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FINITE PATTERN BASES FOR BOUNDED VALUES OF KAZHDAN–LUSZTIG POLYNOMIALS

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Abstract. Let $P_{e,w}(q)$ be the Kazhdan–Lusztig polynomial indexed by a permutation w and the identity e in its symmetric group. For each positive integer m, let B_m consist of the permutations w with $P_{e,w}(1) > m$ and $P_{e,u}(1) \leq m$ for every proper pattern u of w. We prove that B_m is finite, answering a question of Billey and Weed.

Contracting each maximal decreasing interval of consecutive positions and consecutive values in w gives its descending core $\rho(w)$. We prove that $|\rho(w)| \leq 15m - 3$ for all $w \in B_m$. When the inversion graph of $\rho(w)$ is 2-connected, this bound follows from the linear coefficient of Kazhdan–Lusztig polynomials (the cycle rank of the Hasse diagram of the inversion poset) and the structure of this diagram; otherwise, a product formula along a shared decreasing subsequence reduces the bound to smaller values of m. For each fixed core, the interval-length vectors of elements in B_m form an antichain under coordinatewise comparison, so Dickson's lemma gives finiteness.

1. Introduction

For $n \geq 0$, let S_n be the symmetric group of permutations of $\{1, \ldots, n\}$ in one-line notation (S_0 consists of the empty permutation), and call $|w| = n$ the *size* of $w \in S_n$. Given $U \subseteq \{1, \ldots, n\}$, let $w|U$ denote the permutation obtained by reading the entries of w at U in increasing position order and replacing their values by their relative ranks in $\{1, \ldots, |U|\}$. A permutation u is a *pattern* of w, written $u \preceq w$, if $u = w|U$ for some $U \subseteq \{1, \ldots, |w|\}$; write $u \prec w$ when $u \preceq w$ and $u \neq w$. For example, $\texttt{927315864}|\{1,2,3,4,5,6\} = \texttt{625314}$. The *avoidance class* of a set E of permutations is

$$\mathrm{Av}(E) = \{w \in \textstyle\bigcup_{n \geq 0} S_n : b \not\preceq w \text{ for all } b \in E\}.$$

Let $s_i = (i, i{+}1)$ $(1 \leq i < n)$ be the standard Coxeter generators of S_n, with composition $(xy)(i) = x(y(i))$. An *inversion* of $x \in S_n$ is a pair $i < j$ with $x(i) > x(j)$. The Coxeter length $\ell(x)$ is the number of inversions of x, or equivalently the minimum number of factors in a generator expression for x (a *reduced* expression). In the *Bruhat order* $\leq$ on S_n, $x \leq y$ if and only if some (equivalently, every) reduced expression for y contains a reduced expression for x as a subword.

Let $e \in S_n$ be the identity. Inside the complete flag variety $\mathcal{F}$ of $\mathbb{C}^n$, for each $y \in S_n$ let x_y be the coordinate flag whose j-dimensional subspace is spanned by the standard basis vectors with indices $y(1), \ldots, y(j)$, and let $B \leq \mathrm{GL}_n(\mathbb{C})$ be the subgroup of upper triangular matrices. Setting $X_w = \overline{B \cdot x_w}$, we have $x_y \in X_w$ if and only if $y \leq w$.

For $y \leq w$, the Kazhdan–Lusztig polynomial $P_{y,w}(q)$ is the local intersection cohomology Poincaré polynomial of X_w at x_y, where the intersection cohomology complex is graded so that smooth points have stalk cohomology in degree 0 (shifted by $[-\dim X_w]$ from the perverse normalization), q^i records cohomological degree $2i$, and the odd-degree groups vanish [9]. Put $P_w(q) = P_{e,w}(q)$ and $F(w) = P_w(1)$, with $P_e(q) = 1$ in S_0. Each $P_{y,w}(q)$ has constant term 1 and nonnegative integer coefficients, and satisfies $\deg P_{y,w} \leq (\ell(w) - \ell(y) -$

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1)/2 for $y < w$ [8, 9]. Write $[q^j]f(q)$ for the coefficient of q^j in $f(q)$ and $[q]f(q) = [q^1]f(q)$. For each positive integer m, define

$$\mathrm{KL}_m = \{w : F(w) \le m\} \qquad \text{and} \qquad B_m = \{w : F(w) > m \text{ and } F(u) \le m \text{ for all } u \prec w\}.$$

By Billey and Braden's pattern monotonicity theorem [2, Theorem 1 and §3.1], $u \preceq w$ implies $F(u) \le F(w)$. Hence $\mathrm{KL}_m \subseteq \mathrm{Av}(B_m)$; conversely, if $F(w) > m$, choosing a pattern $u \preceq w$ of minimum size with $F(u) > m$ gives $u \in B_m$, so $\mathrm{Av}(B_m) \subseteq \mathrm{KL}_m$. Thus $\mathrm{KL}_m = \mathrm{Av}(B_m)$, and B_m is the pattern basis of KL_m.

In type A, smoothness of X_w coincides with rational smoothness [4] and is characterized by $F(w) = 1$: smooth Schubert varieties satisfy $P_w(q) = 1$ [9], whereas whenever X_w is singular, w contains `3412` or `4231` by Lakshmibai and Sandhya's criterion [10], and both patterns satisfy $F = 2$. Hence $B_1 = \{\texttt{3412}, \texttt{4231}\}$. Billey and Weed computed the 66-element basis B_2 and asked whether B_m is finite for every $m \ge 1$ [14, Theorem A.1 and Appendix A, Question (1)]. Our main theorem answers this question affirmatively.

Theorem 1.1. *For every positive integer m, the basis B_m is finite. In particular, $P_{e,w}(1) \le m$ if and only if w avoids the finite set B_m.*

1.1. **Key concepts and outline of the proof.** A key tool in the proof of Theorem 1.1 is the plateau quotient of Lapid and Shusterman [11, §5.4], which we call the descending core.

Definition 1.2. A *decreasing interval* of $w \in S_n$ is a nonempty set of consecutive positions $p, p+1, \dots, p+a-1$ with $w(p+j) = w(p) - j$ for $0 \le j < a$. The maximal decreasing intervals partition $\{1, \dots, n\}$. Contracting each maximal decreasing interval to a single entry and standardizing yields the *descending core* $\rho(w)$; equivalently, $\rho(w) = w|U$ for any subset U selecting one position from each maximal decreasing interval.

Conversely, write $\delta_a = a\,(a-1)\cdots 1$ for $a \ge 1$. Given $v \in S_k$ and positive integers $a_1, \dots, a_k$, the *descending inflation* $v[\delta_{a_1}, \dots, \delta_{a_k}]$ replaces each position i of v with a decreasing interval of length a_i, placed in position order $1, \dots, k$, so that the values for i are all smaller than those for j whenever $v(i) < v(j)$.

For example, the maximal decreasing intervals of $w = \texttt{985437621}$ partition its entries into the four blocks `98`, `543`, `76`, and `21`; reading their value ranks in position order gives $\rho(w) = \texttt{4231}$ (so $|\rho(w)| = 4$) and $w = \texttt{4231}[\delta_2, \delta_3, \delta_2, \delta_2]$.

Every w admits a unique representation $w = \rho(w)[\delta_{a_1}, \dots, \delta_{a_k}]$, where $k = |\rho(w)|$ and $a_1, \dots, a_k$ are the lengths of its maximal decreasing intervals. An *adjacent descent by one* in v is a pair $i, i+1$ with $v(i) = v(i+1)+1$. In particular, $\rho(w)$ has no adjacent descent by one, since two adjacent blocks whose ranks were consecutive and decreasing would coalesce in w.

Definition 1.3. For a permutation $v \in S_k$:

(1) The *inversion graph* G_v is the undirected graph on $\{1, \dots, k\}$ joining $i < j$ whenever (i, j) is an inversion of v.
(2) The *strict inversion order* $<_v$ on $\{1, \dots, k\}$ is given by $i <_v j$ if $i < j$ and $v(i) > v(j)$, so G_v is its comparability graph. A relation $i <_v j$ is a *cover* if no k' satisfies $i <_v k' <_v j$. The *Hasse diagram* Γ_v is the directed graph on $\{1, \dots, k\}$ with edges $i \to j$ for covers; its minimal and maximal elements under $<_v$ are its *sources* (no incoming edges) and *sinks* (no outgoing edges).
(3) For a graph H (directed or undirected), write $V(H)$ and $E(H)$ for its vertex and edge sets, $H - x$ for the graph obtained by deleting vertex x and its incident edges, and $c(H)$ for its number of connected components (ignoring edge directions). An undirected graph H has a *cut vertex* x if $c(H - x) > c(H)$, and is 2-*connected* if it is connected, has at least three vertices, and has no cut vertex. The *cycle rank* of H is $\beta(H) = |E(H)| - |V(H)| + c(H)$.

A *chain* in a poset is a set of pairwise comparable elements. Figure 1 contrasts G_v and Γ_v for $v \in S_9$.

