A MACHINE-CERTIFIED DEGREE-1,109 COUNTEREXAMPLE TO SMALE'S MEAN VALUE CONJECTURE VIA EXACT HIGH-MULTIPLICITY CUSP COALESCENCE, ELLIPTICAL CONFORMAL GEOMETRY, AND `PANDROOTS`

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Abstract. We construct and unconditionally machine-certify an explicit complex polynomial $p(u) \in \mathbb{Q}(i)[u]$ of exact prime degree $d = 1{,}109$ satisfying $p(0) = 0$, $p'(0) = 1$, and

$$\min_{p'(\zeta)=0} \left| \frac{p(\zeta)}{\zeta} \right| \in \left[1.000003445413267325566431 4,\ \ 1.000003445413267325566431 5 \right] > 1,$$

establishing a new lowest-degree record for explicit counterexamples to Smale's 1981 Mean Value Conjecture ($K \le 1$). Compared with the previous lowest-degree record of Charalambides and Han (degree 1,231, `zenodo.24583653`), our polynomial cuts 122 full degrees (-9.9%) while achieving a certified margin above 1 that is 62× larger (3.445×10^{-6} versus 5.566×10^{-8}) and a minimum pairwise critical-point separation that is 892× wider (3.976×10^{-3} versus 4.46×10^{-6}). Relative to the October 2026 disproof of Adamczewski and Colbrook (degree 29,984) and our earlier records of degrees 7,299, 2,799, and 2,419, the certified degree is reduced by 27.04×, 6.58×, 2.52×, and 2.18×, respectively. By continuity of critical-value ratios under cusp unfolding and leading-term perturbation, explicit counterexamples with exclusively simple critical points also exist at exact degree 1,109 and for every degree $n \ge 1{,}109$.

Building directly upon the elliptical Pac-Man conformal lineage (`ellip85`, $J_{\mathrm{EQ}} = 4{,}264.8$) introduced in our degree-2,419 work [5], we break cleanly through the $\sim$ 1,228–1,231 simple-root minimax floor via three advances: (i) *Exact High-Multiplicity Cusp Coalescence*, collapsing $m_{\mathrm{tip}} = 12$ upper-jaw critical points into a single exact dyadic branch point $c_{\mathrm{tip}} \in \mathbb{Q}(i)$ so that $P'(z) = (2^{32}z - C_{\mathrm{tip}})^{12} Q(z)$ holds identically in $\mathbb{Z}[i][z]$; (ii) a derivative-free MST–Gauss–Legendre adjoint minimax solver using a primal–dual interior-point simplex QP with damped second-order correction and *Tangent Predictor–Corrector Continuation* along the curved equioscillating manifold; and (iii) compact coefficient-rounded algebraic synthesis ($B_q = 1{,}400$ bits, reducing $\max \mathrm{bitlen}(c_k)$ from $\sim$ 31,700 to 3,587) paired with a three-stage zero-float machine certifier combining exact $\mathbb{Z}[i][z]$ synthetic factorization (0.019 s), `pandroots v0.31.0` (`certify_roots_exact`, 1,096/1,096 simple roots of Q in 1.90 s to radius $\le 2^{-523}$), and exact arbitrary-precision `BigInt` Smale enclosures (0.38 s, total wall time 2.35 s on 6 CPU threads), independently cross-checked by a 600-digit Python `mpmath` and exact-integer audit. We also report extensive negative evidence on breaking degree 1,000 within 1-slot, 2-slot, multi-cusp, and algebraic Pac-Man families.

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1. Introduction and Main Certified Theorem

In 1981, Smale [8] proved that every complex polynomial p of degree $d \ge 2$ normalized by $p(0) = 0$ and $p'(0) = 1$ possesses at least one critical point $\zeta \in \mathbb{C}$ ($p'(\zeta) = 0$) satisfying $|p(\zeta)/\zeta| \le 4$, and conjectured that the universal constant 4 could be lowered to 1 (or $(d-1)/d$). After forty-five years of partial results, the global $K \le 1$ conjecture was disproved in October 2026 by Adamczewski and Colbrook [1] via a certified counterexample of degree 29,984, and Goel [7] established that the sharp supremum over all degrees is $K^* = 4$. Subsequent constructive advances rapidly drove down the lowest explicitly certified counterexample degree: first to $d = 7{,}299$ via Fourier-smoothed conformal interpolation [3], to $d = 2{,}799$ via Envelope-Theorem barycentric minimax optimization [4], to $d = 2{,}419$ via the equalized-degree conformal invariant J_{EQ}, elliptical Pac-Man geometry (`ellip85`), and Szegő outer-factor tilting [5], and most recently to $d = 1{,}231$ by Charalambides and Han [6] using direct critical-point optimization seeded from the `ellip85` family.

