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SHARP NEIGHBORHOOD BOUNDS FOR MAXIMUM-LEAF SPANNING TREES AND A PROOF OF WRITTEN ON THE WALL II CONJECTURE 165

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Abstract. Let G be a finite simple connected graph, and let $L_s(G)$ be the maximum number of leaves in a spanning tree of G. For every nonempty set S of vertices of degree at least three, we prove

$$|N_G(S) \setminus S| \le 4L_s(G) - 8, \qquad |N_G[S]| \le 6L_s(G) - 12.$$

The constants are sharp on a family due to Craig Foster. The first inequality proves Written on the Wall II Conjecture 165 and determines the optimal coefficient in Foster's companion question. The proof reduces to graphs with an independent complement of S and uses two potentials on the same sequence of tree extensions. We also prove that a proposed mixed-weight bound has a unique smallest counterexample, of order five.

1. Introduction

All graphs are finite and simple. We use only nonempty connected graphs in statements about spanning trees. A leaf has tree degree one, so $L_s(K_1) = 0$. For $A \subseteq V(G)$, write $N_G(A) = \bigcup_{v\in A} N_G(v)$, $\partial_G A = N_G(A) \setminus A$, and $N_G[A] = A \cup N_G(A)$. Let $M(G)$ be the set of vertices of degree $\Delta(G)$, and put $V_{\ge 3}(G) = \{v : d_G(v) \ge 3\}$.

Conjecture 165 in DeLaViña's *Written on the Wall II* asks whether

$$L_s(G) \ge \left\lceil \sqrt{2|\partial_G M(G)|} \right\rceil \tag{1}$$

for every connected G [4, Conjecture 165]. The source credits Craig Foster's 2005 project, poses the companion question of a constant $c > 0$ with $L_s(G) \ge c|\partial_G M(G)|$, and records Foster's examples. Foster's 2006 conference abstract describes work on these graph invariants and reports results on Graffiti.pc conjectures without identifying their statements or numbers [5]. The accessible record leaves open whether that project contained a proof of (1) or of a linear bound.

Theorem 1.1. *Let G be connected and S a nonempty subset of $V_{\ge 3}(G)$. Then*

$$L_s(G) \ge \tfrac{1}{4}|\partial_G S| + 2, \qquad L_s(G) \ge \tfrac{1}{6}|N_G[S]| + 2. \tag{2}$$

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In each inequality the coefficient and the additive constant are best possible with the other held fixed.

The proof uses the live and dead leaves of a growing tree, a method developed for minimum-degree bounds by Kleitman and West [7]. Here one count charges the vertices of an independent set and another charges every vertex of the growing tree. The two counts have different exceptional terminal steps.

2. Reduction to an independent complement

Lemma 2.1. *Every subtree with at least two vertices in a connected graph extends to a spanning tree with at least as many leaves.*

Proof. While a vertex is missing, connectedness supplies an edge from the tree to a missing vertex. Add that vertex and edge. One leaf is created and at most one old leaf is lost. □

In particular, a star at a vertex of maximum degree extends to a spanning tree, giving $L_s(G) \geq \Delta(G)$ if $\Delta(G) \geq 1$.

Lemma 2.2. *Let G be connected and let $T_1, \ldots, T_c$ be pairwise vertex-disjoint subtrees with at least two vertices each, where $c \geq 1$. Some spanning tree T of G satisfies*

$$L(T) - 2 \;\geq\; \sum_{i=1}^{c} (L(T_i) - 2).$$

Proof. Contract the vertex set of each T_i to a terminal i. Delete loops and replace each parallel class by one edge, recording an original representative for each remaining edge. The resulting graph is connected because each original path maps to a walk after deletion of repeated consecutive vertices. Take a spanning tree of this graph and repeatedly prune leaves outside the set of terminals. The resulting tree Q contains all terminals. If it has a nonterminals, each has degree at least two, and therefore

$$\sum_{i=1}^{c} d_Q(i) \leq 2(c + a - 1) - 2a = 2(c - 1).$$

This also holds for the single-vertex tree when $c = 1$.

