$$

Let N_c be the number of terms in the last sum. Then $|\xi_c^{(u)}|\leqslant N_c$, $\sum_c N_c = (\#\mathcal{A})^u$ and $\sum_c N_c^2 = S_u(\mathcal{A})$. By Hölder's inequality,

$$\Big(\sum_{n\in\mathcal{B}}|v_n|\Big)^u \leqslant (\#\mathcal{B})^{u-1}\sum_{n\in\mathcal{B}}\epsilon_n v_n^u = (\#\mathcal{B})^{u-1}\sum_c \xi_c^{(u)}\, y_c \leqslant (\#\mathcal{B})^{u-1}\sum_c N_c|y_c|,$$

where $y_c := \sum_{n\in\mathcal{B}} \epsilon_n e(\alpha c n^k)$, and, again by Hölder's inequality,

$$\Big(\sum_c N_c|y_c|\Big)^{2w} \leqslant \Big(\sum_c N_c\Big)^{2w-2}\Big(\sum_c N_c^2\Big)\Big(\sum_c |y_c|^{2w}\Big).$$

In the same way as above, $|y_c|^{2w} = \sum_{|l|\leqslant wB^k} \tilde{\epsilon}_l\, e(\alpha c l)$ with coefficients that do not depend on c and satisfy $|\tilde{\epsilon}_l| \leqslant S_w(\mathcal{B})$; indeed, the number of solutions of the equation

$$n_1^k + \cdots + n_w^k - n_{w+1}^k - \cdots - n_{2w}^k = l \qquad (n_i \in \mathcal{B})$$

is $\int_{\mathbb{T}} |\sum_{n\in\mathcal{B}} e(\beta n^k)|^{2w} e(-\beta l)\,\mathrm{d}\beta \leqslant S_w(\mathcal{B})$. Carrying out the summation over c we arrive at

$$\Big(\sum_{n\in\mathcal{B}} |v_n|\Big)^{2uw} \leqslant (\#\mathcal{B})^{2uw-2w}(\#\mathcal{A})^{2uw-2u} S_u(\mathcal{A})\, S_w(\mathcal{B}) \sum_{|l|\leqslant wB^k} \min\Big\{uA^k, \frac{1}{2\|\alpha l\|}\Big\}. \quad (9.11)$$

The term $l = 0$ contributes uA^k to the last sum. For $l \neq 0$ we have $uA^k \leqslant uwA^kB^k/|l|$, and [32, Lemma 2.2] shows that

$$\sum_{1\leqslant l\leqslant wB^k} \min\Big\{\frac{uw(AB)^k}{l}, \frac{1}{\|\alpha l\|}\Big\} \ll (AB)^k\Big(\frac{1}{q_1} + \frac{1}{A^k} + \frac{q_1}{(AB)^k}\Big)\log(2q_1B). \qquad \square$$

A suitable rational approximation is provided by the next lemma.

Lemma 9.5. *Let $X \geqslant 1$ and $\beta \in \mathbb{R}$, and let $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ be coprime with $q \leqslant X^{k/2}$ and $|q\beta - a| \leqslant X^{-k/2}$. Put $\Xi := q + X^k|q\beta - a|$. Then there are coprime $a_1 \in \mathbb{Z}$ and $q_1 \in \mathbb{N}$ with*

$$\Big|\beta - \frac{a_1}{q_1}\Big| \leqslant \frac{1}{q_1^2}, \qquad q_1 \leqslant 4X^k, \qquad \frac{1}{q_1} + \frac{q_1}{X^k} \leqslant \frac{12}{\Xi}.$$

Proof. Note that $\Xi \leqslant 2X^{k/2}$. If $q \geqslant \frac{1}{2}\Xi$, let $a_1 = a$ and $q_1 = q$. Then $1/q \leqslant 2/\Xi$, $q/X^k \leqslant X^{-k/2} \leqslant 2/\Xi$ and $|\beta - a/q| \leqslant 1/(qX^{k/2}) \leqslant q^{-2}$. Now let $q < \frac{1}{2}\Xi$, so that $|q\beta - a| > \Xi/(2X^k)$. By Dirichlet's theorem there are coprime a_1 and q_1 with $q_1 \leqslant 4X^k/\Xi$ and $|q_1\beta - a_1| \leqslant \Xi/(4X^k)$. Then $|\beta - a_1/q_1| \leqslant q_1^{-2}$ and $a_1/q_1 \neq a/q$, and therefore

$$\frac{1}{qq_1} \leqslant \Big|\frac{a}{q} - \frac{a_1}{q_1}\Big| \leqslant \frac{|q\beta - a|}{q} + \frac{|q_1\beta - a_1|}{q_1} \leqslant \frac{\Xi}{qX^k} + \frac{\Xi}{4q_1X^k}.$$

We multiply by qq_1 and use that $q < \frac{1}{2}\Xi$ and $\Xi^2 \leqslant 4X^k$. This gives $1 \leqslant q_1\Xi X^{-k} + \frac{1}{2}$, that is, $q_1 \geqslant X^k/(2\Xi)$. Hence $1/q_1 \leqslant 2\Xi/X^k \leqslant 8/\Xi$ and $q_1/X^k \leqslant 4/\Xi$. $\square$

The bilinear structure of the sums $F_j^{\mathfrak{e}}$ is made explicit in the following lemma.

Lemma 9.6. *Let $1 \leqslant i \leqslant j \leqslant J$, let $\mathfrak{e}$ be a finite set of primes, and let X be attached to j. Put $M_0 := R^i$, $M_1 := (\zeta R)^i$, $Y := X/M_0$ and $\mathcal{B} := \mathcal{C}_{j-i}^{\mathfrak{e}}(Y)$. Then $\mathcal{D}_i^{\mathfrak{e}} \subseteq (M_0, M_1] \cap \mathcal{A}(M_1, \zeta R)$ and $\mathcal{B} \subseteq [1, Y] \cap \mathcal{A}(Y, \zeta R)$, and for all real β*

$$|F_j^{\mathfrak{e}}(\beta; X)| \leqslant (\log P)^2 \sup_{(\xi_d)} \sum_{n\in\mathcal{B}} \Big|\sum_{d\in\mathcal{D}_i^{\mathfrak{e}}} \xi_d\, e\big(\beta(dn)^k\big)\Big|, \quad (9.12)$$

where the supremum is over all families (ξ_d) of complex numbers of modulus 1.

Proof. The assertions about $\mathcal{D}_i^{\mathfrak{e}}$ and $\mathcal{B}$ are clear from the definitions. Every $n \in \mathcal{B}$ has a unique representation $n = d''m$ with $d'' \in \mathcal{D}_{j-i}^{\mathfrak{e}}$ and $m \leqslant R$, as in Lemma 3.1(b); let $p^+(n)$ denote the largest prime factor of d'', with $p^+(n) := 0$ if $d'' = 1$. For $d \in \mathcal{D}_i^{\mathfrak{e}}$ let $p^-(d)$ denote the least prime factor of d.

We claim that the map $(d, n) \mapsto dn$ is a bijection from the set of all pairs with $d \in \mathcal{D}_i^{\mathfrak{e}}$, $n \in \mathcal{B}$, $dn \leqslant X$ and $p^+(n) < p^-(d)$ onto $\mathcal{C}_j^{\mathfrak{e}}(X)$. Indeed, for such a pair we have $n = d''m$ as above, and $dd'' \in \mathcal{D}_j^{\mathfrak{e}}$ because all prime factors of d'' are smaller than those of d; hence $dn \in \mathcal{C}_j^{\mathfrak{e}}(X)$. Conversely, let $v = p_1 \cdots p_j m \in \mathcal{C}_j^{\mathfrak{e}}(X)$ with primes $p_1 < \cdots < p_j$ from $\mathcal{P} \setminus \mathfrak{e}$ and $m \leqslant R$. If $v = dn$ with a pair of the kind described, then $dd'' = p_1 \cdots p_j$ by Lemma 3.1(b), and the condition $p^+(n) < p^-(d)$ holds if and only if $d = p_{j-i+1} \cdots p_j$. For this d the number $n = v/d$ satisfies $n \leqslant X/d < Y$, so that $n \in \mathcal{B}$. This proves the claim, and we conclude that

$$F_j^{\mathfrak{e}}(\beta; X) = \sum_{d \in \mathcal{D}_i^{\mathfrak{e}}} \sum_{\substack{n \in \mathcal{B} \\ dn \leqslant X,\ p^+(n) < p^-(d)}} e\big(\beta(dn)^k\big).$$

The two conditions on d are removed by means of two kernels. For a natural number $d \leqslant M_1$ and real $y > 0$ one has

$$\int_{-1/2}^{1/2} e(d\psi) K_y(\psi)\, \mathrm{d}\psi = \begin{cases} 1, & d \leqslant y, \\ 0, & d > y, \end{cases} \qquad \text{where} \quad K_y(\psi) := \sum_{1 \leqslant l \leqslant \min\{y, M_1\}} e(-l\psi),$$

and for integers c and a with $1 \leqslant c \leqslant \zeta R$ and $a \geqslant 0$

$$\int_{-1/2}^{1/2} e(-c\psi') \widetilde{K}_a(\psi')\, \mathrm{d}\psi' = \begin{cases} 1, & c > a, \\ 0, & c \leqslant a, \end{cases} \qquad \text{where} \quad \widetilde{K}_a(\psi') := \sum_{a < l \leqslant \zeta R} e(l\psi').$$

Also $|K_y(\psi)| \leqslant \min\{M_1, 1/(2|\psi|)\}$ and $|\widetilde{K}_a(\psi')| \leqslant \min\{\zeta R, 1/(2|\psi'|)\}$ for $|\psi|, |\psi'| \leqslant \frac{1}{2}$, and the integral of $\min\{M, 1/(2|\psi|)\}$ over $[-\frac{1}{2}, \frac{1}{2}]$ equals $1 + \log M$. We use the first identity with $y = X/n$ and the second with $c = p^-(d)$ and $a = p^+(n)$. This gives

$$F_j^{\mathfrak{e}}(\beta; X) = \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \sum_{n \in \mathcal{B}} K_{X/n}(\psi) \widetilde{K}_{p^+(n)}(\psi') \sum_{d \in \mathcal{D}_i^{\mathfrak{e}}} e\big(\beta(dn)^k + \psi d - \psi' p^-(d)\big)\, \mathrm{d}\psi\, \mathrm{d}\psi',$$

and (9.12) follows, with $\xi_d = e(\psi d - \psi' p^-(d))$, because $(1 + \log M_1)(1 + \log \zeta R) \leqslant (\log P)^2$ for large P. □

Let $\delta_* := \mathfrak{d}(2k)$; thus $\delta_* + \log \delta_* + 1 = 0$, and δ_* does not depend on k. Numerically $\delta_* = 0.2784645\ldots$ and $4/(1 - 2\delta_*) = 9.0279002\ldots$, so that

$$\varepsilon_* := \frac{1 - 2\delta_*}{4k} - \tau_k > 0. \tag{9.13}$$

Lemma 9.7. *Suppose that J is sufficiently large in terms of k and γ. Let $0 \leqslant j \leqslant J$, let $\mathfrak{e}$ be a finite set of primes, and let $X = P^x$ be attached to j, with $x \geqslant 8\gamma$. Let $\beta \in \mathbb{R}$, and let $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ be coprime with $q \leqslant X^{k/2}$ and $|q\beta - a| \leqslant X^{-k/2}$. Then, with $\Xi := q + X^k|q\beta - a|$,*

$$F_j^{\mathfrak{e}}(\beta; X) \ll X^{1-\tau_k} \Big(\frac{X^{k/2}}{\Xi} \Big)^{1/(2k^2)} R^{1/4}.$$

Proof. By Lemma 3.1(a) and (3.2) we have $(j+1)\eta \geqslant x \geqslant 8\eta$, so that $j \geqslant 7$. Let $i := \lfloor \frac{1}{2}(j+1) \rfloor$; then $1 \leqslant i \leqslant j$ and $\frac{1}{2}j \leqslant i \leqslant \frac{1}{2}(j+1)$. Since $R^{j+7/8} \leqslant X \leqslant R^{j+1}$, it follows that $X^{1/2}R^{-1/2} \leqslant R^i \leqslant X^{1/2}R^{1/16}$. In the notation of Lemma 9.6 we therefore have, by (3.3),

$$\begin{gathered} X^{1/2}R^{-1/2} \leqslant M_0 \leqslant M_1 \leqslant X^{1/2}R^{1/4}, \qquad X^{1/2}R^{-1/2} \leqslant Y \leqslant X^{1/2}R^{1/2}, \\ X \leqslant M_1 Y \leqslant XR^{1/8}, \qquad \min\{M_0, Y\} \geqslant P^{3\gamma}. \end{gathered} \tag{9.14}$$

To the sum on the right of (9.12) we apply Lemma 9.4 with $u = w = k$, $\mathcal{A} = \mathcal{D}_i^{\mathfrak{e}}$, $A = M_1$, $B = Y$, and with a_1, q_1 as in Lemma 9.5. The mean values are bounded by (7.1) with $v = 2k$, and with the number ε_* of (9.13): since $\mathcal{D}_i^{\mathfrak{e}} \subseteq \mathcal{A}(M_1, \zeta R)$ and $\mathcal{B} \subseteq \mathcal{A}(Y, \zeta R)$, by Lemma 9.6, we have

$$S_k(\mathcal{D}_i^{\mathfrak{e}}) \ll M_1^{k+k\delta_*+\varepsilon_*}, \qquad S_k(\mathcal{B}) \ll Y^{k+k\delta_*+\varepsilon_*},$$

provided that $\zeta R \leqslant \min\{M_1, Y\}^{\eta'}$ with a certain $\eta' > 0$ depending only on k; as $\zeta R \leqslant P^{2\eta}$, by (9.14) this holds when $2\eta \leqslant 3\gamma\eta'$. Thus Lemma 9.4 gives

$$\Big(\sum_{n\in\mathcal{B}}\Big|\sum_{d\in\mathcal{D}_i^{\mathfrak{e}}} \xi_d\, e\big(\beta(dn)^k\big)\Big|\Big)^{2k^2} \ll (M_1Y)^{2k^2+k\delta_*+\varepsilon_*}\Big(\frac{1}{q_1} + \frac{q_1}{(M_1Y)^k} + \frac{1}{M_1^k} + \frac{1}{Y^k}\Big)\log P,$$

uniformly in the numbers ξ_d of modulus 1. By Lemma 9.5 and (9.14), the sum of the first two terms in the bracket is at most $12/\Xi$, and the sum of the other two is at most $2X^{-k/2}R^{k/2} \leqslant 4\,\Xi^{-1}R^{k/2}$. Since $M_1Y \leqslant XR^{1/8}$ and

$$X^{1+\delta_*/(2k)}\,\Xi^{-1/(2k^2)} = X^{1-\tau_k-\varepsilon_*}\Big(\frac{X^{k/2}}{\Xi}\Big)^{1/(2k^2)},$$

the lemma follows from (9.12): the powers of R that arise have exponents with sum at most $\frac{1}{8}(1+\frac{1}{k}) + \frac{1}{4k} \leqslant \frac{1}{4}$, and the factors $X^{\varepsilon_*/(2k^2)}$ and $(\log P)^3$ are absorbed by $X^{-\varepsilon_*}$, because $X \geqslant P^{8\gamma}$. □

Lemma 9.8. *Suppose that J is sufficiently large in terms of k and γ. Then, uniformly for $P^{1/4} \leqslant Q \leqslant P^{k/2}$,*

$$\sup_{\alpha\in\mathfrak{N}(Q;P)} |F(\alpha)| \ll P^{1+\eta}\,Q^{-2\tau_k/k}.$$

Proof. Let $\alpha \in \mathfrak{N}(Q;P)$, and let a and q be coprime with $q \leqslant Q$ and $|q\alpha - a| \leqslant QP^{-k}$. By (A3), the number $\Xi_\alpha := q + P^k|q\alpha - a|$ exceeds $\frac{1}{2}Q$. Define μ by $P^\mu = \min\{P, (2Q)^{2/k}\}$, so that $\mu \in [\frac{1}{2k}, 1]$, and let i be the largest integer in $[0, J]$ with $(\zeta R)^i \leqslant P^{1-\mu}$. Let $d \in \mathcal{D}_i$ and $Y := P/d$. Then $Y \geqslant P^\mu > R$, so that $i < J$ by Lemma 3.1(b), and (9.10) holds, as in the proof of Corollary 9.3. In particular

$$Q \leqslant Y^{k/2} < 2Q\,R^{9k/16}.$$

Moreover Y is attached to $J - i$, and $Y \geqslant P^{1/(2k)} \geqslant P^{8\gamma}$ by (3.1). Now let $g := (q, d^k)$, $q_d := q/g$ and $a_d := ad^k/g$. Then $(a_d, q_d) = 1$, and since $Y^k|q_dd^k\alpha - a_d| = P^k|q\alpha - a|/g$, we have

$$q_d \leqslant Q \leqslant Y^{k/2}, \qquad |q_dd^k\alpha - a_d| \leqslant QY^{-k} \leqslant Y^{-k/2}, \qquad q_d + Y^k|q_dd^k\alpha - a_d| = \frac{\Xi_\alpha}{g}.$$

Thus Lemma 9.7 is applicable to $F_{J-i}^{[d]}(d^k\alpha;Y)$, with $\Xi = \Xi_\alpha/g$, and it shows that

$$F_{J-i}^{[d]}(d^k\alpha;P/d) \ll Y^{1-\tau_k}\Big(\frac{g\,Y^{k/2}}{\Xi_\alpha}\Big)^{1/(2k^2)}R^{1/4} \ll \frac{P}{d}\,d_q(d)\,Q^{-2\tau_k/k}R^{1/2}.$$

We used that g divides $d_q(d)^k$, that $Y^{k/2}/\Xi_\alpha < 4R^{9k/16}$, and that $Y^{-\tau_k} \leqslant P^{-\mu\tau_k} \leqslant Q^{-2\tau_k/k}$. Summing over $d \in \mathcal{D}_i$ by means of Lemma 3.1(c) and (9.8), we obtain the lemma. □

For $Q = P^{k/2}$ Lemma 9.8 is a bound of Weyl type, $F(\alpha) \ll P^{1-\tau_k+\eta}$, on the set $\mathfrak{N}(P^{k/2};P) = \mathbb{T} \setminus \mathfrak{M}(\frac{1}{2}P^{k/2};P)$ of extreme minor arcs, in the terminology of [6, §5]; the exponent τ_k is the one obtained for smooth Weyl sums in [6, Lemma 7.1]. For smaller Q it is the analogue of [6, (5.6)]: on arcs of height Q the saving is that of a sum of length $(2Q)^{2/k}$. In (ii) of §1.2 the factor P^η is written as P^ε, in accordance with the convention stated there: $\eta = 1/(J+1)$ is as small as we please when J is large.

Proof of (9.4). Let $P^{1/4} \leqslant Q \leqslant P^{k/2}$. By Lemma 9.8 and Corollary 9.3,

$$\int_{\mathfrak{N}(Q;P)}|F|^s\,\mathrm{d}\alpha \leqslant \Big(\sup_{\mathfrak{N}(Q;P)}|F|\Big)^t\int_{\mathfrak{M}(Q;P)}|F|^r\,\mathrm{d}\alpha \ll P^{s-k}\,Q^{-2(t\tau_k-b)/k}\,P^{4k\gamma+2\varepsilon+2\eta(t+b)}.$$

By (9.2) and (9.3) we have $4k\gamma \leqslant \frac{\nu}{10k}$, $2\varepsilon \leqslant \frac{\nu}{320k^4}$ and $2\eta(t+b) \leqslant \frac{\nu}{50k}$, so the last factor is at most $P^{\nu/(4k)} \leqslant Q^{\nu/k}$. This gives (9.4). □

9.4. **Arcs of small height.** On $\mathfrak{M}(P^{1/4};P)$ a second use of the bilinear structure gives a bound that decays with the height.

Lemma 9.9. *Let $\varepsilon_3 > 0$. Then, for $\alpha \in \mathfrak{M}(P^{1/4};P)$,*

$$F(\alpha) \ll P^{1+\omega+\varepsilon_3}\,\Upsilon(\alpha)^{1/(2k)},$$

where the implied constant depends on ε_3 as well.

Proof. Let a and q be coprime with $q \leqslant P^{1/4}$ and $|q\alpha - a| \leqslant P^{1/4-k}$, and put $\beta := \alpha - a/q$ and $\Xi := q + P^k|q\alpha - a| = \Upsilon(\alpha)^{-1}$; thus $\Xi \leqslant 2P^{1/4}$. We apply Lemma 9.6 with $j = J$, $\mathfrak{e} = \emptyset$, $X = P$ and $i := \lfloor \frac{1}{2}(J+1)\rfloor$. Then M_0 is one of the numbers $P^{1/2}$ and $P^{1/2}R^{-1/2}$, and $M_1 = \zeta^i M_0$, so that by (3.3)

$$\begin{gathered}P^{1/2}R^{-1/2} \leqslant M_0 \leqslant P^{1/2} \leqslant Y \leqslant P^{1/2}R^{1/2},\\ M_1 \leqslant M_0P^\omega, \qquad P \leqslant M_1Y \leqslant P^{1+\omega}.\end{gathered} \tag{9.15}$$

Let (ξ_d) be complex numbers of modulus 1. By Cauchy's inequality, in which we extend the sum over $n \in \mathcal{B}$ to all natural numbers $n \leqslant Y$,

$$\Big(\sum_{n\in\mathcal{B}}\Big|\sum_{d\in\mathcal{D}_i}\xi_d\,e(\alpha d^kn^k)\Big|\Big)^2 \leqslant Y\sum_{n\leqslant Y}\Big|\sum_{d\in\mathcal{D}_i}\xi_d\,e(\alpha d^kn^k)\Big|^2 \leqslant Y\sum_{d_1,d_2\in\mathcal{D}_i}\Big|\sum_{n\leqslant Y}e(\alpha hn^k)\Big|, \tag{9.16}$$

where $h := d_1^k - d_2^k$. By (9.12), $|F(\alpha)|^2$ is at most $(\log P)^4$ times the right hand side of (9.16). The terms with $d_1 = d_2$ contribute at most $M_1Y^2 \leqslant P^{3/2+\omega+\eta/2}$ to it.

Let $d_1 \neq d_2$. Then $\alpha h = ah/q + \beta h$, and the denominator of ah/q in lowest terms is $q_h := q/(q,h)$. The inner sum on the right of (9.16) is a classical Weyl sum of length

Y, to which we apply [32, Theorem 4.1] together with the bounds [32, Theorem 4.2 and Lemma 2.8] for the two factors of the main term. This gives

$$\sum_{n\leqslant Y} e(\alpha h n^k) \ll q_h^{-1/k}\min\bigl\{Y, |\beta h|^{-1/k}\bigr\} + q^{1/2+\varepsilon_3}\bigl(1+Y^k|\beta h|\bigr)^{1/2}. \tag{9.17}$$

Since $|h|\leqslant M_1^k$, we have $q(1+Y^k|\beta h|)\leqslant q+(M_1Y)^k|q\alpha-a|\leqslant P^{k\omega}\,\Xi$ by (9.15). Hence the second term on the right of (9.17) contributes

$$\ll M_1^2 Y\, P^{\varepsilon_3+k\omega/2}\,\Xi^{1/2} \ll P^{13/8+(k+4)\omega/2+\varepsilon_3}$$

to the right hand side of (9.16). For the first term we use Cauchy's inequality once more:

$$\sum_{\substack{d_1,d_2\in\mathcal{D}_i\\ d_1\neq d_2}} q_h^{-1/k}\min\bigl\{Y,|\beta h|^{-1/k}\bigr\} \leqslant (\Sigma_1\Sigma_2)^{1/2},$$

where

$$\Sigma_1 := \sum_{d_1,d_2\leqslant M_1} q_h^{-2/k}, \qquad \Sigma_2 := \sum_{\substack{d_1,d_2\leqslant M_1\\ d_1\neq d_2}} \min\bigl\{Y^2,|\beta h|^{-2/k}\bigr\},$$

and d_1, d_2 now run over all natural numbers up to M_1.

The sum Σ_1. For $g\in\mathbb{N}$ let $N(g)$ be the number of pairs (x,y) modulo g with $x^k\equiv y^k \pmod g$. Since $(q,h)^{2/k}\leqslant\sum_{g|(q,h)} g^{2/k}$, and since $q\leqslant P^{1/4}\leqslant M_1$,

$$\Sigma_1 \leqslant q^{-2/k}\sum_{g|q} g^{2/k}\Bigl(\frac{M_1}{g}+1\Bigr)^2 N(g) \leqslant 4M_1^2 q^{-2/k}\sum_{g|q} g^{2/k-2}N(g).$$

The function N is multiplicative, and we claim that $N(p^e)\leqslant 2k(e+1)\,p^{2e-2e/k}$ for every prime power p^e. Indeed, there are $p^{2(e-\lceil e/k\rceil)}\leqslant p^{2e-2e/k}$ pairs with $p^e\mid x^k$ and $p^e\mid y^k$. In every other pair the exact power p^u of p dividing y satisfies $ku<e$, it is also the exact power dividing x, and $x=p^ux'$, $y=p^uy'$ with $x'^k\equiv y'^k \pmod{p^{e-ku}}$. For each of the at most p^{e-u} classes y' modulo p^{e-u} this leaves at most $2k$ classes for x' modulo p^{e-ku}, because the congruence $z^k\equiv 1$ has at most $2k$ solutions in the group of reduced residues modulo a prime power, and each of these classes splits into $p^{(k-1)u}$ classes modulo p^{e-u}. So there are at most $2k\,p^{e+(k-2)u}\leqslant 2k\,p^{2e-2e/k}$ such pairs for each $u<e/k$, and the claim follows. Consequently $g^{2/k-2}N(g)\leqslant\prod_{p^e\|g}2k(e+1)\ll g^{\varepsilon_3}$, and

$$\Sigma_1 \ll M_1^2 q^{-2/k+\varepsilon_3}.$$

The sum Σ_2. If $d_2<d_1$ then $h\geqslant (d_1-d_2)\,d_1^{k-1}$. Hence, when $(M_1Y)^k|\beta|\geqslant 1$,

$$\Sigma_2 \leqslant \frac{2}{|\beta|^{2/k}}\sum_{d_1\leqslant M_1} d_1^{-2+\frac{2}{k}}\sum_{l<d_1} l^{-\frac{2}{k}} \leqslant \frac{6\,(1+\log M_1)}{|\beta|^{2/k}} \leqslant \frac{12\,(1+\log M_1)\,(M_1Y)^2}{\bigl(1+(M_1Y)^k|\beta|\bigr)^{2/k}},$$

because $\sum_{l<n} l^{-2/k}\leqslant 3n^{1-2/k}$. When $(M_1Y)^k|\beta|<1$ the final bound holds trivially, as $\Sigma_2\leqslant M_1^2Y^2$.

Combining these estimates, and noting that $q\bigl(1+(M_1Y)^k|\beta|\bigr)\geqslant\Xi$ and $(M_1Y)^2\leqslant P^{2+2\omega}$ by (9.15), we see that the first term on the right of (9.17) contributes $\ll Y(\Sigma_1\Sigma_2)^{1/2}$, and hence $\ll P^{2+2\omega+\varepsilon_3}\,\Xi^{-1/k}$, to the right hand side of (9.16). Altogether

$$|F(\alpha)|^2 \ll (\log P)^4\bigl(P^{2+2\omega+\varepsilon_3}\,\Xi^{-1/k} + P^{3/2+\omega+\eta/2} + P^{13/8+(k+4)\omega/2+\varepsilon_3}\bigr).$$

As $\Xi \leqslant 2P^{1/4}$, the first term in the bracket is $\gg P^{2-1/(4k)+2\omega+\varepsilon_3}$, and it dominates the other two, because $\eta \leqslant \frac{1}{2}$ and $k\omega \leqslant \frac{1}{2}$. This proves the lemma. □

Proof of (9.5). Let $R^{1/4} \leqslant Q \leqslant P^{1/4}$ and $\alpha \in \mathfrak{N}(Q;P)$. Then $\Upsilon(\alpha) < 2/Q$ by (A3). Lemma 9.9 with $\varepsilon_3 = \omega$, and (A1), give

$$\int_{\mathfrak{N}(Q;P)} |F|^s \,\mathrm{d}\alpha \ll \big(P^{1+2\omega}Q^{-1/(2k)}\big)^s Q^2 P^{-k} \leqslant P^{s-k}\, Q^{-1/(2k)}\, P^{2s\omega},$$

because $s \geqslant 4k+1$. By (9.3), $2s\omega = \eta/(32k)$, so that $P^{2s\omega} = R^{1/(32k)} \leqslant Q^{1/(4k)}$. □

On the arcs of height at most $R^{1/4}$ neither a factor P^{ε} nor the density Π can be given away. Here the unrestricted factor m of the integers in $\mathcal{C}$ is decisive: the sum over m is a classical Weyl sum, and F has the classical approximation, with the factor Π.

Lemma 9.10. *Let*

$$S(q,a) := \sum_{x=1}^{q} e\Big(\frac{ax^k}{q}\Big), \qquad v(\beta) := \int_0^P e(\beta y^k)\,\mathrm{d}y.$$

Let $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ be coprime with $q \leqslant R^{1/4}$, and let β be a real number with $|\beta| \leqslant R^{1/4}P^{-k}$. Then

$$F\Big(\frac{a}{q}+\beta\Big) = \Pi\, q^{-1} S(q,a)\, v(\beta) + O\big(PR^{-1/2}\big).$$

Consequently $F(\alpha) \ll \Pi\, P\, \Upsilon(\alpha)^{1/k} + PR^{-1/2}$ for $\alpha \in \mathfrak{M}(R^{1/4};P)$.

Proof. Let $\alpha = a/q+\beta$. By Lemma 3.1(b), $F(\alpha) = \sum_{d\in\mathcal{D}_J} \sum_{m\leqslant P/d} e(\alpha d^k m^k)$. Fix $d \in \mathcal{D}_J$ and let $Y := P/d$, so that $Y \leqslant R$. Every prime factor of d exceeds R; hence $(d,q)=1$, and $\alpha d^k = ad^k/q + \beta d^k$ with $(ad^k,q)=1$. By [32, Theorem 4.1], applied with $n = \lfloor Y\rfloor^k$,

$$\sum_{m\leqslant Y} e(\alpha d^k m^k) = q^{-1}S(q,ad^k)\, v_d + O\big(q^{1/2+\varepsilon}(1+Y^k|\beta|d^k)^{1/2}\big),$$

where $v_d := \frac{1}{k}\sum_{l\leqslant \lfloor Y\rfloor^k} l^{-1+1/k} e(\beta d^k l)$. Here $S(q,ad^k) = S(q,a)$, because $x \mapsto dx$ permutes the residues modulo q, and $Y^k|\beta|d^k = P^k|\beta| \leqslant R^{1/4}$. Euler's summation formula, applied on $[1,n]$ to the function $u \mapsto \frac{1}{k}u^{1/k-1}e(\beta d^k u)$, which is bounded by $\frac{1}{k}$ there and has total variation at most $\frac{1}{k} + 2\pi|\beta|d^k n^{1/k}$ on $[1,n]$, and the substitution $u = y^k$ give

$$v_d = \int_0^Y e(\beta d^k y^k)\,\mathrm{d}y + O\big(1+Y|\beta|d^k\big) = \frac{1}{d}\, v(\beta) + O\big(R^{1/4}\big),$$

since $n^{1/k} = \lfloor Y\rfloor \leqslant Y$, since the intervals $[0,1]$ and $[\lfloor Y\rfloor, Y]$ contribute $O(1)$ to the integral, and since $Y|\beta|d^k \leqslant P^k|\beta|$. As $q^{1/2+\varepsilon} \leqslant R^{1/6}$, we conclude that

$$\sum_{m\leqslant P/d} e(\alpha d^k m^k) = \frac{1}{d}\, q^{-1} S(q,a)\, v(\beta) + O\big(R^{3/8}\big).$$

Summing over $d \in \mathcal{D}_J$, and recalling that $\#\mathcal{D}_J \leqslant (\zeta R)^J \leqslant PR^{-7/8}$ by (3.3), we obtain the first assertion.

For the second, let $\alpha \in \mathfrak{M}(R^{1/4}; P)$, and let a and q be as in (A2); then $\beta := \alpha - a/q$ satisfies $|\beta| \leqslant R^{1/4}P^{-k}$, and $\Upsilon(\alpha)^{-1} = q(1+P^k|\beta|)$. By the classical theory [32, Chapters 2 and 4], $S(q,a) \ll q^{1-1/k}$ and $v(\beta) \ll P(1+P^k|\beta|)^{-1/k}$. □

Proof of (9.6). Put $Q_0 := R^{1/4}$. We first note that, for $1 \leqslant Q \leqslant Q_0$ and $\delta > 0$,

$$\int_{\mathfrak{M}(Q_0;P)\setminus\mathfrak{M}(Q;P)} \Upsilon(\alpha)^{2+\delta}\,\mathrm{d}\alpha \ll_\delta P^{-k}Q^{-\delta}. \tag{9.18}$$

If $\alpha = a/q + \beta$ lies in this set, with a and q as in (A2), then $\Upsilon(\alpha)^{-1} = q(1+P^k|\beta|) > Q$, and the integral is at most

$$\sum_{q\leqslant Q_0}\varphi(q)\int_{q(1+P^k|\beta|)>Q}\frac{\mathrm{d}\beta}{\big(q(1+P^k|\beta|)\big)^{2+\delta}} \leqslant \frac{2}{P^k}\sum_{q\leqslant Q_0} q^{-1-\delta}\min\Big\{1,\frac{q}{Q}\Big\}^{1+\delta} \ll_\delta P^{-k}Q^{-\delta}.$$

By Lemma 9.10, $|F|^s \ll \Pi^s P^s\,\Upsilon^{s/k} + P^s R^{-s/2}$ on $\mathfrak{M}(Q_0;P)$. Here $s/k \geqslant 4$, because $s \geqslant 4k+1$, and the measure of $\mathfrak{M}(Q_0;P)$ is at most $2R^{1/2}P^{-k}$, by (A1). Hence (9.18), with $\delta = s/k - 2 \geqslant 2$, gives

$$\int_{\mathfrak{M}(Q_0;P)\setminus\mathfrak{M}(Q;P)} |F|^s\,\mathrm{d}\alpha \ll \Pi^s P^{s-k}Q^{-2} + P^{s-k}R^{(1-s)/2},$$

which is (9.6). □

This completes the proof of Proposition 9.1.

9.5. **The proof of Theorem 2.3.**

Proof of Theorem 2.3. Put $\mathcal{Y} := k\big(C(h_k) - 2T(\Theta) + 2R_k(\Theta) + W_k(\Theta)\big)$, and let s be an integer with $s > \mathcal{Y}$. We shall show that $G(k) \leqslant s$; this gives $G(k) \leqslant \lfloor \mathcal{Y} \rfloor + 1$, which is the first assertion of the theorem.

The choice of the parameters. Let δ_0 and ε_2 be as in (3.1), with $\varepsilon_2 \leqslant \delta_0$, let c_Θ, u_Θ and $\mathcal{G}$ be as in §8, and put

$$b := k\,c_\Theta\,\tau_k, \qquad \text{so that} \qquad a_* = c_\Theta\,\log\frac{k\,c_\Theta\,u_\Theta}{\mathrm{e}\,b} = c_\Theta\,\Big(\log\frac{u_\Theta}{\tau_k} - 1\Big)$$

in (8.8); this is the value of b for which $k\,a_* + b/\tau_k$ is smallest. Then

$$k\big(\mathcal{G}(u_\Theta) + a_*\big) + \frac{b}{\tau_k} = k\Big(\mathcal{G}(u_\Theta) + c_\Theta\,\log\frac{u_\Theta}{\tau_k}\Big). \tag{9.19}$$

We bound $\mathcal{G}(u_\Theta)$ by Lemma 8.1(i) and let $\delta_0 \to 0$. Then $\eta_c \to h_k$, $w_c \to w := \frac{1}{10} - \frac{h_k}{50}$ and $z_c \to z := (1-h_k)w$; moreover $u_\Theta \to \frac12(1-\eta_k(\Theta))\sigma(\Theta) = y_k(\Theta)\,\tau_k$ and $c_\Theta \to 2/\Theta$, so that $c_\Theta\,\log(u_\Theta/\tau_k) \to W_k(\Theta)$. The integral in Lemma 8.1(i) tends to $R_k(\Theta)$: since $1-\delta_0-\eta_k(\theta) = (1-\delta_0)\big(1-\theta\kappa(\theta)/((1-\delta_0)k)\big)$, it is given by the formulae of Lemma 10.4 with $(1-\delta_0)k$ in place of k, and these are continuous in k. As Ψ and I are continuous, the upper limit of the right hand side of (9.19) is at most

$$k\Big(\Psi(z) + 2\,I(2w) - 2\,T(\Theta) + 2R_k(\Theta) + W_k(\Theta)\Big).$$

Now $\frac{1}{16} \leqslant w \leqslant \frac{1}{10}$, so Lemma 10.2 gives $I(2w) = \Lambda - \frac13\log\frac{3-12w}{18w}$. With $h := h_k$ we have $\frac{3-12w}{18w} = \frac{15+2h}{15-3h}$ and $z = \frac{1}{50}(1-h)(5-h)$, hence $\frac{1-z}{z} = \frac{45+6h-h^2}{(1-h)(5-h)}$, and therefore $\Psi(z) + 2I(2w) = C(h_k)$. Thus the last display equals $\mathcal{Y}$. Consequently we can fix δ_0

and ε_2 in such a way that the number in (9.19) is smaller than s. We then choose a real number $t > b/\tau_k$ so close to b/τ_k that $r := s-t$ exceeds $k(\mathcal{G}(u_\Theta)+a_*)$. By Lemma 8.1(v), $\mathcal{G}(u_\Theta)+a_* \geqslant \mathcal{G}(\tau_k) - c_\Theta \geqslant \Psi(\frac{1}{10}) - 1 > 3$, as in §9.1, so that $r > 3k$. With these choices (9.1) holds. Finally we fix γ and ε by (9.2), and then J, η and ω as in §3 and (9.3). The integer H is arbitrary in the range permitted in §3.

The circle method. Let N be a large natural number and $P := N^{1/k}$. By orthogonality, the integral

$$\mathcal{R}(N) := \int_{\mathbb{T}} F(\alpha)^s\, e(-N\alpha)\, \mathrm{d}\alpha$$

is the number of representations of N as a sum of s k-th powers of elements of $\mathcal{C}$. We shall show that

$$\mathcal{R}(N) = \Pi^s\, P^{s-k}\big(\mathfrak{S}_s(N)\, \mathfrak{J}_s + O(R^{-\nu_0/2})\big), \tag{9.20}$$

where $\mathfrak{S}_s(N)$ and $\mathfrak{J}_s$ are the singular series and the singular integral, and ν_0 is as in Proposition 9.1. As $\mathfrak{S}_s(N)\,\mathfrak{J}_s \gg 1$, this implies that $\mathcal{R}(N) > 0$ when N is large, and hence that $G(k) \leqslant s$.

Let $Q := R^{1/10}$, and let $\mathfrak{K}$ be the union of the arcs $\{\alpha \in \mathbb{T} : \|\alpha - a/q\| \leqslant QP^{-k}\}$ with $1 \leqslant a \leqslant q \leqslant Q$ and $(a,q) = 1$. For large P these arcs are disjoint, and $\mathfrak{M}(Q;P) \subseteq \mathfrak{K}$. For $\alpha = a/q + \xi P^{-k}$ in such an arc, Lemma 9.10 gives

$$F(\alpha) = \Pi\, q^{-1} S(q,a)\, P\, V(\xi) + O\big(PR^{-1/2}\big), \qquad \text{where} \quad V(\xi) := \int_0^1 e(\xi y^k)\, \mathrm{d}y,$$

because $v(\xi P^{-k}) = P\, V(\xi)$. Both $F(\alpha)$ and the main term are at most P in modulus, so that

$$F(\alpha)^s = \Pi^s P^s \big(q^{-1}S(q,a)\big)^s V(\xi)^s + O\big(P^s R^{-1/2}\big).$$

As $N = P^k$ and $\operatorname{meas} \mathfrak{K} \leqslant 2Q^3P^{-k}$, integration gives

$$\int_{\mathfrak{K}} F(\alpha)^s\, e(-N\alpha)\, \mathrm{d}\alpha = \Pi^s\, \mathfrak{S}_{s,Q}(N)\, \mathfrak{J}_{s,Q}\, P^{s-k} + O\big(P^{s-k}R^{-1/5}\big),$$

where

$$\mathfrak{S}_{s,Q}(N) := \sum_{q \leqslant Q} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \big(q^{-1}S(q,a)\big)^s e\Big(-\frac{Na}{q}\Big), \qquad \mathfrak{J}_{s,Q} := \int_{-Q}^{Q} V(\xi)^s e(-\xi)\, \mathrm{d}\xi.$$

Recall that $s \geqslant 4k+1$. By the classical theory [32, Chapters 2 and 4], $S(q,a) \ll q^{1-1/k}$ and $V(\xi) \ll \min\{1, |\xi|^{-1/k}\}$, so that $\mathfrak{S}_{s,Q}(N) = \mathfrak{S}_s(N) + O(Q^{-2})$ and $\mathfrak{J}_{s,Q} = \mathfrak{J}_s + O(Q^{-3})$ with the complete singular series $\mathfrak{S}_s(N)$ and singular integral $\mathfrak{J}_s$; moreover $\mathfrak{S}_s(N) \ll 1$, and $\mathfrak{S}_s(N) \gg 1$ uniformly in N when $s \geqslant 4k$ [32, Theorem 4.6], and $\mathfrak{J}_s = \Gamma(1+\frac{1}{k})^s \Gamma(\frac{s}{k})^{-1} > 0$ (this is the value at 1 of the density of the sum of the k-th powers of s independent random variables that are uniformly distributed on $[0,1]$).

On the complement of $\mathfrak{K}$ we use (9.7), noting that $\mathbb{T} \setminus \mathfrak{K} \subseteq \mathbb{T} \setminus \mathfrak{M}(Q;P)$ and that $Q^{-2} = R^{-1/5}$. We find that

$$\mathcal{R}(N) = \Pi^s P^{s-k}\big(\mathfrak{S}_s(N)\, \mathfrak{J}_s + O(R^{-1/5})\big) + O\big(P^{s-k}(R^{-1/5} + R^{-\nu_0})\big).$$

By Lemma 3.2 there is a number $c_J > 0$, depending on J only, with $\Pi \geqslant (c_J/\log P)^J$; hence $\Pi^s \geqslant R^{-\nu_0/2}$ when P is large. Since $\nu_0 < \frac{1}{5}$, this gives (9.20).

The second assertion. We differentiate C term by term. For $0 \leqslant h \leqslant \frac{1}{5}$,

$$\begin{aligned} C'(h) &= \frac{6-2h}{45+6h-h^2} + \frac{1}{1-h} + \frac{1}{5-h} - \frac{2}{3}\Big(\frac{2}{15+2h} + \frac{3}{15-3h}\Big) \\ &\leqslant \frac{6}{45} + \frac{5}{4} + \frac{5}{24} - \frac{2}{3}\Big(\frac{2}{15.4} + \frac{3}{15}\Big) < \frac{7}{5}. \end{aligned}$$

As $h_k \leqslant \frac{1}{5}$ for $k \geqslant 37$, we get $k\,C(h_k) \leqslant C_0 k + \frac{7}{5} k h_k = C_0 k + \frac{252\,k}{25(k-1)} \leqslant C_0 k + 11$. □

By (9.20) and Lemma 3.2, the number of representations that we count satisfies

$$\mathcal{R}(N) \asymp \Big(\frac{H}{\log P}\Big)^{Js} N^{s/k-1}.$$

When $H = 1$, that is, when the primes are taken from the single dyadic interval $(R, 2R]$, this falls short of the order of magnitude $N^{s/k-1}$ by the factor $(\log P)^{Js}$. When H equals $\lfloor \omega \log R/\log 2 \rfloor$, that is, for the union of dyadic intervals of §3.2, we have $\mathcal{R}(N) \asymp N^{s/k-1}$.

