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A SEVENTH-ROOT NEIGHBOURHOOD AND SIX-PERMUTATION MIXTURES FOR GYIRES'S PERMANENT RATIO

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Abstract. We prove the stronger Gram form of Gyires's permanent-ratio inequality on the Frobenius ball of radius $1/(24n^{1/7})$ about the uniform doubly stochastic matrix for every $n \geq 5$. The proof combines a matching expansion with complex Gaussian estimates that separate the large and small singular values of the perturbation. In order five we also prove a radius of $51/100$ by an explicit quartic sum of squares. We record a constant-radius consequence of a contemporary coefficient estimate. A computer-assisted proof establishes the Gram inequality for every five-by-five convex combination of at most six permutation matrices. It accounts for all 256,751 row–column orbits, using 5,723,784 coefficient-certificate leaves and a block reduction for the disconnected supports. In each result equality is attained only at the uniform matrix. The full conjecture remains open.

1. Introduction and main results

Let Ω_n be the set of real nonnegative n-by-n matrices with all row and column sums one. Write $J_n = \mathbf{1}\mathbf{1}^{\mathsf{T}}/n$ and $c_n = n!/n^n$. Gyires's conjecture [6, 3] is

$$\frac{4\,\mathrm{per}(A)^2}{\mathrm{per}(AA^{\mathsf{T}}) + \mathrm{per}(A^{\mathsf{T}}A) + 2\,\mathrm{per}(A^2)} \geq c_n \qquad (A \in \Omega_n), \tag{1}$$

with equality precisely at J_n. We study

$$H_n(A) = \mathrm{per}(A)^2 - c_n\,\mathrm{per}(AA^{\mathsf{T}}).$$

Section 3 explains how simultaneous bounds for $H_n(A)$ and $H_n(A^{\mathsf{T}})$ imply (1).

Theorem 1. *For $n \geq 5$ and $A \in \Omega_n$, suppose $\|A - J_n\|_{\mathrm{F}} \leq 1/(24n^{1/7})$. Then $H_n(A) \geq 0$ and (1) holds. Equality in either statement is equivalent to $A = J_n$.*

Theorem 2. *For $A \in \Omega_5$ with $\|A - J_5\|_{\mathrm{F}} \leq 51/100$, the conclusions of Theorem 1 hold.*

Theorem 3. *If $A \in \Omega_5$ is a convex combination of at most six permutation matrices, then $H_5(A) \geq 0$ and (1) holds. Equality in either statement is equivalent to $A = J_5$.*

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2. Related work

Chang proved the inequality in order three and under the entrywise condition $\min_{ij} A_{ij} \geq (n-2)/(n-1)^2$ [2, 3]. Ozols gives the full order-four Gram and ratio inequalities with their equality cases in Lean [9]. His finite checks use `native_decide`, with compiler and runtime trust. Lavi's public project record [8] reports the radius $3/[4(n-1)]$ for $n \geq 5$, two-permutation mixtures in every order and mixtures of at most three permutations in order five. The contemporary public work [11] gives both Gram inequalities under $\|A - J_n\|_{\mathrm{op}} \leq 1/5000$. Its companion coefficient estimate [10] provides a route to a constant Frobenius radius, as explained below. Theorem 1 gives an independent estimate via mixed coefficients and improves the constant Frobenius guarantee over a finite range of orders. The complex Gaussian representation and positive semidefinite Löwner monotonicity are classical [1]. Han and Niles-Weed [7, Sections 4–5] give closely related centered elementary-symmetric estimates and Gaussian spectral bounds for positive semidefinite doubly stochastic permanents.

The symmetric positive semidefinite class and its two-sided permutation orbit also satisfy the Gram bound by a direct classical argument. A positive semidefinite $S \in \Omega_n$ has eigenvalues in $[0,1]$, so $S^2 \preceq S$ and $\mathrm{per}(S^2) \leq \mathrm{per}(S)$. The van der Waerden inequality with uniqueness [4, 5] implies strictness away from J_n. The Gram bound is invariant under independent row and column permutations, so it extends to $A = PSQ$ and to A^{T}. This class was also recorded in contemporary public work [12].

3. Gram reduction and matching coefficients

For the unit symmetrisation v of $e_1 \otimes \cdots \otimes e_n$, $\mathrm{per}(M) = \langle v, M^{\otimes n} v \rangle$. Apply Cauchy–Schwarz to $(A^{\mathsf{T}})^{\otimes n} v$ and $A^{\otimes n} v$ to obtain $g + h - 2z \geq 0$, where $g = \mathrm{per}(AA^{\mathsf{T}})$, $h = \mathrm{per}(A^{\mathsf{T}}A)$ and $z = \mathrm{per}(A^2)$. The diagonals of AA^{T} and $A^{\mathsf{T}}A$ are at least $1/n$, and all three matrices have nonnegative entries. Thus the denominator is at least $2/n^n$. The numerator in the cross-multiplied difference is

$$4\,\mathrm{per}(A)^2 - c_n(g + h + 2z) = 2H_n(A) + 2H_n(A^{\mathsf{T}}) + c_n(g + h - 2z). \tag{2}$$

All three classes in our theorems are closed under transposition.

