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Alien Hodge Motives and Grothendieck's Standard Conjectures

Levelt Splitting, Sign Reversal, and the Hodge-Type Conjecture for Non-CM Hypergeometric Motives

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hexagon:2611.01265v1

Abstract. Grothendieck proposed his standard conjectures in 1968 as the foundation of the theory of motives. Their two central open cases meet on fourfolds: the Lefschetz conjecture B in characteristic zero, and the Hodge-type conjecture for codimension-two cycles in positive characteristic. We introduce Alien Hodge Motives: systems of realisations over a finite field recording, for every Frobenius block, its quadratic spaces at all finite places and a sign at infinity read from the Newton polygon rather than from a Betti lift. Hilbert reciprocity gives positivity criteria free of the Tate conjecture, a sign law for blocks of rank two, and a dichotomy: local data decide blocks of rank at most three and are blind in rank four. We apply this to three hypergeometric motives of Hodge type $(1,1,1,1,1)$ at the fixed point $t=-1$ of $t \mapsto 1/t$. They are not of CM type, and no algebraic correspondence relating them to abelian varieties or K3 surfaces is known. A Levelt involution splits each orthogonally into a definite part of rank three and a CM part of rank two, negative definite in Betti cohomology. Under stated geometric hypotheses we prove that the intersection form is positive definite on numerical classes in the hypergeometric summand at all primes outside a finite set and a set of density zero, without the Tate conjecture or CM lifts; on the CM part the sign is opposite to the Betti sign. The arithmetic core is verified in Lean 4. Two Alien Hodge Conjectures, whose implications are formal, give both central cases.

2020 Mathematics Subject Classification. 14C25 (primary), 14C15, 14F20, 14F30, 14G15, 11E12, 11G25, 33C20, 03B35 (secondary).

Key words and phrases. standard conjectures, Hodge standard conjecture, motives, hypergeometric motives, Hilbert symbol, supersingular K3 crystals, formal verification.

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1. Introduction

1.1. The standard conjectures and their two summits

In 1968, at the Bombay colloquium on algebraic geometry, Grothendieck stated two conjectures on algebraic cycles "which arose from an attempt at understanding the conjectures of Weil" on the zeta functions of algebraic varieties [Gro69, p. 193]. The first, of Lefschetz type, asserts that the operator Λ of Hodge theory, the inverse of the hard Lefschetz isomorphism, is induced by an algebraic correspondence; the second, of Hodge type, asserts that the intersection form on primitive algebraic classes satisfies the sign conditions that the Hodge–Riemann bilinear relations impose in characteristic zero. Grothendieck saw them as the foundation of the "so-called 'theory of motives' " and called their proof, alongside resolution of singularities, "the most urgent task in algebraic geometry" [Gro69, p. 198]. Deligne's proof of the Weil conjectures [Del74] took another road, and the standard conjectures remain open.

In characteristic zero the Hodge-type conjecture is a theorem of Hodge theory, and the Lefschetz conjecture B implies all the others. In positive characteristic the Hodge-type conjecture holds for divisors by the Hodge index theorem [Gro58], so for varieties of dimension at most three it is known, and the first open case is that of cycles of codimension two on fourfolds. For B the first open case is already in dimension three (Section 3.8); the two problems on which this paper concentrates meet on fourfolds, in the middle degree:

(G) the *geometric summit*: the conjecture $B(X)$ for smooth projective fourfolds X over $\mathbb{C}$, which asks for the *existence* of algebraic cycles on $X \times X$ realising Λ;
(A) the *arithmetic summit*: the conjecture $\mathrm{Hdg}(X)$ in codimension two for smooth projective fourfolds X over $\overline{\mathbb{F}}_q$, which asks for the *positivity* of the intersection form on algebraic classes in H^4.

Section 3 states both precisely and recalls why the cases of lower dimension are settled. The known cases of (A) are, in substance, of abelian or K3 type: abelian fourfolds [Anc21], squares and powers of K3 surfaces [IIK25], squares and powers of certain abelian varieties [Kos24, Agu25, Agu26a], Hermitian varieties [Har26], and the reduction modulo four of the signature for varieties with CM lifts [AM25]. Section 4 surveys this frontier, and that of (G).

1.2. Alien Hodge Motives

Over a finite field the Betti realisation is absent, and with it the archimedean source of positivity. What remains is a system of quadratic spaces: for every prime $\ell \neq q$ the ℓ-adic realisation with its cup product, and at q the crystalline realisation with its Frobenius. Hilbert's reciprocity law ties their local invariants together, and the missing archimedean invariant is the sign that the Hodge-type conjecture predicts. The theory of this paper begins from one observation: for a Frobenius block of a motive over $\mathbb{F}_q$, the sign that the conjecture requires is *not* the sign that the block carries on a Betti lift. It is the sign that the block would carry if its Hodge filtration were its Newton filtration. A block of Newton slopes 0 and w behaves as a Hodge structure of type $(w,0)+(0,w)$; a block of slope $w/2$ behaves as one of type $(w/2, w/2)$, of positive sign. We call this sign the *Newton–Hodge sign* of the block.

An *Alien Hodge Motive* $\mathcal{M}_{\mathrm{AH}}$ (Definition 6.1) is the system formed by the local quadratic and hermitian spaces of all Frobenius blocks of a motive over $\mathbb{F}_q$, its algebraic classes, its Newton–Hodge signs, and isometries between blocks of different motives place by place, which need not come from correspondences. When these isometries are induced by algebraic correspondences, and the mixed structure is trivial, it is a classical motive with additional data. The word *alien* records the origin of the sign: it is read from the Frobenius at q, a structure foreign to the Betti lift. Part II develops the local–global arithmetic of these objects; Part III applies it.

1.3. The main theorems

For multisets $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$ of rational numbers, disjoint modulo $\mathbb{Z}$ and defined over $\mathbb{Q}$ (each stable modulo $\mathbb{Z}$ under $x \mapsto kx$ for every integer k prime to the denominators), let $\mathcal{H}(\alpha;\beta \mid t)$ be the hypergeometric motive over $\mathbb{Q}(t)$ [BH89, Kat90, BCM15, RRV22]. When $\beta \equiv \alpha + \frac{1}{2}$ modulo $\mathbb{Z}$, the inversion $t \mapsto 1/t$ exchanges the local monodromies at 0 and ∞ up to a quadratic twist, and its fixed point $t = -1$ carries an extra symmetry. We consider the three data

$$\begin{aligned} \mathsf{H}_2 &= \mathcal{H}\big(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2};\ 0,0,0,0,0\big), \\ \mathsf{S}_{33} &= \mathcal{H}\big(\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{2},\tfrac{2}{3},\tfrac{2}{3};\ 0,\tfrac{1}{6},\tfrac{1}{6},\tfrac{5}{6},\tfrac{5}{6}\big), \\ \mathsf{S}_3 &= \mathcal{H}\big(\tfrac{1}{3},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{2}{3};\ 0,0,0,\tfrac{1}{6},\tfrac{5}{6}\big), \end{aligned} \tag{1.1}$$

specialised at $t = -1$. Each is a motive of rank 5 and weight 4 with Hodge numbers $(1,1,1,1,1)$, so that $h^{4,0} = 1$. For H_2 a model is the double cover

$$X_{-1}: \quad y^2 = x_1x_2x_3x_4(1-x_1)(1-x_2)(1-x_3)(1-x_4)(1+x_1x_2x_3x_4),$$

and the involution $\sigma(x,y) = (1/x_1,\dots,1/x_4,\ y/(x_1x_2x_3x_4)^2)$ is a birational involution of X_{-1} over $\mathbb{Q}$. We write $K_{\mathcal{M}}$ for $\mathbb{Q}(i)$, $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{-3})$ according as $\mathcal{M}$ is H_2, S_{33}, S_3. The geometric inputs are collected in Hypothesis 12.4: a smooth projective model of the family with an algebraic projector onto the hypergeometric summand, the algebraicity of the inversion symmetry at $t = -1$, and integrality at good primes. For H_2 the involution is the explicit map above; for S_{33} and S_3 its existence is assumed.

Theorem A (The fixed-point theorem). *Let $\mathcal{M}$ be one of H_2, S_{33}, S_3 at $t = -1$, realised as a summand $eH^4(X)$ of a smooth projective fourfold X over $\mathbb{Q}$ as in Hypothesis 12.4. There are a finite set E of primes and a set T of primes of natural density zero such that, for every prime $q \notin E \cup T$ and every prime $\ell \neq q$, the following hold.*

(i) *The intersection form is positive definite on the $\mathbb{Q}$-space of codimension-two cycles on $X_{\overline{\mathbb{F}}_q}$, projected by e, modulo numerical equivalence.*

(ii) *The classes of these cycles in $e\,H^4_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(2))$ lie in an orthogonal sum $L \oplus C$, where L is a line and C has rank two, and they lie in L unless q is inert in $K_{\mathcal{M}}$.*

(iii) *The Betti form on the rank-two part C_{B} of M_{B} is negative definite, whereas every algebraic class in C that is not numerically trivial has positive self-intersection.*

The proof uses neither the Tate conjecture nor a lifting of X with complex multiplication. For H_2 the only prime q with $5 \le q \le 97$ in T is $q = 31$; for S_{33} and S_3 no computed prime $q \le 97$ lies in T.

The set T is the set of primes at which the transcendental plane W of the definite block, or the CM plane C at a prime split in $K_{\mathcal{M}}$, has a Frobenius of finite order. That it has density zero (Theorem 13.9) follows from the Chebotarev density theorem for ℓ-adic Lie groups in Serre's form [Ser81], once the image of Galois on S is shown not to be finite, unipotent or dihedral, which we do by exact computation at finitely many primes. The statement concerns cycles modulo numerical equivalence, which is the form of the Hodge-type conjecture proved in the cases of [Anc21] and [Kos24] (Remark 3.5). The finite set E consists of 2 and 3, the primes of bad reduction of the model, the primes at which the integral structures of Hypothesis 12.4 fail, and the primes dividing a positive rational constant c that compares the cup product with the invariant form of Beukers and Heckman (Proposition 12.6).

Theorem B (Local–global principles). *Let B be a Tate block of a motive over $\mathbb{F}_q$, q a power of an odd prime p, whose ℓ-adic realisations for $\ell \neq p$ are the localisations of one rational quadratic space.*

(i) (Sign law in rank two.) *If B has rank two and its ℓ-adic realisations are those of a definite form $\langle a,b\rangle$ of sign s, with $p \nmid ab$, then an algebraic class of B of self-intersection $r \neq 0$ satisfies $r > 0$ if and only if $(r,-ab)_p = s$.*

(ii) (Decidability.) *If the rational quadratic space of algebraic classes of B has rank at most three and, at every finite place, the local invariants of a positive definite rational form, then it is positive definite. In rank four the finite local invariants do not determine the signature.*

(iii) (One sign.) *At a wall of rank four whose Tate part is a sum of two lines of discriminant $\delta > 0$, the algebraic classes on the two lines have the same sign; one class of known sign decides the block.*

Part (i) is a Hodge parity rule in the sense of Ancona [Anc21] and Ancona–Marmora [AM25], stated class by class. When $s = -1$, the block is of Betti type $(3,1)+(1,3)$, and the rule says that an algebraic class is positive exactly when its p-adic symbol is that of a non-neutral plane; Ogus's theory of supersingular K3 crystals [Ogu79] supplies this symbol, and that is the mechanism of Theorem A(iii). For a motive of rank two over a p-adic ring of integers whose two-dimensional Tate part is spanned by algebraic classes, the same reversal of sign is contained in Ancona's Theorem 8.1 [Anc21]; part (i) gives it one class at a time, so that it applies when only some of the Tate classes are known to be algebraic.

Theorem C (The Alien Hodge Conjectures). *Let* AHC I *and* AHC II *be the conjectures stated in Sections 20 and 21.*

(i) AHC I *for a fourfold X over $\overline{\mathbb{F}}_q$ implies* $\mathrm{Hdg}(X)$ *in codimension two.*
(ii) AHC II *for a smooth projective fourfold X over $\mathbb{C}$ implies $B(X)$.*
(iii) *Part* (a) *of* AHC I*, together with the Tate conjecture, implies* Hdg *on every Tate block of rank at most three.*
(iv) *At a wall point of a fourfold with a lift at which the alien parity of Section 9 holds, the Tate conjecture and* Hdg *on the Tate part imply part* (a) *of* AHC I *for the transcendental block; so* AHC I(a) *cannot fail there unless one of them fails.*
(v) *Theorem A proves* AHC I *for the block C of each of the three motives* (1.1).

Parts (i) and (ii) are formal: the conjectures are formulated so that the summits follow from them by linear algebra, and we do not present these implications as deep. The content of AHC I lies in its part (a), a local–global statement about realisations that makes no reference to algebraic cycles; parts (iii)–(v) are what we can prove about it. We write the two conjectures with their numerals throughout: the abbreviation AHC alone is not used, since in the literature it denotes Deligne's conjecture that Hodge classes are absolute Hodge classes.

1.4. The method

The proof of Theorem A has three steps. First, the involution σ acts on the Betti realisation M_{B} as the intertwiner, unique up to sign, of Levelt's normal form [BH89]: a signed anti-diagonal permutation matrix σ_{H} with $\sigma_{\mathrm{H}}^2 = 1$, $B\sigma_{\mathrm{H}}A = -\sigma_{\mathrm{H}}$ and $A\sigma_{\mathrm{H}}B = -\sigma_{\mathrm{H}}$, where A and B are the companion matrices of the local monodromies. Its eigenspaces split M_{B} orthogonally as $S_{\mathrm{B}} \oplus C_{\mathrm{B}}$, of ranks three and two. For H_2, up to a common positive factor, the Gram matrix on S_{B} is positive definite with leading minors $3, 48, 2048$, and that on C_{B} is $\mathrm{diag}(-4,-64)$. The transcendental part of the Tate classes is thereby confined to S, where it is a binary block W whose discriminant is fixed by the characteristic polynomial of Frobenius.

Second, the local data of $L = W^{\perp} \subset S$ and of C are determined by global rational data. For H_2, on L the square class of the form is $2c(-\Delta) > 0$, where $\Delta < 0$ is the discriminant of Frobenius on W; this uses only that Frobenius is a

proper similitude of W. On C the ℓ-adic forms are those of $\langle -c, -c\rangle$, and at an inert prime the crystalline realisation of $C(1)$ is a supersingular K3 crystal of rank two, whose Tate module is non-neutral by Ogus's theorem.

Third, Hilbert's product formula converts these local data into signs. A class on L has the sign of $2c(-\Delta)$, which is positive; a class on C has the sign forced by the non-neutral crystal, which is positive although the Betti form on C is negative. Since σ is algebraic, every algebraic class splits along $L \oplus C$ orthogonally, and the form is positive definite. Neither the Tate conjecture nor a CM lift enters: each step concerns one algebraic class at a time.

1.5. What is not proved

We do not prove the standard conjectures, nor either summit in general. The obstruction that motivated this work, the walls of the Dwork family of sextic fourfolds, is not resolved; it carries no involution of the kind used here, and Section 23 records it as open together with the other limits. Theorem A rests on the geometric inputs of Hypothesis 12.4. Parts of them are published theorems applied under stated conditions (Appendix C); the existence of a smooth projective model with an algebraic projector onto the summand is part of the hypothesis, and for S_{33} and S_3 so is the algebraicity of the inversion symmetry. The theorem concerns cycles modulo numerical equivalence. We do not know whether the set T of exceptional primes is finite. Principle 8 of Section 11, the location of the walls by a discriminant field, is proved for the three motives (1.1); in rank seven it is not proved in general, and the data, which are consistent with it at every computed prime, determine the relevant determinant only at 14 of the 33 computed primes (Section 18). The two implications of Theorem C(i)–(ii) are formal. The paper has not been refereed.

1.6. Formal verification

The arithmetic of the proofs (the relations satisfied by σ_{H} and the Gram matrices, the Hilbert-symbol computations, the product formula arguments, the sign laws, the decidability dichotomy, and the linear algebra behind Theorem C) is formalised in Lean 4 [dMU21] with Mathlib [mC20]. The core of the formal companion consists of ten modules with 210 theorems, which depend on no axioms other than `propext`, `Classical.choice` and `Quot.sound`; the whole development has been replayed by the kernel checker. The geometric inputs enter the Lean statements as hypotheses. Appendix A lists, for each numbered statement, the declarations it rests on and which of their hypotheses are geometric.

1.7. Relation to other work

The use of Hilbert reciprocity to determine the sign of algebraic classes from local data is due to Ancona [Anc21], whose proof for abelian fourfolds combines it with the computation of p-adic periods of CM lifts, and it is developed further by Ancona and Marmora [AM25]. We take from this work the strategy of comparing quadratic forms place by place. What is different is the source of the global forms and of the invariant at p: we use the Betti realisation of a lift that is not of CM type, made usable by an algebraic involution, and the Tate module of a supersingular K3 crystal in the sense of Ogus [Ogu79], in place of

a CM lift and its p-adic periods. The hypergeometric motives that we treat, and their symmetry under $t \mapsto 1/t$, are classical [BH89, Kat90, BCM15, RRV22]; the factorisation of their zeta functions at $t = -1$ was recorded by Beukers and Delaygue [BD22] in connection with supercongruences. The formalisation of parts of arithmetic geometry in Lean, and of quadratic forms in Mathlib, is part of a broader development [dMU21, mC20].

1.8. Conventions

Varieties are smooth, projective and geometrically connected unless stated otherwise. Cohomology is ℓ-adic étale cohomology with the Tate twist that makes the cup product on the middle degree $\mathbb{Q}_\ell$-valued, or crystalline cohomology at the characteristic. For a motive of weight $2m$ we write $\langle\,,\rangle$ for the cup product on H^{2m} and use $(-1)^m\langle\,,\rangle$ when stating positivity; in weight four, as for the fourfolds of the main theorems, the two agree. Frobenius means the geometric Frobenius acting on $H^{2m}(m)$, where it is an isometry; we call it the normalised Frobenius. We write $\langle a_1, \dots, a_n\rangle$ for the diagonal quadratic form, $(a, b)_v$ for the Hilbert symbol at a place v of $\mathbb{Q}$, and ε_v for the Hasse invariant. Statements marked by a Lean anchor in the source are formalised in the companion (Appendix A).

1.9. Notation

The following symbols are used throughout the paper.

H_2, S_{33}, S_3	the three hypergeometric motives (1.1) at $t = -1$
M_{B}, S_{B}, C_{B}	the Betti realisation of the summand and its Levelt eigenspaces of ranks three and two (Proposition 12.7)
σ_{H}, A, B, S	the Levelt involution, the local monodromies and the invariant form (Proposition 12.2, Appendix D)
c, d_S, $\langle a, b\rangle$	the constant comparing cup product and S; the Gram determinant of S_{B}; the form on C
$K_{\mathcal{M}}$	the CM field of C: $\mathbb{Q}(i)$, $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{-3})$
L, W	the Tate line and the transcendental plane in S at a good prime (Lemma 13.2)
t_W, u	the trace of Frobenius on W; the square class $c\,d_S(4 - t_W^2)$ of the line
T_H	the Tate module of the supersingular K3 crystal of C (Proposition 13.5)
E, T	the finite exceptional set and the set of exceptional primes of density zero (Definition 13.1)
$(a, b)_v$, ε_v, β	the Hilbert symbol, the Hasse invariant, and the finite Hasse product of a block (Section 6.4)
η_ℓ, s_N, κ_ℓ	the hermitian classes of a transcendental block, the sign of its rational model, and the computable local factors at a wall (Section 9.2)
$\mathcal{M}_{\mathrm{AH}}$	an Alien Hodge Motive (Definition 6.1)
$\mathcal{M}$	one of the three motives (1.1)

1.10. Organisation

Part I recalls Grothendieck's motives and his standard conjectures, defines the two summits, and surveys what is known. Part II sets up the local data (Section 5), defines Alien Hodge Motives (Section 6), treats hermitian blocks and CM planes (Section 7), follows these notions through the square of a CM elliptic curve, where everything is classical (Section 8), proves the wall theorems that do not use the Tate conjecture (Section 9) and the general principles (Section 10), and collects nine principles of the theory (Section 11). Part III proves Theorem A: the Levelt splitting (Section 12), the proof (Section 13), the sign reversal (Section 14) and its form for blocks of K3 type (Section 15), the three motives at small primes (Section 16), the comparison with known results (Section 17), the frontier in rank seven (Section 18) and the walls of six hypergeometric families (Section 19). Part IV states AHC I and AHC II, proves Theorem C, collects the evidence (Section 22) and records what remains open (Section 23). The appendices contain the formal companion (Appendix A), the exact computations (Appendix B), the geometric inputs (Appendix C) and the certificates (Appendix D).

Part I

Grothendieck's dream and the two summits

2. Motives: Grothendieck's dream

This section recalls the origin of the standard conjectures in Grothendieck's programme. Its purpose is to fix the statements we shall need and to place the problem of positivity, which is the subject of the paper, where Grothendieck placed it.

2.1. The Weil conjectures

Let X be a smooth projective variety of dimension n over the field $\mathbb{F}_q$ with q elements, and let N_m be the number of its points over $\mathbb{F}_{q^m}$. In 1949 Weil conjectured that the zeta function $Z(X,t) = \exp\bigl(\sum_{m\geq 1} N_m t^m/m\bigr)$ is a rational function, that it satisfies a functional equation under $t \mapsto q^{-n}t^{-1}$, that it factors as an alternating product of polynomials $P_0, \dots, P_{2n}$ whose inverse roots are algebraic integers of absolute value $q^{i/2}$ for P_i, and that the degrees of the P_i are the Betti numbers of a lift to characteristic zero when there is one [Wei49]. For curves he had proved the analogue of the Riemann hypothesis in 1948, by an argument that rests on the positivity of an intersection form on the surface $C \times C$, or equivalently of the Rosati involution on the Jacobian [Wei48].

Weil's conjectures suggested a cohomology theory for varieties over finite fields, with coefficients in a field of characteristic zero, in which the Frobenius endomorphism would act and satisfy a Lefschetz fixed-point formula. Dwork proved rationality by p-adic analysis [Dwo60]. Grothendieck, with Artin and Verdier, constructed ℓ-adic étale cohomology [SGA4], which gives rationality and the functional equation; he also proposed crystalline cohomology, later constructed by Berthelot [Ber74], as the theory for $\ell = p$. In his Bombay lecture he summarised what remained to be proved: that the polynomials $P_i(t)$ have integral coefficients independent of ℓ, and that the inverse roots of P_i have absolute value $q^{i/2}$ [Gro69, §1, p. 194].

Positivity was the guide. Mattuck and Tate observed that Weil's inequality for curves follows from the Castelnuovo–Severi inequality on $C \times C$ [MT58], and Grothendieck showed that it follows from the Hodge index theorem for surfaces, which holds in every characteristic [Gro58]. Serre proved the analogue of the Riemann hypothesis for compact Kähler manifolds: if an endomorphism f satisfies $f^*\xi = q\xi$ for a Kähler class ξ, the eigenvalues of f^* on H^i have absolute value $q^{i/2}$ [Ser60]. His argument uses the Hodge–Riemann relations on primitive cohomology. Grothendieck's plan was to supply, over any field, the two inputs of Serre's argument: a hard Lefschetz theorem whose inverse is algebraic, and a positivity statement for algebraic classes. These are the standard conjectures.

2.2. Positivity for curves

The argument for curves contains, in the simplest case, every element of the later programme: a correspondence, an intersection form, and a positivity statement for it. We recall it in the form given by Mattuck, Tate and Grothendieck [MT58, Gro58].

Theorem 2.1 (Weil). *Let C be a smooth projective geometrically connected curve of genus g over $\mathbb{F}_q$, and $N = \#C(\mathbb{F}_q)$. Then $|N - q - 1| \leq 2g\sqrt{q}$.*

Proof. On the surface $C \times C$ over $\overline{\mathbb{F}}_q$ let Δ be the diagonal and Γ the graph of the q-power Frobenius F. For a divisor D put $d_1(D) = D \cdot (\{x\} \times C)$ and $d_2(D) = D \cdot (C \times \{x\})$. The Hodge index theorem on $C \times C$, applied to the orthogonal projection of D away from the span of the classes of the two fibres, gives the Castelnuovo–Severi inequality

$$D^2 \leq 2\, d_1(D)\, d_2(D).$$

One has $d_1(\Delta) = d_2(\Delta) = 1$, $d_1(\Gamma) = 1$ and $d_2(\Gamma) = q$, since F has degree q; $\Delta^2 = 2 - 2g$ by adjunction, $\Gamma^2 = q(2-2g)$, and $\Delta \cdot \Gamma = N$, the fixed points of F being the rational points, each with multiplicity one because $dF = 0$. For $D = r\Delta + s\Gamma$ with $r, s \in \mathbb{Z}$ the inequality reads

$$r^2(2-2g) + 2rsN + s^2 q(2-2g) \leq 2(r+s)(r+qs),$$

that is, $g\,r^2 - (N - q - 1)\,rs + g q\, s^2 \geq 0$. A binary quadratic form that is nonnegative on $\mathbb{Z}^2$ is nonnegative on $\mathbb{R}^2$, so its discriminant is nonpositive: $(N - q - 1)^2 \leq 4g^2 q$. □

Applied to the extensions $\mathbb{F}_{q^m}$, the theorem gives the Riemann hypothesis for C: the inverse roots of P_1 have absolute value $q^{1/2}$. The proof is a positivity statement on the algebraic classes of a surface; it is the Hodge-type conjecture for divisors, where it is a theorem. Grothendieck's standard conjecture of Hodge type asks for the same positivity for cycles of all codimensions, where no analogue of the index theorem is known.

2.3. Weil cohomology theories

Let k be an algebraically closed field and K a field of characteristic zero. Following Kleiman [Kle68], a *Weil cohomology theory* with coefficients in K is a contravariant functor $X \mapsto H^*(X)$ from smooth projective varieties over k to graded-commutative finite-dimensional K-algebras, satisfying:

- (W1) *Poincaré duality*: for X connected of dimension n, $H^i(X) = 0$ outside $0 \leq i \leq 2n$, there is a trace isomorphism $H^{2n}(X) \simeq K$, and the cup product $H^i(X) \times H^{2n-i}(X) \to K$ is perfect;
- (W2) *Künneth formula*: $H^*(X \times Y) \simeq H^*(X) \otimes H^*(Y)$ through the two projections;
- (W3) *cycle class map*: group homomorphisms $\mathrm{cl}\colon \mathrm{CH}^i(X) \to H^{2i}(X)$, compatible with pull-back, with products and with the Künneth isomorphism, and sending the class of a point to the trace generator;
- (W4) *weak Lefschetz*: for a smooth hyperplane section $Y \subset X$, restriction $H^i(X) \to H^i(Y)$ is an isomorphism for $i < n-1$ and injective for $i = n-1$;
- (W5) *hard Lefschetz*: for the class ξ of a hyperplane section, $L^{n-i}\colon H^i(X) \to H^{2n-i}(X)$ is an isomorphism for $i \leq n$.

The ℓ-adic cohomology $H^*(X_k, \mathbb{Q}_\ell)$ with ℓ prime to the characteristic is a Weil cohomology [SGA4], the hard Lefschetz theorem being Deligne's [Del80]; crystalline cohomology tensored with the fraction field of $W(k)$ satisfies the axioms for varieties defined over finite fields, hard Lefschetz included, by Berthelot's construction and the theorem of Katz and Messing [Ber74, KM74]; Betti and de Rham cohomology are Weil cohomologies over $\mathbb{C}$. Through a Weil cohomology, a correspondence $\gamma \in \mathrm{CH}^n(X \times X)_{\mathbb{Q}}$ acts on $H^*(X)$ by $x \mapsto p_{2*}(p_1^* x \cup \mathrm{cl}\,\gamma)$, and the Lefschetz fixed-point formula expresses the intersection number of γ with

the diagonal as the alternating sum of the traces of this action on the H^i. Applied to the graph of the Frobenius endomorphism of a variety over $\mathbb{F}_q$, it gives the rationality of the zeta function and its expression

$$Z(X,t) = \prod_{i=0}^{2n} \det\bigl(1 - tF \mid H^i(X_{\overline{\mathbb{F}}_q})\bigr)^{(-1)^{i+1}},$$

in which the polynomials $P_i(t) = \det(1 - tF \mid H^i)$ are the factors predicted by Weil.

2.4. Many cohomologies, one motive

The construction of ℓ-adic cohomology produced not one theory but one for each prime ℓ different from the characteristic, to which crystalline cohomology added one for $\ell = p$, and, over a field of characteristic zero, the Betti and de Rham theories. In *Récoltes et Semailles* Grothendieck described the motive as what these theories have in common: "Par ce terme, j'entends suggérer qu'il s'agit du 'motif commun' (ou de la 'raison commune') sous-jacent à cette multitude d'invariants cohomologiques différents associés à la variété" [Gro86, Promenade, §16, p. 62] (by this term he meant the common motif, or common reason, underlying the many cohomological invariants of a variety). He described the cohomology theories by a musical image: "Ces différentes théories cohomologiques seraient comme autant de développements thématiques différents, chacun dans le 'tempo', dans la 'clef' et dans le 'mode' ('majeur' ou 'mineur') qui lui est propre, d'un même 'motif de base' " [Gro86, Promenade, §16, p. 62], each theory a development of one basic theme, in its own tempo, key and mode. The motive would then be the finest of these incarnations: "Ainsi, le motif associé à une variété algébrique constituerait l'invariant cohomologique 'ultime', 'par excellence', dont tous les autres (associés aux différentes théories cohomologiques possibles) se déduiraient, comme autant d''incarnations' musicales, ou de 'réalisations' différentes" [Gro86, Promenade, §16, p. 62]. The word *réalisation* has remained: the Betti, de Rham, ℓ-adic and crystalline cohomologies of a variety are the realisations of its motive.

The earliest written occurrence of the word in this sense is in a letter to Serre of 16 August 1964 [GS01]: "I call a 'motif' over k something like an ℓ-adic cohomology group of an algebraic scheme over k, but considered as independent of ℓ, and with its 'integral' structure, or, let us say for the moment '$\mathbb{Q}$' structure, deduced from the theory of algebraic cycles" (quoted and translated in [Mil09, note 11]). In the same letter Grothendieck wrote that he could not yet define the abelian category of motives, although he was beginning to have a precise "yoga" for it [Mil09, note 11].

2.5. Pure motives

Let k be a field and $\sim$ an adequate equivalence relation on algebraic cycles: rational, homological for a fixed Weil cohomology, or numerical. For smooth projective varieties X and Y over k, with X of pure dimension d, the correspondences of degree r from X to Y are the classes $\mathrm{CH}^{d+r}_{\sim}(X \times Y)_{\mathbb{Q}}$, composed by the usual formula. The category of pure motives over k for $\sim$ has as objects the triples (X, p, m) with p an idempotent correspondence of degree zero on X and $m \in \mathbb{Z}$, and as morphisms $(X, p, m) \to (Y, q, m')$ the correspondences $q \circ f \circ p$

with f of degree $m' - m$. It is additive, pseudo-abelian and has a tensor product; the realisations are functors to graded vector spaces. This construction is due to Grothendieck; early published accounts are Manin's [Man68], Kleiman's [Kle68] and Demazure's [Dem71].

Two properties were expected and are not known in general. The first is that the category for numerical equivalence be abelian and semisimple; Jannsen proved this unconditionally [Jan92], by an argument that avoids the standard conjectures entirely. The second is that the realisations be exact and faithful functors on it, which requires that homological and numerical equivalence coincide, and that the Künneth components of the diagonal, and with them the grading, be algebraic. These are exactly the properties that the standard conjectures were designed to supply. André constructed an unconditional category of "motivated" motives by adjoining the inverse Lefschetz operators formally [And96]; Deligne's theory of absolute Hodge cycles gives another substitute for abelian varieties [Del82]. Neither proves the conjectures.

2.6. The Tannakian expectation

The expected structure of the category of pure motives is that of a Tannakian category [Saa72, DM82]: a rigid abelian tensor category with a fibre functor, equivalent to the category of representations of an affine group scheme, the *motivic Galois group*. Each Weil cohomology would be a fibre functor, and the comparison isomorphisms between them would be the statement that they are fibre functors of one and the same category. For numerical motives the standard conjectures supply exactly the missing pieces of this structure:

(i) the category is abelian and semisimple, by Jannsen's theorem [Jan92];
(ii) if the Künneth projectors are algebraic (conjecture C), the grading by degree is algebraic, and the commutativity constraint can be modified by the sign of the grading, so that the rank of every object becomes nonnegative, a necessary condition for a Tannakian category in characteristic zero;
(iii) if numerical and homological equivalence coincide (conjecture D), a Weil cohomology defines a fibre functor on numerical motives;
(iv) the Hodge-type conjecture provides a *polarisation* of the Tannakian category in the sense of Saavedra Rivano [Saa72], a system of positive symmetric forms compatible with tensor products, which over $\mathbb{C}$ is the polarisation of Hodge structures.

In this picture the Hodge-type conjecture is not an isolated statement about signs. It asserts that the positivity familiar from Hodge theory is a structure of the motive itself, visible in every realisation. Over a finite field, where there is no Hodge structure, it is the only source of such positivity, and the theory of the present paper is an attempt to locate it in the realisations that do exist.

2.7. The heart within the heart

In the *Promenade* that opens *Récoltes et Semailles*, Grothendieck gave the section on motives the title "Les motifs – ou le cœur dans le cœur" [Gro86, Promenade, §16, p. 59]: the motives, or the heart within the heart. He regarded the standard conjectures as the core of the motivic theme, and wrote of the key phenomena isolated in them: "Et les quelques phénomènes-clef dégages dans les conjectures standard peuvent être vus comme formant une sorte de quintessence ultime

du thème motivique, comme le 'souffle' vital de ce thème subtil entre tous, de ce 'cœur dans le cœur' de la géométrie nouvelle" [Gro86, Promenade, §16, p. 61] (the transcription reads *dégages*). He added that the conjectures follow naturally from the yoga of motives and give in turn a way to construct it: "Qu'il me suffise de dire que les 'conjectures standard' découlent le plus naturellement du monde de ce yoga des motifs" [Gro86, Promenade, §16, p. 63], and conversely: "En même temps elles fournissent un principe d'approche pour une des constructions en forme possibles de la notion de motif" [Gro86, Promenade, §16, p. 63]. In the Bombay lecture he put the same thought in the language of 1968: the proof of the two conjectures "would yield results going considerably further than Weil's conjectures. They would form the basis of the so-called 'theory of motives' which is a systematic theory of 'arithmetic properties' of algebraic varieties, as embodied in their groups of classes of cycles for numerical equivalence" [Gro69, p. 198].

He did not regard them as new: "These are not really new, and they were worked out about three years ago independently by Bombieri and myself" [Gro69, p. 193]. And he did not regard them as remote. Of the Lefschetz conjecture he wrote: "I have an idea of a possible approach to Conjecture B, which relies in turn on certain unsolved geometric questions, and which should be settled in any case" [Gro69, p. 197]. In 1985 he still described them as "l'une des deux questions les plus fondamentales qui se posent en géométrie algébrique" [Gro86, Promenade, §16, p. 63], the other being resolution of singularities.

2.8. Weights and Hodge structures

Two structures on cohomology, available in different settings, express the expected properties of motives. Over a finite field, the theorem of Deligne gives the *weights*: the eigenvalues of Frobenius on H^i are q-Weil numbers of weight i [Del74], and more generally on the cohomology of any variety they define an increasing weight filtration [Del80]. Over $\mathbb{C}$, Hodge theory gives the Hodge decomposition and, for smooth projective varieties, the polarisation by the Hodge–Riemann bilinear relations. The weights are an arithmetic shadow of the grading of motives; the polarisation is the archimedean shadow of the positivity that the Hodge-type conjecture asks for.

The two structures do not meet directly. A variety over a number field has both, at different places, and the comparison theorems of p-adic Hodge theory relate the Hodge filtration of de Rham cohomology over a p-adic field to the Frobenius of the crystalline cohomology of the reduction; Mazur's theorem [Maz73] is the first instance, asserting that the Newton polygon of Frobenius lies above the Hodge polygon. What the Hodge-type conjecture asks over a finite field is a polarisation without a Hodge structure. This paper proposes that the Newton polygon, which is the part of the Hodge structure that survives the passage to a finite field, carries the signs of this polarisation (Definition 5.9).

2.9. Absolute Hodge cycles and motivated cycles

Two constructions replace the algebraic cycles of the definition of motives by classes known to behave well, and thereby obtain unconditional categories. Deligne defines *absolute Hodge cycles* on varieties over fields of characteristic zero, classes in the product of all realisations that are compatible with all comparison isomorphisms and conjugations, and proves that on abelian varieties every Hodge class is absolute Hodge [Del82]. André defines *motivated cycles*, the classes obtained from algebraic cycles by the operations of the theory together with the inverse Lefschetz operators, proves that the resulting category of motives is semisimple and Tannakian, and shows that motivated cycles are absolute Hodge [And96]; in these categories the analogue of B holds by construction. Both constructions achieve the structural goals of the theory of motives by enlarging the class of morphisms, and neither says anything about the positivity of the intersection form on algebraic cycles in positive characteristic, which is the subject of this paper.

2.10. The road not taken

Deligne proved the Riemann hypothesis for varieties over finite fields in 1974 [Del74] and its generalisation to sheaves in 1980 [Del80]. His proof uses the cohomological formalism of [SGA4], the theory of Lefschetz pencils and their monodromy, and Rankin's squaring trick; it does not use the standard conjectures, and it does not prove them. Grothendieck's own road, in which the Riemann hypothesis follows from $B(X)$ and $\mathrm{Hdg}(X \times X)$ by Serre's argument [Gro69, §4, Remark 2, p. 197], has not been completed, and the category of pure motives over a finite field remains conditional on it. The positivity of the intersection form on algebraic classes, which Weil used for curves, which Serre used in Kähler geometry, and which Grothendieck made into a conjecture, is the problem to which the rest of this paper is devoted.

3. The standard conjectures and the two summits

3.1. Setting

Let k be an algebraically closed field, ℓ a prime different from its characteristic, and X a smooth projective connected variety of dimension n over k. We write $H^i(X) = H^i_{\text{ét}}(X, \mathbb{Q}_\ell)$, omitting Tate twists when they play no role, and $\xi \in H^2(X)$ for the class of a hyperplane section, with $L = \xi \cup -$. Let $\mathrm{cl}\colon \mathrm{CH}^i(X)_{\mathbb{Q}} \to H^{2i}(X)$ be the cycle class map and $C^i(X)$ its image, a finite-dimensional $\mathbb{Q}$-vector space, the space of *algebraic classes* of codimension i. Grothendieck's statements are made for any Weil cohomology; we use ℓ-adic cohomology throughout, and crystalline cohomology at the characteristic when it is positive. The *primitive part* of $H^i(X)$ for $i \le n$ is $P^i(X) = \ker\big(L^{n-i+1}\colon H^i(X) \to H^{2n-i+2}(X)\big)$ [Gro69, §4, p. 197].

3.2. The Lefschetz decomposition

By the hard Lefschetz theorem, $H^*(X)$ is a representation of $\mathfrak{sl}_2$ in which L raises the degree by two, and every class has a unique decomposition

$$x = \sum_{r \geq \max(0, i-n)} L^r x_r, \qquad x_r \in P^{i-2r}(X), \qquad x \in H^i(X). \tag{3.1}$$

The components $L^r x_r$ are the images of x under the projectors onto the isotypic pieces of the $\mathfrak{sl}_2$-action, which are polynomials in L and Λ. For $2j \leq n$ the bilinear form

$$\langle x, y \rangle_L = (-1)^j \deg\bigl(x \cdot y \cdot \xi^{n-2j}\bigr) \qquad (x, y \in H^{2j}(X)) \tag{3.2}$$

is nondegenerate, and the decomposition (3.1) is orthogonal for it. Over $\mathbb{C}$, the Hodge–Riemann bilinear relations say that on $P^{2j}(X) \cap H^{j,j}$ the form (3.2) is positive definite; more generally, on $P^{2j} \cap H^{p,q}$ it is definite of sign $(-1)^{p-j}$ after the real structure is taken into account. The conjecture $\mathrm{Hdg}(X)$ is the part of this statement that concerns algebraic classes, which are of type (j, j), transported to an arbitrary Weil cohomology.

3.3. The conjectures

The following are the statements of [Gro69, §§2–4], in the notation of [Kle94].

Conjecture 3.1 (Lefschetz type). *$A(X, L)$: for $2j \leq n$, the map $L^{n-2j} \colon C^j(X) \to C^{n-j}(X)$ is an isomorphism.*

Grothendieck stated $A(X, L)$ together with the hard Lefschetz theorem in cohomology, that $L^{n-i} \colon H^i(X) \to H^{2n-i}(X)$ is an isomorphism for $i \leq n$ (his "mild form"); for ℓ-adic cohomology the latter was proved by Deligne [Del80]. He then gave an equivalent formulation.

Conjecture 3.2 (Lefschetz type, second form). *$B(X)$: the operator Λ of Hodge theory is algebraic. That is, there is an algebraic class $\lambda \in H^*(X \times X)$ such that $\Lambda \colon H^*(X) \to H^*(X)$ is the map $x \mapsto p_{2*}(p_1^* x \cup \lambda)$.*

Here Λ is the operator of the $\mathfrak{sl}_2$-triple (L, h, Λ) on $H^*(X)$, with h acting on $H^i(X)$ as multiplication by $i - n$; the third member of an $\mathfrak{sl}_2$-triple is determined by the other two, so Λ is determined by L and h. In Grothendieck's words, "the conjecture seems to be most amenable in the form B. Note that $B(X)$ is stable for products, hyperplane sections and specialisations" [Gro69, p. 196].

Conjecture 3.3 (Künneth type). *$C(X)$: the Künneth components $\pi_i \in H^{2n}(X \times X)$ of the class of the diagonal are algebraic.*

Conjecture 3.4 (Hodge type). $\mathrm{Hdg}(X)$*: for every j with $2j \leq n$, the symmetric bilinear form*

$$(x, y) \longmapsto (-1)^j \deg\bigl(x \cdot y \cdot \xi^{n-2j}\bigr)$$

is positive definite on $C^j_{\mathrm{pr}}(X) = P^{2j}(X) \cap C^j(X)$.

Grothendieck noted that one reduces easily to the case where $\dim X = 2m$ is even and $j = m$ [Gro69, p. 197]; the form is then the intersection pairing itself, up to the sign $(-1)^m$.

Remark 3.5 (*The numerical form*). Grothendieck's $\mathrm{Hdg}(X)$ concerns the space $C^j(X)$ of algebraic classes in cohomology, and its positivity includes the statement that an algebraic class that is numerically trivial is zero in cohomology. The *numerical* form of the conjecture asks only that the intersection form be positive definite on primitive cycles modulo numerical equivalence; it is the form proved in [Anc21] and in [Kos24], and the two forms agree when homological and numerical equivalence agree, that is, under $D(X)$. The results of this paper concern the numerical form.

