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DE RHAM–BETTI GROUPS OF ABELIAN VARIETIES WITH ABELIAN COMPLEX MULTIPLICATION

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Abstract. Let B be an abelian variety over $\overline{\mathbb{Q}}$ that is isogenous to a product of abelian varieties with complex multiplication by CM fields that are abelian over $\mathbb{Q}$. We show that André's de Rham–Betti group of B with rational coefficients equals the Mumford–Tate group of B. Consequently, every rational de Rham–Betti class on the powers of B is a Hodge class, and hence motivated. This applies, for example, to the Jacobians of the Fermat curves of every degree. The proof applies the full faithfulness of the de Rham–Betti realization on H^1 (Bost–Charles, Kreutz–Shen–Vial) to the product of abelian varieties attached to *all* CM types of a single abelian CM field, and combines it with Galois stability and an elementary lemma: for a finite abelian group G and an element c of order 2, every simple summand of the part of $\mathbb{Q}[G]$ on which c acts by -1 contains a vector with coefficients in $\{0, \pm 1\}$. In terms of periods: if a monomial in CM periods lies in $\overline{\mathbb{Q}}^\times (2\pi i)^k$, and so does every monomial obtained from it by letting the Galois group of the CM field permute the periods (with the same k), then the relation is induced by Hodge classes. The corresponding statement with $\overline{\mathbb{Q}}$-coefficients remains open.

1. Introduction

Let X be a smooth projective variety over $\overline{\mathbb{Q}} \subset \mathbb{C}$ and $\mathbb{Q}(k) = (2\pi i)^k \mathbb{Q}$. A *$\mathbb{Q}$-de Rham–Betti class* in $H^n(X)(k)$ is an element of $H^n_{\mathrm{B}}(X^{\mathrm{an}}, \mathbb{Q}(k))$ whose preimage under the period comparison isomorphism lies in $H^n_{\mathrm{dR}}(X/\overline{\mathbb{Q}})$ [13, §1, Def. 2.3]. Algebraic classes are de Rham–Betti classes, and Grothendieck's period conjecture predicts the converse. Following André, these classes are the invariants of the de Rham–Betti group $G_{\mathrm{dRB}}(M) \subset \mathrm{GL}(M_{\mathrm{B}})$ of the corresponding de Rham–Betti object M [13, §2.3].

For an abelian variety A over $\overline{\mathbb{Q}}$ write $G_{\mathrm{dRB}}(A)$ for the de Rham–Betti group of $H^1_{\mathrm{dRB}}(A)$ with $\mathbb{Q}$-coefficients, and $\mathrm{MT}(A)$ for its Mumford–Tate group. Then $G_{\mathrm{dRB}}(A) \subset \mathrm{MT}(A)$ [13, Thm. 5.12(i)], and $G_{\mathrm{dRB}}(A) = \mathrm{MT}(A)$ implies that every $\mathbb{Q}$-de Rham–Betti class on the powers of A is a Hodge class (hence motivated, by André's theorem). This equality is known for

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products of elliptic curves (Kreutz, Shen and Vial [13, Thm. 5.13]; Kahn [11, Thm. 1], who proves all the fullness conjectures of André's book for them), for abelian surfaces with $\mathrm{End}(A) \otimes \mathbb{Q} \neq \mathbb{Q}$ and their powers [13, Thm. 5.16], and, by Ji, for simple CM abelian fourfolds [8, Thm. 1.4], every simple abelian fourfold of type IV, abelian fourfolds whose endomorphism algebra is a quartic CM field, and simple CM abelian varieties of prime dimension [9, Abstract, Thm. 1.1]. Ji also proved that $G_{\mathrm{dRB}}(A)$ contains the homotheties for every A [10, Thm. 1.4]. A basic input in these works is the full faithfulness of the de Rham–Betti realization on H^1 of abelian varieties [4, Thm. 3.8], [13, Thms. 5.4, 5.10].

Theorem A. *Let B be an abelian variety over $\overline{\mathbb{Q}}$ that is isogenous to a product of simple abelian varieties B_i, where each B_i has complex multiplication by a CM field F_i that is abelian over $\mathbb{Q}$ (that is, $F_i \subset \mathrm{End}(B_i) \otimes \mathbb{Q}$ and $[F_i : \mathbb{Q}] = 2 \dim B_i$). Then $G_{\mathrm{dRB}}(B) = \mathrm{MT}(B)$.*

More generally, the proof works whenever $\mathrm{End}(B) \otimes \mathbb{Q}$ contains a commutative subalgebra of dimension $2 \dim B$ that is a product of abelian number fields; see Theorem 4.1. Theorem A covers products of any number of factors in every dimension, but only abelian CM fields; it recovers those CM cases above whose CM fields are abelian.

Corollary B. *Let $N \geq 3$. The equality $G_{\mathrm{dRB}} = \mathrm{MT}$ holds for the Jacobian of the Fermat curve $x^N + y^N = z^N$, and for the Jacobian of the smooth projective model of $y^N = x^a(1-x)^b$, for integers a, b with $\gcd(N, a, b) = 1$ (so that the curve is irreducible), whenever it has positive genus. In particular every $\mathbb{Q}$-de Rham–Betti class in the cohomology of a power of one of these Jacobians is a Hodge class, hence motivated.*

Rational versus algebraic coefficients. For CM abelian varieties, the $\overline{\mathbb{Q}}$-de Rham–Betti classes correspond exactly to the monomial relations among Shimura periods, and the $\overline{\mathbb{Q}}$-realization of the category generated by CM abelian varieties is full if and only if Shimura's monomial relations generate all of them [13, §4.5]. Kreutz, Shen and Vial note that this corrects [2, Prop. 24.6.3.1(1)], where $\overline{\mathbb{Q}}$-coefficients were overlooked and only the realization with $\mathbb{Q}$-coefficients was considered [13, p. 16]. We cite [2] through [13] and through André's own summary [3, Prop. 1], which states [2, §24.6] as an equivalence: for abelian varieties with cyclotomic complex multiplication, Grothendieck's period conjecture is equivalent to the Lang–Rohrlich conjecture. Kahn deduced part of a conjecture of Rohrlich (monomial relations among CM periods are generated by the obvious ones) from the $\mathbb{Q}$-coefficient case for products of elliptic curves [11, Introduction]; Kreutz, Shen and Vial point out that this needs $\overline{\mathbb{Q}}$-coefficients and that the question remains open in general [13, p. 16] (see also [12, Rem. 3.22]). The $\overline{\mathbb{Q}}$-version is open even for a product of two non-isogenous CM elliptic curves [13, Rem. 5.15].

