Alpöge's proof of the generalized Riemann hypothesis for Dirichlet L-functions

A detailed exposition

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Abstract

We present Alpöge's proof of the generalized Riemann hypothesis for Dirichlet L-functions. Its arithmetic assertion is an upper bound for a weighted sum of prime powers in a residue class. A quadratic twist isolates a hypothetical zero in the Mellin transform of this sum. Landau's theorem on Mellin integrals excludes the zero by the sign of a real residue. To prove the upper bound, one estimates the Gram determinant of a family of theta lifts. Divisibility of CM derivatives at finite places gives the prime sum. Bounds for congruences between Hecke eigenforms control the denominators in the reverse determinant inequality. At a cutoff x, the chain contains at least $\sqrt{x}$ lifts, which gives the exponent $1/2$ in the prime-sum bound.

1 Introduction

Let χ be a Dirichlet character modulo a positive integer q. Its L-function is initially defined by

$$L(s,\chi)=\sum_{n\geq 1}\chi(n)n^{-s},\qquad \Re s>1.$$

The zeros of its meromorphic continuation govern the error in counting primes in residue classes modulo q. The Euler product gives nonvanishing to the right of 1. The functional equation of a primitive character pairs a zero in the open critical strip at s with a zero of the conjugate character at $1-s$. The problem is to exclude zeros in the intervening half-strip $1/2<\Re s<1$ [Dav00, DLMF].

Theorem 1.1 (The generalized Riemann hypothesis for Dirichlet characters)**.** *Let χ be any Dirichlet character, including the character of conductor one. Its meromorphic L-function has no zero in $\Re s>1/2$. Every zero in $0<\Re s<1$ has real part $1/2$. For a principal character the point $s=1$ is a simple pole.*

Corollary 1.2 (The Riemann hypothesis)**.** *Every zero ρ of the Riemann zeta function in $0<\Re\rho<1$ satisfies $\Re\rho=\frac{1}{2}$.*

Proof. The unique Dirichlet character χ modulo 1 takes the value 1 at every positive integer. Thus $L(s,\chi)=\zeta(s)$ for $\Re s>1$ by their Dirichlet series, and everywhere by meromorphic continuation. The result follows from Theorem 1.1.

Lean. `DirichletCharacter.LFunction_modOne_eq`. Lean proves `riemannHypothesis` using the functional equation in `New.zeta_zeros_on_line`. □

Levent Alpöge obtained this proof with Claude [Alp26]. Section 7.4 gives the proof of Theorem 1.1. Its arithmetic input is an upper bound for a prime sum in an imaginary quadratic field. Odd powers of inert primes have weight two, while split primes have weight zero. For each rectangle considered below, the same quadratic field works for every character of the chosen modulus.

The prime sum that detects a zero

Put $Q = 18480q$ and choose a prime $d \equiv Q/2 - 1 \pmod{Q}$. Write $\chi_d(n) = (n/d)$ for the Legendre symbol, φ for Euler's function, and Λ for the von Mangoldt function, given by $\Lambda(p^\nu) = \log p$ for primes p and $\nu \geq 1$, with $\Lambda(n) = 0$ elsewhere. For real $y \geq 1$, let $R_{d,Q}(y)$ denote the following remainder, with the constant $c_{d,Q}$ of (2.1):

$$R_{d,Q}(y) = \sum_{\substack{n \leq y \\ n \equiv 1 \pmod{Q}}} \Lambda(n)(1 - \chi_d(n)) \left(\frac{1}{n} - \frac{1}{y}\right) - \frac{\log y}{\varphi(Q)} - c_{d,Q}. \tag{1.1}$$

Equivalently, $c_{d,Q}$ is the unique constant for which $R_{d,Q}(y) \to 0$ as $y \to \infty$. When $\chi_d(p) = -1$, the weight $1 - \chi_d(p^\nu)$ alternates between 2 and 0. At the ramified prime d it is 1. For each fixed Q, d as above and each $\varepsilon > 0$, the arithmetic estimate is

$$R_{d,Q}(y) \leq y^{-1/2+\varepsilon} \qquad (y \text{ sufficiently large}). \tag{1.2}$$

The Mellin transform of the remainder contains the difference between the logarithmic derivatives for $\psi\chi_d$ and for ψ, averaged over the characters $\psi \bmod Q$. At a putative zero ρ of an untwisted function, a twisted zero could cancel its residue. Proposition 3.1 chooses one d for which every twisted function is nonzero in a rectangle containing ρ and the relevant segment of the real axis. The untwisted multiplicities at ρ therefore add. The arithmetic bound makes the Mellin integral of the positive part of $R_{d,Q}$ converge for $\Re s > 1/2$. A pole at ρ would then force a real singularity of the transform of the negative part at its abscissa of convergence, by Landau's theorem. The residue formula there has the wrong sign for a nonnegative integral. Proposition 3.2 shows that the same argument allows gaps between cutoffs and a smaller set of primes d.

How a determinant bounds the prime sum

Fix d. In each interval $[16^k, 16^{k+1}]$ one can select a fourth power x using the inert primes alone. Proposition 7.5 shows that the estimate at this x controls the whole interval. Choose a second cutoff $x_1 > x$ and an auxiliary CM field E containing $K = \mathbb{Q}(\sqrt{-d})$, obtained by adjoining a root of unity to K. A CM field is a totally imaginary quadratic extension of a totally real field. Fix a complex embedding of K. Each of its extensions to E raises a chosen primitive Q-th root of unity to a power in $(\mathbb{Z}/Q\mathbb{Z})^\times$. These exponents label the fibres. Let A be the size of the fibre over 1, B the number of embeddings selected in the fibre over -1, and a the maximal derivative order. Give the selected embeddings weight a/x and all others weight a/x_1, as in Section 4. Let S be the finite-place remainder in (4.1), and define $R_{d,1}$ by (1.1) with $Q = 1$. Counting embeddings (Proposition 4.1) gives

$$\frac{S}{a} = B\{R_{d,Q}(x) - R_{d,Q}(x_1)\} + AR_{d,1}(x_1).$$

The Mellin transforms are regular at 1. Lemma 4.2 uses this to choose $x_1 \leq x^{O_{Q,d}(1)}$ with $|R_{d,Q}(x_1) - (A/B)R_{d,1}(x_1)| \leq x^{-2}$.

The columns of the determinant are successive theta lifts of forms on a definite unitary group. Differentiation of the theta kernel at a CM point gives divisibility at inert primes. In the product

formula these valuations cancel terms in the archimedean estimate, leaving $-mS$ after m lifts. The first and second moments of a Poincaré series bound its source norm from below and the codimension of its translates in the target from above. Lemma 6.3 gives their rank (6.3), and Proposition 6.2 bounds the column norms.

For the reverse bound, complete the columns to a basis. Projection onto Hecke eigenspaces can introduce denominators. Congruences between eigensystems give classes in Selmer groups whose sizes bound these denominators. Proposition 7.3 applies the product formula to the completed Gram determinant. The leading archimedean terms cancel, and the missing columns give a nonnegative term on the left. With $f = [E : K]$ and $b = a/x$, Theorem 7.4 gives, for some constant $C_{Q,d} > 0$,

$$mS \le C_{Q,d}\, f a \log b.$$

Here $m \ge \sqrt{x}$, $f = A\varphi(Q)$, and $\log b \asymp_{Q,d} (\log x)^6$. The average-free set in Proposition 5.2 gives $\log(A/B) \ll \sqrt{\log x \log\log x}$. Dividing by maB in the determinant bound and using the fibre identity gives

$$R_{d,Q}(x) \le x^{-1/2} \exp(C_{Q,d}\sqrt{\log x \log\log x})$$

with a constant $C_{Q,d} > 0$ for all sufficiently large x. This implies (1.2) for every $\varepsilon > 0$ with the same auxiliary construction.

The two twists in this proof have different quantifiers. The quadratic character in the Mellin transform must work for every $\psi \bmod Q$, and is fixed as x grows. In the determinant argument the automorphic twist may be chosen at each coefficient embedding, but its first m powers must all meet the local conditions. The determinant comparison in Section 7 keeps that choice fixed on both sides of the pairing.

Three estimates for one automorphic period

The proof bounds the divisibility of a Bessel period at finite places, its size at complex embeddings, and the denominators introduced by projection onto a Hecke eigenspace. The Bessel period is obtained by taking a Fourier coefficient along a unipotent group and pairing it with a form on a smaller unitary group. Gan, Gross and Prasad describe this restriction problem [GGP12, §§12, 23]. Here the test vector and the measures are fixed. Changing them between estimates would change the determinant.