Conjecture 3.6 (Numerical equivalence). *$D(X)$: an algebraic cycle on X whose class in $H^*(X)$ is zero is numerically equivalent to zero, and conversely.*

3.4. Implications

The relations between the conjectures are recorded in [Gro69, §§3–4] and [Kle68, Kle94]. Conjecture $B(X)$ implies $A(X, L)$, since the algebraicity of Λ gives the algebraicity of the inverse of L^{n-i}, and $A(X \times X, L)$ implies $B(X)$; hence "the equivalence of conjectures A and B" [Gro69, p. 196]. Conjecture $B(X)$ implies the first part of $C(X)$, since the projectors π_i are polynomials with rational coefficients in Λ and L [Gro69, p. 197]. The conjectures $B(X)$ and $\mathrm{Hdg}(X)$ together imply $D(X)$; only the non-degeneracy of the form in $\mathrm{Hdg}(X)$, not its positivity, is used [Gro69, §4, Remark 4, p. 198]. In characteristic zero $\mathrm{Hdg}(X)$ "follows readily from Hodge theory" [Gro69, §4, Remark 1, p. 197], so that over $\mathbb{C}$ the conjecture B implies all the others [CM13]. Finally, $B(X)$ and $\mathrm{Hdg}(X \times X)$ imply the Riemann hypothesis for X over a finite field, by Serre's argument [Ser60]: "Thus, this implies all of Weil's conjectures" [Gro69, §4, Remark 2, p. 197].

These implications explain the shape of the problem. In characteristic zero the positivity is given by Hodge theory and only existence is missing; in positive characteristic both are missing, and positivity has no archimedean source. Over a finite field the Hodge-type conjecture is the one that carries the arithmetic: it is the statement that the intersection form on algebraic classes has the signature it would have if there were a Hodge structure.

3.5. Three implications in detail

We record the arguments behind three of the implications, since the structure they display, an $\mathfrak{sl}_2$-action for B and a positive form for Hdg, is the one that the Alien Hodge Conjectures of Part IV refine.

Proposition 3.7. *$B(X)$ implies that the Künneth projectors π_i are algebraic.*

Proof. Let L be the correspondence $\Delta_*(\xi)$ and Λ the algebraic correspondence given by $B(X)$. Their commutator $h = [L, \Lambda]$ is algebraic, and by the $\mathfrak{sl}_2$-relations it acts on $H^i(X)$ as multiplication by $i - n$. Hence

$$\pi_i = \prod_{j \neq i} \frac{h - (j - n)}{i - j},$$

the product over $0 \leq j \leq 2n$, is a polynomial in h with rational coefficients, and it is algebraic. □

This is the argument behind Grothendieck's remark that the projectors π_i are polynomials with rational coefficients in Λ and L [Gro69, p. 197]. For the equivalence of A and B, he observes that the algebraicity of Λ implies that of Λ^{n-i}, which inverts L^{n-i} on the primitive components up to nonzero rational factors [Gro69, p. 196]; the converse $A(X \times X) \Rightarrow B(X)$ applies the Lefschetz isomorphism on $X \times X$ to the class of L.

Proposition 3.8. *$B(X)$ and $\mathrm{Hdg}(X)$ imply $D(X)$.*

Proof. Let $x \in C^j(X)$ be numerically equivalent to zero; by the Lefschetz isomorphism, which is algebraic under $B(X)$, we may assume $2j \le n$. By Proposition 3.7 and the $\mathfrak{sl}_2$-action, the Lefschetz decomposition $x = \sum_r L^r x_r$, with x_r primitive in $H^{2j-2r}(X)$, has algebraic components x_r, since the projectors onto the components are polynomials in L and Λ. The Lefschetz components of different primitive degree are orthogonal for the pairing $(u, v) \mapsto \deg(u \cdot L^{n-2j} v)$ on H^{2j}, so $\deg(x \cdot L^{n-2j} \cdot L^r x_r) = \deg(L^r x_r \cdot L^{n-2j} \cdot L^r x_r)$ for each r, and the left-hand side vanishes because $L^{n-2j+r} x_r$ is algebraic and x is numerically trivial. By $\mathrm{Hdg}(X)$ applied to $x_r \in C^{j-r}_{\mathrm{pr}}$, the right-hand side is $\pm \deg(x_r \cdot L^{n-2(j-r)} x_r)$, which vanishes only if $x_r = 0$. Hence $x = 0$ in cohomology. □

Only the non-degeneracy of the form is used here, as Grothendieck notes [Gro69, §4, Remark 4, p. 198]. The positivity is what gives the Riemann hypothesis. We give the argument over $\mathbb{C}$ first, where it is due to Serre.

Theorem 3.9 (Serre). *Let X be a compact Kähler manifold of dimension n with Kähler class ω, and $f : X \to X$ a holomorphic map with $f^*\omega = q\,\omega$ for some $q > 0$. Then every eigenvalue of f^* on $H^i(X, \mathbb{C})$ has absolute value $q^{i/2}$.*

Proof. Since f^* commutes with L_ω up to the factor q, it preserves the Lefschetz decomposition, and it suffices to treat the primitive part P^i of H^i; the component $L^r P^{i-2r}$ then has eigenvalues q^r times those on P^{i-2r}, of absolute value $q^{r+(i-2r)/2} = q^{i/2}$. On P^i consider the hermitian form $h(x, y) = \int_X x \wedge \bar{y} \wedge \omega^{n-i}$. By the Hodge–Riemann relations it is definite on each $P^{p,q}$, the decomposition $P^i = \bigoplus_{p+q=i} P^{p,q}$ is h-orthogonal, and f^* preserves it, f being holomorphic. The map f has degree q^n, since $f^*\omega^n = q^n \omega^n$. For $x, y \in P^{p,q}$,

$$h(f^*x, f^*y) = q^{-(n-i)} \int_X f^*\big(x \wedge \bar{y} \wedge \omega^{n-i}\big) = q^{-(n-i)} q^n \, h(x, y) = q^i \, h(x, y).$$

So $q^{-i/2} f^*$ is unitary for a definite hermitian form on each $P^{p,q}$, and its eigenvalues have absolute value 1. □

Grothendieck's algebraic version replaces the hermitian form by a quadratic form on algebraic correspondences. If $B(X)$ holds, the transpose with respect to the Lefschetz involution defines an involution $u \mapsto u'$ on the $\mathbb{Q}$-algebra of correspondences of degree zero modulo homological equivalence, and $\mathrm{Hdg}(X \times X)$ says that the form $(u, v) \mapsto \mathrm{tr}(u \circ v')$ is positive definite on it. For the Frobenius F of a variety over $\mathbb{F}_q$ one has $F' \circ F = q^i$ on H^i, and positivity forces $q^{-i/2} F$ to generate a compact subgroup; its eigenvalues then have absolute value 1 in every complex embedding. This is the argument of Serre's paper [Ser60] transported by Grothendieck [Gro69, §4, Remark 2]; Kleiman gives the details [Kle68, Kle94].

The $\mathfrak{sl}_2$-formalism also explains why B is stable under products, a fact Grothendieck records [Gro69, p. 196].

Proposition 3.10. *If $B(X)$ and $B(Y)$ hold, then $B(X \times Y)$ holds for the ample class $\xi_X \otimes 1 + 1 \otimes \xi_Y$.*

Proof. Under the Künneth isomorphism, $L_{X\times Y} = L_X \otimes 1 + 1 \otimes L_Y$ and the grading operator of $X \times Y$ is $h_X \otimes 1 + 1 \otimes h_Y$. The triples $(h_X \otimes 1, L_X \otimes 1, \Lambda_X \otimes 1)$ and $(1 \otimes h_Y, 1 \otimes L_Y, 1 \otimes \Lambda_Y)$ commute, so by Lemma 21.1 their sum is an $\mathfrak{sl}_2$-triple, and by uniqueness of the third member $\Lambda_{X\times Y} = \Lambda_X \otimes 1 + 1 \otimes \Lambda_Y$. The exterior products of the cycles inducing Λ_X with the diagonal of Y, and of the diagonal of X with the cycles inducing Λ_Y, are algebraic on $(X \times Y)^2$ and induce the two terms. □

The same argument, applied to a fibration instead of a product, is the content of AHC II (Section 21).

3.6. Relation with the Hodge and Tate conjectures

Over $\mathbb{C}$ the operator Λ is a morphism of Hodge structures, so its class in $H^*(X \times X)$ is a Hodge class, and the Hodge conjecture for $X \times X$ implies $B(X)$. Over a finite field the Tate conjecture concerns the same question of existence, for the classes fixed by Frobenius, and it is expected to imply that the motives of varieties over finite fields are of abelian type; for abelian varieties Milne showed that the Hodge-type conjecture in positive characteristic would follow from the Hodge conjecture for complex abelian varieties of CM type [Mil02]. Both summits therefore lie below the Hodge and Tate conjectures, which are harder. The standard conjectures isolate, in Grothendieck's words, "an existence assertion for algebraic cycles (considerably weaker than the Tate conjectures)" and "a statement of positivity" [Gro69, p. 193].

3.7. Over finite fields

Over a finite field some of the conjectures are known in general, as consequences of the Weil conjectures. Katz and Messing proved that for a smooth projective variety over a finite field the Künneth components of the diagonal are algebraic, so that conjecture C holds, and that the characteristic polynomial of Frobenius on crystalline cohomology agrees with that on ℓ-adic cohomology [KM74]; the first statement follows from the Riemann hypothesis, which separates the eigenvalues of Frobenius on different degrees, by applying a suitable polynomial in the graph of Frobenius. The Tate conjecture over finite fields would imply B, since the class of Λ is fixed by Frobenius, and, by Milne's argument [Mil15, 3.22], the Hodge-type conjecture. The Tate conjecture itself is known for divisors on abelian varieties [Tat66], for K3 surfaces [Cha13, MP15], and for abelian fourfolds over finite fields (preprints, [Bro26, BJ26]). What remains is the core of the problem: the existence and the positivity of cycles of codimension two on fourfolds, with no special structure to supply them.

3.8. Known cases

For divisors, Hdg holds in every characteristic: on a surface it is the Hodge index theorem, proved algebraically by Segre and by Bronowski, and the case of divisors on a variety of any dimension reduces to it by hyperplane sections [Gro58, Anc21]. For $n \leq 3$ the only case of $\mathrm{Hdg}(X)$ is $j \leq 1$, so Hdg holds for varieties of dimension at most three in every characteristic. The first open case is $n = 4$, $j = 2$.

For B, the operator Λ is algebraic for curves and surfaces [Kle68], for complete intersections in projective space and for Grassmannians, and for abelian varieties by Lieberman [Gro69, pp. 196–197], [Lie68]. For threefolds over $\mathbb{C}$, B is known for several classes [Tan10, Tan11, CH21], and it is equivalent to the Friedlander–Mazur conjecture in dimension three [CH21]; the first open case of B is thus degree two on threefolds. For fourfolds and higher-dimensional varieties the known cases are those surveyed in Section 4.3.

3.9. Why characteristic zero does not suffice

Let $\mathcal{X}$ be smooth projective over a discrete valuation ring with generic fibre X_η of characteristic zero and special fibre X_0 over a finite field. Specialisation of cycles gives a map $\mathrm{CH}^j(X_{\bar{\eta}}) \to \mathrm{CH}^j(X_{\bar{0}})$, compatible with intersection numbers, and the smooth and proper base change theorem identifies the ℓ-adic cohomologies of the two geometric fibres compatibly with cup products and cycle classes [SGA4, Exp. XVI], [Ful98, Ch. 20]. The intersection form on the image of specialisation is therefore known: it is the form on algebraic classes of $X_{\bar{\eta}}$, which is positive definite on primitive classes by the Hodge–Riemann relations. If every algebraic class of $X_{\bar{0}}$ came from $X_{\bar{\eta}}$, the Hodge-type conjecture for X_0 would follow from Hodge theory.

It does not, and the failure is the rule rather than the exception. The space of algebraic classes can grow under specialisation: the Picard number of a complex K3 surface is at most 20, while that of a supersingular K3 surface is 22; the Picard number of $E \times E$ is at most 4 for an elliptic curve in characteristic zero, and 6 for a supersingular one. The new classes are not specialisations of algebraic classes, and in general not even of Hodge classes of the lift: their cohomology classes lie, in the Betti realisation of the lift, in parts of Hodge type $(j+k, j-k)+(j-k, j+k)$ with $k \neq 0$, on which the Hodge–Riemann relations give the sign $(-1)^{j+k}$ rather than the sign $(-1)^j$ of algebraic classes. For divisors the index theorem settles their sign in every characteristic; in codimension two nothing comparable is known. The graphs D_π of the Frobenius of a supersingular elliptic curve in Section 8 are the simplest example of such classes, and the plane C of Theorem A is a case in codimension two.

The problem is thus to determine the sign of algebraic classes that the lift does not see. The idea of this paper is that the sign is not lost but moved: what the lift records about these classes is a sign at the archimedean place of the wrong parity, and reciprocity, together with an invariant at p of the crystalline realisation, converts it into the sign that the Hodge-type conjecture predicts. Proposition 8.4 shows this mechanism in weight two, and Proposition 13.6 proves it in weight four, one algebraic class at a time.

3.10. The two summits

We take as the two central problems the following cases, which concern the same kind of object, algebraic cycles of codimension two on fourfolds or their self-products.

Definition 3.11 (The arithmetic summit). The *arithmetic summit* is the conjecture $\mathrm{Hdg}(X)$ for $j = 2$, for every smooth projective fourfold X over $\overline{\mathbb{F}}_q$: the intersection form is positive definite on $C^2_{\mathrm{pr}}(X)$.

Definition 3.12 (The geometric summit). The *geometric summit* is the conjecture $B(X)$ for every smooth projective fourfold X over $\mathbb{C}$: the operator Λ is induced by an algebraic class on $X \times X$.

In degree two the operator $\Lambda\colon H^6(X) \to H^2(X)$ inverting L^2 is given by a class in $H^4(X \times X)$, of codimension two; so the geometric summit asks for the *existence* of codimension-two cycles on $X \times X$, and the arithmetic summit for the *positivity* of the intersection form on codimension-two cycles on X. The first is a statement over $\mathbb{C}$, where positivity is known; the second is a statement over $\overline{\mathbb{F}}_q$, where neither is. The two summits are related by specialisation (Section 21): a solution of the geometric summit for a lift of a fourfold X over $\overline{\mathbb{F}}_q$ gives B for X, since B is stable under specialisation, but it gives no information on the sign.

The Alien Hodge Conjectures of Part IV are formulated so as to imply these two summits. Theorem A proves the arithmetic summit for the hypergeometric summand of three fourfolds that lie outside the classes treated so far.

4. The frontier

This section surveys what is known about the two summits, as of November 2026, and identifies the class of fourfolds to which the methods of the paper apply. Results that have appeared only as preprints are marked as such.

4.1. The arithmetic summit

For nearly fifty years after its formulation the Hodge-type conjecture saw little progress beyond divisors. Milne showed that the Hodge conjecture for complex abelian varieties of CM type would imply Hdg for abelian varieties in positive characteristic [Mil02]. The first unconditional case beyond divisors is Ancona's theorem that Hdg holds for abelian fourfolds over any field [Anc21]; Ancona describes it as "the first unconditional result on the conjecture since its formulation". His proof decomposes the motive of the fourfold into motives of rank two attached to quadratic forms, compares the intersection form on algebraic classes with a form on a CM lift place by place, and uses p-adic Hodge theory to compute the invariant at p; the product formula for Hilbert symbols then fixes the sign. His Theorem 8.1 concerns a motive M over the ring of integers of a p-adic field with a quadratic form q: if the Betti realisation V_{B} of M and the space V_Z of numerical classes of its special fibre both have dimension two, and q polarises V_{B}, then q is positive definite on V_Z [Anc21, Thm. 8.1]. For a motive of Hodge type $(1,-1)+(-1,1)$ the polarisation is negative definite on

V_{B}, so the theorem already exhibits a reversal of sign between the Betti form and the form on algebraic classes; Section 14 returns to this.

The method has since been extended in three directions. Ito, Ito and Koshikawa proved Hdg for the square of a K3 surface and for all powers of certain K3 surfaces, using CM liftings and the Kuga–Satake construction [IIK25]. Koshikawa proved the numerical form for the square of a simple abelian variety of prime dimension [Kos24]. Agugliaro constructed, for every prime p and every $g > 4$, infinitely many abelian varieties of dimension g over $\overline{\mathbb{F}}_p$ satisfying Hdg, with Tate classes that are neither Lefschetz classes nor specialisations of Hodge classes of CM lifts [Agu26a], and proved Hdg for powers of abelian threefolds (preprint, [Agu25]). Ancona and Marmora proved that for varieties over finite fields with a CM lifting, such as abelian varieties and products of K3 surfaces, the signature predicted by Hdg holds modulo four: the discriminant and the Hilbert symbol of the intersection form are those predicted [AM25]. Harashita gave an elementary proof of Hdg for Hermitian hypersurfaces, whose middle cohomology is spanned by linear subspaces, with an explicit p-adic bound (preprint, [Har26]).

On the side of the Tate conjecture, which together with Hdg yields D, Broe proved the Tate conjecture for abelian fourfolds over finite fields, deducing all the standard conjectures for them (preprint, [Bro26]), and Broe and Jiang extended it to abelian fourfolds over finitely generated fields (preprint, [BJ26]); both reduce to Markman's algebraicity of Weil classes [Mar25]. Milne's survey [Mil15] describes the relation between the Tate conjecture, the Hodge-type conjecture and the Riemann hypothesis over finite fields.

Every unconditional case above is of abelian or K3 type, or is spanned by linear cycles: the motive in question is reached from abelian varieties or K3 surfaces by known algebraic correspondences, or has all its cohomology algebraic. The methods use either a CM lift, whose Betti realisation supplies the sign at the archimedean place, or a representation-theoretic argument on a motive with large symmetry.

4.2. The methods

The proofs of the arithmetic cases share a structure, which we describe because the method of this paper modifies it.

Reduction to small blocks. Ancona decomposes the motive of an abelian fourfold A over $\overline{\mathbb{F}}_q$ by means of Lefschetz classes and Frobenius into pieces on which the question reduces either to Lefschetz classes, for which positivity follows from the Hodge index theorem and the structure of the Lefschetz ring, or to "exotic" motives of rank two whose algebraic classes are not Lefschetz [Anc21, §§5–7]. This is a decomposition by algebraic correspondences into blocks of small rank, as in Theorem 10.3.

A global form. An abelian variety over a finite field, with a choice of CM structure, lifts to an abelian scheme with complex multiplication over the ring of integers of a p-adic field, by a theorem of Tate in Honda–Tate theory, and the motivic decomposition lifts with it [Anc21, Thm. 6.10, Cor. 6.12]. The exotic motive of rank two thereby becomes the special fibre of a motive over the p-adic ring whose Betti realisation is a rational quadratic plane V_{B} with a polarisation. The ℓ-adic realisations of the special fibre, for $\ell \neq p$, are the localisations of V_{B}, by smooth proper base change. Ancona works with numerical equivalence

throughout, because it is not known that homological equivalence commutes with extension of scalars [Anc21, Rem. 6.9].

The invariant at p. The quadratic space of numerical classes V_Z, of dimension two, and V_B are compared at every place. At $\ell \neq p$ they agree. At ∞, V_B has the sign of the polarisation. At p, the comparison goes through the crystalline realisation and p-adic Hodge theory: Ancona computes the p-adic periods of the lift and shows that the Hasse invariant of $V_Z \otimes \mathbb{Q}_p$ is the one that makes the product formula for V_Z consistent with a positive definite form [Anc21, §§10–13]. Reciprocity then forces V_Z to be positive definite.

The theorem of Ancona and Marmora extends the computation at p to all motives with a CM lift, and obtains the discriminant and the Hasse invariant of the space of algebraic classes in general [AM25]; the statement modulo four is what Lemma 5.3 allows one to deduce from such a computation. Ito, Ito and Koshikawa combine their earlier results on CM liftings of K3 surfaces and on the Tate conjecture for the square of a K3 surface with the positivity of the Rosati involution on the Kuga–Satake abelian variety [IIK25, KS67]. The Tate conjecture for K3 surfaces themselves is due to Charles and to Madapusi Pera [Cha13, MP15], again through the Kuga–Satake construction, which Deligne had used to prove the Weil conjectures for K3 surfaces [Del72].

In these arguments the global form at the archimedean place is supplied by a lift of CM type, and the sign of the algebraic classes is obtained by comparing the lift with the special fibre at p through p-adic Hodge theory; the sign of a block on the lift and the sign of the algebraic classes in its reduction need not agree (Remark 20.3). (Harashita's proof for Hermitian varieties is different in kind: the middle cohomology is spanned by explicit linear cycles and the intersection matrix is computed directly.) The method of Part III keeps the reduction to small blocks and the reciprocity argument, but obtains the global forms from the Betti realisation of a lift that is *not* of CM type, through an algebraic involution, and obtains the invariant at p from Ogus's classification of supersingular K3 crystals rather than from p-adic periods.

4.3. The geometric summit

Over $\mathbb{C}$ the Lefschetz conjecture is known for abelian varieties [Lie68], for varieties whose cohomology is spanned by algebraic classes, and in the following cases among fourfolds and their higher-dimensional relatives. Charles and Markman proved all the standard conjectures for hyper-Kähler varieties deformation equivalent to Hilbert schemes of points on K3 surfaces, by Verbitsky's theory of hyperholomorphic sheaves [CM13]. Foster proved B for hyper-Kähler varieties of generalised Kummer type in certain degrees and then for those of dimension $4d$ (preprints, [Fos23, Fos25]). Voisin proved B in degree two for hyper-Kähler fourfolds covered by Lagrangian subvarieties that have Picard number one and are general in moduli, and showed that for Lagrangian fibrations B in degree two follows from the SYZ conjecture on the divisor classes of such fibrations [Voi22]. Ancona, Cavicchi, Laterveer and Saccà proved B for the hyper-Kähler tenfolds of Laza, Saccà and Voisin, through a relative form of the conjecture for Lagrangian fibrations satisfying the hypotheses of Ngô's support theorem (preprint, [ACLS23]), and Hartlieb and Shah proved it for all hyper-Kähler varieties of OG10 type (preprint, [HS26]). Arapura's notion of

motivation gives B for moduli spaces of sheaves on curves and surfaces that are motivated by the underlying curve or surface [Ara06].

The methods for the geometric summit share a different structure. A criterion recorded by Charles and Markman reduces $B(X)$ in a given degree to the existence of algebraic correspondences whose images span the cohomology in that degree [CM13, §2]. For varieties of $K3^{[n]}$-type the required correspondences come from a moduli space M of sheaves on a K3 surface: the characteristic class of a universal family on $M \times M$, transported along a family of deformations, remains algebraic on every projective fibre, by Markman's theorem relying on Verbitsky's hyperholomorphic sheaves [CM13, Thm. 4.1], and a study of the cohomology ring shows that it generates enough. The results for OG10 type use twisted hyperholomorphic sheaves built from Lagrangian fibrations [HS26]; Arapura's results reduce to curves and surfaces through dominating correspondences [Ara06]. In each case an algebraic cycle is produced by a geometric construction tied to the specific type of the variety.

Two results bear directly on Section 21. Maulik, Shen and Yin proved that the decomposition theorem for the Hitchin fibration is induced by algebraic cycles, and that the inverse of the relative hard Lefschetz isomorphism is induced by a relative algebraic correspondence (preprint, [MSY24]); this is the motivic decomposition conjecture of Corti and Hanamura [CH00] and the relative Lefschetz conjecture in one case. De Cataldo and Migliorini's Hodge-theoretic proof of the decomposition theorem [dCM05] supplies the two commuting Lefschetz actions on which AHC II is built. Beilinson showed that the standard conjectures would follow from the existence of a motivic t-structure, or from a weak form of Suslin's Lawson homology conjecture [Bei12].

For a general fourfold over $\mathbb{C}$ none of these applies, and B is open; it is open already in degree two for general threefolds (Section 3.8).

4.4. A summary

Table 2 summarises the classes of varieties of dimension four or of the relevant degree for which one of the two summits is known, with the method. Entries marked P are preprints.

4.5. The gap

The fourfolds over finite fields for which Hdg in codimension two is unknown include all those whose middle cohomology contains a transcendental part with $h^{4,0} > 0$ that is not of abelian or K3 type in the above sense: the part is not known to be reached from abelian varieties or K3 surfaces by an algebraic correspondence compatible with the intersection forms. For such a part there is no CM lift to supply the archimedean sign, and no known algebraic correspondence to transport positivity from a case already settled. Hypergeometric motives of weight four with Hodge numbers $(1, 1, 1, 1, 1)$ are of this kind; the Dwork family of sextic fourfolds

$$x_0^6 + x_1^6 + \cdots + x_5^6 - 6\psi\, x_0 x_1 \cdots x_5 = 0$$

is the classical example, and its primitive middle cohomology carries such a motive. At a prime where the Tate classes of such a motive form a block of rank three that does not split, the local data do not determine the sign; we call such a prime a *wall* (Definition 6.8). The Dwork walls are not resolved in this paper.

Class	Statement	Method
divisors, any dimension, any characteristic	Hdg for $j = 1$	Hodge index theorem [Gro58]
abelian varieties	B	Lieberman [Lie68]
abelian fourfolds over $\overline{\mathbb{F}}_q$	Hdg	rank-two motives, p-adic periods [Anc21]
squares and powers of K3 surfaces over $\overline{\mathbb{F}}_q$	Hdg	CM lifts, Kuga–Satake [IIK25]
squares of simple abelian varieties of prime dimension	numerical Hdg	[Kos24]
powers of abelian threefolds; examples in every $g > 4$	Hdg	[Agu25] (P), [Agu26a]
CM-liftable varieties over $\overline{\mathbb{F}}_q$	signature mod 4	Hilbert symbol at p [AM25]
Hermitian hypersurfaces	Hdg	linear cycles [Har26] (P)
abelian fourfolds over $\overline{\mathbb{F}}_q$	Tate, hence D	Weil classes [Bro26, BJ26] (P)
$K3^{[n]}$ type	B, all	hyperholomorphic sheaves [CM13]
generalised Kummer type	B	[Fos23, Fos25] (P)
OG10 type	B	Lagrangian fibrations [ACLS23, HS26] (P)
Hitchin systems	relative B, decomposition	[MSY24] (P)
unirational fourfolds, moduli of sheaves on curves and surfaces	B	motivation [Ara06]
hypergeometric summands of three fourfolds of Hodge type $(1,1,1,1,1)$	numerical Hdg outside $E \cup T$, under Hypothesis 12.4	Levelt splitting, Ogus, reciprocity (this paper)

TABLE 2. Known cases of the two summits and nearby statements.

What we show is that when an algebraic involution splits the Tate part into pieces of rank at most two, the local data do determine the sign, and the sign is the Newton–Hodge sign rather than the Betti sign. Among the hypergeometric data of rank at most eight with $\beta \equiv \alpha + \frac{1}{2}$, the three motives (1.1) are the only ones of rank five with Hodge numbers $(1,1,1,1,1)$ (Proposition 12.9).

Part II

Alien Hodge Motives

5. Local data

This section collects the arithmetic of quadratic forms and the structure of Frobenius on the cohomology of varieties over finite fields that the rest of the paper uses. Everything in it is classical except the notion of Newton–Hodge sign in Section 5.5.

5.1. Quadratic forms over $\mathbb{Q}$ and its completions

For a place v of $\mathbb{Q}$ and $a, b \in \mathbb{Q}_v^\times$ the Hilbert symbol $(a,b)_v \in \{\pm 1\}$ is 1 if and only if $ax^2 + by^2 = z^2$ has a nonzero solution in $\mathbb{Q}_v$. It is symmetric and bimultiplicative, depends on a and b only modulo squares, and satisfies $(a,-a)_v = 1$. At $v = \infty$ it is -1 exactly when $a < 0$ and $b < 0$. At an odd prime p, writing $a = p^\alpha u$ and $b = p^\beta w$ with u, w units,

$$(a,b)_p = (-1)^{\alpha\beta\frac{p-1}{2}} \left(\frac{u}{p}\right)^{\beta} \left(\frac{w}{p}\right)^{\alpha}; \tag{5.1}$$

in particular $(a,b)_p = 1$ when a and b are both units. At $p = 2$, with $a = 2^\alpha u$, $b = 2^\beta w$ and u, w odd, and with $\epsilon(u) \equiv \frac{u-1}{2}$ and $\omega(u) \equiv \frac{u^2-1}{8}$ modulo 2,

$$(a,b)_2 = (-1)^{\epsilon(u)\epsilon(w) + \alpha\,\omega(w) + \beta\,\omega(u)} \tag{5.2}$$

[Ser73, Ch. III, Thm. 1]. For example $(-1,-1)_2 = -1$, $(2,u)_2 = (-1)^{\omega(u)}$, and $(-7,-1)_2 = 1$, since -7 is a square in $\mathbb{Q}_2$. For $a, b \in \mathbb{Q}^\times$ the symbol is 1 at all but finitely many places, and Hilbert's reciprocity law

$$\prod_v (a,b)_v = 1 \tag{5.3}$$

holds, the product running over all places of $\mathbb{Q}$ [Ser73, Ch. III, §2].

A nondegenerate quadratic form $f \simeq \langle a_1, \dots, a_n \rangle$ over $\mathbb{Q}_v$ has two invariants: its discriminant $d(f) = a_1 \cdots a_n \in \mathbb{Q}_v^\times / \mathbb{Q}_v^{\times 2}$, and its Hasse invariant $\varepsilon_v(f) = \prod_{i<j} (a_i, a_j)_v$, which does not depend on the diagonalisation. Over $\mathbb{Q}_p$ the rank, the discriminant and the Hasse invariant classify f; over $\mathbb{R}$ the rank and the signature do. A rational form f is classified by its rank, its discriminant in $\mathbb{Q}^\times/\mathbb{Q}^{\times 2}$, its signature and its Hasse invariants at all primes (the Hasse–Minkowski theorem); conversely, local data $(d, \varepsilon_v, \text{signature})$ that satisfy $\prod_v \varepsilon_v = 1$ and the obvious compatibilities at ∞ are those of a rational form [Ser73, Ch. IV, §3]. We shall use three elementary consequences.

Lemma 5.1. *For forms f, g over $\mathbb{Q}_v$, $\varepsilon_v(f \oplus g) = \varepsilon_v(f)\,\varepsilon_v(g)\,(d(f), d(g))_v$ and $d(f \oplus g) = d(f)d(g)$.*

Proof. Diagonalise $f = \langle a_1, \dots, a_m \rangle$ and $g = \langle b_1, \dots, b_k \rangle$; the pairs (i,j) with $i < j$ in $f \oplus g$ are the pairs inside f, the pairs inside g, and the pairs (a_i, b_j), whose symbols multiply to $\prod_{i,j} (a_i, b_j)_v = (d(f), d(g))_v$ by bimultiplicativity. □

Lemma 5.2 (Representation by a binary form). *Let f be a nondegenerate binary form over $\mathbb{Q}_v$ and $r \in \mathbb{Q}_v^\times$. Then f represents r if and only if $(r, -d(f))_v = \varepsilon_v(f)$.*

Proof. This is the case $n = 2$ of the representation criterion of [Ser73, Ch. IV, §2.2]: f represents r if and only if $f \oplus \langle -r \rangle$ represents zero, and for the ternary form $f \oplus \langle -r \rangle$ this condition is expressed through its invariants. For $f = \langle a, b \rangle$ the computation reduces to $(r, -ab)_v = (a,b)_v$. □

Lemma 5.3 (The real invariants). *Let f be a rational form of rank n with s_- negative entries in a diagonalisation over $\mathbb{R}$. Then* $\operatorname{sgn} d(f) = (-1)^{s_-}$ *and* $\varepsilon_\infty(f) = (-1)^{s_-(s_--1)/2}$. *In particular the finite invariants of f determine s_- modulo 4: if f and g have the same rank and the same discriminant and Hasse invariant at every prime, then $s_-(f) \equiv s_-(g) \pmod 4$.*

Proof. The first two formulas follow from the definitions, since $(a,b)_\infty = -1$ only for $a,b<0$. If f and g agree at every prime, the product formula (5.3) applied to the entries of their diagonalisations gives $\varepsilon_\infty(f) = \varepsilon_\infty(g)$, and the discriminants, being equal at every prime, agree in sign. The parities of s_- and of $s_-(s_--1)/2$ determine s_- modulo 4. □

Lemma 5.3 is sharp: the forms $\langle 1,1,1,1\rangle$ and $\langle -1,-1,-1,-1\rangle$ have the same invariants at every prime. Ancona and Marmora proved that for varieties with a CM lifting the signature on algebraic classes is the predicted one modulo four [AM25]; in the language of Lemma 5.3 this is the statement that the finite invariants of the space of algebraic classes are those of a definite form of the predicted sign.

Lemma 5.4 (Square classes). *Let $a,b\in\mathbb{Q}^\times$ with $b>0$. If ab is a square in $\mathbb{Q}_\ell$ for all but finitely many primes ℓ, then $a>0$.*

Proof. If $a<0$, then $ab=-m$ with $m>0$, and $-m$ is not a square modulo infinitely many primes: write $m=s^2m_0$ with m_0 squarefree; by quadratic reciprocity and Dirichlet's theorem there are infinitely many primes $\ell\nmid 2m_0$ with $\left(\frac{-m_0}{\ell}\right)=-1$. □

5.2. Hermitian lines

Let L be a field of characteristic zero, $\Delta\in L^\times$, and $E_L = L[x]/(x^2-tx+1)$ with $t^2-4=\Delta$, an étale quadratic L-algebra with involution $\bar{}$ exchanging the roots; write N for its norm. A binary quadratic space W over L with an isometry ϕ of characteristic polynomial x^2-tx+1 and $\Delta\neq 0$ is a free E_L-module of rank one, x acting by ϕ.

Lemma 5.5 (Hermitian lines). *In this situation there is $a\in L^\times$, unique modulo $N(E_L^\times)$, such that $\langle u,v\rangle = \operatorname{Tr}_{E_L/L}(a\,u\bar v)$ for $u,v\in W\simeq E_L$. In the basis $1,\sqrt{\Delta}$ of E_L the form is $\langle 2a,-2a\Delta\rangle$, so that*

$$d(W)\equiv -\Delta,\qquad \varepsilon(W)=(2a,\Delta)_L$$

when L is a local field, where $(\ ,\ )_L$ is the Hilbert symbol of L.

Proof. A symmetric bilinear form b on E_L with $b(xu,xv)=b(u,v)$ satisfies $b(xu,v)=b(u,x^{-1}v)=b(u,\bar xv)$, since $x\bar x=1$; so multiplication by x, hence by every element of E_L, is adjoint to multiplication by its conjugate. The linear form $\lambda(w)=b(w,1)$ on E_L then determines b, through $b(u,v)=b(u\bar v,1)=\lambda(u\bar v)$. The symmetric condition $b(u,v)=b(v,u)$ says $\lambda(w)=\lambda(\bar w)$, so λ factors through the trace: $\lambda=\operatorname{Tr}(a\,\cdot)$ for a unique $a\in L$, nonzero as b is. Replacing the identification $W\simeq E_L$ by its composite with multiplication by $c\in E_L^\times$ replaces a by $aN(c)$. In the basis $1,\sqrt{\Delta}$, $\operatorname{Tr}(a)=2a$, $\operatorname{Tr}(a\sqrt{\Delta}\cdot\overline{\sqrt{\Delta}})=-2a\Delta$ and $\operatorname{Tr}(a\sqrt{\Delta})=0$. Finally $(2a,-2a\Delta)_L=(2a,-2a)_L(2a,\Delta)_L=(2a,\Delta)_L$. □

The class of a in $L^\times/N(E_L^\times)$, a group of order two when E_L is a field and trivial when it is split, is the *hermitian class* of W; by local class field theory it is detected by $(a,\Delta)_L$. Thus a binary block on which Frobenius acts with irreducible characteristic polynomial carries, at each place, exactly one bit of information beyond its discriminant, and this bit is a norm residue symbol. When $E=\mathbb{Q}(\sqrt{\Delta})$ is imaginary quadratic and the local spaces W_v are the localisations of a global hermitian line $\mathrm{Tr}(a\,u\bar{v})$, $a\in\mathbb{Q}^\times$, the global form is definite of sign $\mathrm{sgn}(a)$, and reciprocity gives $\prod_{v<\infty}(2a,\Delta)_v=(2a,\Delta)_\infty=\mathrm{sgn}(a)$.

5.3. Frobenius on the cohomology of a motive over a finite field

Let q be a power of a prime p, and let X be a smooth projective variety of dimension $2m$ over $\mathbb{F}_q$. Let e be a self-adjoint idempotent correspondence of degree zero on X with rational coefficients, and assume that e acts on $H^*(X_{\overline{\mathbb{F}}_q})$ through H^{2m}; by the theorem of Katz and Messing [KM74] the Künneth projectors of X are algebraic over a finite field, so this can always be arranged by composing with π_{2m}. For a prime $\ell\neq p$ put

$$M_\ell = e\,H^{2m}_{\text{ét}}(X_{\overline{\mathbb{F}}_q},\mathbb{Q}_\ell(m)),\qquad \langle x,y\rangle=\deg(x\cup y),$$

a nondegenerate symmetric space over $\mathbb{Q}_\ell$ (since e is self-adjoint), and let ϕ be the geometric Frobenius over $\mathbb{F}_q$; with the twist by $\mathbb{Q}_\ell(m)$ it is an isometry of M_ℓ. By Deligne's theorem [Del74] and the theorem of Katz and Messing, its characteristic polynomial $P_M(x)$ has rational coefficients independent of ℓ, and its roots have absolute value 1 in every complex embedding.

Definition 5.6 (Frobenius blocks). For a monic irreducible factor f of P_M over $\mathbb{Q}$, the *Frobenius block* $M_{\ell,f}$ is the generalised eigenspace $\ker f(\phi)^N$, $N\gg 0$. The *Tate part* $\mathrm{T}_\ell(M)$ is the sum of the blocks with f cyclotomic, that is, the subspace on which ϕ is unipotent after replacing q by a power. A block, or M itself, is *potentially Tate* if it is contained in the Tate part.

Since ϕ is an isometry, $\langle\ker f(\phi)^N,\ker g(\phi)^N\rangle=0$ unless g is the reciprocal polynomial of f; as the roots of P_M lie on the unit circle, every irreducible factor is self-reciprocal or one of $x\pm1$, so the blocks are pairwise orthogonal and M_ℓ is their orthogonal sum. A block with f of degree two and not cyclotomic is a binary space on which ϕ is a rotation.

Lemma 5.7. *The class of every algebraic cycle of codimension m on $X_{\overline{\mathbb{F}}_q}$, projected by e, lies in $\mathrm{T}_\ell(M)$. Let $Z(M)\subset M_\ell$ be the $\mathbb{Q}$-span of these projected classes. For $z,z'\in Z(M)$ the number $\langle z,z'\rangle$ is rational, and it is the intersection number of the corresponding cycles projected by e.*

Proof. A cycle on $X_{\overline{\mathbb{F}}_q}$ is defined over some $\mathbb{F}_{q^r}$, so its class is fixed by ϕ^r; the projector e is defined over $\mathbb{F}_q$ and commutes with ϕ. The cup product of cycle classes computes intersection numbers, which are integers, and e is a correspondence with rational coefficients, so $e\cdot\mathrm{cl}(Z)$ is the class of the rational cycle e_*Z. □

The Hodge-type conjecture, restricted to a summand e contained in the primitive part, is the statement that $(-1)^m\langle\,,\rangle$ is positive definite on $Z(M)$; it includes the statement that a nonzero class in $Z(M)$ is not numerically trivial.

At p we use the crystalline realisation. Let $k = \overline{\mathbb{F}}_q$, $W = W(k)$, $K_0 = W[1/p]$, and let $D = e\,H^{2m}_{\mathrm{cris}}(X_k/W) \otimes K_0$ with its cup product and its σ-semilinear Frobenius F. Write $q = p^a$; the K_0-linear map F^a has characteristic polynomial $q^{m\cdot\operatorname{rank} M} P_M(x/q^m)$ [KM74], so that the *Newton slopes* of a block, defined as the q-adic valuations of the roots of f plus m, are the slopes of the corresponding sub-isocrystal of D in the normalisation where F^a has slopes in $[0, 2m]$. If X lifts to characteristic zero with torsion-free crystalline cohomology and degenerate Hodge spectral sequence, Mazur's theorem [Maz73] says that the Newton polygon of D lies above its Hodge polygon, with the same end points; the same holds for a summand cut out by a projector compatible with Frobenius and with the Hodge filtration.

Lemma 5.8 (Crystalline cycle classes). *Let Z be a cycle of codimension m on X_k. Its crystalline class* $\mathrm{cl}(Z) \in H^{2m}_{\mathrm{cris}}(X_k/W) \otimes K_0$ *satisfies* $F(\mathrm{cl}\,Z) = p^m\,\mathrm{cl}(Z)$.

Proof. The absolute Frobenius F_X of X_k is the identity on the underlying space and raises functions to the p-th power, so $F_X^{-1}(Z)$ is supported on Z with multiplicity p^m, i.e. $F_X^*[Z] = p^m[Z]$. The crystalline cycle class of Gillet and Messing is compatible with pull-back and with the intersection product [GM87], and F is the map induced by F_X. □

5.4. Isocrystals, slopes and supersingular blocks

By the Dieudonné–Manin classification [Man63], an F-isocrystal over the algebraically closed field k decomposes canonically as a direct sum of isoclinic parts D_λ, $\lambda \in \mathbb{Q}$, on which F has slope λ; the multiset of slopes, each counted with the dimension of its part, is the Newton polygon. Over $\mathbb{F}_q$, F^a is linear and the slopes are the q-adic valuations of its eigenvalues, so the decomposition of D into isoclinic parts refines the decomposition into Frobenius blocks: the part of a block B with polynomial f has the slopes $m + v_q(\alpha)$, α running over the roots of f in $\overline{\mathbb{Q}}_p$. A block is *supersingular* if all its slopes equal m. A potentially Tate block is supersingular. Conversely, a supersingular block is potentially Tate: its eigenvalues α before normalisation are algebraic integers with $\alpha\bar\alpha = q^{2m}$ in every complex embedding, so $q^{-m}\alpha$ is a unit at every finite place and has absolute value one at every archimedean place, and is a root of unity by Kronecker's theorem.

For a supersingular part D_m, the $\mathbb{Q}_p$-space $T = \{x \in D_m : Fx = p^m x\}$ has $T \otimes_{\mathbb{Q}_p} K_0 = D_m$, since $p^{-m}F$ is a σ-linear bijection of slope zero, whose fixed points form a $\mathbb{Q}_p$-structure [Man63]. The cup product restricts to a nondegenerate $\mathbb{Q}_p$-valued form on T, and T is the natural p-adic counterpart of the spaces $\mathrm{T}_\ell(M)$ for $\ell \neq p$; it contains the crystalline classes of algebraic cycles by Lemma 5.8. The theory of supersingular K3 crystals of Ogus, recalled in Section 13.3, computes the invariants of an integral version of this space in weight two.

Katz's Hodge–Newton decomposition [Kat79] says that if a break point of the Newton polygon of an F-crystal lies on its Hodge polygon, the crystal splits accordingly. We use only the inequality of Mazur and its consequence for the first slope: for a summand whose Hodge numbers are $h^{3,1} = h^{1,3} = 1$, the Newton slopes lie in $[1, 3]$, and they are $(2, 2)$ precisely when the summand is supersingular.

