CHIRALITY AND COMPLETE ASYMMETRY IN THE SPATIAL 8-BODY PROBLEM: FIFTY-SIX MACHINE-CERTIFIED EQUAL-MASS CENTRAL CONFIGURATIONS, FORMAL PERTURBATION LOWER BOUNDS, AND THE CUBE–ANTIPRISM TWIST THEOREM

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Abstract. In the planar equal-mass Newtonian N-body problem, Moczurad and Zgliczynski proved that every central configuration with $N \leq 7$ admits reflectional symmetry and that completely asymmetric configurations appear at $N = 8$. In three dimensions, Moczurad and Zgliczynski proved complete classifications up to $N = 6$ (all achiral), and all 20 machine-certified equal-mass spatial 7-body classes known to date are also achiral. Here we formally prove that strict chirality (non-superimposability on the mirror image under $\mathrm{SO}(3) \times S_8$) and complete asymmetry (trivial full O(3) point group C_1) occur in the equal-mass spatial Newtonian N-body problem at $N = 8$—establishing an unconditional upper bound ≤ 8 on the three-dimensional chirality and asymmetry threshold, and an exact onset at $N = 8$ conditional on the completeness of the 20 equal-mass spatial 7-body classes.

Using a pure-integer scaled 17-DOF Banach–Krawczyk certification engine verified inside Lean 4.36.0 + Mathlib4 (zero `sorry`, zero `admit`, zero `native_decide`, all 21 exported theorems depending solely on `[propext, Classical.choice, Quot.sound]`), we prove four main results:

(1) **Fifty-six certified equal-mass spatial 8-body polyhedra:** There exist at least 56 pairwise non-similar (modulo scaling, O(3), translations, and S_8), strictly three-dimensional, strongly nondegenerate equal-mass spatial 8-body central configurations (`P01`–`P56`) spanning 16 distinct point groups, separated by all $\binom{56}{2} = 1{,}540$ formally verified rational spectral cut witnesses.

(2) **Formal proof of chirality and complete asymmetry:** Polyhedron `P27` (C_1, Morse index 4, $U\sqrt{I} \approx 57.57302223$) is formally proved to be both totally asymmetric (`IsAsymmetric8`) and strictly chiral (`IsChiral8`); polyhedron `P35` (C_2, Morse index 5, $U\sqrt{I} \approx 58.85522899$) is formally proved to be strictly chiral (`IsChiral8`) via exhaustive elimination of all $2^4 = 16$ orbit-preserving candidate bijections.

(3) **Formal mass-perturbation lower bounds directly in Lean 4:** Rather than invoking the implicit function theorem only in prose, we formalize the full weighted 17-to-24 Newtonian reconstruction, Banach–Krawczyk mass continuation, kernel-verified stabilizer upper bounds, and open persistence of chirality directly in Lean 4 (`Perturbation693528.lean`). In an explicit open neighborhood of equal masses, we formally prove the existence of at least 56 unlabeled O(3)-classes (with `P27`(m) remaining totally asymmetric and chiral and `P35`(m) remaining chiral), at least 58 unlabeled oriented SO(3)-classes (including both mirror enantiomorphs $\pm$`P27`(m) and $\pm$`P35`(m), breaking the $1:1$ unlabeled $\mathrm{SO}(3):\mathrm{O}(3)$ ratio), at least 693,528 labeled configurations modulo O(3), and at least $1{,}387{,}056 = 2 \times 693{,}528$ labeled oriented configurations modulo SO(3) (restoring the exact universal $2:1$ labeled ratio via reflection doubling).

(4) **Exact Cube (O_h) vs. Square Antiprism (D_{4d}) Twist and Energy Theorem:** Across all heights $h > 0$, the regular cube ($h = 1/\sqrt{2}$, `P03`) is the unique equal-mass square-prism central configuration, while the square antiprism (`P01`, global minimum) is unique with height $h^*_{\mathrm{anti}} \in (0.676, 0.677) < 1/\sqrt{2}$; we formally prove $\partial^2 U/\partial\theta^2|_{\mathrm{Cube}} < 0 < \partial^2 U/\partial\theta^2|_{\mathrm{Antiprism}}$ and $U\sqrt{I}(\mathrm{Antiprism}) < 55.72 < 55.83 < U\sqrt{I}(\mathrm{Cube})$.

The formal library is cross-checked by 12 homotopy-free polynomial certifications via `pandsysroots v0.34.0` (100% complete complex root certification on 8/12 prism, antiprism, and nested-tetrahedra benchmarks, plus certification of the C_{6v} $1 \leftrightarrow 5$ saddle-node cliff across $m^*_{C_{6v}} \approx 1.1520980623$) and an independent 250-digit `mpmath` check against all 24 Cartesian equations (maximum residual $< 10^{-259}$). Global upper-bound completeness ("exactly 56") in 17 dimensions remains open.

1. Introduction and the Spatial Chirality Threshold

A configuration $q = (q_1, \ldots, q_N) \in (\mathbb{R}^3)^N$ of N point masses $m_1, \ldots, m_N > 0$ with center of mass $c = (\sum_k m_k q_k)/(\sum_k m_k)$ is a *Newtonian central configuration* if there exists $\lambda > 0$ such that

$$\sum_{j \neq i} m_j \frac{q_i - q_j}{\|q_i - q_j\|^3} = \lambda(q_i - c), \qquad i = 1, \ldots, N. \tag{1}$$

Central configurations are the critical points of the scale-invariant normalized Newtonian potential $U(q)\sqrt{I(q)}$, where

$$U(q) = \sum_{1\le i<j\le N} \frac{m_i m_j}{\|q_i - q_j\|}, \qquad I(q) = \sum_{i=1}^{N} m_i \|q_i - c\|^2,$$

and govern both homographic self-similar motions and the topology of integral manifolds of the N-body problem [9, 11, 10].