Lift the edges of Q to their representatives and put back the trees T_i. The resulting graph is connected by following paths in Q and connecting their lifted endpoints within each T_i. The edge count is the vertex count minus one, since replacing a terminal by T_i adds $|V(T_i)| - 1$ vertices and edges. Thus the graph is a tree. Each lost old leaf is incident with a lifted edge and costs at most one of the terminal incidences counted above. Lemma 2.1 completes the extension to G. □

Lemma 2.3 (Local tree bounds)**.** *Let C be connected with $V(C) = S \sqcup B$, where $S \neq \varnothing$, every vertex of S has degree at least three in C, and B is*

independent. There is a spanning tree with $4L(T) - |B| \geq 8$*, and there is a spanning tree with* $6L(T) - |V(C)| \geq 12$*.*

To derive (2) from Lemma 2.3, keep the vertices $S \cup \partial_G S$ and precisely the edges of G incident with S. Every retained vertex outside S has a neighbor in S. Each component C_i contains a vertex of S, retains the full degree of each of its vertices in S, and has an independent set $B_i = V(C_i) \setminus S$. For the first bound choose a local tree in each C_i with $4(L(T_i) - 2) \geq |B_i|$. The sum of the $|B_i|$ is $|\partial_G S|$. For the second bound choose local trees with $6(L(T_i) - 2) \geq |V(C_i)|$. The component orders sum to $|N_G[S]|$. Apply Lemma 2.2 to each collection. The argument includes $\partial_G S = \varnothing$. Optimality is proved in Section 5.

3. The tree extensions

For a tree $T \subseteq C$ write $X = V(T)$, $U = V(C) \setminus X$, $\ell = L(T)$, $v_T = |X|$, and $b = |B \cap X|$. A leaf of T is live if it has a neighbor in U, and dead otherwise. Write λ for the number of live leaves. We maintain the invariant that all neighbors of every internal vertex of T belong to X. An initial star at any $r \in S$, with all its neighbors, satisfies it. At any step the old endpoints of edges crossing from X to U are live leaves. Connectedness implies that $\lambda = 0$ exactly when $U = \varnothing$. As U shrinks, an old dead leaf cannot become live.

If $U \neq \varnothing$, apply the first applicable group below. All additions use the listed edges only. In particular, extra edges of C between newly added vertices do not belong to the tree. Primes denote the new counts.

Rule 1.: If a live leaf x has $k \geq 2$ neighbors in U, attach all k to x. The only new internal vertex is x, whose outside neighbors are all added. We have $\Delta\ell = k - 1$, $\Delta v_T = k$, $\Delta b \leq k$, and $\lambda' \leq \lambda + k - 1$.

Rule 2.: Otherwise each live leaf has a unique neighbor in U. If some live leaf x has that neighbor $s \in S$, set $p = |N_C(s) \cap X|$ and $r' = |N_C(s) \cap U|$. Then $p \geq 1$, $p + r' \geq 3$, and the p old neighbors are all live leaves that lose their only outside neighbor when s is added.

2a.: If $r' \geq 2$, add xs and attach all r' outside neighbors to s. Both new internal vertices x, s have no remaining outside neighbor. Then $\Delta\ell = r' - 1$, $\Delta v_T = r' + 1$, $\Delta b \leq r'$, and $\lambda' \leq \lambda - p + r'$.

2b.: If $r' = 0$, add xs. The new leaf s is dead and x is the only new internal vertex. Then $\Delta\ell = 0$, $\Delta v_T = 1$, $\Delta b = 0$, and $\lambda' \leq \lambda - p$, with $p \geq 3$.

2c.: If $r' = 1$, let z be the outside neighbor of s and add xs, sz, as in Figure 1. The new internal vertices x, s are closed to the remaining outside vertices. Then $\Delta\ell = 0$, $\Delta v_T = 2$, $\Delta b = \mathbf{1}_{z \in B}$, and $\lambda' \leq \lambda - p + 1$, with $p \geq 2$. If z is dead, then $\lambda' \leq \lambda - p$.

Rule 3.: In the remaining case all live leaves belong to S, since their unique outside neighbors belong to the independent set B. No vertex of $U \cap S$ has a neighbor in X, by the invariant and the failure of

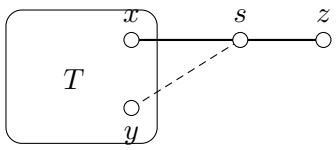

FIGURE 1. Rule 2c. The solid path xsz is added through x. The second old live leaf y loses its only outside neighbor s. A zero-gain 2c step for the closed-neighborhood potential has no edge from z to the old tree.

Rule 2. Choose a live leaf x with unique outside neighbor $w \in B$. Put $t = |N_C(w) \cap X|$ and $r' = |N_C(w) \cap U|$. Each of the $t \geq 1$ old neighbors is a live leaf with unique outside neighbor w; all r' outside neighbors belong to S.