5.5. Newton–Hodge type and sign

Let V be a polarised rational Hodge structure of even weight $2m$, primitive with respect to the polarisation, and let Q be the form $\langle\,,\rangle$ (so that for the primitive middle cohomology of a variety of dimension $2m$, Q is the cup product). The Hodge–Riemann relations say that $i^{p-q}(-1)^m Q(x,\bar{x}) > 0$ for $0 \neq x \in V^{p,q}$; equivalently, the restriction of Q to the real subspace $(V^{p,q} \oplus V^{q,p})_{\mathbb{R}}$ is definite of sign

$$\operatorname{sgn}_{p,q} = (-1)^p \qquad (p+q=2m). \tag{5.4}$$

Indeed $i^{p-q}(-1)^m = (-1)^{(p-q)/2+m} = (-1)^p$. For a fourfold ($m=2$) the types $(4,0)$, $(3,1)$ and $(2,2)$ have signs $+,-,+$; algebraic classes have type (m,m) and sign $(-1)^m$, which is the sign that Hdg predicts.

Over a finite field there is no Hodge structure, but there are Newton slopes, and Mazur's theorem says that they are the best available substitute for the Hodge filtration. This suggests assigning to each slope the sign that (5.4) assigns to the Hodge type of the same index.

Definition 5.9 (Newton–Hodge type and sign). Let B be a Frobenius block of a motive M of weight $2m$ over $\mathbb{F}_q$ whose Newton slopes $\lambda_1,\dots,\lambda_n$ are integers. Its *Newton–Hodge type* is the multiset of pairs $(\lambda_i, 2m-\lambda_i)$, and its *Newton–Hodge signature* is the pair (s_+,s_-) where s_+ (resp. s_-) is the number of i with λ_i even (resp. odd). We say that B has *definite Newton–Hodge sign* $(-1)^\lambda$ when all its slopes are congruent to λ modulo 2.

A Tate block has all slopes equal to m and Newton–Hodge sign $(-1)^m$; the conjecture Hdg asks that algebraic classes carry exactly this sign. An ordinary block of slopes 0 and $2m$ has Newton–Hodge type $(2m,0)+(0,2m)$ and sign $+$. The interest of the definition lies in the blocks where the Newton–Hodge type differs from the Hodge type of a lift: a block of Hodge type $(3,1)+(1,3)$ on a lift, negative in Betti cohomology by (5.4), which becomes supersingular with slopes $(2,2)$ at a prime q, has Newton–Hodge sign $+$. Theorem A(iii) says that in the three motives (1.1) the algebraic classes of such a block follow the Newton–Hodge sign, not the Betti sign.

6. Alien Hodge Motives

6.1. The definition

Throughout, $q = p^a$ is a power of an odd prime, $k = \overline{\mathbb{F}}_q$, and $m \geq 1$. A quadratic space over a field L means a finite-dimensional L-vector space with a nondegenerate symmetric bilinear form. An Alien Hodge Motive is written $\mathcal{M}_{\mathrm{AH}}$; the subscript distinguishes the structure defined below from a motive, for which the letter $\mathcal{M}$ is kept.

Definition 6.1 (Alien Hodge Motive). A pure *Alien Hodge Motive* of weight $2m$ over $\mathbb{F}_q$ is a tuple

$$\mathcal{M}_{\mathrm{AH}} = \big((M_\ell,\phi_\ell)_{\ell\neq p},\ (D,F),\ P,\ Z,\ \mathcal{A}\big)$$

consisting of the following data.

(AHM1) For every prime $\ell \neq p$, a quadratic space M_ℓ over $\mathbb{Q}_\ell$ of rank n with an isometry ϕ_ℓ, whose characteristic polynomial is a fixed polynomial $P \in \mathbb{Q}[x]$ all of whose roots lie on the unit circle in every complex embedding.

(AHM2) A quadratic space D over $K_0 = W(k)[1/p]$ of rank n with a σ-semilinear bijection F such that $\langle Fx, Fy\rangle = p^{2m}\sigma\langle x, y\rangle$, and such that the characteristic polynomial of the linear map F^a is $q^{mn}P(x/q^m)$.

(AHM3) A finite-dimensional $\mathbb{Q}$-vector space Z with a symmetric bilinear form, and for every $\ell \neq p$ a $\mathbb{Q}$-linear map $Z \to \mathrm{T}_\ell(M)$ preserving the forms, together with a $\mathbb{Q}$-linear map $Z \to \{x \in D : Fx = p^m x\}$ preserving the forms. Its elements are the *algebraic classes* of $\mathcal{M}_{\mathrm{AH}}$.

(AHM4) The Newton–Hodge type of each block, determined by (D, F) (Definition 5.9).

(AHM5) A finite set $\mathcal{A}$ of *local isometries*: for another Alien Hodge Motive $\mathcal{M}'_{\mathrm{AH}}$ and blocks B of $\mathcal{M}_{\mathrm{AH}}$ and B' of $\mathcal{M}'_{\mathrm{AH}}$ with the same polynomial, a family of isometries $B_\ell \to B'_\ell$ for $\ell \neq p$ and $B_D \to B'_D$, compatible with Frobenius. We call $\mathcal{A}$ the *atlas* of $\mathcal{M}_{\mathrm{AH}}$.

A *mixed* Alien Hodge Motive carries in addition a finite increasing filtration of each realisation, stable under Frobenius and compatible with the forms, and a nilpotent endomorphism N of the associated graded spaces with $N\phi = q^{-1}\phi N$; the pure case is $N = 0$. This paper uses only pure Alien Hodge Motives.

The definition records what is visible from a finite field. The Hasse invariants of the spaces M_ℓ and of their blocks, and the invariants of D and of its Tate module, are data; the atlas allows blocks of different motives to be compared place by place without a correspondence between them, which is the form in which the comparison with a CM lift enters the work of Ancona [Anc21]. The word *alien* refers to (AHM4): the sign that the theory attaches to a block at the archimedean place is read off the Frobenius at p, and can differ from the sign of the block in any Betti realisation.

Example 6.2 (*The motive of a summand*). Let X be smooth projective of dimension $2m$ over $\mathbb{F}_q$ and e a self-adjoint projector as in Section 5.3, with e acting on the primitive part of H^{2m}. Then $M_\ell = e\,H^{2m}_{\text{ét}}(X_k, \mathbb{Q}_\ell(m))$, the crystalline realisation $D = e\,H^{2m}_{\text{cris}}(X_k/W) \otimes K_0$, the polynomial $P = P_M$, the space $Z = Z(M)$ of Lemma 5.7 with its crystalline classes (Lemma 5.8), and the empty atlas form an Alien Hodge Motive $\mathcal{M}_{\mathrm{AH}}(X, e)$. The conditions (AHM1) and (AHM2) are the theorems of Deligne and of Katz and Messing; (AHM3) is Lemma 5.7 and Lemma 5.8.

Example 6.3 (*A supersingular K3 surface*). Let Y be a K3 surface over $\mathbb{F}_q$ whose reduction to $\overline{\mathbb{F}}_q$ is supersingular, with an ample class ξ, and let e be the projector onto the primitive part of H^2; here $m = 1$ and the relevant form is $-\langle\,,\rangle$. All Newton slopes of H^2 are 1, so P^2_ℓ is potentially Tate, of rank 21, with Newton–Hodge sign $(-1)^1$ for $\langle\,,\rangle$, that is, positive for $-\langle\,,\rangle$. By the Tate conjecture for K3 surfaces [Cha13, MP15] the space Z of primitive algebraic classes has rank 21 over $\mathbb{Q}$ after a finite extension of $\mathbb{F}_q$, and the Hodge index theorem says that $-\langle\,,\rangle$ is positive definite on it. Thus Z itself is a rational form U_{T} whose localisations are the Tate spaces at all places and whose signature is the Newton–Hodge signature.

Suppose Y lifts to a K3 surface $\tilde{Y}$ over the ring of integers of a p-adic field, embedded in $\mathbb{C}$, with ξ lifting to an ample class. Let T_{B} be the transcendental part of the Betti cohomology of $\tilde{Y}_{\mathbb{C}}$, the orthogonal complement of its Néron–Severi group; it is a rational Hodge structure of type $(2,0)+(1,1)+(0,2)$ of rank $r \geq 2$, and $\langle\,,\rangle$ has signature $(2, r-2)$ on it by (5.4). Let V be the orthogonal complement of the specialisation of $\mathrm{NS}(\tilde{Y})$ in $\mathrm{NS}(Y_{\overline{\mathbb{F}}_q})_{\mathbb{Q}}$. Then V and T_{B} are two rational quadratic spaces of rank r with the same localisations at every $\ell \neq p$, both being the orthogonal complement of the specialised Néron–Severi group in H^2_ℓ; but V is negative definite, by the Hodge index theorem, whereas T_{B} has signature $(2, r-2)$. The positive plane of type $(2,0)+(0,2)$ has become, after supersingular reduction, a part of the space of algebraic classes with the sign of type $(1,1)$. This is the phenomenon of Theorem 14.1 in weight two, where it is a consequence of the Hodge index theorem; by Lemma 5.3 the two forms must then differ at p, and indeed $\varepsilon_p(V)$ is computed from Ogus's crystal.

When every local isometry in the atlas is induced by an algebraic correspondence, $\mathcal{M}_{\mathrm{AH}}(X, e)$ together with its atlas is determined by the Chow motive (X, e) and the correspondences; in this sense an Alien Hodge Motive with $N = 0$ and an algebraic atlas is a classical pure motive with additional data. The extra structure is the reading of signs.

6.2. Operations

A *morphism* $\mathcal{M}_{\mathrm{AH}} \to \mathcal{M}'_{\mathrm{AH}}$ of Alien Hodge Motives of the same weight is a family of linear maps of the realisations, $M_\ell \to M'_\ell$ for $\ell \neq p$ and $D \to D'$, commuting with Frobenius and with the forms up to a common rational factor, together with a $\mathbb{Q}$-linear map $Z \to Z'$ compatible with them. Direct sums are formed place by place, with the orthogonal sum of forms; tensor products are formed place by place, with the product forms, the tensor product of Frobenius maps, and $Z \otimes Z' \to Z(\mathcal{M}_{\mathrm{AH}} \otimes \mathcal{M}'_{\mathrm{AH}})$; the weight is additive. For the motive of a summand these operations are those of Chow motives: the Künneth formula gives $\mathcal{M}_{\mathrm{AH}}(X \times X', e \otimes e') = \mathcal{M}_{\mathrm{AH}}(X, e) \otimes \mathcal{M}_{\mathrm{AH}}(X', e')$, the exterior product of cycles providing the map on algebraic classes.

Lemma 6.4. *Let B and B' be Frobenius blocks of $\mathcal{M}_{\mathrm{AH}}$ and $\mathcal{M}'_{\mathrm{AH}}$, of weights $2m$ and $2m'$, with integral Newton slopes. Every Frobenius block of $B \otimes B'$ has the slopes $\lambda + \lambda'$, with λ a slope of B and λ' one of B', and the Newton–Hodge signature of $B \otimes B'$ is the signature of the tensor product of forms of signatures $\mathrm{NH}(B)$ and $\mathrm{NH}(B')$.*

Proof. The eigenvalues of F on the tensor product of isocrystals are the products of eigenvalues, so the slopes add. A slope $\lambda + \lambda'$ is even exactly when λ and λ' have the same parity, and the tensor product of a definite form of sign ϵ with one of sign ϵ' has sign $\epsilon\epsilon'$; counting dimensions gives the statement. □

The lemma is the Newton–Hodge counterpart of the fact that the Hodge–Riemann signs (5.4) are multiplicative under tensor products of polarised Hodge structures, the type $(p, q) \otimes (p', q')$ having sign $(-1)^{p+p'}$. In this sense the Newton–Hodge signature behaves like the signature of a polarised Hodge structure attached to the Newton polygon. It is a structure on the isocrystal and not on a lattice: it does not depend on a lift, and it is defined for every block.

6.3. Betti lifts

The motives of this paper come from varieties over number fields, and the essential additional datum is a global rational quadratic space.

Definition 6.5 (Betti lift). A *Betti lift* of an Alien Hodge Motive $\mathcal{M}_{\mathrm{AH}}$ is a quadratic space V_{B} over $\mathbb{Q}$ together with isometries $V_{\mathrm{B}} \otimes \mathbb{Q}_\ell \simeq M_\ell$ for all $\ell \neq p$. If V_{B} carries a polarised Hodge structure of weight $2m$ whose form is the given one, we call the lift *Hodge*.

Lemma 6.6. *Let $\mathcal{X}$ be smooth projective over $\mathbb{Z}[1/N]$, of relative dimension $2m$, let e be a self-adjoint projector on the generic fibre acting on primitive H^{2m}, and let $q = p^a$ with $p \nmid N$. Then the Betti realisation $V_{\mathrm{B}} = e\,H^{2m}(\mathcal{X}(\mathbb{C}), \mathbb{Q})(m)$ is a Hodge lift of $\mathcal{M}_{\mathrm{AH}}(\mathcal{X}_{\mathbb{F}_q}, e)$.*

Proof. Artin's comparison theorem identifies $V_{\mathrm{B}} \otimes \mathbb{Q}_\ell$ with $e\,H^{2m}_{\text{ét}}(\mathcal{X}_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell(m))$, compatibly with cup products and with the action of correspondences [SGA4, Exp. XI]; the smooth and proper base change theorem identifies the latter with $e\,H^{2m}_{\text{ét}}(\mathcal{X}_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(m))$, again compatibly with cup products, and the specialisation of the correspondence e acts on the special fibre by the specialised cycle, compatibly with cycle classes [Ful98, Ch. 20]. □

A Betti lift fixes the discriminant and all Hasse invariants of the whole space M_ℓ as those of a single rational form. It does not fix the invariants of the individual blocks: the Frobenius decomposition $M_\ell = \bigoplus_f M_{\ell,f}$ is a decomposition over $\mathbb{Q}_\ell$, and there is in general no rational form whose localisations are the blocks. This is the source of the obstructions studied below.

6.4. The invariants of a block

Let B be a block of $\mathcal{M}_{\mathrm{AH}}$, with realisations B_ℓ for $\ell \neq p$. Its *finite invariants* are the discriminants $d_\ell(B)$ and Hasse invariants $\varepsilon_\ell(B)$ for $\ell \neq p$, and, when B is a Tate block, the discriminant and Hasse invariant at p of the $\mathbb{Q}_p$-space $T_p(B) = \{x \in B_D : Fx = p^m x\}$ when the latter is a lattice of full rank, which is the case for supersingular blocks (Proposition 13.5). We write

$$\beta(B) = \prod_{v \text{ finite}} \varepsilon_v(B) \in \{\pm 1\}$$

for the product of the finite Hasse invariants, when it is defined; by (5.3), if B is the localisation of a rational form U, then $\beta(B) = \varepsilon_\infty(U)$. In the motive of a summand, $\varepsilon_\ell(B) = 1$ and $d_\ell(B)$ is a unit for all but finitely many ℓ: for ℓ large, the integral cohomology is a unimodular lattice and the block is cut out of it by an idempotent of $\mathbb{Z}_\ell[\phi]$, so the block is unimodular.

The following elementary observation is the reason binary blocks are special.

Lemma 6.7 (Discriminant of a binary block). *Let W be a binary quadratic space over a field L of characteristic not 2, and $\phi \in \mathrm{SO}(W)$ with characteristic polynomial $x^2 - tx + 1$, $t \neq \pm 2$. Then $d(W) \equiv 4 - t^2$ modulo squares.*

Proof. In a basis, let the Gram matrix be $Q = \left(\begin{smallmatrix} u & r \\ r & s \end{smallmatrix}\right)$ and $\phi = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $\phi^{\mathsf{T}} Q \phi = \mu Q$, $\det \phi = \mu$. The identity

$$(us - r^2)\big(4\mu - (a+d)^2\big) = \big((ub + rd) - (ra + sc)\big)^2$$

holds as a consequence of the three entries of $\phi^{\mathsf{T}} Q \phi = \mu Q$ and of $\det \phi = \mu$: it is the linear combination $-s \cdot (1,1) + 2r \cdot (1,2) - u \cdot (2,2) + (2r^2 - 2us) \cdot \det$ of these relations, as one checks by expansion. With $\mu = 1$ and $a + d = t$, and since $us - r^2 \neq 0$ and $4 - t^2 \neq 0$, the right-hand side is a nonzero square. □

Thus the discriminant of a non-Tate binary block is a rational number determined by P, positive since $|t| < 2$. The Hasse invariant of the block is not determined: a binary space with a rotation of characteristic polynomial $x^2 - tx + 1$ is a hermitian line over $L \otimes \mathbb{Q}(\sqrt{t^2 - 4})$, determined by one norm class.

6.5. Walls

Let $\mathcal{M}_{\mathrm{AH}}$ have a Betti lift V_{B}, and suppose that the non-Tate part of M_ℓ is a sum of binary blocks. If the Tate part T has rank one, its discriminant is $d(V_{\mathrm{B}}) \prod_f d(M_{\ell,f})$, the product over the non-Tate blocks, and by Lemma 6.7 this is a rational number independent of ℓ; by Lemma 5.4 the sign of every algebraic class in T is then determined. If T has rank at least two, its discriminant is still determined, but its Hasse invariants are not: by Lemma 5.1,

$$\varepsilon_\ell(\mathrm{T}) = \varepsilon_\ell(V_{\mathrm{B}})\, \varepsilon_\ell(N)\, (d(\mathrm{T}), d(N))_\ell, \qquad N = \mathrm{T}^{\perp},$$

and $\varepsilon_\ell(N)$ depends on the unknown norm classes of the non-Tate blocks.

Definition 6.8 (Wall). Let $\mathcal{M}_{\mathrm{AH}}$ be an Alien Hodge Motive with a Betti lift whose non-Tate part is a sum of binary blocks. A *wall* of $\mathcal{M}_{\mathrm{AH}}$ is a Tate part of rank at least two that admits no orthogonal decomposition, by algebraic correspondences defined over $\mathbb{F}_q$, into pieces each of which is either of rank one or has all its finite invariants determined by the Betti lift and by P. A prime q is a *wall point* of a motive over a number field when the reduction at q is a wall.

In the terminology of the introduction, a wall is an obstruction locus: the local data of the realisations do not determine the sign of the algebraic classes in it. The theorems of Section 9 make this precise, and identify the missing datum with a statement about the transcendental blocks alone. The Levelt involutions of Section 12 are algebraic correspondences that split certain walls of rank three into a line and a block of rank two whose invariants are global.

7. Hermitian blocks and CM planes

This section treats the blocks that are not potentially Tate. They carry hermitian structures, and the question whether their local hermitian spaces come from one global space is the content of part (a) of AHC I. We also compute the Newton–Hodge signs of CM planes, which explains the walls of the motives of Part III.

7.1. Hermitian blocks

Let B be a block of an Alien Hodge Motive that is not potentially Tate, with irreducible polynomial f of degree $2d$, and suppose ϕ acts semisimply on B, so that B_ℓ is a free module of some rank μ over $E_B \otimes \mathbb{Q}_\ell$, where $E_B = \mathbb{Q}[x]/(f)$. The roots of f lie on the unit circle and are not real, so E_B is a CM field, with totally real subfield F_B of degree d generated by $x + x^{-1}$, and complex conjugation induces the involution $x \mapsto x^{-1}$. As in the proof of Lemma 5.5, the form on B_ℓ is the trace of a unique hermitian form h_ℓ over $E_B \otimes \mathbb{Q}_\ell$ with respect to this involution:

$$\langle u, v\rangle = \mathrm{Tr}_{E_B\otimes\mathbb{Q}_\ell/\mathbb{Q}_\ell}\, h_\ell(u, v).$$

A hermitian space of rank μ over the CM extension E_w/F_w at a finite place w of F_B is classified by its determinant in $F_w^\times/N(E_w^\times)$, a group of order two if w is inert or ramified in E_B and trivial if it splits; at a real place ι of F_B it is classified by its signature (r_ι, s_ι), $r_\iota + s_\iota = \mu$. Landherr's theorem [Lan35] says that a global hermitian space over E_B is determined by its localisations, and that local hermitian spaces of rank μ, almost all with unit determinant, are the localisations of a global one if and only if their determinants satisfy the product formula

$$\prod_{w<\infty} \big(\det h_w, E_B/F_B\big)_w \prod_\iota (-1)^{s_\iota} = 1, \tag{7.1}$$

where $(\,\cdot\,, E_B/F_B)_w$ is the norm residue symbol [Sch85, Ch. 10].

Proposition 7.1. *Let B be as above, and prescribe signatures (r_ι, s_ι) at the real places of F_B. If some place of F_B above p is not split in E_B, there is a global hermitian space over E_B with the localisations h_ℓ at the places above $\ell \neq p$ and the prescribed signatures. If every place of F_B above p splits in E_B, such a space exists if and only if*

$$\prod_{w\nmid p\infty} \big(\det h_w, E_B/F_B\big)_w = \prod_\iota (-1)^{s_\iota}.$$

Proof. At the places above p no localisation is prescribed. If one of them, w_0, is not split, choose the determinant at w_0 so that (7.1) holds and the determinants at the other places above p arbitrarily; Landherr's theorem gives the global space. If all places above p split, the norm residue symbols there are trivial, and (7.1) is the stated condition. □

For the blocks of a motive of geometric origin that are not potentially Tate, the places above p are often split: a block of a binary non-Tate type is always split at p (Proposition 20.5), and more generally an ordinary block is. For such blocks AHC I(a) is a genuine condition on the local data, a reciprocity law for the hermitian classes of the realisations. For blocks with a non-split place above

p it holds automatically, with any signatures, and the conjecture asks nothing beyond the existence of the right signature at infinity.

7.2. CM planes

Let K be an imaginary quadratic field and C a rank-two motive of weight $2m$ over a number field, of CM type with multiplication by K, whose Betti realisation is a Hodge structure of type $(m+k, m-k) + (m-k, m+k)$ with $k \geq 1$. Examples are the motives of CM newforms of weight $2k+1$, twisted by $\mathbb{Q}(k-m)$; the plane C of the motive H_2 is of this kind, with $K = \mathbb{Q}(i)$, $k = 1$ and the form $\eta(4\tau)^6$ of weight three (Section 14.4).

Proposition 7.2. *Let p be a prime of good reduction for C, unramified in K, and let ϕ be the normalised Frobenius at a prime above p of a field of definition, of residue degree one.*

(i) *If p splits in K, the Newton slopes of C are $m-k$ and $m+k$; the Newton–Hodge type is the Hodge type, and the Newton–Hodge sign of C for $(-1)^m\langle\,,\rangle$ is $(-1)^k$, the Betti sign.*

(ii) *If p is inert in K, ϕ is potentially Tate, with slopes (m, m), and the Newton–Hodge sign of C for $(-1)^m\langle\,,\rangle$ is $+1$.*

In particular the Newton–Hodge sign differs from the Betti sign exactly at the inert primes and for k odd.

Proof. By the theory of complex multiplication [Shi98], the Frobenius eigenvalues of C before normalisation are the values $\pi, \bar{\pi} \in K$ of an algebraic Hecke character, and the ideal generated by π is determined by the CM type: if $p = \mathfrak{p}\bar{\mathfrak{p}}$ splits, $(\pi) = \mathfrak{p}^{m+k}\bar{\mathfrak{p}}^{m-k}$ for a suitable labelling, so the p-adic valuations of π and $\bar{\pi}$ at $\mathfrak{p}$ are $m+k$ and $m-k$. If p is inert, $(\pi) = (p)^m$, so $\pi = \zeta p^m$ with ζ a unit of K at every place, of absolute value one, hence a root of unity, and the normalised eigenvalues are roots of unity. The signs follow from Definition 5.9 and (5.4): the type $(m \pm k, m \mp k)$ has sign $(-1)^{m+k}$ for $\langle\,,\rangle$, that is $(-1)^k$ for $(-1)^m\langle\,,\rangle$, and the slopes $m \pm k$ give the same Newton–Hodge sign; the slope m gives $+1$. □

Thus for a CM plane of odd k, the walls are the primes inert in K, and at each of them AHC I predicts the reversal of Theorem 14.1: the algebraic classes are positive for $(-1)^m\langle\,,\rangle$, while the Betti form of the plane is negative. By the sign law (Corollary 10.2) this is equivalent to the p-adic condition $(r, -ab)_p = -1$ for every algebraic class of square r, where $\langle a, b\rangle$ is the Betti form. For $k = 1$ the crystal of C, with its Frobenius divided by p^{m-1}, has Hodge slopes 0 and 2 and is a K3 crystal of rank two in the sense of Ogus, and Ogus's theorem supplies the condition (Proposition 13.5). For odd $k \geq 3$ the crystal is of a different type and the condition is open; for even k there is no reversal and the condition reads $(r, -ab)_p = +1$.

7.3. Products of CM elliptic curves

Let E be an elliptic curve over $\mathbb{Q}$ with complex multiplication by K, and consider the surface $E \times E$. Its cohomology H^2 contains $H^1(E) \otimes H^1(E)$, which over K splits into the Hodge structures $H^{1,0} \otimes H^{1,0}$, $H^{0,1} \otimes H^{0,1}$ and the two classes of type $(1,1)$ given by the graph of the identity and of the complex multiplication. The transcendental plane T is of type $(2,0) + (0,2)$, a CM plane with $m = 1$ and $k = 1$. By Proposition 7.2, at a prime inert in K the curve E is supersingular, T becomes potentially Tate, and its Newton–Hodge sign is that of $(1,1)$-classes. Here the Hodge index theorem on the surface $E \times E$ gives the sign of the algebraic classes, which is negative for $\langle\,,\rangle$ on primitive classes: the reversal holds, as it must. This is the simplest instance of the phenomenon, and the case of the plane C of H_2, in weight four, is its first instance beyond divisors. Section 8 works the example out in full.

8. A model case: the square of a CM elliptic curve

Before the walls of Section 9 we follow the notions of this part through one surface on which every quantity can be computed by hand and every statement can be checked against a classical theorem. The surface is the square of the elliptic curve with complex multiplication by $\mathbb{Z}[i]$; the sign reversal of Section 7.3 becomes an explicit computation with graphs of endomorphisms, and the sign law (Theorem 10.1, proved in Section 10) can be seen at work on an algebraic class whose square is known. Since $m = 1$, the form of the Hodge-type conjecture is $Q = -\langle\,,\rangle$ (Section 5.5), and the statements of Sections 9 and 10, which are written for m even, are applied to Q.

8.1. The curve and its reductions

Let E be the elliptic curve $y^2 = x^3 - x$ over $\mathbb{Q}$. It has good reduction at every odd prime, and complex multiplication by $\mathbb{Z}[i]$, the automorphism $[i]$ being $(x, y) \mapsto (-x, iy)$; it is defined over $\mathbb{Q}(i)$. For an odd prime p write $a_p = p + 1 - \#E(\mathbb{F}_p)$ and π for the p-power Frobenius endomorphism of the reduction, so that $\pi^2 - a_p\pi + p = 0$ in $\mathrm{End}(E_{\mathbb{F}_p})$.

Lemma 8.1. *If $p \equiv 3 \pmod 4$, then $a_p = 0$, $\pi^2 = -p$, and $\pi\,[i] = -[i]\,\pi$; the reduction is supersingular. If $p \equiv 1 \pmod 4$, the reduction is ordinary and π commutes with $[i]$.*

Proof. Let $p \equiv 3 \pmod 4$ and $g(x) = x^3 - x$. As $g(-x) = -g(x)$ and -1 is not a square modulo p, for each $x \in \mathbb{F}_p$ with $g(x) \neq 0$ exactly one of $g(x)$ and $g(-x)$ is a nonzero square, so the pair $\{x, -x\}$ carries exactly two affine points; the three roots $0, \pm 1$ of g carry one point each. Hence E has p affine points over $\mathbb{F}_p$, $\#E(\mathbb{F}_p) = p + 1$ and $a_p = 0$, so $\pi^2 = -p$; a curve whose trace of Frobenius is divisible by p is supersingular. The automorphism $[i]$ involves i, and $\pi \circ [i] = [i]^{(p)} \circ \pi$, where $[i]^{(p)}$ is $(x, y) \mapsto (-x, i^p y)$ and $i^p = -i$; so $\pi[i] = -[i]\pi$. If $p \equiv 1 \pmod 4$, then $i \in \mathbb{F}_p$, $[i]$ is defined over $\mathbb{F}_p$ and commutes with π, and p splits in $\mathbb{Q}(i)$; by the theorem of Deuring [Shi98], the reduction is ordinary and π is an element of $\mathbb{Z}[i]$ of norm p. □

For example $\#E(\mathbb{F}_5) = 8$, so $a_5 = -2$ and $\pi = -1 \pm 2i$; and $\#E(\mathbb{F}_3) = 4$, so $a_3 = 0$.

8.2. Graphs of endomorphisms

On the surface $X = E \times E$ over any algebraically closed field of characteristic different from 2, let $\theta_1 = [E \times \{0\}]$ and $\theta_2 = [\{0\} \times E]$, and for an endomorphism f of E let Γ_f be its graph and

$$D_f = \Gamma_f - \theta_1 - (\deg f)\,\theta_2.$$

For an endomorphism f write $\hat{f}$ for its dual, so that $f\hat{f} = \deg f$, and $\mathrm{Tr}(f) = f + \hat{f} \in \mathbb{Z}$.

Lemma 8.2. *The classes D_f are orthogonal to θ_1 and θ_2, and for endomorphisms f, g,*

$$\langle D_f, D_g \rangle = -\mathrm{Tr}(f\hat{g}), \qquad \langle D_f, D_f \rangle = -2\deg f.$$

Proof. One has $\theta_1^2 = \theta_2^2 = 0$ and $\theta_1 \cdot \theta_2 = 1$. The graph Γ_f meets $\{0\} \times E$ in one point, transversally, so $\Gamma_f \cdot \theta_2 = 1$; its intersection with $E \times \{0\}$ is the pull-back of the origin by f, of degree $\deg f$, so $\Gamma_f \cdot \theta_1 = \deg f$. Hence $D_f \cdot \theta_1 = \deg f - \deg f = 0$ and $D_f \cdot \theta_2 = 1 - 1 = 0$. The intersection $\Gamma_f \cdot \Gamma_g$ is the degree of the pull-back of the origin by $f - g$, that is $\deg(f-g)$, for $f \neq g$; and $\Gamma_f^2 = 0$ by adjunction, since $\Gamma_f \simeq E$ has genus one and the canonical class of X is trivial, which is $\deg(f - f)$ as well. Expanding,

$$\begin{aligned}\langle D_f, D_g \rangle &= \deg(f-g) - (\deg f + \deg g) - (\deg f + \deg g) + (\deg f + \deg g) \\ &= \deg(f-g) - \deg f - \deg g.\end{aligned}$$

Finally $\deg(f-g) = (f-g)(\hat{f} - \hat{g}) = \deg f + \deg g - (f\hat{g} + g\hat{f})$, and $f\hat{g} + g\hat{f} = \mathrm{Tr}(f\hat{g})$, since the dual of $f\hat{g}$ is $g\hat{f}$. □

8.3. The Betti lift

Over $\mathbb{C}$, the four classes $\theta_1, \theta_2, D_1, D_{[i]}$ are algebraic and of type $(1,1)$. By Lemma 8.2, with $\deg 1 = \deg[i] = 1$ and $\mathrm{Tr}(1 \cdot \widehat{[i]}) = \mathrm{Tr}(-[i]) = 0$, their Gram matrix is that of

$$N = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \langle -2, -2 \rangle,$$

nondegenerate of signature $(1,3)$. Let $T \subset H^2(X(\mathbb{C}), \mathbb{Q})$ be the orthogonal complement of N.

Proposition 8.3. *T is a plane of Hodge type $(2,0) + (0,2)$, stable under $[i]^*$, and as a rational quadratic space $T \simeq \langle 1, 1 \rangle$. The primitive part of N for the ample class $\theta_1 + \theta_2$ is negative definite.*

Proof. By the Künneth formula, $H^2(X) = H^2(E) \otimes H^0(E) \oplus H^1(E) \otimes H^1(E) \oplus H^0(E) \otimes H^2(E)$. The outer summands are spanned by θ_2 and θ_1 and form a hyperbolic plane. If α, β is a symplectic basis of $H^1(E, \mathbb{Q})$, the cup product on $H^1 \otimes H^1$ is $\langle a \otimes b, c \otimes d \rangle = -(a \cdot c)(b \cdot d)$, which pairs $\alpha \otimes \alpha$ with $\beta \otimes \beta$ and $\alpha \otimes \beta$ with $\beta \otimes \alpha$ and vanishes otherwise; so $H^2(X, \mathbb{Q})$ is an orthogonal sum of three hyperbolic planes. Its discriminant is -1 and its Hasse invariant at every place is that of $\langle 1, -1, 1, -1, 1, -1 \rangle$, namely $(-1,-1)_v^3 = (-1,-1)_v$.

The space $N \otimes \mathbb{C}$ lies in $H^{1,1}$, which has dimension four, and is nondegenerate of dimension four; so $N \otimes \mathbb{C} = H^{1,1}$ and $T \otimes \mathbb{C} = H^{2,0} \oplus H^{0,2}$. Hence $N = H^{1,1} \cap H^2(X, \mathbb{Q})$, which every automorphism of X preserves; so $1 \times [i]$ preserves N and T. The discriminant of T is $d(H^2)/d(N) = (-1)/(-4) \equiv 1$. By Lemma 5.1, with $d(T) \equiv 1$, $\varepsilon_v(T) = \varepsilon_v(H^2)\,\varepsilon_v(N)$, and $N \simeq \langle 1, -1, -2, -2\rangle$ has $\varepsilon_v(N) = (-1,-2)_v^2(-2,-2)_v = (-2,-1)_v$; hence $\varepsilon_v(T) = (-1,-1)_v(-2,-1)_v = (2,-1)_v = 1$ at every place, since $2 = 1^2 + 1^2$ is a norm from $\mathbb{Q}(i)$. The Hodge–Riemann relations (5.4) give the sign $(-1)^2 = +1$ to the type $(2,0)$, so T is positive definite. A positive definite binary form of discriminant 1 and Hasse invariant 1 everywhere is $\langle 1, 1\rangle$ by the Hasse–Minkowski theorem. Finally, the primitive part of N is spanned by $\theta_1 - \theta_2$, D_1 and $D_{[i]}$, with squares $-2, -2, -2$ and orthogonal; it is negative definite, as the Hodge index theorem requires. □

Thus $V_{\mathrm{B}} = H^2(X(\mathbb{C}), \mathbb{Q})$ is a Betti lift, in the sense of Definition 6.5, of the motive of the reduction of X at every odd prime, and the decomposition $V_{\mathrm{B}} = N \oplus T$ is defined by algebraic classes over $\mathbb{Q}(i)$. The plane T is the CM plane of Section 7.3: it has multiplication by $\mathbb{Q}(i)$ through $1 \times [i]$, and the form $Q = -\langle\,,\,\rangle$ is negative definite on it, $T \simeq \langle -1, -1\rangle$ for Q.

8.4. A split prime

Let $p \equiv 1 \pmod 4$, and work over $\mathbb{F}_p$. By Lemma 8.1, $\pi \in \mathbb{Z}[i]$ has norm p, and the classes θ_1, θ_2, D_1, $D_{[i]}$ are defined over $\mathbb{F}_p$. The Frobenius of X acts on $H^1(E) \otimes H^1(E)$ with the eigenvalues $\pi\otimes\pi$, $\pi\otimes\bar\pi$, $\bar\pi\otimes\pi$, $\bar\pi\otimes\bar\pi$; after normalisation by p, these are $\pi/\bar\pi$, 1, 1, $\bar\pi/\pi$. The two eigenvalues 1 belong to the span of D_1 and $D_{[i]}$, which is fixed by Frobenius; so the reduction of T is the block with the eigenvalues $\zeta = \pi/\bar\pi$ and $\bar\zeta$. As π and $\bar\pi$ are non-associate primes of $\mathbb{Z}[i]$, ζ is not an algebraic integer, and in particular not a root of unity: T is a binary block that is not potentially Tate. Its Newton slopes are the p-adic valuations of π^2 and $\bar\pi^2$ at a prime above p, which are 0 and 2; its Newton–Hodge type is $(2,0) + (0,2)$, the Hodge type of T, and its Newton–Hodge sign for Q is -1, the sign of T for Q.

For $p = 5$, with $\pi = -1 + 2i$, one finds $\pi^2 = -3 - 4i$, $\zeta + \bar\zeta = (\pi^2 + \bar\pi^2)/5 = -\frac{6}{5}$, and $4 - (\zeta + \bar\zeta)^2 = \frac{64}{25}$, a square; by Lemma 6.7 the discriminant of the block is a square, in accordance with $d(T) \equiv 1$. The Tate part is N, spanned by algebraic classes, positive definite on its primitive part for Q by the Hodge index theorem. In the notation of Conjecture 20.1 below, part (a) holds for the transcendental block with $U_T = T$, since its Newton–Hodge sign is the Betti sign; this is the situation in which a lift with complex multiplication supplies the global form (Remark 20.3).

8.5. An inert prime

Let $p \equiv 3 \pmod 4$, and work over $\mathbb{F}_{p^2}$, over which $[i]$ is defined. By Lemma 8.1 the Frobenius of $\mathbb{F}_{p^2}$ is $\pi^2 = -p$, which acts on $H^1(E)$ as the scalar $-p$; the normalised Frobenius of X acts on H^2 as the identity, so the whole of H^2 is a Tate block, and the reduction of T is a binary Tate block.

Proposition 8.4. *Let $p \equiv 3 \pmod 4$. On $X_{\mathbb{F}_{p^2}}$ the classes D_π and $D_{\pi[i]}$ are orthogonal to N, and*

$$\langle D_\pi, D_\pi\rangle = \langle D_{\pi[i]}, D_{\pi[i]}\rangle = -2p, \qquad \langle D_\pi, D_{\pi[i]}\rangle = 0.$$

They span the reduction T_ℓ of T for every $\ell \neq p$. For $Q = -\langle\,,\rangle$, the algebraic classes of T_ℓ form the positive definite plane $\langle 2p, 2p\rangle$, while the Betti form of T is the negative definite plane $\langle -1, -1\rangle$.

Proof. By Lemma 8.1, $\pi\hat\pi = p$ and $\hat\pi = -\pi$, since $\pi^2 = -p$; and $\widehat{[i]} = [i]^{-1} = -[i]$. The endomorphism $\pi[i]$ satisfies $(\pi[i])^2 = \pi[i]\pi[i] = -\pi^2[i]^2 = \pi^2 = -p$, so $\operatorname{Tr}(\pi[i]) = 0$ and $\deg(\pi[i]) = p$. By Lemma 8.2: $\langle D_\pi, D_1\rangle = -\operatorname{Tr}(\pi) = -a_p = 0$; $\langle D_\pi, D_{[i]}\rangle = -\operatorname{Tr}(\pi\cdot(-[i])) = \operatorname{Tr}(\pi[i]) = 0$; $\langle D_{\pi[i]}, D_1\rangle = -\operatorname{Tr}(\pi[i]) = 0$; $\langle D_{\pi[i]}, D_{[i]}\rangle = -\operatorname{Tr}(\pi[i](-[i])) = -\operatorname{Tr}(\pi) = 0$. Next, $\widehat{\pi[i]} = \widehat{[i]}\,\hat\pi = (-[i])(-\pi) = [i]\pi = -\pi[i]$, so $\pi\,\widehat{\pi[i]} = -\pi^2[i] = p[i]$, whose trace is 0; hence $\langle D_\pi, D_{\pi[i]}\rangle = 0$. The squares are $-2\deg\pi = -2\deg(\pi[i]) = -2p$.

The classes D_π and $D_{\pi[i]}$ are orthogonal to $\theta_1, \theta_2, D_1, D_{[i]}$, whose ℓ-adic classes are the specialisations of those of N (Lemma 6.6); so their ℓ-adic classes lie in the orthogonal complement of N_ℓ, which is the specialisation T_ℓ of T. Their Gram matrix is nondegenerate, so they span the plane T_ℓ. The sign statements follow. □

The proposition is the sign reversal of Section 7.3 made explicit: the plane T, positive for $\langle\,,\rangle$ in the Betti realisation, becomes after reduction modulo an inert prime a plane spanned by algebraic classes on which $\langle\,,\rangle$ is negative, as the Hodge index theorem on $X_{\mathbb{F}_{p^2}}$ requires. We now read it through the local invariants.

The two rational forms. The plane T_ℓ carries two rational structures: the Betti form $\langle -1, -1\rangle$ for Q, and the form $\langle 2p, 2p\rangle$ of the algebraic classes. They are isometric over $\mathbb{Q}_\ell$ for every $\ell \neq p$, as they must be. Both have discriminant 1 modulo squares. Their Hasse invariants are $(-1,-1)_v$ and $(2p,2p)_v = (2p,-1)_v = (2,-1)_v(p,-1)_v = (p,-1)_v$. At $v = 2$, $(p,-1)_2 = (-1)^{(p-1)/2} = -1 = (-1,-1)_2$, since $p \equiv 3 \pmod 4$; at an odd prime $\ell \neq p$ both symbols are 1; at $v = p$ one has $(-1,-1)_p = 1$ and $(p,-1)_p = \left(\frac{-1}{p}\right) = -1$; at ∞ the two forms have opposite signs. Thus the two forms differ exactly at p and at ∞, as Lemma 5.3 and the product formula require of two forms of signatures $(0,2)$ and $(2,0)$ that agree at all $\ell \neq p$.

The sign law. Theorem 10.1 applies to T in its odd case, with $a = b = -1$ for Q, and says that an algebraic class of square $r = Q(z) \neq 0$ is positive if and only if $(r,-1)_p = -1$. For $z = D_\pi$, $r = 2p$ and $(2p,-1)_p = (p,-1)_p = -1$: the law is satisfied, and the class is positive.

Ogus's normal form. For $\langle\,,\rangle$ the algebraic classes of T_ℓ form the plane $\langle -2p, -2p\rangle = \langle pu_2, pu_3\rangle$ with $u_2 = u_3 = -2$, and $-u_2u_3 = -4$ is not a square modulo p. This is the normal form of the Tate module of a supersingular K3 crystal of rank two that Proposition 13.5 gives for the plane C of Part III, here obtained directly from cycles. For $p = 3$, the classes D_π and $D_{\pi[i]}$ have square -6, and $\langle -6, -6\rangle$ is anisotropic over $\mathbb{Q}_3$.

8.6. What the model case shows

On $E \times E$ every ingredient of the later arguments is visible and every conclusion is a classical theorem. The Betti lift supplies the global form $\langle -1, -1\rangle$ of the transcendental plane; at a split prime the plane stays transcendental, with Newton–Hodge sign equal to its Betti sign; at an inert prime it becomes a Tate block, its Newton–Hodge sign is the opposite of its Betti sign, and the algebraic classes in it obey the Newton–Hodge sign. The local invariants of the two rational forms differ at exactly two places, p and ∞, and the p-adic invariant is that of Ogus's normal form.

In weight two the positivity of the algebraic classes is the Hodge index theorem, and the classes themselves are graphs of endomorphisms. For the plane C of the hypergeometric motives of Part III, of Hodge type $(3,1)+(1,3)$ in weight four, neither is available: there is no index theorem for cycles of codimension two, and no algebraic class in C is known. Proposition 13.6 replaces both by the argument read off above: the ℓ-adic invariants come from the Betti lift, the p-adic invariant from Ogus's theorem, and reciprocity fixes the sign of any algebraic class there may be.

