hexagon:2611.01284v1

POSITIVE FIRST-HOMOLOGY TORSION GROWTH IN CHARACTERISTIC TOWERS OF CLOSED HYPERBOLIC THREE-MANIFOLDS

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Abstract. We construct profinitely cofinal towers of finite covers in which ordinary integral first-homology torsion grows exponentially. For each $k \geq 5$ a closed hyperbolic three-manifold admits such a tower fully invariant in its fundamental group, with normalized lower rate $\frac{1}{2}\log((k-1)^{k-1}/(2k^{k-2}))$. We obtain the same lower bound over closed hyperbolic rational homology three-spheres. For these bases the cut class lives in a double cover, and mod-two abelianization gives full invariance in the original group. An estimate on the relative Fox differential yields the lower bound. We also obtain regular towers with a positive rate over the alternating n-component cyclic chain-link exterior for $n \geq 4$. For the first closed family we compute the leading L^2-Alexander coefficient of the cut class and bound the normalized Mahler measure of the corresponding finite-cover initial faces.

Owing to time constraints, the search for overlapping results was limited, and readers who find substantial overlap with their own published work are kindly asked to contact the author.

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1. Main results

For a connected finite cover $Y \to X$ let

$$T(Y) = |\operatorname{Tor} H_1(Y;\mathbb{Z})|, \qquad q_X(Y) = \frac{\log T(Y)}{[Y:X]}.$$

All coefficients in this definition are trivial. Let a tower be *profinally cofinal* if its subgroups are eventually contained in every finite-index subgroup of the fundamental group of the original base. A fully invariant subgroup is preserved by every endomorphism, in the sense of inclusion. In particular, it is characteristic and normal.

Theorem 1.1. *For every $k \geq 5$ there exist a closed connected orientable hyperbolic three-manifold X_k and a nested profinitely cofinal tower of connected covers $Y_i \to X_k$ with strictly increasing degrees, fully invariant in $\pi_1 X_k$, such that*

$$\liminf_{i\to\infty} q_{X_k}(Y_i) \;\geq\; c_k, \qquad c_k = \tfrac{1}{2}\log\left(\frac{(k-1)^{k-1}}{2k^{k-2}}\right) > 0. \tag{1}$$

Date: 28 November 2026.

A subsequence has a positive limit. Every fixed connected finite cover $V \to X_k$ has a profinitely cofinal tower regular over V with lower limit at least $[V : X_k]c_k$ in the normalization q_V.

Theorem 1.2. *For each $k \geq 5$ there is a closed connected orientable hyperbolic rational homology three-sphere M_k with a nested profinitely cofinal tower of connected finite covers $M_{k,i}$, fully invariant in $\pi_1 M_k$, of strictly increasing degrees, such that*

$$\liminf_i q_{M_k}(M_{k,i}) \;\geq\; c_k > 0.$$

The bases can have arbitrarily large hyperbolic volume. All levels of the tower factor through a connected double cover whose fundamental group maps onto the free group of rank $k-1$.

1.1. **Relation to torsion growth conjectures.** The explicit constant satisfies $c_5 = \frac{1}{2}\log(128/125) > 0$. For real $x > 1$, the derivative of $2c_x$ is $\log(1-1/x) + 2/x$. The inequality $\log(1-u) > -u/(1-u)$ for $0 < u < 1$ makes this derivative positive for $x \geq 2$. Also $c_k = \frac{1}{2}\log(k/(2e)) + o(1)$, since $(1-1/k)^{k-1} \to e^{-1}$.

Lackenby asked for a finitely presented group with a nested chain of finite-index normal subgroups of trivial intersection along which normalized first integral homology torsion has a positive limit [4, Problem 51 in the published version]. See also the discussion in [2, Section 1]. Both theorems above give closed hyperbolic examples. The limit can be obtained by taking a subsequence, since Lê's upper bound gives

$$\limsup_i q_X(Y_i) \leq \frac{\mathrm{vol}(X)}{6\pi}$$

for residual normal towers over a closed hyperbolic base [28, Theorem 1.2]. The stronger conjecture in the closed hyperbolic case predicts equality as a limit for every cofinal regular tower [32, Conjecture 5.3]. For arithmetic congruence towers see Bergeron and Venkatesh [12] and the criterion of Bergeron, Şengün and Venkatesh [11]. The present results establish a positive rate for specific towers. In higher real rank, the setting of a different question in [4], Slutsky proves vanishing for normalized first-homology torsion along Benjamini–Schramm convergent sequences [44].

Liu constructs an exhausting nested tower with positive first homology torsion growth over each closed hyperbolic three-manifold [30][32, Theorem 5.5]. Those covers can be irregular over the base. Liu also proves that a single regular finite cover can have positive logarithmic Alexander Mahler measure [31, Corollary 1.3]. A single such cover gives a positive rate in its abelian refinements. A residual tower obtained by diagonal refinement requires a positive lower bound on the degree-normalized Mahler measure at arbitrarily deep intermediate covers.

Baik et al. prove exponential torsion growth in cyclic covers of random three-manifolds of positive first Betti number [7]. Alexandrov constructs

closed hyperbolic examples with exponential growth in a family of cyclic covers [6, Sections 3–4]. All covers pulled back from the same integral character retain its kernel. Champanerkar and Kofman [13] treat abelian link towers. In a different direction, Guo constructs a vanishing-torsion tower over a cusped manifold in response to [8, Problem 3.6]. Each bonding map in that tower is regular, while regularity of the composite covers over the original base is not asserted [21, Remark 4.2].

1.2. **Proof overview.** Sections 2 and 3 construct a closed hyperbolic base X_k by identifying the ends of a cobordism with a single relative one-handle and two-handle. The relative Fox differential is $1 + u_1 + \cdots + u_{k-1}$, where the u_j freely generate a retract of $\pi_1 X_k$. On the finite covers in Section 4, restriction to the free subgroup turns character averaging into integration over independent edge phases. A matching-count estimate bounds the logarithmic determinant in proportion to the degree.

Section 5 identifies this determinant with an initial face of the Alexander polynomial. Section 6 controls the loss of ordinary integral torsion in the passage from the Alexander module to abelian covers and constructs the tower. For the rational homology spheres, Section 7 performs the cut in a double cover. Its two cobordism differentials each contribute to the estimate. A common mod-two characteristic subgroup makes both retractions available on subgroups that are fully invariant in the original group.

Section 8 gives a criterion involving only three terms in an exposed Fox face. Section 9 applies the criterion to the alternating cyclic chain links of Duan [15, Example 6.7]. Section 10 compares the initial-face bound with the leading L^2-Alexander coefficient on X_k. In Section 11 a single closed fibered base admits regular towers whose normalized primary torsion can be made negligible at any prescribed finite set of primes at each level.

2. A relative handle presentation

We compose paths in the order in which they are traversed. Take a planar graph with vertices P, Q joined by $k+1$ parallel edges $e_0, e_1, \ldots, e_k$, all oriented from Q to P. At the two vertices the planar cyclic orders of the numbered edges are opposite. Its closed regular neighborhood L in the sphere is a sphere with $k+1$ holes. Put $a_j = e_0^{-1} e_j$ based at P and $a_0 = 1$. The $a_1, \ldots, a_k$ freely generate $\pi_1 L$. Let S be the oriented double of L and write L' for the second half, identified with L by reflection. The surface S has genus k.

On the top of $S \times [0,1]$ attach an orientable three-dimensional one-handle, using small discs about P, Q in the interior of L. Let b be a core path from P to Q, and put $x = be_0$. On the new top surface truncate each of the disjoint edge interiors $e_1, \ldots, e_k$ at the attaching discs. In the tube join the arrival end of e_j to the departure end of e_{j+1}, with the index taken cyclically in $1, \ldots, k$. These joining arcs can be chosen disjoint. Indeed, the two boundary circles of the tube, parametrized with the same annular

coordinate, read the cyclic orders of the incident planar half-edges in opposite surface orientations. Parametrize the tube as $[0,1]\times(\mathbb{R}/k\mathbb{Z})$, placing the end of e_j at angle $j-1$ on each boundary circle. The curves $t\mapsto(t,j-1+t)$ are disjoint and realize exactly the required cyclic shift. Their union with the truncated edges is one simple closed curve γ. All its traversals of the tube run from P to Q. In the three-dimensional one-handle each is homotopic through its attaching discs to b. With an appropriate choice of initial point its word is

$$w=(xa_1)(xa_2)\cdots(xa_k). \tag{2}$$

The belt meridian has algebraic intersection number $\pm k$ with γ, so γ is non-separating. Attach a two-handle along it. The resulting oriented manifold W has two closed boundary components S_-,S_+, each of genus k. Here $S_-=S\times\{0\}$. Relative to S_- we used just one one-handle and one two-handle. Put

(3)

$$u_0=1,\qquad u_j=(xa_1)\cdots(xa_j)\quad(1\le j<k),\qquad B=\sum_{j=0}^{k-1}u_j\in\mathbb{Z}[\pi_1W].$$

The boundary of the relative two-cell in the lifted handle complex is multiplication by B. Each passage along the positive x-cell has coefficient its initial prefix, or equivalently $B=\partial w/\partial x$ in relative Fox calculus. The other pieces of the attaching word are in the old surface. These bases fix all lift choices up to signed group elements.

Lemma 2.1. *The manifold W is irreducible, has incompressible boundary, contains no incompressible torus and is not an interval bundle.*

Proof. The handles were attached in the half corresponding to L. Thus W is the union of $L'\times[0,1]$ and a manifold W_0 formed from $L\times[0,1]$ using those same two handles. They meet in the $k+1$ annuli $\partial L\times[0,1]$. Before attachment of the two-handle the modified $L\times[0,1]$ is a handlebody. A band transverse to the interior of e_k has a properly embedded cocore arc in L. The product of this arc with $[0,1]$ is a meridian disc of the handlebody. Its boundary meets γ once, on the truncated e_k, and misses the new feet. Compression along the geometrically primitive attaching curve cancels a handle. Thus W_0 is a handlebody of genus k. In its free group one can eliminate a_k using (2), leaving the free basis $x,a_1,\dots,a_{k-1}$.

Write $v_j=[a_j]$, $v_x=[x]$ in the abelianization of this group. We have $v_0=0$ and $v_k=-kv_x-\sum_{j=1}^{k-1}v_j$. The boundary circle between two consecutive edges is represented up to orientation and conjugation by $a_j^{-1}a_{j+1}$ (indices cyclic in $0,\dots,k$). Its homology class up to sign is $v_{j+1}-v_j$. In the displayed basis these $k+1$ vectors are nonzero, primitive and pairwise nonproportional. For the last two this follows from their v_x coordinates and their different v_{k-1} coefficients. For the others it follows from their distinct supports. Consequently no core is a proper power, and no two cores have

conjugate nontrivial powers. They generate a malnormal family of maximal cyclic subgroups in the free group of W_0. The same is true of the distinct peripheral subgroups of $\pi_1 L'$.