Theorem A illustrates this correction: with $\mathbb{Q}$-coefficients the realization of the category generated by CM abelian varieties with abelian CM fields is

full (Remark 4.2), by the inputs already used in [13], while the $\overline{\mathbb{Q}}$-version is out of reach. In terms of periods (Proposition 5.2): if a monomial P_n in CM periods lies in $\overline{\mathbb{Q}}^\times(2\pi i)^k$ *together with all monomials* P_{sn} *obtained by letting* $s \in \mathrm{Gal}(E/\mathbb{Q})$ *permute the periods*, then it is a Hodge relation; for a single monomial this is the open statement.

The method. Let E be the compositum of the CM fields involved, with group G, and replace B by the product A_E of abelian varieties attached to *all* CM types of E. Rationally, $X^*(\mathrm{MT}(A_E)) \cong \mathbb{Q} \oplus \mathbb{Q}[G]^-$, where $\mathbb{Q}[G]^-$ is the part of the group algebra on which complex conjugation acts by -1. The characters trivial on $G_{\mathrm{dRB}}(A_E)$ form a Galois-stable lattice Λ. The summand $\mathbb{Q}$ is excluded by the surjectivity of the determinant; as $\mathbb{Q}[G]^-$ is multiplicity-free, a nonzero Λ then contains a simple summand of $\mathbb{Q}[G]^-$, which by Lemma C contains a difference of two weights of $H^1(A_E)$. Full faithfulness forbids this, by Ji's criterion [9, Cor. 2.5] (cf. [8, Cor. 3.17]); the use of Galois stability follows Ribet's method as in [9, Lemma 2.10]. What is new is the passage to A_E, the multiplicity-freeness of $\mathbb{Q}[G]^-$ and Lemma C. Section 6 explains why the argument fails for non-abelian CM fields and for $\overline{\mathbb{Q}}$-coefficients.

2. Preliminaries

Throughout, $\overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$, ρ denotes complex conjugation, and abelian varieties are over $\overline{\mathbb{Q}}$ and of positive dimension; $\mathrm{End}(A)$ and $\mathrm{Hom}(A, A')$ are taken over $\overline{\mathbb{Q}}$, and $\mathrm{End}^0(A) = \mathrm{End}(A) \otimes \mathbb{Q}$.

2.1. **De Rham–Betti objects.** Let $\mathcal{C} = \mathcal{C}_{\mathrm{dRB},\overline{\mathbb{Q}},\mathbb{Q}}$ be the category of triples $M = (M_{\mathrm{dR}}, M_{\mathrm{B}}, c_M)$ with M_{dR} a finite-dimensional $\overline{\mathbb{Q}}$-vector space, M_{B} a finite-dimensional $\mathbb{Q}$-vector space and $c_M \colon M_{\mathrm{dR}} \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \xrightarrow{\sim} M_{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{C}$ [13, Def. 2.1]. It is a neutral Tannakian category with fibre functor $M \mapsto M_{\mathrm{B}}$, and $G_{\mathrm{dRB}}(M) \subset \mathrm{GL}(M_{\mathrm{B}})$ denotes the Tannakian group of the subcategory $\langle M \rangle$ generated by M [13, Def. 2.4]. We use two formal facts [13, §2.3]: (T1) for $N \in \langle M \rangle$, the action of $G_{\mathrm{dRB}}(M)$ on N_{B} is an epimorphism $G_{\mathrm{dRB}}(M) \to G_{\mathrm{dRB}}(N)$; (T2) an element of $M_{\mathrm{B}}^{\otimes a} \otimes (M_{\mathrm{B}}^\vee)^{\otimes b}$ is a de Rham–Betti class if and only if it is fixed by $G_{\mathrm{dRB}}(M)$. Let $\mathbf{1}(k) = (\overline{\mathbb{Q}}, \mathbb{Q}(k), \mathrm{incl})$. For an abelian variety A put $V_A = H^1_{\mathrm{B}}(A^{\mathrm{an}}, \mathbb{Q})$, $H^1_{\mathrm{dRB}}(A) = (H^1_{\mathrm{dR}}(A/\overline{\mathbb{Q}}), V_A, c_A)$ and $G_{\mathrm{dRB}}(A) = G_{\mathrm{dRB}}(H^1_{\mathrm{dRB}}(A)) \subset \mathrm{GL}(V_A)$. Since $H^m_{\mathrm{dRB}}(A) \cong \wedge^m H^1_{\mathrm{dRB}}(A)$ (see the proof of [10, Cor. 5.6]), the object $h(A)$ used in [13, Thm. 5.12] generates the same Tannakian category as $H^1_{\mathrm{dRB}}(A)$, so the two de Rham–Betti groups coincide. A polarization gives a de Rham–Betti morphism $H^1_{\mathrm{dRB}}(A)^{\otimes 2} \to \mathbf{1}(-1)$ [10, (5)], so $\mathbf{1}(k) \in \langle H^1_{\mathrm{dRB}}(A) \rangle$ for all $k \in \mathbb{Z}$. Applying (T2) to $H^1_{\mathrm{dRB}}(A) \oplus \mathbf{1}(1) \oplus \mathbf{1}(-1)$, which generates the same category, (T2) also holds for the spaces $T^{a,b}(V_A) \otimes \mathbb{Q}(k)$ defined below. A homomorphism $A \to A'$ induces a morphism $H^1_{\mathrm{dRB}}(A') \to H^1_{\mathrm{dRB}}(A)$ in $\mathcal{C}$, by functoriality of the comparison isomorphism; so does every element of $\mathrm{Hom}(A, A') \otimes \mathbb{Q}$.

2.2. **Mumford–Tate groups.** For a rational Hodge structure V let $\mu\colon \mathbb{C}^\times \to \mathrm{GL}(V\otimes\mathbb{C})$ act on $V^{p,q}$ by z^p; $\mathrm{MT}(V)$ is the smallest $\mathbb{Q}$-subgroup of $\mathrm{GL}(V)$ whose complex points contain $\mu(\mathbb{C}^\times)$, as in [8, Rem. 2.25] (the image of the Deligne torus is generated by $\mu(\mathbb{C}^\times)^{-1}$ and its complex conjugate). A *Hodge class* in $T^{a,b}(V)(k) = V^{\otimes a}\otimes (V^\vee)^{\otimes b}\otimes \mathbb{Q}(k)$ is a rational element of type $(0,0)$; it is fixed by $\mu(z)$, hence by $\mathrm{MT}(V)$. In fact $\mathrm{MT}(V)$ is the subgroup fixing all Hodge classes [8, Def. 2.30] and the Tannakian group of $\langle V\rangle$ [8, Def. 2.29], so (T1) holds for MT. We write $\mathrm{MT}(A) = \mathrm{MT}(V_A)$ and $\mathrm{End}_{\mathrm{Hdg}}(V_A)$ for the endomorphisms of the Hodge structure V_A.