A classic question in the equal-mass problem ($m_1 = \cdots = m_N = 1$) is how much geometric symmetry is forced at small N and at what threshold completely asymmetric or chiral central configurations first appear. In the *planar* ($d = 2$) equal-mass problem, Albouy [1] classified $N = 4$, Ferrario [4] and Moeckel [10] noted asymmetric numerical examples at $N = 8$, and Moczurad and Zgliczynski [7] proved complete computer-assisted classifications for $N = 5, 6, 7$: every planar equal-mass central configuration for $N \le 7$ admits at least one line of reflectional symmetry, whereas at $N = 8, 9, 10$ the first totally asymmetric planar equal-mass central configurations appear [7].

In the *spatial* ($d = 3$) equal-mass problem, Moczurad and Zgliczynski [8] proved complete classifications up to $N = 6$:

- $N = 4$: 1 class (regular tetrahedron T_d);
- $N = 5$: 4 classes ($D_{3h}, C_{4v}, C_{3v}, C_{2v}$);
- $N = 6$: 9 classes ($O_h, C_{5v}, D_{3h}, C_{4v}, D_{2d}, C_{3v}, 2\,C_{2v}, C_s$).

Every spatial equal-mass central configuration for $N \le 6$ is therefore rigorously achiral (admitting an improper symmetry in $\mathrm{O}(3) \setminus \mathrm{SO}(3)$). In our preceding work on spatial $N = 7$ [2], we machine-certified 20 equal-mass spatial 7-body similarity classes spanning 9 point groups ($O_h, D_{5h}, C_{6v}, 2\,D_{3h}, 2\,C_{5v}, C_{4v}, 4\,C_{3v}, 5\,C_{2v}, 3\,C_s$), every one of which is also achiral (with no 21st class detected in 157,000 multi-seed starts), though global upper-bound completeness for spatial $N = 7$ remains an open numerical conjecture.

Prior published literature on spatial 8-body central configurations has focused on specific symmetric families such as regular cubes, square prisms and antiprisms, twisted or stacked polygons, and nested regular tetrahedra [12, 5, 3, 6]. While unpublished numerical lists may exist in informal working notes (such as unpublished tables of D. Ferrario), to our knowledge neither a published equal-mass spatial 8-body atlas nor a rigorous (let alone kernel-verified) proof of strict chirality (C_2, C_1) and complete asymmetry (C_1) for equal-mass spatial 8-body central configurations has previously appeared in the literature.

TABLE 1. Symmetry-breaking progression in the planar ($d = 2$) and spatial ($d = 3$) equal-mass Newtonian N-body problems. Complete classifications for $d = 2$ ($N \le 7$) and $d = 3$ ($N \le 6$) are due to Albouy [1] and Moczurad–Zgliczynski [7, 8]; lower bounds for spatial $N = 7$ and $N = 8$ are machine-certified in Lean 4 ([2] and this paper). The spatial chirality threshold is unconditionally ≤ 8 (certified here) and strictly $N = 8$ conditional on completeness of the 20 spatial $N = 7$ classes.

N	Planar ($d = 2$)	Planar C_1?	Spatial ($d = 3$, $\mathrm{O}_3/\mathrm{SO}_3$)	Spatial chiral / C_1?
3	2 (exact)	No	—	—
4	4 (exact)	No	1/1 (exact)	No (proved)
5	5 (exact)	No	4/4 (exact)	No (proved)
6	9 (exact)	No	9/9 (exact)	No (proved)
7	14 (exact)	No	≥ 20/ ≥ 20 (Lean 4)	None in 20 (conj.)
8	≥ 31	**Yes (C_1)**	**≥ 56/ ≥ 58 (Lean 4)**	**Yes (C_2, C_1 proved)**

The main contribution of this paper is to prove—with kernel-checked formal proofs in Lean 4.36.0 + Mathlib4—that **strict chirality and complete asymmetry exist in three dimensions at** $N = 8$ (Table 1), establishing an unconditional upper bound ≤ 8 on the 3D

equal-mass chirality threshold (and exact onset at $N = 8$ conditional on the completeness of the 20 equal-mass spatial 7-body classes). Among 56 newly certified equal-mass spatial 8-body polyhedra (`P01`–`P56`), two are formally certified to be strictly chiral:

- `P27` (point group C_1, $|O(3)| = |SO(3)| = 1$, Morse index 4, $U\sqrt{I} \approx 57.57302223$): completely asymmetric and strictly chiral;
- `P35` (point group C_2, $|O(3)| = |SO(3)| = 2$, Morse index 5, $U\sqrt{I} \approx 58.85522899$): purely rotational chiral, possessing a single two-fold rotation axis and no improper symmetries.

Furthermore, we formalize the full mass-perturbation continuation directly inside Lean 4 (`Perturbation693528.lean`), proving that in an explicit open neighborhood of equal masses both chirality results persist, producing ≥ 56 unlabeled O(3)-classes, ≥ 58 unlabeled SO(3)-classes, $\geq 693{,}528$ labeled configurations mod O(3), and $\geq 1{,}387{,}056 = 2 \times 693{,}528$ labeled oriented configurations mod SO(3).

Remark 1.1 (Honest Scope Distinction Regarding Global Completeness). We distinguish strictly between what is formally proved and what remains open:

- Theorems 3.1, 4.2, 4.3, and 3.2 establish formally verified *existence* and *lower bounds* (≥ 56 unlabeled O(3)-classes, ≥ 58 unlabeled SO(3)-classes, $\geq 693{,}528$ labeled O(3)-classes, and $\geq 1{,}387{,}056$ labeled SO(3)-classes near equal masses, plus chirality of `P27` and `P35`). Although an intensive unconstrained and symmetry-seeded Rust numerical search exceeding 400,000 starts discovered zero classes beyond these 56 (with zero new classes after the first 80,000 starts), proving global upper-bound completeness in 17 real dimensions remains an open problem. Likewise, absence of chirality at spatial $N = 7$ depends on the unproved completeness of the 20 certified $N = 7$ classes [2].
- Theorem 5.1 is *globally complete* across all heights $h > 0$ for equal-mass square prisms and square antiprisms.
- Among the 12 `pandsysroots v0.34.0` benchmarks (Section 6), 8 attain 100% complete complex root certification (`12/12` and `16/16`), whereas the 4 C_{6v} benchmarks certify a partial affine bound (`102/120`) and are reported strictly as certified lower counts.

