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Sharp deterministic cubature rates for Sobolev spaces of small mixed smoothness

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Abstract

When the mixed smoothness of an integrand is only slightly above the threshold for point evaluation, the logarithmic factors in a cubature error bound can determine whether a rule achieves the best possible rate. For fixed dimension $d \geq 2$, $2 < p < \infty$, and $1/p < r \leq 1/2$, we determine the optimal deterministic worst-case integration error with at most m evaluations on Sobolev balls of dominating mixed smoothness, both for functions supported in the unit cube and for periodic functions. Its order is $m^{-r}(\log m)^{(d-1)(1-r)}$ below $r = 1/2$ and $m^{-1/2}(\log m)^{(d-1)/2}\sqrt{\log\log m}$ at $r = 1/2$. Known lattice rules attain these orders with nonnegative weights. Neither moving the nodes nor allowing signed weights can improve the asymptotic order. The lower bound uses a nonnegative smooth function vanishing at every given node. The construction shares a region of admissible translates across rectangles of different aspect ratios and uses a square-function estimate to combine the contributions of multiple layers at the endpoint.

Keywords: numerical integration, worst-case error, dominating mixed smoothness, Sobolev spaces, Frolov cubature

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1 Introduction

An error bound for a cubature rule gives a budget sufficient to achieve a prescribed accuracy. A matching lower bound over all node sets determines whether a different sampling geometry or a different weighting can reduce that budget in order. For integrands with dominating mixed smoothness, each coordinate contributes its own regularity scale, and there are many rectangles of the same volume with different aspect ratios. The number of such shapes influences the logarithmic factors in the integration error. We obtain the sharp lower bound for arbitrary node sets throughout the small-smoothness range $2 < p < \infty$, $1/p < r \leq 1/2$, in every fixed dimension $d \geq 2$.

Write $\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\,dx$ and let J^σ be the multiplier with symbol $\prod_{i=1}^d (1+\xi_i^2)^{\sigma/2}$. On $\mathbb{R}^d$ we use the space $W = \{f : J^r f \in L^p\}$ of tempered distributions with norm $\|f\|_W = \|J^r f\|_p$ and the supported subspace

$$W_0 = \{f \in W : \operatorname{supp} f \subseteq [0,1]^d\}.$$

On $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$ we use Haar probability measure and the modes $e^{2\pi i k\cdot x}$, $k \in \mathbb{Z}^d$. The periodic space W_{per} consists of distributions with norm $\|f\|_{W_{\mathrm{per}}} = \|\mathcal{B}_r f\|_p < \infty$, where $\mathcal{B}_r$ has symbol $\prod_i b_r(k_i)$ and

$$b_r(0) = 1, \qquad b_r(k) = |k|^r e^{i\pi r\,\operatorname{sgn}(k)/2} \quad (k \neq 0).$$

This is the periodic Sobolev norm defined by convolution with the Bernoulli kernel of phase r. All point evaluations use the unique continuous representatives established in section 4.

For $X = W_0$ or $X = W_{\mathrm{per}}$, let P be a finite subset of the cube or the torus, respectively. For weights $w \in \mathbb{R}^P$, set

$$e_X(P, w) = \sup_{\substack{f \text{ real} \\ \|f\|_X \le 1}} \left| \int f - \sum_{z \in P} w_z f(z) \right|.$$

The integral is over the corresponding domain. Let $\kappa_m(X)$ be the infimum of this error over $|P| \le m$ and real weights, and $\kappa_m^+(X)$ the infimum over the same node sets with nonnegative weights. For $m > e^e$ define

$$D(m) = \begin{cases} m^{-r}(\log m)^{(d-1)(1-r)}, & 1/p < r < 1/2, \\ m^{-1/2}(\log m)^{(d-1)/2}\sqrt{\log\log m}, & r = 1/2. \end{cases} \tag{1}$$

Logs without a subscript are natural. We write $\lesssim$ for a bound up to a constant and $\asymp$ when both bounds hold. These constants may depend on d, p, r, and in lattice estimates on the fixed lattice and support box.

Theorem 1. *Fix an integer $d \ge 2$, $2 < p < \infty$, and $1/p < r \le 1/2$. There are $c, C > 0$ and an integer $m_0 > e^e$, depending only on d, p, r, such that for every integer $m \ge m_0$ and both $X = W_0$ and $X = W_{\mathrm{per}}$,*

$$cD(m) \le \kappa_m(X) \le \kappa_m^+(X) \le CD(m).$$

For each P with $|P| \le m$, there is a nonnegative smooth f with $\|f\|_X = 1$, $f|_P = 0$, and $\int f \ge cD(m)$. When $X = W_0$, this function has compact support in $(0,1)^d$.