2.3. **The inputs.**

Theorem 2.1 ([13, Thm. 5.12])**.** *Let A be an abelian variety over $\overline{\mathbb{Q}}$.*

(I1) *$G_{\mathrm{dRB}}(A)$ is a connected reductive subgroup of $\mathrm{MT}(A)$.*
(I2) *$\mathrm{End}_{G_{\mathrm{dRB}}(A)}(V_A) = \mathrm{End}^0(A)$.*
(I3) *$\det\colon G_{\mathrm{dRB}}(A)\to \mathbb{G}_{\mathrm{m}}$ is surjective.*

The inclusion in (I1) rests on André's theorem $\mathrm{MT}(A) = G_{\mathrm{And}}(A)$ [1, Thm. 0.6.2], as recalled in the proof of [13, Thm. 5.12]; connectedness is [13, Thm. 2.6]. Statement (I2) is the full faithfulness theorem [13, Thm. 5.10]; with $\mathbb{Q}$-coefficients it is due to André, Bost and Wüstholz [13, Thm. 5.4, Rem. 5.5] and also follows from [4, Thm. 3.8(1)]. Statement (I3) uses the transcendence of π. No simplicity assumption on A is made in Theorem 2.1.

Lemma 2.2. *For every abelian variety A over $\overline{\mathbb{Q}}$, $\mathrm{End}_{\mathrm{Hdg}}(V_A) = \mathrm{End}^0(A)$.*

Proof. Endomorphisms of A induce Hodge endomorphisms. Conversely, a Hodge endomorphism is a Hodge class in $\mathrm{End}(V_A)$, so it is fixed by $\mathrm{MT}(A)$, hence by $G_{\mathrm{dRB}}(A)$ by (I1), and it lies in $\mathrm{End}^0(A)$ by (I2). (The lemma is of course classical: Riemann's theorem and rigidity.) □

Lemma 2.3. *Let A, B be abelian varieties over $\overline{\mathbb{Q}}$, $m\ge 1$, and suppose that the Hodge structure V_B is isomorphic to a direct summand of $V_A^{\oplus m}$. If $G_{\mathrm{dRB}}(A) = \mathrm{MT}(A)$, then $G_{\mathrm{dRB}}(B) = \mathrm{MT}(B)$.*

Proof. Let $\iota\colon V_B\to V_A^{\oplus m}$ and $\pi\colon V_A^{\oplus m}\to V_B$ be morphisms of Hodge structures with $\pi\iota = \mathrm{id}$. Put $X = B\times A^m$, so $V_X = V_B\oplus V_A^{\oplus m}$, and, via the projections, $H^1_{\mathrm{dRB}}(X) = H^1_{\mathrm{dRB}}(B)\oplus H^1_{\mathrm{dRB}}(A)^{\oplus m}$ in $\mathcal{C}$. The endomorphisms $(v,w)\mapsto(0,\iota v)$ and $(v,w)\mapsto(\pi w,0)$ of V_X are Hodge, so they lie in $\mathrm{End}^0(X)$ by Lemma 2.2. Hence ι and π are the Betti realizations of morphisms in $\mathcal{C}$, and $H^1_{\mathrm{dRB}}(B)$ is a direct summand of $H^1_{\mathrm{dRB}}(A)^{\oplus m}$. The group $G_{\mathrm{dRB}}(A)$ acts diagonally on $V_A^{\oplus m}$, preserves $\iota(V_B)$, and by (T1) its image under $g\mapsto \pi g\iota$ is $G_{\mathrm{dRB}}(B)$. The same argument in the category of Hodge structures shows that the image of $\mathrm{MT}(A)$ under the same map is $\mathrm{MT}(B)$. Since $G_{\mathrm{dRB}}(A) = \mathrm{MT}(A)$, the images agree. □

2.4. **CM types.** Let $E\subset\overline{\mathbb{Q}}$ be a CM field that is Galois over $\mathbb{Q}$, with group $G = \mathrm{Gal}(E/\mathbb{Q})$. Then $\mathrm{Hom}(E,\overline{\mathbb{Q}}) = G$, and complex conjugation induces

an element $c \in G$ of order 2 that is central in G [10, Lemma 3.3]. A *CM type* of E is a subset $\Psi \subset G$ with $G = \Psi \sqcup c\Psi$. We say that an abelian variety A has *CM type* (E, Ψ) if $E \subset \mathrm{End}^0(A)$, $[E : \mathbb{Q}] = 2 \dim A$, and E acts on $H^{1,0}(A) = H^0(A, \Omega^1)$ through the embeddings in Ψ. Then V_A is an E-vector space of dimension 1, and

$$V_A \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} = \bigoplus_{\tau \in G} V_{A,\tau}, \qquad V_{A,\tau} = \{v : ev = \tau(e)v \text{ for } e \in E\}, \tag{2.1}$$

with every $V_{A,\tau}$ a line, and $V_{A,\tau} \otimes \mathbb{C} \subset V^{1,0}$ if $\tau \in \Psi$, $\subset V^{0,1}$ if $\tau \notin \Psi$.

Lemma 2.4. *For every CM type Ψ of E there is an abelian variety A_Ψ over $\overline{\mathbb{Q}}$ with CM type (E, Ψ).*

Proof. The complex torus $Y = \mathbb{C}^\Psi / \Psi(\mathcal{O}_E)$ is an abelian variety with complex multiplication, so it has a model A over $\overline{\mathbb{Q}}$ [14, Prop. 7.9, Cor. 7.10]; see also [6, §2.1], [7, §6.5]. The ring $\mathcal{O}_E$ acts on Y, hence by Hodge endomorphisms on $V_A = H^1(Y, \mathbb{Q})$, so $E \subset \mathrm{End}^0(A)$ by Lemma 2.2. An element e acts on the tangent space $\mathbb{C}^\Psi$ of Y by the diagonal matrix $(\psi(e))_{\psi \in \Psi}$, hence on the space $H^0(Y, \Omega^1)$ of invariant differentials, the dual of the tangent space, with the eigenvalues $\psi(e)$, $\psi \in \Psi$. □