In particular the gluing annuli are incompressible in both handlebodies. Gluing irreducible pieces along incompressible annuli gives an irreducible manifold by the innermost-circle argument. Its graph of groups has these two free vertex groups and the $k+1$ annular edge groups. Malnormality on both sides implies that the pointwise stabilizer of a segment of two edges in its Bass–Serre tree is trivial. A subgroup $\mathbb{Z}^2$ cannot fix a vertex in a free vertex group. If it acts on an axis, the kernel of its action there fixes a two-edge segment and the image of a torsion-free abelian group acting by translations is cyclic. These alternatives exhaust the actions of a finitely generated abelian group on a simplicial tree without inversion [47]. Thus $\pi_1 W$ contains no $\mathbb{Z}^2$.

Choose a prime p not dividing k. With constant $\mathbb{F}_p$ coefficients the relative handle differential is k, so $H_*(W, S_-; \mathbb{F}_p) = 0$. Poincaré–Lefschetz duality gives $H_*(W, S_+; \mathbb{F}_p) = 0$ as well. Each boundary inclusion induces an isomorphism on H_1 and a surjection on H_2 with these coefficients, also in group homology by naturality of the surjection from space to group homology in degree two. Stallings's mod-p central series theorem [48] identifies all successive lower mod-p central quotients of the surface group with those of $\pi_1 W$. A closed orientable surface group is residually a finite p-group [38], so each boundary inclusion is injective on fundamental groups. Finally $H_1(W, S_-; \mathbb{Z}) = \mathbb{Z}/k\mathbb{Z}$. A connected interval bundle with these two closed boundary components would be the product of one of the surfaces with an interval and its relative homology would vanish. □

Lemma 2.2. *Let F be free on $z_1, \dots, z_{k-1}$. There is an epimorphism $r_W : \pi_1 W \to F$ taking u_j to z_j. Its restrictions to the two boundary surface groups are epimorphisms.*

Proof. Put $z_0 = z_k = 1$. On $\pi_1 S$ first fold the double onto L, then send a_j to $z_{j-1}^{-1} z_j$. Send the new generator x to 1. Relation (2) telescopes to 1 and the prefix u_j maps to z_j. Both boundaries contain a copy of L' in the untouched product $L' \times [0,1]$. The image of either copy is already all of F. □

3. Closing the manifold and retaining the retraction

3.1. **Annular slopes and gluing.**

Lemma 3.1. *On either end $S_\pm$ of W, there is an essential curve with zero geometric intersection with every boundary slope of an essential annulus. This includes annuli with both ends on the same surface. Suppose copies of W are glued in pairs along their incompressible boundary surfaces. If the two sets of annular slopes are disjoint under each gluing, the resulting closed manifold is atoroidal.*

Proof. Use the characteristic submanifold of the irreducible, boundary-incompressible W [24, 25]. Every essential annulus is carried by its characteristic pair, with boundary homotoped in ∂W. The part of the pair on an end consists of horizontal surfaces of interval bundles and boundary annuli. If a horizontal surface is an entire closed end, its bundle component has empty frontier and hence is all of W. This is excluded by Lemma 2.1. If the pair meets an end in a nonempty proper subsurface, its frontier curves there are essential by the incompressibility of the frontier annuli. Choose one such curve. Every annular slope is carried by the subsurface, so has zero intersection with this curve. If there are no slopes, any essential curve suffices. In particular, each nonempty slope set has diameter at most two in the curve graph.

For the last assertion, let T be an incompressible torus and isotope it to minimize intersections with all the gluing surfaces. The innermost disk argument, using incompressibility and irreducibility of the glued manifold, makes every intersection essential in both surfaces. A component of the cut torus is therefore an incompressible annulus. If such a component is boundary-parallel, its parallelism gives a product solid torus between the component and an annulus of the cut boundary. Any further pieces of the cut torus in this region have longitudinal boundary on the latter annulus. Incompressible annuli of this kind in a solid torus are boundary-parallel, as is seen by cutting along a meridian disc. Choose a parallelism containing no further piece. Pushing across it removes two intersections. Thus the annuli in minimal position are essential. Each intersection circle would then give a common slope across its gluing surface. If there are no intersections, T lies in a copy of W, whose fundamental group has no $\mathbb{Z}^2$ by Lemma 2.1. □

3.2. **A gluing compatible with the free quotient.**

Lemma 3.2. *There is a gluing of S_- to S_+ for which the closed orientable manifold X is hyperbolic and r_W extends to $r : \pi_1 X \to F$. In particular, $A = \langle u_1, \dots, u_{k-1} \rangle$ is a free group on the displayed generators and a retract of $G = \pi_1 X$.*

Proof. Give the incoming and outgoing surfaces their cobordism orientations. By strong equivalence of surface-group epimorphisms [33, Theorem 3.15], each epimorphism in Lemma 2.2 is modeled by a genus-k handlebody projection followed by the projection of F_k onto the marked F_{k-1}. The marking can have either orientation. In the model handlebody $L \times [-1, 1]$, reflection in $L \times \{0\}$ reverses the boundary orientation and acts trivially on the fundamental group. Composing one marking with this reflection changes the orientation without changing the epimorphism. Thus there is a gluing $h_0 : S_- \to S_+$ preserving the cobordism orientations and matching the epimorphisms up to conjugation in F. There is a pseudo-Anosov β on S_+ preserving this marked epimorphism up to conjugation. To see this, take a meridian d of the standard handlebody and a pseudo-Anosov f extending

over that handlebody [23]. For large a, d and $f^a d$ have curve graph distance at least three [34], so together they fill the surface. Penner's construction [39] makes $T_d T_{f^a d}^{-1}$ pseudo-Anosov. Both disc twists act trivially in the outer automorphism group of the handlebody fundamental group [23]. Transfer this mapping class by the marking of S_+.

Write $E_\pm$ for the annular slopes on $S_\pm$. By Lemma 3.1 the two sets have bounded diameter. For $h = \beta^m h_0$ the sets $h(E_-)$ and E_+ are disjoint for all sufficiently large m, since a pseudo-Anosov acts loxodromically on the curve graph [34]. Glue by such an h. The identified surface is incompressible and the resulting X irreducible by gluing along incompressible boundary. Lemma 3.1 also gives atoroidality.

The epimorphisms still match, so the HNN presentation extends r_W to r by sending the stable letter to the conjugator in F. The map $z_j \mapsto u_j$ splits r. The glued surface is a nonseparating incompressible surface of genus k. In a Seifert fibered manifold such a surface would be horizontal, since vertical closed surfaces have Euler characteristic zero. Cutting a Seifert fibration along a horizontal surface leaves an interval bundle, contrary to Lemma 2.1. Hyperbolization for closed atoroidal Haken manifolds [36] now gives a hyperbolic metric on X. □

4. Free-group characters and a determinant estimate

Write $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and give all compact tori probability Haar measure. For a nonzero Laurent polynomial P on a torus let $m(P) = \int \log|P|$ be its logarithmic Mahler measure. The logarithm is integrable, also when P has zeros on the torus. The following graph estimate was used in the one-relator construction of [37]. We give its proof and then pass from edge phases to characters of finite-index subgroups of the three-manifold group.

Lemma 4.1. *Let $H = (h_{ab})$ be a square s-by-s nonnegative integer matrix with positive permanent. Replace each entry by a sum of h_{ab} independent uniform circle variables, distinct for all labelled occurrences. Let P be the determinant of the resulting matrix. Then*

$$m(P) \geq \tfrac{1}{2} \log \operatorname{per} H - \tfrac{s}{2} \log 2.$$

If every row and column sum of H is $k \geq 2$, then $m(P) \geq s c_k$, where c_k is given by (1).

Proof. The logarithmic Khintchine inequality for independent Steinhaus variables Z_j and complex coefficients b_j, not all zero, is

$$\mathbb{E} \log \left| \sum b_j Z_j \right| \geq \tfrac{1}{2} \log \sum |b_j|^2 - \tfrac{1}{2} \log 2. \tag{4}$$

For the precise input, the sharp negative-moment inequality of Rapaport, Tkocz and Wu [41, Theorem 1] bounds the moment of order $p \in (-1, 0)$ of a normalized sum by the corresponding moment of $(Z_1 + Z_2)/\sqrt{2}$. Divide

the logarithm of that inequality by the negative number p and let $p \uparrow 0$. This gives (4). The limit is justified by finiteness of a fixed negative moment and boundedness of the sum. Jensen's formula gives $\mathbb{E}\log|(Z_1+Z_2)/\sqrt{2}| = -\frac{1}{2}\log 2$. The published positive-moment argument of König [26] and its correction are discussed in [41, Section 3].

Expand the determinant on its first row. Conditional on the other rows, the coefficient of each independent edge phase is the signed complementary minor P_e, repeated for parallel edges. A minor with a supported perfect matching is a nonzero polynomial, because distinct labelled matchings give distinct monomials. Omit the other minors. Applying (4) and then Jensen's inequality to the convex function $(v_j) \mapsto \log \sum e^{v_j}$ on the remaining variables gives

$$e^{2m(P)} \;\geq\; \tfrac{1}{2}\sum_{e} e^{2m(P_e)}.$$

Integrability of the nonzero minors justifies the inequalities. The empty minor has determinant one. Induction and expansion of the permanent now give $e^{2m(P)} \geq 2^{-s}\operatorname{per} H$. For constant row and column sums Schrijver's theorem for nonnegative integral matrices, allowing parallel edges [43], gives $\operatorname{per} H \geq ((k-1)^{k-1}/k^{k-2})^s$. This yields sc_k. □

Proposition 4.2. *Suppose G is a finitely generated group with a retraction onto the subgroup A freely generated by $u_1,\dots,u_{k-1}$. Let $D \leq G$ be fully invariant of finite index N. Put $\Gamma = D_{\mathrm{ab}}/\operatorname{Tor} D_{\mathrm{ab}}$ and $\widehat{\Gamma} = \operatorname{Hom}(\Gamma,\mathbb{T})$. For $\chi \in \widehat{\Gamma}$ inflate χ to D and set $\rho_\chi = \operatorname{Ind}_D^G \chi$. With $u_0 = 1$ and $B = \sum_{j=0}^{k-1} u_j$,*

$$\int_{\widehat{\Gamma}} \log|\det \rho_\chi(B)|\, d\chi \;\geq\; Nc_k. \tag{5}$$

The determinant is a nonzero Laurent polynomial in χ after choosing coset representatives and an integral basis of Γ. The same conclusion holds for any finite-index normal $D \lhd G$ if the inclusion $D \cap A \hookrightarrow D$ has a retraction.

Proof. Let $E = D \cap A$, of index s in A. Under the first hypothesis the retraction is an endomorphism of G after inclusion of A. Full invariance of D gives a retraction $D \to E$. Under the second hypothesis use the given retraction $D \to E$. The free group E has torsion-free abelianization, which embeds as a direct summand of Γ. Restriction from $\widehat{\Gamma}$ onto $\widehat{E_{\mathrm{ab}}}$ is thus a surjective continuous homomorphism sending Haar measure to Haar measure.

