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PRIME-POWER CAPACITY, OPTIMAL APPROXIMATION AND CRITICAL HEIGHT OBSTRUCTIONS FOR EGYPTIAN FRACTIONS

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Abstract. Let $J(N)$ be the largest integer representable by distinct reciprocals of integers at most N, and let $k(N)$ be the maximum number of disjoint unit-sum subsets. We study the precision and limitations of local capacity approximations to these extrema. After subtracting the exact prime-square loss from the large-prime capacity, we give upper and lower integer-part bounds for the common value of $k(N)$ and $J(N)$ with a capacity error $O(N^{-1/2}/\log N)$ before rounding. The same precision holds for the highest-prime-power capacity and for the capacity imposing every prime-power constraint above $\sqrt{N}$. The optimal uniform polynomial exponent for the original, uncorrected capacity is $1/2$, including the endpoint. We also prove an in-place decomposition theorem for integer-sum supports in a logarithmic near-capacity window. The gap between the highest-layer and all-high-layer relaxations is of order $N^{-2/3}/\log N$ outside $O(x^{8/9})$ heights up to x. The exceptional translated threshold phases have Hausdorff dimension at most $2/3$. For a set of integer critical indices of positive lower density, the highest-layer relaxation overestimates the representable integer and the true least height exceeds its prediction by an unbounded, quantified delay. The last assertion uses variable prime-power exponents and a macroscopic nonresonance estimate for harmonic zeros, not a conjecture about harmonic primes. Finally we transfer the refined deletion estimates from a companion entropy manuscript to a sharper uniform error scale and an iterated-logarithmic expansion of the common minimum height. The prime-power precision and obstruction theorems do not require these companion entropy estimates. We give an exact integer formulation of the capacity hierarchy and, as separate transfers from the companion short-cover bounds, strengthen the quantitative critical-height delay.

1. Introduction and statements of results

For a finite set A of positive integers put

$$\mathcal{H}(A)=\sum_{n\in A}\frac{1}{n},\qquad H_N=\mathcal{H}([1,N]).$$

Intervals of integers are understood to contain their integer points. Define

$$k(N)=\max\{m:\ A_1,\dots,A_m\subseteq[1,N],\ A_i\cap A_j=\varnothing\ (i\neq j),\ \mathcal{H}(A_i)=1\},$$
$$J(N)=\max\{\mathcal{H}(A):A\subseteq[1,N],\ \mathcal{H}(A)\in\mathbb{Z}\}.$$

Taking unions gives $k(N)\le J(N)$. The converse does not follow from the definition: a given integer-sum set need not have an evident decomposition into unit sums. We seek a quantitative description of the extremal values, a structural condition guaranteeing such a decomposition, and obstructions to exact prediction by progressively stronger local relaxations.

For a prime p and $0\le q<p$, let

$$d(p,q)=\min\{\mathcal{H}(D):D\subseteq[1,q],\ \mathcal{H}(D)\equiv H_q\pmod p\},\tag{1}$$

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where $d(p,0) = H_0 = 0$. Rational congruences are taken in the localization of $\mathbb{Z}$ at p. The minimum exists, since deleting $[1,q]$ is admissible. Define the large-prime capacity and its prime-square correction by

$$\mathcal{C}(N) = H_N - \sum_{\sqrt{N}<p\le N} \frac{d(p,\lfloor N/p\rfloor)}{p}, \tag{2}$$

$$\Delta_2(N) = \sum_{N^{1/3}<p\le\sqrt{N}} \frac{d(p,\lfloor N/p^2\rfloor)}{p^2}, \qquad C_2(N) = \mathcal{C}(N) - \Delta_2(N). \tag{3}$$

All sums indexed by primes are over primes only. All logarithms are natural; write $\ell = \log N$ when using the height parameter N. The symbol $\log_j$ denotes the j-fold iterated logarithm, and all formulas using it are for sufficiently large arguments.

For every prime $p \le N$ define

$$a_p = \lfloor \log N/\log p\rfloor, \qquad Q_p = p^{a_p}, \qquad q_p = \lfloor N/Q_p\rfloor.$$

Then $Q_p > \sqrt{N}$ and $q_p < p$. The highest-layer capacity is

$$F(N) = \mathcal{C}_{\mathrm{pp}}(N) = H_N - \sum_{p\le N} \frac{d(p,q_p)}{Q_p}. \tag{4}$$

For the all-high-layer relaxation, put $b_p = \min\{b \ge 1 : p^b > \sqrt{N}\}$ and $V_p^* = \{n \le N : p^{b_p} \mid n\}$. Define

$$D_p^*(N) = \min\{\mathcal{H}(V_p^* \setminus A) : A \subseteq V_p^*,\ Q_p\mathcal{H}(A) \equiv 0 \pmod{p^{a_p-b_p+1}}\},$$
$$\mathcal{C}_{\mathrm{high}}(N) = H_N - \sum_{p\le N} D_p^*(N).$$

Write $D(j)$ for the least height representing exactly the integer j, and $P(j)$ for the least height with $k(N) \ge j$. Also put $D_{\ge}(j) = \min\{N : J(N) \ge j\}$. The first crossing heights of the highest-layer capacity are

$$M_j(\alpha) = \min\{N : F(N) \ge j + \alpha\}, \qquad M_j = M_j(0), \quad 0 \le \alpha < 1.$$

Their finiteness will follow below. We write γ for Euler's constant.

Precision of the capacity hierarchy.

Theorem 1.1 (Prime-square precision)**.** *There is an absolute constant $C > 0$ such that, for sufficiently large N,*

$$\left\lfloor C_2(N) - \frac{C}{\sqrt{N}\log N}\right\rfloor \le k(N) = J(N) \le \lfloor C_2(N)\rfloor. \tag{5}$$

The same assertion, with the same order of error, holds with $C_2(N)$ replaced by either $F(N)$ or $\mathcal{C}_{\mathrm{high}}(N)$. In particular,

$$\left\lfloor \mathcal{C}_{\mathrm{high}}(N) - \frac{C}{\sqrt{N}\log N}\right\rfloor \le k(N) = J(N) \le \lfloor \mathcal{C}_{\mathrm{high}}(N)\rfloor.$$

Theorem 1.2 (Optimal uniform polynomial exponent)**.** *Define*

$$c_* = \sup\{c > 0 :\ \exists C > 0\ \forall N \ge 1,\ k(N) \ge \lfloor \mathcal{C}(N) - CN^{-c}\rfloor\}.$$

Then $c_* = 1/2$, and the endpoint is admissible. More precisely, for an absolute constant $C > 0$ and all large N,

$$k(N) \geq \lfloor \mathcal{C}(N) - C\Phi(N) \rfloor, \qquad \Phi(N) = \frac{(\log_2 N)^{5/2}(\log_3 N)^2}{\sqrt{N \log N}} = o(N^{-1/2}). \tag{6}$$

In the other direction, no nonnegative error function $e(N) = o(\log_2 N/\sqrt{N \log N})$ can satisfy $k(N) \geq \lfloor \mathcal{C}(N) - e(N) \rfloor$ for all sufficiently large N.

Theorem 1.3 (In-place stability in a near-capacity window)**.** *For all sufficiently large N, if $A \subseteq [1, N]$ satisfies*

$$\mathcal{H}(A) = j \in \mathbb{Z}, \qquad 0 \leq \mathcal{C}(N) - j \leq (\log N)^{-36},$$

then A itself has a partition $A = \bigsqcup_{i=1}^{j} A_i$ with $\mathcal{H}(A_i) = 1$. Consequently (also a consequence of the exact threshold equality in [6]*),*

$$k(N) = J(N) \qquad (N \text{ sufficiently large}). \tag{7}$$

The stability assertion concerns the entire given support, not just existence of some disjoint packing of the same total mass. It applies throughout a near-capacity window, without requiring the height to be minimal for the integer sum.

Limitations of local relaxation.

Theorem 1.4 (The capacity gap)**.** *For all sufficiently large N,*

$$0 \leq F(N) - \mathcal{C}_{\text{high}}(N) \ll \frac{1}{N^{2/3} \log N}. \tag{8}$$

There is an absolute $c > 0$ such that

$$F(N) - \mathcal{C}_{\text{high}}(N) \geq \frac{c}{N^{2/3} \log N} \tag{9}$$

except for $O(x^{8/9})$ positive integers $N \leq x$ (with a fixed finite initial range omitted). In particular the two capacities have a difference of this order on a set of natural density one.

Theorem 1.5 (Exceptional phases)**.** *There is an absolute $c > 0$ such that the set of $\alpha \in [0, 1)$ for which*

$$F(M_j(\alpha)) - \mathcal{C}_{\text{high}}(M_j(\alpha)) < \frac{c}{M_j(\alpha)^{2/3} \log M_j(\alpha)}$$

for infinitely many j has Hausdorff dimension at most $2/3$.

Theorem 1.6 (Critical failures and height delay)**.** *There are absolute constants $c_1, c_2 > 0$ such that for every sufficiently large T there are at least $c_1 T$ integers $j \in [T, 5T]$ for which*

$$J(M_j) \leq j - 1 < \lfloor F(M_j) \rfloor = j, \qquad D_{\geq}(j) - M_j \geq c_2 \frac{j^{1/3}}{(\log j)^{10}(\log\log j)^4}. \tag{10}$$

Consequently the set of critical indices at which the highest-layer capacity overestimates the representable integer has positive lower density.

In Theorem 1.6 one also has $k(M_j) = J(M_j) = j - 1$ and $D_{\geq}(j) = D(j) = P(j)$ for all sufficiently large indices under consideration. The obstructions to integer representation will be proved independently of the packing arguments. Theorem 1.4 compares unrounded capacities and does not assert exactness of the rounded all-high-layer relaxation.

Relation to earlier results and scope. The local capacity framework in [4] permits simultaneous rational targets and controls cardinalities of the groups. In particular, Theorem 16 there gives a polynomial error of order $N^{-1/12}$ for unit targets with a length bound, and the paper proves the two logarithmic terms in the inverse-height expansion. The scalar rational-grid paper [5] and the prescribed-critical-support entropy paper [6] establish a refined scalar threshold, exact equality of the least heights for scalar and disjoint representations, and the entropy of partitions. We neither claim those results as new nor impose the length constraints of [4]. Our main advances over those results are the square-layer precision, sharp polynomial exponent, broader near-capacity in-place stability and arithmetic obstructions at the critical indices of stronger local relaxations. The public preprints [5, 6] have a Lean supplement with a declaration-to-statement map. Their scalar and partition thresholds are denoted there by $T_*(m)$ and $N_*(m)$; these are our $D(m)$ and $P(m)$ respectively. The companion harmonic-entropy manuscript [7] already proves the refined weighted deletion expansion and estimates the square-layer loss at the entropy scale. In the final section we use those estimates explicitly to obtain new capacity and height consequences; the core precision, stability, and obstruction theorems here are independent of them.

The questions of Erdős and Graham on integer representations by distinct reciprocals motivated Croot's results [1] and Yokota's improvement [2]. The only external representation theorem in the core proofs is the powersmooth interval lemma of Liu and Sawhney [3, Lemma 4.1 of arXiv v1]. The nonresonance method uses elementary two-point bounds for the zero set of partial harmonic sums modulo a prime. The derivative-polynomial method and the bound $O(p^{2/3})$ already occur in [8, Lemma 2.2]; we use the ordered-pair version to prove the critical-height incidence results. See also [9, 10]. No conjectural distribution of harmonic primes is assumed. No exhaustive literature priority search or independent refereeing is claimed.