2. Exact 24-Equation Model, 17-DOF Gauge Reduction, and Pure-Integer Contraction

2.1. **Exact Definitions in Lean 4.** In `lean/Spatial8/NewtonianModel8.lean`, we define the full 24-equation Cartesian predicate `IsNewtonian8CC q m lam`, strict three-dimensionality via nonzero signed tetrahedron volume,

$$\texttt{Strictly3D8}(q) \iff \exists a, b, c, d \in \mathrm{Fin}\ 8, \quad \det\big(q_b - q_a,\ q_c - q_a,\ q_d - q_a\big) \neq 0,$$

and prove `not_coplanar_of_strictly3D8`: no affine plane in $\mathbb{R}^3$ contains all 8 points of a `Strictly3D8` configuration. We define exact geometric equivalence predicates (unlabeled and labeled):

$$\begin{aligned}
\texttt{AreSimilar8}(p, q) &\iff \exists s > 0,\ \sigma \in S_8,\ \forall i, j,\ \|p_{\sigma(i)} - p_{\sigma(j)}\|^2 = s\|q_i - q_j\|^2,\\
\texttt{AreDirectlySimilar8}(p, q) &\iff \exists s > 0,\ R \in \mathrm{SO}(3),\ t \in \mathbb{R}^3,\ \sigma \in S_8,\ \forall i,\ p_{\sigma(i)} = sRq_i + t,\\
\texttt{IsChiral8}(q) &\iff \neg\, \texttt{AreDirectlySimilar8}(q, \mathrm{reflect}_3 \circ q),\\
\texttt{IsAsymmetric8}(q) &\iff \forall s > 0,\ R \in \mathrm{O}(3),\ t \in \mathbb{R}^3,\ \sigma \in S_8,\\
&\qquad \big(\forall i,\ q_{\sigma(i)} = sRq_i + t\big) \implies (s = 1 \wedge R = I \wedge t = 0 \wedge \sigma = \mathrm{id}),
\end{aligned}$$

where $\mathrm{reflect}_3(x, y, z) = (x, y, -z) \in \mathrm{O}(3) \setminus \mathrm{SO}(3)$. In `Perturbation693528.lean`, we also define exact fixed-label similarity relations `AreLabeledSimilarO3` (matching pairwise squared distances up to $s > 0$) and `AreLabeledSimilarSO3` (matching both pairwise squared distances up to $s > 0$ and all signed tetrahedron volumes up to $s^{3/2}$), and formally prove their faithfulness under arbitrary rigid similarities in $\mathbb{R}_{>0} \times \mathrm{O}(3) \times \mathbb{R}^3$ and $\mathbb{R}_{>0} \times \mathrm{SO}(3) \times \mathbb{R}^3$ (`areLabeledSimilarO3_of_rigid`, `areLabeledSimilarSO3_of_rigid`).

2.2. **Universal 17-to-24 Reconstruction (Equal and Arbitrary Positive Masses).** Removing the 7 continuous symmetries of $\mathbb{R}^3$ (3 translations, 1 dilation, 3 rotations) leaves $24-7=17$ real degrees of freedom. For any positive mass vector $m=(m_0,\dots,m_7)\in\mathbb{R}^8_{>0}$ and $u\in\mathbb{R}^{17}$, we unpack $q=\texttt{gauge17PosM}(m,u)\in(\mathbb{R}^3)^8$ by fixing

$$q_0=(1,0,0),\qquad q_1=(u_0,u_1,0),$$
$$q_k=(u_{3k-4},u_{3k-3},u_{3k-2})\quad(2\le k\le 6),\qquad q_7=-\frac{1}{m_7}\sum_{i=0}^{6}m_iq_i,$$

reducing to $\texttt{gauge17Pos}(u)$ when $m=(1,\dots,1)$. Setting $\lambda(m,u)=f^{(x)}_{0,m}(q)$ where $f_{i,m}(q)=\sum_{j\neq i}m_j(q_i-q_j)\|q_i-q_j\|^{-3}$, the reduced 17-component map $\texttt{gauge17FM}(m,u)\in\mathbb{R}^{17}$ enforces $f^{(d)}_{i,m}(q)-\lambda(m,u)q_i^{(d)}=0$ on the 17 free coordinates $(1,x),(1,y)$ and $(k,x),(k,y),(k,z)$ for $2\le k\le 6$.

Theorem 2.1 (Universal 17-to-24 Weighted Reconstruction)**.** *Let $m\in\mathbb{R}^8_{>0}$ and let $u\in\mathbb{R}^{17}$ be collision-free ($\|q_i-q_j\|^2>0$ for $i\neq j$) with $\texttt{gauge17FM}(m,u)=0$ (`gauge17_reconstruction_24eq_M`). If the anchor nondegeneracy conditions*

$$1-q_{7,x}\neq 0\qquad\text{and}\qquad(1-q_{7,x})(u_1-q_{7,y})+q_{7,y}(u_0-q_{7,x})\neq 0$$

*hold, then $\lambda(m,u)>0$ and $q=\texttt{gauge17PosM}(m,u)$ satisfies **all** 24 **weighted Cartesian Newtonian equations** (1). (The equal-mass case is `gauge17_reconstruction_24eq`.)*

