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# TOWARDS MOCZURAD–ZGLICZYNSKI AT $N = 7$: TWENTY MACHINE-CERTIFIED EQUAL-MASS SPATIAL 7-BODY CENTRAL CONFIGURATIONS, FORMAL PERTURBATION LOWER BOUNDS, AND THE EXACT CENTERED-OCTAHEDRON PITCHFORK CURVE

IVAN BESEVIC

Abstract. In their computer-assisted classifications of equal-mass Newtonian central configurations, Moczurad and Zgliczynski established exact similarity counts in the plane up to $N = 7$ (2019) and in three dimensions for $N = 4$ (1 class), $N = 5$ (4 classes), and $N = 6$ (9 classes) (2020), leaving the spatial $N = 7$ classification open. Moving towards $N = 7$ in three dimensions, we establish machine-verified formal lower bounds and global symmetric-family results along three complementary axes. First, combining a 157,000-start gauge-fixed Newton census with a universal 14-degree-of-freedom rational interval Newton–Banach engine in Lean 4.36.0, we formally prove the existence of at least 20 pairwise non-similar (modulo scaling, $O(3)$ isometries, translations, and arbitrary $S_7$ relabelings), strictly three-dimensional, and 14-dimensionally nondegenerate equal-mass spatial 7-body central configurations satisfying all 21 Cartesian Newtonian equations. Rather than invoking the implicit function theorem only in prose, we formalize the full mass-perturbation continuation and kernel-verified stabilizer upper bounds directly in Lean 4 (`Perturbation20475.lean`), proving rigorously that in an explicit open neighborhood of the equal-mass point there exist at least 20 unlabeled similarity classes, at least 20,475 unoriented ($\mathrm{mod}\, O(3)$) labeled spatial 7-body central configurations, and at least 40,950 oriented ($\mathrm{mod}\, SO(3)$) labeled spatial 7-body central configurations. Second, for the centered square bipyramid on masses $(1, 1, 1, 1, m, m, \mu)$, we prove in Lean 4 the global existence and uniqueness of the centered $D_{4h}$ branch and of the exact $\mu$-parameterized $D_{4h} \to D_{2d}$ pitchfork curve $(z^*(\mu), m^*(\mu))$ for all $\mu \geq 0$, establish strict monotonicity in $\mu$, prove the universal closed-form lower bounds $z^*(\mu) > 1$ and $m^*(\mu) > (4 - \sqrt{2})/6$ together with the exact $\mu \to \infty$ limit $(1, (4 - \sqrt{2})/6)$, and certify the equal-central-mass enclosure $1.05548 < z^*(1) < 1.05549$, $0.44432 < m^*(1) < 0.44444$. Third, we run 12 homotopy-free polynomial certifications via `pandsysroots v0.34.0` across the $C_{5v}$, centered-$D_{2h}$, and centered-$D_{2d}$ 7-body families, achieving complete 504/504 complex root certification on three asymmetric centered-$D_{2h}$ mass triples and certifying both the $C_{5v}$ saddle-node cliff ($m^*_{C_{5v}} \approx 1.1384268451$) and the centered-$D_{2d}$ pitchfork jump ($m^*(1) \approx 0.4443752621$). All 19 exported Lean 4 theorems compile with zero `sorry`, zero `admit`, and zero `native_decide`, depending solely on the standard axioms `[propext, Classical.choice, Quot.sound]`, and all numerical values are independently verified to 250-digit precision. A global upper-bound completeness proof ("exactly 20") in 14 dimensions remains open.

## 1. Introduction

A configuration $q = (q_0, \ldots, q_{N-1}) \in (\mathbb{R}^3)^N$ of $N$ point masses $m_0, \ldots, m_{N-1} > 0$ with center of mass $c = (\sum_k m_k q_k)/(\sum_k m_k)$ is a *Newtonian central configuration* if there exists a multiplier $\lambda > 0$ such that

$$\sum_{j \neq i} m_j \frac{q_i - q_j}{\|q_i - q_j\|^3} = \lambda(q_i - c), \qquad i = 0, \ldots, N-1. \tag{1}$$

Central configurations govern self-similar homographic solutions, total-collapse asymptotics, and topological bifurcations of the integral manifolds of the Newtonian $N$-body problem [13, 11, 8]. Despite more than a century of investigation since Moulton's collinear theorem [9] and Chazy–Wintner–Smale's finiteness conjecture [12], exact classifications in three dimensions remain rare. Even for equal masses $m_0 = \cdots = m_{N-1} = 1$, Hampton and Moeckel [3] and Albouy and Kaloshin [1] established generic finiteness in the plane ($N = 4, 5$), while Moczurad and Zgliczynski achieved

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complete computer-assisted classifications for equal masses in the plane up to $N = 7$ [6] and in three dimensions up to $N = 6$ [7]:

| Equal masses $N$ | 4 | 5 | 6 | 7 (this work) |
|---|---|---|---|---|
| 3D classes (mod $O(3), S_N$) | 1 (complete) | 4 (complete) | 9 (complete) | $\geq 20$ (Lean 4 lower bound) |

Prior published work on spatial 7-body central configurations has focused on particular symmetric or stacked families, such as double pyramids, nested polyhedra, twisted-triangle constructions, and stacked $(6 + 1)$ configurations [14, 4, 5, 2, 10]. While unpublished numerical lists may exist in informal working notes (such as unpublished notes of D. Ferrario), to our knowledge neither a published equal-mass spatial 7-body list with full nondegeneracy analysis nor a kernel-verified formal proof of existence, non-coplanarity, pairwise non-similarity, and near-equal-mass perturbative continuation has previously been established.

1.1. **Summary of Main Contributions.** This paper advances towards an equal-mass spatial 7-body classification along three rigorously verified fronts:

(1) **Twenty Lean 4-certified equal-mass spatial 7-body polyhedra and formal perturbation bounds (Section 3).** Using a 157,000-start gauge-fixed Newton exploration (which reproduces all 9/9 equal-mass spatial 6-body classes of [7] in 0.5 s on our reference machine as a validation check), we identify 20 distinct, strictly 3D, strongly nondegenerate equal-mass spatial 7-body central configurations (`P01`–`P20`) distributed across 9 achiral point groups ($O_h$, $D_{5h}$, $C_{6v}$, $D_{3h}$, $C_{5v}$, $C_{4v}$, $C_{3v}$, $C_{2v}$, $C_s$). Using a universal 14-degree-of-freedom rational interval Newton–Banach engine in Lean 4.36.0 (`Gauge14Cert.lean`), we formally prove (Theorem 3.1) that all 20 exist as exact solutions of all 21 Cartesian equations (1), are strictly non-coplanar, have injective $14 \times 14$ gauge Jacobians, and are pairwise non-similar across all $\binom{20}{2} = 190$ pairs modulo uniform scaling, arbitrary $O(3)$ isometries, translations, and arbitrary $S_7$ body relabelings. In addition, we formally prove the complete mass-perturbation continuation theorem directly inside Lean 4 (`Perturbation20475.lean`, Theorem 3.2), establishing that in an open neighborhood of $(1, \ldots, 1)$ every mass vector admits at least 20 unlabeled similarity classes, at least 20,475 labeled configurations modulo $O(3)$, and at least 40,950 oriented labeled configurations modulo $SO(3)$.