Proof structure. We first establish the local capacities, their increments, and a preliminary packing bound. We then construct partitions inside prescribed near-capacity sets and obtain square-layer precision. The optimal polynomial exponent requires a rounding argument at suitably chosen crossings. The comparison of the two higher-power relaxations is proved through harmonic-zero pairs; the same bounds, combined with nonresonance across two macroscopic index ranges, give a positive density of integer critical failures and a quantitative delay. We end with consequences of the companion entropy results and the remaining open questions.

2. Local capacities and modular representation

We record an auxiliary packing estimate. It is useful for the proof of in-place stability and gives an independent, self-contained route to the equality of the extremal values without using the known threshold equality from [6].

Theorem 2.1 (Capacity and a preliminary packing bound)**.** *As $N \to \infty$,*

$$H_N - \mathcal{C}(N) \sim \frac{(\log_2 N)^2}{2 \log N}. \tag{11}$$

For every fixed $0 < c < 1/4$ and all sufficiently large N,

$$\lfloor \mathcal{C}(N) - N^{-c} \rfloor \le k(N) \le J(N) \le \lfloor \mathcal{C}(N) \rfloor . \tag{12}$$

Put $\mathcal{L}(x) = \operatorname{lcm}(1, \dots, \lfloor x \rfloor)$, and call an integer Y-powersmooth if every prime power dividing it is at most Y, or equivalently it divides $\mathcal{L}(Y)$. For a rational number v let $\operatorname{den}(v)$ be its positive reduced denominator. We use the convention $P^+(1) = 1$, where $P^+(n)$ denotes the largest prime factor of n.

Lemma 2.2 (Local necessary conditions)**.** *The capacity $\mathcal{C}(N)$ is the maximum mass of a subset of $[1, N]$ subject to the separate conditions*

$$\sum_{a:\ pa\in A} \frac{1}{a} \equiv 0 \pmod{p} \qquad (\sqrt{N} < p \le N).$$

In particular $k(N) \le J(N) \le \lfloor \mathcal{C}(N) \rfloor$, and $\mathcal{C}(N)$ is nondecreasing and unbounded.

Proof. For $p > \sqrt{N}$, the block of its multiples contains no multiple of p^2 and is disjoint from every other such block. Multiplying an integer-sum representation by p and reducing modulo p proves the displayed condition. The maximum contribution of the block is $(H_q - d(p, q))/p$, $q = \lfloor N/p \rfloor$. The maxima can be attained simultaneously, since the blocks are disjoint and all other denominators can be retained. This proves the capacity characterization. Every admissible subset at height N stays admissible at height $N + 1$: existing prime restrictions either persist or disappear, and a newly appearing prime divides no old element. Finally all denominators up to $\sqrt{N}$ may be retained, proving $\mathcal{C}(N) \ge H_{\lfloor \sqrt{N} \rfloor} \to \infty$. □

Lemma 2.3 (Liu–Sawhney interval representation)**.** *There is an absolute $c_0 > 0$ such that, for all sufficiently large X, if*

$$1/3 \le v \le 1, \qquad \operatorname{den}(v) \mid \mathcal{L}(S(X)), \qquad S(X) = \frac{c_0 X}{(\log X)^3 (\log\log X)^4},$$

then some $B \subseteq [X/16, X]$ satisfies $\mathcal{H}(B) = v$.

This is [3, Lemma 4.1 and the definition of smoothness in Section 1.3 of arXiv v1]. In particular, the restriction is on the prime powers of the denominator, not on the denominator itself.

Lemma 2.4 (Short modular covers and scaled deletions)**.** *Let $\ell = \log N$. Choose either $Z_0 = \lceil \ell^2 \rceil$ or*

$$Z = \lceil A\ell (\log \ell)^3 (\log\log \ell)^4 \rceil \tag{13}$$

with an absolute constant A sufficiently large. Write z for either choice. For every prime $z < p \le N$ and $h \in \mathbb{F}_p$, there is $B \subseteq [z/16, z]$ with

$$\mathcal{H}(B) \equiv h \pmod{p}, \qquad \mathcal{H}(B) \le 1.$$

If $z \le q < p$, then $d(p, q) \le 2z/q$. If $Q = p^a \le N$ with $q = \lfloor N/Q \rfloor < p$, then (including the case $p \le z$, where $q < z$)

$$\frac{d(p, q)}{Q} \ll \frac{z \log(2z)}{N}, \tag{14}$$

and the stronger bound $O(z/N)$ holds when $q \ge z$.

Proof. The prime number theorem gives $\log \mathcal{L}(x) \sim x$. For A large enough, or for the choice Z_0, it follows that $D = \mathcal{L}(S(z)) > 3N$. Since $p > z > S(z)$, we may select an integer $a \in [D/3, D]$ with $a \equiv hD \pmod{p}$. Lemma 2.3 represents a/D, proving the first assertion. When $q \ge z$, put $v = \lfloor q/z \rfloor$. Apply the first assertion with target vH_q and delete $vB \subseteq [1, q]$. Since $v < p$, this set has the required residue, and mass at most $1/v \le 2z/q$. For $q < z$ delete the entire block and use $H_q \ll \log(2z)$. Since $Qq > N/2$ for $q \ge 1$, these estimates imply (14). □

Lemma 2.5 (Capacity loss)**.** *Equation* (11) *holds. In particular $\mathcal{C}(N) = \log N + \gamma - o(1)$.*

Proof. Fix a small absolute $a > 0$. For $p > N/(a \log N)$, put $q = \lfloor N/p \rfloor$. The prime number theorem for $\log \mathcal{L}(q)$ gives $\mathcal{L}(q)H_q < p$ for all large N, uniformly in this range. A nonempty retained subset has a positive integer numerator after multiplication by $\mathcal{L}(q)$, strictly less than p, so it cannot have residue zero. Thus $d(p, q) = H_q$.

For $\sqrt{N} < p \le N/Z$, Lemma 2.4 gives total loss at most $O(Z\pi(N/Z)/N) = O(1/\log N)$. For the other blocks we use $d \le H_q$. It remains to estimate, for $z = a\log N$ and $z = Z$,

$$\sum_{N/z<p\le N} \frac{H_{\lfloor N/p\rfloor}}{p}.$$

By $H_{\lfloor N/x\rfloor} = \log(N/x) + O(1)$, partial summation and the classical prime number theorem with its error term, this equals

$$\int_{N/z}^{N} \frac{\log(N/x)}{x\log x}\,dx + O\left(\frac{\log z}{\log N}\right) + o(1/\log N).$$

Indeed, $x \ge N/z$ with z of polylogarithmic size, so the classical error term is negligible after partial summation; the bounded harmonic approximation contributes $O(\log z/\log N)$. Substituting $u = \log(N/x)$ gives $\int_0^{\log z} u/(\log N - u)\,du$. In both cases $\log z \sim \log_2 N$, so the integral is asymptotic to $(\log_2 N)^2/(2\log N)$. These bounds prove the assertion. □

Lemma 2.6 (Capacity increments and highest powers)**.** *The capacities satisfy*

$$\begin{gathered} k(N) \le J(N) \le \lfloor \mathcal{C}_{\text{high}}(N)\rfloor \le \lfloor F(N)\rfloor \le \lfloor C_2(N)\rfloor \le \lfloor \mathcal{C}(N)\rfloor, \\ \mathcal{C}_{\text{high}}(N) \le F(N) \le C_2(N) \le \mathcal{C}(N). \end{gathered} \tag{15}$$

The capacities $\mathcal{C}$ and F are nondecreasing and unbounded. For either $K = \mathcal{C}$ or $K = F$,

$$0 \le K(N) - K(N-1) \ll \frac{\Lambda(N)}{N}, \qquad \Lambda(N) = \ell(\log\ell)^4(\log\log\ell)^4. \tag{16}$$

The cruder bound $O(\ell^3/N)$ also holds.

Proof. Since $pQ_p > N$ and $Q_p \ge p$, we have $Q_p^2 > N$ and $q_p < p$. The blocks of multiples of Q_p for distinct primes are disjoint. Multiplying an integer representation by Q_p and reducing modulo p imposes the local condition corresponding to $d(p, q_p)$. Thus $F = \mathcal{C}_{\text{pp}}$ is the maximum mass under those separate conditions. It includes every constraint in C_2. The blocks V_p^* are disjoint, because their defining prime powers exceed $\sqrt{N}$ and have distinct bases. For any integer-sum set, multiplication by Q_p and reduction modulo $p^{a_p-b_p+1}$ annihilates the terms outside V_p^*. Hence the all-high-layer capacity is the maximum mass subject to these stronger independent conditions. Equivalently an admissible support has reciprocal sum with denominator dividing $\mathcal{L}(\sqrt{N})$; indeed the local conditions are exactly the assertions that the p-adic valuation of that sum is at least $-(b_p-1)$ for each p. This proves (15). All denominators up to $\sqrt{N}$ are admissible for all capacities, which proves unboundedness. Monotonicity holds because when a highest power changes at $N = p^a$, its new block consists of the single new integer, and old subsets automatically satisfy the new condition.

If a new integer lies in a fixed block $Q[1, q]$ with $N = Qq$, the increase is $1/N + (d(p, q-1) - d(p, q))/Q$. This is bounded above by $1/N + d(p, q-1)/Q$, and Lemma 2.4 at height $N-1$ gives (16). At a prime-power transition $N = p^a$ ($a \ge 2$), the previous highest block is released and the new singleton is necessarily deleted; the increase of $\mathcal{C}_{\text{pp}}$ is $d(p, p-1)/p^{a-1}$, bounded by the same lemma. At a new prime the increment of $\mathcal{C}_{\text{pp}}$ is zero. Outside these cases the increment is $1/N$. For $\mathcal{C}$ the only additional type of transition is $N = p^2$, where the old large-prime restriction is released and the increment is $1/N + d(p, p-1)/p$. All cases obey the claimed bound; using $z = Z_0$ gives the cruder estimate. □

2.1. **An exact integer formulation of the capacity hierarchy.** For formalization, no reduction of a rational with a nonunit denominator is necessary. Put $L_N = \mathcal{L}(N)$, $L_0 = 1$, and, for $A \subseteq [1, N]$, put

$$W_N(A) = \sum_{n \in A} L_N/n \in \mathbb{Z}_{\geq 0}, \qquad \mathcal{H}(A) = W_N(A)/L_N. \tag{17}$$

For any positive divisor M of L_N, reduction to lowest terms gives the exact equivalence

$$M \mid W_N(A) \quad \Longleftrightarrow \quad \operatorname{den}(\mathcal{H}(A)) \mid L_N/M. \tag{18}$$

Indeed, the second condition is equivalent to $(L_N/M)\mathcal{H}(A)$ being an integer, and this number equals $W_N(A)/M$. Define four positive integer moduli

$$M_1 = \prod_{\sqrt{N} < p \leq N} p, \quad M_2 = \prod_{N^{1/3} < p \leq N} p, \quad M_3 = \prod_{p \leq N} p, \quad M_4 = L_N/\mathcal{L}(\sqrt{N}). \tag{19}$$

Then the four capacities $K_i = \mathcal{C}, C_2, F, \mathcal{C}_{\text{high}}$, in that order, satisfy

$$L_N K_i(N) = \max\{W_N(A) : A \subseteq [1, N],\ M_i \mid W_N(A)\}. \tag{20}$$

For $i = 1, 2, 3$, a prime p in the modulus has highest power $Q_p = p^{a_p}$ at height N. A term L_N/n is zero modulo p unless $Q_p \mid n$; writing $n = Q_p b$ in the remaining block gives $L_N/n = (L_N/Q_p)/b$, interpreted modulo p using the inverse of b. The common factor L_N/Q_p is a unit modulo p. The local condition is thus precisely the one defining the retained optimum $H_{q_p} - d(p, q_p)$. For M_2 the only possible highest layers for its primes have exponents one or two, giving exactly the large-prime and prime-square blocks in (3). For $i = 4$, the modulus has p-adic exponent $a_p - b_p + 1$. A term outside the corresponding block V_p^* is divisible by that power; the remaining terms give the congruence in the definition of D_p^*. Equivalently one may use (18) and the high-layer characterization of Lemma 2.6. Distinct active blocks are disjoint, so a maximum of the global numerator is attained by taking the independent local maxima and every unrestricted denominator. The chain $M_1 \mid M_2 \mid M_3 \mid M_4 \mid L_N$ proves all capacity inequalities. For finite verification the square and cube tests can be written as $p^2 > N$ and $p^3 > N$, without real roots.