The divisibility estimate uses the theta kernel. The Weil representation and its splitting on a unitary dual pair define this kernel [Wei64, Kud94]. Kudla's seesaw identity interchanges two period integrals, and the convergent Siegel–Weil formula and the Rallis inner-product formula identify the pairing of lifts [Kud84, Wei65, GQT14]. These identities allow the local derivative estimate of Section 20 to be applied to the Bessel period. At a complex embedding, the Chowla–Selberg formula evaluates a product of CM discriminants over the ideal classes of the imaginary quadratic field K [SC67][How19, Theorem 2.1.1]. The determinant uses a specified algebraic period root. Theorem 19.6 uses isogenies to identify its absolute value at each complex embedding fixing K and proves that the same root is a unit at every finite place. The comparison of these two kinds of size is the determinant-line calculation of Arakelov theory [Bos01, Proposition 4.5].

The projector can introduce denominators within a residual Hecke eigenspace. For a semisimple operator T with spectrum $\{\alpha, \beta\}$, $\alpha \ne \beta$, the projector onto the α-eigenspace is $(T - \beta I)/(\alpha - \beta)$. Congruent eigenvalues can therefore introduce a denominator at the coefficient prime. For the unitary groups here, Labesse's base change and the Galois attachment theorems identify the characteristic-zero eigensystems [Lab11, CH13, BLGGT14b, BLGGT14a]. Ribet's method relates congruences to extension classes [Rib76]. Poitou–Tate duality compares the resulting Selmer conditions with their annihilators, and the Bloch–Kato dimension formula describes the finite

local condition [Mil06, BK90]. At a place above the coefficient prime the Breuil–Kisin lattice comparison furnishes the integral realization [BMS18]. Part IV computes the length that bounds the denominator of the projector.

For the odd cyclotomic cocycles that enter the denominator estimate, Theorem 14.5 gives a finite-level length bound uniform in the coefficient prime ℓ. For odd ℓ, Poitou–Tate duality and the refined Coates–Sinnott theorem give the bound from generalized Bernoulli values. At 2 we use Stickelberger's theorem on the cyclotomic class group and count square classes in E [GP15, Was97]. At a place above ℓ, the local cup-product defect vanishes if each pair of test lattices becomes unramified over an extension of degree prime to ℓ (Corollary 15.8).

Part I proves the analytic criterion. Part II deduces Theorem 1.1 from the theta estimates. The remaining parts prove the dimension estimate by period and trace calculations (Part III), bound the index of the integral Hecke eigenspaces (Part IV), prove local divisibility (Part V), and estimate the completed determinant (Part VI). Each proof identifies the corresponding Lean source [Lean26]. The heading *Lean, formal context* gives the formal result to which the argument contributes. Appendix A records the source revisions and classical inputs.

Use of AI. The author used ChatGPT (OpenAI) to draft and revise this exposition, explore shorter arguments, and compare the written proofs with the cited Lean sources.

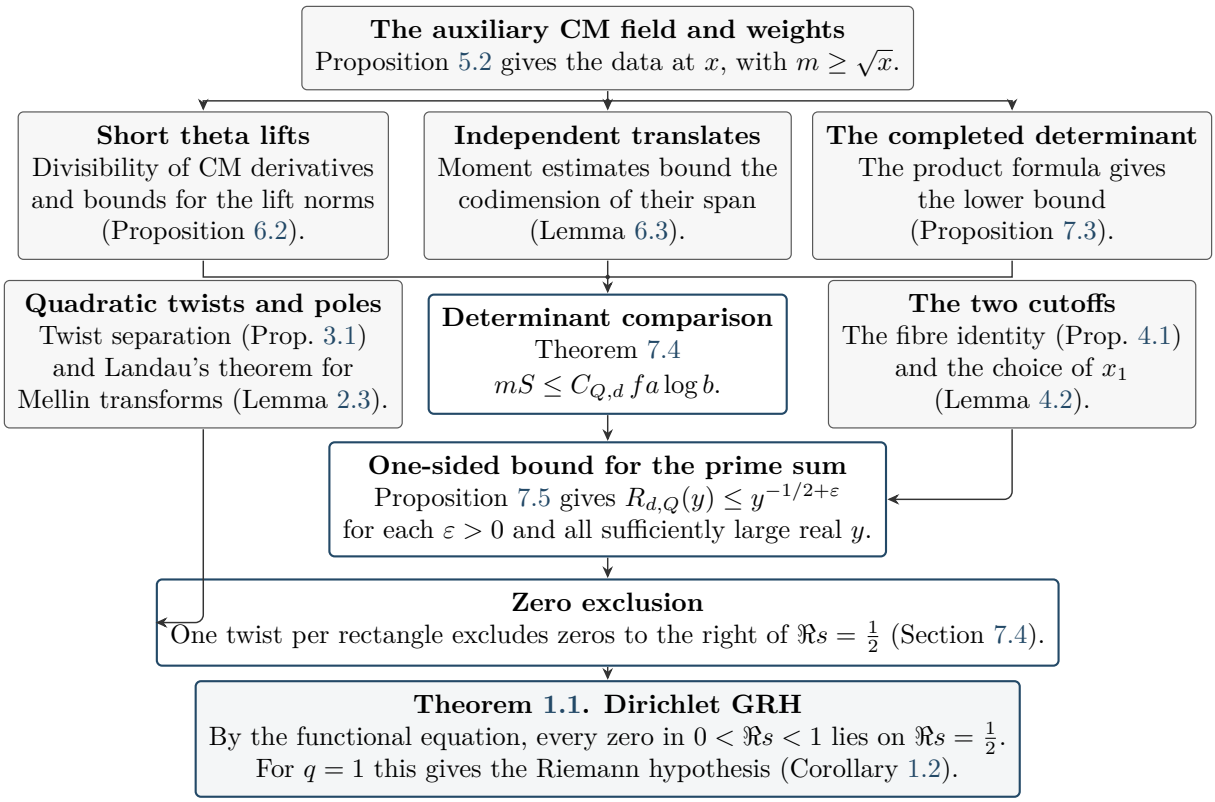

Figure 1: The arithmetic and analytic estimates in the proof of Theorem 1.1.

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Notation for the main argument

The analytic criterion uses q, Q, d and $R_{d,h}$. The remaining symbols refer to the field, weights and theta lifts chosen for a fixed integer cutoff x. The implicit constants in the final prime estimate may depend on the fixed modulus Q and the quadratic twist d.

Symbol	Meaning and first use
q, Q, d	Character modulus, enlarged modulus $18480q$, and the prime of the quadratic twist in (1.1).
R_h, c_h	The normalized prime remainder $R_{d,h}$ and its constant, for $h = 1, Q$, in (1.1) and (2.1).
x, x_1	The cutoff being estimated and the second cutoff in the fibre identity (Proposition 4.1, Lemma 4.2).
a, b, κ	The maximal derivative order $a = xb$ and the two weights $\kappa \in \{b, a/x_1\}$ (Section 4).
Φ, f, A, B	The chosen CM embeddings, their number, the size of the fibre over 1, and the number selected over -1. Thus $f = A\varphi(Q)$ (Section 4).
m, S	The number of theta lifts and the signed sum specified in (4.1). Their product occurs in Theorem 7.4.
D, n	The target dimension and the number of independent short columns. Their difference is at most Dx^{-204} by (6.3).
$\mathcal{A}$	The common leading term in the logarithms of the two determinant bounds. Its sign is proved in Section 6.
c, δ	An upper bound for the logarithm of the short-column Gram determinant, and the finite-place index bound in (7.1) and (7.2).

In Lean the modulus Q is q_0. For a choice of arithmetic data, the weight function `weight` is κ, `primeSum` is s_{fin}, and `jetSize` is s_∞. These functions are defined in Sections 4 and 6. The quantities S and $\mathcal{A}$ are `totalRemainder` and `mainAmount`, respectively.

Part I
A prime-sum criterion for zeros

We first assume the upper bound (1.2). The residue in (2.3) shows which zeros it can detect, and the choice of a quadratic character removes the only possible cancellation.