Upon restriction to A the induced representation splits according to the A-orbits on the cosets of D. Normality of D identifies every stabilizer in A with E. There are N/s blocks, each isomorphic to $\operatorname{Ind}_E^A \xi$ with $\xi(e) = \chi(geg^{-1})$ for a suitable coset representative g. Conjugation by g preserves D and its free abelianization, so each ξ is individually Haar on $\widehat{E_{\mathrm{ab}}}$. Dependence between different blocks does not affect the integral of the sum of their logarithms.

Identify E with the fundamental group of the s-vertex covering graph of the rose marked by $u_1,\dots,u_{k-1}$. Assign an independent Haar phase to each lifted oriented generator edge and use its inverse for backtracking.

Products around closed paths define a character of E. This produces Haar measure on $\widehat{E_{\mathrm{ab}}}$. Indeed, after choosing a spanning tree, all its phases can be set to one by vertex gauges and the remaining fundamental cycles can be assigned arbitrarily. This gives a surjective homomorphism of tori. Such vertex gauges act by diagonal conjugation on all generator matrices, and the edge-weighted permutation matrices are exactly the induced representation for the character of the loops, up to these gauges.

In these bases the block for B is the identity plus the $k-1$ weighted permutation matrices. Multiply each row by an additional independent uniform phase. The absolute determinant is unchanged. In a given row its k labelled entries now have phases $(U, UZ_1, \dots, UZ_{k-1})$. These are independent Haar phases, since this is an automorphism of the product torus. Variables in distinct rows are independent as well. Parallel edges count separately. The support matrix has row and column sums k and has the identity matching. Lemma 4.1 gives an integral at least sc_k for each block. Adding the N/s bounds proves (5). The representation in a coset basis is monomial in the characters. The nonzero determinant almost everywhere on each of the finitely many blocks proves the last assertion. □

5. The initial face of the Alexander polynomial

Let X be a compact connected orientable three-manifold with empty or toroidal boundary, $G = \pi_1 X$, and $D \le G$ have finite index. Write X_D for the corresponding cover, $\Gamma = D_{\mathrm{ab}}/\operatorname{Tor} D_{\mathrm{ab}}$, and $R = \mathbb{Z}[\Gamma]$. The first homology of the free abelian cover of X_D is the R-module $H_1(\ker(D \to \Gamma); \mathbb{Z})$. Write Δ_D for its order, which is the greatest common divisor of its zeroth Fitting minors, up to a signed monomial. The order is zero for a module of positive rank.

Theorem 5.1. *For the manifold of Lemma 3.2 and every fully invariant finite-index subgroup $D \le G$, the polynomial Δ_D is nonzero and satisfies*

$$m(\Delta_D) \;\ge\; [G : D] c_k. \tag{6}$$

The following comparison proves the theorem and also applies to the double-cover and link constructions.

Lemma 5.2 (Relative determinant and initial face)**.** *Suppose $b_1(X_D) \ge 2$. Cut X along a connected, properly embedded, oriented two-sided surface S with connected complement. Denote the cut manifold by W and the cohomology class dual to S by ψ. Use the sign of ψ for which the outgoing attachment in the mapping cone has positive weight. Suppose the based relative complex of (W, S_-), with its group-ring coefficients extended to $\mathbb{Z}G$, reduces by simple cell cancellations and elementary basis changes to a square differential P in degrees two and one. Put*

$$\rho = \operatorname{Ind}_D^G(d \mapsto [d]), \qquad P_D = \det \rho(P) \in R,$$

where ρ is extended linearly and evaluated entry by entry. If $P_D \neq 0$, then $\Delta_D \neq 0$. The sum I_D of its terms of least weight with respect to $\psi|_D$ satisfies

$$I_D \doteq P_D, \qquad m(\Delta_D) \geq m(I_D) = m(P_D).$$

Here $\doteq$ means equality up to a signed monomial in R. The statement uses finite index, and applies to subgroups D that are non-normal in G.

Proof. The character ψ vanishes on the vertex group of the cut decomposition. Write $\psi_D : \Gamma \to \mathbb{Z}$ for the map induced by its restriction, and $\mathcal{K} = \operatorname{Frac} R$. In the coset basis the matrices of $\rho(g)$ are monomial and their determinants are signed elements of Γ. We use the torsion convention in which an acyclic two-term complex in degrees two and one has torsion $\det \partial_2$.

Twist ρ by $\rho(g) \mapsto t^{\psi(g)}\rho(g)$. The cellular complex of X can be obtained from a cylinder on the glued surface, as in the surface-cutting description of twisted Alexander invariants [19]. Over $\mathcal{K}(t)$ it is the based mapping cone of

$$i_- - tj_+ : C_*(S_-; \rho) \otimes \mathcal{K}(t) \longrightarrow C_*(W; \rho) \otimes \mathcal{K}(t). \tag{7}$$

The second map includes the gluing homeomorphism, the outgoing inclusion and parallel transport along the stable letter in ρ. To see the power of t, lift the cells of W to one fundamental region of the cyclic cover. The bases of each prism lie in that region and its next translate. They give the two terms of (7). Its side faces give the shifted surface differential. Take i_- to be a cellular inclusion, using a cellular approximation for the second attachment. In these bases all cone differentials are defined at $t = 0$.

At zero, cancelling the identical pairs of surface cells in the cone of i_- gives $C_*(W, S_-; \rho)$. The assumed reductions give the square matrix $\rho(P)$ in degrees two and one. Each prism cancellation is a unit pivot, and elementary basis changes have determinant one. Relative handle CW structures and their cellular subdivisions give the same simple structure. Changing lifts contributes only determinants of group elements, hence signed monomials in R. In the opposite module convention the relative matrix is also transposed and its group elements inverted. On the unitary torus this takes the adjoint and leaves its absolute determinant unchanged. Fix one convention throughout.

Since $P_D \neq 0$, the complex at zero is acyclic. Choose boundary bases using minors nonzero at zero. The same minors prove generic acyclicity and compute the torsion $\mathcal{T}(t)$ of the cone over $\mathcal{K}(t)$. Their denominators are units in the local ring $\mathcal{K}[t]_{(t)}$, so torsion can be evaluated there and

$$\mathcal{T}(0) \doteq P_D. \tag{8}$$

Here $\doteq$ denotes equality up to a signed element of Γ.

On the other hand, the coefficient system on the cover is $d \mapsto [d]t^{\psi(d)}$. To compare its induced system with the twisted system in (7), choose representatives a_i for the cosets G/D. If $ga_j = a_i d$, the coefficient of the induced system has exponent $\psi(d) = \psi(g) + \psi(a_j) - \psi(a_i)$. Conjugating $t^{\psi(g)}\rho(g)$ by

the diagonal matrix with entries $t^{-\psi(a_i)}$ gives precisely this exponent. The cellular induction isomorphism groups the lifts of each cell into these cosets. The resulting basis changes contribute only signed monomials in Γ and t. Finally, the cover character is obtained by applying the automorphism

$$\Theta : \mathcal{K}(t) \longrightarrow \mathcal{K}(t), \qquad \Theta(t) = t, \qquad \Theta(\gamma) = \gamma t^{\psi_D(\gamma)} \quad (\gamma \in \Gamma)$$

to the ordinary universal character of D. Thus the ordinary universal complex is acyclic over $\mathcal{K}$. The Milnor–Turaev identification for $b_1 \geq 2$ [35, Theorem 5.4][20, Propositions 1 and 2] therefore gives $\Delta_D \neq 0$ and

$$\mathcal{T}(t) = \pm\gamma_0\, t^j \Theta(\Delta_D) \qquad (\gamma_0 \in \Gamma,\ j \in \mathbb{Z}). \tag{9}$$

To compare the Alexander-module conventions, the based Alexander module surjects onto the augmentation ideal of R with kernel M_D. At every height-one prime the augmentation ideal is free of rank one, so the first Alexander polynomial in the based convention is the order of M_D used here. The right-hand side is Laurent polynomial in t. By (8) it is regular and nonzero at zero. Its constant coefficient is therefore, up to a signed monomial, the sum of the terms of Δ_D at minimum ψ_D-weight. Writing this sum as I_D, we obtain $m(I_D) = m(P_D)$.

For almost every $\chi \in \widehat{\Gamma}$ the value $I_D(\chi)$ is nonzero. After removing the lowest power of t, the specialization $\Delta_D(\chi t^{\psi_D})$ is then an ordinary polynomial with constant term $I_D(\chi)$. Jensen's circle formula gives

$$\int_{\mathbb{T}} \log|\Delta_D(\chi t^{\psi_D})|\, dt \ \geq\ \log|I_D(\chi)|.$$

For each fixed $t \in \mathbb{T}$ the translation $\chi \mapsto \chi t^{\psi_D}$ preserves Haar measure. Integrability of $\log|\Delta_D|$ on the character torus and Fubini give $m(\Delta_D) \geq m(I_D) = m(P_D)$. □

Proof of Theorem 5.1. The free quotient and rational transfer give $b_1(X_D) \geq b_1(X) \geq k-1 \geq 2$. The relative handle complex has the single Fox differential B in (3). With ρ as in Lemma 5.2, Proposition 4.2 shows that $\det\rho(B)$ is nonzero and has Mahler measure at least $[G : D]c_k$. Lemma 5.2 transfers this bound to Δ_D. □

6. Abelian covers and a profinitely cofinal tower

The following comparison between Alexander module coinvariants and ordinary homology also appears in the study of prescribed primary torsion [40]. We include the integral argument to track its effect on the exponential rate.