This also shows that C_2 and $\mathcal{C}_{\text{high}}$ are nondecreasing, not only $\mathcal{C}$ and F. For $i = 1, 2$, the allowed denominator L_N/M_i has exponent a_p or $a_p - 1$ according as p is below or above the relevant cutoff. As N grows, a_p never decreases and the cutoff can only release a constraint for a fixed p. For $i = 3$ the allowed exponent is $\max(a_p - 1, 0)$, and for $i = 4$ it is the exponent in $\mathcal{L}(\sqrt{N})$. Thus each allowed denominator divides the corresponding allowed denominator at the next height. An old admissible support stays admissible by (18) (using the new common denominator to form its numerator). These identities are convenient finite lemmas to prove before importing any asymptotics.

By Lemma 2.4, the extra highest-layer losses for primes at most $\sqrt{N}$ have total mass $O(\pi(\sqrt{N})\Lambda(N)/N)$. Consequently Lemma 2.5 and $H_N = \log N + \gamma + O(1/N)$ give

$$F(N),\ \mathcal{C}(N) = \log N + \gamma - \left(\frac{1}{2} + o(1)\right) \frac{(\log_2 N)^2}{\log N}. \tag{21}$$

In particular, the first crossing satisfies

$$0 \leq F(M_j) - j \ll \Lambda(M_j)/M_j, \qquad \log M_j = j - \gamma + \eta_j, \quad \eta_j = \left(\frac{1}{2} + o(1)\right) \frac{(\log j)^2}{j}. \tag{22}$$

For the second formula first deduce $\log M_j = j - \gamma + o(1)$ from (21), and then substitute it into the lower-order term. The jump error is exponentially small in j. This argument needs only the capacity asymptotic, not a theorem about attainable integer sums.

3. Packing with polynomial precision

We prove the lower bound of Theorem 2.1. In the following estimates, c is fixed with $0 < c < 1/4$. Choose constants u, τ, y, x, t, v satisfying

$$\begin{aligned} &c < u < \tau < 1/2 - c, \qquad \max(\tau, 2c) < y < 1/2, \\ &y < x < t < 1/2, \qquad x + t < 1, \qquad t/2 < v < y. \end{aligned} \tag{23}$$

These conditions are feasible by first choosing τ, u, y, then choosing x, t close to y, and finally v. Set

$$T = N^{1-\tau}, \quad Y = N^y, \quad \eta = N^{-c}, \quad X = \lfloor N^x \rfloor, \quad K = \lfloor \mathcal{C}(N) - \eta \rfloor, \quad m = K - 1.$$

By Lemma 2.5, $m \sim \ell$. We set aside $\{1\}$ as the first unit-sum group.

3.1. **Three disjoint resources.** For every prime $p > T$, take an optimal retained block $U_p \subseteq p[1, \lfloor N/p \rfloor]$ attaining the local maximum. By the mod-p condition, its total mass has denominator dividing $\mathcal{L}(N/p)$, hence $\mathcal{L}(Y)$. Treat each nonempty U_p as an indivisible atom. Its mass is at most $H_N/T = o(\eta)$, and all these atoms together have mass at most $H_N - H_{\lfloor T \rfloor} \le \tau\ell + O(1)$.

Choose m distinct primes $u_i \in [N^u, 2N^u]$ and m distinct primes $t_i \in [N^t, 2N^t]$, possible by the prime number theorem. Reserve the following sets:

$$\begin{aligned} \mathcal{R}_{p,i} &= pu_i[Z_0/16, Z_0] && (Y < p \le T), \\ \mathcal{P}_i &= t_i[X/16, X], \\ \mathcal{F} &= \{rs : N^v \le r < s \le 2N^v,\ r, s \text{ prime}\}. \end{aligned}$$

Here and below aI is the set of integer multiples ab with $b \in I$. The sets $\mathcal{R}_{p,i}$ are the modular repair pools, $\mathcal{P}_i$ the exact closure pools, and $\mathcal{F}$ the balancing pool.

All these sets consist of denominators in $[2, N]$: this follows from $Tu_iZ_0 < N$, $t_iX < N$ and $4N^{2v} < N$. They do not meet any atom U_p with $p > T$, since their largest prime factors are at most T. The repair pools are pairwise disjoint because $p > Y > u_i > Z_0$ makes the first two prime factors identifiable. The closure pools are pairwise disjoint because $t_i > X$. If a repair denominator equals a closure denominator, comparison of largest prime factors gives $p = t_j$ and then $u_ib = b'$ with $b \le Z_0$, $b' \ge X/16$. This is impossible because $N^uZ_0 = o(X)$. The balancing pool consists of Y-powersmooth integers, whereas the other pools have a prime factor larger than Y. Thus it is disjoint from them as well.

Their masses satisfy

$$E_{\mathrm{mod}} := \sum_{p,i} \mathcal{H}(\mathcal{R}_{p,i}) \ll N^{-u}\ell^2, \quad E_{\mathrm{cl}} := \sum_i \mathcal{H}(\mathcal{P}_i) \ll N^{-t}\ell, \quad W := \mathcal{H}(\mathcal{F}) \asymp \ell^{-2}. \tag{24}$$

For the first bound sum $O(1/(pu_i))$ over p, i, using $\sum_{p \le N} 1/p \le H_N$; for the last, the prime number theorem in $[N^v, 2N^v]$ gives $\asymp N^v/\ell$ primes, hence $\asymp N^{2v}/\ell^2$ distinct pair-products. Each balancing term has mass $O(N^{-2v}) = o(1/t_i)$ uniformly.

From the ordinary denominators outside these pools and atoms, delete those divisible by a prime power $p^a > Y$ with $a \ge 2$. Counting prime powers by $O(\sqrt{z}\log z)$ up to z, then using partial summation, bounds the total mass by

$$E_{\mathrm{pow}} \le H_N \sum_{p^a > Y,\ a \ge 2} (p^a)^{-1} \ll \ell^2 Y^{-1/2}. \tag{25}$$

By (23), $E_{\mathrm{mod}} + E_{\mathrm{cl}} + E_{\mathrm{pow}} = o(\eta)$. We do not delete from the atoms, whose reduced denominators are already powersmooth. The remaining ordinary denominators are singleton atoms. If M is the total mass of all atoms (excluding $\{1\}$ and all reserved pools), then

$$M + W \ge \mathcal{C}(N) - 1 - o(\eta) \ge m. \tag{26}$$

The first inequality is valid because only the large-prime blocks with $p > T$ have been optimized; the sum of those losses is at most the full loss defining $\mathcal{C}$. No local optimizer of a damaged block is assumed to remain available.

3.2. **A balancing and closure principle.** Suppose m groups have masses $s_i \le 1-\eta$, total deficit at most W, and reduced denominators whose only prime powers exceeding Y are primes $Y < p \le T$ to the first power. Suppose the pools $\mathcal{R}_{p,i}$ and $\mathcal{P}_i$ described above are disjoint from the groups and each other, and the balancing pool has mass W and each of its terms has mass $o(1/t_i)$. Assume $E_{\mathrm{mod}} = o(\eta)$, $1/t_i = o(\eta)$ and $S(X) > Y$. Then all groups can be completed to unit sums.

Indeed, for group i and each such prime p, use Lemma 2.4 with Z_0 to choose $B_{p,i}$ with

$$\mathcal{H}(B_{p,i}) \equiv -u_i p s_i \pmod{p}.$$

Add $pu_iB_{p,i}$. This cancels the p-part of the reduced denominator. Every other prime power in these newly added denominators is at most Y for large N; in particular another prime $p' > Y$ is unaffected, since the new mass is a p'-integer. Thus the repaired masses s_i' satisfy $\mathrm{den}(s_i') \mid \mathcal{L}(Y)$, $s_i' \le 1-\eta/2$, and total deficit at most W. This argument applies unchanged when the marker u_i depends on p and when some primes have no residue to correct.

Process the groups one by one using unused balancing terms, stopping for group i at the first mass at least $1 - 1/(2t_i)$. Initially it lies below this target. Because each term is $o(1/t_i)$, the first crossing is still strictly below one. Each completed group uses less balancing mass than its initial deficit. The total deficit bound therefore guarantees enough remaining mass for every unprocessed group. The final deficit Δ_i belongs to $[1/(3t_i), 1/t_i]$ and has denominator dividing $\mathcal{L}(Y)$. Apply Lemma 2.3 to $t_i\Delta_i$, and add the resulting denominators in $t_i[X/16, X] = \mathcal{P}_i$. This completes the groups exactly, with no repeated denominator.

3.3. **Least-load allocation.** Allocate atoms to the currently least-loaded of the m empty groups, with ceiling $1-\eta$. First allocate all the indivisible large-prime atoms, then the singleton atoms in decreasing mass order. Stop before adding the first atom that would exceed the ceiling in the least-loaded group, or when all atoms have been used.

The large-prime atoms all fit: the current minimum is at most their total mass divided by m, at most $\tau + o(1)$, and each atom has mass $o(\eta)$. If the next singleton has mass greater than η, all earlier singletons have denominators at most N^c. At that moment the least load is at most $\tau + c + o(1)$, and the next mass is at most $1/2$. By (23) this cannot cause a stop. Hence any stopping atom has mass at most η, and all loads at a premature stop exceed $1-2\eta$. If the atoms are exhausted, (26) bounds the sum of deficits by W. If the allocation stops early, the sum is at most $2m\eta = o(W)$. The hypotheses of the balancing and closure principle in Section 3.2 are now met, since the only nonsmooth factors left in the group denominators are first powers of primes in $(Y, T]$. It yields m disjoint unit sums in addition to $\{1\}$. This proves (12), and Lemma 2.5 completes Theorem 2.1.

4. Stability: adaptive repair and a final absorbing group

We now work inside the prescribed set A of Theorem 1.3. Put $\delta = \mathcal{C}(N) - j$. For each prime $p > \sqrt{N}$ let

$$e_p(A) = \sum_{a \le N/p,\ pa \notin A} \frac{1}{pa}.$$

The local condition from Lemma 2.2 implies the nonnegative decomposition

$$\delta = \sum_{n\notin A,\ n\le N,\ P^+(n)\le\sqrt{N}} \frac{1}{n} + \sum_{\sqrt{N}<p\le N} \left(e_p(A) - \frac{d(p,\lfloor N/p\rfloor)}{p} \right). \tag{27}$$

Every summand is nonnegative. In particular every integer up to $\lfloor \ell^{35} \rfloor$ belongs to A: otherwise a missing denominator in this range would alone cost at least $\ell^{-35} > \delta$.