Remark 9.11. Lemma 9.4 is [41, Lemma 3.1] with the sets of that paper replaced by arbitrary sets of integers, and with bounded coefficients attached to one of the variables; the proof is the same. For smooth Weyl sums the passage to a bilinear sum costs a factor $P^{\varepsilon}R$ [41, Lemma 2.3], whereas in Lemma 9.6 it costs a factor $(\log P)^2$. In this section a factor d of a suitable size is split off from the integers counted by F in Lemmas 9.2, 9.7 and 9.9, and the unrestricted factor m gives the classical approximation of Lemma 9.10; for the variables of [10], in which the complementary factor is smooth, the theory of smooth Weyl sums on the major arcs [31, 33] would be needed instead.

9.6. **Almost all natural numbers.** We now prove Theorem 2.4. The argument is the familiar one, in which the contribution of the minor arcs is estimated in mean square over n by means of Bessel's inequality, so that a mean value of order 2ℓ controls the representations by ℓ k-th powers; compare [6, §9].

Let $\mathcal{Y}$ be the number defined in the proof of Theorem 2.3, let ℓ be an integer with $2\ell > \mathcal{Y}$, and let all parameters be chosen as in that proof with $s := 2\ell$. Then (9.1), (9.2) and (9.3) hold, so that Proposition 9.1 is available with $s = 2\ell$; moreover $2\ell > 4.19\,k$, by §9.1, so that $\ell \geqslant 2k+1$. For a large real number N let $P := N^{1/k}$, and for $n \in \mathbb{N}$ let

$$\mathcal{R}_\ell(n) := \int_{\mathbb{T}} F(\alpha)^\ell\, e(-n\alpha)\,\mathrm{d}\alpha,$$

which is the number of representations of n as a sum of ℓ k-th powers of elements of $\mathcal{C}$. Let $\mathfrak{S}_\ell(n)$ and $\mathfrak{S}_{\ell,Q}(n)$ be the singular series and its partial sums, as in §9.5 with ℓ and n in place of s and N, and put

$$\mathfrak{J}_\ell(u) := \Gamma\big(1+\tfrac{1}{k}\big)^\ell\, \Gamma\big(\tfrac{\ell}{k}\big)^{-1} u^{\ell/k-1} \qquad (0 < u \leqslant 1).$$

Proposition 9.12. *In this situation all integers n with $\frac{1}{2}N < n \leqslant N$, with at most $O(NR^{-\nu_0/4})$ exceptions, satisfy*

$$\mathcal{R}_\ell(n) = \Pi^\ell P^{\ell-k}\Big(\mathfrak{S}_\ell(n)\,\mathfrak{J}_\ell(n/N) + O\big(R^{-\nu_0/8}\big)\Big).$$

Proof. Let $Q := R^{1/10}$, and let $\mathfrak{K}$ be the set of major arcs defined in the proof of Theorem 2.3. For $\alpha = a/q + \xi P^{-k}$ in one of these arcs we have $e(-n\alpha) = e(-na/q)\, e(-\xi n/N)$, and the argument given there, with ℓ in place of s, shows that

$$\int_{\mathfrak{K}} F(\alpha)^\ell\, e(-n\alpha)\, \mathrm{d}\alpha = \Pi^\ell\, \mathfrak{S}_{\ell,Q}(n)\, \mathfrak{J}_{\ell,Q}(n/N)\, P^{\ell-k} + O\big(P^{\ell-k}R^{-1/5}\big),$$

where $\mathfrak{J}_{\ell,Q}(u) := \int_{-Q}^{Q} V(\xi)^\ell e(-\xi u)\, \mathrm{d}\xi$. Note that $\ell/k > 2.09$. As $S(q,a) \ll q^{1-1/k}$ and $V(\xi) \ll \min\{1, |\xi|^{-1/k}\}$, the singular series converges absolutely, $\mathfrak{S}_\ell(n) \ll 1$, and the difference $\mathfrak{S}_{\ell,Q}(n) - \mathfrak{S}_\ell(n)$ is $O(Q^{2-\ell/k})$ uniformly in n, and

$$\mathfrak{J}_{\ell,Q}(u) = \int_{-\infty}^{\infty} V(\xi)^\ell e(-\xi u)\, \mathrm{d}\xi + O(Q^{1-\ell/k})$$

uniformly in u. The last integral is the value at u of the density of the sum of the k-th powers of ℓ independent random variables that are uniformly distributed on $[0,1]$. For $0 < u \leqslant 1$ the upper bound 1 for these variables is immaterial, the density is the ℓ-fold convolution of the function $\frac{1}{k}y^{1/k-1}$ on $(0,\infty)$, and its value is $\mathfrak{J}_\ell(u)$. As $Q^{2-\ell/k} \leqslant Q^{-1/12} = R^{-1/120}$, we find that

$$\int_{\mathfrak{K}} F(\alpha)^\ell\, e(-n\alpha)\, \mathrm{d}\alpha = \Pi^\ell P^{\ell-k}\Big(\mathfrak{S}_\ell(n)\, \mathfrak{J}_\ell(n/N) + O\big(R^{-1/120}\big)\Big) + O\big(P^{\ell-k}R^{-1/5}\big) \quad (9.21)$$

uniformly for $\frac{1}{2}N < n \leqslant N$.

For $n \in \mathbb{Z}$ let $\mathcal{W}(n) := \int_{\mathbb{T}\setminus\mathfrak{K}} F(\alpha)^\ell\, e(-n\alpha)\, \mathrm{d}\alpha$. These are the Fourier coefficients of the function that equals F^ℓ on $\mathbb{T}\setminus\mathfrak{K}$ and vanishes on $\mathfrak{K}$. By Bessel's inequality and (9.7) with $s = 2\ell$, since $\mathbb{T}\setminus\mathfrak{K} \subseteq \mathbb{T}\setminus\mathfrak{M}(Q;P)$ and $Q^{-2} = R^{-1/5}$,

$$\sum_{n\in\mathbb{Z}} |\mathcal{W}(n)|^2 \leqslant \int_{\mathbb{T}\setminus\mathfrak{M}(Q;P)} |F(\alpha)|^{2\ell}\, \mathrm{d}\alpha \ll \Pi^{2\ell}P^{2\ell-k}R^{-1/5} + P^{2\ell-k}R^{-\nu_0} \ll \Pi^{2\ell}P^{2\ell-k}R^{-\nu_0/2}.$$

For the last step recall that $\Pi^{2\ell} \geqslant R^{-\nu_0/2}$ when P is large, and that $\nu_0 < \frac{1}{5}$. Hence the number of integers n with $|\mathcal{W}(n)| > \Pi^\ell P^{\ell-k}R^{-\nu_0/8}$ is $\ll P^kR^{-\nu_0/4} = NR^{-\nu_0/4}$. For all other n with $\frac{1}{2}N < n \leqslant N$ the assertion follows from (9.21), because $\mathcal{R}_\ell(n)$ is the sum of $\mathcal{W}(n)$ and the integral over $\mathfrak{K}$, and because $R^{-1/5} \leqslant \Pi^\ell R^{-1/6}$ and $\frac{1}{8}\nu_0 < \frac{1}{120}$. □

The lower bound for $G^+(k)$ when k is a power of 2 comes from the following elementary fact.

Lemma 9.13. *Let $k \geqslant 4$ be a power of 2, and let $n = x_1^k + \cdots + x_s^k$ with integers x_i, exactly j of which are odd. Then $n \equiv j \pmod{4k}$. If $k \geqslant 8$ and $n \equiv 4k \pmod{8k}$, then $s \geqslant 4k$. Consequently $G^+(k) \geqslant 4k$ when $k \geqslant 8$.*

Proof. If x is odd, then $x^{2^i} \equiv 1 \pmod{2^{i+2}}$ for all $i \geqslant 1$, by induction on i; in particular $x^k \equiv 1 \pmod{4k}$. If x is even, then 2^k divides x^k, and $4k$ divides 2^k, because both are powers of 2 and $4k \leqslant 2^k$. This gives the first assertion. Now let $k \geqslant 8$. Then $8k \leqslant 2^k$, so that $8k$ divides 2^k. If $n \equiv 4k \pmod{8k}$, then $j \equiv 0 \pmod{4k}$ by the first assertion, and $j \neq 0$, since otherwise $8k$ would divide n. Hence $s \geqslant j \geqslant 4k$. The integers $n \equiv 4k \pmod{8k}$ have density $1/(8k)$, and none of them is the sum of fewer than $4k$ k-th powers; so $G^+(k) \geqslant 4k$. □

For $k = 4$ the second step fails, because then $2^k = 4k$; and indeed $G^+(4) = 15$ [18]. By the first assertion of the lemma, an integer $n \equiv j \pmod{4k}$ with $1 \leqslant j < 4k$ is not the sum of fewer than j k-th powers.

Proof of Theorem 2.4. Let ℓ be as in the theorem. Then $2\ell > \mathcal{Y}$, so that Proposition 9.12 is available, and $\ell > 2.09\,k$. Let $\mathcal{N}$ denote the set of all natural numbers when k is not a power of 2, and the set of all natural numbers $n \equiv j \pmod{4k}$ in the situation of part (ii). We claim that

$$\mathfrak{S}_\ell(n) \gg 1 \qquad \text{uniformly for } n \in \mathcal{N}. \tag{9.22}$$

When k is not a power of 2 this is [32, Theorem 4.6], because $\ell \geqslant \frac{3}{2}k$. When k is a power of 2 it follows from the classical theory of the singular series [32, Lemma 2.13 and Theorem 4.5], as in the proof of [8, Lemma 3.2]; we recall the facts on which this rests. Since $\ell \geqslant 2k+1$ and $S(q,a) \ll q^{1-1/k}$, the singular series is an absolutely convergent product $\prod_p \chi_p(n)$ of local densities, and $\chi_p(n) = 1 + O(p^{-1-1/k})$ uniformly in n; hence the product over the primes $p > p_0$ is at least $\frac{1}{2}$ for every n, if p_0 is large in terms of k. For each prime p the density $\chi_p(n)$ is bounded below by a positive number that depends only on p, k and ℓ, provided that the congruence

$$x_1^k + \cdots + x_\ell^k \equiv n$$

has a solution with $p \nmid x_1$ modulo $4k$ when $p = 2$, and modulo p when p is odd: the modulus $4k$ is the one that [32, Lemma 2.13] prescribes for $p = 2$ when k is a power of 2, and for odd p one has $p \nmid k$, so that a solution modulo p with $p \nmid x_1$ is non-singular. For $n \equiv j \pmod{4k}$ with $1 \leqslant j \leqslant \ell$ a solution modulo $4k$ with x_1 odd is $x_1 = \cdots = x_j = 1$, $x_{j+1} = \cdots = x_\ell = 0$. For odd p let $g := (k, p-1)$. There are $(p-1)/g$ non-zero k-th power residues modulo p, so that every residue class modulo p is a sum of g k-th powers, by the Cauchy–Davenport theorem; as $\ell - 1 \geqslant k \geqslant g$, we may take $x_1 = 1$ and represent $n-1$ by the remaining variables. (When $\ell \geqslant 4k$ one may quote [32, Theorem 4.6] again.)

Let N be large. For $\frac{1}{2}N < n \leqslant N$ we have $\mathfrak{J}_\ell(n/N) \geqslant 2^{1-\ell/k}\mathfrak{J}_\ell(1) > 0$. By Proposition 9.12 and (9.22), all $n \in \mathcal{N}$ with $\frac{1}{2}N < n \leqslant N$, with at most $O(NR^{-\nu_0/4})$ exceptions, satisfy $\mathcal{R}_\ell(n) > 0$, and are therefore sums of ℓ k-th powers of natural numbers. Here $R = P^\eta = N^{\eta/k}$, so that the number of exceptions is $O(N^{1-\delta})$ with $\delta := \eta\nu_0/(4k)$. We apply this with $2^{-i}N$ in place of N for all integers $i \geqslant 0$ with $2^{-i}N \geqslant N^{1/2}$, and conclude that the number of $n \leqslant N$ in $\mathcal{N}$ that are not sums of ℓ k-th powers of natural numbers is $O(N^{1-\delta})$. This proves the first assertions of (i) and (ii). The bound for $G^+(k)$ in (i) follows on taking for ℓ the least integer that exceeds $\frac{1}{2}\mathcal{Y}$. For the last assertion of (ii) we apply what has been proved with $\max\{\ell, 4k\}$ in place of ℓ, and with all j in the range $1 \leqslant j \leqslant 4k$. □

9.7. **The mean value of F.** Theorem 2.5 is contained in what has been proved, as we now show.

Proof of Theorem 2.5. Let s_1 be as in the theorem. In the proof of Theorem 2.3 the number s is an integer that exceeds $\mathcal{Y}$, but the choice of the parameters that is described there does not depend on s being an integer. So the parameters can be chosen as in that proof with s_1 in place of s, and then (9.1), (9.2) and (9.3) hold with $s = s_1$. Let P be

large. By Lemma 3.2 we have $\frac{1}{2}\Pi P \leqslant \#\mathcal{C} \leqslant 2\,\Pi P$ and $\Pi \gg (H/\log P)^J$, uniformly in H, and $\Pi^{s_1} \geqslant R^{-\nu_0/2}$, as in the proof of Theorem 2.3. Let $1 \leqslant Q \leqslant R^{1/4}$. By (9.7),

$$\int_{\mathbb{T}\setminus\mathfrak{M}(Q;P)} |F(\alpha)|^{s_1}\,\mathrm{d}\alpha \ll \Pi^{s_1}P^{s_1-k}Q^{-2} + P^{s_1-k}R^{-\nu_0} \ll (\#\mathcal{C})^{s_1}P^{-k}\big(Q^{-2}+R^{-\nu_0/2}\big).$$

As $|F| \leqslant \#\mathcal{C}$, the same bound holds with s in place of s_1 for every real $s \geqslant s_1$. This is (ii) with $\nu_1 := \frac{1}{2}\eta\,\nu_0$, since $R^{-\nu_0/2} = P^{-\nu_1}$ and $P^{\nu_1} \leqslant R^{1/4}$. By (A1) the set $\mathfrak{M}(1;P)$ has measure at most $2P^{-k}$, so that (ii) with $Q = 1$ gives the upper bound in (i). For the lower bound we note that $|e(\alpha n^k) - 1| \leqslant 2\pi|\alpha|P^k$ for $n \leqslant P$. Hence $|F(\alpha)| \geqslant \frac{1}{2}\#\mathcal{C}$ when $|\alpha| \leqslant (4\pi P^k)^{-1}$, and the integral in (i) is at least $(\frac{1}{2}\#\mathcal{C})^s(2\pi P^k)^{-1}$. Finally, every element of $\mathcal{C}$ is at most P, and its prime factors are at most $\zeta R \leqslant R^2 = P^{2\eta}$; together with the bounds for $\#\mathcal{C}$ and Π just quoted this gives (iii). □

10. Evaluation of the bound

In this section the functions of Definition 2.1 are made explicit, Theorems 1.1 and 1.2 are deduced from Theorem 2.3, and Theorem 1.3 is deduced from Theorem 2.4. Recall the notation v_D and U_D from (4.6).

10.1. **The functions σ, θ^* and I.** We begin with the monotonicity of ς and with closed formulae for σ and θ^*.

Lemma 10.1. (i) *The function ς is continuous on $[1,\infty)$. It decreases on $[1,\frac{50}{13}]$, increases on $[\frac{50}{13},4]$, decreases on $[4,\frac{540}{121}]$ and increases on $[\frac{540}{121},5]$. For each integer $D \geqslant 5$ one has $D < U_{D+1} < D+1$, and ς decreases on $[D,U_{D+1}]$ from $(D+1)^{-2}$ to v_{D+1} and increases on $[U_{D+1},D+1]$ from v_{D+1} to $(D+2)^{-2}$.*

(ii) *One has $\sigma = \varsigma$ on $[1,\frac{50}{13}]$, and*

$$\sigma(\theta) = \begin{cases} \frac{1}{26}, & \frac{50}{13} \leqslant \theta \leqslant \frac{105}{26}, \\ (5-\theta)/25, & \frac{105}{26} \leqslant \theta \leqslant \frac{160}{37}, \\ (6-\theta)/62, & \frac{160}{37} \leqslant \theta \leqslant \frac{540}{121}, \\ \frac{3}{121}, & \frac{540}{121} \leqslant \theta \leqslant \frac{618}{121}, \\ (6-\theta)/36, & \frac{618}{121} \leqslant \theta \leqslant U_6, \\ v_D, & U_D \leqslant \theta \leqslant D+1-(D+1)^2v_D,\ D \geqslant 6, \\ (D+1-\theta)/(D+1)^2, & D+1-(D+1)^2v_D \leqslant \theta \leqslant U_{D+1},\ D \geqslant 6. \end{cases}$$

In particular $\sigma(N) = v_N$ for every integer $N \geqslant 6$.

(iii) *One has*

$$\theta^*(s) = \begin{cases} 2-2s, & \frac{1}{4} \leqslant s \leqslant \frac{1}{2}, \\ 3-6s, & \frac{1}{8} \leqslant s \leqslant \frac{1}{4}, \\ 4-14s, & \frac{1}{16} \leqslant s \leqslant \frac{1}{8}, \\ 5-30s, & \frac{1}{26} < s \leqslant \frac{1}{16}, \\ 5-25s, & \frac{1}{37} \leqslant s \leqslant \frac{1}{26}, \\ 6-62s, & \frac{3}{121} < s \leqslant \frac{1}{37}, \\ 6-36s, & \frac{1}{55} < s \leqslant \frac{3}{121}, \\ D+1-(D+1)^2 s, & v_{D+1} < s \leqslant v_D,\ D \geqslant 6. \end{cases}$$

Proof. (i) Each branch in Definition 2.1 is linear, and one checks that consecutive branches agree at the common end point; for instance both $(5-U)/25$ and $(6-U)/62$ equal $\frac{1}{37}$ at $U = \frac{160}{37}$. On $[D, D+1]$ with $D \geqslant 5$ the first term of the maximum decreases from $(D+1)^{-2}$ to 0 and the second increases to $(D+2)^{-2}$; they cross where $(D+1-U)(D+2)^2 = U(D+1)$, that is, at $U = U_{D+1}$, with common value v_{D+1}. The inequalities $D < U_{D+1} < D+1$ are equivalent to $3D+4 > 0$ and $(D+2)^2 < D^2+5D+5$. At $U = 5$ the first term equals $\frac{1}{36} = \frac{5}{180}$ and exceeds the second.

(ii) By (i), the local minima of ς are $\frac{1}{26}$ (at $\frac{50}{13}$), $\frac{3}{121}$ (at $\frac{540}{121}$) and v_{D+1} (at U_{D+1}, $D \geqslant 5$); these values form a decreasing sequence. Hence the running minimum σ is constant from a local minimum up to the point of the next decreasing branch at which ς returns to that value, and coincides with ς from there to the next local minimum. The points of return are $\frac{105}{26}$ (where $(5-U)/25 = \frac{1}{26}$), $\frac{618}{121}$ (where $(6-U)/36 = \frac{3}{121}$), and $D+1-(D+1)^2 v_D$ for $D \geqslant 6$. The last assertion holds because $U_N < N \leqslant N+1-(N+1)^2 v_N$.

(iii) This follows from (ii) by inverting each decreasing branch; at a value s taken on an interval of constancy, $\theta^*(s)$ is the right end point of that interval. □

Lemma 10.2. (i) *For every integer $N \geqslant 6$,*

$$I(v_N) = \sum_{D \geqslant N} \frac{1}{D+1} \log\Bigl(1+\frac{2}{D}\Bigr) \qquad \text{and} \qquad \frac{2}{N+1} \leqslant I(v_N) \leqslant \frac{2}{N}.$$

(ii) *If $\theta^*(t) = A - Bt$ for $s_1 < t \leqslant s_2$, then*

$$I(s_2) - I(s_1) = \frac{1}{A} \log \frac{s_2\,(A - Bs_1)}{s_1\,(A - Bs_2)}.$$

In particular, with $I(\frac{1}{55}) = I(v_6)$ given by (i),

$$I\bigl(\tfrac{3}{121}\bigr) - I\bigl(\tfrac{1}{55}\bigr) = \tfrac{1}{6}\log\tfrac{147}{103}, \qquad I\bigl(\tfrac{1}{37}\bigr) - I\bigl(\tfrac{3}{121}\bigr) = \tfrac{1}{6}\log\tfrac{9}{8},$$
$$I\bigl(\tfrac{1}{26}\bigr) - I\bigl(\tfrac{1}{37}\bigr) = \tfrac{1}{5}\log\tfrac{32}{21}, \qquad I\bigl(\tfrac{1}{16}\bigr) - I\bigl(\tfrac{1}{26}\bigr) = \tfrac{1}{5}\log 2,$$
$$I\bigl(\tfrac{1}{8}\bigr) - I\bigl(\tfrac{1}{16}\bigr) = \tfrac{1}{4}\log\tfrac{25}{9}, \qquad I(s) - I\bigl(\tfrac{1}{8}\bigr) = \tfrac{1}{3}\log\tfrac{18s}{3-6s} \quad (\tfrac{1}{8} \leqslant s \leqslant \tfrac{1}{4}).$$

(iii) *For $\frac{1}{16} \leqslant w \leqslant \frac{1}{10}$ one has $I(2w) = \Lambda - \frac{1}{3}\log\frac{3-12w}{18w}$, with Λ as in* (1.2). *In particular $I(\frac{1}{5}) = \Lambda$.*

Proof. The first formula in (ii) follows from $\int \frac{\mathrm{d}t}{t(A-Bt)} = \frac{1}{A}\log\frac{t}{A-Bt}$, and the others from Lemma 10.1(iii). For $D \geqslant 6$ the range $v_{D+1} < t \leqslant v_D$, on which $A = D+1$ and

$B=(D+1)^2$, contributes

$$\frac{1}{D+1}\log\frac{v_D\,(1-(D+1)v_{D+1})}{v_{D+1}\,(1-(D+1)v_D)}=\frac{1}{D+1}\log\frac{(D+1)(D+2)^2}{D(D+1)(D+2)}=\frac{1}{D+1}\log\Big(1+\frac{2}{D}\Big),$$

and summing over $D\geqslant N$ gives the series in (i). The bounds in (i) follow from the inequalities $\frac{2x}{2+x}\leqslant\log(1+x)\leqslant x$ for $x\geqslant 0$: the general term lies between $\frac{2}{(D+1)^2}$ and $\frac{2}{D(D+1)}$. Finally, (iii) is obtained by adding up the increments in (ii): for $s=\frac15$ they give the six logarithms in (1.2), and for $s=2w$ the last one is replaced by $\frac13\log\frac{36w}{3-12w}=\frac13\log 2-\frac13\log\frac{3-12w}{18w}$. □

Numerically, the sum $\sum_{D\geqslant 6}\frac{1}{D+1}\log(1+\frac{2}{D})$ equals $0.307897\ldots$, whence $\Lambda=1.096145\ldots$ and $C_0=2+\log 9+2\Lambda=6.389515\ldots$.

Lemma 10.3. *For every real $\Theta\geqslant 5$ one has $\sigma(\Theta)\leqslant 1/(\Theta^2+\Theta-1)$ and $T(\Theta)\geqslant 2/(\Theta+2)$.*

Proof. If $N\leqslant\Theta<N+1$ with an integer $N\geqslant 6$, then $\sigma(\Theta)\leqslant\sigma(N)=v_N$ and $\Theta^2+\Theta-1<(N+1)^2+N=1/v_N$. If $5\leqslant\Theta\leqslant U_6$ then $\sigma(\Theta)\leqslant\frac{3}{121}<\frac{1}{33}\leqslant 1/(\Theta^2+\Theta-1)$, and if $U_6\leqslant\Theta<6$ then $\sigma(\Theta)=\frac{1}{55}<\frac{1}{41}<1/(\Theta^2+\Theta-1)$. For the second assertion let $N:=\max\{\lceil\Theta\rceil,6\}\leqslant\Theta+1$. Then $\sigma(\Theta)\geqslant\sigma(N)=v_N$, so that, by Lemma 10.2(i),

$$T(\Theta)=I(\sigma(\Theta))\geqslant I(v_N)\geqslant\frac{2}{N+1}\geqslant\frac{2}{\Theta+2}.$$ □

Lemma 10.4. *Let $k>0$ and $\Theta>\frac95$ be real numbers with $\eta_k(\Theta)<1$. Put $t_0=\frac95$, $t_1=\frac94$, $t_2=\frac{25}{8}$, $t_3=\frac{160}{37}$, $t_4=16$ and $\kappa_1=4$, $\kappa_2=8$, $\kappa_3=13$, $\kappa_4=21$. If $\Theta\leqslant 16$ then, with $t_i':=\min\{t_i,\Theta\}$,*

$$R_k(\Theta)=\sum_{\substack{1\leqslant i\leqslant 4\\ t_{i-1}<\Theta}}\frac{\kappa_i}{k}\log\frac{t_i'\,(k-\kappa_i t_{i-1})}{t_{i-1}\,(k-\kappa_i t_i')}+\sum_{\substack{1\leqslant i\leqslant 3\\ t_i<\Theta}}\frac{1}{t_i}\log\frac{k-\kappa_i t_i}{k-\kappa_{i+1}t_i}.$$

If $\Theta>16$ then, with $a_k:=\frac12\big(\sqrt{4k+25}-5\big)$,

$$\begin{aligned}R_k(\Theta)&=R_k(16)+\int_{16}^{\Theta}\frac{(2\theta+5)\,\mathrm{d}\theta}{\theta\,(k-\theta^2-5\theta)}\\&=R_k(16)+\frac{5}{k}\log\frac{\Theta}{16}+\frac{1}{a_k}\log\frac{a_k-16}{a_k-\Theta}+\frac{1}{a_k+5}\log\frac{a_k+5+\Theta}{a_k+21}.\end{aligned}$$

Moreover $k\,R_k(16)$ is a decreasing function of k, and $k\,R_k(16)<60$ for $k\geqslant 1000$.

Proof. On $(t_{i-1},t_i]$ we have $-\log(1-\eta_k(\theta))=-\log(1-\kappa_i\theta/k)$, with differential $\frac{\kappa_i\,\mathrm{d}\theta}{k-\kappa_i\theta}$, and $\int\frac{\kappa\,\mathrm{d}\theta}{\theta(k-\kappa\theta)}=\frac{\kappa}{k}\log\frac{\theta}{k-\kappa\theta}$. At t_i $(i\leqslant 3)$ the function jumps by $\log\frac{k-\kappa_i t_i}{k-\kappa_{i+1}t_i}$. The number $a=a_k$ is the positive root of $a(a+5)=k$. For $\theta\geqslant 16$ we have $1-\eta_k(\theta)=(k-\theta^2-5\theta)/k$, which equals $(a-\theta)(a+5+\theta)/k$, and

$$\frac{2\theta+5}{\theta\,(a-\theta)(a+5+\theta)}=\frac{5}{k}\cdot\frac{1}{\theta}+\frac{1}{a}\cdot\frac{1}{a-\theta}+\frac{1}{a+5}\cdot\frac{1}{a+5+\theta},$$

which gives the formula for $\Theta>16$. Each summand of $k\,R_k(16)$ decreases as k grows: this is clear for the first sum, and for the jumps it follows from the fact that $y\mapsto y\log(1+\frac{a}{y-b})$ is decreasing for $y>b$ when $b>a>0$. Finally $1000\,R_{1000}(16)=59.96\ldots$. □

10.2. **Moderate k.** For $k \geqslant 37$ let

$$\Theta_{\max}(k) := \begin{cases} \min\{\frac{k}{16}, \frac{25}{8}\}, & 37 \leqslant k \leqslant 81, \\ \min\{\frac{k}{26}, \frac{160}{37}\}, & 82 \leqslant k \leqslant 174, \\ \frac{3k}{121}, & k \geqslant 175, \end{cases} \qquad \Theta^\sharp(k) := \min\bigl\{\Theta_{\max}(k), \tfrac{9}{10}\,a_k\bigr\},$$

where $a_k = \frac{1}{2}(\sqrt{4k+25} - 5)$ is the positive root of $a(a+5) = k$, as in Lemma 10.4. Thus $\Theta_{\max}(k)$ is the largest number Θ with $\Theta \leqslant \varpi(\Theta)k$, and every $\Theta \in [\frac{9}{4}, \Theta_{\max}(k)]$ has this property. Moreover $\Theta^\sharp(k) \geqslant \frac{9}{4}$, because $\Theta_{\max}(k) \geqslant \frac{37}{16}$ and $\frac{9}{10}a_k \geqslant \frac{9}{10}a_{37} > \frac{9}{4}$; and $\Theta(\Theta+5) < a_k(a_k+5) = k$ for $\Theta = \Theta^\sharp(k)$. Hence $\Theta^\sharp(k)$ is admissible for every $k \geqslant 37$. One has $\Theta^\sharp(k) = \Theta_{\max}(k)$ for $k \leqslant 1136$ and $\Theta^\sharp(k) = \frac{9}{10}a_k$ for $k \geqslant 1137$; for $k \geqslant 1318$ the latter holds because $\frac{9}{10}a_k < \frac{9}{10}\sqrt{k} \leqslant \frac{3k}{121}$. The factor $\frac{9}{10}$ is suggested by Proposition 10.7 below. We write

$$B(k,\Theta) := C(h_k) - 2\,T(\Theta) + 2R_k(\Theta) + W_k(\Theta) + \frac{1}{k},$$

so that Theorem 2.3 asserts that $G(k) \leqslant \lfloor k\,B(k,\Theta)\rfloor$ for admissible Θ.

Proposition 10.5. *For $37 \leqslant k \leqslant 5000$ let $B^\sharp(k) := B(k,\Theta^\sharp(k))$ and $G^\sharp(k) := \lfloor k\,B^\sharp(k)\rfloor$.*
(i) *One has $B^\sharp(37) \leqslant 7.3898$ and $G^\sharp(37) = 273$, and $B^\sharp(k) \leqslant 7.3567$ for $k \geqslant 38$.*
(ii) *One has $B^\sharp(k) < 7$ for $k \geqslant 212$, and $G^\sharp(k) < 6.5\,k$ for $k \geqslant 927$.*
(iii) *One has $G^\sharp(k) < C_0k + 10\sqrt{k}$ for all k, $G^\sharp(k) < C_0k + 4\sqrt{k}$ for $k \geqslant 697$, and $G^\sharp(k) < C_0k + 3\sqrt{k}$ for $k \geqslant 1339$.*

Proof. By Lemmas 10.1–10.4 the number $B^\sharp(k)$ is given by a closed expression in elementary functions, apart from the series in Lemma 10.2(i), whose tail is controlled by the inequalities stated there. We have evaluated it for each of the 4964 values of k (see §10.4). The maximum of $B^\sharp$ is 7.38976, attained at $k = 37$, and the maximum over $k \geqslant 38$ is 7.35659, attained at $k = 38$. The maximum over $212 \leqslant k \leqslant 5000$ is 6.99812, attained at $k = 212$, and $6.5\,k - G^\sharp(k) \geqslant \frac{1}{2}$ for $927 \leqslant k \leqslant 5000$. The largest value of $(G^\sharp(k) - C_0k)/\sqrt{k}$ is 9.8146, attained at $k = 175$, where $G^\sharp(k) = 1248$ and $C_0k + 10\sqrt{k} = 1250.45\ldots$; the largest value for $697 \leqslant k \leqslant 5000$ is 3.9990, attained at $k = 699$; and the largest value for $1339 \leqslant k \leqslant 5000$ is 2.9984, attained at $k = 1343$. □

To illustrate the computation, let $k = 37$, so that $\Theta^\sharp = \frac{37}{16}$, $h_k = \frac{1}{5}$ and $\eta_k(\Theta^\sharp) = \frac{1}{2}$. Then $\sigma(\Theta^\sharp) = \frac{27}{224}$, and Lemmas 10.2 and 10.4 give

$$T = \Lambda - \frac{1}{3}\log 2 - \frac{1}{4}\log\frac{25}{9} + \frac{1}{4}\log\frac{50\,\sigma}{4-14\sigma} = 0.84915\ldots,$$
$$R_k = \frac{4}{37}\log\frac{149}{112} + \frac{4}{9}\log\frac{28}{19} + \frac{8}{37}\log\frac{19}{18} = 0.21489\ldots,$$

and $C(\frac{1}{5}) = 6.63417\ldots$. The number $y_k(\Theta^\sharp)$ of (2.3) is $\frac{1}{4}\cdot\frac{27}{224}\cdot 9.027901\cdot 37 = 10.0657\ldots$, so that $W_k = \frac{32}{37}\log 10.0657\ldots = 1.99709\ldots$. Altogether

$$B = 6.63417 - 1.69831 + 0.42978 + 1.99709 + 0.02703 = 7.38976,$$

and $G(37) \leqslant \lfloor 37B\rfloor = 273$.

TABLE 3. The bound $G(k) \leqslant \lfloor k\,B(k,\Theta)\rfloor$ of Theorem 2.3 with $\Theta = \Theta^\sharp(k)$, compared with (1.1).

k	Θ	$B(k,\Theta)$	$\lfloor kB\rfloor$	(1.1)	k	Θ	$B(k,\Theta)$	$\lfloor kB\rfloor$	(1.1)
37	2.3125	7.3898	273	290	212	5.2562	6.9981	1483	2027
38	2.3750	7.3566	279	298	250	6.1983	6.8993	1724	2431
39	2.4375	7.3249	285	307	330	8.1818	6.7632	2231	3300
40	2.5000	7.2946	291	316	500	12.397	6.6186	3309	5208
41	2.5625	7.2656	297	325	700	17.355	6.5408	4578	7526
50	3.1250	7.0622	353	406	927	22.983	6.5000	6025	10227
67	3.1250	7.1902	481	564	1000	24.793	6.4922	6492	11109
81	3.1250	7.2843	590	697	2000	38.062	6.4515	12902	23603
100	3.8462	7.0917	709	881	5000	61.429	6.4232	32116	63588
150	4.3243	7.0851	1062	1382	10^4	87.778	6.4113	64112	134107
174	4.3243	7.1344	1241	1629	10^5	282.36	6.3952	639520	1571325
200	4.9587	7.0385	1407	1900	10^6	897.75	6.3912	6391170	18015831

10.3. **Large k.** For $k \geqslant 5000$ the three terms in Theorem 2.3 that depend on Θ are estimated by elementary functions of k.

Proposition 10.6. *Let $k \geqslant 5000$ and $\Theta := \frac{9}{10}a_k$, with a_k as in Lemma 10.4. Then Θ is admissible for k, and*

$$G(k) \leqslant C_0\,k + c_3\sqrt{k} + 5\log k + 55,$$
$$\text{where}\quad c_3 := 2\log 19 - \tfrac{40}{9} + \tfrac{20}{9}\log\frac{19\cdot 9.027901}{162} = 1.5714\ldots.$$

In particular $G(k) < C_0\,k + 3\sqrt{k}$.

Proof. Write $a := a_k$, and note that $\sqrt{k} \geqslant 70.71$. Since $a + \frac{5}{2} = \sqrt{k}\,(1 + \frac{25}{4k})^{1/2}$, we have $\sqrt{k} - \frac{5}{2} < a < \sqrt{k} - 2.45$, and in particular $a > 68.2$. We have seen in §10.2 that $\Theta = \Theta^\sharp(k)$, and that this number is admissible. As $\Theta > 16$, we have $\kappa(\Theta) = \Theta + 5$, and since $k = a(a+5)$,

$$1 - \eta_k(\Theta) = \frac{(a-\Theta)(a+5+\Theta)}{k} = \frac{19a+50}{100\,(a+5)}.$$

The term T. By Lemma 10.3,

$$-2k\,T(\Theta) \leqslant -\frac{4k}{\Theta+2} = -\frac{40\,a(a+5)}{9a+20} = -\frac{40}{9}\,a - \frac{1000}{81} + \frac{20000}{729\,(a+\frac{20}{9})}$$
$$< -\frac{40}{9}\sqrt{k} + \frac{100}{9} - \frac{1000}{81} + 0.39 < -\frac{40}{9}\sqrt{k} - 0.84.$$

The term R_k. By Lemma 10.4, and since $k/a = a+5$,

$$2k\,R_k(\Theta) = 2k\,R_k(16) + 10\log\frac{\Theta}{16} + 2(a+5)\log\frac{a-16}{a-\Theta} + 2a\log\frac{a+5+\Theta}{a+21}.$$

Here $2kR_k(16) < 120$, and $10\log\frac{\Theta}{16} < 10\log\frac{9\sqrt{k}}{160} = 5\log k - 10\log\frac{160}{9} < 5\log k - 28.77$. Next, $\frac{a-16}{a-\Theta} = 10\big(1-\frac{16}{a}\big)$, so that the third term is smaller than

$$2(a+5)\Big(\log 10 - \frac{16}{a}\Big) < 2a\log 10 + 10\log 10 - 32 < 2a\log 10 - 8.97.$$

Further $\frac{a+5+\Theta}{a+21} = \frac{19}{10}\big(1-\frac{21-50/19}{a+21}\big)$ and $21-\frac{50}{19} > 18.368$, so that the fourth term is smaller than

$$2a\log\frac{19}{10} - 36.736\,\frac{a}{a+21} < 2a\log\frac{19}{10} - 28.08,$$

because $\frac{a}{a+21} > \frac{68.2}{89.2}$. Finally $2a\log 19 < 2\sqrt{k}\log 19 - 4.9\log 19 < 2\sqrt{k}\log 19 - 14.42$. Altogether

$$2k\,R_k(\Theta) < 2\sqrt{k}\log 19 + 5\log k + 39.76.$$

The term W_k. By Lemma 10.3, $\sigma(\Theta) \leqslant 1/(\Theta^2+\Theta-1)$. Hence

$$\begin{aligned} y_k(\Theta) &\leqslant \frac{9.027901\,a(a+5)}{2}\cdot\frac{19a+50}{100\,(a+5)}\cdot\frac{100}{81a^2+90a-100} \\ &= \frac{19\cdot 9.027901}{162}\Big(1+\frac{\frac{260}{171}a+\frac{100}{81}}{a^2+\frac{10}{9}a-\frac{100}{81}}\Big) \leqslant \frac{19\cdot 9.027901}{162}\Big(1+\frac{1.53}{a}\Big), \end{aligned}$$

the last step being equivalent to $\big(1.53-\frac{260}{171}\big)a^2 + \big(1.7-\frac{100}{81}\big)a \geqslant \frac{17}{9}$, which holds for $a \geqslant 4$. With $L_0 := \log\frac{19\cdot 9.027901}{162} = 0.05716\ldots$ we obtain $\log y_k(\Theta) \leqslant L_0 + \frac{1.53}{a}$, and since $2k/\Theta = \frac{20}{9}(a+5)$,

$$k\,W_k(\Theta) = \frac{20}{9}(a+5)\log y_k(\Theta) \leqslant \frac{20}{9}\Big(L_0\,a + 5L_0 + 1.53 + \frac{7.65}{a}\Big) < \frac{20}{9}L_0\sqrt{k} + 3.98;$$

here we used that $a < \sqrt{k} - 2.45$ and $a > 68.2$.

By the second inequality of Theorem 2.3,

$$G(k) \leqslant C_0 k + k\big(-2T(\Theta) + 2R_k(\Theta) + W_k(\Theta)\big) + 12.$$

The three estimates now give the bound of the proposition, as $12-0.84+39.76+3.98 < 55$. Finally $c_3 < 1.572$, and

$$1.572\sqrt{k} + 5\log k + 55 < 3\sqrt{k} \qquad (k \geqslant 5000),$$

since this holds for $k = 5000$ and $1.428\sqrt{k} - 5\log k$ increases with k. □

Proposition 10.7. *Let $0 < \lambda < 1$ be fixed, and put*

$$\phi(\lambda) := -\frac{4}{\lambda} + 2\log\frac{1+\lambda}{1-\lambda} + \frac{2}{\lambda}\,\log\frac{9.027901\,(1-\lambda^2)}{2\lambda^2}.$$

Then $G(k) \leqslant C_0\,k + \phi(\lambda)\sqrt{k} + 5\log k + O_\lambda(1)$. The function ϕ attains its minimum at $\lambda_0 := (9.027901/11.027901)^{1/2} = 0.90478\ldots$, and

$$\phi(\lambda_0) = -\frac{4}{\lambda_0} + 2\log\frac{1+\lambda_0}{1-\lambda_0} = 1.57112\ldots, \qquad \phi\big(\tfrac{9}{10}\big) = c_3 = 1.57146\ldots.$$

Proof. Let k be large and $\Theta := \lambda\sqrt{k}$. Then Θ is admissible, $\eta_k(\Theta) = \lambda^2 + 5\lambda/\sqrt{k}$ and $a_k = \sqrt{k} - \frac{5}{2} + O(k^{-1/2})$. We argue as in the proof of Proposition 10.6. First, $-2kT(\Theta) \leqslant -4k/(\Theta+2) = -4\sqrt{k}/\lambda + O_\lambda(1)$. Secondly, $\frac{a_k-16}{a_k-\Theta} = \frac{1}{1-\lambda}\big(1 + O_\lambda(k^{-1/2})\big)$ and $\frac{a_k+5+\Theta}{a_k+21} = (1+\lambda)\big(1+O(k^{-1/2})\big)$, so that $2kR_k(\Theta) = 2\sqrt{k}\log\frac{1+\lambda}{1-\lambda} + 5\log k + O_\lambda(1)$. Thirdly, $1-\eta_k(\Theta) \leqslant 1-\lambda^2$ and $\sigma(\Theta) \leqslant \Theta^{-2}$, so that $y_k(\Theta) \leqslant 9.027901(1-\lambda^2)/(2\lambda^2)$, and $kW_k(\Theta) = (2\sqrt{k}/\lambda)\log y_k(\Theta) \leqslant (2\sqrt{k}/\lambda)\log\big(9.027901(1-\lambda^2)/(2\lambda^2)\big)$. The first assertion now follows from Theorem 2.3. A short computation gives

$$\phi'(\lambda) = -\frac{2}{\lambda^2}\,\log\frac{9.027901\,(1-\lambda^2)}{2\lambda^2},$$

which is negative for $\lambda < \lambda_0$ and positive for $\lambda > \lambda_0$, the logarithm being zero at λ_0. This gives the remaining assertions. □

Thus the bound is smallest, to this order, when Θ is such that the term $W_k(\Theta)$ vanishes, in accordance with the remarks following Theorem 2.3.

Proof of Theorems 1.1 and 1.2. For $k > 5000$ we have $G(k) < C_0k + 3\sqrt{k} < 6.45\,k$ by Proposition 10.6. For $37 \leqslant k \leqslant 5000$, Theorem 2.3 gives $G(k) \leqslant G^\sharp(k) \leqslant k\,B^\sharp(k)$, and Proposition 10.5 shows that $G(k) < C_0k + 10\sqrt{k}$, that $G(k) < C_0k + 4\sqrt{k}$ if $k \geqslant 697$, that $G(k) < C_0k + 3\sqrt{k}$ if $k \geqslant 1339$, that $G(37) \leqslant 273 < 7.38 \cdot 37$ and $G(k) \leqslant 7.3567\,k$ if $k \geqslant 38$, that $G(k) < 7k$ if $k \geqslant 212$, and that $G(k) < 6.5\,k$ if $k \geqslant 927$. The last bound in Theorem 1.1 is contained in Proposition 10.6, because $c_3 < 1.572$. Finally, for $k \leqslant 36$, (1.1) gives $G(k) \leqslant \lceil k(\log k + 4.20032)\rceil \leqslant 7.806\,k$, the largest value of the ratio being $\frac{281}{36}$, for $k = 36$. □

Proof of Theorem 1.3. Let $\mathcal{Y} = k\big(C(h_k) - 2T(\Theta) + 2R_k(\Theta) + W_k(\Theta)\big)$, as in §9.5. For $37 \leqslant k \leqslant 5000$ we take $\Theta = \Theta^\sharp(k)$. Then $\mathcal{Y} = k\,B^\sharp(k) - 1$, and the least integer ℓ with $2\ell > \mathcal{Y}$ is $\lceil \frac{1}{2}G^\sharp(k)\rceil$, because $k\,B^\sharp(k)$ is not an integer. By Proposition 10.5 and its proof, $\mathcal{Y} < 7.39\,k$ and $G^\sharp(k) \leqslant C_0k + 9.82\sqrt{k}$, so that

$$\big\lceil \tfrac{1}{2}G^\sharp(k)\big\rceil \leqslant \tfrac{1}{2}\big(G^\sharp(k)+1\big) \leqslant \tfrac{1}{2}C_0k + 4.91\sqrt{k} + \tfrac{1}{2} < \tfrac{1}{2}C_0k + 5\sqrt{k};$$

note that $0.09\sqrt{k} > \frac{1}{2}$ for $k \geqslant 37$. For $k \geqslant 5000$ we take $\Theta = \frac{9}{10}a_k$. Then, by the inequality $k\,C(h_k) \leqslant C_0k + 11$ of Theorem 2.3 together with the three estimates in the proof of Proposition 10.6, we have $\mathcal{Y} \leqslant C_0k + c_3\sqrt{k} + 5\log k + 54$, since $11 - 0.84 + 39.76 + 3.98 < 54$. Hence $\mathcal{Y} < C_0k + 3\sqrt{k} < 8k$ and, as $c_3 < 1.572$,

$$\big\lfloor \tfrac{1}{2}\mathcal{Y}\big\rfloor + 1 \leqslant \tfrac{1}{2}C_0k + 0.786\sqrt{k} + \tfrac{5}{2}\log k + 28 < \tfrac{1}{2}C_0k + 5\sqrt{k};$$

the last inequality holds because $0.786\sqrt{k} + \frac{5}{2}\log k + 28 < 5\sqrt{k}$ for $k \geqslant 5000$. Part (ii) now follows from Theorem 2.4(i).

