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Composite Fibonacci Shifts and Integer Parts of Pisot Powers at Sparse Exponents

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Abstract

For every integer h, we prove that $F_{2^n}+h$ is composite for infinitely many n, where F_k is the kth Fibonacci number. For every Pisot number α, we also prove that $\lfloor \alpha^{n+2^n} \rfloor$ is composite for all sufficiently large n in some infinite arithmetic progression. Here a Pisot number is a real algebraic integer greater than one whose other conjugates have absolute value less than one. The exponent sequence $R(n)=n+2^n$ satisfies a linear recurrence with integer coefficients and a constant inhomogeneous term, but none whose characteristic polynomial has constant term ± 1. These results answer two questions of Saito. For his question about $\lfloor \beta^{3^n} \rfloor$, we prove that these numbers are composite for infinitely many n whenever β is a cubic Pisot number with nonreal conjugates. Our divisibility conditions apply to a proportion of cubic Pisot numbers $\beta \le X$ tending to one as $X \to \infty$. Some individual totally real cases remain open.

Keywords. Fibonacci numbers; Pisot numbers; composite values; linear recurrences; congruences.
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1 Introduction and main results

We prove compositeness results for shifted Fibonacci numbers and integer parts of powers of Pisot numbers at exponentially growing indices and exponents.

Throughout, a composite number is a positive integer greater than one that is not prime, and $\lfloor x \rfloor$ is the greatest integer at most x. A *Pisot number* is a real algebraic integer $\alpha > 1$ all of whose other conjugates over $\mathbb{Q}$ have absolute value less than one. Its degree is $[\mathbb{Q}(\alpha):\mathbb{Q}]$; it is *totally real* if all these conjugates are real. This convention includes integers greater than one as Pisot numbers of degree one. A *quadratic unit* is an algebraic integer of degree two whose minimal polynomial has constant term 1 or -1.

Forman and Shapiro proved that $\lfloor \alpha^n \rfloor$ is composite infinitely often for $\alpha = 3/2$ and $\alpha = 4/3$ [8], and Cass established this for every quadratic unit $\alpha > 1$ [4]. Dubickas proved infinite compositeness of $\lfloor \alpha^k \rfloor$ for Pisot numbers in [6] and obtained simultaneous divisibility of bounded shifts of recurrence values in [7]. Saito extended this approach to compositions of recurrence sequences [11]. His results impose a reversibility condition on the inner sequences. To state the questions addressed here, an integer sequence $R(n)$, $n \ge 1$, satisfies an *integer inhomogeneous linear recurrence* if, for some $d \ge 1$ and integers $a_0, \dots, a_{d-1}, b$ with $a_0 \ne 0$,

$$R(n+d) = \sum_{j=0}^{d-1} a_j R(n+j) + b \qquad (n \ge 1). \tag{1.1}$$

A representation of this form is *reversible* when $|a_0| = 1$. We will exhibit a sequence that has no reversible representation.

Let $F_0 = 0$, $F_1 = 1$, and $F_{k+2} = F_{k+1} + F_k$ define the Fibonacci sequence. The problem numbers below refer to Saito's arXiv:2504.14968v1 (21 April 2025) [11]. Saito asks whether every fixed integer shift of F_{2^n} is composite infinitely often (Problem 1.8), asks for an integer inhomogeneous linear recurrence sequence $R(n)$ admitting no reversible representation and giving infinite compositeness of $\lfloor \alpha^{R(n)} \rfloor$ for every Pisot number (Problem 1.7), and asks whether $\lfloor \beta^{3^n} \rfloor$ is composite infinitely often for every cubic Pisot number β (Problem 1.1). We answer the first two questions and give partial results on the third.

Theorem 1.1 (shifted Fibonacci values)**.** *For every integer h, the number $F_{2^n} + h$ is composite for infinitely many integers $n \geq 1$. More precisely, there are a prime p and integers $n_0, P \geq 1$ such that*

$$p \mid F_{2^{n_0+jP}} + h \qquad (j \geq 0).$$

Consequently all sufficiently large terms on this progression are composite.

A classical analogue is the infinitude of composite values of $2^{2^n} + k$ for each integer $k \neq 1$, proved by an argument of Schinzel [13, Problem 115, pp. 10, 62–63]. In the Fibonacci setting, prime divisors may also be congruent to -1 modulo 2^n; the adjacent-index congruence (2.4) handles this additional possibility.

Theorem 1.2 (a non-reversible exponent sequence)**.** *The sequence $R(n) = n + 2^n$, $n \geq 1$, is positive and strictly increasing, satisfies an integer inhomogeneous linear recurrence, and admits no reversible representation* (1.1)*. For every Pisot number α, there are integers $n_0, P \geq 1$ such that $\lfloor \alpha^{R(n_0+jP)} \rfloor$ is composite for all sufficiently large integers j.*

The proof of Theorem 1.2 uses a useful property of its exponents: for every $m \geq 1$ and every residue r modulo m, the congruence $n + 2^n \equiv r \pmod{m}$ holds on an infinite arithmetic progression. Thus a suitable trace exponent can be selected first and then revisited by $R(n)$.

Theorem 1.3 (cubic Pisot numbers with nonreal conjugates)**.** *Let β be a cubic Pisot number whose two other conjugates are nonreal. Then $\lfloor \beta^{3^n} \rfloor$ is composite for infinitely many integers $n \geq 1$.*

Earlier constructions give special cases with compositeness at all sufficiently large integer exponents. Alkauskas and Dubickas constructed Pisot numbers of arbitrary degree for which every $\lfloor \alpha^m \rfloor$, $m \geq 1$, is composite [1]. Hahn, Ismailescu, Kim and Kim give further explicit families, including cubic Pisot numbers with nonreal conjugates and totally real cubics [9, Theorems 4, 5 and 7]. Theorem 1.3 treats every cubic Pisot number with nonreal conjugates at the sparse exponents 3^n; the residue criteria and density results below also treat totally real cubics.