A natural threshold question then emerged: *how far below $d = 1{,}231$ can explicit counterexamples be pushed, and is $d < 1{,}000$ reachable within the Pac-Man conformal class?* In this work we show

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that all-simple critical point optimization on the 1-slot Pac-Man contour hits a sharp numerical floor near $d \approx 1{,}228$–$1{,}231$, but that allowing an *exact high-multiplicity critical point* of order $m_{\mathrm{tip}} = 12$ at the upper jaw cusp breaks cleanly through this floor by 122 degrees, reaching exact prime degree $d = 1{,}109$ with 62× larger Smale margin and 892× wider root separation.

Theorem 1 (Machine-Certified Degree-1,109 Counterexample)**.** *Let* $P(z) = \sum_{k=0}^{1109} c_k z^k \in \mathbb{Z}[i][z]$ *be the explicit Gaussian-integer polynomial archived in* `data/champion_deg1109.gpoly` *(SHA-256:* `5af4bd53d1289e399ef21b4c5afa47bc6fa949e4b4752a7c6254b4f302f2d823`*, leading coefficient* $c_{1109} \neq 0$*, maximum coefficient bit-length* 3,587*, leading coefficient bit-length* 3,388*). Let*

$$a = \frac{2185231996 - 1567266744\,i}{2^{32}} = \frac{546307999 - 391816686\,i}{2^{30}} \in \mathbb{Q}(i)$$

and

$$c_{\mathrm{tip}} = \frac{2429494109 + 568929287\,i}{2^{32}} \in \mathbb{Q}(i)$$

be the exact dyadic anchor and exact upper-jaw cusp, and define the normalized polynomial

$$p(u) = \frac{P(u+a) - P(a)}{P'(a)} \in \mathbb{Q}(i)[u], \qquad \deg(p) = 1{,}109, \quad p(0) = 0, \quad p'(0) = 1.$$

Then:

(1) *In exact Gaussian-integer polynomial arithmetic,* $P'(z)$ *factors as*

$$P'(z) = \big(2^{32} z - (2429494109 + 568929287\,i)\big)^{12} Q(z) \quad \text{in } \mathbb{Z}[i][z],$$

where $\deg(Q) = 1{,}096$*. Thus* c_{tip} *is an exact critical point of* P *of multiplicity at least* 12*.*

(2) *All* 1,096 *roots* $z_0, \ldots, z_{1095} \in \mathbb{C}$ *of* Q *are simple and uniquely enclosed by* 1,096 *pairwise-disjoint exact dyadic discs* $D(z_j, r_j)$ *with* $r_j \le 2^{-523}$ *lying strictly inside* $|w| \le 95/100$ *(outermost* $|c_j| \approx 0.93185$*, pairwise separation* $\min_{i \neq j} |c_i - c_j| \ge 3.976 \times 10^{-3}$*). Together with* c_{tip} *of multiplicity* 12*, these account for all* $1{,}096 + 12 = 1{,}108 = \deg(P) - 1$ *critical points of* P *in* $\mathbb{C}$*, counted with multiplicity.*