For a matrix M let $e_m(M)$ be the weighted m-matching sum, the sum of the permanents of all m-by-m submatrices, with $e_0 = 1$. For a vector x, $e_m(x)$ denotes the elementary symmetric polynomial. Let $w_{n,m} = n^m/(n)_m$, with $(n)_m = n(n-1)\cdots(n-m+1)$. Expanding along the positions selected from M gives

$$\frac{\mathrm{per}(J_n + M)}{c_n} = \sum_{m=0}^{n} w_{n,m} e_m(M). \tag{3}$$

The product $w_{n,m} = \prod_{j=0}^{m-1}(1 - j/n)^{-1}$ decreases in n and is at most $m^m/m! \leq e^m$, by integrating $\log x$ from 1 to m.

Let $U = \mathbf{1}^{\perp}$, $\Pi = I - J_n$, and let $E = A - J_n$. Set $r = \|E\|_{\mathrm{F}}$, $G = EE^{\mathsf{T}}$, and use $R_i = \sum_j E_{ij}^2$, $C_j = \sum_i E_{ij}^2$. Thus $E = \Pi E \Pi$ and $AA^{\mathsf{T}} = J_n + G$.

On the space of zero row and column sums,

$$e_1(E) = 0, \quad e_2(E) = r^2/2, \quad e_3(E) = \tfrac{2}{3}\sum_{ij} E_{ij}^3,$$

$$e_4(E) = \tfrac{1}{8}r^4 - \tfrac{3}{4}\Big(\sum_i R_i^2 + \sum_j C_j^2\Big) + \tfrac{1}{4}\|G\|_{\mathrm{F}}^2 + \tfrac{3}{2}\sum_{ij} E_{ij}^4. \tag{4}$$

To see these identities, write $m!e_m$ as a sum over two injective m-tuples and apply partition-lattice inclusion–exclusion separately to the two tuples. A partition π has weight $\prod_{B\in\pi}(-1)^{|B|-1}(|B|-1)!$. A singleton kills the summand by a vanishing line sum. In degree four the single block paired with itself gives weight 36, its pairings with the three two-block matchings give -18 in either order, and equal and unequal pairs of matchings give 3 and 6, respectively.

The same expansion gives, with d_m the number of derangements of m,

$$|e_m(E)| \le \frac{d_m^2}{m!} r^m. \tag{5}$$

Indeed, a pair of partitions produces a bipartite multigraph of minimum degree two whose edges carry $|E_{ij}|$. At each left vertex apply Hölder with all exponents the degree, then use the monotonicity of counting ℓ^p norms to replace each by its ℓ^2 norm. The resulting edge functions depend only on the right endpoint. Applying the same argument there yields r^m. The total absolute weight of the singleton-free partitions is d_m, by cycle decomposition.

We also have $e_4(E) \ge -r^4/2$. In fact, $\|G\|_{\mathrm{F}}^2 \ge (\sum_i R_i^2 + \sum_j C_j^2)/2$, and $\sum_i R_i^2 + \sum_j C_j^2 \le r^4 + \sum_{ij} E_{ij}^4$. The last inequality follows by expanding the squares of the row and column sums of E_{ij}^2.

For a probability row x and $v_i = x_i - 1/n$,

$$\sum_i v_i^3 \ge -\frac{n-2}{n(n-1)} \sum_i v_i^2. \tag{6}$$

With $p_j = \sum_i x_i^j$, the difference is $p_3 - (1/n + 1/(n-1))p_2 + 1/[n(n-1)]$. At the boundary $p_3 \ge p_2^2$ and $p_2 \ge 1/(n-1)$ by Cauchy–Schwarz. At an interior minimum the derivatives of the same cubic in each coordinate agree. They form a quadratic, with at most one coordinate at its smaller root by the second-order condition along a coordinate difference. At $(1-(n-1)b, b, \dots, b)$ the difference is $(n-2)(nb-1)^2(1-(n-1)b)/n$, which is nonnegative.

4. Gaussian estimates

A standard complex Gaussian g has $\mathbb{E}[g^a \overline{g}^b] = \delta_{ab} a!$. Expanding independent coordinates gives the Gaussian matching identity $\mathbb{E}[e_m(\alpha)e_m(\beta)] = e_m(M)$ whenever α is linear in independent complex Gaussians, β is conjugate linear in the same Gaussians, and $\mathbb{E}[\alpha_i \beta_j] = M_{ij}$. In particular $e_m(M) = \mathbb{E}|e_m(\alpha)|^2 \ge 0$ for a positive semidefinite covariance M.