For a cubic Pisot number, write the minimal polynomial as $X^3 - aX^2 + bX - c$; its coefficient triple is $(a, b, c) \in \mathbb{Z}^3$, and its norm is c, the product of the three conjugates. For totally real cubics, the same divisibility argument gives explicit sufficient conditions. Tests modulo 2, 3 and 5 cover 25 188 of the 27 000 coefficient triples modulo 30, irrespective of the conjugates' signs (Proposition 5.2). More substantially, Theorem 6.1 proves that a proportion tending to one of cubic Pisot numbers satisfy the required divisibility conditions when ordered by their real value in any fixed coefficient residue class. This remains true at each compatible fixed nonzero norm and separately for either possible sign of the sum of the two small real conjugates. These density statements do not decide every individual totally real cubic.

The cubic question is motivated by Mills' constant, the least real number $\xi > 1$ for which $\lfloor \xi^{3^n} \rfloor$ is prime for every $n \geq 1$ [10]. Saito's later work [12, Theorem 1.7] shows that if this constant were algebraic, then, for some integer $m \geq 1$, the power $\beta = \xi^{3^m}$ would be a totally real cubic Pisot number with additional restrictions on its conjugates. An affirmative answer to the cubic question

would exclude this algebraic possibility, since $\lfloor \beta^{3^n} \rfloor = \lfloor \xi^{3^{m+n}} \rfloor$ is prime for every $n \geq 1$. Saito proved that the Riemann Hypothesis implies the transcendence of Mills' constant [12, Theorem 1.8]. Combined with Alpöge's proof of the Riemann Hypothesis [2], this establishes the transcendence of Mills' constant. The proofs of our compositeness results do not use the Riemann Hypothesis (RH) or the Generalized Riemann Hypothesis (GRH).

A fixed prime divisor can prove compositeness at infinitely many indices of an integer sequence. The difficulty for a sequence sampled at exponents 2^n or 3^n is to make a divisor return: periodicity in the exponent does not by itself imply that the starting exponent is visited again. We identify divisors that return by examining the largest power of the exponential base, 2 or 3, that divides the period. When every prime divisor fails this test, the value of the sequence satisfies a restrictive congruence.

Section 2 proves Theorem 1.1. Section 3 proves Theorem 1.2. Sections 4 and 5 develop the cubic trace argument and prove Theorem 1.3 and the finite residue classification. Section 6 proves the density theorem and explains a limitation of tests using a prescribed finite set of primes. Appendix A gives exact verification procedures for the finite tables and supplementary certificates.

2 Shifted Fibonacci values

For odd h, parity suffices, and for $h = 0$ a doubling identity gives the fixed divisor three. For the other shifts, we prove that any sufficiently late pair of consecutive indices contains a prime divisor that returns on a progression. The contradiction uses two congruences for the Fibonacci and Lucas numbers; no distribution of primes is needed.

For a prime q and a positive integer d, write $v_q(d)$ for the largest integer $e \geq 0$ such that $q^e \mid d$. If $\gcd(b, d) = 1$, write $\mathrm{ord}_d(b)$ for the least positive P with $b^P \equiv 1 \pmod{d}$, and put $\mathrm{ord}_1(b) = 1$.

Lemma 2.1 (returning exponents)**.** *Let b be a prime and let an integer sequence $U(k)$, $k \geq 1$, be periodic modulo a prime p with period d. Write $d = b^e t$ with $\gcd(b, t) = 1$. If $n \geq e$, then*

$$U(b^{n+jP}) \equiv U(b^n) \pmod{p} \qquad (j \geq 0), \qquad P = \mathrm{ord}_t(b).$$

In particular, any divisor $p \mid U(b^n) - h$ at such an index divides $U(b^{n+jP}) - h$ for all $j \geq 0$.

Proof. Both exponents vanish modulo b^e and agree modulo t. The Chinese remainder theorem gives their equality modulo d, so periodicity applies. □

Put $Q = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. The identity

$$Q^k = \begin{pmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{pmatrix} \qquad (k \geq 1) \tag{2.1}$$

follows by induction. Its reduction modulo p is invertible. Let $\pi(p)$ be the order of this matrix in the finite group $\mathrm{GL}_2(\mathbb{F}_p)$; this is a period of F_k modulo p.

The following classical bounds for Fibonacci periods may be found in Wall [15]; we include a short proof.

Lemma 2.2 (the even part of the Fibonacci period)**.** *One has $\pi(2) = 3$ and $\pi(5) = 20$. For every odd prime $p \neq 5$,*

$$\pi(p) \mid p - 1 \quad \text{if } 5 \text{ is a square modulo } p, \qquad \pi(p) \mid 2(p + 1) \quad \text{otherwise.}$$

Consequently, if $n \geq 2$ and $v_2(\pi(p)) > n$, then $p \equiv 1$ or $-1 \pmod{2^n}$.

Proof. The characteristic polynomial of Q is $X^2 - X - 1$. For $p \neq 2, 5$ it has distinct nonzero roots r, s in $\mathbb{F}_{p^2}$. If 5 is a square, both roots lie in $\mathbb{F}_p$ and their orders divide $p-1$, so $Q^{p-1} = I$. Otherwise Frobenius interchanges the roots: $r^p = s$ and $s^p = r$. Since $rs = -1$, each root has $(p+1)$st power -1, giving $Q^{2(p+1)} = I$. For $p = 2$, direct multiplication gives order three. Modulo 5, $Q = 3I + N$ with $N \neq 0$ and $N^2 = 0$, so $Q^k = 3^k I + k3^{k-1}N$. Equality with I requires $4 \mid k$ and $5 \mid k$, and is attained at $k = 20$. The final claim now follows: in the split case $2^{n+1} \mid p-1$, and in the other case $2^n \mid p+1$. The primes $2, 5$ cannot satisfy the assumed inequality when $n \geq 2$. □

Define the Lucas numbers by $L_0 = 2$, $L_1 = 1$ and $L_{k+2} = L_{k+1} + L_k$. If r, s are the roots of $X^2 - X - 1$ over $\mathbb{R}$, induction gives $L_k = r^k + s^k$ and $F_k = (r^k - s^k)/(r - s)$. In particular,

$$F_{2k} = F_k L_k, \qquad L_{2k} = L_k^2 - 2(-1)^k. \tag{2.2}$$

Since $L_2 = 3$, the second identity implies by induction that

$$L_{2^n} \equiv -1 \pmod{2^{n+1}} \qquad (n \geq 1). \tag{2.3}$$

Indeed, if $L_{2^n} = -1 + 2^{n+1}u$, then $L_{2^{n+1}} + 1 = -2^{n+2}u + 2^{2n+2}u^2$. Writing $A_n = F_{2^n}$, we obtain

$$A_{n+1} \equiv -A_n \pmod{2^{n+1}} \qquad (n \geq 1). \tag{2.4}$$

Proof of Theorem 1.1. Modulo 2 the Fibonacci sequence repeats $0, 1, 1$, so A_n is odd for every $n \geq 0$. If h is odd, $A_n + h$ is even and exceeds two for all sufficiently large n, proving both assertions with $p = 2$. For $h = 0$, $F_4 = 3$ and (2.2) show inductively that $3 \mid F_{2^n}$ for every $n \geq 2$. Thus $p = 3$, $n_0 = 2$, and $P = 1$ give the required progression, and the terms are composite for $n \geq 3$.

