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Sharp Bounds on Total Domination and Mean Center Distance in Four-Cycle-Free Graphs

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Abstract. Written on the Wall II Conjectures 268 and 269 compare the total domination number with one plus the mean distance between center vertices. We prove the floor inequality of Conjecture 268. We refute the ceiling inequality of Conjecture 269 by explicit infinite families of four-cycle-free planar graphs, one of which is 2-connected and self-centered. A quotient on the edges of a minimum total dominating set reduces every counterexample to the case in which its total domination number and radius are both four. This gives the ceiling inequality for triangle-free and for minimum-degree-three graphs in the prescribed class. An exact finite calculation on the resulting neighborhood structure verifies the strict and sharp bound $\overline{d}_C < \gamma_t - 29/30$. It also establishes that the least counterexample order is 170 and classifies the three counterexamples of that order.

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1. Introduction

For a connected graph G, the center records those vertices of minimum eccentricity. Their mutual distances, measured in G, can differ substantially from the mean distance over all vertices.

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Bounds for the latter in terms of total domination were studied by DeLaViña, Pepper, and Waller [4]. We study the center version proposed in two entries of *Written on the Wall II* [2, Conjectures 268–269].

Throughout, graphs are finite, simple, undirected, connected, and have at least two vertices. Write $d(u, v)$ for graph distance, $\mathrm{ecc}(u) = \max_v d(u, v)$, $r = \mathrm{rad}(G) = \min_v \mathrm{ecc}(v)$, and $C = \{v : \mathrm{ecc}(v) = r\}$. Set

$$M = \binom{|C|}{2}, \qquad S = \sum_{\{u,v\} \subseteq C} d(u, v), \qquad \overline{d}_C = S/M \quad (M > 0).$$

We use $\overline{d}_C = 0$ when $M = 0$. A total dominating set contains a neighbor of every vertex, including each vertex in the set. Its minimum size is γ_t [1].

Both conjectures were announced on February 23, 2007. Conjecture 268 asserts $\gamma_t \geq \lfloor 1 + \overline{d}_C \rfloor$. Conjecture 269 asserts

$$\gamma_t \geq \lceil 1 + \overline{d}_C \rceil \tag{1}$$

when G is C_4-free. Here C_4-free forbids a four-cycle even if it has chords. We put $\Delta(G) = \overline{d}_C - (\gamma_t - 1)$, so a violation of the ceiling inequality is exactly $\Delta(G) > 0$.

Theorem 1.

(1) *Every graph in the stated domain satisfies $\overline{d}_C < \gamma_t$. In particular Conjecture 268 holds.*
(2) *Conjecture 269 is false. Every C_4-free counterexample has $\gamma_t = r = 4$ and contains a triangle and a vertex of degree 2.*
(3) *Over connected C_4-free graphs, $\Delta(G) < 1/30$, with supremum $1/30$. Equivalently $\overline{d}_C < \gamma_t - 29/30$ is strict and sharp.*
(4) *The smallest order is 170, with exactly three nonisomorphic counterexamples of that order, of sizes 271, 272, and 273.*

The conjecture resolutions and the structural exclusion of larger total domination numbers are analytic. Assertions (3) and (4) use exact arithmetic after the analytic reduction to four dominators.

2. Spanning trees and Conjecture 268

The following argument gives in one place the known radius bound and the equality information we require. It follows the spanning tree approach of DeLaViña et al. [3, Lemmas 2–3 and Theorems 3–4]. Component-count preservation for a prescribed minimum total dominating set also appears in [7, Lemma 3.1].

Lemma 2. *If D is a minimum total dominating set of size t and $G[D]$ has c components, then*

$$\mathrm{rad}(G) \leq \lfloor t/2 \rfloor + c \leq 2\lfloor t/2 \rfloor.$$

If $\mathrm{rad}(G) = t$, then $t = 2k$, every minimum total dominating set induces kK_2, and C contains two adjacent vertices.

Proof. Take a spanning tree of each component of $G[D]$ and an edge from each vertex outside D to a chosen neighbor in D. This is a spanning forest, and it can be extended to a spanning tree R of G. Its restriction to D has exactly the original components. The set D remains total dominating. It is still minimum, since any total dominating set of a spanning subgraph also works in G.

For each component K of $R[D]$, its closed neighborhood in R is a connected subtree of diameter at most $|K| + 1$. It intersects a diametral path in at most $|K| + 2$ vertices. These subtrees cover the path, so $\mathrm{diam}(R) + 1 \leq t + 2c$. Also $c \leq \lfloor t/2 \rfloor$, since each component has at

least two vertices. Using the tree radius formula $\mathrm{rad}(R) = \lceil \mathrm{diam}(R)/2 \rceil$ and the monotonicity of distances under adding edges gives the assertion.

In the equality case t is even, $c = t/2$, and all components of $G[D]$ are edges. Also $\mathrm{rad}(R) = t$ and the diameter bound forces $\mathrm{diam}(R) = 2t - 1$. The centers of an odd-diameter tree are adjacent [6]. Both have eccentricity at most t in G, so both are centers of G. □

For any center pair $d(u, v) \le r$. If $r < \gamma_t$, this proves $\overline{d}_C < \gamma_t$. When $r = \gamma_t$, Lemma 2 provides an adjacent pair in C, of distance $1 < r$. Averaging then gives the strict inequality again. The one-center convention is immediate from $\gamma_t \ge 2$. Taking the floor of $1 + \overline{d}_C < 1 + \gamma_t$ proves Conjecture 268. The transfer of radius-preserving tree centers is also useful for ordinary domination [5].