- **3a.:** If $r' = 0$, add xw. The new leaf w is dead; x is the only new internal vertex. Then $\Delta\ell = 0$, $\Delta v_T = \Delta b = 1$, and $\lambda' = \lambda - t$.
- **3b.:** If $r' \geq 2$, add xw and attach the r' outside neighbors to w. Neither x nor w has a remaining outside neighbor. Then $\Delta\ell = r' - 1$, $\Delta v_T = r' + 1$, $\Delta b = 1$, and $\lambda' \leq \lambda - t + r'$.
- **3c.:** If $r' = 1$ and $t \geq 2$, add xw. The new leaf w is live and x has no remaining outside neighbor. Then $\Delta\ell = 0$, $\Delta v_T = \Delta b = 1$, and $\lambda' = \lambda - t + 1$.
- **3d.:** If $r' = t = 1$, let $s \in U \cap S$ be the unique outside neighbor of w. This s has no neighbor in X, so all $q = d_C(s) - 1 \geq 2$ neighbors of s other than w belong to $U \setminus \{w, s\}$. Add xw, ws and attach those q vertices to s. The outside neighbors of x, w, s are all added. Then $\Delta\ell = q - 1$, $\Delta v_T = q + 2$, $\Delta b \leq 1 + q$, and $\lambda' \leq \lambda - 1 + q$. If a new leaf lies in S, then $\Delta b \leq q$.

In each operation the added vertices form a tree attached by one edge to x. Thus the result is a tree and the neighbor invariant holds by the checks above. The bounds on live leaves count the new leaves and subtract the old live leaves whose only outside neighbor was added, or which became internal. For 3a and 3c, w is the only new vertex, so the count is exact. The rules exhaust all possibilities and every step adds a vertex. The process therefore ends at a spanning tree.

4. The two potentials

Put $g(a) = \max(0, a - 1)$. For nonnegative integers, g is nondecreasing and $g(a + d) - g(a) \leq d$ for $d \geq 0$. If $a' \leq a - j$ and $a \geq j + 1$, then $g(a) - g(a') \geq j$. Define

$$\Psi = 4\ell - b - g(\lambda), \qquad \Phi = 6\ell - v_T - 2g(\lambda).$$

Let $B_1 = \{w \in B : d_C(w) = 1\}$. In Case I, $B_1 \neq \varnothing$, choose as the center of the initial star the neighbor of a vertex in B_1; it belongs to S. In Case II,

$B_1 = \varnothing$, choose any center in S. If the center has degree $k \geq 3$, the initial counts are $\ell = k$, $v_T = k + 1$, and $b \leq k$. In Case I the initial star has a dead leaf, so $\lambda \leq k - 1$. Hence

$$\Psi_0 \geq 2k + 2 \geq 8, \qquad \Phi_0 \geq 3k + 3 \geq 12.$$

If an initial leaf belongs to S, then $b \leq k - 1$ and $\Psi_0 \geq 9$. Equality $\Phi_0 = 12$ forces $k = 3, \lambda = 2$. In Case II, $\lambda \leq k$, giving $\Psi_0 \geq 2k + 1 \geq 7$ and $\Phi_0 \geq 3k + 1 \geq 10$. Equality $\Psi_0 = 7$ forces $k = b = \lambda = 3$. The only situation with $\Phi_0 < 12$ is $k = \lambda = 3$, when $\Phi_0 = 10$. These exceptional equalities follow by substitution when $k = 3$.

For the rules of Section 3, substitution gives the following lower bounds. In rows with an equality sign, the entry is the exact change.

Rule	$\Delta\Psi$	$\Delta\Phi$
1	$\geq 2k - 3$	$\geq 3k - 4$
2a	$\geq 2r' - 3$	$\geq 3r' - 5$
2b	≥ 2	≥ 3
2c	$= d - \mathbf{1}_{z \in B} \geq 0$	$= 2d - 2 \geq 0$
3a, $\lambda \geq 2$	$= d - 1 \geq 0$	$= 2d - 1 \geq 1$
3a, $\lambda = 1$	$= -1$	$= -1$
3b	$\geq 3r' - 4$	$\geq 3r' - 5$
3c	$= t - 2 \geq 0$	$= 2t - 3 \geq 1$
3d	$\geq 2q - 4$	$\geq 3q - 6$