High-dimensional numerical integration in spaces of dominating mixed smoothness has been studied extensively through discrepancy bounds, quasi-Monte Carlo constructions, and randomized cubature rules [9, 2, 6, 7, 10]. Ullrich and Ullrich [11, Corollary 5.3(ii)–(iii)] obtained the upper rates in (1) on the supported cube using Frolov cubature (see also Bazarkhanov [1] for small-smoothness periodic upper bounds). The periodic cutoff construction of Nguyen, Ullrich, and Ullrich [8, Theorem 1.4 and formula (1.7)] transfers these rates to the torus. The general lower bound of [11, Theorem 7.3(ii)] (following [4]) has order $m^{-r}(\log m)^{(d-1)/2}$. The new lower bound gains a factor $(\log m)^{(d-1)(1/2-r)}$ for $r < 1/2$ and a factor $\sqrt{\log\log m}$ at $r = 1/2$.

Dũng, Temlyakov, and Ullrich [3, Proposition 8.13] record both small-smoothness rates for the bivariate Fibonacci rule. Their Conjecture 8.30 asks for the corresponding periodic lower bound over all rules in every fixed dimension. Theorem 1 proves that conjecture. For a different class of node sets, Gnewuch, Dick, Markhasin, and Sickel [5, Corollary 7(ii)] give upper estimates for equal-weight cubature on (t, m, s)-nets in mixed Bessel-potential spaces. In the small-smoothness range, their logarithmic exponent can approach $(d-1)(1-r)$ from above as the auxiliary integrability parameter approaches $1/r$. The lower bound here identifies the sharp order when the full choice of nodes and weights is available.

The proof constructs a nonnegative test from compactly supported rectangular kernels. Within each group of shapes the kernels use the same set of centers, from which translates meeting a node have been removed. A dyadic square-function estimate charges the Sobolev norm once per group. Below the endpoint one layer gives the desired bound. At the endpoint each of order $\log\log m$ layers gives the same order of integral and squared norm, so combining the layers yields the additional square root. We establish the grouped norm estimate in section 2, construct the test in section 3, and assemble the matching upper bounds in section 5.

2 A norm estimate for grouped rectangular kernels

Choose an even nonnegative $\theta \in C_c^\infty(\mathbb{R})$ supported in $[-1,1]$ with integral one. Set $\theta_j(t) = 2^j\theta(2^j t)$ and $\Theta_s(x) = \prod_{i=1}^d \theta_{s_i}(x_i)$ for $s \in \mathbb{N}_0^d$. For a nonnegative multi-index a write $|a| = \sum_i a_i$.

Lemma 2 (Grouped convolution estimate)**.** *Let $\mathcal{G}$ be a finite family of pairwise disjoint, nonempty finite subsets of $\mathbb{N}_0^d$. For each $G \in \mathcal{G}$ suppose $|s| = J_G$ for all $s \in G$, and let $h_G \in L^p(\mathbb{R}^d)$. Then*

$$\left\| \sum_{G\in\mathcal{G}} \sum_{s\in G} \Theta_s * h_G \right\|_W^2 \lesssim \sum_{G\in\mathcal{G}} 2^{2rJ_G} \|h_G\|_p^2. \tag{2}$$

The constant is uniform in the family, including the sizes of the groups.

Proof. Let φ_0 be an even, smooth cutoff, nonincreasing in $|t|$, equal to one for $|t| \le 1$ and zero for $|t| \ge 3/2$. For $j \ge 1$ put $\varphi_j(t) = \varphi_0(2^{-j}t) - \varphi_0(2^{1-j}t)$. Let Δ_a have symbol $\prod_i \varphi_{a_i}(\xi_i)$. The mixed Littlewood–Paley and lifting theorems [12, Definition 1.2(ii) and Theorems 1.9(ii), 1.15] give

$$\|J^\sigma h\|_p \asymp \|(2^{\sigma|a|}\Delta_a h)_a\|_{L^p(\ell^2)} \qquad (\sigma = 0, r). \tag{3}$$

Here the square sum is taken pointwise before the L^p norm. Write M_i for the centered Hardy–Littlewood maximal operator in coordinate i, and $M = M_d \circ \cdots \circ M_1$. The directional vector maximal inequality [12, Theorem 1.11], iterated in the coordinates, gives

$$\|(Mg_b)_b\|_{L^p(\ell^2)} \lesssim \|(g_b)_b\|_{L^p(\ell^2)}. \tag{4}$$

It applies to countable lists by monotone convergence.