Suppose now that h is even and nonzero. Choose $n \geq 2$ so large that $A_n + h > 1$ and $2^n > 2|h| + 2$. At an index k, call a prime divisor $p \mid A_k + h$ suitable if $v_2(\pi(p)) \leq k$. If no prime divisor at k is suitable, Lemma 2.2 shows that every prime factor of $A_k + h$ is 1 or -1 modulo 2^k. Multiplying these residues, with their multiplicities, gives

$$A_k + h \equiv \varepsilon_k \pmod{2^k}, \qquad \varepsilon_k \in \{-1, 1\}.$$

If neither n nor $n+1$ has a suitable divisor, adding their congruences and using (2.4) yields

$$2h \equiv \varepsilon_n + \varepsilon_{n+1} \pmod{2^n}.$$

The absolute value of the difference is at most $2|h| + 2 < 2^n$, so $2h = \varepsilon_n + \varepsilon_{n+1} \in \{-2, 0, 2\}$. This is impossible for an even nonzero h.

There is therefore a suitable prime divisor at n or $n+1$. Lemma 2.1, with $b = 2$ and $d = \pi(p)$, makes this fixed prime recur on an infinite progression. The values tend to infinity, so this divisor is proper at every sufficiently large index on it. □

3 An exponent sequence reaching every residue

The trace of a Pisot power differs from that power by a quantity tending to zero. The floor is consequently one of two consecutive integers. We will choose one prime divisor for each possibility at a single exponent and arrange for both to return.

Sun [14, Theorem 1.1] proved that, for integers a, b and a positive integer m with $\gcd(b, m) = 1$, the set $\{a^n + bn : 1 \leq n \leq m^2\}$ contains every residue modulo m. For the case $a = 2$, $b = 1$, the following lemma gives uniform distribution and recurrence on infinite arithmetic progressions.

Lemma 3.1 (residues of $n + 2^n$)**.** *For every positive integer m, the sequence $R(n) = n + 2^n$ is uniformly distributed modulo m: each residue occurs with limiting frequency $1/m$. Moreover, each residue is attained on an infinite arithmetic progression of indices.*

Proof. The assertion is immediate for $m = 1$. For $m \geq 2$, put $D = \operatorname{lcm}(1, 2, \ldots, m)$. If p^e is the exact power of a prime p dividing D, then $p - 1 \mid D$. For odd p, also $(p-1)p^{e-1} \mid D$, since p is coprime to $p - 1$. Euler's theorem gives $2^D \equiv 1 \pmod{p^e}$. For $n \geq \max(1, v_2(D))$, the value 2^n vanishes modulo the power of two dividing D. Thus $R(n + D) \equiv R(n) \pmod{D}$ whenever $n \geq \max(1, v_2(D))$. For such indices, $R(n) \bmod D$ depends only on $n \bmod D$, so R induces a well-defined map on $\mathbb{Z}/D\mathbb{Z}$.

We show that this map is injective. Suppose sufficiently large u, v satisfy $R(u) \equiv R(v) \pmod{D}$. Then $u \equiv v$ modulo the full power of two dividing D. Process the odd primes dividing D in increasing order. Every prime-power factor of $p - 1$ has already been processed, because it divides D and its prime is less than p. Hence $u \equiv v \pmod{p-1}$. For $j = 1, \ldots, e$, assume inductively that $u \equiv v \pmod{p^{j-1}}$. Thus their difference is divisible by $(p-1)p^{j-1}$, and Euler's theorem gives $2^u \equiv 2^v \pmod{p^j}$. Subtracting from the assumed equality of $R(u)$ and $R(v)$ yields $u \equiv v \pmod{p^j}$. This proves $u \equiv v \pmod{D}$.

The finite map is therefore a permutation. Every block of D sufficiently late indices visits each residue modulo m exactly D/m times. Each particular residue modulo D recurs at indices differing by D, which proves the progression assertion as well. □

Lemma 3.2 (power sums modulo a prime)**.** *Let $f \in \mathbb{Z}[X]$ be monic of degree d, with complex roots $\alpha_1, \ldots, \alpha_d$ counted with multiplicity, and put $S(k) = \sum_{i=1}^{d} \alpha_i^k$. Then $S(k)$ is an integer for every $k \geq 1$. For every prime p, let d_p be the least common multiple of the multiplicative orders of the nonzero roots of $f \bmod p$ in a finite splitting field, with the empty least common multiple equal to one. The sequence $S(k) \bmod p$ has period d_p for all $k \geq 1$.*

Proof. Let C be the integral matrix of multiplication by X on $\mathbb{Z}[X]/(f)$ in the basis $1, X, \ldots, X^{d-1}$. Its characteristic polynomial is f. The equality $S(k) = \operatorname{tr}(C^k)$ gives integrality. After reduction modulo p, triangularize C over a finite splitting field. Its diagonal consists of the roots with their multiplicities, so the trace of C^k is their kth power sum. Zero roots contribute zero for every positive k, and every other root has order dividing d_p. This proves periodicity, including when $f \bmod p$ has repeated roots or zero roots. □

Proof of Theorem 1.2. The identity

$$R(n + 2) = 3R(n + 1) - 2R(n) - 1$$

gives the required recurrence, and positivity and strict increase are immediate. Suppose another representation (1.1) holds and set $P(X) = X^d - \sum_{j=0}^{d-1} a_j X^j$. For the shift operator E, defined by $(EU)(n) = U(n+1)$, this means $P(E)R = b$. Taking a first difference and using $R(n+1) - R(n) = 1 + 2^n$ gives

$$P(1) + P(2)2^n = 0 \qquad (n \geq 1).$$

Subtracting consecutive equalities proves $P(2) = 0$. Reduction modulo two then shows that a_0 is even. In particular no reversible representation exists.