Here $d = g(\lambda) - g(\lambda')$ in rows 2c and 3a. For Rules 1, 2a, and 3b put $h = k - 1$ or $r' - 1$, respectively. The increase in g is at most h, so the two gains are at least $3h - \Delta b$ and $4h - \Delta v_T$. For 2b the penalty drops by at least two. For 2c the penalty drops by at least one. If $\lambda \geq 3$ and either $p \geq 3$ or z is dead, then $\lambda' \leq \lambda - 2$, dropping the penalty by at least two. Rule 3a with $\lambda \geq 2$ drops the penalty by at least one. With $\lambda = 1$ it has $t = 1$, so $w \in B_1$ and the new count $\lambda' = 0$ makes the step terminal. In Rule 3c, $\lambda' \geq 1$ yields $d = t - 1$. In 3d, the penalty increases by at most $q - 1$. Its first gain is at least $2q - 3$ if a new leaf belongs to S, by the improved bound on Δb.

4.1. **Open-neighborhood potential.** In Case I, all steps are nonnegative except a possible terminal 3a with $\lambda = 1$. Suppose this step occurs, and let $x \in S$ be its unique live leaf. It was a live leaf when it first entered the tree: its tree degree could only increase and its final outside neighbor had not entered the tree. It is distinct from the initial center. If x was an initial leaf, then $\Psi_0 \geq 9$. If it entered later, inspection of the rules shows its entry increased Ψ by at least one. Rules 1, 2a and 3b do so unconditionally. Rule 2c adds a leaf in S only with $\Delta b = 0$, and 3d uses the improved bound just given. The other rules cannot introduce a live leaf in S. Thus the terminal loss leaves $\Psi \geq 8$.

In Case II all steps are nonnegative. If $\Psi_0 = 7$, the initial star has three live leaves in B. Rule 3 is unavailable. If the first step has zero gain, it must be 2c, adding $s_1 \in S$ and $z_1 \in B$, with $p = 2$ and z_1 live. Indeed $z_1 \in S$, $p = 3$, or a dead z_1 would force a gain. The new tree has two live leaves, the third old leaf and z_1, both in B. Its only two vertices in S are internal. If the second step also has zero gain it must be 2c again, adding $s_2 \in S$ and $z_2 \in B$. Both old live leaves lose their outside neighbor. The new leaf z_2 is not adjacent to the old tree: the old B vertices are independent and the old S vertices are internal. Since $d_C(z_2) \geq 2$ in Case II, z_2 is the only live leaf. The next step is Rule 1 if it has at least two outside neighbors. Otherwise its unique outside neighbor s_3 belongs to S, has just one neighbor in the tree, and has at least two outside neighbors, triggering 2a. Either step gives a positive gain. Hence the final $\Psi \geq 8$, proving the first assertion of Lemma 2.3.

4.2. **Closed-neighborhood potential.** The only possible neutral rules for Φ are 2c and 3d. For a neutral 2c, $g(\lambda) - g(\lambda') = 1$ and $\lambda \geq 2$. Consequently $\lambda' = \lambda - 1$ or $(\lambda, \lambda') = (2, 0)$; the latter ends the process. A neutral 3d requires $q = 2$ and $\lambda' = \lambda + 1$, since the upper bound $q - 1$ on the penalty increase must be attained.

Lemma 4.1. *In a neutral 2c step, z has no neighbor in the old tree. If the step changes the live count from two to one, the next step has positive gain and is neither 3a nor 3d.*

Proof. If z had a neighbor in the old tree, that neighbor would be a live leaf with unique outside neighbor z. It is distinct from the $p \geq 2$ live leaves pointing to s. Thus $\lambda \geq 3$ and at least $p + 1$ old live leaves cease to be live, giving $\lambda' \leq \lambda - 2$. This would increase Φ by at least two. Hence s is the only neighbor of z among the vertices of the new tree.

When two live leaves drop to one, the $p \geq 2$ old live leaves pointing to s disappear from the live set and z is the sole live leaf. If $z \in S$ it has at least two outside neighbors, giving Rule 1. If $z \in B$ it has either at least two outside neighbors, again giving Rule 1, or a unique outside neighbor $s' \in S$. The latter has precisely one neighbor in the tree because z is its only possible live neighbor. Since $d_C(s') \geq 3$, Rule 2a applies. Both rules give positive gain. □

In Case I the initial potential is at least 12. The only negative step is terminal 3a with $\lambda = 1$. If the potential just before that step were 12, then initially it was 12 with two live leaves, and every earlier step was neutral. The first entrance to one live leaf along that nonterminal prefix must be a neutral 2c from two live leaves, since neutral 3d increases the count. Lemma 4.1 forces a positive next step even if that step would be the last, a contradiction. The final potential is at least 12.