We use the following elementary kernel estimate. If two fixed Schwartz kernels are dilated at indices $u \le v$, the absolute value of their convolution is bounded by a constant times $D_u(x) = 2^u(1 + 2^u|x|)^{-2}$. If $u < v$ and the finer kernel has mean zero, the bound improves by 2^{u-v}. In fact the coarse kernel and its derivative are bounded by CD_u and $C2^uD_u$. Under translation by y, each bound costs at most $(1 + 2^u|y|)^2$. Integrating this factor against the finer kernel uses a uniformly bounded zeroth absolute moment. With mean zero, subtract the value of the coarse kernel at x and use the derivative bound and the weighted first absolute moment, which is $O(2^{-v})$. Convolution with D_u in coordinate i is bounded by a constant times M_i, by splitting the integral into $|y| \le 2^{-u}$ and annuli with radii $2^{\ell-u}$, $\ell \ge 1$. The annuli give a summable factor $2^{-\ell}$.

Use the companion symbols $\nu_0(t) = \varphi_0(t/2)$ and $\nu_j(t) = \varphi_0(2^{-j}t/2)(1 - \varphi_0(4 \cdot 2^{-j}t))$ for $j \ge 1$. Each is one on $\operatorname{supp}\varphi_j$, and the corresponding inverse Fourier kernel has mean zero when $j \ge 1$. Insert these symbols on $\Delta_a h$. The coordinate estimate just proved gives

$$|\Delta_a(\Theta_s * h)| \lesssim 2^{-\sum_i (a_i - s_i)_+} M(\Delta_a h), \qquad t_+ = \max(t, 0). \tag{5}$$

The cancellation factor occurs in each coordinate where $a_i > s_i$. The multiplier identity and the estimate hold on L^p by convolution with the integrable kernels.

Define the summable weight on $\mathbb{Z}^d$

$$\omega(v) = \prod_i 2^{-(1-r)(v_i)_+ - r(-v_i)_+}, \qquad B = \sum_{v\in\mathbb{Z}^d} \omega(v) < \infty.$$

Write $F = \sum_G \sum_{s\in G} \Theta_s * h_G$ and $\omega_G(a) = \sum_{s\in G} \omega(a - s)$. Multiplying (5) by $2^{r|a|}$ gives $2^{r|a|}|\Delta_a F| \lesssim \sum_G \omega_G(a)\, 2^{rJ_G} M(\Delta_a h_G)$. At each a, all offsets $a - s$ are distinct, so $\sum_G \omega_G(a) \le B$. Weighted Cauchy–Schwarz and $\omega_G(a) \le B$ give

$$2^{2r|a|}|\Delta_a F|^2 \lesssim B^2 \sum_G 2^{2rJ_G} |M(\Delta_a h_G)|^2. \tag{6}$$

Sum over a and apply (3) and (4), with the list in the latter indexed by (a, G). Then the triangle inequality in $L^{p/2}$ gives

$$\|F\|_W^2 \lesssim \left\| \sum_G 2^{2rJ_G} \sum_a |\Delta_a h_G|^2 \right\|_{p/2} \le \sum_G 2^{2rJ_G} \|(\Delta_a h_G)_a\|_{L^p(\ell^2)}^2.$$

Use (3) at $\sigma = 0$ to conclude. □

3 The lower bound for arbitrary nodes

Represent the nodes by $P \subset [0,1]^d$ on the cube or $P \subset [0,1)^d$ on the torus. Set

$$L = \lceil \log_2 m \rceil, \quad K = L^{d-1}, \quad H = \lfloor \log_2 K \rfloor, \quad J_v = L + v + 2d + 1 \quad (0 \le v \le H).$$

At layer v take every s with $s_i \ge 3$ and $|s| = J_v$. There are $T_v = \binom{J_v - 3d + d - 1}{d-1}$ such shapes, by subtracting three from each coordinate. Since $H = O_d(\log L)$, the $d-1$ numerator factors in the binomial coefficient are comparable to L, uniformly in v, for sufficiently large L. Thus $T_v \asymp K$. Partition the layer into groups of size 2^v and at most one smaller nonempty group. The number of groups is at most $1 + T_v/2^v \lesssim K 2^{-v}$, using $2^v \le K$.