Fix a Pisot number α and let $S(k)$ be the power sum of all its conjugates. The other conjugates have modulus less than one, so $S(k) - \alpha^k \to 0$. For every sufficiently large k,

$$\lfloor \alpha^k \rfloor \in \{S(k), S(k) - 1\}. \tag{3.1}$$

This also holds in degree one, when $S(k) = \alpha^k$ exactly. Choose a positive integer k_0 with $S(k_0) \geq 3$, and choose primes $p_0 \mid S(k_0)$ and $p_1 \mid S(k_0) - 1$. Let m be the least common multiple of the two periods in Lemma 3.2. By Lemma 3.1, $R(n) \equiv k_0 \pmod{m}$ on an infinite arithmetic progression. On that progression $p_0 \mid S(R(n))$ and $p_1 \mid S(R(n)) - 1$. For sufficiently large indices, (3.1) applies and the floor exceeds both primes. It is then composite whichever of its two possible values occurs. □

4 Cubic traces and recurring divisors

Let β be a cubic Pisot number, and write its minimal polynomial and conjugates as

$$f(X) = X^3 - aX^2 + bX - c, \qquad \beta, \gamma, \eta. \tag{4.1}$$

Thus $a, b, c \in \mathbb{Z}$, $c \neq 0$, and $|\gamma|, |\eta| < 1$. The integer $c = \beta\gamma\eta$ is the norm of β. Put

$$T(k) = \beta^k + \gamma^k + \eta^k, \qquad t_n = T(3^n) \quad (n \geq 0).$$

These are integers by Lemma 3.2. Once $|\gamma|^k + |\eta|^k < 1$, we have the exact identity

$$\lfloor \beta^k \rfloor = T(k) - \mathbf{1}_{\gamma^k + \eta^k > 0}, \tag{4.2}$$

where $\mathbf{1}$ is one when its subscripted condition holds and zero otherwise. The condition on the conjugates holds for every sufficiently large k; it need not hold at the initial exponents.

If γ, η are real, the sign of their sum of odd powers equals the sign of $\gamma + \eta = a - \beta$. Indeed, for odd k the function $x \mapsto x^k$ is strictly increasing, so the signs of $\gamma^k - (-\eta)^k$ and $\gamma - (-\eta)$ agree. The sum $a - \beta$ is nonzero because β has degree three. Thus in the totally real case the shift in (4.2) at $k = 3^n$ is the fixed value

$$h = \mathbf{1}_{a > \beta} \in \{0, 1\}. \tag{4.3}$$

With nonreal conjugates the shift may vary. We first obtain conditions that allow both possibilities.

4.1 The divisibility test

For a prime p, let d_p be the period supplied by Lemma 3.2 for f. Every nonzero root of an irreducible factor of degree $j \leq 3$ lies in $\mathbb{F}_{p^j}$, and hence

$$d_p \mid \operatorname{lcm}(p - 1, p^2 - 1, p^3 - 1). \tag{4.4}$$

Call p a *recurring divisor for shift h at index n* if

$$p \mid t_n - h, \qquad v_3(d_p) \leq n. \tag{4.5}$$

This definition is a sufficient test for recurrence, not a necessary condition for all possible repetitions of the trace residue. Lemma 2.1 shows that such a prime divides $t_{n+jP} - h$ for every $j \geq 0$, for a suitable $P \geq 1$.

Lemma 4.1 (absence of recurring divisors)**.** *Let h be an integer and $n \geq 1$. If $t_n - h > 1$ has no recurring divisor at n, then*

$$t_n - h \equiv 1 \quad \text{or} \quad -1 \pmod{3^n}.$$

Proof. Each prime $p \mid t_n - h$ must have $v_3(d_p) > n$. For $p = 3$ the right side of (4.4) is prime to three, so this is impossible. If $p \equiv 1 \pmod 3$, the three valuations on the right are $v_3(p-1)$, $v_3(p-1)$ and $v_3(p-1)+1$. For the last equality, write $p = 1 + 3u$ and note that $p^2 + p + 1 = 3(1 + 3u + 3u^2)$ has valuation one. Thus failure of the test implies $3^n \mid p - 1$. If $p \equiv -1 \pmod 3$, only $p^2 - 1$ has a factor of three, with valuation $v_3(p+1)$; failure implies $3^{n+1} \mid p + 1$. Every prime factor is therefore 1 or -1 modulo 3^n. Multiplication, retaining prime-factor multiplicities, proves the result. □

4.2 A trace limit depending only on the coefficients modulo three

We use $\mathbb{Z}_3$ for the ring of compatible residues modulo all powers of three. In this ring $x \equiv y \pmod{3^j}$ means that their residues modulo 3^j agree. A sequence converges when each such residue is eventually constant. Cubing three roots changes their elementary symmetric functions by the integral polynomial map

$$\Phi(a,b,c) = \bigl(a^3 - 3ab + 3c,\ b^3 - 3abc + 3c^2,\ c^3\bigr). \tag{4.6}$$

For example, the first coordinate is the sum of the cubes; the second follows by applying the same identity to the three products of pairs of roots. The first coordinate of $\Phi^n(a,b,c)$ is t_n.

Lemma 4.2 (the cubic trace limit)**.** *The iterates of Φ converge in $\mathbb{Z}_3^3$ to a fixed triple that depends only on (a,b,c) mod 3. If $\lambda \in \mathbb{Z}_3$ is its first coordinate, then*

$$t_n \equiv \lambda \pmod{3^{n+1}} \qquad (n \geq 0). \tag{4.7}$$

Proof. The map Φ fixes each residue triple modulo three. If two triples agree modulo 3^j, with $j \geq 1$, their images agree modulo 3^{j+1}. For a cube this follows by factoring $x^3 - y^3$; the factor $x^2 + xy + y^2$ is divisible by three when $x \equiv y \pmod 3$. In every remaining nonconstant term of (4.6), the coefficient is divisible by three.