For part (i) let $k \geqslant 8$ be a power of 2. The lower bound $G^+(k) \geqslant 4k$ is contained in Lemma 9.13. If $k \geqslant 64$, then $\frac{1}{2}\mathcal{Y} < 4k$ by what has been shown, so that Theorem 2.4(ii) gives $G^+(k) \leqslant 4k$. For $k = 8$ the upper bound is due to Vaughan [30], and for $k = 16$ and $k = 32$ to Wooley [39]. □

For $k = 64$, 128, 256, 512, 1024, 2048 and 4096 the least integer ℓ that is admissible in Theorem 2.4 with $\Theta = \Theta^\sharp(k)$ is 229, 450, 881, 1693, 3323, 6605 and 13164, respectively; these numbers are to be compared with $4k$.

It follows that $G^+(k) \leqslant 4k$ for every $k \in \mathbb{N}$, with equality if and only if $k \geqslant 8$ is a power of 2, as was stated in §1. Indeed, let k be a natural number that is not a power of 2. If $37 \leqslant k \leqslant 5000$, then by the proof just given $G^+(k) \leqslant \lceil \frac12 G^\sharp(k) \rceil \leqslant \frac12(7.39\,k+1) < 4k$. If $k \geqslant 5000$, then $G^+(k) < \frac12 C_0 k + 5\sqrt{k} < 4k$. If $k \leqslant 36$, then [6, Theorem 9.1] gives $G^+(k) \leqslant \lceil \frac12 k(\log k + 4.20032) \rceil < 4k$; the largest value of the ratio is $\frac{47}{12}$, for $k = 36$. Finally, for the powers of 2 below 8 one has $G^+(1) = 1$, $G^+(2) = 4$ and $G^+(4) = 15$.

Proof of Corollary 2.6 and of Theorem 1.4. Let $\mathcal{Y}^\sharp(k) := k\,B(k, \Theta^\sharp(k)) - 1$; this is the number $\mathcal{Y}$ of §9.5 for $\Theta = \Theta^\sharp(k)$. By Theorem 2.5 it suffices to show that $s_1 > \mathcal{Y}^\sharp(k)$ in each case. For $37 \leqslant k \leqslant 5000$ we have $\mathcal{Y}^\sharp(k) = k\,B^\sharp(k) - 1 < G^\sharp(k)$, and the assertions are contained in Proposition 10.5 and its proof: $B^\sharp(k) < 7.39$ for all k, $B^\sharp(k) < 7$ for $k \geqslant 212$, and $G^\sharp(k)$ is smaller than $6.5\,k$ for $k \geqslant 927$, smaller than $C_0k + 10\sqrt{k}$ for all k, smaller than $C_0k + 4\sqrt{k}$ for $k \geqslant 697$, and smaller than $C_0k + 3\sqrt{k}$ for $k \geqslant 1339$. Now let $k \geqslant 5000$, so that $\Theta^\sharp(k) = \frac{9}{10}a_k$. We saw in the proof of Theorem 1.3 that

$$\mathcal{Y}^\sharp(k) \leqslant C_0k + c_3\sqrt{k} + 5\log k + 54 < C_0k + 1.572\sqrt{k} + 5\log k + 54 < C_0k + 3\sqrt{k} < 6.45\,k.$$

This gives the assertion on $s_1 = C_0k + 1.572\sqrt{k} + 5\log k + 54$, and all columns of the table except the last. The last one holds because $0.428\sqrt{k} > 5\log k + 54$ for $k \geqslant 70000$. The final assertion of the corollary follows from the proof of Proposition 10.7, which shows that $k\big(C(h_k) - 2T(\Theta) + 2R_k(\Theta) + W_k(\Theta)\big) \leqslant C_0k + \phi(\lambda)\sqrt{k} + 5\log k + O_\lambda(1)$ for $\Theta = \lambda\sqrt{k}$; we take $\lambda = 0.90478$, for which $\phi(\lambda) < 1.5712$. Theorem 1.4 is the case $s_1 = C_0k + 10\sqrt{k}$ of the corollary, respectively the case $s_1 = C_0k + 3\sqrt{k}$, with H the largest integer that is permitted in §3. For this H the set $\mathcal{C}$ is the set (1.4), in which J is any natural number that is sufficiently large in terms of k, as in §3, and $\omega = \eta/(64ks_1)$, by (9.3) with $s = s_1$ (see §9.7); the bound $\#\mathcal{C} \gg P$ is Theorem 2.5(iii). □

Remark 10.8. The number $G^\sharp(k)$ is smaller than the bound (1.1) for $37 \leqslant k \leqslant 5000$ (see Table 3). The function $B^\sharp$ is not monotone: for instance, it has local maxima at $k = 81$ and $k = 174$, the last values of k for which $\Theta_{\max}(k)$ equals $\frac{25}{8}$ and $\frac{160}{37}$; this is why the largest value of $(G^\sharp(k) - C_0k)/\sqrt{k}$ occurs at $k = 175$. The choice of $\Theta^\sharp$ is close to optimal. Indeed, for fixed k the function $\Theta \mapsto B(k, \Theta)$ is continuous on the interval of admissible Θ, and

$$\frac{\partial B}{\partial \Theta}(k, \Theta) = -\frac{2}{\Theta^2}\,\log y_k(\Theta)$$

wherever the derivative exists. This follows from the definitions: $\mathrm{d}T(\Theta) = \Theta^{-1}\mathrm{d}\log\sigma(\Theta)$, because $\theta^*(\sigma(\Theta)) = \Theta$ unless σ is constant near Θ; $\mathrm{d}R_k(\Theta) = -\Theta^{-1}\mathrm{d}\log(1 - \eta_k(\Theta))$; and the jumps of $2R_k$ and W_k at the discontinuities of η_k cancel. As y_k is decreasing, $B(k, \cdot\,)$ decreases as long as $y_k(\Theta) > 1$ and increases thereafter. For $k \leqslant 1132$ its least value is therefore attained at the largest admissible Θ, which is $\Theta^\sharp(k)$, and for $k \geqslant 1133$ at the solution of $y_k(\Theta) = 1$, which is $(\lambda_0 + o(1))\sqrt{k}$ as $k \to \infty$. For $1133 \leqslant k \leqslant 5000$ this solution exceeds $\Theta^\sharp(k)$ for 3623 of the 3868 values of k, among them all $k \geqslant 2313$, in accordance with $\lambda_0 > \frac{9}{10}$. For $37 \leqslant k \leqslant 5000$ the least value of $B(k, \cdot\,)$ is smaller than $B^\sharp(k)$ by less than 10^{-5}; the corresponding integer is $G^\sharp(k) - 1$ for 18 values of k, and $G^\sharp(k)$ for all others. For $k = 10^6$ the optimal choice is $\Theta = 902.36$, and it gives 6391169 in place of the last entry of Table 3. The numbers for small k are not optimised in other

respects. The step function κ is a crude majorant for the rounding losses, and no use is made of the sharper mean value estimates for smooth Weyl sums that are available for moderate k. The term $5\log k$ in Propositions 10.6 and 10.7 is an artefact of the choice $\kappa(\theta) = \theta + 5$ for large θ.

10.4. **Numerical verification.** The paper is accompanied by a Python program (using NumPy), which is available as the ancillary file `verify_explicit_waring.py`. Its rôle is twofold.

First, it evaluates $B^\sharp(k)$ for $37 \leqslant k \leqslant 5000$ from the formulae of Lemmas 10.1–10.4; this is the only computation on which the proofs depend (Proposition 10.5; the numbers $E^\sharp(k)$ of Proposition 11.10(i) are obtained from these values by a few arithmetical operations). The series in Lemma 10.2(i) is summed up to $D = 10^6$ and the remainder is bounded by that lemma, with the upper bound used for Λ and the lower bound for T. The computation is done in double precision arithmetic. The quantities involved are of moderate size, so that the accumulated rounding errors in $k\,B^\sharp(k)$ are of order 10^{-9} at most. The bounds for $B^\sharp$ stated in Proposition 10.5 hold with a margin of at least $4 \cdot 10^{-5}$; the numbers $k\,B^\sharp(k)$ differ from the nearest integer by at least $1.3 \cdot 10^{-4}$, so that the integers $G^\sharp(k)$ are determined reliably; and the inequalities for $G^\sharp(k)$ in parts (ii) and (iii) hold with a margin of at least 0.025. As a safeguard against programming and rounding errors, the program contains a second, independently written implementation of the formulae of Lemmas 10.1–10.4 in 40-digit decimal arithmetic; the two sets of values agree to within 10^{-10} and give the same integers $G^\sharp(k)$. As both implementations use the same closed formulae, these are compared with the definitions separately (see below). The complete list of values is printed by the option `--list`. The numbers $\frac{1}{2}\mathcal{Y}^\sharp(k) - \chi_0 k/\Theta^\sharp(k)$, whose integer parts enter the definition of $E^\sharp(k)$ in §11.5, differ from the nearest integer by at least $1.3 \cdot 10^{-5}$, and the inequalities of Proposition 11.10(i) hold with a margin of at least 0.5. The program also confirms the elementary numerical inequalities stated in the proofs of Theorems 2.3 and 1.3, of Corollary 2.6, of Propositions 10.6, 10.7 and 11.10, and of Lemma 10.4, and the numbers quoted after Theorem 1.3 and after its proof, after Theorem 1.5, and in Table 4; in particular the three inequalities in the proof of Proposition 10.6 are compared with the exact values of the three terms at 45 001 values of k between 5000 and 10^{12}.

Secondly, it carries out consistency checks which are not needed for the proofs but guard against errors in the bookkeeping. The closed formulae of Lemmas 10.1, 10.2 and 10.4 are compared with the definitions; Lemma 4.5 is tested at $5.4 \cdot 10^5$ parameters, chosen at random and at the boundary of the admissible region, and the exponents of Lemmas 4.3 and 4.4 are tested on explicit exponential sums; the statements of §8 are tested on tables of the potential for thirteen pairs (k, Θ); the identities and inequalities of §§3, 9 and 11 are tested on random data and on a small example of the set of variables, in part in exact arithmetic; and the numerical statements of §§11.4, 12.1 and 12.3 are reproduced. One of these checks concerns the method rather than the arithmetic. The recursion that the potential is designed to handle,

$$\widetilde{\Phi}(x,\lambda) = \min\Big\{kx,\ k\lambda\Psi(\lambda/x),\ \min_{\theta}\max_{\mu\leqslant\lambda}\big[3\lambda - \mu + \widetilde{\Phi}\big(x(1-\theta/k),\mu\big)\big]\Big\},$$

with the steps $\theta \in [\frac{9}{5}, \Theta]$ permitted by Corollary 5.2, is solved by dynamic programming for $(k, \Theta) = (40, \frac{5}{2})$, $(100, \frac{50}{13})$ and $(250, \frac{750}{121})$. With b as in the proof of Theorem 2.3, the least value of r/k with $\sup_\lambda(\widetilde{\Phi}(1, \lambda) - r\lambda) \leqslant b$ comes out as 6.385, 6.531 and 6.566, against the upper bounds 6.470, 6.562 and 6.573 for $\mathcal{G}(u_\Theta) + a_*$ on which Theorem 2.3 rests; thus the potential loses little. The complete list of the checks, with their outcomes, is part of the documentation of the program. The part of the program that concerns Appendix A is described at the end of that appendix.

11. A large sieve estimate

The bound of Weyl type of §9.3 rests on moments of order $2k$ of two sets of integers of size about $P^{1/2}$, and at that order nothing better is known for these sets than for the smooth numbers that contain them. On the other hand the moments of order about $C_0 k$ of the sums $F_j^{\mathfrak{e}}$ exceed the expected size by a small power of P only (Corollary 8.3), whereas for a smooth Weyl sum the admissible exponent at this order is $k\,\mathfrak{d}(C_0 k) = 0.0045\,k$. In this section we show that these moments yield a second bound of Weyl type for F, by the variant of Vinogradov's method that is due to Vaughan [31, §10], in the form given to it by Wooley [41, Lemma 4.1]: Hölder's inequality is applied to the outer variable of a bilinear sum, and the large sieve to the points αm^k. For smooth Weyl sums the moment must have order about $k \log k$, and one arrives at the exponent of [41]; for F the order $C_0 k$ suffices. The new bound is stronger than Lemma 9.8 on arcs of height below $P^{0.41\,k}$, roughly, and it gives a bound with an exponent of order $1/k$ on all minor arcs in the classical sense. It is weaker on the arcs of larger height, and these turn out to decide the number of variables (§11.4); so the results of this section are not used in the proofs of Theorems 1.1 and 1.2. They are what is needed for the fractional parts of αn^k, however: Theorems 1.5 and 2.7 are deduced from Corollary 11.2 in §11.5.

Throughout this section $k \geqslant 37$, Θ is admissible for k, and the parameters are as in §3; the choices (9.2) and (9.3) are not assumed, so that ω is any number with $0 < \omega \leqslant \frac{1}{8}\eta$. For $b > 0$ we write $a_*(b)$ for the number defined in (8.8).

11.1. **Statement.** The main result of this section is the following estimate.

Theorem 11.1. *Let $b_0 > 0$, and let $s_0 \in \mathbb{N}$ satisfy $2s_0 \geqslant \max\{k(\mathcal{G}(u_\Theta) + a_*(b_0)),\, 2k\}$. Then, uniformly for $R^2 \leqslant Q \leqslant P^{k/2}$ and $\alpha \in \mathfrak{N}(Q; P)$,*

$$F(\alpha) \ll P^{1+\omega+\varepsilon}\Big(\frac{P^{\,b_1}}{\min\{Q,\, P(2Q)^{-1/k}\}}\Big)^{1/(2s_0)}, \qquad \text{where} \quad b_1 := b_0 + 4k\gamma + 2\varepsilon + 2\eta.$$

The implied constant depends on b_0 and s_0 as well.

The parameters γ, ε and η are at our disposal, subject to (3.1) and (3.2), so that b_1 is as close to b_0 as we please.

Corollary 11.2. *Let b_0, s_0 and b_1 be as in Theorem 11.1, suppose that $b_1 < 1$, and put*

$$\xi_1 := \frac{1 - b_1}{1 + 4s_0\tau_k}, \qquad \tau^{(1)} := 2\,\xi_1\tau_k = \frac{1 - b_1}{2s_0 + 1/(2\tau_k)} = \frac{1 - b_1}{2s_0 + 4.5139505\,k}.$$

(i) *Let $\frac{1}{k+1} \leqslant \xi \leqslant \frac{1}{2}$. Then $F(\alpha) \ll P^{1-(1-\xi-b_1)/(2s_0)+\omega+\varepsilon}$ for $\alpha \in \mathfrak{N}(P^{k\xi}; P)$. For $\xi < \xi_1$ the exponent $(1 - \xi - b_1)/(2s_0)$ is larger than the exponent $2\xi\tau_k$ of Lemma 9.8.*

(ii) *Suppose that J is so large that Lemma* 9.8 *is applicable. Then $F(\alpha) \ll P^{1-\tau^{(1)}+\varepsilon+\eta}$ uniformly for $\alpha \in \mathbb{T} \setminus \mathfrak{M}(P^{k/(k+1)}; P)$.*

As $k \to \infty$ the parameters can be chosen in such a way that $2s_0 = (C_0 + o(1))k$ and $b_1 = o(1)$; see the end of §11.3. Then, with $\xi = \log Q/(k \log P)$,

$$\frac{1-\xi-b_1}{2s_0} \sim \frac{1-\xi}{C_0\, k}, \qquad \xi_1 \to \frac{1}{1+2C_0/9.027901} = 0.4139\ldots,$$
$$\big(\tau^{(1)}\big)^{-1} = (C_0 + 4.51395\ldots + o(1))\, k = (10.9034\ldots + o(1))\, k.$$

Thus the large sieve exponent exceeds τ_k itself when $\xi < 1 - C_0/9.027901 = 0.2922\ldots$, and it is about $1/(6.39\, k)$ on arcs of height P. At $\xi = \frac{1}{2}$ it is $1/(2C_0k) = 1/(12.78\, k)$, which is smaller than τ_k. Part (ii) is a bound of Weyl type with an exponent of order $1/k$ on a set that contains all minor arcs in the classical sense. For comparison, Lemma 9.8 alone gives the exponent $2\tau_k/(k+1)$ on this set, and for smooth Weyl sums the exponent known on such a set is the one of [41], of order $1/(k \log k)$.

11.2. **Three lemmas.** We need a large sieve inequality for points that are not well spaced, a bound for the number of points βm^k in a short interval, and a variant of Lemma 9.6 in which the roles of the two variables are interchanged.

Lemma 11.3 (large sieve with multiplicities)**.** *Let $N \in \mathbb{N}$, let $z_1, \ldots, z_N$ be complex numbers and $Z(x) := \sum_{y \leqslant N} z_y\, e(xy)$. Let $0 < \delta \leqslant 1$ and $A \geqslant 1$, and let $x_1, \ldots, x_n$ be real numbers such that for every real x at most A of the indices i satisfy $\|x_i - x\| \leqslant \delta/2$. Then*

$$\sum_{i \leqslant n} |Z(x_i)|^2 \leqslant A\big(\delta^{-1} + \pi N\big) \sum_{y \leqslant N} |z_y|^2.$$

Proof. This is Gallagher's proof of the large sieve inequality [12]. If f is continuously differentiable on $\mathcal{I} = [x - \delta/2, x + \delta/2]$, then

$$f(x) = \delta^{-1} \int_{\mathcal{I}} f(u)\, \mathrm{d}u + \delta^{-1} \int_{\mathcal{I}} w(u) f'(u)\, \mathrm{d}u,$$

where $w(u) = u - x + \delta/2$ for $u < x$ and $w(u) = u - x - \delta/2$ for $u > x$, so that $|f(x)| \leqslant \delta^{-1} \int_{\mathcal{I}} |f| + \frac{1}{2} \int_{\mathcal{I}} |f'|$. Let $Z_0(x) := e(-\frac{N+1}{2}x) Z(x)$; we apply the last inequality to $f = Z_0^2$ at the points x_i. The functions $|Z_0| = |Z|$ and $|Z_0'|$ have period 1, and every point of $\mathbb{T}$ lies in at most A of the intervals $[x_i - \delta/2, x_i + \delta/2]$ modulo 1. Hence

$$\sum_i |Z(x_i)|^2 \leqslant A\Big(\delta^{-1} \int_0^1 |Z|^2\, \mathrm{d}x + \int_0^1 |Z_0 Z_0'|\, \mathrm{d}x\Big),$$

and $\int_0^1 |Z_0 Z_0'| \leqslant (\int_0^1 |Z_0|^2)^{1/2} (\int_0^1 |Z_0'|^2)^{1/2} \leqslant \pi N \sum_y |z_y|^2$, because the frequencies $y - \frac{N+1}{2}$ of Z_0 are at most $\frac{N}{2}$ in modulus. □

In the next lemma, and in the remainder of this section, we write $\Omega_k(q) := (2k)^l$ when q has exactly l distinct prime factors.

Lemma 11.4 (spacing)**.** *Let $0 < V_0 < V_1$ and $Y \geqslant 1$ be real numbers, let $\beta \in \mathbb{R}$, and let $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ be coprime with $|q\beta - a| \leqslant \frac{1}{2} V_1^{-k}$. Let $\mathcal{V}$ be a set of integers in $(V_0, V_1]$*

that are coprime to q. Then $\mathcal{V}$ is the union of at most $\Omega_k(q)$ sets $\mathcal{V}_l$ such that for every real x the number of $m \in \mathcal{V}_l$ with $\|\beta m^k - x\| \leqslant \frac12\delta_1$ is at most

$$1 + \frac{2V_1}{q + (YV_0)^k|q\beta - a|}, \qquad \text{where} \quad \delta_1 := \min\Big\{\frac{1}{4q},\, Y^{-k}\Big\}.$$

Proof. For u coprime to q the congruence $x^k \equiv u \pmod q$ has at most $\Omega_k(q)$ solutions modulo q, since $z^k \equiv 1 \pmod{p^e}$ has at most $2k$ solutions for every prime power p^e. We number the solutions as $r_1(u), r_2(u), \dots$, and let $\mathcal{V}_l$ be the set of $m \in \mathcal{V}$ with $m \equiv r_l(m^k) \pmod q$. Then $m_1, m_2 \in \mathcal{V}_l$ and $m_1^k \equiv m_2^k \pmod q$ imply that $m_1 \equiv m_2 \pmod q$.

Let $m_1 > m_2$ be elements of $\mathcal{V}_l$ with $\|\beta m_1^k - x\| \leqslant \frac12\delta_1$ and $\|\beta m_2^k - x\| \leqslant \frac12\delta_1$, and put $h := m_1^k - m_2^k$. Then $\|\beta h\| \leqslant \delta_1 \leqslant 1/(4q)$, and $|(\beta - a/q)h| \leqslant V_1^k/(2qV_1^k) = 1/(2q)$. If $q \nmid h$, then $\|ah/q\| \geqslant 1/q$ and therefore $\|\beta h\| \geqslant 1/(2q)$, which is not the case. So $q \mid h$; then $m_1 \equiv m_2 \pmod q$, and $\|\beta h\| = |\beta - a/q|\, h \leqslant Y^{-k}$. Since $h \geqslant (m_1 - m_2)V_0^{k-1}$, this gives $(m_1 - m_2)\,|q\beta - a| \leqslant q\,Y^{-k}V_0^{1-k}$. Thus the integers $m \in \mathcal{V}_l$ with $\|\beta m^k - x\| \leqslant \frac12\delta_1$ lie in one residue class modulo q, and in an interval whose length is at most V_1 and at most $q\,Y^{-k}V_0^{1-k}|q\beta - a|^{-1}$. So their number is at most

$$1 + \min\Big\{\frac{V_1}{q},\ \frac{V_0}{(YV_0)^k|q\beta - a|}\Big\} \leqslant 1 + \frac{2V_1}{q + (YV_0)^k|q\beta - a|}. \qquad \square$$

Lemma 11.5 (separation of the variables)**.** *Let $1 \leqslant i \leqslant j \leqslant J$, let $\mathfrak{e}$ be a finite set of primes, and let X be attached to j. Put $M_0 := R^i$, $Y := X/M_0$ and $\mathcal{B} := \mathcal{C}^{\mathfrak{e}}_{j-i}(Y)$. Then, for all real β,*

$$|F_j^{\mathfrak{e}}(\beta; X)| \leqslant (\log P)^2 \sup_{(\epsilon_n)} \sum_{d \in \mathcal{D}_i^{\mathfrak{e}}} \Big|\sum_{n \in \mathcal{B}} \epsilon_n\, e\big(\beta(dn)^k\big)\Big|,$$

where the supremum is over all families (ϵ_n) of complex numbers of modulus 1.

Proof. The set $\mathcal{B}$ is the one that occurs in Lemma 9.6, and it was shown in the proof of that lemma that

$$F_j^{\mathfrak{e}}(\beta; X) = \sum_{d \in \mathcal{D}_i^{\mathfrak{e}}} \sum_{\substack{n \in \mathcal{B} \\ n \leqslant X/d,\ p^+(n) < p^-(d)}} e\big(\beta(dn)^k\big),$$

where $p^+(n)$ is the largest prime factor of n that exceeds R (or 0 if there is none), and $p^-(d)$ is the least prime factor of d. This time the two conditions are removed by kernels that depend on d. For a natural number $n \leqslant Y$ and real $y > 0$ one has

$$\int_{-1/2}^{1/2} e(n\psi) K_y(\psi)\, \mathrm{d}\psi = \begin{cases} 1, & n \leqslant y, \\ 0, & n > y, \end{cases} \qquad \text{where} \quad K_y(\psi) := \sum_{1 \leqslant l \leqslant \min\{y, Y\}} e(-l\psi),$$

and for integers $a \geqslant 0$ and $c \geqslant 1$

$$\int_{-1/2}^{1/2} e(a\psi') K_c'(\psi')\, \mathrm{d}\psi' = \begin{cases} 1, & a < c, \\ 0, & a \geqslant c, \end{cases} \qquad \text{where} \quad K_c'(\psi') := \sum_{0 \leqslant l < c} e(-l\psi').$$

We use these identities with $y = X/d$, $a = p^+(n)$ and $c = p^-(d) \leqslant \zeta R$. Therefore

$$F_j^{\mathfrak{e}}(\beta; X) = \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} \sum_{d \in \mathcal{D}_i^{\mathfrak{e}}} K_{X/d}(\psi) K'_{p^-(d)}(\psi') \sum_{n \in \mathcal{B}} e\big(\beta(dn)^k + \psi n + \psi' p^+(n)\big)\, \mathrm{d}\psi\, \mathrm{d}\psi',$$

and the lemma follows, with $\epsilon_n = e(\psi n + \psi' p^+(n))$, since $|K_{X/d}(\psi)| \leqslant \min\{Y, 1/(2|\psi|)\}$ and $|K'_{p^-(d)}(\psi')| \leqslant \min\{\zeta R, 1/(2|\psi'|)\}$ for $|\psi|, |\psi'| \leqslant \frac{1}{2}$, and $(1+\log Y)(1+\log \zeta R) \leqslant (\log P)^2$. □

In Lemma 9.6 the conditions that link d and n were removed at the cost of coefficients attached to d. Here the coefficients are attached to n, so that d becomes the outer variable and the inner sum is taken over a fixed set, whose moments are those of $F^{\mathfrak{e}}_{j-i}(\,\cdot\,;Y)$.

11.3. **The estimate for the sums $F^{\mathfrak{e}}_j$, and the proofs.** We first prove an estimate for all the sums $F^{\mathfrak{e}}_j$, from which Theorem 11.1 will be deduced.

Proposition 11.6. *Let b_0 and s_0 be as in Theorem 11.1. Let $1 \leqslant i \leqslant j \leqslant J$, let $\mathfrak{e}$ be a finite set of primes, let X be attached to j, and put $M_0 := R^i$, $M_1 := (\zeta R)^i$ and $Y := X/M_0$. Let $\beta \in \mathbb{R}$, and let $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ be coprime with $q \leqslant P^k$ and $|q\beta - a| \leqslant \frac{1}{2}M_1^{-k}$. Then, with $\Xi := q + X^k|q\beta - a|$,*

$$F^{\mathfrak{e}}_j(\beta;X) \ll X\, P^{\omega+\varepsilon}\Big(\Big(\frac{1}{M_0} + \frac{1}{\Xi}\Big)\Big(1 + \frac{q}{Y^k}\Big)Y^{\,b_0}P^{\,4k\gamma+2\varepsilon}\Big)^{1/(2s_0)},$$

where the implied constant depends on b_0 and s_0 as well.

Proof. Recall that $\mathcal{D}^{\mathfrak{e}}_i \subseteq (M_0, M_1]$. By Lemma 11.5 it suffices to bound

$$\Sigma := \sum_{d \in \mathcal{D}^{\mathfrak{e}}_i} |f(d^k\beta)|, \qquad \text{where} \quad f(\alpha) := \sum_{n \in \mathcal{B}} \epsilon_n\, e(\alpha n^k),$$

uniformly in the complex numbers ϵ_n of modulus 1.

The inner sum. Every $n \in \mathcal{B}$ satisfies $n \leqslant Y$. Hence $f(\alpha)^{s_0} = \sum_{y \leqslant N} z_y\, e(\alpha y)$ with $N := \lfloor s_0 Y^k \rfloor$, where $|z_y|$ is at most the number of solutions of $n_1^k + \cdots + n_{s_0}^k = y$ with $n_i \in \mathcal{B}$. The number Y is attached to $j-i$, and (8.7) holds with $r = 2s_0$ and $\Delta = b_0$, by the second assertion of Corollary 8.3. So that corollary gives

$$\sum_{y \leqslant N} |z_y|^2 \leqslant S_{s_0}(\mathcal{B}) = \int_{\mathbb{T}} \big|F^{\mathfrak{e}}_{j-i}(\alpha;Y)\big|^{2s_0} \mathrm{d}\alpha \ll Y^{\,2s_0-k+b_0}P^{\,4k\gamma+2\varepsilon}. \tag{11.1}$$

Common factors with q. For $d \in \mathcal{D}^{\mathfrak{e}}_i$ let $c := d_q(d)$ be the product of the primes that divide both d and q, as in §9.2, and $d' := d/c$. For a given c, let $\mathcal{V}_c$ be the set of the numbers d' that arise; they are coprime to q and lie in $(M_0/c, M_1/c]$. If c has i' prime factors, then $\mathcal{V}_c \subseteq \mathcal{D}^{\mathfrak{e}}_{i-i'}$, whence $\#\mathcal{V}_c \leqslant (\zeta R)^{i-i'} \leqslant M_1/c$. With $\beta_c := c^k\beta$ we have

$$\Sigma = \sum_c \Sigma_c, \qquad \Sigma_c := \sum_{d' \in \mathcal{V}_c} |f(d'^{\,k}\beta_c)|.$$

Every prime in $\mathcal{P}$ exceeds $R = P^\eta$, and $q \leqslant P^k$; so at most $k(J+1)$ of these primes divide q, there are at most $2^{k(J+1)}$ numbers c, and each of them other than $c = 1$ exceeds R.

The points. Fix c, and let $g_c := (q, c^k)$, $q_c := q/g_c$ and $a_c := ac^k/g_c$. Then $(a_c, q_c) = 1$ and $q_c\beta_c - a_c = (c^k/g_c)(q\beta - a)$, so that

$$|q_c\beta_c - a_c| \leqslant \tfrac{1}{2}(M_1/c)^{-k}, \qquad q_c + (X/c)^k|q_c\beta_c - a_c| = \Xi/g_c.$$

We apply Lemma 11.4 to $\mathcal{V}_c$, with β_c, a_c, q_c in place of β, a, q, with $V_0 = M_0/c$ and $V_1 = M_1/c$, and with the present Y; note that $YV_0 = X/c$. It shows that $\mathcal{V}_c$ is the union of at most $\Omega_k(q)$ sets $\mathcal{V}_{c,l}$ such that at most $A_c := 1 + 2M_1g_c/(c\,\Xi)$ of the points $d'^{\,k}\beta_c$

with $d' \in \mathcal{V}_{c,l}$ lie within $\frac12\delta_1$ of any given point of $\mathbb{T}$, where $\delta_1 = \min\{1/(4q_c), Y^{-k}\}$. By Lemma 11.3, applied to the trigonometric polynomial f^{s_0},

$$\sum_{d'\in\mathcal{V}_{c,l}} |f(d'^k\beta_c)|^{2s_0} = \sum_{d'\in\mathcal{V}_{c,l}} \Big|\sum_{y\leqslant N} z_y\, e(d'^k\beta_c\, y)\Big|^2 \leqslant A_c\big(4q + Y^k + \pi s_0 Y^k\big) S_{s_0}(\mathcal{B}).$$

Conclusion. By Hölder's inequality, $\Sigma_c^{2s_0} \leqslant (M_1/c)^{2s_0-1} \sum_l \sum_{d'\in\mathcal{V}_{c,l}} |f(d'^k\beta_c)|^{2s_0}$. Since $g_c \leqslant c^k$, we have $(c/M_1)A_c \leqslant 2c^k(1/M_1 + 1/\Xi)$. Hence, by (11.1),

$$\Sigma_c \ll \frac{M_1 Y}{c^{1-k/(2s_0)}} \Big(\Omega_k(q)\Big(\frac{1}{M_1} + \frac{1}{\Xi}\Big)\Big(1 + \frac{q}{Y^k}\Big) Y^{b_0} P^{4k\gamma+2\varepsilon}\Big)^{1/(2s_0)}.$$

As $2s_0 \geqslant 2k$, the sum of $c^{-1+k/(2s_0)}$ over all c is at most $1 + 2^{k(J+1)}R^{-1/2} \leqslant 2$. Finally $M_1 Y = \zeta^i X \leqslant XP^\omega$ by (3.3), and $\Omega_k(q)(\log P)^{4s_0} \ll P^\varepsilon$, because $q \leqslant P^k$ has $O(\log P/\log\log P)$ distinct prime factors. □

Remark 11.7. The factor $1/M_0 + 1/\Xi$ shows that the saving is limited by the height of β as well as by the length of the outer variable, and the factor $1 + q/Y^k$ that the inner variable must have length at least $q^{1/k}$. For β of height $X^{k\xi}$ this forces $M_0 \leqslant X^{1-\xi}$, which is the source of the exponent $(1-\xi)/(2s_0)$, and of the loss at $\xi = \frac12$. The second restriction is inherent in the large sieve: the points βm^k are spaced by $1/q$ at best, and f^{s_0} has length $s_0 Y^k$. For smooth Weyl sums the analogous restriction is the reason why [41] combines the large sieve estimate [41, Lemma 4.1] with a second estimate for large q.

Proof of Theorem 11.1. Let $R^2 \leqslant Q \leqslant P^{k/2}$ and $\alpha \in \mathfrak{N}(Q;P)$, and let a and q be coprime with $q \leqslant Q$ and $|q\alpha - a| \leqslant QP^{-k}$. By (A3), $\Xi := q + P^k|q\alpha - a| > \frac12 Q$. Define $\tilde\mu$ by $P^{\tilde\mu} = \min\{Q, P(2Q)^{-1/k}\}$; then $R^2 \leqslant P^{\tilde\mu} \leqslant P^{k/(k+1)}$. Let i be the largest integer with $(\zeta R)^i \leqslant P^{\tilde\mu}$. Then $i \geqslant 1$, as $\zeta R \leqslant R^2$, and $i < J$, as $(\zeta R)^J > R^J = P^{1-\eta} > P^{k/(k+1)}$. Hence $(\zeta R)^{i+1} > P^{\tilde\mu}$, which gives $R^i > P^{\tilde\mu}/(\zeta^J R) \geqslant P^{\tilde\mu}R^{-9/8}$ as in the proof of Corollary 9.3. Thus, in the notation of Proposition 11.6 with $j = J$, $\mathfrak{e} = \emptyset$ and $X = P$,

$$P^{\tilde\mu}R^{-9/8} < M_0 \leqslant M_1 \leqslant P^{\tilde\mu} \leqslant P(2Q)^{-1/k}, \qquad Y = P/M_0 \geqslant (2Q)^{1/k}.$$

The hypotheses of Proposition 11.6 hold, since $q \leqslant P^k$ and $|q\alpha - a| \leqslant QP^{-k} \leqslant \frac12 M_1^{-k}$. Moreover $q/Y^k \leqslant \frac12$, $1/M_0 + 1/\Xi \leqslant 3R^{9/8}P^{-\tilde\mu} \leqslant P^{2\eta-\tilde\mu}$ and $Y \leqslant P$. This gives the theorem. □

Proof of Corollary 11.2. (i) When $Q = P^{k\xi}$ and $\xi \geqslant \frac{1}{k+1}$, the number $\min\{Q, P(2Q)^{-1/k}\}$ equals $2^{-1/k}P^{1-\xi}$; and $(1-\xi-b_1)/(2s_0) > 2\xi\tau_k$ if and only if $\xi < \xi_1$.

(ii) Lemma 9.8 and its proof do not depend on the choices (9.2) and (9.3); so that lemma is available here. Let $\alpha \in \mathbb{T} \setminus \mathfrak{M}(P^{k/(k+1)};P)$. By (A1), $\alpha \in \mathfrak{N}(Q;P)$ for some $Q = P^{k\xi}$ of the form $2^{-i}P^{k/2}$ with $\frac{1}{k+1} < \xi \leqslant \frac12$. If $\xi \leqslant \xi_1$ we use (i), noting that $(1-\xi-b_1)/(2s_0) \geqslant (1-\xi_1-b_1)/(2s_0) = \tau^{(1)}$; if $\xi > \xi_1$ we use Lemma 9.8, noting that $Q^{-2\tau_k/k} = P^{-2\xi\tau_k} \leqslant P^{-\tau^{(1)}}$. □

It remains to justify the remark on the limit $k \to \infty$ that follows Corollary 11.2. Let $\Theta = \Theta(k)$ tend to infinity in such a way that $\Theta/\log k \to \infty$ and $\Theta/\sqrt{k} \to 0$, let $\delta_0, \varepsilon_2 \to 0$, and let $b_0 = 1/\log k$. Then $\mathcal{G}(u_\Theta) \leqslant C_0 + o(1)$ by Lemma 8.1(i) and the formulae of §10, and $a_*(b_0) \leqslant c_\Theta \log(k\log k) = o(1)$, so that $2s_0 = (C_0 + o(1))k$ is admissible in Theorem 11.1; and $b_1 \to 0$ if we take $\gamma \leqslant 1/k^2$ and $\varepsilon \leqslant \gamma$, since $\eta \leqslant \gamma$ by (3.2).

11.4. **The arcs that decide the number of variables.** We return to the setting of §9: let b, r and t be as in (9.1), and let b_0, s_0 and b_1 be as in Theorem 11.1. In the proof of Proposition 9.1 the arcs of height $Q \leqslant P^{1/4}$ are harmless, by Lemma 9.9, and the condition $t\tau_k > b$ of (9.1) is needed only for the arcs with $P^{1/4} \leqslant Q \leqslant P^{k/2}$. If Theorem 11.1 is used there in place of Lemma 9.8, then together with Corollary 9.3 it gives

$$\begin{aligned}\int_{\mathfrak{N}(Q;P)} |F|^{r+t}\,\mathrm{d}\alpha &\leqslant \sup_{\mathfrak{N}(Q;P)} |F|^t \int_{\mathfrak{M}(Q;P)} |F|^r\,\mathrm{d}\alpha\\ &\ll P^{r+t-k}\Big(\frac{P^{b_1}}{\min\{Q, P(2Q)^{-1/k}\}}\Big)^{\frac{t}{2s_0}} Q^{\frac{2b}{k}} P^{\varepsilon''},\end{aligned}$$

where $\varepsilon'' := t(\omega+\varepsilon) + 4k\gamma + 2\varepsilon + 2\eta b$ is as small as we please. For $Q = P^{k\xi}$ with $\frac{1}{k+1} \leqslant \xi \leqslant \frac{1}{2}$ the condition on t therefore becomes $t(1-\xi-b_1)/(2s_0) > 2\xi b$, apart from small terms, and in the limit described after Corollary 11.2 the two conditions read

$$\frac{t}{k\,b} > 9.027901 \quad \text{(Lemma 9.8)}, \qquad\qquad \frac{t}{k\,b} > \frac{2C_0\,\xi}{1-\xi} \quad \text{(Theorem 11.1)}. \tag{11.2}$$

The right hand side of the second one is 0.67, 1.42, 3.19, 5.48, 9.03, 12.78 for $\xi = 0.05$, 0.1, 0.2, 0.3, 0.414, 0.5. So Theorem 11.1 would allow one to prune the arcs with $\xi < \xi_1$ with fewer variables, but for $\xi \geqslant \xi_1$ it is weaker than Lemma 9.8, and the bound of Theorem 2.3 is not improved.

Remark 11.8. Let $P^{1/4} \leqslant Q \leqslant P^{k/2}$ and $Y := (2Q)^{2/k}$, the length of the sums to which the change of variables reduces the moments over $\mathfrak{M}(Q;P)$ (Corollary 9.3), and suppose that $|F| \leqslant PY^{-\tau}$ on $\mathfrak{N}(Q;P)$ for some $\tau \in (0, \frac{1}{5}]$. If $s > k\,\mathcal{G}(\tau)$, then a supporting line at $z = \tau$ of the concave function $k\,z\,\mathcal{G}(z)$ (Lemma 8.1(v)) provides numbers r and Δ with $2k < r < s$, $0 \leqslant \Delta < (s-r)\tau$ and (8.7); here $r > 2k$ because $\mathcal{G} - \mathfrak{g} > 2$ on $(0, \frac{1}{5}]$, by the proof of Lemma 8.1(ii). By Corollary 8.3 and the proof of Corollary 9.3, $\int_{\mathfrak{M}(Q;P)} |F|^r\,\mathrm{d}\alpha \ll P^{r-k}Y^{\Delta}P^{\varepsilon''}$ with ε'' as small as we please, and therefore $\int_{\mathfrak{N}(Q;P)} |F|^s\,\mathrm{d}\alpha \ll P^{s-k}Y^{\Delta-(s-r)\tau}P^{\varepsilon''}$. Conversely, when $s < k\,\mathcal{G}(\tau)$ no choice of r and Δ gives a saving in this way, because $r + \Delta/\tau \geqslant k\,\mathcal{G}(\tau)$ whenever (8.7) holds. Thus arcs on which F saves a factor Y^{τ} are pruned with $k\,\mathcal{G}(\tau)$ variables, and by this argument not with fewer.

On arcs of height P^h with fixed $h \geqslant 1$, Theorem 11.1 gives $\tau \approx 1/(2C_0h)$ for large k, and with the limit $\mathcal{G}_\infty$ of the potential (see below) the number $k\,\mathcal{G}_\infty(1/(2C_0h))$ is $4.45\,k$, $5.02\,k$, $5.55\,k$, $5.82\,k$, $6.15\,k$ for $h = 1, 2, 5, 10, 50$. On the arcs with $\xi \geqslant \xi_1$ one has $\tau = \tau_k$, and $k\,\mathcal{G}(\tau_k) = C_0k + O(\sqrt{k})$ for suitable Θ. So the variables beyond about $4.5\,k$ are needed for arcs of large height, and the second order term in Theorem 1.1 is determined by the exponent that is available there. If Lemmas 9.7 and 9.8 held with $1/(\mathsf{D}k)$ in place of τ_k, for a constant D with $4 \leqslant \mathsf{D} \leqslant 9.027901$, then Theorem 2.3 and Proposition 10.7 would hold with D in place of 9.027901 in (2.3) and in the definition of ϕ, and the constant $1.57112\ldots$ of Proposition 10.7 would become

$$c_2(\mathsf{D}) := -\frac{4}{\lambda_0} + 2\log\frac{1+\lambda_0}{1-\lambda_0}, \qquad \lambda_0 = \sqrt{\frac{\mathsf{D}}{\mathsf{D}+2}};$$

for instance $c_2(8) = 1.302$, $c_2(C_0) = 0.794$ and $c_2(4) = -0.314$. The leading constant C_0 is not affected by any exponent of order $1/k$.