9. Wall theorems without the Tate conjecture

The theorems of this section are statements about one algebraic class at a time. None of them assumes that a Tate block is spanned by algebraic classes. Throughout, $\mathcal{M}_{\mathrm{AH}}$ is an Alien Hodge Motive of weight $2m$ over $\mathbb{F}_q$, $q = p^a$, and all Tate blocks B that occur have supersingular crystalline realisation, so that their finite invariants (Section 6.4) are defined at every finite place. To fix signs we take m even, as for the middle cohomology of a fourfold, so that Hdg asks for positivity; for m odd, replace the form by its negative.

9.1. Binary Tate blocks

Theorem 9.1 (The Tate-free criterion). *Let B be a binary Tate block of $\mathcal{M}_{\mathrm{AH}}$ whose discriminants at all finite places are the images of one rational number $d > 0$, and let $z \in Z \cap B$ be an algebraic class with $r = \langle z, z\rangle \neq 0$. Then $B_v \simeq \langle r, rd\rangle$ at every finite place v, and*

$$r > 0 \iff \beta(B) = 1.$$

Proof. At a finite place v, the vector z spans a line of B_v with square r; its orthogonal complement is a line of discriminant $d/r \equiv rd$, so $B_v \simeq \langle r, rd\rangle$ and $\varepsilon_v(B) = (r, rd)_v$. The numbers r and rd are rational, so by reciprocity $\beta(B) = \prod_{v<\infty}(r, rd)_v = (r, rd)_\infty$, which is -1 exactly when $r < 0$ and $rd < 0$, that is, since $d > 0$, exactly when $r < 0$. □

Corollary 9.2 (The dichotomy). *In the situation of Theorem 9.1, if $\beta(B) = 1$ then every algebraic class of B with nonzero square is positive; if $\beta(B) = -1$, then every algebraic class of B with nonzero square is negative. In the second case, the Tate conjecture for B (that B is spanned by algebraic classes) and* Hdg *on B cannot both hold.*

Proof. The first two statements are Theorem 9.1, applied to each class separately. If B is spanned by algebraic classes, then, B being nondegenerate, some algebraic class z of B has $\langle z, z\rangle \neq 0$; when $\beta(B) = -1$ it is negative, and Hdg fails on B. □

The point of the dichotomy is that $\beta(B)$ is a product of local invariants. When these are known, the sign of every algebraic class of a binary Tate block is known, whether or not the Tate conjecture holds for it.

Lemma 9.3 (The line correction). *Let $A = L \oplus V$ be an orthogonal sum of Tate blocks, L a line containing an algebraic class of square $r_L > 0$, and suppose the discriminant of A is the image of $d_A \in \mathbb{Q}_{>0}$ at every finite place. Then $\beta(A) = \beta(V)$.*

Proof. By Lemma 5.1, $\varepsilon_v(A) = \varepsilon_v(V)\,(r_L, d(V))_v$, and $d(V) \equiv d_A/r_L \equiv d_A r_L$. The product of $(r_L, d_A r_L)_v$ over the finite places is $(r_L, d_A r_L)_\infty = 1$, since $r_L > 0$. □

9.2. The wall equivalence

Let $\mathcal{M}_{\mathrm{AH}}$ now have a Betti lift V_{B}, and consider a wall of rank three: the Tate part is $A = L \oplus V^+$ with L a line and V^+ binary, and its orthogonal complement in M_ℓ is a binary non-Tate block W with $\phi|_W$ of characteristic polynomial $x^2 - tx + 1$. Put $\Delta = t^2 - 4 < 0$ and fix a rational binary form N^0 of discriminant $-\Delta$, for instance the norm form $\langle 1, -\Delta\rangle$ of $E_W = \mathbb{Q}(\sqrt{\Delta})$; we call it a *rational model* of W and write $s_N = \varepsilon_\infty(N^0)$. By Lemma 6.7, $d(W_\ell) \equiv -\Delta \equiv d(N^0)$ for every $\ell \neq p$, so W_ℓ and $N^0 \otimes \mathbb{Q}_\ell$ differ only in their Hasse invariants, and we write

$$\varepsilon_\ell(W_\ell) = \varepsilon_\ell(N^0)\,\eta_\ell, \qquad \eta_\ell \in \{\pm 1\}.$$

The sign η_ℓ is the class of the hermitian line W_ℓ over $E_W \otimes \mathbb{Q}_\ell$ relative to the hermitian line of N^0; it is 1 for all but finitely many ℓ. With $d_A = d(V_{\mathrm{B}})/d(N^0)$, Lemma 5.1 gives

$$\varepsilon_\ell(A) = \kappa_\ell\,\eta_\ell, \qquad \kappa_\ell = \varepsilon_\ell(V_{\mathrm{B}})\,\varepsilon_\ell(N^0)\,(d_A, d(N^0))_\ell \quad (\ell \neq p). \tag{9.1}$$

The numbers κ_ℓ are Hasse invariants of rational forms and are computable; the η_ℓ are not.

Definition 9.4 (The Alien Hodge condition at a wall). The transcendental block W satisfies the *Alien Hodge condition* at the wall if

$$\prod_{\ell \neq p} \eta_\ell = s_N.$$

The condition is what reciprocity requires if the local hermitian lines W_ℓ are the localisations of one global hermitian line U_W whose real place carries the Newton–Hodge sign of W. Indeed, W is ordinary (it is not Tate, and lies in a part of Hodge type $(2m, 0) + (m, m) + (0, 2m)$ in the cases of interest), so its Newton–Hodge type is $(2m, 0) + (0, 2m)$, of sign $+$, and U_W would be positive definite. At p the eigenvalues of Frobenius on W are a unit and a non-unit, so p splits in E_W and the local hermitian classes at p are trivial. The product formula for U_W and for N^0 then gives $\prod_{\ell \neq p} \eta_\ell = \varepsilon_\infty(U_W)\varepsilon_\infty(N^0) = s_N$.

Theorem 9.5 (The wall equivalence). *In the situation above, let $z \in Z \cap V^+$ be an algebraic class with $r = \langle z, z\rangle \neq 0$, suppose that L contains an algebraic class of positive square and that $d_A > 0$, and suppose the local identity*

$$\Big(\prod_{\ell \neq p} \kappa_\ell\Big)\, \varepsilon_p(A)\, s_N = 1. \tag{9.2}$$

Then $r > 0$ if and only if W satisfies the Alien Hodge condition.

Proof. By Theorem 9.1 and Lemma 9.3, $r > 0$ if and only if $\beta(V^+) = \beta(A) = 1$. By (9.1), $\beta(A) = \varepsilon_p(A) \prod_{\ell \neq p} \kappa_\ell \eta_\ell$, and by (9.2) this is $s_N \prod_{\ell \neq p} \eta_\ell$. As $s_N = \pm 1$, it equals 1 if and only if $\prod_{\ell \neq p} \eta_\ell = s_N$. □

The last two sentences of the proof do not use the algebraic class z; we record them separately.

Corollary 9.6. *At a wall of rank three as above, under the local identity (9.2), $\beta(A) = 1$ if and only if W satisfies the Alien Hodge condition.*

Proof. By (9.1) and (9.2), $\beta(A) = \varepsilon_p(A) \prod_{\ell \neq p} \kappa_\ell \eta_\ell = s_N \prod_{\ell \neq p} \eta_\ell$, which is 1 exactly when $\prod_{\ell \neq p} \eta_\ell = s_N$. □

The finite product in (9.2) can be removed. Choose diagonal forms representing V_{B} and N^0; then κ_ℓ is the value at ℓ of a product of Hasse invariants and Hilbert symbols of rational numbers, and the product formula gives

$$\prod_{\ell} \kappa_\ell = \kappa_\infty, \qquad \kappa_\infty = \varepsilon_\infty(V_{\mathrm{B}})\, \varepsilon_\infty(N^0)\, (d_A, d(N^0))_\infty, \tag{9.3}$$

the product over all primes. Since $\kappa_\ell = \pm 1$, $\prod_{\ell \neq p} \kappa_\ell = \kappa_\infty \kappa_p$.

Theorem 9.7 (The general wall theorem). *In the situation of Theorem 9.5, replace (9.2) by the* alien parity

$$\varepsilon_p(A)\, \kappa_\infty\, \kappa_p\, s_N = 1. \tag{9.4}$$

Then $r > 0$ if and only if W satisfies the Alien Hodge condition. The alien parity involves only the crystalline invariant $\varepsilon_p(A)$, the signatures of V_{B} and N^0, and Hilbert symbols at p of rational numbers.

Proof. By (9.3), (9.4) is (9.2). □

The alien parity is a local statement of the kind that Ancona and Marmora compute for varieties with a CM lifting [AM25]: a relation between the archimedean signature, the discriminant and the p-adic Hilbert symbol. For the Tate module of a supersingular K3 crystal the Hasse invariant is -1 (Proposition 13.5). If $\varepsilon_p(A) = -1$, if V_{B} has signature $(3, 2)$ and N^0 is positive, so that $\kappa_\infty = -1$ and $s_N = 1$, and if the rational numbers entering κ_p are p-adic units, so that $\kappa_p = 1$, then (9.4) holds automatically.

Remark 9.8 (*The hypergeometric wall points*). Consider the six hypergeometric families $\mathcal{H}(\alpha; 0^5 \mid t)$ of weight four with Hodge numbers $(1, 1, 1, 1, 1)$, α one of $(\frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6})$ (the Dwork family of sextic fourfolds), $(\frac{1}{2})^5$, $(\frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4})$, $(\frac{1}{8}, \frac{3}{8}, \frac{1}{2}, \frac{5}{8}, \frac{7}{8})$, $(\frac{1}{10}, \frac{3}{10}, \frac{1}{2}, \frac{7}{10}, \frac{9}{10})$ and $(\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4})$. Among the pairs (q, t) with $q < 80$ prime and $t \in \mathbb{F}_q$, we found 77 wall points of rank three, at which the

Tate part has rank three and the transcendental block is ordinary (the family with $\alpha = (\frac{1}{10}, \dots, \frac{9}{10})$ contributes none). At each of them, with the value $\varepsilon_p(A) = -1$, the local identity (9.2) is a statement about finitely many explicit Hilbert symbols, and it has been verified by the Lean kernel (Appendix D); by the preceding paragraph it holds for a structural reason. Consequently, at each of the 77 points at which $\varepsilon_p(A) = -1$, Hdg on the Tate part is equivalent to the Alien Hodge condition for the transcendental block. This is an equivalence, not evidence for either side; none of the 77 walls is resolved by it.

9.3. The Alien Hodge condition is implied by Tate and Hdg

Theorem 9.9. *In the situation of Theorem 9.5, assume the local identity (9.2). Suppose that the Tate part A is spanned by algebraic classes over $\mathbb{Q}_\ell$ for every $\ell \neq p$ and over $\mathbb{Q}_p$ (the Tate conjecture for A), and that the intersection form on $Z \cap A$ is positive definite* (Hdg *on A). Then W satisfies the Alien Hodge condition.*

Proof. Diagonalise the rational form on $Z \cap A$ as $\langle a_1, a_2, a_3 \rangle$ with $a_i > 0$. Under the Tate conjecture, $A_v \simeq \langle a_1, a_2, a_3 \rangle \otimes \mathbb{Q}_v$ at every finite place, so $\beta(A) = \prod_{v<\infty} \varepsilon_v(\langle a_1, a_2, a_3 \rangle) = \varepsilon_\infty(\langle a_1, a_2, a_3 \rangle) = 1$ by reciprocity. By (9.1) and (9.2), $1 = \beta(A) = s_N \prod_{\ell \neq p} \eta_\ell$. □

Thus, at a wall where the local identity holds, the Alien Hodge condition can fail only if the Tate conjecture or the Hodge-type conjecture fails on the Tate part. The converse direction, that the condition implies Hdg on the wall, is Theorem 9.5, and it needs only one algebraic class.

10. General principles

This section proves the general statements on which the case analysis of Part III rests. As in Section 9, m is even; $q = p^a$ with p odd.

10.1. The sign law in rank two

Theorem 10.1 (Sign law). *Let B be a binary Tate block of $\mathcal{M}_{\mathrm{AH}}$ and p odd. Suppose that for every prime $\ell \neq p$ the realisation B_ℓ is isometric to $\langle a, b \rangle \otimes \mathbb{Q}_\ell$, where $a, b \in \mathbb{Q}^\times$ have the same sign $s \in \{\pm 1\}$ and $p \nmid ab$. Let z be an algebraic class of B with $r = \langle z, z \rangle \neq 0$. Then*

$$r > 0 \iff (r, -ab)_p = s.$$

Proof. For $\ell \neq p$, B_ℓ represents r, so $(r, -ab)_\ell = (a, b)_\ell$ by Lemma 5.2. Since a and b are units at p, $(a, b)_p = 1$, and $(a, b)_\infty = s$; reciprocity for (a, b) gives $\prod_{\ell \neq p} (a, b)_\ell = s$, the product over the finite places other than p. Reciprocity for $(r, -ab)$ gives

$$(r, -ab)_\infty = (r, -ab)_p \prod_{\ell \neq p} (a, b)_\ell = (r, -ab)_p \, s.$$

As $-ab < 0$, the left side is -1 exactly when $r < 0$. □

Corollary 10.2. *In the situation of Theorem 10.1:*

(i) *if* $a, b < 0$ (*the* odd *case, for instance a block of Betti type* $(m+1, m-1) + (m-1, m+1)$ *with* m *even*)*, then* $r > 0$ *if and only if* $(r, -ab)_p = -1$*;*

(ii) *if* $a, b > 0$ (*the* even *case, for instance Betti type* $(2m, 0) + (0, 2m)$)*, then* $r > 0$ *if and only if* $(r, -ab)_p = 1$*.*

Proof. This is Theorem 10.1 with $s = -1$ and $s = 1$. For the examples: by (5.4) the type $(m \pm 1, m \mp 1)$ has sign $(-1)^{m+1} = -1$ for $\langle\,,\rangle$ when m is even, and the type $(2m, 0) + (0, 2m)$ has sign $(-1)^{2m} = 1$; a definite rational binary form of sign s is isometric to $\langle a, b\rangle$ with a, b of sign s. □

Thus on a binary block whose ℓ-adic realisations come from a definite rational form, Hdg is equivalent to one p-adic parity: the p-adic Hilbert symbol of the algebraic class against $-ab$ must equal the Betti sign of the block. In the odd case, this asks that the algebraic classes at p be represented by a $\mathbb{Q}_p$-plane with Hasse invariant opposite to that of $\langle a, b\rangle \otimes \mathbb{Q}_p$. For a plane of discriminant $-ab$ that is a unit nonsquare, such a plane is anisotropic; this is the role of the non-neutral Tate modules of supersingular K3 crystals (Proposition 13.5). The criterion is the class-by-class form of the Hodge parity that Ancona establishes for orthogonal motives of rank two when their numerical classes span a plane [Anc21, §§8–13], and that Ancona and Marmora compute in general for CM lifts [AM25].

10.2. The decidability dichotomy

Theorem 10.3 (Decidability in rank at most three)**.** *Let* f *be a rational quadratic form of rank at most three, and suppose that at every prime* ℓ *the form* $f \otimes \mathbb{Q}_\ell$ *has the discriminant and the Hasse invariant of a positive definite rational form* g *of the same rank. Then* f *is positive definite.*

Proof. By Lemma 5.3, $s_-(f) \equiv s_-(g) = 0 \pmod 4$. Since $0 \le s_-(f) \le 3$, $s_-(f) = 0$. □

Corollary 10.4. *An orthogonal sum of rational forms, each of rank at most three and each satisfying the hypothesis of Theorem 10.3, is positive definite.*

Proof. Each summand is positive definite by Theorem 10.3, and an orthogonal sum of positive definite forms is positive definite. □

Proposition 10.5 (Blindness in rank four)**.** *The rational forms* $\langle 1, 1, 1, 1\rangle$ *and* $\langle -1, -1, -1, -1\rangle$*, of signatures* $(4, 0)$ *and* $(0, 4)$*, have the same discriminant and the same Hasse invariant at every prime.*

Proof. The discriminants are 1. At a prime ℓ, $\varepsilon_\ell(\langle -1, -1, -1, -1\rangle) = (-1, -1)_\ell^6 = 1 = \varepsilon_\ell(\langle 1, 1, 1, 1\rangle)$. □

Theorem 10.3 and Proposition 10.5 describe exactly what the local data of the realisations can decide. A Tate part that splits, by algebraic correspondences, into orthogonal pieces of rank at most three whose local invariants are known is decided by them: its algebraic classes form a positive definite space as soon as they have the local invariants of one. A piece of rank four or more is not decided

by its finite local invariants, whatever they are. In the motives of Part III the Levelt involution brings the Tate part to pieces of rank one and two; the frontier in rank seven (Section 18) is a piece of rank four.

10.3. The discriminant field of an even block

Let B be a Betti block of even rank n, that is, a $\mathbb{Q}$-subspace of the Betti lift V_{B} cut out by an algebraic correspondence defined over $\mathbb{Q}$, and let ϕ_B be the Frobenius at a good prime q on its ℓ-adic realisation. Its determinant $\det \phi_B = \pm 1$ is the value at Frob_q of a quadratic character of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Lemma 10.6. *If $\phi \in \mathrm{O}(B_\ell)$ with n even and $\det \phi = -1$, then the characteristic polynomial of ϕ vanishes at 1 and at -1. Hence B contains two orthogonal potentially Tate lines L_+ and L_-, on which ϕ acts by $+1$ and -1.*

Proof. The characteristic polynomial $c(x) = \sum_k c_k x^k$ of an orthogonal map with determinant -1 satisfies $x^n c(1/x) = -c(x)$, that is, $c_{n-k} = -c_k$. Summing gives $c(1) = 0$; summing with the signs $(-1)^k$, which satisfy $(-1)^{n-k} = (-1)^k$ for n even, gives $c(-1) = 0$. The eigenspaces for 1 and -1 are orthogonal, since ϕ is an isometry. □

Definition 10.7 (Discriminant field). The *discriminant field* of an even Betti block B of rank n with Gram determinant d_B is $F_B = \mathbb{Q}\big(\sqrt{(-1)^{n/2} d_B}\big)$.

For the whole middle cohomology of a smooth projective variety of even dimension over $\mathbb{Q}$, the determinant of Frobenius on ℓ-adic cohomology is the quadratic character attached to the de Rham discriminant, by a theorem of T. Saito [Sai94]; compare also Elsenhans and Jahnel [EJ11] for the value of the characteristic polynomial at -1. For a rank-two block with complex multiplication the analogous statement is elementary (Lemma 13.7), and with the Betti discriminant in place of the de Rham one it gives the walls of the three motives of Part III. In rank four we only know that the prediction $\det \phi_B = \chi_{F_B}(q)$ agrees with the data (Section 18).

10.4. One bit and the anchor

Proposition 10.8 (The two lines have the same sign). *Let $B = L_+ \oplus L_- \oplus U$ be an orthogonal decomposition of an even Betti block at a prime where $\det \phi_B = -1$, with U a non-Tate binary block, and suppose $d_B > 0$. If $z_\pm \in L_\pm$ are algebraic classes with squares $x_\pm \neq 0$, then $x_+ x_- > 0$. Hence the form on the algebraic classes of $L_+ \oplus L_-$ is definite, and it is positive definite if and only if $x_+ > 0$.*

Proof. At every $\ell \neq p$ the plane $L_+ \oplus L_-$ has discriminant $x_+ x_-$, and also $d_B / d(U) \equiv d_B(4 - t_U^2)$ by Lemma 6.7, where $x^2 - t_U x + 1$ is the characteristic polynomial on U. The rational number $d_B(4 - t_U^2)$ is positive. By Lemma 5.4, $x_+ x_- > 0$. □

Theorem 10.9 (The anchor principle). *Let f be a rational form of rank at most three satisfying the hypothesis of Theorem 10.3, and $x_0 > 0$. Then $\langle x_0 \rangle \oplus f$ is positive*

definite. Without the class x_0 of known sign, the finite local invariants of a block of rank four do not determine its signature (*Proposition 10.5*).

Proof. By Theorem 10.3, f is positive definite, and so is its orthogonal sum with the positive line $\langle x_0\rangle$. For the second statement, the forms $\langle 1\rangle \oplus \langle 1,1,1\rangle$ and $\langle -1\rangle \oplus \langle -1,-1,-1\rangle$ of Proposition 10.5 have the same finite invariants and opposite signatures; they are distinguished by the sign of one vector of the line, which is what the class x_0 supplies. □

Proposition 10.8 reduces a Tate part $L_+ \oplus L_-$ of a rank-four wall to one sign, and Theorem 10.9 says that one algebraic class of known sign settles it. When the block B carries real multiplication by a real quadratic field F, the two lines are the traces of one F-line, the product formula over F sees only the product of the two real signs, and the missing sign is the sign at one real place of F. We record the elementary fact behind this.

Lemma 10.10. *For $D > 0$ the form $(x_1, x_2) \mapsto x_1^2 + Dx_2^2$, which is half the trace form $x \mapsto \mathrm{Tr}_{F/\mathbb{Q}}(x^2)$ of $F = \mathbb{Q}(\sqrt{D})$ written in the basis $1, \sqrt{D}$, is positive definite. If $s_1, s_2 \in \{\pm 1\}$ are the signs of a definite F-quadratic form at the two real places and their product P is known, then $s_2 = s_1 P$.*

Proof. For $x = x_1 + x_2\sqrt{D}$ with $x_1, x_2 \in \mathbb{Q}$, $x^2 = x_1^2 + Dx_2^2 + 2x_1x_2\sqrt{D}$, so $\mathrm{Tr}(x^2)/2 = x_1^2 + Dx_2^2$, a form with positive coefficients. The second statement is $s_1 s_2 = P$ and $s_1^2 = 1$. □

11. The nine principles

We collect the results of Part II and Part III as nine principles; Figure 1 shows how they depend on one another. Table 3 gives, for each, the statement in brief, the place where it is proved, and its status: *theorem* means proved in this paper from the stated hypotheses; *theorem under Hypothesis 12.4* means proved for the three motives (1.1) from the geometric inputs listed there; *observation* means supported by exact computation and not proved. Two general criteria that are used throughout, the sign law (Theorem 10.1) and the decidability dichotomy (Theorem 10.3), are not counted among the nine.

11.1. How the principles fit together

The principles fall into three groups. Principles 1, 2, 7 and 9 are statements about the arithmetic of quadratic forms over $\mathbb{Q}$ applied to the realisations of a motive over a finite field; they hold for any Alien Hodge Motive satisfying their hypotheses, and their proofs use only Hilbert reciprocity and Lemmas 5.1 to 5.3. They show that the local data of a Tate block decide the sign of its algebraic classes exactly when the block is of rank at most three and its local invariants are known, and they locate the missing information, at a wall, in the transcendental part.

Principles 3, 4, 5 and the rank-two case of 8 are the content of Theorem A. They concern the three motives (1.1), and they show that an algebraic correspondence (the Levelt involution) can remove the obstruction of principle 2 by making the invariants of the Tate part global. The mechanism is that the involution splits a

	Principle	Content	Status
1	Tate-free criterion	On a binary Tate block with global discriminant, one algebraic class is positive iff the product of the finite Hasse invariants is 1	theorem (Theorem 9.1 and Corollary 9.2)
2	Wall equivalence	At a wall of rank three, Hdg on the Tate part is equivalent to the Alien Hodge condition on the transcendental block, given the alien parity at p and ∞	theorem (Theorems 9.5 and 9.7)
3	Levelt splitting	At the fixed point $t = -1$ of $t \mapsto 1/t$, the Levelt involution splits the motive orthogonally into a definite part S and a CM part C, and confines the transcendental part to S	theorem under Hypothesis 12.4 (Propositions 12.2 and 12.7)
4	Sign reversal	On C the Betti form is negative definite and every algebraic class is positive: the sign of an algebraic class is its Newton–Hodge sign	theorem under Hypothesis 12.4 (Theorem 14.1)
5	Inert-prime prediction	The walls of the three motives are at the primes inert in the CM field of C	theorem under Hypothesis 12.4 (Lemma 13.7)
6	Reduction of the summits	AHC I implies the arithmetic summit and AHC II the geometric summit	theorem, formal (Theorems 20.6 and 21.5)
7	Unfalsifiability	Where the local identity holds, the Tate and Hodge-type conjectures imply the Alien Hodge condition	theorem (Theorem 9.9)
8	Discriminant field	The determinant of Frobenius on an even Betti block is the character of its discriminant field; at the inert primes the block acquires the Tate lines $L_{\pm}$	theorem in rank two for the three motives (Lemma 13.7); observation in rank four (Section 18)
9	One bit and the anchor	A rank-four wall of the form $L_+ \oplus L_-$ is decided by one sign; one class of known sign decides it, and nothing less does	theorem (Proposition 10.8 and Theorem 10.9)

TABLE 3. The nine principles of the theory of Alien Hodge Motives.

wall of rank three into a line, whose square class is fixed by the discriminant of a binary block, and a CM plane, whose p-adic invariant is fixed by Ogus's theory and whose ℓ-adic invariants are fixed by the Betti lift. Principle 4 says what the outcome looks like: the sign of the algebraic classes in the CM plane is the sign that the Newton polygon at p predicts, and it is the opposite of the sign of the plane in the Betti realisation.

Principle 6 connects the theory to the two summits through the Alien Hodge Conjectures of Part IV, and principle 7 bounds the risk in them: the arithmetic conjecture can fail at a wall only where the Tate or the Hodge-type conjecture

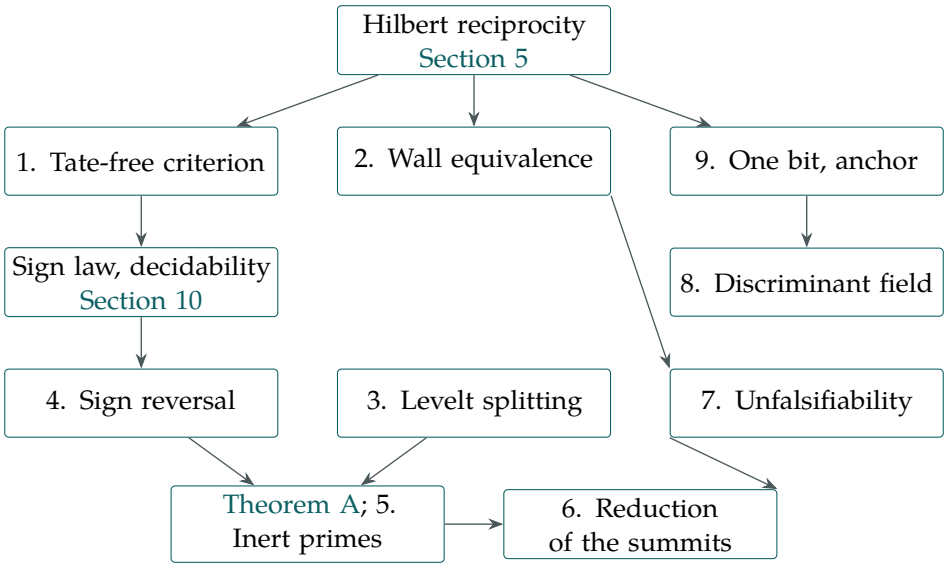

FIGURE 1. Logical dependence among the nine principles and the main theorem. Principle 8 in rank two is Lemma 13.7, which enters the proof of Theorem A; in rank four it is an observation.

fails. Principles 8 and 9 describe what happens beyond the rank-five motives: in rank seven the Levelt splitting leaves a block of rank four, the determinant of Frobenius on it seems to be governed by its discriminant field, and at the inert primes of that field the obstruction reduces to one sign, which an anchor class would supply.

11.2. Scope and limits of each principle

We comment on each principle in turn: what it needs, where it applies, and where it stops.

Principle 1 needs a binary Tate block whose discriminant is the image of one positive rational number at all finite places. The discriminant is the cheap invariant: when the block is the orthogonal complement of non-Tate binary blocks in a space with a Betti lift, Lemma 6.7 provides it. The Hasse invariants are the expensive ones; the principle converts the question of positivity of one algebraic class into a question about them, and nothing more.

Principle 2 requires, in addition, the alien parity at p and ∞. For supersingular Tate parts of K3 type it follows from Ogus's theorem; for Tate parts of other crystalline types we do not know the value of ε_p, and the equivalence is then conditional on it. The principle is an equivalence; it neither proves nor refutes Hdg at a wall.

Principle 3 is the one place where geometry enters. It needs an algebraic correspondence that commutes with Frobenius and splits the Tate part into pieces whose local invariants are global. The Levelt involution at the fixed point of $t \mapsto 1/t$ is such a correspondence for the three motives (1.1). For the Dwork family no such correspondence is known, and the scan of Proposition 12.9 shows that among hypergeometric motives with the inversion symmetry, the three are the only ones of rank five and Hodge type $(1, 1, 1, 1, 1)$.

Principle 4 is a theorem about three motives. Its weight-two analogue is the Hodge index theorem on a supersingular K3 surface (Example 6.3). We expect

it to hold whenever a block of Hodge type $(m+1, m-1) + (m-1, m+1)$ on a lift becomes supersingular, and this expectation is a case of AHC I.

Principle 5 locates the walls by the CM field. It rests on the identification of the determinant of Frobenius on C with the character of K (Lemma 13.7), which in turn uses the Hodge type of C and a finite computation.

Principle 6 is formal. It records that the Alien Hodge Conjectures were formulated so as to imply the two summits, and it should be read together with principle 7.

Principle 7 limits the risk in AHC I(a) at walls: a counterexample there would be a counterexample to the Tate conjecture or to the Hodge-type conjecture. It does not say that AHC I(a) is easier than those conjectures.

Principle 8 is a theorem in rank two and an observation in rank four. In rank two the determinant of Frobenius on a CM plane is the character of the CM field, by Hodge theory and a finite computation; in rank four we have no proof, and the data determine the determinant only at 14 of the 33 computed primes.

Principle 9 is a statement about quadratic forms, combined with the square-class lemma. It says exactly how much information is missing at a wall of rank four of the shape $L_+ \oplus L_-$: one sign. It does not provide the sign.

11.3. Terminology

For the reader who meets the theory in other sources we record the correspondence between the terms used here and some informal names: the *wall* of Definition 6.8 is an obstruction locus; the invariant $\beta(B)$ is the finite Hasse product of the block; the Newton–Hodge sign of Definition 5.9 is the slope signature; the Alien Hodge condition of Definition 9.4 is a hermitian local–global condition on a transcendental Frobenius block; the alien parity (9.4) is a p-adic Hodge parity rule of the kind computed by Ancona and Marmora; and the Levelt splitting is the splitting at the fixed point of the base involution.

Part III

The fixed-point theorem

12. Symmetric hypergeometric motives and the Levelt splitting

12.1. Hypergeometric motives

Let $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$ be multisets of rational numbers modulo $\mathbb{Z}$, disjoint, and each a union of full Galois orbits (so that $\prod_j (x - e^{2\pi i \alpha_j})$ and $\prod_j (x - e^{2\pi i \beta_j})$ have rational coefficients). Let A and B be the companion matrices of these two polynomials. Levelt's theorem [BH89, Thm. 3.5] says that a representation of the free group on two generators in which the generators act with these characteristic polynomials and the quotient $A^{-1}B$ acts as a pseudo-reflection is unique up to conjugation, and it is irreducible because α and β are disjoint [BH89, Prop. 3.3]. The local system on $\mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$ with this monodromy is the hypergeometric local system $\mathcal{H}(\alpha; \beta)$, and its fibres form a variation of Hodge structure; when α and β are defined over $\mathbb{Q}$ it underlies a family of motives over $\mathbb{Q}(t)$, the hypergeometric motive $\mathcal{H}(\alpha; \beta \mid t)$, whose trace of Frobenius at a good prime is a finite hypergeometric sum [Kat90, BCM15, RRV22]. When α and β are each stable under $x \mapsto -x$, the group $\langle A, B \rangle \subset \mathrm{GL}_n(\mathbb{Q})$ preserves a nondegenerate symmetric or alternating form, unique up to a scalar [BH89, §4]; for n odd it is symmetric.

The Hodge numbers of $\mathcal{H}(\alpha; \beta \mid t)$ are read off from the interlacing pattern of α and β on the circle [RRV22]. For the three data (1.1) the motive has rank 5, weight 4 and Hodge numbers $(1, 1, 1, 1, 1)$; the Betti realisation is therefore a polarised Hodge structure of signature $(3, 2)$ by (5.4).

12.2. Hodge numbers

The Hodge numbers of $\mathcal{H}(\alpha; \beta \mid t)$ are given by the zigzag rule [RRV22]: arrange the elements of α and β, taken in $[0, 1)$, in increasing order, with β before α at equal values; walk through them, lowering a counter by one at each element of β and raising it by one at each element of α, and record the value of the counter after each element of α. The recorded values, shifted so that the smallest is zero, are the Hodge indices p of the n Hodge numbers $h^{p,w-p}$, and w is their range. For H_2 the five elements of β come first and bring the counter to -5; the five elements of α then record $-4, -3, -2, -1, 0$, so $w = 4$ and the Hodge numbers are $(1, 1, 1, 1, 1)$. For S_{33} the order is $0, \frac{1}{6}, \frac{1}{6}$ (counter -3), then $\frac{1}{3}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{2}{3}$, recording $-2, -1, 0, 1, 2$, then $\frac{5}{6}, \frac{5}{6}$; for S_3 it is $0, 0, 0, \frac{1}{6}$ (counter -4), then $\frac{1}{3}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{2}{3}$, recording $-3, -2, -1, 0, 1$, then $\frac{5}{6}$. In all three cases the Hodge numbers are $(1, 1, 1, 1, 1)$ and the weight is four.

The local monodromies are also visible in the data. At $t = 0$ the monodromy has the eigenvalues $e^{2\pi i \beta_j}$; for H_2 it is unipotent with a single Jordan block of size five, the point of maximal unipotent monodromy. At $t = \infty$ it has the eigenvalues $e^{2\pi i \alpha_j}$, here -1 with multiplicity five, again in a single Jordan block, since a companion matrix is cyclic. At $t = 1$ the monodromy $A^{-1}B$ is a pseudo-reflection; for the three data it has determinant -1 (as $\det A = -1$ and $\det B = 1$), and as it preserves the invariant symmetric form it is the reflection in a vector of nonzero square, the vanishing cycle.

12.3. The inversion symmetry

Suppose $\beta \equiv \alpha + \frac{1}{2}$ modulo $\mathbb{Z}$. Pulling back by $\iota(t) = 1/t$ exchanges the punctures 0 and ∞, so $\iota^*\mathcal{H}(\alpha;\beta)$ has local monodromies with eigenvalues $e^{2\pi i\beta_j}$ at ∞ and $e^{2\pi i\alpha_j}$ at 0, and the pseudo-reflection at 1. Tensoring with the rank-one system with monodromy -1 at 0 and at ∞ multiplies both local monodromies by -1, which shifts both parameter sets by $\frac{1}{2}$, and $\beta + \frac{1}{2} \equiv \alpha$. By Levelt's theorem the twisted pull-back is isomorphic to $\mathcal{H}(\alpha;\beta)$, and on the fibre at the fixed point $t = -1$ of ι, where the rank-one system is trivialised, the isomorphism becomes an automorphism σ_* of the fibre that intertwines the monodromy representation with its transform. In terms of A and B this is the condition

$$\sigma A \sigma^{-1} = -B^{-1}, \qquad \sigma B \sigma^{-1} = -A^{-1}. \tag{12.1}$$

The symmetry is visible in the traces of Frobenius. Let q be a power of an odd prime with $q \equiv 1$ modulo the denominators of α and β, put $Q = q - 1$, fix a generator ω of the character group of $\mathbb{F}_q^\times$, and let $g(k) = \sum_{x \in \mathbb{F}_q^\times} \omega(x)^k \psi(x)$ be the Gauss sums for a nontrivial additive character ψ. The finite hypergeometric function of Beukers, Cohen and Mellit [BCM15] is

$$H_q(\alpha;\beta \mid t) = \frac{1}{1-q} \sum_{m=0}^{Q-1} \prod_{j=1}^{n} \frac{g(m + \alpha_j Q)\, g(-m - \beta_j Q)}{g(\alpha_j Q)\, g(-\beta_j Q)}\, \omega\big((-1)^n t\big)^m, \tag{12.2}$$

which is, up to a power of q fixed by the data, the trace of Frobenius on the fibre of the hypergeometric motive at $t \in \mathbb{F}_q^\times \smallsetminus \{1\}$.

Proposition 12.1 (Inversion of traces). *Suppose α and β are stable under $x \mapsto -x$ modulo $\mathbb{Z}$, $\beta \equiv \alpha + \frac{1}{2}$, n is odd, and q is as above. Let φ be the quadratic character of $\mathbb{F}_q^\times$. Then for every $t \in \mathbb{F}_q^\times$,*

$$H_q(\alpha;\beta \mid t) = \varphi(-t)\, H_q(\alpha;\beta \mid t^{-1}).$$

In particular the trace at the fixed point $t = -1$ is invariant, and the identity expresses the twist by $-t$ of the inversion symmetry of Section 12.3.

Proof. Substituting $m \mapsto -m$ in (12.2) replaces the numerator by $\prod_j g(-m + \alpha_j Q) g(m - \beta_j Q)$ and the character by $\omega((-1)^n t)^{-m} = \omega((-1)^n t^{-1})^m$. As α and β are stable under negation, the numerator is $\prod_j g(m + \beta_j Q) g(-m - \alpha_j Q)$, and the denominator $\prod_j g(\alpha_j Q) g(-\beta_j Q)$ equals $\prod_j g(\beta_j Q) g(-\alpha_j Q)$. Hence $H_q(\alpha;\beta \mid t) = H_q(\beta;\alpha \mid t^{-1})$. Next, substituting $m \mapsto m + \frac{Q}{2}$ in the sum defining $H_q(\alpha;\beta \mid s)$, and using $\alpha + \frac{1}{2} \equiv \beta$ and $\beta + \frac{1}{2} \equiv \alpha$, transforms the numerator into that of $H_q(\beta;\alpha \mid s)$ and multiplies the character by $\omega((-1)^n s)^{Q/2} = \varphi((-1)^n s)$; the denominators agree by the same stability under negation. Hence $H_q(\beta;\alpha \mid s) = \varphi((-1)^n s) H_q(\alpha;\beta \mid s)$. With $s = t^{-1}$ and n odd, $\varphi((-1)^n t^{-1}) = \varphi(-t)$, and the two identities give the claim. □

The identity has been checked numerically for the three data at $q = 13$ and $q = 37$ and all t. It is the arithmetic counterpart of the relations (12.1): the twist by $\varphi(-t)$ is the twist by the rank-one system with monodromy -1 at 0 and ∞, trivial at $t = -1$.

Proposition 12.2 (The Levelt involution). *For each of the three data* (1.1)*, with A and B as in Appendix D, the signed anti-diagonal permutation matrix*

$$\sigma_{\mathrm{H}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

satisfies $\sigma_{\mathrm{H}}^2 = 1$ and (12.1)*, that is $B\sigma_{\mathrm{H}}A = -\sigma_{\mathrm{H}}$ and $A\sigma_{\mathrm{H}}B = -\sigma_{\mathrm{H}}$, and it preserves the invariant form S. Every $\sigma \in \mathrm{GL}_5(\mathbb{Q})$ with $\sigma^2 = 1$ satisfying* (12.1) *is $\pm\sigma_{\mathrm{H}}$.*

Proof. The identities are equalities of integer matrices, verified by the Lean kernel. If σ also satisfies (12.1), then $\sigma_{\mathrm{H}}^{-1}\sigma$ commutes with A and with B, hence is a scalar λ by irreducibility and Schur's lemma, and $\sigma^2 = \lambda^2\sigma_{\mathrm{H}}^2 = 1$ forces $\lambda = \pm 1$. □

The same matrix serves for all three data. In each case the eigenvalues of B are those of $-A$, since $\beta \equiv \alpha + \frac{1}{2}$, so that the characteristic polynomial of B is $-f(-x)$ for the characteristic polynomial f of A; the relations (12.1) are consistent because α is stable under $x \mapsto -x$, so that $-B^{-1}$ and A have the same eigenvalues.

Remark 12.3 (*The other intertwiner*). Whether the pulled-back loops around 0 and ∞ appear as the local monodromies or as their inverses depends on orientation conventions. With the other convention the relations become $\sigma A\sigma^{-1} = -B$ and $\sigma B\sigma^{-1} = -A$. For each of the three data these also have a solution with $\sigma^2 = 1$, unique up to sign, namely $\mathrm{diag}(1, -1, 1, -1, 1)$, and it preserves S; but its eigenspaces have signatures $(2, 1)$ and $(1, 1)$ for S. These signatures are impossible for sub-Hodge structures of a Hodge structure of type $(1, 1, 1, 1, 1)$, since by (5.4) every such sub-Hodge structure is a sum of pieces $(4, 0) + (0, 4)$, $(3, 1) + (1, 3)$ and $(2, 2)$, of signatures $(2, 0)$, $(0, 2)$ and $(1, 0)$. Hence that candidate cannot be the action of an algebraic correspondence, and whichever convention applies, an algebraic intertwiner acts by $\pm\sigma_{\mathrm{H}}$.

12.4. The geometric inputs

We now state precisely what we assume about the geometric realisation of the three motives.

Hypothesis 12.4 (Geometric inputs). Let $\mathcal{M}$ be one of H_2, S_{33}, S_3.

(H1) *Model.* There are an open subset $U \subset \mathbb{P}^1_{\mathbb{Q}} \smallsetminus \{0, 1, \infty\}$ containing -1 and stable under $t \mapsto 1/t$, a smooth projective morphism $f\colon \mathcal{X} \to U$ of relative dimension four, and a self-adjoint relative idempotent correspondence e on $\mathcal{X}/U$ with rational coefficients, acting on the cohomology of the fibres through their primitive H^4, such that $e\,R^4 f_*\mathbb{Q}(2)$ is isomorphic to the hypergeometric local system $\mathcal{H}(\alpha; \beta)$ as a polarised variation of Hodge structure, the polarisation corresponding to the cup product. We put $X = \mathcal{X}_{-1}$, a smooth projective fourfold over $\mathbb{Q}$, and $M_{\mathrm{B}} = e\,H^4(X(\mathbb{C}), \mathbb{Q})(2)$. The identification is also arithmetic: for every prime ℓ, the realisation $M_\ell = e\,H^4_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell(2))$ is, as a representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, the fibre at -1 of the ℓ-adic hypergeometric local system of

Katz [Kat90, Ch. 8]. In particular M_ℓ is unramified outside ℓ and the *bad primes of the datum*, namely 2 for H_2 and 2, 3 for S_{33} and S_3, and at every other prime $q \neq \ell$ the trace of the r-th power of Frobenius on M_ℓ is given by the finite hypergeometric sum (12.2) over $\mathbb{F}_{q^r}$ at $t = -1$, for every r with $q^r \equiv 1$ modulo the denominators of the datum, up to the twist by a quadratic character unramified outside the bad primes of the datum.

(H2) *Inversion.* There is an algebraic correspondence σ on X, defined over $\mathbb{Q}$, with $\sigma e = e\sigma$ and $(e\sigma e)^2 = e$, whose action on M_{B} is an intertwiner of the monodromy as in (12.1), in the basis of M_{B} in which the monodromy at the base point -1 is generated by A and B.