2 The Mellin transform of the prime sum

We determine the constant in (1.1). Fix Q, d as above and, for $h = 1$ or Q, write $R_h = R_{d,h}$, $c_h = c_{d,h}$, and

$$\lambda_h(n) = \mathbf{1}_{n\equiv 1 \pmod h}\Lambda(n)(1-\chi_d(n)), \qquad A_h(t) = \sum_{n\le t}\lambda_h(n), \qquad \mu_h = \frac{1}{\varphi(h)}.$$

Set

$$c_h = \int_1^\infty \frac{A_h(t)-\mu_h t}{t^2}\,dt. \tag{2.1}$$

Lemma 2.1. *The integral (2.1) converges absolutely. For every $y \ge 1$,*

$$R_h(y) = -\int_y^\infty \frac{A_h(t)-\mu_h t}{t^2}\,dt.$$

For some $c = c_{h,d} > 0$ one has $R_h(y) \ll_{h,d} \exp(-c\sqrt{\log y})$ as $y \to \infty$. If only the terms $2\log p$ for primes $p \equiv 1 \bmod h$ with $\chi_d(p) = -1$ are retained, and c_h is recomputed by (2.1) with this count, the resulting remainder $\widetilde{R}_h$ satisfies $0 \le \widetilde{R}_h(y) - R_h(y) \ll_{h,d} y^{-2/3}$.

Proof. On the units modulo dh above $1 \bmod h$, the residues modulo d run through all nonzero classes, and their Legendre symbols sum to zero. The prime number theorem in fixed progressions [MV07, Corollary 11.17] therefore gives $A_h(t) = \mu_h t + O_{h,d}(te^{-c\sqrt{\log t}})$, apart from powers of primes dividing dh. These powers contribute $O_{h,d}(\log(2t))$. Now $1/n - 1/y = \int_n^y t^{-2}\,dt$. Summing this equality and subtracting $\int_1^y \mu_h\,dt/t$ gives the identity. The substitution $u = \sqrt{\log t}$ bounds the tail by $O_{h,d}((1+\sqrt{\log y})e^{-c\sqrt{\log y}})$, and decreasing c absorbs the prefactor.

For $p \nmid d$ the even powers have weight zero. By $\sum_{p\le u}\log p \ll u$, the discarded cubes contribute $O(t^{1/3})$, and the odd powers of exponent at least five contribute $O(t^{1/5}\log(2t))$. The powers of d contribute $O_d(\log(2t))$. Integrating the discarded nonnegative count against $t^{-2}\,dt$ from y to infinity gives the bound for $\widetilde{R}_h - R_h$.

Lean. Lean treats the full prime-power sum by a Tauberian estimate. Here the fixed-modulus prime number theorem and the same tail integral also give the comparison with primes. See
`primeRemainder_eq_integral_countError_sub`, `eventually_abs_primeRemainder_le`. □

2.1 The poles of the Mellin transform

Form the meromorphic quotient

$$\mathcal{D}_h(s) = \prod_{\psi \bmod h}\frac{L(s,\psi)}{L(s,\psi\chi_d)}.$$

The product is over the Dirichlet characters modulo h. All the twists in its denominator are nonprincipal, since $d \nmid h$ and χ_d is ramified at d.

Lemma 2.2. *For $\Re s > 1$ the Mellin transform of the remainder is*

$$\mathcal{M}_h(s) := \int_1^\infty R_h(y) y^{-s}\, dy = -\frac{\mathcal{D}_h'(s)/\mathcal{D}_h(s)}{\varphi(h)s(s-1)} - \frac{\mu_h}{(s-1)^2} - \frac{c_h}{s-1}. \tag{2.2}$$

Its meromorphic continuation is regular at 1. If $z \neq 0, 1$ and the twisted functions are nonzero at z, then

$$\operatorname*{res}_{s=z} \mathcal{M}_h(s) = \frac{\sum_{\psi \bmod h} \operatorname{ord}_z L(\cdot, \psi)}{\varphi(h) z(1-z)}. \tag{2.3}$$

Proof. Character orthogonality in the logarithmic derivative of the Euler product gives $-\mathcal{D}_h'/\mathcal{D}_h = \varphi(h)\sum_n \lambda_h(n) n^{-s}$. Write $u_+ = \max(u, 0)$ for the positive part of a real number. The Mellin transform of $(1/n - 1/y)_+$ on $y \geq 1$ is $n^{-s}/(s(s-1))$. Termwise integration in the half-plane of absolute convergence, followed by subtraction of the transforms of $\mu_h \log y + c_h$, proves (2.2). By Lemma 2.1, $\int_1^\infty |R_h(y)|\, dy/y < \infty$. The transform therefore has a finite limit as real $s \downarrow 1$, so its possible pole there is removable. At $z \neq 0, 1$ the logarithmic derivative has residue $\operatorname{ord}_z \mathcal{D}_h$, giving (2.3).

Lean. The Euler-product quotient collects the individual orders summed in `isPolar_primeRemainder`. Lean uses the coefficients `lambdaKResidueClass` and the remainder `primeRemainder`. Its `logDerivMean` uses the pole-removed L-functions, subtracts 1 from the sum of twisted minus untwisted logarithmic derivatives, and divides by $\varphi(h)$. Its real value at 1 equals c_h, since both constants normalize the remainder to tend to zero. Lean computes the integral and cancellation at 1 in `integral_primeRemainder` and `ofReal_re_logDerivMean_one`. □

2.2 A sign criterion for Mellin poles

Lemma 2.3 (Landau's positivity principle)**.** *Let R be real and locally integrable on $[1, \infty)$. Suppose that $M(s) = \int_1^\infty R(y) y^{-s}\, dy$ converges absolutely for $\Re s > 1$ and extends meromorphically to $\Re s > \beta$, where $\beta < 1$. Suppose further that the Mellin integral of $R_+ = \max(R, 0)$ converges there and that every real pole of M there is simple with positive residue. Then M is holomorphic on $\Re s > \beta$.*

Proof. Let α be the abscissa of convergence for $R_- = \max(-R, 0)$. Its transform equals the transform of R_+ minus M, initially for $\Re s > 1$. If $\alpha > \beta$, Landau's theorem puts a singularity at α [Lay15, Theorem 3]. Indeed, otherwise the Taylor series of the transform at real $a > \alpha$ sufficiently near α, evaluated at $a - z$, would converge for some $z > a - \alpha$. Its coefficients as a series in z are $\int_1^\infty R_-(y) y^{-a} (\log y)^j\, dy/j!$. Monotone convergence would then give a finite integral at $a - z < \alpha$. Thus M would have a pole at α. Its positive residue would force the transform of R_- to tend to $-\infty$ from the right, contrary to nonnegativity. Hence $\alpha \leq \beta$, and both parts of R have holomorphic transforms in the asserted half-plane.

Lean. The formal criterion assumes a pointwise upper bound. Here integrability of the positive part suffices for the same abscissa argument. See `IsPolar.landau` and `mem_integrableExpSet_of_summable_taylor`. □

Lemma 2.4 (Boundary nonvanishing and reflection)**.** *For every Dirichlet character χ, $L(s, \chi)$ has no zero in $\Re s \geq 1$. If $0 < \Re \rho < 1$, then $L(\rho, \chi) = 0$ exactly when $L(1 - \rho, \bar{\chi}) = 0$.*

Proof. Let χ^* be the primitive character inducing χ. Passing to χ^* changes only Euler factors $1 - \chi^*(p) p^{-s}$, nonzero in $\Re s > 0$. A primitive character is a finite-order Hecke character of $\mathbb{Q}$. Boundary nonvanishing follows from [RV99, Theorem 7-28] and nonvanishing to its right from the Euler product. For conductor h and parity $j \in \{0, 1\}$, where $\chi^*(-1) = (-1)^j$, the factor completing $L(s, \chi^*)$ is $(h/\pi)^{(s+j)/2}\Gamma((s+j)/2)$, finite and nonzero in the open strip. The completed functional equation reflects (s, χ) to $(1-s, \bar{\chi})$ with nonzero root number, including the zeta case $h = 1$ [DLMF, §25.15(i)] [Dav00].

Lean. Mathlib proves the boundary statement directly in `LFunction_ne_zero_of_one_le_re`. The primitive reflection uses `gaussSum_mulShift_of_isPrimitive` and `completedLFunction_one_sub`. □

3 A quadratic twist avoiding a rectangle

A single prime must work for every character modulo Q, even when the rectangle meets the real axis. A second moment suffices for the full progression. The sharper count below also applies to sparse sets of primes.

Proposition 3.1. *Let $Q \geq 1$, $(a_0, Q) = 1$, $1/2 < \beta < 1$, and $T_0 \geq 0$. For $X \geq 2$ and every $\kappa > 0$, the number of zeros of $L(s, \psi\chi_d)$ in $\Re s \geq \beta$, $|\Im s| \leq T_0$, counted with multiplicity and summed over characters $\psi \bmod Q$ and odd primes $d \leq X$ with $d \equiv a_0 \bmod Q$, is*

$$O_{Q,T_0,\beta,\kappa}\Big(X^{E(\beta)+\kappa}\Big), \qquad E(\beta) = \min\left\{\frac{3(1-\beta)}{2-\beta}, \frac{8(1-\beta)}{3}\right\}.$$

Infinitely many primes in the class therefore avoid the rectangle for all ψ simultaneously.

