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Positive rational quadrature for polynomials

GPT-6.2 Astra, with prompting by Darij Grinberg

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Abstract. If a polynomial is nonnegative on an open interval, why is the difference of its antiderivative at the two endpoints nonnegative? Over $\mathbb{R}$, this follows from basic measure theory. We prove this constructively over any totally ordered field, and even over a partially ordered commutative $\mathbb{Q}$-algebra in which $1/1,\ 1/2,\ 1/3,\ \ldots$ are positive. The proof uses a finite quadrature rule: For each degree bound d, we express the integral $\int_0^1 p(x)dx$ of a polynomial p of degree $\leq d$ as a positive weighted sum of values at rational fractions in the interval.

We also ask whether nonnegativity inside the interval implies nonnegativity at its endpoints. We prove this over every totally ordered field and, more generally, over any partially ordered field in which every square is nonnegative. We give a counterexample over a totally ordered $\mathbb{Q}$-algebra.

Manifest. This note was written by GPT-6.2 Astra (Researcher subscription), written in response to my prompts. I have fully proofread it. I have preserved the original "olympiad worksheet" format, as this is a problem that has caught my attention back in my high school days. [1]

1 The theory

All polynomials here will be univariate; the variable is called x. The base ring is assumed to be a $\mathbb{Q}$-algebra. Given a polynomial p, write

$$\mathcal{I}(p) = F(1) - F(0), \quad \text{where } F \text{ is an antiderivative of } p \text{ (that is, a polynomial with } F' = p).$$

We call this value $\mathcal{I}(p)$ the *integral* of p, as it is an algebraic definition of $\int_0^1 p(x)dx$. In particular, $\mathcal{I}(x^k) = 1/(k+1)$. We first work over $\mathbb{Q}$. The following is our main result:

Theorem 1.1. *For each integer $d \geq 0$, one can construct rational numbers $0 < t_1 < t_2 < \cdots < t_s < 1$ and positive rational numbers $w_1, w_2, \ldots, w_s$ with $\sum_i w_i = 1$ such that*

$$\text{every polynomial } p \text{ of degree at most } d \text{ satisfies} \qquad \mathcal{I}(p) = \sum_{i=1}^{s} w_i p(t_i).$$

[1] This work is in the public domain. –Darij

Such a choice of rational numbers $t_1, t_2, \ldots, t_s$ and $w_1, w_2, \ldots, w_s$ will be called a *positive rational quadrature rule exact through degree d*, or, for short, a *rule in degree $\leq d$*. The numbers t_i are called its *nodes*, and the numbers w_i are called its *weights*. From the viewpoint of numerical analysis (where this notation originates), the rule is a formula for computing $\mathcal{I}(p)$ by sampling p at finitely many points (the nodes) and taking a fixed linear combination of the sampled values (using the weights as the coefficients).

Proof of Theorem 1.1. The cases $d \leq 1$ are easy: The midpoint rule $\mathcal{I}(p) = p(1/2)$ holds for $p = 1$ and $p = x$, hence for every polynomial of degree at most 1.

Now fix $d \geq 2$ and suppose we have a rule in degree $\leq d-1$ with nodes t_i and weights w_i. Let D be a common positive denominator of its nodes and set $N = D + 1$.

Given a polynomial p of degree at most d, we seek a polynomial q of degree at most $d-1$ with the same integral. Subdivide the interval $[0, 1]$ into N equal parts and form the average

$$(A_N p)(x) = \frac{1}{N} \sum_{j=0}^{N-1} p\left(\frac{x+j}{N}\right).$$

This averaging operation $p \mapsto A_N p$ preserves the integral. Indeed, if $F' = p$, then $NF((x+j)/N)$ is an antiderivative of $p((x+j)/N)$ (by the chain rule). Consequently, for each j, we have $\mathcal{I}(p((x+j)/N)) = NF((j+1)/N) - NF(j/N)$. Therefore,

$$\begin{aligned}
\mathcal{I}(A_N p) &= \frac{1}{N} \sum_{j=0}^{N-1} \left(NF\left(\frac{j+1}{N}\right) - NF\left(\frac{j}{N}\right)\right) = \sum_{j=0}^{N-1} \left(F\left(\frac{j+1}{N}\right) - F\left(\frac{j}{N}\right)\right) \\
&= F(1) - F(0) \qquad \text{(by the telescope principle)} \\
&= \mathcal{I}(p). \qquad (1)
\end{aligned}$$