(H3) *Integrality.* There is a finite set E of primes, containing the bad primes of the datum and the primes of bad reduction of $\mathcal{X}$ at $t = -1$, such that for every prime $q \notin E$: X has a smooth projective model over $\mathbb{Z}_{(q)}$ whose crystalline cohomology is torsion-free with degenerate Hodge–de Rham spectral sequence; e and σ extend to correspondences over $\mathbb{Z}_{(q)}$ acting on integral crystalline cohomology; the two eigen-summands of $e\sigma e$ in $e\,H^4_{\mathrm{cris}}(X_{\overline{\mathbb{F}}_q}/W)$ are lattices on which the cup product is perfect; and $q \geq 5$.

Part (H1) is the statement that the hypergeometric motive is realised geometrically, with an algebraic projector, in a smooth projective family. For hypergeometric motives defined over $\mathbb{Q}$ this is due to Katz [Kat90, Ch. 8], with explicit models in terms of the parameters by Beukers, Cohen and Mellit [BCM15]; the smooth projective model and the algebraicity of the projector onto the summand, as a relative correspondence, are the conditions under which we use it. The arithmetic part of (H1) is what makes the finite hypergeometric sums the traces of Frobenius on the realisations of X, and it confines the ramification of M_ℓ to the bad primes of the datum and ℓ; the computations of Appendix B and the exclusion of quadratic characters in Lemma 13.7 and Theorem 13.9 use it in this form. A quadratic twist multiplies the normalised Frobenius by ± 1; it changes neither which blocks are potentially Tate, nor the determinant of Frobenius on a plane, nor the numbers $4 - t^2$ that enter the square classes below, and these are the only features of the traces that the proofs use. For H_2 the family is the double cover

$$\mathcal{X}_t : \quad y^2 = x_1x_2x_3x_4(1-x_1)(1-x_2)(1-x_3)(1-x_4)(1-t\,x_1x_2x_3x_4)$$

of $\mathbb{A}^4$, suitably compactified and resolved, and e projects onto a summand of the part on which $y \mapsto -y$ acts by -1; its period over the cube $[0,1]^4$ is a hypergeometric function (Proposition 12.5), which identifies the summand with $\mathcal{H}((\frac{1}{2})^5; 0^5 \mid t)$. The existence of e as an algebraic correspondence is part of (H1).

Proposition 12.5 (The Euler period). *For $|t| < 1$, write $P_t(x) = x_1 \cdots x_4(1 - x_1) \cdots (1 - x_4)(1 - t\,x_1 \cdots x_4)$. The integral of the holomorphic form $\omega_t = dx_1 \wedge \cdots \wedge dx_4/y$ of $\mathcal{X}_t$ over the real cycle lying above the cube $[0,1]^4$ is*

$$\int_{[0,1]^4} \frac{dx_1 \cdots dx_4}{\sqrt{P_t(x)}} = \pi^4\, {}_5F_4\Big(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}; 1, 1, 1, 1; t\Big).$$

Proof. Expand $(1 - t x_1 \cdots x_4)^{-1/2} = \sum_{k \geq 0} \frac{(1/2)_k}{k!} t^k (x_1 \cdots x_4)^k$, which converges uniformly on the cube for $|t| < 1$, and integrate term by term. Each factor gives a Beta integral,

$$\int_0^1 x^{k-1/2}(1-x)^{-1/2} dx = B\big(k + \tfrac{1}{2}, \tfrac{1}{2}\big) = \frac{\Gamma(k+\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(k+1)} = \pi \, \frac{(1/2)_k}{k!},$$

using $\Gamma(k + \frac{1}{2}) = (1/2)_k \sqrt{\pi}$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Hence the integral is $\pi^4 \sum_k \frac{(1/2)_k^5}{k!^5} t^k$, which is the stated series. □

The function ${}_5F_4((\frac{1}{2})^5; 1^4; t)$ satisfies the hypergeometric differential equation of order five whose local exponents are $0, 0, 0, 0, 0$ at $t = 0$ and $\frac{1}{2}, \ldots, \frac{1}{2}$ at $t = \infty$, and whose monodromy is the hypergeometric group with parameters $\alpha = (\frac{1}{2})^5$ and $\beta = 0^5$ [BH89]. Its periods over the other cycles of the summand span the solution space. This identifies, over the disc $|t| < 1$ and then by continuation over $\mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$, the variation of Hodge structure of the summand of (H1) with $\mathcal{H}(\alpha; \beta)$ for H_2, once one knows that the summand has rank five; the rank is part of (H1).

Part (H2) is the geometric form of the inversion symmetry. For H_2 it is explicit: the substitution

$$\sigma(x_1, \ldots, x_4, y) = \Big(\frac{1}{x_1}, \ldots, \frac{1}{x_4}, \frac{y}{(x_1 x_2 x_3 x_4)^2}\Big)$$

transforms the right-hand side of the equation of $\mathcal{X}_t$ into $-t\,(x_1x_2x_3x_4)^{-4}$ times the right-hand side of the equation of $\mathcal{X}_{1/t}$. It therefore maps $\mathcal{X}_t$ birationally to the quadratic twist of $\mathcal{X}_{1/t}$ by $-t$, and at $t = -1$, where the twist is trivial, it is a birational involution of $\mathcal{X}_{-1}$ over $\mathbb{Q}$. The closure of its graph is a correspondence on the resolution X. In the family it covers $t \mapsto 1/t$, so on the summand it induces a morphism of variations of Hodge structure from $\iota^*\mathcal{H}(\alpha; \beta)$, twisted, to $\mathcal{H}(\alpha; \beta)$; contributions from the exceptional loci of the resolution factor through the cohomology of varieties of smaller dimension, which carries no constituent isomorphic to the irreducible variation $\mathcal{H}(\alpha; \beta)$ of Hodge level four, and so vanish on the summand. On the fibre at -1 the induced map is therefore an intertwiner as in (12.1). For S_{33} and S_3 we have not written down a model on which the symmetry is visible, and (H2) is an assumption.

Part (H3) holds for all but finitely many primes for any smooth projective model: the de Rham cohomology of a smooth projective scheme over $\mathbb{Z}[1/N]$ is finitely generated, so its torsion and the failure of degeneration are supported at finitely many primes; crystalline cohomology is de Rham cohomology at good primes; the denominators of e and σ are finitely many primes; and the discriminant of the cup product on each integral eigen-summand is a nonzero rational number. We let E be the set of these primes, together with 2, 3 and the primes dividing the constant c of Proposition 12.6.

12.5. The splitting

Proposition 12.6 (The cup product). *Under (H1), the cup product on M_{B} is $c \cdot S$ for a rational number $c > 0$, where S is the invariant form of Appendix D.*

Proof. The cup product on the fibres of $e\,R^4 f_* \mathbb{Q}(2)$ is a flat symmetric form, so on M_{B} it is invariant under the monodromy group $\langle A, B \rangle$. The invariant symmetric forms form a line, by irreducibility and Schur's lemma, so the cup product is

cS with $c \in \mathbb{Q}^\times$, both forms being rational. The cup product has signature $(3,2)$ (Section 12.1); the form S has signature $(3,2)$ by Proposition 12.7 below. Hence $c > 0$. □

The value of c is not needed. If the Picard–Lefschetz formula is normalised so that the vanishing cycle at $t = 1$ has self-intersection ± 2, one expects c to be a power of 2; we do not use this, and include the primes dividing c in E.

Proposition 12.7 (The Levelt splitting). *Let S_B and C_B be the $(+1)$- and (-1)-eigenspaces of σ_H in $M_B = \mathbb{Q}^5$. They are orthogonal for S, of ranks three and two, with bases*

$$S_B = \langle e_3,\ e_4 - e_2,\ e_1 + e_5 \rangle, \qquad C_B = \langle e_2 + e_4,\ e_5 - e_1 \rangle,$$

and Gram matrices given in Table 4. The form is positive definite on S_B and negative definite on C_B. Under (H1) and (H2), S_B and C_B are sub-Hodge structures of M_B, of types $(4,0)+(2,2)+(0,4)$ and $(3,1)+(1,3)$ respectively, and C_B has complex multiplication by the imaginary quadratic field $K_{\mathcal{M}} = \mathbb{Q}(\sqrt{-d(C_B)})$.

$\mathcal{M}$	Gram on S_B	minors	Gram on C_B	$d(C_B)$	$K_{\mathcal{M}}$
H_2	$\begin{pmatrix} 3 & 0 & -10 \\ 0 & 16 & 0 \\ -10 & 0 & 76 \end{pmatrix}$	$3, 48, 2048$	$\mathrm{diag}(-4,-64)$	$256 \equiv 1$	$\mathbb{Q}(i)$
S_{33}	$\begin{pmatrix} 4 & 0 & -10 \\ 0 & 18 & 0 \\ -10 & 0 & 34 \end{pmatrix}$	$4, 72, 648$	$\mathrm{diag}(-2,-18)$	$36 \equiv 1$	$\mathbb{Q}(i)$
S_3	$\begin{pmatrix} 5 & 0 & -14 \\ 0 & 24 & 0 \\ -14 & 0 & 68 \end{pmatrix}$	$5, 120, 3456$	$\mathrm{diag}(-4,-48)$	$192 \equiv 3$	$\mathbb{Q}(\sqrt{-3})$

TABLE 4. The Levelt splitting: Gram matrices of the invariant form S in the bases of Proposition 12.7, the leading principal minors on S_B, the discriminant of C_B modulo squares, and the CM field.

Proof. The eigenvector relations, the orthogonality and the Gram matrices are identities of integer matrices verified by the kernel. The leading minors on S_B are positive, so the form is positive definite there (Sylvester), and the Gram matrix on C_B is diagonal with negative entries. By (H2) the action of σ on M_B is $\pm\sigma_H$ (Proposition 12.2), and it is a morphism of Hodge structures, being induced by an algebraic correspondence; so its eigenspaces are sub-Hodge structures. By (5.4) and Proposition 12.6, a sub-Hodge structure of rank three on which the polarisation is positive definite is of type $(4,0)+(2,2)+(0,4)$, and one of rank two on which it is negative definite is of type $(3,1)+(1,3)$. A rational Hodge structure V of rank two and type $(p,q)+(q,p)$ with $p \neq q$, polarised by a form of discriminant d, has its Mumford–Tate group inside the orthogonal similitude group of the form, whose identity component is the torus attached to $\mathbb{Q}(\sqrt{-d})$; the centraliser of this torus in $\mathrm{End}(V)$ is $\mathbb{Q}(\sqrt{-d})$, acting by Hodge endomorphisms. □

Under the comparison isomorphisms of Lemma 6.6, σ splits every realisation of $\mathcal{M}$ at a prime $q \notin E$: for $\ell \neq q$, $M_\ell = S_\ell \oplus C_\ell$ orthogonally, with $S_\ell \simeq cS_B \otimes \mathbb{Q}_\ell$ and $C_\ell \simeq cC_B \otimes \mathbb{Q}_\ell$, and Frobenius commutes with the splitting, since σ is

defined over $\mathbb{Q}$. We write $d_S \in \mathbb{Q}_{>0}$ for the Gram determinant of S_B and $a, b < 0$ for the diagonal entries of the Gram matrix of C in the cup product, so that $C_\ell \simeq \langle a, b\rangle \otimes \mathbb{Q}_\ell$ with $a = c\cdot(-4)$, $b = c\cdot(-64)$ for H_2, and similarly for the others.

12.6. The vanishing cycle

The splitting of Proposition 12.7 is a splitting of the fibre at -1, not of the local system: the monodromy group $\langle A, B\rangle$ is irreducible, and σ_H does not commute with it but intertwines it with its transform (12.1). The monodromy at $t = 1$ shows this concretely.

Proposition 12.8. *For each of the three data* (1.1)*, let $R = A^{-1}B$ be the monodromy at $t = 1$, and let*

$$\delta = (x, 0, y, 0, 1)^{\mathsf{T}}, \qquad (x, y) = (5, 10),\ (3, 5),\ (4, 7)$$

for H_2, S_{33} and S_3. Then R is the orthogonal reflection $v \mapsto v - 2\frac{S(v,\delta)}{S(\delta,\delta)}\delta$, with $S(\delta, \delta) =$ 128, 18 and 72 respectively, and $\sigma_H\delta = -A\delta$. In the decomposition $\delta = \delta_S + \delta_C$ along $S_B \oplus C_B$ both components are nonzero, with

$$\big(S(\delta_S, \delta_S), S(\delta_C, \delta_C)\big) = (384, -256), \quad (36, -18), \quad (180, -108).$$

Proof. The companion matrices A and B differ only in their last columns, so $B - A = w\, e_5^{\mathsf{T}}$ for the difference w of these columns, and $R - 1 = A^{-1}w\, e_5^{\mathsf{T}}$ has rank one: R fixes the hyperplane $e_5^{\perp}$ (for the standard pairing) pointwise. Its determinant is $\det B/\det A = 1/(-1) = -1$. An isometry of a nondegenerate quadratic space that fixes a hyperplane pointwise and has determinant -1 is the reflection in a vector orthogonal to that hyperplane, and this vector spans the image of $R - 1$, the line through $A^{-1}w$. For H_2, $w = (2, 0, 20, 0, 10)^{\mathsf{T}}$, and one checks that $A\delta = (-1, 0, -10, 0, -5)^{\mathsf{T}} = -\frac{1}{2}w$, so the image is spanned by δ; similarly $A\delta = -(1, 0, 5, 0, 3)^{\mathsf{T}}$ and $A\delta = -(1, 0, 7, 0, 4)^{\mathsf{T}}$ for the two other data, which are again multiples of w. With the matrices S of Appendix D, $S(\delta, \delta) = 3\cdot 25 + 3\cdot 100 + 3 + 2(-250 + 175 - 50) = 128$ for H_2, and similarly 18 and 72. The matrix σ_H reverses the order of the coordinates and changes the signs of the second and fourth, so $\sigma_H\delta = (1, 0, y, 0, x)^{\mathsf{T}}$, which is $-A\delta$ in each case. Finally, in the bases of Proposition 12.7, $xe_1 + e_5 = \frac{x+1}{2}(e_1 + e_5) - \frac{x-1}{2}(e_5 - e_1)$, so $\delta_S = ye_3 + \frac{x+1}{2}(e_1 + e_5)$ and $\delta_C = -\frac{x-1}{2}(e_5 - e_1)$; the squares follow from Table 4: for H_2, $S(\delta_C, \delta_C) = 4\cdot(-64) = -256$ and $S(\delta_S, \delta_S) = 128 + 256 = 384$. □

The relation $\sigma_H\delta = -A\delta$ is forced by (12.1): conjugation by σ_H takes $R = A^{-1}B$ to $(-B)(-A^{-1}) = BA^{-1} = ARA^{-1}$, the reflection in $A\delta$, so $\sigma_H\delta = \pm A\delta$. Since δ has a nonzero component in each eigenspace, R preserves neither S_B nor C_B. The Galois group, on the other hand, acts on the fibre at -1 through automorphisms that commute with σ, because σ is defined over $\mathbb{Q}$. The splitting is therefore invisible to the monodromy of the family and visible to arithmetic and to the Hodge structure of one fibre: it is a property of the fixed point of the inversion, of the Hodge structure there, and of the action of Frobenius on its realisations.

12.7. The scan

Proposition 12.9. *There are* 146 *hypergeometric data* (α, β) *defined over* $\mathbb{Q}$ *with* $\beta \equiv \alpha + \frac{1}{2}$, α *a union of Galois orbits of roots of unity of order at most* 30, *and rank at most eight. Up to the exchange of* α *and* β, *which corresponds to* $t \mapsto 1/t$, *exactly three of them have rank five and Hodge numbers* $(1,1,1,1,1)$: *the data* (1.1). *Four of them, up to the exchange, have rank seven and weight four; their Hodge numbers are* $(2,1,1,1,2)$ *for three of them and* $(1,1,3,1,1)$ *for one.*

Proof. This is a finite enumeration, with the Hodge numbers computed from the interlacing of α and β; the list is in the deposit (Appendix B). □

The rank-seven data are the subject of Section 18. In rank five, the three motives are the only ones in this range to which the argument of the next section applies.

13. Proof of the fixed-point theorem

Throughout this section $\mathcal{M}$ is one of H_2, S_{33}, S_3, Hypothesis 12.4 is assumed, X, e, σ and E are as there, and $q \notin E$ is a prime; ℓ denotes a prime different from q. We write $M_\ell = S_\ell \oplus C_\ell$ for the Levelt splitting of the ℓ-adic realisation at q (Section 12.5), ϕ for the normalised Frobenius, $K = K_{\mathcal{M}}$, $d_S > 0$ for the Gram determinant of S_{B} and $a, b < 0$ for the diagonal entries of the cup product on C, so that $C_\ell \simeq \langle a, b\rangle \otimes \mathbb{Q}_\ell$ for every $\ell \neq q$.

The space of algebraic classes is taken modulo numerical equivalence: we write $\mathcal{Z}$ for the $\mathbb{Q}$-vector space of codimension-two cycles on $X_{\overline{\mathbb{F}}_q}$ with rational coefficients, projected by e, modulo numerical equivalence. Its elements are represented by their classes in M_ℓ and in the crystalline realisation, and the intersection form on $\mathcal{Z}$ is computed in either. This is the numerical form of Hdg (Remark 3.5).

13.1. The structure of Frobenius

The map ϕ preserves S_ℓ and C_ℓ, since σ is defined over $\mathbb{Q}$, and is orthogonal on each. Its characteristic polynomials on S_ℓ and on C_ℓ have rational coefficients independent of ℓ: they are computed by the traces of the correspondences $\Gamma_{\mathrm{Frob}^n} \circ e(1 \pm \sigma)/2$, which are rational numbers by the Lefschetz trace formula, together with the theorem of Katz and Messing.

On S_ℓ, of rank three, ϕ has the eigenvalue $\lambda = \det(\phi|S_\ell) = \pm 1$, and its other two eigenvalues ζ, ζ^{-1} are the roots of $x^2 - tx + 1$ with $t \in \mathbb{Q}$.

Definition 13.1 (Exceptional primes). The prime q is *W-exceptional* if ζ is a root of unity, and *C-exceptional* if q splits in K and C_ℓ is potentially Tate. We write T for the set of exceptional primes.

Since $x^2 - tx + 1$ has rational coefficients, ζ is a root of unity exactly when $t \in \{-2, -1, 0, 1, 2\}$.

Lemma 13.2. *Let* $q \notin E \cup T$. *Then the* λ*-eigenspace* L *of* ϕ *on* S_ℓ *is a line, its orthogonal complement* $W = L^\perp \cap S_\ell$ *is a binary block on which* $\phi \in \mathrm{SO}(W)$ *has characteristic*

polynomial $x^2 - tx + 1$ with $t^2 < 4$, and the Tate part of M_ℓ is L if C_ℓ is not potentially Tate, and $L \oplus C_\ell$ otherwise.

Proof. Since q is not W-exceptional, $\zeta \neq \pm 1$ and ζ is not a root of unity, so the eigenvalue λ has multiplicity one and $W = \ker(\phi^2 - t\phi + 1)$ is a ϕ-stable plane orthogonal to L (Section 5.3), on which $\det \phi = \zeta\zeta^{-1} = 1$. The roots lie on the unit circle and are not real, so $t^2 < 4$. The Tate part of S_ℓ is L, and C_ℓ is either potentially Tate or contains no potentially Tate vector at all: if $\phi|C_\ell$ had one eigenvalue a root of unity, the other, being its inverse or its conjugate over $\mathbb{Q}$, would be one too. □

Lemma 13.3 (The splitting of algebraic classes). *Let $q \notin E \cup T$. Then $\mathcal{Z} = \mathcal{Z}^+ \oplus \mathcal{Z}^-$ orthogonally, where $\mathcal{Z}^\pm$ is the (± 1)-eigenspace of the action of σ; the classes of $\mathcal{Z}^+$ lie in L and those of $\mathcal{Z}^-$ lie in C_ℓ. If C_ℓ is not potentially Tate, $\mathcal{Z}^- = 0$.*

Proof. The correspondence $e\sigma e$ acts on cycles and on numerical classes, compatibly with the cycle class maps, and its square is e; the projectors $\frac{1}{2}(e \pm e\sigma e)$ have rational coefficients, so $\mathcal{Z} = \mathcal{Z}^+ \oplus \mathcal{Z}^-$. The action of σ on M_ℓ is $\pm\sigma_{\mathrm{H}}$, an isometry of S, hence of the cup product cS; so the two eigenspaces are orthogonal, in M_ℓ and in $\mathcal{Z}$. By Lemma 5.7 the classes lie in the Tate part, which is $L \oplus C_\ell$ or L by Lemma 13.2; the classes of $\mathcal{Z}^+$ lie in the Tate part of S_ℓ, which is L, and those of $\mathcal{Z}^-$ in the Tate part of C_ℓ. □

13.2. The line

Proposition 13.4 (The line). *Let $q \notin E \cup T$. Then $\dim \mathcal{Z}^+ \leq 1$, and every nonzero $z \in \mathcal{Z}^+$ has $\langle z, z\rangle > 0$.*

Proof. For every $\ell \neq q$, $d(S_\ell) \equiv c^3 d_S \equiv c\, d_S$, and $d(W) \equiv 4 - t^2$ by Lemma 6.7; so $L \simeq \langle u\rangle \otimes \mathbb{Q}_\ell$ with

$$u = c\, d_S\, (4 - t^2) \in \mathbb{Q}_{>0}.$$

The line L is nondegenerate, so a nonzero vector of it has nonzero square. Let $z \in \mathcal{Z}^+$ be nonzero, with $v = \langle z, z\rangle$. If $v = 0$, the class of z in L is zero for every ℓ, and then z is orthogonal to all of $\mathcal{Z}$ (by Lemma 13.3 and since z has zero image in L), so $z = 0$ in $\mathcal{Z}$; hence $v \neq 0$. The class of z spans L, so $v \equiv u$ in $\mathbb{Q}_\ell^\times / \mathbb{Q}_\ell^{\times 2}$ for every $\ell \neq q$, that is, uv is a square in $\mathbb{Q}_\ell$ for all $\ell \neq q$. By Lemma 5.4, $v > 0$. If $z, z' \in \mathcal{Z}^+$, their classes are proportional in L, and the ratio is $\langle z, z'\rangle / \langle z, z\rangle \in \mathbb{Q}$, so z' is a rational multiple of z modulo the classes with zero image, which are zero in $\mathcal{Z}$; hence $\dim \mathcal{Z}^+ \leq 1$. □

The proposition uses no information at q beyond the characteristic polynomial: the sign of the algebraic classes on the line is fixed by the rational number $c\, d_S(4 - t^2)$, that is, by the Betti form on S_{B} and by the discriminant of the transcendental plane W. This is where the Levelt splitting does its work. Without σ, the Tate part at an inert prime would be the rank-three space $L \oplus C$, whose square class is still global but whose Hasse invariants are not (Section 6.5).

13.3. The crystalline input

Let $H \subset e\,H^4_{\mathrm{cris}}(X_k/W)$, $k = \overline{\mathbb{F}}_q$, $W = W(k)$, be the integral (-1)-eigensummand of $e\sigma e$, with the cup product and with $\Phi = q^{-1}F$, where F is the σ-semilinear crystalline Frobenius. (Here q is a prime, so F is the Frobenius of Lemma 5.8, with $p = q$.)

Proposition 13.5 (Ogus). *Suppose $q \notin E$ and that C_ℓ is potentially Tate. Then (H, Φ) is a supersingular K3 crystal of rank two in the sense of* [Ogu79, Def. 3.1]. *Its Tate module $T_H = \{x \in H : \Phi x = qx\}$ is a free $\mathbb{Z}_q$-module of rank two, and there are units $u_2, u_3 \in \mathbb{Z}_q^\times$ such that*

$$T_H \simeq \langle q\,u_2,\ q\,u_3\rangle, \qquad -u_2u_3 \text{ is not a square modulo } q,$$

so that $T_H \otimes \mathbb{Q}_q$ is anisotropic and $\varepsilon_q(T_H \otimes \mathbb{Q}_q) = -1$. The crystalline class of every element of $\mathcal{Z}^-$ lies in $T_H \otimes \mathbb{Q}_q$.

Proof. The conditions of [Ogu79, Def. 3.1] are: (a) $q^2H \subset \Phi(H)$, (b) $\Phi \otimes k$ has rank one, (c) the form is perfect, (d) $\langle \Phi x, \Phi y\rangle = q^2\sigma\langle x, y\rangle$. Condition (d) holds because the cup product $H^4 \times H^4 \to H^8$ is compatible with F, and F acts on H^8 of the fourfold by q^4. Condition (c) is part of (H3). For (a) and (b): C_{B} has Hodge type $(3,1)+(1,3)$ (Proposition 12.7), so the de Rham realisation of the summand has Hodge numbers $h^{3,1} = h^{1,3} = 1$; by (H3) and Mazur's theorem [Maz73], in the form of [BO78, Thm. 8.26] for a summand cut out by a projector compatible with the Hodge filtration and with Frobenius, the image of F on H is $qH' + q^3H''$ for a splitting $H = H' \oplus H''$ into rank-one summands, so that $\Phi(H) = H' + q^2H''$, which gives (a) and (b). The crystal is supersingular: the eigenvalues of ϕ on C_ℓ are roots of unity, so by the theorem of Katz and Messing all slopes of F on H are 2, and all slopes of Φ are 1.

For a supersingular K3 crystal of rank n over k, $p = q > 2$, Ogus proves that T_H is free of rank n with nondegenerate form [Ogu79, Thm. 3.3], that the q-adic valuation of its discriminant is $2\sigma_0$ with $\sigma_0 \geq 1$ and its Hasse invariant is -1 [Ogu79, Prop. 3.13], and that $T_H \simeq (T_0, q\langle\,,\,\rangle_0) \oplus (T_1, \langle\,,\,\rangle_1)$ with perfect forms, T_0 of rank $2\sigma_0$, and the discriminant d_0 of $\langle\,,\,\rangle_0$ satisfying $\left(\frac{d_0}{q}\right) = -\left(\frac{-1}{q}\right)^{\sigma_0}$ [Ogu79, Thm. 3.4]. For $n = 2$ we have $\sigma_0 = 1$ and $T_1 = 0$; diagonalising the perfect binary form $\langle\,,\,\rangle_0$ over $\mathbb{Z}_q$ as $\langle u_2, u_3\rangle$ gives the stated shape, and $\left(\frac{u_2u_3}{q}\right) = -\left(\frac{-1}{q}\right)$ says that $-u_2u_3$ is a nonsquare. A binary form whose discriminant is minus a nonsquare is anisotropic.

Finally, if Z is a cycle of codimension two on X_k, its crystalline class satisfies $F(\mathrm{cl}\,Z) = q^2\,\mathrm{cl}(Z)$ by Lemma 5.8, and the projection to $H \otimes \mathbb{Q}_q$ commutes with F, so Φ fixes it up to the factor q: $\Phi x = qx$. An element $x \in H \otimes \mathbb{Q}_q$ with $\Phi x = qx$ lies in $T_H \otimes \mathbb{Q}_q$, since $q^N x \in H$ for N large and then $q^N x \in T_H$. □

13.4. The CM block

Proposition 13.6 (The CM block). *Let $q \notin E \cup T$ be inert in K. Then every nonzero $z \in \mathcal{Z}^-$ has $r = \langle z, z\rangle > 0$.*

Proof. If C_ℓ is not potentially Tate, $\mathcal{Z}^- = 0$. Otherwise let $z \in \mathcal{Z}^-$ be nonzero. Its crystalline class z_{cris} lies in the anisotropic plane $T_H \otimes \mathbb{Q}_q$ (Proposition 13.5). If $z_{\mathrm{cris}} = 0$, then z is orthogonal to every algebraic class, the intersection form being

computed crystalline, so $z = 0$ in $\mathcal{Z}$; hence $z_{\mathrm{cris}} \neq 0$ and $r = \langle z_{\mathrm{cris}}, z_{\mathrm{cris}} \rangle \neq 0$. For the same reason the ℓ-adic class of z is nonzero for every $\ell \neq q$.

At $\ell \neq q$: $C_\ell \simeq \langle a, b \rangle \otimes \mathbb{Q}_\ell$ represents r, so $(r, -ab)_\ell = (a, b)_\ell$ by Lemma 5.2.

At q: $T_H \otimes \mathbb{Q}_q \simeq \langle qu_2, qu_3 \rangle$ represents r, so $(r, -q^2 u_2 u_3)_q = \varepsilon_q(T_H \otimes \mathbb{Q}_q) = -1$ by Lemma 5.2 and Proposition 13.5. Since q is inert in $K = \mathbb{Q}(\sqrt{-ab})$ and $q \nmid ab$ (as $q \notin E$), $-ab$ is a unit nonsquare modulo q; so is $-u_2 u_3$; their ratio is a square in $\mathbb{Q}_q$, and therefore $(r, -ab)_q = (r, -u_2 u_3)_q = (r, -q^2 u_2 u_3)_q = -1$.

Reciprocity: $(a, b)_q = 1$ since a, b are units at q, and $(a, b)_\infty = -1$ since $a, b < 0$; so $\prod_{\ell \neq q} (a, b)_\ell = -1$, the product over the finite places other than q. Then

$$(r, -ab)_\infty = (r, -ab)_q \prod_{\ell \neq q} (r, -ab)_\ell = (-1) \prod_{\ell \neq q} (a, b)_\ell = 1.$$

Since $-ab < 0$, this says $r > 0$. □

The proof is the odd case of the sign law (Corollary 10.2), with the q-adic symbol supplied by Ogus. It treats one class at a time: it does not use that C_ℓ is spanned by algebraic classes. The Lean statement `AlienHodgeWorlds.cm_block_pos_ab` is the reciprocity argument with the local symbols as hypotheses, and `ogus_bridge_ab` is the step from Ogus's normal form to the symbol at q.

13.5. The determinant on the CM block

Let ε_C be the character $\mathrm{Frob}_q \mapsto \det(\phi | C_\ell)$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$; it is quadratic, and unramified outside ℓ and the bad primes of the datum, by (H1).

Lemma 13.7. *$\varepsilon_C = \chi_K$, the quadratic character of K. Consequently, at a prime $q \notin E$ inert in K, ϕ acts on C_ℓ as a reflection, with eigenvalues 1 and -1, and C_ℓ is potentially Tate; at a prime split in K, ϕ acts on C_ℓ as a rotation.*

Proof. Complex conjugation acts on $C_{\mathrm{B}} \otimes \mathbb{C} = H^{3,1} \oplus H^{1,3}$ through the real Frobenius F_∞, which exchanges $H^{3,1}$ and $H^{1,3}$; so it is an involution of trace zero on a plane, with determinant -1. By the comparison isomorphism, ε_C is odd. Next, if C_ℓ is not potentially Tate at a prime q, then $\phi | C_\ell$ is not a reflection (a reflection has eigenvalues ± 1), so $\varepsilon_C(q) = 1$. The computations of Appendix B exhibit, for every quadratic character $\chi \neq 1, \chi_K$ unramified outside ℓ and the bad primes of the datum, a prime q with $\chi(q) = -1$ at which C_ℓ is not potentially Tate; so $\varepsilon_C \notin \{\chi\}$. Hence $\varepsilon_C \in \{1, \chi_K\}$, and as ε_C is odd, $\varepsilon_C = \chi_K$. An element of $\mathrm{O}(2)$ of determinant -1 is a reflection; one of determinant 1 is a rotation. □

The lemma is the rank-two case of the discriminant-field principle (Section 10.3): $F_C = \mathbb{Q}(\sqrt{-d(C_{\mathrm{B}})}) = K$. It locates the walls: at the primes inert in K the Tate part is $L \oplus C$, of rank three, and at the split primes it is the line L, except at the exceptional primes.

13.6. The density of the exceptional primes

We first record a lemma on the orthogonal group in three variables. Let V be a nondegenerate quadratic space of dimension three over $\mathbb{Q}_\ell$, $G \subset \mathrm{O}(V)$ a compact ℓ-adic Lie subgroup with Lie algebra $\mathfrak{g} \subset \mathfrak{so}(V)$, and $\psi = \mathrm{tr} - \det$, a locally analytic function on G, invariant under conjugation.

Lemma 13.8. *If $\dim \mathfrak{g} \geq 2$, every fibre of ψ has empty interior in G, and so has Haar measure zero.*

Proof. Suppose ψ is constant on a neighbourhood of $g \in G$. Since det is locally constant, so is tr, and for $X \in \mathfrak{g}$ and s small $\mathrm{tr}(g\exp(sX)) = \sum_k \frac{s^k}{k!}\mathrm{tr}(gX^k)$ is constant; hence $\mathrm{tr}(gX^k) = 0$ for all $X \in \mathfrak{g}$ and $k \geq 1$. These are polynomial identities in X, so they hold on $\mathfrak{g} \otimes \overline{\mathbb{Q}}_\ell$. We work over $\overline{\mathbb{Q}}_\ell$, where V has a basis e_1, e_2, e_3 with Gram matrix antidiagonal, $\langle e_1, e_3\rangle = \langle e_2, e_2\rangle = 1$, and $\mathfrak{so}(V) \simeq \mathfrak{sl}_2$ has the basis $H = \mathrm{diag}(1, 0, -1)$, N_+ and N_-, with $N_+e_1 = 0$, $N_+e_2 = e_1$, $N_+e_3 = -e_2$ and N_- its transpose with respect to the form. A subalgebra of dimension at least two of $\mathfrak{sl}_2$ is conjugate to the Borel subalgebra spanned by H and N_+, or is everything; so after conjugation $\mathfrak{g}\otimes\overline{\mathbb{Q}}_\ell$ contains H and N_+. From $\mathrm{tr}(gH^k) = 0$ for $k = 1, 2$ we get $g_{11} - g_{33} = 0$ and $g_{11} + g_{33} = 0$, so $g_{11} = g_{33} = 0$; from $N_+^2e_3 = -e_1$ and $N_+^2e_1 = N_+^2e_2 = 0$ we get $\mathrm{tr}(gN_+^2) = -g_{31} = 0$. Therefore $ge_1 = g_{21}e_2$. But e_1 is isotropic and e_2 is not, and g is an isometry, so $g_{21} = 0$ and $ge_1 = 0$, which is impossible. A closed subset with empty interior that is the fibre of a locally analytic function is a locally analytic subset of positive codimension, and has measure zero [Ser81, n°6.4]. □

Theorem 13.9. *The set T of exceptional primes has natural density zero.*

Proof. Fix a prime ℓ ($\ell = 3$ for H_2 and $\ell = 5$ for S_{33} and S_3), and let G_S and G_C be the closures of the images of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ in $\mathrm{O}(S_\ell)$ and $\mathrm{O}(C_\ell)$, compact ℓ-adic Lie groups. Frobenius elements at primes outside $E \cup \{\ell\}$ are defined up to conjugation. By the Chebotarev density theorem in the form of [Ser81, n°2.1, Th. 1], applied to the finite quotients of G_S and G_C, the set of primes whose Frobenius lies in a closed subset of Haar measure zero stable under conjugation has density zero; this is the argument of [Ser81, Introduction] for supersingular primes of elliptic curves.

W-exceptional primes. These are the primes whose Frobenius lies in $\Psi_S = \{g \in G_S : \mathrm{tr}(g) - \det(g) \in \{-2, -1, 0, 1, 2\}\}$, a closed set stable under conjugation: for g with eigenvalues $\lambda, \zeta, \zeta^{-1}$, $\lambda = \det g$, one has $\mathrm{tr}\, g - \det g = \zeta + \zeta^{-1}$. We show that Ψ_S has measure zero by examining the Lie algebra $\mathfrak{g}$ of G_S.

- If $\mathfrak{g} = 0$, then G_S is finite and every Frobenius has finite order, so every prime is W-exceptional.
- If $\mathfrak{g}$ is spanned by a nilpotent element, then an open subgroup of G_S consists of unipotent elements, and every eigenvalue of every Frobenius is a root of unity.
- If $\mathfrak{g}$ is spanned by a semisimple element X, then X has eigenvalues $0, \pm\mu$ with $\mu \neq 0$; it fixes a line L' and acts on the plane $W' = L'^{\perp}$ with eigenvalues $\pm\mu$. Every $g \in G_S$ normalises $\mathbb{Q}_\ell X$, so $gXg^{-1} = \pm X$. If the sign is $+$, g preserves L' and commutes with X on W', so it acts on W' as an element of the torus T_0 centralising X; if it is $-$, g exchanges the $\pm\mu$-eigenvectors

and acts on W' as a reflection. The sign is a continuous homomorphism $G_S \to \{\pm 1\}$, hence a quadratic character χ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ unramified outside ℓ and the bad primes of the datum, and every prime with $\chi(q) = -1$ is W-exceptional. On the kernel of χ, which is open, $\psi(g) = \zeta + \zeta^{-1}$ where g rotates W' by ζ, a non-constant analytic function on each coset of $T_0 \cap G_S$, whose fibres have measure zero.

- If $\dim \mathfrak{g} \geq 2$, the fibres of ψ have measure zero by Lemma 13.8.

The computations of Appendix B exclude the first two cases and the third case with $\chi \neq 1$: there is a prime at which W is not potentially Tate, and for every quadratic character $\chi \neq 1$ unramified outside ℓ and the bad primes of the datum there is a prime q with $\chi(q) = -1$ at which W is not potentially Tate. In the remaining cases Ψ_S, a finite union of fibres of ψ, has measure zero.

C-exceptional primes. By Lemma 13.7, at a split prime ϕ acts on C_ℓ as a rotation, and it is potentially Tate only if its trace lies in $\{-2, -1, 0, 1, 2\}$. The group $G_C^0 = G_C \cap \mathrm{SO}(C_\ell)$ has index at most two in G_C; it is a compact subgroup of the one-dimensional ℓ-adic Lie group $\mathrm{SO}(C_\ell)$, and it is infinite, since C_ℓ is not potentially Tate at some split prime (Appendix B), where Frobenius is a rotation of infinite order. A compact subgroup of dimension zero is finite, so G_C^0 has dimension one and is open in $\mathrm{SO}(C_\ell)$. The trace is a non-constant analytic function on G_C^0, so the elements of G_C^0 with trace in a finite set form a closed set, stable under conjugation, of Haar measure zero in G_C, and the C-exceptional primes have density zero. □

Remark 13.10. We do not know whether T is finite. For H_2, among the primes $5 \leq q \leq 97$ the only exceptional prime is $q = 31$, where W is Tate; for S_{33} and S_3 no prime $q \leq 97$ that we computed is exceptional (Appendix B). The W-exceptional primes are the primes at which the binary motive W, which is not of CM type by the theorem just proved, has a Frobenius of finite order; for motives of this kind the expected number of such primes up to x is bounded, but we do not know a proof.

13.7. Examples

We follow the proof at the first inert primes, in the normalisation of Appendix B and with $c = 1$ for definiteness; a different value of c multiplies every square class below by c and changes nothing in the signs.

The motive H_2 at $q = 7$. Here $q \equiv 3 \pmod 4$ is inert in $K = \mathbb{Q}(i)$. The exact computation gives $t_W = -\frac{30}{49}$ on W, so $4 - t_W^2 = \frac{8704}{2401}$, and $8704 = 2^9 \cdot 17$; the Gram determinant of S_{B} is $d_S = 2048 = 2^{11}$. The square class of the line is

$$u = d_S(4 - t_W^2) \equiv 2 \cdot 2 \cdot 17 \equiv 17,$$

and Proposition 13.4 says that every nonzero algebraic class on L has a square in $17\mathbb{Q}^{\times 2}$, which is positive. The block C is $\langle -4, -64\rangle \simeq \langle -1, -1\rangle$. A nonzero algebraic class $z \in C$, if there is one, has square r represented by $\langle -1, -1\rangle$ at every $\ell \neq 7$, so $(r, -1)_\ell = (-1, -1)_\ell$, which is -1 at $\ell = 2$ and 1 at the odd $\ell \neq 7$; and by Proposition 13.5, $(r, -1)_7 = -1$. Then $(r, -1)_\infty = (-1)(-1) = 1$, and $r > 0$. The class $r = 7$ satisfies all these conditions, by the certificates of Appendix D.2.

The same conclusion can be reached without separating L from C at the level of classes, by treating the Tate part $A = L \oplus C$ as one ternary block. For $\ell \neq 7$, $A_\ell \simeq G \otimes \mathbb{Q}_\ell$ with $G = \langle 17, -1, -1\rangle$; reciprocity gives $\prod_{\ell \neq 7} \varepsilon_\ell(A) =$

$\varepsilon_\infty(G)\,\varepsilon_7(G) = (-1)\cdot 1 = -1$, and at 7 the crystalline Tate space has Hasse invariant -1 (the line contributes a symbol $(x, u_2u_3)_7 = 1$, since u_2u_3 is a square modulo 7). Hence $\beta(A) = \prod_{v<\infty} \varepsilon_v(A) = 1$. This is the computation that the formal companion records at 24 inert primes (Appendix D.2). It shows that A has the local invariants of a positive definite form; since A has rank three, Theorem 10.3 then gives positivity on the algebraic classes, provided they span A. The class-by-class argument of Proposition 13.4 and Proposition 13.6 does not need this proviso.

The motive H_2 *at* $q = 5$. Here q splits in $\mathbb{Q}(i)$, the CM plane is a rotation and not Tate, and the Tate part is the line. The computation gives $t_W = \frac{34}{25}$, $4 - t_W^2 = \frac{1344}{625}$, $1344 = 2^6 \cdot 21$, and $u \equiv 2 \cdot 21 = 42$. Algebraic classes on L have squares in $42\mathbb{Q}^{\times 2}$.

The motive S_{33} *at* $q = 7$. Again q is inert in $\mathbb{Q}(i)$. The CM plane is $\langle -2, -18\rangle \simeq \langle -2, -2\rangle$, of discriminant $36 \equiv 1$, and $-ab = -36 \equiv -1$ is a nonsquare modulo 7. For a class of square r the local conditions are $(r, -36)_\ell = (-2, -18)_\ell$ at $\ell \neq 7$ and $(r, -36)_7 = -1$, and they force $r > 0$; the value $r = 7$ satisfies them at 2 and 3 (Appendix D.2).

The motive S_3 *at* $q = 5$ *and* $q = 11$. These primes are inert in $\mathbb{Q}(\sqrt{-3})$. The CM plane is $\langle -4, -48\rangle$, of discriminant $192 \equiv 3$, and $-192 \equiv -3$ is a nonsquare modulo 5 and modulo 11. The local conditions read $(r, -192)_\ell = (-4, -48)_\ell$ at $\ell \neq q$ and $(r, -192)_q = -1$, and $r = q$ satisfies them at $\ell = 2, 3$. At $q = 7$, which splits in $\mathbb{Q}(\sqrt{-3})$, the computation shows that C is not Tate, in accordance with Lemma 13.7.

13.8. The exceptional prime 31

At $q = 31$, which is inert in $\mathbb{Q}(i)$, the motive H_2 has $t_W = 2$ in the normalisation of Appendix B: Frobenius acts on W with characteristic polynomial $(x-1)^2$, and the whole motive is potentially Tate. The Levelt involution still splits it, as $S \oplus C$, and the argument of Proposition 13.6 applies to C without change. The part S, however, is now a Tate block of rank three. Its ℓ-adic realisations are those of the positive definite rational form $c\,S_{\mathrm{B}}$, so by Theorem 10.3 and reciprocity its algebraic classes form a positive definite space if they span S and if the crystalline space $T(S) = \{x \in S_D : Fx = q^2 x\}$ has the Hasse invariant of $c\,S_{\mathrm{B}} \otimes \mathbb{Q}_{31}$. The second condition is a statement about a crystal of a type not covered by Ogus's theory: the Hodge numbers of S are $h^{4,0} = h^{2,2} = h^{0,4} = 1$, so after division by q^2 the Frobenius is not integral, and S is not a K3 crystal. Without the second condition, and without the Tate conjecture for S, the method gives no conclusion at $q = 31$, and 31 belongs to the exceptional set T.