3. Explicit counterexamples

For positive integers t, s, begin with four vertices a_1, b_1, a_2, b_2 and edges a_1b_1, a_2b_2. Add t paths $a_1p_iq_ia_2$ for $1 \le i \le t$, and s paths $b_1u_jv_jb_2$ for $1 \le j \le s$. For each i, add x_i adjacent to a_1, p_i and z_i adjacent to a_2, q_i. All new vertices are distinct. Call the result $H(t, s)$. Adding two fresh leaves ℓ_1, ℓ_2 at a_1, a_2, respectively, gives $H^+(t, s)$. These edge lists specify the graphs completely. The orders are $4t + 2s + 4$ and $4t + 2s + 6$, respectively. The two block types are shown in Figure 1.

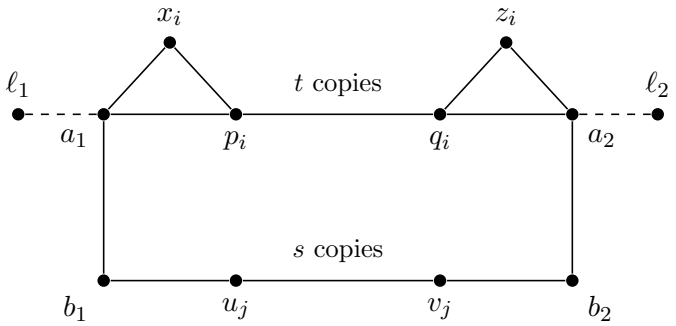

Figure 1. The two types of repeated blocks between the four shared hubs. The dashed leaf edges occur only in $H^+(t, s)$.

Theorem 3. *For $t, s \ge 1$, both graphs are simple, connected, and C_4-free and have total domination number 4. The graph $H(t, s)$ is 2-connected.*

Proof. Simplicity follows from the edge lists. One path of each kind together with the two initial edges forms an 8-cycle. Add the other three-edge paths and then the two-edge paths through x_i, z_i. Each added path has distinct old endpoints and fresh internal vertices. Deleting any one vertex from the enlarged graph leaves the old graph connected, or leaves it intact when the deleted vertex is new. Every surviving segment of the added path is still attached to an old endpoint. This proves 2-connectivity by induction. Adding leaves gives a connected graph.

Before the x_i, z_i are added, a cycle avoiding the initial two edges uses two paths between the same pair of hubs and has length 6. All others use both initial edges and have length 8. In $H(t, s)$, a cycle using x_i and the edge a_1p_i is the triangle on those vertices. Otherwise its segment $a_1x_ip_i$ can be replaced by a_1p_i. The same argument applies to z_i. For a cycle other than one of the added triangles, all the replacements preserve distinctness of the surviving vertices and produce a simple cycle. Indeed, each replacement edge is distinct and cannot already be on the cycle. Thus a nontriangle cycle has length at least 6. The leaves are on no cycle.

The open neighborhoods of the four hubs form a partition of the vertex set in both graphs. The hubs therefore totally dominate. Every total dominating set must meet all four parts in order to dominate the four hubs, so it has size at least 4. □

3.1. **Center distances and the witnesses.** Write $A_i = (p_i, x_i, q_i, z_i)$, $B_j = (u_j, v_j)$. In H, deleting the hubs leaves exactly these blocks. Between hubs, a three-edge path has weight 3 and a hub edge has weight 1. The triangular detours cannot improve those distances. Thus

$$d(a_1, a_2) = d(b_1, b_2) = 3, \qquad d(a_i, b_i) = 1, \qquad d(a_i, b_{3-i}) = 4.$$

In the vertex orders displayed above, the shortest lengths to first reach the two hubs of the block are

A_i to (a_1, a_2)	$(1,2), (1,3), (2,1), (3,1)$
B_j to (b_1, b_2)	$(1,2), (2,1)$

Write the corresponding length as $\lambda(v, h)$. Between distinct blocks the distance is

$$d(v, w) = \min_{h,k} \{\lambda(v, h) + d(h, k) + \lambda(w, k)\}, \tag{2}$$

where each hub ranges over the two of its own block. The first and last hubs on a path give the lower bound, and concatenating paths for a minimizing pair gives a walk attaining the upper bound. Within a block, take the smaller of the minimum in (2) and the distance along the internal path. For a block vertex and a fixed hub k, use $\min_h\{\lambda(v, h) + d(h, k)\}$.

These rules give the frequency vectors in Table 1. For example, the four rows of the A_i-to-B_j matrix are $(3,4), (3,4), (4,3), (4,3)$, so each such block pair gives four distances 3 and four distances 4. All distances in H are at most 4. Each block vertex has a vertex at distance 4 in every block of the other kind, and the hub pairs $\{a_1, b_2\}, \{a_2, b_1\}$ are at distance 4. Thus every eccentricity in H is 4, so its radius is 4 and $C = V(H)$.