(2) **Exact $\mu$-parameterized centered-octahedron pitchfork curve (Section 4).** Adding a central mass $\mu \geq 0$ at the origin of the 6-body square bipyramid $(1, 1, 1, 1, m, m)$ produces a 7-body family $(1, 1, 1, 1, m, m, \mu)$ with exact $D_{4h}$ and $D_{2d}$ strata. Surprisingly, the transverse $D_{2d}$ balance equation $G_1(m, \zeta, z) = 0$ is *identically independent of* $\mu$. In Lean 4 (`CenteredOctahedronPitchfork.lean`), we prove global existence and uniqueness of the centered $D_{4h}$ branch for all $m > 0, \mu \geq 0$, global existence and uniqueness of the $D_{4h} \to D_{2d}$ pitchfork curve $(z^*(\mu), m^*(\mu))$ for all $\mu \geq 0$, strict monotonicity of both $z^*(\mu)$ and $m^*(\mu)$, universal closed-form lower bounds $1 < z^*(\mu)$ and $(4 - \sqrt{2})/6 < m^*(\mu)$, exact convergence $(z^*(\mu), m^*(\mu)) \to (1, (4 - \sqrt{2})/6)$ as $\mu \to \infty$, and a formal tight enclosure at equal central mass $\mu = 1$: $1.05548 < z^*(1) < 1.05549$ and $0.44432 < m^*(1) < 0.44444$ (Theorem 4.1).

(3) **Homotopy-free polynomial subsystem certification via `pandsysroots v0.34.0` (Section 5).** We formulate three 7-body spatial polynomial subsystems ($C_{5v}$ pentagon plus two axial poles, centered-$D_{2h}$ generalized octahedron, and centered-$D_{2d}$ octahedron) and run 12 homotopy-free benchmarks using `pandsysroots v0.34.0` (simultaneous sweeps of the Pandrosion rectangle step and the law of residues without homotopy). On three asymmetric centered-$D_{2h}$ mass triples we obtain *complete* 504/504 complex root certification (in 0.12–0.52 s on our reference Apple M5 workstation; wall-clock times depend on hardware, whereas root counts are deterministic); on $C_{5v}$ we certify the exact $1 \leftrightarrow 5$ saddle-node cliff across $m^*_{C_{5v}} \approx 1.1384268451$ (recovering `P01`, `P11`, and `P13` at $m = 1$); and on centered $D_{2d}$ at $\mu = 1$ we certify the $1 \to 3$ pitchfork transition across $m^*(1) \approx 0.4443752621$. Every certified real root satisfies all 21 Cartesian equations to $\leq 2.66 \times 10^{-15}$.

1.2. **Honest Scope Distinction.** We emphasize clearly what is formally proved versus what remains open:

- Theorems 3.1 and 3.2 are formally proved *lower bounds* ($\geq 20$ equal-mass spatial 7-body similarity classes, and in a neighborhood of equal masses $\geq 20$ unlabeled classes, $\geq 20{,}475$ unoriented labeled configurations modulo $O(3)$, and $\geq 40{,}950$ oriented labeled configurations modulo $SO(3)$). Unlike Moczurad and Zgliczynski's complete classifications for $N \leq 6$ [7], we do *not* prove a global upper bound in 14 dimensions: the assertion that there are *exactly* 20 equal-mass spatial 7-body classes remains a numerical conjecture (supported by our 157,000-start census hitting each of the 20 classes between 366 and 21,959 times with no 21st class detected).
- Theorem 4.1 is *globally complete* over the entire half-plane $(m, \mu) \in (0, \infty) \times [0, \infty)$ for the centered $D_{4h}$ branch and its transverse $D_{2d}$ pitchfork locus.
- Among the 12 `pandsysroots` benchmarks, three centered-$D_{2h}$ slices achieve complete 504/504 complex root certification against the affine total-degree bound; the remaining nine slices certify partial fractions of the affine bound because of solutions at infinity (e.g., 102/144 on $C_{5v}$ and 878–894/1200 on centered $D_{2d}$), and are reported strictly as certified lower counts rather than algebraic completeness proofs.

## 2. Newtonian 7-Body Model and Universal 14-DOF Gauge Reduction

Let $q : \{0, \ldots, 6\} \to \mathbb{R}^3$ denote seven body positions and $d_{ij}^2 = \|q_i - q_j\|^2$. We write $\mathrm{invCube}(r^2) = (r^2)^{-3/2}$ and denote the equal-mass Newtonian force on body $i$ by

$$g_i(q) = \sum_{j \neq i} (q_i - q_j)\, \mathrm{invCube}(d_{ij}^2) \in \mathbb{R}^3.$$

Two basic identities hold identically for any collision-free configuration $q$: vanishing total force $\sum_{i=0}^{6} g_i(q) = 0$ and vanishing total torque $\sum_{i=0}^{6} q_i \times g_i(q) = 0$, together with the virial positivity identity $\sum_{i=0}^{6} q_i \cdot g_i(q) = \sum_{i<j} \|q_i - q_j\|^{-1} > 0$.

**Definition 2.1** (Strict 3D Non-Coplanarity and Exact Similarity)**.** A 7-body configuration $q$ is *strictly* $3D$ (`Strictly3D7`) if there exist indices $a, b, c, d \in \{0, \ldots, 6\}$ such that the signed six-fold tetrahedron volume

$$\mathrm{tetDet}_6(q_a, q_b, q_c, q_d) = \det\big(q_b - q_a,\ q_c - q_a,\ q_d - q_a\big) \neq 0.$$

Two configurations $p, q : \{0, \ldots, 6\} \to \mathbb{R}^3$ are *similar* (`AreSimilar7`) if there exist a scale factor $s > 0$ and a permutation $\sigma \in S_7$ such that $\|p_{\sigma(i)} - p_{\sigma(j)}\|^2 = s\, \|q_i - q_j\|^2$ for all $i, j \in \{0, \ldots, 6\}$.

By elementary linear algebra (formalized in `NewtonianModel7.lean` as `not_coplanar_of_strictly3D7`), $\mathrm{tetDet}_6 \neq 0$ implies that no affine plane $n \cdot x = c$ ($n \neq 0$) contains all seven bodies. Moreover, `AreSimilar7` captures the full equivalence relation generated by uniform dilations, arbitrary orthogonal transformations $O(3)$ (both rotations and reflections), translations, and all $7! = 5040$ body permutations.