The phenomenon is general: a W-exceptional prime is one at which the transcendental plane, of Newton slopes 0 and 4 at the ordinary primes, becomes supersingular, and the block S becomes a Tate block of the Hodge type $(4,0) + (2,2) + (0,4)$. The invariant at q that would decide it is the Hasse invariant of the Tate module of a supersingular crystal of this type. An analogue of Ogus's theorem for such crystals would remove the W-exceptional primes from Theorem A.

13.9. Where the hypotheses are used

For the reader who wishes to check the dependence of Theorem A on the geometric inputs, we list the steps of the proof and what each uses.

- The structure of Frobenius (Lemma 13.2) uses (H1), to identify the ℓ-adic realisations with the Betti lift (Lemma 6.6); the action of σ (H2), to split them; and Deligne's theorem and the theorem of Katz and Messing, for the rationality of the characteristic polynomials.
- The splitting of algebraic classes (Lemma 13.3) uses only that σ is an algebraic correspondence defined over $\mathbb{Q}$ that acts on the summand by an isometry (H2), and Lemma 5.7.
- The line (Proposition 13.4) uses the comparison of ℓ-adic and Betti forms (H1), Lemma 6.7, and Lemma 5.4; no information at q.
- The crystalline input (Proposition 13.5) uses (H3): the integral model, torsion-free crystalline cohomology, degeneration, integrality of the projectors and perfectness of the form on the summand H; and Mazur's and Ogus's theorems.
- The CM block (Proposition 13.6) uses Proposition 13.5, the comparison at $\ell \neq q$ (H1), and $q \nmid ab$ (H3).
- The determinant on C (Lemma 13.7) uses the Hodge type of C_{B} (H1, H2) and the computations of Appendix B.
- The density (Theorem 13.9) uses the computations of Appendix B and Serre's form of the Chebotarev theorem.

No step uses the Tate conjecture, a CM lift of X, or the Hodge conjecture.

13.10. Proof of Theorem A

Let $q \notin E \cup T$. By Lemma 13.3, $\mathcal{Z} = \mathcal{Z}^+ \oplus \mathcal{Z}^-$ orthogonally, with the classes of $\mathcal{Z}^+$ in the line L and those of $\mathcal{Z}^-$ in C_ℓ, which gives (ii) together with Lemma 13.7: at a prime split in K, C_ℓ is not potentially Tate (as q is not C-exceptional) and $\mathcal{Z}^- = 0$. By Proposition 13.4 the form is positive definite on $\mathcal{Z}^+$, and by Proposition 13.6 it is positive on every nonzero element of $\mathcal{Z}^-$ when q is inert, so positive definite on $\mathcal{Z}^-$. An orthogonal sum of positive definite spaces is positive definite, which is (i). The Betti form on C_{B} is c times a negative definite form (Proposition 12.7), and Proposition 13.6 gives (iii). The exceptional set T has density zero by Theorem 13.9, and for H_2 the statement about the primes $q \leq 97$ is Appendix B.2. No step uses the Tate conjecture or a lifting with complex multiplication: Proposition 13.4 uses only ℓ-adic realisations and the characteristic polynomial, Proposition 13.6 uses one algebraic class at a time and the crystalline realisation of X itself. □

14. Sign reversal under supersingular specialisation

14.1. The phenomenon

Part (iii) of Theorem A deserves a separate statement, because it is the reason for the name of the theory.

Theorem 14.1 (Sign reversal). *Let $\mathcal{M}$ be one of the three motives (1.1), under Hypothesis 12.4, and let $q \notin E \cup T$ be a prime inert in $K_{\mathcal{M}}$. Then:*

(i) *every nonzero vector of C_{B} has negative square for the cup product;*
(ii) *C_ℓ is potentially Tate, of Newton slopes $(2,2)$, Newton–Hodge type $(2,2)+(2,2)$ and Newton–Hodge sign $+$;*
(iii) *every algebraic class in C that is not numerically trivial has positive square.*

Proof. (i) is Proposition 12.7 and $c > 0$. (ii) is Lemma 13.7 and Proposition 13.5: the slopes of a potentially Tate block of weight four are all 2. (iii) is Proposition 13.6. □

The sign of an algebraic class is therefore not the sign of the block in which it lies in the Betti realisation of a lift. In the Betti realisation, C_{B} is a rational Hodge structure of type $(3,1)+(1,3)$, and by (5.4) it is negative. After reduction modulo an inert prime it becomes supersingular, and the algebraic classes that it may contain are positive, as Hdg requires, and as the Newton–Hodge sign of Definition 5.9 predicts. The Newton polygon, not the Hodge filtration of the lift, carries the sign.

This is consistent with the general expectation. On a lift $\mathcal{X}$ over a q-adic ring, a class that becomes algebraic on the special fibre need not lift as a Hodge class; the Hodge structure of the generic fibre and the algebraic classes of the special fibre are related only through the crystalline realisation, and the comparison of quadratic forms across this specialisation is the subject of Ancona's work [Anc21]. In his Theorem 8.1, for an orthogonal motive M of rank two over the ring of integers of a q-adic field whose Betti form is a polarisation and whose space of numerical classes on the special fibre has dimension two, the conclusion is that the numerical classes form a positive definite plane. When the polarisation is negative definite, as for Hodge type $(3,1)+(1,3)$ in weight four, this is a sign reversal of the same kind, obtained when the whole plane is algebraic.

14.2. The reversal in weight two

In weight two the reversal is classical, and it helps to see why the Newton polygon is the right guide. Let Y be a K3 surface over $\mathbb{F}_q$ with supersingular reduction, lifted to characteristic zero, as in Example 6.3. On the lift, the transcendental lattice contains a plane of Hodge type $(2,0)+(0,2)$ on which the intersection form is positive; after reduction it is part of the space of algebraic classes, on which the Hodge index theorem forces the form to be negative on primitive classes. The two rational forms, the transcendental lattice of the lift and the corresponding space of algebraic classes of the reduction, have the same ℓ-adic localisations for $\ell \neq p$, and their signatures differ by two; by Lemma 5.3 they differ at p, and the difference is the non-neutrality of the Tate module of the supersingular K3 crystal [Ogu79, Prop. 3.13]. The Newton slopes of the

reduction are all 1, which is the Hodge index of the type $(1,1)$, and the sign of the algebraic classes is the sign of that type.

Theorem 14.1 is the same phenomenon in weight four, for a plane of Hodge type $(3,1)+(1,3)$ that is not part of a family of algebraic classes in characteristic zero. The weight-two case follows from the Hodge index theorem, which is available for divisors in every characteristic; in weight four there is no such theorem, and the reversal has to be proved. In our case the proof is the sign law, with the crystalline symbol supplied by Ogus's theorem applied to the twist $C(1)$, which has the shape of a K3 crystal of rank two.

14.3. The mechanism in one case

For H_2 at $q=7$ the reversal can be followed by hand. The block C_ℓ is $\langle -c,-c\rangle\otimes\mathbb{Q}_\ell$, the norm form of $\mathbb{Q}(i)$ scaled by $-c$. Since -1 is not a square modulo 7, a non-neutral binary $\mathbb{Z}_7$-form of the shape of Proposition 13.5 is $\langle 7u_2, 7u_3\rangle$ with u_2u_3 a square, for instance $\langle 7,7\rangle$. Suppose an algebraic class has square r. Then r is represented by $\langle -c,-c\rangle$ at every $\ell\neq 7$ and by the anisotropic plane at 7. Take for illustration $c=1$ and $r=7$: then $(7,-1)_7=\left(\frac{-1}{7}\right)=-1$, $(7,-1)_2=(-1,-1)_2=-1$, and the symbols at all other places are trivial, so $r=7$ is locally represented everywhere it has to be, and $r>0$. A negative value such as $r=-7$ satisfies the condition at 7, since $(-7,-1)_7=\left(\frac{-1}{7}\right)=-1$, but fails at 2: -7 is a square in $\mathbb{Q}_2$, so $(-7,-1)_2=1\neq(-1,-1)_2$. Reciprocity is what forbids every negative value at once: by the computation in the proof of Proposition 13.6, the conditions at the finite places force $(r,-1)_\infty=1$.

14.4. Numerical illustrations

The following facts about H_2 were found by exact computation (Appendix B) and are not used in the proofs. They concern the normalisation of the traces of Frobenius given by the finite hypergeometric sums of Beukers, Cohen and Mellit [BCM15], which may differ from the geometric normalisation on X by a quadratic twist; such a twist does not change which blocks are potentially Tate.

(a) For all 166 primes $5\le q<1000$ the exact first and second power sums of ϕ are those of a decomposition $L\oplus C\oplus W$ in which ϕ acts on L by $\left(\frac{-2}{q}\right)$ and on C with characteristic polynomial $x^2-\frac{a_q}{q}x+\chi_{-4}(q)$, where $\sum a_nq^n=\eta(4\tau)^6\in S_3(\Gamma_0(16),\chi_{-4})$ is the CM newform of weight three and level 16.

(b) In this decomposition W is Tate only at $q=31$, where $\phi|_W$ has characteristic polynomial $(x-1)^2$.

For all three motives, Table 5 lists, at the primes computed over $\mathbb{F}_q$, the trace of Frobenius on W and the square class of the line L, $u=d_S(4-t_W^2)$, with $c=1$. In every case $u>0$, as Proposition 13.4 requires: the sign of the line is fixed by the global data and does not depend on the prime. At the inert primes the plane C is Tate as well, and its algebraic classes are positive by Proposition 13.6, although C_{B} is negative definite.

TABLE 5. The line L at the primes computed over $\mathbb{F}_q$: the trace t_W of ϕ on W, the squarefree part of $u = d_S(4 - t_W^2)$ (with $c = 1$), which is positive in every case, and whether q is inert in $K_{\mathcal{M}}$, so that C is Tate.

$\mathcal{M}$	q	t_W	u	q inert
H_2	5	34/25	42	no
	7	−30/49	17	yes
	11	−42/121	71	yes
	13	−62/169	138	no
	17	−478/289	33	no
	19	−522/361	311	yes
	23	930/529	497	yes
	29	1282/841	1482	no
	37	2338/1369	282	no
	41	−2462/1681	182	no
	43	−3690/1849	1847	yes
	47	−190/2209	1057	yes
	53	2018/2809	3818	no
	59	−5162/3481	3031	yes
	61	6658/3721	282	no
	67	−2250/4489	2807	yes
	71	6882/5041	4241	yes
	73	−10558/5329	663	no
	79	−318/6241	3041	yes
	83	−13130/6889	7	yes
	89	−15358/7921	39	no
	97	3682/9409	473	no
S_{33}	7	26/49	31	yes
	13	14/169	11	no
	19	−434/361	1	yes
	31	1850/961	943	yes
	37	1442/1369	2090	no
	43	910/1849	697	yes
	61	6146/3721	6794	no
	67	5134/4489	1	yes
	73	−7522/5329	1010	no
	79	11834/6241	6079	yes
	97	−8002/9409	1490	no
S_3	7	−34/49	22	no
	13	290/169	314	no
	19	134/361	107	no
	31	−898/961	470	no
	37	2690/1369	2714	no
	43	2726/1849	803	no
	61	−670/3721	3386	no
	67	2630/4489	1451	no
	73	−7294/5329	187	no
	79	−12226/6241	4118	no
	97	−18142/9409	385	no

The identification of C with the motive of $\eta(4\tau)^6$ agrees with the factorisation of the zeta function at $t = -1$ recorded by Beukers and Delaygue [BD22, §4]. They also identify, conjecturally, the factor of the zeta function belonging to W: its trace at p is $\left(\frac{-8}{p}\right)(\gamma_p^2 - 2p^2)$, where γ_p is the p-th coefficient of a newform $g \in S_3(\Gamma_0(256), \chi_{-4})$ [BD22, Conj. 1.4 and §4]. The shape of this formula suggests that S is related to a twist of the symmetric square of the motive of g; that W is not of CM type is part of the proof of Theorem 13.9. We return to this in Section 17.3.

15. Supersingular blocks of K3 type

The plane C of Theorem A is the simplest case of a general situation: a summand of K3 type of a motive with a Betti lift becomes supersingular at a prime, and its Betti signature differs from the signature that the Hodge-type conjecture predicts for algebraic classes. This section shows that Ogus's theorem and reciprocity always predict the reversal modulo four, and decide it in rank at most three, provided the Tate conjecture holds for the block.

15.1. The setting

Let M be a motive of weight $2m$, with m even, over a number field, realised in the cohomology of a smooth projective variety, with a Betti lift V_{B} that is a polarised Hodge structure of *K3 type*: of Hodge type $(m+1, m-1)+(m, m)^{\oplus(n-2)}+(m-1, m+1)$, with $h^{m+1,m-1} = 1$ and rank $n \geq 2$. As m is even, the form of the Hodge-type conjecture is $Q = (-1)^m\langle\,,\rangle = \langle\,,\rangle$, and by (5.4) V_{B} has signature $(n-2, 2)$ for it. (For m odd the same argument applies to $-\langle\,,\rangle$ after the corresponding change of Ogus's invariants; we do not need it.) Let p be a prime of good reduction for M, with $p > 2$, at which

- (K1) a lattice $\Lambda \subset V_{\mathrm{B}}$ preserved by the integral structure is unimodular for Q at p;
- (K2) the reduction of M is supersingular, and the crystalline realisation H of the reduction, with $\Phi = p^{-(m-1)}F$, is a K3 crystal of rank n in the sense of Ogus [Ogu79, Def. 3.1].

Condition (K2) holds for the reduction of a smooth projective model with torsion-free crystalline cohomology and degenerate Hodge spectral sequence, by Mazur's theorem, as in the proof of Proposition 13.5. The block is then potentially Tate, and its ℓ-adic realisations are $\Lambda \otimes \mathbb{Q}_\ell$, by smooth proper base change.

15.2. The reversal modulo four

Theorem 15.1. *In the setting of Section 15.1, suppose that the space Z of numerical classes in the reduction of M spans the Tate spaces, that is, $Z \otimes \mathbb{Q}_\ell \simeq M_\ell$ for $\ell \neq p$ and $Z \otimes \mathbb{Q}_p \simeq T_H \otimes \mathbb{Q}_p$. Then the form Q on Z has positive discriminant and Hasse invariant 1 at the real place, so that its number of negative entries is $s_-(Z) \equiv 0 \pmod 4$. If $n \leq 3$, Q is positive definite on Z.*

Proof. For $\ell \neq p$, $Z \otimes \mathbb{Q}_\ell \simeq \Lambda \otimes \mathbb{Q}_\ell$, so $\varepsilon_\ell(Z) = \varepsilon_\ell(V_{\mathrm{B}})$ and $d(Z) \equiv d(V_{\mathrm{B}})$ in $\mathbb{Q}_\ell^\times/\mathbb{Q}_\ell^{\times 2}$. A rational number that is a square in $\mathbb{Q}_\ell$ for all $\ell \neq p$ is a square (Lemma 5.4 and

its proof), so $d(Z) \equiv d(V_{\mathrm{B}})$ in $\mathbb{Q}^{\times}/\mathbb{Q}^{\times 2}$; as V_{B} has two negative entries, $d(Z) > 0$. By (K1), $\varepsilon_p(V_{\mathrm{B}}) = 1$, and by reciprocity for V_{B},

$$\prod_{\ell \neq p} \varepsilon_\ell(V_{\mathrm{B}}) = \varepsilon_\infty(V_{\mathrm{B}})\, \varepsilon_p(V_{\mathrm{B}}) = (-1)^{2\cdot 1/2} = -1.$$

By (K2) and Ogus's theorem [Ogu79, Prop. 3.13], $\varepsilon_p(Z) = \varepsilon_p(T_H \otimes \mathbb{Q}_p) = -1$. Reciprocity for Z then gives $\varepsilon_\infty(Z) = \varepsilon_p(Z) \prod_{\ell \neq p} \varepsilon_\ell(Z) = (-1)(-1) = 1$. By Lemma 5.3, $s_-(Z)$ is even because $d(Z) > 0$, and $s_-(Z)(s_-(Z) - 1)/2$ is even because $\varepsilon_\infty(Z) = 1$; together, $s_-(Z) \equiv 0 \pmod 4$. If $n \leq 3$ this forces $s_-(Z) = 0$. □

The Betti form has $s_- = 2$, and the theorem says that the algebraic classes have $s_- \equiv 0 \pmod 4$: the signature of the algebraic classes is never the Betti signature, and it agrees modulo four with the signature that the Hodge-type conjecture predicts. This is the modulo-four statement of Ancona and Marmora [AM25] for this class of blocks, obtained here without a CM lift, with the invariant at p supplied by Ogus. For $n = 2$ it is the conclusion of Proposition 13.6 under the additional hypothesis that both Tate classes are algebraic; Proposition 13.6 does not need it.

15.3. Without the Tate conjecture

Without the hypothesis that Z spans the Tate spaces, the argument of Proposition 13.6 extends to the following statement, whose proof is the same.

Proposition 15.2. *In the setting of Section 15.1, suppose $n = 2$, $V_{\mathrm{B}} \simeq \langle a, b\rangle$ with $a, b < 0$ and $p \nmid ab$, and $-ab$ not a square modulo p, that is, p inert in the CM field $\mathbb{Q}(\sqrt{-ab})$ of V_{B}. Then every algebraic class of the reduction that is not numerically trivial has positive square.*

Proof. Ogus's normal form gives $T_H \simeq \langle pu_2, pu_3\rangle$ with $-u_2u_3$ a nonsquare unit, so $u_2u_3 \equiv ab$ in $\mathbb{Q}_p^{\times}/\mathbb{Q}_p^{\times 2}$. For an algebraic class of square r, Lemma 5.2 at p gives $(r, -ab)_p = (r, -p^2u_2u_3)_p = \varepsilon_p(T_H) = -1$, and at $\ell \neq p$ it gives $(r, -ab)_\ell = (a, b)_\ell$. The sign law (Corollary 10.2) concludes. □

For $n = 3$ the same argument would need two algebraic classes, or one class and the square class of its complement; we do not know how to obtain the latter in general. In the motives of Part III the line L provides it, but L lies in the definite part S, not in a block of K3 type.

16. A complete example

We follow the motive H_2 at the primes 5 and 7 through every notion of the paper, and then the motives S_{33} and S_3 at their first split and inert primes. All numbers are exact; they are computed in the normalisation of Beukers, Cohen and Mellit (Appendix B), with $c = 1$.

16.1. The Betti lift

By Proposition 12.7, the Betti realisation splits as $S_{\mathrm{B}} \oplus C_{\mathrm{B}}$, with Gram matrices

$$\begin{pmatrix} 3 & 0 & -10 \\ 0 & 16 & 0 \\ -10 & 0 & 76 \end{pmatrix} \quad \text{and} \quad \operatorname{diag}(-4, -64).$$

Diagonalising, $S_{\mathrm{B}} \simeq \langle 3, 16, \frac{128}{3} \rangle \simeq \langle 3, 1, 6 \rangle$ and $C_{\mathrm{B}} \simeq \langle -1, -1 \rangle$ up to squares, so that

$$V_{\mathrm{B}} \simeq \langle 3, 1, 6, -1, -1 \rangle, \qquad d(V_{\mathrm{B}}) \equiv 2, \qquad \text{signature } (3, 2).$$

Its Hasse invariant at ∞ is $(-1, -1)_\infty = -1$; at every odd prime $\ell \neq 3$ all entries are units and $\varepsilon_\ell(V_{\mathrm{B}}) = 1$.

16.2. The prime 5

The characteristic polynomial of the normalised Frobenius on the summand is

$$x^5 + \tfrac{21}{25}x^4 + \tfrac{26}{125}x^3 + \tfrac{26}{125}x^2 + \tfrac{21}{25}x + 1 = (x+1)\Big(x^2 + \tfrac{6}{5}x + 1\Big)\Big(x^2 - \tfrac{34}{25}x + 1\Big).$$

The factor $x + 1$ is the line L, with eigenvalue $\left(\frac{-2}{5}\right) = -1$. The factor $x^2 + \frac{6}{5}x + 1$ is the plane C: its trace $-\frac{6}{5} = \frac{a_5}{5}$ with $a_5 = -6$ the fifth coefficient of $\eta(4\tau)^6$; its roots have 5-adic valuations ± 1, so its Newton slopes are 1 and 3, and its Newton–Hodge type is the Hodge type $(3, 1) + (1, 3)$, as Proposition 7.2 predicts at a prime split in $\mathbb{Q}(i)$. The factor $x^2 - \frac{34}{25}x + 1$ is the transcendental plane W, of slopes 0 and 4. The Tate part is L alone, and by Proposition 13.4 an algebraic class on it has square in $u\,\mathbb{Q}^{\times 2}$ with

$$u = d_S(4 - t_W^2) = 2048 \cdot \tfrac{1344}{625} \equiv 42.$$

So the Hodge-type conjecture holds for the summand at 5, whatever the Tate conjecture says. The blocks C and W are not Tate; AHC I(a) predicts for each of them a reciprocity law for its hermitian classes (Proposition 20.5), which is open (Section 22.4).

16.3. The prime 7

At $q = 7$, inert in $\mathbb{Q}(i)$,

$$x^5 + \tfrac{79}{49}x^4 + \tfrac{30}{49}x^3 - \tfrac{30}{49}x^2 - \tfrac{79}{49}x - 1 = (x+1)(x-1)(x+1)\Big(x^2 + \tfrac{30}{49}x + 1\Big).$$

The line L has eigenvalue $\left(\frac{-2}{7}\right) = -1$; the plane C has eigenvalues ± 1, a reflection, as Lemma 13.7 predicts; W has trace $-\frac{30}{49}$ and slopes 0 and 4. The Tate part $A = L \oplus C$ has rank three: this is a wall.

Without the involution. The Tate part does not split under Frobenius, and the local data of A are not determined by the Betti lift alone. In the notation of

Section 9.2, $\Delta = t_W^2 - 4 = -\frac{8704}{2401}$, so $-\Delta \equiv 34$ and the rational model of W is $N^0 = \langle 1, 34\rangle$, positive, with $s_N = 1$. Then $d_A = d(V_{\mathrm{B}})/d(N^0) \equiv 2 \cdot 34 \equiv 17$, and

$$\kappa_\infty = \varepsilon_\infty(V_{\mathrm{B}})\,\varepsilon_\infty(N^0)\,(17,34)_\infty = -1, \qquad \kappa_7 = \varepsilon_7(V_{\mathrm{B}})\,\varepsilon_7(N^0)\,(17,34)_7 = 1,$$

the latter because all the numbers involved are units at 7. With the crystalline invariant $\varepsilon_7(A) = -1$ (Proposition 13.5 and the computation in the proof of Corollary 19.1), the alien parity (9.4) reads $(-1)(-1)(1)(1) = 1$ and holds. By Theorem 9.7, an algebraic class in the plane V^+ of A is positive if and only if $\prod_{\ell\neq 7}\eta_\ell = 1$, the Alien Hodge condition for W. This is where the theory stood before the Levelt splitting: an equivalence, with neither side known.

With the involution. The Levelt involution splits A as $L \oplus C$, with global local data on each piece. On L the square class is $u = 2048 \cdot \frac{8704}{2401} \equiv 17$, positive. On C the ℓ-adic forms are those of $\langle -1, -1\rangle$ and the crystalline Tate module is $\langle 7u_2, 7u_3\rangle$ with $-u_2u_3$ a nonsquare modulo 7; the sign law gives, for an algebraic class of square r, $(r,-1)_7 = -1$, $(r,-1)_2 = (-1,-1)_2 = -1$ and $(r,-1)_\ell = 1$ elsewhere, hence $(r,-1)_\infty = 1$ and $r > 0$. The form on algebraic classes is positive definite, although the Betti form on C_{B} is negative. Reading the wall equivalence backwards (Corollary 19.1), $\prod_{\ell\neq 7}\eta_\ell = 1$: the transcendental block W at 7 satisfies the Alien Hodge condition, a statement about the hermitian classes of W at the finitely many primes where they can be nontrivial, among them 2, 3 and 17, which we do not know how to verify directly.

16.4. The motive S_{33}

For S_{33} the Gram matrix of S_{B} has the leading minors 4, 72 and 648. Hence $S_{\mathrm{B}} \simeq \langle 4, 18, 9\rangle$, which is $\langle 1, 2, 1\rangle$ up to squares, and $d_S \equiv 2$; the plane C_{B} is $\langle -2, -18\rangle \simeq \langle -2, -2\rangle$, of discriminant $36 \equiv 1$, with $K = \mathbb{Q}(i)$. Thus $V_{\mathrm{B}} \simeq \langle 1, 2, 1, -2, -2\rangle$, of signature $(3, 2)$ and discriminant $8 \equiv 2$.

The split prime 13. The characteristic polynomial of ϕ is

$$(x+1)\Big(x^2 + \tfrac{10}{13}x + 1\Big)\Big(x^2 - \tfrac{14}{169}x + 1\Big).$$

The line L has eigenvalue -1. The plane C has trace $-\frac{10}{13}$, of 13-adic valuation -1, so its slopes are 1 and 3; it is a rotation, as Lemma 13.7 requires at a prime split in $\mathbb{Q}(i)$. The plane W has $t_W = \frac{14}{169}$, of valuation -2, slopes 0 and 4. Here $4 - t_W^2 = \frac{114048}{28561}$ with $114048 = 2^7 \cdot 3^4 \cdot 11$, so $u = d_S(4 - t_W^2) \equiv 2 \cdot 22 \equiv 11$: algebraic classes on the Tate line have squares in $11\mathbb{Q}^{\times 2}$.

The inert prime 7. The characteristic polynomial is

$$(x-1)(x+1)^2\Big(x^2 - \tfrac{26}{49}x + 1\Big).$$

The plane C is a reflection, with eigenvalues 1 and -1, and the line has eigenvalue -1; the Tate part $L \oplus C$ has rank three. On L, $4 - t_W^2 = \frac{8928}{2401}$ with $8928 = 2^5 \cdot 3^2 \cdot 31$, so $u \equiv 2 \cdot 62 \equiv 31$. On C, with $c = 1$, an algebraic class of square r satisfies $(r,-36)_\ell = (-2,-18)_\ell$ at $\ell \neq 7$. As $-36 \equiv -1$ and $(-2,-18)_\ell = (-2,-2)_\ell = (-2,-1)_\ell$, which is -1 at $\ell = 2$ and 1 at every odd ℓ, the conditions read $(r,-1)_2 = -1$ and $(r,-1)_\ell = 1$ at the odd $\ell \neq 7$; and $(r,-1)_7 = -1$ by Proposition 13.5, -1 being a nonsquare modulo 7. Reciprocity gives $(r,-1)_\infty = 1$, so $r > 0$.

Without the involution, the wall would be treated as in Section 16.3. The rational model of W is $N^0 = \langle 1, -\Delta\rangle \simeq \langle 1, 62\rangle$, since $-\Delta = 4 - t_W^2 \equiv 62$; then $d_A \equiv 2 \cdot 62 \equiv 31$, $\kappa_\infty = \varepsilon_\infty(V_{\mathrm{B}}) = (-2,-2)_\infty = -1$, and $\kappa_7 = 1$, all the numbers

involved being units at 7. With $\varepsilon_7(A) = -1$ the alien parity (9.4) holds, and the wall equivalence applies; the Levelt splitting resolves the wall, and, as for H_2, the positivity transfers back to the Alien Hodge condition for W at 7.

The inert prime 11. Here $11 \not\equiv 1 \pmod 6$ and the traces are computed over $\mathbb{F}_{121}$ (Appendix B): the characteristic polynomial of ϕ^2 is $(x-1)^3\big(x^2 - \frac{21794}{14641}x + 1\big)$. The trace of ϕ^2 on W is $t_W^2 - 2$, so $t_W^2 = \frac{21794}{14641} + 2 = \big(\frac{226}{121}\big)^2$ and $t_W = \pm\frac{226}{121}$. The sign is not determined by ϕ^2, and it is not needed: the square class of the line depends only on t_W^2, and $4 - t_W^2 = \frac{7488}{14641}$ with $7488 = 2^6 \cdot 3^2 \cdot 13$ gives $u \equiv 2 \cdot 13 = 26$. The plane C is a reflection (Lemma 13.7), so ϕ^2 is the identity on it, in accordance with the factor $(x-1)^3$.

16.5. The motive S_3

For S_3 the leading minors of the Gram matrix of S_{B} are $5, 120, 3456$, so $S_{\mathrm{B}} \simeq \langle 5, 24, \frac{144}{5}\rangle$ and $d_S = 3456 = 2^7 \cdot 3^3 \equiv 6$; the plane C_{B} is $\langle -4, -48\rangle \simeq \langle -1, -3\rangle$, of discriminant $192 \equiv 3$, with $K = \mathbb{Q}(\sqrt{-3})$. A prime $q \geq 5$ is inert in K if and only if $q \equiv 2 \pmod 3$, and then $q \not\equiv 1 \pmod 6$, so at the inert primes the traces are always computed through ϕ^2.

The split prime 7. The characteristic polynomial of ϕ is

$$(x-1)\Big(x^2 - \tfrac{2}{7}x + 1\Big)\Big(x^2 + \tfrac{34}{49}x + 1\Big).$$

The line has eigenvalue $+1$; the plane C has trace $\frac{2}{7}$ and slopes 1 and 3; the plane W has $t_W = -\frac{34}{49}$, and $4 - t_W^2 = \frac{8448}{2401}$ with $8448 = 2^8 \cdot 3 \cdot 11$, so $u \equiv 6 \cdot 33 \equiv 22$.

The split prime 13. The characteristic polynomial is $(x+1)\big(x^2 + \frac{22}{13}x + 1\big)\big(x^2 - \frac{290}{169}x + 1\big)$; here $4 - t_W^2 = \frac{30144}{28561}$ with $30144 = 2^6 \cdot 3 \cdot 157$, and $u \equiv 6 \cdot 471 \equiv 314$.

The inert prime 11. The characteristic polynomial of ϕ^2 is $(x-1)^3\big(x^2 - \frac{23618}{14641}x + 1\big)$, so $t_W^2 = \big(\frac{230}{121}\big)^2$, $4 - t_W^2 = \frac{5664}{14641}$ with $5664 = 2^5 \cdot 3 \cdot 59$, and $u \equiv 6 \cdot 354 \equiv 59$. On the plane C, an algebraic class of square r satisfies $(r, -3)_\ell = (-1, -3)_\ell$ at $\ell \neq 11$, since $-192 \equiv -3$; the right-hand side is 1 at $\ell = 2$, by (5.2) and $-3 \equiv 1 \pmod 4$, and -1 at $\ell = 3$, where $(-1, -3)_3 = (-1, 3)_3 = \big(\frac{-1}{3}\big)$. At 11, -3 is not a square, and Proposition 13.5 gives $(r, -3)_{11} = -1$. Reciprocity gives $(r, -3)_\infty = (-1)(-1) = 1$, so $r > 0$. The value $r = 11$ satisfies all the conditions (Appendix D.2).

16.6. What the example shows

At the split prime 5 the Hodge-type conjecture on the summand reduces to a square-class computation on a line. At the inert prime 7 it involves a wall of rank three, which the local data alone do not decide, and which the involution splits into a line and a CM plane. On the plane the sign of algebraic classes is the Newton–Hodge sign, opposite to the Betti sign. The positivity, once proved, gives information about the transcendental block, through reciprocity. All four phenomena (the line, the reversal, the wall equivalence, and the information transferred to W) are visible at one prime of one motive.

17. Comparison with known results

17.1. The rank-two theorem of Ancona

The closest known result is Theorem 8.1 of [Anc21], recalled in Section 4.1. Applied to the block C, it would require C to be the realisation of a relative Chow motive over the ring of integers of a q-adic field, with a quadratic form, and the space of numerical classes of C on the special fibre to have dimension two. The first condition holds under Hypothesis 12.4, since C is cut out by the algebraic projector $\frac{1}{2}(e - e\sigma e)$ from a smooth projective model. The second condition says that both Tate classes of C are algebraic, which is the Tate conjecture for C. When it holds, Ancona's theorem and Proposition 13.6 give the same conclusion. Proposition 13.6 is proved one class at a time, by the sign law; it applies when the numerical classes of C span a line, and it applies without knowing their dimension at all. In this sense the rank-two part of Theorem A extends Ancona's theorem from $\dim V_Z = 2$ to $\dim V_Z \leq 2$ for these motives, by a different route: Ancona computes the q-adic periods of a lift through p-adic Hodge theory, whereas Proposition 13.5 reads the q-adic invariant from the crystalline Tate module of the special fibre.

17.2. CM lifts, K3 surfaces and Fermat varieties

Ancona and Marmora [AM25] determine the discriminant and the Hilbert symbol of the intersection form for varieties with a CM lifting; this gives the signature modulo four. In the language of Lemma 5.3, their theorem computes the finite invariants of the space of algebraic classes, and Theorem 10.3 turns such a computation into positivity whenever the space has rank at most three. Our motives have no CM lift: the part S is not of CM type, by the proof of Theorem 13.9. Ito, Ito and Koshikawa [IIK25] treat self-products of K3 surfaces, using CM liftings and the Kuga–Satake construction; Harashita [Har26] treats Hermitian varieties, whose middle cohomology is spanned by linear subspaces. The motives (1.1) are not Fermat or Jacobi-sum motives: their transcendental part W is not of CM type.

17.3. The question of abelian type

The definite part S_{B}, twisted by $\mathbb{Q}(1)$, is a rational Hodge structure of K3 type of rank three. By the Kuga–Satake construction it embeds, as a Hodge structure, into the endomorphisms of the cohomology of an abelian surface with quaternionic multiplication. If this embedding were induced by an algebraic correspondence between X and an abelian variety, compatible with the intersection forms, then Hdg for the summand S could in principle be reduced to Ancona's theorem for abelian fourfolds or to its extensions. No such correspondence is known. The conjectural relation of S with the symmetric square of a weight-three newform of level 256 (Section 14.4) points in a different direction: if it were realised by a correspondence with the square of an elliptic modular surface, the problem for S would become Hdg for the square of a surface whose transcendental part is not of CM type and which is not a K3 surface, and this is not known either. We know no reduction of Hdg for the summand $\mathcal{M}$ of X to a case settled before. Since our arguments do not need one, we do not pursue this.

17.4. The Tate conjecture and rational Tate classes

Over the algebraic closure of a finite field, the Hodge-type conjecture for all varieties follows from the full Tate conjecture, by an argument of Milne through the motives of abelian varieties [Mil15, 3.22]; for abelian varieties it follows from the Hodge conjecture for complex abelian varieties of CM type [Mil02]. His theory of rational Tate classes [Mil09b] constructs, for abelian varieties over finite fields and assuming only the Hodge conjecture for CM abelian varieties, a $\mathbb{Q}$-structure on the spaces of Tate classes that is compatible with the theories over all fields and that would contain the algebraic classes. Part (b) of AHC I is a conjecture of the same shape for arbitrary motives, and part (a) asks for $\mathbb{Q}$-structures, with the right signatures, on the blocks that are not potentially Tate as well. For the three motives of Part III our argument proves the positivity without constructing such a structure on the whole Tate part; for the plane C it then produces one (Theorem 20.11).

17.5. The Hodge conjecture

The Tate part of the motives (1.1) at an inert prime is the reduction of a part of the Betti realisation that is not of Hodge type $(2,2)$: the plane C_{B} is of type $(3,1)+(1,3)$. The algebraic classes in C therefore do not lift to algebraic classes in characteristic zero, even as Hodge classes, and no argument through the Hodge conjecture on a lift can reach them; they are exotic in the sense of Ancona [Anc21], Tate classes of the reduction that are not reductions of Hodge classes. The line L is different: the eigenvalue of Frobenius on it is $\left(\frac{-2}{q}\right)$ in the normalisation of Appendix B, a quadratic character, which is what one expects if L is the reduction of a Hodge class of S_{B} defined over $\mathbb{Q}(\sqrt{-2})$. We do not use this, and the argument of Proposition 13.4 does not depend on it.

17.6. What is new

The following statements are, to our knowledge and within the literature searched for Section 4, not previously in the literature.

(1) The Levelt splitting of Proposition 12.7: at the fixed point of the inversion symmetry, an algebraic involution splits a wall of rank three of a hypergeometric motive into a line and a CM plane whose local invariants are global, so that the line is decided by its square class and the plane by the sign law.
(2) Theorem A: the numerical Hodge-type conjecture for the summand of three hypergeometric motives of Hodge type $(1,1,1,1,1)$ that are not of CM type and are not known to be reached from abelian varieties or K3 surfaces by an algebraic correspondence, at all primes outside a finite set and a set of density zero, without the Tate conjecture and without a CM lift, under Hypothesis 12.4.
(3) The location of the walls of these motives at the primes inert in the CM field of the plane C (Lemma 13.7), and the sign reversal of Theorem 14.1 as a phenomenon in which the sign of algebraic classes is the Newton–Hodge sign.
(4) The framework of Alien Hodge Motives and its principles: the Tate-free criterion and the wall equivalence of Section 9, the sign law in rank two and the decidability dichotomy of Section 10, and the one-bit reduction of rank-four walls.

Item (2) rests on Hypothesis 12.4; for H_2 its part (H2) is explicit, and for S_{33} and S_3 it is assumed.

18. The frontier in rank seven

18.1. The four motives

By Proposition 12.9 there are four symmetric hypergeometric data of rank seven and weight four, up to the exchange of α and β. Table 6 lists them in the normalisation of the scan, with the Levelt splitting $M_{\mathrm{B}} = S_{\mathrm{B}} \oplus R_{\mathrm{B}}$ of the Betti realisation into eigenspaces of ranks three and four, given by diagonal representatives of the invariant form.

	parameters $\alpha;\beta$	Hodge numbers
0	$0^5, 1/3, 2/3;\ 1/6, (1/2)^5, 5/6$	$(2,1,1,1,2)$
1	$0^3, 1/10, 3/10, 7/10, 9/10;\ 1/5, 2/5, (1/2)^3, 3/5, 4/5$	$(2,1,1,1,2)$
2	$0, (1/3)^3, (2/3)^3;\ (1/6)^3, 1/2, (5/6)^3$	$(1,1,3,1,1)$
3	$0, 1/10, 1/6, 3/10, 7/10, 5/6, 9/10;\ 1/5, 1/3, 2/5, 1/2, 3/5, 2/3, 4/5$	$(2,1,1,1,2)$

	S_{B}	R_{B}	signature of R_{B}	F_R
0	$\langle 85, 17, 85\rangle$	$\langle 1, 51, -2, -51\rangle$	$(2,2)$	$\mathbb{Q}(\sqrt{2})$
1	$\langle 7, 7, 7\rangle$	$\langle 1, 42, -1, -105\rangle$	$(2,2)$	$\mathbb{Q}(\sqrt{10})$
2	$\langle 1, -1, -2\rangle$	$\langle 1, 3, 1, 6\rangle$	$(4,0)$	$\mathbb{Q}(\sqrt{2})$
3	$\langle 34, 510, 510\rangle$	$\langle 1, 255, -30, -255\rangle$	$(2,2)$	$\mathbb{Q}(\sqrt{30})$

TABLE 6. The symmetric hypergeometric motives of rank seven and weight four, their Levelt splitting, and the discriminant field $F_R = \mathbb{Q}(\sqrt{d(R_{\mathrm{B}})})$ of the rank-four block.

The Levelt involution exists in all four cases, and its two eigenspaces are orthogonal. In contrast with rank five, the blocks are not both definite. For data 0, 1, 3, of Hodge type $(2,1,1,1,2)$ and Betti signature $(5,2)$, the block S_{B} is positive definite and R_{B} has signature $(2,2)$; for datum 2, of Hodge type $(1,1,3,1,1)$, R_{B} is positive definite and S_{B} has signature $(1,2)$. The argument of Section 13 applies to pieces of rank at most three with known invariants, and the block R of rank four is not of this kind: its finite invariants do not determine its signature (Proposition 10.5), and its Hasse invariants as a sum of Frobenius blocks are not global.

18.2. The discriminant field and the walls

At a good prime q, let $\lambda_S = \det(\phi|S_\ell)$, the eigenvalue of ϕ on the Tate line of S. The determinant of ϕ on the whole motive is $\lambda_S \det(\phi|R_\ell)$, and it is the sign of the functional equation. By Lemma 10.6, when $\det(\phi|R_\ell) = -1$ the block R contains two potentially Tate lines L_+ and L_-, and the Tate part of the motive is $L_S \oplus L_+ \oplus L_-$, a wall of rank three.

Observation 18.1. For each of the four data, the exact characteristic polynomials of Frobenius at the primes listed in Appendix B are consistent with

$$\det(\phi|R_\ell) = \chi_{F_R}(q),$$

the quadratic character of the discriminant field. More precisely: at every computed prime with $\chi_{F_R}(q) = 1$ the Tate multiplicity of the motive is one,

which forces $\det(\phi|R_\ell) = 1$; at every computed prime with $\chi_{F_R}(q) = -1$ the Tate multiplicity is three and the three eigenvalues ± 1 are not all equal, which is consistent with $\det(\phi|R_\ell) = -1$ but does not determine it, since the eigenvalues ± 1 cannot be attributed to S and R from the characteristic polynomial alone.

The primes in question are, for data 0 to 3: 13, 6, 12 and 2 primes at which the sign of the functional equation is determined by the computed power sums, of which 6, 3, 5 and 0 have $\chi_{F_R}(q) = 1$ (Table 7). For datum 3 the two computed primes $q = 31$ and $q = 61$ both have $\chi_{F_R}(q) = -1$; at $q = 31$ the eigenvalues ± 1 are $1, -1, -1$, and the observation holds there only if $\lambda_S = -1$. Principle 8 in rank four is therefore an observation and not a theorem. Saito's theorem [Sai94] concerns the determinant on the whole middle cohomology of a variety, with the de Rham discriminant; it does not apply blockwise to a summand cut out by a correspondence, and the Betti discriminant of a block need not agree with the de Rham one.

18.3. One bit

At a prime where $\det(\phi|R_\ell) = -1$, write $R_\ell = L_+ \oplus L_- \oplus U$ with U a binary non-Tate block. Since R_{B} has positive determinant in all four cases, Proposition 10.8 applies: algebraic classes on L_+ and L_- have the same sign. The line L_S is decided by Proposition 13.4, which only uses that S_{B} is definite; this holds for data 0, 1, 3. At such a prime, therefore, the Tate part is decided up to one sign, the common sign of the algebraic classes on L_+ and L_-, and by Theorem 10.9 one algebraic class of known sign on $L_+ \oplus L_-$ would decide it. Without such a class, the finite local data are blind (Proposition 10.5).

An anchor could come from a cycle on the special fibre that lifts to characteristic zero as an algebraic cycle, whose sign is then fixed by the Hodge–Riemann relations on the lift, or from a correspondence with a motive whose Tate part is already decided. When R has real multiplication by a real quadratic field F, the two lines would be the traces of one F-line and the missing sign would be the sign at one real place of F (Lemma 10.10). The exact computations show that R does not have real multiplication by a fixed field: the discriminant of the quadratic factor of the trace polynomial on R varies with q. The relevant quadratic field is the discriminant field F_R, which governs the determinant and not a multiplication.