Lemma 4. *For $E = \Pi E \Pi$, nuclear norm S, and operator norm s,*

$$|e_m(E)| \le (e(S/m + s))^m, \tag{7}$$

$$|e_m(E)|^2 \le e_m(EE^{\mathsf{T}}) e_m(\Pi), \tag{8}$$

$$w_{n,m}|e_m(E)| \le \left[e^{5/4}\sqrt{1 + 1/m}\sqrt{1 + \sqrt{2n/m}\, r}\right]^m. \tag{9}$$

Proof. Use a singular value decomposition with singular vectors $u_j, v_j \in U$ and singular values σ_j. Couple $\alpha = \sum_j \sqrt{\sigma_j} g_j u_j$ and $\beta = \sum_j \sqrt{\sigma_j} \bar{g}_j v_j$. Phase orthogonality gives the generating function $\sum_m e_m(E) z^m$ as the expectation of $\prod_i (1 + \sqrt{\rho} e^{i\theta} \alpha_i) \prod_i (1 + \sqrt{\rho} \beta_i)$, where $z = \rho e^{i\theta}$. Use $|1 + y| \le e^{\Re y + |y|^2/2}$ and cancel the linear sums. The modulus is bounded in expectation by $\prod_j (1 - \rho\sigma_j)^{-1}$ for $\rho < 1/s$. Its logarithm is at most $\rho S/(1 - \rho s)$. Cauchy's coefficient formula at $\rho = m/(S + ms)$ proves (7); $E = 0$ is immediate. For (8), replace the two weights $\sqrt{\sigma_j}$ by σ_j and 1, respectively, completing the right basis through the zero singular subspace. Apply Cauchy–Schwarz in Gaussian L^2.

Finally, expanding $xI + (1 - x)J_n$ along the diagonal gives

$$\begin{aligned} \frac{\operatorname{per}(J_n + x\Pi)}{c_n} &= (1 - x)^n \sum_{k=0}^{n} \frac{(nx/(1 - x))^k}{k!} \\ &\le \exp(nx^2/(1 - x)), \qquad 0 < x < 1. \end{aligned} \tag{10}$$

The last step uses $\log(1 - x) + x/(1 - x) \le x^2/(1 - x)$. Every coefficient on the left is nonnegative. Take $x = \sqrt{m}/(\sqrt{2n} + \sqrt{m})$ to bound $w_{n,m} e_m(\Pi)$ by $[e^{1/2}(1 + \sqrt{2n/m})]^m$. Use (7) on EE^{T}, whose nuclear and operator norms are r^2 and at most r^2, and $w_{n,m} \le e^m$. This gives (9). □

Lemma 5. *Let $0 \le x \le 1/10$, and let $B = \Pi B \Pi \succeq 0$ have positive eigenvalues λ_j. If $z \ge 0$ and $6z\lambda_j < 1$ for all j, then*

$$\frac{\operatorname{per}(J_n + x\Pi + zB)}{c_n} \le 2 \exp(nx^2/(1 - x)) \prod_j (1 - 6z\lambda_j)^{-1}.$$

Proof. Integrate first over the independent Gaussian coordinates of covariance xI. The remaining integrand is $\prod_i (x + |\sqrt{(1 - x)/n}\, g_0 + \sqrt{z}\, h_i|^2)$, where h is independent of g_0, has covariance B, and has zero coordinate sum. Write $t = |g_0|^2$ and $a = x + (1 - x)t/n$. On $t \ge n/5$, $a \ge 9/50$. Factor out a^n and bound each factor $1 + y$ by e^y. The cross terms sum to zero and the product is at most $a^n \exp(6z\|h\|^2)$. By (10), the expectation of a^n/c_n is at most $\exp(nx^2/(1 - x))$. On $t < n/5$, arithmetic–geometric mean bounds the product by $(3/10 + z\|h\|^2/n)^n$. Dividing by $c_n \ge e^{-n}$ bounds this by $\exp(6z\|h\|^2)$, because $3e/10 < 1$ and $10/3 < 6$. Each $|g_j|^2$ is exponential of mean one, giving the stated product. □

Lemma 6. *If $5 \le m \le n/162$, then*

$$w_{n,m}|e_m(E)| \le 4(m + 1)^2 e^{\sqrt{m}/e} \max\left\{\frac{9rn^{1/4}}{m^{3/4}},\, 32e^{3/4} r\right\}^m. \tag{11}$$

Proof. Assume $r > 0$, and split the singular values at $r/\sqrt{m}$. On the large part the nuclear sum S_B is at most $r\sqrt{m}$ and its largest value s_B at most r. On the small part the sum of squared singular values is at most r^2 and each is at most r^2/m. In the Gaussian construction of Lemma 4 use weights $(\sqrt{\sigma_j}, \sqrt{\sigma_j})$ on the large part and $(\sigma_j, 1)$ on the small part. For $a+b=m$, let $F_{a,b}(u,v) = [\xi^a\eta^b]\prod_i(1+\xi u_i+\eta v_i)$. Rotating the large Gaussian coordinates annihilates all cross terms with different large degrees. If $I_{a,b}$ and $K_{a,b}$ are the two squared Gaussian L^2 norms of $F_{a,b}$, each multiplied by $w_{n,m}$, then the matching sum is at most $\sum_{a+b=m}\sqrt{I_{a,b}K_{a,b}}$.