3. The combinatorial lemma

Let G be a finite abelian group and $c \in G$ an element of order 2. We identify $\mathbb{Q}[G]$ with the space $\mathrm{Fun}(G, \mathbb{Q})$ of functions $f \colon G \to \mathbb{Q}$ via $f \mapsto \sum_t f(t)t$; then left multiplication by $s \in G$ becomes the translation $(sf)(t) = f(s^{-1}t)$. Put $\mathbb{Q}[G]^- = \{x \in \mathbb{Q}[G] : cx = -x\}$. For a character $\chi \colon G \to \overline{\mathbb{Q}}^\times$ and $x = \sum x_t t$ put $\hat{\chi}(x) = \sum_t x_t \chi(t)$. Recall that $\mathbb{Q}[G] = \prod_O \mathbb{Q}[G]e_O$ is a product of fields, indexed by the orbits O of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the characters of G, with $\mathbb{Q}[G]e_O \cong \mathbb{Q}(\chi)$ for $\chi \in O$; and $x \in \mathbb{Q}[G]e_O$ if and only if $\hat{\chi}(x) = 0$ for all $\chi \notin O$. These summands are pairwise non-isomorphic $\mathbb{Q}[G]$-modules, and

$$\mathbb{Q}[G]^- = \bigoplus_{O \colon \chi(c) = -1} C_O, \qquad C_O = \mathbb{Q}[G]e_O.$$

A *CM type* in G is a subset S with $G = S \sqcup cS$, and $\mathbf{1}_S$ is its indicator function.

Lemma C. *Let O be an orbit of characters with $\chi(c) = -1$. Then C_O contains a nonzero element whose coefficients lie in $\{0, \pm 1\}$. Every such element has the form $\mathbf{1}_S - \mathbf{1}_{S'}$ with CM types $S \neq S'$.*

Proof. Fix $\chi \in O$ and let $H = \ker \chi$. Then $G/H \cong \chi(G)$ is cyclic, of some order n. Since $\chi(c) = -1$, the image $\bar{c}$ of c has order 2, so n is even. For each odd prime $q \mid n$ choose $h_q \in G/H$ of order q, and put

$$y = (1 - \bar{c}) \prod_{q \mid n,\ q \text{ odd}} (1 - h_q) \in \mathbb{Z}[G/H].$$

Fourier support. For a character ψ of G/H we have $\hat{\psi}(y) = (1-\psi(\bar{c}))\prod_q(1-\psi(h_q))$. This is nonzero if and only if ψ is nontrivial on every subgroup of G/H of prime order (these are $\langle\bar{c}\rangle$ and the $\langle h_q\rangle$), that is, if and only if $\ker\psi$ contains no element of prime order, that is, if and only if ψ is faithful. The faithful characters of a cyclic group of order n form one Galois orbit, by the irreducibility of the n-th cyclotomic polynomial. Pulled back to G, they are the characters with kernel exactly H, which are precisely the elements of O.

Coefficients. Expanding the product, $y = \sum_{\varepsilon,T}(-1)^{\varepsilon+|T|}\,\bar{c}^{\,\varepsilon}\prod_{q\in T}h_q$, over $\varepsilon\in\{0,1\}$ and sets T of odd primes dividing n. The group elements occurring are pairwise distinct, since the component of $\bar{c}^{\,\varepsilon}\prod_{q\in T}h_q$ in the 2-primary part of G/H is $\bar{c}^{\,\varepsilon}$ and its component in the q-primary part is h_q or 1 according as $q\in T$ or not. So the coefficients of y lie in $\{0,\pm1\}$.

Pullback. Let $x\in\mathbb{Z}[G]$ have coefficient $x_t = y_{tH}$ at $t\in G$. Its coefficients lie in $\{0,\pm1\}$. For a character χ' of G we have $\hat{\chi}'(x) = |H|\hat{\psi}(y)$ if $\chi' = \psi\circ(G\to G/H)$, and $\hat{\chi}'(x) = 0$ if $\chi'|_H\neq 1$, since then $\sum_{h\in H}\chi'(h) = 0$. By the first step $\hat{\chi}'(x)\neq 0$ exactly for $\chi'\in O$, so $0\neq x\in C_O$.

Finally let $x\in\mathbb{Q}[G]^-$ have coefficients in $\{0,\pm1\}$. On each pair $\{t,ct\}$ either x vanishes, or it takes the values $+1$ and -1. Let S (resp. S') contain from each pair the point where $x=1$ (resp. $x=-1$), and the same chosen point of each pair on which x vanishes. Then $x = \mathbf{1}_S - \mathbf{1}_{S'}$, and $S\neq S'$ if $x\neq 0$. □

4. Proof of Theorem A

We prove the following slightly more general statement.

Theorem 4.1. *Let B be an abelian variety over $\overline{\mathbb{Q}}$. Assume that $\mathrm{End}^0(B)$ contains a commutative $\mathbb{Q}$-subalgebra R with $\dim_{\mathbb{Q}} R = 2\dim B$ which is a product of number fields, each abelian over $\mathbb{Q}$. Then $G_{\mathrm{dRB}}(B) = \mathrm{MT}(B)$.*

Theorem A is the special case $R = \prod_i F_i$, acting factorwise on $B\sim\prod_i B_i$ (the B_i may repeat), where $\dim_{\mathbb{Q}} R = \sum_i[F_i:\mathbb{Q}] = \sum_i 2\dim B_i = 2\dim B$.

Step 0: reduction to one abelian CM field. Write $R = \prod_j K_j$ with idempotents e_j, and put $V = V_B$ and $V_j = e_jV$. Since R acts faithfully on V, each V_j is a nonzero K_j-vector space; since $\sum_j\dim_{\mathbb{Q}}V_j = \dim V = \dim R = \sum_j[K_j:\mathbb{Q}]$, each V_j has dimension one over K_j. As e_j is a Hodge endomorphism, V_j is a sub-Hodge structure. The Hodge decomposition is K_j-stable, and the eigenspaces $V_{j,\varphi}$ ($\varphi\in\mathrm{Hom}(K_j,\mathbb{C})$) of $V_j\otimes\mathbb{C}$ are lines, so $V_j^{1,0} = \bigoplus_{\varphi\in\Phi_j}V_{j,\varphi}$ for some set Φ_j. Complex conjugation on $V_j\otimes\mathbb{C}$ maps $V_{j,\varphi}$ onto $V_{j,\rho\varphi}$ and $V_j^{1,0}$ onto $V_j^{0,1}$; hence $\rho\varphi\neq\varphi$ for all φ, and $\mathrm{Hom}(K_j,\mathbb{C}) = \Phi_j\sqcup\rho\Phi_j$. So K_j is totally imaginary. Since K_j is abelian, there is an automorphism c_j with $\rho\circ\varphi = \varphi\circ c_j$ for every φ (if $\rho\varphi = \varphi c'$, then $\rho\varphi g = \varphi c' g = (\varphi g)c'$), and its fixed field is totally real of index 2. Thus K_j is a CM field and Φ_j is a CM type.