2.1. **Universal 14-DOF Gauge Reduction.** Modulo the 7-dimensional similarity group $\mathbb{R}_{>0} \times (SO(3) \ltimes \mathbb{R}^3)$, a generic spatial 7-body configuration has $21 - 7 = 14$ degrees of freedom. For $u = (u_0, \ldots, u_{13}) \in \mathbb{R}^{14}$, define the gauge-fixed positions $q(u) = \texttt{gauge14Pos}(u)$ by

$$\begin{aligned} q_0 &= (1, 0, 0), & q_1 &= (u_0, u_1, 0), \\ q_2 &= (u_2, u_3, u_4), & q_3 &= (u_5, u_6, u_7), \\ q_4 &= (u_8, u_9, u_{10}), & q_5 &= (u_{11}, u_{12}, u_{13}), \\ q_6 &= -(q_0 + q_1 + q_2 + q_3 + q_4 + q_5), \end{aligned}$$

so that the center of mass $\frac{1}{7} \sum_{i=0}^{6} q_i$ vanishes identically. Fix the multiplier by the $x$-component of body 0,

$$\lambda(u) = g_{0,x}(q(u)),$$

and define the 14-component reduced residual map $F : \mathbb{R}^{14} \to \mathbb{R}^{14}$ (`gauge14F`) by evaluating $r_{i,d}(u) = g_{i,d}(q(u)) - \lambda(u)\, q_{i,d}(u)$ on the 14 active coordinates

$$(i,d) \in \{(1,x),(1,y)\} \cup \big(\{2,3,4,5\} \times \{x,y,z\}\big).$$

**Theorem 2.2** (Universal 14-to-21 Reconstruction)**.** *(`gauge14_reconstruction_21eq`.) Let $u \in \mathbb{R}^{14}$ be a collision-free zero of $F$ ($F(u) = 0$) such that the two gauge-anchor determinants do not vanish:*

$$a_x := 1 - q_{6,x} \neq 0, \qquad \Delta_2 := (1 - q_{6,x})(u_1 - q_{6,y}) + q_{6,y}(u_0 - q_{6,x}) \neq 0. \tag{2}$$

*Then $\lambda(u) > 0$ and $q(u)$ satisfies all 21 Cartesian Newtonian central-configuration equations (1) with equal masses $m_i = 1$.*

*Proof.* By construction, $r_{0,x} = 0$, $r_{1,x} = r_{1,y} = 0$, and $r_{i,d} = 0$ for $i \in \{2,3,4,5\}$ and all $d \in \{x,y,z\}$. Since $\sum_{i=0}^{6} g_i = 0$ and $\sum_{i=0}^{6} q_i = 0$, we have $\sum_{i=0}^{6} r_{i,d} = 0$, hence $r_{6,d} = -(r_{0,d} + r_{1,d})$ for each $d \in \{x,y,z\}$. Vanishing total torque of $g$ implies $\sum_{i=0}^{6}(q_{i,a} r_{i,b} - q_{i,b} r_{i,a}) = 0$ for all $a, b \in \{x,y,z\}$. Evaluating at $(a,b) = (x,y)$ collapses all terms except bodies 0 and 6, yielding $(1 - q_{6,x})\, r_{0,y} = 0$, so $r_{0,y} = 0$ by $a_x \neq 0$. Evaluating at $(x,z)$ and $(y,z)$ gives a $2 \times 2$ homogeneous linear system for $(r_{0,z}, r_{1,z})$ whose determinant is precisely $\Delta_2$:

$$\begin{aligned} (1 - q_{6,x})\, r_{0,z} + (u_0 - q_{6,x})\, r_{1,z} &= 0, \\ -q_{6,y}\, r_{0,z} + (u_1 - q_{6,y})\, r_{1,z} &= 0. \end{aligned}$$

Since $\Delta_2 \neq 0$, we conclude $r_{0,z} = r_{1,z} = 0$, and therefore $r_{6,d} = -(r_{0,d} + r_{1,d}) = 0$ for all $d$. Finally, the virial identity yields $0 < \sum_i q_i \cdot g_i = \lambda(u) \sum_i \|q_i\|^2$, and $\|q_0\|^2 = 1$ gives $\sum_i \|q_i\|^2 \geq 1$, forcing $\lambda(u) > 0$. □

## 3. Twenty Machine-Certified Equal-Mass Spatial 7-Body Polyhedra

3.1. **Numerical Census Protocol.** To locate candidate equal-mass spatial 7-body central configurations prior to formal certification, we implemented a multi-start damped Newton–Raphson solver in Rust on the 14-DOF gauge map $F$ (together with symmetry-reduced searches in $C_s$, $C_2$, and $C_i$ subspaces). As a benchmark check, running the identical pipeline on $N = 6$ equal masses recovered all 9/9 spatial similarity classes of Moczurad and Zgliczynski [7] in 0.5 s. Across 157,000 independent starts for $N = 7$ equal masses (115,000 unconstrained 14-DOF and symmetry-seeded starts plus 42,000 exact $C_s/C_2/C_i$-reduced Newton starts), we discovered exactly 20 distinct, strictly 3D similarity classes (`P01`–`P20`), every single one hit between 366 and 21,959 times. Exact full-sweep cyclic-Jacobi diagonalization of the $21 \times 21$ augmented potential Hessian confirmed that every one of the 20 polyhedra has exactly 3 zero eigenvalues (pure $SO(3)$ rotations) and minimum nonzero eigenvalue $|\mu_4| \geq 2.24 \times 10^{-2}$ (strong nondegeneracy). Polishing every root to 260 decimal digits in `mpmath` (`verify_250digits.py`) gave maximum 21-equation Cartesian residual $\leq 3.66 \times 10^{-260}$ and minimum pairwise normalized 21-distance spectrum gap 0.052798666 across all $\binom{20}{2} = 190$ pairs.

3.2. **Lean 4.36.0 Full-3D 14-DOF Certification Engine.** To convert these 20 numerical discoveries into kernel-verified mathematical theorems without trusting floating-point arithmetic, we built a reusable 14-DOF certification engine in Lean 4.36.0 (`BanachEngine.lean` and `Gauge14Cert.lean`):

(1) **Affine Jacobian factorization.** Every coordinate $q_{i,d}(u)$ of `gauge14Pos` is affine in $u$ with exact integer sensitivity matrix $\partial q_{i,d}/\partial u_k \in \{-1, 0, 1\}$. This yields a compact exact analytical $14 \times 14$ Jacobian $J(u) =$ `gauge14Jac`$(u)$ and a 20-line directional-derivative proof (`gauge14_hasDerivAt_seg`) along every collision-free segment in $\mathbb{R}^{14}$.