The moments of F do not improve the exponent τ_k itself. Lemma 9.4 bounds a bilinear sum in terms of the moments $S_u(\mathcal{A})$ and $S_w(\mathcal{B})$. Suppose that $\mathcal{A} \subseteq [1, P^\mu]$ and $\mathcal{B} \subseteq [1, P^{1-\mu}]$ have $P^{\mu+o(1)}$ and $P^{1-\mu+o(1)}$ elements, that their moments of orders $2u = v_A k$ and $2w = v_B k$ have the exponents $k\,d_A$ and $k\,d_B$, that is, $S_u(\mathcal{A}) \ll P^{\mu(2u-k+kd_A)}$ and $S_w(\mathcal{B}) \ll P^{(1-\mu)(2w-k+kd_B)}$ with $0 \leqslant d_A, d_B \leqslant 1$, and that $q_1 = P^{k/2+o(1)}$ in Lemma 9.4. Then the lemma gives the bound $P^{1-\mathcal{E}/k+o(1)}$ for the bilinear sum, where

$$\mathcal{E} = \mathcal{E}(\mu; v_A, v_B) := \frac{2}{v_A v_B}\Bigl(\min\{\mu, 1-\mu\} - \mu\,d_A - (1-\mu)\,d_B\Bigr). \tag{11.3}$$

Lemma 9.7 is the case $\mu = \frac12$, $v_A = v_B = 2$, $d_A = d_B = \delta_*$, in which $\mathcal{E} = \frac14(1 - 2\delta_*)$, which is $1/9.02790\ldots$. Let $\mathcal{G}_\infty$ be the limit of the potential as $k \to \infty$ (§2.2): $\mathcal{G}_\infty(z) = 1/z$ for $z \geqslant \frac12$, $\mathcal{G}_\infty(z) = \Psi(z)$ for $\frac{1}{10} \leqslant z \leqslant \frac12$, and $\mathcal{G}_\infty(z) = \Psi(\frac{1}{10}) + 2\int_z^{1/10} \frac{\mathrm{d}u}{u\,\theta^*(2u)} = C_0 - 2I(2z)$ for $0 < z \leqslant \frac{1}{10}$. By Corollary 8.3, the exponent for the moment of order vk of the sums $F_j^{\mathfrak{e}}$ that follows from Proposition 8.2 is $(d_\infty(v) + o(1))\,k$ as $k \to \infty$, for $v > 1$ and suitable Θ, where

$$d_\infty(v) := \sup_{0<z\leqslant 1} z\,\bigl(\mathcal{G}_\infty(z) - v\bigr) \qquad (v \geqslant 0).$$

Proposition 11.9. (i) *Let* $v_c := 1 - \frac19 + \log 9 = 3.0861\ldots$. *Then* $d_\infty(v) = \mathfrak{d}(vk)$ *for* $0 \leqslant v \leqslant v_c$, $0 < d_\infty(v) < \mathfrak{d}(vk)$ *for* $v_c < v < C_0$, *and* $d_\infty(v) = 0$ *for* $v \geqslant C_0$.

(ii) *For all* $\mu \in (0,1)$ *and* $v_A, v_B > 0$, $\mathcal{E}(\mu; v_A, v_B) \leqslant \mathcal{E}(\frac12; v_A, v_B) = (1-d_A-d_B)/(v_A v_B)$. *With* $d_A = d_\infty(v_A)$ *and* $d_B = d_\infty(v_B)$, *this is the same number as with the exponents* $\mathfrak{d}(v_A k)$ *and* $\mathfrak{d}(v_B k)$ *of smooth Weyl sums when* $v_A, v_B \leqslant v_c$, *and it is at most* $v_c^{-2} < 1/9.52$ *when* $v_A, v_B \geqslant v_c$.

Thus F has better moments than a smooth Weyl sum only at orders above $3.09\,k$, and these do not help in Lemma 9.4. In the mixed range, where one of v_A, v_B exceeds v_c and the other does not, a numerical evaluation gives $\mathcal{E}(\frac12; v_A, v_B) < 1/9.46$; for $v_A, v_B \leqslant v_c$ the largest value is the one found for smooth Weyl sums in [6, §7], namely $1/9.026725\ldots$ at $v_A = v_B = 1.98464\ldots$, which exceeds the value $1/9.02790$ at $v_A = v_B = 2$ by an amount that is of no consequence here. An improvement of τ_k by this route would require moments of order about $2k$ for the sets $\mathcal{D}_i^{\mathfrak{e}}$ and $\mathcal{B}$ of Lemma 9.6 with an exponent δk smaller than $\delta_* k$; the exponent would then be $(1-2\delta)/(4k)$, and the limit of the method is $1/(4k)$. The fibre method does not provide such moments: at the level $z = \delta_*/(1+\delta_*) = 0.218$ that governs the moment of order $2k$, the permitted steps are $\theta \leqslant \theta^*(2z) = 1.13$, and the resulting bound is weaker than Ψ (§2.2).

Proof of Proposition 11.9. (i) Let $\mathfrak{g}_\infty(z) := -z\mathcal{G}'_\infty(z)$; thus $\mathfrak{g}_\infty(z) = 2/\theta^*(2z)$ for $z < \frac{1}{10}$, $\mathfrak{g}_\infty(z) = 1/(1-z)$ for $\frac{1}{10} < z < \frac12$, and $\mathfrak{g}_\infty(z) = 1/z$ for $z > \frac12$. The function θ^* is nonincreasing with $\theta^*(\frac15) = \frac95$, so that $\mathfrak{g}_\infty$ is nondecreasing on $(0, \frac12)$, and $\mathcal{G}_\infty - \mathfrak{g}_\infty$ is nonincreasing on $(0,1]$, with the value 0 on $[\frac12, 1]$. The derivative of $z \mapsto z(\mathcal{G}_\infty(z) - v)$ is $\mathcal{G}_\infty(z) - \mathfrak{g}_\infty(z) - v$. Hence this function is concave, and it attains its maximum at a point where $\mathcal{G}_\infty - \mathfrak{g}_\infty - v$ changes sign. For $\frac{1}{10} < z < \frac12$ write $\delta := z/(1-z)$; then $\mathcal{G}_\infty(z) - \mathfrak{g}_\infty(z) = 1 - \delta - \log\delta$, and $z\,(\mathcal{G}_\infty(z) - v) = \delta$ when $v = 1 - \delta - \log\delta$. So for $0 \leqslant v \leqslant v_c$ the maximum is attained at $z = \delta/(1+\delta)$ with $\delta = \mathfrak{d}(vk) \geqslant \frac19$, and

$d_\infty(v) = \mathfrak{d}(vk)$. The same computation with Ψ in place of $\mathcal{G}_\infty$ on all of $(0, \frac{1}{2}]$ shows that $\sup_z z(\Psi(z) - v) = \mathfrak{d}(vk)$ for all $v \geqslant 0$. For $v > v_c$ the maximum of $z(\mathcal{G}_\infty(z) - v)$ is attained at a point $z < \frac{1}{10}$, where $\mathcal{G}_\infty(z) < \Psi(z)$ because $2/\theta^*(2u) < 1/(1-u)$ for $u < \frac{1}{10}$ (§2.2); hence $d_\infty(v) < \mathfrak{d}(vk)$. Finally $\mathcal{G}_\infty < C_0$ on $(0, 1]$, so $d_\infty(v) = 0$ for $v \geqslant C_0$, while $d_\infty(v) > 0$ for $v < C_0$.

(ii) For $\mu \leqslant \frac{1}{2}$ the bracket in (11.3) equals $\mu(1-d_A+d_B)-d_B$, which is nondecreasing in μ because $d_A \leqslant 1$; for $\mu \geqslant \frac{1}{2}$ it equals $(1-\mu)(1-d_B+d_A)-d_A$, which is nonincreasing in μ. This gives the first assertion. The second follows from (i), and the third holds because $d_A, d_B \geqslant 0$. □

11.5. **Small fractional parts of αn^k.** We now deduce Theorem 2.7 from Corollary 11.2, and then prove Theorem 1.5. The argument is the standard one; we follow [40, §§4 and 5]. If $\|\alpha n^k\|$ is not small for any $n \leqslant N$, then by a lemma of Harman the sum F is large at one of the points $l\alpha$ with $l \leqslant L$. By the bound of Weyl type, $l\alpha$ then lies close to a rational number with a small denominator q, and this leads to a contradiction on considering $n = ql$.

Proof of Theorem 2.7. The choice of the parameters. Let δ_0 and ε_2 be as in (3.1), with $\varepsilon_2 \leqslant \delta_0$, and let c_Θ, u_Θ and $\mathcal{G}$ be as in §8. We saw in the proof of Theorem 2.3 that, as $\delta_0 \to 0$, one has $c_\Theta \to 2/\Theta$ and $u_\Theta \to y_k(\Theta)\,\tau_k$, and that the upper limit of $k\,\mathcal{G}(u_\Theta)$ is at most $k\big(C(h_k) - 2T(\Theta) + 2R_k(\Theta)\big)$. Hence the number $a_*(b) = c_\Theta \log\big(k\,c_\Theta\,u_\Theta/(\mathrm{e}\,b)\big)$ of (8.8) tends to

$$\frac{2}{\Theta}\Big(\log y_k(\Theta) + \log\frac{2k\tau_k}{\Theta\,b} - 1\Big) = W_k(\Theta) - \frac{2}{\Theta}\Big(1 + \log\frac{9.027901\,\Theta\,b}{2}\Big),$$

and the upper limit of $k\big(\mathcal{G}(u_\Theta)+a_*(b)\big)$ is at most the right hand side of the first inequality in the theorem. We can therefore fix δ_0 and ε_2 in such a way that $2s_0 \geqslant k\big(\mathcal{G}(u_\Theta)+a_*(b)\big)$. As $s_0 \geqslant k$, the hypotheses of Theorem 11.1 then hold with $b_0 := b$. Next we choose γ and ε subject to (3.1), then J so large that Lemma 9.8 is applicable, and η and ω as in (3.2), in such a way that the numbers $b_1 = b+4k\gamma+2\varepsilon+2\eta$ and $\tau^{(1)} = (1-b_1)/(2s_0+4.5139505\,k)$ of Corollary 11.2 satisfy $b_1 < 1$ and

$$\rho + \varepsilon + \eta < \tau^{(1)}. \tag{11.4}$$

This is possible because γ and ε may be taken arbitrarily small, and $\eta \leqslant \gamma$ by (3.2). Finally we take $H = 1$.

The argument. We may suppose that $\rho > 0$. Let N be a large real number, let $P := N$, and let $\mathcal{C}$ and F be as in §3. Put $L := \lceil P^\rho \rceil$, and suppose that $\|\alpha n^k\| \geqslant L^{-1}$ for all natural numbers $n \leqslant N$. This holds in particular for all $n \in \mathcal{C}$. By a lemma of Harman [20, Lemma 5], which we use in the form [40, Lemma 4.1] with the weight 1 on $\mathcal{C}$ and 0 elsewhere, it follows that

$$\sum_{l=1}^{L} |F(l\alpha)| \geqslant \tfrac{1}{6}\,\#\mathcal{C}.$$

Hence there is a natural number $l \leqslant L$ with $|F(l\alpha)| \geqslant \#\mathcal{C}/(6L)$. By Lemma 3.2 we have $\#\mathcal{C} \geqslant P(\log P)^{-J-1}$, so that $|F(l\alpha)| \geqslant P^{1-\rho}(\log P)^{-J-2}$. In view of (11.4), this is

larger than the bound of Corollary 11.2(ii) when P is large. Consequently $l\alpha$ is congruent modulo 1 to an element of $\mathfrak{M}(P^{k/(k+1)}; P)$: there are coprime integers a and q with

$$1 \leqslant q \leqslant P^{k/(k+1)}, \qquad |q\,l\alpha - a| \leqslant P^{k/(k+1)-k}.$$

Let $n := ql$. Since $\tau^{(1)} < 1/(2s_0) \leqslant 1/(2k)$, we have $\rho < 1/(2k)$, and therefore $L < P^{1/(k+1)}$. Thus $n \leqslant P^{k/(k+1)}L \leqslant P = N$, and

$$\begin{aligned}\|\alpha n^k\| \leqslant \bigl|\alpha\, q^k l^k - a\, q^{k-1}l^{k-1}\bigr| &= (ql)^{k-1}\,|q\,l\alpha - a| \\ &\leqslant L^{k-1}P^{(k-1)k/(k+1)}P^{k/(k+1)-k} = L^{k-1}P^{-k/(k+1)} < L^{-1},\end{aligned}$$

contrary to our assumption. Hence $\|\alpha n^k\| < L^{-1} \leqslant N^{-\rho}$ for some natural number $n \leqslant N$. All estimates are uniform in α. □

For the evaluation we take $\Theta = \Theta^\sharp(k)$, as in §10.2, and $b = 2/(11\,\Theta)$, which is close to the best choice. Let $\chi_0 := 1 + \log(9.027901/11) = 0.80242\ldots$, and for $k \geqslant 37$ put

$$b^\sharp(k) := \frac{2}{11\,\Theta^\sharp(k)}, \qquad E^\sharp(k) := \frac{2s_0^\sharp(k) + 4.5139505\,k}{1 - b^\sharp(k)},$$

where $s_0^\sharp(k)$ is the least natural number $s_0 \geqslant k$ with $2s_0 > \mathcal{Y}^\sharp(k) - 2\chi_0\,k/\Theta^\sharp(k)$, and $\mathcal{Y}^\sharp(k) = k\,B(k, \Theta^\sharp(k)) - 1$, as in §10.3. As $9.027901\,\Theta\,b/2 = 9.027901/11$ for $b = 2/(11\,\Theta)$, Theorem 2.7 shows that for every real number $\rho < 1/E^\sharp(k)$ there is a number N_0 with

$$\min_{1\leqslant n\leqslant N} \|\alpha n^k\| < N^{-\rho} \qquad (\alpha \in \mathbb{R},\ N \geqslant N_0). \tag{11.5}$$

Proposition 11.10. (i) *Let $37 \leqslant k \leqslant 5000$. Then $E^\sharp(k) < 10.9035\,k + 12\sqrt{k}$. Moreover $E^\sharp(k) < 10.9035\,k + 4\sqrt{k}$ for $k \geqslant 900$, $E^\sharp(k) < 12\,k$ for $k \geqslant 42$, $E^\sharp(k) < 11\,k$ for $k \geqslant 1339$, and $E^\sharp(k) < k(\log k + 8.02113)$ for $k \geqslant 46$.*

(ii) *Let $k \geqslant 5000$. Then $E^\sharp(k) < 10.9035\,k + 2\sqrt{k} + 5\log k + 58$.*

Proof. (i) We have evaluated $E^\sharp(k)$ for each of these k from the values of $B^\sharp(k)$ computed for Proposition 10.5 (see §10.4). In all cases $s_0^\sharp(k) > 3k$. The largest value of $(E^\sharp(k) - 10.9035\,k)/\sqrt{k}$ is $11.5025\ldots$, attained at $k = 175$, and the largest value for $900 \leqslant k \leqslant 5000$ is $3.9808\ldots$. The largest value of $E^\sharp(k)/k$ for $k \geqslant 42$ is $11.9863\ldots$, attained at $k = 80$. For $1339 \leqslant k \leqslant 5000$ one has $11k - E^\sharp(k) > 1.5$, and for $46 \leqslant k \leqslant 5000$ one has $k(\log k + 8.02113) - E^\sharp(k) > 1$, whereas $E^\sharp(k) > k(\log k + 8.02113)$ for $37 \leqslant k \leqslant 45$.

(ii) Let $a := a_k$, so that $\Theta := \Theta^\sharp(k) = \frac{9}{10}a$. Recall from the proof of Proposition 10.6 that $\sqrt{k} - \frac{5}{2} < a < \sqrt{k} - 2.45$ and $a > 68.2$. Thus $b := b^\sharp(k) = \frac{20}{99a} < 0.003$, and since $k = a(a+5)$, we have $k/\Theta = \frac{10}{9}(a+5) < \frac{10}{9}(\sqrt{k} + 2.55)$. By the proof of Corollary 2.6, $\mathcal{Y}^\sharp(k) \leqslant C_0 k + c_3\sqrt{k} + 5\log k + 54$. The right hand side exceeds $2k + 2\chi_0 k/\Theta$, so that $2s_0^\sharp(k) \leqslant C_0 k + c_3\sqrt{k} + 5\log k + 56 - 2\chi_0 k/\Theta$ by the definition of $s_0^\sharp(k)$. As $C_0 + 4.5139505 < 10.9035$, the number $\mathcal{Z} := 2s_0^\sharp(k) + 4.5139505\,k$ satisfies

$$\mathcal{Z} \leqslant 10.9035\,k + c_3\sqrt{k} + 5\log k + 56 - 2\chi_0\,\frac{k}{\Theta}.$$

Now $1/(1-b) < 1 + 1.0031\,b$, and $b \cdot 10.9035\,k = \frac{2\cdot 10.9035}{11} \cdot \frac{k}{\Theta}$. Further $\sqrt{k}/a < 1.037$ and $(5\log k + 56)/a < 1.446$, the latter because $(5\log k + 56)/(\sqrt{k} - \frac{5}{2})$ decreases for $k \geqslant 5000$;

hence $b\,(c_3\sqrt{k}+5\log k+56)<\frac{20}{99}(1.572\cdot 1.037+1.446)<0.622$. It follows that

$$E^\sharp(k)=\frac{\mathcal{Z}}{1-b}<\mathcal{Z}+1.0031\,b\,\mathcal{Z}$$
$$<10.9035\,k+c_3\sqrt{k}+5\log k+56.63+\Big(\frac{1.0031\cdot 2\cdot 10.9035}{11}-2\chi_0\Big)\frac{k}{\Theta}.$$

The bracket is smaller than 0.3838, and $0.3838\cdot\frac{10}{9}(\sqrt{k}+2.55)<0.42645\sqrt{k}+1.09$. Since $c_3+0.42645<2$ and $56.63+1.09<58$, this proves (ii). □

TABLE 4. The number $E^\sharp(k)$ of (11.5): one has $\min_{n\leqslant N}\|\alpha n^k\|<N^{-\rho}$ for $\rho<1/E^\sharp(k)$ and large N. The last two columns belong to the localised bound of [9] and to the unlocalised bound of [40].

k	$\Theta^\sharp(k)$	$b^\sharp(k)$	$2s_0^\sharp(k)$	$E^\sharp(k)$	$k(\log k+8.02113)$	$9.028\,k$
37	2.3125	0.0786	248	450.43	430.39	334.0
46	2.8750	0.0632	302	544.05	545.09	415.3
50	3.1250	0.0582	328	587.90	596.66	451.4
100	3.8462	0.0473	668	1174.94	1262.63	902.8
200	4.9587	0.0367	1342	2330.23	2663.89	1805.6
500	12.397	0.0147	3244	5582.86	7117.87	4514.0
1000	24.793	0.0073	6428	11022.78	14928.89	9028.0
2000	38.062	0.0048	12818	21950.76	31244.06	18056.0
5000	61.429	0.0030	31986	54717.71	82691.62	45140.0
10^4	87.778	0.0021	63930	109295.89	172314.70	90280.0
10^5	282.36	0.0006	638952	1091049.59	1953405.55	902800.0
10^6	897.75	0.0002	6389382	10905541.15	21836640.56	9028000.0

Proof of Theorem 1.5. By (11.5) it suffices to show that $E^\sharp(k)$ is smaller than the reciprocal of the exponent in question. For $37\leqslant k\leqslant 5000$ this is Proposition 11.10(i). Let $k\geqslant 5000$. Then $E^\sharp(k)<10.9035\,k+2\sqrt{k}+5\log k+58$ by Proposition 11.10(ii). Moreover $2\sqrt{k}+5\log k+58<3.5\sqrt{k}$, since this holds for $k=5000$ and $1.5\sqrt{k}-5\log k$ increases with k; and $10.9035\,k+3.5\sqrt{k}<11\,k$, since $0.0965\sqrt{k}>3.5$. □

Remark 11.11. The exponent $\tau^{(1)}$ of Corollary 11.2 is the least saving on the arcs of height $P^{k\xi}$ with $\frac{1}{k+1}\leqslant\xi\leqslant\frac12$; it is attained at $\xi=\xi_1$, where the large sieve exponent $(1-\xi-b_1)/(2s_0)$ and the exponent $2\xi\tau_k$ of Lemma 9.8 agree. A decomposition into two factors of size $P^{1/2}$ for all α, as in [40, §5], corresponds to the case $\xi=\frac12$ of Corollary 11.2(i), and by itself it gives only $1/\rho=(2C_0+o(1))k=(12.78+o(1))k$. The unlocalised bound of [40] corresponds to the exponent τ_k on the arcs of height $P^{k/2}$: if a/q is a convergent to the continued fraction of an irrational number α and $P=N:=q^{2/k}$, then for $l\leqslant L$ the point $l\alpha$ does not lie in $\mathfrak{M}(Q;P)$ when $Q<P^{k/2}/(L+1)$, and the argument above, with Lemma 9.8 in place of Corollary 11.2, gives a natural number $n\leqslant N$ with $\|\alpha n^k\|<N^{-\tau_k+\varepsilon'}$, for any fixed $\varepsilon'>0$ and large q. For numbers α with bounded partial quotients every large N is of this kind, up to a bounded factor. Thus the arcs of

height near $P^{k/2}$ limit the localised and the unlocalised bounds alike, and by §11.4 the moments of F do not improve the exponent τ_k that is available there.

12. Concluding remarks

12.1. **Intermediate coefficients.** In the proof of Lemma 4.4 only the leading coefficient α_D is used to bound $\mathcal{N}$, and its denominator $q_D \leqslant q^{1/(D+1)}$ is small. For $2 \leqslant r \leqslant D$ the denominator of $\alpha_r(n_0)$ is $q_r = \prod_p p^{\max\{e_p - rs_p, 0\}}$, which is much larger. One may therefore bound the sum over each residue class modulo A separately by a Weyl-type inequality for the coefficient α_r: by [32, Theorem 5.2] with the optimal mean value [4, 44], that is, by [3, Theorem 5],

$$\mathcal{S} \ll X^{1+\varepsilon}\bigl(q_r^{-1} + Y^{-1} + q_r Y^{-r}\bigr)^{1/(D(D-1))} \tag{12.1}$$

in place of the bound $X^{1+\varepsilon}(Y^{-D}+q_D^{-1})^{1/(D(D+1))}$ of Lemma 4.4; the case $r = D$ is the device of [10, Lemma 3.1]. Ignoring the rounding of moduli, $A = q^{1/(D+1)}$ gives $Y = X^{1-U/(D+1)}$ and $q_r = X^{U(D+1-r)/(D+1)}$, so that the saving in (12.1) is

$$\frac{1}{D(D-1)}\min\Bigl\{1 - \frac{U}{D+1},\ \frac{U(D+1-r)}{D+1},\ r - U\Bigr\} \leqslant \frac{D+1-U}{(D-1)D(D+1)}, \tag{12.2}$$

and a larger A is worse. Thus $r > U$ is needed, and however large q_r is, the saving is limited by the term Y^{-1}, because a single residue class is short; Lemma 4.4 applies the mean value theorem to all A classes together, which replaces Y^{-1} by Y^{-D}. For a fixed degree $D \geqslant 2U$ the saving (12.2) is larger than that of Lemma 4.4 with the same D, but Lemma 4.4 is applied with $D \approx U$, and then the comparison goes the other way. The maximum of (12.2) over all D and r never exceeds $\varsigma(U)$; the two agree at $U = \frac{3}{2}$, where both give $\mathcal{S} \ll X^{3/4+\varepsilon}$, their ratio is at most 0.59 for $U \geqslant 2$ and at most 0.52 for $U \geqslant \frac{50}{13}$, which is the range in which Lemma 4.4 is used, and it tends to $\frac{4}{27}$ as $U \to \infty$, the best degree being $D \approx \frac{3}{2}U$. The same holds without the term $q_r Y^{-r}$ in (12.1). If Lemma 4.4 were replaced by (12.1), the constant C_0 would become 6.916, and without Lemma 4.3(ii) as well it would be 7.404. Using α_r with $r < D$ inside the proof of Lemma 4.4 does not help either: the count becomes $\mathcal{N}/A \ll (q_r Y^{-r} + 1)(q_r^{-1} + A^{-1})$, and for $A = q^\tau$ the resulting saving $\min\{r(1-U\tau),\, U(1-r\tau),\, r - U + U\tau,\, U\tau\}/(D(D+1))$ is largest for $r = D$ and $\tau = \frac{1}{D+1}$. These statements were verified numerically for $U \leqslant 600$ (and $U \leqslant 30$ for the last one). They concern progressions running over the residue classes modulo A. In Appendix A the progressions are allowed to overlap, and then the coefficient α_{D-1}, used together with α_D, does help (Lemma A.9).

12.2. **What limits the constant.** (1) The term 2Λ depends on ς only through an average of $1/\theta^*$. With square-root cancellation for all U, that is, with $\varsigma = \frac{1}{2}$ identically, the iteration would give $\Psi(\frac{1}{4}) = 2 + \log 3 = 3.10$ in place of C_0, and the number of variables would be limited by other parts of the argument: the pruning of §9.4 and, when k is a power of 2, the singular series both require $s \geqslant 4k$. More realistically, a version of Lemma 4.3 with more flexible differencing, or a bound for $\mathcal{N}$ in Lemma 4.4 that uses several coefficients at once, would improve ς. For $D < U < D+1$ the leading coefficient leaves $\mathcal{N}$ as large as X^{U-D}, roughly; if $\mathcal{N}$ could be shown to be as small as the pigeonhole principle allows, C_0 would drop to about 6.30, but we do not know how to prove this.

In Appendix A the progressions modulo A are allowed to overlap, and the pairs of progressions that interact are counted by means of two coefficients; this route leads to the constant $6.2821\ldots$.

(2) The term $2+\log 9$ reflects the admissible exponents $k\mathfrak{d}(t)$ for smooth Weyl sums; any improvement of these for $t \asymp k$ feeds in directly.

(3) The minor arcs and the pruning cost $O(\sqrt{k})$ variables. In [10] this task is given to $2k$ smooth Weyl sums. If the moments of F were used over the whole of $\mathbb{T}$ only, then $k+O(1)$ smooth Weyl sums would suffice, by [7, 9], and the constant would be C_0+1. The difference is made by Corollary 9.3 and Lemmas 9.8 and 9.9: on arcs of height Q the moments of F are essentially those of a sum of length $Q^{2/k}$, F obeys a bound of Weyl type, and it decays on the major arcs up to height $P^{1/4}$, so that the loss $P^{4k\gamma}$ in Corollary 8.3, which comes from the trivial bound for sums of length at most $P^{4\gamma}$, does no harm. The exponent τ_k affects only the term $W_k(\Theta)$; see Remark 11.8.

(4) In Theorem 2.3 the sum $-2T(\Theta)+W_k(\Theta)$ is about $\frac{2}{\Theta}(\log y_k(\Theta)-2)$, which for large k is negative only when Θ exceeds $0.62\sqrt{k}$, roughly, while the rounding term $2R_k(\Theta)$ becomes infinite as $\Theta(\Theta+5) \to k$. The balance between them is the source of the term of order $\sqrt{k}$ in Theorem 1.1. Smaller rounding losses in Lemma 4.5 for $\theta \asymp \sqrt{k}$ would reduce this term. In Appendix A the losses are of smaller order, the restriction $\Theta(\Theta+5)<k$ disappears, and the term of order $\sqrt{k}$ becomes negative (Proposition A.19 and Remark A.25).

12.3. **The constants in the paper of Chow and Dietmann.** In [10] one has $c=10^{-3}$ (Lemma 3.1), $0<b<\min\{1,c/2\}$ and an integer $A_0 \geqslant \max\{100/b, b^{-2}\}$ (§5), $r=3A_0k+1$ (Corollary 5.2) and $s=2k+r$ (§6). Since $b<1/2000$ forces $A_0>4\cdot 10^6$, the least admissible value is $A_0=4\,000\,001$, with $b=A_0^{-1/2}$, giving $s=12\,000\,005\,k+1$. The requirements "k large" in [10] are explicit inequalities, the most restrictive one that we found being (3.5) there, which holds for $k \geqslant 122\,878$; for smaller k, (1.1) gives a far better bound. In this sense the argument of [10], with the parameters chosen there, yields $G(k) \leqslant 12\,000\,005\,k+1$ for all k.

The assertion in [10] that $G(k) \leqslant 8.4\,k$ for all k, which is stated there without proof, is contained in Theorem 1.2. We do not know how the figure 8.4 was obtained. Two observations may be relevant. First, (1.1) gives $G(k) \leqslant 8.4\,k$ precisely for $k \leqslant 66$, so the assertion requires a bound of the present type for every $k \geqslant 67$. Secondly, if in the present argument the saving ς is replaced by what (12.1) gives, with the degree and the coefficient chosen optimally and with Lemma 4.3(i) for $U<2$, then C_0 becomes 7.404, and it becomes 8.404 if, in addition, $k+O(1)$ smooth Weyl sums are used for the minor arcs and the pruning (§12.2). With the saving X^{-10^{-3}/θ^2} of [10, Lemma 3.1], the constant C_0 becomes $4+\log(8\cdot 10^3)=12.99$.

Appendix A. A mean value with two coefficients, a recursion, and the constant 6.2821

In the proof of Lemma 4.4 the sum $\mathcal{S}$ is split into A residue classes, and the number $\mathcal{N}$ of classes whose coefficient vectors nearly agree is bounded by means of the leading coefficient alone. In the classical theory of exponential sums, Heath-Brown [23] used Vinogradov's mean value theorem to prove a k-th derivative test in which van der Corput's exponent

$1/(2^k-2)$ is replaced by $1/(k(k-1))$; the new ingredient there is a count of the pairs of points at which the Taylor coefficients of two consecutive orders nearly agree modulo 1. For the fourth derivative test, Robert and Sargos [29] had improved the exponent $\frac{1}{14}$ to $\frac{1}{13}$, and Robert [28] reached every exponent below $\frac{1}{12}$. In this appendix the idea of [23] is carried over to the sums $\mathcal{S}$ of §4 (Lemma A.9). In comparison with Lemma 4.4 two things change: the progressions modulo A are allowed to overlap, so that their starting points run over an interval of integers of length about X and not over A residue classes, and the two leading coefficients are used to count the pairs of progressions that interact. For each degree $D \geqslant 3$ this gives the saving $X^{-1/(D(D+1))}$ throughout the range $\frac{1}{2}(D+1) \leqslant U \leqslant D$, apart from a factor X^{ε} and the effects of rounding, whereas the saving in Lemma 4.4 is at most $X^{-U/(D(D+1)^2)}$.

The new estimate replaces Lemma 4.4 and the cases $j \geqslant 4$ of Lemma 4.3(ii), and it lowers the constant C_0 to $C_0^\star = 6.2821\ldots$. Three further changes concern the passage from the estimates for the sums $\mathcal{S}$ to the bounds for $G(k)$. First, the rounding of moduli is treated more carefully than in Lemma 4.5: each estimate is used with the loss that it actually suffers, wherever it is the best one, and the conditions of Definition 2.2 on the largest step Θ are no longer needed (§A.3). In consequence the term of order $\sqrt{k}$ in Theorem 1.1 becomes negative. Secondly, the potential of §8 is formed with the exact length of the steps, which lowers the bounds for $G(k)$ by about $\log k$ (Theorem A.17). Thirdly, the steps are treated as a recursion, and the arcs of large height are pruned by means of large values of exponential sums instead of moments, so that the concavity of the potential is no longer needed (Theorem A.21); for $37 \leqslant k \leqslant 5000$ this lowers the bounds by a further 1 to 5. The recursion applies to all $k \geqslant 13$. For $13 \leqslant k \leqslant 36$ it gives bounds for $G(k)$ and for $G^+(k)$ that improve on those in the literature; for $13 \leqslant k \leqslant 20$ we combine it with the tables of permissible exponents of Vaughan and Wooley [34].

The results are stated in §A.1, and the appendix is organised as follows. The estimate is proved in §A.2. Section A.3 contains the saving function that results from it, with its rounding losses, and the chains of estimates by which this function is bounded from below. In §A.4 we isolate the properties of a potential on which the induction of §8 rests, and we record what is used of §§3–9 for exponents below 37. Two potentials are then constructed. The first is the analogue of the one in §8, with a largest step Θ (§A.5); it gives the bounds for $k \geqslant 5000$, by an explicit evaluation, and the result on fractional parts (§A.8). The second is the recursive one (§A.6); with the computations of §A.7 it gives the bounds for $G(k)$, for $G^+(k)$ and for the mean value of F when $k \leqslant 5000$. The theorems are proved in §A.9, §A.10 contains remarks, and §A.11 describes the part of the program of §10.4 that concerns this appendix. Sections 3–11 are used with the changes stated in §A.4 and in the proofs of Theorems A.17 and A.21.

A.1. **Results.** For integers $D \geqslant 4$ let $\Lambda_D := \frac{1}{D+1}\log\bigl(1+\frac{2(D+1)}{D^2}\bigr)$, and put

$$\begin{aligned} \Lambda^\star &:= \tfrac{1}{3}\log 2 + \tfrac{1}{4}\log\tfrac{17}{9} + \tfrac{1}{4}\log\tfrac{5}{3} + \tfrac{1}{5}\log\tfrac{7}{6} + \sum_{D\geqslant 4}\Lambda_D = 1.042453\ldots, \\ C_0^\star &:= 2 + \log 9 + 2\Lambda^\star = 6.282131\ldots. \end{aligned} \tag{A.1}$$

Thus $C_0 - C_0^\star = 0.10738\ldots$, with C_0 as in (1.3).

Theorem A.1. *For every $k \geqslant 37$ one has $G(k) < C_0^\star k + 0.32\sqrt{k}$; for every $k \geqslant 56$ one has $G(k) < C_0^\star k$; and for every $k \geqslant 78$ one has $G(k) < C_0^\star k - \frac{1}{4}\sqrt{k}$. Moreover*

$$G(k) \leqslant C_0^\star k - 0.9856\,\sqrt{k} + \log k + 20 \qquad (k \geqslant 5000).$$

In particular $\limsup_{k\to\infty} G(k)/k \leqslant C_0^\star$.

Theorem A.2. *One has $G(k) < 6.43\,k$ for every $k \in \mathbb{N}$. Moreover $G(k) < 6.34\,k$ for every $k \geqslant 37$, and $G(k) < 6.3\,k$ for every $k \geqslant 47$.*

The method applies to smaller exponents as well (Theorem A.21). For $21 \leqslant k \leqslant 36$ it gives the bounds of Table 6, which are smaller than those of (1.1); for example, $G(21) \leqslant 135$, $G(30) \leqslant 191$ and $G(36) \leqslant 228$, in place of 153, 229 and 281. For $13 \leqslant k \leqslant 20$ the mean value estimate (7.1) can be replaced by the permissible exponents that were tabulated by Vaughan and Wooley [34], and then the method gives

$$G(13) \leqslant 80, \quad G(14) \leqslant 86, \quad G(15) \leqslant 93, \quad G(16) \leqslant 99,$$
$$G(17) \leqslant 106, \quad G(18) \leqslant 112, \quad G(19) \leqslant 119, \quad G(20) \leqslant 125$$

(Proposition A.23 and Table 7). The bounds known before are $G(13) \leqslant 81$ [42, Theorem 1.1] and 89, 97, 105, 113, 121, 129 and 137 for $14 \leqslant k \leqslant 20$ [6, Theorem 1.3]. For $k \leqslant 12$ we use the bounds of the literature. For $37 \leqslant k \leqslant 5000$ the new bounds are smaller than the numbers $G^\sharp(k)$ of Proposition 10.5 by at least 38, and by 2.47 to 14.29 per cent; for example, we obtain $G(37) \leqslant 234$, $G(50) \leqslant 314$, $G(100) \leqslant 624$ and $G(1000) \leqslant 6248$, in place of 273, 353, 709 and 6492 (see Table 6).

Theorem A.3. (i) *Let $k \geqslant 37$ be a natural number that is not a power of 2. Then one has $G^+(k) < \frac{1}{2}C_0^\star k + 0.24\sqrt{k}$, and $G^+(k) < \frac{1}{2}C_0^\star k$ if $k \geqslant 65$. Moreover*

$$G^+(k) \leqslant \tfrac{1}{2}C_0^\star k - 0.4928\,\sqrt{k} + \tfrac{1}{2}\log k + 11 \qquad (k \geqslant 5000).$$

(ii) *One has*

$$G^+(13) \leqslant 40, \quad G^+(14) \leqslant 43, \quad G^+(15) \leqslant 47,$$
$$G^+(17) \leqslant 53, \quad G^+(18) \leqslant 56, \quad G^+(19) \leqslant 60, \quad G^+(20) \leqslant 63,$$

and $G^+(k) \leqslant \ell^\flat(k)$ for $21 \leqslant k \leqslant 36$ with $k \neq 32$, where the numbers $\ell^\flat(k)$ are those of Table 6.

Thus the upper limit of $G^+(k)/k$ over the numbers k that are not powers of 2 is at most $\frac{1}{2}C_0^\star = 3.1410\ldots$, and Theorem 1.3(i) is best possible already. In part (ii) the numbers $\ell^\flat(k)$ lie between 68 and 114. The bounds for $G^+(k)$ that are recorded in [6, Theorem 9.1] are 45, 49, 57, 61, 65 and 69 for $k = 14$, 15, 17, 18, 19 and 20, and $\lceil \frac{1}{2}k(\log k + 4.20032)\rceil$ for the other k in part (ii); this is 44 for $k = 13$, and it lies between 77 and 141 for $21 \leqslant k \leqslant 36$. So part (ii) improves on these bounds by 2 to 6 when $k \leqslant 20$, and by 9 to 27 when $21 \leqslant k \leqslant 36$.

When k is a power of 2, the integers ℓ that are admissible in Theorem 2.4(ii) become smaller as well (Theorem A.21(ii)). For $k = 16$ and $k = 32$ one may take $\ell = 50$ and $\ell = 102$, where [6, Theorem 9.2] has 53 and 123; and for $k = 64$, 128, 256, 512, 1024, 2048 and 4096 the numbers 229, 450, 881, 1693, 3323, 6605 and 13164 of §10.3 can be replaced

by 201, 399, 798, 1597, 3199, 6407 and 12827. For instance, almost all natural numbers $n \equiv j \pmod{256}$ with $1 \leqslant j \leqslant 201$ are sums of 201 sixty-fourth powers.

Theorem A.4. *The conclusions of Theorem* 1.4*, in which ω is to be formed with the present number s_1, hold with $s_1 := C_0^\star k + 0.23\sqrt{k}$ for every $k \geqslant 37$, with $s_1 := C_0^\star k$ for every $k \geqslant 50$, and with $s_1 := C_0^\star k - \frac{1}{4}\sqrt{k}$ for every $k \geqslant 69$.*

Theorem A.5. *Let $k \geqslant 37$, and let α be a real number. Then, whenever N is sufficiently large in terms of k,*

$$\min_{1 \leqslant n \leqslant N} \|\alpha n^k\| < N^{-1/(10.7961\,k + \sqrt{k})}.$$

If $k \geqslant 135$, this holds with $10.7961\,k$ in place of $10.7961\,k + \sqrt{k}$, and if $k \geqslant 5000$, with $10.7961\,k - 0.62\sqrt{k} + \log k + 24$. Moreover, the minimum is smaller than $N^{-1/(11k)}$ for every $k \geqslant 37$.

Here 10.7961 is a rounded-up value of $C_0^\star + \frac{1}{2} \cdot 9.027901 = 10.79608\ldots$. The exponents are larger than the one of [9] that is displayed in §1.1 for every $k \geqslant 37$ (for $k \leqslant 5000$ by Proposition A.24(i), and for larger k because $10.7961 < \log k + 8.02113$); for example, when $k = 100$ and $k = 1000$ we obtain the exponents $1/1082$ and $1/10776$, in place of the exponents $1/1175$ and $1/11023$ that were mentioned after Theorem 1.5 (see Table 8). For large k the exponent is about 84 per cent of the one in the unlocalised bound of [40].

Table 5 shows on which results of this appendix the five theorems rest in each range of k. Up to $k = 5000$ the bounds for $G(k)$, for $G^+(k)$ and for the mean value come from the recursion of §A.6, evaluated numerically; beyond, they come from the potential of Theorem A.17 and an explicit estimate, which is weaker (at $k = 5000$ the two give $G(k) \leqslant 31323$ and $G(k) \leqslant 31369$). The result on fractional parts rests on Theorem A.17 throughout, for the reason explained at the beginning of §A.8.

TABLE 5. The sources of the bounds in Theorems A.1–A.5.

bound for	range of k	potential	evaluation
$G(k)$	$k \leqslant 8$		(1.1)
	$9 \leqslant k \leqslant 12$		[42, Theorem 1.1]
	$13 \leqslant k \leqslant 20$	Theorem A.21, with [34]	Proposition A.23
	$21 \leqslant k \leqslant 36$	Theorem A.21	Proposition A.22(i)
$G^+(k)$	$13 \leqslant k \leqslant 20$	Theorem A.21, with [34]	Proposition A.23
	$21 \leqslant k \leqslant 36$	Theorem A.21	Proposition A.22(i)
G, G^+, mean value	$37 \leqslant k \leqslant 5000$	Theorem A.21	Proposition A.22
	$k \geqslant 5000$	Theorem A.17	Proposition A.19
fractional parts	$37 \leqslant k \leqslant 5000$	Theorem A.17	Proposition A.24(i)
	$k \geqslant 5000$	Theorem A.17	Proposition A.24(ii)

These theorems realise two improvements that were anticipated in §12.2. In (1) there, it was observed that a bound for the number $\mathcal{N}$ of Lemma 4.4 that is as small as the pigeonhole principle allows would lead to the constant 6.30. The constant $C_0^\star$ is smaller because that lemma is limited, in addition, by the number A of residue classes, a limitation that is removed by the overlapping progressions. In (4) the term of order $\sqrt{k}$ in

Theorem 1.1 was traced to the rounding losses of Lemma 4.5 for steps of size about $\sqrt{k}$. These losses are now of smaller order, and the term becomes negative (Proposition A.19); steps larger than $\sqrt{k}$, which are available as well, improve it further (Remark A.25). The exact length of the steps was not considered in §12.2; it accounts for the term $-\log(z_k/\tau_k)$ in Theorem A.17 below, and the recursion of §A.6 goes further in the same direction.

A.2. **The estimate.** Let k, J, X, m, ϑ, q, U, e_p, a and $\mathcal{S}$ be as in §4. Let D be an integer with $2 \leqslant D \leqslant k-1$, and put

$$u := k - D, \qquad s_p := \max\{1, \lceil e_p/(D+1)\rceil\}, \qquad A := \prod_{p\mid m} p^{s_p},$$

$$q_r := \prod_{p\mid m} p^{\max\{e_p - rs_p, 0\}} \quad (r = D-1,\, D).$$

Thus $1 \leqslant u < u+1 \leqslant k-1$, with $u = 1$ when $D = k-1$, and s_p, A and q_D are the numbers that occur in the proof of Lemma 4.4. One has $e_p/(D+1) \leqslant s_p \leqslant (e_p+D)/(D+1)$ when $e_p \geqslant 1$, and $s_p = 1$ when $e_p = 0$. Hence

$$\begin{gathered} q^{\frac{1}{D+1}} \leqslant A \leqslant q^{\frac{1}{D+1}} m, \qquad q^{\frac{1}{D+1}} m^{-D} \leqslant q_D \leqslant q^{\frac{1}{D+1}}, \qquad q_{D-1} \geqslant q^{\frac{2}{D+1}} m^{1-D}, \\ q_D \mid A, \qquad q_D \mid q_{D-1}, \qquad q_{D-1} \leqslant q_D^2\, m^D. \end{gathered} \tag{A.2}$$

For the last inequality note that if $e_p \geqslant Ds_p$ then $v_p(q_{D-1}) - 2v_p(q_D) = (D+1)s_p - e_p \leqslant D$, and that otherwise $v_p(q_D) = 0$ and $v_p(q_{D-1}) < s_p < D$; here $s_p < D$ because $s_p = 1$ when $e_p = 0$, and $(D+1)s_p \leqslant e_p + D < Ds_p + D$ when $e_p \geqslant 1$.

Lemma A.6 (classes)**.** *Let $p > k$ be a prime. The reduced residue classes modulo p can be partitioned into fewer than k^2 sets on each of which both maps $x \mapsto x^u$ and $x \mapsto x^{u+1}$ are injective.*

Proof. Let Γ be the group of the residues ω modulo p with $\omega^{u(u+1)} = 1$; it has at most $u(u+1) < k^2$ elements. Let $\mathcal{X}$ be a set of representatives of the cosets of Γ in $(\mathbb{Z}/p\mathbb{Z})^\times$. The sets $\omega\mathcal{X}$ with $\omega \in \Gamma$ form a partition of $(\mathbb{Z}/p\mathbb{Z})^\times$. If $x, x' \in \omega\mathcal{X}$ and $x^u = x'^u$ or $x^{u+1} = x'^{u+1}$, then $x'/x \in \Gamma$, so that x/ω and x'/ω are elements of $\mathcal{X}$ in the same coset of Γ, whence $x = x'$. □

We fix such a partition for every prime $p \mid m$; it depends on $u = k - D$. Two integers coprime to m, positive or negative, are called *equivalent* if for every $p \mid m$ their residues modulo p lie in the same set of the partition. There are at most k^{2J} classes of equivalent integers.