Fix embeddings $K_j \hookrightarrow \overline{\mathbb{Q}}$; their images are Galois over $\mathbb{Q}$. Let $E \subset \overline{\mathbb{Q}}$ be the compositum of these images. It is abelian over $\mathbb{Q}$, and it is a CM field [10, Lemma 3.3]. Let Ψ_j be the set of $\psi \in G = \mathrm{Gal}(E/\mathbb{Q})$ with $\psi|_{K_j} \in \Phi_j$; it is a CM type of E. Let A_{Ψ_j} be as in Lemma 2.4 and $W_j = V_{A_{\Psi_j}}$, a free K_j-module of rank $d_j = [E : K_j]$. The φ-eigenspace of K_j in $W_j \otimes \mathbb{C}$ is $W_{j,\varphi} = \bigoplus_{\psi|_{K_j}=\varphi} W_{j,\psi}$, so $W_j^{1,0} = \bigoplus_{\varphi\in\Phi_j} W_{j,\varphi}$, and likewise for $V_j^{\oplus d_j}$. A K_j-linear isomorphism $V_j^{\oplus d_j} \to W_j$ maps φ-eigenspaces to φ-eigenspaces, hence is an isomorphism of Hodge structures. Hence $V = \bigoplus_j V_j$ is a direct summand of $\bigoplus_j W_j$.

Let $\mathcal{T}$ be the set of CM types of E, $|\mathcal{T}| = 2^d$ where $|G| = 2d$, and put $A_E = \prod_{\Psi\in\mathcal{T}} A_\Psi$ and $V_E = V_{A_E} = \bigoplus_\Psi V_\Psi$, $V_\Psi = V_{A_\Psi}$. Then $\bigoplus_j W_j$ is a direct summand of $V_E^{\oplus m}$, where m is the number of factors of R. By Lemma 2.3 it suffices to prove $G_{\mathrm{dRB}}(A_E) = \mathrm{MT}(A_E)$.

Step 1: the characters of $\mathrm{MT}(A_E)$**.** *Conventions.* $\mathrm{Hom}(E, \overline{\mathbb{Q}}) = G$, and $s \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on it by $\tau \mapsto s|_E\,\tau$; as G is abelian, left and right translations agree. We use the eigenlines $V_{\Psi,\tau}$ of (2.1) and μ acting on $V^{p,q}$ by z^p. Other conventions (types on the tangent space, or μ^{-1}) replace each weight $\mathbf{1}_{\tau\Psi^{-1}}$ below by $\mathbf{1}_{c\tau\Psi^{-1}}$; the set of weights, which is all that is used, is unchanged.

The algebra $\prod_\Psi E \subset \mathrm{End}^0(A_E)$ consists of Hodge classes, so $\mathrm{MT}(A_E)$ lies in its centralizer

$$T^* = \prod_{\Psi\in\mathcal{T}} \mathrm{Res}_{E/\mathbb{Q}}\,\mathbb{G}_\mathrm{m} \subset \mathrm{GL}(V_E).$$

The character group is $X^*(T^*) = \bigoplus_\Psi \mathbb{Z}[G]$, with basis $[\tau]_\Psi$; the torus T^* acts on $V_{\Psi,\tau}$ through $[\tau]_\Psi$, and $s \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ maps $[\tau]_\Psi$ to $[s\tau]_\Psi$. The cocharacter μ factors through T^*, and $\mu = \sum_\Psi \sum_{\tau\in\Psi} [\tau]_\Psi^\vee$ in the dual basis.

By the anti-equivalence between tori over $\mathbb{Q}$ and lattices with Galois action [8, Thm. 2.41], subtori of T^* correspond to Galois-stable saturated subgroups of $X_*(T^*)$. The smallest $\mathbb{Q}$-subgroup of T^* containing $\mu(\mathbb{C}^\times)$ is connected, hence a torus, so $\mathrm{MT}(A_E)$ is the subtorus with $X_*(\mathrm{MT}(A_E))\otimes\mathbb{Q}$ spanned by the conjugates $s\mu$, $s \in G$. Consequently the map

$$w\colon X^*(\mathrm{MT}(A_E)) \to \mathrm{Fun}(G, \mathbb{Z}), \qquad w(\chi)(s) = \langle \chi, s\mu\rangle,$$

is injective, and it is G-equivariant for the translation action on $\mathrm{Fun}(G,\mathbb{Z})$, because $\langle s\chi, t\mu\rangle = \langle \chi, s^{-1}t\mu\rangle$. Since $s\mu = \sum_\Psi \sum_{\tau\in\Psi} [s\tau]_\Psi^\vee$, we find

$$w([\tau]_\Psi)(s) = \begin{cases} 1, & s^{-1}\tau \in \Psi \\ 0, & \text{otherwise} \end{cases} \qquad \text{i.e.} \qquad w([\tau]_\Psi) = \mathbf{1}_{\tau\Psi^{-1}}. \tag{4.1}$$

Here $\tau\Psi^{-1}$ is a CM type, because c is central of order 2; and every CM type S occurs, with $\tau = 1$ and $\Psi = S^{-1}$. As $X^*(T^*) \to X^*(\mathrm{MT}(A_E))$ is surjective, $w(X^*(\mathrm{MT}(A_E)))$ is spanned by the indicators $\mathbf{1}_S$ of all CM types

S. Under the identification of Section 3,

$$X^*(\mathrm{MT}(A_E))\otimes\mathbb{Q} \;\cong\; \{f : f + f\circ c \text{ is constant}\} \;=\; \mathbb{Q}\cdot\mathbf{1}_G \oplus \mathbb{Q}[G]^-. \tag{4.2}$$

Indeed $\mathbf{1}_S + \mathbf{1}_S\circ c = \mathbf{1}_G$; conversely the differences $\mathbf{1}_S - \mathbf{1}_{S'}$ of CM types that differ on one pair $\{t, ct\}$ are $\pm(t - ct)$, and these span $\mathbb{Q}[G]^-$. The decomposition (4.2) is multiplicity-free: c acts by $+1$ on $\mathbb{Q}\cdot\mathbf{1}_G$ and by -1 on every C_O.