On the other hand, denoting by $[x^m]f$ the x^m-coefficient of any polynomial f, we have $[x^d](A_N p) = N^{-d}[x^d]p$, since substituting $(x+j)/N$ for x in a polynomial of degree $\leq d$ multiplies its x^d-coefficient by N^{-d}. Thus the polynomial

$$q = \frac{N^d A_N p - p}{N^d - 1} \qquad (2)$$

has degree at most $d-1$ (since the x^d-coefficients of $N^d A_N p$ and p cancel out in the difference) and satisfies $\mathcal{I}(q) = \mathcal{I}(p)$ (by (1)). Applying the old rule to q, we thus obtain

$$\mathcal{I}(p) = \sum_i w_i q(t_i). \qquad (3)$$

It remains to see why each $q(t_i)$ is a positive weighted average of values of p. This is where our choice of N as $D+1$ (with D being a common positive denominator of the t_i) enters. For each i, the integer $j_i := (N-1)t_i = Dt_i$ satisfies $0 \leq j_i < N$ (since $0 < t_i < 1$)

and $(t_i + j_i)/N = t_i$. To compute $q(t_i)$, recall the definition of q:

$$\begin{aligned}
(N^d - 1)q(t_i) &= N^d (A_N p)(t_i) - p(t_i) \qquad \text{(by (2))} \\
&= N^{d-1} \sum_{j=0}^{N-1} p\left(\frac{t_i + j}{N}\right) - p(t_i) \qquad \text{(by the definition of } A_N p\text{)} \\
&= N^{d-1} \sum_{j=0}^{N-1} p\left(\frac{t_i + j}{N}\right) - p\left(\frac{t_i + j_i}{N}\right) \qquad \text{(since } t_i = (t_i + j_i)/N\text{)} \\
&= \sum_{j=0}^{N-1} p\left(\frac{t_i + j}{N}\right) \begin{cases} N^{d-1} - 1, & \text{if } j = j_i; \\ N^{d-1}, & \text{if } j \neq j_i. \end{cases} \qquad (4)
\end{aligned}$$

In the last equality we subtracted one copy of $p\left(\frac{t_i + j_i}{N}\right)$ from the $j = j_i$ addend. Its coefficient is now $N^{d-1} - 1$, which is still positive since $d \geq 2$ and $N = D + 1 > 1$. Since $N = D + 1 > 1$ and $d > 0$, we can solve (4) for $q(t_i)$ by dividing by $N^d - 1$; we thus obtain

$$q(t_i) = \sum_{j=0}^{N-1} p\left(\frac{t_i + j}{N}\right) \cdot \frac{1}{N^d - 1} \begin{cases} N^{d-1} - 1, & \text{if } j = j_i; \\ N^{d-1}, & \text{if } j \neq j_i. \end{cases} \qquad (5)$$

We can now write down the new rule in degree $\leq d$. For $1 \leq i \leq s$ and $0 \leq j < N$, take the node $u_{i,j}$ and its weight $v_{i,j}$ to be

$$u_{i,j} = \frac{t_i + j}{N}, \qquad v_{i,j} = \frac{w_i}{N^d - 1} \begin{cases} N^{d-1} - 1, & \text{if } j = j_i; \\ N^{d-1}, & \text{if } j \neq j_i \end{cases}$$

(note that these do not depend on p). Substituting the expression (5) for $q(t_i)$ into the old rule (3) gives

$$\mathcal{I}(p) = \sum_{i=1}^{s} w_i q(t_i) = \sum_{j=0}^{N-1} \sum_{i=1}^{s} v_{i,j}\, p(u_{i,j}). \qquad (6)$$

All weights $v_{i,j}$ are positive rationals since $w_i > 0$, $N > 1$ and $d \geq 2$. Their sum is 1, as we see by taking $p = 1$ in (6). The nodes $u_{i,j}$ are rational and occur in increasing order in this double sum: for fixed j we have

$$\frac{j}{N} < u_{1,j} < u_{2,j} < \cdots < u_{s,j} < \frac{j+1}{N},$$

because $0 < t_1 < t_2 < \cdots < t_s < 1$. Thus (6) gives the required rule in degree $\leq d$. This completes the induction, and with it, the proof of Theorem 1.1. □