The first product has modulus at most $\exp(|\xi|^2\|u\|^2 + |\eta|^2\|v\|^2)$ by the zero sums. Cauchy on the two circles, optimised separately, bounds $|F_{a,b}|^2$ by $(2e/a)^a(2e/b)^b\|u\|^{2a}\|v\|^{2b}$. A zero degree omits its factor. A zero vector makes any positive degree in it vanish. Independently for the two norms, a weighted sum of independent exponentials with weights d_j has p-th moment $p!h_p(d)$. Here $h_p(d) \le [e(\sum_j d_j/p+\max_j d_j)]^p$, by bounding $\log\prod(1-td_j)^{-1}$ by $t\sum_j d_j/(1-t\max_j d_j)$ and selecting $t = p/(\sum_j d_j+p\max_j d_j)$. For $p \ge 1$, integration gives $\log(p!) \le \int_1^p \log t\,dt+\log p$, hence $p!(e/p)^p \le ep$. The bound $e(p+1)$ also covers degree zero. It follows that

$$I_{a,b} \le e^2(a+1)(b+1)[2e^2(S_B/a+s_B)]^a[2e^2(r^2/b+r^2/m)]^b.$$

For the second norm, add an independent Gaussian to complete the small covariance to Π. Its degree-zero term in the added Gaussian is the original polynomial, orthogonal to the terms of higher added degree. The completed norm is the coefficient $[z^ax^b]$ in $w_{n,m}e_m(z|B| + x\Pi)$, where $|B|$ is the right positive semidefinite square root of $B^{\mathsf{T}}B$ for the large singular part of E. All such coefficients in the permanent expansion are nonnegative by the Gaussian identity. In Lemma 5 choose $x = \sqrt{b}/(\sqrt{2n}+\sqrt{b}) \le 1/19$ and $z = a/[6(S_B + as_B)]$. The corresponding exponent and logarithm are at most $b/2$ and a. A zero degree is handled by the limit of its parameter to zero. If the large part is zero, all positive large degrees vanish. Thus

$$K_{a,b} \le 2[6e(S_B/a+s_B)]^a[e^{1/2}(1+\sqrt{2n/b})]^b.$$

After taking square roots, the large part is bounded by $(32r)^a e^{\sqrt{m}/e}$, using $(\sqrt{m}/a)^a \le e^{\sqrt{m}/e}$ and $4\sqrt{3}\,e^{3/2} < 32$. The small part is bounded by $(9rn^{1/4}/b^{3/4})^b$, using $b \le m \le n/162$, $1+\sqrt{2n/b} \le (19/18)\sqrt{2n/b}$, and $2e^{5/4}2^{1/4}\sqrt{19/18} < 9$. For $\gamma = 32r$ and $d = 9rn^{1/4}$, maximise $(m-b)\log\gamma + b\log(d/b^{3/4})$ on $[0,m]$. Its stationary point is $b_* = (d/\gamma)^{4/3}/e$. If $b_* \ge m$ the maximum is at m. Otherwise it is $m\log\gamma + 3b_*/4 \le m\log(\gamma e^{3/4})$. The remaining sum of prefactors is at most $4(m+1)^2$. □

5. Proof of the universal radius

Let $B_m = w_{n,m}|e_m(E)|$ and $g_r = (74/5)r^2$. In the domain $g_r < 1$, (3), (6), and (7) give

$$\frac{H_n(A)}{c_n^2 r^2} \ge P(n) - w_{n,4}r^2 - \frac{n}{2(n-1)}r^2 - \frac{g_r^3}{(1-g_r)r^2} - 2\sum_{m=5}^{n}\frac{B_m}{r^2}. \tag{12}$$

Here $P(n) = n/(n-1) - 4n/[3(n-1)(n-2)]$. Indeed, write the normalized square as $1+2(p-1)+(p-1)^2$, where $p = \mathrm{per}(A)/c_n$, and discard the square. The Gram quadratic term uses $\|G\|_{\mathrm{F}}^2 \le r^4$. For $m \ge 3$ its remaining terms are at most g_r^m, since $e^2 < 37/5$, $\|G\|_{S_1} = r^2$ and $\|G\|_{\mathrm{op}} \le r^2$. All error bounds divided by r^2 can be evaluated at the upper endpoint for r. For the matching terms this follows from their homogeneity, or from the positive powers r^{m-2} in the bounds. Also $P(n) = 1 - (n+6)/[3(n-1)(n-2)]$ increases for $n \ge 5$. The other explicit errors in (12) decrease with n when evaluated at a nonincreasing endpoint.

For the low-degree range $5 \le m \le K < n$, use (5). Writing $K_+ \ge K$, the ratio of consecutive bounds is at most

$$q = \frac{n}{n-K_+} K_+ r (265/264)^2. \tag{13}$$

This follows from $d_{m+1} = (m+1)d_m + (-1)^{m+1}$ and $d_m \ge 44$ for $m \ge 5$. For $q < q_* < 1$, the contribution after division by r^2 is at most

$$\frac{135}{8(1-q_*)} \left(\frac{n}{n-K_+}\right)^5 r^3. \tag{14}$$

Here $d_5/5! < 3/8$.

Orders $5 \le n \le 242$. Take $r_0 = 1/24$. For $5 \le m \le \min(n, 10)$ apply (5) with $w_{n,m} \le w_{m,m}$. For $m \ge 11$, (9) has base below $43/100$, since $\sqrt{484/11} < 20/3$ and $e^{5/4} < 3491/1000$. The bound in (12) evaluated with $P(5), w_{5,4}$ and the Gram error at 5 is greater than $1/4$. Explicitly the matching error uses

$$\sum_{m=5}^{10} w_{m,m} \frac{d_m^2}{m!} r_0^{m-2} + \frac{(43/100)^{11}}{(1-43/100) r_0^2}.$$

This covers the required seventh-root radius in these orders.