Proof outline. By Newton's third law, total weighted force $\sum_i m_if_{i,m}=0$ and total weighted torque $\sum_i m_i(q_i\times f_{i,m})=0$ vanish identically, and $\sum_i m_iq_i=0$ by construction. Letting $r_i=m_i(f_{i,m}-\lambda q_i)$, the 17 reduced equations plus $r_{0,x}=0$ leave only $r_{0,y},r_{0,z},r_{1,z}$ and $r_7=-(r_0+r_1)$. The xy-torque identity gives $(1-q_{7,x})r_{0,y}=0$, hence $r_{0,y}=0$; the xz- and yz-torque identities yield a 2×2 linear system in $(r_{0,z},r_{1,z})$ whose determinant is $(1-q_{7,x})(u_1-q_{7,y})+q_{7,y}(u_0-q_{7,x})\neq0$, forcing $r_{0,z}=r_{1,z}=0$ and $r_7=0$. Dividing by $m_i>0$ gives $f_{i,m}=\lambda q_i$ for all i. Finally, $\lambda\sum_i m_i\|q_i\|^2=\sum_i m_iq_i\cdot f_{i,m}=\frac12\sum_{i\neq j}m_im_j\|q_i-q_j\|^{-1}>0$ implies $\lambda>0$. □

2.3. **Pure-Integer Kernel Contraction Engine.** In `lean/Spatial8/BanachEngine.lean`, we formalize a dimension-generic Banach–Krawczyk fixed-point theorem (`banach_krawczyk_fin`) proving existence of a unique root $u^*\in\overline{B}_\infty(x_0,\delta)$ with an injective Jacobian $J(u^*)$ from three ℓ_∞ bounds:

$$\|AF(x_0)\|_\infty\le\frac{\delta}{2},\qquad\sup_{u\in\overline{B}_\infty(x_0,\delta)}\|I-AJ(u)\|_\infty\le\frac12,\qquad\|I-B_0A\|_\infty\le\frac12.$$

To verify all 56 certificates in 17 dimensions purely by `decide +kernel` (without `native_decide`), `lean/Spatial8/Gauge17Cert.lean` stores the preconditioning matrices $A=A_{\text{num}}/D_A$, $B_0=B_{0,\text{num}}/D_B$ and pre-rounded residual/Jacobian enclosures $[F_{0,\text{lo}},F_{0,\text{hi}}]/D_F$, $[J_{\text{lo}},J_{\text{hi}}]/D_J$ as **pure integers** with positive natural scale factors. All $17\times17\times17$ matrix contractions execute strictly in $\mathbb{Z}$ arithmetic with **zero `Nat.gcd` calls**, while shared box enclosures (`dicB`, `lamB`, `dLamB`) are validated once per certificate and the 56 polyhedra are partitioned across eight parallel modules `CertsBatch1`–`CertsBatch8`.

3. Fifty-Six Certified Polyhedra and Formal Perturbation Lower Bounds

Theorem 3.1 (Fifty-Six Equal-Mass Spatial 8-Body Central Configurations)**.** *Formally verified in `FiftySixEqualMassCertified.lean` (`fifty_six_equal_mass_3D_8body_central_configurations`): there exist* 56 *vectors $U_0,\dots,U_{55}\in\mathbb{R}^{17}$ such that:*

1. *For every $k\in\{0,\dots,55\}$, $q^{(k)}=\texttt{gauge17Pos}(U_k)$ satisfies all* 24 *exact Cartesian Newtonian equal-mass central-configuration equations (`IsNewtonian8CC`) with $\lambda_k>0$;*
2. *Every $q^{(k)}$ is strictly three-dimensional (`Strictly3D8`), hence non-coplanar;*
3. *Every $q^{(k)}$ is strongly nondegenerate: the exact 17×17 analytical gauge Jacobian $\texttt{gauge17Jac}(U_k)$ has trivial kernel;*

(4) *For all* $0 \le i < j \le 55$, $\neg$ `AreSimilar8`$(q^{(i)}, q^{(j)})$ *(no uniform scaling,* O(3) *isometry, translation, or* S_8 *relabeling maps* $q^{(i)}$ *to* $q^{(j)}$*);*
(5) *Polyhedron* $q^{(26)}$ *(P27) satisfies* `IsAsymmetric8`$(q^{(26)}) \wedge$ `IsChiral8`$(q^{(26)})$*, and polyhedron* $q^{(34)}$ *(P35) satisfies* `IsChiral8`$(q^{(34)})$*.*

The pairwise non-similarity proof relies on `not_areSimilar8_of_cut`: for any two similar configurations, the normalized squared-distance cumulative distribution

$$N_q(t) = \left|\left\{(i,j) \in (\mathrm{Fin}\ 8)^2 : \frac{2\|q_i - q_j\|^2}{\sum_{a,b}\|q_a - q_b\|^2} < t\right\}\right|$$

is invariant under $\mathbb{R}_{>0} \times \mathrm{O}(3) \times \mathbb{R}^3 \times S_8$. Every one of the $\binom{56}{2} = 1{,}540$ pairs is separated by an explicit rational threshold $t_{ij} > 0$ checked in Lean 4 by `allCuts56Valid_eq_true`.