2 Example: the first step

Let us see what the rule constructed in the above proof looks like in degree ≤ 2. Start with the midpoint $t_1 = 1/2$ and the weight $w_1 = 1$ (which define a rule in degree ≤ 1), and take $d = 2$ and $D = 2$, so that $N = 3$. The three nodes of the refined rule are $u_{1,0} = 1/6$,

$u_{1,1} = 1/2$ and $u_{1,2} = 5/6$. Here $j_1 = (N-1)t_1 = 1$, so the middle node $u_{1,1}$ coincides with the old node t_1. To see what happens to its weight, specialize (2) at $x = 1/2$. This gives

$$8q(1/2) = 3\big(p(1/6) + p(1/2) + p(5/6)\big) - p(1/2).$$

Thus two of the three copies of $p(1/2)$ remain, giving the midpoint weight $2/8 = 1/4$. Since the old rule gives $\mathcal{I}(p) = q(1/2)$ by (3), we obtain

$$\mathcal{I}(p) = \frac{3}{8}\,p\left(\frac{1}{6}\right) + \frac{1}{4}\,p\left(\frac{1}{2}\right) + \frac{3}{8}\,p\left(\frac{5}{6}\right).$$

This is also exact in degree 3, since both sides vanish on $(x-1/2)^3$ by symmetry, and the rule is already exact in degree ≤ 2.

3 Using the rule for proving inequalities

We use the order conventions of Fuchs [1, Ch. VI, §1, pp. 105–106]. In a partially ordered ring, addition preserves the order and the product of two nonnegative elements is nonnegative. The order is total if every two elements are comparable. In either case $a < b$ means $a \leq b$ and $a \neq b$.

Corollary 3.1. *Let K be a totally ordered field. Let $a \leq b$ in K, and let $p \in K[x]$ be a polynomial nonnegative on the open interval (a,b) (that is, satisfying $p(t) \geq 0$ for all $t \in (a,b)$). If F is a polynomial satisfying $F' = p$, then $F(b) - F(a) \geq 0$.*

Proof. If $a = b$, then this is obvious, so we assume that $a < b$.

Choose an integer $d \geq \deg p$ and a rule in degree $\leq d$ (this exists by Theorem 1.1). This rule holds for all polynomials of degree $\leq d$ over $\mathbb{Q}$, thus in particular for the polynomials $x^0, x^1, \ldots, x^d$ over $\mathbb{Q}$. Since $\mathbb{Q}$ embeds into K as a field (since K is totally ordered), this rule therefore holds for the polynomials $x^0, x^1, \ldots, x^d$ over K as well, and hence (by K-linearity) also for any polynomial of degree $\leq d$ over K. In particular, we can thus apply the rule to the polynomial $(b-a)p(a+(b-a)x)$. Since $\mathcal{I}((b-a)p(a+(b-a)x)) = F(b) - F(a)$ (because the formal derivative of $F(a+(b-a)x)$ is $(b-a)p(a+(b-a)x)$), this gives the equality

$$F(b) - F(a) = \sum_{i=1}^{s} w_i\,(b-a)\,p\big(a + t_i(b-a)\big).$$

Each sampling point $a + t_i(b-a)$ lies in (a,b) (here we use $a < b$), and every factor in each term on the right is nonnegative; hence the equality yields $F(b) - F(a) \geq 0$. □

Remark 3.2. *Let R be a commutative $\mathbb{Q}$-algebra equipped with a partial order making it a partially ordered ring. Its nonnegative cone $R_{\geq 0} = \{r \in R \mid 0 \leq r\}$ is then closed under addition and multiplication, because of the ordered-ring axioms. Assume also that $\mathbb{Q}_{\geq 0} \cdot 1_R \subseteq R_{\geq 0}$. Then, the above proof of Corollary 3.1 works in R. Indeed, for a sampling point $y_i = a + t_i(b-a)$, both $y_i - a = t_i(b-a)$ and $b - y_i = (1-t_i)(b-a)$ belong to the cone. If $a < b$, then $b - a \neq 0$, and multiplication by either of the nonzero rational scalars t_i*

and $1 - t_i$ is invertible on R. Thus both differences are nonzero, so $a < y_i < b$ even when R has zero divisors.