The leaves in H^+ preserve distances between the original vertices. Every block vertex and each of a_1, a_2 is within 3 of both a-hubs, so these vertices retain eccentricity 4. Each b_i has distance 5 from the opposite leaf, and $d(\ell_1, \ell_2) = 5$. No distance exceeds 5. Thus H^+ also has radius 4, and its center is the set of all block vertices and the two a-hubs.

Pairs counted per block choice	Number of choices	Counts at distances $1, 2, 3, 4$
Within one A block	t	$(3, 2, 1, 0)$
Between distinct A blocks	$\binom{t}{2}$	$(0, 8, 6, 2)$
Within one B block	s	$(1, 0, 0, 0)$
Between distinct B blocks	$\binom{s}{2}$	$(0, 2, 2, 0)$
Between A and B blocks	ts	$(0, 0, 4, 4)$
The a-hubs to an A block	t	$(4, 2, 2, 0)$
The a-hubs to a B block	s	$(0, 2, 2, 0)$
The pair $\{a_1, a_2\}$	1	$(0, 0, 1, 0)$
The b-hubs to an A block	t	$(0, 4, 2, 2)$
The b-hubs to a B block	s	$(2, 2, 0, 0)$
Remaining hub pairs	1	$(2, 0, 1, 2)$

TABLE 1. Unordered center pairs. The first eight rows apply to both graphs. The last three apply to H only.

Taking the scalar product of each row with $(-2,-1,0,1)$ and applying its multiplicity gives $S-3M$. With $Q=4ts-s^2-3t^2$, the resulting exact sums are

$$S(H^+)=3\binom{4t+2s+2}{2}+Q-15t-3s, \tag{3}$$

$$S(H)=3\binom{4t+2s+4}{2}+Q-17t-9s-2. \tag{4}$$

Since $\gamma_t=4$, either graph violates (1) exactly when its $S-3M$ is positive. In that case $3<S/M\le 4$, so the right-hand side is 5. In particular,

Graph	$\lvert V\rvert$	$\lvert E\rvert$	$\lvert C\rvert$	M	S	$S-3M$
$H^+(21,40)$	170	271	166	13695	41087	2
$H(35,62)$	268	433	268	35778	107340	6

These substitutions give the two promised counterexamples. Along $s=2t$, the excesses are $t(t-21)$ and $t(t-35)-2$, respectively. They are positive for $t\ge 22$ in H^+ and $t\ge 36$ in H. The orders increase, and for both sequences $S/M\to 97/32$.

There is a planar embedding of either family. Start with a quadrilateral on the four hubs. Replace each of two opposite sides by the parallel three-edge routes drawn in a thin strip along that side. Each triangle can be inserted in a thin region alongside its designated edge. Adding the two leaves also preserves planarity.

For a second limiting sequence put $t=4m, s=7m$. Then $w=2t+s=15m$, $Q=w^2/15$, and in either family $M=2w^2+O(w)$. The excess $S-3M=Q+O(w)$ is positive for all sufficiently large m. Thus $(S-3M)/M\longrightarrow 1/30$.

4. Structure of every violation

Suppose $\overline{d}_C>\gamma_t-1$. Since $\overline{d}_C\le r\le\gamma_t$, Lemma 2 forces $r=\gamma_t=2k$. Fix a minimum total dominating set D and write $G[D]=e_1\cup\cdots\cup e_k$ as an induced matching. Define the quotient H_D on these edges by joining e_i,e_j when $d_G(e_i,e_j)\le 3$, where the distance between two edges is the minimum between their endpoints. The closed neighborhoods of the edges cover G. Overlap makes two edges have distance at most 2, and an edge across two neighborhoods makes their distance at most 3. A separation of the quotient would therefore separate G, so H_D is connected.

Lemma 4. *For any spanning tree T of H_D, $\operatorname{rad}(G)\le 2\operatorname{diam}(T)+2$. At $\operatorname{rad}(G)=2k$, every choice of shortest connectors for T forms a path on $4k-2$ vertices together with the matching edges. Each connector has length 3, its internal vertices lie outside D, and the internal vertex sets are disjoint. The quotient is a path or a cycle.*

Proof. For the radius bound, make an abstract tree by replacing each vertex of T by its matching edge and each tree edge by a shortest connector of length 2 or 3. Use separate internal copies even if their images in G coincide. Add a leaf copy of every vertex outside D at a chosen neighbor in D. The resulting tree maps onto G with all edges preserved. A longest path in the tree has endpoints among the matching vertices and added leaves. Traversing q quotient links uses at most $3q+(q+1)+2=4q+3$ edges. Taking its radius bounds the eccentricity of the image of a tree center in G. This gives the claimed inequality.