3.1. **Formal Mass-Perturbation Lower Bounds in Lean 4.** In `lean/Spatial8/Perturbation693528.lean`, we eliminate any informal prose appeal to the implicit function theorem by formalizing the entire continuation, symmetry-stabilizer, and reflection-doubling pipeline directly inside Lean 4:

(1) **Banach–Krawczyk mass continuation (`continue_cert_near_equal_mass`):** From each certified equal-mass root $u^* \in \overline{B}_\infty(x_0, \delta)$ with contraction factor $\le 1/2$, continuity of the weighted 17-DOF map `gauge17FM` and its analytical Jacobian `gauge17JacM` yields an open neighborhood of $(1, \dots, 1)$ on which a (3/4)-contractive Banach–Krawczyk engine (`banach_krawczyk_fin_34`) produces exact continued solutions $u(m)$ satisfying all 24 weighted equations (via Theorem 2.1), strict 3D non-coplanarity, and 17-dimensional nondegeneracy.
(2) **Kernel-verified** O(3) **stabilizer upper bounds (`all56_stab_O3_bounds`):** A depth-first backtracking checker over S_8 executes strictly inside `decide +kernel` (11,187 interval comparisons total across all 56 polyhedra, partitioned into 8 batches of 7) to prove that the number of S_8 permutations compatible with each polyhedron's certified normalized distance intervals `ndB` is at most $|G_k^{\mathrm{O}(3)}|$, matching all 56 point-group orders in Table 2 with sum $\sum_{k=1}^{56} 8!/|G_k^{\mathrm{O}(3)}| = 693{,}528$ (`sum_orbit_O3_56`).
(3) **Open persistence of chirality and universal** $2:1$ **reflection doubling (`labeled_SO3_double_of_O3`):** Because the defining inequalities of `P27`'s asymmetry/chirality and `P35`'s C_2 chirality are strictly open, both properties persist throughout an open neighborhood of equal masses. Moreover, for *any* strictly 3D configuration q and mass vector m, its mirror image $\mathrm{reflect}_3 \circ q$ satisfies the exact same 24 weighted Newtonian equations (`isNewtonian8CC_reflect3`) while reversing the sign of $\mathrm{tetDet}_6 \neq 0$, so q and $\mathrm{reflect}_3 \circ q$ are never labeled-SO(3)-similar.

Theorem 3.2 (Formal Mass-Perturbation Lower Bounds Near Equal Masses)**.** *Formally verified in* `Perturbation693528.lean`*: there exists* $\eta > 0$ *such that for EVERY mass vector* $m : \mathrm{Fin}\ 8 \to \mathbb{R}$ *satisfying* $|m_i - 1| < \eta$ *for all* i*:*

(1) (`perturbation_56_unlabeled_O3_and_chiral_near_equal_mass`) *There exist at least* 56 *pairwise non-similar (*$\neg$ `AreSimilar8`*), strictly 3D, 17D-nondegenerate spatial* 8*-body central configurations for masses* m*, of which* `P27`(m) *remains totally asymmetric and strictly chiral and* `P35`(m) *remains strictly chiral.*
(2) (`perturbation_58_unlabeled_SO3_near_equal_mass`) *There exist at least* 58 *strictly 3D spatial* 8*-body central configurations for masses* m *that are pairwise non-directly-similar (*$\neg$ `AreDirectlySimilar8`*, modulo scaling,* SO(3)*, translations, and* S_8*), consisting of the* 56 *branches together with the two mirror enantiomorphs* $\mathrm{reflect}_3 \circ$ `P27`(m) *and* $\mathrm{reflect}_3 \circ$ `P35`(m)*.*
(3) (`perturbation_693528_labeled_O3_near_equal_mass`) *There exist at least* 693,528 *strictly 3D spatial* 8*-body central configurations for masses* m *that are pairwise non-labeled-*O(3)*-similar (*$\neg$ `AreLabeledSimilarO3`*).*

(4) (`perturbation_1387056_labeled_SO3_near_equal_mass`) *There exist at least* 1,387,056 = 2 × 693,528 *strictly 3D spatial 8-body central configurations for masses m that are pairwise non-labeled*-SO(3)*-similar* (¬ `AreLabeledSimilarSO3`).

Remark 3.3 (Correcting the Oriented Labeled Count: Unlabeled 58 : 56 vs. Labeled 2 : 1). In three dimensions, the naive single-representative sum $\sum_{k=1}^{56} 8!/|G_k^{\mathrm{SO}(3)}| = 1{,}326{,}576$ omits the mirror enantiomorphs reflect$_3 \circ$ `P27` (contributing an additional $8!/1 = 40{,}320$ oriented labeled configurations) and reflect$_3 \circ$ `P35` (contributing an additional $8!/2 = 20{,}160$ oriented labeled configurations). Because reflection acts freely on strictly 3D labeled SO(3)-classes (`labeled_SO3_double_of_O3`), the ratio of strictly 3D *labeled* central configurations modulo SO(3) to modulo O(3) is *universally* 2 : 1, giving

$$1{,}326{,}576 + 40{,}320 + 20{,}160 = 1{,}387{,}056 = 2 \times 693{,}528.$$

What strict chirality breaks at $N = 8$ is instead the 1 : 1 ratio between *unlabeled* SO(3)-classes and *unlabeled* O(3)-classes: ≥ 58 unlabeled oriented SO(3)-classes versus ≥ 56 unlabeled O(3)-classes (Theorem 3.2(2)).

Table 2 lists all 56 machine-certified equal-mass polyhedra ordered by their scale-invariant potential $U\sqrt{I}$, spanning 16 point groups: 1 O_h, 1 T_d, 1 D_{6h}, 1 D_{5h}, 1 D_{4d}, 1 D_{4h}, 1 C_{7v}, 3 D_{3h}, 2 C_{6v}, 1 C_{5v}, 2 D_{2d}, 5 C_{3v}, 13 C_{2v}, 21 C_s, 1 C_2, and 1 C_1.

TABLE 2. Complete atlas of all 56 machine-certified equal-mass spatial 8-body central configurations (`P01`–`P56`), sorted by scale-invariant potential $U\sqrt{I}$. Columns give point group G, orders $|O(3)|/|SO(3)|$, Morse index m of $U\sqrt{I}$ on the 17-dimensional shape space, and $U\sqrt{I}$. Markers: `(min)` = global minimum square antiprism, `(cube)` = regular cube, `(∗)` = formally certified totally asymmetric and chiral `P27` (C_1), `(†)` = formally certified strictly chiral `P35` (C_2), `(tet)` = nested regular tetrahedra.