The condition $\mathbb{Q}_{\geq 0} \cdot 1_R \subseteq R_{\geq 0}$ is necessary if $1_R \geq 0$ and the assertion is to hold for all polynomials. Applying it on $(0, 1)$ to x^{n-1} gives $(1/n)1_R \geq 0$ for every $n \geq 1$.

For example, on the ring $\mathbb{Q}$ define $u \preceq v$ to mean that $v - u$ is a nonnegative integer. This is a ring partial order with nonnegative cone $\mathbb{Z}_{\geq 0}$. Its interval $[0, 1]$ consists of 0 and 1, so the open interval is empty. In particular, $p(x) = x$ is nonnegative on $(0, 1)$. Yet $\mathcal{I}(p) = 1/2$ lies outside the cone.

Application to a Polish Mathematical Olympiad problem. Problem 3 of the final round of the 43rd Polish Mathematical Olympiad (1991–92) asks for an inequality over the reals.[2] We prove it over any totally ordered field K. Let n be a positive integer and let $a_1, a_2, \ldots, a_n \in K$. Then

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{a_i a_j}{i + j} \geq 0. \tag{7}$$

To prove (7), define $P(x) = \sum_{i=1}^{n} a_i x^{i-1}$. Then, $xP(x)^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j x^{i+j-1}$ (by expanding the square). Since $\mathcal{I}(x^{i+j-1}) = 1/(i + j)$, this yields

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{a_i a_j}{i + j} = \mathcal{I}\big(xP(x)^2\big) = \sum_{k=1}^{s} w_k t_k P(t_k)^2 \ \geq \ 0,$$

where the last equality uses a quadrature rule in degree $\leq 2n - 1$. Each coefficient $w_k t_k$ is a positive rational number and thus ≥ 0 in K.

In fact, equality holds only if $a_1 = a_2 = \cdots = a_n = 0$. To see this, apply the same rule on each of the n intervals $[(r - 1)/n, r/n]$, $1 \leq r \leq n$. Their integrals are nonnegative and add up to $\mathcal{I}(xP(x)^2)$. If the sum is zero, every sampled value of P is zero, since every sampling point is strictly positive and every weight is positive. There is at least one sampling point in the interior of each interval, giving n distinct roots of a polynomial of degree at most $n - 1$. Thus $P = 0$. This proves positive definiteness of the shifted Hilbert matrix $\big(1/(i+j)\big)_{1 \leq i,j \leq n}$ over K.

4 Nonnegativity at the endpoints

For $a < b$, the open interval (a, b) suffices in the hypothesis of Corollary 3.1. In fact, one can recover the sign of a nonzero endpoint value from a single interior sample.

Proposition 4.1. *Let K be a totally ordered field, let $a < b$ in K, and let $p \in K[x]$. If $p(x) \geq 0$ for every $a < x < b$, then $p(a) \geq 0$ and $p(b) \geq 0$.*

Proof. For the left endpoint, assume $p(a) \neq 0$, since the conclusion is immediate if $p(a) = 0$. Normalize the polynomial by writing

$$q(t) = \frac{p(a + (b - a)t)}{p(a)} = 1 + \sum_{i=1}^{d} c_i t^i.$$

[2] The problem sheet is available at https://imomath.com/othercomp/Pol/PolMO392.pdf.

Thus, for each $s \in K$, we have $p(a) = p(a + (b-a)s)/q(s)$ if $q(s)$ is nonzero. We seek $0 < s < 1$ such that $q(s) > 0$. Take

$$s = \frac{1}{2 + d + \sum_{i=1}^{d} c_i^2}.$$

Then $0 < s \leq 1/2$. For each $1 \leq i \leq d$, expanding $(c_i + s^{i-1})^2 \geq 0$ gives

$$2c_i s^{i-1} \geq -c_i^2 - s^{2i-2} \geq -c_i^2 - 1.$$

Multiplying this by $s/2$ and summing over i, and then adding 1, we obtain

$$1 + \sum_{i=1}^{d} c_i s^i \geq 1 - \frac{s}{2}\left(\sum_{i=1}^{d} c_i^2 + d\right) = \frac{1}{2} + s$$

by the definition of s. Since the left-hand side is $q(s)$, we thus have found

$$q(s) \geq \frac{1}{2} + s > 0.$$

But $a < a + (b-a)s < b$ (because $0 < s \leq 1/2 < 1$), so that $p(a + (b-a)s) \geq 0$, since we assumed that $p(x) \geq 0$ for every $a < x < b$. Since $q(s) > 0$, we can divide by it and obtain $p(a) = p(a + (b-a)s)/q(s) \geq 0$. Applying this conclusion to the polynomial $p(a + b - x)$ gives $p(b) \geq 0$ as well. □

Partially ordered fields. The same proof works in a partially ordered field in which every square is nonnegative. Indeed, in such a field, the reciprocal of any $z > 0$ is positive: $z^{-1} = z(z^{-1})^2$ is nonzero and nonnegative. This justifies both taking the reciprocal in the definition of s and dividing by $q(s)$.