Lemma A.7 (valuations)**.** *If n and n' are equivalent, then for every prime $p \mid m$ one has $v_p(n^u - n'^u) = v_p(n^{u+1} - n'^{u+1}) = v_p(n - n')$.*

Proof. Let $i \in \{u, u+1\}$. If $p \nmid n - n'$, then $n^i \not\equiv n'^i \pmod p$ by the choice of the partition, and all three valuations are 0. If $p \mid n - n'$, then $n^i - n'^i = (n-n')\sum_{l<i} n^l n'^{i-1-l}$, and the sum is congruent to $i\, n^{i-1}$ modulo p, which is not divisible by p because $p \nmid n$ and $i \leqslant k < p$. □

Lemma A.8 (pairs)**.** *Let $\mathcal{V}$ be a set of integers coprime to m that are pairwise equivalent, and suppose that $\mathcal{V}$ is contained in a set of L consecutive integers. Let $r_1 \mid r_2$ be natural*

numbers all of whose prime factors divide m (the case $r_1 = 1$ is included), let b_1 and b_2 be integers with $(b_1, r_1) = (b_2, r_2) = 1$, and let $\delta_1, \delta_2 > 0$. Then the number of pairs $(n, n') \in \mathcal{V}^2$ with

$$\Big\| \frac{b_1(n^u - n'^u)}{r_1} \Big\| \leqslant \delta_1 \qquad \text{and} \qquad \Big\| \frac{b_2(n^{u+1} - n'^{u+1})}{r_2} \Big\| \leqslant \delta_2 \tag{A.3}$$

is at most

$$L + \mathcal{N}_1(\delta_2) + 2\delta_1 \Big(r_1 L + (r_1 + L)\, \mathcal{N}_1(2\delta_2) \Big), \quad \text{where} \quad \mathcal{N}_1(\delta) := \frac{2L^2}{r_2} + 4k^J d(r_2)\, \delta \Big(\frac{Lr_2}{r_1^2} + \frac{L^2}{r_1} \Big)$$

and $d(r_2)$ is the number of divisors of r_2. The elements of $\mathcal{V}$ may be negative, and the bound increases with L.

Proof. The hypothesis $r_1 \mid r_2$ is used only to see that the number c below is a multiple of r_1, and the equivalence of the elements of $\mathcal{V}$ only through Lemma A.7. There are at most L pairs with $n = n'$. Next we count, for a given $\delta > 0$, the pairs with $n \neq n'$ and $r_1 \mid n - n'$ that satisfy the second condition in (A.3) with δ in place of δ_2; let $\mathcal{N}_1^*(\delta)$ be their number. Put $h := n' - n$, so that $0 < |h| < L$, and $c := (h, r_2)$, a multiple of r_1. If $c = r_2$, there are at most $2L/r_2$ possibilities for h and at most L for n. Let $c \neq r_2$, and put $r' := r_2/c > 1$. The polynomial

$$f_h(t) := \frac{(t+h)^{u+1} - t^{u+1}}{c}$$

has integer coefficients, its degree is u, its leading coefficient is $(u+1)h/c$, and one has $b_2(n'^{u+1} - n^{u+1})/r_2 = b_2 f_h(n)/r'$. Let $p \mid r'$. Then $v_p(h) = v_p(c)$, so that the leading coefficient of f_h is not divisible by p; moreover, by Lemma A.7, neither $f_h(n)$ nor $f_h'(n) = (u+1)\big((n+h)^u - n^u\big)/c$ is divisible by p when n and $n+h$ are equivalent. Hence $b_2 f_h(n) \equiv l \pmod{r'}$ for an integer l with $0 < |l| \leqslant \delta r'$, which leaves at most $2\delta r'$ values of l. For each of them the integers n in question lie in at most u^J residue classes modulo r': for each prime power p^b dividing r' exactly, the congruence $b_2 f_h(t) \equiv l \pmod{p}$ has at most u solutions, and by Hensel's lemma a solution modulo p^b at which f_h' is not divisible by p is determined by its residue modulo p. A residue class modulo r' contains at most $1 + L/r'$ elements of $\mathcal{V}$. For a given c there are at most $2L/c$ numbers h, and $c \geqslant r_1$. Therefore

$$\mathcal{N}_1^*(\delta) \leqslant \frac{2L^2}{r_2} + \sum_c \frac{2L}{c} \cdot 2u^J \delta\, (r' + L) \leqslant \mathcal{N}_1(\delta).$$

Now consider the pairs counted in the lemma with $r_1 \nmid n - n'$. Let r be the residue of $b_1(n'^u - n^u)$ modulo r_1 of least absolute value. Then $|r| \leqslant \delta_1 r_1$ by (A.3), and $r \neq 0$ by Lemma A.7; so there are at most $2\delta_1 r_1$ possibilities for r. For given n and r let $\mathcal{R}(n, r)$ be the set of the numbers n' that occur. If $n_1', n_2' \in \mathcal{R}(n, r)$ are distinct, then $n_1'^u \equiv n_2'^u \pmod{r_1}$, hence $r_1 \mid n_1' - n_2'$ by Lemma A.7, and $\|b_2(n_1'^{u+1} - n_2'^{u+1})/r_2\| \leqslant 2\delta_2$; so the pair (n_1', n_2') is counted by $\mathcal{N}_1^*(2\delta_2)$. Conversely, for given n_1' and r the congruence $b_1(n_1'^u - n^u) \equiv r \pmod{r_1}$ determines n modulo r_1, again by Lemma A.7, so that at most $2\delta_1 r_1(1 + L/r_1)$ pairs (n, r) have $n_1' \in \mathcal{R}(n, r)$. Since $N \leqslant 1 + N(N-1)$ for every integer

$N \geqslant 0$, we conclude that

$$\sum_{(n,r)} \#\mathcal{R}(n,r) \leqslant 2\delta_1 r_1 L + \sum_{(n,r)} \#\mathcal{R}(n,r)\big(\#\mathcal{R}(n,r)-1\big) \leqslant 2\delta_1 r_1 L + 2\delta_1(r_1+L)\,\mathcal{N}_1^*(2\delta_2).$$

The lemma follows on adding the three contributions. □

Lemma A.9 (mean value with two coefficients)**.** *Let $A \leqslant X$, and put $Y := \lfloor X/A \rfloor$. Then, for every $\varepsilon > 0$,*

$$\mathcal{S} \ll X^{1+\varepsilon}\bigg(\frac{1}{X} + \Big(1+\frac{X}{Y^D}\Big)\Big(\frac{1}{q_{D-1}} + \frac{q_{D-1}}{Y^{D-1}Xq_D^2} + \frac{1}{Y^{D-1}q_D}\Big) + \frac{q_D}{Y^D X}\bigg)^{\frac{1}{D(D+1)}}.$$

The implied constant depends on k, J and ε only.

Proof. Every prime factor of m divides A, so that $(n+Ay,m) = (n,m)$ for all integers n and y. Counting the pairs (n,y) with $0 \leqslant y < Y$ and $1 \leqslant n + Ay \leqslant X$ in two ways, we find that

$$Y\,\mathcal{S} = \sum_{\substack{-AY < n \leqslant X \\ (n,m)=1}} \sum_{y \in \mathcal{K}_n} e\Big(\frac{a(n+Ay)^k}{q}\Big),$$

where $\mathcal{K}_n$ is a set of consecutive integers in $[0,Y)$. Here n may be negative (the value $n=0$ does not occur, since $(0,m) = m > 1$); Lemmas A.7 and A.8 do not require that the integers be positive. As in the proof of Lemma 4.4, the inner sum is the sum of $e\big(\sum_{i \leqslant D} \alpha_i(n) y^i\big)$ over $y \in \mathcal{K}_n$, where $\alpha_i(n) := \frac{a}{q}\binom{k}{i} A^i n^{k-i}$ for all integers n, and it is at most $2g^*(\boldsymbol{\alpha}(n))$ in modulus; here $\boldsymbol{\alpha}(n) := (\alpha_1(n), \dots, \alpha_D(n))$, and g_t and g^* are defined as in that proof, but with the present number $Y = \lfloor X/A \rfloor$ in place of the number $\lfloor X/A \rfloor + 1$ used there. Let $w := \frac{1}{2}D(D+1)$. The integers n in the outer sum fall into at most k^{2J} sets $\mathcal{V}$ of pairwise equivalent integers, each contained in an interval with at most $2X$ integers, and by Hölder's inequality

$$Y\,|\mathcal{S}| \leqslant 2\sum_{\mathcal{V}} (2X)^{1-1/w}\Big(\sum_{n \in \mathcal{V}} g^*(\boldsymbol{\alpha}(n))^w\Big)^{1/w}. \tag{A.4}$$

Let $\mathcal{Q} := \prod_{i \leqslant D}[-Y^{-i}, Y^{-i}]$, a box of volume $2^D Y^{-w}$. As in the proof of Lemma 4.4, $g^*(\boldsymbol{\alpha}) \leqslant (1+2\pi D)\, g^*(\boldsymbol{\beta})$ for $\boldsymbol{\beta} \in \boldsymbol{\alpha} + \mathcal{Q}$. Averaging over $\boldsymbol{\beta}$ and using the periodicity of g^*, and then Cauchy's inequality, we obtain

$$\sum_{n \in \mathcal{V}} g^*(\boldsymbol{\alpha}(n))^w \ll Y^w \int_{\mathbb{T}^D} g^*(\boldsymbol{\beta})^w\, \nu(\boldsymbol{\beta})\, \mathrm{d}\boldsymbol{\beta} \leqslant Y^w \Big(\int_{\mathbb{T}^D} g^*(\boldsymbol{\beta})^{2w}\, \mathrm{d}\boldsymbol{\beta}\Big)^{1/2}\Big(\int_{\mathbb{T}^D} \nu(\boldsymbol{\beta})^2\, \mathrm{d}\boldsymbol{\beta}\Big)^{1/2},$$

where $\nu(\boldsymbol{\beta})$ is the number of pairs $(n,\mathbf{l}) \in \mathcal{V} \times \mathbb{Z}^D$ with $\boldsymbol{\beta} + \mathbf{l} \in \boldsymbol{\alpha}(n) + \mathcal{Q}$. The first integral on the right is $\ll Y^{w+\varepsilon}$. Indeed, the argument in the proof of Lemma 4.4, which does not depend on the degree, bounds it by $(\log 2Y)^{2w}$ times the mean value of $|g_Y|^{2w}$ over $\mathbb{T}^D$, and the latter is $\ll Y^{w+\varepsilon}$ by Vinogradov's mean value theorem at the critical exponent $2w = D(D+1)$; this is classical for $D = 2$, it is due to Wooley [43] for $D = 3$ and to Bourgain, Demeter and Guth [4] for $D \geqslant 4$, and [44] covers all degrees. For the second integral, note that

$$\int_{\mathbb{T}^D} \nu(\boldsymbol{\beta})^2\, \mathrm{d}\boldsymbol{\beta} = \sum_{n,n' \in \mathcal{V}} \sum_{\mathbf{d} \in \mathbb{Z}^D} \mathrm{vol}\Big(\big(\boldsymbol{\alpha}(n) + \mathcal{Q}\big) \cap \big(\boldsymbol{\alpha}(n') + \mathcal{Q} + \mathbf{d}\big)\Big).$$

The intersection is empty unless $\boldsymbol{\alpha}(n)-\boldsymbol{\alpha}(n')\in\mathbf{d}+2\mathcal{Q}$, which for given n and n' leaves at most 5^D values of $\mathbf{d}$, and none unless $\|\alpha_i(n)-\alpha_i(n')\|\leqslant 2Y^{-i}$ for $1\leqslant i\leqslant D$. So the second integral is at most $5^D\operatorname{vol}(\mathcal{Q})\,\mathcal{M}$, where $\mathcal{M}$ is the number of pairs $(n,n')\in\mathcal{V}^2$ with this property. Hence the left hand side is $\ll Y^{w+\varepsilon}\mathcal{M}^{1/2}$, and by (A.4)

$$\mathcal{S}\ll X^{1+\varepsilon}\max_{\mathcal{V}}\bigl(\mathcal{M}/X^2\bigr)^{1/(2w)}.$$

In $\mathcal{M}$ we keep the conditions for $i=D$ and $i=D-1$ only. As $p\nmid a\binom{k}{i}$ for $p\mid m$, the numbers $\frac{a}{q}\binom{k}{D}A^D$ and $\frac{a}{q}\binom{k}{D-1}A^{D-1}$ are, in lowest terms, fractions b_1/q_D and b_2/q_{D-1} with $(b_1,q_D)=(b_2,q_{D-1})=1$; thus $\alpha_D(n)=b_1n^u/q_D$ and $\alpha_{D-1}(n)=b_2n^{u+1}/q_{D-1}$. So Lemma A.8 applies with $r_1=q_D$, $r_2=q_{D-1}$, $\delta_1=2Y^{-D}$ and $\delta_2=2Y^{1-D}$, and since its bound increases with L, we may take $L=2X$. Let $\mathcal{B}$ denote the sum of the three terms in the inner bracket in the lemma, and put $C_1':=4k^J(k+1)^J$. As $d(q_{D-1})\leqslant(k+1)^J$, one finds that $\mathcal{N}_1(\delta_2)\leqslant 8C_1'X^2\mathcal{B}$ and $\mathcal{N}_1(2\delta_2)\leqslant 16C_1'X^2\mathcal{B}$. Since $q_D\leqslant A\leqslant X$, it follows that

$$\frac{\mathcal{M}}{X^2}\leqslant\frac{2}{X}+8C_1'\mathcal{B}+\frac{8q_D}{Y^DX}+\frac{192\,C_1'X}{Y^D}\,\mathcal{B}\leqslant 192\,C_1'\biggl(\frac{1}{X}+\Bigl(1+\frac{X}{Y^D}\Bigr)\mathcal{B}+\frac{q_D}{Y^DX}\biggr),$$

which proves the lemma. □

Corollary A.10. *Let $2\leqslant D\leqslant k-1$ and $U/(D+1)+\vartheta<1$, and put*

$$E_1:=\tfrac{2U}{D+1}-(D-1)\vartheta,\qquad E_2:=D-1-\tfrac{(D-2)U}{D+1}-(2D-1)\vartheta,$$
$$E_3:=(2D-2)\bigl(1-\tfrac{U}{D+1}\bigr)-(3D-1)\vartheta,\qquad E_4:=D+1-U-D\vartheta.$$

Then, for every $\varepsilon>0$,

$$\mathcal{S}\ll X^{1+\varepsilon-\varrho^\star},\qquad\text{where}\quad\varrho^\star:=\frac{1}{D(D+1)}\min\{1,E_1,E_2,E_3,E_4\}.$$

The implied constant depends on k, J and ε only.

Proof. If the minimum is not positive, there is nothing to prove. Write $U_1:=U/(D+1)$. By (A.2), $A\leqslant X^{U_1+\vartheta}<X$, so that $Y\geqslant X/(2A)\geqslant\frac12X^{1-U_1-\vartheta}$; also $X^{U_1-D\vartheta}\leqslant q_D\leqslant X^{U_1}$, $q_{D-1}\geqslant X^{2U_1-(D-1)\vartheta}$ and $q_{D-1}/q_D^2\leqslant X^{D\vartheta}$. On multiplying out, the expression in large brackets in Lemma A.9 consists of $1/X$ and the seven terms

$$\frac{1}{q_{D-1}},\quad\frac{q_{D-1}}{Y^{D-1}Xq_D^2},\quad\frac{1}{Y^{D-1}q_D},\quad\frac{X}{Y^Dq_{D-1}},\quad\frac{q_{D-1}}{Y^{2D-1}q_D^2},\quad\frac{X}{Y^{2D-1}q_D},\quad\frac{q_D}{Y^DX},$$

which are $\ll X^{-E}$ with E equal to E_1, E_2+1-U_1, E_2, E_2, E_3+1-U_1, E_3 and E_4, respectively; the powers of 2 that come from the lower bound for Y are at most 2^{2k}, and they are absorbed in the implied constants. □

When $\vartheta=0$ and $D\geqslant3$, the minimum in Corollary A.10 equals $\min\{1,\frac{2U}{D+1},D+1-U\}$ for $U\leqslant D+1$. So the saving is $1/(D(D+1))$ for $\frac12(D+1)\leqslant U\leqslant D$; this is the limit of the method, as the term $1/X$ in Lemma A.9 shows, which comes from the pairs with $n=n'$. For $D=2$ and $\vartheta=0$ the saving is $\min\{1,\frac23U,\frac23(3-U)\}/6\leqslant\frac19(3-U)$, less than the saving $\frac16(3-U)$ of Lemma 4.3(ii) with $j=1$, and we shall use the corollary for $D\geqslant3$ only. For $D=3$ the mean value theorem that enters is the cubic case of the main conjecture, which was proved in [43] (see also [44]).

A.3. **The saving function, rounding, and chains.** In Lemma 4.5 each estimate of §4 is assigned to a fixed range of U, and the losses that are caused by the rounding of moduli are majorised by the common factor $1-\eta_k(\theta)$. This is convenient but wasteful. For example, the exponent of Lemma 4.3(ii) is $\varrho-\frac{1}{2}\vartheta$, whereas Lemma 4.5 uses $\varrho-\kappa(\theta)\vartheta\varrho$, in which the loss can be twice as large; and the conditions of Definition 2.2 only serve to make every estimate applicable throughout the range assigned to it. We now let the estimates compete.

Definition A.11. Let $k \geqslant 13$. For $U > 1$ and $\vartheta \geqslant 0$ let $\varsigma^\star(U,\vartheta)$ denote the largest of the following numbers, a number being admitted only when the conditions stated with it hold:

(a) 0 and $1-\frac{1}{2}U$;
(b) $\varrho_j - \frac{1}{2}\vartheta$, where $\varrho_j := (j+2-U)/(2^{j+2}-2)$, for $j \in \{1,2,3\}$ with $U \geqslant j+2^{-j}$ and $\varrho_j \geqslant \vartheta$;
(c) $\min\{1, E_1, E_2, E_3, E_4\}/(D(D+1))$, with $E_1, \dots, E_4$ as in Corollary A.10, for integers D with $3 \leqslant D \leqslant k-1$ and $U/(D+1)+\vartheta < 1$.

Further let

$$\sigma_k(\theta) := \inf\{\varsigma^\star(U,\theta/k) : \ 1 < U \leqslant \theta\} \quad (1 < \theta \leqslant \tfrac{1}{2}k), \qquad \sigma_k(1) := \tfrac{1}{2},$$
$$\theta_k(t) := \sup\{\theta \in [1, \tfrac{1}{2}k] : \ \sigma_k(\theta) \geqslant t\} \quad (0 < t \leqslant \tfrac{1}{2}).$$

The set in the definition of θ_k contains 1. Lemma A.13 below shows that σ_k is nonincreasing and left-continuous, so that the supremum is attained and θ_k is nonincreasing. The number $\sigma_k(\theta)$ is the saving in the sums $\mathcal{S}$ that is guaranteed for all moduli q when $m \leqslant X^{\theta/k}$, with all losses included; it takes the place of $(1-\eta_k(\theta))\sigma(\theta)$, and $\theta_k(t)$ is the largest step for which the saving t is guaranteed.

To see how the constant $C_0^\star$ arises, let $\vartheta = 0$. For $U \geqslant 1$ put

$$\varsigma^\star(U) := \begin{cases} \varsigma(U), & 1 \leqslant U \leqslant \frac{17}{6}, \\ \frac{1}{12}, & \frac{17}{6} \leqslant U \leqslant 3, \\ (4-U)/12, & 3 \leqslant U \leqslant \frac{10}{3}, \\ (5-U)/30, & \frac{10}{3} \leqslant U \leqslant \frac{7}{2}, \\ \max\Big\{\dfrac{D+1-U}{D(D+1)}, \ \dfrac{1}{(D+1)(D+2)}\Big\}, & U \geqslant \frac{7}{2}, \end{cases} \tag{A.5}$$

where ς is the function of Definition 2.1, and in the last line $D \geqslant 3$ is an integer with $D \leqslant U \leqslant D+1$.

Lemma A.12. *Let $k \geqslant 13$.*

(i) *The function $\varsigma^\star$ of* (A.5) *is continuous and nonincreasing.*

(ii) *For $U > 1$ one has $\varsigma^\star(U,0) \leqslant \varsigma^\star(U)$, with equality when $U \leqslant k-1$.*

(iii) *For $\vartheta = 0$ the exponents saved by Lemma* 4.4*, and by Lemma* 4.3(ii) *with $j \geqslant 4$, do not exceed $\varsigma^\star(U)$. Consequently $\varsigma \leqslant \varsigma^\star$.*

Proof. For an integer $D' \geqslant 3$ and $1 < U < D'+1$ let

$$c_{D'}(U) := \frac{1}{D'(D'+1)} \min\Big\{1, \frac{2U}{D'+1}, D'+1-U\Big\}.$$

By the remark that follows Corollary A.10, this is the number in (c) of Definition A.11 for $\vartheta = 0$ and the degree D'; for $U \geqslant D'+1$ that number is not admitted. Let $D \geqslant 3$ and $D \leqslant U \leqslant D+1$. Then $c_D(U) = (D+1-U)/(D(D+1))$ if $U < D+1$, $c_{D+1}(U) = 1/((D+1)(D+2))$, and $c_{D'}(U) \leqslant 1/((D+2)(D+3))$ for $D' \geqslant D+2$. For $1 < U \leqslant 3$ one has $c_3(U) = \min\{1, \frac12 U\}/12$ and $c_{D'}(U) \leqslant \frac{1}{20}$ for $D' \geqslant 4$. The numbers in (a) and (b) are $1 - \frac12 U$, and $\frac16(3-U)$ for $U \geqslant \frac32$, $\frac{1}{14}(4-U)$ for $U \geqslant \frac94$ and $\frac{1}{30}(5-U)$ for $U \geqslant \frac{25}{8}$, as far as these are not negative; on $[1, \frac{17}{6}]$ the largest of them is $\varsigma(U)$, which is at least $\varsigma(\frac{17}{6}) = \frac{1}{12}$ there. A comparison of these numbers shows that the largest of them is the one in (A.5); the points $\frac{17}{6}$, $\frac{10}{3}$ and $\frac72$ are those at which $\frac{1}{14}(4-U) = \frac{1}{12}$, at which $\frac{1}{12}(4-U) = \frac{1}{30}(5-U)$, and at which $\frac{1}{30}(5-U) = \frac{1}{20}$, and for $U \geqslant 4$ one has $\frac{1}{30}(5-U) \leqslant \frac{1}{20}(5-U) = c_4(U)$. This proves the inequality in (ii), and it shows that there is equality when the degrees $D' = D$ and $D' = D+1$ are admitted in Definition A.11, that is, when $D+1 \leqslant k-1$, which can be arranged for $U \leqslant k-1$. Part (i) is read off from (A.5): for an integer $D \geqslant 4$ both descriptions give $\varsigma^\star(D) = 1/(D(D+1))$.

(iii) Let $4 \leqslant D' \leqslant k-1$, $U < D'+1$ and $U_1 := U/(D'+1)$. For $\vartheta = 0$ the exponent of Lemma 4.4 is $\min\{U_1, D'(1-U_1)\}/(D'(D'+1))$, and this is at most $c_{D'}(U) \leqslant \varsigma^\star(U)$, because $U_1 \leqslant \min\{1, 2U_1\}$ and $D'(1-U_1) \leqslant D'+1-U$. Now let $j \geqslant 4$ and $U \geqslant j + 2^{-j}$. The exponent $\varrho = (j+2-U)/(2^{j+2}-2)$ of Lemma 4.3(ii) is at most 2^{-j-1}, with equality for $U = j + 2^{-j}$, and $2^{j+1} \geqslant (j+1)(j+2)$. As $\varsigma^\star(U) \geqslant 1/((j+1)(j+2))$ for $U \leqslant j+1$, and $\varsigma^\star(U) \geqslant (j+2-U)/((j+1)(j+2))$ for $j+1 \leqslant U \leqslant j+2$, we get $\varrho \leqslant \varsigma^\star(U)$. Finally, each value of the function ς of Definition 2.1 is the exponent saved for $\vartheta = 0$ by Lemma 4.3(i), by Lemma 4.3(ii) or by Lemma 4.4 (compare Table 2), and for the first of these and for the second with $j \leqslant 3$ the exponent is one of the numbers in (a) and (b). In the comparison just made the restriction $D' \leqslant k-1$ plays no part, so that $\varsigma \leqslant \varsigma^\star$. □

The function $\varsigma^\star$ is larger than ς for $\frac{17}{6} < U < \frac{10}{3}$ and for $U > \frac72$. Let $D \geqslant 4$. For $\frac{1}{(D+1)(D+2)} < t \leqslant \frac{1}{D(D+1)}$ the largest U with $\varsigma^\star(U) \geqslant t$ is $(D+1)(1-Dt)$, and it is $5-30t$, $4-12t$, $4-14t$ and $3-6t$ for t in $(\frac{1}{20}, \frac{1}{18}]$, $[\frac{1}{18}, \frac{1}{12}]$, $(\frac{1}{12}, \frac18]$ and $[\frac18, \frac14]$, respectively. By the formula of Lemma 10.2(ii), the integral $I(\frac15)$ of Definition 2.1, formed with $\varsigma^\star$ in place of ς, is the sum of $\frac13 \log 2$, $\frac14 \log \frac{17}{9}$, $\frac14 \log \frac53$, $\frac15 \log \frac76$ and the numbers Λ_D with $D \geqslant 4$; that is, it equals $\Lambda^\star$. Since $\varsigma^\star = \varsigma$ on $[1, \frac{17}{6}]$, the heuristic argument of §2.2 applies as it stands, and gives the constant $2 + \log 9 + 2\Lambda^\star = C_0^\star$.

Lemma A.13. *Let $k \geqslant 13$.*

(i) *In the situation of §4, suppose that $m \leqslant X$. Then $\mathcal{S} \ll X^{1-\varsigma^\star(U,\vartheta)+\varepsilon}$ for every $\varepsilon > 0$, with an implied constant that depends on k, J and ε only.*

(ii) *The function $\varsigma^\star$ is nonincreasing in ϑ and upper semicontinuous, and $\varsigma^\star \leqslant \frac12$. The function σ_k is nonincreasing and left-continuous on $[1, \frac12 k]$.*

(iii) *For $\frac32 \leqslant \theta \leqslant \min\{\frac94, \frac{k}{8}\}$ one has*

$$\sigma_k(\theta) \geqslant \frac{3-\theta}{6} - \frac{\theta}{2k} \geqslant \Big(1 - \frac{4\theta}{k}\Big)\frac{3-\theta}{6}.$$

Moreover $\sigma_k(\frac94) \leqslant \frac18$, and $\theta_k(2\tau_k) \geqslant \frac94$ if $k \geqslant 18$; here $\tau_k = 1/(9.027901\,k)$, as in §2.

Proof. (i) The number $1 - \frac12 U$ is the exponent saved by Lemma 4.3(i), the numbers in (b) are those of Lemma 4.3(ii), with the conditions stated there, and the numbers in (c)

are those of Corollary A.10. The factors $\log(2qm)$ in Lemma 4.3 are $\ll \log X$, because $q \leqslant m^k$ and $m \leqslant X$. For a given k only finitely many estimates are involved.

(ii) Each of the numbers in (b) and (c) is nonincreasing in ϑ, and the conditions attached to it remain valid when ϑ decreases. Consider one of these numbers as a function of (U, ϑ), with the value 0 where it is not admitted or negative. In case (b) this function is upper semicontinuous, because the conditions define a closed set, on which the number is at least $\frac{1}{2}\vartheta \geqslant 0$. In case (c) it is continuous, because $E_3 = -(D+1)\vartheta \leqslant 0$ when $U/(D+1) + \vartheta = 1$. Hence $\varsigma^\star$ is upper semicontinuous. The numbers in (a), (b) and (c) are smaller than $\frac{1}{2}$, at most $\frac{1}{4}$ and at most $\frac{1}{12}$, respectively. If $1 < \theta < \theta'$, then $\sigma_k(\theta') \leqslant \inf_{U \leqslant \theta} \varsigma^\star(U, \theta'/k) \leqslant \sigma_k(\theta)$. Finally let $\theta_n \uparrow \theta$ with $\theta > 1$, and let $1 < U \leqslant \theta$. Then $U_n := \min\{U, \theta_n\}$ tends to U, and $\limsup \sigma_k(\theta_n) \leqslant \limsup \varsigma^\star(U_n, \theta_n/k) \leqslant \varsigma^\star(U, \theta/k)$. Taking the infimum over U we get $\lim \sigma_k(\theta_n) \leqslant \sigma_k(\theta)$, and the reverse inequality holds because σ_k is nonincreasing.

(iii) Let $\frac{3}{2} \leqslant \theta \leqslant \min\{\frac{9}{4}, \frac{k}{8}\}$, $\vartheta := \theta/k$ and $1 < U \leqslant \theta$; then $\vartheta \leqslant \frac{1}{8}$. If $U \leqslant \frac{3}{2}$, then $\varsigma^\star(U, \vartheta) \geqslant 1 - \frac{1}{2}U \geqslant \frac{1}{4} \geqslant \frac{1}{6}(3-\theta)$. If $U \geqslant \frac{3}{2}$, then $\varrho_1 = \frac{1}{6}(3-U) \geqslant \frac{1}{8} \geqslant \vartheta$, so that the number $\varrho_1 - \frac{1}{2}\vartheta \geqslant \frac{1}{6}(3-\theta) - \frac{1}{2}\vartheta$ is admitted in (b). This gives the first inequality, and the second one is equivalent to $3 \leqslant 4(3-\theta)$. For $U = \frac{9}{4}$ the numbers in (a) are not positive, those in (b) are $\frac{1}{8} - \frac{1}{2}\vartheta$ for $j = 1$ and $j = 2$, if they are admitted, while $j = 3$ is not admitted, and those in (c) are at most $\frac{1}{12}$; hence $\sigma_k(\frac{9}{4}) \leqslant \varsigma^\star(\frac{9}{4}, \frac{9}{4k}) \leqslant \frac{1}{8}$. Finally, if $k \geqslant 18$, then $\frac{9}{4} \leqslant \frac{k}{8}$ and $\sigma_k(\frac{9}{4}) \geqslant \frac{1}{8} - \frac{9}{8k} > 2\tau_k$, so that $\theta_k(2\tau_k) \geqslant \frac{9}{4}$. □

Lower bounds for θ_k are obtained from the following lemma, in which the estimates are arranged in a chain. Its last two parts concern the end point $v/(1-\theta_k(2v)/k)$ of a step that begins at v (see §A.6).

Lemma A.14 (chains). *Let $k \geqslant 13$, $t > 0$ and $\vartheta \geqslant 0$. Let*

$$\mathcal{I}_0 := (1,\, 2-2t],$$

$$\mathcal{I}_j := \big[\, j + 2^{-j},\ \min\{j + 2 - (2^{j+2}-2)(t + \tfrac{1}{2}\vartheta),\ j + 2 - (2^{j+2}-2)\vartheta\}\big] \quad (j = 1, 2, 3),$$

and for integers D with $3 \leqslant D \leqslant k-1$ and $t_D := D(D+1)\,t \leqslant 1$ let $\mathcal{T}_D := [l_D, r_D]$ (we call these intervals tents*), where*

$$l_D := \tfrac{1}{2}(D+1)\big(t_D + (D-1)\vartheta\big),$$

$$r_D := \min\Big\{\frac{(D+1)\big(D-1-t_D-(2D-1)\vartheta\big)}{D-2},\ (D+1)\Big(1 - \frac{t_D + (3D-1)\vartheta}{2D-2}\Big),$$
$$D + 1 - t_D - D\vartheta\Big\}.$$

(i) *A number $U > 1$ satisfies $\varsigma^\star(U, \vartheta) \geqslant t$ if and only if it lies in one of these intervals. Hence, if $\theta := k\vartheta$ satisfies $1 < \theta \leqslant \frac{1}{2}k$, then $\sigma_k(\theta) \geqslant t$ if and only if the intervals cover $(1, \theta]$.*

(ii) *Let $\mathcal{J}_1, \dots, \mathcal{J}_n$ be intervals from this list with $\mathcal{J}_1 = \mathcal{I}_0$, such that the left end point of $\mathcal{J}_{i+1}$ does not exceed the right end point of $\mathcal{J}_i$ for $1 \leqslant i < n$, and let $\theta := k\vartheta$. If $1 < \theta \leqslant \frac{1}{2}k$, and if θ does not exceed the right end point of $\mathcal{J}_n$, then $\sigma_k(\theta) \geqslant t$, and consequently $\theta_k(t) \geqslant \theta$.*

(iii) *For a given choice of the intervals in* (ii)*, each of the conditions imposed there either does not involve ϑ, or has the form $\vartheta \leqslant a_1 - a_2 t$ with real numbers a_1 and $a_2 \geqslant 0$ that do*

not depend on t and ϑ. Let $\tilde\theta(t)$ be k times the least of the numbers $a_1 - a_2t$ that belong to the conditions of the second kind, the condition $\theta \leqslant \frac12 k$ being left out. Then, on an interval of t on which the intervals chosen are defined and the conditions of the first kind hold, $\tilde\theta$ is a concave, piecewise linear and nonincreasing function of t, and $\theta_k(t) \geqslant \tilde\theta(t)$ wherever $1 < \tilde\theta(t) \leqslant \frac12 k$.

(iv) *In* (iii) *each of the numbers a_1 is smaller than* 1. *Consequently, if $\mathcal{I} \subseteq (0, \frac12]$ is an interval as in* (iii) *on which $\tilde\theta \leqslant \frac12 k$, then the function $t \mapsto t/(1 - \tilde\theta(t)/k)$ is positive and increasing on $\mathcal{I}$.*

(v) *Let $\mathcal{I}$ be as in* (iv), *and let $0 < a < b$ with $[2a, 2b] \subseteq \mathcal{I}$. Then*

$$\frac{v}{1 - \theta_k(2v)/k} \geqslant \frac{a}{1 - \tilde\theta(2a)/k} \qquad (a \leqslant v \leqslant b).$$

Proof. (i) As $t > 0$, the number 0 in (a) of Definition A.11 is irrelevant, and it suffices to show that each interval of the list is the set of the numbers $U > 1$ at which one of the other numbers of the definition is admitted and at least t. For $\mathcal{I}_0$ and the number $1 - \frac12 U$ this is clear. For $\mathcal{I}_j$ and $\varrho_j - \frac12\vartheta$ note that the left end point expresses the condition $U \geqslant j + 2^{-j}$, and the two terms of the minimum the conditions $\varrho_j - \frac12\vartheta \geqslant t$ and $\varrho_j \geqslant \vartheta$. For $\mathcal{T}_D$ and the number in (c) with this D, the inequality $U \geqslant l_D$ is equivalent to $E_1 \geqslant t_D$, and $U \leqslant r_D$ to $\min\{E_2, E_3, E_4\} \geqslant t_D$; moreover $E_3 \geqslant t_D > 0$ implies that $U/(D+1) + \vartheta < 1$, and $\min\{1, E_1, \dots, E_4\} \geqslant t_D$ requires $t_D \leqslant 1$. The second assertion of (i) follows, since $\sigma_k(\theta) \geqslant t$ means that $\varsigma^\star(U, \vartheta) \geqslant t$ for $1 < U \leqslant \theta$.

(ii) By induction on i, the union of $\mathcal{J}_1, \dots, \mathcal{J}_i$ contains the interval from 1, exclusive, to the largest of the right end points of these intervals; an interval $\mathcal{J}_i$ may be empty, but then its right end point is smaller than its left end point, and the step from $i-1$ to i is trivial. So $(1, \theta]$ is contained in the union of $\mathcal{J}_1, \dots, \mathcal{J}_n$, and by (i) we have $\varsigma^\star(U, \theta/k) \geqslant t$ for $1 < U \leqslant \theta$.

(iii) Every end point is of the form $a_3 - a_4t - a_5\vartheta$, where $a_4, a_5 \geqslant 0$ for the right end points and $a_4, a_5 \leqslant 0$ for the left end points. So, after the terms have been collected, a condition between two end points reads $a_6\vartheta \leqslant a_7 - a_8t$ with $a_6, a_8 \geqslant 0$, and the condition on $\theta = k\vartheta$ and the right end point of $\mathcal{J}_n$ has the same shape with $a_6 \geqslant k$; the condition $\theta \leqslant \frac12 k$ is $\vartheta \leqslant \frac12$. The minimum of finitely many functions $a_1 - a_2t$ with $a_2 \geqslant 0$ is concave, piecewise linear and nonincreasing. If $1 < \tilde\theta(t) \leqslant \frac12 k$, then $\theta = \tilde\theta(t)$ satisfies all conditions of (ii), so that $\theta_k(t) \geqslant \tilde\theta(t)$.

(iv) Write a term of a right end point as $\rho_0 - \rho_1 t - \rho_2\vartheta$ and a left end point as $\lambda_0 + \lambda_1 t + \lambda_2\vartheta$; all the coefficients are nonnegative. For $\mathcal{I}_0$ one has $(\rho_0, \rho_2) = (2, 0)$. For $\mathcal{I}_j$ with $j \geqslant 1$ the two terms have $(\rho_0, \rho_2) = (j+2,\ 2^{j+1} - 1)$ and $(j+2,\ 2^{j+2} - 2)$, and for $\mathcal{T}_D$ the three terms have

$$(\rho_0, \rho_2) = \Big(\frac{(D+1)(D-1)}{D-2},\ \frac{(D+1)(2D-1)}{D-2}\Big),\ \Big(D{+}1,\ \frac{(D+1)(3D-1)}{2D-2}\Big),\ (D{+}1, D).$$

So $\rho_2 \geqslant 3$ and $\rho_0 \leqslant \rho_2 + 1$ for every term that does not belong to $\mathcal{I}_0$. For the left end points, (λ_0, λ_2) is $(j + 2^{-j}, 0)$ for $\mathcal{I}_j$, where $j + 2^{-j} \geqslant \frac32$, and $(0, \frac12(D+1)(D-1))$ for $\mathcal{T}_D$, where $\frac12(D+1)(D-1) \geqslant 4$. The condition between the left end point of an interval and a term of the right end point of its predecessor reads $(\lambda_2 + \rho_2)\vartheta \leqslant \rho_0 - \lambda_0 - (\lambda_1 + \rho_1)t$. It involves ϑ only if $\lambda_2 + \rho_2 > 0$, and then $a_1 = (\rho_0 - \lambda_0)/(\lambda_2 + \rho_2)$. If the predecessor is

$\mathcal{I}_0$, the interval is a tent, and $a_1 = 2/\lambda_2 \leqslant \frac{1}{2}$. Otherwise $a_1 \leqslant (\rho_2 - \frac{1}{2})/\rho_2 < 1$ when the interval is one of $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3$, and $a_1 \leqslant (\rho_2+1)/(\rho_2+4) < 1$ when it is a tent. The condition on $\theta = k\vartheta$ and a term of the right end point of the last interval reads $(k+\rho_2)\vartheta \leqslant \rho_0 - \rho_1 t$, and here $a_1 = \rho_0/(k+\rho_2) < 1$ because $\rho_0 \leqslant \rho_2 + 2$. Thus $\tilde{\theta}(t)/k$ is the least of finitely many numbers $a_1 - a_2 t$ with $a_1 < 1$ and $a_2 \geqslant 0$, so that $t/(1 - \tilde{\theta}(t)/k)$ is the least of the numbers $t/(1 - a_1 + a_2 t)$, and each of these is positive and increasing for $t > 0$.

(v) Let $a \leqslant v \leqslant b$ and $t := 2v$. If $\tilde{\theta}(t) > 1$, then $\theta_k(t) \geqslant \tilde{\theta}(t)$ by (iii), and otherwise $\theta_k(t) \geqslant 1 \geqslant \tilde{\theta}(t)$. As $\theta_k(t) \leqslant \frac{1}{2}k$, the left hand side of the inequality in (v) is at least $v/(1 - \tilde{\theta}(2v)/k)$, and by (iv) this number is at least its value at $v = a$. □

A.4. **Potentials.** In §8 the potential $\mathcal{G}$ enters the induction through Lemma 8.1, and the proofs there draw on its particular shape, on the admissibility of Θ and on the hypothesis $k \geqslant 37$. In this appendix two other potentials are used, for all $k \geqslant 13$. We therefore record what is used of §§3–9 when $k < 37$, and then isolate the properties of a potential on which Proposition 8.2 rests (Lemma A.16).

The range of k. Let $k \geqslant 13$. Sections 3–7 and 9 were written for $k \geqslant 37$ and for a number Θ that is admissible for k. Apart from §8, and from the evaluation of the potential of that section in §9.5, the admissibility of Θ is used in (3.1) and (9.2), in Lemma 4.5 and in Corollary 5.2, and the hypothesis $k \geqslant 37$ is used in two places only: in the inequality $s > 4.19\,k \geqslant 4k + 1$ of §9.1, which requires $k \geqslant 6$, and in the inequality $h_k \leqslant \frac{1}{5}$, on which the second assertion of Theorem 2.3 depends. The results from outside the paper that are quoted in these sections hold for $k \geqslant 13$: [9, Theorem 2.2] is stated for $k \geqslant 6$ and for all real orders $t \geqslant 4$, with arbitrary weights, so that (7.1) is available as explained in §7; and §4 holds for $k \geqslant 3$. Lemma 9.7 is proved in §9.3 from (7.1) with $v = 2k$. In §8 both hypotheses are used at several further points; that section is replaced by Lemma A.16 below.

The parameters. Let Θ be a real number with $1 < \Theta \leqslant \frac{1}{2}k$ and $\sigma_k(\Theta) > 0$. In place of (3.1) we suppose that

$$0 < \delta_0 < \tfrac{1}{40}, \qquad 0 < \varepsilon_2 \leqslant \min\{\tfrac{1}{100}, \delta_0\}, \qquad 0 < \gamma \leqslant \frac{1}{100\,k}, \qquad 0 < \varepsilon \leqslant \tfrac{1}{4}\delta_0\,\sigma_k(\Theta); \tag{A.6}$$

thus $\sigma(\Theta)$ is replaced by $\frac{1}{2}\sigma_k(\Theta)$, and the condition $\delta_0 < \frac{1}{2}(1 - \eta_k(\Theta))$ is omitted. The other parameters are chosen as in §3.1; in particular (3.2) holds. With these parameters Corollary 5.2 holds for $1 < \theta \leqslant \Theta$ with $\sigma_k(\theta)$ in place of $(1 - \eta_k(\theta))\sigma(\theta)$. Indeed, if $m \leqslant X^{\theta/k}$, and if q is a divisor of m^k with $q > X$, then $\vartheta \leqslant \theta/k \leqslant \frac{1}{2}$ and $1 < U \leqslant k\vartheta \leqslant \theta$, so that $\varsigma^\star(U, \vartheta) \geqslant \varsigma^\star(U, \theta/k) \geqslant \sigma_k(\theta)$ by Lemma A.13(ii); and Lemma A.13(i) takes the place of Lemma 4.5 in the proof. Next put

$$M(\theta) := \tfrac{1}{2}(1 - \delta_0)\,\sigma_k(\theta) \quad (1 \leqslant \theta \leqslant \Theta), \qquad u_\Theta := M(\Theta),$$
$$\theta(z) := \max\{\theta \in [1, \Theta] : \ M(\theta) \geqslant z\} \quad (0 < z \leqslant \tfrac{1}{5}).$$

By Lemma A.13(ii) the function M is positive, nonincreasing and left-continuous, and $M(1) = \frac{1}{4}(1 - \delta_0) > \frac{1}{5}$. So $\theta(z)$ is well defined, the function $\theta(\cdot)$ is nonincreasing with values in $[1, \Theta]$, $M(\theta(z)) \geqslant z$, and $\theta(z) = \Theta$ exactly when $z \leqslant u_\Theta$. By the definition of θ_k,

$$\theta(z) = \min\big\{\Theta,\ \theta_k\big(2z/(1 - \delta_0)\big)\big\} \qquad (0 < z \leqslant \tfrac{1}{5}). \tag{A.7}$$

In place of (8.1) we have

$$\sigma_k\big(\theta(z)\big) \geqslant 2z + 2\delta_0\, u_\Theta \geqslant 2z + 2\varepsilon \qquad (0 < z \leqslant \tfrac15), \tag{A.8}$$

because $\sigma_k(\theta(z)) \geqslant 2z/(1-\delta_0) \geqslant 2z + 2\delta_0 z$ when $z \geqslant u_\Theta$, while for $z < u_\Theta$ we have $\sigma_k(\Theta) = 2u_\Theta/(1-\delta_0) \geqslant 2u_\Theta + 2\delta_0 u_\Theta$; moreover $2\delta_0 u_\Theta \geqslant \frac12\delta_0\sigma_k(\Theta) \geqslant 2\varepsilon$, by (A.6).