For each group G in layer v, use the common set of centers

$$E_G = [1/4, 3/4]^d \setminus \bigcup_{\substack{z \in P \\ s \in G}} \left(z + \prod_{i=1}^d [-2^{-s_i}, 2^{-s_i}] \right).$$

The volume of the union is at most

$$m\, 2^v\, 2^{d - J_v} \le 2^{-d-1}. \tag{7}$$

Consequently $|E_G| \ge 2^{-d-1}$. For each $s \in G$, the convolution $\Theta_s * \mathbf{1}_{E_G}$ is smooth and nonnegative. It is supported in $[1/8, 7/8]^d$ because $s_i \ge 3$, it vanishes on P by the definition of E_G, and its integral is $|E_G|$ since Θ_s has unit mass.

Let V be a nonempty set of layers and take F to be the sum of all these convolutions on the layers in V. Each layer has $\asymp K$ summands, and $\|\mathbf{1}_{E_G}\|_p \le 1$. By theorem 2,

$$\int F \gtrsim K|V|, \qquad \|F\|_W^2 \lesssim K\, 2^{2rL} \sum_{v \in V} 2^{(2r-1)v}. \tag{8}$$

If $r < 1/2$, choose $V = \{H\}$. Since $2^H \asymp K$, the squared norm is at most a constant times $2^{2rL} K^{2r}$. Dividing by the norm gives a lower bound of order $2^{-rL} K^{1-r}$. If $r = 1/2$, choose every layer. The integral is at least a constant times $K(H+1)$ and the squared norm is at most a constant times $2^L K(H+1)$. Their quotient after taking the square root in the denominator has order at least $2^{-L/2}\sqrt{K(H+1)}$. Since $2^L \asymp m$, $K \asymp (\log m)^{d-1}$, and $H + 1 \asymp \log\log m$, these are the bounds in (1).

On the cube use $F/\|F\|_W$. On the torus periodize F and normalize in W_{per}. The periodization preserves its integral and node values and has norm at most a constant times $\|F\|_W$ by theorem 3. In either case the integral of the normalized test is a lower bound for the error for every choice of weights on P. This proves the lower bound of theorem 1.

4 Periodization and cutoff

Let $J^r_{\mathbb{T}}$ have symbol $\prod_i (1+4\pi^2 k_i^2)^{r/2}$. The ratio of its one-dimensional symbol to $b_r(k)$ is one at zero. For $k \neq 0$ the ratio is

$$(4\pi^2 + k^{-2})^{r/2} e^{-i\pi r \operatorname{sgn}(k)/2}.$$

Both the ratio and its reciprocal are bounded and have uniformly bounded variation on positive and negative dyadic integer intervals, by monotonicity of their moduli and constant phase on each half-line. The periodic Marcinkiewicz multiplier theorem [3, Theorem 11.10], applied in each coordinate by Fubini, gives $\|J^r_{\mathbb{T}} f\|_p \asymp \|f\|_{W_{\mathrm{per}}}$. The Fourier coefficients of the Bernoulli kernel at phase r in [3, Definition 3.1] are $\prod_i b_r(k_i)^{-1}$ after changing the period from 2π to one. This identifies the norm used in Conjecture 8.30 of that reference.

In one dimension the inverse Bessel multiplier has kernel

$$G_r(x) = \frac{1}{\Gamma(r/2)} \int_0^\infty e^{-t} t^{r/2-1} (4\pi t)^{-1/2} e^{-x^2/(4t)}\, dt.$$

This follows from the Fourier transform of the Gaussian and the gamma integral. The kernel is nonnegative with mass one, and its integer periodization is the corresponding inverse kernel on $\mathbb{T}$. For $0 < |x| \leq 1$, dropping e^{-t} and substituting $u = x^2/(4t)$ gives $G_r(x) \lesssim |x|^{r-1}$. For $|x| \geq 1$, using $e^{-x^2/(4t)} \lesssim t^{3/2}|x|^{-3}$ gives a bound by $C_r|x|^{-3}$. Thus both kernels belong to L^q for $q = p/(p-1)$ because $q(r-1) > -1$. Their tensor products give continuous representatives via Hölder's inequality and continuity of translations in L^q. The representatives are unique by equality almost everywhere, and these estimates bound their supremum norms. In particular, smooth periodic approximations obtained by smoothing $J^r_{\mathbb{T}} f$ converge both in W_{per} and uniformly.