Step 2: the relation lattice. By (I1), $G_{\mathrm{dRB}} := G_{\mathrm{dRB}}(A_E)$ is a connected subgroup of the torus $\mathrm{MT}(A_E)$, hence a subtorus. Let $\Lambda \subset X^*(\mathrm{MT}(A_E))$ be the group of characters trivial on G_{dRB}. Then $X^*(\mathrm{MT}(A_E))/\Lambda$ embeds in $X^*(G_{\mathrm{dRB}})$, which is torsion-free, so Λ is saturated; Λ is Galois-stable because G_{dRB} is defined over $\mathbb{Q}$; and $G_{\mathrm{dRB}} = \mathrm{MT}(A_E)$ if and only if $\Lambda = 0$, since a subtorus on which no nontrivial character vanishes has full dimension. By Step 1, $\Lambda_{\mathbb{Q}} = \Lambda\otimes\mathbb{Q}$ is a $\mathbb{Q}[G]$-submodule of a multiplicity-free semisimple module, so it is a sum of some of the summands in (4.2).

The summand $\mathbb{Q}\cdot\mathbf{1}_G$ does not occur. The determinant of V_E is $\det = \sum_{\Psi,\tau}[\tau]_\Psi$, and by (4.1), $w(\det)(s) = \sum_\Psi |\{\tau : s^{-1}\tau \in \Psi\}| = 2^d\cdot d$, a nonzero constant. If $\mathbb{Q}\cdot\mathbf{1}_G \subset \Lambda_{\mathbb{Q}}$, then some power $\det^r$, $r \geq 1$, is trivial on G_{dRB}, contradicting (I3). Hence, if $\Lambda \neq 0$, then $C_O \subset \Lambda_{\mathbb{Q}}$ for some orbit O with $\chi(c) = -1$.

Step 3: a contradiction in degree $(1,1)$**.** This step is the criterion of [9, Cor. 2.5] (cf. [8, Cor. 3.17]), applied to A_E. Suppose $\Lambda \neq 0$ and let O be as above. By Lemma C there are CM types $S \neq S'$ with $\mathbf{1}_S - \mathbf{1}_{S'} \in C_O$. Let $\chi = [1]_{S^{-1}} - [1]_{S'^{-1}} \in X^*(T^*)$; by (4.1), the restriction of χ to $\mathrm{MT}(A_E)$ has w-image $\mathbf{1}_S - \mathbf{1}_{S'}$, so it lies in $\Lambda_{\mathbb{Q}} \cap X^*(\mathrm{MT}(A_E)) = \Lambda$. Consider the lines $\ell_1 = V_{S^{-1},1}$ and $\ell_2 = V_{S'^{-1},1}$, which lie in the distinct summands $V_{S^{-1}}$ and $V_{S'^{-1}}$ of V_E, and the line $t = \mathrm{Hom}(\ell_2, \ell_1) \subset \mathrm{End}(V_E)\otimes\overline{\mathbb{Q}}$ of maps that send ℓ_2 to ℓ_1 and kill all other eigenlines. The torus T^* acts on t through χ, which is trivial on G_{dRB}. Since taking invariants commutes with extension of scalars, (I2) gives $t \subset \mathrm{End}_{G_{\mathrm{dRB}}}(V_E)\otimes\overline{\mathbb{Q}} = \mathrm{End}^0(A_E)\otimes\overline{\mathbb{Q}}$. Elements of $\mathrm{End}^0(A_E)$ are Hodge classes, hence fixed by $\mathrm{MT}(A_E)$. So $\mathrm{MT}(A_E)$ acts trivially on t, that is, $\chi|_{\mathrm{MT}(A_E)} = 0$ and $\mathbf{1}_S - \mathbf{1}_{S'} = 0$, a contradiction. Therefore $\Lambda = 0$ and $G_{\mathrm{dRB}}(A_E) = \mathrm{MT}(A_E)$, which proves Theorem 4.1 and Theorem A. □

Remark 4.2. Let B be any abelian variety over $\overline{\mathbb{Q}}$ with $G_{\mathrm{dRB}}(B) = \mathrm{MT}(B)$, for example as in Theorem 4.1. By (T2), every $\mathbb{Q}$-de Rham–Betti class in $T^{a,b}(V_B)\otimes\mathbb{Q}(k)$ is fixed by $G_{\mathrm{dRB}}(B) = \mathrm{MT}(B)$, hence is a Hodge class, hence motivated by André's theorem; this includes the classes in $H^j_{\mathrm{B}}(B^n, \mathbb{Q}(k))$. Since $G_{\mathrm{And}}(B) = \mathrm{MT}(B)$ [1, Thm. 0.6.2], [13, proof of Thm. 5.12], the de Rham–Betti torsor of periods of $h(B)$ is contained in the motivated one and is a torsor under the same group [13, Def. 2.4, §4.2, Def. 4.3], so they are equal: $h(B)$ satisfies the motivated de Rham–Betti conjecture [13, Conj. 4.7(i)] with $L = \mathbb{Q}$, and the realization is full on $\langle h(B)\rangle$ [13, Prop. 4.8].

The stronger [13, Conj. 4.7(ii)] would follow from the Hodge conjecture for the powers of B (which gives $\Omega^{\mathrm{And}} = \Omega^{\mathrm{mot}}$).

5. Consequences

5.1. **Fermat curves.**

Proof of Corollary B. Let C_N be the Fermat curve $x^N + y^N = z^N$, of genus $g = (N-1)(N-2)/2$, and $J = J(C_N)$. The group μ_N^2 acts on C_N over $\overline{\mathbb{Q}}$ by $(\zeta, \zeta') \cdot (x : y : z) = (\zeta x : \zeta' y : z)$, hence on J and on $V_J = H^1_{\mathrm{B}}(C_N, \mathbb{Q})$. By [12, (4.1)], with $K = \mathbb{Q}(\mu_N)$,

$$V_J \otimes_{\mathbb{Q}} K = \bigoplus_{(a,b)\in\mathfrak{A}} X_{\mathrm{B}}^{a,b}, \qquad \mathfrak{A} = \{(a,b) \in (\mathbb{Z}/N)^2 : a, b, a+b \neq 0\},$$

where μ_N^2 acts on $X_{\mathrm{B}}^{a,b}$ through the character $(\zeta, \zeta') \mapsto \zeta^a \zeta'^b$, and each $X^{a,b}$ has rank one (it is an element of the Picard group in [12, Def. 4.2, (4.5)–(4.7)]). Since $|\mathfrak{A}| = 2g = \dim V_J$, the image R of $\mathbb{Q}[\mu_N^2]$ in $\mathrm{End}^0(J)$ has dimension $2g$. It is a quotient of $\mathbb{Q}[\mu_N^2]$, hence a product of cyclotomic fields $\mathbb{Q}(\mu_e)$ with $e \mid N$. So Theorem 4.1 applies to J.