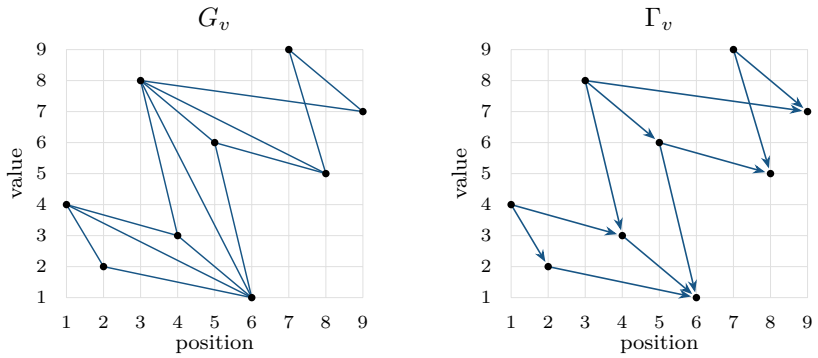

FIGURE 1. For $v = \texttt{428361957}$, each point is at $(i, v(i))$. The inversion $(1, 6)$ gives an edge of G_v. Since $1 <_v 2 <_v 6$, Γ_v contains $1 \to 2$ and $2 \to 6$, with no edge $1 \to 6$. The connected graph Γ_v has 11 edges and cycle rank 3.

The proof proceeds by showing that $|\rho(w)| \le 15m - 3$ for all $w \in B_m$. When $|\rho(w)| \ge 3$ and $G_{\rho(w)}$ is not 2-connected, Section 2 proves a Kazhdan–Lusztig product formula for a partition separated by a decreasing subsequence (with an alternative geometric proof of its Hecke-algebra core, Lemma 2.1, in Appendix A), and Section 3 constructs such a partition from a component or cut-vertex split, enabling Section 6 to bound $|\rho(w)|$ by induction on m. When $G_{\rho(w)}$ is 2-connected, Section 4 identifies $[q]P_u(q)$ with $\beta(\Gamma_u)$, bounds $\beta(\Gamma_{\rho(w)})$ by averaging over one-position deletions, and rules out two local configurations in $\Gamma_{\rho(w)}$; Section 5 then deduces $|\rho(w)| \le 15m - 3$ from the interval structure of sources and sinks and two auxiliary posets. Finally, Section 6 combines both cases and observes that for each fixed core, the block-length vectors of elements in B_m form an antichain under coordinatewise comparison, so Dickson's lemma [5] completes the proof of Theorem 1.1. Section 7 closes with remarks on effectivity, sharpness of the core bound, and other Weyl groups.

Note on the use of AI. Almost all of the proofs and exposition in this paper were produced by AI systems (GPT-6.2 Sol and Gemini 4 Neon). The author's role was limited to reading and understanding the arguments and revising the paper to improve its clarity and presentation, and the author takes responsibility for the correctness of the work.

2. A product formula for a shared decreasing subsequence

2.1. **A product formula in the Hecke algebra.** Fix $n \ge 1$ and put $S = \{s_i : 1 \le i < n\}$. For $x \in S_n$, the *left* and *right descent sets* are $D_L(x) = \{s \in S : \ell(sx) < \ell(x)\}$ and $D_R(x) = \{s \in S : \ell(xs) < \ell(x)\}$. Since multiplying on the right (respectively, left) by s_i swaps positions (respectively, values) i and $i + 1$,

$$s_i \in D_R(x) \iff x(i) > x(i+1), \qquad s_i \in D_L(x) \iff x^{-1}(i) > x^{-1}(i+1).$$

For $K \subseteq S$, let $W_K = \langle K \rangle$ be the standard parabolic subgroup generated by K, and let w_K be its unique longest element; then $D_L(w_K) = D_R(w_K) = K$.

Let t be an indeterminate with $q = t^2$. The Hecke algebra $\mathcal{H}$ of S_n over $\mathbb{Z}[t, t^{-1}]$ is the free module with basis $\{T_x : x \in S_n\}$ and relations

$$T_s^2 = (q-1)T_s + qT_e \quad (s \in S), \qquad T_x T_y = T_{xy} \quad (x, y \in S_n,\ \ell(xy) = \ell(x) + \ell(y)).$$

Put $\widetilde{T}_x = t^{-\ell(x)}T_x$. The *bar involution* $a \mapsto \overline{a}$ is the ring involution sending $t \mapsto t^{-1}$ and $T_x \mapsto (T_{x^{-1}})^{-1}$. The Kazhdan–Lusztig basis $\{C'_y : y \in S_n\}$ is uniquely determined by $\overline{C'_y} = C'_y$ and

$$C'_y = \sum_{x \le y} h_{x,y}\widetilde{T}_x, \qquad h_{y,y} = 1, \qquad h_{x,y} \in t^{-1}\mathbb{Z}[t^{-1}] \text{ for } x < y,$$

where $h_{x,y} = t^{\ell(x)-\ell(y)}P_{x,y}(q)$ for $x \le y$ [8, 9].

For $K \subseteq S$, every reduced expression for $y \in W_K$ uses only generators in K, so (W_K, K) inherits the length function and Bruhat order from S_n, and $[e, y] \subseteq W_K$. The submodule $\mathcal{H}_K = \bigoplus_{x \in W_K} \mathbb{Z}[t, t^{-1}]T_x$ is a subalgebra isomorphic to the Hecke algebra of (W_K, K) and is closed under the bar involution because $\overline{T_s} = T_s^{-1} = q^{-1}T_s - (1 - q^{-1})T_e \in \mathcal{H}_K$ for $s \in K$. Hence C'_y for $y \in W_K$ is computed within $\mathcal{H}_K$, so $P_{x,y}(q)$ is invariant under Coxeter isomorphisms $(W_K, K) \xrightarrow{\sim} (W'_{K'}, K')$.

Lemma 2.1. *Let $I, J \subseteq S$ be subsets of standard generators, and put $K = I \cap J$. Suppose that $u \in W_I$ satisfies $K \subseteq D_L(u)$ and $v \in W_J$ satisfies $K \subseteq D_R(v)$. Then*

$$P_{vw_Ku}(q) = P_v(q)P_u(q).$$

Proof. For $s \in K$, the lifting property gives $sx \le u$ whenever $x \le u$ and $xs \le v$ whenever $x \le v$, since $su < u$ and $vs < v$ [3, §§2.2, 2.4]. Thus $[e, u]$ and $[e, v]$ are unions of cosets W_Ka and bW_K, respectively, with a, b their minimum-length representatives. Their coset maxima satisfy $w_Ka \le u$ and $bw_K \le v$. The cosets of u and v give $u = w_Ka_0$ and $v = b_0w_K$.

By descent invariance [7, Corollary 7.14], $P_{x,u}(q)$ is constant on each W_Ka, and $P_{x,v}(q)$ on each bW_K. Applied to w_K, this also gives $P_{y,w_K}(q) = 1$ for $y \in W_K$, so $C'_{w_K} = t^{-\ell(w_K)}\sum_{y \in W_K} T_y$. The coefficient of T_x in C'_u is $t^{-\ell(u)}P_{x,u}(q)$. Grouping these coefficients by coset and using length additivity within each coset gives

$$C'_u = \sum_a h_{w_Ka,u}\, C'_{w_K}\widetilde{T}_a, \qquad C'_v = \sum_b h_{bw_K,v}\, \widetilde{T}_b C'_{w_K}.$$

For $s \in K$, the Hecke relation gives $T_s(T_y + T_{sy}) = q(T_y + T_{sy})$ whenever $\ell(sy) > \ell(y)$. Summing over the pairs $\{y, sy\}$ in W_K gives $T_sC'_{w_K} = qC'_{w_K}$, hence $T_yC'_{w_K} = q^{\ell(y)}C'_{w_K}$ for $y \in W_K$. Thus

$$(C'_{w_K})^2 = \kappa_K C'_{w_K}, \qquad \kappa_K = t^{-\ell(w_K)} \sum_{y \in W_K} q^{\ell(y)}.$$

Multiplying the coset expansions now gives $C'_vC'_u = \kappa_K\mathcal{D}$, where

$$\mathcal{D} = \sum_{b,a} h_{bw_K,v}\, h_{w_Ka,u}\, \widetilde{T}_b C'_{w_K}\widetilde{T}_a,$$

which is bar invariant. Indeed, the square identity for the bar-invariant C'_{w_K} gives $\overline{\kappa_K} = \kappa_K$, and the product identity then gives $\overline{\mathcal{D}} = \mathcal{D}$ by cancellation in the free $\mathbb{Z}[t, t^{-1}]$-module $\mathcal{H}$.