The a priori bound. We write (Ψ°, τ°) for one of the following two pairs. In the *first case* $\Psi^\circ := \Psi$ on $(0, \frac12]$, where $\Psi(z) = 2 + \log\frac{1-z}{z}$ as in §7, $\Psi^\circ(z) := 1/z$ for $z > \frac12$, and $\tau^\circ := \tau_k$. In the *second case* $13 \leqslant k \leqslant 20$, and Ψ° and τ° are formed with the permissible exponents that Vaughan and Wooley [34, §§15–22] have tabulated, as follows. We write $\lambda_s = \lambda_s^{(k)}$ $(3 \leqslant s \leqslant S_k)$ for the numbers that are recorded in the table for k in [34]; here $S_k = 36$, 40, 43, 47, 50, 54, 58, 62 for $k = 13, \dots, 20$. Further let $\lambda_0 := 0$, $\lambda_1 := 1$ and $\lambda_2 := 2$. These numbers are permissible exponents in the sense of [34, §2]: for every $\varepsilon' > 0$ there is $\eta' > 0$ such that the number of solutions of the equation

$$n_1^k + \cdots + n_s^k = n_{s+1}^k + \cdots + n_{2s}^k \qquad \big(n_i \in \mathcal{A}(Y, Z)\big)$$

is $\ll Y^{\lambda_s + \varepsilon'}$ whenever $2 \leqslant Z \leqslant Y^{\eta'}$. For $s \leqslant 2$ this is trivial, or Hua's lemma, as in §7; for $s \geqslant 3$ the recorded numbers are stated in [34, §8] to be upper bounds for the exponents that were computed there. Put

$$\Delta_s := \lambda_s - 2s + k \quad (0 \leqslant s \leqslant S_k), \qquad z^{(s)} := \tfrac12(\Delta_s - \Delta_{s+1}) \quad (0 \leqslant s < S_k),$$
$$\Psi_k^{\mathrm{VW}}(z) := \frac{1}{k} \min_{0 \leqslant s \leqslant S_k} \Big(2s + \frac{\Delta_s}{z}\Big) \quad (z > 0), \qquad \tau_k^{\mathrm{VW}} := \frac{1}{T_k},$$

where the numbers T_k for $k = 13, \dots, 20$ are

$$104.95, \quad 114.19, \quad 123.40, \quad 132.60, \quad 141.78, \quad 150.95, \quad 160.07, \quad 169.18,$$

and let $\Psi^\circ := \Psi_k^{\mathrm{VW}}$ and $\tau^\circ := \tau_k^{\mathrm{VW}}$. The following facts are verified by the program of §10.4, in exact rational arithmetic:

(V1) the sequence $(\lambda_s)_{0 \leqslant s \leqslant S_k}$ is convex, $\lambda_{s+1} - \lambda_s < 2$, and $\Delta_{S_k} > 0$;
(V2) $\lambda_k - \lambda_{k-1} \leqslant \frac85$, that is, $z^{(k-1)} \geqslant \frac15$;
(V3) with $u := k-1$ one has $\tau_k^{\mathrm{VW}} < (k - 2\Delta_u)/(4u^2)$, $9k \leqslant 8u^2$ and $\frac18\big(1 + \frac{\Delta_u}{2u^2}\big) + \frac{k}{4u^2} < \frac14$.

By (V1) the numbers Δ_s are positive and decrease, the numbers $z^{(s)}$ are positive and do not increase with s, and the minimum that defines $\Psi_k^{\mathrm{VW}}(z)$ is attained at s when $z^{(s)} \leqslant z \leqslant z^{(s-1)}$, where $z^{(-1)} := \infty$ and $z^{(S_k)} := 0$; note that $z^{(0)} = \frac12$, so that $\Psi_k^{\mathrm{VW}}(z) = 1/z$ for $z \geqslant \frac12$. The number T_k is a rounded-up value of $4u^2/(k - 2\Delta_u)$; for instance $1/\tau_{17} = 153.47\ldots$, whereas $T_{17} = 141.78$.

Lemma A.15. *Let $k \geqslant 13$, and let (Ψ°, τ°) be as above.*

(i) *The function Ψ° is continuous, decreasing and locally Lipschitz continuous on $(0, \infty)$, and $\Psi^\circ(z) = 1/z$ for $z \geqslant \frac12$. The function $\mathfrak{g}^\circ(z) := -z\,\frac{\mathrm{d}}{\mathrm{d}z}\Psi^\circ(z)$ satisfies $0 \leqslant \mathfrak{g}^\circ \leqslant \Psi^\circ$ almost everywhere, and*

$$\Psi^\circ(z) - \mathfrak{g}^\circ(z) \geqslant 2 \qquad \textit{for almost all } z \in (0, \tfrac15]. \tag{A.9}$$

In particular $\Psi^\circ \geqslant 2$ on $(0, \frac15]$.

(ii) *Let $0 < z_* \leqslant \frac12$. In the situation of Lemma 7.1 one has $D_j(X, W) \ll P^{k\lambda\Psi^\circ(z) + \varepsilon}$ for all $z = \lambda/x \geqslant z_*$.*

(iii) *Suppose that J is sufficiently large in terms of k and γ. Then Lemmas* 9.7 *and* 9.8 *hold with τ° in place of τ_k. More precisely, let $P^{1/4} \leqslant Q \leqslant P^{k/2}$, define $\mu \in [\frac{1}{2k}, 1]$ by $P^\mu = \min\{P, (2Q)^{2/k}\}$, and let i be the largest integer in $[0, J]$ with $(\zeta R)^i \leqslant P^{1-\mu}$. Then $i < J$; for every $d \in \mathcal{D}_i$ the number P/d is attached to $J - i$ and satisfies* (9.10)*; and there is a constant C_3, which does not depend on Q, d, q and H, such that*

$$\big|F_{J-i}^{[d]}(d^k\alpha; P/d)\big| \leqslant C_3\, \frac{P}{d}\, d_q(d)\, Q^{-2\tau^\circ/k} R^{1/2} \qquad (d \in \mathcal{D}_i) \tag{A.10}$$

whenever $\alpha \in \mathfrak{N}(Q; P)$ and a, q are coprime integers with $1 \leqslant q \leqslant Q$ and $|q\alpha - a| \leqslant QP^{-k}$.

Proof. (i) In the first case $\mathfrak{g}^\circ(z)$ equals $1/(1-z)$ for $z < \frac12$ and $1/z$ for $z > \frac12$. So $\Psi^\circ - \mathfrak{g}^\circ$ vanishes for $z > \frac12$, and for $z < \frac12$ it equals $\Psi(z) - \frac{1}{1-z}$, which is decreasing, vanishes at $z = \frac12$, and is at least $2 + \log 4 - \frac54 > 2$ for $z \leqslant \frac15$. In the second case let $z^{(s)} < z < z^{(s-1)}$. Then

$$\mathfrak{g}^\circ(z) = \frac{\Delta_s}{k\, z} \geqslant 0 \qquad \text{and} \qquad \Psi^\circ(z) - \mathfrak{g}^\circ(z) = \frac{2s}{k} \geqslant 0,$$

and if $z < \frac15$, then $s \geqslant k$ by (V2). By (V1) each of the functions $2s + \Delta_s/z$ is decreasing, and hence so is their minimum.

(ii) In the first case this is Lemma 7.1 when $z \leqslant \frac12$, and for $z \geqslant \frac12$ it is the trivial bound $D_j(X, W) \leqslant X^k = P^{kx}$, because $k\lambda\Psi^\circ(z) = kx$. In the second case let $0 \leqslant s \leqslant S_k$. The mean value of order $2s$ of the sum $F_j^{\mathfrak{e}}(\alpha; X)$ is the number of solutions of the equation displayed above with $n_i \in \mathcal{C}_j^{\mathfrak{e}}(X)$, and this set is contained in $\mathcal{A}(X, \zeta R)$. So the mean value is $\ll X^{2s-k+\Delta_s+\varepsilon'}$, uniformly in $\mathfrak{e}$, when J is large, and the argument in the proof of Lemma 7.1, applied with $v = 2s$, gives $D_j(X, W) \ll W^{2s} X^{\Delta_s} P^{\varepsilon/2} = P^{\lambda(2s+\Delta_s/z)+\varepsilon/2}$. It remains to choose s so that the exponent is smallest.

(iii) The assertions on i and on P/d, and the bound (A.10) with τ_k, are contained in the proof of Lemma 9.8; the constant comes from Lemma 9.7, whose implied constant does not depend on β, a, q, $\mathfrak{e}$ and H. This settles the first case. In the second case we apply Lemma 9.4 in the proof of Lemma 9.7 with $u = w = k - 1$ (in the notation of that lemma), and bound $S_u(\mathcal{D}_i^{\mathfrak{e}})$ and $S_u(\mathcal{B})$ by means of the permissible exponent λ_u. This gives

$$\Big(\sum_{n\in\mathcal{B}}\Big|\sum_{d\in\mathcal{D}_i^{\mathfrak{e}}} \xi_d\, e\big(\beta(dn)^k\big)\Big|\Big)^{2u^2} \ll (M_1 Y)^{2u^2+\Delta_u+\varepsilon'}\, \frac{R^{k/2}}{\Xi}\, \log P,$$

and since $X^{1+\Delta_u/(2u^2)}\, \Xi^{-1/(2u^2)} = X^{1-(k-2\Delta_u)/(4u^2)} (X^{k/2}/\Xi)^{1/(2u^2)}$, we obtain

$$F_j^{\mathfrak{e}}(\beta; X) \ll X^{1-\tau^\circ} \Big(\frac{X^{k/2}}{\Xi}\Big)^{1/(2u^2)} R^{1/4},$$

by (V3); the powers of R that arise have exponents whose sum is at most

$$\frac18\Big(1 + \frac{\Delta_u + \varepsilon'}{2u^2}\Big) + \frac{k}{4u^2} < \frac14.$$

In the proof of Lemma 9.8 the factor $(g\, Y^{k/2}/\Xi_\alpha)^{1/(2u^2)}$ is at most $4\, d_q(d)\, R^{1/4}$, because g divides $d_q(d)^k$, $k \leqslant 2u^2$ and $9k \leqslant 8u^2$; so (A.10) holds with τ°. □

Lemma A.16 (potentials)**.** *Let $k \geqslant 13$, let (Ψ°, τ°) be as above, and let Θ, the parameters and the function $\theta(\cdot)$ be as in the paragraph on the parameters. Let u and z_0 be real numbers with*

$$0 < u \leqslant z_0 < \tfrac{1}{10} \qquad \text{and} \qquad \theta(z_0) \geqslant \tfrac{3}{2},$$

and let $\mathcal{G}$ be a real function on $(0,\infty)$ with the following properties:

(G1) *$\mathcal{G}(z) = \Psi^\circ(z)$ for $z \geqslant z_0$;*

(G2) *$\mathcal{G}$ is nonincreasing and locally Lipschitz continuous, and $-z\,\mathcal{G}'(z) \leqslant \mathcal{G}(z) - 2$ for almost all $z \in (0, z_0)$;*

(G3) *$\mathcal{G}(z') \leqslant \mathcal{G}(z) - \frac{2}{k}$ whenever $0 < z < z_0$ and $z' \geqslant z/\bigl(1 - \theta(z)/((1+\varepsilon_2)k)\bigr)$.*

For $\gamma \leqslant x \leqslant 1$ put $\Phi(x,\lambda) := k\lambda\,\mathcal{G}(\max\{\lambda, \gamma u\}/x)$ for $\lambda > 0$ and $\Phi(x,0) := 0$, and let $\mathcal{G}_\gamma := \mathcal{G}(\gamma u)$.

(i) *For $\gamma \leqslant x \leqslant 1$ the functions $\lambda \mapsto \Phi(x,\lambda)$ and $\lambda \mapsto \mathcal{G}_\gamma k\lambda - \Phi(x,\lambda)$ are nondecreasing on $[0,\infty)$, and the function $\lambda \mapsto \Phi(x,\lambda) - 2k\lambda$ is nondecreasing on $[0, \frac{1}{5}x]$.*

(ii) *If $\lambda/x \leqslant \frac{1}{5}$ and $0 \leqslant t \leqslant \min\{\lambda, x - \gamma\}$, then $\Phi(x-t, \lambda - t) \leqslant \Phi(x,\lambda) - 2kt$.*

(iii) *Let $z := \lambda/x < z_0$ and $\theta := \theta(z)$, and let $(1+\varepsilon_2)^{-1}\theta x/k \leqslant y \leqslant \min\{\theta x/k,\, x - \gamma\}$. Then $\Phi(x,\lambda) - \Phi(x-y,\lambda) \geqslant 2\lambda$.*

(iv) *The assertion of Proposition 8.2 holds with this function Φ, when J is sufficiently large.*

Parts (i)–(iii) are the assertions of Lemma 8.1(ii)–(iv) in the form in which they are used in the proof of Proposition 8.2; z_0 plays the part of z_c, and u that of u_Θ.

Proof. (i) Let $\mathfrak{g}(z) := -z\,\mathcal{G}'(z)$, which is defined almost everywhere and nonnegative, and $\mathfrak{h}(z) := z\,\mathcal{G}(z)$. The function $\mathfrak{h}$ is locally Lipschitz continuous, and $\mathfrak{h}' = \mathcal{G} - \mathfrak{g}$ almost everywhere. By (G1) and Lemma A.15(i) we have $\mathcal{G} - \mathfrak{g} = \Psi^\circ - \mathfrak{g}^\circ \geqslant 0$ almost everywhere on (z_0, ∞), and $\mathcal{G} - \mathfrak{g} \geqslant 2$ almost everywhere on $(z_0, \frac{1}{5})$; by (G2) the latter holds on $(0, z_0)$ as well. Hence $0 \leqslant \mathfrak{h}' \leqslant \mathcal{G}$ almost everywhere, and $\mathfrak{h}' \geqslant 2$ almost everywhere on $(0, \frac{1}{5})$. Moreover $\mathcal{G}(z) \geqslant \mathcal{G}(\frac{1}{5}) = \Psi^\circ(\frac{1}{5}) \geqslant 2$ for $z \leqslant \frac{1}{5}$. Now fix x. For $0 \leqslant \lambda \leqslant \gamma u$ the function $\Phi(x,\lambda) = k\lambda\,\mathcal{G}(\gamma u/x)$ is linear, and its slope lies between $2k$ and $\mathcal{G}_\gamma k$, because $\gamma u \leqslant \gamma u/x \leqslant u < \frac{1}{5}$. For $\lambda \geqslant \gamma u$ we have $\Phi(x,\lambda) = kx\,\mathfrak{h}(\lambda/x)$, a locally Lipschitz continuous function of λ whose derivative $k\,\mathfrak{h}'(\lambda/x)$ lies between 0 and $k\,\mathcal{G}(\lambda/x) \leqslant \mathcal{G}_\gamma k$ almost everywhere, and is at least $2k$ for almost all $\lambda < \frac{1}{5}x$. A locally Lipschitz continuous function is the integral of its derivative, and (i) follows.

(ii) As $\mathcal{G}$ is nonincreasing, $\Phi(x',\mu) \leqslant \Phi(x,\mu)$ for $\gamma \leqslant x' \leqslant x$ and $\mu \geqslant 0$. Hence $\Phi(x-t, \lambda - t) \leqslant \Phi(x, \lambda - t)$, and this is at most $\Phi(x,\lambda) - 2kt$ by (i).

(iii) Let $z_1 := \max\{\lambda, \gamma u\}/x$ and $z_1' := \max\{\lambda, \gamma u\}/(x-y) = z_1/(1 - y/x)$. Since $0 < y \leqslant x - \gamma$, we have $x > \gamma$ and therefore $\gamma u/x < u \leqslant z_0$; thus $z \leqslant z_1 < z_0$. As $\theta(\cdot)$ is nonincreasing,

$$\frac{y}{x} \geqslant \frac{\theta(z)}{(1+\varepsilon_2)k} \geqslant \frac{\theta(z_1)}{(1+\varepsilon_2)k}, \qquad \text{so that} \qquad z_1' \geqslant \frac{z_1}{1 - \theta(z_1)/((1+\varepsilon_2)k)}.$$

By (G3), applied at z_1, it follows that $\mathcal{G}(z_1') \leqslant \mathcal{G}(z_1) - \frac{2}{k}$, that is,

$$\Phi(x,\lambda) - \Phi(x-y,\lambda) = k\lambda\bigl(\mathcal{G}(z_1) - \mathcal{G}(z_1')\bigr) \geqslant 2\lambda.$$

(iv) We follow the proof of Proposition 8.2, with z_0 in place of z_c and u in place of u_Θ. Case 1 is unchanged. Case 2 is now the case $z \geqslant z_0$. Then $\lambda = zx \geqslant z_0\gamma \geqslant \gamma u$, so that

$\Phi(x,\lambda) = k\lambda\,\Psi^\circ(z)$ by (G1), and the assertion is Lemma A.15(ii) with $z_* = z_0$. Case 3 is the case $x > 4\gamma$ and $z < z_0$. Here $\theta = \theta(z)$ lies in $[\frac{3}{2}, \Theta]$, because $\theta(\cdot)$ is nonincreasing and $\theta(z_0) \geqslant \frac{3}{2}$, and $\Theta \leqslant \frac{1}{2}k$. In the proof of (8.6) the bound $\theta x > \frac{3}{2}\cdot 4\gamma$ shows that the right hand side of the inequality that is required there exceeds $5\gamma\varepsilon_2/k$, while the left hand side is at most $4\gamma\varepsilon_2/k$, as before; the remaining assertions of (8.6) follow from $\theta \leqslant \frac{1}{2}k$. By (A.8) we have $W^2 = X^{2z} \leqslant X^{\sigma_k(\theta)-2\varepsilon}$, so that Corollary 5.2 applies in the form stated above, and Lemma 6.1 gives (6.1). The estimates for Σ_1 and Σ_2 use Lemma 8.1(ii)–(iv) in the form of the present parts (i)–(iii), and in addition the inequalities $\lambda^+/(x-y) \leqslant \frac{1}{5}$, $\lambda^\dagger/x \leqslant \frac{1}{5}$ and $\lambda < \frac{1}{10}$. These hold for large P, because $\lambda^+/(x-y) \leqslant 2z + \log 4/(2\gamma\log P)$ and $\lambda^\dagger/x \leqslant z + \log K_0/(4\gamma\log P)$, while $z < z_0 < \frac{1}{10}$ and z_0 does not depend on P. □

A.5. **A potential with a largest step, and large k.** The first of our two potentials is the analogue of the one in §8. Let $k \geqslant 37$, let $h_k = \frac{36}{5(k-1)}$ and $\tau_k = 1/(9.027901\,k)$ be as in §2, and put

$$z_k := (1-h_k)\big(\tfrac{1}{10} - \tfrac{1}{50}h_k\big), \qquad \Theta_k := \theta_k(2\tau_k);$$

thus $\Theta_k \geqslant \frac{9}{4}$, by Lemma A.13(iii). For real numbers Θ with $\frac{9}{4} \leqslant \Theta \leqslant \Theta_k$ let

$$\mathcal{Y}_k(\Theta) := k\Big(\Psi(z_k) + 2\int_{\tau_k}^{z_k} \frac{\mathrm{d}v}{v\,\min\{\Theta, \theta_k(2v)\}}\Big) - \log\frac{z_k}{\tau_k}, \qquad \text{where} \quad \Psi(z) = 2 + \log\frac{1-z}{z}.$$

The number $\mathcal{Y}_k(\Theta)$ is nonincreasing in Θ; for $\Theta = \Theta_k$ the minimum in the integrand equals $\theta_k(2v)$. The term $-\log(z_k/\tau_k) = -\log(9.027901\,k\,z_k)$, which is about $-\log(0.9\,k)$, has no counterpart in Theorem 2.3: it comes from the exact length of the steps in the potential (see the remark that follows the proof of Theorem A.17).

Theorem A.17. *Let $k \geqslant 37$ and $\frac{9}{4} \leqslant \Theta \leqslant \Theta_k$.*

(i) *One has $G(k) \leqslant \lfloor \mathcal{Y}_k(\Theta)\rfloor + 1$.*

(ii) *Theorems 2.4 and 2.5 hold for k and Θ when $\mathcal{Y}_k(\Theta)$ takes the place of the number $k\big(C(h_k) - 2T(\Theta) + 2R_k(\Theta) + W_k(\Theta)\big)$ wherever it occurs in them.*

(iii) *Theorem 2.7 holds for k and Θ when the condition on s_0 is replaced by*

$$2s_0 > \mathcal{Y}_k(\Theta) - \frac{2k}{\Theta'}\Big(1 + \log\frac{9.027901\,\Theta'\,b}{2}\Big), \qquad \textit{where} \quad \Theta' := \frac{\Theta}{1 - \Theta/(2k)}.$$

In these statements Θ need not be admissible in the sense of Definition 2.2. Any positive function $\tilde\theta$ with $\tilde\theta \leqslant \theta_k$ on $[2\tau_k, 2z_k]$ yields an upper bound for $\mathcal{Y}_k(\Theta)$, namely the number that is obtained when θ_k is replaced by $\tilde\theta$ in the integral, and it shows that $\tilde\theta(2\tau_k) \leqslant \Theta_k$. It is in this way that the theorem will be applied. For $k \leqslant 5000$ parts (i) and (ii) are superseded by Theorem A.21; we use them for $k \geqslant 5000$ (Proposition A.19), and part (iii) for all $k \geqslant 37$ (§A.8).

Proof. We go through §§3–11 with k and Θ as in the theorem. In these sections the hypothesis that Θ is admissible for k, which is made in §3.1 and again at the beginning of §11, is replaced by $\frac{9}{4} \leqslant \Theta \leqslant \Theta_k$. It was used in (3.1) and (9.2); in Lemma 4.5 and Corollary 5.2; throughout §8.1, and in the proofs of Lemma 8.1 and Proposition 8.2, namely for the positivity and the strict monotonicity of M, for (8.1) and (8.3), and through the inequality $\theta \leqslant \Theta < \sqrt{k} \leqslant \frac{1}{2}k$; and in Lemma 8.1(i), which is quoted in §§9.5 and 11.5. We have $\Theta \leqslant \Theta_k \leqslant \frac{1}{2}k$ and $\sigma_k(\Theta) \geqslant \sigma_k(\Theta_k) \geqslant 2\tau_k > 0$. So the

paragraph of §A.4 on the parameters applies: the parameters are subject to (A.6), with $\varepsilon := \min\{\frac{1}{4}\delta_0\sigma_k(\Theta), \gamma/(16k^2s)\}$ in (9.2), Corollary 5.2 holds in the form stated there, and M, u_Θ and $\theta(\cdot)$ are defined as there.

The potential. The numbers η_c, w_c, z_c and θ_0 are defined as in §8.1, so that $\eta_c < \frac{1}{4}$, $\frac{9}{5} < \theta_0 < 2$ and $\frac{1}{6}(3-\theta_0) = 2w_c$. By Lemma A.13(iii),

$$M(\theta_0) \geqslant \tfrac{1}{2}(1-\delta_0)\Big(1-\frac{4\theta_0}{k}\Big)\,2w_c \geqslant \Big(1-\delta_0-\frac{4\theta_0}{k}\Big)w_c \geqslant (1-\eta_c)\,w_c = z_c,$$

the last step by the inequality $4\theta_0/k \leqslant \eta_c - \delta_0$ of §8.1. Hence $\theta(z_c) \geqslant \theta_0$, and (8.2) holds. Moreover $u_\Theta \leqslant M(\frac{9}{4}) \leqslant \frac{1}{16}$, again by Lemma A.13(iii), which gives (8.3). We define $\mathcal{G}$ as in §8.1, with one change: for $0 < z \leqslant z_c$ we put

$$\mathcal{G}(z) := \mathcal{G}(z_c) + \int_z^{z_c} \Big(\frac{2(1+\varepsilon_2)}{\theta(v)} - \frac{1}{k}\Big)\frac{\mathrm{d}v}{v}, \qquad \text{and accordingly} \qquad c_\Theta := \frac{2(1+\varepsilon_2)}{\Theta} - \frac{1}{k}$$

in (8.4). The integrand is positive, because $\theta(v) \leqslant \frac{1}{2}k$, and it is smaller than the one in §8.1. The term $-\frac{1}{k}$ expresses that a step of length y replaces the ratio $z = \lambda/x$ by $z/(1-y/x)$, which is larger than $z(1+y/x)$. The function $\theta(\cdot)$ is monotone, so that it is continuous outside a countable set, and at each point $z < z_c$ of continuity the function $\mathcal{G}$ is differentiable, with

$$\mathfrak{g}(z) := -z\,\mathcal{G}'(z) = \frac{2(1+\varepsilon_2)}{\theta(z)} - \frac{1}{k}.$$

We verify the hypotheses of Lemma A.16, in the first case, with $z_0 := z_c$ and $u := u_\Theta$. By (8.3) and (8.2) we have $u_\Theta < z_c < \frac{1}{10}$ and $\theta(z_c) \geqslant \theta_0 > \frac{3}{2}$, and (G1) holds by the definition of $\mathcal{G}$. The function $\mathcal{G}$ is nonincreasing and locally Lipschitz continuous. For almost all $z < z_c$ we have $\mathfrak{g}(z) \leqslant 2(1+\varepsilon_2)/\theta_0 \leqslant \frac{2.02}{1.8}$, while $\mathcal{G}(z) \geqslant \Psi(z_c) \geqslant \Psi(\frac{1}{10}) > 4.19$; this gives (G2). For (G3) let $0 < z < z_c$, and put $\theta := \theta(z)$, $w_0 := \theta/((1+\varepsilon_2)k)$ and $z' := z/(1-w_0)$; as $\mathcal{G}$ is nonincreasing, it suffices to consider this number z'. Here $0 < w_0 \leqslant \frac{1}{2}$, so that $z' \leqslant 2z < \frac{1}{5}$. For $0 \leqslant w < 1$ one has

$$-\log(1-w) \geqslant \frac{2w}{2-w},$$

because both sides vanish at $w = 0$, and the difference of the two sides has the derivative $w^2/((1-w)(2-w)^2)$. Hence $\log(z'/z) \geqslant 2w_0/(2-w_0)$. On the other hand, for almost all $v \in (z, z')$ we have

$$\mathfrak{g}(v) \geqslant \frac{2(1+\varepsilon_2)}{\theta} - \frac{1}{k} = \frac{2-w_0}{k\,w_0}\,:$$

if $v < z_c$, this holds because $\theta(v) \leqslant \theta$; and if $z_c < v < \frac{1}{5}$, then $\mathfrak{g}(v) = 1/(1-v) \geqslant 1/(1-z_c)$ is at least $2(1+\varepsilon_2)/\theta(z_c) \geqslant 2(1+\varepsilon_2)/\theta$, by (8.2). Therefore

$$\mathcal{G}(z) - \mathcal{G}(z') = \int_z^{z'} \mathfrak{g}(v)\,\frac{\mathrm{d}v}{v} \geqslant \frac{2-w_0}{k\,w_0}\cdot\frac{2w_0}{2-w_0} = \frac{2}{k},$$

which is (G3). Thus Lemma A.16 applies: parts (ii)–(iv) of Lemma 8.1 hold in the form in which they are used, and so does Proposition 8.2. Part (v) of Lemma 8.1 holds with the same proof. That proof involves the function $\mathfrak{g}$ on $(0, z_c)$ through three properties only: it is nondecreasing, it is at most $2(1+\varepsilon_2)/\theta(z_c)$, and it is at least c_Θ. In addition

it requires that $\Theta \geqslant \frac{9}{4}$ and that $\mathcal{G}$ is the integral of its derivative. Part (i) of that lemma is not needed: by the definition of $\mathcal{G}$,

$$\mathcal{G}(u_\Theta) = \Psi(z_c) + \int_{u_\Theta}^{z_c} \Big(\frac{2(1+\varepsilon_2)}{\theta(v)} - \frac{1}{k} \Big) \frac{\mathrm{d}v}{v}. \tag{A.11}$$

The proof of Corollary 8.3 is unchanged.

Sections 9 and 11. These sections use the potential only through Lemma 8.1(v), Corollary 8.3, (8.3) and the numbers $\mathcal{G}(u_\Theta)$, u_Θ and c_Θ. So they apply as they stand, and the proofs in §§9.5–9.7 and §11.5 show that the assertions of Theorems 2.3, 2.4 and 2.5 hold with the upper limit of the number (9.19), as δ_0 and ε_2 tend to 0, in place of $k\big(C(h_k) - 2T(\Theta) + 2R_k(\Theta) + W_k(\Theta)\big)$, and that Theorem 2.7 holds when $2s_0$ exceeds the upper limit of $k\big(\mathcal{G}(u_\Theta) + a_*(b)\big)$. (In §9.5 the inequality $c_\Theta \leqslant 1$ is used; it holds for the new number c_Θ as well.) It remains to evaluate these limits. We have $z_c \to z_k$, $c_\Theta \to \frac{2}{\Theta} - \frac{1}{k} = \frac{2}{\Theta'}$, with Θ' as in (iii), and $u_\Theta \to \bar{u} := \frac{1}{2}\sigma_k(\Theta)$, where $\bar{u} \geqslant \tau_k$. By (A.7), as δ_0 decreases to 0 the number $\theta(v)$ does not decrease, and it tends to $\min\{\Theta, \theta_k(2v)\}$ for all v outside a countable set, because θ_k is monotone. On the range of integration in (A.11) the integrand is at most $\frac{2.02}{1.8\,v}$, by (8.2), and $v \geqslant u_\Theta \geqslant \frac{39}{80}\sigma_k(\Theta)$. By dominated convergence, the number (9.19) therefore tends to

$$k\Big(\Psi(z_k) + \int_{\bar{u}}^{z_k} \Big(\frac{2}{\min\{\Theta, \theta_k(2v)\}} - \frac{1}{k} \Big) \frac{\mathrm{d}v}{v} + \Big(\frac{2}{\Theta} - \frac{1}{k} \Big) \log \frac{\bar{u}}{\tau_k} \Big) = \mathcal{Y}_k(\Theta);$$

the last equation holds because $\theta_k(2v) \geqslant \Theta$ for $\tau_k \leqslant v \leqslant \bar{u}$. Similarly $a_*(b)$ tends to $\frac{2}{\Theta'}\big(\log \frac{\bar{u}}{\tau_k} - 1 - \log \frac{9.027901\,\Theta' b}{2}\big)$, by (8.8), and so $k\big(\mathcal{G}(u_\Theta) + a_*(b)\big)$ tends to the right hand side of the condition in (iii). □

The new term $-\log(z_k/\tau_k)$ arises because a step of length θ is as effective as one of length $\theta' = \theta/(1 - \theta/(2k))$ was taken to be in §8: one has $\frac{2}{\theta'} = \frac{2}{\theta} - \frac{1}{k}$. If the logarithm in the proof of (G3) were kept in place of its lower bound, the integrand $2/\theta - 1/k$ would become $2/(-k\log(1-\theta/k))$, with $\theta = \min\{\Theta, \theta_k(2v)\}$. A quadrature, with θ_k computed from Definition A.11, shows that for $\Theta = \Theta_k$ this would lower $\mathcal{Y}_k(\Theta_k)$ by 0.057, 0.045, 0.018 and 0.008 for $k = 37$, 100, 1000 and 5000; we have preferred the simpler expression, and the recursion of §A.6 takes the exact length of the steps into account in any case.

When $\sigma_k(\Theta) = 2\tau_k$, the saving that is available for the largest step is twice the Weyl exponent, and $\mathcal{Y}_k(\Theta)$ is k times the value at τ_k of the limit of the potential, in accordance with the remarks at the end of §2.2. The term $W_k(\Theta)$ of Theorem 2.3 corresponds to the part of the integral over the range in which $\theta_k(2v) > \Theta$.

Large k. We now make the bound of Theorem A.17 explicit for $k \geqslant 5000$, by means of the chains of Lemma A.14. Let $D_\star = D_\star(k)$ be the largest integer D with $D^2+6D+32 \leqslant k$. Then $D_\star = \lfloor \sqrt{k-23} \rfloor - 3$, and $D_\star \geqslant 67$ for $k \geqslant 5000$.

Lemma A.18. *Let $k \geqslant 5000$, and put $\hat{t} := \frac{k-9}{8k}$, $\check{t} := \frac{k-20}{6(3k+10)}$ and $\Theta := D_\star/(1+D_\star/k)$. Then $\frac{1}{18} > \check{t} > \frac{1}{20}$, $\Theta \leqslant \Theta_k$, and*

$$\theta_k(t) \geqslant \begin{cases} (3-6t)/(1+\frac{3}{k}), & \hat{t} \leqslant t \leqslant \frac{1}{5}, \\ (4-14t)/(1+\frac{7}{k}), & \frac{1}{12} < t \leqslant \hat{t}, \\ (4-12t)/(1+\frac{8}{k}), & \check{t} \leqslant t \leqslant \frac{1}{12}, \\ (5-30t)/(1+\frac{15}{k}), & \frac{1}{20} < t \leqslant \check{t}, \\ (D+1)(1-Dt)/(1+\frac{D}{k}), & \frac{1}{(D+1)(D+2)} < t \leqslant \frac{1}{D(D+1)},\ 4 \leqslant D < D_\star, \\ \Theta, & 0 < t \leqslant \frac{1}{D_\star(D_\star+1)}. \end{cases}$$

Proof. Let θ denote the number on the right for a given t, and $\vartheta := \theta/k$. In all cases $\theta < D_\star \leqslant \sqrt{k} - 3$, so that $\vartheta < k^{-1/2} < 0.0142$. We apply Lemma A.14(ii).

The first four lines. We use the chains $\mathcal{I}_0, \mathcal{I}_1$; then $\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2$; then $\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2, \mathcal{T}_3$; and $\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2, \mathcal{T}_3, \mathcal{I}_3$. The interval $\mathcal{I}_1$ may follow $\mathcal{I}_0$, since $\frac{3}{2} \leqslant 2 - 2t$. In the first line $\theta = 3 - 6t - 3\vartheta$ by the choice of θ, and this is at most $3 - 6\vartheta$ because $\vartheta \leqslant 2t$; so θ is the right end point of $\mathcal{I}_1$. The interval $\mathcal{I}_2$ may follow $\mathcal{I}_1$ when $t + \frac{1}{2}\vartheta \leqslant \frac{1}{8}$, as $\vartheta \leqslant \frac{1}{8}$. In the second line $t + \frac{1}{2}\vartheta = t + (2-7t)/(k+7)$ increases with t and equals $\frac{1}{8}$ for $t = \hat{t}$, when $\theta = \frac{9}{4}$; and $\theta = 4 - 14t - 7\vartheta \leqslant 4 - 14\vartheta$ is the right end point of $\mathcal{I}_2$. From now on $t \leqslant \frac{1}{12}$, so that $t + \frac{1}{2}\vartheta < \frac{1}{8}$ and $t_3 = 12t \leqslant 1$. Here

$$l_3 = 24t + 4\vartheta, \qquad r_3 = \min\{8 - 48t - 20\vartheta,\ 4 - 12t - 8\vartheta,\ 4 - 12t - 3\vartheta\} = 4 - 12t - 8\vartheta,$$

because $36t + 12\vartheta \leqslant 4$. The interval $\mathcal{T}_3$ may follow $\mathcal{I}_2$, since $38t + 11\vartheta \leqslant 4$ and $24t + 18\vartheta \leqslant 4$. In the third line $\theta = r_3$. In the fourth line $\mathcal{I}_3$ may follow $\mathcal{T}_3$, since $r_3 \geqslant \frac{25}{8}$: here $\vartheta = (5-30t)/(k+15)$, so that $12t + 8\vartheta$ increases with t, and at $t = \check{t}$ the numbers in the third and the fourth line coincide, so that $r_3 = (4 - 12\check{t})/(1 + \frac{8}{k}) > 3.3$ there, as $\check{t} < \frac{1}{18}$. Moreover $\theta = 5 - 30t - 15\vartheta \leqslant 5 - 30\vartheta$ is the right end point of $\mathcal{I}_3$.

The last two lines. Let $4 \leqslant D \leqslant D_\star$ and $\frac{1}{(D+1)(D+2)} < t \leqslant \frac{1}{D(D+1)}$, so that $\frac{D}{D+2} < t_D \leqslant 1$, and let $\theta := (D + 1 - t_D)/(1 + D/k)$; for $D < D_\star$ this is the number in the fifth line. We show that the chain $\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2, \mathcal{T}_3, \mathcal{T}_4, \dots, \mathcal{T}_D$ is admitted, and that $\theta \leqslant r_D$, when $D < D_\star$, and also when $D = D_\star$ and $t_D = 1$. In the latter case $\theta = \Theta$, and since the right end points do not decrease and the left end points do not increase when t decreases while ϑ is fixed, the last line of the lemma follows for all $t \leqslant 1/(D_\star(D_\star+1))$. In particular $\theta_k(2\tau_k) \geqslant \Theta$, because $D_\star(D_\star+1) < k$.

The intervals $\mathcal{I}_1$, $\mathcal{I}_2$ and $\mathcal{T}_3$ may follow their predecessors, as before. As $t \leqslant \frac{1}{20}$, we have $l_4 = 50t + \frac{15}{2}\vartheta \leqslant 4 - 12t - 8\vartheta = r_3$. Let $4 \leqslant D' < D$. The number $l_{D'+1}$ does not decrease and the number $r_{D'}$ does not increase when t increases, and $t \leqslant 1/((D'+1)(D'+2))$. So it suffices to verify that $l_{D'+1} \leqslant r_{D'}$ when $t_{D'+1} = 1$ and $t_{D'} = D'/(D'+2)$. A computation shows that this holds if and only if

$$\vartheta \leqslant \min\Big\{ \frac{D'^3 + 4}{(D'+2)(D'^3 + 4D'^2 - 2D' - 2)},\ \frac{D'(D'^2 - 3)}{(D'+2)(D'^3 + 4D'^2 - 1)},\ \frac{D'}{(D'+2)(D'+4)} \Big\},$$

the three terms corresponding to those of $r_{D'}$. For $4 \leqslant D' \leqslant 7$ the minimum exceeds 0.066, and for $D' \geqslant 8$ it is at least $1/(D'+8)$: for the three terms this is equivalent to $2D'^3 - 6D'^2 + 10D' + 36 \geqslant 0$, to $2D'^3 - 11D'^2 - 23D' + 2 \geqslant 0$ and to $D' \geqslant 4$, respectively. Thus

it is enough to show that $\vartheta \leqslant 1/(D+7)$, since $\vartheta < 0.0142 < 0.066$ by the first paragraph of the proof. For $D < D_\star$ this holds because $\theta(D+7) < (D+1)(D+7) \leqslant D_\star(D_\star+6) < k$; for $D = D_\star$ and $t_D = 1$ we have $\vartheta = D_\star/(k+D_\star)$, and $D_\star(D_\star+7) \leqslant k + D_\star$.

It remains to show that $\theta \leqslant r_D$. The third term of r_D equals θ, by the choice of θ. On inserting $\theta = D + 1 - t_D - D\vartheta$, the inequalities between θ and the second and the first term of r_D become

$$(D^2+4D-1)\,\vartheta \leqslant (D-3)\,t_D \qquad \text{and} \qquad (D^2+3D-1)\,\vartheta \leqslant D+1-3t_D.$$

Since $\vartheta = (D+1-t_D)/(k+D)$, the first one is most restrictive for the smallest value of t_D, and the second one for $t_D = 1$; for the second note that both of its sides decrease as t_D grows, the right hand side at the rate 3 and the left hand side at the rate $(D^2+3D-1)/(k+D) < 3$, because $D^2 < k$. For $t_D = 1$ the two inequalities read $k + D \geqslant D(D^2+4D-1)/(D-3)$ and $k + D \geqslant D(D^2+3D-1)/(D-2)$, that is,

$$k \geqslant D^2 + 6D + 20 + \frac{60}{D-3} \qquad \text{and} \qquad k \geqslant D^2 + 4D + 9 + \frac{18}{D-2}.$$

The right hand sides increase with D for $D \geqslant 5$, their derivatives being $2D+6-60/(D-3)^2$ and $2D+4-18/(D-2)^2$, and both inequalities hold for $D = D_\star$, as $k \geqslant D_\star^2+6D_\star+32$ and $D_\star \geqslant 67$; for $D = 4$ the right hand sides are 120 and 50. So both hold for $4 \leqslant D \leqslant D_\star$. This settles the case $D = D_\star$, and the second inequality in general. For $D < D_\star$ and $t_D = D/(D+2)$ the first inequality reads

$$k + D \geqslant \frac{(D^2+2D+2)(D^2+4D-1)}{D(D-3)} = D^2 + 9D + 36 + \frac{114D-2}{D(D-3)}.$$

If $D \geqslant 41$, the fraction on the right is smaller than 3, and $D^2+8D+39 \leqslant D_\star^2+6D_\star+32 \leqslant k$ because $D \leqslant D_\star - 1$. If $D \leqslant 40$, the right hand side is smaller than 2000. □

Proposition A.19. *Let $k \geqslant 5000$, and let $D_\star$ and Θ be as in Lemma* A.18. *Put*

$$\begin{aligned} C^\star(k) := \Psi(z_k) &+ \tfrac{2}{3}\big(1+\tfrac{3}{k}\big)\log\frac{6(k+3)(1-h_k)(5-h_k)}{(k-9)(15+12h_k-2h_k^2)} + \tfrac{1}{2}\big(1+\tfrac{7}{k}\big)\log\frac{17(k-9)}{9(k+7)} \\ &+ \tfrac{1}{2}\big(1+\tfrac{8}{k}\big)\log\frac{5(k+8)}{3(k-20)} + \tfrac{2}{5}\big(1+\tfrac{15}{k}\big)\log\frac{7(k-20)}{6(k+15)} + 2\sum_{D\geqslant 4}\Lambda_D. \end{aligned}$$

Then

$$\begin{aligned} \mathcal{Y}_k(\Theta) &\leqslant k\Big(C^\star(k) - 2\sum_{D\geqslant D_\star}\Lambda_D + \frac{2}{k}\sum_{D=4}^{D_\star-1} D\,\Lambda_D + \frac{2}{\Theta}\log\frac{9.027901\,k}{2D_\star(D_\star+1)}\Big) - \log\frac{z_k}{\tau_k} \\ &< C_0^\star k - c_\star\sqrt{k} + \log k + 18.84, \end{aligned}$$

where $c_\star := 4 - 2\log\frac{9.027901}{2} = 0.98565\ldots$. Thus $G(k) \leqslant C_0^\star k - 0.9856\sqrt{k} + \log k + 20$.

Proof. By Lemma A.18 we have $\Theta \leqslant \Theta_k$, and the function on the right hand side there is a lower bound for $\min\{\Theta, \theta_k(t)\}$ on $[2\tau_k, 2z_k]$; note that $\hat{t} < 2z_k < \frac{1}{5}$, that $2\tau_k < 1/(D_\star(D_\star+1))$, and that the numbers in the first five lines are smaller than Θ (in the fifth line the number is smaller than $(D+1)/(1+\frac{D+1}{k}) \leqslant \Theta$, because $t > \frac{1}{(D+1)(D+2)}$

and $2D+2 \leqslant Dk$). We insert this bound in the definition of $\mathcal{Y}_k(\Theta)$, after the substitution $t = 2v$. Since

$$\int \frac{\mathrm{d}t}{t(c-c't)} = \frac{1}{c}\log\frac{t}{c-c't} \qquad \text{and} \qquad \frac{2z_k}{1-4z_k} = \frac{(1-h_k)(5-h_k)}{15+12h_k-2h_k^2},$$

the first four ranges contribute to $2\int \mathrm{d}t/(t\,\theta_k(t))$ at most the four logarithmic terms in the definition of $C^\star(k)$. The range belonging to D contributes $2(1+\frac{D}{k})\Lambda_D$, as in the proof of Lemma 10.2, and the range $2\tau_k \leqslant t \leqslant 1/(D_\star(D_\star+1))$ contributes $\frac{2}{\Theta}\log\frac{9.027901\,k}{2D_\star(D_\star+1)}$. This gives the first inequality, the last term of $\mathcal{Y}_k(\Theta)$ being carried along.

For the second one write $n := D_\star$, $\xi := k/n$ and $L_1 := \log\frac{9.027901}{2} = 1.50717\ldots$, so that $c_\star = 4 - 2L_1$. From $n^2+6n+32 \leqslant k < (n+1)^2+6(n+1)+32$ and $n \geqslant 67$ we get

$$n+6 < \xi < \bar{\xi} := n+8+\frac{39}{n} \qquad \text{and} \qquad \sqrt{k}-4.164 < n < \sqrt{k}-3;$$

for the last assertion note that $n+4 > \sqrt{k-23} > \sqrt{k}-0.164$.