Lemma 6.1. *Let D be the fundamental group of a connected finite complex. Put $\Gamma = D_{\mathrm{ab}}/\operatorname{Tor} D_{\mathrm{ab}} \cong \mathbb{Z}^r$ with $r \geq 1$, $K = \ker(D \to \Gamma)$, and $M = H_1(K;\mathbb{Z})$. Suppose its order Δ over $\mathbb{Z}[\Gamma]$ is nonzero. For $D(n) = \ker(D \to \Gamma/n\Gamma)$ one has*

$$\lim_{n\to\infty} \frac{\log|\operatorname{Tor} H_1(D(n);\mathbb{Z})|}{n^r} = m(\Delta).$$

Proof. The cellular chain complex of the free abelian cover and Noetherianity show that M is a finitely generated $\mathbb{Z}[\Gamma]$-module. For the dual algebraic Γ-action its entropy is $m(\Delta)$ by the entropy formula and the addition theorem [29, 42]. The group of $n\Gamma$-periodic points is dual to $M_n = M_{n\Gamma}$, the coinvariants by $n\Gamma$. It has $|\operatorname{Tor} M_n|$ connected components. Dimitrov's periodic component theorem for these grids [16, Theorem 1(A)] gives

$$\lim_{n\to\infty} n^{-r} \log |\operatorname{Tor} M_n| = m(\Delta). \tag{10}$$

The five-term integral homology exact sequence of $1 \to K \to D(n) \to n\Gamma \to 1$ gives

$$\bigwedge^2 (n\Gamma) \longrightarrow M_n \longrightarrow H_1(D(n);\mathbb{Z}) \longrightarrow n\Gamma \longrightarrow 0.$$

Write J_n for the transgression image and $Q_n = \Gamma/n\Gamma$. Conjugation by D acts trivially on $n\Gamma$. Equivariance of the five-term sequence thus makes J_n pointwise fixed by Q_n. Furthermore J_n is generated by at most $\binom{r}{2}$ elements. The norm on M_n factors as

$$M_n \longrightarrow (M_n)_{Q_n} = M_\Gamma \longrightarrow M_n, \qquad [x] \longmapsto \sum_{g\in Q_n} gx \quad \text{in the second map.}$$

The first map takes torsion to torsion. Let e be the exponent of $\operatorname{Tor} M_\Gamma$, which is finite since M is finitely generated over $\mathbb{Z}[\Gamma]$. If $y \in \operatorname{Tor} J_n$, the norm sends y to $n^r y$, and the displayed factorization gives $en^r y = 0$. Therefore

$$|\operatorname{Tor} J_n| \le (en^r)^{\binom{r}{2}}.$$

The kernel of the homology map on $\operatorname{Tor} M_n$ is exactly $\operatorname{Tor} J_n$. Its image consists of torsion elements, so this comparison loses at most $O_D(\log n)$ in the logarithm. This proves the lower limit by (10). Additional torsion can arise from quotienting by the free part of J_n, so we obtain the upper limit separately from Lê's abelian covering formula [27][28, Equation (1.5)]. Its hypothesis on the shortest nonzero vector holds for $n\Gamma$. The Alexander polynomial in that formula is Δ, since M has rank zero. This proves the limit in the stated normalization. □

Proof of Theorem 1.1. Let $G = \pi_1 X$ for X in Lemma 3.2 and put

$$C_a = \bigcap_{[G:L]\le a} L.$$

This is a finite-index subgroup because G is finitely generated. For any endomorphism $\phi : G \to G$ the inverse image of a subgroup of index at most a has index at most a. Thus $\phi(C_a) \subset C_a$. The decreasing sequence (C_a) is profinitely cofinal. Its intersection is trivial since a finitely generated linear group in characteristic zero is residually finite and G is a hyperbolic three-manifold group.

Start with $U_0 = G$. After U_{i-1} has been chosen take $a \ge \max(i, [G : U_{i-1}])$ and $D = C_a$. Then $D \subseteq U_{i-1}$. For this fixed group apply Lemma 6.1 and

Theorem 5.1. Since $[G : D(n)] = [G : D]n^{b_1(D)}$, we obtain

$$\liminf_{n\to\infty} q_X(X_{D(n)}) \;\geq\; \frac{m(\Delta_D)}{[G:D]} \;\geq\; c_k.$$

Choose n sufficiently large that the degree strictly increases and $q_X(X_{D(n)}) \geq c_k - 1/i$, and put $U_i = D(n)$. This subgroup is fully invariant in G. Indeed every endomorphism preserves D, and its restriction induces an endomorphism on the functorial quotient $D_{\mathrm{ab}}/\operatorname{Tor} D_{\mathrm{ab}}$ and on its quotient modulo n. The inclusions $U_i \leq C_i$ give profinite cofinality and hence trivial intersection. This proves (1). Lê's upper bound [28] bounds the sequence from above, so it has a convergent cofinal subsequence. For a fixed finite-index $V \leq G$ the tower is eventually contained in V and is normal there. Any subgroup of finite index in V has finite index in G. In this tail the degree relative to V is divided by $[G : V]$, proving the last assertion. □

Corollary 6.2. *The base with positive growth in a fully invariant tower can also be taken to be the mapping torus of an orientation-preserving pseudo-Anosov on a closed surface. More precisely, there is a fully invariant finite-index fibered subgroup $V \leq G = \pi_1 X_k$. For $v = [G : V]$, every fully invariant finite-index $D \leq V$ satisfies*

$$m(\Delta_D) \geq v c_k [V : D].$$

The base X_V has a tower fully invariant in V with lower rate at least vc_k.

Proof. Virtual fibering [5] gives a finite-index surface bundle subgroup of G. Some C_a is contained in it. A finite cover of a closed surface bundle fibers with connected fiber by dividing the restricted fibered class by its divisibility. Take $V = C_a$. Its hyperbolicity forces the monodromy to be pseudo-Anosov. Full invariance is transitive, so Theorem 5.1 applies to every fully invariant $D \leq V$. Repeat the construction of Theorem 1.1 using the intersections of bounded-index subgroups of V as intermediate groups. □

7. Characteristic towers over rational homology spheres

7.1. **The two caps and the marked free quotient.** We use the cobordism W with boundary $S_- \sqcup S_+$ from Section 2. Write $F = \langle z_1, \ldots, z_{k-1} \rangle$ and let $\iota \in \mathrm{Aut}(F)$ send each z_j to z_j^{-1}. Put $Q = F \rtimes_\iota C_2$, where C_2 is the group of order two. For a free orientation-reversing involution τ on a closed oriented surface S, set

$$J_\tau = (S \times [-1, 1])/\big((x, t) \sim (\tau x, -t)\big).$$

This is the orientable interval bundle over the nonorientable surface S/τ. We identify ∂J_τ with S at $t = 1$. Its double cover $S \times [-1, 1]$ is classified by the orientation character of S/τ. Ferrari, Kolpakov and Reid describe the related boundary identification for a free orientation-preserving involution [18, Example 3.3].

Lemma 7.1. *There are free orientation-reversing involutions $\tau_\pm$ of $S_\pm$ such that $M = J_{\tau_-} \cup W \cup J_{\tau_+}$ is a rational homology three-sphere, and the map $r_W : \pi_1 W \to F$ of Lemma 2.2 extends to $\theta : \pi_1 M \to Q$. On each cap the composite with $Q \to C_2$ is its orientation character. If $E_\pm$ is the set of slopes of essential annuli of W on $S_\pm$, the involutions may also be chosen so that $\tau_\pm(E_\pm) \cap E_\pm$ is empty.*

Proof. The marked involutions. Thicken a graph with vertices $v_0, \dots, v_k$ and two edges between v_0 and v_j for each $1 \le j \le k$ to an oriented genus-k handlebody H. Use the antipodal involution on each vertex ball and exchange the two one-handles for each pair of edges, placing their attaching discs in antipodal pairs. The result is an orientation-reversing involution of H with fixed points only at the ball centres. Its restriction to ∂H is free. On $\pi_1(H, v_0) = F_k$ it inverts each member of the basis formed by going out on one edge and returning on its mate. The marked epimorphism of either boundary of W in Lemma 2.2 can be written, after a surface marking, as

$$\pi_1(\partial H) \longrightarrow F_k \longrightarrow F,$$

where the second map forgets the last basis element. This is the strong equivalence theorem for epimorphisms of closed orientable surface groups [33, Theorem 3.15].

The model gives a homomorphism $\pi_1(\partial H/\tau) \to Q$. Indeed the group of lifts of the involution and the identity to the universal cover of H is $F_k \rtimes C_2$. The lift fixing a chosen point over v_0 has order two and gives the indicated basis-inverting action. A loop of $\partial H/\tau$ lifts to a path of ∂H from x to x or τx. Lifting that path further to the universal cover of H gives its element of this extension. Equivalently, this is the map on fundamental groups of the C_2 homotopy quotients induced by the equivariant boundary inclusion. Forgetting the last generator is equivariant, giving the map to Q. On the boundary subgroup it restricts to the marked epimorphism, up to a base path conjugation. On the quotient C_2 it records the two sheets of the orientation cover. Changing the surface marking by a disk twist preserves the marked epimorphism up to inner automorphism [23]. Hence such changes on the two ends are available independently. The maps of the two caps and r_W patch by van Kampen, after conjugating the cap maps to agree on their boundary groups.

Rational homology. We next choose the markings to kill rational first cohomology. Both boundary inclusions identify $H^1(W; \mathbb{Q})$ with $H^1(S_\pm; \mathbb{Q})$ by the relative handle differential k and Poincaré–Lefschetz duality. The two induced surface pairings agree up to orientation sign, since the oriented boundary of W represents zero in W. For either end, restriction of $H^1(J_\tau; \mathbb{Q})$ to its boundary identifies it with the $+1$ eigenspace of τ^* by transfer. Mayer–Vietoris for the tree of three pieces thus identifies $H^1(M; \mathbb{Q})$ with the intersection of the two transported $+1$ eigenspaces. Each is Lagrangian, as the involution is anti-symplectic.

At the positive end the -1 eigenspace is $L = \mathrm{im}(H^1(H;\mathbb{Q}) \to H^1(\partial H;\mathbb{Q}))$. It has dimension k and the model involution acts on it by -1. Write P for the complementary $+1$ eigenspace and identify $L = P^*$ using the symplectic form. Twist equally about the boundaries of a complete system of k disjoint meridian discs. Their Poincaré duals on ∂H form a basis of L. The transvection formula therefore sends P to the graphs of $mT : P \to P^*$ as the common twist power m varies. Up to an overall sign, T is the sum of the squares of this basis of covectors and is positive definite. If U is the Lagrangian transported from the other end, let P_0 be its projection to P. Then $U \cap L$ is the annihilator of P_0, and U modulo this annihilator is the graph of a symmetric form T_0 on P_0. The intersection with the graph of mT projects isomorphically onto the kernel of the symmetric form $mT|_{P_0} - T_0$ on P_0. The leading term in its determinant is nonzero by positive definiteness of $T|_{P_0}$, so the kernel vanishes for all sufficiently large $|m|$. If $P_0 = 0$, the intersection is already zero. Conjugating the positive-end involution by the chosen twist makes the two eigenspaces transverse. The resulting M is closed and orientable, so it is a rational homology three-sphere.

Annular slopes. It remains to move annulus slopes while fixing these eigenspaces. In the marked handlebody choose a separating meridian d. A pseudo-Anosov handlebody mapping class sends d a distance greater than two in the curve graph after a sufficiently large power [23, 34]. The resulting two separating meridians fill the surface. The positive twist about one and the negative twist about the other have pseudo-Anosov product β_0 by [39]. Each twist acts trivially on surface homology and preserves the marked epimorphism. We may choose a conjugate β of β_0 by a power of T_d whose backward limiting projective measured lamination λ is moved by the current involution τ. To see this, let λ_0 be the corresponding lamination for β_0. If τ fixes λ_0 and d, it sends $T_d^j\lambda_0$ to $T_d^{-j}\lambda_0$, a different projective class for $j \neq 0$ since λ_0 fills. If $\tau(d) \neq d$, the classes $T_d^j\lambda_0$ converge to $[d]$ as $j \to +\infty$, and their τ-images converge to $[\tau(d)]$. If τ already moves λ_0, take $\beta = \beta_0$.