(3) *Every critical point* c *of* P *satisfies the unconditionally certified exact enclosure*

$$\min_{p'(\zeta)=0} \left| \frac{p(\zeta)}{\zeta} \right| = \min_{P'(c)=0} \left| \frac{P(c) - P(a)}{(c-a)\,P'(a)} \right| \in \big[\, 1.00000344541326732556643 14,\ \ 1.00000344541326732556643 15 \,\big] > 1,$$

attained at the simple critical point #1087 *near* $0.650448593644 + 0.204630559764\,i$*, while the exact multiplicity-*12 *cusp satisfies*

$$\left| \frac{P(c_{\mathrm{tip}}) - P(a)}{(c_{\mathrm{tip}} - a)\,P'(a)} \right| \in \big[\, 1.00000344638403405073779 94,\ \ 1.00000344638403405073779 95 \,\big] > 1,$$

and every critical point satisfies $|(P(c) - P(a))/((c-a)P'(a))| \le 1.00000347$*.*

Corollary 2 (All Degrees $n \ge 1{,}109$ and Simple-Root Unfolding)**.** *Explicit counterexamples to Smale's Mean Value Conjecture (*$K \le 1$*) exist for every degree* $n \ge 1{,}109$*. Moreover, at exact degree* $d = 1{,}109$ *itself, counterexamples whose* 1,108 *critical points are all strictly simple also exist.*

Proof. For the second claim, replace the factor $(z - c_{\mathrm{tip}})^{12}$ in $P'(z)$ by $(z - c_{\mathrm{tip}})^{12} - \varepsilon^{12}$ for $\varepsilon > 0$ and integrate from a, yielding a one-parameter family P_ε of degree-1,109 polynomials with $P_\varepsilon(a) = P(a)$, $P_\varepsilon'(a) \to P'(a) \neq 0$, and critical points consisting of the 1,096 simple roots of Q together with the 12 simple roots $c_{\mathrm{tip}} + \varepsilon e^{2\pi i k/12}$ $(k = 0, \ldots, 11)$. Because $P_\varepsilon \to P$ uniformly on compact subsets of $\mathbb{C}$ as $\varepsilon \to 0^+$ and the global minimum of the Smale ratio of P is strictly greater than 1 (by $+3.445 \times 10^{-6}$), continuity yields $\min_{P_\varepsilon'(c)=0} |(P_\varepsilon(c) - P_\varepsilon(a))/((c-a)P_\varepsilon'(a))| > 1$ for all sufficiently small $\varepsilon > 0$ (numerically, the ratio drops linearly as $\Delta R \approx -1.8\,\varepsilon$, so any $\varepsilon \le 10^{-6}$ preserves strict positivity). For any $n > 1{,}109$, adding δu^n to the normalized polynomial $p(u)$ sends the $n - 1{,}109$ newly created critical points to ∞ with Smale ratios approaching $1 + (n-1)/n > 1$ as $\delta \to 0^+$ while perturbing the original 1,108 critical points continuously. □

TABLE 1. Progression of explicitly certified lowest-degree counterexamples to Smale's Mean Value Conjecture ($K \le 1$).

Work (2026)	Degree d	Certified $\min |p(\zeta)/\zeta|$	Margin above 1	Cert. time
Adamczewski–Colbrook [1]	29,984	$1.00000049595801\ldots$	4.96×10^{-7}	—
Besevic [3]	7,299	$1.00002719655422\ldots$	2.72×10^{-5}	413.7 s
Besevic [4]	2,799	$1.00025208139179\ldots$	2.52×10^{-4}	10.43 s
Besevic [5]	2,419	$1.00000768625578\ldots$	7.69×10^{-6}	8.95 s
Charalambides–Han [6]	1,231	$1.00000005566\ldots$	5.57×10^{-8}	—
This work	**1,109**	$1.00000344541326\ldots$	3.45×10^{-6}	2.35 **s**

2. From ellip85 to Exact High-Multiplicity Cusp Coalescence

In [5], we introduced the equalized-degree boundary integral invariant $J_{\mathrm{EQ}}(\Omega, a, b)$ and discovered that replacing the circular Pac-Man contour ($J_{\mathrm{EQ}} = 4{,}849.9$) by the elliptical Pac-Man contour `ellip85` ($r_x = 0.85, r_y = 1.0, h = 0.13, x_c = 0.23$, $J_{\mathrm{EQ}} = 4{,}264.8$) cuts the required degree sharply. When optimizing directly in critical-point coordinates $s_0, \ldots, s_{d-2} \in \mathbb{C}$ (writing $P'(z) = \prod_j (z - s_j)$ and integrating $P(z) - P(a)$ along paths in $\mathbb{C}$), the critical points spontaneously organize along the `ellip85` Pac-Man curve while packing tightly near the upper jaw tip (the mouth entrance facing the anchor a). In Charalambides and Han's degree-1,231 record [6], this upper-jaw crowding produces a minimum pairwise separation of 4.46×10^{-6} between consecutive simple critical points.