The term $C^\star(k)$. When $k \to \infty$, each of the four logarithmic terms tends to the corresponding term of $2\Lambda^\star$ in (A.1), and $\Psi(z_k) \to 2+\log 9$. With $h := h_k$ the five differences are $\log\bigl(1+\frac{6h-h^2}{45}\bigr) - \log(1-h) - \log\bigl(1-\frac{h}{5}\bigr)$ and

$$\tfrac{2}{3}\Bigl(\bigl(1+\tfrac{3}{k}\bigr)\log\Bigl(\bigl(1+\tfrac{12}{k-9}\bigr)\bigl(1-\tfrac{30h-5h^2}{15+12h-2h^2}\bigr)\Bigr)+\tfrac{3}{k}\log 2\Bigr),$$
$$\tfrac{1}{2}\Bigl(\bigl(1+\tfrac{7}{k}\bigr)\log\bigl(1-\tfrac{16}{k+7}\bigr)+\tfrac{7}{k}\log\tfrac{17}{9}\Bigr), \qquad \tfrac{1}{2}\Bigl(\bigl(1+\tfrac{8}{k}\bigr)\log\bigl(1+\tfrac{28}{k-20}\bigr)+\tfrac{8}{k}\log\tfrac{5}{3}\Bigr),$$
$$\tfrac{2}{5}\Bigl(\bigl(1+\tfrac{15}{k}\bigr)\log\bigl(1-\tfrac{35}{k+15}\bigr)+\tfrac{15}{k}\log\tfrac{7}{6}\Bigr).$$

We use that $\log(1+x) \leqslant x$ for $x > -1$, that $-\log(1-x) \leqslant x/(1-x)$ for $0 \leqslant x < 1$, that $\frac{30h-5h^2}{15+12h-2h^2} \geqslant 2h-2h^2$, and that $7.2 < kh \leqslant 7.2015$ for $k \geqslant 5000$. In this way the five differences, multiplied by k, are found to be at most 9.62, -0.18, -5.77, 16.13 and -13.07, respectively. Hence

$$k\,C^\star(k) \leqslant C_0^\star k + 6.73.$$

The sums. For $D \geqslant n$ we have $\Lambda_D \geqslant \frac{2}{D^2} - \frac{2}{D^3} - \frac{2}{D^4}$, because $\log(1+x) \geqslant x - \frac{1}{2}x^2$ for $x \geqslant 0$. The function x^{-2} is convex, so that $\sum_{D\geqslant n} D^{-2} \geqslant \frac{1}{n} + \frac{1}{2n^2}$, by comparison with the trapezoidal rule, while $\sum_{D\geqslant n} D^{-3} \leqslant \frac{1}{2}(n-\frac{1}{2})^{-2}$ and $\sum_{D\geqslant n} D^{-4} \leqslant \frac{1}{3}(n-\frac{1}{2})^{-3}$, by comparison with the midpoint rule. For $n \geqslant 67$ this gives $\sum_{D\geqslant n}\Lambda_D \geqslant \frac{2}{n} - \frac{2}{n^3}$, and therefore

$$-2k\sum_{D\geqslant n}\Lambda_D \leqslant -4\xi + \frac{4\xi}{n^2} \leqslant -4\xi + 0.07.$$

Further $D\,\Lambda_D \leqslant \frac{2}{D}$, so that $2\sum_{D=4}^{n-1} D\,\Lambda_D \leqslant 4\log\frac{n-1}{3} < 2\log k - 4.394$.

The last term. Here $2k/\Theta = 2\xi+2$ and $k/(n(n+1)) = \xi/(n+1)$. The function

$$F(\xi) := -4\xi + (2\xi+2)\Bigl(L_1 + \log\frac{\xi}{n+1}\Bigr)$$

is increasing for $\xi > n+1$, because $F'(\xi) = 2L_1 - 2 + 2\log\frac{\xi}{n+1} + \frac{2}{\xi} > 0$. Hence $F(\xi) < F(\bar\xi)$, and since $\bar\xi/(n+1) = 1 + \frac{7n+39}{n(n+1)}$,

$$F(\bar\xi) \leqslant -c_\star\Big(n+8+\frac{39}{n}\Big) + 2L_1 + \frac{(2n^2+18n+78)(7n+39)}{n^2(n+1)} < -c_\star n - 7.885 + 3.015 + 17.08;$$

the last fraction equals $14 + (190n^2+1248n+3042)/(n^2(n+1))$, so that it decreases as n grows, and it is smaller than 17.08 for $n = 67$. Finally $-c_\star n < -c_\star\sqrt{k} + 4.105$.

Adding up we obtain

$$\mathcal{Y}_k(\Theta) < C_0^\star k - c_\star\sqrt{k} + 2\log k + 6.73 + 0.07 - 4.394 + 12.21 + 4.105 - \log\frac{z_k}{\tau_k}.$$

Here $z_k/\tau_k = 9.027901\,k\,z_k$, and z_k increases with k, so that $z_k \geqslant z_{5000} > 0.0998$ and $-\log(z_k/\tau_k) < -\log k + 0.105$. This gives the second inequality. The assertion on $G(k)$ follows from Theorem A.17(i), since $c_\star > 0.9856$. □

The coefficient $c_\star$ belongs to the choice $\Theta = (1+o(1))\sqrt{k}$: the three terms that depend on Θ are then about $-4k/\Theta$, from the tail of the series, $4\log\Theta$, from the rounding, and $(2k/\Theta)\log(9.027901\,k/(2\Theta^2))$; the second of them, which is about $2\log k$, is halved by the term $-\log(z_k/\tau_k)$. In Theorem 2.3 the term $2R_k(\Theta)$ is of order $k^{-1/2}$ when Θ is of order $\sqrt{k}$, and becomes infinite as $\Theta(\Theta+5) \to k$; here its place is taken by a term of order $k^{-1}\log k$. Larger steps are considered in Remark A.25.

A.6. **The steps of the potential as a recursion.** Condition (G3) of Lemma A.16 requires that the potential decrease by $\frac{2}{k}$ between the two ends of a step, that is, between z and $z/(1-\theta(z)/k)$, apart from the effect of the parameters δ_0 and ε_2. The potentials of §8 and of Theorem A.17 achieve this by means of a differential inequality: their logarithmic derivative at a point v is governed by $\theta(v)$, and since $\theta(\cdot)$ decreases along a step, they decrease by more than $\frac{2}{k}$ along most steps. In this section the requirement is imposed as it stands. The value of the potential at z is defined to be $\frac{2}{k}$ plus its value at the end of the step that begins at z; more precisely, at the least of the end points of the steps that begin at points $v \geqslant z$, because the potential has to be monotone. Then $z\,\mathcal{G}(z)$ is no longer concave, and Lemma 8.1(v) and the second assertion of Corollary 8.3 are lost. They were needed in §9 only for (9.4), where the arcs of height $Q \geqslant P^{1/4}$ are pruned by means of a moment of F and a bound for its supremum (compare Remark 11.8). We prove (9.4) in a different way. After the change of variables of Lemma 9.2 the bound of Weyl type holds for each of the shorter sums, so that only the values of these sums below that bound have to be considered, and the measure of the set on which such a sum is large is bounded by Proposition 8.2 itself (Proposition A.20). In this way arcs on which F saves a factor Y^τ are pruned with any number of variables that exceeds $k\,\mathcal{G}(z_*)$ for some $z_* < \tau$, whether $z\,\mathcal{G}(z)$ is concave or not. The outcome is Theorem A.21: it bounds $G(k)$ in terms of a single sequence of steps, which begins just below the exponent of the bound of Weyl type and ends below $\frac{1}{10}$, and no integral has to be estimated. The theorem holds for all $k \geqslant 13$; its numerical consequences are worked out in §A.7.

Pruning by large values. We turn to (9.4). The proposition below takes the place of Corollary 9.3 and of the proof of (9.4) in §9.3; the numbers b, r and t of (9.1) do not

occur in it. Let $k \geqslant 13$, and let (Ψ°, τ°) be one of the two pairs of §A.4; in both cases $\tau^\circ < \frac{1}{100}$.

Proposition A.20. *In the situation of §9, with the parameters as in §A.4, let $\mathcal{G}$ be a nonincreasing function from $(0,\infty)$ to $[0,\infty)$, let $0 < u \leqslant 1$, and put*

$$\Phi(x,\lambda) := k\lambda\,\mathcal{G}\big(\max\{\lambda,\gamma u\}/x\big) \quad (\lambda > 0), \qquad \Phi(x,0) := 0.$$

Suppose that the assertion of Proposition 8.2 holds with this function Φ, and that for $\gamma \leqslant x \leqslant 1$ the function $\lambda \mapsto \Phi(x,\lambda) - 2k\lambda$ is nondecreasing on $[0, \frac{1}{5}x]$. Let $s \geqslant 4k+1$ and z_* be real numbers with

$$u \leqslant z_* < \tau^\circ \qquad \text{and} \qquad \xi := s - k\,\mathcal{G}(z_*) > 0,$$

put $\nu := \xi z_*$, and suppose that, in place of (9.1), (9.2) and (9.3),

$$\gamma \leqslant \min\Big\{\frac{1}{100k}, \frac{\nu}{40k^2}\Big\}, \qquad \varepsilon \leqslant \gamma, \qquad \eta \leqslant \frac{\tau^\circ - z_*}{8k}, \qquad \omega = \frac{\eta}{64\,k\,s},$$

and that J is so large that Lemma A.15(iii) applies. Then the estimate (9.4) holds with this number ν, the estimates (9.5) and (9.6) hold, and so does (9.7), in which ν_0 is formed with ν.

Proof. The proofs of (9.5) and (9.6) in §9.4 depend on the parameters only through the conditions $s \geqslant 4k+1$ and $\omega = \eta/(64ks)$, and (9.7) is deduced from (9.4)–(9.6) as in §9.1. So only (9.4) has to be proved, and for this the only property of s that is needed is $s > 2k$. Let $P^{1/4} \leqslant Q \leqslant P^{k/2}$, and let μ, i and C_3 be as in Lemma A.15(iii). For $d \in \mathcal{D}_i$ we write $Y := P/d = P^y$ and $F_d(\beta) := F_{J-i}^{[d]}(\beta; Y)$; thus $\mu \leqslant y < \mu + \frac{9}{8}\eta$ by (9.10). For a divisor c of d let $V_c := C_3\,Y\,c\,Q^{-2\tau^\circ/k}R^{1/2}$ and

$$f_{d,c}(\beta) := \begin{cases} |F_d(\beta)|^s, & |F_d(\beta)| \leqslant V_c, \\ 0, & \text{otherwise}, \end{cases} \qquad T(d,c) := \int_{\mathbb{T}} f_{d,c}(\beta)\,\mathrm{d}\beta.$$

The large values of the shorter sums. We claim that

$$T(d,c) \ll Y^{s-k}\,c^{s-2k}\,P^{-\nu y + 4k\gamma + \varepsilon}(\log P)^{2J}. \tag{A.12}$$

Define λ_c by $V_c = YP^{-\lambda_c}$, and let $\lambda^0 := \max\{\lambda_c, 0\}$. Since $|F_d| \leqslant Y$,

$$\begin{aligned} T(d,c) &\leqslant \int_0^{\min\{V_c, Y\}} s\,v^{s-1}\,\mathrm{meas}\{\beta : |F_d(\beta)| \geqslant v\}\,\mathrm{d}v \\ &= s\log P \int_{\lambda^0}^{\infty} (YP^{-\lambda})^s\,\mathrm{meas}\{\beta : |F_d(\beta)| \geqslant YP^{-\lambda}\}\,\mathrm{d}\lambda. \end{aligned}$$

The number Y is attached to $J-i$, and $y \geqslant \mu \geqslant \frac{1}{2k} \geqslant \gamma$. So Proposition 8.2 bounds the measure in the integrand by $Y^{-k}C_*^{J+1}(\log P)^{2J}P^{4k\gamma+\varepsilon+\Phi(y,\lambda)}$, and

$$T(d,c) \ll Y^{s-k}(\log P)^{2J+1}P^{4k\gamma+\varepsilon}\int_{\lambda^0}^{\infty} P^{\Phi(y,\lambda)-s\lambda}\,\mathrm{d}\lambda.$$

Let $\lambda_1 := z_*y$. If $\lambda \geqslant \lambda_1$, then $\lambda \geqslant z_*/(2k) > \gamma u$, and since $\mathcal{G}$ is nonincreasing,

$$\Phi(y,\lambda) = k\lambda\,\mathcal{G}(\lambda/y) \leqslant k\lambda\,\mathcal{G}(z_*) = (s-\xi)\lambda.$$

If $0 \leqslant \lambda \leqslant \lambda_1$, then, by the hypothesis on Φ and since $\lambda_1 = z_*y \leqslant \frac{1}{5}y$,

$$\Phi(y,\lambda) \leqslant \Phi(y,\lambda_1) - 2k(\lambda_1 - \lambda), \qquad \text{whence} \qquad \Phi(y,\lambda) - s\lambda \leqslant -\xi\lambda_1 + (s-2k)(\lambda_1 - \lambda).$$

Therefore

$$\int_{\lambda^0}^{\infty} P^{\Phi(y,\lambda)-s\lambda}\,\mathrm{d}\lambda \leqslant \frac{P^{-\xi\lambda_1}}{\log P}\Big(\frac{1}{\xi} + \frac{P^{(s-2k)\max\{\lambda_1-\lambda^0,0\}}}{s-2k}\Big).$$

Now $\frac{2}{k}\log Q \geqslant \mu\log P - \log 2$, so that $\lambda_c \log P \geqslant (\tau^\circ\mu - \frac{1}{2}\eta)\log P - \log c - \log(2C_3)$, while $\lambda_1 < z_*\mu + \frac{9}{8}\eta$. Hence

$$(\lambda_1 - \lambda_c)\log P \leqslant \big(-(\tau^\circ - z_*)\mu + 2\eta\big)\log P + \log c + \log(2C_3) \leqslant \log c$$

for large P, because $(\tau^\circ - z_*)\mu \geqslant (\tau^\circ - z_*)/(2k) \geqslant 4\eta$. Consequently the power of P in the last bound for the integral is at most c^{s-2k}, and (A.12) follows, since $\xi\lambda_1 = \nu y$.

The change of variables. Suppose first that $Q \leqslant \frac{1}{2}P^{k/2}$, so that $P^{1-\mu} = P(2Q)^{-2/k}$, and let $q \leqslant Q$. For $d \in \mathcal{D}_i$ put $d' := d/d_q(d)$ and $q'' := q/(q,d^k)$, as in the proof of Lemma 9.2. Let $\alpha \in \mathfrak{N}(Q;P) \cap \mathfrak{M}_q(Q;P)$. By Lemma 3.1(c), Hölder's inequality and (9.8), as in that proof, and then by (A.10),

$$|F(\alpha)|^s \leqslant 2^{kJ(s-1)} \sum_{d\in\mathcal{D}_i} {d'}^{s-1}\big|F_d(d^k\alpha)\big|^s = 2^{kJ(s-1)} \sum_{d\in\mathcal{D}_i} {d'}^{s-1} f_{d,d_q(d)}(d^k\alpha).$$

The right hand side is defined and nonnegative for every α. We integrate it over all of $\mathfrak{M}_q(Q;P)$, and apply (9.9) to each of the functions $f_{d,d_q(d)}$, which are nonnegative, bounded and 1-periodic. This gives

$$\int_{\mathfrak{N}(Q;P)\cap\mathfrak{M}_q(Q;P)} |F(\alpha)|^s\,\mathrm{d}\alpha \leqslant 2^{kJ(s-1)} \sum_{d\in\mathcal{D}_i} {d'}^{s-1-k} \int_{\mathfrak{M}_{q''}(Q;P/d)} f_{d,d_q(d)}(\beta)\,\mathrm{d}\beta.$$

Now we sum over $q \leqslant Q$. For given d, a given divisor c of d and given q'' there are at most $(k+1)^J$ numbers q with $d_q(d) = c$ and $q/(q,d^k) = q''$, and the sets $\mathfrak{M}_{q''}(Q;P/d)$ with $q'' \leqslant Q$ are pairwise disjoint, by (A2), as in the proof of Lemma 9.2. Since d has at most 2^J divisors, we obtain

$$\int_{\mathfrak{N}(Q;P)} |F(\alpha)|^s\,\mathrm{d}\alpha \leqslant 2^{kJ(s-1)}(k+1)^J \sum_{d\in\mathcal{D}_i} \sum_{c\mid d} \Big(\frac{d}{c}\Big)^{s-1-k} T(d,c). \tag{A.13}$$

If $Q > \frac{1}{2}P^{k/2}$, then $\mu = 1$, $i = 0$ and $F_1 = F$, and (A.10) shows that $|F| \leqslant V_1$ on $\mathfrak{N}(Q;P)$; so the integral is at most $T(1,1)$, and (A.13) holds again.

Conclusion. By (A.12), and since $Y = P/d$ and $y \geqslant \mu$,

$$\Big(\frac{d}{c}\Big)^{s-1-k} T(d,c) \ll \frac{P^{s-k}}{d\,c^{k-1}}\,P^{-\nu\mu+4k\gamma+\varepsilon}(\log P)^{2J}.$$

The sum of d^{-1} over $d \in \mathcal{D}_i$ is at most 1, by Lemma 3.1(c), and each d has at most 2^J divisors. Moreover $P^\mu \geqslant Q^{2/k}$, and $P^{4k\gamma+\varepsilon}(\log P)^{2J} \leqslant P^{\nu/(4k)} \leqslant Q^{\nu/k}$ for large P, because $4k\gamma+\varepsilon \leqslant \frac{\nu}{10k} + \frac{\nu}{40k^2}$ and $Q \geqslant P^{1/4}$. Thus the left hand side of (A.13) is $\ll P^{s-k}Q^{-2\nu/k}Q^{\nu/k}$, which is (9.4). □

When $z\,\mathcal{G}(z)$ is concave, the conclusion is the one that is described in Remark 11.8; the point of the proposition is that the concavity is not needed. The truncation at V_c is what makes this possible: in Corollary 9.3 the moments of the shorter sums are taken over all

of $\mathbb{T}$, and they are controlled by the values of $\mathcal{G}$ at all points of $(0,1]$, whereas $T(d,c)$ depends on its values at points $z \geqslant z_*$ only, apart from the terms with $c > 1$, which are harmless.

The theorem. As before, (Ψ°, τ°) denotes one of the two pairs of §A.4.

Theorem A.21. *Let $k \geqslant 13$. Let $n \geqslant 1$ be an integer and $c_0 < c_1 < \cdots < c_n$ real numbers with the following properties:*

(R1) $0 < c_0 < \tau^\circ$, $c_n < \frac{1}{10}$ *and* $\theta_k(2c_n) \geqslant \frac{3}{2}$;
(R2) $\inf\bigl\{v/(1-\theta_k(2v)/k) : \ c_{j-1} \leqslant v < c_j\bigr\} > c_j$ *for* $1 \leqslant j \leqslant n$;
(R3) $\Psi^\circ(c_{n-1}) - \Psi^\circ(c_n) \geqslant \frac{2}{k}$.

Put $\mathcal{Y} := k\,\Psi^\circ(c_n) + 2n$, and in the second case suppose that $\mathcal{Y} \geqslant 4k+1$.

(i) *One has $G(k) \leqslant \lfloor \mathcal{Y} \rfloor + 1$.*

(ii) *The conclusions of Theorem 2.5 hold for k and every real number $s_1 > \mathcal{Y}$, the parameters being chosen in terms of k and s_1. If $\mathcal{Y} \geqslant 4.19\,k$, then the conclusions of Theorem 2.4 hold for k and every integer $\ell > \frac{1}{2}\mathcal{Y}$; in the first case this condition is always satisfied.*

Theorems 2.4 and 2.5 are stated for $k \geqslant 37$ and for an admissible number Θ; part (ii) requires neither. For $k \geqslant 15$ the condition $\theta_k(2c_n) \geqslant \frac{3}{2}$ follows from $c_n < \frac{1}{10}$, since $\sigma_k(\frac{3}{2}) \geqslant \frac{1}{4} - \frac{3}{4k} \geqslant \frac{1}{5}$ by Lemma A.13(iii). Condition (R2) says that every step that begins in $[c_{j-1}, c_j)$ ends beyond c_j. It can be verified by means of Lemma A.14(v): if $\tilde\theta$ is as in that lemma, on an interval that contains $[2c_{j-1}, 2c_j]$, then (R2) holds for j as soon as $c_j < c_{j-1}/(1-\tilde\theta(2c_{j-1})/k)$. Thus the points c_j may be defined recursively, and each step is as long as the best chain at its starting point allows. Condition (R3) says that the last step does not increase $\mathcal{Y}$: it is equivalent to $k\,\Psi^\circ(c_n) + 2n \leqslant k\,\Psi^\circ(c_{n-1}) + 2(n-1)$.

Proof. The parameters. By (R2) there is a number $\epsilon \in (0, \frac{1}{k}]$ such that

$$\frac{v}{1-\theta_k(2v)/k} \geqslant \frac{c_j}{1-\epsilon} \qquad (c_{j-1} \leqslant v < c_j,\ 1 \leqslant j \leqslant n). \tag{A.14}$$

By Lemma A.15(i) the function Ψ° is continuous and decreasing, so that by (R3) there is exactly one number z_0 with

$$c_{n-1} \leqslant z_0 < c_n \qquad \text{and} \qquad \Psi^\circ(z_0) = \Psi^\circ(c_n) + \frac{2}{k}.$$

Let $\Theta := \theta_k(2c_0)$. Then $\frac{3}{2} \leqslant \theta_k(2c_n) \leqslant \Theta \leqslant \frac{1}{2}k$, and $\sigma_k(\Theta) \geqslant 2c_0 > 0$, because σ_k is left-continuous. So the paragraph of §A.4 on the parameters applies with this Θ: the parameters δ_0, ε_2, γ and ε are subject to (A.6), and $\theta(\cdot)$ is defined as there. We suppose that δ_0 and ε_2 are so small that

$$\delta_0 + \varepsilon_2 \leqslant \epsilon \qquad \text{and} \qquad z_0 \leqslant (1-\delta_0)\,c_n. \tag{A.15}$$

Then $\theta(z_0) \geqslant \theta_k(2c_n) \geqslant \frac{3}{2}$, by (A.7).

The steps. For $0 < v \leqslant \frac{1}{5}$ let $S(v) := v/(1-\theta(v)/((1+\varepsilon_2)k))$, the end point of the shortest step that is admitted at v in Lemma A.16(iii), and let

$$\underline{S}(z) := \inf\bigl\{S(v) : \ z \leqslant v \leqslant \tfrac{1}{5}\bigr\} \qquad (0 < z \leqslant \tfrac{1}{5}).$$

We need two facts. First,

$$\frac{z}{1-1/(2k)} \leqslant \underline{S}(z) \leqslant S(z) \leqslant 2z, \qquad \underline{S}(z) \leqslant \underline{S}(z') \leqslant \frac{z'}{z}\,\underline{S}(z) \tag{A.16}$$

for $0 < z < z' \leqslant \frac{1}{5}$. Indeed $1 \leqslant \theta(v) \leqslant \frac{1}{2}k$ and $1 + \varepsilon_2 \leqslant 2$, which gives the first chain of inequalities, and $\underline{S}$ is nondecreasing by its definition. If $\underline{S}(z) < \underline{S}(z')$, then $\underline{S}(z)$ is the infimum of $S(v)$ over $z \leqslant v < z'$; for these v one has $\theta(v) \geqslant \theta(z')$, and therefore $S(v) \geqslant (z/z')\,S(z') \geqslant (z/z')\,\underline{S}(z')$. In particular $\underline{S}$ is continuous. Secondly,

$$\underline{S}(z) \geqslant c_j \qquad \text{whenever } c_{j-1} \leqslant z \leqslant \tfrac{1}{5} \text{ and } 1 \leqslant j \leqslant n. \tag{A.17}$$

To see this, let $z \leqslant v \leqslant \frac{1}{5}$ and $v' := v/(1-\delta_0)$. Then $v' > c_{j-1} \geqslant c_0$, so that $\theta_k(2v') \leqslant \Theta$, and $\theta(v) = \theta_k(2v')$ by (A.7). For $0 \leqslant a \leqslant \frac{1}{2}$ one has

$$\frac{1-a}{1-a/(1+\varepsilon_2)} = 1 - \frac{a\,\varepsilon_2}{1+\varepsilon_2-a} \geqslant 1-\varepsilon_2;$$

with $a = \theta(v)/k$ this shows that

$$S(v) \geqslant (1-\delta_0)(1-\varepsilon_2)\,\frac{v'}{1-\theta_k(2v')/k}.$$

If $v' < c_j$, the last fraction is at least $c_j/(1-\epsilon)$ by (A.14), and if $v' \geqslant c_j$, it is at least $c_j/(1-\frac{1}{k}) \geqslant c_j/(1-\epsilon)$. As $(1-\delta_0)(1-\varepsilon_2) \geqslant 1-\epsilon$ by (A.15), we get $S(v) \geqslant c_j$, which is (A.17).

The potential. We put

$$\mathcal{G}(z) := \Psi^\circ(z) \quad (z \geqslant z_0), \qquad \mathcal{G}(z) := \max\Big\{\Psi^\circ(z_0),\ \frac{2}{k} + \mathcal{G}\big(\underline{S}(z)\big)\Big\} \quad (0 < z < z_0).$$

By the first inequality in (A.16) this defines $\mathcal{G}$ successively on the intervals $[z_0(1-\frac{1}{2k})^m, \infty)$, $m = 1, 2, \dots$, and hence on $(0, \infty)$. Let $L := \Psi^\circ(z_0) - 2$. We claim that $\mathcal{G}$ is continuous and nonincreasing, that

$$0 \leqslant \mathcal{G}(z) - \mathcal{G}(z') \leqslant L \log \frac{z'}{z} \qquad (0 < z < z' \leqslant \tfrac{1}{5}), \tag{A.18}$$

and that

$$\mathcal{G}(z) \leqslant \Psi^\circ(c_n) + \frac{2(n-j+1)}{k} \qquad (z \geqslant c_{j-1},\ 1 \leqslant j \leqslant n+1). \tag{A.19}$$

On $[z_0, \infty)$ the function $\mathcal{G}$ is continuous and decreasing. For almost all $v \in (z_0, \frac{1}{5})$ the inequality (A.9) gives $\mathfrak{g}^\circ(v) \leqslant \Psi^\circ(v) - 2 \leqslant L$, so that (A.18) holds when $z \geqslant z_0$. Now suppose that $z_1 \leqslant z_0$, and that $\mathcal{G}$ is continuous and nonincreasing on $[z_1, \infty)$ and satisfies (A.18) for $z_1 \leqslant z < z' \leqslant \frac{1}{5}$; we show that the same holds with $z_1(1-\frac{1}{2k})$ in place of z_1. For $z_1(1-\frac{1}{2k}) \leqslant z < z_0$ we have $z_1 \leqslant \underline{S}(z) \leqslant 2z < \frac{1}{5}$. So, by (A.16), the function $z \mapsto \mathcal{G}(\underline{S}(z))$ is continuous and nonincreasing on this range, and for $z < z'$ in it

$$\mathcal{G}(\underline{S}(z)) - \mathcal{G}(\underline{S}(z')) \leqslant L \log \frac{\underline{S}(z')}{\underline{S}(z)} \leqslant L \log \frac{z'}{z}.$$

These properties are preserved when $\frac{2}{k}$ is added and the maximum with a constant is taken. As $z \to z_0$ from below, $\mathcal{G}(z)$ tends to $\max\{\Psi^\circ(z_0), \frac{2}{k} + \Psi^\circ(\underline{S}(z_0))\}$, and this is $\Psi^\circ(z_0) = \mathcal{G}(z_0)$, because $\underline{S}(z_0) \geqslant c_n$ by (A.17) and $\frac{2}{k} + \Psi^\circ(c_n) = \Psi^\circ(z_0)$. So $\mathcal{G}$ is

continuous at z_0, and the claim follows for $[z_1(1-\frac{1}{2k}),\infty)$; for $z<z_0<z'$ the two bounds (A.18) for the pairs z,z_0 and z_0,z' are added. By induction, $\mathcal{G}$ is continuous and nonincreasing on $(0,\infty)$, and (A.18) holds. Finally (A.19) is proved by induction on $n+1-j$. For $j=n+1$ it holds because $c_n>z_0$. Let $j\leqslant n$ and $z\geqslant c_{j-1}$. If $z\geqslant z_0$, then $\mathcal{G}(z)\leqslant\Psi^\circ(z_0)=\Psi^\circ(c_n)+\frac{2}{k}$. If $z<z_0$, then $\underline{S}(z)\geqslant c_j$ by (A.17), so that $\mathcal{G}(\underline{S}(z))\leqslant\Psi^\circ(c_n)+\frac{2(n-j)}{k}$ by the case $j+1$, and (A.19) follows from the definition of $\mathcal{G}(z)$. For $j=1$ and $z=c_0$ we obtain

$$k\,\mathcal{G}(c_0)\leqslant k\,\Psi^\circ(c_n)+2n=\mathcal{Y}. \tag{A.20}$$

The induction. We apply Lemma A.16 with $u:=c_0$ and with the number z_0. We have $c_0\leqslant c_{n-1}\leqslant z_0<c_n<\frac{1}{10}$ and $\theta(z_0)\geqslant\frac{3}{2}$, and (G1) holds by the definition of $\mathcal{G}$. By what has been shown, $\mathcal{G}$ is nonincreasing, and it is locally Lipschitz continuous, because of (A.18) and since it agrees with Ψ° on $[z_0,\infty)$. Moreover

$$-z\,\mathcal{G}'(z)\leqslant L=\Psi^\circ(z_0)-2\leqslant\mathcal{G}(z)-2\qquad\text{for almost all } z\in(0,z_0),$$

by (A.18) and since $\mathcal{G}\geqslant\Psi^\circ(z_0)$ on $(0,z_0]$. This is (G2). For (G3) let $0<z<z_0$ and $z'\geqslant S(z)$. Then $z'\geqslant\underline{S}(z)$, and as $\mathcal{G}$ is nonincreasing, the definition of $\mathcal{G}(z)$ gives

$$\mathcal{G}(z')\leqslant\mathcal{G}\big(\underline{S}(z)\big)\leqslant\mathcal{G}(z)-\frac{2}{k}.$$

So Lemma A.16 applies. With $\Phi(x,\lambda):=k\lambda\,\mathcal{G}(\max\{\lambda,\gamma c_0\}/x)$ the assertion of Proposition 8.2 holds, and for $\gamma\leqslant x\leqslant 1$ the function $\lambda\mapsto\Phi(x,\lambda)-2k\lambda$ is nondecreasing on $[0,\frac{1}{5}x]$.

The circle method. Let s be a real number with $s>\mathcal{Y}$. Then $s>4k+1$: in the first case $\mathcal{Y}>k\,\Psi(\frac{1}{10})>4.19\,k$, and in the second case this is the hypothesis. Fix z_* with $c_0<z_*<\tau^\circ$. By (A.20), $k\,\mathcal{G}(z_*)\leqslant k\,\mathcal{G}(c_0)\leqslant\mathcal{Y}<s$. Thus the hypotheses of Proposition A.20 on $\mathcal{G}$, s and z_* hold, with $u=c_0$ and with $\nu\geqslant(s-\mathcal{Y})\,c_0$. The potential depends on δ_0 and ε_2, but not on the parameters that are still to be chosen. We take $\gamma:=\min\{\frac{1}{100k},\nu/(40k^2)\}$ and $\varepsilon:=\min\{\frac{1}{4}\delta_0\sigma_k(\Theta),\gamma/(16k^2s)\}$, then J so large that $\eta\leqslant(\tau^\circ-z_*)/(8k)$, in addition to the requirements of §3 and of Lemmas A.15 and A.16, and $\omega:=\eta/(64ks)$. By Proposition A.20 the estimate (9.7) is available. The part of the proof of Theorem 2.3 that is headed 'The circle method' in §9.5 uses (9.7) and the inequality $s\geqslant 4k+1$ only, and it shows that $G(k)\leqslant s$ when s is an integer. This proves (i). The proofs of Theorems 2.4 and 2.5 in §§9.6 and 9.7 depend on the choice of the parameters and on the potential only through (9.7), for $s=2\ell$ and for $s=s_1$, and, in the case of Theorem 2.4, through the inequality $2\ell>4.19\,k$, which holds when $2\ell>\mathcal{Y}\geqslant 4.19\,k$; in the first case $\mathcal{Y}>4.19\,k$, as we have seen. These proofs do not use the hypothesis $k\geqslant 37$, and they are the same in both cases. As $2\ell>\mathcal{Y}$ and $s_1>\mathcal{Y}$, what has just been shown applies with $s=2\ell$ and with $s=s_1$. This gives (ii). □

The number $k\,\mathcal{G}(c_0)$ in this proof cannot be lowered by another choice of the potential, as long as the potential is to be nonincreasing, to agree with Ψ° from some point z_0 on, and to satisfy (G3) below z_0: the function $\mathcal{G}$ is the least one with these properties. In comparison with Theorem A.17 there are two gains. The potential of that theorem decreases along a step that begins at v by an amount that is formed with the values of $\theta(\cdot)$ on the whole step, and these are smaller than $\theta(v)$; summed over the steps between τ_k and z_k, this costs roughly $\log\big(\Theta_k/\theta_k(2z_k)\big)$ in $\mathcal{Y}_k(\Theta_k)$. Secondly, the sequence may end

at any point below $\frac{1}{10}$, whereas z_k is prescribed; for small k the number z_k is considerably smaller than the best end point (for $k = 37$ one has $z_{37} = 0.0768$, and the sequence that is used below ends at $c_n = 0.0940\ldots$).

A.7. **Numerical values.** *The computation.* The sequences of Theorem A.21 are constructed by the program of §10.4, with directed rounding, as follows. Let

$$D_k := \min\big\{\lceil 1.45\sqrt{k}\rceil + 7,\ \lfloor \tfrac{1}{2}k\rfloor - 1\big\}.$$

The chains that we use are $\mathcal{I}_0, \mathcal{I}_1$ and $\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2$, and those that continue one of these two by $\mathcal{T}_{D_1}, \mathcal{T}_{D_1+1}, \ldots, \mathcal{T}_{D_2}$, where $D_1 \in \{3, 4\}$ and $D_1 \leqslant D_2 \leqslant D_k$; in addition $\mathcal{I}_3$ may be inserted between $\mathcal{T}_3$ and $\mathcal{T}_4$, or may follow $\mathcal{T}_3$ as the last interval. Fix such a chain, let $p := \frac{1}{5}$ if it contains no tent and $p := 1/(D_2(D_2+1))$ otherwise, so that its tents are defined for $0 < t \leqslant p$, and let $\tilde{\theta}$ be the function of Lemma A.14(iii) that belongs to the chain. On $(0, p]$ this function is concave and nonincreasing, because the only condition there that does not involve ϑ is $\frac{3}{2} \leqslant 2 - 2t$. The condition on θ and the last interval of the chain shows that $\tilde{\theta}(t)$ is at most the right end point of that interval, which is smaller than $\max\{5, D_2 + 1\} \leqslant \frac{1}{2}k$; moreover $D_2 \leqslant k - 1$, as the tents require. Hence $\tilde{\theta} \leqslant \theta_k$ on $(0, p]$: where $\tilde{\theta}(t) > 1$ this is Lemma A.14(iii), and elsewhere it is trivial. For an integer $D \geqslant 3$ let $\mathcal{K}_D$ be the set of those among these chains whose tents $\mathcal{T}_d$ have $d \leqslant D$, and let $\mathcal{K}_2$ be the set of those without tents. Put

$$p_D := \frac{1}{D(D+1)} \quad (D \geqslant 3), \qquad p_2 := \frac{1}{5},$$

and for $2 \leqslant D \leqslant D_k$ and $0 < t \leqslant p_D$ let $\tilde{\theta}_D(t)$ be the largest of the numbers $\tilde{\theta}(t)$ for the chains in $\mathcal{K}_D$, or 1 if these are smaller. Parts (iv) and (v) of Lemma A.14 apply to each of these chains with $\mathcal{I} = (0, p_D]$.

The starting point c_0 is a floating point number below τ°, within two units in the last place. Suppose that $c = c_{j-1}$ has been found. Let D be the largest integer with $2 \leqslant D \leqslant D_k$ and $2c \leqslant p_D$. By Lemma A.14(v), and since $\theta_k \geqslant 1$,

$$\frac{v}{1 - \theta_k(2v)/k} \geqslant \frac{c}{1 - \tilde{\theta}_D(2c)/k} \qquad (c \leqslant v \leqslant \tfrac{1}{2}p_D).$$

If the right hand side exceeds $\frac{1}{2}p_D$, the range of v is continued beyond $\frac{1}{2}p_D$ by means of the chains in $\mathcal{K}_{D-1}$, which are used with their values at p_D: in the same way,

$$\frac{v}{1 - \theta_k(2v)/k} \geqslant \frac{\frac{1}{2}p_D}{1 - \tilde{\theta}_{D-1}(p_D)/k} \qquad (\tfrac{1}{2}p_D \leqslant v \leqslant \tfrac{1}{2}p_{D-1});$$

and so on, if necessary. The number c_j is a floating point number below the least of the bounds obtained in this way, so that (R2) holds for j. The second bound matters because θ_k jumps downwards at the points p_D (for the limit function (A.5) and $D \geqslant 5$ the jump at p_D is $\frac{D-1}{D+1}$): a step that begins shortly before $\frac{1}{2}p_D$ would otherwise end beyond the end points of steps that begin after it. The sequence is continued as long as $c_j < \frac{1}{10}$. Among the indices $n \geqslant 1$ for which (R3) holds the program takes the one for which its upper bound for $k\,\Psi^\circ(c_n) + 2n$ is least. We denote this upper bound by

$$\mathcal{Y}^\flat(k) \quad (17 \leqslant k \leqslant 5000) \qquad \text{and} \qquad \mathcal{Y}^{\mathrm{VW}}(k) \quad (13 \leqslant k \leqslant 20)$$

in the first and in the second case, respectively, and put

$$G^\flat(k) := \lfloor \mathcal{Y}^\flat(k) \rfloor + 1, \qquad \ell^\flat(k) := \lfloor \tfrac{1}{2}\mathcal{Y}^\flat(k) \rfloor + 1,$$
$$G^{\mathrm{VW}}(k) := \lfloor \mathcal{Y}^{\mathrm{VW}}(k) \rfloor + 1, \qquad \ell^{\mathrm{VW}}(k) := \lfloor \tfrac{1}{2}\mathcal{Y}^{\mathrm{VW}}(k) \rfloor + 1.$$

In the second case the numbers $\Psi_k^{\mathrm{VW}}(c_j)$ are computed exactly, as fractions, the points c_j being floating point numbers. In the first case the logarithms in (R3) and in $\Psi(c_n)$ are evaluated with directed rounding, and both are certified a second time in decimal arithmetic with 40 digits; see §A.11.

Proposition A.22. (i) *Let $17 \leqslant k \leqslant 5000$. Then $G(k) \leqslant G^\flat(k)$. If k is not a power of 2, then $G^+(k) \leqslant \ell^\flat(k)$. The conclusions of Theorem 2.5 hold for every real number $s_1 > \mathcal{Y}^\flat(k)$.*

(ii) *One has $G^\flat(k) < 6.34\,k$ for $37 \leqslant k \leqslant 5000$, and $G^\flat(k) < 6.3\,k$ for $47 \leqslant k \leqslant 5000$.*

(iii) *Let $37 \leqslant k \leqslant 5000$. Then $G^\flat(k) < C_0^\star k + 0.32\sqrt{k}$. Moreover $G^\flat(k) < C_0^\star k$ if $k \geqslant 56$, and $G^\flat(k) < C_0^\star k - \frac{1}{4}\sqrt{k}$ if $k \geqslant 78$. Also $\mathcal{Y}^\flat(k) < C_0^\star k + 0.23\sqrt{k}$, with $\mathcal{Y}^\flat(k) < C_0^\star k$ if $k \geqslant 50$, and $\mathcal{Y}^\flat(k) < C_0^\star k - \frac{1}{4}\sqrt{k}$ if $k \geqslant 69$.*

(iv) *Let $37 \leqslant k \leqslant 5000$. Then $\ell^\flat(k) < \frac{1}{2}C_0^\star k + 0.24\sqrt{k}$, and $\ell^\flat(k) < \frac{1}{2}C_0^\star k$ if $k \geqslant 64$.*

Proof. (i) By construction there is a sequence that satisfies (R1)–(R3) for (Ψ, τ_k) and has $k\,\Psi(c_n) + 2n \leqslant \mathcal{Y}^\flat(k)$; for $k \geqslant 15$ the condition $\theta_k(2c_n) \geqslant \frac{3}{2}$ holds of itself. So the assertions follow from Theorem A.21, since $2\ell^\flat(k) > \mathcal{Y}^\flat(k)$.

(ii)–(iv) We have computed $\mathcal{Y}^\flat(k)$ for each k. The assertions are inequalities between rational numbers and rigorous bounds for $C_0^\star$, and they were checked, in exact arithmetic, for each k in the ranges stated; the inequalities with the constants 6.3, $C_0^\star$ and $C_0^\star - \frac{1}{4}k^{-1/2}$ fail for $k = 46$, 55, 77, 49, 68 and 63, respectively, in the order in which they are stated. For $37 \leqslant k \leqslant 5000$ the largest values of $G^\flat(k)/k$, of $(G^\flat(k) - C_0^\star k)/\sqrt{k}$ and of $(\ell^\flat(k) - \frac{1}{2}C_0^\star k)/\sqrt{k}$ are $\frac{19}{3}$, $0.3197\ldots$ and $0.2399\ldots$, all attained at $k = 39$, where $G^\flat(39) = 247$ and $\ell^\flat(39) = 124$; the largest value of $(\mathcal{Y}^\flat(k) - C_0^\star k)/\sqrt{k}$ is $0.2266\ldots$, attained at $k = 37$. □

For example, $G(21) \leqslant 135$, $G(37) \leqslant 234$, $G(50) \leqslant 314$, $G(100) \leqslant 624$, $G(1000) \leqslant 6248$ and $G(5000) \leqslant 31323$ (Table 6), and $\ell^\flat(k) = 117$, 157, 312 and 3124 for $k = 37$, 50, 100 and 1000. For $21 \leqslant k \leqslant 36$ the numbers $G^\flat(k)$ are smaller than the bounds of (1.1) by 18 to 53, and the largest value of $G^\flat(k)/k$ in this range is $\frac{135}{21} = 6.4285\ldots$, for $k = 21$. For $37 \leqslant k \leqslant 5000$ the number n of steps lies between k and $1.04\,k$, and the end points c_n lie between 0.0932 and 0.09996; for $21 \leqslant k \leqslant 36$ they lie between 0.0846 and 0.0938. There is no subdivision on which the result depends, and the effect of the rounding on $\mathcal{Y}^\flat(k)$ is of the order of 10^{-8} at most; so the integers $G^\flat(k)$ are those that the chains give. In a few cases $\mathcal{Y}^\flat(k)$ lies just below an integer, or just above one: for $37 \leqslant k \leqslant 5000$ its fractional part exceeds 0.999 for seven values of k, the largest being $0.99969\ldots$ for $k = 513$, and it is smaller than 0.001 for ten values of k, the smallest being $0.00005\ldots$ for $k = 2885$; for $21 \leqslant k \leqslant 36$ the numbers $\mathcal{Y}^\flat(k)$ lie below the next integer by at least 0.10. We have not carried the recursion beyond $k = 5000$; for larger k the bounds of this appendix rest on Theorem A.17 and Proposition A.19.