Replace τ by $\tau_l = \beta^l\tau\beta^{-l}$. Its homological action and the boundary marking into F are preserved. If E is nonempty, Lemma 3.1 gives an essential curve a with zero geometric intersection with every member of E. If $\tau_l(E) \cap E$ were nonempty for arbitrarily large l, there would be a curve disjoint from both a and $\tau_l(a)$. After applying β^{-l} and taking a subsequence in projective measured lamination space, its limit would have zero intersection with both λ and $\tau\lambda$. The backward iterates of a converge to λ, and this filling lamination is uniquely ergodic. The only projective class with zero intersection with it is λ itself. This contradicts $\tau\lambda \neq \lambda$. Here we use the usual projective lamination compactification, continuity of intersection after normalizing by length, and pseudo-Anosov dynamics [17]. Choose l on each end separately. □

7.2. **Full invariance and the two determinant contributions.**

Lemma 7.2. *Let p be a prime and write $\Phi_p(B) = [B, B]B^p$ for the kernel of the mod-p abelianization of a group B. Let G be finitely generated and $\theta : G \twoheadrightarrow F \rtimes E$ an epimorphism, where E is a finite elementary abelian p-group acting trivially on $F/\Phi_p(F)$. Put $K = \theta^{-1}(F)$ and $J = \Phi_p(G)$. There are cofinally many finite-index subgroups D fully invariant in G, contained in K, with the following simultaneous property. For every section $s : F \to K$ of $\theta|_K$, the restriction of $s\theta|_K$ retracts D onto $D \cap s(F)$. Every fully invariant finite-index subgroup of J has this property.*

Proof. The quotient G/K is elementary abelian of exponent p, so $J \leq K$. The semidirect product has mod-p abelianization $(F/\Phi_p(F)) \times E$. In particular $\theta(J) \subseteq \Phi_p(F)$. It follows that $s\theta(J) \subseteq \Phi_p(s(F)) \subseteq J$ for every such section. Each $s\theta|_J$ is an endomorphism of J, and J itself is fully invariant of finite index in G. Full invariance is transitive. Thus any fully invariant finite-index $D \leq J$ is fully invariant in G and is preserved by all these retractions. Each fixes $D \cap s(F)$ pointwise and maps D into that intersection. The intersections of all subgroups of J of bounded index supply cofinally many choices even relative to G, since J is finitely generated of finite index. □

Proof of Theorem 1.2. Let $G = \pi_1 M$ and let K be the kernel of $G \xrightarrow{\theta} Q \to C_2$. The associated connected double cover X consists of two copies W_1, W_2 of W, glued at each end by $\tau_\pm$ through product collars. This follows because the orientation character on a cap is onto C_2 and is trivial on its boundary group, and its double cover is the product with its two boundary identifications related by $\tau_\pm$. The character on W is trivial. Both seams are incompressible and X is irreducible by the incompressible gluings. Since $\tau_\pm(E_\pm) \cap E_\pm = \varnothing$, Lemma 3.1 gives atoroidality.

Cut along the negative seam only. The result is the connected cobordism $C = W_1 \cup_{S_+} W_2$ with two negative ends. If X were Seifert fibered, its nonseparating closed incompressible seam of genus k would be horizontal and its complement would be an interval bundle. With these two boundary components that bundle would be a product. In fact $H_1(C, S_-^{(1)}; \mathbb{Z})$ has order k^2. The relative groups for (W, S_-) and (W, S_+) each consist of $\mathbb{Z}/k$ in degree one, by the handle differential and integral duality. Excision in the exact sequence of the triple $S_-^{(1)} \subset W_1 \subset C$ gives

$$0 \longrightarrow \mathbb{Z}/k \longrightarrow H_1(C, S_-^{(1)}; \mathbb{Z}) \longrightarrow \mathbb{Z}/k \longrightarrow 0,$$

which proves the claim. Hyperbolization now makes X hyperbolic. The manifold M is irreducible by gluing the interval bundles and W along their incompressible boundaries. Its fundamental group has no $\mathbb{Z}^2$ since K has none, and a Seifert fibration would lift to X. Its incompressible boundary gluing surfaces make M Haken, so the same hyperbolization theorem applies to M [36].

We give the group argument ensuring full invariance over M. Within K, let A_j be generated by the $k - 1$ prefixes of (3) in W_j, for $j = 1, 2$. On each

A_j the map θ is an isomorphism onto F. On the first copy this follows from Lemma 2.2. On the second, the two vertex groups are conjugate in G by an element mapping to the nonidentity coset of F in Q, which normalizes F. The map θ is onto Q, since W maps onto F and each cap supplies the other coset. Let $s_j : F \to A_j$ be the two inverse markings. The action of ι on $F/\Phi_2(F)$ is trivial. By Lemma 7.2, with $J = \Phi_2(G)$, every fully invariant finite-index subgroup $D \le J$ is fully invariant in G, and $s_j\theta$ retracts it onto $D \cap A_j$ for both j.

Put $N = [K : D]$ and let B_j denote the relative Fox element (3) of W_j. For the induced representation $\rho_\chi = \operatorname{Ind}_D^K \chi$, with χ on the free abelian character torus of D, the proof of Proposition 4.2 gives

$$\int \log |\det \rho_\chi(B_j)| \, d\chi \;\ge\; N c_k \qquad (j = 1, 2).$$

Indeed that proof uses normality in K and the retraction $D \to D \cap A_j$ to give Haar characters on every A_j-orbit. The two estimates are integrated over the same torus. Additivity of the integral of their logarithms requires no independence between them.

The based relative complex of $(C, S_-^{(1)})$ has a filtration by its first cobordism. The two relative handle complexes are those of $(W_1, S_-^{(1)})$ and $(W_2, S_+^{(2)})$. Their differentials are B_1 and B_2^* up to signed group units, where $*$ inverts group elements. For the second complex one turns the handle decomposition upside down. In dimension three the original two-handle becomes a one-handle and conversely. In the intersection formula for the differential each lifted attaching–belt intersection is read in the opposite direction, inverting its deck element, with an overall orientation sign. Concatenating the relative handle decompositions gives two relative one-cells and two relative two-cells. The first copy is a subcomplex. With its cells listed first, the differential after inducing coefficients has the form

$$\begin{pmatrix} \rho_\chi(B_1) & * \\ 0 & \rho_\chi(B_2^*) \end{pmatrix}$$

up to basis changes of unit determinant in absolute value. The off-diagonal block records attaching paths in the first copy. For unitary ρ_χ, the second diagonal block has the same absolute determinant as $\rho_\chi(B_2)$. Consequently the logarithmic Mahler measure of the relative determinant is at least $2Nc_k$.

The negative seam is dual to a primitive class on X. In the surface-cutting cone for that class the specialization at zero is the relative complex just computed. Its determinant is a nonzero Laurent polynomial. Also $b_1(D) \ge b_1(K) \ge k - 1$ by transfer and the free quotient. Lemma 5.2 applies and gives

$$m(\Delta_D) \ge 2[K : D]c_k = [G : D]c_k.$$

Finally, inside J take the intersections of all subgroups of index at most a. These are fully invariant and cofinal. At each stage choose such a D inside

the previous level and inside every subgroup of G of index at most the stage number. This is possible since J has finite index in G. Lemma 6.1 supplies an abelian congruence refinement $D(n)$ whose logarithmic torsion divided by $[G : D(n)]$ is at least $c_k - 1/i$ at stage i, with strictly larger degree. This refinement is fully invariant in D and hence in G. Residual finiteness gives trivial intersection. Since $c_k \to \infty$, Lê's upper bound $\mathrm{vol}(M_k)/(6\pi)$ for these residual normal towers proves the volume assertion [28]. □

The original group surjects onto $Q/\Phi_2(Q) \cong (\mathbb{Z}/2)^k$. In particular the bases have $\dim_{\mathbb{F}_2} H_1(M_k; \mathbb{F}_2) \geq k$ even though $b_1(M_k) = 0$. The tower levels have $b_1 \geq k - 1$ by the free quotient of K and transfer. For sequences whose members are themselves rational homology spheres, Abdurrahman, Adve, Giri, Lowe and Zung give an exponential torsion bound under a bounded-rank assumption and a uniform spectral gap on coclosed one-forms [1].

8. An exposed triangle suffices

For two free generators the estimate obtained by inserting the permanent of a trivalent graph into Lemma 4.1 has a negative right-hand side. We give a positive bound after a fixed finite cover. Throughout this section $c_5 = \frac{1}{2}\log(128/125)$ and $\eta = c_5/5$.

Lemma 8.1. *Let R be the rose on a, b. There is a connected five-sheeted cover $R_0 \to R$ with the following property. Let $R' \to R$ be any connected s-sheeted cover factoring through R_0. Write $E = \pi_1 R'$ and let ξ range over its unitary character torus. Then*

$$\int \log \left| \det \mathrm{Ind}_E^{\pi_1 R}\, \xi(1 + a + b) \right| \, d\xi \;\geq\; s\eta.$$

The notation inside the determinant denotes the linear extension of the induced representation to the group ring.

Proof. The five vertices are o, L_1, L_2, R_1, R_2. Retain the following arrows:

$$o \to L_i, \qquad L_i \to R_j, \qquad R_j \to o \quad (1 \leq i, j \leq 2).$$

Use label a on $oL_1, R_1o, L_1R_1, L_2R_2$ and label b on the other four arrows. Complete the two permutations by R_2L_2 with label a and R_1L_1 with label b. This specifies R_0 and its retained directed subgraph. Each vertex has $s/5$ lifts in R'.

As in Proposition 4.2, assign independent Haar phases to all lifted generator edges. Multiply the weights above the two completion arrows by a common complex variable t. For fixed $|t| = 1$ the joint law is unchanged. Jensen's formula shows that the expected logarithmic determinant at $|t| = 1$ is at least its expectation at $t = 0$, once the latter determinant is nonzero almost everywhere.

At zero the matrix is the identity plus the retained arrows. Use the convention that a path product follows its arrows, or transpose the entire matrix. Let A_i, B_{ij}, C_j be the weighted permutations in the three successive

layers. Eliminating the four nonroot diagonal blocks, which are identities in an upper triangular submatrix, leaves

$$I + \sum_{i,j} A_i B_{ij} C_j.$$

Here the plus sign is due to the three arrows in each return path. Conditional on all A_i, C_j, the four middle arrow families use disjoint independent phases. The four summands are permutations with independent Haar weights in every row of each summand. Multiplication of each row by a fresh independent phase randomizes the identity entry as in Proposition 4.2. The support has five labels in each row and column. Lemma 4.1 gives a nonzero determinant almost everywhere and an expected logarithm at least $(s/5)c_5$. All nonzero Laurent polynomials used in the Jensen step have integrable logarithm. □

We next arrange the character surjectivity needed for these estimates in covers regular over the original group.