A fifth-root estimate for $n \ge 243$. Write $x = n^{1/5} \ge 3$ and $r_0 = 1/(9x)$. Use $K = \lceil 3x \rceil$, $K_+ = 3x+1$. Both K_+/n and $K_+ r_0$ decrease, and (13) at $x = 3$ is below $2/5$. Consequently (14) is largest at $x = 3$. For $m \ge K+1$, the square of the base in (9) is at most

$$\frac{e^{5/2}}{81} \left(1 + \frac{1}{3x+1}\right) \left(x^{-2} + \sqrt{2/3}\right) < (41/100)^2.$$

The upper bound decreases in x. At $x = 3$ the inequality follows from the rational upper bounds $3491/1000$ for $e^{5/4}$ and $817/1000$ for $\sqrt{2/3}$. The tail divided by r_0^2 is bounded by $81x^2(41/100)^{3x+1}/(1-41/100)$. Its logarithmic derivative $2/x + 3\log(41/100)$ is negative, since $(100/41)^9 > 37/5 > e^2$. Evaluation of (12) at $x = 3$ gives a lower bound greater than $13/20$. For $243 \le n \le 10^7$, the fifth-root radius contains the seventh-root radius, since $10^{14} 9^{35} < 24^{35}$.

Orders $n \ge 10^7$. Put $x = n^{1/7} \ge 10$, $r_0 = 1/(24x)$, $K = \lceil 5x \rceil$, $K_+ = 5x+1$. As before, (13) and (14) decrease in x; now $q < 1/4$ at $x = 10$. For $K+1 \le m \le n/162$, we have $m \ge 5x+1 \ge 51$ and

$$9 r_0 n^{1/4}/m^{3/4} < 1/8, \qquad 32 r_0 \le 2/15.$$

Also $e^{\sqrt{m}/e} \le (11/10)^m$ because $e > 27/10,\ 15^2 < 51(27/10)^2$, and $e^{1/15} < 11/10$. The estimate $4(m+1)^2 \le (6/5)^m$ holds at $m = 51$ and propagates by the ratio $((m+2)/(m+1))^2 \le (53/52)^2 < 6/5$. With $e^{3/4} < 1059/500$, (11) has effective base below $2/5$. The normalized sum is bounded by $576x^2(2/5)^{5x+1}/(1-2/5)$, decreasing for $x \ge 10$ since $(5/2)^{25} > e$.

For $m > n/162$, (7), $w_{n,m} \le e^m$ and $\|E\|_{S_1} \le \sqrt{n}\, r_0$ give a base at most

$$\frac{37}{5}\left(1 + \frac{162}{\sqrt{n}}\right) r_0 < 1/20.$$

The normalized sum is at most $576x^2(1/20)^{x^7/162}/(1-1/20)$. Its logarithmic derivative is $2/x - 7x^6 \log(20)/162 < 0$. At $x = 10$ we may bound it by using exponent 100. Inserting the three bounds into (12) at $x = 10$ gives a lower bound greater than $999/1000$.

The endpoint comparisons in the three cases are rational inequalities. In the order displayed their lower bounds exceed 0.288, 0.655 and 0.9999. They can be checked using $d_m = \sum_{j=0}^{m} (-1)^j m!/j!$ and a Taylor sum for each exponential upper bound. The remainder after degree 28 is at most the first omitted term divided by $1 - x/30$ for the positive arguments x used here. This proves strict positivity for the punctured ball. At $A = J_n$ the normalized difference vanishes. Equation (2) then completes the proof of Theorem 1.

A constant-radius consequence of contemporary work. The public coefficient theorem [10] gives $|e_m(E)| \le 2 \cdot 200^m r^m$ for centered complex matrices, in particular for our real E. In (12) at $r_0 = 1/1000$, bound B_m for $5 \le m \le 60$ by $m!(3r_0)^m$ from (5) and $w_{n,m} \le e^m$. Consecutive bounds have ratio at most $60(3r_0) < 1/5$. For $m \ge 61$, use $B_m \le 2(600r_0)^m$. The sum of both geometric bounds, divided by r_0^2, is at most

$$\frac{5!(3r_0)^5}{(1-1/5)r_0^2} + \frac{2(600r_0)^{61}}{(1-600r_0)r_0^2}.$$

With $n = 5$ in the remaining bounds of (12), the result exceeds $2/3$. Thus the constant radius $1/1000$ holds for every $n \ge 5$, as a consequence of [10]. Combining this with Theorem 1 gives the pointwise guarantee $\max\{1/1000, 1/(24n^{1/7})\}$.