ID	G	$\lvert O_3\rvert/\lvert SO_3\rvert$	m	$U\sqrt{I}$	ID	G	$\lvert O_3\rvert/\lvert SO_3\rvert$	m	$U\sqrt{I}$
`P01` (min)	D_{4d}	16/8	0	55.65011787	`P29`	C_{3v}	6/3	4	57.64083172
`P02`	D_{2d}	8/4	1	55.66746814	`P30`	C_s	2/1	5	57.64100732
`P03` (cube)	O_h	48/24	2	55.83534085	`P31`	C_{6v}	12/6	5	58.27909627
`P04`	D_{2d}	8/4	1	56.14349646	`P32`	C_{6v}	12/6	4	58.33006180
`P05`	C_s	2/1	2	56.15165525	`P33`	D_{4h}	16/8	4	58.43207555
`P06`	C_{2v}	4/2	3	56.15878991	`P34`	C_{2v}	4/2	4	58.71665431
`P07`	D_{6h}	24/12	3	56.40673525	`P35` (†)	C_2	2/2	5	58.85522899
`P08`	C_{5v}	10/5	2	56.73130706	`P36`	C_s	2/1	5	59.10164838
`P09`	D_{5h}	20/10	1	56.75773922	`P37`	C_s	2/1	5	59.11190490
`P10`	C_{2v}	4/2	2	56.77043799	`P38`	C_s	2/1	5	59.11563698
`P11`	C_s	2/1	2	56.77213991	`P39`	C_s	2/1	6	59.12208124
`P12`	C_s	2/1	3	56.77230168	`P40`	C_{7v}	14/7	6	59.15547554
`P13`	C_{3v}	6/3	4	56.77234701	`P41`	C_{2v}	4/2	5	59.31616394
`P14`	C_{3v}	6/3	3	56.78474230	`P42`	C_s	2/1	6	59.44541226
`P15`	C_s	2/1	3	56.79066032	`P43`	C_s	2/1	6	59.50355789
`P16`	C_{2v}	4/2	3	56.79643458	`P44`	C_s	2/1	6	60.16217477
`P17`	C_{2v}	4/2	4	56.83168581	`P45`	D_{3h}	12/6	6	60.79727146
`P18`	C_{2v}	4/2	3	57.32124954	`P46`	C_s	2/1	6	61.15593528
`P19`	D_{3h}	12/6	3	57.36039160	`P47`	C_s	2/1	7	61.17399193
`P20`	C_{2v}	4/2	4	57.37062256	`P48`	C_s	2/1	7	61.39490448
`P21`	C_{2v}	4/2	3	57.39677969	`P49`	C_{2v}	4/2	6	62.54740830
`P22`	C_s	2/1	3	57.40744403	`P50`	D_{3h}	12/6	6	62.60803111
`P23`	C_{2v}	4/2	4	57.40749993	`P51`	C_{2v}	4/2	7	62.61396061
`P24`	C_s	2/1	4	57.41493165	`P52`	C_s	2/1	7	63.17764230
`P25`	C_{3v}	6/3	5	57.41671980	`P53` (tet)	T_d	24/12	8	63.38165516
`P26`	C_s	2/1	4	57.51558772	`P54`	C_s	2/1	8	65.48860549
`P27` (∗)	C_1	1/1	4	57.57302223	`P55`	C_{3v}	6/3	9	65.49135494
`P28`	C_s	2/1	5	57.59220352	`P56`	C_{2v}	4/2	8	68.61165074

4. Formal Proof of Complete Asymmetry (P27) and Chirality (P27, P35)

For any configuration $q \in (\mathbb{R}^3)^8$, define the body distance-sum invariant

$$\rho_i(q) = \sum_{j=0}^{7} \|q_i - q_j\|^2 = 8\|q_i - \bar{q}\|^2 + \sum_{j=0}^{7} \|q_j - \bar{q}\|^2, \qquad i \in \mathrm{Fin}\ 8.$$

Lemma 4.1 (`distSq_perm_forces_scale_one_and_rho8`). *Let $q \in (\mathbb{R}^3)^8$ satisfy $\sum_{i,j} \|q_i - q_j\|^2 > 0$. If $s > 0$ and $\sigma \in S_8$ satisfy $\|q_{\sigma(i)} - q_{\sigma(j)}\|^2 = s^2\|q_i - q_j\|^2$ for all i, j, then $s = 1$, $\|q_{\sigma(i)} - q_{\sigma(j)}\|^2 = \|q_i - q_j\|^2$, and $\rho_{\sigma(i)}(q) = \rho_i(q)$ for every $i \in \mathrm{Fin}\ 8$.*

This elementary invariant reduces both symmetry-group triviality and chirality to finite interval arithmetic on the certified box (and, by openness of strict inequality, throughout a neighborhood of equal masses in Theorem 3.2):

Theorem 4.2 (Formal Asymmetry and Chirality of P27, `p27_asymmetric_and_chiral`). *Let $q = \texttt{gauge17Pos}(u)$ be the certified polyhedron `P27`. On its certified box ($\delta = 10^{-6}$), all eight rational enclosures $\texttt{rho8Ivl}_i$ are pairwise strictly disjoint (verified by `p27_rho_disjoint`). Consequently:*

1. *Every self-similarity (s, R, t, σ) in $\mathbb{R}_{>0} \times \mathrm{O}(3) \times \mathbb{R}^3 \times S_8$ satisfies $s = 1$ and $\sigma = \mathrm{id}$ by Lemma 4.1, whence $(R - I)(q_k - q_a) = 0$ on the certified nondegenerate tetrahedron (a, b, c, d). By 3×3 Cramer's rule, $R = I$ and $t = 0$, proving $\texttt{IsAsymmetric8}(q)$.*
2. *Any direct similarity (s, R, t, σ) with $R \in \mathrm{SO}(3)$ from $\mathrm{reflect}_3 \circ q$ to q also forces $s = 1$ and $\sigma = \mathrm{id}$, giving the contradiction*

$$\begin{aligned}\det(q_b - q_a, q_c - q_a, q_d - q_a) &= s^3(\det R)\big(-\det(q_b - q_a, q_c - q_a, q_d - q_a)\big) \\ &= -\det(q_b - q_a, q_c - q_a, q_d - q_a) \neq 0,\end{aligned}$$

proving $\texttt{IsChiral8}(q)$.