Lemma 8.2. *Let B be a finitely generated subgroup of G and let $p : G \to F$ be a homomorphism to a finitely generated free group, injective on B. Given a finite-index normal $E \lhd G$ and a finite-index subgroup $B_0 \le B$, there is a finite-index normal $D \lhd G$, contained in E, such that $D \cap B \le B_0$ and $D \cap B$ is a retract of D.*

Proof. Write $B' = p(B)$ and $C = p(B \cap E \cap B_0)$, of finite index in B'. Hall's theorem gives a finite-index subgroup $U \le F$ containing C as a free factor [22]. Subgroup separability of C in F, also part of Hall's theorem, lets us shrink U by intersection with finitely many finite-index subgroups containing C until $U \cap B' = C$: exclude representatives of the nonidentity cosets of C in B'. The free factor property persists. Indeed, in a covering graph of a rose for a basis adapted to a free factor, the component above its subrose is an embedded subgraph, whose fundamental group is a free factor by extending a spanning tree. Take the normal core K of U in F. The same graph argument makes $K \cap C = K \cap B'$ a free factor of K. Set $D = E \cap p^{-1}(K)$. Since $p|_B$ is injective and $K \cap B' \le C$, it identifies $D \cap B$ with $K \cap B'$ and puts $D \cap B$ in B_0. Compose $p|_D$ with the free factor retraction of K and the inverse of this identification. □

Theorem 8.3. *Let M be a compact connected orientable three-manifold with empty or toroidal boundary, $b_1(M) \ge 2$, and residually finite fundamental group G. Cut M along a connected, properly embedded, oriented two-sided surface dual to a primitive class, with connected complement W and incoming surface S. Suppose that the relative based complex of (W, S), extended to $\mathbb{Z}G$, reduces by simple cell cancellations and elementary basis changes to a two-term complex in degrees two and one with scalar differential $P = \sum_{j=0}^{s} \epsilon_j g_j$, where $g_0 = 1$, $\epsilon_j = \pm 1$. Signed group units in the bases are allowed. Suppose an integral character of G has weight zero on $g_0, \dots, g_r$ and strictly positive weight on all remaining terms, for some $r \ge 2$. If all terms have weight zero*

the character may be zero. Suppose also that $g_1, \dots, g_r$ freely generate B, and that some homomorphism from G to a finite-rank free group is injective on B. Then M has a nested profinitely cofinal tower, regular over M, satisfying

$$\liminf_i q_M(M_i) \;\geq\; \begin{cases} \eta & r = 2, 3, \\ \max\{\eta, c_{r+1}\} & r \geq 4. \end{cases}$$

Proof. Apply Lemma 8.2 to B. For B_0 use the stabilizer of o in the five-point action of Lemma 8.1 on the first two generators, letting the others act trivially. Put $N = [G : D]$. Since D is normal and retracts to $D \cap B$, the induced characters on each B-orbit in $\mathrm{Ind}_D^G \chi$ are Haar on the character torus of $D \cap B$, by the stabilizer and conjugation argument of Proposition 4.2. That argument uses only these two properties of D. Within each orbit all lifted generator-edge phases can therefore be integrated independently. Apply Lemma 4.1 to all $r+1$ terms to obtain the bound Nc_{r+1} when $r \geq 4$. For the bound $N\eta$, first give the edges of the last $r-2$ generators a common variable and specialize it to zero by Jensen's formula. The graph on the first two generators splits into covers of the five-vertex graph. To see this, project the marked B-cover to its B_0-cover; every such component maps locally bijectively onto the connected five-vertex graph. Lemma 8.1 on the components gives $N\eta$. Generator signs are absorbed into the Haar edge phases after a common overall sign. In particular the face determinant is a nonzero Laurent polynomial.

Write α for the exposing character. In a coset basis $\mathrm{Ind}_D^G(\chi z^{\alpha|_D})$ is conjugate to $g \mapsto z^{\alpha(g)}(\mathrm{Ind}_D^G \chi)(g)$. For $|z| = 1$ the change in χ preserves Haar measure. At $z = 0$ the determinant of the latter evaluation of P is the face determinant just bounded. Jensen's formula thus gives the same lower bound for $m(\det \mathrm{Ind}_D^G(P))$, where the induction here uses the universal free abelian character of D. Rational transfer gives $b_1(M_D) \geq 2$. Lemma 5.2 now gives the same bound for $m(\Delta_D)$. Finally, at stage i choose E inside the preceding level and the intersection of all subgroups of index at most i. These can be taken normal in G. Choose D as above, then use Lemma 6.1 to take a sufficiently large free abelian congruence cover $D(n)$ losing at most $1/i$ in the normalized estimate and increasing the index strictly. It is characteristic in D and hence normal in G. This gives the asserted tower. □

Corollary 8.4. *The closed construction of Sections 2 and 3 also gives fully invariant profinitely cofinal towers of positive ordinary torsion rate for $k = 3, 4$, with lower rate at least η.*

Proof. The topology of the two sections applies unchanged for $k \geq 3$: the consecutive boundary vectors are primitive and pairwise nonproportional, the surfaces have genus at least three, and the boundary epimorphisms have rank $k - 1$. The subgroup A is a retract of G. Take the intermediate bounded-index intersections D of Section 6 inside the preimage under this retraction of the five-sheeted subgroup used in Theorem 8.3. Full invariance then gives a retraction from D to $D \cap A$, and $D \cap A$ is contained in the chosen

five-sheeted subgroup. The two-generator part of the proof of Theorem 8.3 gives $m(\Delta_D) \geq [G : D]\eta$ at every such stage. Its free abelian congruence refinements are fully invariant in G by the proof of Theorem 1.1. □

9. Alternating chain links

Let L_n be the alternating cyclic chain of n unknotted components with the link diagram and orientation of Duan [15, Example 6.7 and Figure 7]. Write $M_n = S^3 \setminus \operatorname{int} \nu L_n$ for its compact exterior. The specified diagram fixes the twisting convention.

Theorem 9.1. *For every $n \geq 4$, the alternating chain-link exterior M_n has a nested profinitely cofinal tower of finite covers regular over M_n with*

$$\liminf_i \frac{\log |\operatorname{Tor} H_1(M_{n,i}; \mathbb{Z})|}{[M_{n,i} : M_n]} \geq \begin{cases} \eta, & 4 \leq n \leq 7, \\ c_{\lfloor n/2 \rfloor + 1}, & n \geq 8. \end{cases}$$

In particular, the lower bounds are unbounded as n varies. The case of three components after a fixed finite cover is addressed in Corollary 9.2.

Proof. The connected Seifert surface S of [15, Example 6.7] cuts M_n open to a genus-$(n-1)$ handlebody W. Choose the cut direction so the incoming surface is Duan's Σ_+. Duan writes the two free groups of Σ_+ and W using bases of size $n-1$. Cancelling the vertex pair in the relative mapping cylinder of their based graph models leaves the Fox Jacobian in degrees two and one. These are spines of the surface and handlebody with their simple structures. In the matrix in [15, Lemma 6.8], row $i < n-1$ has 1 in column i and $-s_i$ in column $i+1$, and the last row starts as $(f_1, \ldots, f_{n-1})$. Subtracting the current entry in column i times row i from the last row clears that entry and adds its right product with s_i to the next entry. Iteration leaves the final pivot

$$f_1 s_1 \cdots s_{n-2} + f_2 s_2 \cdots s_{n-2} + \cdots + f_{n-2} s_{n-2} + f_{n-1}.$$

The first $n-2$ unit pivots can then be cancelled. With the values in Duan's example, $s_i \cdots s_{n-2} = x_i x_{n-1}^{-1}$ and this last pivot is $x_{n-1} P x_{n-1}^{-1}$, where

$$P = 1 + y_1 + \cdots + y_{n-1},$$

and the $y_i = x_{n-1} \cdots x_i$ form a free basis of $\pi_1 W$. All additions and cancellations took place over the integral group ring. They apply before taking determinants in the finite-dimensional monomial representations.

Let ψ be the class of S. The surface-cutting sequence contains

$$H_1(S; \mathbb{Z}) \xrightarrow{i_+ - i_-} H_1(W; \mathbb{Z}) \longrightarrow H_1(M_n; \mathbb{Z}) \xrightarrow{\psi} \mathbb{Z} \longrightarrow 0.$$

The middle groups have ranks $n-1$ and n; the latter is freely generated by the meridians. The first arrow has rank zero and its target is torsion-free. Thus the $[y_j]$ are an integral basis of the direct summand $\ker \psi$. Every subset of $\{0, [y_1], \ldots, [y_{n-1}]\}$ containing zero is exposed by a character taking values zero and one on these basis elements.

Number the components around the chain and retain a set of $m = \lfloor n/2 \rfloor$ mutually nonadjacent components. In the chain diagram crossings occur only between adjacent components and there are no self-crossings, so the subdiagram is an m-component unlink. Filling back the deleted components along their meridians gives an epimorphism $p : \pi_1 M_n \to F_m$, the group of its exterior. In homology this drops the deleted meridians and preserves the others. Since ψ takes value one on each oriented meridian, the differences between retained meridians and one deleted meridian lie in $\ker \psi$ and map to a basis of $H_1(F_m; \mathbb{Z})$. Accordingly, among the $p(y_j)$ one can choose m with independent images over $\mathbb{Q}$ in $H_1(F_m; \mathbb{Q})$. The subgroup they generate is free, has rank at least m by this independence, and is generated by m elements. The map from the free group on the selected y_j is therefore an isomorphism onto its image, by Hopficity of finite-rank free groups. Expose precisely these terms and the identity. Theorem 8.3 with $r = m$ proves the estimate. The link exteriors in question are hyperbolic [15, Remark 6.9]. □

Duan computed the relative universal and L^2-torsions of the cut handlebody. Finite abelian refinements of a single exterior recover the Mahler measure of its Alexander polynomial [27, 45]. The regular towers in Theorem 9.1 refine arbitrary prescribed finite covers of the original exterior. Champanerkar and Kofman computed volume-related rates in abelian covering towers of specific links in the thickened torus [13]. Their abelian towers have nontrivial intersection in the fundamental group of the hyperbolic base. The free quotient of the chosen face provides the required estimates at these successive stages.

Corollary 9.2. *The alternating three-chain-link exterior has a connected finite cover Y with a profinitely cofinal tower of covers fully invariant in $\pi_1 Y$ and with lower rate at least η measured over Y.*

Proof. Use the handlebody W and Seifert surface from the proof of Theorem 9.1 for $n = 3$. That surface is norm-minimizing [15, Example 6.7 and Remark 6.9], hence incompressible, and van Kampen in the cut decomposition embeds $A = \pi_1 W = F(y_1, y_2)$ in $G = \pi_1 M_3$. The subgroup is geometrically finite. In fact the tameness and covering theorems give geometric finiteness or a virtual surface fiber [3, Theorem 4.1.2]. In the latter case there is a finite-index bundle subgroup G' with fiber subgroup commensurable with A up to conjugation. The cut character ψ vanishes on A. A homomorphism to $\mathbb{Z}$ that vanishes on a finite-index subgroup of the fiber vanishes on the fiber itself. The nonzero restriction of ψ to G' therefore factors through the cyclic bundle quotient, so $\ker \psi \cap G'$ is the finitely generated fiber group. Then $\ker \psi$ is finitely generated, and Stallings's fibering theorem [49] would make ψ fibered. In that case the relative torsion of the cut product along an actual fiber would give $C(M_3, \psi) = 1$ by [14, Theorem 1.1]. Duan computes $C(M_3, \psi) = 2/\sqrt{3} > 1$ in [15, Remark 6.9]. This excludes the virtual fiber

case. A geometrically finite subgroup of a finite-volume hyperbolic three-manifold group is a virtual retract [3, (H.9)]. Take a finite-index $V \leq G$ containing A with a retraction $r : V \to A$, and let Y be its cover.