Let C' be the smooth projective model of $y^N = x^a(1-x)^b$; it is irreducible because $\gcd(N, a, b) = 1$, so that $x^a(1-x)^b$ is not a d-th power in $\mathbb{C}(x)$ for any $d > 1$ dividing N. The map $(x, y) \mapsto (x^N, x^a y^b)$ from the affine curve $x^N + y^N = 1$ defines a dominant rational map, hence a nonconstant morphism $f\colon C_N \to C'$. Since $f_* f^* = \deg f$, the Hodge structure $H^1(C')$ is a direct summand of $H^1(C_N)$, and Lemma 2.3 gives $G_{\mathrm{dRB}}(J(C')) = \mathrm{MT}(J(C'))$ when C' has positive genus. The last assertion of Corollary B is Remark 4.2. □

5.2. **One-dimensional de Rham–Betti objects.**

Ji conjectured that for every abelian variety A over $\overline{\mathbb{Q}}$, every one-dimensional object of $\langle H^1_{\mathrm{dRB}}(A, \mathbb{Q})\rangle$ is isomorphic to some $\mathbf{1}(k)$ [10, Conj. 5.7].

Corollary 5.1. *Ji's conjecture holds for every abelian variety A over $\overline{\mathbb{Q}}$ with $G_{\mathrm{dRB}}(A) = \mathrm{MT}(A)$, in particular for every B as in Theorem 4.1.*

Proof. One-dimensional objects of $\langle H^1_{\mathrm{dRB}}(A)\rangle$ correspond to characters of $G_{\mathrm{dRB}}(A) = \mathrm{MT}(A)$ defined over $\mathbb{Q}$, hence to one-dimensional objects of the category of Hodge structures generated by V_A [8, Def. 2.29]. A one-dimensional rational Hodge structure has a single type (p, q), and $p = q$ by complex conjugation, so it is $\mathbb{Q}(-p)$. The object $\mathbf{1}(-p) \in \langle H^1_{\mathrm{dRB}}(A)\rangle$ has the same character, since $\mathbf{1}(-1)$ and $\mathbb{Q}(-1)$ are both realized as the quotient of $V_A^{\otimes 2}$ given by a polarization [10, (5)]. Hence the object is isomorphic to $\mathbf{1}(-p)$. □

5.3. **Periods.** Keep the notation of Section 4. For $\Psi \in \mathcal{T}$ and $\tau \in G$ choose a nonzero $\omega_{\Psi,\tau} \in H^1_{\mathrm{dR}}(A_\Psi/\overline{\mathbb{Q}})$ on which E acts through τ, and $0 \neq \gamma_{\Psi,\tau} \in V_{\Psi,\tau}$. The comparison isomorphism is E-equivariant, so $c(\omega_{\Psi,\tau}) = p(\Psi,\tau)\gamma_{\Psi,\tau}$ for a period $p(\Psi,\tau) \in \mathbb{C}^\times$, well defined up to $\overline{\mathbb{Q}}^\times$; these are the Shimura periods of [13, §4.5]. For $n = (n_{\Psi,\tau}) \in X^*(T^*) = \bigoplus_\Psi \mathbb{Z}[G]$ put

$$P_n = \prod_{\Psi,\tau} p(\Psi,\tau)^{n_{\Psi,\tau}} \in \mathbb{C}^\times/\overline{\mathbb{Q}}^\times, \qquad (sn)_{\Psi,s\tau} = n_{\Psi,\tau} \quad (s \in G).$$

Call $(n,k) \in X^*(T^*) \times \mathbb{Z}$ a *Hodge relation* if $\sum_{\Psi,\tau} n_{\Psi,\tau} \mathbf{1}_{\tau\Psi^{-1}} = k\,\mathbf{1}_G$, i.e. $w(n) = k\mathbf{1}_G$.

Proposition 5.2. *Let $n \in X^*(T^*)$ and $k \in \mathbb{Z}$.*

(i) *If (n,k) is a Hodge relation, then $P_n \in \overline{\mathbb{Q}}^\times(2\pi i)^k$.*
(ii) *If $P_{sn} \in \overline{\mathbb{Q}}^\times(2\pi i)^k$ for every $s \in G$, then (n,k) is a Hodge relation.*

Proof. Write $n = n^+ - n^-$ with $n^\pm \geq 0$, and let $T_{\mathrm{B}} = T^{a,b}(V_E)$ and $T_{\mathrm{dR}} = T^{a,b}(H^1_{\mathrm{dR}}(A_E/\overline{\mathbb{Q}}))$ with $a = |n^+|$, $b = |n^-|$. The monomials in the $\gamma_{\Psi,\tau}$ and in the dual basis vectors $\gamma^\vee_{\Psi,\tau}$ form a basis of $T_{\mathrm{B}} \otimes \overline{\mathbb{Q}}$ by eigenvectors of T^*; let $T_{\mathrm{B},n'} \subset T_{\mathrm{B}} \otimes \overline{\mathbb{Q}}$ be the $\overline{\mathbb{Q}}$-span of those of weight n', and define $T_{\mathrm{dR},n'} \subset T_{\mathrm{dR}}$ in the same way with the $\omega_{\Psi,\tau}$. Since the comparison isomorphism on duals is the inverse transpose, a monomial in the ω's of weight n' is mapped to $P_{n'}$ times the corresponding monomial in the γ's. So $c = P_{n'}c^0_{n'}$ on $T_{\mathrm{dR},n'} \otimes \mathbb{C}$, where $c^0_{n'} \colon T_{\mathrm{dR},n'} \to T_{\mathrm{B},n'}$ is an isomorphism of $\overline{\mathbb{Q}}$-vector spaces. Hence, for $0 \neq v \in T_{\mathrm{B},n'}$, the element $c^{-1}((2\pi i)^k v) = (2\pi i)^k P_{n'}^{-1}(c^0_{n'})^{-1}(v)$ lies in T_{dR} if and only if $P_{n'} \in \overline{\mathbb{Q}}^\times(2\pi i)^k$.