For equality, take the actual union $K\subseteq G$ of the chosen connectors and $G[D]$. It is connected with $\lvert V(K)\rvert\le 2k+2(k-1)=4k-2$. Take a spanning tree R of K and attach the remaining vertices as leaves at neighbors in D, obtaining a spanning tree R' of G. Then

$$4k-1\le\operatorname{diam}(R')\le\operatorname{diam}(R)+2\le\lvert V(K)\rvert+1\le 4k-1.$$

Here the first step follows from $\mathrm{rad}(R') \geq \mathrm{rad}(G) = 2k$. Equality in the order count gives length 3 and two fresh internal vertices for every connector. The union K has $4k-3$ edges, so is a tree, and its diameter $4k-3$ certifies it is a path. At an edge of D, any two incident connectors attach to its different endpoints, since an endpoint already has its matching edge in this path. In particular T has maximum degree 2. Three quotient edges incident to one vertex would be an acyclic star and extend to a spanning tree, which is impossible. The alternative follows from connectedness. □

Every quotient edge belongs to some spanning tree; hence all its edge distances are exactly 3. Other pairs have distance at least 4. In particular the sets $V_i = N_G[e_i]$ partition $V(G)$, and any cross-edge joins adjacent quotient clusters. At a degree-two quotient vertex $e_i = a_i b_i$, label the endpoints so that incoming connectors attach at a_i and outgoing ones at b_i. These orientations hold for *all* cross-edges. To see this, fix a connector for one of the two links. Any cross-edge of the other link gives a shortest connector via any incident hub of its crossing endpoint. The two links extend together to a tree, and the path condition forbids them attaching at the same hub. This also forbids a common neighbor of a_i and b_i from making a crossing edge.

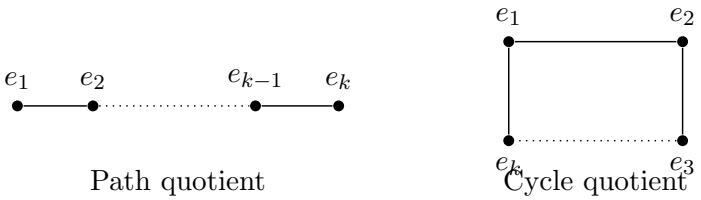

FIGURE 2. Longer path and cycle quotients. Each vertex represents a matching edge, and each link lifts to a three-edge connector. Dotted runs indicate the intermediate links.

4.1. **Path quotients.** Assume $H_D = P_k$, $k \geq 3$, with clusters in order. For an internal i, put the earlier clusters and the exclusive neighbors of a_i on the left of e_i, and the later clusters and exclusive neighbors of b_i on the right. Include a_i and b_i in the left and right sides, respectively. A four-cycle would be formed by an edge joining the two exclusive parts within V_i, or by an edge from a common neighbor to one of those parts. Thus every path from the left to the right must pass through a_i before reaching b_i. A common neighbor allows a two-step passage instead of the edge, but never shortens a geodesic. Therefore

$$d(u,v) = d(u,a_i) + 1 + d(b_i,v) \tag{5}$$

for u on the left and v on the right: a path gives the lower bound, and a concatenated walk gives the upper bound.

Use a path from Lemma 4, with matching edges in quotient order and connectors $b_i p_i q_i a_{i+1}$. If $k = 2m+1$, the left hub of the middle matching edge, a_{m+1}, is within $2k-2$ of all left matching endpoints along this path, and hence within $2k-1$ of all left vertices. Its common neighbors are at distance 1. Radius $2k$ gives a right-hand witness z with $d(a_{m+1}, z) \geq 2k$. For a vertex in an earlier cluster, its distance to a_{m+1} is at least 2, so (5) excludes it from the center. Reflecting the argument gives $C \subseteq V_{m+1}$, where all distances are at most 3.

If $k = 2m \geq 4$, the middle connector vertices p_m, q_m are adjacent. The distance along the path from p_m to any matching endpoint on its side is at most $2k-2$, and to any on the far side at most $2k-1$. Its eccentricity is $2k$ and the far side contains a witness z at that distance. Thus $d(b_m, z) \geq 2k-1$. Equation (5) at $i = m$ rules out earlier clusters, since their vertices are at distance at least 2 from a_m. Reflection using q_m rules out the later clusters. Hence

$C \subseteq V_m \cup V_{m+1}$. Across these two clusters the distance is at most $2+3+2 = 7 \le 2k-1$, and within one at most 3. This excludes a positive excess in both parity cases.

4.2. **Cycle quotients.** Suppose $H_D = C_k$, $k \ge 3$. Each pair of quotient edges lies in a common spanning tree. Thus the full system of connectors has pairwise disjoint internal vertices and uses different endpoints at every matching edge. The lifted cycle L has length $4k$. Its positions in cyclic order are a_i, b_i, p_i, q_i, where $q_i \in V_{i+1}$. Any cycle vertex whose antipode on L is outside D is within $2k-1$ of all of D. It is therefore central. For odd k this includes all of D, and for even k it includes p_i, q_i for every i. In either case $n_i = |C \cap V_i| \ge 2$.

Within a cluster, distances are at most 3. At cyclic quotient distance j, matching endpoints on the lifted cycle are within $4j+1$, so two cluster vertices are within $4j+3$. For $j < \lfloor k/2 \rfloor$ this gives at most $2k-1$. In the remaining cluster pairs, center distances are bounded by $2k$, with excess at most 1. Every cluster has at most two farthest clusters, so $2n_in_j \le n_i^2 + n_j^2$ bounds their total positive contribution by $\sum_i n_i^2$. For $k \ge 4$ we obtain

$$S - (2k-1)M \le \sum_i n_i\big(-(k-3)n_i + (k-2)\big) \le 0.$$

The last step uses $n_i \ge 2$. For $k = 4$ the middle expression can vanish, which suffices. The central connector edges also give strict negative terms from nearer clusters.