Failure in partially ordered rings. Order the commutative $\mathbb{Q}$-algebra $\mathbb{Q}(T)$ (the field of univariate rational functions with rational coefficients) by eventual sign at positive infinity, so that T exceeds every integer. This makes $\mathbb{Q}(T)$ a totally ordered field. The polynomial $p(x) = Tx - 1$ satisfies $p(r) > 0$ for every rational $0 < r < 1$, yet $p(0) = -1 < 0$. In the field, the interior sample $x = 1/(2T)$ gives $p(x) = -1/2 < 0$, so Proposition 4.1 does not apply. However, in the ordered subring $\mathbb{Q}[T]$, every element between 0 and 1 is rational, since the absolute value of every nonconstant polynomial in T exceeds 1. Thus the same p is positive on the entire open interval of this ring and negative at its left endpoint. This shows that Proposition 4.1 cannot be generalized to partially ordered $\mathbb{Q}$-algebras R (even with the property $\mathbb{Q}_{\geq 0} \cdot 1_R \subseteq R_{\geq 0}$), and that there is no positive-quadrature-like rule for $p(a)$ in terms of $p(a + t(b-a))$ with rational $t \in (0, 1)$.

5 A question

Let K be a totally ordered field, and let $p(t) = \sum_{i=0}^{d} c_i t^i$ with $c_0, c_1, \ldots, c_d \in K$. Suppose that $p(t) \geq 0$ for every $t \in K$ with $0 < t < 1$. Corollary 3.1, applied with $a = 0$ and $b = 1$, gives $\sum_{i=0}^{d} c_i/(i+1) \geq 0$.

Over $\mathbb{R}$ we have, more generally, the weighted inequality

$$\sum_{i=0}^{d} \frac{c_i}{i+y} \geq 0 \qquad \text{for every } y > 0. \tag{8}$$

Indeed, the sum equals $\int_0^1 t^{y-1} p(t)\, dt$. The integrand is nonnegative on $(0,1)$, and the integral converges also for $0 < y < 1$.

Tarski's transfer principle gives (8) over every real closed field [2, §4.2, p. 139].[3] To see why it applies, fix d and clear the positive denominator $\prod_{j=0}^{d}(y+j)$. The implication from $p(t) \geq 0$ for all $0 < t < 1$ to (8), quantified over $c_0, \ldots, c_d$ and $y > 0$, is then a first-order sentence in the language of ordered fields.

For a general totally ordered field K, passing to its real closure can enlarge the interval on which the hypothesis must hold. For example, take $K = \mathbb{Q}(T)$ with the order used in Section 4, and put $g(t) = (2t^2-1)^2 - 1/T$. Every $t \in K$ with $0 < t < 1$ is a bounded rational function of T, so its limit as $T \to +\infty$ is a rational number r. Since $2r^2 \neq 1$, the limit of $g(t)$ is $(2r^2-1)^2 > 0$. Thus $g(t) > 0$ throughout $(0,1)$ in K. In the real closure, however, $t = 1/\sqrt{2}$ belongs to $(0,1)$ and gives $g(t) = -1/T < 0$.

Can one give an elementary proof of (8) that works directly over an arbitrary totally ordered field K?

References

[1] László Fuchs, *Partially Ordered Algebraic Systems*, Akadémiai Kiadó, 1963. https://real-eod.mtak.hu/17986/.

[2] Alexander Prestel and Charles N. Delzell, *Mathematical Logic and Model Theory: A Brief Introduction*, Springer, 2011. https://doi.org/10.1007/978-1-4471-2176-3.

[3] The same text gives Artin's solution of Hilbert's seventeenth problem in Theorem 4.2.5. That theorem represents globally nonnegative polynomials over a real closed field as sums of squares of rational functions.