In Case II all changes are nonnegative, and it suffices to consider the exceptional initial potential 10 with three live leaves. There is at least one

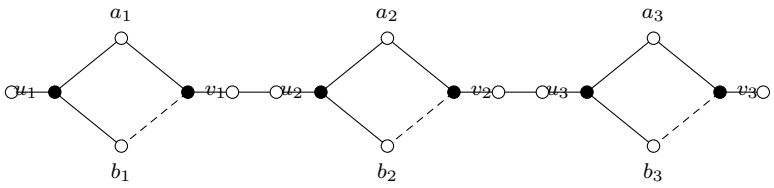

FIGURE 2. Foster's graph F_3. The vertices of degree three are filled. Deleting the dashed edges gives five leaves.

step. At termination $\lambda' = 0$. Rule 1 then gains $5k - 6 + 2g(\lambda) \geq 4$. Rules 2a and 3b gain $5r' - 7 + 2g(\lambda) \geq 3$. Rule 2b gains at least three by the table. Rule 2c gains at least two if $\lambda \geq 3$. A terminal 2c with $\lambda = 2$ would be neutral, and by Lemma 4.1 its new leaf z would have only the neighbor s in the entire graph. This forces $z \in B_1$, excluded in Case II. Rule 3c cannot terminate, and 3d gains $5q - 8 + 2g(\lambda) \geq 2$. For a terminal 3a the gain is at least three if $\lambda \geq 3$, and exactly one if $\lambda = 2$.

In the latter case the preceding steps must have accrued at least one unit. Otherwise they were all neutral and at least one preceded the last step, because the live count started at three and is now two. The immediately preceding step was 2c from three live leaves, or 3d from one. In the 2c case z is live afterwards, since at most one old live leaf survives. If $z \in S$, Lemma 4.1 gives at least two outside neighbors and forces Rule 1. If $z \in B$, Rule 3 cannot apply with such a live leaf. In the 3d case look at the first earlier entrance to one live leaf. It must have been a neutral 2c from two live leaves, forcing positive gain on the next step, still in the supposedly neutral prefix. This contradiction proves the terminal bound $\Phi \geq 12$ and completes Lemma 2.3.

5. The conjecture and sharpness

If $\Delta(G) \geq 3$, apply Theorem 1.1 with $S = M(G)$, so

$$L_s(G)^2 - 2|\partial_G M(G)| \geq (L_s(G) - 4)^2 \geq 0.$$

For $\Delta(G) \leq 2$, the connected graph is a path, a cycle, or a single vertex. The external neighborhood is empty in the regular cases. A path on at least three vertices has two leaves and its two endpoints form the external neighborhood of its maximum-degree set. This proves (1), including the ceiling since the leaf count is integral. It also gives $L_s(G) \geq |\partial_G M(G)|/4$ in the low-degree cases.

For $k \geq 1$, Foster's graph F_k consists of squares $u_i a_i v_i b_i u_i$, for $1 \leq i \leq k$. For $i < k$, connect v_i to u_{i+1} by a three-edge path with two new internal vertices. Attach a pendant vertex to u_1 and another to v_k. See Figure 2. This is the family recorded in the note to Conjecture 165 [4]. It has $6k$ vertices, the $2k$ vertices u_i, v_i are precisely the degree-three vertices, and all other vertices are adjacent to one of them.

Deleting the edge b_iv_i from each square produces a spanning tree with $k+2$ leaves. Conversely, the u_i, v_i and the internal vertices of the connecting paths are cut vertices of F_k and must be internal in any spanning tree. In each square, a_i, b_i cannot both be leaves: deleting both would preserve connectivity of the tree on its remaining vertices, whereas their deletion separates the two pendant ends in F_k. Thus a spanning tree has at most one leaf from each square and at most the two pendant leaves in addition. Consequently

$$L_s(F_k) = k + 2, \qquad |\partial_{F_k} M(F_k)| = 4k, \qquad |N_{F_k}[M(F_k)]| = 6k.$$

These equalities prove optimality in Theorem 1.1. They give $(k+2)/(4k) \to 1/4$, which proves that $1/4$ is the best coefficient in Foster's companion question over all connected graphs.