6. A quartic sum of squares in order five

Use $n = 5$, and define

$$F = r^4, \quad R_2 = \sum_i R_i^2, \quad C_2 = \sum_j C_j^2, \quad T = \|EE^{\mathsf{T}}\|_{\mathrm{F}}^2, \quad S_4 = \sum_{ij} E_{ij}^4.$$

Let $P = I - J_5,\ b_i = Pe_i,\ d_i = b_i b_i^{\mathsf{T}} - P/5$. On $U \otimes U$, use orthogonal projections P_0, P_1, P_2, P_a onto the scalar line spanned by $P/2$, the span of the d_i, its remaining symmetric complement, and the alternating subspace. The Gram matrix of the d_i is $(3/5)P$. Thus $P_1 = (5/3)\sum_i d_i\langle d_i, \cdot\rangle$. These four equivariant projectors have ranks $1, 4, 5, 6$. On $E^{\otimes 2}$, after grouping the row and column indices, the tensor products of the first three on both sides

and $P_a \otimes P_a$ give ten orthogonal components, since simultaneous exchange fixes $E^{\otimes 2}$. Write p_{ij} for the squared norm of each component. Contracting against $P/2$ and the d_i gives

$$p_{00} = F/16, \quad p_{10} = 5R_2/12 - F/12, \quad p_{20} = T/4 - p_{00} - p_{10},$$
$$p_{11} = \tfrac{25}{9} \sum_{ij} (E_{ij}^2 - R_i/5 - C_j/5 + r^2/25)^2,$$
$$p_{12} = \tfrac{5}{3}(R_2 - T/5) - p_{10} - p_{11}, \quad p_{aa} = (F - T)/2.$$

For p_{12} the total one-sided squared norm is $(5/3)\sum_i \|E^{\mathsf{T}} d_i E\|_{\mathrm{F}}^2 = (5/3)(R_2 - T/5)$. For p_{aa} the inner product with the row swap is T. Transpose gives the missing off-diagonal components, and p_{22} is F minus the other nine. The invariant quartics are spanned by (F, R_2, C_2, T, S_4): average over the row and column groups independently, write their coordinate equalities as set partitions of four slots, discard singletons by the zero sums, and quotient by simultaneous slot permutation. The single-block partition and the three pairings give five orbits of pairs. These are precisely the displayed invariants. Their independence follows from the rank-five evaluation matrix in the accompanying round-six check. Direct substitution gives for any $X = PXP$

$$\tfrac{27}{200}\|X\|_{\mathrm{F}}^4 - e_4(X) = \tfrac{27}{100}(p_{10} + p_{01} + p_{12} + p_{21}) + \tfrac{7}{50} p_{aa} \geq 0,$$

where the projectors on the right now act on $X^{\otimes 2}$.

Ryser inclusion–exclusion with complementary subsets gives

$$\operatorname{per}(E) = 2\sum_j \prod_i E_{ij} - 2\sum_{a<b} \prod_i (E_{ia} + E_{ib}).$$

For a zero-sum five-vector y, $|\prod_i y_i| \leq [2/(25\sqrt{30})]\|y\|^5$. Indeed, if none vanish, use arithmetic–geometric mean on its p positive and $q = 5 - p$ negative magnitudes, with common group sum at most $\sqrt{pq/5}\|y\|$. The larger bound is at $p = 2, 3$. Put $s_{ab} = \|E_{:a} + E_{:b}\|^2$. Then $C_j \leq 4r^2/5$ and $s_{ab} \leq 6r^2/5$, by applying $\|E\|_{\mathrm{F}}$ to the projected coordinate or pair indicator. Expanding the squares with $\sum_j E_{:j} = 0$ gives $\sum s_{ab} = 3r^2$ and $\sum s_{ab}^2 = F - 3C_2 + 2T$.

Take $t_0 = 9/20$, $t_1 = 137/125$. Let $q(x) = ax^2 + bx + c$ match $x^{5/2}$ and its derivative at t_0^2 and its value at t_1^2. Then

$$q(t^2) - t^5 = (t - t_0)^2 (t_1 - t)(t^2 + ut + v),$$
$$u = t_1 + 2t_0 - a > 0, \quad v = c/(t_0^2 t_1) > 0.$$

The coefficients are rational. For example,

$$a = \frac{t_1^5 - t_0^5 - (5/2)t_0^3(t_1^2 - t_0^2)}{(t_1^2 - t_0^2)^2}.$$

Since $t_1^2 > 6/5$, the bound applies to s_{ab}/r^2. Using $k = 1461/50000 > 4/(25\sqrt{30})$ and $s = 1789/2000 > \sqrt{4/5}$, and averaging the resulting bounds for E and E^{T}, gives

$$|\operatorname{per}(E)| \leq rL(E), \qquad L = k\big[(a + 3b + 10c)F + (s - 3a)(R_2 + C_2)/2 + 2aT\big].$$

In particular $L \geq 0$. Set $\rho = 51/100$. The quartic $Q = (125/12)e_4(E) - (5/8)T$ has coefficient vector $(125/96, -125/16, 95/48, 125/8)$ in the basis $(F, R_2 + C_2, T, S_4)$. For the coefficients (q_0, q_1, q_2, q_3) of $Q - (625/12)\rho L$, use

$$\beta = -\tfrac{4}{5}(q_1 + q_3/5), \quad \alpha = \tfrac{9}{25}q_3 - \beta, \quad \gamma = \tfrac{4}{3}\beta - 2q_2,$$
$$\lambda = q_0 - \alpha/9 - 73\beta/144 - \gamma/2.$$

Substituting gives the identity

$$Q - (625/12)\rho L = \lambda F + \alpha p_{11} + \beta p_{22} + \gamma p_{aa}. \tag{15}$$

All three projector weights are positive. The constant is

$$\lambda = -\frac{55353904747509023}{19120928000000000}.$$

For $r \leq \rho$, the left side with ρ replaced by r is at least λF, since $L \geq 0$.