Use the parameters

$$\begin{aligned} &T = N/\ell^{12}, \quad Y = \ell^{14}, \quad Z_0 = \lceil \ell^2 \rceil, \quad U = \ell^6, \quad V = \ell^{10}, \\ &X = \left\lfloor \ell^{16} \right\rfloor, \qquad t_i \in [\ell^{18}, 2\ell^{18}] \text{ distinct primes}, \qquad \eta = \ell^{-2}. \end{aligned} \tag{28}$$

By (27) and Lemma 2.4, the missing mass outside the very large prime blocks satisfies

$$E_A := \sum_{n\notin A,\ n\le N,\ P^+(n)\le T} \frac{1}{n} \le \delta + \sum_{\sqrt{N}<p\le T} \frac{d(p,\lfloor N/p\rfloor)}{p} \ll \ell^{-36} + \frac{TZ_0}{N} \ll \ell^{-10}. \tag{29}$$

Here every such prime has $\lfloor N/p \rfloor \gg \ell^{12} > Z_0$. For $p > T$, retain the actual block $A \cap p[1, \lfloor N/p \rfloor]$ as an indivisible atom. By the integer condition its reduced denominator divides $\mathcal{L}(N/p) \mid \mathcal{L}(Y)$, and the combined mass of these atoms is $O(\log \ell)$.

4.1. **Adaptive modular pools.** Set aside $\{1\}$ and put $m = j - 2 \sim \ell$, since we intend to construct m further groups and leave the last group to absorb all unused terms. For each prime $Y < p \le T$ and every prime $u \in [U, 2U]$, consider

$$\mathcal{R}_{p,u} = pu[Z_0/16, Z_0].$$

All candidate pools are pairwise disjoint by uniqueness of the two large prime factors, and they lie in $[2, N]$ since $2TUZ_0 < N$. There are $\asymp U/\log \ell$ markers, more than $2m$ for large N. Call p good if at least m of its pools are completely contained in A, and bad otherwise. If p is bad, at least half its pools each lack a denominator of mass at least $1/(2pUZ_0)$. The missing mass for a bad prime is therefore $\gg 1/(pZ_0 \log \ell)$. All such missing terms are distinct and have largest prime at most T, so (29) implies

$$\sum_{p \text{ bad}} \frac{1}{p} \ll E_A Z_0 \log \ell \ll \ell^{-8} \log \ell. \tag{30}$$

Consequently the mass of ordinary denominators in A divisible by any bad prime is at most $H_N \sum_{p \text{ bad}} 1/p = o(1)$. We temporarily exclude those ordinary denominators from the active construction. For every good p, reserve m distinct complete pools, one per group; the marker $u_{p,i}$ can vary with p.

The balancing pool is $\mathcal{F} = \{rs : V \le r < s \le 2V,\ r, s \text{ prime}\}$ and the closure pools are $\mathcal{P}_i = t_i[X/16, X]$. Their maximum denominators are at most $4\ell^{20}$ and $2\ell^{34}$, respectively. Hence both are contained in A. The size comparisons of Section 3 show that all selected pools are mutually disjoint and avoid the very large prime atoms. In particular $u_{p,i}Z_0 \ll \ell^8 = o(X)$ rules out collisions with the closure pools even if their largest prime factors coincide. The exclusion for bad primes acts only on ordinary denominators outside the reserved pools: a closure marker need not be good.

Reserve these pools before excluding ordinary denominators divisible by bad primes or nonfirst prime powers larger than Y. The estimates of Sections 3 and (30) show that the mass set aside (excluding $\{1\}$) is $o(1)$: the repair pools cost $O(mU^{-1}\ell) = O(\ell^{-4}) = o(\eta)$, the closure pools $O(\ell^{-17})$, nonsmooth higher powers $O(\ell^2 Y^{-1/2}) = O(\ell^{-5})$, and the balancing pool has

$W \asymp (\log \ell)^{-2} = o(1)$. No denominators are actually discarded from A; they remain available for the last group. The remaining atom mass M therefore satisfies

$$M \geq j - 1 - o(1) > j - 2 = m.$$

4.2. **The last group.** Allocate the very large prime atoms first, then the ordinary atoms in decreasing mass order, to the least-loaded group under ceiling $1 - \eta$. The very large prime atoms have total mass $O(\log \ell)$ and individual mass $o(\eta)$, so they all fit. Before an ordinary atom of mass greater than η is assigned, the combined mass assigned is $O(\log \ell)$ (the ordinary denominators used up to that point are at most ℓ^2). The least-loaded group has mass $o(1)$, and the new atom has mass at most $1/2$. Thus no large atom can trigger a stop. As $M > m$, a premature stop must occur, and all loads lie in $(1 - 2\eta, 1 - \eta]$. Their total deficit is $O(m\eta) = O(\ell^{-1}) = o(W)$.

Only the good primes in $(Y, T]$ can now appear to the first power outside $\mathcal{L}(Y)$ in the group denominators. Apply Section 3.2 with their adaptive repair pools. All its conditions hold: $S(X) > Y$ from (28), the balancing terms have mass $O(\ell^{-20}) = o(1/t_i)$, and $1/t_i = o(\eta)$. We obtain $m = j - 2$ disjoint unit sums inside A. Together with $\{1\}$ they are $j - 1$ groups. Put every unused denominator in A into the last group. Its mass is exactly $j - (j - 1) = 1$. This proves the stability assertion.

To establish eventual equality, let $j = J(N)$ and fix $c \in (0, 1/4)$. If $\mathcal{C}(N) - j \geq N^{-c}$, Theorem 2.1 gives $k(N) \geq j$. Otherwise $\mathcal{C}(N) - j < N^{-c} < (\log N)^{-36}$ for all large N. An extremizer for $J(N)$ then satisfies the stability hypothesis, giving $k(N) \geq j$ again. This proves Theorem 1.3.

For comparison, Corollary 1.2 of [6] states $D(j) = P(j)$ for all sufficiently large integers j. It gives a short independent implication to equality of extrema: $J(N) \to \infty$ by monotonicity and Theorem 2.1 (or by the finiteness of each scalar threshold); hence for $j = J(N)$ sufficiently large, $P(j) = D(j) \leq N$, so $k(N) \geq j$. This argument proves only equality of extrema, not the prescribed-support assertion of Theorem 1.3, whose independent proof above is retained.

Remark 4.1. The result $k(N) = J(N)$ is an equality of extrema. The proof does not assert that *every* extremizer for $J(N)$ decomposes. Only the prescribed near-capacity sets are proved to have the latter property.

For completeness we record the leading height consequence of the auxiliary packing estimates. It is weaker than the expansion already established in [4, 5]; a further refinement will be deduced in Section 11.

Corollary 4.2 (Sharp heights)**.** *For every sufficiently large integer j one has $D(j) = P(j)$, and*

$$D(j) = P(j) = e^{j-\gamma}\left(1 + \left(\frac{1}{2} + o(1)\right)\frac{(\log j)^2}{j}\right). \tag{31}$$

Moreover, for every fixed $\varepsilon > 0$ and all sufficiently large N, the common value $k(N) = J(N)$ lies between the integer parts of

$$H_N - \left(\frac{1}{2} + \varepsilon\right)\frac{(\log_2 N)^2}{\log N} \quad \text{and} \quad H_N - \left(\frac{1}{2} - \varepsilon\right)\frac{(\log_2 N)^2}{\log N}.$$

Proof of Corollary 4.2. Combining (11), (12) and (7) proves the stated rounded inequalities. For every sufficiently large height N, any integer from 0 to $J(N)$ can be represented by taking a suitable union of disjoint unit-sum groups. Since $D(j) \to \infty$, we have $k(D(j)) = J(D(j)) \geq j$ for large j, so $P(j) \leq D(j)$. The reverse inequality is immediate by taking a union.

Fix $0 < \varepsilon < 1/2$, and take the integer parts of

$$e^{j-\gamma}\left(1+\left(\frac{1}{2}\pm\varepsilon\right)\frac{(\log j)^2}{j}\right)$$

as test heights $N_\pm$. The integer rounding in these heights changes their logarithms by $O(e^{-j})$. At each, $H_{N_\pm} = j + (1/2 \pm \varepsilon + o(1))(\log j)^2/j$, while $(\log_2 N_\pm)^2/\log N_\pm \sim (\log j)^2/j$. Using the rounded capacity bounds with a smaller tolerance, e.g. $\varepsilon/2$, shows $J(N_-) < j \le k(N_+)$. Thus $N_- < D(j) = P(j) \le N_+$, and letting $\varepsilon \downarrow 0$ proves (31). □

5. Isolating prime-square losses

We prove Theorem 1.1. The upper bound follows from applying the local conditions to all blocks of multiples of $p > \sqrt{N}$ and of p^2 with $N^{1/3} < p \le \sqrt{N}$. All their defining prime powers exceed $\sqrt{N}$, so they are disjoint. At a square block we multiply by p^2 and reduce modulo p, since $p^3 > N$. The maximum permitted mass is therefore exactly $C_2(N)$ under these conditions alone. For the lower bound we shall construct a single integer-sum set and use Theorem 1.3 only at the end.

5.1. **Initial optimizers and a closure marker.** Let Z be as in (13). Use

$$Y = \frac{\sqrt{N}}{\ell^6}, \qquad X = \left\lfloor \frac{\sqrt{N}}{4\ell} \right\rfloor, \qquad Q_* = \frac{N}{16Z}, \qquad t \in [\sqrt{N}\ell, 2\sqrt{N}\ell] \text{ prime}.$$

Reserve $\mathcal{P} = t[X/16, X]$. It lies in $[1, N]$, has mass $O(1/t)$, and $S(X) > Y$ for sufficiently large N. Optimize all the prime and prime-square blocks separately, except the block with marker t. In that block first remove the reserved pool from the working set. If $q_t = \lfloor N/t \rfloor$, the remaining auxiliary denominators are $[1, q_t] \setminus [X/16, X]$, where $q_t \ge X$. Put $v = \lfloor X/(32Z) \rfloor$. By Lemma 2.4, we can delete vB within $[1, X/32]$ so that the remaining auxiliary mass has residue zero modulo t: use the modular target v times the mass to be corrected. This is legal since $v < t$. The total mass removed from the original t-block, including the closure pool, is

$$O(1/t + 1/(tv)) = O(1/(\sqrt{N}\ell) + Z/N) = O(1/(\sqrt{N}\ell)).$$

Thus the additional loss relative to its optimizer is no larger. All other optimizers are untouched. For a square block the resulting sum has p-adic denominator exponent at most one, and the auxiliary denominator sizes are below $N^{1/3} < Y$. All prime blocks have had their large prime factor removed from the reduced denominator of their sum. The ordinary auxiliary factors in prime blocks need not themselves be powersmooth; the following steps correct the total rational mass.

5.2. **Small-base higher powers.** First remove all denominators divisible by $p^a > Q_*$ for $p \le N^{1/3}$. Since $Q_* > \sqrt{N}$, these do not meet any optimized block or the reserved closure pool. There are $O(N^{1/3}\ell)$ relevant prime powers and each has a multiple-block mass $O(Z\log(2Z)/N)$. Hence this deletion costs $O(N^{-2/3}\ell Z \log(2Z)) = o(1/(\sqrt{N}\ell))$.

Fix a prime $p \le N^{1/3}$ and process its remaining layers $Q = p^a$ in decreasing order, for $Y < Q \le Q_*$. We require available denominators Qwb where $1 \le b \le Z$, $p \nmid b$, $p \nmid w$, and every prime power of wb is at most Y. We also require $Qwb \le N$ and $w \asymp N/(QZ)$. Here is a construction valid uniformly in p, Q.