Since the first two iterates agree modulo three, successive iterates at steps n and $n+1$ agree modulo 3^{n+1}. They consequently converge, and the same congruence with the limit gives (4.7). Continuity of the polynomial map makes the limit fixed. Starting from two triples that agree modulo three gives iterates agreeing modulo 3^{n+1}, so their limits are equal. □

Proposition 4.3 (two possible floor shifts)**.** *Fix $h \in \{0,1\}$. If $\lambda - h \notin \{-1,1\}$ in $\mathbb{Z}_3$, then $t_n - h$ has a recurring divisor at every sufficiently large n. If this holds for both shifts, $\lfloor \beta^{3^n} \rfloor$ is composite for every sufficiently large index on an infinite arithmetic progression. The same conclusion holds if one shift satisfies this limit condition and the other has a fixed prime divisor throughout some infinite arithmetic progression of indices.*

Proof. The two nonzero 3-adic integers $\lambda - h - 1$ and $\lambda - h + 1$ are nonzero modulo some fixed power of three. For all sufficiently large n, (4.7) therefore excludes both congruences in Lemma 4.1. Also $t_n - h > 1$, since $t_n \to \infty$. This proves the first assertion.

If both shifts satisfy the limit condition, choose a recurring divisor for each at a common sufficiently large index and take a common multiple of their return periods. For the final assertion, choose a sufficiently large index n_0 on the given progression for the second shift. This is possible because the first shift has a recurring divisor at every sufficiently large index. Choose such a divisor at n_0, with return period P. If the given progression has step Q, both fixed divisibilities hold at $n_0 + j\,\mathrm{lcm}(P,Q)$ for every $j \geq 0$. Equation (4.2) makes one of these two fixed primes a divisor of the floor; it is proper once the floor exceeds both primes. □

The limit is determined by only 27 residue triples. Table 1 lists its centered residue modulo 81, meaning the representative between -40 and 40. Three iterations of (4.6) suffice by (4.7). For entries in $\{-1,0,1,2\}$ the displayed value is also the exact 3-adic integer λ. To see this, lift the reduced polynomial either to a product of three factors with roots in $\{0,1,-1\}$, or to $(X-r)(X^2+1)$ with $r \in \{0,1,-1\}$. For each of those ten entries one of these lifts has the prescribed coefficient residues and the displayed trace. Cubing permutes its multiset of roots, so its coefficient triple is fixed by Φ. Lemma 4.2 then identifies the limit of every lift. The ten exact values are certified

Table 1: The cubic trace limit modulo 81. Column labels for c are residues modulo 3.

$a \bmod 3$	$b \bmod 3$	$c=0$	$c=1$	$c=2$
0	0	0	3	−3
0	1	0	21	−21
0	2	0	39	−39
1	0	1	−23	7
1	1	−2	1	4
1	2	22	−29	1
2	0	−1	−7	23
2	1	2	−4	−1
2	2	−22	−1	29

by the integral fixed triples in Table 2 of Appendix A. All other entries already exclude the four exceptional values modulo 81.

There are seventeen classes in which both shifts satisfy the limit condition. In the other ten, exactly one shift needs an additional argument: $h=0$ when $\lambda=-1$ or 1, and $h=1$ when $\lambda=0$ or 2. In particular, every totally real cubic with $a>\beta$ and $3\nmid c$ is covered, because its required shift is $h=1$, whereas the entries 0 and 2 occur only in the column $c=0$.

5 Nonreal conjugates and finite residue tests

5.1 What parity supplies

There are eight monic cubic polynomials over $\mathbb{F}_2$. The two irreducible ones have roots of order seven in $\mathbb{F}_8$. Let $\zeta\in\mathbb{F}_8$ be a root of the irreducible reduction $f \bmod 2$. Its conjugates are ζ,ζ^2,ζ^4, so

$$t_n \bmod 2 = \mathrm{Tr}_{\mathbb{F}_8/\mathbb{F}_2}(\zeta^{3^n}) = \zeta^{3^n}+\zeta^{2\cdot 3^n}+\zeta^{4\cdot 3^n}.$$

Since 3 has order six modulo seven, the powers ζ^{3^n} run through $\mathbb{F}_8\setminus\{0,1\}$. The map $x\mapsto x+x^2+x^4$, which is the field trace, is a nonzero linear map from the three-dimensional $\mathbb{F}_2$-space $\mathbb{F}_8$ to $\mathbb{F}_2$, and takes value one at 1. On $\mathbb{F}_8\setminus\{0,1\}$ it therefore takes each value three times. Both trace parities recur on progressions of indices. In particular, the prime two supplies whichever shift is not covered by Proposition 4.3.

For the other six reductions, t_n has constant parity for $n\geq 1$. If the polynomial splits, all roots are zero or one, with multiplicity, so this is immediate. Otherwise it is a linear factor times X^2+X+1. The two roots of the quadratic have order three; their 3^nth powers contribute $1+1=0$ for $n\geq 1$. Of these six polynomials, three have constant parity zero and three have constant parity one. These observations hold even for repeated roots, by Lemma 3.2.

5.2 The signs of a nonreal conjugate pair

Lemma 5.1 (sign changes under cubing)**.** *Suppose the two small conjugates of the cubic Pisot number β are γ and $\overline{\gamma}$, with $\gamma\notin\mathbb{R}$. Then* $\mathrm{Re}(\gamma^{3^n})$ *is positive for infinitely many n and negative for infinitely many n.*

The irrationality of the argument of a nonreal Pisot conjugate, divided by 2π, is proved in [12, Lemma 5.10]. A related argument under repeated cubing appears in [12, Lemma 10.2 and the proof of Theorem 1.7]. We give a direct proof of both signs in the cubic case.

Proof. Put $z = \gamma/|\gamma|$. First z is not a root of unity. Otherwise $\gamma^k = \overline{\gamma}^k$ for some $k > 0$. The degree of the subfield $\mathbb{Q}(\beta^k) \subseteq \mathbb{Q}(\beta)$ is one or three. In degree three it equals $\mathbb{Q}(\beta)$, whose three embeddings send its generator β^k to distinct conjugates, contradicting that equality. In degree one, every embedding fixes β^k, forcing $\beta^k = \gamma^k$, impossible in absolute value. In particular, no z^{3^n} has real part zero.