(2) **Sound rational interval arithmetic with pre-rounded containment (`subIvl`).** Naive $14 \times 14$ rational interval matrix multiplication in Lean's kernel would cause severe denominator growth. Each certificate `Gauge14Cert` therefore stores small-denominator rational matrices $A, B_0 \in \mathbb{Q}^{14 \times 14}$ together with pre-rounded interval enclosures `F0B` $\in$ QIvl$^{14}$, `JB` $\in$ QIvl$^{14 \times 14}$, and `ndB` $\in$ QIvl$^{21}$. The decidable validator `Gauge14Cert.valid`

verifies: (a) polynomial enclosures for all 21 values of invCube($r_{ij}^2$) at both the center $x_0$ and across the box $B_\infty(x_0, \delta)$; (b) strict nonvanishing of $a_x$, $\Delta_2$, and $\mathrm{tetDet}_6$ over the entire box; (c) sound containment `rawF0Ivl` $\subseteq$ `F0B`, `rawJacIvl` $\subseteq$ `JB`, and `normDistIvls7` $\subseteq$ `ndB`; (d) the Newton step bound $\|A\,\mathtt{F0B}\|_\infty \le \delta/2$; (e) the contraction bound $\|I - A\,\mathtt{JB}\|_\infty \le 1/2$; and (f) the non-singularity certificate $\|I - B_0 A\|_\infty \le 1/2$.

(3) **Simultaneous existence and 14D nondegeneracy.** Applying `banach_krawczyk_fin` produces a unique fixed point $u^* \in B_\infty(x_0, \delta)$ with $F(u^*) = 0$. Crucially, the same bound $\|I - AJ(u^*)\|_\infty \le 1/2$ immediately implies injectivity of $J(u^*)$ ($J(u^*)v = 0 \implies \|v\|_\infty = \|(I - AJ(u^*))v\|_\infty \le \frac{1}{2}\|v\|_\infty \implies v = 0$), formally certifying full 14-dimensional nondegeneracy in a single step!

(4) **Scale-, $O(3)$-, translation-, and $S_7$-invariant cut witnesses.** Under `AreSimilar7`, the count of ordered pairs $(i, j) \in \{0, \dots, 6\}^2$ whose normalized squared distance $2d_{ij}^2 / \sum_{a,b} d_{ab}^2$ lies strictly below a rational threshold $t > 0$ is an exact invariant (`areSimilar7_card_lt_eq`). Storing one small-denominator rational cut $t_{ij} \in \mathbb{Q}_{>0}$ for each of the 190 pairs ($i < j$) lets `not_areSimilar7_of_cut` certify all 190 non-similarity inequalities via a single `decide +kernel` check.

**Theorem 3.1** (Twenty Equal-Mass Spatial 7-Body Central Configurations)**.** *There exist* 20 *configurations* $u^{(1)}, \dots, u^{(20)} \in \mathbb{R}^{14}$ *(certified in* `TwentyEqualMassCertified.lean`*) such that, setting* $q^{(k)} = \mathtt{gauge14Pos}(u^{(k)})$ *and* $\lambda^{(k)} = \mathtt{gauge14Lam}(u^{(k)})$*:*

(1) *For every* $k \in \{1, \dots, 20\}$*,* $(q^{(k)}, \lambda^{(k)})$ *satisfies all* 21 *Cartesian equal-mass Newtonian central-configuration equations* (1) *with* $\lambda^{(k)} > 0$*.*

(2) *For every* $k \in \{1, \dots, 20\}$*,* $q^{(k)}$ *is strictly three-dimensional (*`Strictly3D7`*), hence non-coplanar.*

(3) *For every* $k \in \{1, \dots, 20\}$*, the* $14 \times 14$ *gauge Jacobian* $\mathtt{gauge14Jac}(u^{(k)})$ *is injective* ($J(u^{(k)})v = 0 \implies v = 0$*).*

(4) *For all* $1 \le i < j \le 20$*,* $q^{(i)}$ *and* $q^{(j)}$ *are non-similar (*$\neg$ `AreSimilar7`*) modulo uniform scaling, arbitrary* $O(3)$ *rotations and reflections, translations, and arbitrary* $S_7$ *body relabelings.*

**Theorem 3.2** (Formally Verified Perturbation Lower Bounds Near Equal Masses)**.** *(Formalized in* `Perturbation20475.lean`*.) There exists* $\eta > 0$ *such that for* every *mass vector* $m = (m_0, \dots, m_6) \in \mathbb{R}^7$ *satisfying* $|m_i - 1| < \eta$ *for all* $i$*:*

(1) *(*`perturbation_20_non_similar_near_equal_mass`*) There exist at least* 20 *pairwise non-similar (modulo scaling,* $O(3)$*, translations, and* $S_7$*), strictly* $3D$*,* $14D$*-nondegenerate spatial* 7*-body central configurations for masses* $m$*.*

(2) *(*`perturbation_20475_labeled_O3_near_equal_mass`*) There exist at least*

$$\sum_{k=1}^{20} \frac{7!}{|G_k|} = 20{,}475$$

*strictly* $3D$ *spatial* 7*-body central configurations solving all* 21 *weighted Cartesian equations* (1) *for masses* $m$*, pairwise non-similar modulo uniform scaling, arbitrary* $O(3)$ *isometries, and translations with fixed body labels (*`AreLabeledSimilarO3`*).*

(3) *(*`perturbation_40950_labeled_SO3_near_equal_mass`*) There exist at least*

$$\sum_{k=1}^{20} \frac{7!}{|G_k \cap SO(3)|} = 40{,}950$$

*strictly* $3D$ *spatial* 7*-body central configurations solving* (1) *for masses* $m$*, pairwise non-similar modulo uniform scaling, proper rotations* $SO(3)$*, and translations with fixed body labels (*`AreLabeledSimilarSO3`*).*

*Architecture of the Formal Lean 4 Proof.* Rather than appealing to the implicit function theorem informally in prose, `Perturbation20475.lean` verifies every step inside Lean 4's kernel: (i) `areLabeledSimilarO3_of_rigid` and `areLabeledSimilarSO3_of_rigid` prove that the distance-ratio and signed-tetrahedron-volume predicates faithfully capture rigid $O(3)$ and $SO(3)$ similarity

TABLE 1. The 20 machine-certified equal-mass spatial 7-body central configurations (`P01`–`P20`), ordered by scale-invariant normalized potential $U\sqrt{I}$ (where $U = \sum_{i<j} d_{ij}^{-1}$ and $I = \sum_i \|q_i - c\|^2$). Columns give point group $G_k$, order $|G_k|$ in $O(3)$, Morse index $m$ of $U\sqrt{I}$ (computed from the $21 \times 21$ Hessian with exactly 3 rotational zero modes), Lean 4 certification box radius $\delta$, 250-digit 21-equation residual, and geometric description. Asterisks mark stacked $(6+1)$ configurations.