Put $U_* = N/(QZ) \ge 16$ and $V_* = Y/Z$. Then $U_*/V_*^2 \le NZ/Y^3 = o(1)$. Choose two distinct primes $r, s \in \{2, 3, 5\} \setminus \{p\}$. Let r^e be the largest power of r not exceeding $\min(U_*, V_*)$ and let

s^f be the largest power of s not exceeding U_*/r^e. If $U_* \le V_*$, the second power is bounded; otherwise $U_*/r^e \le rU_*/V_* < V_*$ for large N. Thus

$$w = r^e s^f \asymp U_*, \quad w \le U_*, \quad r^e, s^f \le V_*.$$

Every prime power dividing wb is then at most $V_* Z = Y$. The full denominator Qwb has exactly p-adic exponent a and has no other prime-power divisor above Y.

These candidates are all present: they cannot meet an optimized prime or square block (each of those has a different prime base and defining power greater than Y), the closure pool, the initial top-power deletions, or earlier repairs of higher p-adic exponent. They also cannot meet earlier repairs for a different prime base. In particular the available set of candidates does not depend on which deletions were chosen at earlier stages.

Let S_A be the current total mass. Its reduced denominator has p-exponent at most a, by the initial deletion and the descending order of repairs. If $p > Z$, use Lemma 2.4 to take $B \subseteq [Z/16, Z]$ with $\mathcal{H}(B) \equiv wQS_A \pmod p$ and $\mathcal{H}(B) \le 1$. If $p \le Z$, take the empty set for zero target or a singleton $b \in [1, p-1]$ whose inverse has the required residue. Delete QwB. Then

$$Q(S_A - \mathcal{H}(QwB)) \equiv QS_A - \frac{\mathcal{H}(B)}{w} \equiv 0 \pmod p,$$

so the p-exponent of the reduced denominator falls to at most $a-1$. The cost is at most $1/(Qw) \ll Z/N$. No other prime's nonsmooth layer can be reintroduced. The number of layers is $O(N^{1/3}\ell)$, so their total cost is $O(N^{-2/3}\ell Z) = o(1/(\sqrt{N}\ell))$.

At the end, primes above $\sqrt{N}$ have already been cancelled, small-base nonsmooth powers have been corrected, and the square blocks for $N^{1/3} < p \le \sqrt{N}$ retain at most one p in the reduced denominator. The only remaining powers above Y are single primes $Y < p \le \sqrt{N}$ (note that $Y > N^{1/3}$ eventually).

5.3. **Two-denominator repair at medium primes.**

Lemma 5.1. *Uniformly for primes $Y < p \le \sqrt{N}$ and $q = \lfloor N/p \rfloor$, there exist at least $(3/4 - o(1))p$ Y-powersmooth integers in $[q-p+1, q] \cap [q/4, q]$. They give distinct nonzero residues modulo p. Every $h \in \mathbb{F}_p$ is a sum of inverses of two distinct such integers.*

Proof. The interval $[q-p+1, q]$ is a complete residue system, since $q \ge p$. At least $3p/4 - O(1)$ of its elements are at least $q/4$. We show that only $o(p)$ elements have a prime power exceeding Y.

If an element b has a prime factor $r > Y$, write $b = rd$ with $d \le q/Y \le \ell^{12}$. For a fixed d, r is restricted to an interval of length at most p/d. The classical Brun–Titchmarsh bound in modulus one (or the standard sieve upper bound for primes in a translated interval) gives $O((p/d)/\log(p/d)) = O(p/(d\ell))$ primes, because $p/d \ge \sqrt{N}/\ell^{18}$. Summing over d gives $O(p \log \ell/\ell) = o(p)$. Prime powers $r^a > Y$ with $a \ge 2$ account for at most

$$\sum_{Y < r^a \le q,\ a \ge 2} \left(\frac{p}{r^a} + 1 \right) \ll pY^{-1/2} \log Y + \sqrt{q} \log q = o(p).$$

The last estimate is uniform since $q \le \sqrt{N}\ell^6$ and $p > \sqrt{N}/\ell^6$. A remaining powersmooth element cannot be divisible by $p > Y$. Their inverses form a set $\mathcal{A}_p \subseteq \mathbb{F}_p$ of size greater than $2p/3$ for large N. For any h, $|\mathcal{A}_p \cap (h - \mathcal{A}_p)| \ge 2|\mathcal{A}_p| - p > p/3$. At most one solution in the intersection has equal summands because p is odd. Select a solution with distinct summands. □

For each prime $Y < p \le \sqrt{N}$, apply Lemma 5.1 with target $pS_A \pmod p$ to choose b_1, b_2. Delete pb_1, pb_2. They are available: their sole prime-power divisor above Y is p itself, to exponent one, so they belong to none of the already optimized or repaired layers or closure pool. They

have mass $O(1/N)$ since $b_i \geq q/4$ and $pq > N/2$. The deletion cancels p from the reduced denominator and does not affect any other nonsmooth prime. The combined cost of these repairs is $O(\pi(\sqrt{N})/N) = O(1/(\sqrt{N}\ell))$.

5.4. **Integer closure with two balancing scales.** After all repairs we have a working set A (excluding the reserved pool) with

$$C_2(N) - O(1/(\sqrt{N}\ell)) \leq \mathcal{H}(A) \leq C_2(N), \qquad \mathrm{den}(\mathcal{H}(A)) \mid \mathcal{L}(Y). \tag{32}$$

The upper bound holds because all deletions after initial optimization were outside the optimized blocks, preserving their local conditions. Define coarse and fine balancing sets

$$\mathcal{F}_0 = [N^{1/4}, 10N^{1/4}],$$
$$\mathcal{F}_1 = \{rs : N^{1/3} \leq r < s \leq 2N^{1/3},\ r, s \text{ prime}\}.$$

Both consist entirely of Y-powersmooth integers, hence have not been touched by any removal or reservation, and they are disjoint for large N. The prime-square blocks and the reserved t-block have defining prime powers exceeding Y. The first has mass $W_0 = \log 10 + o(1) > 2$ and individual mass $O(N^{-1/4})$; the second has mass $W_1 \asymp \ell^{-2}$ and individual mass $O(N^{-2/3})$.

Choose a sufficiently large absolute constant C and let $K = \left\lfloor C_2(N) - C/(\sqrt{N}\ell) \right\rfloor$. Then $\mathcal{H}(A) \geq K$ and $\mathcal{H}(A) - K < 1 + o(1)$. Temporarily take out both balancing sets. The remaining mass is below $K - 1$. Aim at $K - 1/(2t)$. Add coarse terms one at a time until the next would exceed that target or the coarse pool is exhausted. In the first case the remaining gap to the target is $O(N^{-1/4}) = o(W_1)$. In the second case the total available mass bound $\mathcal{H}(A) \geq K$ ensures that the fine pool can bridge the gap. Add fine terms until first attaining the target, using no term if it has already been attained. Each fine term is $o(1/t)$, so the new gap Δ to K is in $[1/(3t), 1/t]$. As all balancing terms are powersmooth, its denominator divides $\mathcal{L}(Y)$. Apply Lemma 2.3 to $t\Delta$, and use the reserved pool $t[X/16, X]$ to supply Δ exactly. The resulting set represents K. It follows that $J(N) \geq K$ and, by Theorem 1.3, $k(N) = J(N)$. This proves (5). Since $\mathcal{C}_{\mathrm{high}} \leq F = \mathcal{C}_{\mathrm{pp}} \leq C_2$ and $J \leq \lfloor \mathcal{C}_{\mathrm{high}} \rfloor$ by (15), the lower bound for C_2 immediately supplies the same two-sided estimate for both F and $\mathcal{C}_{\mathrm{high}}$. This completes Theorem 1.1.

6. The optimal exponent and rounding obstructions

6.1. **Bounds on the square correction.**

Lemma 6.1. *The correction in* (3) *satisfies*

$$\Delta_2(N) \ll \Phi(N), \qquad \liminf_{N\to\infty} \frac{\sqrt{N \log N}}{\log_2 N} \Delta_2(N) \geq \sqrt{2}. \tag{33}$$

Proof. For $p \in (N^{1/3}, \sqrt{N}]$ put $q = \lfloor N/p^2 \rfloor$. If $q \geq Z$, Lemma 2.4 gives $d(p,q)/p^2 \ll Z/N$, and there are at most $\pi(\sqrt{N/Z})$ such primes. If $q < Z$, use $d \leq H_q \ll \log(2Z)$ and $p > \sqrt{N/Z}$. By the prime number theorem and partial summation,

$$\Delta_2(N) \ll \frac{\sqrt{Z}\log(2Z)}{\sqrt{N}\log N} \ll \Phi(N).$$

For the lower bound fix $0 < \alpha < 1/2$ and $B > 1$, and put $P_0 = \sqrt{N/(\alpha \log N)}$. For primes $P_0 < p \leq BP_0$ (all in the square-layer range eventually) we have $q \leq \alpha \log N$ and $\mathcal{L}(q)H_q < p$. Hence no nonempty retained subset has residue zero, and $d(p,q) = H_q = \log_2 N + O_{\alpha,B}(1)$. Partial summation gives

$$\sum_{P_0 < p \leq BP_0} p^{-2} \sim \frac{1 - 1/B}{P_0 \log P_0}.$$

The resulting liminf is at least $2\sqrt{\alpha}(1-1/B)$. Let $B \to \infty$ and then $\alpha \uparrow 1/2$. This proves the lower bound. It is not an assertion that this normalization has a finite limit. □

6.2. **Optimality despite integer rounding.**

Proof of Theorem 1.2. Theorem 1.1 and Lemma 6.1 give (6). Since $\Phi(N) = o(N^{-1/2})$, the endpoint $c = 1/2$ is admissible, after enlarging a constant for finitely many small N.

To prove optimality put $f(N) = \log_2 N/\sqrt{N \log N}$. There is an absolute $a > 0$ such that $J(N) \le \lfloor \mathcal{C}(N) - af(N) \rfloor$ by the lower bound in (33). Lemma 2.6 gives $\mathcal{C}(N) - \mathcal{C}(N-1) = o(f(N))$. Define the first crossing heights

$$n_j = \min\{N : \mathcal{C}(N) \ge j\}, \qquad n'_j = \min\{N : \mathcal{C}(N) \ge j + af(n_j)/4\}.$$

Both exist and tend to infinity by Lemma 2.2. Their capacities are $j + o(1)$, hence by Lemma 2.5 their logarithms are $j - \gamma + o(1)$, and $n'_j/n_j \to 1$. In particular $f(n'_j) \sim f(n_j)$. Using the jump bound at the second crossing gives

$$\mathcal{C}(n'_j) - j = (a/4 + o(1))f(n'_j).$$

Therefore $J(n'_j) \le j - 1$, whereas $\left\lfloor \mathcal{C}(n'_j) - e(n'_j) \right\rfloor \ge j$ for any $e(N) = o(f(N))$ and all large j. Such a uniform approximation is impossible. For $c > 1/2$, $CN^{-c} = o(f(N))$, so $c_* \le 1/2$, proving the theorem. □

Corollary 6.2 (Failure of the uncorrected capacity formula)**.** *Let $n_j = \min\{N : \mathcal{C}(N) \ge j\}$. For every sufficiently large j, one has*

$$k(n_j) = J(n_j) = j - 1 < \lfloor \mathcal{C}(n_j) \rfloor = j.$$

In fact, with $f(N) = \log_2 N/\sqrt{N \log N}$ and Λ as in (16)*, this persists for all integer heights in*

$$n_j \le N \le n_j + c_1 \frac{n_j f(n_j)}{\Lambda(n_j)} \tag{34}$$

for an absolute $c_1 > 0$. In particular this includes intervals of length $\gg \sqrt{n_j}/(\log n_j)^4$.