Choose fully invariant finite-index $D \leq V$ inside the inverse image under r of the five-sheeted subgroup from Lemma 8.1. Such D may be taken inside every prescribed finite-index subgroup of V by the bounded-index intersections of Section 6. In the A-orbit decomposition of $\mathrm{Ind}_D^G \chi$, the sheets $D\backslash V$ give a total expected log determinant at least $[V : D]\eta$ for $P = 1+y_1+y_2$, by the retraction and Lemma 8.1. Each remaining orbit has a nonzero integral Laurent determinant in the universal character of D. Indeed, specialize to the restriction of $g \mapsto z^{\alpha(g)}$, where $\alpha(y_1) = \alpha(y_2) = 1$. Such α exists by the integral splitting in Theorem 9.1. On an orbit the resulting matrix is diagonally conjugate to the identity plus z times the sum of two permutation matrices, so its determinant has constant term one. Every nonzero integral Laurent polynomial has nonnegative Mahler measure, by exposing a single vertex monomial and applying Jensen's formula. The total determinant is nonzero. Rational transfer gives $b_1(D) \geq b_1(M_3) = 3$, so Lemma 5.2 gives $m(\Delta_D) \geq [V : D]\eta$. Lemma 6.1 and the diagonal construction inside V finish the proof. This assertion uses regularity over Y. □

10. Comparison with the leading L^2-Alexander coefficient

For the closed examples, the same relative differential occurs in two measures of torsion. Write I_D for the lowest face in the cut direction in Theorem 5.1. Set

$$h_k = \left(\frac{(k-1)^{k-1}}{k^{\,k-2}} \right)^{1/2}.$$

Let $C(X, \psi)$ denote the leading coefficient of the L^2-Alexander torsion, normalized as in [9, 14].

Theorem 10.1. *For the closed examples of Theorem 1.1 and the primitive class ψ dual to the construction surface,*

$$C(X_k, \psi) = h_k.$$

For every nested residual chain of fully invariant subgroups D_i of finite index in $G = \pi_1 X_k$, with $N_i = [G : D_i]$, one has

$$\log h_k - \tfrac{1}{2}\log 2 \;\leq\; \liminf_i \frac{m(I_{D_i})}{N_i} \;\leq\; \limsup_i \frac{m(I_{D_i})}{N_i} \;\leq\; \log h_k.$$

In particular, for the ordinary torsion towers of Theorem 1.1 the lower bound is within $\frac{1}{2}\log 2$ of the logarithm of this leading coefficient for every k. The volume-normalized approximation problem concerns the full L^2-torsion.

Proof. Both inclusions $S_\pm \to W$ induce rational homology isomorphisms: on the incoming relative complex the differential is multiplication by k, and on the outgoing side the assertion follows by duality. In the infinite cyclic cover for ψ, every finite stack of copies of W has the rational homology of one copy

of S, by Mayer–Vietoris. The maps between stacks induce isomorphisms. Passing to their union shows that its homology has dimensions $1, 2k, 1$ in degrees $0, 1, 2$, respectively, with invertible deck action. A d-dimensional rational vector space with an invertible deck action A has order $\det(tI - A)$ over $\mathbb{Q}[t^{\pm 1}]$, of degree d. Here degree means the difference of highest and lowest exponent. The degree of the one-variable rational torsion of X_k is therefore $2k - 2$. The Alexander lower bound for the Thurston norm [19, Theorem 1.1] gives $\|\psi\|_T \geq 2k-2$. The genus-k surface S attains this bound.

Duan's relative torsion formula [14, Theorem 1.1] now identifies $C(X_k, \psi)$ with the L^2-torsion of (W, S_-). By (3) this is the Fuglede–Kadison determinant of B. Since the displayed generators freely generate $A \leq \pi_1 W$, induction from the subgroup preserves the trace spectral measure of B^*B: the group ring trace takes the coefficient of the identity in either group. Ben Aribi's free group formula [10] gives $\det_{\mathcal{N}(A)}(B) = h_k$. Duan previously obtained the same numbers for his alternating chain links and thereby answered [9, Conjecture 1.8(3)] [15, Remark 6.9].

For the upper bound let μ_i be the averaged eigenvalue counting measure of $\rho_\chi(B)^*\rho_\chi(B)$ divided by N_i. All measures are probabilities on $[0, k^2]$. For $g \in G$, the averaged normalized trace of $\rho_\chi(g)$ equals the indicator of $K_i = \ker(D_i \to (D_i)_{\mathrm{ab}}/\operatorname{Tor})$. Indeed, outside D_i the coset permutation has no fixed point; inside D_i the characters of the conjugates average to zero unless $g \in K_i$, since $K_i \lhd G$. As $K_i \subset D_i$ and the latter are nested with trivial intersection, this trace tends to the coefficient of the identity on every group ring element. Applying this to the powers of B^*B gives moment, and hence weak, convergence to its group trace spectral measure μ. Apply weak convergence to the continuous functions $\log\max(x, \delta)$ and then decrease δ to zero. This yields

$$\limsup_i \frac{m(\det \rho_\chi(B))}{N_i} = \limsup_i \tfrac{1}{2} \int \log x \, d\mu_i(x) \leq \tfrac{1}{2} \int \log x \, d\mu(x) = \log h_k.$$

On the left the Mahler measure is in the universal character variables. On the right the operator is injective and its logarithm is integrable by the cited free group determinant formula. Theorem 5.1 identifies $m(I_D)$ with that finite determinant integral. The lower bound is Lemma 4.1 and Proposition 4.2. □

11. Torsion supported outside prescribed sets of primes

For a finite abelian group T and a finite set of primes $\mathcal{S}$, write $T_{\mathcal{S}}$ for the sum of its primary subgroups at those primes. The closed examples admit a refinement which distinguishes the primes dividing the covering degree from those supplying torsion.

Theorem 11.1. *Fix $k \geq 5$. There is a single closed fibered hyperbolic manifold $Y \to X_k$ of degree v with the following property. Given any sequence of finite sets of primes $\mathcal{S}_i$, there are nested profinitely cofinal covers Y_i regular*

over Y such that, writing $N_i = [Y_i : Y]$ and $T_i = \operatorname{Tor} H_1(Y_i;\mathbb{Z})$, one has

$$\frac{\log|(T_i)_{\mathcal{S}_i}|}{N_i} \longrightarrow 0, \qquad \liminf_i \frac{\log|T_i/(T_i)_{\mathcal{S}_i}|}{N_i} \geq vc_k.$$

Alternatively, for any specified prime p, the tower may be chosen with finite p-group deck groups, cofinal for the pro-p topology on $\pi_1 Y$ and with trivial intersection, satisfying the same two bounds. In particular one may take $\mathcal{S}_i$ to be all primes up to any prescribed threshold tending to infinity.

Proof. In Corollary 6.2 choose $V = \pi_1 Y$ also inside a compact special finite-index subgroup of G, using virtual specialness and one more bounded-index intersection [5, 3]. It lies in a right-angled Artin group and is residually p for every prime p: right-angled Artin groups are residually torsion-free nilpotent, and finitely generated torsion-free nilpotent groups are residually p [3]. Fix a primitive fibration $\phi : V \to \mathbb{Z}$. All fully invariant finite-index subgroups D of V satisfy $m(\Delta_D) \geq [V : D]vc_k$.

First fix such a D. Divide $\phi|_D$ by its positive divisibility to obtain ϕ_D, write $H = \ker\phi_D$, and choose $s \in D$ with $\phi_D(s) = 1$. In the free abelianization write $\Gamma_D = \Lambda \oplus \mathbb{Z}[s]$ with Λ the image of H. Put $b = \operatorname{rank}\Lambda$. The subgroup $D(n)$ of Lemma 6.1 has fiber $H_n = \ker(H \to \Lambda/n\Lambda)$ and time generator s^n. Let B_n denote the action of s on $H_1(H_n;\mathbb{Z})$, a free abelian group. For an automorphism A of a finite-dimensional complex vector space put $J(A) = \sum_\lambda \log\max(1,|\lambda|)$, with algebraic multiplicity. The following limit is for fixed D:

$$\frac{J(B_n)}{n^b} \longrightarrow m(\Delta_D). \tag{11}$$

Indeed, let ξ be a unitary character of Λ. In the infinite Λ-cover of the fiber, the chosen lift of s commutes with Λ. At ξ let $B_{j,\xi}$ be its action on the jth twisted fiber homology group. The mapping-torus cone, with variable z in the time direction, has torsion $\prod_{j=0}^{2}\det(1 - zB_{j,\xi})^{(-1)^{j+1}}$. At chain level all determinants of $1 - zB$ have constant coefficient one, so all denominators remain nonzero polynomials at every ξ. The universal Milnor–Turaev formula ($b_1(D) \geq 2$) normalizes Δ_D as a polynomial in z with constant term one and identifies all these specializations. For $\xi \neq 1$ the zeroth and second twisted homology groups vanish; for $\xi = 1$ their factors $1 - z$ have Mahler measure zero. Hence $h(\xi) = m_z(\Delta_D(\xi, z)) = J(B_{1,\xi})$. This function is continuous: the reciprocal polynomials of fixed formal degree are monic, and Jensen's formula sums the continuous function $\log\max(1,|\lambda|)$ over their roots. The finite $\Lambda/n\Lambda$-cover splits in complex homology into these character spaces. Thus $J(B_n)/n^b$ is the average of h on the n-torsion grid. Riemann sums and Fubini give (11), also along $n = p^a$.