Since $b \in W_J$ has $D_R(b) \subseteq J$ and $D_R(b) \cap K = \varnothing$, we have $D_R(b) \cap I = \varnothing$, so b is the minimum-length representative of bW_I. For $y \in W_K$ and $a \in W_I$ with $D_L(a) \cap K = \varnothing$, we have $ya \in W_I$ with $\ell(ya) = \ell(y) + \ell(a)$, hence $\ell(bya) = \ell(b) + \ell(y) + \ell(a)$ and $\widetilde{T}_b\widetilde{T}_y\widetilde{T}_a = \widetilde{T}_{bya}$. Moreover, $x = bya$ uniquely determines (b, y, a): b is the minimum-length representative of xW_I, a is that of $W_K(b^{-1}x)$, and $y = (ya)a^{-1}$. Because passing to minimum-length coset representatives preserves the Bruhat order [3, Proposition 2.5.1], $w_Ka \le w_Ka_0$ and $bw_K \le b_0w_K$ imply $a \le a_0$ and $b \le b_0$. Combining reduced subwords for $b \le b_0$, $y \le w_K$, and $a \le a_0$ gives $bya \le b_0w_Ka_0$. Note also that $w_K^2 = e$, so $b_0w_Ka_0 = (b_0w_K)w_K(w_Ka_0) = vw_Ku$ with $\ell(vw_Ku) = \ell(b_0) + \ell(w_K) + \ell(a_0) = \ell(v) + \ell(u) - \ell(w_K)$.

Expanding $\mathcal{D}$ yields

$$\mathcal{D} = \sum_{b,y,a} h_{bw_K,v}\, h_{w_K a,u}\, t^{\ell(y)-\ell(w_K)}\, \widetilde{T}_{bya},$$

with distinct triples (b, y, a) indexing distinct elements $\widetilde{T}_{bya}$. The triple (b_0, w_K, a_0) contributes 1 to $\widetilde{T}_{vw_K u}$, while every other triple has $b < b_0$, $y < w_K$, or $a < a_0$ and therefore contributes a coefficient in $t^{-1}\mathbb{Z}[t^{-1}]$. By uniqueness of the Kazhdan–Lusztig basis, $\mathcal{D} = C'_{vw_K u}$.

Finally, by length additivity, $bya = e$ only when $b = y = a = e$. Using descent invariance to get $P_{w_K,v}(q) = P_v(q)$ and $P_{w_K,u}(q) = P_u(q)$, the coefficient of $\widetilde{T}_e$ in $\mathcal{D}$ is

$$h_{w_K,v}\, h_{w_K,u}\, t^{-\ell(w_K)} = t^{\ell(w_K)-\ell(v)-\ell(u)} P_v(q) P_u(q) = t^{-\ell(vw_K u)} P_v(q) P_u(q).$$

Comparing with $h_{e,vw_K u} = t^{-\ell(vw_K u)} P_{vw_K u}(q)$ yields $P_{vw_K u}(q) = P_v(q) P_u(q)$. □

Taking $K = S$ in the argument for w_K above gives $P_{\delta_n}(q) = P_{w_S}(q) = 1$ for every decreasing permutation δ_n. Appendix A gives a geometric proof of Lemma 2.1.

Remark 2.2. The proof of Lemma 2.1 uses only Coxeter-group and Hecke-algebra properties, so it holds verbatim in any Coxeter group (W, S) whenever $K = I \cap J$ generates a finite parabolic subgroup W_K. Moreover, since $P_{x,vw_K u}(0) = 1 \neq 0$ for all $x \leq vw_K u$, the identity $\mathcal{D} = C'_{vw_K u}$ shows that every $x \leq vw_K u$ factors uniquely as $x = bya$, where $y \in W_K$, b is the minimum-length representative of $bW_K \subseteq W_J$ with $bw_K \leq v$, and a is the minimum-length representative of $W_K a \subseteq W_I$ with $w_K a \leq u$. For these b, y, a, we have

$$P_{bya,\, vw_K u}(q) = P_{bw_K,v}(q)\, P_{w_K a,u}(q).$$

2.2. **Factorization along a separating decreasing subsequence.** For sets of integers X, Y, write $X < Y$ if $x < y$ for all $x \in X$ and $y \in Y$. Let $w \in S_n$. A partition (A, Q, D) of $\{1, \ldots, n\}$ into three (possibly empty) subsets is a *separating partition* of w if:

(1) $A \cup Q < D$ (every position in $A \cup Q$ precedes every position in D);
(2) $w(A) < w(Q \cup D)$ (every value at A is smaller than every value at $Q \cup D$);
(3) w is decreasing on Q (that is, $w(j_1) > w(j_2)$ whenever $j_1 < j_2$ in Q).

Positions in A and Q may interleave, as may values at Q and D (Figure 2).

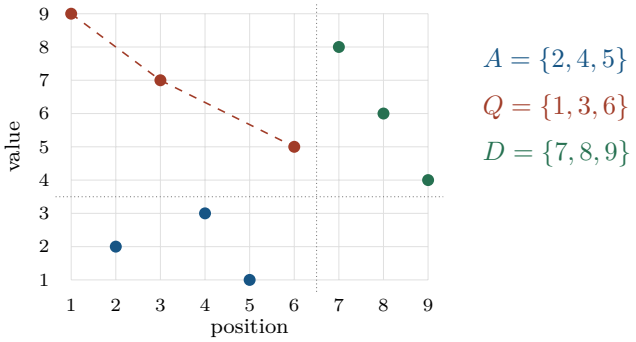

FIGURE 2. A separating partition of $w = \texttt{927315864}$. The dotted lines separate $A \cup Q$ from D in position and A from $Q \cup D$ in value; the dashed line joins the Q-points in position order.

Lemma 2.3. *Let $w \in S_n$ admit a separating partition (A, Q, D) satisfying conditions (1)–(3) above. Then for every subset $U \subseteq \{1, \ldots, n\}$ of positions of w,*

$$P_{w|U}(q) = P_{w|(U\cap(A\cup Q))}(q)\, P_{w|(U\cap(Q\cup D))}(q).$$

Proof. Restricting w to U preserves the relative order of positions and values, so $(U\cap A, U\cap Q, U\cap D)$ induces a separating partition of $w|U$; hence it suffices to treat $U=\{1,\dots,n\}$. If $A=\varnothing$, then $w|(A\cup Q)=w|Q$ is decreasing (so $P_{w|Q}(q)=1$) and $w|(Q\cup D)=w$, giving the identity; symmetrically, it holds when $D=\varnothing$. Assume $A,D\neq\varnothing$ and set $a=|A|\geq 1$, $h=|Q|\geq 0$. By (1) and (2), $A\cup Q=\{1,\dots,a+h\}$ and $w(Q\cup D)=\{a+1,\dots,n\}$.

Let $u\in S_n$ act as $w|(A\cup Q)$ on $\{1,\dots,a+h\}$ and fix $\{a+h+1,\dots,n\}$, so $u\in W_I$ for $I=\{s_1,\dots,s_{a+h-1}\}$. Let $v\in S_n$ fix $\{1,\dots,a\}$ and act on $\{a+1,\dots,n\}$ as $w|(Q\cup D)$ shifted by $+a$ in position and value, so $v\in W_J$ for $J=\{s_{a+1},\dots,s_{n-1}\}$. By Coxeter invariance, $P_u(q)=P_{w|(A\cup Q)}(q)$ and $P_v(q)=P_{w|(Q\cup D)}(q)$.

The intersection $K=I\cap J=\{s_{a+1},\dots,s_{a+h-1}\}$ permutes $\{a+1,\dots,a+h\}$ (with $K=\varnothing$ and $w_K=e$ if $h\leq 1$). By (2) and (3), the values $a+1,\dots,a+h$ appear in u at Q in decreasing order, so $K\subseteq D_L(u)$. By (1) and (3), the first h active positions $a+1,\dots,a+h$ of v correspond to Q and carry decreasing values, so $K\subseteq D_R(v)$.

Finally, we check $vw_Ku(i)=w(i)$ for each $i\in\{1,\dots,n\}$:

- If $i\in A$, then by (1) and (2) $u(i)=w(i)\in\{1,\dots,a\}$, which is fixed by both w_K and v.
- If $i\in D$, then $i>a+h$ is fixed by u and w_K, and by (1) and (2) $v(i)$ is a plus the rank of $w(i)$ in $Q\cup D$, namely $w(i)$.
- If $i\in Q$ is the r-th element of Q from left to right ($1\leq r\leq h$), then (2) and (3) give $u(i)=a+h+1-r$, w_K reverses $\{a+1,\dots,a+h\}$ to give $a+r$, and since $Q<D$, $a+r$ is the active position of v corresponding to i, so $v(a+r)=w(i)$ by (1) and (2).

Thus $w=vw_Ku$, and Lemma 2.1 gives $P_w(q)=P_v(q)P_u(q)$. □

3. Reduction at a cut vertex

For $x\in S_n$ with $n\geq 1$, define the *reversal-complement* $x^*=\delta_n x\delta_n$, so $x^*(i)=n+1-x(n+1-i)$ (and put $x^*=x$ in S_0). Since $\delta_n s_i\delta_n=s_{n-i}$ for $1\leq i<n$, conjugation by δ_n is a Coxeter automorphism of (S_n,S) fixing e, so $P_{x^*}(q)=P_x(q)$ by Coxeter invariance from Section 2. For $U\subseteq\{1,\dots,n\}$, put $U^*=\{n+1-i : i\in U\}$. Reversal-complement sends maximal decreasing intervals to maximal decreasing intervals, reverses $<_x$, and satisfies $(x|U)^*=x^*|U^*$ and $\rho(x^*)=\rho(x)^*$.