| ID | $G_k$ | $\lvert G_k\rvert$ | $m$ | $U\sqrt{I}$ | $\delta$ (Lean 4) | Geometric description |
|---|---|---|---|---|---|---|
| `P01` | $D_{5h}$ | 20 | 0 | 38.2355925634 | $10^{-5}$ | Pentagonal bipyramid (global min) |
| `P02` | $C_{2v}$ | 4 | 1 | 38.2515985812 | $10^{-6}$ | Capped trigonal prism (index-1 saddle) |
| `P03` | $C_{3v}$ | 6 | 2 | 38.2534483884 | $10^{-6}$ | Capped octahedron ($C_{3v}$) |
| `P04` | $C_{2v}$ | 4 | 2 | 38.8816105974 | $10^{-5}$ | Deformed centered octahedron ($C_{2v}$) |
| `P05` | $C_{3v}$ | 6 | 3 | 38.8950387216 | $10^{-5}$ | Off-center staggered triangles + pole |
| `P06` | $C_{4v}$ | 8 | 2 | 39.1506797828 | $10^{-5}$ | Square + 3 axial poles (off-center) |
| `P07`* | $O_h$ | 48 | 3 | 39.1557827617 | $10^{-5}$ | Centered regular octahedron (stacked $6+1$) |
| `P08`* | $D_{3h}$ | 12 | 3 | 39.4266616869 | $10^{-5}$ | Centered triangular prism (stacked $6+1$) |
| `P09` | $C_s$ | 2 | 3 | 39.4452088496 | $10^{-5}$ | Mirror-symmetric 7-polyhedron I |
| `P10` | $C_{3v}$ | 6 | 4 | 39.4472829541 | $10^{-5}$ | Aligned coaxial triangles + pole I |
| `P11` | $C_{5v}$ | 10 | 4 | 39.7675448620 | $10^{-5}$ | Nested pentagonal pyramids I |
| `P12`* | $C_{6v}$ | 12 | 3 | 39.7917870909 | $10^{-5}$ | Regular hexagonal pyramid (stacked $6+1$) |
| `P13` | $C_{5v}$ | 10 | 3 | 39.8076537558 | $10^{-5}$ | Nested pentagonal pyramids II |
| `P14` | $C_s$ | 2 | 4 | 39.9643719919 | $10^{-5}$ | Mirror-symmetric 7-polyhedron II |
| `P15` | $C_{2v}$ | 4 | 4 | 41.1408395443 | $10^{-5}$ | Two orthogonal pairs + 3 axial poles I |
| `P16` | $D_{3h}$ | 12 | 4 | 41.2195859674 | $10^{-6}$ | Equilateral triangle + 4 symmetric poles |
| `P17` | $C_{2v}$ | 4 | 5 | 41.2240902506 | $10^{-6}$ | Two orthogonal pairs + 3 axial poles II |
| `P18` | $C_s$ | 2 | 5 | 41.5153700886 | $10^{-5}$ | Mirror-symmetric 7-polyhedron III |
| `P19` | $C_{3v}$ | 6 | 6 | 43.2723806872 | $10^{-5}$ | Aligned coaxial triangles + pole II |
| `P20` | $C_{2v}$ | 4 | 6 | 44.8556455358 | $10^{-5}$ | Elongated $C_{2v}$ 7-polyhedron |

actions; (ii) `gauge14_reconstruction_21eq_M` generalizes the 14-to-21 reconstruction to arbitrary positive masses $m$, and `continue_cert_near_equal_mass` applies a (3/4)-contractive Banach–Krawczyk fixed-point argument centered at each equal-mass root $u^{(k)}$ (using continuity of $F_m$ and $J_m$ together with the already-certified equal-mass contraction $\|I - AJ(u^{(k)})\|_\infty \le 1/2 < 3/4$) to continue all 20 branches to every $m$ with $\|m - \mathbf{1}\|_\infty < \eta$; (iii) for any $\sigma \in S_7$, applying branch $k$ to the permuted mass vector $m \circ \sigma^{-1}$ and relabeling by $\sigma$ produces a solution for the original mass vector $m$; (iv) a fast backtracking enumerator checked by `decide +kernel` (6,724 interval comparisons across all 20 polyhedra) proves that the already-certified `ndB` intervals permit at most $|G_k|$ distance-compatible permutations in $S_7$ (`all20_stab_O3_bounds`), and checking the interval sign of `tetDet6Ivl` on those 176 permutations (`all20_stab_SO3_bounds`, zero ambiguous cases) proves that at most $|G_k|/2$ preserve orientation; (v) an elementary greedy independent-set bound on $S_7$ combined with shifted pairwise cuts (`cutShift`) separating distinct branches yields the exact sums 20,475 (mod $O(3)$) and 40,950 (mod $SO(3)$). □

*Remark* 3.3 (Achirality of All Observed Equal-Mass 7-Body Polyhedra). Every single one of the 20 certified polyhedra in Table 1 is achiral, possessing at least one mirror reflection plane ($C_s \subseteq G_k$). In particular, the point-group counts are $1\,O_h$, $1\,D_{5h}$, $1\,C_{6v}$, $2\,D_{3h}$, $2\,C_{5v}$, $1\,C_{4v}$, $4\,C_{3v}$, $5\,C_{2v}$, and $3\,C_s$. The global minimum of $U\sqrt{I}$ is the $D_{5h}$ pentagonal bipyramid `P01` ($U\sqrt{I} \approx 38.2355925634$), and the lowest transition state (Morse index 1) is the $C_{2v}$ capped trigonal prism `P02` ($U\sqrt{I} \approx 38.2515985812$). Whether any chiral equal-mass spatial 7-body central configuration exists remains an open question.

## 4. Exact Centered-Octahedron $D_{4h} \to D_{2d}$ Pitchfork Curve

Consider seven point masses $(1, 1, 1, 1, m, m, \mu)$ with $m > 0$ and $\mu \geq 0$ placed in the centered $D_{2d}$ ansatz

$$q_0, q_1 = (\pm 1, 0, \zeta), \qquad q_2, q_3 = (0, \pm 1, -\zeta),$$
$$q_4, q_5 = (0, 0, \pm z), \qquad q_6 = (0, 0, 0),$$

with $z > 0$ and $\zeta \in \mathbb{R}$. The plane $\zeta = 0$ is the centered $D_{4h}$ square bipyramid stratum; when $\mu = 0$, body 6 decouples and the system reduces to the 6-body octahedral family. Direct evaluation of the 21 Cartesian equations (`centeredD2d_reconstruction_21eq` in `NewtonianModel7.lean`) shows that $(q, \lambda)$ with

$$\lambda = \frac{1}{4} + 2(2 + 4\zeta^2)^{-3/2} + \mu(1 + \zeta^2)^{-3/2} + m\big((1 + (\zeta - z)^2)^{-3/2} + (1 + (\zeta + z)^2)^{-3/2}\big)$$

solves all 21 Cartesian Newtonian equations whenever two scalar equations hold:

(3)
$$G_1(m, \zeta, z) := 2\zeta(2 + 4\zeta^2)^{-3/2} - \frac{\zeta}{4} + mz\big((1 + (\zeta + z)^2)^{-3/2} - (1 + (\zeta - z)^2)^{-3/2}\big) = 0,$$

(4)
$$G_2(m, \mu, \zeta, z) := \frac{m/4 + \mu}{z^2} + 2\big((z - \zeta)(1 + (\zeta - z)^2)^{-3/2} + (z + \zeta)(1 + (\zeta + z)^2)^{-3/2}\big) - \lambda z = 0.$$

Remarkably, the central mass $\mu$ cancels identically from the transverse force equation (3): the term $\mu\zeta(1 + \zeta^2)^{-3/2}$ coming from the central body matches the $\mu$-contribution in $\lambda\zeta$!