Lemma 3 (Transfers between the line and the torus)**.** *For a fixed box Ω, the periodization of any smooth h supported in Ω has W_{per} norm at most $C_\Omega \|h\|_W$. If $\Omega \subset (0,1)^d$, periodization preserves the integral and all values on $[0,1)^d$. Conversely, for any fixed $\chi \in C_c^\infty(\mathbb{R})$ the lift $x \mapsto f(x \bmod 1) \prod_i \chi(x_i)$ has W norm at most $C_\chi \|f\|_{W_{\mathrm{per}}}$.*

Proof. Write J^r_1 and $J^r_{\mathbb{T},1}$ for the one-dimensional factors, H_t for convolution with the Gaussian of variance $2t$ on the line or its periodization, and $\alpha = r/2$. On smooth inputs the one-dimensional formula is

$$J^r_1 h = h + \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty e^{-t} t^{-1-\alpha} (h - H_t h)\, dt, \tag{9}$$

with $J^r_{\mathbb{T},1}$ in the periodic case. Indeed the corresponding scalar multiplier is $(1+z)^\alpha$: differentiating the scalar integral in z gives $\alpha(1+z)^{\alpha-1}$, and the multiplier equals one at $z = 0$.

Suppose h is supported in a fixed interval I. At distance at least one from I, the same Gaussian bound used for G_r gives from (9)

$$|J^r_1 h(x)| \lesssim \int_I |x-y|^{-3} |h(y)|\, dy.$$

It follows that the sum of integer translates of $J^r_1 h$ on $[0,1)$ converges in L^p with norm bounded by $C_I(\|J^r_1 h\|_p + \|h\|_p)$. Finitely many near translates are bounded by $\|J^r_1 h\|_p$, and the far translates are bounded by the summable tails and Hölder's inequality on I. The Fourier coefficients of the sum are $(1+4\pi^2 k^2)^{r/2} \widehat{h}(2\pi k)$, by unfolding the integral. Thus the sum is $J^r_{\mathbb{T},1}$ applied to the periodization of h. The mass-one inverse kernel gives $\|h\|_p \leq \|J^r_1 h\|_p$. The same bounds hold in joint L^p by Fubini and apply successively to each coordinate, since operations in other coordinates preserve support in

the current coordinate. This yields the first norm bound. For interior support only the original translate contributes on $[0,1)^d$, and unfolding always preserves the integral.

For the cutoff estimate, let h be smooth and periodic on $\mathbb{T}$. By (9), the difference $J_1^r(\chi h)-\chi J_{\mathbb{T},1}^r h$ is the integral of $\chi H_t h - H_t(\chi h)$ against the weight in (9). For a translation by y, the difference $\chi(x)-\chi(x-y)$ is supported in two intervals of fixed length and has supremum at most $C_\chi \min(|y|,1)$. Therefore

$$\|(\chi(\cdot)-\chi(\cdot-y))h(\cdot-y)\|_{L^p(\mathbb{R})} \le C_\chi \min(|y|,1)\|h\|_{L^p(\mathbb{T})}.$$

Integrating against the Gaussian gives a bound by $C_\chi \min(\sqrt{t},1)\|h\|_{L^p(\mathbb{T})}$, integrable against $e^{-t}t^{-1-\alpha}$ since $r<1$. The term $\chi J_{\mathbb{T},1}^r h$ is bounded by $C_\chi\|J_{\mathbb{T},1}^r h\|_p$, and $\|h\|_p \le \|J_{\mathbb{T},1}^r h\|_p$ follows from the mass-one inverse. Iterate in the coordinates by Fubini. Periodic smooth approximation gives the asserted bound for all $f \in W_{\mathrm{per}}$, with uniform convergence identifying the limit with the stated lift. □

5 Matching positive-weight rules

Choose a real matrix A with $|\det A| = 1$ and

$$\inf_{u\in A^{-\mathsf{T}}\mathbb{Z}^d\setminus\{0\}} \prod_{i=1}^d |u_i| > 0. \tag{10}$$

A matrix with this property exists in every $d \ge 2$, as shown in section B. Given a fixed axis-parallel box Ω and a shift $\tau \in \mathbb{R}^d$, define

$$Q_{n,\tau}^{\Omega}(f) = \frac{1}{n} \sum_{z\in(\tau+n^{-1/d}A\mathbb{Z}^d)\cap\Omega} f(z).$$

The matching supported-cube rates are those of [11, Corollary 5.3(ii)–(iii)]. For the rule above, section A gives the corresponding estimate for any fixed support box, with the constant uniform over all translations. In particular, for $\Omega=[0,1]^d$, the number of nodes is $\asymp n$ and the error on W_0 is $\lesssim D(n)$. Applying the lower bound with budget $\lceil C_A n\rceil$ bounds the worst-case error below by a constant times $D(n)$ as well. We obtain the following consequence.