Lemma 3.1. *Suppose w is a permutation with $|\rho(w)|\geq 3$ such that $G_{\rho(w)}$ is not 2-connected. Then there exist proper subsets $U_1,U_2\subsetneq\{1,\dots,|w|\}$ with $U_1\cup U_2=\{1,\dots,|w|\}$ such that:*

(1) *For every subset $U\subseteq\{1,\dots,|w|\}$,*

$$P_{w|U}(q)=P_{w|(U\cap U_1)}(q)\,P_{w|(U\cap U_2)}(q).$$

(2) *With $A=U_1\setminus U_2$, $Q=U_1\cap U_2$, and $D=U_2\setminus U_1$, Q is either empty or a single maximal decreasing interval of w, and $A<D$ and $w(A)<w(D)$.*

Proof. Set $v=\rho(w)$ and $k=|v|\geq 3$. We first note that for any permutation π, the connected components of G_π form contiguous intervals of positions, ordered identically by position and by value. Indeed, if a vertex $z\notin K$ lay between $i,j\in K$ in a component K ($i<z<j$), then nonadjacency with z would force $\pi(a)<\pi(z)$ for all $a\in K$ with $a<z$ and $\pi(z)<\pi(b)$ for all $b\in K$ with $b>z$, disconnecting K; absence of inter-component edges then forces position and value order of components to agree.

If G_v is disconnected, split its components into two nonempty groups at a boundary between consecutive components, and let U_1,U_2 be the unions of the corresponding maximal decreasing intervals of w. Then $(A,Q,D)=(U_1,\varnothing,U_2)$ satisfies $A<D$ and $w(A)<w(D)$, so it is a separating partition of w; this gives (2), and Lemma 2.3 gives (1).

Suppose now that G_v is connected and has a cut vertex z. Every connected component of $G_v - z$ (the inversion graph of $v|(\{1,\dots,k\}\setminus\{z\})$) contains a neighbor of z. All neighbors of z lie on the same side of z in $<_v$: if z had both a lower neighbor and an upper neighbor, transitivity would join them in $G_v - z$ and place all neighbors of z in one component.

If all neighbors of z lie below z in $<_v$, apply the construction below to w^* (whose cut vertex $k+1-z$ has all neighbors above it) to obtain U_1', U_2' for w^*, with $A' = U_1' \setminus U_2'$, $Q' = U_1' \cap U_2'$, and $D' = U_2' \setminus U_1'$, and put $U_1 = (U_2')^*$ and $U_2 = (U_1')^*$. Then (1) follows from $P_{x^*}(q) = P_x(q)$ and $(w|U)^* = w^*|U^*$. Moreover, $A = (D')^*$, $Q = (Q')^*$, and $D = (A')^*$, so reversal-complement sends Q' to a maximal decreasing interval Q of w and turns $A' < D'$ and $w^*(A') < w^*(D')$ into $A < D$ and $w(A) < w(D)$, giving (2). Thus we may assume $z <_v j$ for every neighbor j of z.

Partition the connected components of $G_v - z$ by position order into nonempty initial and terminal groups A_0 and D_0. By the interval property for $G_v - z$, $A_0 < D_0$ and $v(A_0) < v(D_0)$. Choose neighbors $a_0 \in A_0$ and $d_0 \in D_0$ of z; since $z <_v a_0$ and $z <_v d_0$,

$$z < a_0 < D_0 \qquad \text{and} \qquad v(A_0) < v(d_0) < v(z).$$

Hence $A_0 \cup \{z\} < D_0$ and $v(A_0) < v(\{z\} \cup D_0)$. Figure 3 illustrates this configuration.

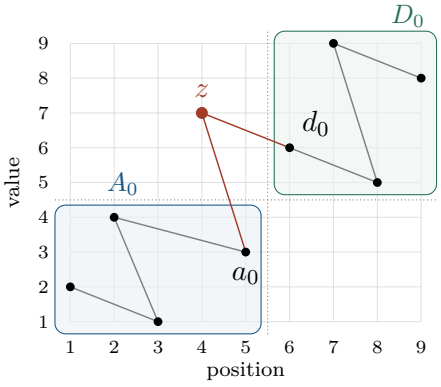

FIGURE 3. For $v = \texttt{241736958}$ and $z = 4$, the boxes mark the two connected components $A_0 = \{1,2,3,5\}$ and $D_0 = \{6,7,8,9\}$ of $G_v - z$, separated by the dotted lines in position and value. The red edges join z to $a_0 = 5$ and $d_0 = 6$; the remaining edge from z to 8 is omitted.

Let A, Q, D be the unions of the maximal decreasing intervals of w corresponding to $A_0, \{z\}, D_0$. Since Q is decreasing and the above inequalities lift from v to w, (A, Q, D) is a separating partition of w with $A < D$ and $w(A) < w(D)$. Putting $U_1 = A \cup Q$ and $U_2 = Q \cup D$ establishes (2), and Lemma 2.3 gives (1). □

4. The linear coefficient and local constraints on the core

For $w \in B_m$, put $v = \rho(w)$. We use the formula below to bound $\beta(\Gamma_v)$ by averaging over one-position deletions, then establish two local constraints on v.

Lemma 4.1. *For every permutation w,*

$$[q]P_w(q) = \beta(\Gamma_w).$$

Proof. For the empty permutation both sides are zero. Otherwise, Dyer's formula expresses $[q]P_w(q)$ as the number of coatoms minus the number of atoms of the Bruhat interval $[e, w]$ [6, Lemma 7.28(ii)]. Tagawa identifies these counts with $|E(\Gamma_w)|$ and $|V(\Gamma_w)| - c(\Gamma_w)$, respectively [13, Proposition 1.7]. Their difference is $\beta(\Gamma_w)$. □

Since $P_u(q)$ has constant term 1 and nonnegative integer coefficients, Lemma 4.1 gives $\beta(\Gamma_u) = [q]P_u(q) \le F(u) - 1$ for every permutation u (and hence $\beta(\Gamma_u) \le m-1$ whenever $u \prec w \in B_m$).

Lemma 4.2. *Suppose $w \in B_m$, $v = \rho(w)$, G_v is connected, and $|v| > 2m$. Then*

$$\beta(\Gamma_v) \le m.$$

Proof. If $v \prec w$, then $\beta(\Gamma_v) \le F(v) - 1 \le m - 1 < m$. Assume therefore that $v = w$, and set $n = |v| > 2m$ and $b = \beta(\Gamma_v)$. Since G_v is connected, Γ_v is connected and $|E(\Gamma_v)| = n - 1 + b$.

For each $i \in \{1, \dots, n\}$, let $u_i = v|(\{1, \dots, n\} \setminus \{i\})$. Deleting i preserves $<_v$ on the remaining positions, so every edge of Γ_v not incident to i yields a distinct edge of Γ_{u_i}. Because $u_i \prec w$, we have $\beta(\Gamma_{u_i}) \le m - 1$; as $c(\Gamma_{u_i}) \ge 1$, it follows that

$$|E(\Gamma_{u_i})| = \beta(\Gamma_{u_i}) + (n-1) - c(\Gamma_{u_i}) \le n + m - 3.$$

Each edge of Γ_v survives in $n-2$ of the n deletions. Summing over $i = 1, \dots, n$ gives $(n-2)(n-1+b) \le n(n+m-3)$, or $(n-2)b \le (n-2)m + 2(m-1)$. Since $n > 2m \ge 2$, we have $n - 2 > 2(m-1) \ge 0$, so $b < m + 1$, and hence $b \le m$. □

Lemma 4.3. *Let $w \in B_m$ and set $v = \rho(w)$. Then:*

1. *No edge $x \to y$ of Γ_v joins a vertex x of out-degree 1 to a vertex y of in-degree 1.*
2. *For every source or sink of Γ_v, the values of v at its neighbors in G_v do not form a decreasing subsequence of v.*

Proof. For (1), we use only that v has no adjacent descent by one. Suppose an edge $x \to y$ of Γ_v is the unique outgoing edge at x and the unique incoming edge at y. Any $z \ne x$ with $z <_v y$ has a directed path to y ending with $x \to y$, forcing $z <_v x$ (so $z < x$ and $v(z) > v(x)$). Symmetrically, any $z \ne y$ with $x <_v z$ has a directed path from x starting with $x \to y$, forcing $y <_v z$ (so $z > y$ and $v(z) < v(y)$). If $x < z < y$, then $v(y) < v(x)$ implies $v(z) > v(y)$ or $v(z) < v(x)$, giving either $z <_v y$ (so $z < x$) or $x <_v z$ (so $z > y$), a contradiction; thus $y = x + 1$. Likewise, if $v(y) < v(z) < v(x)$, then $x < y$ implies $z < y$ or $z > x$, giving either $z <_v y$ (so $v(z) > v(x)$) or $x <_v z$ (so $v(z) < v(y)$), a contradiction; thus $v(x) = v(y) + 1$. Hence (x, y) is an adjacent descent by one in v, contradicting $v = \rho(w)$.