Proof. Fix the primitive ξ inducing ψ. Since $d \nmid Q$, $c_d = \xi\chi_d$ is primitive of conductor $d\,\mathrm{cond}(\xi)$ [MV07, §9.1]. The omitted Euler factors are nonzero in $\Re s > 0$. With μ the Möbius function, put

$$M_d(s) = \sum_{n \leq X} \mu(n) c_d(n) n^{-s}, \qquad a_n = \xi(n) \sum_{k \mid n,\, k \leq X} \mu(k).$$

Then $L(s, c_d)M_d(s)$ has coefficients $a_n\chi_d(n)$, with $a_1 = 1$, $a_n = 0$ for $1 < n \leq X$ and $|a_n| \leq \tau(n)$, the number of divisors of n.

Choose one zero ρ for each exceptional prime d, and let $X \leq Y \leq X^{3/2}$. The Mellin transform of $(1-u)_+^2$ is $2/[w(w+1)(w+2)]$. Apply Mellin inversion to $L(\rho+w, c_d)M_d(\rho+w)$ with cutoff Y and move the line to $\Re w = 1/2 - \Re\rho$ [CZ23, §§2,4]. The zero cancels the pole at $w = 0$ and partial summation of the bounded character sums gives $L(s, c_d) = O_d(1 + |\Im s|)$ in the strip of integration, so the horizontal integrals vanish. Since $(1 - 1/Y)^2 \geq 1/4$, at least one of the two terms on the right of

$$\begin{aligned} 1 \ll_{\beta,T_0} Y^{1/2-\beta} \int_{\mathbb{R}} \frac{|L(1/2+iv, c_d)M_d(1/2+iv)|}{(1+|v|)^3}\, dv \\ + \left| \sum_{X < n \leq Y} a_n \chi_d(n) n^{-\rho} (1 - n/Y)^2 \right| \end{aligned} \tag{3.1}$$

is bounded below. The height of ρ is bounded and the new line stays at distance at least $\beta - 1/2$ from $w = 0$.

Write $\|\cdot\|_2$ for the norm over the primes d under consideration. Heath-Brown's quadratic large sieve [HB95, Corollary 3] gives

$$\left\| \sum_{n \leq N} b_n \chi_d(n) \right\|_2 \ll_\eta (XN)^\eta \sqrt{(X+N)N}\, \max_{n \leq N} |b_n|.$$

At $s = 1/2 + iv$, applying this to dyadic blocks up to X bounds both $\|M_d(s)\|_2$ and $\|\sum_{n \leq X} c_d(n) n^{-s}\|_2$ by $O_\eta(X^{1/2+\eta})$, on starting with a smaller exponent. The Pólya–Vinogradov inequality [MV07, Theorem 9.18] bounds interval sums of c_d by $O_Q(\sqrt{X}\log(2X))$. Partial summation from X to infinity therefore bounds the tail of each $L(s, c_d)$ by $O_Q((1+|v|)\log(2X))$. Thus $\|L(s, c_d)\|_2 \ll_{Q,\eta} (1+|v|)X^{1/2+\eta}$.

With $Y = X$ the finite sum in (3.1) is empty. Cauchy–Schwarz in d under the integral bounds the number of exceptional primes by $O_{Q,T_0,\beta,\eta}(X^{3/2-\beta+\eta})$. Take $\eta < \beta - 1/2$ and sum over the finitely many ψ. The prime number theorem in the progression proves the existence assertion.

For the sharper count, let H count the primes detected by the integral at the cutoff Y. Dahl's fourth moment for the fixed primitive twist ξ [Dah18, Theorem 3 in the author's version] and the Weyl bound [PY23, Theorem 1.1] give

$$\sum_d |L(s,c_d)|^4 \ll_{Q,\eta} X^{1+\eta}(1+|v|)^2, \qquad \max_d |L(s,c_d)| \ll_{Q,\eta} (X(1+|v|))^{1/6+\eta}.$$

Summing the integral over these primes and applying Hölder with exponents $(4,2,4)$ or $(\infty,2,2)$ gives, respectively,

$$H \ll_{Q,T_0,\beta,\eta} X^\eta \min\{XY^{(2/3)(1-2\beta)},\ X^{4/3}Y^{1-2\beta}\}.$$

For $X \le t \le Y \le X^{3/2}$, the same large sieve with $|a_n| \ll_\varepsilon n^\varepsilon$ gives $\|\sum_{X<n\le t} a_n\chi_d(n)\|_2 \ll_\eta X^\eta t$. The weight $t^{-\rho}(1-t/Y)^2$ vanishes at Y and its derivative is $O_{T_0}(t^{-\beta-1})$ for every selected zero. Partial summation and the triangle inequality bound the norm of the finite sum in (3.1) by $O_{\beta,T_0,\eta}(X^\eta Y^{1-\beta})$. Its square bounds the number of primes detected by that sum. Take $Y = X^{3/(2(2-\beta))}$ with the fourth moment and $Y = X^{4/3}$ with the Weyl bound. The two resulting bounds give $E(\beta)$. The zero-counting formula at height $\max(7, T_0+2)$ bounds the multiplicity count for each d by $O_{Q,T_0}(\log(2X))$ [BMOR21, Theorem 1.1]. Choosing η sufficiently small and summing over $\psi \bmod Q$ proves the stated multiplicity bound.

Lean. Lean uses Gaussian theta and Möbius sums in `sum_thetaTailIntegral_le`. For $8 \mid Q$ and $a_0 \equiv 7 \bmod 8$ its exponent is $1-(\beta-1/2)/18$ in `zero_density_quadTwist`, with existence in `exists_entireL_quadTwist_ne_zero`. Here the equal cutoffs and two second moments give existence for the full progression. Varying Y gives $E(\beta)$ for a fixed complex twist. □

3.1 The zero-exclusion criterion

Proposition 3.2. *Fix $q \ge 1$, put $Q = 18480q$, and let $\mathcal{G}$ be a set of primes $d \equiv Q/2-1 \pmod Q$. Write*

$$\delta = \limsup_{X\to\infty} \frac{\log(1+\#\{d\in\mathcal{G} : d \le X\})}{\log X}.$$

For each $d \in \mathcal{G}$, write $R = R_{d,Q}$, $R_+ = \max(R,0)$, and put $P(Y) = Y^{-1}\int_1^Y R(y)\,dy$. Suppose $\int_1^\infty \max(P(Y),0)Y^{-\sigma}\,dY < \infty$ for every real $\sigma > 1/2$. One sufficient condition is that there are cutoffs $1 \le a_1 < a_2 < \cdots \to \infty$, with $h_j = a_{j+1}-a_j$, such that for each $\varepsilon > 0$ and all sufficiently large $X = 2^k$,

$$\sum_{X\le a_j<2X}\left[h_j\{R_+(a_j)+R_+(a_{j+1})\} + \frac{\mu_Q h_j^3}{12X^2}\right] \ll_{d,Q,\varepsilon} X^{1/2+\varepsilon}. \tag{3.2}$$

Then no $L(s,\chi)$ with $\chi \bmod q$ has a zero in

$$\Re s > \min\left\{\frac{3-2\delta}{3-\delta}, 1-\frac{3\delta}{8}\right\}.$$

For $\delta = 1$, all its zeros in the open critical strip lie on $\Re s = 1/2$.

Proof. Fix d and write $R = R_{d,Q}$. The function

$$yR(y) + \mu_Q y\log y + c_{d,Q}y = \sum_n \lambda_Q(n)(y/n-1)_+$$

is convex. On $[a,b]$ its chord therefore bounds it from above. The chord of $y\log y$ exceeds $y\log y$ by at most $(y-a)(b-y)/(2a)$, since its second derivative is $1/y \le 1/a$. After division by y, the endpoint weights on $R(a), R(b)$ sum to one, giving

$$R(y) \le \max\{R(a),R(b)\} + \frac{\mu_Q}{2a^2}(y-a)(b-y). \tag{3.3}$$

On each $[a_j, a_{j+1}]$ this bounds $\int R_+(y)y^{-\sigma}\,dy$ by $X^{-\sigma}$ times the corresponding summand in (3.2), since $\int_0^h t(h-t)\,dt = h^3/6$. For real $\sigma > 1/2$, choose $\varepsilon < \sigma - 1/2$ and sum over dyadic X. This proves the stated condition on P, since Fubini's theorem bounds the transform of $\max(P, 0)$ at real $\sigma > 1/2$ by $1/\sigma$ times the transform of R_+.