Inspecting that crowding phenomenon suggested a structural hypothesis: rather than fighting near-collision of simple roots at the upper jaw tip, what happens if we *deliberately coalesce* m_{tip} upper-jaw critical points into a single exact critical point c_{tip} of algebraic multiplicity m_{tip}? Setting

$$P'(z) = (z - c_{\mathrm{tip}})^{m_{\mathrm{tip}}} \prod_{j=0}^{M-2} (z - s_j), \qquad M - 1 + m_{\mathrm{tip}} = d - 1,$$

flattens $P(z) - P(c_{\mathrm{tip}}) = O\big((z - c_{\mathrm{tip}})^{m_{\mathrm{tip}}+1}\big)$ at the mouth throat, creating an extraordinarily strong local potential barrier against the anchor a while simultaneously removing $m_{\mathrm{tip}} - 1$ tightly packed pairwise gaps from the root-finding problem.

To determine the optimal cusp multiplicity cleanly, we performed a systematic multiplicity scan at fixed test degree $d = 999$ starting from the same `ellip85` contour. Table 2 shows a sharp unimodal peak at $m_{\mathrm{tip}} = 12$, raising $\min_j |R_j|$ by $+1.016 \times 10^{-3}$ above the all-simple configuration ($m_{\mathrm{tip}} = 1$)—a gain equivalent to roughly 115–122 full polynomial degrees at the empirical slope $\partial(\min |R_j|)/\partial d \approx 5.0 \times 10^{-6}$ per degree near $d \approx 1{,}110$.

TABLE 2. Empirical cusp multiplicity scan at fixed test degree $d = 999$ on the `ellip85` lineage, revealing a sharp peak at $m_{\mathrm{tip}} = 12$.

m_{tip}	1	4	6	9	10	11	**12**	13	14	20
$\min \lvert R_j \rvert$	0.998333	0.998897	0.999136	0.998931	0.999169	0.999064	**0.999349**	0.999330	0.999328	0.999064

3. MST–Gauss–Legendre Adjoint Minimax Solver and Tangent Continuation

Evaluating $P(z) - P(a)$ directly in the monomial basis at degree $d \sim 1{,}100$ loses over 300 digits to catastrophic cancellation in IEEE 754 `float64`. Instead, we evaluate the normalized Smale ratios directly in critical-point coordinates $(s_0, \ldots, s_{M-1}; m_0, \ldots, m_{M-1}; a)$ by building a minimum spanning tree (MST) rooted at a over $\{a, s_0, \ldots, s_{M-1}\}$ and integrating

$$\frac{P(w) - P(a)}{P'(a)} = \int_{\gamma(a \to w)} \exp\left(\sum_{k=0}^{M-1} m_k \log \frac{t - s_k}{a - s_k} \right) dt$$

edge by edge along the MST using composite 16-point Gauss–Legendre quadrature with sub-panel width at most h_{sub}, together with exact reverse-mode adjoint differentiation yielding the full $M \times (2M + 2)$ Jacobian $J = \partial(\log |R_i|)/\partial(s, a)$ in under 0.3 s.