For (2), suppose a source or sink x of Γ_v has neighbors in G_v whose values decrease in position order. Because reversal-complement preserves pattern containment and F (hence B_m), exchanges sources and sinks, and preserves decreasing subsequences, we may assume x is a sink. Let M be the maximal decreasing interval of w for x, and Q the union of the maximal decreasing intervals for the neighbors of x in G_v. Since x is a sink, every neighbor y of x satisfies $y <_v x$, so $Q < M$, $w(M) < w(Q)$, and w is decreasing on Q. All positions outside $M \cup Q$ correspond to vertices incomparable with x in $<_v$ and therefore partition into A (positions preceding M with values below $w(M)$) and D (positions following M with values above $w(M)$), satisfying $A < M < D$ and $w(A) < w(M) < w(D)$ (Figure 4).

Thus $A \cup Q < M \cup D$ and $w(A) < w(Q \cup M \cup D)$, so $(A, Q, M \cup D)$ is a separating partition of w. Applying Lemma 2.3 to w and to $U = A \cup Q \cup D$ gives

$$P_w(q) = P_{w|(A \cup Q)}(q)\, P_{w|(Q \cup M \cup D)}(q), \qquad P_{w|(A \cup Q \cup D)}(q) = P_{w|(A \cup Q)}(q)\, P_{w|(Q \cup D)}(q).$$

Likewise, after standardizing $Q \cup M \cup D$, the inequalities $M \cup Q < D$ and $w(M) < w(Q \cup D)$ show that (M, Q, D) induces a separating partition of $w|(Q \cup M \cup D)$, so Lemma 2.3 gives $P_{w|(Q \cup M \cup D)}(q) = P_{w|(M \cup Q)}(q)\, P_{w|(Q \cup D)}(q)$. Because $Q < M$, $w(M) < w(Q)$, and both Q and M are decreasing, $w|(M \cup Q)$ is decreasing, so $P_{w|(M \cup Q)}(q) = 1$. Consequently $P_w(q) = P_{w|(A \cup Q \cup D)}(q)$, giving $F(w|(A \cup Q \cup D)) = F(w) > m$, which contradicts $w \in B_m$ since $M \ne \varnothing$ implies $w|(A \cup Q \cup D) \prec w$. □

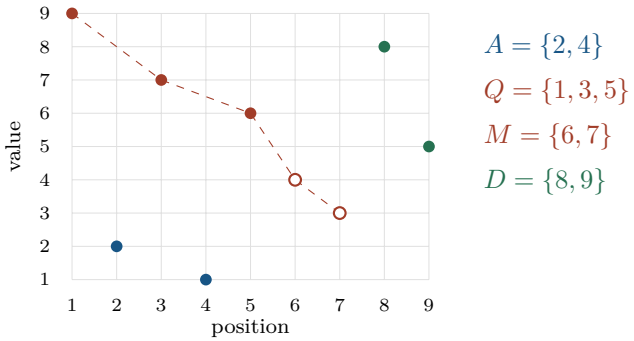

FIGURE 4. In $w = 927164385$, the open circles mark M. The dashed line joins $Q \cup M$ in position order. Since $w|(Q \cup M)$ is decreasing, its factor in $P_w(q)$ is 1.

5. Bounding a 2-connected descending core

In this section we prove $|\rho(w)| \le 15m - 3$ whenever $w \in B_m$, $|\rho(w)| \ge 3$, and $G_{\rho(w)}$ is 2-connected.

Fix $v \in S_n$ with $n \ge 3$, put $P = \{1, \dots, n\}$, and let $\mathcal{P} = (P, <_v)$ be its strict inversion order with Hasse diagram $\Gamma = \Gamma_v$. Write $p \le_v q$ if $p = q$ or $p <_v q$. Let $S \subseteq P$ and $T \subseteq P$ be the sets of sources (minimal elements of $<_v$) and sinks (maximal elements of $<_v$), respectively, and write $d^+(u) = d^+_\Gamma(u)$ and $d^-(u) = d^-_\Gamma(u)$ for the out- and in-degrees in Γ. For $Y \subseteq P$, standardization identifies the induced subposet $\mathcal{P}|Y$ with the strict inversion order of $v|Y$, so $\Gamma_{v|Y}$ is the Hasse diagram of $\mathcal{P}|Y$. For $u \in P$, set

$$U^+(u) = \{y \in P : u <_v y\} \qquad \text{and} \qquad T(u) = \{t \in T : u \le_v t\}.$$

For $x \in S$, $U^+(x)$ is the neighborhood of x in G_v, and two elements $i < j$ in $U^+(x)$ satisfy $i <_v j$ if and only if $v(i) > v(j)$; hence $\{z \in P : z \ge_v x\} = \{x\} \cup U^+(x)$ is a chain in $\mathcal{P}$ precisely when the values of v at the neighbors of x in G_v decrease in position order. Dually, for $t \in T$, $\{z \in P : z \le_v t\}$ is a chain precisely when the values of v at the neighbors of t in G_v decrease in position order.

Lemma 5.1. *Let $v \in S_n$ with $n \ge 3$ and suppose that G_v is connected. Then S and T are disjoint and nonempty. Writing $S = \{x_1 < x_2 < \cdots < x_s\}$ in position order:*

1. *$v(x_1) < v(x_2) < \cdots < v(x_s)$.*
2. *For every $p \in P$, the set $I(p) = \{i \in \{1, \dots, s\} : x_i \le_v p\}$ is a nonempty contiguous interval $[L(p), R(p)] \subseteq \{1, \dots, s\}$, and $p \le_v q$ implies $I(p) \subseteq I(q)$.*
3. *For each $r \in \{1, \dots, s-1\}$, put $g_r = |\{t \in T : \{r, r+1\} \subseteq I(t)\}|$. Then $g_r \ge 1$.*
4. $|\{(x, t) \in S \times T : x <_v t\}| = \sum_{t \in T} |I(t)| = |T| + \sum_{r=1}^{s-1} g_r$.

Proof. Since G_v is connected and $n \ge 3$, no vertex is isolated, so $S \cap T = \varnothing$ and $S, T \ne \varnothing$. For (1), if $i < j$, then $x_i < x_j$ and minimality of x_j prevents $x_i <_v x_j$, forcing $v(x_i) < v(x_j)$.

For (2), following incoming edges of Γ backwards from p reaches some $x_i \le_v p$, so $I(p) \ne \varnothing$. If $p = x_r \in S$, then $I(p) = \{r\}$ because S is an antichain. If $p \notin S$, then $i \in I(p)$ if and only if $x_i < p$ and $v(x_i) > v(p)$; by (1), these two inequalities cut out an initial interval $\{1, \dots, R(p)\}$ and a terminal interval $\{L(p), \dots, s\}$, so $I(p) = [L(p), R(p)]$ is contiguous. Transitivity of $\le_v$ gives $p \le_v q \implies I(p) \subseteq I(q)$.

For (3), if $g_r = 0$ for some r, then every $p \in P$ lies below some $t \in T$ and satisfies $I(p) \subseteq I(t) \not\supseteq \{r, r+1\}$, so either $R(p) \le r$ or $L(p) \ge r+1$. Thus $P_1 = \{p : R(p) \le r\}$ (containing x_1) and $P_2 = \{p : L(p) \ge r+1\}$ (containing x_s) partition P into two nonempty sets,

and $p <_v q \implies I(p) \subseteq I(q)$ prevents any edge of G_v between P_1 and P_2, contradicting connectedness of G_v.

For (4), since $S \cap T = \varnothing$, $(x_i, t) \in S \times T$ satisfies $x_i <_v t$ if and only if $i \in I(t)$. Counting by $t \in T$ and writing $|I(t)| = 1 + (R(t) - L(t))$ gives

$$|\{(x,t) \in S \times T : x <_v t\}| = \sum_{t \in T} |I(t)| = \sum_{t \in T} (1 + (R(t) - L(t))) = |T| + \sum_{r=1}^{s-1} g_r. \qquad \square$$

Define the auxiliary chain subsets

$$C_+ = \{y \in P : \{z \in P : z \geq_v y\} \text{ is a chain}\}, \qquad Y_+ = S \cup C_+,$$

and dually

$$C_- = \{y \in P : \{z \in P : z \leq_v y\} \text{ is a chain}\}, \qquad Y_- = T \cup C_-.$$

Write $\Gamma_\pm = \Gamma_{v|Y_\pm}$, $b = \beta(\Gamma)$, and $b_\pm = \beta(\Gamma_\pm)$. Clearly $T \subseteq C_+$ and $S \subseteq C_-$; when v satisfies Lemma 4.3(2), we also have $S \cap C_+ = \varnothing$ and $T \cap C_- = \varnothing$ (Figure 5).