It remains to bound the Gram terms of degree six and higher in E. For a zero-sum five-vector y, $|\sum y_i^3| \leq 3\|y\|^3/(2\sqrt{5})$. On the unit sphere in U, Lagrange multipliers give two coordinate values of multiplicities $p, 5-p$, with cubic sum $(5-2p)/\sqrt{5p(5-p)}$. Applying the bound to each row of G and using $\|G_{i:}\| \leq (2/\sqrt{5})\|G\|_{\mathrm{F}}$ yields $e_3(G) \leq (2/5)r^6$. The quartic SOS gives $e_4(G) \leq (27/200)r^8$. The Gaussian identity and arithmetic–geometric mean give

$$\operatorname{per}(G) \leq 5^{-5}\mathbb{E}\|\alpha\|^{10} = 5^{-5}5!h_5(\sigma_1^2, \ldots, \sigma_4^2) \leq c_5 r^{10},$$

where α has covariance G. Combining (6), (15) and (3), and discarding the normalized square as in (12), gives

$$\frac{H_5(A)}{c_5^2 r^2} \geq \tfrac{5}{6} + \lambda r^2 - \tfrac{5}{6}r^4 - \tfrac{45}{64}r^6 - r^8.$$

This expression decreases for $r \geq 0$ and exceeds $7/1000$ at $r = 51/100$ by exact substitution. This proves Theorem 2.

7. Computational certification in order five

We represent a permutation by its list of images, ordered lexicographically with the identity first. A mixture of fewer than six distinct permutations can be extended to a six-element set by assigning zero to the new weights. Under independent row and column permutations, H_5 is invariant. We use the action $(a, b) : p \mapsto apb$, equivalently the usual left–right action with inverse in the right coordinate.

7.1. **Enumeration and completeness.** For each of the 14,400 actions on the 120 permutations, its cycles of lengths ℓ give the fixed-subset polynomial $\prod_\ell (1 + z^\ell)$. Burnside's lemma, applied to the coefficient of z^6, gives 3,697,214,400/14,400 = 256,751. To canonicalize an individual subset, choose in turn each member to map to identity, let the left permutation range over all 120 possibilities, and solve for the unique right permutation. Take the lexicographically least sorted image. It must contain the identity, since some image does. The number of actions producing that least image is

the stabilizer size. All supplied representatives were recanonicalized into one common set. The stabilizers were recomputed and the orbit sizes summed.

A disconnected support can be assumed to contain identity. Its components then become components on the five positions after contracting the identity matching, and a smallest component has size one or two. After conjugation it is a fixed initial block of that size. The remaining five permutations can therefore be enumerated in $S_1 \times S_4$ or $S_2 \times S_3$, with the identity already included. The independent orbit check uses these two enumerations and removes duplicates after canonicalization. This yields 336 orbits of total size 3,456,900. The classes and independently checked counts are

Support class	Orbits	Subsets	Leaves
Contains a Latin set	19	154,560	7,448
Disconnected	336	3,456,900	
Other connected supports	256,396	3,649,134,000	5,716,336
Total	256,751	3,652,745,460	5,723,784

The orbit-size total equals $\binom{120}{6}$; disjointness and this total prove completeness.

7.2. **Disconnected supports.** A disconnected union of permutation supports can be brought to a nontrivial block sum by row and column permutations. For a block of size d, let $H_d \geq 0$ be the corresponding Gram bound. Independently permuting rows and columns leaves both permanents in H_d invariant. For a split into orders j and $5-j$, multiplicativity gives the lower bound $c_j c_{5-j} > c_5$ for $\operatorname{per}(A)^2/\operatorname{per}(AA^{\mathsf{T}})$. The strict inequality follows because $\binom{5}{j}(j/5)^j(1-j/5)^{5-j} < 1$. The blocks of orders one and two satisfy the Gram bound directly. For order two, with $A_{11} = a$, the difference is $(2a-1)^2/2$. For order three, subtract the least entry times the all-ones matrix and permute it to position $(1,1)$. The remainder is a nonnegative combination of the four permutations avoiding that position, by the Birkhoff decomposition. Sort the four weights and use their successive differences as new coordinates. The resulting 24 cones have an exact coefficient check for $9\operatorname{per}(A)^2 - 2\operatorname{per}(AA^{\mathsf{T}})$. Every coefficient is nonnegative, each pure sixth power for the four differences is positive, and the pure sixth power along the uniform direction vanishes. The stand-alone order-three certificate is replayed by the round-two verifier. The block of order four uses the stronger Gram theorem `Gyires.permanent_gram_le` of Ozols [9].