For P35 (C_2), the eight normalized squared radii $\|q_i\|^2/I$ split into four strictly separated pairs,

$$\{0.028017,\ 0.132553,\ 0.142014,\ 0.197416\},$$

with minimum inter-pair gap $> 9.46 \times 10^{-3} \gg \delta = 10^{-6}$.

Theorem 4.3 (Formal C_2 Chirality of P35, `p35_strictly_chiral`). *Let $q = \texttt{gauge17Pos}(u)$ be the certified polyhedron `P35`, with orbit-pairing involution $\tau = [1, 0, 5, 6, 7, 2, 3, 4]$ and orbit representatives $(0, 2, 3, 4)$. On its certified box ($\delta = 10^{-6}$), `checkC2Chiral8` certifies via `decide +kernel` that:*

1. *Any two bodies outside the same orbit pair have strictly disjoint `rho8Ivl` enclosures, forcing any self-isometry or mirror-isometry permutation $\sigma \in S_8$ to satisfy $\sigma(i) \in \{i, \tau(i)\}$ for all i. By injectivity, σ is uniquely determined by 4 swap bits ($2^4 = 16$ candidate bijections).*
2. *Every one of the 16 candidate bijections f either violates a pairwise squared-distance enclosure ($\texttt{distSqIvl}(i, j) \cap \texttt{distSqIvl}(f(i), f(j)) = \emptyset$) or preserves the strict sign of $\mathrm{tetDet}_6(q_a, q_b, q_c, q_d)$ (contradicting orientation reversal under $\mathrm{reflect}_3$).*

Therefore $\texttt{IsChiral8}(q)$ holds: `P35` is strictly chiral.

5. The Exact Cube (O_h) vs. Square Antiprism (D_{4d}) Twist and Energy Theorem

Consider eight unit masses arranged as two parallel unit squares at $z = \pm h$ ($h > 0$) with relative twist angle $\theta \in [0, \pi/4]$. At $\theta = 0$ (D_{4h} square prism), the radial-to-axial balance reduces to the one-variable equation

$$E_{\mathrm{prism}}(h) = \frac{1}{4} + \frac{1}{\sqrt{2}} - \frac{1}{4h^3} - 2(2 + 4h^2)^{-3/2} = 0.$$

At $\theta = \pi/4$ (D_{4d} square antiprism), the balance equation is

$$E_{\text{anti}}(h) = \frac{1}{4} + \frac{1}{\sqrt{2}} - (2+\sqrt{2})(2-\sqrt{2}+4h^2)^{-3/2} - (2-\sqrt{2})(2+\sqrt{2}+4h^2)^{-3/2} = 0.$$

Differentiating the cross-layer potential twice with respect to the twist angle θ at fixed h gives exact closed-form expressions at $\theta = 0, h = 1/\sqrt{2}$ and at $\theta = \pi/4$:

$$\left.\frac{\partial^2 U}{\partial \theta^2}\right|_{\text{Cube}} = 4\left(-2^{-3/2} + 6\cdot 4^{-5/2} + 6^{-3/2}\right) \approx -0.3920480352,$$

$$\left.\frac{\partial^2 U}{\partial \theta^2}\right|_{\text{Antiprism}}(h) = 8\left(-\frac{r_1^{-3/2}}{\sqrt{2}} + \frac{3}{2}r_1^{-5/2} + \frac{r_2^{-3/2}}{\sqrt{2}} + \frac{3}{2}r_2^{-5/2}\right),$$

where $r_{1,2} = 2 \mp \sqrt{2} + 4h^2$. All five parts of the following theorem are formally proved in `lean/Spatial8/CubeAntiprismTwist.lean`.

Theorem 5.1 (Exact Cube–Antiprism Twist and Energy Theorem)**.** *For the equal-mass spatial 8-body problem:*

(1) (`regular_cube_is_cc`, `regular_cube_unique_d4h`) *The regular cube ($h = 1/\sqrt{2}$, `P03`) satisfies all* 24 *Cartesian Newtonian equations and is the* unique *equal-mass square-prism (D_{4h}) central configuration across all $h > 0$.*

(2) (`square_antiprism_exists_unique`) *There exists a* unique *equal-mass square-antiprism (D_{4d}, `P01`) height $h^*_{\text{anti}} > 0$, satisfying*

$$\frac{676}{1000} < h^*_{\text{anti}} < \frac{677}{1000} < \frac{1}{\sqrt{2}},$$

so the equilibrium square antiprism is strictly flatter than the regular cube. (To 250 *digits, $h^*_{\text{anti}} = 0.676694817399777497\ldots$.)*

(3) (`cube_twist_unstable`, `antiprism_twist_stable`) *The regular cube is strictly twist-unstable ($\partial^2 U/\partial\theta^2|_{\text{Cube}} < 0$), whereas the square antiprism is strictly twist-stable ($\partial^2 U/\partial\theta^2|_{\text{Antiprism}}(h^*_{\text{anti}}) > 0$, $\approx +0.4768402863$).*

(4) (`antiprism_energy_lt_cube`) *The scale-invariant potential satisfies the strict energy ordering*

$$U\sqrt{I}(\text{Antiprism}) < 55.72 < 55.83 < U\sqrt{I}(\text{Cube}).$$

6. Homotopy-Free Polynomial Certification Across Four 8-Body Subsystems

To complement our Lean 4 proofs with global algebraic root certification on low-dimensional symmetric families, `src/main.rs` runs 12 homotopy-free polynomial benchmarks via `pandsysroots v0.34.0` (simultaneous sweeps of the Pandrosion rectangle step and the law of residues without homotopy) across four spatial 8-body families (reference timings measured on an Apple M5 10-core workstation with `rustc --release`; wall-clock durations depend on hardware, whereas complex root certification counts are deterministic):

(1) **Square Prism (D_{4h}) and Square Antiprism (D_{4d}):** Rationalizing $s_1 = r_1^{-1/2}, s_2 = r_2^{-1/2}$ produces 3-variable algebraic systems of affine bound 12; `pandsysroots` certifies **all** 12/12 **complex roots** in 1–2 ms, confirming global uniqueness of the regular cube `P03` ($h = 1/\sqrt{2}$) and square antiprism `P01` ($h = h^*_{\text{anti}}$).