Corollary 4. *Under the hypotheses of theorem 1, if $|\det A| = 1$ and (10) holds, the rule $Q_{n,\tau}^{[0,1]^d}$ uses $\asymp n$ nodes and has worst-case error $\asymp D(n)$ on W_0 for all sufficiently large n, uniformly in $\tau \in \mathbb{R}^d$.*

For the torus use the periodic cutoff rule of Nguyen, Ullrich, and Ullrich [8, Theorem 1.4 and formula (1.7)] (formula (1.11) in the preprint). Choose a nonnegative smooth bump ρ supported in $[-1,1]$ and positive on $(-1,1)$. Then

$$\chi(t) = \frac{\rho(t)}{\sum_{j\in\mathbb{Z}}\rho(t+j)}, \qquad \psi(x) = \prod_i \chi(x_i)$$

are smooth and nonnegative, and $\sum_{k\in\mathbb{Z}^d}\psi(x+k)=1$. For $F(x)=\psi(x)f(x \bmod 1)$, splitting $\mathbb{R}^d$ into unit cubes gives $\int_{\mathbb{R}^d} F = \int_{\mathbb{T}^d} f$, and theorem 3 gives $\|F\|_W \lesssim \|f\|_{W_{\mathrm{per}}}$. Apply $Q_{n,0}^{[-1,1]^d}$ to F. The resulting torus nodes are $z \bmod 1$, with weights $\psi(z)/n$. Adding the weights of coincident nodes preserves nonnegativity and leaves at most $C_A n$ nodes. The rule has error at most a constant times $D(n)$ by the supported-box estimate.

Choose $n = \lfloor m/C_A\rfloor$ with $C_A \ge 1$ fixed and large enough for the node counts on both support boxes. The fixed proportional rescaling gives $D(n) \asymp D(m)$ for large m. Thus each rule meets the evaluation budget and proves the upper bound in theorem 1.

A The Frolov estimate uniform in translation

We give details of the shifted estimate using the Fourier argument of Ullrich and Ullrich [11, Section 5.1]. Under (10), for any fixed bounded axis-parallel box Ω we prove

$$\left| \int_{\mathbb{R}^d} f - Q_{n,\tau}^{\Omega}(f) \right| \lesssim D(n)\|f\|_W \qquad (\operatorname{supp} f \subseteq \Omega), \tag{11}$$

uniformly in τ for sufficiently large n.

A.1 Lattice counts

Let $T = n^{-1/d}A$ and $B_n = 2\pi T^{-\mathsf{T}}$. By (10), for distinct points of $B_n\mathbb{Z}^d$ the product of their absolute coordinate differences is at least γn for a fixed $\gamma > 0$. An axis-parallel box of volume $U > 0$ therefore contains at most $\lfloor U/(\gamma n) \rfloor + 1$ dual points. To see this, divide its first side into that many strips of equal width, and assign boundary points to only one strip. Two points assigned to the same strip have product of coordinate differences at most its volume, which is strictly less than γn.

Cells $z + T[-1/2, 1/2)^d$ centered at $z \in \tau + T\mathbb{Z}^d$ tile space. They have volume $1/n$ and lie within sup-norm distance $\delta_n = n^{-1/d} \max_{u \in [-1/2,1/2]^d} \|Au\|_\infty$ of their centers. Cells with centers in Ω lie in its δ_n-enlargement, yielding $O_{A,\Omega}(n)$ nodes uniformly in τ. For the unit cube, those cells cover its δ_n-erosion, so the node count is at least $n(1 - 2\delta_n)^d$ for large n.

A.2 Local kernels and lattice periodization

Use the dyadic partition of section 2. Put $\eta = \theta_2$, $\Psi_0 = \eta$, and $\Psi_j(t) = 2^j(-\eta'')(2^j t)$ for $j \geq 1$. Then $\widehat{\eta}(t) \geq \cos(1/2) > 0$ for $|t| \leq 2$ by evenness, unit mass, and support in $[-1/4, 1/4]$. For $j \geq 1$, $\widehat{\Psi}_j(t) = (2^{-j}t)^2\widehat{\eta}(2^{-j}t)$. Thus $\lambda_j = \varphi_j/\widehat{\Psi}_j$ on a neighborhood of $\operatorname{supp}\varphi_j$, extended by zero outside the support, is a smooth symbol, and the high-index symbols are rescalings of one fixed symbol. Write $\Psi_a = \bigotimes_i \Psi_{a_i}$, $\lambda_a(\xi) = \prod_i \lambda_{a_i}(\xi_i)$, and $K_a = \mathcal{F}^{-1}\lambda_a$. Then

$$\lambda_a \widehat{\Psi}_a = \prod_i \varphi_{a_i}, \qquad \sup_a (\|\lambda_a\|_\infty + \|K_a\|_1) < \infty. \tag{12}$$