7.3. **Exact coefficient certificates for connected supports.** For $A(t) = \sum_{i=0}^{5} t_i P_i$, use the homogeneous polynomial

$$625\operatorname{per}(A(t))^2 - 24\operatorname{per}(A(t)A(t)^{\mathsf{T}}). \tag{16}$$

On the simplex it is $625H_5(A(t))$. A tree node labelled by an edge $i < j$ makes two substitutions. One replaces t_i by $y_i + y_j$, the other replaces t_j by $y_i + y_j$, leaving the other variables at y_k in each substitution. The two maps cover $t_i \geq t_j$ and $t_j \geq t_i$, and together cover the entire orthant. After normalizing the rays this is a bisection of the simplex at the chosen edge midpoint in the current coefficient coordinates. The nonnegative monomial

coefficients at leaves are equivalent to nonnegative homogeneous Bernstein coefficients, whose basis monomials differ by positive multinomial factors. For an ordinary leaf all six pure tenth powers are positive, which proves strict positivity at all nonzero nonnegative coordinates.

For a subset containing J_5, the five disjoint permutation matrices in a Latin set give the uniform vertex after averaging. Conversely each row must have five occupied columns in any representation of J_5. If six distinct weights were positive, each row would consist of four singleton weights $1/5$ and one pair of weights totalling $1/5$. The two weights below $1/5$ would occur together in every row, so their permutations would agree. Thus the positive support is a Latin set of five with uniform weights. Two different Latin sets cannot share four permutations because their fifth is the complement of the same four permutation supports.

In the 19 Latin orbits use the 120 decreasing orders of the five Latin weights. Each ordering cone has generators $12\mathbf{11}^{\mathsf{T}}$, the four prefix sums of lengths $4, 3, 2, 1$ scaled by $15, 20, 30, 60$, respectively, and the extra permutation scaled by 60. Each has line sum 60. Coefficient trees record which leaves retain the uniform ray. On a leaf that retains it, the pure uniform power is zero, each mixed power $y_0^8 y_k^2$ for $k \neq 0$ is positive, and so is every nonuniform pure power y_k^{10}. These last powers certify the face $y_0 = 0$. If $y_0 > 0$, the mixed powers also give strictness off the uniform ray. Every other leaf has all pure powers positive. The uniform ray is retained by the first child of a split, and by the second child precisely when the smaller edge index i is nonzero.

The coefficient generator used Ryser expansion. The independent replays reconstruct (16) by direct summation over the 120 permutations, followed by integer convolutions and binomial substitutions on each branch. Positive common divisors may be removed without affecting any sign. Readers can recover the full binary tree from its prefix string; the verifier requires exactly two children at each split and consumes the entire string. The audited signed 128-bit implementation checks input coefficients before each split. A target coefficient receives at most 3003 summands, each bounded in absolute value by 2^{10} times the largest input coefficient, so the input limit 2^{104} makes all intermediate sums smaller than 2^{126}. At the root $(625 + 24)360^{10} < 2^{96}$ bounds absolute coefficients and the preceding convolution sums, by evaluating the two nonnegative permanent polynomials at all variables one and using the line sum at most 360. The large replays reported leaf widths up to 39 bits for ordinary cones and 74 bits for the Latin cones after content reduction. Negative controls reject a replaced normalization constant and a one-leaf truncation of a certificate requiring subdivision.

This proves the Gram bound and strict equality condition on every six-permutation simplex. Transposition preserves the class of mixtures, hence (2) proves Theorem 3. The order-five Birkhoff polytope has dimension 16. Its Carathéodory bound is 17 summands. The union of six-vertex simplices has empty relative interior in the full polytope.

8. Reproducibility

The accompanying audit includes a zero-dependency Python coefficient and endpoint check and a separate C++ enumeration. From the problem directory, run

```
python final_audit/verify_analytic.py
python final_audit/prepare_replays.py /tmp/gyires
c++ -O3 -std=c++17 final_audit/orbit_audit.cpp -o orbits
```

The orbit check takes the five-permutation certificate and the prepared `k6-files.txt` manifest as arguments. The full commands, including the coefficient replays, appear in the audit's reproducibility guide. The native runs used Python 3.12 and Clang 22.1.8 with exact arithmetic. The independent orbit check also recovers all 13,710 five-permutation orbits and 190,578,024 subsets. The historical five-permutation replay covers 149,771 non-Latin leaves, with the two Latin classes verified separately. The validation is computer-assisted; the order-five statements have not been formalized in a proof assistant.

9. Open problems

Does the Gram inequality hold on all of Ω_5? Does the original ratio inequality hold on all of Ω_n for $n \geq 5$? A local question is how large a uniform Frobenius radius one can guarantee. Another is whether a finite family of larger permutation simplices can cover the remaining order-five cases with feasible exact certificates.

Code and Data Availability

The exact certificate inputs, independent verifiers and replay instructions are available in the public reproducibility repository at the following fixed revision:

```
9fa4f117e192f29a11aa216c5392aea9d633ce93
```

The repository is distributed under CC BY 4.0.

AI use disclosure

Google Gemini developed the ten research rounds and their proof candidates and certificate code. OpenAI Codex independently audited the arguments, strengthened the certificate checks, ran the exact replays, and prepared the manuscript. The human project team selected the research problem and commissioned the final audit. The authors are responsible for reviewing the manuscript and approving any submission. The authors confirm that they have checked and understood some parts of the contribution.

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