(2) **Nested Regular Tetrahedra (T_d, dual and aligned):** For two concentric regular tetrahedra with mass ratio $m \in \{1, 0.2, 5\}$ and scale ratio ρ, rationalizing the inter-tetrahedron distances yields 3-variable systems of bound 16; `pandsysroots` certifies **all** 16/16 **complex roots** on all 6 instances in 1–4 ms, recovering both the cube `P03` (dual, $\rho = 1$) and the aligned nested tetrahedra `P53` at $m = 1$ with $\rho^* = 0.44447864092266465\ldots$ (verified to 250 digits with residual 3.05×10^{-261}).

(3) **Hexagon + 2 Poles (C_{6v}):** For a unit regular hexagon of unit masses plus two axial poles of masses (m_1, m_2) at heights (u, v), rationalizing $(1+u^2)^{-1/2}$ and $(1+v^2)^{-1/2}$

gives a 4-variable system. At equal pole masses $m_1 = m_2 = m$, an exact saddle-node cliff occurs at

$$m^*_{C_{6v}} = 1.152098062285408843\ldots \quad (u^* = 1.2798850246\ldots,\ v^* = 0.3740549350\ldots),$$

switching between 1 real root (D_{6h} hexagonal bipyramid) above $m^*_{C_{6v}}$ and 5 real roots (3 similarity classes: D_{6h} `P07` plus two asymmetric C_{6v} configurations `P31`, `P32` at $m = 1$) below it.

TABLE 3. All 12 homotopy-free `pandsysroots v0.34.0` benchmarks in `src/main.rs`. Eight of twelve benchmarks achieve 100% complete complex root certification (`12/12` and `16/16`); every real root is independently verified against all 24 Cartesian Newtonian equations. Reference wall-clock timings are measured on an Apple M5 10-core CPU (`--release`); timings depend on hardware, while root certification counts are deterministic.

Family	Parameters	Certified / Bound	Real	Shapes	Time (s)	Max 24-eq err
D_{4h} Square Prism	equal mass	**12 / 12 (complete)**	1	1 (`P03`)	0.002	2.22×10^{-16}
D_{4d} Square Antiprism	equal mass	**12 / 12 (complete)**	1	1 (`P01`)	0.001	6.66×10^{-16}
T_d Dual Nested Tet.	$m = 1.0$	**16 / 16 (complete)**	1	1 (`P03`)	0.004	8.88×10^{-16}
T_d Dual Nested Tet.	$m = 0.2$	**16 / 16 (complete)**	1	1	0.002	4.44×10^{-16}
T_d Dual Nested Tet.	$m = 5.0$	**16 / 16 (complete)**	1	1	0.001	6.66×10^{-16}
T_d Aligned Nested Tet.	$m = 1.0$	**16 / 16 (complete)**	2	1 (`P53`)	0.004	6.22×10^{-15}
T_d Aligned Nested Tet.	$m = 0.2$	**16 / 16 (complete)**	2	2	0.002	8.88×10^{-16}
T_d Aligned Nested Tet.	$m = 5.0$	**16 / 16 (complete)**	2	2	0.002	1.11×10^{-15}
C_{6v} Hexagon + 2 Poles	$m = (1.20, 1.20)$	102 / 120	1	1	0.35	5.55×10^{-16}
C_{6v} Hexagon + 2 Poles	$m = (1.14, 1.14)$	102 / 120	5	3	0.41	2.00×10^{-15}
C_{6v} Hexagon + 2 Poles	$m = (1.00, 1.00)$	102 / 120	5	3 (`P07,P31,P32`)	0.37	1.22×10^{-15}
C_{6v} Hexagon + 2 Poles	$m = (0.80, 0.30)$	102 / 120	5	5	0.38	6.66×10^{-16}

7. Trust Boundary, Axiom Audit, and Reproducibility

Every formal theorem in the Lean 4 library `Spatial8` (`NewtonianModel8.lean`, `Banach Engine.lean`, `CubeAntiprismTwist.lean`, `Gauge17Cert.lean`, `CertsBatch1`–`CertsBatch8`, `FiftySixEqualMassCertified.lean`, and `Perturbation693528.lean`) is verified by the Lean 4.36.0 kernel with zero `sorry`, zero `admit`, zero `native_decide`, and zero custom axioms. Running `lake env lean Spatial8/Audit.lean` confirms that all 21 exported theorems depend exclusively on Lean's three standard axioms `[propext, Classical.choice, Quot.sound]`.

In addition, `verify_250digits.py` independently refines all 56 polyhedra to 250-digit precision and verifies all 24 Cartesian Newtonian equations directly (maximum residual 4.065×10^{-260}, minimum pairwise normalized 28-distance spectrum gap 0.0135580759 at `(P22, P23)`), together with the 250-digit cube/antiprism twist values, nested-tetrahedra ratio ρ^*, and C_{6v} saddle-node cliff $m^*_{C_{6v}}$. All four verification tiers (Lean 4 build + 21-theorem axiom audit, Rust `pandsysroots` suite, Python 250-digit check, and pure-ASCII Tectonic PDF compilation) are reproduced automatically in GitHub Actions CI.

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