Along the centered $D_{4h}$ stratum $\zeta = 0$, $G_1(m, 0, z) \equiv 0$ identically, while $G_2(m, \mu, 0, z) = 0$ is equivalent to $E(m, \mu, z) = 0$, where $(\alpha := \frac{1}{4} + \frac{1}{\sqrt{2}})$

$$E(m, \mu, z) := (\alpha + \mu) + 2m(1 + z^2)^{-3/2} - \left(\frac{m}{4z^3} + \frac{\mu}{z^3} + 4(1 + z^2)^{-3/2}\right) = 0.$$

Multiplying by $(1 + z^2)^{3/2}$ solves explicitly for $m$ as a rational-power function of $(z, \mu)$:

$$m(z, \mu) = \frac{N(\mu, z)}{D(z)} := \frac{4 - \alpha(1 + z^2)^{3/2} + \mu\big((1 + z^{-2})^{3/2} - (1 + z^2)^{3/2}\big)}{2 - \frac{1}{4}(1 + z^{-2})^{3/2}}. \tag{5}$$

Notice that at $z = 1$, $(1 + z^{-2})^{3/2} - (1 + z^2)^{3/2} = 0$, so $m(1, \mu) = 1$ for *every* $\mu \geq 0$, corresponding to the centered regular octahedron (`P07` at $\mu = 1$). Differentiating (3) at $\zeta = 0$ yields the exact $\mu$-independent transverse stability derivative

$$\Phi(m, z) := \left.\frac{\partial G_1}{\partial \zeta}\right|_{\zeta=0} = \left(\frac{1}{\sqrt{2}} - \frac{1}{4}\right) - 6mz^2(1 + z^2)^{-5/2}.$$

**Theorem 4.1** (Global Centered-$D_{4h}$ Branch and Exact Pitchfork Curve)**.** *For the centered 7-body family $(1, 1, 1, 1, m, m, \mu)$ with $m > 0$ and $\mu \geq 0$ (formalized in `CenteredOctahedron Pitchfork.lean`):*

1. ***Regular centered octahedron:*** *For every $\mu \geq 0$, $(m = 1, \zeta = 0, z = 1)$ satisfies all 21 Cartesian Newtonian central-configuration equations (`centered_regular_octahedron_is_cc`).*
2. ***Global $D_{4h}$ existence and uniqueness:*** *For every $\mu \geq 0$, there is a unique upper height $z_{\max}(\mu) \in (1, z_{\max}(0)]$ (where $z_{\max}(0) = \sqrt{(4/\alpha)^{2/3} - 1} \approx 1.262765$) such that $N(\mu, z_{\max}(\mu)) = 0$. For every $m > 0$, there is a unique $z > 0$ solving $E(m, \mu, z) = 0$, lying strictly inside $(1/\sqrt{3}, z_{\max}(\mu))$ (`centered_d4h_exists_unique`).*
3. ***Global pitchfork existence and uniqueness:*** *For every $\mu \geq 0$, there exists a unique pair $(z^*(\mu), m^*(\mu)) \in (0, \infty)^2$ solving $E(m^*(\mu), \mu, z^*(\mu)) = 0$ and $\Phi(m^*(\mu), z^*(\mu)) = 0$ (`centered_pitchfork_exists_unique`).*

TABLE 2. 250-digit values along the exact centered-octahedron $D_{4h} \to D_{2d}$ pitchfork curve $(z^*(\mu), m^*(\mu))$ on $(1,1,1,1,m,m,\mu)$, verified by `verify_250digits.py` to residual $< 1.2 \times 10^{-259}$.

| $\mu$ | $z^*(\mu)$ | $m^*(\mu)$ | 250-digit residual |
|---|---|---|---|
| 0 | 1.13979097617105981423… | 0.47002582667566173409… | $1.27 \times 10^{-261}$ |
| 0.25 | 1.10255396834226766680… | 0.45793882606020191351… | $5.08 \times 10^{-262}$ |
| 0.5 | 1.08027272476639643395… | 0.45127517036691126228… | $1.27 \times 10^{-261}$ |
| 1 | 1.05548569592542351761… | 0.44437526212943942714… | $3.81 \times 10^{-261}$ |
| 2 | 1.03403948614190857002… | 0.43885201993014512348… | $1.78 \times 10^{-261}$ |
| 5 | 1.01566846992415537986… | 0.43445880055390744277… | $4.32 \times 10^{-261}$ |
| 100 | 1.00086252626186077734… | 0.43115062586869801290… | $1.18 \times 10^{-259}$ |
| $\infty$ | 1 (exact) | $(4-\sqrt{2})/6 \approx 0.4309644062711508\ldots$ | exact limit |

(4) ***Strict monotonicity, universal lower bounds, and exact $\mu \to \infty$ limit:*** *Both $\mu \mapsto z^*(\mu)$ and $\mu \mapsto m^*(\mu)$ are strictly decreasing on $[0,\infty)$, satisfy the universal closed-form bounds*

$$1 < z^*(\mu) \le z^*(0) \approx 1.1397909762, \qquad \frac{4-\sqrt{2}}{6} < m^*(\mu) \le m^*(0) \approx 0.4700258267 \tag{6}$$

*for every $\mu \ge 0$, and satisfy the exact asymptotic limit*

$$\lim_{\mu\to\infty} z^*(\mu) = 1, \qquad \lim_{\mu\to\infty} m^*(\mu) = \frac{4-\sqrt{2}}{6} \approx 0.4309644062711508$$

*(`centered_pitchfork_strictAnti_and_bounds`).*

(5) ***Equal-central-mass enclosure ($\mu = 1$):*** *At $\mu = 1$,*

$$\frac{105548}{100000} < z^*(1) < \frac{105549}{100000}, \qquad \frac{44432}{100000} < m^*(1) < \frac{44444}{100000},$$

*and the transverse eigenvalue $\Phi(m(z,1),z)$ is strictly negative for $z \in (1/\sqrt{3}, z^*(1))$ and strictly positive for $z \in (z^*(1), z_{\max}(1))$ (`centered_pitchfork_mu1_enclosure`).*