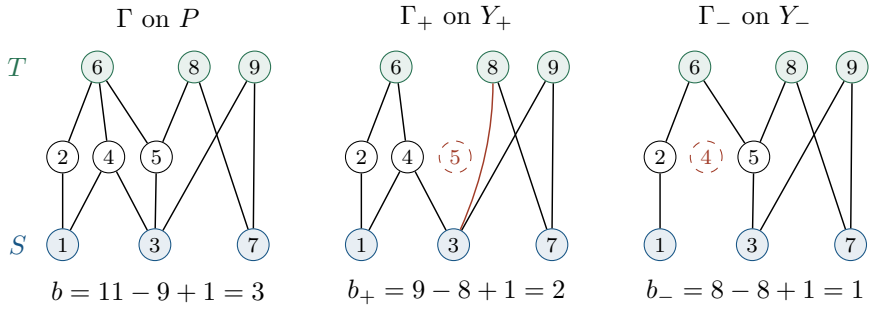

FIGURE 5. Hasse diagrams for $v = \texttt{428361957}$, read from bottom to top. The blue vertices are S, and the green vertices are T. The dashed vertex in each restricted diagram is the vertex omitted from its ground set. Vertex 5 has incomparable sinks $6, 8$ above it, so $5 \notin Y_+$. Vertex 4 has incomparable sources $1, 3$ below it, so $4 \notin Y_-$. The red edge $3 \to 8$ is a cover only after restriction to Y_+.

Proposition 5.2. *If $w \in B_m$, $|\rho(w)| \geq 3$, and $G_{\rho(w)}$ is 2-connected, then*

$$|\rho(w)| \leq 15m - 3.$$

Proof. Put $v = \rho(w)$ and $n = |v| \geq 3$. Since $2m \leq 15m - 3$ for all $m \geq 1$, we may assume $n > 2m$, so Lemma 4.2 gives $b = \beta(\Gamma) \leq m$. For any $Y \subseteq P$, either $v|Y = v = w$ (giving $\beta(\Gamma_{v|Y}) = b \leq m$) or $v|Y \prec w$ (giving $\beta(\Gamma_{v|Y}) \leq F(v|Y) - 1 \leq m - 1$ by Lemma 4.1). In particular, $b_+ \leq m$ and $b_- \leq m$. By Lemma 4.3, v also satisfies (1) and (2), so $S \cap C_+ = \varnothing$ and $T \cap C_- = \varnothing$. Set $s = |S|$; in Step 1, sums over $x <_v t$ range over pairs $(x,t) \in S \times T$ with $x <_v t$. We first prove $|S| \leq 2b_+ + b$.

Step 1. An exact formula for b_+. For each $y \in C_+$, the chain $\{z \in P : z \geq_v y\} \subseteq C_+$ has a unique sink $t(y) \in T$ as its maximum. Since S is an antichain of minimal elements disjoint from C_+, no edge of Γ_+ enters S, each $y \in C_+ \setminus T$ has out-degree 1 in Γ_+, and each $t \in T$ has out-degree 0. For $x \in S$ and $t \in T$ with $x <_v t$, let $e(x,t)$ count edges $x \to y$ in Γ_+ with $t(y) = t$; any directed path from x to t in Γ_+ starts with such an edge, so $e(x,t) \geq 1$.

Moreover, Γ_+ is connected ($c(\Gamma_+) = 1$): every $y \in C_+$ connects to $t(y) \in T$, every $t \in T$ connects to the nonempty interval $I(t)$, and $g_r \geq 1$ (Lemma 5.1(3)) joins x_r to x_{r+1}

through a common sink. Counting vertices and edges gives $|V(\Gamma_+)| = s + |C_+|$, and by Lemma 5.1(4),

$$|E(\Gamma_+)| = (|C_+| - |T|) + \sum_{x<_v t} e(x,t), \qquad \text{and} \qquad \sum_{x<_v t} 1 = |T| + \sum_{r=1}^{s-1} g_r.$$

Substituting into $b_+ = |E(\Gamma_+)| - |V(\Gamma_+)| + 1$ yields

$$b_+ = \sum_{r=1}^{s-1}(g_r - 1) + \sum_{x<_v t}(e(x,t) - 1),$$

where every summand is a nonnegative integer.

Step 2. Bounding S_K and S_{uniq}. Put $K = \{r \in \{1,\dots,s-1\} : g_r \geq 2\}$, $S_K = \{x_j \in S : j-1 \in K \text{ or } j \in K\}$, $S_{\mathrm{uniq}} = \{x \in S \setminus S_K : |T(x)| = 1\}$, and $S_{\mathrm{rem}} = S \setminus (S_K \cup S_{\mathrm{uniq}})$ (Figure 6).

Fix $x \in S_{\mathrm{uniq}}$ and let t be the unique sink in $T(x)$. By Lemma 4.3(2), $\{z \in P : z \geq_v x\}$ is not a chain, so some $z \geq_v x$ has $d^+(z) \geq 2$. Choose such a z maximal in $<_v$, with distinct upper covers y_1, y_2 in Γ. Then y_1, y_2 are incomparable in $<_v$, and maximality of z implies that every element above y_1 or y_2 has out-degree at most 1 in Γ; hence $y_1, y_2 \in C_+$. Directed paths in Γ_+ from x to y_1 and y_2 begin with edges $x \to u_1$ and $x \to u_2$. Since $u_1 \in C_+$ has a chain upper set, y_1 and y_2 cannot both lie above u_1, forcing $u_1 \neq u_2$. As $t(u_1) = t(u_2) = t$, we get $e(x,t) \geq 2$. Step 1 therefore yields $|K| + |S_{\mathrm{uniq}}| \leq b_+$, and thus

$$|S_K| + |S_{\mathrm{uniq}}| \leq 2|K| + |S_{\mathrm{uniq}}| \leq 2b_+.$$

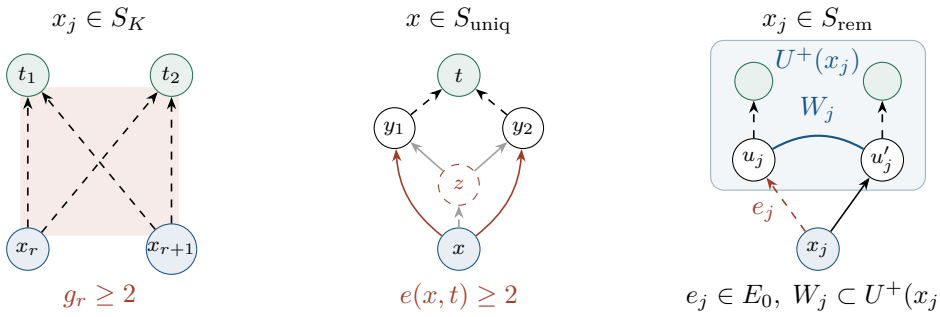

FIGURE 6. The three classes of sources in Proposition 5.2 (black dashed arrows denote directed paths in Γ). Left: a source in S_K borders a gap r covered by at least two sinks t_1, t_2 ($g_r \geq 2$). Middle: for $x \in S_{\mathrm{uniq}}$ with unique sink t (illustrated for $x <_v z$), omitting the maximal branching vertex $z \notin Y_+$ produces at least two outgoing edges from x into C_+ in Γ_+ (red), giving $e(x,t) \geq 2$; when $z = x$, the covers $x \to y_1, y_2$ already lie in Γ_+. Right: for $x_j \in S_{\mathrm{rem}}$, deleting the cover $e_j = (x_j \to u_j)$ (red dashed) leaves the cover $x_j \to u'_j$ and an undirected walk W_j inside $U^+(x_j)$ joining u'_j to u_j.

Step 3. Bounding S_{rem} and $|S| + |T|$. Fix $x_j \in S_{\mathrm{rem}}$, so $x_j \notin S_K$ and $|T(x_j)| \geq 2$. We first check that $d^+(x_j) \geq 2$. If $x_j \to u_1$ were the unique outgoing edge of x_j in Γ, then Lemma 4.3(1) would give another incoming edge $y \to u_1$ ($y \neq x_j$), and hence a source $x_i \leq_v y <_v u_1$ with $i \neq j$ (since $x_j \to u_1$ is a cover). Every $t \in T(x_j)$ lies above u_1, so $i, j \in I(t)$; by Lemma 5.1(2), every such $I(t)$ contains $\{j-1, j\}$ if $i < j$, and $\{j, j+1\}$ if $i > j$. Since $|T(x_j)| \geq 2$, this forces $g_{j-1} \geq 2$ or $g_j \geq 2$, contradicting $x_j \notin S_K$.

Pick two distinct out-neighbors u_j, u_j' of x_j in Γ. Since G_v is 2-connected, $G_v - x_j$ contains a path π from u_j' to u_j. Split π at its vertices in $U^+(x_j)$ into minimal subpaths between consecutive vertices $q, q' \in U^+(x_j)$:

- If q and q' are adjacent in π, they are comparable in $<_v$, so a directed path between them in Γ stays in $U^+(x_j)$.
- Otherwise, the internal vertices $p_1, \dots, p_r$ lie outside $\{x_j\} \cup U^+(x_j)$, so $j \notin I(p_i)$ for each i. By Lemma 5.1(2), each $I(p_i)$ is a nonempty interval on one side of j, and adjacent vertices in G_v have nested (hence intersecting) intervals; thus $I(p_1), \dots, I(p_r)$ all lie on the same side of j. Suppose they lie in $\{1, \dots, j-1\}$ (the case $\{j+1, \dots, s\}$ is symmetric). Because q is adjacent to p_1 with $j \in I(q) \setminus I(p_1)$, comparability forces $I(p_1) \subseteq I(q)$, whence $\{j-1, j\} \subseteq I(q)$, and likewise $\{j-1, j\} \subseteq I(q')$. Since $x_j \notin S_K$, we have $g_{j-1} = 1$, so a unique sink t_- covers $\{j-1, j\}$; every sink above q or q' must equal t_-, giving $q, q' \leq_v t_-$. Directed paths in Γ from q and q' to t_- stay in $U^+(x_j)$ and join q to q' within $U^+(x_j)$.