Proof. Lemma 6.1 gives $\Delta_2(N) \ge af(N)$ for a fixed $a > 0$ and all large N. By the jump estimate, the excess $\mathcal{C}(n_j) - j$ is $O(\Lambda(n_j)/n_j) = o(f(n_j))$. The upper endpoint in (34) is $n_j + o(n_j)$. Uniformly on this interval $f(N) \sim f(n_j)$ and $\Lambda(N)/N \sim \Lambda(n_j)/n_j$. Summing the jump bound gives $\mathcal{C}(N) - j \le (Cc_1 + o(1))f(n_j)$ for an absolute C. For sufficiently small c_1 this is less than $\Delta_2(N)$. Thus $J(N) \le \lfloor C_2(N) \rfloor \le j - 1$, while monotonicity and Theorem 2.1 give $k(N) \ge j - 1$. Since $\mathcal{C}(N) = j + o(1)$ and $\mathcal{C}(N) \ge j$, its floor is j. The older singleton-square obstruction is also immediate: all p^2 with $\sqrt{N/2} < p \le \sqrt{N}$ are necessarily excluded and belong to no large-prime block. Their total mass is asymptotic to $2(\sqrt{2} - 1)/(\sqrt{N} \log N)$. The full square-layer bound gives the longer interval (34). □

Corollary 6.3 (Exceptional heights)**.** *For $x \to \infty$, all but $O(\sqrt{x}(\log x)^2)$ positive integers $N \le x$ satisfy $k(N) = J(N) = \lfloor \mathcal{C}(N) \rfloor$.*

Proof. Let $x \le a < b \le 2x$ be integers with $b - a \ge 18\sqrt{x}$. Take distinct primes $r, s \in [2x^{1/4}, 3x^{1/4}]$. There are $\gg (b-a)/(rs)$ multiples of rs in $(a, b]$, because $rs \le 9\sqrt{x}$. Every such multiple has all prime factors at most $\sqrt{x}$: its remaining cofactor is at most $2x/(rs) \le \sqrt{x}/2$. Each integer is counted at most three times, as it has at most three distinct prime factors in this interval. Summing over prime pairs by the prime number theorem gives $\gg (b-a)/(\log x)^2$

distinct integers. They can be added to a maximizing set for $\mathcal{C}(a)$ without violating any large-prime condition at height b. Thus

$$\mathcal{C}(b) - \mathcal{C}(a) \gg \frac{b-a}{x(\log x)^2}.$$

By the endpoint of Theorem 1.2, an exceptional height in $[x, 2x]$ must have $\mathcal{C}(N)$ in a band of width $O(x^{-1/2})$ immediately above an integer. There are only $O(1)$ such bands since $\mathcal{C}(N) = \log N + \gamma - o(1)$. By monotonicity, the integer indices in each band form an interval; the displayed lower bound makes its length $O(\sqrt{x}(\log x)^2)$. Summing over dyadic intervals proves the corollary. □

7. Harmonic zeros and the high-layer capacity gap

For an odd prime p put

$$Z_p = \{0 \le r < p : H_r \equiv 0 \pmod p\}.$$

In particular $0, p-1 \in Z_p$. The following two-point estimate is the key to both the metric result and the unshifted critical result.

Lemma 7.1 (Short-distance zero pairs)**.** *For $1 \le D < p$ the number of ordered pairs $(r, s) \in Z_p^2$ with $1 \le |r - s| \le D$ is at most $D(D-1)$. In particular $|Z_p| \ll p^{2/3}$.*

Proof. If $s = r + d$ and both are zeros, then $\sum_{i=1}^{d} (r+i)^{-1} = 0$ in $\mathbb{F}_p$. Since $r + d < p$, clearing the unit denominators gives a root of

$$P_d(X) = \sum_{i=1}^{d} \prod_{\substack{1 \le h \le d \\ h \ne i}} (X + h).$$

This is the derivative of $\prod_{i=1}^{d}(X+i)$, as in Sanna [8, Lemma 2.2]. It has degree $d-1$ and leading coefficient $d \not\equiv 0 \pmod p$. There are at most $d-1$ such pairs at distance d in the increasing direction. Summing and doubling proves the pair bound. Order the zeros. The number of consecutive gaps at most D is at most $D(D-1)/2$ and the number of larger gaps at most p/D. Take $D = \left\lfloor p^{1/3} \right\rfloor$. □

Define the fixed-cubic and variable-exponent counts

$$G_3(N) = \#\{p \text{ odd prime} : N/2 < p^3 \le N,\ \left\lfloor N/p^2 \right\rfloor - p \notin Z_p\},$$
$$G_*(N) = \#\{p \text{ odd prime} : N/2 < p^{a_p} \le N,\ p^{a_p - 1} > \sqrt{N},\ \left\lfloor N/p^{a_p - 1} \right\rfloor - p \notin Z_p\}.$$

Lemma 7.2 (Isolated second-layer obstruction)**.** *For all sufficiently large N, $G_*(N) \ge G_3(N)$, and*

$$F(N) - \mathcal{C}_{\text{high}}(N) \ge G_*(N)/N, \qquad J(N) \le \lfloor F(N) - G_*(N)/N \rfloor. \tag{35}$$

Proof. For a prime counted by G_* write $a = a_p$ and $r = \lfloor N/p^{a-1} \rfloor - p$, so $0 \le r < p$. The highest block consists just of p^a and its optimal retained part is empty. All denominators in the next layer have the form $p^{a-1}b$ with $1 \le b \le p + r$, $p \nmid b$. If retained in their entirety, their sum multiplied by p^{a-1} is $H_{p-1} + H_r \equiv H_r \not\equiv 0 \pmod p$. Thus a high-layer admissible set must omit at least one such denominator. It costs at least $1/N$. These next-layer blocks are disjoint from every highest-layer block and from each other, because their defining powers exceed $\sqrt{N}$

and their bases differ. The argument therefore adds costs to $F(N)$ without double counting. The cubic primes satisfy the variable-exponent conditions for large N. □

Lemma 7.3 (Exceptional heights in short grids)**.** *There exists $a > 0$ such that in each interval $[X, 1.1X]$ the integers satisfying $G_3(N) < aN^{1/3}/\log N$ can be covered by $O(X^{2/9})$ integer intervals of length $O(X^{2/3})$.*

Proof. Take the primes $0.9X^{1/3} \le p \le X^{1/3}$, whose number P_X is $\asymp X^{1/3}/\log X$. Their cubes are in $(N/2, N]$ for every height under consideration. For any such prime, the condition $\lfloor N/p^2 \rfloor - p = r$ holds on an interval of length p^2, and there are $O(X^{2/9})$ zero values of r by Lemma 7.1. Divide the height interval into consecutive grids of integer length $\lceil X^{2/3} \rceil$. Each bad interval for a given prime meets at most two grids (allowing three at endpoints also suffices). Hence the total number of incidences of a prime with a grid that contains some zero for that prime is $O(P_X X^{2/9})$. Any grid containing a height with $G_3(N) < aN^{1/3}/\log N$ must have at least $P_X/2$ such incidences, for a sufficiently small. The assertion follows by division. □

Summing the grid estimates over a geometric partition of $[1, x]$ gives $O(x^{8/9})$ exceptional integers. Together with (35), this proves (9).

Proof of (8). Start with a highest-layer optimal set, retaining all denominators outside the highest blocks. For each prime p repair the still unprocessed layers $Q = p^a > \sqrt{N}$, $a < a_p$, in descending order. Write S for the current reciprocal sum and $q = \lfloor N/Q \rfloor \ge p$. Because the higher layers have been repaired, QS is a p-integer. Every denominator Qb with $p \nmid b$, $b \le q$, still belongs to the set: it is at the current layer and cannot meet any other prime's high layer.

For $p \ge 11$, among the last p consecutive integers up to q at least $3p/4 - 2$ are at least $q/4$ and prime to p. Their inverse residues form a set U of size greater than $(p+1)/2$. For every residue h, $|U \cap (h - U)| \ge 2|U| - p > 1$, so two distinct elements sum to h (there is at most one diagonal solution). Delete the corresponding two denominators Qb with $h = QS \bmod p$. Each denominator is at least $Qq/4 > N/8$. For the finitely many smaller primes, use one inverse residue from the last p integers if $q \ge 2p$ (then the selected integer is at least $q/2$), or from $[1, p-1]$ otherwise (then $q < 2p$ is bounded), with a uniform $O(1/N)$ cost; the zero target uses no deletion. In all cases the reduced denominator of the new sum has p-adic exponent at most $a - 1$. The disjointness of different high-layer blocks ensures that no repaired constraint is reintroduced. At the end the denominator divides $\mathcal{L}(\sqrt{N})$, exactly the high-layer admissibility condition.

There are no nonhighest layers above $\sqrt{N}$ for $p > N^{1/3}$. For $N^{1/4} < p \le N^{1/3}$ there is at most one, and the total number for $p \le N^{1/4}$ is $O(N^{1/4} \log N)$. Thus there are $O(N^{1/3}/\log N)$ repairs. Their total cost is $O(N^{-2/3}/\log N)$, giving the upper bound and completing Theorem 1.4. □

8. Translated critical phases

Proof of Theorem 1.5. If $M_j(\alpha) = N$, then $F(N-1) < j + \alpha \le F(N)$. A bad grid $[u, v] \subset [X, 1.1X]$ from Lemma 7.3 therefore maps into the interval $(F(u-1), F(v)]$ modulo one. Its length is $O(X^{-1/3}\Lambda(X))$ by (16). Modulo one this is covered by at most two intervals. The grid covering and (35) show that at scale X the bad phases are covered by $O(X^{2/9})$ intervals of this length. Take $X = (1.1)^h$ and sum the s-powers of the covering diameters. For every $s > 2/3$, the sum

$$\sum_{h \ge 1} O(X^{2/9 - s/3} \Lambda(X)^s)$$

converges. The Hausdorff covering criterion for a limsup set proves the bound. In particular Lebesgue-almost every phase satisfies the lower bound eventually. A fixed phase, especially

zero, need not be outside a set of Hausdorff dimension less than one; the next section treats the integer phase separately. □

9. Nonresonance at integer critical indices

We now prove an unconditional obstruction at $\alpha = 0$, replacing the fixed cubic exponent by a variable exponent. We first prove a quantitative incidence estimate.

Theorem 9.1 (Variable-power incidence). *Set $P = T^{4/3}/(\log T)^4$ and let $G_{*,P}$ denote G_* restricted to $P \le p \le 2P$. Then for all large T,*

$$\sum_{T\le j\le 5T} G_{*,P}(M_j) \gg \frac{TP}{(\log T)^2}, \qquad \max_{T\le j\le 5T} G_{*,P}(M_j) \ll \frac{P}{(\log T)^2}.$$

Throughout the proof put $E = (\log T)^2/T$. Equation (22) implies the uniform asymptotic

$$\eta_j/E = \frac{T}{2j} + o(1) \qquad (T \le j \le 5T). \tag{36}$$

We do not differentiate this relation.