*Sketch of the Formal Proof.* For every $\mu \ge 0$, $N(\mu,z)$ is strictly decreasing in $z > 0$ with $N(\mu,1) > 0$ and $N(\mu, z_{\max}(0)) \le 0$, while $D(z)$ is strictly increasing with unique zero $1/\sqrt{3}$. Hence $m(z,\mu)$ is strictly decreasing on $(1/\sqrt{3}, z_{\max}(\mu))$ from $+\infty$ to 0, proving (2). On $(1/\sqrt{3}, 1]$, $m(z,\mu) \ge 1$ and $6z^2(1+z^2)^{-5/2} > 1/2 > 1/\sqrt{2} - 1/4$, forcing $\Phi(m(z,\mu),z) < 0$. On $[1, z_{\max}(\mu))$, both $m(z,\mu) > 0$ and the bump function $b(z) = z^2(1+z^2)^{-5/2}$ are strictly decreasing (verified for $b$ by exact polynomial factorization of $a^4(1+b^2)^5 - b^4(1+a^2)^5$), so $\Phi(m(z,\mu),z)$ is strictly increasing from $\Phi(1,1) < 0$ to $\frac{1}{\sqrt{2}} - \frac{1}{4} > 0$, proving (3). For $z > 1$, $(1+z^{-2})^{3/2} - (1+z^2)^{3/2} < 0$, so $m(z,\mu)$ is strictly decreasing in $\mu$ and $\Phi(m(z,\mu),z)$ is strictly increasing in $\mu$, forcing $z^*(\mu)$ to be strictly decreasing in $\mu$. From $\Phi(m^*(\mu), z^*(\mu)) = 0$ we obtain

$$m^*(\mu) = \frac{1/\sqrt{2} - 1/4}{6\, b(z^*(\mu))}.$$

Because $b$ is strictly decreasing on $[1,\infty)$ with $b(1) = 2^{-5/2} = (4\sqrt{2})^{-1}$, $m^*(\mu)$ is strictly decreasing in $\mu$ and strictly exceeds $(1/\sqrt{2} - 1/4)/(6\,b(1)) = (4-\sqrt{2})/6$. Finally, for any $\varepsilon > 0$ and $z_\varepsilon = 1 + \varepsilon/2 > 1$, taking $\mu > 4/((1+z_\varepsilon^2)^{3/2} - (1+z_\varepsilon^{-2})^{3/2})$ forces $N(\mu, z_\varepsilon) < 0$, hence $1 < z^*(\mu) < z_{\max}(\mu) < 1 + \varepsilon/2$, establishing $\lim_{\mu\to\infty} z^*(\mu) = 1$ and by continuity $\lim_{\mu\to\infty} m^*(\mu) = (4-\sqrt{2})/6$. Part (5) follows from elementary rational square-root bounds at $z = 1.05548$ and $z = 1.05549$. □

## 5. Homotopy-Free Polynomial Certification via `pandsysroots v0.34.0`

To complement the Lean 4 theorems with global complex polynomial root certification across symmetric 7-body slices, we formulate three algebraic 7-body subsystems in Rust using `pandsysroots v0.34.0` (`src/main.rs`). Unlike homotopy continuation, `pandsysroots` locates

TABLE 3. All 12 homotopy-free `pandsysroots v0.34.0` benchmarks in `src/main.rs`. Boldface marks complete complex root certification (504/504). Every certified real solution satisfies all 21 Cartesian Newtonian equations with $\max|r| \le 2.66 \times 10^{-15}$. Wall-clock times are measured in `--release` mode on an Apple M5 (10-core CPU, `rustc 1.100.0`); runtimes depend on hardware, whereas all certification counts are deterministic.

| Family | Mass slice | Complex cert / bound | Real CC | Time (s) | Notes |
|---|---|---|---|---|---|
| $C_{5v}$ (4v) | $(m_5, m_6) = (1.15, 1.15)$ | 102/144 | 1 | 3.21 | Above cliff $m^*_{C_{5v}}$ ($D_{5h}$ only) |
| $C_{5v}$ (4v) | $(m_5, m_6) = (1.13, 1.13)$ | 102/144 | 5 | 2.15 | Below cliff ($1\,D_{5h} + 4\,C_{5v}$) |
| $C_{5v}$ (4v) | $(m_5, m_6) = (1.00, 1.00)$ | 102/144 | 5 | 1.68 | Recovers `P01`, `P11`, `P13` |
| $C_{5v}$ (4v) | $(m_5, m_6) = (0.05, 0.05)$ | 102/144 | 5 | 1.36 | Light axial poles |
| $C_{5v}$ (4v) | $(m_5, m_6) = (0.50, 0.20)$ | 102/144 | 5 | 1.64 | Asymmetric $C_{5v}$ (5 distinct) |
| $D_{2h}$ (6v) | $(1, 1, 1;\ \mu = 1.0)$ | 288/360 | 1 | 1.22 | Centered octahedron `P07` |
| $D_{2h}$ (6v) | $(1, 2, 5;\ \mu = 1.0)$ | **504 / 504** | 1 | 0.52 | **Complete complex cert.** |
| $D_{2h}$ (6v) | $(1, 0.3, 0.05;\ \mu = 0.5)$ | **504 / 504** | 1 | 0.12 | **Complete complex cert.** |
| $D_{2h}$ (6v) | $(1, 0.3, 0.05;\ \mu = 5.0)$ | **504 / 504** | 1 | 0.12 | **Complete complex cert.** |
| $D_{2d}$ (6v) | $m = 1.00,\ \mu = 1.0$ | 886/1200 | 1 | 2.14 | Equal mass `P07` ($O_h$) |
| $D_{2d}$ (6v) | $m = 0.46,\ \mu = 1.0$ | 878/1200 | 1 | 2.55 | Above $m^*(1) \approx 0.444375$ |
| $D_{2d}$ (6v) | $m = 0.43,\ \mu = 1.0$ | 894/1200 | 3 | 3.00 | Below $m^*(1)$ ($1\,D_{4h} + 2\,D_{2d}$) |

and certifies polynomial system roots directly via simultaneous sweeps of the Pandrosion rectangle step and the law of residues without homotopy. Every real solution returned by the solver is independently verified directly against all 21 Cartesian Newtonian equations (1).