Fix one d and let $1/2 < \beta < 1$, $T_0 \geq 0$. Suppose all denominator twists are nonzero on $\Re s \geq \beta$, $|\Im s| \leq T_0$, and the positive part of P has convergent Mellin integral for every real $\sigma > \beta$. Then

$$\operatorname{ord}_s L(\,\cdot\,, \psi) \leq 0 \qquad (\psi \bmod Q,\ \Re s > \beta,\ |\Im s| \leq T_0). \tag{3.4}$$

Indeed, for $\Re s > 1$, Fubini gives $\int_1^\infty P(Y)Y^{-s}\,dY = \mathcal{M}_Q(s)/s$. By (2.3) every real pole of this quotient in $(\beta, 1)$ is simple with positive residue. It is regular on $[1, \infty)$. Lemma 2.3 thus gives holomorphy on $\Re s > \beta$. At a numerator zero in the indicated rectangle the sum of multiplicities in (2.3) would be positive, proving (3.4).

Now suppose $L(\rho, \chi) = 0$ to the right of the asserted boundary. Lemma 2.4 gives $\Re\rho < 1$, so $\delta > 0$. Solving $E(\beta) < \delta$ in Proposition 3.1 gives the displayed boundary for β. Choose such a $\beta < \Re\rho$, $T_0 = |\Im\rho|$ and $0 < \kappa < \delta - E(\beta)$. The definition of the limsup gives a $d \in \mathcal{G}$ outside the exceptional set. For $\delta = 1$, the bound $O(X^{3/2-\beta+\eta})$ with $0 < \eta < \beta - 1/2$ already suffices. Inducing χ to modulus Q preserves ρ, since the extra Euler factors are nonzero in $\Re s > 0$. This contradicts (3.4). For $\delta = 1$ apply Lemma 2.4 to $\bar\chi$ for the other half of the strip.

Lean. Lean uses pointwise bounds for every prime in the class in `New.LFunction_ne_zero_of_primeRemainder_le`. Here convexity and the exceptional-prime count weaken both hypotheses. The real-axis test is `IsPolar.landau`. □

Part II
The arithmetic proof of the prime-sum bound

We now fix the prime d. At each large x, two weights on the complex embeddings of a CM field produce the exact identity (4.2). Estimates for the same family of theta lifts at all places give the bound for its left-hand side.

4 Two weights on the complex embeddings

Take integers $1 < x < x_1 \leq a$ and $b > 1$ with $a = xb$ and $x_1 \mid a$. For $n \geq 1$, let ζ_n denote a primitive n-th root of unity. We shall construct a cyclotomic extension $E = \mathbb{Q}(\sqrt{-d}, \zeta_{q_1})$ of the imaginary quadratic field $K = \mathbb{Q}(\sqrt{-d})$, with $Q \mid q_1$, $(q_1, d) = 1$, and q_1, Q having the same prime divisors. Fixing the embedding of K into $\mathbb{C}$ identifies one embedding from every conjugate pair of E with an element of $\Phi = (\mathbb{Z}/q_1\mathbb{Z})^\times$. Put $f = |\Phi|$. Let Φ_a be the whole fibre over $1 \bmod Q$, and let Φ_b be a nonempty subset of the fibre over $-1 \bmod Q$. Write $A = |\Phi_a|$, $B = |\Phi_b|$. Thus $f = A\varphi(Q)$. Define

$$\kappa(\sigma) = \begin{cases} a/x = b, & \sigma \in \Phi_b, \\ a/x_1, & \sigma \notin \Phi_b. \end{cases}$$

For an integer v, write $[v]$ for its class in Φ if it is a unit and for the identity of Φ otherwise. The finite contribution of a derivative of integer order $0 \leq j \leq a$ at $\tau \in \Phi$ is

$$s_{\mathrm{fin}}(j, \tau) = \sum_{n \leq a} \Lambda(n)(1 - \chi_d(n))\big(j/n - \kappa([-n]\tau)\big)_+.$$

The determinant estimate bounds the sum

$$S = \sum_{\tau \in \Phi_a} \left(s_{\mathrm{fin}}(a, \tau) - s_{\mathrm{main}} \right), \qquad s_{\mathrm{main}} = a \left(\log a - \frac{1}{f} \sum_{\sigma \in \Phi} \log \kappa(\sigma) + c_1 \right). \tag{4.1}$$

Proposition 4.1 (The two cutoffs)**.** *With these choices,*

$$\frac{S}{a} = B(R_Q(x) - R_Q(x_1)) + AR_1(x_1). \tag{4.2}$$

Proof. For a unit $n \bmod q_1$, multiplication by $[-n]$ is a bijection from Φ_a onto the fibre over $-n \bmod Q$. Precisely $B\mathbf{1}_{n \equiv 1 \pmod Q}$ of its images lie in Φ_b. For a nonunit both sides of this count vanish since q_1, Q have the same prime divisors and the two fibres are disjoint.

Write $F_h(z) = R_h(z) + \mu_h \log z + c_h$. Giving every place the cutoff x_1 first, then changing it to x on the counted fibre, gives

$$a^{-1} \sum_{\tau \in \Phi_a} s_{\mathrm{fin}}(a, \tau) = AF_1(x_1) + B\{F_Q(x) - F_Q(x_1)\}.$$

The mean of $\log \kappa$ is $\log(a/x_1) - (B/f)\log(x/x_1)$. Thus $As_{\mathrm{main}}/a = A(\log x_1 + c_1) + B\mu_Q \log(x/x_1)$. Subtracting leaves (4.2). The constant c_Q has cancelled between the two cutoffs.

Lean. Here s_{main} is `mainPrimeSum`. Lean counts the marked fibre for each integer and then sums. Here the common cutoff is subtracted before taking the sum. See `card_filter_mem_bPlaces`, `totalRemainder_eq`. □

Lemma 4.2 (Choice of the second cutoff)**.** *Fix Q, d and $J > 0$. Let I be a nonempty set of integers in $[X, 2X]$, and let $0 \le H_x \le x^{13}$ for $x \in I$. For all sufficiently large X there is an integer $X^{10^6+1} \le C \le X^{O_{Q,d,J}(1)}$ such that*

$$\left| \sum_{x \in I} (R_Q(Cx) - H_x R_1(Cx)) \right| \le X^{-J}.$$

Proof. Write $F(v) = \sum_{x \in I} \{R_Q(xv) - H_x R_1(xv)\}$ for $v \ge 1$. Both remainders are bounded by Lemma 2.1, so $|F(v)| \ll_{Q,d} X^{14}$. By Lemma 2.2 the two Mellin transforms $\mathcal{M}_h$, $h = 1, Q$, are holomorphic on $|s - 2| \le 1$, including at 1, hence on a common disk $|s - 2| \le r$ with $1 < r < 3/2$. Substitution $y = xv$ gives

$$\int_1^\infty R_h(xv) v^{-s}\, dv = x^{s-1} \left(\mathcal{M}_h(s) - \int_1^x R_h(y) y^{-s}\, dy \right) \qquad (\Re s > 1).$$

The right side is holomorphic and $O_{Q,d}(X^4)$ on the disk: there $\Re s > 1/2$ and $\Re(s-1) < 5/2$. Thus the continued Mellin transform of F is $O_{Q,d}(X^{18})$ there.

Let U_k have density $e^{-u}u^{k-1}/(k-1)!$ on $u > 0$. Substituting $v = e^u$ shows that $\mathbb{E}F(e^{U_k})$ is the coefficient of z^{k-1} in the transform of F at $s = 2 - z$. Cauchy's estimate bounds it by $O_{Q,d}(X^{18} r^{1-k})$. Since $\mathbb{E}e^{tU_k} = (1-t)^{-k}$ for $t < 1$, Markov's inequality gives

$$\Pr\{U_k \notin [k/2, 2k]\} \le (e^{1/2}/2)^k + (2/e)^k.$$

The conditional mean on this interval is therefore $O_{Q,d}(X^{18} e^{-ck})$ for some $c > 0$. By continuity it equals $F(v)$ at some $e^{k/2} \le v \le e^{2k}$. By the integral in Lemma 2.1 and $A_h(y) \ll_{Q,d} y$, rounding v up to C changes $F(v)$ by $O_{Q,d}(X^{14}/v)$. Taking $k = \lceil K \log X \rceil$ for a sufficiently large $K = K_{Q,d,J}$ proves the assertion.