18.4. An example

Take datum 0 of Table 6 at $q = 13$, where $\chi_{F_R}(13) = \left(\frac{2}{13}\right) = -1$. The exact characteristic polynomial of the normalised Frobenius on the fibre at $t = -1$ is

$$(x-1)^2(x+1)\Big(x^2 + \tfrac{6}{13}x + 1\Big)\Big(x^2 + \tfrac{334}{169}x + 1\Big).$$

The factor U with trace $-\frac{6}{13}$, of 13-adic valuation -1, has Newton slopes 1 and 3; the factor W with trace $-\frac{334}{169}$ has slopes 0 and 4. Each of S and R contains one of them: S has rank three, and its Tate part has rank at most one at a prime where its transcendental plane is not potentially Tate, while R has rank four and its Tate part has even rank. Mazur's inequality on each summand, with the Hodge polygons $(0, 2, 4)$ for S and $(0, 1, 3, 4)$ for R, allows both attributions, $W \subset S$ and $U \subset R$, or $U \subset S$ and $W \subset R$, and the characteristic polynomial does not decide between them. Nor does it decide how the eigenvalues $1, 1, -1$

are distributed: either $\lambda_S = 1$ and the Tate plane $\mathrm{T}(R)$ is $L_+ \oplus L_-$, which is the prediction $\det(\phi|R_\ell) = \chi_{F_R}(13) = -1$; or $\lambda_S = -1$ and $\phi = 1$ on $\mathrm{T}(R)$. This is why Observation 18.1 is an observation.

Two conclusions hold in every case. First, the line L_S of S is decided: $S_{\mathrm{B}} \simeq \langle 85, 17, 85\rangle$ is positive definite with $d_S \equiv 17$, and the complement of L_S in S_ℓ is a non-Tate binary block, so Proposition 13.4 applies and the square class of an algebraic class on L_S is $17(4 - t^2)$, with t the trace of that block. As $4 - t_W^2 = \frac{2^7 \cdot 3 \cdot 7}{13^4}$ and $4 - t_U^2 = \frac{2^7 \cdot 5}{13^2}$, this is $17 \cdot 42 \equiv 714$ or $17 \cdot 10 = 170$, positive in both cases. Second, if $\mathrm{T}(R) = L_+ \oplus L_-$, its square class is $d(R_{\mathrm{B}})$ times that of the non-Tate block of R, namely $2 \cdot 10 \equiv 5$ or $2 \cdot 42 \equiv 21$, positive in both cases, so by Proposition 10.8 algebraic classes on L_+ and L_- have the same sign. Nothing in the local data at $\ell \neq 13$ fixes that sign: $\mathrm{T}(R)$ is cut out of R_ℓ by Frobenius, not by a correspondence over $\mathbb{Q}$, and its Hasse invariants depend on the unknown hermitian class of the non-Tate block of R_ℓ. This is the one bit of Section 18.3, at one prime.

18.5. The four data one by one

For the three data of Hodge type $(2, 1, 1, 1, 2)$ the Betti realisation has signature $(5, 2)$: by (5.4) the types $(4, 0) + (0, 4)$ contribute four positive directions, $(3, 1) + (1, 3)$ two negative ones, and $(2, 2)$ one positive. The splitting puts a positive definite part of rank three into S and a part of signature $(2, 2)$ into R. As Hodge structures, S is then of type $(4, 0) + (2, 2) + (0, 4)$ and R of type $(4, 0) + (3, 1) + (1, 3) + (0, 4)$, by the same argument as in Proposition 12.7. The part S behaves as in rank five: at a prime where the transcendental plane of S is not potentially Tate, its Tate line is decided by Proposition 13.4. The part R contains both a plane of type $(4, 0) + (0, 4)$ and one of type $(3, 1) + (1, 3)$, and is not split further by any algebraic correspondence that we know.

For the datum of Hodge type $(1, 1, 3, 1, 1)$ the roles are exchanged: R is positive definite of rank four, and S has signature $(1, 2)$. As R is definite, it is of type $(4, 0) + (2, 2) + (0, 4)$ with $h^{2,2} = 2$, and S of type $(3, 1) + (2, 2) + (1, 3)$. At a prime where $\det(\phi|R_\ell) = -1$, the Tate part of R contains $L_+ \oplus L_-$, of positive discriminant, and Proposition 10.8 applies; the line of S, however, is no longer decided by its square class alone, since S is indefinite.

In all four cases the Hodge structure of R has $h^{4,0} = 1$, so R contains a transcendental part whose reduction cannot be entirely Tate at a prime where it is ordinary.

datum	$\chi_{F_R}(q) = 1$: Tate multiplicity 1	$\chi_{F_R}(q) = -1$: Tate multiplicity 3
0	7, 31, 73, 79, 103, 127	13, 19, 37, 43, 61, 67, 139
1	31, 41, 71	11, 61, 101
2	7, 31, 73, 97, 127	13, 19, 37, 43, 61, 67, 139
3	none	31, 61

TABLE 7. The rank-seven data: the computed primes at which the sign of the functional equation is determined, sorted by the value of the character of the discriminant field.

19. The walls of the hypergeometric families

19.1. The list

Remark 9.8 records 77 wall points of rank three in six hypergeometric families of Hodge type $(1,1,1,1,1)$, among the pairs (q,t) with $q<80$ prime and $t\in\mathbb{F}_q$. Table 8 lists them. At each of them the Tate part of the reduction of the motive at t is a space of rank three, the transcendental part is an ordinary binary block, and the local identity (9.2) holds with $\varepsilon_q=-1$; by Theorem 9.5, Hdg on the Tate part is equivalent to the Alien Hodge condition for the transcendental block.

TABLE 8. The wall points (q,t) of rank three, by family; t is given as an integer modulo q.

family	wall points (q,t)
$(k/6)_{k=1}^{5}$	$(13,12)$, $(19,8)$, $(37,31)$, $(37,36)$, $(43,16)$, $(61,34)$, $(61,41)$, $(73,10)$, $(73,64)$, $(79,41)$
$(1/2)^5$	$(7,3)$, $(7,5)$, $(7,6)$, $(11,10)$, $(17,4)$, $(17,13)$, $(19,2)$, $(19,10)$, $(19,18)$, $(23,17)$, $(23,19)$, $(23,22)$, $(29,16)$, $(29,20)$, $(31,5)$, $(31,6)$, $(31,25)$, $(31,26)$, $(37,9)$, $(37,33)$, $(43,32)$, $(43,39)$, $(43,42)$, $(47,10)$, $(47,33)$, $(47,35)$, $(47,39)$, $(47,41)$, $(47,43)$, $(47,46)$, $(53,13)$, $(53,28)$, $(53,36)$, $(53,49)$, $(59,58)$, $(67,7)$, $(67,48)$, $(67,50)$, $(67,63)$, $(67,66)$, $(71,26)$, $(71,41)$, $(71,70)$, $(73,18)$, $(73,69)$, $(79,3)$, $(79,6)$, $(79,14)$, $(79,17)$, $(79,27)$, $(79,41)$, $(79,53)$, $(79,63)$, $(79,66)$, $(79,74)$, $(79,78)$
$(1/4,1/2,1/2,1/2,3/4)$	$(13,9)$, $(29,20)$, $(37,7)$, $(41,18)$, $(73,32)$
$(1/8,3/8,1/2,5/8,7/8)$	$(73,9)$
$(1/4,1/3,1/2,2/3,3/4)$	$(37,30)$, $(37,33)$, $(61,25)$, $(73,12)$, $(73,69)$

19.2. The walls at the fixed point

Ten of the walls of the family $(\frac{1}{2})^5$ lie at $t\equiv -1$: the points $(q,q-1)$ for

$$q\in\{7,11,19,23,43,47,59,67,71,79\}.$$

These are exactly the primes $q<80$ with $q\equiv 3 \pmod 4$, except $q=31$. This is what Theorem A predicts: at $t=-1$ the Tate part is $L\oplus C$, of rank three, at the primes inert in $\mathbb{Q}(i)$, unless $q\in T$; and $31\in T$, where the Tate part has rank five (Section 13.8). At the ten points the wall is resolved by Theorem A, under Hypothesis 12.4 and for $q\notin E$: the Levelt involution splits the Tate part into L and C, and both carry positive algebraic classes.

Corollary 19.1. *Assume Hypothesis 12.4 for* H_2. *At each of the ten wall points* $(q,-1)$ *above with* $q\notin E$, *the intersection form is positive definite on the numerical classes of codimension two in the hypergeometric summand of the reduction, and the Alien Hodge condition holds for the transcendental block.*

Proof. The first statement is Theorem A, since the ten primes are not exceptional (Appendix B). For the second we compute $\beta(A)$ for the Tate part $A=L\oplus C$, without using algebraic classes. For $\ell\neq q$, $A_\ell\simeq G\otimes\mathbb{Q}_\ell$ with $G=\langle u,a,b\rangle$, where

$u = c\,d_S(4 - t_W^2) > 0$ and $\langle a, b\rangle$ is the form on C, $a, b < 0$ (Propositions 12.7 and 13.4). Since $q \notin E$, the numbers a, b, c, d_S are q-adic units, and so is $4 - t_W^2$ up to the factor q^{-4}, because W is ordinary and its trace $q^2 t_W$ is prime to q; so $\varepsilon_q(G) = 1$, and by reciprocity $\prod_{\ell \neq q} \varepsilon_\ell(A) = \varepsilon_\infty(G)\varepsilon_q(G) = -1$. At q, the crystalline Tate space of A is the orthogonal sum of a line and of $T_H \otimes \mathbb{Q}_q \simeq \langle qu_2, qu_3\rangle$ (Proposition 13.5), whose Hasse invariant is -1 and whose discriminant u_2u_3 is a unit square modulo q, since $q \equiv 3 \pmod 4$; by Lemma 5.1 the Hasse invariant of the sum is -1, whatever the line. Hence $\beta(A) = (-1)(-1) = 1$, and by Corollary 9.6, whose local identity holds at these points, W satisfies the Alien Hodge condition. □

The wall equivalence here runs in the direction opposite to the one in which it is usually applied: the positivity, proved by the Levelt splitting, implies a statement about the hermitian classes of the transcendental block W, namely that $\prod_{\ell \neq q} \eta_\ell = 1$ (Proposition 20.5). This is a non-trivial reciprocity statement about W that we would not know how to prove directly.

19.3. The other walls

The remaining 67 walls are not at a fixed point of an involution that we know. For the family $(\frac{1}{2})^5$ at $t \neq -1$, and for the other five families, the Tate part of rank three does not split. The Dwork family accounts for ten of them. A search for further symmetries, by a scan for partial complex multiplication among the rational points of height at most twelve in the six families (and at most forty for the Dwork family), found none outside the fixed point of the family $(\frac{1}{2})^5$. At these 67 points Hdg on the Tate part remains equivalent to the Alien Hodge condition, and is open.

19.4. An open wall in detail

We follow one of the open walls, the point $(q, t) = (13, 12)$ of the Dwork family $\alpha = (\frac{1}{6}, \dots, \frac{5}{6})$, $\beta = 0^5$, through the invariants of Section 9.2. The exact computation of Appendix B gives the characteristic polynomial

$$(x-1)^2(x+1)\Big(x^2 + \tfrac{175}{169}x + 1\Big)$$

of the normalised Frobenius on the fibre at $t \equiv 12 \equiv -1$. The quadratic factor is the transcendental block W: its trace $t_W = -\frac{175}{169}$ has 13-adic valuation -2, so W is ordinary, of Newton slopes 0 and 4. The Tate part A has rank three, and Frobenius splits it into the plane V^+ on which it acts as the identity and the line L on which it acts by -1. This is the shape of Section 9.2, but the splitting is by the eigenvalues of Frobenius on ℓ-adic spaces, not by a correspondence defined over $\mathbb{Q}$: the Betti lift fixes the invariants of A, and not those of L and V^+ separately.

The numbers are as follows. One has $4 - t_W^2 = \frac{3^3 \cdot 19 \cdot 163}{13^4}$, so $-\Delta \equiv 3 \cdot 19 \cdot 163 = 9291$, and the rational model of W is $N^0 = \langle 1, 9291\rangle$, positive, with $s_N = 1$. The invariant symmetric form of the hypergeometric group of the datum, for the companion matrices of $x^5 + x^4 + x^3 + x^2 + x + 1$ and $(x-1)^5$ and scaled to be integral, is the symmetric Toeplitz matrix $S = (s_{|i-j|})$ with $(s_0, \dots, s_4) = (265, 151, -119, -329, -119)$, of signature $(3, 2)$ and discriminant $\equiv 1$; the cup product is cS with $c > 0$ (Proposition 12.6 applies verbatim). The kernel certificate uses the representative $V_{\mathrm{B}} \simeq \langle 1590, 130910, -247, -1, 2\rangle$, which is $6S$ up to

isometry: it has discriminant $\equiv 6$ and the same Hasse invariants as S at every place, as a scaling of a form of rank five must. A change of c multiplies κ_ℓ by $(c, 9291)_\ell$, and the product of these over $\ell \neq 13$ is $(c, 9291)_\infty (c, 9291)_{13} = 1$ when $c > 0$ is a unit at 13; so nothing below depends on c. With the representative of the certificate, $d_A = d(V_{\mathrm{B}})/d(N^0) \equiv 6 \cdot 9291 \equiv 2 \cdot 19 \cdot 163 = 6194 > 0$. The invariants κ_ℓ of (9.1) are 1 at every prime outside $\{2, 3, 5, 19, 53, 163\}$, and

v	2	3	5	19	53	163	∞
$\varepsilon_v(V_{\mathrm{B}})$	-1	1	1	1	1	1	-1
$(d_A, 9291)_v$	-1	-1	1	-1	1	-1	1
κ_v	1	-1	1	-1	1	-1	-1

with $\varepsilon_v(N^0) = 1$ at every place listed, and $\kappa_{13} = 1$, all the numbers involved being units at 13. The product of the κ_ℓ over $\ell \neq 13$ is -1, in accordance with (9.3), and with $\varepsilon_{13}(A) = -1$ the local identity (9.2) reads $(-1)(-1)\cdot 1 = 1$. This is the identity that the kernel checks at this point (`AlienHodgeWalls.wall_D6_13_12`, Appendix D.3), here recomputed independently.

By Theorem 9.5, if L carries an algebraic class of positive square, then an algebraic class of V^+ is positive if and only if $\prod_{\ell \neq 13} \eta_\ell = 1$, where η_ℓ compares the hermitian line W_ℓ over $\mathbb{Q}_\ell(\sqrt{-9291})$ with that of N^0. The signs η_ℓ are 1 at every odd prime $\ell \neq 13$ that is unramified in $\mathbb{Q}(\sqrt{-9291})$ and at which W_ℓ is unimodular, since two unimodular binary lattices of the same discriminant have the same Hasse invariant; so they can be nontrivial only at $2, 3, 19, 163$ and at the primes at which the integral structure of the family fails. We cannot compute them: they depend on the ℓ-adic Galois representation W_ℓ, not only on its characteristic polynomial. There is no Levelt involution here, since $\alpha + \frac{1}{2} \not\equiv \beta$, and we know no other correspondence that would separate L from V^+ with global invariants. This is the obstruction that Section 23.1 records.

Part IV

The Alien Hodge Conjectures and the two summits

20. Alien Hodge Conjecture I

20.1. The statement

Let X be a smooth projective variety of dimension $2m$ over $\mathbb{F}_q$, $q = p^a$ with p odd, and let ξ be an ample class. Write $P_\ell \subset H^{2m}_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(m))$ and $P_D \subset H^{2m}_{\text{cris}}(X_{\overline{\mathbb{F}}_q}/W) \otimes K_0$ for the primitive parts, the kernels of $\cup\,\xi$, with the forms $(-1)^m\langle\,,\rangle$, and decompose them into Frobenius blocks as in Section 5.3, taking the potentially Tate part T as one block. For a block B that is not potentially Tate, with irreducible polynomial f, the field $E_B = \mathbb{Q}[x]/(f)$ is a CM field (the roots of f lie on the unit circle and are not real), and B_ℓ is a free $E_B \otimes \mathbb{Q}_\ell$-module of rank one on which the form is the trace of a hermitian form, ϕ acting as the class of x. Let $\mathcal{Z}_{\text{pr}}$ be the $\mathbb{Q}$-space of primitive codimension-m cycles on $X_{\overline{\mathbb{F}}_q}$ modulo numerical equivalence, with the form $(-1)^m(\,\cdot\,)$.

Conjecture 20.1 (Alien Hodge Conjecture I: Newton–Hodge signs).

(a) *For every Frobenius block B of P there is a rational quadratic space U_B with $U_B \otimes \mathbb{Q}_\ell \simeq B_\ell$ for every $\ell \neq p$, whose signature is the Newton–Hodge signature of B (Definition 5.9, taken for the form $(-1)^m\langle\,,\rangle$). If B is not potentially Tate, U_B is the trace form of an E_B-hermitian space whose localisations are the hermitian spaces B_ℓ. If $B = \mathrm{T}$, then moreover $U_{\mathrm{T}} \otimes \mathbb{Q}_p$ is isometric to the $\mathbb{Q}_p$-space $\{x \in \mathrm{T}_D : Fx = p^m x\}$.*

(b) (Rational skeleton.) *The space $\mathcal{Z}_{\text{pr}}$ embeds isometrically into U_{T}.*

For a Tate block the Newton–Hodge signature is definite and positive for the form $(-1)^m\langle\,,\rangle$, since all slopes equal m; so (a) asserts in particular that the local invariants of T, at all finite places, are those of a positive definite rational form. Part (b) is in the spirit of Milne's rational Tate classes [Mil09b]: a rational structure on the Tate classes for which the algebraic classes are rational. For the blocks that are not potentially Tate, (a) is a statement about realisations only, and it is the part of the conjecture that goes beyond the Hodge-type conjecture. At a wall of rank three it implies the Alien Hodge condition of Definition 9.4, by the computation that follows that definition.

Example 20.2 (*Supersingular K3 surfaces*). The analogue of AHC I in weight two holds for a supersingular K3 surface Y over $\mathbb{F}_q$ (Example 6.3). Its primitive cohomology is one potentially Tate block, of Newton–Hodge signature positive definite for $-\langle\,,\rangle$; the Tate conjecture [Cha13, MP15] gives a rational space of algebraic classes whose localisations are the Tate spaces, and the Hodge index theorem says that it is positive definite for $-\langle\,,\rangle$. Hence U_{T} can be taken to be the space of primitive algebraic classes itself, and both parts hold. The example is not evidence for part (a) in its non-trivial form, since there are no blocks that are not potentially Tate; it shows that the normalisations of the conjecture are the right ones.

Remark 20.3 (*Evidence*). Suppose X admits a lift with complex multiplication, in the sense that its motive over $\overline{\mathbb{F}}_q$ is the reduction of a CM motive over a number field. Then each block B is the reduction of a piece V_B of the Betti realisation of the lift on which the CM field acts, and V_B is a global hermitian space whose localisations at $\ell \neq p$ are the B_ℓ. Its signature is the Hodge signature of the lift. By the theorem of Shimura and Taniyama [Shi98], the Newton slopes of B are

determined by the CM type of the lift and the decomposition of p in the CM field; they agree with the Hodge indices of the lift when p splits completely, and not in general: for the plane C of Theorem A, which is of CM type, the Hodge type is $(3,1)+(1,3)$ and at an inert prime the slopes are $(2,2)$. When the two signatures differ, AHC I(a) asks for a global space other than V_B, with the same localisations at $\ell \neq p$ and with the Newton–Hodge signature; by reciprocity such a space differs from V_B at p. For Tate blocks this is the computation of Ancona and Marmora, who determine the discriminant and the Hasse invariant at p of the algebraic classes for CM-liftable varieties [AM25], and of Ancona's Theorem 8.1 in rank two. The motives of Part III show the same phenomenon for a block C whose ambient motive has no CM lift.

20.2. Local–global reformulation

Part (a) of AHC I is an existence statement for rational quadratic and hermitian spaces with prescribed local invariants, and the classical local–global theorems turn it into conditions on the local data alone. We record this for Tate blocks and for binary blocks, which are the cases met in this paper.

Proposition 20.4 (Tate blocks). *Let* T *be the potentially Tate part of P, of rank n, and suppose that its discriminants at all finite places, including the $\mathbb{Q}_p$-space $\mathrm{T}_p = \{x \in \mathrm{T}_D : Fx = p^m x\}$, are the images of one rational number d. Then* AHC I(a) *holds for* T *if and only if $d > 0$, $\beta(\mathrm{T}) = \prod_{v<\infty} \varepsilon_v(\mathrm{T}_v) = 1$, and, when $n \leq 2$, the local spaces satisfy the conditions that every form of rank n satisfies: $\varepsilon_v = 1$ for all v if $n = 1$, and $\varepsilon_v = 1$ at every place where $-d$ is a square if $n = 2$.*

Proof. A rational quadratic space with the localisations T_v and positive definite has discriminant d modulo squares, so $d > 0$, and Hasse invariant 1 at the real place, so $\prod_{v<\infty} \varepsilon_v = 1$ by reciprocity; the conditions for $n \leq 2$ are the conditions satisfied by any form of rank n [Ser73, Ch. IV, §2.2]. Conversely, local data satisfying these conditions, together with the positive definite real form of discriminant $d > 0$ and Hasse invariant 1, are those of a rational form by the existence theorem [Ser73, Ch. IV, §3.3, Prop. 7], and a rational form is determined by its localisations (Hasse–Minkowski). □

Thus for the Tate part, AHC I(a) is the statement that the finite Hasse product of Section 6.4 is trivial; this is the form in which the invariant β entered Section 9. Combined with Theorem 10.3, it gives Theorem 20.8 below.

Proposition 20.5 (Binary blocks). *Let B be a binary block of P that is not potentially Tate, with polynomial $x^2 - tx + 1$, $\Delta = t^2 - 4$ and $E_B = \mathbb{Q}(\sqrt{\Delta})$, an imaginary quadratic field, and suppose its Newton slopes are integers. Then p splits in E_B. For $\ell \neq p$ let $a_\ell \in \mathbb{Q}_\ell^\times$ be the hermitian class of B_ℓ (Lemma 5.5), and let $\varsigma \in \{\pm 1\}$ be the Newton–Hodge sign of B for the form $(-1)^m\langle\,,\rangle$. Then* AHC I(a) *holds for B if and only if*

$$\prod_{\ell \neq p} (a_\ell, \Delta)_\ell = \varsigma.$$

Proof. If p were not split in E_B, the two roots $\zeta, \bar{\zeta}$ of $x^2 - tx + 1$ would be conjugate over $\mathbb{Q}_p$, hence have the same p-adic valuation, which is then zero since $\zeta\bar{\zeta} = 1$; the block would be supersingular and so potentially Tate (Section 5.4), contrary

to the hypothesis. Since p splits, the slopes of B are $m \pm \lambda$ with $\lambda \neq 0$; when λ is an integer, as in Definition 5.9, the two slopes have the same parity and B has a definite Newton–Hodge sign ς.

A global hermitian line over E_B is $\mathrm{Tr}(a\, u\bar{v})$ with $a \in \mathbb{Q}^\times$, and by Lemma 5.5 its localisation at ℓ is isometric to B_ℓ, compatibly with the action of E_B, if and only if $a \equiv a_\ell$ modulo norms from $E_B \otimes \mathbb{Q}_\ell$. Its signature is that of $\langle 2a, -2a\Delta\rangle$, definite of sign $\mathrm{sgn}(a)$. Thus AHC I(a) asks for $a \in \mathbb{Q}^\times$ with $\mathrm{sgn}(a) = \varsigma$ and $a \equiv a_\ell$ modulo local norms for all $\ell \neq p$; at p every element is a local norm, since p splits. By global class field theory for the quadratic extension $E_B/\mathbb{Q}$ the sequence

$$\mathbb{Q}^\times/N(E_B^\times) \longrightarrow \bigoplus_v \mathbb{Q}_v^\times/N(E_{B,v}^\times) \xrightarrow{\prod_v (\cdot,\Delta)_v} \{\pm 1\} \longrightarrow 1$$

is exact, the sum running over all places, so a family of local classes, almost all trivial, comes from a global a if and only if the product of their norm residue symbols is 1 [Lan35], [Sch85, Ch. 10]. The local classes here are a_ℓ at $\ell \neq p$, trivial at p, and ς at ∞, where $(\varsigma, \Delta)_\infty = \varsigma$ because $\Delta < 0$. The condition is $\prod_{\ell \neq p} (a_\ell, \Delta)_\ell \cdot \varsigma = 1$. □

For the transcendental plane W of a wall of rank three in weight four, ordinary of slopes 0 and 4, $\varsigma = +1$, and with the rational model $N^0 = \langle 1, -\Delta\rangle$, for which $2a = 1$, one has $\eta_\ell = (2a_\ell, \Delta)_\ell$ and $s_N = 1$. Since $\prod_{\ell \neq p} (2, \Delta)_\ell = (2, \Delta)_\infty (2, \Delta)_p = 1$, the condition of Proposition 20.5 is exactly the Alien Hodge condition of Definition 9.4.

20.3. The implications

Theorem 20.6. *If* AHC I *holds for a smooth projective fourfold X over $\overline{\mathbb{F}}_q$, then the numerical Hodge-type conjecture holds for X in codimension two. In particular* AHC I *for all smooth projective fourfolds over finite fields implies the arithmetic summit, in its numerical form.*

Proof. By (b), $\mathcal{Z}_{\mathrm{pr}}$ embeds isometrically in U_{T}, which is positive definite by (a). A subspace of a positive definite space is positive definite. The case of divisors is the Hodge index theorem. □

The proof is formal, and (b) alone, granted that U_{T} is positive definite, implies the conclusion. The following remark makes the relation between (b) and the Hodge-type conjecture precise.

Remark 20.7. Assume (a) for T. If $\mathcal{Z}_{\mathrm{pr}}$ is positive definite and $\mathcal{Z}_{\mathrm{pr}} \otimes \mathbb{Q}_v$ embeds isometrically into $U_{\mathrm{T}} \otimes \mathbb{Q}_v$ for every finite place v, then (b) holds, by the Hasse–Minkowski theorem for the representation of one rational form by another [Ser73, Ch. IV, §3]; at the real place the embedding exists because both spaces are positive definite. Thus, granted (a) and the local embeddings, (b) is equivalent to the numerical Hodge-type conjecture. The content of AHC I beyond Hdg lies in part (a).

Theorem 20.8. *Let X be a smooth projective fourfold over $\mathbb{F}_q$. Suppose that* AHC I(a) *holds for the potentially Tate part* T *of P, that* T *has rank at most three, and that the Tate conjecture holds for* T *in the form that $\mathcal{Z}_{\mathrm{pr}} \otimes \mathbb{Q}_v \simeq \mathrm{T}_v$ for every finite place v (with $\mathrm{T}_p = \{x \in \mathrm{T}_D : Fx = p^2 x\}$). Then the intersection form on $\mathcal{Z}_{\mathrm{pr}}$ is positive definite.*

Proof. Under the hypotheses, $\mathcal{Z}_{\mathrm{pr}}$ is a rational form of rank at most three whose localisations at all finite places are those of the positive definite form U_{T}. By Theorem 10.3 it is positive definite. □

Here part (a) is used in an essential way, and part (b) is not assumed. In rank four the theorem fails in this form, by Proposition 10.5.

Theorem 20.9. *At a wall of rank three of a motive with a Betti lift, at which the transcendental block W is ordinary and the alien parity* (9.4) *holds,* AHC I(a) *for W implies that every algebraic class of the Tate part is positive. Conversely, the Tate conjecture and the Hodge-type conjecture on the Tate part imply the Alien Hodge condition for W.*

Proof. AHC I(a) for W gives a global hermitian line U_W with the localisations W_ℓ and positive signature; the computation following Definition 9.4 shows that W then satisfies the Alien Hodge condition, and Theorem 9.7, applied with the line L (which is decided by Lemma 5.4 as in Section 6.5) and any algebraic class of V^+, gives the first statement. The second is Theorem 9.9. □

20.4. Sums and products

Part (a) of AHC I is a statement about realisations only, and it behaves well under the operations of Section 6.2. We record this for Tate parts, where the statement is cleanest. For an Alien Hodge Motive $\mathcal{M}_{\mathrm{AH}}$ of weight $2m$ over $\mathbb{F}_q$ with potentially Tate part T, we say that T *has a rational form* if there is a positive definite rational quadratic space U, for the form $(-1)^m\langle\,,\rangle$, with $U\otimes\mathbb{Q}_\ell \simeq \mathrm{T}_\ell$ for every $\ell\neq p$ and $U\otimes\mathbb{Q}_p \simeq \mathrm{T}_p = \{x\in \mathrm{T}_D : Fx = p^m x\}$; this is part (a) of Conjecture 20.1 for the Tate block.

Proposition 20.10. *Let $\mathcal{M}_{\mathrm{AH}}$ and $\mathcal{M}'_{\mathrm{AH}}$ be Alien Hodge Motives over $\mathbb{F}_q$ of weights $2m$ and $2m'$, with potentially Tate parts* T *and* T′ *that have rational forms U and U'.*

(i) *If $m=m'$, the potentially Tate part* $\mathrm{T}\oplus\mathrm{T}'$ *of $\mathcal{M}_{\mathrm{AH}}\oplus\mathcal{M}'_{\mathrm{AH}}$ has the rational form $U\oplus U'$.*

(ii) *The summand* $\mathrm{T}\otimes\mathrm{T}'$ *of $\mathcal{M}_{\mathrm{AH}}\otimes\mathcal{M}'_{\mathrm{AH}}$ is potentially Tate, $(\mathrm{T}\otimes\mathrm{T}')_p = \mathrm{T}_p\otimes\mathrm{T}'_p$, and $U\otimes U'$ is a positive definite rational form of it for $(-1)^{m+m'}\langle\,,\rangle$. If the potentially Tate part of $\mathcal{M}_{\mathrm{AH}}\otimes\mathcal{M}'_{\mathrm{AH}}$ is* $\mathrm{T}\otimes\mathrm{T}'$*, it has a rational form.*

Proof. (i) The Frobenius blocks of a direct sum are the sums of the blocks with the same polynomial, so the potentially Tate part of $\mathcal{M}_{\mathrm{AH}}\oplus\mathcal{M}'_{\mathrm{AH}}$ is $\mathrm{T}\oplus\mathrm{T}'$, at every ℓ and at p; an orthogonal sum of positive definite spaces is positive definite, and localisation commutes with orthogonal sums.

(ii) The eigenvalues of ϕ on $\mathrm{T}_\ell\otimes\mathrm{T}'_\ell$ are products of roots of unity, so the summand is potentially Tate. The form on $\mathcal{M}_{\mathrm{AH}}\otimes\mathcal{M}'_{\mathrm{AH}}$ is the product form, and $(-1)^{m+m'}\langle x\otimes x', y\otimes y'\rangle = (-1)^m\langle x,y\rangle\cdot(-1)^{m'}\langle x',y'\rangle$; the tensor product of two positive definite spaces is positive definite, as one sees in orthogonal bases, and $(U\otimes U')\otimes\mathbb{Q}_\ell = (U\otimes\mathbb{Q}_\ell)\otimes(U'\otimes\mathbb{Q}_\ell)\simeq \mathrm{T}_\ell\otimes\mathrm{T}'_\ell$. At p, if $Fx=p^m x$ and $Fx'=p^{m'}x'$ then $F(x\otimes x') = p^{m+m'}x\otimes x'$, so $\mathrm{T}_p\otimes\mathrm{T}'_p\subset(\mathrm{T}\otimes\mathrm{T}')_p$. Both sides are $\mathbb{Q}_p$-structures of the same K_0-space: by Section 5.4, $\mathrm{T}_p\otimes K_0 = \mathrm{T}_D$ and $\mathrm{T}'_p\otimes K_0 = \mathrm{T}'_D$, so the left side spans $\mathrm{T}_D\otimes\mathrm{T}'_D$, and the right side, being the space of fixed points of the slope-zero σ-linear bijection $p^{-m-m'}F$ on the supersingular space $\mathrm{T}_D\otimes\mathrm{T}'_D$, is a

$\mathbb{Q}_p$-structure of it as well. Hence they have the same dimension and are equal, and $U \otimes U' \otimes \mathbb{Q}_p \simeq \mathrm{T}_p \otimes \mathrm{T}'_p$. □

The proposition does not extend to part (b), and this is as it should be. The algebraic classes of $X \times X'$ are not all products of algebraic classes of X and X', and the positivity of the new ones is not a formal consequence of the positivity on the factors: the Hodge-type conjecture for $X \times X$ is the hypothesis of Grothendieck's derivation of the Riemann hypothesis (Section 2.10), not a consequence of $\mathrm{Hdg}(X)$. Part (a), on the other hand, concerns the local invariants of the realisations, which are compatible with tensor products by Lemma 6.4; for blocks that are not potentially Tate, the same argument shows that $U_B \otimes U_{B'}$ is a rational form of $B \otimes B'$ with the Newton–Hodge signature, although $B \otimes B'$ is in general a sum of several blocks, and part (a) asks for a rational form of each of them.

20.5. A case of the conjecture

Theorem 20.11. *Let $\mathcal{M}$ be one of the three motives (1.1), under Hypothesis 12.4, and let $q \notin E \cup T$ be inert in $K_{\mathcal{M}}$. Then the block C satisfies both parts of* AHC I*: there is a positive definite rational binary form U_C with $U_C \otimes \mathbb{Q}_\ell \simeq C_\ell$ for all $\ell \neq q$ and $U_C \otimes \mathbb{Q}_q \simeq T_H \otimes \mathbb{Q}_q$, and the space $\mathcal{Z}^-$ of numerical classes in C embeds isometrically into U_C.*

Proof. We prescribe the local invariants of a binary form: discriminant ab; Hasse invariant $(a,b)_\ell$ at $\ell \neq q$; Hasse invariant -1 at q; positive definite at ∞. The discriminants are consistent: at q, $d(T_H \otimes \mathbb{Q}_q) \equiv u_2u_3 \equiv ab$, since $-u_2u_3$ and $-ab$ are both unit nonsquares (Proposition 13.5 and q inert). The product of the Hasse invariants is $\big(\prod_{\ell \neq q}(a,b)_\ell\big) \cdot (-1) \cdot 1 = (-1)(-1) = 1$, by the reciprocity computation in the proof of Proposition 13.6. At every place where $-ab$ is a square the prescribed Hasse invariant is that of a hyperbolic plane, as required: at $\ell \neq q$ because it is that of $\langle a, b\rangle$, and at q and ∞ because $-ab$ is not a square there. By the existence theorem for rational quadratic forms with prescribed invariants [Ser73, Ch. IV, §3] there is a rational binary form U_C with these invariants; it is positive definite, which is the Newton–Hodge signature of C at an inert prime (Theorem 14.1), and its localisations are C_ℓ for $\ell \neq q$ and $T_H \otimes \mathbb{Q}_q$ at q, binary forms over local fields being classified by these invariants. This is (a).

For (b): by Proposition 13.6 every nonzero element of $\mathcal{Z}^-$ has positive square, so the form on $\mathcal{Z}^-$ is positive definite, and its Gram matrix in a basis $z_1, \dots, z_k$ is nondegenerate. The crystalline classes of the z_i have the same Gram matrix, hence are linearly independent over $\mathbb{Q}_q$; they lie in $T_H \otimes \mathbb{Q}_q$ (Proposition 13.5), which has dimension two. Hence $\dim \mathcal{Z}^- \leq 2$. If $\dim \mathcal{Z}^- = 2$, then $\mathcal{Z}^-$ is a positive definite binary form (Proposition 13.6) isometric to U_C at q (both are $T_H \otimes \mathbb{Q}_q$), at every $\ell \neq q$ (choose representative cycles; their ℓ-adic classes have the nondegenerate Gram matrix of $\mathcal{Z}^-$, so they span C_ℓ), and at ∞; by Hasse–Minkowski, $\mathcal{Z}^- \simeq U_C$. If $\dim \mathcal{Z}^- = 1$, spanned by a class of square $r > 0$, then r is represented by U_C at every place, hence globally. If $\mathcal{Z}^- = 0$ there is nothing to prove. □

The theorem of Part III is thus, for the block C, a proof of AHC I. For the line L the ℓ-adic part of the global form is visible from the start: $L_\ell \simeq \langle u\rangle \otimes \mathbb{Q}_\ell$

for $\ell \neq q$ with $u = c\, d_S(4 - t^2) > 0$ (Proposition 13.4). If L contains a nonzero numerical class z, then $U_L = \mathbb{Q}z$ satisfies both parts of the conjecture, since the crystalline class of z spans the crystalline Tate line; without such a class, part (a) for L would require the discriminant of the crystalline Tate line, which we have not computed.

21. Alien Hodge Conjecture II

21.1. Two commuting Lefschetz actions

Let V be a finite-dimensional vector space over a field of characteristic zero. An $\mathfrak{sl}_2$-triple (h, e, f) in $\mathrm{End}(V)$ satisfies $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$, with $h \neq 0$; the element f is determined by h and e. For the cohomology of a smooth projective variety of dimension n with an ample class ξ, $e = L_\xi$, h acts on H^i by $i - n$, and $f = \Lambda$ is the operator of Conjecture 3.2.

Lemma 21.1 (The diagonal triple). *Let (h_1, e_1, f_1) and (h_2, e_2, f_2) be $\mathfrak{sl}_2$-triples in $\mathrm{End}(V)$ such that e_1, h_1, f_1 commute with e_2, h_2, f_2 in the six combinations $[e_1, f_2] = [e_2, f_1] = [h_1, e_2] = [h_2, e_1] = [h_1, f_2] = [h_2, f_1] = 0$, and $h_1 + h_2 \neq 0$. Then $(h_1 + h_2,\ e_1 + e_2,\ f_1 + f_2)$ is an $\mathfrak{sl}_2$-triple. Consequently, if $(h_1 + h_2, e_1 + e_2, f)$ is an $\mathfrak{sl}_2$-triple, then $f = f_1 + f_2$, and f lies in every subalgebra of $\mathrm{End}(V)$ containing f_1 and f_2.*

Proof. $[h_1 + h_2, e_1 + e_2] = [h_1, e_1] + [h_2, e_2] = 2(e_1 + e_2)$, and similarly for f; $[e_1 + e_2, f_1 + f_2] = [e_1, f_1] + [e_2, f_2] = h_1 + h_2$. The third member of an $\mathfrak{sl}_2$-triple is determined by the other two. □

Lemma 21.2 (Block reduction). *Let (h, e, f) be an $\mathfrak{sl}_2$-triple in $\mathrm{End}(V)$ and π an idempotent commuting with h and e. Then π commutes with f, and $f = \pi f \pi + (1 - \pi) f (1 - \pi)$. Hence f lies in every subalgebra containing $\pi f \pi$ and $(1 - \pi) f (1 - \pi)$.*

Proof. Put $f' = \pi f \pi + (1 - \pi) f (1 - \pi)$. As π commutes with h, $[h, f'] = \pi[h, f]\pi + (1-\pi)[h, f](1-\pi) = -2f'$; as π commutes with e, $[e, f'] = \pi[e, f]\pi + (1-\pi)[e, f](1-\pi) = \pi h \pi + (1-\pi)h(1-\pi) = h$. Hence (h, e, f') is an $\mathfrak{sl}_2$-triple, and $f' = f$ by the uniqueness of the third member: the difference $f - f'$ commutes with e and has weight -2 for $\mathrm{ad}\, h$, while in the finite-dimensional representation ad of the triple on $\mathrm{End}(V)$ the kernel of $\mathrm{ad}\, e$ consists of highest weight vectors, whose weights are nonnegative. Then $\pi f = \pi f' = \pi f \pi = f' \pi = f \pi$. □

21.2. The perverse–Leray decomposition

Let $f : X \to Y$ be a projective morphism from a smooth projective variety X of dimension n over $\mathbb{C}$ to a projective variety Y, let η be an ample class on X and A an ample class on Y. By the decomposition theorem, in the Hodge-theoretic form of de Cataldo and Migliorini [dCM05], the perverse Leray filtration P on $H^*(X, \mathbb{Q})$ has an associated graded space on which two commuting Lefschetz operators act: L_η, which satisfies the relative hard Lefschetz theorem with respect to the perverse degree, and L_{f^*A}, which satisfies the hard Lefschetz theorem for the intersection cohomology of Y with coefficients in the perverse cohomology sheaves. Together they give an action of $\mathfrak{sl}_2 \times \mathfrak{sl}_2$ on the graded

space, whose diagonal is the action of $L_{\eta+f^*A}$ up to the passage to the graded. The motivic decomposition conjecture of Corti and Hanamura [CH00] asks that the decomposition be induced by algebraic cycles; Maulik, Shen and Yin have proved it, together with the algebraicity of the inverse relative Lefschetz operator, for the Hitchin system [MSY24], and Ancona, Cavicchi, Laterveer and Saccà have studied the relative form of the Lefschetz conjecture for Lagrangian fibrations [ACLS23].

We recall the form of the decomposition theorem that we use. Let $f\colon X \to Y$ be a projective morphism from a smooth projective complex variety of dimension n, and η an f-ample class. Then

$$R f_*\mathbb{Q}_X[n] \simeq \bigoplus_i {}^{p}\mathcal{H}^i(R f_*\mathbb{Q}_X[n])[-i], \tag{21.1}$$

each perverse cohomology sheaf is semisimple, a direct sum of intersection complexes of local systems on strata of Y, and cup product with η^i induces isomorphisms ${}^{p}\mathcal{H}^{-i} \simeq {}^{p}\mathcal{H}^i$ (relative hard Lefschetz). The perverse Leray filtration P on $H^*(X)$ is the filtration induced by the perverse truncations; the decomposition (21.1) splits it, non-canonically. De Cataldo and Migliorini prove these statements by Hodge-theoretic means and show that, for A ample on Y, the graded space of P carries commuting actions of the Lefschetz operators of η and of f^*A, each satisfying hard Lefschetz in its own grading [dCM05].

21.3. The statement

Conjecture 21.3 (Alien Hodge Conjecture II: perverse–Leray algebraicity). *Let X be a smooth projective variety of dimension $n \geq 2$ over $\mathbb{C}$. There are a smooth projective variety $\tilde{X}$ with a surjective morphism $\tilde{X} \to X$, a projective morphism $f\colon \tilde{X} \to Y$ onto a projective variety Y with $0 < \dim Y < \dim \tilde{X}$, an f-ample class η on $\tilde{X}$ and an ample class A on Y such that $\eta + f^*A$ is ample, and a splitting $H^*(\tilde{X}, \mathbb{Q}) = \bigoplus_{i,j} H^{i,j}$ of the perverse Leray filtration of f, such that:*

(i) *the projectors of the splitting are induced by algebraic cycles on $\tilde{X} \times \tilde{X}$;*
(ii) *on $H^*(\tilde{X}, \mathbb{Q})$, the operators $L_1 = L_\eta$ and $L_2 = L_{f^*A}$ belong to $\mathfrak{sl}_2$-triples (h_1, L_1, Λ_1) and (h_2, L_2, Λ_2) that commute as in Lemma 21.1, where h_1 and h_2 act on $H^{i,j}$ by scalars depending on (i, j) and $h_1 + h_2$ acts on $H^k(\tilde{X}, \mathbb{Q})$ as multiplication by $k - \dim \tilde{X}$;*
(iii) *the operators Λ_1 and Λ_2 are induced by algebraic cycles on $\tilde{X} \times \tilde{X}$.*

Part (ii) is a requirement on the splitting: on the associated graded space the two triples exist by the relative hard Lefschetz theorem and the hard Lefschetz theorem for intersection cohomology, and (ii) asks for a splitting on which they are realised by the cup products themselves. Parts (i) and (iii) are the algebraicity statements: the perverse–Leray projectors, the inverse of the relative Lefschetz operator, and the inverse of the Lefschetz operator of the base, each induced by a cycle. For $\tilde{X}$ one may take the blow-up of X along the base locus of a Lefschetz pencil, which maps to $\mathbb{P}^1$; then Λ_1 is the inverse of the relative hard Lefschetz isomorphism for the family of hyperplane sections, and Λ_2 concerns only the base curve. In this case the conjecture expresses $B(X)$ through the hyperplane sections of X, in the spirit of an induction on the dimension.