Suppose all sufficiently late real parts are positive. Choose their arguments $u_n \in (-\pi/2, \pi/2)$. Then $3u_n \in (-3\pi/2, 3\pi/2)$. Within this interval the only arguments with positive cosine lie in $(-\pi/2, \pi/2)$. Thus $u_{n+1} = 3u_n$ for every sufficiently large n. Boundedness forces $u_n = 0$, making z a root of unity, a contradiction. If all sufficiently late real parts were negative, apply the same argument to $-z$, using $(-z)^{3^n} = -z^{3^n}$. □

Proof of Theorem 1.3. If $f \bmod 2$ is irreducible, the parity argument above and Proposition 4.3 give composite floors on a progression. In each remaining case the parity of t_n is a constant $e \in \{0, 1\}$ for $n \geq 1$. By Lemma 5.1, the indicator in (4.2) equals e infinitely often. The floor is then even and, for all sufficiently large such indices, greater than two. This proves infinite compositeness. □

5.3 Coefficient classes modulo six and thirty

For the seventeen nonexceptional classes modulo three, all eight reductions modulo two are covered. For each of the other ten classes, the two irreducible reductions and the three constant-parity reductions matching the required shift are covered. By the Chinese remainder theorem this gives

$$17 \cdot 8 + 10 \cdot 5 = 186 \tag{5.1}$$

covered coefficient classes modulo six. Of the thirty remaining classes, eighteen need a divisor for shift zero and twelve need one for shift one. These numbers follow from the six entries $\lambda \in \{-1, 1\}$ and the four entries $\lambda \in \{0, 2\}$ in Table 1, with three unhelpful parity reductions each.

For any prime p, further trace residues can be found by cubing the class of X in the finite ring $\mathbb{F}_p[X]/(f)$. Once a state repeats, the subsequent states form a cycle. A trace residue occurring on the cycle occurs on an infinite arithmetic progression of indices, whether or not it satisfies the sufficient test (4.5). For $p = 5$ there are 125 possible reduced cubics. Exact enumeration gives the following counts:

trace residues on the cycle	neither 0 nor 1	1 only	0 only	both 0 and 1
number of cubics	40	20	21	44.

(5.2)

Appendix A supplies the ring arithmetic and complete enumeration procedure, also included in executable form.

Proposition 5.2 (a finite family of sufficient conditions)**.** *There is an explicitly computable set of* 25 188 *triples modulo* 30 *such that every cubic Pisot number whose minimal polynomial* (4.1) *has a coefficient triple in that set satisfies infinite compositeness of* $\lfloor \beta^{3^n} \rfloor$*. For each such number, all sufficiently large indices on some infinite arithmetic progression give composite floors.*

Proof. By the Chinese remainder theorem, each coefficient triple modulo 30 corresponds uniquely to a pair of coefficient triples modulo 6 and modulo 5. Include every pair whose class modulo six is already covered. For each remaining class modulo six, include the pairs whose cycle modulo five contains the required shift. The other shift has a recurring divisor at every sufficiently large index by Proposition 4.3; hence the common-start argument in its proof still applies. Of the thirty remaining classes modulo six, eighteen require shift zero and twelve require shift one. The table

modulo five in (5.2) has $40+20=60$ classes whose cycles omit zero and $40+21=61$ whose cycles omit one. Thus the tests miss

$$18 \cdot 60 + 12 \cdot 61 = 1812$$

classes modulo thirty, leaving $30^3 - 1812 = 25\,188$. □

For a cubic Pisot number with nonreal conjugates, Theorem 1.3 applies directly. In the totally real case, put $h = 1$ if $a > \beta$ and $h = 0$ otherwise. Use Table 1, together with the exact limits in Appendix A, to test whether $\lambda - h \notin \{-1, 1\}$. If so, the first assertion of Proposition 4.3 supplies a recurring divisor. Otherwise, compute the eventual cycles modulo 2 and 5 as in Appendix A. If h occurs among the trace residues of either cycle, the corresponding prime divides $t_n - h$ on an infinite arithmetic progression. Either success proves compositeness of the floors at all sufficiently large indices on a progression. Failure of these tests gives no conclusion.

The 25 188 classes in Proposition 5.2 are those certified by these tests for both shifts on a common progression, using the common starting index construction of Proposition 4.3. This is a sufficient set, not a complete classification, and its size does not give the proportion of cubic Pisot numbers covered. Additional totally real cases can be certified by testing only their actual shift.

For example, the totally real cubic polynomials

$$X^3 - 31X^2 - 17X - 1, \qquad X^3 - 42X^2 + 24X + 6$$

belong to the remainder modulo six and are covered by five. For the first polynomial the signs at $-1, -1/4, 0$ are respectively $-, +, -$, and the third root exceeds 31. For the second the signs at $-1, 0, 1$ are $-, +, -$, and the third root lies between 1 and 42. Thus both have two nonzero roots in $(-1, 1)$. Neither has an integer root among the two small roots. An integer third root would leave a monic integral quadratic whose nonzero constant term has absolute value less than one, which is impossible. Both polynomials are therefore irreducible and define Pisot numbers. Their shifts (4.3) are respectively zero and one, and $T(3) - h$ is respectively 31 375 and 71 045. In each quotient ring modulo five, the state X^3 returns after thirty cubings. Hence five divides every sufficiently large floor with $n \equiv 1 \pmod{30}$.

6 Density in every coefficient progression

The finite tests above can be extended by allowing more primes. For the density theorem it is enough to use the single trace exponent three and primes in two fixed residue classes modulo nine. The triples that fail these divisibility conditions form a vanishingly small exceptional set. We then compare that set with the number of admissible Pisot polynomials.