Lean. The Mellin identity is `integral_primeRemainder`. Lean bounds each remainder at its prescribed second cutoff in `eventually_abs_primeRemainder_le`. Here averaging the sum gives one ratio C for all $x \in I$. □

5 The auxiliary field at a prescribed scale

The field must serve two purposes at each large integer x. Some of its complex embeddings must be sparse along every short Frobenius orbit, and two auxiliary norms must lie in specified intervals. We obtain the places from a set of integers in which no element is the average of a short list of other elements. The norms come from primes in progressions.

5.1 Primes in the auxiliary progressions

Lemma 5.1 (Primes in the auxiliary progressions)**.** *Let the prime divisors of $q = q(z)$ belong to a fixed finite set, and suppose $\log q = o(\sqrt{\log z})$. Then*

$$\#\{p \text{ prime} : z < p \le 2z,\ p \equiv 1 \bmod q\} \sim \frac{z}{\varphi(q)\log z} \qquad (z \to \infty).$$

Proof. The conductor of each primitive quadratic character that induces a character modulo q is the absolute value of a fundamental discriminant. It divides eight times the product of the primes in the fixed set. Only finitely many primitive characters can therefore supply an exceptional zero. Their L-functions are nonzero at 1, so every such zero is at most $1-\eta$ for some fixed $\eta > 0$. In the prime number theorem for progressions [MV07, Corollary 11.20, (11.32) and (11.34)], the exceptional term is therefore $o(z/\varphi(q))$. The error $O(ze^{-c\sqrt{\log z}})$ is also $o(z/\varphi(q))$ by the hypothesis on q. Subtract the estimates at $2z$ and z. Prime powers of exponent at least two contribute $O(\sqrt{z}\,(\log z)^2) = o(z/\varphi(q))$ to the von Mangoldt sum. Dividing the remaining sum by $\log z + O(1)$ proves the formula.

Lean. Lean obtains primes in logarithmic intervals in `New.exists_prime_modEq_one_log_mem`. Here the moduli have fixed prime divisors. □

5.2 Weights, places and levels

Proposition 5.2. *Fix Q, d. At every sufficiently large integer x there are auxiliary data as in Section 4, admissible for the theta estimates of Sections 6 and 7, with*

$$\begin{gathered}\sqrt{x} \le m < 2\sqrt{x}, \qquad \log(A/B) \ll \sqrt{\log x \log\log x},\\ \log x_1 \asymp_{Q,d} \log x, \qquad \log b \asymp_{Q,d} (\log x)^6.\end{gathered} \tag{5.1}$$

Here m is the length of the chain of theta lifts. Fix a nonempty piece of a dyadic interval $[X, 2X]$ on which m, r are constant, and prescribe an integer x_* in it. One can choose E, the marked embeddings, κ, and p_0, p_1, N once on the piece so that (5.1) holds throughout and $|R_Q(x_1) - (A/B)R_1(x_1)| \le x_*^{-2}$ at $x = x_*$. Alternatively, for any nonempty subset I of the integers on the piece, the common data can be chosen so that the sum of these signed errors over I has absolute value at most X^{-3}.

Proof. Choose m to be the least power of two at least $\sqrt{x}$, and r the least power of three at least x^5. Set

$$2^s = m^{22}, \qquad t = 5r, \qquad k = r + 1 - m, \qquad l_0 = 3^{\lfloor \log_3 s \rfloor + 4}, \qquad q_1 = 3l_0\, 2^s Q.$$

The field K has discriminant $-d$ and is disjoint from $\mathbb{Q}(\zeta_{q_1})$, since $d \nmid q_1$ ramifies in K. Thus $A = 3l_0\, 2^s$, and $E = K(\zeta_{q_1})$ has the indicated infinite places. In particular $i \in E$.

Put $o = 2^s$, $J = 2l_0$, so that $o \asymp x^{11}$ and $J \asymp \log x$. A set is J-average-free if an element equal to the average of at most J elements of the set equals every summand, repetitions allowed. By the Behrend construction [Beh46] in the form of Abboud, Bringmann, Hermelin and Shabtay [ABHS22, Lemma 3.2 of the conference version], such a set of n elements exists in $[0, J^{c/\varepsilon} n^{1+\varepsilon}]$ for an absolute

c and every $0 < \varepsilon < 1$. Take $\varepsilon = \sqrt{\log J/\log o}$ and n maximal with $J^{c/\varepsilon} n^{1+\varepsilon} \le (o-1)/J$. The resulting set T satisfies

$$\log(o/|T|) \le \varepsilon \log o + (1 + c/\varepsilon)\log J + O(1) \ll \sqrt{\log o \log J}.$$

Indeed n is the integer part of $((o-1)/J^{1+c/\varepsilon})^{1/(1+\varepsilon)}$, which tends to infinity. If $c_1, c_2 > 0$ are integers with $c_1 + c_2 \le J$, then $(c_1 + c_2)e_2 \equiv c_1 e_1 + c_2 e_3 \bmod o$ in T is an equality in $[0, o-1]$. Average-freeness gives $e_1 = e_2 = e_3$. Since $A = 3l_0 o$, we obtain $\log(A/|T|) \ll \sqrt{\log x \log\log x}$.

Put $g = 1 + 3l_0 Q$ and $\Phi_b = \{-g^e \bmod q_1 : e \in T\}$. For $v = v_2(Q) \ge 4$, lifting the exponent gives g order 2^s modulo 2^{s+v}. Thus $B = |T|$, and all the places lie over -1. The congruences to exclude for $\sigma_1, \sigma_2, \sigma_3 \in \Phi_b$ are

$$\sigma_2/\sigma_1 \equiv \pm p^i, \qquad \sigma_3/\sigma_1 \equiv \pm p^j \pmod{(q_1)_{p'}}, \tag{5.2}$$

where $(q_1)_{p'}$ is the prime-to-p part of q_1, $p \le kt$ is prime, and $0 < i < j < l_0$. For $p = 2$ the first relation gives $4^i \equiv 1 \bmod 3l_0$, whereas 4 has order l_0 there. For odd p, square the two relations and eliminate p modulo 2^{s+v}. The exponents in T satisfy $2je_2 \equiv 2(j-i)e_1 + 2ie_3 \bmod 2^s$. Thus they coincide. This would give $p^{2i} \equiv 1 \bmod 2^{s+v}$. But $2^{v_2(p^{2i}-1)} \le 2(p+1)i$, whereas $2(kt+1)l_0 < 2^s$ eventually since $kt = O(x^{10})$ and $2^s \ge x^{11}$.

Fix a dyadic interval $[X, 2X]$ and a piece on which m, r are constant. Then q_1, E, Φ_a, Φ_b can be chosen once on that piece, so $H = A/B$ is fixed before either scale is chosen. For each x on the piece, $H \le f \le q_1 = O_Q(x^{11}\log x) \le x^{13}$. Apply Lemma 4.2 to the singleton at $X = x_*$ with $J = 2$, or to the given I in $[X, 2X]$ with $J = 3$ and $H_x = H$. In either case set $x_1 = Cx$ throughout the piece. Put $U = 1 + \log(2CX)$, let b be the first multiple of $q_1 C$ strictly above e^{U^6}, and set $a = xb$. The two weights $b, b/C$ are multiples of q_1, and $q_1 x_1 \mid a$. Since $\log(q_1 C) = O_Q(U)$, rounding gives

$$U^6 < \log b \le U^6 + 1, \qquad U - 1 - \log 2 \le \log x_1 \le U - 1. \tag{5.3}$$

Here $\log C \asymp_{Q,d} \log X$, which proves (5.1).

To make the tail small at every point with a single family one may take $C = \lceil \exp(K(\log(2X))^2)\rceil$ instead. By Lemma 2.1, the tail is $O_{Q,d}((1 + A/B)e^{-c\sqrt{\log x_1}})$. Taking $c\sqrt{K} > 16$ makes it at most x^{-2} on the whole piece. This common choice has $\log x_1 \asymp_{Q,d} (\log x)^2$ and $\log b \asymp_{Q,d} (\log x)^{12}$. Both choices satisfy the same hierarchy below.

It remains to choose the auxiliary levels at these same scales. Write $L = \log b$ and $q_2 = (Qd)^{s+6}$. Every prime divisor of q_2 divides Qd, and $\log(q_2^2) = O_{Q,d}(\log x) = o(\sqrt{L})$. Let h_K denote the class number of K. Applying Lemma 5.1 at $z = b^{10}$ gives more than h_K primes $p \equiv 1 \bmod q_2^2$ in $(b^{10}, 2b^{10}]$. All these primes split in K. Choose one ideal above each of $h_K + 1$ of them. Two chosen prime ideals have the same class. Multiplying one by the conjugate of the other gives a principal ideal, so for their rational primes p_0, p_1 we have $M = p_0 p_1 = \alpha\bar{\alpha}$ with $\alpha \in \mathcal{O}_K$.