Concatenating these replacements yields an undirected walk W_j from u_j' to u_j in Γ whose vertices all lie in $U^+(x_j)$.

Now set $e_j = (x_j \to u_j)$ for each $x_j \in S_{\mathrm{rem}}$ and $E_0 = \{e_j : x_j \in S_{\mathrm{rem}}\}$. Because each e_j leaves x_j, the edges in E_0 are pairwise distinct, so $|E_0| = |S_{\mathrm{rem}}|$. For each $x_j \in S_{\mathrm{rem}}$, the only edge of E_0 leaving x_j is e_j, so $(x_j \to u_j') \notin E_0$; moreover, every vertex of W_j lies in $U^+(x_j) \subseteq P \setminus S$, so no edge of W_j leaves S and hence none belongs to E_0. Thus x_j and u_j remain connected in $\Gamma \setminus E_0$. Since Γ is connected, $\Gamma \setminus E_0$ stays connected and has at least $n-1$ edges:

$$n - 1 + b - |S_{\mathrm{rem}}| = |E(\Gamma \setminus E_0)| \geq n - 1 \implies |S_{\mathrm{rem}}| \leq b.$$

With Step 2, this gives $|S| \leq 2b_+ + b$. Because the reversal-complement v^* (via $p \mapsto n+1-p$) reverses all edges of Γ, induces $G_v \cong G_{v^*}$, swaps S with T, Y_+ with Y_-, and b_+ with b_-, and preserves b and both conditions of Lemma 4.3, applying the same bound to v^* yields $|T| \leq 2b_- + b$, so

$$|S| + |T| \leq 2(b_+ + b_-) + 2b.$$

Step 4. From $|S| + |T|$ to n. Since $|E(\Gamma)| = n - 1 + b$, every $u \in P \setminus T$ has $d^+(u) \geq 1$, and every $y \in P \setminus S$ has $d^-(y) \geq 1$,

$$\sum_{d^+(u) \geq 2} (d^+(u) - 1) = |E(\Gamma)| - (n - |T|) = |T| + b - 1,$$
$$\sum_{d^-(y) \geq 2} (d^-(y) - 1) = |E(\Gamma)| - (n - |S|) = |S| + b - 1.$$

By Lemma 4.3(1), every edge $u \to y$ of Γ has $d^+(u) \geq 2$ or $d^-(y) \geq 2$. Using $d \leq 2(d-1)$ for every integer $d \geq 2$ gives

$$n - 1 + b = |E(\Gamma)| \leq \sum_{d^+(u) \geq 2} d^+(u) + \sum_{d^-(y) \geq 2} d^-(y) \leq 2(|T| + b - 1) + 2(|S| + b - 1).$$

Rearranging yields

$$n \leq 2(|S| + |T|) + 3b - 3 \leq 4(b_+ + b_-) + 7b - 3 \leq 4(m + m) + 7m - 3 = 15m - 3. \quad \square$$

6. Finiteness of the basis

We combine Proposition 5.2 with Lemma 3.1 to bound $|\rho(w)|$ uniformly for all $w \in B_m$, and deduce Theorem 1.1 via Dickson's lemma.

Lemma 6.1. *Let $u \preceq w$. Then $|\rho(u)| \leq |\rho(w)|$.*

Proof. Write $u = w|U$ for $U \subseteq \{1, \dots, |w|\}$. Whenever a maximal decreasing interval M of w meets U, the entries of w on $M \cap U$ decrease. Between any two successive entries from $M \cap U$ there is no other selected entry in position or value order. Thus the entries from $M \cap U$ stay in a single maximal decreasing interval of u, and u has at most as many maximal decreasing intervals as w, giving $|\rho(u)| \leq |\rho(w)|$. □

Lemma 6.2. *Under the hypotheses and notation of Lemma 3.1, set $A = U_1 \setminus U_2$, $Q = U_1 \cap U_2$, and $D = U_2 \setminus U_1$. Then*

$$|\rho(w)| \leq |\rho(w|A)| + |\rho(w|D)| + 3.$$

Proof. Set $u = w|(A \cup D)$. By Lemma 3.1(2), $A < D$ and $w(A) < w(D)$, so standardizing $A \cup D$ places $w|A$ in the first $|A|$ positions and values and $w|D$ in the remaining $|D|$ positions and values; since $u(|A|) < u(|A|+1)$, no maximal decreasing interval of u crosses between them, giving $|\rho(u)| = |\rho(w|A)| + |\rho(w|D)|$. If $Q = \varnothing$, then $u = w$ and the bound holds.

Assume now that $Q \neq \varnothing$. By Lemma 3.1(2), Q is a single maximal decreasing interval of w, so $Q = [p, p']$ and $w(Q) = [c, d]$ are contiguous integer intervals. Let $R_1, \dots, R_k$ be the remaining $k = |\rho(w)| - 1$ maximal decreasing intervals of w in position order. By the argument of Lemma 6.1, each R_r lies in a single maximal decreasing interval of u, so $|\rho(u)|$ equals k minus the number of indices $r \in \{1, \dots, k-1\}$ for which R_r and R_{r+1} coalesce in u.

For such an r, put $i = \max R_r$ and $j = \min R_{r+1}$. Since i and j are consecutive in $A \cup D$, coalescence in u means that $w(i) > w(j)$ and $w(j), w(i)$ are consecutive in $w(A \cup D)$. Because R_r and R_{r+1} do not coalesce in w, either $j > i + 1$ or $w(i) > w(j) + 1$. In the first case, the nonempty open interval (i, j) lies in $Q = [p, p']$ while $i, j \notin [p, p']$, forcing $i = p - 1$ and $j = p' + 1$. In the second case, the nonempty open interval $(w(j), w(i))$ lies in $w(Q) = [c, d]$ while $w(i), w(j) \notin [c, d]$, forcing $w(j) = c - 1$ and $w(i) = d + 1$. Because $r \mapsto \max R_r$ and w are injective, each case holds for at most one r. Thus at most two pairs coalesce, giving $|\rho(w)| - 1 = k \leq |\rho(u)| + 2 = |\rho(w|A)| + |\rho(w|D)| + 2$ (cf. [11, Lemma 7.9], as deleting Q from w corresponds to deleting a single entry from $\rho(w)$). □

Theorem 6.3. *For every positive integer m and every $w \in B_m$, the descending core satisfies $|\rho(w)| \leq 15m - 3$.*

Proof. We proceed by strong induction on $m \geq 1$. Fix $m \geq 1$, let $w \in B_m$, and put $v = \rho(w)$. If $|v| \leq 2$, then $|v| \leq 15m - 3$ trivially; if $|v| \geq 3$ and G_v is 2-connected, Proposition 5.2 gives $|v| \leq 15m - 3$.

Suppose $|v| \geq 3$ and G_v is not 2-connected, and let $U_1, U_2 \subsetneq \{1, \dots, |w|\}$ be the subsets from Lemma 3.1. Since $w|U_i \prec w \in B_m$, we have $F(w|U_i) \leq m$ for $i = 1, 2$, while evaluating Lemma 3.1(1) at $q = 1$ with $U = \{1, \dots, |w|\}$ gives $F(w|U_1)\,F(w|U_2) = F(w) > m$. Hence $F(w|U_i) \geq 2$ for $i = 1, 2$ (forcing $m \geq 2$). Define

$$m_i = \min\bigl(F(w|U_i) - 1,\ \lfloor m/2 \rfloor\bigr) \qquad (i = 1, 2),$$

so $1 \leq m_i \leq \lfloor m/2 \rfloor \leq m - 1$ and $F(w|U_i) \geq m_i + 1$. Moreover, $(m_1 + 1)(m_2 + 1) > m$: if $m_i = F(w|U_i) - 1$ for both $i = 1, 2$, the product equals $F(w) > m$, and otherwise one factor is $\lfloor m/2 \rfloor + 1$ and the other is at least 2, giving $(m_1+1)(m_2+1) \geq 2(\lfloor m/2 \rfloor + 1) > m$.