(1) **Pentagon + 2 axial poles ($C_{5v}$, 4 polynomial variables).** Five unit-mass bodies form a regular unit pentagon in $z = 0$, and two axial bodies of masses $m_5, m_6 > 0$ sit at $(0, 0, z_5)$ and $(0, 0, z_6)$. Introducing inverse-distance variables $s_5 = (1 + z_5^2)^{-1/2}$ and $s_6 = (1 + z_6^2)^{-1/2}$ yields a 4-variable polynomial system with affine total-degree bound 144. For $m_5 = m_6 = m$, we locate the exact $C_{5v}$ saddle-node cliff at

$$m^*_{C_{5v}} = 1.13842684511857884161\ldots$$

(verified in `verify_250digits.py` to residual $9.80 \times 10^{-216}$): for $m > m^*_{C_{5v}}$ there is 1 real solution (the $D_{5h}$ pentagonal bipyramid), whereas for $0 < m < m^*_{C_{5v}}$ two pairs of $C_{5v}$ nested pentagonal pyramids bifurcate into existence, yielding 5 labeled real solutions (3 non-congruent shapes; at $m = 1$ these are precisely `P01`, `P11`, and `P13`).

(2) **Centered generalized octahedron ($D_{2h}$, 6 polynomial variables).** Three antipodal pairs of masses $(m_x, m_y, m_z)$ sit at $(\pm 1, 0, 0)$, $(0, \pm y, 0)$, $(0, 0, \pm z)$ together with a central mass $\mu > 0$ at $(0, 0, 0)$. Clearing radicals via the three mutual distances $(r_{xy}, r_{xz}, r_{yz})$ gives a 6-variable polynomial system. On three asymmetric mass triples ($(1, 2, 5; \mu = 1)$, $(1, 0.3, 0.05; \mu = 0.5)$, and $(1, 0.3, 0.05; \mu = 5.0)$), `pandsysroots` achieves *complete* 504/504 complex root certification (in 0.12–0.52 s on our Apple M5 reference workstation; timings vary with hardware), proving rigorously that only the unique real centered-$D_{2h}$ configuration exists on each slice.

(3) **Centered $D_{2d}$ octahedron (6 polynomial variables, $\mu = 1$).** Rationalizing (3)–(4) with four inverse-distance variables gives a 6-variable polynomial system (affine bound 1200). Across $m = 1.00$, $m = 0.46$, and $m = 0.43$, `pandsysroots` directly certifies the $1 \to 3$ pitchfork transition across $m^*(1) \approx 0.4443752621$ proved in Theorem 4.1.

## 6. Formal Verification, Axiom Audit, and Reproducibility

The accompanying repository provides four fully automated, independent verification tiers executed on every commit via GitHub Actions:

(1) **Lean 4.36.0 + Mathlib4 formal library (`lean/Spatial7/`):** `NewtonianModel7.lean` (21-equation model, 14-to-21 reconstruction, `Strictly3D7`, `AreSimilar7`, centered-$D_{2d}$ reconstruction), `BanachEngine.lean` (dimension-independent Newton–Banach

contraction and Jacobian injectivity), `CenteredOctahedronPitchfork.lean` (Theorem 4.1), `Gauge14Cert.lean` (sound `QIvl`, analytical $14 \times 14$ Jacobian, `certify_gauge14_root`, `not_areSimilar7_of_cut`), `TwentyEqualMassCertified.lean` (Theorem 3.1: all 20 certificates and all 190 pairwise cuts checked purely by `decide +kernel`), `Perturbation20475.lean` (Theorem 3.2: rigid-action faithfulness, weighted 14-to-21 reconstruction, Banach–Krawczyk mass continuation, kernel-verified stabilizer upper bounds, and the 20,475 mod$O(3)$ / 40,950 mod$SO(3)$ perturbation theorems), and `Audit.lean` (`#print axioms` over all 19 top-level theorems). A comment-stripped static check confirms zero occurrences of `sorry`, `admit`, `native_decide`, or custom `axiom` declarations, and `Audit.lean` confirms that all 19 theorems depend solely on `[propext, Classical.choice, Quot.sound]`.

(2) **Rust `pandsysroots v0.34.0` suite (`src/main.rs`):** executes all 12 homotopy-free benchmarks in Table 3 and verifies all 21 Cartesian Newtonian equations.

(3) **Independent 250-digit `mpmath` verifier (`verify_250digits.py`):** polishes and verifies all 20 equal-mass polyhedra ($< 3.66 \times 10^{-260}$), all 7 slices of the pitchfork curve ($< 1.2 \times 10^{-259}$), and the $C_{5v}$ saddle-node cliff ($< 10^{-215}$).

(4) **Pure-7-bit-ASCII LaTeX preprint build:** verifies byte-level 7-bit ASCII compliance and compiles this manuscript with `tectonic`.

## References


[1] A. Albouy and V. Kaloshin, *Finiteness of central configurations of five bodies in the plane*, Ann. of Math. (2) **176** (2012), no. 1, 535–588.

[2] A. C. Fernandes and L. F. Mello, *Relative equilibria in the spatial n-body problem: stacked central configurations*, Discrete Contin. Dyn. Syst. **34** (2014), no. 10, 4059–4074.

[3] M. Hampton and R. Moeckel, *Finiteness of relative equilibria of the four-body problem*, Invent. Math. **163** (2006), no. 2, 289–312.

[4] M. Hampton and M. Santoprete, *Seven-body central configurations: a family of central configurations in the spatial seven-body problem*, Celestial Mech. Dynam. Astronom. **99** (2007), no. 4, 293–305 (arXiv:0909.4887).

[5] L. F. Mello and A. C. Fernandes, *Co-circular and co-spherical central configurations in the n-body problem*, J. Math. Anal. Appl. **376** (2011), 243–255.

[6] M. Moczurad and P. Zgliczynski, *Central configurations in planar n-body problem with equal masses for* $n = 5, 6, 7$, Celestial Mech. Dynam. Astronom. **131** (2019), Article 46 (arXiv:1812.07279).

[7] M. Moczurad and P. Zgliczynski, *Central configurations in spatial n-body problems for* $n = 5, 6$ *with equal masses*, Celestial Mech. Dynam. Astronom. **132** (2020), Article 56 (arXiv:2006.06768).

[8] R. Moeckel, *Central configurations*, in *Central Configurations, Periodic Orbits, and Hamiltonian Systems*, Advanced Courses in Mathematics – CRM Barcelona, Birkhauser, 2015, pp. 105–167.

[9] F. R. Moulton, *The straight line solutions of the problem of n bodies*, Ann. of Math. (2) **12** (1910), no. 1, 1–17.

[10] A. Oliveira, *Central configurations of the spatial seven-body problem: two twisted triangles plus one*, preprint, arXiv:1506.04666 (2015).

[11] D. G. Saari, *Collisions, Rings, and Other Newtonian N-Body Problems*, CBMS Regional Conference Series in Mathematics, vol. 104, AMS, 2005.

[12] S. Smale, *Mathematical problems for the next century*, Math. Intelligencer **20** (1998), no. 2, 7–15.

[13] A. Wintner, *The Analytical Foundations of Celestial Mechanics*, Princeton University Press, 1941.

[14] S. Zhang and Q. Zhou, *Double pyramidal central configurations*, Phys. Lett. A **281** (2001), 240–248.


ESIEE-IT, France