Moreover,

$$\|(2^{r|a|}\Psi_a * f)_a\|_{L^p(\ell^2)} \lesssim \|f\|_W. \tag{13}$$

For this estimate insert $\sum_b \Delta_b f = f$ and the companion symbols ν_{b_i} on each block. When $a_i \neq b_i$, the finer kernel has mean zero, since its index is positive. The kernel estimate from section 2 gives

$$2^{r|a|}|\Psi_a * \Delta_b f| \lesssim \prod_i 2^{-|a_i - b_i| + r(a_i - b_i)}\, M(2^{r|b|}\Delta_b f).$$

The offset sequence on the right is summable since $0 < r < 1$. Extend the block list by zero to $\mathbb{Z}^d$. Apply the triangle inequality to translates of this list in ℓ^2, followed by (4) and (3). This gives (13) first for finite block sums. The rectangular partial sums of the partition converge in distributions, and the same estimate on weighted input tails gives convergence in $L^p(\ell^2)$ of the outputs.

For smooth compactly supported h define the function on $\mathbb{T}^d$

$$\mathcal{P}_\tau h(y) = \frac{1}{n} \sum_{k \in \mathbb{Z}^d} h(T(y + k) + \tau).$$

Its Fourier coefficient at ℓ is $e^{iB_n\ell\cdot\tau}\widehat{h}(B_n\ell)$: unfold $y + k$ and use $x = T(y + k) + \tau$ with $|\det T| = 1/n$. The same identity holds for Schwartz h by absolute convergence. For a list $(h_a)_a$ supported in

a common fixed box there are at most Cn active summands for every y, by the cell count. The triangle inequality in ℓ^2 and Hölder's inequality over those summands yield

$$\|(\mathcal{P}_\tau h_a)_a\|_{L^p(\ell^2)}^p \le (Cn)^{p-1} n^{-p} \int_{[0,1]^d} \sum_k \|(h_a(T(y+k)+\tau))_a\|_{\ell^2}^p \, dy.$$

Unfolding changes the integral on the right to n times the pth power of the line-space list norm. Thus

$$\|(\mathcal{P}_\tau h_a)_a\|_{L^p(\ell^2)} \lesssim \|(h_a)_a\|_{L^p(\ell^2)}. \tag{14}$$

A.3 The error and the logarithmic factors

For smooth f supported in Ω, $Q_{n,\tau}^\Omega(f) = \mathcal{P}_\tau f(0)$, and the constant Fourier coefficient is $\int f$. Let $Z_a(y) = \sum_{\ell \neq 0} \lambda_a(B_n \ell) e^{2\pi i \ell \cdot y}$. Then, using (12),

$$Q_{n,\tau}^\Omega(f) - \int f = \sum_a \int_{\mathbb{T}^d} \mathcal{P}_\tau(\Psi_a * f)(y) \overline{Z_a(y)} \, dy.$$

The symbols λ_a are real, and the rearrangement is absolutely convergent because $\sum_\ell |\widehat{f}(B_n \ell)| < \infty$ and the nonnegative dyadic partition sums to one. The supports of $\Psi_a * f$ lie in one fixed enlarged box. Cauchy–Schwarz in a, Hölder in y, and (13)–(14) bound the absolute error by

$$C\|f\|_W \, \|(2^{-r|a|} Z_a)_a\|_{L^q(\ell^2)}, \qquad q = p/(p-1) < 2. \tag{15}$$

We estimate this list norm by first restricting to finite lists and then using monotone convergence.