For a fixed $A \in \mathrm{GL}_d(\mathbb{Z})$ let $T(h) = \operatorname{Tor}\operatorname{coker}(A^h - I)$. Two one-variable facts are

$$\lim_{h\to\infty} \frac{\log|T(h)|}{h} = J(A), \qquad \log|T(h)_{\{q\}}| = O_{A,q}(\log(h+1)) \tag{12}$$

for every fixed prime q. The first follows directly from [16, Theorem 1(A)] applied to the module $\mathbb{Z}^d$ with z acting by A; its order is $\det(zI - A)$. For the second, split the rational lattice into the cyclotomic and noncyclotomic primary subspaces and take their intersection lattices. Their direct sum has fixed finite index I. The kernel and cokernel of the induced map on cokernels of $A^h - I$ have order bounded by I by the snake lemma. The image of torsion from the smaller lattice then has index at most I in the torsion of the larger cokernel: a torsion element lying in the image lifts to a torsion element because the kernel is finite. The quotient by the full image has order at most I. On the cyclotomic part the entries of $A^h - I$ grow polynomially in h, by Jordan form, so all its minors are polynomially bounded. The torsion order divides a nonzero minor of maximal rank (the zero-rank convention gives order one). On the noncyclotomic part the cokernel has order $|\det(A^h - I)|$. All eigenvalues are q-adic units, since A is integral unimodular. For any such algebraic unit λ which is not a root of unity, in a finite extension of $\mathbb{Q}_q$ one has $v_q(\lambda^h - 1) \leq C_\lambda + v_q(h)$ whenever $\lambda^h \neq 1$. Indeed, first require divisibility of h by the order of its residue, else the valuation is zero. After removing that order, raising to a power prime to q does not change the valuation of the difference from one. For sufficiently large q-power exponents the q-adic logarithm is isometric and each further qth power increases the normalized valuation by one. Summing over the eigenvalues proves (12). Related primary growth in cyclic covers is studied by Silver and Williams [46].

At step i take a fully invariant $D \leq V$ inside the preceding level and the ith term of a neighborhood basis of the identity. In the profinite case use bounded-index intersections. In the pro-p case use intersections of kernels of all maps to p-groups of bounded order; these have finite p-power index and are fully invariant. Choose n sufficiently large in (11), a p-power in the latter case. Now take the further subgroup

$$U = D(n) \cap \ker(\phi_D \bmod nh).$$

It is normal in V: $D(n)$ is fully invariant in V, and conjugation by V preserves ϕ_D on D. Its fiber is H_n, its time generator s^{nh}, and its index in V is $[V : D]n^{b+1}h$. The abelianization exact sequence for the bundle identifies its torsion with $T(h)$ for $A = B_n^n$. Since $J(B_n^n) = nJ(B_n)$, choose h sufficiently large in (12) that the total normalized logarithm is at least $vc_k - 2/i$ and the normalized contribution of the finite set $\mathcal{S}_i$ is at most $1/i$. Again h can be a p-power; normality and the index formula then give a p-group deck group. The choice of D ensures nesting and the stated cofinality. The index can be made strictly larger at each step by increasing h. □

Use of generative AI

ChatGPT was used in developing and checking arguments, searching the literature, and preparing and revising the LaTeX manuscript.

References

[1] A. Abdurrahman, A. Adve, V. Giri, B. Lowe and J. Zung, *Hyperbolic 3-manifolds with uniform spectral gap for coclosed 1-forms*, arXiv:2404.19039.

[2] M. Abért, N. Bergeron, M. Frączyk and D. Gaboriau, *On homology torsion growth*, J. Eur. Math. Soc. **27** (2025), 2293–2357. doi:10.4171/JEMS/1411.

[3] M. Aschenbrenner, S. Friedl and H. Wilton, *3-Manifold Groups*, EMS Series of Lectures in Mathematics, European Mathematical Society, 2015. doi:10.4171/154.

[4] M. Abért, T. Gelander and N. Nikolov, *Rank, combinatorial cost, and homology torsion growth in higher rank lattices*, Duke Math. J. **166** (2017), 2925–2964. doi:10.1215/00127094-2017-0020.

[5] I. Agol, *The virtual Haken conjecture*, with an appendix by I. Agol, D. Groves and J. Manning, Doc. Math. **18** (2013), 1045–1087.

[6] S. Alexandrov, *Family of hyperbolic manifolds with exponential homology torsion growth*, arXiv:2512.08915.

[7] H. Baik, D. Bauer, I. Gekhtman, U. Hamenstädt, S. Hensel, T. Kastenholz, B. Petri and D. Valenzuela, *Exponential torsion growth for random 3-manifolds*, Int. Math. Res. Not. IMRN (2018), no. 21. doi:10.1093/imrn/rnx076.

[8] R. İ. Baykur, R. C. Kirby and D. Ruberman (eds.), *K3: A New Problem List in Low-Dimensional Topology*, Mathematical Surveys and Monographs, vol. 295, Amer. Math. Soc., 2026. doi:10.1090/surv/295.

[9] F. Ben Aribi, S. Friedl and G. Herrmann, *The leading coefficient of the L^2-Alexander torsion*, Ann. Inst. Fourier **72** (2022), 1993–2035. doi:10.5802/aif.3509.

[10] F. Ben Aribi, *Fuglede–Kadison determinants over free groups and Lehmer's constants*, Confluentes Math. **14** (2022), no. 1, 3–22. doi:10.5802/cml.79.

[11] N. Bergeron, M. H. Şengün and A. Venkatesh, *Torsion homology growth and cycle complexity of arithmetic manifolds*, Duke Math. J. **165** (2016), 1629–1693. doi:10.1215/00127094-3450429.

[12] N. Bergeron and A. Venkatesh, *The asymptotic growth of torsion homology for arithmetic groups*, J. Inst. Math. Jussieu **12** (2013), 391–447. doi:10.1017/S1474748012000667.

[13] A. Champanerkar and I. Kofman, *Examples of homological torsion and volume growth*, Acta Math. Vietnam. **46** (2021), 301–308. arXiv:1901.07494.

[14] J. Duan, *Guts determine the leading coefficients of L^2-Alexander torsions*, Trans. Amer. Math. Soc. **378** (2025), 3699–3720. doi:10.1090/tran/9384.

[15] J. Duan, *Universal L^2-torsion and sutured decomposition for 3-manifolds*, arXiv:2512.01305.

[16] V. Dimitrov, *Convergence to the Mahler measure and the distribution of periodic points for algebraic Noetherian $\mathbb{Z}^d$-actions*, arXiv:1611.04664.

[17] B. Farb and D. Margalit, *A Primer on Mapping Class Groups*, Princeton Mathematical Series, vol. 49, Princeton University Press, 2012.

[18] L. Ferrari, A. Kolpakov and A. W. Reid, *Infinitely many arithmetic hyperbolic rational homology 3-spheres that bound geometrically*, Trans. Amer. Math. Soc. **376** (2023), 1979–1997. doi:10.1090/tran/8816.

[19] S. Friedl and T. Kim, *The Thurston norm, fibered manifolds and twisted Alexander polynomials*, Topology **45** (2006), 929–953.

[20] S. Friedl and S. Vidussi, *A survey of twisted Alexander polynomials*, in *The Mathematics of Knots*, Contrib. Math. Comput. Sci. **1**, Springer, 2011, 45–94. arXiv:0905.0591.

[21] Q. Guo, *Cofinal towers with vanishing homology torsion*, arXiv:2608.06601.

[22] J. R. Stallings, *Topology of finite graphs*, Invent. Math. **71** (1983), 551–565.

[23] S. Hensel, *A primer on handlebody groups*, in *Handbook of Group Actions*, vol. V, International Press, 2020, 143–177.

[24] W. Jaco and P. Shalen, *Seifert fibered spaces in 3-manifolds*, Mem. Amer. Math. Soc. **21** (1979), no. 220.
[25] K. Johannson, *Homotopy Equivalences of 3-Manifolds with Boundaries*, Lecture Notes in Mathematics, vol. 761, Springer, 1979.
[26] H. König, *On the best constants in the Khintchine inequality for Steinhaus variables*, Israel J. Math. **203** (2014), 23–57. doi:10.1007/s11856-013-0006-y.
[27] T. T. Q. Lê, *Homology torsion growth and Mahler measure*, Comment. Math. Helv. **89** (2014), 719–757. doi:10.4171/CMH/332.
[28] T. T. Q. Lê, *Growth of homology torsion in finite coverings and hyperbolic volume*, Ann. Inst. Fourier **68** (2018), 611–645. doi:10.5802/aif.3173.
[29] D. Lind, K. Schmidt and T. Ward, *Mahler measure and entropy for commuting automorphisms of compact groups*, Invent. Math. **101** (1990), 593–629.
[30] Y. Liu, *Immersing quasi-Fuchsian surfaces of odd Euler characteristic in closed hyperbolic 3-manifolds*, J. Differential Geom. **111** (2019), 457–493.
[31] Y. Liu, *Virtual homological spectral radii for automorphisms of surfaces*, J. Amer. Math. Soc. **33** (2020), 1167–1227.
[32] Y. Liu and H. Sun, *Toward and after virtual specialization in 3-manifold topology*, Surv. Differ. Geom. **25**, 215–252.
[33] W. Marzantowicz and Ł. P. Michalak, *Relations between Reeb graphs, systems of hypersurfaces and epimorphisms onto free groups*, Fund. Math. **265** (2024), 97–140. doi:10.4064/fm263-1-2024.
[34] H. A. Masur and Y. N. Minsky, *Geometry of the complex of curves. I. Hyperbolicity*, Invent. Math. **138** (1999), 103–149.
[35] G. Massuyeau, *An introduction to the abelian Reidemeister torsion of three-dimensional manifolds*, Ann. Math. Blaise Pascal **18** (2011), 61–140. doi:10.5802/ambp.294.
[36] J.-P. Otal, *Thurston's hyperbolization of Haken manifolds*, in *Surveys in Differential Geometry*, vol. III, International Press, 1998, 77–194.
[37] *Exponential first-homology torsion in a hyperbolic one-relator group*, working manuscript, 28 November 2026, Section 2.
[38] L. Paris, *Residual p properties of mapping class groups and surface groups*, Trans. Amer. Math. Soc. **361** (2009), 2487–2507. doi:10.1090/S0002-9947-08-04573-X.
[39] R. C. Penner, *A construction of pseudo-Anosov homeomorphisms*, Trans. Amer. Math. Soc. **310** (1988), 179–197.
[40] *Primary torsion in characteristic covers of three-manifolds and the growth of mod p homology*, working manuscript, 28 November 2026.
[41] M. Rapaport, T. Tkocz and I. Wu, *Negative moments of Steinhaus sums*, arXiv:2512.09077v2, 2025.
[42] K. Schmidt, *Dynamical Systems of Algebraic Origin*, Progress in Mathematics, vol. 128, Birkhäuser, 1995.
[43] A. Schrijver, *Counting 1-factors in regular bipartite graphs*, J. Combin. Theory Ser. B **72** (1998), 122–135. doi:10.1006/jctb.1997.1798.
[44] R. Slutsky, *Sparse random covers and growth of torsion in first homology*, arXiv:2608.05037v2, 2026.
[45] D. S. Silver and S. G. Williams, *Mahler measure, links and homology growth*, Topology **41** (2002), 979–991.
[46] D. S. Silver and S. G. Williams, *Torsion numbers of augmented groups with applications to knots and links*, Enseign. Math. (2) **48** (2002), 317–343. arXiv:math/0202197.
[47] J.-P. Serre, *Trees*, Springer, 1980.
[48] J. Stallings, *Homology and central series of groups*, J. Algebra **2** (1965), 170–181.
[49] J. R. Stallings, *On fibering certain 3-manifolds*, in *Topology of 3-Manifolds and Related Topics*, Prentice-Hall, 1962, 95–100.