Fix $m \geq 1$ and $\mathbf{r} \in (\mathbb{Z}/m\mathbb{Z})^3$. Let $\mathcal{P}_{m,\mathbf{r}}(X)$ be the set of cubic Pisot numbers $1 < \beta \leq X$ for which the coefficient triple in (4.1) is congruent to $\mathbf{r}$ modulo m. Define $\mathcal{G}$ to be the set of cubic Pisot numbers for which there exist primes p_0, p_1 and integers $n_0, P \geq 1$ such that

$$p_0 \mid t_{n_0+jP}, \qquad p_1 \mid t_{n_0+jP} - 1 \qquad (j \geq 0). \tag{6.1}$$

The primes and the progression may depend on β. Every member of $\mathcal{G}$ has composite floors on the sufficiently late part of this progression, by (4.2).

Theorem 6.1 (density in coefficient classes)**.** *For every m and $\mathbf{r}$ as above,*

$$\frac{\#(\mathcal{G} \cap \mathcal{P}_{m,\mathbf{r}}(X))}{\#\mathcal{P}_{m,\mathbf{r}}(X)} \longrightarrow 1 \qquad (X \to \infty). \tag{6.2}$$

The same conclusion holds after restricting to totally real cubic Pisot numbers with $a > \beta$, or to those with $a < \beta$. Each of these three assertions also holds with c fixed at any nonzero integer congruent to the third coordinate of $\mathbf{r}$ modulo m.

6.1 Many coefficients have a divisor returning from the first index

Let $\mathcal{S}$ be the set of primes $p > 3$ with $p \equiv 2$ or $5 \pmod 9$. For these primes $v_3(p+1) = 1$, whereas $p-1$ and p^3-1 are prime to three. Equation (4.4) therefore gives $v_3(d_p) \le 1$. Every such prime dividing

$$t_1 - h = a^3 - 3ab + 3c - h \tag{6.3}$$

is a recurring divisor for shift h at $n = 1$. Thus one prime from $\mathcal{S}$ dividing each of the two values in (6.3), for $h = 0, 1$, proves (6.1) with a common starting index one.

For any fixed $p \in \mathcal{S}$, exactly a fraction $1/p$ of all triples modulo p satisfy (6.3) congruent to zero: each pair (a, b) determines a unique c, since $p \neq 3$. With c fixed, every nonzero a mod p determines a unique b, so at least a fraction $(p-1)/p^2$ of pairs (a, b) mod p satisfy the same condition.

Take a finite set $S \subset \mathcal{S}$ containing no prime divisor of m. In a box with side lengths proportional to X, the fraction of triples in the prescribed class modulo m that miss every divisibility condition for a fixed shift tends to

$$\prod_{p \in S} (1 - 1/p).$$

This follows by counting each residue class modulo $m \prod_{p \in S} p$; the count in each coordinate is its interval length divided by that modulus, with an error of at most one. The Chinese remainder theorem makes the conditions at different primes independent for this finite counting problem. In the plane of a fixed c, the corresponding limiting fraction is at most

$$\prod_{p \in S} \left(1 - \frac{p-1}{p^2}\right).$$

Dirichlet's theorem in its reciprocal-sum form states that $\sum_{p \equiv r \pmod d} 1/p$ diverges whenever $\gcd(r, d) = 1$; see [5, §§9–11, especially (15)]. For a modern quantitative formulation, see [3, Theorem 12.8]. Applying it to $r = 2, d = 9$ shows that both displayed products tend to zero as S increases. Removing the finitely many primes dividing m does not affect this conclusion. First let $X \to \infty$ for each fixed S, and only then enlarge S. The triples that miss all primes of $\mathcal{S}$ for one shift consequently number $o(X^3)$ in any containing box of side lengths $O(X)$. With c fixed the analogous count is $o_c(X^2)$. The union of the exceptional sets for shifts zero and one has the same bounds. Here $o(X^d)$ means a quantity whose ratio to X^d tends to zero; all limits allow dependence on the fixed modulus, residue class, and, when specified, norm.

6.2 Passing from coefficient boxes to Pisot numbers

If $\beta \le X$, the conjugate bounds in (4.1) give

$$|a| \le X + 2, \qquad |b| \le 2X + 1, \qquad |c| < X. \tag{6.4}$$

The exceptional bounds just proved therefore apply to all the Pisot numbers under consideration. To obtain a proportion, we need lower bounds for the number of Pisot polynomials in each class.

First let all three coefficients vary. Choose

$$\frac{X}{2} < a < \frac{2X}{3}, \qquad \frac{a}{3} < b < \frac{a}{2} \quad \text{or} \quad -\frac{a}{2} < b < -\frac{a}{3}, \qquad -\frac{a}{8} < c < -\frac{a}{16}. \tag{6.5}$$

For all sufficiently large X, these inequalities give $f(-1) < 0 < f(0)$ and $f(1) < 0$. There are roots in $(-1, 0)$ and $(0, 1)$, and a third root $\beta > 1$. Their sum is a, so $|\beta - a| < 2$ and $\beta < X$. No small root is an integer. If the third root were an integer, division by its monic linear factor would leave a monic integral quadratic with nonzero constant term of absolute value less than one. This is impossible. The polynomial is therefore irreducible, and β is a cubic Pisot number.

At the argument a, the other two factors of f are positive, so

$$\operatorname{sgn}(a - \beta) = \operatorname{sgn} f(a) = \operatorname{sgn}(ab - c).$$

For sufficiently large a this is positive in the positive-b region of (6.5) and negative in its negative-b region. For each choice of sign of b, an explicit rectangular box inside that region is given by

$$\frac{X}{2} < a < \frac{3X}{5}, \qquad \frac{X}{5} < |b| < \frac{X}{4}, \qquad -\frac{X}{16} < c < -\frac{3X}{80}.$$

Its side lengths are $X/10$, $X/20$, and $X/40$, so its volume is $X^3/8000$. Every fixed residue class modulo m consequently contains at least $C_m X^3$ such polynomials for all sufficiently large X, for some $C_m > 0$, in either sign.

Next fix $c \neq 0$, keeping the ranges for a, b in (6.5). On $|z| = 1$ and for all sufficiently large a we have

$$| - az^2| = a > 1 + |b| + |c| \geq |z^3 + bz - c|.$$

Rouché's theorem therefore gives exactly two roots in the open unit disk. The discriminant is

$$a^2b^2 - 4b^3 - 4a^3c - 27c^2 + 18abc.$$

Its leading term is at least $a^4/9$, whereas the absolute value of the remaining terms is $O_c(a^3)$. It is positive for large a, so the cubic has three distinct real roots. Since $f(1) < 0$, the root outside the disk is positive and greater than one. The same irreducibility, size and sign arguments now apply. Each compatible coefficient class contains at least $C_{m,c} X^2$ such polynomials in either sign, for some $C_{m,c} > 0$.