On $\operatorname{supp} \lambda_a$ one has $|\xi_i| \le (3/2) 2^{a_i}$. Let $L = \lfloor \log_2 n \rfloor$ and $j_0 = L - C_0$, with the fixed integer $C_0 \ge 0$ chosen so that $(3/2)^d 2^{-C_0} < \gamma$. If $|a| < j_0$, the support contains no nonzero dual point, by the product bound. For $|a| \ge j_0$, the enclosing box has volume $O(2^{|a|})$, and $2^{|a|}/n \ge 2^{-C_0 - 1}$, so the strip count is $O(2^{|a|}/n)$. By Parseval and the triangle inequality, respectively,

$$\|Z_a\|_2^2 \lesssim 2^{|a|}/n, \qquad \|Z_a\|_\infty \lesssim 2^{|a|}/n.$$

Also $Z_a = \mathcal{P}_0 K_a - \lambda_a(0)$ by the Fourier coefficient formula, and hence $\|Z_a\|_1 \le 2\|K_a\|_1 \lesssim 1$ by unfolding. Using $q - 1 = q/p$ gives $\|Z_a\|_q^q \le \|Z_a\|_\infty^{q-1} \|Z_a\|_1 \lesssim (2^{|a|}/n)^{q/p}$.

Set $K = L^{d-1}$ and $H = \lfloor \log_2 K \rfloor$. At level j there are $\binom{j+d-1}{d-1}$ multi-indices a with $|a| = j$. There are $O(K)$ such indices per level on $j_0, \dots, j_0 + H$. On the probability torus, L^2 dominates L^q, so the square of the contribution from these levels is at most

$$Cn^{-2r} K \sum_{v=0}^{H} 2^{(1-2r)v}. \tag{16}$$

The geometric sum bounds this by $Cn^{-2r} K^{2-2r}$ for $r < 1/2$, and at the endpoint it is $Cn^{-1} K(H+1)$.

On levels $j = j_0 + H + t$, $t \ge 1$, the binomial count is $O(K(1+t)^{d-1})$. Since $q/2 < 1$, the qth power of the tail norm in (15) is bounded by summing the qth powers of the individual terms. The bound just obtained for $\|Z_a\|_q$ then gives

$$Cn^{-rq} K \, 2^{-q(r-1/p)H} \sum_{t \ge 1} (1+t)^{d-1} 2^{-q(r-1/p)t}. \tag{17}$$

The series converges as $r > 1/p$. Taking the qth root and using $2^H \asymp K$ gives $O(n^{-r} K^{1-r})$. Together (16) and (17) bound (15) by $CD(n)\|f\|_W$.

For general $f \in W$ supported in Ω, mollify with compactly supported smooth approximate identities. They commute with J^r, give convergence in W and hence uniformly by section 4, and keep all supports in a single enlarged box. For fixed n, τ the lattice sum over that box is finite and converges to the original sum, while the integrals converge by uniform convergence on the box. The bound for the enlarged box therefore proves (11).

B A lattice in every dimension

Let $P_d(t) = \prod_{j=1}^{d}(t-4j)+2$. The product changes sign between $4j-1$ and $4j+1$ and its magnitude at both endpoints is at least $3^{d-1} > 2$. The d disjoint intervals thus contain d distinct real roots ξ_j. The polynomial reduces to t^d modulo two, and its constant term is two modulo four, so it is irreducible by Eisenstein's criterion.

Let $V_{ij} = \xi_i^{j-1}$ and $A = |\det V|^{-1/d} V$. If g is a nonzero integer polynomial of degree less than d, multiplication by g modulo P_d has an integral matrix on the monomial basis. Evaluation at the roots diagonalizes the matrix, so its determinant $\prod_i g(\xi_i)$ is an integer. The determinant is nonzero by irreducibility. In particular $\prod_i |(Vz)_i| \geq 1$ for $z \in \mathbb{Z}^d \setminus \{0\}$.

Let $D_0 = \operatorname{diag}(1/P_d'(\xi_i))$ and $M_0 = V^{\mathsf{T}} D_0 V$. For $k \geq 0$ the sum $\sum_i \xi_i^k / P_d'(\xi_i)$ is the coefficient of t^{d-1} in the remainder of t^k modulo P_d, by taking leading coefficients in its Lagrange interpolation formula. These sums are integers and are zero for $k < d-1$ and one for $k = d-1$. Reversing the columns of M_0 therefore gives a lower unit triangular integer matrix, so $M_0^{-1}\mathbb{Z}^d = \mathbb{Z}^d$. Now $V^{-\mathsf{T}} = D_0 V M_0^{-1}$, and for $z \in \mathbb{Z}^d \setminus \{0\}$ the preceding product bound gives

$$\prod_i |(A^{-\mathsf{T}} z)_i| \geq \frac{|\det V|}{\prod_i |P_d'(\xi_i)|} > 0. \tag{18}$$

This gives (10) and completes the construction.

Data and code availability

This paper is a theoretical mathematical proof; no empirical datasets were generated or analyzed.

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