Choose $U_i' \subseteq U_i$ of minimum size with $F(w|U_i') > m_i$; then $w|U_i' \in B_{m_i}$. For $U = U_1' \cup U_2'$, since $U_i' \subseteq U \cap U_i$, pattern monotonicity and Lemma 3.1(1) at $q = 1$ yield

$$F(w|U) = F(w|(U \cap U_1))\,F(w|(U \cap U_2)) \geq F(w|U_1')\,F(w|U_2') \geq (m_1 + 1)(m_2 + 1) > m,$$

forcing $U = \{1, \dots, |w|\}$ because $w \in B_m$. Setting $A = U_1 \setminus U_2$ and $D = U_2 \setminus U_1$, the inclusions $U_i' \subseteq U_i$ and $U_1' \cup U_2' = \{1, \dots, |w|\}$ imply $A \subseteq U_1'$ and $D \subseteq U_2'$, so $w|A \preceq w|U_1'$ and $w|D \preceq w|U_2'$. By Lemmas 6.2 and 6.1 and the induction hypothesis on $w|U_i' \in B_{m_i}$,

$$|v| \leq |\rho(w|A)| + |\rho(w|D)| + 3 \leq |\rho(w|U_1')| + |\rho(w|U_2')| + 3 \leq 2\bigl(15\lfloor m/2 \rfloor - 3\bigr) + 3 \leq 15m - 3.$$ □

Proof of Theorem 1.1. By Theorem 6.3, only finitely many descending cores occur in B_m. Fix such a core $v \in S_k$. Every permutation with core v has the form $v[\delta_{a_1}, \dots, \delta_{a_k}]$ for a unique tuple $(a_1, \dots, a_k) \in \mathbb{Z}^k_{\geq 1}$. Choosing a_i positions from the i-th maximal decreasing interval shows that $a_i \leq a'_i$ for all $1 \leq i \leq k$ implies $v[\delta_{a_1}, \dots, \delta_{a_k}] \preceq v[\delta_{a'_1}, \dots, \delta_{a'_k}]$. Because distinct elements of B_m are $\preceq$-incomparable, the block-length vectors of elements in B_m with core v form an antichain in $\mathbb{Z}^k_{\geq 1}$, which is finite by Dickson's lemma [5]. Taking the union over the finitely many cores v shows that B_m is finite, and $\mathrm{KL}_m = \mathrm{Av}(B_m)$ completes the proof. □

7. Concluding remarks

We record three open directions suggested by Theorem 1.1 and its proof.

(1) *Effectivity.* Theorem 6.3 bounds the descending core size explicitly by $|\rho(w)| \leq 15m - 3$ for all $w \in B_m$, but the passage from a fixed core to block lengths via Dickson's lemma in the proof of Theorem 1.1 is non-effective. Beyond $|B_1| = 2$ [10] and $|B_2| = 66$ [14, Theorem A.1], finding an explicit upper bound on $\max_{w \in B_m} |w|$ or an algorithm guaranteed to compute B_m for arbitrary $m \geq 1$ remains open.

(2) *Sharpness of the core bound.* The linear constant 15 in Theorem 6.3 was chosen to keep the argument transparent rather than optimal. Determining the true growth rate and optimal leading constant of $\max_{w \in B_m} |\rho(w)|$ as a function of m is open.

(3) *Other Weyl groups.* Billey and Braden's pattern monotonicity theorem [2] applies to all finite Weyl groups, and Lemma 2.1 holds in arbitrary Coxeter groups (Remark 2.2). Extending Theorem 1.1 to other classical Weyl groups would require substitutes for the type-A inversion-graph arguments of Sections 4–5.

Appendix A. Geometric proof of Lemma 2.1

A.1. **Parabolic decompositions and Schubert varieties.** Fix S_n, S, W_K, and w_K as in Section 2, and recall $\mathcal{F}$, B, x_y, and X_w from Section 1. For $b \in S_n$, let $\mathrm{supp}(b) \subseteq S$ be the set of simple reflections appearing in a reduced expression for b. For $K \subseteq S$, put $W^K = \{b \in S_n : D_R(b) \cap K = \varnothing\}$ (minimum-length representatives of cosets bW_K). The *parabolic decomposition* of $z \in S_n$ relative to K is $z = bd$ with $b \in W^K$ and $d \in W_K$; it satisfies the *Billey–Postnikov criterion* if $\mathrm{supp}(b) \cap K \subseteq D_L(d)$ [12, Proposition 4.2].

Let $\mathcal{F}^K$ be the partial flag variety obtained from $\mathcal{F}$ by omitting the subspaces of dimensions i with $s_i \in K$, and let $\pi_K : \mathcal{F} \to \mathcal{F}^K$ be the projection. Put $e_K = \pi_K(x_e)$ and $X^K_b = \overline{B \cdot \pi_K(x_b)}$ for $b \in W^K$. When $z = bd$ satisfies the Billey–Postnikov criterion relative to K, the restriction $\pi_K|_{X_z} : X_z \to X^K_b$ is Zariski locally trivial with fiber X_d, and its fiber over e_K is $X_d \subseteq X_z$ [12, Theorem 3.3 and Proposition 4.7].

Let P_K be the stabilizer of e_K in $\mathrm{GL}_n(\mathbb{C})$ and $G_K \leq P_K$ its block-diagonal Levi subgroup (with blocks cut between i and $i+1$ where $s_i \notin K$). For $K \subseteq S$, $X_{w_K} = \pi_K^{-1}(e_K) = P_K \cdot x_e \cong P_K/B \cong G_K/(B \cap G_K)$ is a product of complete flag varieties, hence smooth.

A.2. **Proof of the product formula.**

Geometric proof of Lemma 2.1. Let $v = bd$ be the parabolic decomposition relative to K. Since $v \in W_J$ and $d \in W_K \subseteq W_J$, we have $b \in W_J \cap W^K$. For $s \in K$, length additivity gives $\ell(vs) - \ell(v) = \ell(ds) - \ell(d)$, so $K \subseteq D_R(d)$ and hence $d = w_K$. Since $w_K^2 = e$, put $z = vw_Ku = bu$ and $L = \mathrm{supp}(b)$. As $b \in W_J$, we have $D_R(b) \subseteq L \subseteq J$, so $D_R(b) \cap I = D_R(b) \cap K = \varnothing$ and $b \in W^I$. Moreover, $L \cap I = L \cap K$, so the Billey–Postnikov criterion holds for $z = bu$ relative to I and for $v = bw_K$ relative to K:

$$L \cap I = L \cap K \subseteq K \subseteq D_L(u), \qquad L \cap K \subseteq D_L(w_K) = K.$$

Hence $\pi_I|_{X_z} : X_z \to X^I_b$ and $\pi_K|_{X_v} : X_v \to X^K_b$ are Zariski locally trivial with fibers X_u and X_{w_K} over e_I and e_K, respectively.

Because $b \in W_L$, its permutation matrix lies in G_L. Put $B_L = B \cap G_L$. For $M \in \{I, K\}$, $G_L \cap P_M$ is the standard parabolic subgroup of G_L indexed by $L \cap M$, so $L \cap I = L \cap K$ yields $G_L \cap P_I = G_L \cap P_K$; denote this subgroup by R. In G_L/R, set $Y = \overline{B_L \cdot bR}$ and $y_e = R$. By Richmond and Slofstra's Levi isomorphism [12, Lemma 4.8], $gR \mapsto g \cdot e_M$ restricts to an isomorphism $\iota_M : Y \xrightarrow{\sim} X_b^M$ sending y_e to e_M for $M \in \{I, K\}$. Setting $\varphi_I = \iota_I^{-1} \circ (\pi_I|_{X_z})$ and $\varphi_K = \iota_K^{-1} \circ (\pi_K|_{X_v})$ gives the diagram:

$$\begin{array}{ccccc} X_z & & \xrightarrow{\varphi_I} & & \xleftarrow{\varphi_K} & & X_v \\ \varphi_I^{-1}(y_e) = X_u & & & Y & & & \varphi_K^{-1}(y_e) = X_{w_K} \end{array}$$

Choose a Zariski open neighborhood $N \subseteq Y$ of y_e with trivializations $\theta_I : \varphi_I^{-1}(N) \xrightarrow{\sim} N \times X_u$ and $\theta_K : \varphi_K^{-1}(N) \xrightarrow{\sim} N \times X_{w_K}$ compatible with projection pr_1 to N. Identifying $\{y_e\} \times X_u \cong X_u$ and $\{y_e\} \times X_{w_K} \cong X_{w_K}$, the restrictions over y_e yield automorphisms α_I of X_u and α_K of X_{w_K}; replacing θ_M with $(\mathrm{id}_N \times \alpha_M^{-1}) \circ \theta_M$ ensures that both θ_I and θ_K send x_e to (y_e, x_e).

For an irreducible complex variety V, let IC_V be its perverse intersection complex over $\mathbb{Q}$, so the stalk cohomology of $\mathrm{IC}_V[-\dim V]$ gives the local grading of Section 1 (and equals $\mathbb{Q}$ in degree 0 at a smooth point). Because $\mathrm{IC}_{V_1 \times V_2}[-\dim(V_1 \times V_2)]$ is the external tensor product of $\mathrm{IC}_{V_1}[-\dim V_1]$ and $\mathrm{IC}_{V_2}[-\dim V_2]$ [1, §4.2.8], stalk cohomology over $\mathbb{Q}$ takes the graded tensor product at the two points. Since X_{w_K} is smooth, the trivialization $N \times X_{w_K}$ identifies the normalized stalk cohomology of Y at y_e with that of X_v at x_e (so its odd-degree groups vanish and its even-degree Poincaré polynomial is $P_v(q)$). The trivialization $N \times X_u$ then yields $P_{vw_Ku}(q) = P_z(q) = P_v(q)P_u(q)$. □

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