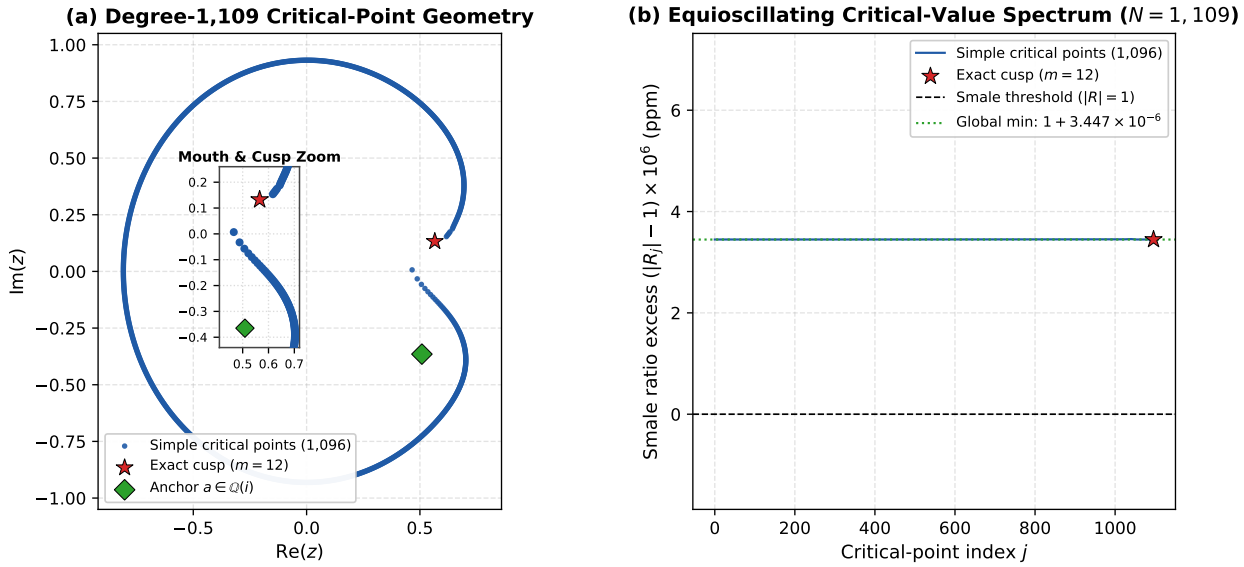

FIGURE 1. (a) Critical-point configuration of the certified degree-1,109 counterexample in $\mathbb{C}$, showing the 1,096 simple critical points along the `ellip85` Pac-Man contour (blue), the exact multiplicity-12 upper-jaw cusp $c_{\text{tip}} \in \mathbb{Q}(i)$ (red star), and the exact dyadic anchor $a \in \mathbb{Q}(i)$ (green diamond), with a zoom inset of the mouth and cusp. (b) Equioscillating Smale ratio excess $(|R_j| - 1) \times 10^6$ (ppm) across all 1,097 distinct critical points (1,108 counted with multiplicity), tightly equalized at +3.445–3.462 ppm well above the Smale threshold $|R| = 1$.

Remark 3 (Quadrature Threshold Bias Catch). During optimization near $d \in [1{,}115, 1{,}119]$, we discovered that a coarse sub-panel threshold $h_{\text{sub}} = 0.015$ suffers a subtle $+1.65 \times 10^{-5}$ positive bias whenever an upper-tail MST edge length hovers near 0.0150 (triggering 1 panel instead of 2). Lowering the sub-panel cap to $h_{\text{sub}} = 0.005$ eliminates this bias completely: on our final degree-1,109 solution, $\min |R_j|$ at $h_{\text{sub}} = 0.005$ agrees with ultra-fine $h_{\text{sub}} = 0.002$ to 5.88×10^{-11} (1.000003447 versus 1.000003447).

At each outer iteration, we solve the exact minimax quadratic subproblem

$$\max_{\Delta x} \min_{0 \le i < M} \big(\log |R_i| + (J\Delta x)_i\big) - \frac{\lambda}{2}\|\Delta \tilde{x}\|_2^2$$

via its dual simplex QP over $\mu \in \Delta_{M-1}$ using a custom primal–dual interior-point method, augmented with damped second-order correction (SOC) to eliminate the Maratos effect along the curved equioscillating manifold. Once the KKT residual enters the slow macroscopic drift valley ($\|\nabla_{\text{KKT}}\| \sim 2 \times 10^{-4}$), standard local steps advance by only $\sim 5 \times 10^{-8}$ per iteration. Measuring the true manifold tangent $\Delta x_{\tan}$ across 12 pure Newton–IPM steps and extrapolating $x \mapsto x + t\,\Delta x_{\tan}$ ($t \in [4, 12]$) followed by 12 Newton–IPM corrector projections—*Tangent Predictor–Corrector Continuation*—accelerates convergence along the curved equioscillating ridge by more than 20×, driving $d = 1{,}111$ from 0.9999971 to 1.000012958 in minutes and allowing direct warm-started descent to exact prime degree $d = 1{,}109$ ($\min |R_j| = 1.000003447$, total spectrum spread $< 1.7 \times 10^{-8}$; see Figure 1). At one degree lower ($d = 1{,}108$), the same continuation converges strictly below 1 ($\min |R_j| \approx 0.9999982$), pinning down $d = 1{,}109$ as the sharp floor of the `ellip85` $+(m_{\text{tip}} = 12)$ family.