6. A mixed-weight counterexample

Proposition 6.1. *The inequality* $2|S| + |\partial_G S| \leq 8(L_s(G) - 2)$ *fails for a connected graph* G *with* $\varnothing \neq S \subseteq V_{\geq 3}(G)$. *Its minimum counterexample order is five, and the pair* (G, S) *of that order is unique up to isomorphism.*

Proof. The complement of $P_3 \sqcup K_2$, shown in Figure 3, has degree sequence $2, 3, 3, 3, 3$, with its degree-two vertex adjacent to two degree-three vertices. It is connected, since every component of a graph of minimum degree two has order at least three. For $S = V_{\geq 3}(G)$ the left side is 9. Its maximum leaf number is three, by the star extension bound and the absence of a vertex of degree four, so the right side is 8.

There is no admissible S on at most three vertices. A four-vertex graph with admissible S has a spanning star and the left side is at most $2|V(G)| = 8$. On five vertices a vertex of degree four gives $L_s(G) = 4$, with right side 16 and left side at most 10. A five-vertex counterexample therefore has maximum degree three and $L_s(G) = 3$. Since $2|S| + |\partial_G S| \leq |S| + 5$, a counterexample needs at least four vertices in S. Five vertices of degree three violate the handshaking identity, so $|S| = 4$ and the exterior set is the fifth vertex. This fifth vertex has positive degree below three and even degree by the handshaking identity, so its degree is two. The complementary degree sequence $2, 1, 1, 1, 1$ forces the unique degree-two vertex to form a path with its two neighbors, with a separate edge on the other two vertices. This forces the graph and the choice of S. □

7. Relation to degree bounds and verification

Kleitman and West proved $L_s \geq n/4+2$ at minimum degree at least three [7]. Bonsma obtained stronger bounds with triangle or diamond restrictions [3]. Bonsma and Zickfeld use the number of vertices of degree at least three under forbidden-configuration hypotheses [2]. Bankevich and Karpov treat vertices of degree different from two [1], and Karpov gives distinct

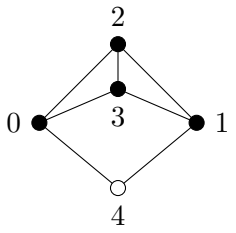

FIGURE 3. The unique smallest mixed-weight counterexample. The filled vertices form S, and the open vertex has degree two.

weights for several degree classes [6]. For example, the minimum-degree-three bound is stronger than the closed-neighborhood result on that class. In the neighborhood inequalities, degree-two vertices can contribute and the set S is arbitrary. In F_k, the count of vertices of degree different from two is $2k+2$, so the bound in [1, Theorem 1] gives $k/2+2$. Both neighborhood bounds give the exact value $k+2$ for $S = M(F_k)$.

The proofs give polynomial-time constructions. For example, with an adjacency matrix one can recompute all counts and find an applicable rule by scans costing $O(n^2)$ per extension, where $n = |V(G)|$. There are at most n extensions across the local components because their vertex sets are disjoint. Component and quotient construction, spanning trees, pruning, lifting, and the final extension can each be implemented within $O(n^3)$, even if the last step repeatedly scans all pairs for a crossing edge. No optimized running time is needed for the proof.

The archived scan covered 14,096,635 nonisomorphic connected graphs in specified order and degree ranges. It checked the exterior bound with $S = M(G)$. An independent computation checked all 1,314,381 eligible pairs (G, S) within 12,113 graphs of order at most eight. The closed-neighborhood constructor was executed 22,630 times, and a separate example exercised Rule 3c. All unbounded claims above follow from the proofs.

Data and code availability. The C and Python sources, enumeration parameters, independent verification code, and run logs are archived with project 0006 in the `ai-math-research` repository. The private archive is pinned to commit `a54bad6`. Materials for reproducibility may be supplied through the review process subject to the maintainers' permission. The proofs in this paper do not require the computation.

Contributions and AI assistance. Qirui Zhu is one of the principal contributors to this research project. This manuscript is primarily AI-generated. The human authors report understanding some parts of the work.

Generative AI tools contributed to the development and drafting of the arguments, implementation of the computational checks, independent AI-assisted audits, literature searches, and preparation of the manuscript. The formalization archive contains verified components and documents the remaining graph-theoretic obligations. The paper gives the mathematical arguments in full so that its claims can be assessed independently.

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