Example 21.4 (*Products*). Let $X = C \times S$ with C a curve and S a surface, and $f = \mathrm{pr}_C\colon X \to C$, with $\eta = \mathrm{pr}_S^*\xi_S$ for an ample class ξ_S on S, which is f-ample,

and A an ample class on C. For a smooth projective morphism the perverse Leray filtration is the Leray filtration up to a shift, and here it is split by the Künneth decomposition $H^{a,b} = H^a(C) \otimes H^b(S)$, with perverse degree given by b. The projectors are $\pi_a^C \otimes \pi_b^S$, algebraic since the Künneth components of curves and surfaces are algebraic. On $H^{a,b}$ put $h_1 = b - 2$ and $h_2 = a - 1$; then $(h_1, 1 \otimes L_{\xi_S}, 1 \otimes \Lambda_S)$ and $(h_2, L_A \otimes 1, \Lambda_C \otimes 1)$ are commuting $\mathfrak{sl}_2$-triples with $h_1 + h_2$ the grading operator shifted by $\dim X$, and Λ_S, Λ_C are algebraic because B holds for surfaces and curves. Therefore AHC II holds for X, and Theorem 21.5 recovers $B(C \times S)$, which also follows from the stability of B under products.

The example shows the intended mechanism: the problem for the total space is split into a problem in the fibre direction and a problem on the base. For a general fourfold no fibration with algebraic perverse–Leray projectors is known, and this is where the conjecture has content.

21.4. The implication

Theorem 21.5. *If* AHC II *holds for a smooth projective variety X over $\mathbb{C}$, then $B(X)$ holds. In particular* AHC II *for all smooth projective fourfolds over $\mathbb{C}$ implies the geometric summit.*

Proof. By (ii) and Lemma 21.1, $(h, L, \Lambda_1 + \Lambda_2)$ is an $\mathfrak{sl}_2$-triple, with $h = h_1 + h_2$ the shifted grading operator and $L = L_1 + L_2$ the cup product with the ample class $\eta + f^*A$. By uniqueness of the third member, the operator Λ of $\tilde{X}$ for this ample class is $\Lambda_1 + \Lambda_2$, which is algebraic by (iii). Hence $B(\tilde{X})$ holds; the conjecture does not depend on the choice of the ample class [Kle94]. The conjecture B is stable under products [Gro69, p. 196], so $B(\tilde{X}^k)$ holds for all k. Since $\tilde{X} \to X$ is surjective, X is motivated by $\tilde{X}$ in the sense of Arapura [Ara06, Cor. 1.2], and by [Ara06, Lemma 4.2], $B(X)$ holds. □

Part (i) is not used in this argument: it enters through Lemma 21.2, which reduces the algebraicity of Λ to that of its diagonal blocks with respect to an algebraic projector commuting with h and L, and it is the natural companion of (iii) in the motivic decomposition conjecture. The implication of Theorem 21.5 is formal, like that of Theorem 20.6: the conjecture is designed so that the summit follows. Its content is that the inverse Lefschetz operator of the total space splits, under a fibration, into a relative part and a part on the base, which can be attacked separately. The results quoted in Section 21.2 prove the relative part in some cases.

21.5. Lefschetz pencils

Let X be a smooth projective variety of dimension n over $\mathbb{C}$, embedded by an ample class ξ, and let $\{X_s\}_{s \in \mathbb{P}^1}$ be a Lefschetz pencil of hyperplane sections, with base locus Z of codimension two. The blow-up $\tilde{X}$ of X along Z maps to $\mathbb{P}^1$, with fibres the X_s; the fibres are smooth except for finitely many with one ordinary double point. The cohomology of $\tilde{X}$ is that of X plus a copy of the

cohomology of Z shifted by two, and $\tilde{X} \to X$ is surjective. For $f : \tilde{X} \to \mathbb{P}^1$, the decomposition theorem gives

$$Rf_*\mathbb{Q}_{\tilde{X}}[n] \simeq \bigoplus_i {}^pR^i f_*\mathbb{Q}_{\tilde{X}}[n][-i],$$

where the perverse cohomology sheaves are, up to shift, the intermediate extensions of the local systems $R^j f_*\mathbb{Q}$ on the smooth locus, together with skyscraper contributions at the singular fibres when the vanishing cycles do not vanish in cohomology. The relative hard Lefschetz theorem identifies ${}^pR^{-i} f_*$ with ${}^pR^i f_*$ through the class η of a relatively ample divisor, and the base $\mathbb{P}^1$ contributes the Lefschetz action of a point class.

In this situation AHC II asks for three things: that the splitting of $H^*(\tilde{X})$ by the perverse filtration be induced by cycles; that the inverse of the relative Lefschetz operator be induced by a relative correspondence; and that the two Lefschetz actions be realised by commuting $\mathfrak{sl}_2$-triples. The third is a statement about the cohomology of $\tilde{X}$ alone. The first two concern the family of hyperplane sections, which are varieties of dimension $n-1$; for $n = 4$ they are threefolds, where the Lefschetz conjecture is open in degree two in general but known in many cases (Section 3.8). The conjecture thus organises the geometric summit as an induction on the dimension through Lefschetz pencils, in the form suggested by the relative results of [ACLS23, MSY24].

21.6. Relation with the Hodge conjecture

Over $\mathbb{C}$, every operator in AHC II that is a morphism of Hodge structures is given by a Hodge class on $\tilde{X} \times \tilde{X}$. The operators Λ_1 and Λ_2 of (ii) are of this kind when the splitting of the perverse filtration is chosen compatibly with the Hodge structures, since they are then determined by L_1, L_2 and the gradings through the $\mathfrak{sl}_2$-relations; the projectors of such a splitting are Hodge classes as well. Therefore the Hodge conjecture for $\tilde{X} \times \tilde{X}$ implies AHC II for X, given a splitting compatible with Hodge structures satisfying (ii). In this sense AHC II lies between the Lefschetz conjecture and the Hodge conjecture, as B itself does; its role is to divide the existence problem along a fibration.

21.7. The two conjectures together

The two conjectures concern the two halves of Grothendieck's pair: existence of algebraic cycles in characteristic zero and positivity in characteristic p. They meet under specialisation. If X over $\overline{\mathbb{F}}_q$ is the reduction of a variety X_0 over a number field, then $B(X_0)$ implies $B(X)$, since B is stable under specialisation [Gro69, p. 196]. Hence AHC II for X_0 and AHC I for X give $B(X)$ and the numerical form of $\mathrm{Hdg}(X)$. Grothendieck's form of $\mathrm{Hdg}(X)$, for classes modulo homological equivalence, would give in addition $D(X)$ [Gro69, §4, Remark 4]; for this one needs the embedding of part (b) of AHC I for homological classes.

22. Evidence and tests

We collect what is known in favour of the two conjectures, what would count against them, and how they can be tested in examples. Nothing in this section is used elsewhere.

22.1. Cases of Alien Hodge Conjecture I

The following cases of AHC I, or of its part (a), are known.

(1) *Divisors.* For the primitive part of H^2 of a surface whose Tate part is spanned by algebraic classes, for instance a supersingular K3 surface (Example 20.2) or the square of a supersingular elliptic curve (Section 7.3), the space of algebraic classes is the global form of part (a) for the Tate block, and the Hodge index theorem gives its signature.

(2) *The plane C.* For the three motives (1.1), at every prime $q \notin E \cup T$ inert in $K_{\mathcal{M}}$, both parts hold for the block C (Theorem 20.11).

(3) *The ten walls.* For H_2 at the ten wall points $(q, -1)$ of Section 19.2, part (a) holds for the transcendental block W: the Alien Hodge condition of Definition 9.4 holds (Corollary 19.1), and it is the condition of Proposition 20.5. Here part (a) is proved for a block that is not potentially Tate, of Newton slopes 0 and 4, without knowing the hermitian classes η_ℓ individually.

(4) *Tate blocks under the Tate conjecture with a CM lift.* Let X admit a CM lifting in the sense of [AM25], and suppose that the Tate conjecture holds for X over $\overline{\mathbb{F}}_q$, so that the space of algebraic classes Z spans the Tate part at every finite place. Ancona and Marmora prove that the discriminant and the Hasse invariants of Z are those of a form with the signature predicted by the Hodge-type conjecture. Then the localisations of Z are the Tate spaces, and their invariants are those of a positive definite form; by Proposition 20.4, part (a) holds for the Tate block, with U_{T} any positive definite form with these invariants. Part (b) is then the statement that Z is positive definite, which is the Hodge-type conjecture itself.

Cases (2) and (3) are the new ones. They are of different kinds: in (2) the conjecture is proved for a Tate block by producing its global form from algebraic classes and local data; in (3) it is proved for a transcendental block, and the argument passes through the positivity of algebraic classes in the complementary Tate block. This interplay between the Tate part and its complement is the content of the wall equivalence: at a wall, part (a) for the transcendental block is equivalent to the positivity on the Tate part.

22.2. What would count against them

By Theorem 9.9, a counterexample to AHC I(a) at a wall where the local identity holds would be a counterexample to the Tate conjecture or to the Hodge-type conjecture on the Tate part. Away from walls, a counterexample to AHC I(a) for a transcendental binary block would be a block whose hermitian classes violate the reciprocity of Proposition 20.5. Such a block would have, in every Betti lift, local hermitian classes that cannot be the localisations of a global line of the Newton–Hodge sign. We know of no mechanism that produces such blocks, but we also know no general argument that excludes them.

A counterexample to AHC II would be a fourfold X for which, for every fibration of every dominating variety, the perverse–Leray projectors or the

relative inverse Lefschetz operators fail to be algebraic, while $B(X)$ might still hold by other means. AHC II is stronger than B, and its failure would not refute B.

22.3. Tests

Part (a) of AHC I for a binary block is, by Proposition 20.5, a statement about finitely many norm residue symbols. For a block of a motive with a Betti lift, the classes a_ℓ are determined by the integral structure of the Betti realisation and by the action of Frobenius, and they can be computed when the decomposition of the lattice into Frobenius blocks can be computed. The difficulty is that the Frobenius decomposition is defined over $\mathbb{Q}_\ell$ and not over $\mathbb{Q}$, so the local classes a_ℓ at the finitely many primes ℓ that matter must be computed one prime at a time, from the ℓ-adic Galois representation. For hypergeometric motives this is possible in principle through the finite hypergeometric sums over extensions of $\mathbb{F}_q$ and the action on the integral lattice of the monodromy group.

At the 67 walls of Section 19.3 the conjecture predicts $\prod_{\ell \neq q} \eta_\ell = 1$, and the equivalence of Theorem 9.5 says that this is the same as the positivity of algebraic classes. A computation of the η_ℓ at one of these walls would therefore either confirm the prediction or produce a counterexample to the Tate or the Hodge-type conjecture. We have not carried out such a computation.

22.4. An explicit prediction

For the motive H_2 at a prime q that splits in $\mathbb{Q}(i)$, the Tate part is the line L, and the transcendental block W is a binary block that is not Tate, of Newton slopes 0 and 4. Proposition 20.5 turns AHC I(a) for W into the statement $\prod_{\ell \neq q} (a_\ell, \Delta_q)_\ell = 1$, where $\Delta_q = t_W^2 - 4$ and a_ℓ is the hermitian class of W_ℓ. At the inert primes this was proved through the positivity of the Tate part (Corollary 19.1); at the split primes there is no Tate part of rank greater than one, the wall equivalence does not apply, and the statement is a prediction of the conjecture about the Galois representation W_ℓ alone. Since Δ_q varies with q, the prediction involves infinitely many imaginary quadratic fields.

22.5. The conjectures and the motivic picture

In the language of Section 2.6, AHC I asks that the category of motives over a finite field carry, on each Frobenius block, a polarisation whose archimedean signs are read from the Newton polygon. For the potentially Tate part this is the polarisation that the Hodge-type conjecture asks for. For the other blocks it is an additional structure, and Lemma 6.4 shows that it is compatible with tensor products, as a polarisation of a Tannakian category must be. AHC II asks that the Lefschetz structure of a variety in characteristic zero be generated, through fibrations, by algebraic cycles. Together they describe the two halves of Grothendieck's standard conjectures, positivity and existence, in a form adapted to the methods of this paper.

23. Open problems

We list the questions left open by this paper, beginning with the obstruction that it does not remove.

23.1. The Dwork walls

The family of sextic fourfolds $x_0^6 + \cdots + x_5^6 = 6\psi\, x_0 \cdots x_5$ carries the hypergeometric motive with $\alpha = (\frac{1}{6}, \ldots, \frac{5}{6})$, $\beta = 0^5$, of Hodge type $(1,1,1,1,1)$. Its parameters do not satisfy $\beta \equiv \alpha + \frac{1}{2}$, so there is no inversion symmetry and no Levelt involution of the kind used in Part III. At its walls of rank three (Remark 9.8) the Tate part does not split, and Theorem 9.5 shows only that Hdg there is equivalent to the Alien Hodge condition for the transcendental block. A proof of Hdg at these walls needs either a new algebraic correspondence splitting the Tate part, or a proof of the Alien Hodge condition, which is a statement about the norm classes of a hermitian line at finitely many primes.

23.2. General fourfolds

The arithmetic summit for general fourfolds is open, and so are the two Alien Hodge Conjectures. AHC I(a) for the non-Tate blocks is a local–global statement about realisations; we know it for CM-liftable motives and for the block C of Theorem 20.11, and we know no other case in which a transcendental block is shown to be the localisation of a global hermitian space with the Newton–Hodge signature. For AHC II the cases known are those of Section 21.2.

23.3. The geometric inputs

For S_{33} and S_3, part (H2) of Hypothesis 12.4 is assumed: one needs a model of the motive at $t = -1$ on which the inversion symmetry is induced by an algebraic correspondence. For all three motives, (H1) asks for a smooth projective model with an algebraic projector onto the hypergeometric summand. Explicit constructions would make Theorem A unconditional. The normalisation of the constant c of Proposition 12.6 would remove its prime divisors from the exceptional set E.

23.4. The exceptional primes

We do not know whether the set T of Theorem 13.9 is finite. For H_2 it contains $q = 31$. A prime in T is a prime at which the transcendental plane W, or the CM plane C at a split prime, has Frobenius of finite order; at such a prime the Tate part has rank at least three and does not split under σ into pieces whose invariants are global, and the method of Section 13 does not apply.

23.5. Rank seven

In the four motives of Section 18, the walls of the block R are decided up to one sign (Proposition 10.8), and an anchor class would decide them (Theorem 10.9). Finding such a class, for instance a cycle on the special fibre lifting to an algebraic cycle on a lift, would extend Theorem A to these motives. The determinant of Frobenius on R is consistent with the discriminant field F_R at every computed prime, and is determined by the data at 14 of the 33; a proof of Observation 18.1,

or a computation that separates the eigenvalues on S and on R, is needed before principle 8 can be used in rank four.

23.6. Crystals of higher level

The arithmetic input of Proposition 13.6 is Ogus's theorem that the Tate module of a supersingular K3 crystal is non-neutral. The walls left open in this paper call for analogues for other crystalline types: for the definite block S at the W-exceptional primes, a supersingular crystal of Hodge type $(4,0)+(2,2)+(0,4)$ (Section 13.8); for CM planes with odd $k \geq 3$, a supersingular crystal of type $(m+k, m-k)+(m-k, m+k)$ (Section 7.2). In each case the question is the Hasse invariant of the $\mathbb{Q}_p$-space of Frobenius-fixed vectors, normalised by the weight, with the cup product. We expect a non-neutrality statement whenever the Newton–Hodge sign differs from the Betti sign, which is what AHC I predicts through reciprocity.

23.7. Beyond hypergeometric motives

The method of Part III needs three things: a global quadratic space (a Betti lift), an algebraic involution splitting the Tate part into pieces of rank at most two, and, for the pieces of rank two, a crystalline symbol at p. Any motive with an automorphism of finite order defined over $\mathbb{Q}$ whose eigenspaces reduce walls to such pieces is a candidate. The scan of Proposition 12.9 covers only one symmetry of one class of motives.

Appendix A. The formal companion

A.1. Scope

The formal companion is a Lean 4 package of 66 modules, about 10 600 lines. Its core is formed by the ten modules

```
AlienHodge        AlienHodgeNode1   AlienHodgePrinciples
AlienHodgeWalls   AlienHodgeWorlds  AlienHodgeOverF
AlienHodgeII      AlienHodgeTheory  AlienHodgeDiscriminant
AlienHodgeIII
```

in the namespace `GaiMua.Research.StandardConjectures`. The other modules are those of the GAI-MUA library on which these depend: Hilbert symbols and their product formula, Hasse invariants, the determination of signatures modulo four, and $\mathfrak{sl}_2$-triples. It is built with the toolchain `leanprover/lean4:v4.36.0-rc2`, a release candidate, and with Mathlib at the commit `b7fb16a940e9d6383e4376cecea18bb818a2d2e3`.

The Lean statements formalise the arithmetic of the proofs. The geometric inputs, namely the realisations, their comparison isomorphisms, the action of correspondences, the crystalline cycle classes and Ogus's classification, enter as hypotheses: a Lean theorem such as `AlienHodgeWorlds.cm_block_pos_ab` says that a rational number r satisfying the local symbol conditions at a finite set of primes is positive, and the paper supplies, from the geometry, the fact that the square of an algebraic class satisfies these conditions. Table 9 lists, for the main statements of the paper, the declarations they rest on and the hypotheses of these declarations that are geometric.

TABLE 9. Main statements, their Lean declarations (in the namespace `GaiMua.Research.StandardConjectures`), and the hypotheses of the declarations that the paper supplies from the geometry.

Statement	Lean declarations	Geometric hypotheses
Lemma 5.3	`ModFour.local_data_mod4`, `mod4_sharp`	none
Lemma 6.7	`AlienHodgeNode1.disc_similitude`	none
Theorem 9.1 and Corollary 9.2	`AlienHodge.ahm_T1`, `ahm_dichotomy`	local invariants of the block
Theorems 9.5 and 9.7	`AlienHodge.wall_hr_iff_ahc`, `AlienHodgeII.hr_iff_ahc_local`	the decomposition (9.1), the crystalline invariant $\varepsilon_p(A)$
Theorem 9.9	`AlienHodge.ahc_of_tate_hr`	the Tate part is spanned by algebraic classes
Theorem 10.1	`AlienHodgePrinciples.rank2_sign_law`	$B_\ell \simeq \langle a, b\rangle \otimes \mathbb{Q}_\ell$ for $\ell \neq p$
Theorem 10.3 and Proposition 10.5	`AlienHodgePrinciples.decide_le3`, `blind_rank4`	none
Proposition 10.8 and Theorem 10.9	`AlienHodgeDiscriminant.rank4_wall_bit`, `AlienHodgeOverF.anchor_principle`	square classes of the lines at almost all ℓ
Props. 12.2, 12.7	`AlienHodgeNode1.sigma_A`, …, `gram_minus`; `AlienHodgeWorlds.S33.*`, `S3.*`	none (identities of integer matrices)
Proposition 13.4	`AlienHodgeNode1.line_pos`	$L \simeq \langle u\rangle \otimes \mathbb{Q}_\ell$ for almost all ℓ
Proposition 13.5	`AlienHodgeNode1.ogus_rank2`	Ogus's normal form of T_H
Proposition 13.6	`AlienHodgeWorlds.cm_block_pos_ab`, `ogus_bridge_ab`	representation of r by C_ℓ and by $T_H \otimes \mathbb{Q}_q$
Theorem A(i)	`AlienHodgeNode1.node1_hr`, `AlienHodgeWorlds.world_hr`, `AlienHodgeTheory.summand_hr_split`, `summand_hr_inert`	the orthogonal splitting of the algebraic classes by σ
Theorem 14.1	`AlienHodgeTheory.alien_sign_flip`	as for Proposition 13.6
Theorem 20.6	`AlienHodge.asl_hr`, `AlienHodgeII.pos_of_blocks`	the embedding of AHC I(b)
Theorem 21.5	`AlienHodgeII.lambda_diag`, `AlienHodge.lambda_of_blocks`	the triples of AHC II(ii)

Not formalised: the density theorem (Theorem 13.9), whose proof uses the Chebotarev density theorem for ℓ-adic Lie groups, and the classification of the Lie algebra of the image; the existence theorem for rational quadratic forms used in Theorem 20.11; and the computations of Appendix B, which are carried out in exact rational and modular arithmetic outside Lean.

A.2. Verification record

The package was built from a clean build directory with the pinned toolchain and Mathlib. All 8186 build jobs completed without errors. For each of the 210 theorems of the ten modules, the command `#print axioms` was run; each depends on no axioms other than `propext`, `Classical.choice` and `Quot.sound`, and two of them depend on none. The environment of the package was then replayed by the kernel checker `leanchecker` in fresh mode, which re-checks every declaration of the imported environment; it terminated with exit status zero and no error. The axiom report, the build log and the replay log are part of the deposit.

A.3. Anchors

Every statement of the paper that rests on the formal companion carries an anchor, a list of the declarations it rests on, which the build of the paper checks against the declarations of the package. The table below gives, for each section, the number of anchored statements and of distinct declarations, and the module folders in which they lie. The statement-by-statement list is in the deposit.

§	Section	Page	Stat.	Decl.	Lean modules
1	Introduction	5	3	16	`AlienHodgeNode1`, `AlienHodgeWorlds`, `AlienHodgeTheory`, `AlienHodgePrinciples`, `AlienHodgeDiscriminant`, `AlienHodgeOverF`, `AlienHodge`, `AlienHodgeII`
3	The standard conjectures and the two summits	18	1	2	`AlienHodgeII`
5	Local data	30	2	6	`ModFour`, `Hilbert`, `SpinorTate`
6	Alien Hodge Motives	34	1	1	`AlienHodgeNode1`
9	Wall theorems without the Tate conjecture	45	8	10	`AlienHodge`, `AlienHodgeII`, `AlienHodgeWalls`, `RSLTheory`
10	General principles	48	9	16	`AlienHodgePrinciples`, `ModFour`, `AlienHodgeDiscriminant`, `AlienHodgeOverF`
12	Symmetric hypergeometric motives and the Levelt splitting	56	3	28	`AlienHodgeNode1`, `AlienHodgeWorlds`
13	Proof of the fixed-point theorem	63	4	15	`AlienHodgeNode1`, `SpinorTate`, `AlienHodgeWorlds`, `AlienHodgePrinciples`, `AlienHodgeTheory`

§	Section	Page	Stat.	Decl.	Lean modules
14	Sign reversal under supersingular specialisation	71	1	3	`AlienHodgeTheory`
15	Supersingular blocks of K3 type	74	1	2	`ModFour`
20	Alien Hodge Conjecture I	88	3	6	`AlienHodge`, `AlienHodgeII`, `AlienHodgePrinciples`, `ModFour`
21	Alien Hodge Conjecture II	93	3	5	`AlienHodgeII`, `Sl2Unique`, `AlienHodge`

Appendix B. Exact computations

B.1. The method

For a hypergeometric datum (α, β) defined over $\mathbb{Q}$, a prime p and $t \in \mathbb{F}_{p^r}$, the trace of Frobenius on the motive is given, in the normalisation of Beukers, Cohen and Mellit [BCM15], by a finite hypergeometric sum, a polynomial in Gauss sums of $\mathbb{F}_{p^r}$. When $p^r \not\equiv 1$ modulo the denominators of the datum, the sum is not defined over $\mathbb{F}_{p^r}$ and we pass to $\mathbb{F}_{p^{2r}}$. Floating-point evaluation is not reliable for p^r beyond a few thousand, and all traces used in the paper were computed exactly as follows.

(1) Every Gauss sum $g(\chi^k) = \sum_x \chi^k(x)\zeta_p^{\mathrm{Tr}\,x}$ of $\mathbb{F}_{p^r}$ is computed in a prime field $\mathbb{F}_\lambda$ with $\lambda \equiv 1$ modulo $2(p^r-1)p$ and $\lambda < 2^{40}$, which contains the required roots of unity. The discrete Fourier transform of length $p^r - 1$ is converted into a convolution by Bluestein's chirp transform, and the convolution is computed exactly over $\mathbb{Z}$ by floating-point fast Fourier transforms on 8-bit limbs, the partial sums staying far below the precision of double-precision arithmetic, and then reduced modulo λ.

(2) The normalised trace, multiplied by a power of p, is an integer N with $|N| \le n\, p^{rw/2}$ by the Weil bound. It is reconstructed by the Chinese remainder theorem from enough moduli λ, and the bound is checked: a wrong residue produces a reconstruction far outside it, so that each value certifies itself.

(3) From the exact power sums p_1, p_2, p_3 of the normalised Frobenius ϕ and the functional equation $x^n c(1/x) = \delta\, c(x)$, with $\delta = \det\phi = \pm 1$, the characteristic polynomial c is determined when only one value of δ gives roots on the unit circle; for ϕ^2 the sign is $\det(\phi)^2 = 1$.

(4) The polynomial is factored over $\mathbb{Q}$. Monic factors with integer coefficients are cyclotomic, by Kronecker's theorem. A non-cyclotomic irreducible quadratic $x^2 - sx + 1$ is attributed to W if the q-adic valuation of s is -2 (Newton slopes 0 and 4) and to C if it is -1 (slopes 1 and 3); by Mazur's inequality the Newton polygon of each summand lies above its Hodge polygon, which is $(0, 2, 4)$ for S and $(1, 3)$ for C, so this attribution is forced.

The scripts (`m5_hgmexact.py` for the Gauss sums, and `density_check.py`, `h2_table.py`, `noncm_check.py`, `rank7_table.py` and `make_tables.py` for the computations of this appendix) are in the deposit with their output;

the tables below are generated from that output. The self-test of the Gauss-sum code compares the exact traces with floating-point values at small primes.

B.2. The classification at small primes

Table 11 lists, for each of the three motives, the primes $q \le 97$ (excluding the bad primes and ℓ) at which the characteristic polynomial was determined, sorted by whether W and C are potentially Tate. For H_2 the only prime at which W is potentially Tate is $q = 31$; for the other two motives there is none among the computed primes. At every computed prime, C is potentially Tate exactly when q is inert in $K_{\mathcal{M}}$, in accordance with Lemma 13.7.

TABLE 11. Classification of the factors of the characteristic polynomial of Frobenius at $t = -1$: the primes at which the transcendental plane W and the CM plane C are not potentially Tate. A prime marked * was computed through ϕ^2 over $\mathbb{F}_{q^2}$.

$\mathcal{M}$	W not potentially Tate	C not potentially Tate
H_2	5, 7, 11, 13, 17, 19, 23, 29, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97	5, 13, 17, 29, 37, 41, 53, 61, 73, 89, 97
S_{33}	7, 11*, 13, 17*, 19, 23*, 29*, 31, 37, 41*, 43, 47*, 53*, 61, 67, 73, 79, 97	13, 17*, 29*, 37, 41*, 53*, 61, 73, 97
S_3	7, 11*, 13, 17*, 19, 23*, 29*, 31, 37, 41*, 43, 47*, 53*, 61, 67, 73, 79, 97	7, 13, 19, 31, 37, 43, 61, 67, 73, 79, 97

Table 12 gives the witnesses used in the proofs of Lemma 13.7 and Theorem 13.9. For H_2 the fixed prime is $\ell = 3$ and the bad prime is 2; for S_{33} and S_3 the fixed prime is $\ell = 5$ and the bad primes are 2 and 3. The characters are those of the quadratic fields $\mathbb{Q}(\sqrt{D})$ unramified outside these primes.

TABLE 12. Exclusion of the dihedral image on S and of the characters $\varepsilon_C \ne 1, \chi_K$: for each quadratic character χ_D unramified outside the bad primes and ℓ, the first computed primes with $\chi_D(q) = -1$ at which W (resp. C) is not potentially Tate.

$\mathcal{M}, \ell$	D	W	C
H_2, $\ell = 3$	-24	13, 17, 19, 23	13, 17, 37, 41
	-8	5, 7, 13, 23	5, 13, 29, 37
	-4	7, 11, 19, 23	(the character of K)
	-3	5, 11, 17, 23	5, 17, 29, 41
	8	5, 11, 13, 19	5, 13, 29, 37
	12	5, 7, 17, 19	5, 17, 29, 41
	24	7, 11, 13, 17	13, 17, 37, 41
S_{33}, $\ell = 5$	-120	7, 19, 41, 53	41, 53, 61, 73
	-40	17, 29, 31, 43	17, 29, 61, 73
	-24	13, 17, 19, 23	13, 17, 37, 41
	-20	11, 13, 17, 19	13, 17, 37, 53

$\mathcal{M}, \ell$	D	W	C
	-15	7, 11, 13, 29	13, 29, 37, 41
	-8	7, 13, 23, 29	13, 29, 37, 53
	-4	7, 11, 19, 23	(the character of K)
	-3	11, 17, 23, 29	17, 29, 41, 53
	5	7, 13, 17, 23	13, 17, 37, 53
	8	11, 13, 19, 29	13, 29, 37, 53
	12	7, 17, 19, 29	17, 29, 41, 53
	24	7, 11, 13, 17	13, 17, 37, 41
	40	7, 11, 17, 19	17, 29, 61, 73
	60	13, 19, 23, 29	13, 29, 37, 41
	120	11, 23, 31, 41	41, 53, 61, 73
S_3, $\ell = 5$	-120	7, 19, 41, 53	7, 19, 61, 73
	-40	17, 29, 31, 43	31, 43, 61, 67
	-24	13, 17, 19, 23	13, 19, 37, 43
	-20	11, 13, 17, 19	13, 19, 31, 37
	-15	7, 11, 13, 29	7, 13, 37, 43
	-8	7, 13, 23, 29	7, 13, 31, 37
	-4	7, 11, 19, 23	7, 19, 31, 43
	-3	11, 17, 23, 29	(the character of K)
	5	7, 13, 17, 23	7, 13, 37, 43
	8	11, 13, 19, 29	13, 19, 37, 43
	12	7, 17, 19, 29	7, 19, 31, 43
	24	7, 11, 13, 17	7, 13, 31, 37
	40	7, 11, 17, 19	7, 19, 61, 73
	60	13, 19, 23, 29	13, 19, 31, 37
	120	11, 23, 31, 41	31, 43, 61, 67

B.3. The motive H_2 up to 1000

For all 166 primes $5 \le q \le 997$ the exact power sums $p_1 = \operatorname{tr}\phi$ and $p_2 = \operatorname{tr}\phi^2$ were computed. In every case they are those of the decomposition described in Section 14.4: with $\varepsilon = (\frac{-2}{q})$ on L, the polynomial $x^2 - \frac{a_q}{q}x + \chi_{-4}(q)$ on C, and $t_W = p_1 - \varepsilon - \frac{a_q}{q}$, the identity $p_2 = 1 + (\frac{a_q^2}{q^2} - 2\chi_{-4}(q)) + (t_W^2 - 2)$ holds exactly (0 failures). In this decomposition W is potentially Tate only at $q = 31$, where $t_W = 2$. Two power sums do not determine the characteristic polynomial, so this computation is not a proof of the decomposition beyond $q \le 97$, where Table 11 applies; it is a consistency check over a larger range. Table 13 gives the values for $q < 100$.

TABLE 13. H_2 at $t = -1$, primes $5 \le q < 100$: the line eigenvalue $\varepsilon = (\frac{-2}{q})$, the coefficient a_q of $\eta(4\tau)^6$, and the trace t_W of ϕ on W, in the normalisation of [BCM15].

q	ε	a_q	t_W	q	ε	a_q	t_W
5	-1	-6	34/25	47	-1	0	$-190/2209$
7	-1	0	$-30/49$	53	-1	90	2018/2809
11	$+1$	0	$-42/121$	59	$+1$	0	$-5162/3481$

q	ε	a_q	t_W	q	ε	a_q	t_W
13	-1	10	$-62/169$	61	-1	-22	$6658/3721$
17	$+1$	-30	$-478/289$	67	$+1$	0	$-2250/4489$
19	$+1$	0	$-522/361$	71	-1	0	$6882/5041$
23	-1	0	$930/529$	73	$+1$	-110	$-10558/5329$
29	-1	42	$1282/841$	79	-1	0	$-318/6241$
31	-1	0	2	83	$+1$	0	$-13130/6889$
37	-1	-70	$2338/1369$	89	$+1$	-78	$-15358/7921$
41	$+1$	18	$-2462/1681$	97	$+1$	130	$3682/9409$
43	$+1$	0	$-3690/1849$				

B.4. The scan and rank seven

The enumeration of Proposition 12.9 runs over all multisets α that are unions of Galois orbits of roots of unity of order at most 30, of total size at most eight, with $\beta = \alpha + \frac{1}{2}$ and $\alpha \cap \beta = \emptyset$; for each, the Hodge numbers are computed from the interlacing of α and β, and the Levelt involution, the eigenspaces and their Gram matrices are computed in exact rational arithmetic. Of the 146 data, 14 have rank five; six of these have Hodge numbers $(1, 1, 1, 1, 1)$, forming three pairs exchanged by $\alpha \leftrightarrow \beta$. The rank-seven data and the classification of Table 7 are computed from exact characteristic polynomials in the same way, with the eigenvalues ± 1 counted with multiplicity.

Appendix C. The geometric inputs

This appendix lists the published theorems on which the proof of Theorem A rests, the form in which they are used, and the conditions of their application. The hypotheses (H1)–(H3) are those of Hypothesis 12.4.

TABLE 14. The inputs of the proof of Theorem A.

	Statement used	Source	Where used
I1	Hypergeometric local systems over $\mathbb{Q}$ are of geometric origin, with finite hypergeometric sums as traces of Frobenius	[Kat90, Ch. 8], [BCM15]	(H1), Appendix B
I2	Levelt's normal form and irreducibility of the hypergeometric group; uniqueness of the invariant form	[BH89, Prop. 3.3, Thm. 3.5, §4]	Propositions 12.2 and 12.6
I3	Artin comparison and smooth proper base change, compatible with cup products and correspondences; specialisation of cycles	[SGA4, Exp. XI, XVI], [Ful98, Ch. 20]	Lemma 6.6

	Statement used	Source	Where used
I4	Rationality and independence of ℓ of characteristic polynomials of Frobenius on summands cut out by correspondences; algebraicity of Künneth projectors over finite fields	[Del74], [KM74]	Sections 5.3 and 13.1
I5	Hodge–Riemann relations	classical	(5.4), Propositions 12.6 and 12.7
I6	Mazur's theorem: Newton above Hodge; the image of Frobenius is governed by the Hodge filtration	[Maz73], [BO78, Thm. 8.26]	Proposition 13.5, Appendix B
I7	Crystalline cycle classes, compatible with pull-back and intersection	[GM87]	Lemma 5.8
I8	Supersingular K3 crystals: definition, Tate module, its discriminant, Hasse invariant and normal form	[Ogu79, Def. 3.1, 3.2, Thm. 3.3, 3.4, Prop. 3.13]	Proposition 13.5
I9	Hilbert reciprocity, invariants of quadratic forms, representation, Hasse–Minkowski and existence	[Ser73, Ch. III, IV]	Sections 5, 9, 10 and 20
I10	Chebotarev density for ℓ-adic images; analytic sets with empty interior have measure zero	[Ser81, n°2.1, n°6.4]	Theorem 13.9
I11	B is stable under products and specialisations; Arapura's lemma on motivated varieties	[Gro69, p. 196], [Ara06, Cor. 1.2, Lemma 4.2]	Theorem 21.5

C.1. Conditions of application

Inputs I3, I4 and I7 are applied to the smooth projective model X of (H1) at primes of good reduction, and to the correspondences e and σ of (H1) and (H2), specialised to the special fibre. Input I6 is applied to the summand H of Proposition 13.5, under the torsion-freeness and degeneration of (H3); the passage from the whole crystalline cohomology to the summand uses that the projector is integral at q and compatible with the Hodge filtration, which holds because it is induced by a correspondence defined over $\mathbb{Z}_{(q)}$. Input I8 needs $q > 2$ and an algebraically closed residue field, both satisfied. Input I2 is used in the form stated in Section 12.1, for the companion matrices of Appendix D; the identification of the monodromy at the fibre over -1 with the group generated by these matrices is part of (H1).

C.2. What is assumed

For H_2, the double cover of Section 12.4, its involution σ, and the Euler integral representation make (H2) explicit; the existence of a smooth projective model of the family over an open set stable under $t \mapsto 1/t$ and of an algebraic relative projector onto the hypergeometric summand is the content of (H1). For S_{33} and S_3 the same is assumed, and in addition (H2). Hypothesis (H3) holds after excluding finitely many primes for any model satisfying (H1) and (H2), as explained in Section 12.4.

Appendix D. Certificates

D.1. The three motives

For each of the three data, A and B are the companion matrices of $\prod_j (x - e^{2\pi i \alpha_j})$ and $\prod_j (x - e^{2\pi i \beta_j})$, with the coefficients of the polynomial in the last column, and S is the invariant symmetric form, scaled to be integral. The relations $A^{\mathsf{T}} S A = S$, $B^{\mathsf{T}} S B = S$, $B \sigma_{\mathrm{H}} A = -\sigma_{\mathrm{H}}$, $A \sigma_{\mathrm{H}} B = -\sigma_{\mathrm{H}}$, $\sigma_{\mathrm{H}}^{\mathsf{T}} S \sigma_{\mathrm{H}} = S$ and the Gram matrices of Table 4 are verified by the Lean kernel; the matrices are listed in Table 15. The characteristic polynomials of A are $(x+1)^5$, $(x+1)(x^2+x+1)^2$ and $(x+1)^3(x^2+x+1)$, and those of B are $(x-1)^5$, $(x-1)(x^2-x+1)^2$ and $(x-1)^3(x^2-x+1)$.

$\mathcal{M}$	A	B	S
H_2	$\begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & -5 \\ 0 & 1 & 0 & 0 & -10 \\ 0 & 0 & 1 & 0 & -10 \\ 0 & 0 & 0 & 1 & -5 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -5 \\ 0 & 1 & 0 & 0 & 10 \\ 0 & 0 & 1 & 0 & -10 \\ 0 & 0 & 0 & 1 & 5 \end{pmatrix}$	$\begin{pmatrix} 3 & 0 & -5 & 0 & 35 \\ 0 & 3 & 0 & -5 & 0 \\ -5 & 0 & 3 & 0 & -5 \\ 0 & -5 & 0 & 3 & 0 \\ 35 & 0 & -5 & 0 & 3 \end{pmatrix}$
S_{33}	$\begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 & -5 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & -3 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & 3 \end{pmatrix}$	$\begin{pmatrix} 4 & 0 & -5 & 0 & 13 \\ 0 & 4 & 0 & -5 & 0 \\ -5 & 0 & 4 & 0 & -5 \\ 0 & -5 & 0 & 4 & 0 \\ 13 & 0 & -5 & 0 & 4 \end{pmatrix}$
S_3	$\begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & -4 \\ 0 & 1 & 0 & 0 & -7 \\ 0 & 0 & 1 & 0 & -7 \\ 0 & 0 & 0 & 1 & -4 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -4 \\ 0 & 1 & 0 & 0 & 7 \\ 0 & 0 & 1 & 0 & -7 \\ 0 & 0 & 0 & 1 & 4 \end{pmatrix}$	$\begin{pmatrix} 5 & 0 & -7 & 0 & 29 \\ 0 & 5 & 0 & -7 & 0 \\ -7 & 0 & 5 & 0 & -7 \\ 0 & -7 & 0 & 5 & 0 \\ 29 & 0 & -7 & 0 & 5 \end{pmatrix}$

TABLE 15. The companion matrices A and B and the invariant form S of the three data (1.1).

D.2. Local certificates

The following identities of Hilbert symbols are verified by the kernel; they are instances of the hypotheses of Proposition 13.6 at the first inert primes, for a class of the shape given by Proposition 13.5, and they show that the hypotheses of the formal statements are not vacuous.

- H_2, $q = 7$, $C_{\mathrm{B}} \simeq \langle -1, -1 \rangle$, $r = 7$: $(7, -1)_7 = -1$ and $(7, -1)_2 = (-1, -1)_2$.

- S_{33}, $q = 7$, $C_{\mathrm{B}} \simeq \langle -2, -18\rangle$, $r = 7$: $(7, -36)_7 = -1$, and $(7, -36)_\ell = (-2, -18)_\ell$ for $\ell = 2, 3$.
- S_3, $q = 5$ and $q = 11$, $C_{\mathrm{B}} \simeq \langle -4, -48\rangle$, $r = q$: $(q, -192)_q = -1$, and $(q, -192)_\ell = (-4, -48)_\ell$ for $\ell = 2, 3$.

An earlier form of the argument for H_2 treated the Tate part $L \oplus C$ at an inert prime as one ternary block with the global form $\langle u, -1, -1\rangle$, $u = c\, d_S(4 - t^2)$ up to squares with $c = 1$, and computed the finite Hasse product β from the crystalline invariant -1. The kernel verifies the corresponding identity at the 24 inert primes $7 \le q \le 307$ at which the data were tabulated; for instance at $q = 7$ one has $u \equiv 17$ and $\varepsilon_\infty(\langle 17, -1, -1\rangle)\varepsilon_7(\langle 17, -1, -1\rangle) \cdot (-1) = 1$.

D.3. The 77 wall points

Table 16 lists the number of wall points of rank three, among the points tabulated, in each of the six families of Remark 9.8, at which the local identity (9.2) was verified by the kernel with $\varepsilon_p = -1$.

family α (with $\beta = 0^5$)	wall points
$(1/6, 2/6, 3/6, 4/6, 5/6)$, the Dwork family	10
$(1/2, 1/2, 1/2, 1/2, 1/2)$	56
$(1/4, 1/2, 1/2, 1/2, 3/4)$	5
$(1/8, 3/8, 1/2, 5/8, 7/8)$	1
$(1/10, 3/10, 1/2, 7/10, 9/10)$	0
$(1/4, 1/3, 1/2, 2/3, 3/4)$	5
total	77

Table 16. The wall points of rank three of Remark 9.8, by family.

Declarations

Authorship and the use of AI. WiseMotif.ai is an AI-native company, and the research reported here was carried out throughout with AI assistance, including the GAI-MUA engine for reasoning and the discovery of mathematical structure, for search and for formal verification. The ideas, the framework and the theory of Alien Hodge Motives originate with the author, who is responsible for the paper. The work is protected by copyright.

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Data and code. The Lean companion, the scripts of the exact computations and their output are deposited with the paper; Appendix A and Appendix B describe them.

Competing interests. The author declares no competing interests.

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