Lemma 9.2 (Good rational approximations). *Let $Q = \lfloor T/(100\log(2P))\rfloor$ and $q_0 = \left\lfloor \sqrt{P}/\log T \right\rfloor$. For all but $o(P/\log P)$ primes in $[P, 2P]$, there exist coprime integers q, h with*

$$q_0 \le q \le Q, \quad 1 \le h \le T/50, \quad q\log p = h + \epsilon_p, \qquad |\epsilon_p| \le 1/Q = o(E).$$

Proof. For fixed $1 \le q < q_0$, the condition $|q\log n - h| \le 1/Q$ on integers $P \le n \le 2P$ involves $O(q)$ values of h. Each gives a real interval of length $O(P/(qQ))$. There are $O(Pq_0/Q + q_0^2) = o(P/\log P)$ such integers in total, using $q_0 \asymp T^{2/3}/(\log T)^3$. For any prime outside this set, Dirichlet approximation gives $1 \le q \le Q$ with the displayed error. Dividing q, h by their common divisor preserves the error bound, so the reduced q cannot be smaller than q_0. The bound on h follows from the definition of Q. □

Lemma 9.3 (Macroscopic nonresonance). *For each prime of Lemma 9.2 there are positive integers s, d, $d \le T$, with $s\log p = d + \delta$, $|\delta| \le 4E$, and an interval I equal to $[T, 1.1T]$ or $[3T, 3.1T]$, such that for all integers $j \in I$,*

$$cE \le |\eta_{j+d} - \eta_j - \delta| \le 5E \tag{37}$$

for an absolute $c > 0$.

Proof. Put $m_0 = \lfloor T/h\rfloor$. If $m_0|\epsilon_p| \ge 4E$, choose $m = \lceil 2E/|\epsilon_p|\rceil \le m_0$ and set $(s, d, \delta) = (mq, mh, m\epsilon_p)$. Then $2E \le |\delta| \le 3E$ for all large T, and (36) gives $|\eta_{j+d} - \eta_j| \le E$ on the first interval. Otherwise take $m = m_0$. In this case $0.98T \le d \le T$ and $|\delta| < 4E$. Writing $u = j/T$ and $v = d/T$, we have

$$(\eta_{j+d} - \eta_j)/E = -v/(2u(u+v)) + o(1).$$

Its ranges on the two candidate intervals are contained respectively in $[-0.27, -0.19]$ and $[-0.06, -0.02]$ for large T. These intervals are separated by a positive absolute distance. For any value of δ/E one of the ranges is at distance at least 0.05 from that value. This proves the assertion with a fixed small c. □

Proof of Theorem 9.1, lower bound. Fix a good prime and the parameters of Lemma 9.3. For positive integer a, let

$$j(a) = \lceil \gamma + a\log p\rceil, \qquad \theta(a) = j(a) - \gamma - a\log p \in [0, 1).$$

Keep the exponents for which $j = j(a) \in I$ and $1/10 \le \theta(a) \le 2/5$. For these exponents, $M_j/p^a = \exp(\theta(a) + \eta_j) \in (1, 2)$ for large T. Moreover $j(a + s) = j(a) + d$, since $s\log p = d + \delta$

and the phases are bounded away from the endpoints. At the paired critical point $M_{j+d}/p^{a+s} = \exp(\theta(a) - \delta + \eta_{j+d}) \in (1, 2)$. Both are legitimate singleton highest layers, and their next layers exceed the respective square roots, since the heights are exponential in T while p is polynomial in T.

The corresponding indices of harmonic sums are

$$r = \lfloor p(\exp(\theta(a) + \eta_j) - 1) \rfloor ,$$
$$r' = \lfloor p(\exp(\theta(a) - \delta + \eta_{j+d}) - 1) \rfloor .$$

They lie in $[0, p-1]$. The mean value theorem, (37) and the unit rounding error yield $1 \le |r - r'| \le D = \lceil CpE \rceil < p$, since $pE \asymp T^{1/3}/(\log T)^2 \to \infty$.

It remains to show that the pairs are not excessively repeated. The exponents with $j(a) \in I$ form a consecutive interval of length $\asymp T/\log p$; it contains complete consecutive blocks of length q retaining $\gg T/\log p$ exponents since $q \le Q$. On each block the phases form a shifted uniform q-point grid modulo one with perturbation at most $1/Q \le 1/q$, by the coprimality of q, h. Thus $\gg q$ phases lie in the desired interval. A fixed value of r can occur at most $O(1 + qE + q/p) = O(1 + qE)$ times in such a block: the phase must lie in an interval of length $O(E + 1/p)$ by (36). By Lemma 7.1, at most $O(D^2(1 + qE))$ of the selected pairs can have both endpoints in Z_p. The parameter choices give

$$D^2/q \le D^2/q_0 \ll 1/\log T = o(1), \qquad D^2 E = o(1).$$

So a positive proportion of each block has at least one nonzero harmonic value. The map $a \mapsto j(a)$ is injective; for a fixed prime each critical index occurs at most once as a starting point and once as an endpoint of these pairs. It follows that this prime is counted by $G_*(M_j)$ for $\gg T/\log p$ different indices in $[T, 5T]$. Summing over the $\gg P/\log P$ good primes proves the lower bound. □

Proof of Theorem 9.1, upper bound. Fix j and put $w = \log M_j \asymp T$. If a prime $p \in [P, 2P]$ is counted then some integer a satisfies $w - \log 2 < a \log p \le w$. The possible a form an interval of length $O(T/(\log T)^2)$, and each a is $\asymp T/\log T$. For fixed a the allowed prime interval has length $O(P \log T/T)$ after intersecting with $[P, 2P]$. Enlarge if necessary to an interval of this length with an absolute constant. The classical Brun–Titchmarsh upper bound for primes in an interval of length y is $O(y/\log y)$ (see e.g. [11]). Here $\log y \asymp \log T$, hence the number of primes per a is $O(P/T)$. Summing over the possible a proves the uniform upper bound. □

Proof of Theorem 1.6. The two estimates in Theorem 9.1 imply that for $\gg T$ indices in $[T, 5T]$,

$$g = G_{*,P}(M_j) \gg T^{4/3}/(\log T)^6.$$

With $N = M_j$, (22) gives $\Lambda(N) \asymp j(\log j)^4(\log\log j)^4$. Therefore $g/\Lambda(N) \to \infty$ on these indices. Equation (35) and the jump bound imply $J(N) \le j - 1$ and $\lfloor F(N) \rfloor = j$.

We check persistence to prove the delay. Fix the primes counted in g and increase the height by an integer $u \le cg/\Lambda(N)$ with $c > 0$ small. The old second-highest powers are all far greater than $\sqrt{N + u}$, since N grows exponentially and $p \asymp P$ is polynomial in j. For any such prime the defining harmonic index, the upper endpoint $2p^{a_p}$ of its singleton window, or the value of its highest exponent can change only when a new multiple of its old second-highest power is crossed. Different such powers have different prime bases and product greater than $N + u$. Thus each newly admitted integer invalidates at most one of the original counted primes, giving $G_*(N + u) \ge g - u$. Meanwhile

$$F(N + u) - j \ll (u + 1)\Lambda(N)/N < (g - u)/(N + u)$$

for large j and a sufficiently small fixed c. Hence $J(N+u) \le j-1$ by (35). This establishes the stated delay and proves (10). Positive lower density follows by taking $T = \lfloor x/5 \rfloor$ in the incidence conclusion. To obtain the equality $k(M_j) = J(M_j) = j-1$, apply the preliminary packing estimate of Theorem 2.1 to $\mathcal{C}(M_j) \ge F(M_j) \ge j$. No packing estimate was required for the obstruction itself. □

10. Conditional cubic criterion and open problems

The variable-exponent method bypasses, rather than solves, the fixed-cubic problem. For clarity we retain the conditional connection to harmonic primes. Write $\pi_0(x)$ for the number of odd primes at most x with $Z_p = \{0, p-1\}$. The two trivial zeros can occur at most once each among primes in the cubic window: the functions $N/x^2 - x$ and $N/x^2 - 2x$ decrease with derivative at most -3 and -4 there. The corresponding integer conditions are membership in $[0,1)$ and $[-1,0)$, respectively. Thus

$$G_3(N) \ge \pi_0(N^{1/3}) - \pi_0((N/2)^{1/3}) - 2.$$

A positive asymptotic relative density for this set of primes implies $G_3(N) \gg N^{1/3}/\log N$ uniformly. The harmonic-prime density conjecture discussed in [9, 10] is a stronger sufficient condition, not an input to Theorem 1.6.

A weaker conditional statement also follows. Set $x_j = M_j^{1/3}$. By (22), $x_{j+1}/x_j \to e^{1/3}$, so the intervals $(2^{-1/3}x_j, x_j]$ are eventually disjoint. The prime number theorem and geometric summation show that their union up to x_j occupies asymptotic proportion

$$\vartheta = \frac{1 - 2^{-1/3}}{1 - e^{-1/3}}$$

of the primes up to x_j. The summation is justified by first keeping a fixed number of last intervals and then bounding the earlier geometric tail. If $G_3(M_j) = O(j^3)$ for all sufficiently large j, the total number of primes counted by π_0 in those intervals is $O(j^4)$, negligible compared with $\pi(x_j)$. Hence the assumption $\liminf_{x\to\infty} \pi_0(x)/\pi(x) > 1 - \vartheta$ implies $\limsup_j G_3(M_j)/j^3 = \infty$. This is sufficient for unshifted critical failures using (16) and (35); it remains a conditional statement.

The questions whether the unshifted cubic counts alone force infinitely many failures, whether all sufficiently large critical indices fail for the highest-layer capacity, and whether the full high-layer capacity is eventually exact after rounding remain open here. The proof of Theorem 1.4 addresses the unrounded difference of two relaxations and cannot settle the last question by itself. The separate harmonic subset-sum entropy results are not a premise of the obstruction theorems above; we turn to their consequences for precision and inverse heights next.

11. Consequences of the entropy-scale deletion estimates

The weighted deletion problem of [7] is exactly the loss $\mathcal{D}(N) = H_N - \mathcal{C}(N)$ in the present notation. In this section only, we use the two asymptotic estimates proved in that companion manuscript. We spell out the transfer to the rounded capacity and to inverse heights; no result of this section enters the earlier proofs.

Let $s(n)$ be the logarithm of the number of distinct rational reciprocal subset sums from $[1,n]$, and let $b(P) = \min\{n : s(n) \ge \log P\}$. Fix the same constant cutoff $A_{\mathrm{cut}} > 3$ as in [7], and write

$$\mathcal{C}_j(x) = \prod_{i \ge j:\, \log_i x \ge A_{\mathrm{cut}}} \log_i x, \qquad L = \log N, \quad U = \log_2 N, \quad V = \log_3 N,$$

$$B_N = b(\sqrt{N}), \qquad a_N = \frac{\sqrt{B_N}\log B_N}{\sqrt{N}\log N}.$$

An empty product is one. To avoid confusion, $\mathcal{C}_j$ denotes an iterated-logarithm product rather than a local capacity. The companion estimates are

$$b(P^t) \sim tb(P) \quad (t > 0 \text{ fixed}), \qquad b(P) \asymp \frac{\log P \log_2 P}{\mathcal{C}_4(P)}, \tag{38}$$

$$\mathcal{D}(N) = \frac{(U + V - \log \mathcal{C}_4(N))^2}{2L} + O(U/L), \tag{39}$$

$$2 \le \liminf_{N\to\infty} \frac{\Delta_2(N)}{a_N} \le \limsup_{N\to\infty} \frac{\Delta_2(N)}{a_N} \le 2\sqrt{\tau} < 2\sqrt{2}, \tag{40}$$

where, by the full-coverage and prime-square-loss theorems of [7], we may use the explicit sufficient factor

$$\tau = \frac{9200}{4601} < 2, \qquad 2\sqrt{\tau} = 2\sqrt{\frac{9200}{4601}}. \tag{41}$$

This is an imported bound, not a proof of optimality of the modular threshold or of the exact constant for Δ_2. Since $\mathcal{C}_4(\sqrt{N}) \asymp \mathcal{C}_4(N)$ (by the fixed-cutoff convention and the iterated-logarithm comparisons), in particular $a_N \asymp U^{3/2}/\sqrt{NL\mathcal{C}_4(N)}$ and $a_N\sqrt{N}\log N = \sqrt{B_N}\log B_N \to \infty$. Formula (39) is the refined weighted deletion expansion originating in the reciprocal-subsets manuscript, reproved via the entropy threshold in [7]. We use it as an imported theorem rather than repeat its proof. Likewise (40) does not determine an exact square-loss constant; the directed low-cost covering condition for constant 2 remains open.