For $k = 3$, split C into D and the remaining core with cluster sizes r_i and total r_0. On the 12-cycle, the three matching-edge pairs have distance 1, three other hub pairs have distance at most 3, six at most 4, and three at most 5. Their total excess relative to 5 is at most -24. A remaining vertex is adjacent to a hub and its distances to the six hubs are bounded, in a suitable order, by $1, 2, 4, 5, 5, 6$. Its aggregate excess against the hubs is at most -7. Within the remaining core the excess is at most

$$\sum_{i<j} r_ir_j - 2\sum_i \binom{r_i}{2} = r_0 + \tfrac{1}{2}r_0^2 - \tfrac{3}{2}\sum_i r_i^2 \le r_0.$$

The total is at most $-24 - 6r_0$. This completes the cycle case.

4.3. **The remaining small quotients.** If $k = 1$, a dominating edge uv has at most one common neighbor in a C_4-free graph. When $r = 2$, both exclusive neighbor sets are nonempty. A four-cycle would be formed by any edge between the exclusive sets or from either to a common neighbor. Thus each exclusive vertex has an opposite exclusive vertex at distance 3, while u, v and their common neighbor, if present, are centers. The centers form a clique, so $\overline{d}_C \le 1$. When $r = 1$, all centers are universal and the same bound holds. Consequently a violation requires $k = 2$, that is $\gamma_t = r = 4$.

5. The four-dominator case

Assume G is C_4-free and $\gamma_t = r = 4$. Use the matching $D = \{a_1, b_1, a_2, b_2\}$ and the two clusters of Lemma 4. On each side, separate the exclusive neighbors of the two hubs and their common neighbors. Call the four exclusive parts A_1, B_1, A_2, B_2, in that order. Every exclusive part has induced maximum degree one, because two neighbors of one of its vertices would make a four-cycle through the hub. Edges between the two exclusive parts of the same side, or between the common and exclusive neighbors, also form four-cycles. The common-neighbor set has size at most one. There are no hub-to-opposite-side edges, since the two matching edges have distance 3.

A cross-edge at a common neighbor would put all four hubs within distance 2 of its opposite endpoint, contradicting $r = 4$. Thus a common neighbor has exactly the two hub neighbors. No exclusive vertex has two cross-neighbors. If they have the same hub, they form a four-cycle with

that vertex and the hub. If they have different hubs, all four hubs are within 2 of that vertex and its eccentricity is at most 3. The cross-edges therefore form a single matching.

Every cross-matched vertex is central, since all four hubs are within distance 3. Any other central exclusive vertex must have a cross-matched neighbor in its own part. Otherwise it and its possible within-part mate form a branch attached to its hub alone. A vertex at distance at least 4 from the hub then has distance at least 5 from the branch vertex. The remaining central exclusive vertices are therefore degree-two triangle partners of cross-matched vertices.

5.1. **Counting the central core.** Index a cross-matching edge by the row of its left hub and column of its right hub. The counts for types $11, 12, 21, 22$ are a, b, c, d. In each type count the left and right triangle partners separately:

type	matched pairs	left partners	right partners
11	a	x_{11}	z_{11}
12	b	x_{12}	w_{12}
21	c	y_{21}	z_{21}
22	d	y_{22}	w_{22}

Each partner count is at most its row's number of matched pairs, since the same-part pairs are matchings. These give eight matched blocks and eight partner blocks, covering the central core outside the hubs and common neighbors. Write $X = x_{11} + x_{12}$, $Y = y_{21} + y_{22}$, $Z = z_{11} + z_{21}$, $W = w_{12} + w_{22}$, $J = a + b + c + d$, $P = X + Y + Z + W$, and $N = 2J + P$. Connectedness gives $J \geq 1$.

There are four incompatibilities:

$$x_{11}w_{22} = x_{12}z_{21} = y_{21}w_{12} = y_{22}z_{11} = 0.$$

For example, take a partner x of type 11 on the left and a partner w of type 22 on the right. Their neighbor sets are $\{a_1, p\}$, $\{b_2, v\}$ with p matched to the a_2-part and v matched from the b_1-part. The pairs a_1, b_2, a_1, v, and p, b_2 are at distance at least 3 by exclusivity and the matching rule. Also p, v are neither adjacent nor have a common neighbor: their neighbors are their own hubs, their separate cross-mates, and the two separate triangle partners. Hence x, w are at distance at least 5, contradicting centrality. The other cases are relabelings.