For the last prime, set $\bar{w} = f^{-1}\sum_{\sigma\in\Phi} \log\kappa(\sigma)$ and $\mathrm{rd}_E = |\operatorname{disc} E|^{1/[E:\mathbb{Q}]}$. Put

$$U_N = m\bar{w} + (m-1)\log\Big(\frac{M\,\mathrm{rd}_E}{2\pi}\Big) - \frac{15}{2}L.$$

The cyclotomic and relative discriminant formulas applied to $E \subset \mathbb{Q}(\zeta_{q_1 d})$ give $1 \le \mathrm{rd}_E \le q_1 d \le b$. As $b^{20} \le M \le b^{22}$, we have $20L - 2 \le \log(M\,\mathrm{rd}_E/(2\pi)) \le 23L$. Together with $0 \le \bar{w} \le L$, this gives $mL \le U_N \le 24mL$ for large x. Lemma 5.1 at e^{U_N} therefore gives a prime $N \equiv 1 \bmod q_2^2$ with $0 \le \log N - U_N \le \log 2 \le \sqrt{L}$. Since $N \equiv 1 \bmod 4$, Fermat's two-square theorem gives $\pi \in \mathbb{Z}[i] \subset \mathcal{O}_E$ with $\pi\bar{\pi} = N$. Choose p_0, p_1 once on the piece. Then M, $\bar{w}$, and every term in U_N are fixed, so one prime N and its generator work throughout. The auxiliary depth in Lemma 24.11 may also be fixed by evaluating its lower bound at the largest integer in the piece. This integer is

at most $2x$, so the resulting depth still satisfies the lemma's upper bound x^{40} for all sufficiently large X.

For later use the same choices give

$$(\log b)^{100} \le x, \quad x^{10^6} \le x_1, \quad x_1^{10^6} \le b, \quad (\log x_1)^6 \le \log b. \tag{5.4}$$

Indeed $L^{100} \ll_{Q,d} (\log x)^{1200} = o(x)$, while $C \ge X^{10^6+1}$ and $L > U^6$. Also $\log L \to \infty$, so it eventually exceeds any prescribed constant depending on Q, d. Also $s \le 32 \log x$, $27s < l_0 \le 81s$, so $q_1 = O_Q(x^{11} \log x)$, $f \le x^{13}$, and $5040 \mid q_1$.

Finally $n = m/2 \asymp \sqrt{x}$, $k \asymp x^5$, and the local depth is $s_{\mathrm{loc}} = k - 3n - 6$. Hence $n \ge 10^8$, $s_{\mathrm{loc}} \ge 10^{10} n^4$ and $k \ge 512000 n^9$ eventually. Since $\log b / \log x \to \infty$, also $f \le b/1000$, $k \le b$, $\mathrm{rd}_E \le b$, and $b^{6n+2} \ge 8r$. All thresholds depend only on Q, d.

Lean. The Lean datum includes $A \le B \log b$ (`Data.towerDeg_le_card_bPlaces_mul`). We keep A/B in (7.11). In Section 6 the sign of $\mathcal{A}$ follows term by term, independently of this ratio. The Lean record requires $q_1 x_1 \mid b$. Its weight formulas use $q_1 x_1 \mid a$ and $q_1 \mid b$, both supplied by the common choice on each piece. See `theoff-placeweight` (L90-L98) and `weightdivisibility` (L125-L131). We take $h_K + 1$ primes in one dyadic interval. The formal prime-pair lemma takes one from each of $h_K + 1$ disjoint intervals. See `Parameters.exists_prime_pair_norm` and `exists_three_term_free`, `Parameters.exists_large_isSparse`, `eventually_exists_data`. □

6 Divisibility and the number of independent lifts

Fix the field and levels of Proposition 5.2, and let $\Omega > 0$ be the CM period fixed in Section 19. For each integer $j \ge 0$, the archimedean contribution of a derivative of order j is

$$s_\infty(j) = \log(j!) + j \log(M/\Omega).$$

The integral and archimedean realizations use the same period. Write $L = \log b$ and define

$$\mathcal{A} = mk \sum_{\tau \in \Phi} \{ s_\infty(\kappa(\tau)) - s_{\mathrm{fin}}(\kappa(\tau), \tau) \}.$$

On each piece of Proposition 5.2, $\mathcal{A}$ is constant. Indeed, for $j = \kappa(\tau)$, a nonzero finite summand requires $n < j/\kappa([-n]\tau) \le C < a$, so its cutoff imposes no further restriction. The archimedean terms depend only on the common weights, field, and M. To see that $\mathcal{A} \ge 0$, observe that $\Omega \, \mathrm{rd}_E$ depends only on d by Lemma 19.8. Since $\mathbb{Q}(\zeta_{2^s}) \subset E$, the cyclotomic and relative discriminant formulas give $\mathrm{rd}_E \ge 2^{s-1}$. Thus $\Omega \le 1$ for large x, and $M \ge b^{20}$ gives $\log(M/\Omega) \ge 20L$. For $0 \le j \le a$ the summands defining $s_{\mathrm{fin}}(j, \tau)$ vanish at $n > x_1$. Thus $s_{\mathrm{fin}}(j, \tau) \le 4j \log x_1 \le jL$. Here $\sum_{n \le X} \Lambda(n)/n \le 2 \log X$ follows by counting the prime factors of $\lfloor X \rfloor !$.

Lean. `primeSum_le`, `mainAmount_nonneg`.

For each prime p put

$$s_p(j, \tau) = \frac{1}{2} \sum_{e \ge 1} (1 - \chi_d(p^e)) \big(j/p^e - \kappa([-p^e]\tau) \big)_+, \qquad g_p = m \left\{ \sum_{\tau \in \Phi_a} s_p(a, \tau) + k \sum_{\tau \in \Phi} s_p(\kappa(\tau), \tau) \right\}.$$

Only powers $p^e < x_1$ can contribute. At an inert prime one counts odd exponents. At the ramified prime d, every exponent enters with factor $1/2$. The number g_p is the valuation the local construction seeks to gain before allowing for losses at p. Since $s_{\mathrm{fin}}(j, \tau) = 2 \sum_p s_p(j, \tau) \log p$, the definitions give the exact identity

$$\mathcal{A} - mS + 2 \sum_p g_p \log p = mA s_{\mathrm{main}} + mk \sum_{\tau \in \Phi} s_\infty(\kappa(\tau)). \tag{6.1}$$

The right side is the leading archimedean term. The finite valuations will supply $2\sum_p g_p \log p$ in the product formula, up to the local losses.

Lean. `two_mul_sum_gain_mul_log`, `mainAmount_sub_add_two_mul_sum_gain`.

We first estimate the valuation of an ordinary derivative. On the completed algebraic closure $\mathbb{C}_p$ of $\mathbb{Q}_p$ normalize v_p by $v_p(p) = 1$ and $v_p(0) = +\infty$.

Lemma 6.1. *Let F be a power series over $\mathbb{C}_p$ in ν variables whose coefficients of total degree $\ell \geq 1$ are integral. If P is homogeneous of degree ℓ with integral coefficients, then*

$$v_p(P(\partial)F(0)) \geq \frac{\ell}{p-1} - \nu(1 + \lfloor \log_p \ell \rfloor).$$

Proof. For a multi-index α of degree ℓ, let $[z^\alpha]F$ denote the coefficient of z^α. Then $\partial^\alpha F(0) = (\prod_i \alpha_i!)[z^\alpha]F$. Legendre's formula gives $v_p(j!) = (j - \sigma_p(j))/(p-1)$, where $\sigma_p(j)$ is the sum of the base-p digits of j. Each $\alpha_i \leq \ell$ has at most $1 + \lfloor \log_p \ell \rfloor$ digits, each at most $p-1$. Summing over i gives the asserted lower bound for every monomial separately. The ultrametric inequality gives it for their sum.

Lean. The code bounds each factorial in the p-adic norm. Legendre's digit formula gives the same bound directly in valuations. See `norm_prod_factorial_le`, `norm_jet_le_of_isHomogeneous`. □

For the theta kernel the differential operators are products of minors. In the local coordinates supplied by Frobenius, the factorial estimate applies to the integral part of the expansion. Passing back to the original coordinates introduces the weights at the partner places $[-p^e]\tau$. Counting the minors with these weights and subtracting the losses from coincident indices gives the contribution g_p to the chain. Section 20 computes the loss at each prime and bounds their sum. Earlier work on the p-adic sigma function exhibits inert-prime divisibility for critical Hecke values of imaginary quadratic fields [BKY17].