Each of these polynomials determines a unique root greater than one, and conversely that root determines its minimal polynomial. Dividing the $o(X^3)$ exceptional count by the first lower bound, or the $o_c(X^2)$ count by the second, proves every assertion of Theorem 6.1.

6.3 Why a fixed finite set of primes does not finish the cubic problem

The density theorem allows the witness primes to depend on the polynomial. This freedom is necessary for these divisibility tests.

Proposition 6.2. *For every finite set of primes S, there is a totally real cubic Pisot number β of norm one such that no prime in S divides $\lfloor \beta^{3^n} \rfloor$ for any sufficiently large n.*

Proof. Choose an integer $W \geq 12$ divisible by six and by all primes in S, and put

$$f(X) = X^3 - (2W + 1)X^2 + (1 - W)X - 1.$$

Its values at $-1, -1/4, 0$ have signs $-, +, -$; specifically $f(-1/4) = (8W - 85)/64 > 0$. Writing $a = 2W + 1$ and $b = 1 - W$, we have $f(a) = ab - 1 < 0$, so the third root satisfies $\beta > a > 1$. The two small roots are nonzero and lie in $(-1, 0)$. The irreducibility argument above applies, proving that β is a cubic Pisot number of norm one with shift $h = 0$.

For every $p \mid W$ the reduction is $f(X) \equiv (X - 1)(X^2 + 1) \pmod{p}$. Its odd power sums equal one: the roots of $X^2 + 1$ cancel in pairs, also in characteristic two when counted with multiplicity. Thus $t_n \equiv 1 \pmod{p}$ for all $n \geq 0$. For sufficiently large n, the floor equals t_n by (4.2), and no prime in S divides it. □

This proposition does not assert primality of any of these floors. The unresolved question is whether every individual totally real cubic Pisot number has infinitely many composite floors at exponents 3^n, with divisors allowed to vary with the polynomial.

A Finite tables and supplementary certificates

The infinite statements above use the written proofs. We describe the finite calculations underlying the coefficient-class counts completely here. The verification supplement is distributed as `supplement-v3.tar.gz`. It contains `verify_saito.py`, `fib_certs.json`, and `README.md` in the `supplement-v3` directory. The supplement is archived together with this paper on Zenodo at https://doi.org/10.5281/zenodo.24630676. The verifier checks the finite tables, examples, and Fibonacci certificates by exact integer arithmetic using the Python standard library. Each check compares its computed result with the stated target; a failed check exits with nonzero status, and PASS is reported only after all checks succeed.

A.1 The cubic tables

Table 2 supplies the ten exact limits used in Table 1. In each row, $\Phi(A,B,C) = (A,B,C)$. By Lemma 4.2, every integral coefficient triple in the same residue class modulo 3 therefore has exact trace limit $\lambda = A$. These auxiliary polynomials need not be irreducible or define Pisot numbers; A is the sum of their roots, counted with multiplicity.

Table 2: Integral fixed triples certifying the ten exact trace limits.

(a,b,c) mod 3	Integral fixed triple (A,B,C)	Exact λ
$(0,0,0)$	$(0,0,0)$	0
$(0,1,0)$	$(0,1,0)$	0
$(0,2,0)$	$(0,-1,0)$	0
$(1,0,0)$	$(1,0,0)$	1
$(1,1,1)$	$(1,1,1)$	1
$(1,2,2)$	$(1,-1,-1)$	1
$(2,0,0)$	$(-1,0,0)$	-1
$(2,1,0)$	$(2,1,0)$	2
$(2,1,2)$	$(-1,1,-1)$	-1
$(2,2,1)$	$(-1,-1,1)$	-1

Represent an element of $(\mathbb{Z}/q\mathbb{Z})[X]/(X^3-aX^2+bX-c)$ by the triple (u,v,w) for $u+vX+wX^2$. Multiply polynomials and reduce using

$$X^3 = aX^2 - bX + c.$$

The sum of the values of this element at the three roots, counted with multiplicity, is

$$3u + av + (a^2 - 2b)w \pmod{q}. \tag{A.1}$$

Start from $(0,1,0)$ and repeatedly cube. For each of the 27 triples $(a,b,c) \in \{0,1,2\}^3$, three cubings modulo 81 and (A.1) produce Table 1. This checks the table without iterating the coefficient map Φ.

For (5.2), do the same modulo five for each of the 125 triples $(a, b, c) \in \{0, 1, 2, 3, 4\}^3$. Record each state until the first repeated state. Discard states before its first occurrence and collect the traces of the remaining cycle. Count whether that set contains zero, one, both or neither. There are only 125 ring elements, so each orbit terminates after at most 126 visits. This is a complete finite procedure, including repeated-root and zero-root reductions. The identical procedure modulo two checks the parity counts used in (5.1).

A.2 Fibonacci certificates

The supplementary file `fib_certs.json` gives one tuple (p, n_0, P) for each even nonzero integer h with $|h| \le 20\,000$. Each tuple asserts a prime divisor on an entire progression, not just at the first few terms. The verifier proves primality of p by trial division, computes $M = Q^{2^{n_0}} \bmod p$ by repeated squaring, and checks

$$M_{12} + h \equiv 0 \pmod{p}, \qquad M^{2^P} = M \pmod{p}. \tag{A.2}$$

The second equality makes the entire matrix return after P squarings. Induction then gives the first congruence at every index $n_0 + jP$. By (2.1), this proves divisibility of the Fibonacci value throughout the progression. The divisor is proper for all sufficiently large terms.

All 20 000 supplied records satisfy (A.2); they use 141 distinct matrix returns and primes at most 137. For example, $h = 110$ has $(p, n_0, P) = (37, 10, 18)$, and $h = -160$ has $(37, 13, 18)$. These records illustrate the theorem and are not used to extend a finite verification to all shifts. The file's SHA-256 digest is recorded in the supplement README and checked by the verifier before reading its certificates.

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