For center-pair excess relative to distance 3, same-part pairs contribute at most -1, while same-side pairs from different parts contribute at most zero. Across the sides, either matched endpoint supplies a path of length at most 3 whenever its cross-mate belongs to the other endpoint's part. All remaining cross-pairs contribute at most 1. The baseline cross count is

$$2ad + 2bc + X(c + d) + Y(a + b) + Z(b + d) + W(a + c) + (X + Y)(Z + W).$$

Subtract the four within-part binomial counts. The quadratic part of the resulting upper bound, after collecting terms, is

$$\begin{aligned} Q = {} & (X + Y)(Z + W) - (a + d)(b + c) - XY - ZW \\ & - \tfrac{1}{2}(a + b - c - d + X - Y)^2 - \tfrac{1}{2}(a + c - b - d + Z - W)^2. \end{aligned} \tag{6}$$

The within-part binomials give the linear term $N/2$. Subtract a further $2J$ for matched pairs (distance 1 instead of 3), and $2P$ for partner–mate edges (1 instead of 2) and partner-to-cross-mate pairs (2 instead of 3). If both endpoints of a matching edge have partners, their partners are at distance at most 3 rather than 4. Let o count these overlaps. The core excess E_0 therefore satisfies

$$E_0 \leq Q - J - \tfrac{3}{2}P - o, \qquad o \geq \sum_{\text{types}} \max(0, \text{left count} + \text{right count} - \text{matched count}). \tag{7}$$

5.2. **The exceptional center vertices.** The two common neighbors, when both exist, are at distance 5, so at most one is central. Put h for the number of central hubs and common neighbors. Every such exceptional vertex has nonpositive aggregate excess against the core. On its own side the J matched vertices are within 2. On the opposite side all matched vertices are within 3, using their cross-mates. The at most J opposite triangle partners are at distance at most 4 since both vertices are central. The same-side partners give no positive contribution. Within a side the central exceptional vertices form a clique. The two side counts satisfy $l \le 3$ and $r' \le 2$ after reflection. Their total pair excess η obeys

$$\eta \le lr' - 2\binom{l}{2} - 2\binom{r'}{2} \le 1, \qquad \eta = 0 \quad \text{if } h \le 1.$$

If there are no central triangle partners then $Q \le 0$ by (6), so $S - 3M \le -J + 1 \le 0$. Therefore every violation contains a triangle partner, which supplies a triangle and a vertex of degree 2. This proves the stated girth, bipartite, and minimum-degree conclusions.

6. The sharp quantitative bound

Lemma 5. *For the compatible nonnegative parameters above, $Q \le N^2/60$. On the first two support orbits in Table 2, $Q \le 0$.*

Proof. Each incompatibility pairs disjoint parameters, so a maximal support chooses one from each of four pairs. The 16 choices are partitioned by the eight row/column/side symmetries as in Table 2. The symmetry action preserves (6) and N. One can verify the table directly by applying row swap, column swap, and transposition to the listed representatives. Their orbit sizes sum to 16.

Orbit	Size	Partner variables allowed	Cones
1	2	$z_{11}, w_{12}, z_{21}, w_{22}$	1
2	2	$z_{11}, x_{12}, y_{21}, w_{22}$	1
3	8	$x_{11}, z_{11}, w_{12}, z_{21}$	2
4	4	$x_{11}, x_{12}, z_{11}, w_{12}$	4

Table 2. Maximal supports and order cones for their representatives.

For each type, order its allowed partner counts between zero and its matching count. Replace the ordered quantities by cumulative sums of nonnegative increments. For example $0 \le z_{11} \le x_{11} \le a$ is parameterized by $z_{11} = u_1,\ x_{11} = u_1 + u_2,\ a = u_1 + u_2 + u_3$. A type with no partners has its matching count as its single coordinate. Taking every order gives the eight cones, each a linear image of $\mathbb{R}^8_{\ge 0}$. In particular, the cone with $x_{11} \le z_{11}$ and $x_{12} \le w_{12}$ is included. Use the matrix of $-2Q$ on the first two cones, and the matrix of $N^2 - 60Q$ on the other six.

For a real symmetric matrix A, the form $u^{\mathsf{T}}Au$ is nonnegative on the nonnegative orthant if and only if no invertible nonempty principal submatrix A_I has $A_I^{-1}\mathbf{1}$ strictly negative componentwise. Indeed, choose an inclusion-minimal simplex face with a negative minimum. The minimizer u is interior and satisfies $A_I u = \lambda \mathbf{1}$, with $\lambda < 0$. If A_I has a nonzero kernel vector z, multiplying by z^{T} gives $\mathbf{1}^{\mathsf{T}} z = 0$. One can then move along z to the boundary at constant negative value, contradicting minimality. Thus $A_I^{-1}\mathbf{1} = u/\lambda < 0$. Conversely a strictly negative solution, normalized by its coordinate sum, is a simplex vector with negative value.

The certificate reconstructs each integer matrix from the displayed polynomial by evaluation at coordinate vectors and their pairwise sums. It evaluates the condition just proved on all $8(2^8 - 1) = 2040$ nonempty faces using exact integer determinants and Cramer's rule. No

invertible face has a strictly negative solution. The companion independent verifier also runs the face test without orbit compression. The tests are reproducible with the supplied standard-library code. □

For any violation, the preceding reduction gives $\gamma_t = r = 4$. Let $E = S - 3M$. Equations (7) and Lemma 5, together with the exceptional pair bounds, imply

$$60(E - M/30) \le -58J - 89P + 60\eta - 4hJ - 2hP - h(h-1).$$

This follows by expanding $2M = (N+h)(N+h-1)$. It is negative if $h \le 1$, since $J \ge 1$ and $\eta = 0$. If $h \ge 2$, it is at most $60 - 58 - 8 - 2 = -8$. This analytic finite step also agrees with the finite configuration check in the reproduction archive. Hence $\Delta(G) < 1/30$. The planar family of Section 3 with $t = 4m, s = 7m$ approaches $1/30$, proving sharpness.