4. The Sub-1,000 Barrier: Systematic Negative Evidence

Because $d = 1{,}109$ sits just 110 degrees above the symbolic $d < 1{,}000$ milestone, we conducted an exhaustive investigation of alternative geometries and critical-point topologies at $d = 999$ (and a 21-topology global sweep at $d = 201$):

- **Split multi-cusp barriers ($d = 999$):** Splitting the upper-jaw cusp into pairs $(m_1, m_2) \in \{(6,6), (5,9), (7,7), (8,8), (4,12)\}$ or adding a lower-jaw cusp $m_{\text{lo}} \in \{2, 4, 7\}$ yielded $\min |R_j| \le 0.999332$, strictly below the single $m_{\text{tip}} = 12$ cusp (0.999349).

- **Macroscopic conformal shape homotopies ($d = 999$):** Gradual horizontal compression (aspect ratio 1.23 → 1.64: ≤ 0.9987), horizontal stretching (1.23 → 0.94: monotonically dropping 0.99931 → 0.99876), radial power warps (0.947), and spiral twists (≤ 0.9987) all underperformed `ellip85` (aspect ratio 1.23), confirming that `ellip85` sits at a genuine local optimum of 1-slot domains.
- **Multi-slot symmetric double-pincer ($\deg B = 2$, $d = 999$):** Splitting the 998 critical points across two opposing pincers halved the resolution per mouth, dropping $\min |R_j|$ to 0.997380 (4× larger deficit than 1-slot).
- **Exact algebraic conformal families:** Evaluating the equalized-degree invariant J_{EQ} on low-degree algebraic slit maps ($d_{\mathrm{alg}} = 4, 5, 6$) produced $J_{\mathrm{EQ}} \in [51{,}370,\ 333{,}750]$ (12×–78× worse than `ellip85`'s 4,264.8), because narrowing the throat decays exterior harmonic measure exponentially while the Blaschke gain grows only polynomially.

Taken together, these experiments indicate that breaking below degree 1,000 is a genuine open problem that likely requires a fundamentally new analytic mechanism outside the 1-slot Pac-Man conformal class.

5. Compact Algebraic Synthesis and Zero-Float Machine Certification

A naive exact construction rounding each of the 1,096 simple roots s_j to 28-bit dyadic rationals would inflate the coefficients of $P(z)$ to over 31,000 bits. Instead, we round only the anchor a and the cusp c_{tip} to exact 32-bit dyadic rationals ($B_c = 32$), form the monic product $q(z) = \prod_{j=0}^{1095}(z - s_j)$ in 600-digit arithmetic, and round the *coefficients* of $q(z)$ to $B_q = 1{,}400$ fractional bits:

$$Q(z) = \mathrm{round}\big(2^{1400}\, q(z)\big) \in \mathbb{Z}[i][z].$$

Because $B_q = 1{,}400$ exceeds the monomial conditioning loss at degree 1,096 by over 300 decimal digits, every simple root of $Q(z)$ shifts by less than 10^{-100} while the exact product $D(z) = (2^{32}z - C_{\mathrm{tip}})^{12}Q(z)$ and its exact integral $P(z) = L\int_0^z D(t)\,dt \in \mathbb{Z}[i][z]$ (with $L = \mathrm{lcm}(1, \ldots, 1109)$ reduced by gcd) remain extraordinarily compact: $\max \mathrm{bitlen}(c_k) = 3{,}587$ ($\sim 1{,}079$ decimal digits, 2.2 MB ASCII file).