Corollary 11.1 (Optimal entropy-scale order of uniform approximation)**.** *Using* (38) *and* (40)*, for every fixed $\varepsilon > 0$ and all sufficiently large N,*

$$\lfloor \mathcal{C}(N) - (2\sqrt{\tau} + \varepsilon)a_N \rfloor \le k(N) = J(N) \le \lfloor \mathcal{C}(N) - (2 - \varepsilon)a_N \rfloor . \tag{42}$$

No nonnegative $e(N) = o(a_N)$ can satisfy $k(N) \ge \lfloor \mathcal{C}(N) - e(N) \rfloor$ for all sufficiently large N. Thus a_N is the optimal order of the uniform uncorrected-capacity error in the sense of this rounded inequality. This does not identify a best uniform constant.

Proof. Use $C_2(N) = \mathcal{C}(N) - \Delta_2(N)$ in Theorem 1.1. The residual error $O(1/(\sqrt{N}\log N))$ is $o(a_N)$, giving (42). For optimality, choose a fixed $0 < a < 1$ such that $\Delta_2(N) \ge aa_N$ for large N. With the first crossings n_j of Corollary 6.2, take the first crossing n'_j of $j + aa_{n_j}/4$. Both capacities are $j + o(1)$ by the increment bound, so $n'_j/n_j \to 1$ by (21). Monotonicity of b and the first relation in (38), applied with fixed exponents $1 \pm \delta$ and then $\delta \downarrow 0$, imply $b(\sqrt{n'_j})/b(\sqrt{n_j}) \to 1$. Hence $a_{n'_j} \sim a_{n_j}$. The increment bound is $o(a_N)$, because $\Lambda(N)$ is polylogarithmic and $a_N\sqrt{N}\log N \to \infty$. We obtain

$$\mathcal{C}(n'_j) - j = (a/4 + o(1))a_{n'_j}.$$

But $J(n'_j) \le \left\lfloor \mathcal{C}(n'_j) - aa_{n'_j} \right\rfloor \le j - 1$, while $\left\lfloor \mathcal{C}(n'_j) - e(n'_j) \right\rfloor \ge j$ whenever $e = o(a_N)$. This gives the claimed obstruction for infinitely many heights. □

Corollary 11.2 (Iterated-logarithmic refinement of the common height)**.** *Using the deletion expansion* (39)*, the common minimum height has the expansion*

$$\log D(j) = \log P(j) = j - \gamma + \frac{(\log j)^2}{2j} + \frac{\log j \, \log_2 j}{j} - \frac{\log j}{j}\log \mathcal{C}_3(j) + O\left(\frac{\log j}{j}\right). \tag{43}$$

Equivalently, the correction after $j - \gamma$ is

$$\frac{(\log j + \log_2 j - \log \mathcal{C}_3(j))^2}{2j} + O(\log j/j).$$

Thus, after centering by the two terms in the scalar threshold theorem, the coefficient at the scale $(\log j)\log_3 j/j$ *is zero, and the coefficient at the next scale* $(\log j)\log_4 j/j$ *is* -1*. The analogous assertion for each subsequent fixed iterated logarithm uses successive centering; the constant at scale* $(\log j)/j$ *is not determined.*

Proof. Let $N = D(j) = P(j)$ for large j, as in Corollary 4.2. By minimality, $k(N-1) < j \le k(N)$. Necessity gives $\mathcal{C}(N) \ge j$. The lower bound in Theorem 1.1 and Lemma 6.1 give $k(N-1) \ge \lfloor \mathcal{C}(N-1) - C\Phi(N-1) \rfloor$ for some absolute C. Consequently $\mathcal{C}(N-1) < j + C\Phi(N-1)$, and (16) shows $\mathcal{C}(N) = j + O(\Phi(N) + \Lambda(N)/N)$. This error is exponentially small in j, since $L = j + O(1)$ by (21). It follows from $H_N = L + \gamma + O(1/N)$ that

$$L = j - \gamma + \mathcal{D}(N) + O(\Phi(N) + \Lambda(N)/N).$$

Substitute (39). For $L = j + O(1)$ we have $U = \log j + O(1/j)$, $V = \log_2 j + O(1/(j \log j))$, and $\mathcal{C}_4(N) = \mathcal{C}_3(L)$. The logarithms of $\mathcal{C}_3(L)$ and $\mathcal{C}_3(j)$ differ by $O(1)$: away from the cutoff this follows by the mean value theorem and summing the rapidly decreasing derivatives of the iterated logarithms; at a cutoff crossing the only extra factor is bounded. In detail, the successive derivative ratios are at most $1/A_{\mathrm{cut}}$ while the iterates remain above the cutoff, and at most one final factor can cross that cutoff when $|L - j| = O(1)$ and j is large. Thus replacing L by j changes the expression in (39) by $O(\log j/j)$. Expanding the square, its last quadratic term is $O((\log_2 j)^2/j) = o(\log j/j)$. This proves (43). The identification of the iterated-log coefficients follows from the definition of $\mathcal{C}_3$. □

11.1. **Improved increment and delay bounds from companion short covers.** We record a further transfer, separate from the independent proof of Theorem 1.6. Paper I, Theorem 1.3 of [5], gives bounded-mass full coverage for all sufficiently large primes at height $O(\log p \log_2 p (\log_3 p)^{12})$. Its weight bound is $1/2$. In the notation $\ell = \log N$, take

$$z_I = \left\lceil A_I \ell (\log \ell)(\log\log \ell)^{12} \right\rceil, \quad \Lambda_I(N) = \ell (\log \ell)^2 (\log\log \ell)^{12},$$

with A_I a sufficiently large absolute constant. The defining function for z_I is eventually increasing. For every prime $z_I < p \le N$, every residue has a representation $B \subseteq [1, z_I]$ with $\mathcal{H}(B) \le 1/2$. We emphasize that this conclusion does not place B in $[z_I/16, z_I]$, so it is used only for the scaled-deletion estimate and the increments, not for the reserved interval pools in the earlier completion proofs.

Here are the details of the scaling and its uniformity. For any power $Q = p^a \le N$ with $q = \lfloor N/Q \rfloor < p$, if $q \ge z_I$, then $v = \lfloor q/z_I \rfloor \ge 1$, $v < p$, and $v \ge q/(2z_I)$. Choose B with residue vH_q and delete vB. It is a subset of $[1, q]$, since $vz_I \le q$, and its weight is at most $1/(2v) \le z_I/q$. If $q < z_I$, delete all of $[1, q]$, at cost $H_q \le 1 + \log(2z_I)$. In both cases $Qq > N/2$, so uniformly

$$\frac{d(p,q)}{Q} \ll \frac{z_I \log(2z_I)}{N} \ll \frac{\Lambda_I(N)}{N}. \tag{44}$$

The argument of Lemma 2.6, including the prime-power transitions, uses only this scaled bound. In its application at $N - 1$ the corresponding functions are asymptotic to those at N. Consequently Paper I alone gives

$$0 \le K(N) - K(N-1) \ll \frac{\Lambda_I(N)}{N} \qquad (K = \mathcal{C}, F). \tag{45}$$

This is asymptotically stronger than (16), because $\Lambda_I(N)/\Lambda(N) = (\log\log \ell)^8/(\log \ell)^2 \to 0$.

If the all-prime bounded-mass coverage bound of [7] is also used, $g(p) \ll \log p \log_2 p \log_3 p/\mathcal{C}_5(p)$. Dropping the divisor $\mathcal{C}_5(p) \ge 1$, choose instead $z_E = \lceil A_E \ell (\log \ell)(\log\log \ell) \rceil$. The same proof

with $\mathcal{H}(B) \le 1$ gives

$$0 \le K(N) - K(N-1) \ll \frac{\Lambda_E(N)}{N}, \qquad \Lambda_E(N) = \ell(\log \ell)^2 \log\log \ell. \tag{46}$$

This last estimate requires the all-prime result of [7], not its density-one coverage theorem. The explicit factor in (41) is not used in either jump estimate.

For clarity, the persistence argument has the following explicit form. At a height $N = M_j$ supplied by Theorem 9.1, there are $g \gg j^{4/3}/(\log j)^6$ counted primes. Write $\Lambda_\bullet$ for either of the improved increment bounds, under its corresponding input. For all integer $u \ge 0$ with $u \le cg/\Lambda_\bullet(N)$, the original second-layer powers remain greater than $\sqrt{N+u}$, and the proof of Theorem 1.6 gives $G_*(N+u) \ge g - u$. The excess over the integer crossing is bounded by

$$F(N+u) - j \le C_\bullet(u+1)\Lambda_\bullet(N)/N.$$

Here $N \asymp e^j$, $u = o(N)$, and the ratio of $\Lambda_\bullet$ at these heights to its value at N tends uniformly to one. Since $\Lambda_\bullet(N) = o(g)$, choosing a sufficiently small fixed c makes the right side strictly less than $(g-u)/(N+u)$ for every such u and all sufficiently large j. By (35) this implies $J(N+u) \le j - 1$. The positive-density selection of indices is unchanged; it used only (22) and the incidence theorem. Thus under Paper I's short-cover theorem the delay in (10) can be replaced by

$$D_{\ge}(j) - M_j \gg \frac{j^{1/3}}{(\log j)^8 (\log\log j)^{12}}, \tag{47}$$

and with the additional all-prime input from [7] it can be strengthened further to

$$D_{\ge}(j) - M_j \gg \frac{j^{1/3}}{(\log j)^8 \log\log j}. \tag{48}$$

The constants and the initial thresholds may depend on the indicated companion bounds. Neither refinement asserts exactness of the all-high-layer relaxation or failure at every critical index. They are transfer consequences, rather than changes to the independent main proof.

12. Concluding remarks

The polynomial endpoint and the refined local relaxations show that a sharp logarithmic threshold does not by itself determine the integer at any particular critical crossing. At the highest-layer crossings the harmonic-zero obstruction forces positive lower density of failures, although the unrounded gap between the two stronger relaxations is extremely small. The uniform rounded estimate for $\mathcal{C}_{\text{high}}$ leaves an error of order $1/(\sqrt{N}\log N)$, larger than the typical gap of order $N^{-2/3}/\log N$; it cannot settle exactness of $\lfloor \mathcal{C}_{\text{high}} \rfloor$. It would be of interest to improve the residual completion error after imposing more prime-power layers, to settle the directed threshold and exact square-loss constant, and to determine whether the critical failures for F hold at every sufficiently large index. No such claim is made here. The general decomposition of integer-sum supports far from the near-capacity window also remains open in this work.

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