7. Minimum order and classification

7.1. **An exact bounded parameter problem.** To classify all counterexamples of order at most 170, use the positive excess condition $E > 0$. The core bound and the exceptional bounds show that $Q > 0$, since $J \ge 1$. Lemma 5 rules out the first two support orbits. In the remaining representatives $Y = 0$ and $X, Z + W > 0$. A partner forces its own hub to be central. For a target other than the partner itself, a hub-to-mate edge can replace the first step from the partner to its mate without increasing length. For the partner itself the hub is at distance 1. Thus a_1 is central, a_2 is central if $Z > 0$, and b_2 is central if $W > 0$.

There must be at least two positive bridge types. If only one exists, relabel it as type 11, with $a > 0$. The quadratic reduces to

$$XZ - \tfrac{1}{2}(a+X)^2 - \tfrac{1}{2}(a+Z)^2 = -a^2 - a(X+Z) - \tfrac{1}{2}(X-Z)^2 < 0.$$

With two distinct positive types, any pair of opposite hubs is at distance at most 4. A bridge type either directly joins their parts or shares at least one of the two designated parts. Indeed only one type shares neither. In either case a route of length at most 4 follows.

Consequently, if b_1 is not central, some vertex on the right side outside D and the core is at distance at least 5 from it. If either right hub is not central, the same applies on the left side. The two witnesses are distinct if both are needed. For the central hub subset $B \subseteq D$, this proves

$$n \ge N + 4 + \mathbf{1}[b_1 \notin B] + \mathbf{1}[\{a_2, b_2\} \not\subseteq B]. \tag{8}$$

Any central common neighbor contributes at most zero against the core. Its incident hub subset on its side includes at least one central hub, by $X > 0$ and $Z + W > 0$. The pair contributions with other exceptional vertices are at most $-2 + 1 + 1 = 0$. We may omit its terms from the upper bound for E, keeping (8).

The four hub–core upper bounds g_v, obtained from paths of lengths $1, 2, 3, 4$ through the hubs and cross-mates, are

$$\begin{aligned}
g_{a_1} &= -3(a+b) - 2X - (c+d+Y) + z_{21} + w_{22},\\
g_{b_1} &= -3(c+d) - 2Y - (a+b+X) + z_{11} + w_{12},\\
g_{a_2} &= -3(a+c) - 2Z - (b+d+W) + x_{12} + y_{22},\\
g_{b_2} &= -3(b+d) - 2W - (a+c+Z) + x_{11} + y_{21}.
\end{aligned}$$

For example, from a_1 there are $a+b+X$ same-hub neighbors at distance 1 and $c+d+Y$ in the other left group at distance 2. The $a+b$ right-side matches connected to the a_1-part are within distance 2. Other right matches and right partners of types $11, 12$ are within distance 3. Only $z_{21} + w_{22}$ may contribute $+1$. The other three formulas are the corresponding relabelings.

Among hubs in B, a same-side pair contributes -2. An opposite pair contributes 0 if its type is present and 1 otherwise. These distances are exactly 3 and 4 respectively: the cross-edge route realizes distance 3 precisely for a present type, and the two-type argument above covers the other pairs. Let their sum be θ_B. Using the lower bound o_0 for o from (7), we get the exact necessary test

$$Q - J - \tfrac{3}{2}P - o_0 + \sum_{v \in B} g_v + \theta_B \geq 1. \tag{9}$$

The left side is an integer or half-integer upper bound on the integer E.

The included finite certificate enumerates the nonnegative integers in the two remaining supports subject to the partner capacity bounds, the hub implications, (8) with right side at most 170, and (9). In particular $2J + P \leq 166$ and $P \geq 2$, so $J \leq 82$. The implementation uses integer arithmetic on twice (9). For fixed a, b, c, d, let P_* be the smaller of $166 - 2J$ and the sum of the partner capacities. Bound $2Q$ by the smaller of $\lfloor (2J + P_*)^2/30 \rfloor$ and $2 \max X(Z+W) - 2(a+d)(b+c)$, dropping the nonpositive squares and $-2ZW$. The maximum is over the remaining capacities and $X + Z + W \leq P_*$. Fixing X further allows subtraction of the first square and use of the largest feasible $Z + W$. The forced hub a_1 contributes at most $-3(a + b) - 2X - d$. At least one of the two right hubs is forced, with contribution bounded by the larger of $-3(a + c) - d - 2$ and $-3(b + d) - c - 2$, as follows from the hub formulas and the positive partner count there. Other hub terms are nonpositive and $\theta_B \leq 1$. At either pruning step, $-2J - 3P$ is bounded above using the smallest possible P; the overlap term can be dropped. A remaining state is then tested with the full expression (9) against the order bound. The companion archive contains the runnable enumeration and all survivors. This is an exhaustion of the necessary parameter region, rather than graph generation up to order 170.