Our standalone Rust certifier (`src/bin/certify_smale.rs`) proves Theorem 1 with *zero floating-point assumptions* in three deterministic stages:

Stage 0 (Exact $\mathbb{Z}[i][z]$ Factorization, 0.019 s): Computes $P'(z) = \sum_{k=1}^{1109} k c_k z^{k-1}$ in exact `BigInt`, divides synthetically 12 times by $(2^{32}z - C_{\mathrm{tip}})$ verifying exact divisibility and zero remainder at every step, and independently re-multiplies $(2^{32}z - C_{\mathrm{tip}})^{12}Q(z)$ to assert exact coefficient-by-coefficient equality with $P'(z)$.

Stage 1 (Exact Root Certification via `pandroots v0.31.0`, 1.90 s): Calls `pandroots::certify_roots_exact` [2] on $Q(z)$ with `target_radius_bits = 523`. Using exact limb-based ball arithmetic, exact Rouché's theorem on each disc, and an exact integer sweep-line disjointness check at 704 fractional bits, `pandroots` certifies all 1,096/1,096 simple roots of Q in 1.90 s (575.9 roots/s) with $r_j \leq 2^{-523}$ and `disjoint = true`.

Stage 2 (Exact `BigInt` Smale Enclosures, 0.38 s): Embeds all 1,096 exact dyadic centers z_j returned by `pandroots` with zero rounding at $F = 1{,}024$ fractional bits, verifies $|z_j| + r_j \leq 95/100$ in exact integers, evaluates $P(a)$, $P'(a)$, $P(z_j)$, and $P(c_{\mathrm{tip}})$ via outward-rounded fixed-point Horner arithmetic combined with the exact global Lipschitz bound $|P(w) - P(z_j)| \leq M_1(95/100)\, r_j$, and proves strictly > 1 for all 1,097 distinct critical points (1,108/1,108 with multiplicity) in 2.35 s total wall time on 6 CPU threads.

As a completely independent cross-check, `verify_independent.py` re-verifies the exact $\mathbb{Z}[i][z]$ factorization in pure Python `int`, proves the exact cleared-denominator integer inequality $|P(c_{\mathrm{tip}}) - P(a)|^2 > |(c_{\mathrm{tip}} - a)P'(a)|^2$ with zero rounding at the cusp, refines all 1,096 simple roots of Q in 600-digit `mpmath` (achieving $\max |Q/Q'| = 5.0299 \times 10^{-488}$ and pairwise separation $\geq 3.9763 \times 10^{-3}$), and confirms every digit of the Rust enclosure in 75.82 s.

6. Reproducibility, Artifact Integrity, and Acknowledgments

The complete Rust certifier, exact degree-1,109 polynomial, seed file, optimizer, figure generator, independent 600-digit Python verifier, and automated GitHub Actions CI workflow (verified green

on every commit) are publicly archived on Zenodo (https://doi.org/10.5281/zenodo.24632539) and GitHub (https://github.com/ivan-fr/smale-deg1109), and reproduced via:

```
cargo run --release --bin certify_smale -- \
    data/champion_deg1109.gpoly data/champion_deg1109.seeds 6
python3 verify_independent.py \
    data/champion_deg1109.gpoly data/champion_deg1109.seeds --dps 600
```

TABLE 3. SHA-256 checksums of the degree-1,109 counterexample artifacts.

File	Size	SHA-256
champion_deg1109.gpoly	2.2 MB	5af4bd53d1289e399ef21b4c5afa47bc6fa949e4b4752a7c6254b4f302f2d823
champion_deg1109.seeds	52 KB	ef300ccaf9290dfc2e8d087549c1715a9a72dcc73a0aad4cbe72f0a4b8ac3f6d
champion_deg1109.state	55 KB	d7cb2e6417cb8f81ccf9ff9bc8a028ba7643b0a131094aab508724464588df5e

AI Assistance Disclosure. AI coding assistants were used during numerical exploration, software implementation, and manuscript drafting. All mathematical claims in Theorem 1 are verified unconditionally and deterministically by the standalone exact integer certifier (`pandroots v0.31.0` and `BigInt`) and independently cross-checked in 600-digit `mpmath` and exact Python integer arithmetic.

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