Now let $v \in T_{\mathrm{B}}$ be rational, with components $v_{n'} \in T_{\mathrm{B},n'}$. Since $T_{\mathrm{dR}} = \bigoplus T_{\mathrm{dR},n'}$, the element $(2\pi i)^k v \in T_{\mathrm{B}} \otimes \mathbb{Q}(k)$ is a $\mathbb{Q}$-de Rham–Betti class if and only if $P_{n'} \in \overline{\mathbb{Q}}^\times(2\pi i)^k$ for every n' with $v_{n'} \neq 0$. For $s \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ we have $s(v_{n'}) = v_{sn'}$, so the support of v is Galois-stable. On the other hand $\mathrm{MT}(A_E)$ acts on $T_{\mathrm{B},n'} \otimes \mathbb{Q}(k)$ through $n'|_{\mathrm{MT}} + k\nu$, and $w(n'|_{\mathrm{MT}} + k\nu) = w(n') - k\mathbf{1}_G$; so $(2\pi i)^k v$ is a Hodge class if and only if (n',k) is a Hodge relation for every n' in its support.

(i) The $\mathrm{MT}(A_E)$-invariants of $T_{\mathrm{B}} \otimes \mathbb{Q}(k) \otimes \overline{\mathbb{Q}}$ form the sum of the $T_{\mathrm{B},n'}$ over the Hodge relations (n',k). They are defined over $\mathbb{Q}$, so they are spanned by rational Hodge classes, which are $\mathbb{Q}$-de Rham–Betti classes by (I1) and (T2). A rational Hodge class with $v_n \neq 0$ exists, and the first paragraph gives $P_n \in \overline{\mathbb{Q}}^\times(2\pi i)^k$.

(ii) Since E splits T^*, the $\gamma_{\Psi,\tau}$ may be chosen in $V_E \otimes E$, so the spaces $T_{\mathrm{B},n'}$ have bases in $T_{\mathrm{B}} \otimes E$. Choose $0 \neq y \in T_{\mathrm{B},n} \cap (T_{\mathrm{B}} \otimes E)$, and $\lambda \in E$, and put $v = \sum_{s \in G} s(\lambda y) \in T_{\mathrm{B}}$. Then $v_n = \sum_{s \in H} s(\lambda)s(y)$ with H the stabilizer of n in G. By Dedekind's independence of characters, λ can be chosen with $v_n \neq 0$. The support of v is then the orbit Gn. By hypothesis and the previous paragraph, $(2\pi i)^k v$ is a $\mathbb{Q}$-de Rham–Betti class. By Theorem A

for A_E and (T2) it is fixed by $\mathrm{MT}(A_E)$, hence a Hodge class, so (n,k) is a Hodge relation. □

Without the Galois orbit ("$P_n \in \overline{\mathbb{Q}}^\times (2\pi i)^k$ alone forces a Hodge relation") this is the open $\overline{\mathbb{Q}}$-coefficient statement of [13, §4.5]. As $p(\Psi,\tau)p(\Psi,c\tau) \in \overline{\mathbb{Q}}^\times 2\pi i$ [13, §4.5], relations may be written in the holomorphic periods $p(\Psi,\tau)$, $\tau \in \Psi$.

Remark 5.3. Up to algebraic factors, the periods of the Fermat motives are the values $B(\langle ua/N\rangle, \langle ub/N\rangle)$ of the Beta function [12, Lemma 4.35]. We expect that, for Fermat Jacobians, Proposition 5.2(ii) then reads: *if* $\prod_a \Gamma(\langle ua/N\rangle)^{m_a} \in \overline{\mathbb{Q}}^\times \pi^k$ *for every* $u \in (\mathbb{Z}/N)^\times$*, with the same* k*, then* $\sum_a m_a \langle ua/N\rangle$ *is independent of* u. This is the condition in the Koblitz–Ogus theorem [5, Appendix], as recalled in [12, §4.4]. We have not carried out the translation, including the distribution relations and the normalization of k, and do not claim this statement.

6. Remarks and limits

Remark 6.1 (Non-abelian CM fields). Steps 2 and 3 use that every nonzero Galois-stable subspace of $X^*(\mathrm{MT}) \otimes \mathbb{Q}$ meets the finite set of weight differences $\mathbf{1}_S - \mathbf{1}_{S'}$. For a non-abelian Galois group G, the odd part of the relevant $\mathbb{Q}[G]$-module can contain a simple module U with multiplicity ≥ 2; the submodules isomorphic to U then form an infinite family, while a nonzero vector lies in at most one of them, so all but finitely many avoid the weight differences. This is consistent with the case analysis, half-twists and positivity arguments used by Ji for non-abelian CM fourfolds [8, 9]. Some non-abelian groups are still multiplicity-free: for $G = Q_8$, $c = -1$, we have $\mathbb{Q}[Q_8] \cong \mathbb{Q}^4 \times \mathbb{H}_\mathbb{Q}$, the odd part $\mathbb{H}_\mathbb{Q}$ is simple and contains $1-c$. We expect the argument to extend to this case, but have not written out the left/right bookkeeping.

Remark 6.2 ($\overline{\mathbb{Q}}$-coefficients). For $L = \overline{\mathbb{Q}}$ the inputs (I1)–(I3) still hold [13, Thm. 5.12], but the characters of $\mathrm{MT}(A_E)$ trivial on $G_{\mathrm{dRB}}(h(A_E)\otimes\overline{\mathbb{Q}})$ need not form a Galois-stable lattice, and Step 2 fails. If E is the compositum of two imaginary quadratic fields, $\mathbb{Q}[G]^-$ has dimension 2, and a generic rational line in it avoids the finitely many directions of weight differences. Correspondingly, $E_1 \times E_2$ with $\overline{\mathbb{Q}}$-coefficients is open [13, Rem. 5.15].

Remark 6.3 (The transcendence input). In terms of periods, Step 3 reduces any Galois-stable family of monomial relations to one relation $p(S^{-1},1)/p(S'^{-1},1) \in \overline{\mathbb{Q}}^\times$ between two periods, which (I2) excludes. So the only transcendence input is (I2), together with the transcendence of π in (I3). In [13], (I2) is derived from Wüstholz's analytic subgroup theorem [15]; with $\mathbb{Q}$-coefficients it is also a consequence of the older transcendence theorems of Schneider–Lang type [4, Thm. 3.1, Thm. 3.8],

[13, Rem. 5.5]. As said in the introduction, what is new is the passage to A_E, multiplicity-freeness and Lemma C, not a transcendence result.

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