Up to the quotient symmetries, the only surviving parameters are

(a, b, c, d)	(x_{11}, z_{11})	other partners	B	upper excess
$(21, 0, 0, 40)$	$(21, 21)$	0	$\{a_1, a_2\}$	2
$(22, 0, 0, 38)$	$(22, 22)$	0	$\{a_1, a_2\}$	4.

In both rows the lower bound in (8) is 170. The first row is realized by $H^+(21, 40)$.

7.2. **Reconstructing the graphs at order 170.** In either row, the 164 core vertices and four hubs leave exactly two vertices. These are the required distance-five witnesses for b_2 and b_1 on opposite sides. The witness for b_2 cannot be attached to b_1: the type 22 bridges give $d(b_1, b_2) = 3$. It cannot be a common neighbor for the same reason. It is therefore an exclusive neighbor of a_1. All the other vertices in that part have their within-part neighbor already fixed by the saturated partner counts. The extra vertex has no cross-edge, since it is noncentral. Thus it is a leaf at a_1. By reflection the other is a leaf at a_2.

The reconstruction gives $H^+(a, d)$, with possible extra edges only within $U = \{u_j\}$ or $V = \{v_j\}$ of its type 22 bridges. Each added set is a matching. The same index pair $\{i, j\}$ cannot occur in both, since the two new edges and the bridge edges $u_i v_i, u_j v_j$ would be a four-cycle. Conversely these conditions suffice. A four-cycle using $u_i u_j$ would need a three-edge route between its endpoints avoiding that edge. Inspecting their remaining neighbors b_1, v_i and b_1, v_j shows that the only possible route uses $v_i v_j$; the case of a new edge in V is symmetric.

For any fixed vertex outside $U \cup V$, its original distance is constant on U and on V separately. Those distances are unchanged by the extra edges. To see this for the full distance function, it differs by at most one across every original edge and by zero across every new edge, so it still lower-bounds every new path length. The former eccentricity-four witnesses for vertices of $U \cup V$ in the other path family remain at distance four. Consequently the radius and center are unchanged. The four hubs' disjoint neighborhoods continue to give $\gamma_t = 4$.

An edge u_iu_j lowers the center sum by exactly 3: the pairs $\{u_i, u_j\}, \{u_i, v_j\}, \{u_j, v_i\}$ each decrease by one. Indeed, original distances within one group U or V are 2 for distinct vertices, and across groups they are 1 for a bridge pair and 3 for different indices. A new length-two path between unmatched indices must use one of the two within-group edges incident with the bridge endpoints. The same index pair cannot be used by both new matchings, so these decreases are disjoint. Edges in V have the symmetric effect.

For the first parameter row only the graph without any added edge still has positive excess. For the second row one may add at most one edge. All single-edge choices are isomorphic by permuting the type 22 routes and reflecting the two sides of the construction. Thus the three graphs, with 271, 272, and 273 edges, are pairwise nonisomorphic and exhaust the minimum order. Their excesses are $2, 4, 1$. The third graph is also planar: draw its added edge in the face between the first two parallel b-side routes.

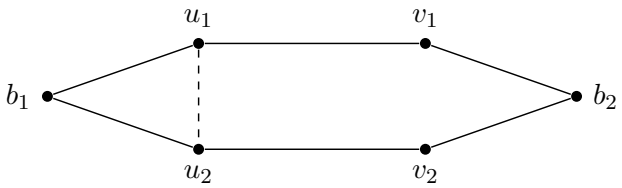

FIGURE 3. The extra edge in the 273-edge minimum-order graph. Only two of the 38 type-22 routes are shown.

8. Consequences and further questions

The regular case combines the minimum-degree statement with the small degrees. A connected 1-regular graph is K_2. A connected 2-regular graph is a cycle. Of the four-cycle-free cycles, C_3 has center mean 1 and total domination number 2, and cycles of length at least 5 are covered by the girth statement. Connectivity at least three implies minimum degree at least three. The construction shows that planarity, even together with 2-connectivity and self-centeredness, permits violations. The 170-vertex witnesses include a planar member, and the sharpness sequence can be chosen planar. The order and quantitative bounds thus also apply sharply within the connected planar four-cycle-free class.

The proof of Conjecture 268 gives a strict bound without the four-cycle restriction. Determining the largest possible real excess in that wider class is a separate problem. The classification given here concerns minimum order. It leaves open the classification of larger graphs whose excess is close to the supremum.

Code and data availability

The pinned fourth-round record contains the upstream certificate. The pinned first independent audit records the original construction checks. The repository is private. The reproduction archive accompanies this manuscript and contains the necessary graph data, exact verifiers, proof audit, and instructions for offline execution using Python's standard library. The enclosed verification logs state the commands and actual scopes. The conjecture resolutions themselves can be checked without code.

Contributions and AI use

Qirui Zhu and Jiayuan Wang used AI systems in the research and prepared the results for submission. Jiayuan Wang coordinated the work, checked the three explicit 170-vertex counterexamples with a separate program, and proposed the matching-edge quotient radius argument and the spanning-tree equality structure used in Section 4. Gemini developed the graph families, the conjecture proofs, and the successive structural and extremal arguments and certificate programs across four rounds. OpenAI Codex independently audited the proofs and the finite calculations, implemented separate verification, and prepared this manuscript. The manuscript was primarily AI-generated. The results are submitted for independent expert review.

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