TABLE 6. The numbers $\mathcal{Y}^\flat(k)$, $G^\flat(k)$ and $\ell^\flat(k)$ of Proposition A.22, with the number n of steps of the sequence that is used: one has $G(k) \leqslant G^\flat(k)$, and $G^+(k) \leqslant \ell^\flat(k)$ when k is not a power of 2. The last column contains the bound for $G(k)$ known before: that of (1.1) for $k \leqslant 36$, and that of Table 3 for $k \geqslant 37$.

k	n	$\mathcal{Y}^\flat(k)$	$G^\flat$	$\ell^\flat$	before	k	n	$\mathcal{Y}^\flat(k)$	$G^\flat$	$\ell^\flat$	before
21	21	134.0179	135	68	153	41	42	258.4626	259	130	297
22	22	140.2882	141	71	161	50	51	313.9727	314	157	353
23	23	146.6033	147	74	169	56	56	350.9112	351	176	398
24	25	152.8904	153	77	178	67	67	418.7500	419	210	481
25	26	159.2111	160	80	186	78	78	486.9647	487	244	566
26	27	165.5140	166	83	194	81	81	505.5671	506	253	590
27	28	171.7246	172	86	203	100	100	623.5525	624	312	709
28	29	177.8940	178	89	211	150	151	933.7193	934	467	1062
29	30	184.1304	185	93	220	174	175	1083.1920	1084	542	1241
30	31	190.3041	191	96	229	200	201	1245.1492	1246	623	1407
31	32	196.5224	197	99	237	212	214	1320.0039	1321	661	1483
32	33	202.7877	203	102	246	250	252	1556.6930	1557	779	1724
33	34	209.0098	210	105	254	330	334	2056.2439	2057	1029	2231
34	35	215.2477	216	108	263	500	509	3118.9115	3119	1560	3309
35	36	221.4729	222	111	272	700	715	4370.0634	4371	2186	4578
36	37	227.6492	228	114	281	927	948	5790.3206	5791	2896	6025
37	38	233.8173	234	117	273	1000	1024	6247.6151	6248	3124	6492
38	39	239.9664	240	120	279	2000	2057	12511.3081	12512	6256	12902
39	40	246.0702	247	124	285	3000	3093	18780.3738	18781	9391	19309
40	41	252.3324	253	127	291	5000	5167	31322.5609	31323	15662	32116

Proposition A.23. *For* $13 \leqslant k \leqslant 20$ *one has* $G(k) \leqslant G^{\mathrm{VW}}(k)$*, where the numbers* $G^{\mathrm{VW}}(k)$ *are those of Table* 7*; that is,*

$$G(13) \leqslant 80, \quad G(14) \leqslant 86, \quad G(15) \leqslant 93, \quad G(16) \leqslant 99,$$
$$G(17) \leqslant 106, \quad G(18) \leqslant 112, \quad G(19) \leqslant 119, \quad G(20) \leqslant 125.$$

Moreover, the conclusions of Theorem 2.4 *hold for every integer* $\ell \geqslant \ell^{\mathrm{VW}}(k)$*, and those of Theorem* 2.5 *for every real number* $s_1 > \mathcal{Y}^{\mathrm{VW}}(k)$*. In particular* $G^+(k) \leqslant \ell^{\mathrm{VW}}(k)$ *when* $k \neq 16$*; the numbers* $\ell^{\mathrm{VW}}(k)$ *are shown in Table* 7*.*

Proof. This is Theorem A.21 in the second case. The program verifies (V1)–(V3) and (R3) exactly; it confirms (R1), including the inequality $\theta_k(2c_n) \geqslant \frac{3}{2}$, for which it uses the value of a chain at $2c_n$; the points c_j are constructed in such a way that (R2) holds; and the numbers $\mathcal{Y}^{\mathrm{VW}}(k)$ exceed $4.19\,k$, and hence $4k+1$. Finally $2\ell^{\mathrm{VW}}(k) > \mathcal{Y}^{\mathrm{VW}}(k)$. □

The tables of [34] enter in two places: through the function Ψ_k^{VW}, which is smaller than Ψ in the range that matters, and through the exponent τ_k^{VW}, which is larger than τ_k; the first of the two is the more important one. The numbers $\mathcal{Y}^{\mathrm{VW}}(k)$ lie below the next

TABLE 7. The bounds $G(k) \leqslant G^{\mathrm{VW}}(k)$ and $G^+(k) \leqslant \ell^{\mathrm{VW}}(k)$ of Proposition A.23, with the number T_k of §A.4, the number n of steps and the end point c_n of the sequence that is used; for $k = 16$ the number $\ell^{\mathrm{VW}}(16)$ is admissible in Theorem 2.4(ii). The columns headed 'before' contain the bounds known before: for $G(k)$ those of [42, Theorem 1.1] for $k = 13$ and of [6, Theorem 1.3] for $k \geqslant 14$, and for $G^+(k)$ those of [6, Theorem 9.1], with the number of [6, Theorem 9.2] for $k = 16$.

k	T_k	n	c_n	$\mathcal{Y}^{\mathrm{VW}}(k)$	$G^{\mathrm{VW}}(k)$	before	$\ell^{\mathrm{VW}}(k)$	before
13	104.95	10	0.0583	79.0746	80	81	40	44
14	114.19	11	0.0590	85.7394	86	89	43	45
15	123.40	12	0.0602	92.2842	93	97	47	49
16	132.60	14	0.0696	98.7888	99	105	50	53
17	141.78	15	0.0701	105.2755	106	113	53	57
18	150.95	16	0.0704	111.8246	112	121	56	61
19	160.07	17	0.0707	118.3697	119	129	60	65
20	169.18	19	0.0788	124.7279	125	137	63	69

integer by at least 0.17. For $k = 13$ the number $\mathcal{Y}^{\mathrm{VW}}(13) = 79.0745\ldots$ exceeds 79 only slightly, so that the bound $G(13) \leqslant 79$ is missed narrowly. Without the tables, that is, in the first case, the recursion gives $G^\flat(k) = 109$, 116, 122 and 128 and $\ell^\flat(k) = 55$, 58, 61 and 64 for $17 \leqslant k \leqslant 20$; the numbers $\ell^{\mathrm{VW}}(k)$ are smaller by 1 or 2.

A.8. **Fractional parts.** Theorem A.5 rests on Theorem A.17(iii), and not on the recursion of §A.6. The reason is that the proof of Theorem 2.7 uses the potential through Corollary 8.3, with numbers $r = 2s_0$ that are smaller than $k\,\mathcal{G}(\tau_k)$: it requires an upper bound for $k\,z\,\mathcal{G}(z) - r\,z$ for all $z \in (0,1]$, and thus bounds for the potential at all points, whereas Theorem A.21 controls it along a single sequence of steps. For the potential of Theorem A.17 these bounds are supplied by Lemma 8.1(v). The recursive potential could be bounded at all points by means of several sequences; one would expect to gain a few units in $2s_0$, as for $G(k)$, and we have not carried this out.

The numbers $B^\natural(k)$ and $\Theta^\natural(k)$. Let $k \geqslant 37$. We bound $\mathcal{Y}_k(\Theta)$ from above, for a suitable Θ, by means of the chains that are described at the beginning of §A.7; for $k \geqslant 37$ one has $D_k = \lceil 1.45\sqrt{k} \rceil + 7$. Recall that each of these chains has $\tilde\theta \leqslant \theta_k$ on $(0,p]$. Let $0 < t' < t'' \leqslant p$, and suppose that $\tilde\theta(t'') > 1$. The linear function $c - c't$ that agrees with $\tilde\theta$ at t' and t'' is a positive minorant of $\tilde\theta$, and hence of θ_k, on $[t', t'']$, and

$$\int_{t'}^{t''} \frac{\mathrm{d}t}{t\,\theta_k(t)} \leqslant \int_{t'}^{t''} \frac{\mathrm{d}t}{t\,(c - c't)} = \frac{1}{c}\,\log\frac{t''\,\tilde\theta(t')}{t'\,\tilde\theta(t'')}.$$

The program of §10.4 divides $[2\tau_k, 2z_k]$ into subintervals $[t', t'']$, and on each of them it uses one chain in this way, chosen so as to make the right hand side small. The points of division are the numbers $p_D = 1/(D(D+1))$ with $3 \leqslant D \leqslant D_k$ that exceed $2\tau_k$; the range $(\frac{1}{12}, 2z_k]$ is divided further into 64 parts, each range between two consecutive numbers p_D into 32 parts, and the remaining range, from $2\tau_k$ to the smallest of these numbers, into 256 parts, in geometric progression in each case. Finally the numbers $\hat t$

and $\check{t}$ of Lemma A.18, which are defined by the same formulae for all k, are added as points of division when they lie in $(\frac{1}{12}, 2z_k)$ and in $(\frac{1}{20}, \frac{1}{12})$, respectively. Near $\hat{t}$ the best chain changes from $\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2$ to $\mathcal{I}_0, \mathcal{I}_1$, and near $\check{t}$ from a chain that ends with $\mathcal{T}_3$ to one that ends with $\mathcal{I}_3$; there the largest of the functions $\tilde{\theta}$ has a convex corner, which a chord of a single chain cannot follow. On each subinterval the chains that are considered are those whose tents are defined at its right end point. All chains are considered on the last subinterval, which contains $2\tau_k$, and the program computes a lower bound $\Theta^\natural(k)$ for the largest of the numbers $\tilde{\theta}(2\tau_k)$ over all chains. Then $\Theta^\natural(k) \leqslant \theta_k(2\tau_k) = \Theta_k$. The chords are formed with the numbers $\min\{\Theta^\natural(k), \tilde{\theta}(t)\}$ in place of $\tilde{\theta}(t)$; as the minimum of two concave functions is concave, they are minorants of $\min\{\Theta^\natural(k), \theta_k\}$, and the program obtains an upper bound $B^\natural(k) - \frac{1}{k}$ for

$$\Psi(z_k) + 2\int_{2\tau_k}^{2z_k} \frac{\mathrm{d}t}{t\,\min\{\Theta^\natural(k), \theta_k(t)\}} - \frac{1}{k}\log\frac{z_k}{\tau_k} = \frac{1}{k}\,\mathcal{Y}_k\big(\Theta^\natural(k)\big).$$

Therefore

$$\tfrac{9}{4} \leqslant \Theta^\natural(k) \leqslant \Theta_k \qquad \text{and} \qquad \mathcal{Y}_k\big(\Theta^\natural(k)\big) \leqslant \mathcal{Y}^\natural(k) := k\,B^\natural(k) - 1 \qquad (37 \leqslant k \leqslant 5000); \tag{A.21}$$

the inequality $\Theta^\natural(k) \geqslant \frac{9}{4}$ is seen from the values computed, the smallest of which is $\Theta^\natural(37) = 5.6507\ldots$. Any subdivision and any choice of chains give valid bounds. When the fractional part of $k\,B^\natural(k)$ is smaller than 0.01, which happens for 45 values of k, the computation is repeated with a subdivision eight times as fine, and $B^\natural(k)$ denotes the smaller of the two numbers obtained. The same construction applies to every $k \geqslant 37$, and (A.21) holds for the larger values of k in Table 8 as well. The computations are carried out with directed rounding, so that the numbers obtained are rigorous bounds, provided that the logarithm of the computer's library is accurate to within four units in the last place; see §A.11.

The exponents. Let $\chi_0 = 1 + \log(9.027901/11)$, as in §11.5. For $37 \leqslant k \leqslant 5000$ let $\Theta := \Theta^\natural(k)$ and $\Theta' := \Theta/(1 - \Theta/(2k))$, as in Theorem A.17(iii), so that $k/\Theta' = k/\Theta - \frac{1}{2}$, and put

$$b^\natural(k) := \frac{2}{11\,\Theta'}, \qquad E^\natural(k) := \frac{2s_0^\natural(k) + 4.5139505\,k}{1 - b^\natural(k)},$$

where $s_0^\natural(k)$ is a natural number $s_0 \geqslant k$ with $2s_0 > \mathcal{Y}^\natural(k) - 2\chi_0\,k/\Theta'$, namely the least integer that exceeds one half of the upper bound that the program computes for the right hand side. Note that $9.027901\,\Theta'\,b^\natural(k)/2 = 9.027901/11$. So by (A.21) and Theorem A.17(iii), applied with $\Theta = \Theta^\natural(k)$ and $b = b^\natural(k)$, the bound (11.5) holds for every real number $\rho < 1/E^\natural(k)$.

Proposition A.24. (i) *Let $37 \leqslant k \leqslant 5000$. Then $E^\natural(k) < 10.7961\,k + \sqrt{k}$, $E^\natural(k) < 11\,k$ and $E^\natural(k) < k(\log k + 8.02113) - 25$. Moreover $E^\natural(k) < 10.7961\,k$ for $k \geqslant 135$.*

(ii) *Let $k \geqslant 5000$, and let Θ be as in Lemma* A.18. *Put $\Theta' := \Theta/(1 - \Theta/(2k))$ and $b := 2/(11\,\Theta')$, and let s_0 be the least integer with $2s_0 > \mathcal{Y}_k(\Theta) - 2\chi_0\,k/\Theta'$. Then $s_0 \geqslant k$, and*

$$\frac{2s_0 + 4.5139505\,k}{1 - b} < 10.7961\,k - 0.62\sqrt{k} + \log k + 24.$$

Proof. (i) We have evaluated an upper bound for $E^\natural(k)$ for each of these k from the numbers $B^\natural(k)$ and $\Theta^\natural(k)$, and the inequalities were checked for these bounds. In all cases $s_0^\natural(k) > 3k$. The largest value of $(E^\natural(k) - 10.7961\,k)/\sqrt{k}$ is $0.9910\ldots$, attained at $k = 45$, and the largest value of $E^\natural(k)/k$ is $10.9474\ldots$, at $k = 37$. One has $10.7961\,k - E^\natural(k) > 0.03$ for $135 \leqslant k \leqslant 5000$, whereas the computed value of this difference is negative for $k = 134$; and $11k - E^\natural(k) > 1.9$ and $k(\log k + 8.02113) - E^\natural(k) > 25.3$ for all k.

(ii) Write $n := D_\star$. Then $\Theta = n/(1+n/k)$, so that $k/\Theta = k/n + 1$ and $k/\Theta' = k/n + \frac{1}{2}$. In the proof of Proposition A.19 we saw that $n \geqslant 67$, that $k/n < n + 8 + \frac{39}{n}$, and that $\sqrt{k} - 4.164 < n < \sqrt{k} - 3$. Hence

$$\Theta' > \Theta > n - 1 > \sqrt{k} - 5.17, \qquad b < \frac{2}{11 \cdot 66} < 0.00276, \qquad \frac{k}{\Theta'} < \frac{k}{\Theta} < \sqrt{k} + 6.59.$$

We have $s_0 \geqslant k$, because $\mathcal{Y}_k(\Theta) \geqslant k\,\Psi(z_k) > 4k$ and $2\chi_0 k/\Theta' < 2k$, and

$$2s_0 \leqslant \mathcal{Y}_k(\Theta) + 2 - 2\chi_0\,\frac{k}{\Theta'}.$$

By Proposition A.19, and as $C_0^\star + 4.5139505 < 10.7961$, the number $\mathcal{Z} := 2s_0 + 4.5139505\,k$ satisfies

$$\mathcal{Z} < 10.7961\,k - c_\star\sqrt{k} + \log k + 20.84 - 2\chi_0\,\frac{k}{\Theta'}.$$

Now $1/(1-b) < 1 + 1.0028\,b$ and $b \cdot 10.7961\,k = \frac{2\cdot 10.7961}{11} \cdot \frac{k}{\Theta'}$, while

$$b\,(\log k + 20.84) < \frac{2}{11} \cdot \frac{\log k + 20.84}{\sqrt{k} - 5.17} < 0.09,$$

because the last quotient decreases for $k \geqslant 5000$. It follows that

$$\frac{\mathcal{Z}}{1-b} < \mathcal{Z} + 1.0028\,b\,\mathcal{Z} < 10.7961\,k - c_\star\sqrt{k} + \log k + 20.94 + \Big(\frac{1.0028 \cdot 2 \cdot 10.7961}{11} - 2\chi_0\Big)\frac{k}{\Theta'}.$$

The bracket is smaller than 0.3636, and $0.3636\,(\sqrt{k} + 6.59) < 0.3636\sqrt{k} + 2.40$. Since $c_\star - 0.3636 > 0.62$ and $20.94 + 2.40 < 24$, this proves (ii). □

The exponent of [9] is $1/(k(\log k + 8.02113))$. By Proposition A.24(i) it is smaller than $1/E^\natural(k)$ for $37 \leqslant k \leqslant 5000$, and for $k > 5000$ it is smaller than $1/(10.7961\,k)$. The remarks that follow Corollary 11.2 apply with $C_0^\star$ in place of C_0. Indeed, let Θ be as in Lemma A.18 and $b_0 = 1/\log k$. The proof of Proposition A.19 shows that, in the limit $\delta_0, \varepsilon_2 \to 0$, one has $\mathcal{G}(u_\Theta) \leqslant C_0^\star + O(k^{-1}\log k)$, while $a_*(b_0) = O(k^{-1/2}\log k)$; so $2s_0 = (C_0^\star + o(1))k$ is admissible in Theorem 11.1. For instance, the number ξ_1 now tends to $1/(1 + 2C_0^\star/9.027901) = 0.4181\ldots$. The conclusions of §11.4 remain valid as well. The second condition in (11.2) becomes $t/(kb) > 2C_0^\star\xi/(1-\xi)$, and this is stronger than the first one exactly when $\xi > 0.4181\ldots$; so Theorem 11.1 does not reduce the number of variables. In the proof of Proposition 11.9(i) the inequality $2/\theta^*(2u) < 1/(1-u)$ for $u < \frac{1}{10}$ persists, because the function θ^* that belongs to $\varsigma^\star$ is at least the one that belongs to ς; and the leading constant is not affected by the exponent τ_k. The numerical values quoted in §11.4 refer to C_0 and change accordingly.

TABLE 8. The number $E^\natural(k)$ of §A.8: one has $\min_{n\leqslant N}\|\alpha n^k\| < N^{-\rho}$ for $\rho < 1/E^\natural(k)$ and large N. The last column contains the number $E^\sharp(k)$ of Table 4. For $k > 5000$ the numbers are defined and computed in the same way as for $k \leqslant 5000$, with directed rounding, and (11.5) holds for $\rho < 1/E^\natural(k)$ in these cases as well.

k	$\Theta^\natural(k)$	$b^\natural(k)$	$2s_0^\natural(k)$	$E^\natural(k)$	$E^\sharp(k)$
37	5.651	0.0297	226	405.05	450.43
46	6.423	0.0263	282	502.88	544.05
50	6.788	0.0250	306	545.31	587.90
100	10.560	0.0163	612	1081.02	1174.94
200	15.614	0.0112	1228	2154.90	2330.23
500	26.006	0.0068	3092	5385.65	5582.86
1000	37.615	0.0047	6210	10775.05	11022.78
2000	54.460	0.0033	12458	21556.89	21950.76
5000	87.573	0.0021	31236	53916.71	54717.71
10^4	125.397	0.0014	62572	107866.93	109295.89
10^5	401.897	0.0005	627394	1079276.33	1091049.59
10^6	1276.717	0.0001	6279514	10795000.84	10905541.15

A.9. **Proofs of the theorems.**

Proof of Theorems A.1 and A.2. For $37 \leqslant k \leqslant 5000$ we have $G(k) \leqslant G^\flat(k)$, by Proposition A.22(i), and all assertions are contained in parts (ii) and (iii) of that proposition. Let $k \geqslant 5000$. By Proposition A.19, $G(k) \leqslant C_0^\star k - 0.9856\sqrt{k} + \log k + 20$. Now $0.7356\sqrt{k} > \log k + 20$, since this holds for $k = 5000$ and $0.7356\sqrt{k} - \log k$ increases with k. Hence $G(k) < C_0^\star k - \frac{1}{4}\sqrt{k} < 6.2822\,k$. It remains to prove the first assertion of Theorem A.2 for $k \leqslant 36$. For $21 \leqslant k \leqslant 36$ it follows from Proposition A.22(i) and Table 6, where $G^\flat(k) \leqslant \frac{135}{21}k < 6.43\,k$, and for $13 \leqslant k \leqslant 20$ from Proposition A.23, where one has $G^{\mathrm{VW}}(k) \leqslant \frac{119}{19}k$. For $k \leqslant 8$, (1.1) gives $G(k) \leqslant \lceil k(\log k + 4.20032)\rceil \leqslant \frac{51}{8}k < 6.43\,k$, the largest value of the ratio being attained at $k = 8$. Finally $G(9) \leqslant 47$, $G(10) \leqslant 55$, $G(11) \leqslant 63$ and $G(12) \leqslant 72$ by [42, Theorem 1.1]. □

Proof of Theorem A.3. (i) For $37 \leqslant k \leqslant 5000$ this is Proposition A.22(i) and (iv); recall that 64 is a power of 2. Let $k \geqslant 5000$, let Θ be as in Lemma A.18, and $\ell := \lfloor \frac{1}{2}\mathcal{Y}_k(\Theta)\rfloor + 1$. By Theorem A.17(ii) and Theorem 2.4(i), $G^+(k) \leqslant \ell$ when k is not a power of 2, and by Proposition A.19

$$\ell \leqslant \tfrac{1}{2}\mathcal{Y}_k(\Theta) + 1 < \tfrac{1}{2}C_0^\star k - 0.4928\sqrt{k} + \tfrac{1}{2}\log k + 11 < \tfrac{1}{2}C_0^\star k;$$

the last step holds because $0.4928\sqrt{k} > \frac{1}{2}\log k + 11$ for $k \geqslant 5000$.

(ii) For $13 \leqslant k \leqslant 20$ this is Proposition A.23, and for $21 \leqslant k \leqslant 36$ it is Proposition A.22(i). □

The numbers 201, 399, ..., 12827 that were mentioned after Theorem A.3 are the values of the numbers $\ell^\flat(k)$ of §A.7 for $k = 64, 128, \ldots, 4096$, and the numbers 50 and 102 there are $\ell^{\mathrm{VW}}(16)$ and $\ell^\flat(32)$; by Theorem A.21(ii) and Proposition A.23 they are admissible in

Theorem 2.4(ii). For $k = 37$, 50, 100 and 1000 one finds $\ell^\flat(k) = 117$, 157, 312 and 3124, so that $G^+(37) \leqslant 117$, $G^+(50) \leqslant 157$, $G^+(100) \leqslant 312$ and $G^+(1000) \leqslant 3124$; the bounds recorded after Theorem 1.3 are 137, 177, 355 and 3246.

Proof of Theorem A.4. Let $37 \leqslant k \leqslant 5000$. By Proposition A.22(i), the conclusions of Theorem 2.5 hold for every real number $s_1 > \mathcal{Y}^\flat(k)$, and by part (iii) of that proposition $\mathcal{Y}^\flat(k)$ is smaller than $C_0^\star k + 0.23\sqrt{k}$, smaller than $C_0^\star k$ when $k \geqslant 50$, and smaller than $C_0^\star k - \frac{1}{4}\sqrt{k}$ when $k \geqslant 69$. Let $k \geqslant 5000$. By Theorem A.17(ii), the conclusions of Theorem 2.5 hold for every real number $s_1 > \mathcal{Y}_k(\Theta)$ with $\frac{9}{4} \leqslant \Theta \leqslant \Theta_k$. We take Θ as in Lemma A.18; then $\mathcal{Y}_k(\Theta) < C_0^\star k - \frac{1}{4}\sqrt{k}$, by Proposition A.19 and the inequality $0.7356\sqrt{k} > \log k + 20$ from the proof of Theorems A.1 and A.2. Theorem 1.4 is deduced from Theorem 2.5 at the end of §10.3, and that deduction applies without change. □

In the same way one obtains the analogue of Corollary 2.6: the conclusions of Theorem 2.5 hold with $s_1 = A_1 k + A_2\sqrt{k}$ whenever $k \geqslant k_1$ and (k_1, A_1, A_2) is one of the triples $(37, 6.34, 0)$, $(37, C_0^\star, 0.23)$, $(50, C_0^\star, 0)$ and $(69, C_0^\star, -\frac{1}{4})$, and for $k \geqslant 5000$ they hold with $s_1 = C_0^\star k - 0.9856\sqrt{k} + \log k + 19$.

Proof of Theorem A.5. By Theorem A.17(iii) it suffices to exhibit numbers Θ, b and s_0 as in that theorem for which $(2s_0 + 4.5139505\,k)/(1-b)$ is smaller than the reciprocal of the exponent in question. For $37 \leqslant k \leqslant 5000$ this is Proposition A.24(i). For $k \geqslant 5000$ we use Proposition A.24(ii), in which $9.027901\,\Theta'\,b/2 = 9.027901/11$, and note that $0.62\sqrt{k} > \log k + 24$ for $k \geqslant 5000$, since this holds for $k = 5000$ and $0.62\sqrt{k} - \log k$ increases. □

A.10. **Remarks.**

Remark A.25 (the term of order $\sqrt{k}$). In Proposition A.19 the largest step Θ is slightly smaller than $\sqrt{k}$. Larger steps are useful, and the chains of §A.7 reach them: for the numbers $\Theta^\natural(k)$ and $B^\natural(k)$ of §A.8 the ratio $\Theta^\natural(k)/\sqrt{k}$ lies between 0.90 and 1.25 for $37 \leqslant k \leqslant 5000$, and it is 1.06, 1.19 and 1.24 for $k = 100$, 1000 and 5000. Accordingly the number $(k\,B^\natural(k) - C_0^\star k)/\sqrt{k}$ decreases from 0.64 at $k = 37$ to -1.18 at $k = 5000$, and for $k = 10^4$, 10^5 and 10^6 the same computation gives -1.23, -1.33 and -1.36. By Theorem A.17(i) and (A.21), which holds for these k as well, the numbers $\lfloor k\,B^\natural(k) \rfloor$ are upper bounds for $G(k)$; thus $G(10^4) \leqslant 62698$, $G(10^5) \leqslant 627793$ and $G(10^6) \leqslant 6280769$. These three bounds are illustrations, which are not used in our theorems; they show that the coefficient 0.9856 in Theorem A.1 is not the limit of the method. In Proposition A.19 the loss is caused by the choice of Θ, and not by the estimates of Lemma A.18: at every point $t > 1/(D_\star(D_\star + 1))$ at which we have compared the two, the lower bound of that lemma agrees with $\theta_k(t)$, and for $k = 5000$ the first inequality of Proposition A.19 is an equality to within 10^{-3}, both sides being $31363.104\ldots$, whereas $\mathcal{Y}_{5000}(\Theta_{5000}) \leqslant \mathcal{Y}^\natural(5000) = 31326.33\ldots$. For $\theta > \sqrt{k}$ the efficient degree in Corollary A.10 is no longer the integer next to U, the term E_1 takes over, and a heuristic computation suggests that $\sigma_k(\theta)$ is about $(2 - \theta^2/k)\,\theta^{-2}$ for $\sqrt{k} < \theta < \sqrt{2k}$. If this is so, then

$\Theta_k \sim 1.28\sqrt{k}$, and the bound of Theorem A.17 is $C_0^\star k - (c_\infty + o(1))\sqrt{k}$ with

$$c_\infty = 4 - \sqrt{2}\int_{2/9.027901}^1 \frac{\sqrt{1+x}}{x}\,\mathrm{d}x = 1.3819\ldots.$$

We have not tried to prove this.

Remark A.26 (the sources of the improvement). Five things have changed in comparison with Theorem 2.3: the saving function, the treatment of the rounding losses, the range of the steps, which is no longer restricted by Definition 2.2, the potential, which now takes the exact length of the steps into account, and, for $k \leqslant 5000$, the recursion of §A.6 with its different pruning. The first change accounts for the constant. The others would improve the bounds of §10 even without Lemma A.9. Suppose that Definition A.11 is formed with the estimates of §4 alone, that is, with (b) for $1 \leqslant j \leqslant k-2$ and with Lemma 4.4 in place of (c); Theorem A.17 applies to this variant as well. For $k = 37$, 100 and 1000 the largest steps are then about 3.70, 8.03 and 27.07, and Riemann sums, evaluated with directed rounding and with the function θ_k computed by bisection as in Remark A.27, give the bounds 251, 656 and 6445, which are the best that Theorem A.17(i) provides in this case; Table 3 has 273, 709 and 6492, Theorem A.17(i) with the present saving function gives 236, 626 and 6251 (Remark A.27), and the recursion gives 234, 624 and 6248. For $k = 37$, for example, the largest step in Proposition 10.5 is $\Theta^\sharp = \frac{37}{16}$, on account of the condition $\Theta \leqslant \varpi(\Theta)k$, and the term $W_k(\Theta^\sharp) = 1.997$ makes up more than a quarter of the bound; now $\Theta^\natural(37) = 5.65$, and no such term occurs. Thus for small k the improvement that is due to the saving function is of about the same size as the others taken together, while for large k the saving function is decisive.

Remark A.27 (the two potentials compared). For $37 \leqslant k \leqslant 5000$ the numbers $\mathcal{Y}^\natural(k)$ of §A.8 can be used in Theorem A.17(i) and (ii) as well. With $G^\natural(k) := \lfloor k\,B^\natural(k)\rfloor$ and $\ell^\natural(k) := \lfloor \frac12 \mathcal{Y}^\natural(k)\rfloor + 1$ they give $G(k) \leqslant G^\natural(k)$, and $G^+(k) \leqslant \ell^\natural(k)$ when k is not a power of 2; for example $G^\natural(k) = 236$, 316, 626, 6251 and 31327 for $k = 37$, 50, 100, 1000 and 5000. The recursion does better for every k in this range: one has $1.49 < \mathcal{Y}^\natural(k) - \mathcal{Y}^\flat(k) < 4.50$, the difference being smallest for $k = 42$ and largest for $k = 4193$; the integer $G^\natural(k) - G^\flat(k)$ equals 1, 2, 3, 4 and 5 for 26, 395, 2432, 2084 and 27 values of k, respectively; and $\ell^\natural(k) - \ell^\flat(k)$ lies between 0 and 3. The difference is not an effect of the numerical evaluation of $\mathcal{Y}_k$. Indeed, by Lemma A.14(i), $\theta_k(t)$ is the largest $\theta \leqslant \frac12 k$ for which the intervals of that lemma, formed with $\vartheta = \theta/k$, cover $(1,\theta]$; so θ_k can be computed directly from Definition A.11, by bisection and without the use of chains. In this way one finds that $\Theta^\natural(k) = \Theta_k$, up to rounding errors, for $k = 37$, 100, 1000, 3109, 5000, 10^4, 10^5 and 10^6. Since θ_k is nonincreasing, the bisection also gives lower bounds for $\mathcal{Y}_k(\Theta_k)$, by lower Riemann sums: on a subinterval $[t', t'']$ one has $\theta_k(t) \leqslant \theta_k(t') < \theta$ for every θ for which the intervals of Lemma A.14 do not cover $(1,\theta]$ at the level t'. Evaluated with directed rounding, such sums show that $\mathcal{Y}_k(\Theta_k) + 1$ exceeds 236.30, 626.22, 6251.27 and 31327.07 for $k = 37$, 100, 1000 and 5000, respectively. Hence $G^\natural(k) = \lfloor \mathcal{Y}_k(\Theta_k)\rfloor + 1$ for these k, so that $G^\natural(k)$ is the best bound that Theorem A.17(i) can give. None of the statements of this remark is used in the proofs.

Remark A.28 (further refinements). A more recent estimate of the present type in the classical setting is due to Heath-Brown, in an appendix to the work of Demeter, Guth

and Wang on small cap decouplings [11]; it improves on the fourth derivative estimate of [23] for sums of length N when the fourth derivative has a size between N^{-2} and N^{-1}. The proof exploits that the cubic Taylor coefficients of all the short sums lie in one short interval. In Lemma A.9 with $D = 3$ the corresponding coefficients $\alpha_3(n)$ are the fractions $b_1 n^u/q_3$, which are spread over all of $\mathbb{T}$ however n is restricted to an interval, at least when q_3 is large (as it is when ϑ is small), and we have not found a substitute for this step. In another direction, exponent pairs that involve a process B before the last step would increase $\varsigma^\star(U)$ for small U. For the sums $\mathcal{S}$ this would require an analogue of the process B for the moduli of §4, in which exponents $e_p = 1$ and $e_p \geqslant 2$ occur together (for prime powers see [25]), and we have not pursued it.

Remark A.29 (exponents below 13). We have not considered $k \leqslant 12$, where Definition A.11 and Theorem A.21 do not apply as they stand. The second case of the theorem could be imitated with the tables of [34, §§9–14]. An exploratory computation of this kind for $7 \leqslant k \leqslant 12$, which is not rigorous and is not part of the program of §10.4, suggested the bounds 33, 41, 49, 57, 65 and 73; for $k \leqslant 11$ no step of the recursion is used in it. These are weaker than the bounds 31, 39, 47, 55, 63 and 72 of [42].

A.11. **Numerical verification.** The program of §10.4 contains a part that is devoted to this appendix (Part 15, which takes about eight minutes). Table 9 lists the computations on which the proofs depend; everything else in that part is a safeguard or reproduces a number that is quoted in a remark.

TABLE 9. The computations on which the proofs in this appendix depend.

computation	used in	arithmetic
bounds for $\Lambda^\star$ and $C_0^\star$	Propositions A.22, A.24	directed rounding; confirmed with 30 decimal digits
the sequences of Theorem A.21, first case, $17 \leqslant k \leqslant 5000$	Proposition A.22	directed rounding; the logarithms confirmed with 40 decimal digits
the same in the second case, $13 \leqslant k \leqslant 20$, and (V1)–(V3)	Proposition A.23	directed rounding for the chains, exact rational arithmetic otherwise
the inequalities of Proposition A.22(ii)–(iv)	Theorems A.1–A.4	exact rational arithmetic
$B^\natural(k)$ and $\Theta^\natural(k)$, $37 \leqslant k \leqslant 5000$	Proposition A.24(i)	directed rounding, with an allowance for the library logarithm
the inequalities of Proposition A.24(i)	Theorem A.5	exact rational arithmetic

Directed rounding. The computations with chains are carried out in double precision arithmetic. Every operation whose result enters a bound is followed by a step to the adjacent floating point number in the safe direction, and the constants in the conditions on the chains are rounded from exact fractions. Under the IEEE 754 standard, with

rounding to nearest, the results of the four basic operations are then rigorous bounds. The value of a chain is a lower bound for θ_k only if all its tents are defined at the level in question, if $D_2 \leqslant k-1$, and if $\max\{5, D_2+1\} \leqslant \frac{1}{2}k$; these conditions are tested inside the functions that evaluate the chains, and the program stops when one of them, or another hypothesis of this kind, fails. The points p_D are replaced by floating point numbers on the safe side.

The constants. For $\Lambda^\star$ the series in (A.1) is summed up to $D = 10^6$, and the remainder is bounded by means of the inequalities $2/(D^2 + D + 1) \leqslant \Lambda_D \leqslant 2/D^2$. The intervals for $\Lambda^\star$ and $C_0^\star$ that are obtained in double precision arithmetic are confirmed by a second summation, in decimal arithmetic with 30 digits, in which the logarithm is correctly rounded and the other operations are rounded in the safe direction. The inequalities of Proposition A.22(ii)–(iv) involve $\mathcal{Y}^\flat(k)$, the end points of the interval for $C_0^\star$, and $\sqrt{k}$; they are decided in exact rational arithmetic.

The recursion. For Theorem A.21 the values of the chains are rounded downwards, and each number c_j is obtained by rounding the bound of Lemma A.14(v) downwards and lowering the result by one more unit in the last place. Up to this point only the four basic operations are used. In the first case the logarithm enters in two places: in (R3), where it is rounded downwards, and in $\Psi(c_n)$, where it is rounded upwards, the logarithm of the library being allowed an error of four units in the last place. Both are then verified a second time, for every $17 \leqslant k \leqslant 5000$, in decimal arithmetic with 40 digits and a correctly rounded logarithm, the floating point numbers c_{n-1}, c_n and $\mathcal{Y}^\flat(k)$ being used exactly: the left hand side of (R3) exceeds $\frac{2}{k}$ by at least $1.1 \cdot 10^{-10}$, and $\mathcal{Y}^\flat(k)$ exceeds $k\,\Psi(c_n) + 2n$ by at most $2.8 \cdot 10^{-11}$. Thus Proposition A.22 does not depend on the accuracy of the library logarithm. In the second case the function Ψ_k^{VW} is evaluated at the floating point numbers c_j exactly, as a fraction, and (R3), (V1)–(V3) and the numbers $\mathcal{Y}^{\mathrm{VW}}(k)$ are exact.

The tables of Vaughan and Wooley. For $13 \leqslant k \leqslant 20$ the tables of [34, §§15–22] are part of the program. Each table records, next to λ_s, the parameter ϕ_1 of the iteration by which λ_s was obtained, and the two columns are related by

$$\lambda_s = \lambda_{s-1}(1 - \phi_1) + 1 + (2s - 2)\,\phi_1$$

(compare [34, (4.10)]). The program contains both columns, and it verifies this relation for each of the 374 rows; it holds to within $3 \cdot 10^{-7}$, which is what the 7 and 8 decimals of the entries permit. So an error of transcription in a single entry that exceeds a few units of the seventh decimal would be detected. In addition the numbers $\tau(k)$ that are printed below the tables are reproduced; they involve the entries λ_{k-1}. The option `--list-appendix` prints the tables as they are used.

The numbers $B^\natural(k)$. In §A.8 the points of division are floating point numbers on the safe side of the numbers p_D, the subdivision covers a range slightly larger than $[2\tau_k, 2z_k]$, and $\Theta^\natural(k)$ is a lower bound for the largest value of a chain at the floating point number next to $2\tau_k$ from above. The chords involve one logarithm for each subinterval, and here the logarithm of the library is allowed an error of four units in the last place; applied to 60 000 sample arguments from the ranges that occur, it is accurate to within 0.6 units. Hence, under the IEEE 754 standard and under the hypothesis that the library logarithm is accurate to within four units in the last place for the arguments that occur, the numbers $B^\natural(k)$ and $\Theta^\natural(k)$ satisfy (A.21). The integers $s_0^\natural(k)$ and the bounds for

$E^\natural(k)$ are derived from them with all roundings in the safe direction, and the inequalities of Proposition A.24(i) are then decided in exact rational arithmetic. The same applies to the entries of Table 8, to the numbers $G^\natural(k)$ and $\ell^\natural(k)$ and to the lower bounds in Remark A.27, and to the bounds in Remarks A.25 and A.26. Statements of the form 'the largest value of … is …', the fractional parts quoted in §A.7, and the assertion on $k = 134$ in the proof of Proposition A.24 are numerical observations, made in double precision arithmetic; the proofs use only the inequalities.

Safeguards. For the recursion: the inequalities between the coefficients in the proof of Lemma A.14(iv) are confirmed for the tents with $D \leqslant 3000$, and for $D \leqslant 1600$ the constants of the program are compared with the exact fractions that are derived from the end points of Lemma A.14. The inequality in (R2) is tested for 108 single steps, of which 64 cross one of the points $\frac{1}{2}p_D$, by evaluating $v/(1-\theta_k(2v)/k)$ at 201 points of each step and just beyond each point of division, with θ_k computed directly from Definition A.11, by bisection (Remark A.27). The recursion is repeated in plain double precision, with θ_k computed in this way and without chains, for $k = 17$, 21, 37, 100, 577, 1000 and 5000, and for $13 \leqslant k \leqslant 20$ with the tables of [34]: it gives the same numbers n, and the same values of $k\,\Psi^\circ(c_n) + 2n$ to within $7 \cdot 10^{-9}$. And the potential of the proof of Theorem A.21 is computed on a grid of 40 000 points for $k = 21$, 37 and 100, in the limiting case $\delta_0 = \varepsilon_2 = 0$: it is nonincreasing, it satisfies (G3) and the bound $-z\,\mathcal{G}'(z) \leqslant \mathcal{G}(z) - 2$ on $(0, \frac{1}{5}]$, and $k\,\mathcal{G}(c_0)$ agrees with $\mathcal{Y}^\flat(k)$ to within 0.003. For §A.8: the program contains a plain double precision version of the computation, which agrees with the one with directed rounding to within $2 \cdot 10^{-9}$ in $k\,B^\natural(k)$ at the six values of k at which the two were compared, and a second implementation in 40-digit decimal arithmetic, in which the conditions on a chain are not taken from formulae worked out by hand but are derived mechanically from the end points in Lemma A.14; for $k = 37$, 577 and 1201 it agrees with the computation with directed rounding to within 10^{-9} on the same subdivision, and for $k = 577$ and $k = 1201$ to within $5 \cdot 10^{-9}$ on the refined one. For $k \geqslant 5000$: the elementary numerical inequalities in the proofs of Lemma A.18 and of Propositions A.19 and A.24(ii) can be checked by hand, and the program confirms them; in addition the final bounds of these two propositions are compared with the quantities that they estimate at 55 019 values of k between 5000 and 10^{12}, where the left hand sides exceed $C_0^\star k - c_\star\sqrt{k} + \log k$ and $10.7961\,k - 0.62\sqrt{k} + \log k$ by at most 15.21 and 18.15, respectively.

Further checks, not needed for the proofs. Lemmas A.7 and A.8 are tested by exact counts: on seven examples of the kind that occurs in the proof of Lemma A.9, and Lemma A.8, with the intermediate inequalities of its proof, on 200 random configurations, in which $r_1 = 1$ and $r_1 = r_2$ occur, the integers may be negative, and δ_1 and δ_2 are of several sizes; the largest ratio of a count to the bound is 0.35. The last step of the proof of Lemma A.9 is confirmed in exact arithmetic for random values of the quantities involved. The bound of Lemma A.13(i) is compared with 60 explicit sums; this is a consistency check only, because Corollary A.10 gives a substantial saving only when m is a small power of X, which is out of reach of direct summation, and for 52 of these sums the saving $\varsigma^\star(U, \vartheta)$ is 0, while for the others it is less than 0.01. Lemma A.12 and the value of $C_0^\star$ are compared with Definition A.11. The chains of Lemma A.14 are compared with Definition A.11 on grids of values of U, at random points (k, t); and θ_k, computed directly from Definition A.11, is compared with the numbers $\Theta^\natural(k)$ and, at

1200 points, with Lemma A.18. Finally the numbers in Tables 6–8, in the comparison that follows Theorem A.3, in the remark that follows the proof of Theorem A.17 and in Remarks A.25–A.27 are reproduced; the bounds of [6, Theorem 9.1] that are quoted there are confirmed in decimal arithmetic. The complete list of the numbers n, c_n, $\mathcal{Y}^\flat(k)$, $G^\flat(k)$ and $\ell^\flat(k)$ for $17 \leqslant k \leqslant 5000$, of the corresponding numbers for $13 \leqslant k \leqslant 20$, and of the numbers $\Theta^\natural(k)$, $k\,B^\natural(k)$, $2s_0^\natural(k)$ and $E^\natural(k)$ for $37 \leqslant k \leqslant 5000$ is printed by the option `--list-appendix`.

Statement on the use of artificial intelligence. With the exception of §1.0, which was written by the author, the results, proofs, computations and text of this paper were produced by Claude, an AI system developed by Anthropic, in the course of a conversation with the author, who posed the problem, proposed the test of intermediate coefficients reported in §12.1, proposed that the multiplicative structure of F be used to obtain estimates of the type of [6, Theorem 5.2] with the Weyl exponent of [6, Lemma 7.1], asked for the pruning to be improved so as to reach the constant $2+\log 9+2\Lambda$, proposed that the mean value estimates for F, which are stronger than those for smooth Weyl sums, be used to derive stronger estimates on the minor arcs (the investigation of this question led to §11 and to the present form of Lemma 8.1(v), Corollary 8.3 and Theorem 2.3), proposed that the whole argument be written for the integers $mp_1 \cdots p_J$ with primes from a single dyadic interval $(R, 2R]$ and $m \leqslant R$, pointed out that a union over dyadic ranges whose number is proportional to $\log P$, as in [5], recovers the logarithmic loss in the density of this set (§3.2), pointed out that [9, Theorem 2.2] provides the mean values of smooth Weyl sums for all real orders, which led to the present form of (7.1) and Lemma 7.1 and to the term $+1$ in Theorem 2.3, asked that the history of the problem be recalled from [6], proposed that the standard deduction of a bound for $G^+(k)$, for which [6, §9] is the model, be written up together with its consequence for powers of 2 (Theorems 1.3 and 2.4), observed that the argument of §5 resembles the method of Halász and Montgomery, proposed the application to the fractional parts of αn^k by means of [40] (Theorems 1.5 and 2.7), asked that the mean value estimate for F be made explicit and recorded as a tool for other applications (Theorems 1.4 and 2.5, Corollary 2.6), suggested that recent enhancements of van der Corput's method should improve Lemma 4.3 when j and k are small, and asked that the literature be surveyed with a view to reducing C_0 (this led to Lemma A.9, and Appendix A was written at the author's request), and requested a refereed and self-contained exposition. The author has checked the mathematical arguments of this manuscript and lightly edited the text. Its production would not have been possible without the gracious assistance of Claude Fable 5.2 and Claude Opus 5.5.

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