Lean. `norm_jet_minorProd_le_of_hasNormalForm`, `le_divisibilityBound`.

6.1 Norms and valuations of the lifted columns

Let X be the space of forms on the definite unitary group of rank k with the prescribed weight and level, $X_{\mathrm{ram}} \subset X$ its ramified subspace at N, and V the space of good target forms on the unitary group of rank r at principal level. Here the good forms are the image of the central orthogonal projector e for the selected local types at p_0, p_1. The finite group Γ, a product of general linear groups over the chosen residue fields above these primes, acts unitarily on the ambient target. Write ρ for this action and $D = \dim V$. The target parameters in (17.1) depend only on r, m, κ. With the same finite type labels on each piece of Proposition 5.2, the target space and D are therefore fixed as $a = xb$ varies. Let E^+ be the maximal totally real subfield of E. Throughout, a twist ξ is a tame character of weight zero of the norm-one torus of E/E^+, admissible for m lifts. Its tame components ψ_w are supported on a finite set Q_ξ disjoint from its conjugate. Writing $k(w)$ for the residue field at w, the level has norm $\mathrm{N}Q_\xi = \prod_{w \in Q_\xi} \#k(w)$, prime to MN. At every $w \in Q_\xi$ one has $\psi_w^j \neq 1$ for $1 \leq j \leq m$. The trivial twist has empty support. The operator Θ_ξ is the composite of m theta lifts starting at V. Each lift integrates a form on one member of a unitary dual pair against the Weil theta kernel to give a form on the other member. The subscript 1 denotes the trivial twist. These automorphic twists are distinct from the Dirichlet character χ_d in the prime sum.

Lean. `MFoldTwist`, `injective_lift_targetLattice`.

The Poincaré map Π is determined by a period. With the Bessel subgroup R, the prescribed measure on its adelic quotient $[R]$, and a fixed real translate g_0, denote by $\mathcal{W}(u;\cdot)$ the Bessel

coefficient of a source form u for the prescribed test vector. For a target form y, the pairing is

$$\langle y, \Pi u\rangle = \int_{[R]} \mathcal{W}(u; r')\, \overline{y(r'g_0)}\, dr'.$$

Unfolding identifies this period with the Poincaré series in the basis of definite classes. On the ramified source, a seesaw expresses the lower theta chain applied to Πu, up to its normalization and restriction, by a finite combination of iterated theta kernels on the definite source. The restriction of the lower chain to V is Θ_ξ. Kudla's seesaw identity explains the interchange of the two period integrals. The level, support, coefficients and common scalar in this particular interchange are proved in Section 23, culminating in Theorem 23.37 [Kud84].

Lean. `poincare_eq_poincareMap_comp`, `seesaw_identity`.

The local divisibility estimates of Section 20 and Corollary 20.38 give the following bounds. There are losses $\epsilon_p \geq 0$ such that $\sum_p \epsilon_p \log p \ll faL$ on the constructed scales. They vanish off the nonsplit primes below x_1 and the level prime N. For each p-adic realization one can pass from the integral ramified source to a module of colength cost at most $\epsilon_p \dim X$ on which the lift has valuation at least $g_p - \epsilon_p$, simultaneously for twists of level prime to p. Colength cost e' means that an integral matrix of row operations of determinant valuation at most e' carries any given finite family into that module. This formulation also works over a nondiscrete valuation ring. The estimates at nonsplit primes come from the elementary divisors of Frobenius on CM derivatives. When indices in a Frobenius orbit coincide, an invariant average and the sparsity of Φ_b bound the resulting loss. At N the ramified source is used. Section 20 proves the nonsplit estimate and Corollary 20.38 treats the level prime. The sum of the losses is (11.9).

Lean. `exists_err_gaugeLe_compositeLift`, `exists_colengthDvd_gaugeLe_divisibilityError`.

Let $c_{\mathrm{ss}} \neq 0$ be the common seesaw scalar. For $g \in \Gamma$ put

$$B_g = |\Gamma| e\rho(g) c_{\mathrm{ss}}^{-1} \Pi\big|_{X_{\mathrm{ram}}}, \qquad h_p = g_p - \epsilon_p. \tag{6.2}$$

Both g_p and h_p vanish outside the nonsplit primes below x_1 and the level prime N.

Proposition 6.2 (Short lifts)**.** *There is a spanning set $\mathcal{S}$ of X_{ram} consisting of forms integral at all finite realizations such that, for every $u \in \mathcal{S}$, $g \in \Gamma$ and $\sigma \in \mathrm{Aut}(\mathbb{C})$,*

$$\|\Theta_1 B_g u\|_\sigma^2 \leq \exp\left(\mathcal{A} - mS + 2\sum_p g_p \log p + C_1 faL\right).$$

The constant C_1 depends only on Q, d. Here $\|\cdot\|_\sigma$ is the norm after applying σ to the value coordinates in the normalized coefficient representation. For a fixed g, any finite family from $B_g\mathcal{S}$ admits the row operations of (11.8) at each finite embedding. After these operations every lift at an allowed twist has valuation at least h_p in its integral gauge.

Proof. Apply the ramifying operator at N to the integral spanning set in Section 11. By (11.4) each resulting form has squared norm at most $\exp(72faL)$ at every embedding. Equation (11.16) and the finite projector add $\mathcal{A} - mS + 2\sum_p g_p \log p + O(faL) + x^{99}$ to the log norm; $x^{99} \leq b \leq faL$. At each finite embedding apply the row operations to the source family. The seesaw (11.11) identifies its lifted values, and $|\Gamma| e\rho(g)$ preserves the valuation bound by integral equivariance.

Lean. Lean multiplies the short columns by a rational product of prime powers. Here their finite valuations enter the determinant directly. See `span_isShort_ramifiedSourceForms_eq_top`, `weylNormSq_compositeLift_le`, `span_short_integralRamified_eq_top`. □

6.2 The dimension of the span of translates

We use the second moment of the Poincaré map to bound the codimension of the span of its translates. Write $\|C\|_{\mathrm{HS}}^2 = \operatorname{tr}(C^*C)$ for the squared Hilbert–Schmidt norm. For a self-adjoint operator A, A_+ is its positive part, defined by the spectral theorem.

Lemma 6.3. *Let a finite group Γ act unitarily on a complex Hilbert space V of dimension $D > 0$ by ρ. Let $F : X \to V$ be linear, where X is a complex Hilbert space of dimension $v > 0$, and put $\mu = v/D$. Write $\mathbb{E}_g$ for the uniform average over Γ, and put $C = \mathbb{E}_g\, \rho(g)FF^*\rho(g)^*$. Suppose that*

$$\operatorname{tr}(I_X - F^*F)_+ \le \epsilon v, \qquad D\operatorname{tr}(C^2) \le (1+\epsilon)v^2.$$

If $0 \le 3\epsilon < \eta < 1$, a set of at most

$$\left\lceil \frac{\log(1/\eta)}{\mu(1-\sqrt{3\epsilon/\eta})} \right\rceil$$

translates of FX spans a subspace of codimension at most ηD. Moreover $\mu \le (1-\epsilon)^{-1}$.

Proof. Truncate the eigenvalues of FF^* at 1, and denote the resulting positive contraction by Q_0. Put $Q_g = \rho(g)Q_0\rho(g)^*$ and $M = \mathbb{E}_g Q_g$. The singular values of F give

$$\operatorname{tr} M = v - \operatorname{tr}(I_X - F^*F)_+ \ge (1-\epsilon)v.$$

As $M \le I$, this gives the bound on μ. Since $0 \le M \le C$, eigenvalue monotonicity gives $\operatorname{tr} M^2 \le \operatorname{tr} C^2$. Consequently

$$\|I - M/\mu\|_{\mathrm{HS}}^2 = D - 2\operatorname{tr} M/\mu + \operatorname{tr} M^2/\mu^2 \le 3\epsilon D.$$

Let R project orthogonally onto the complement of the translates already chosen, and let $s = \operatorname{tr} R$. Adding $\rho(g)FX$ gains $\dim R\rho(g)FX$ dimensions. This is at least $\operatorname{tr}(RQ_gR)$, since RQ_gR is a positive contraction with range contained in $R\rho(g)FX$. Averaging and applying Cauchy–Schwarz gives a translate gaining at least

$$\operatorname{tr}(RM) \ge \mu\big(s - \sqrt{3\epsilon D s}\big).$$

While $s > \eta D$, this gain is at least $\mu(1-\sqrt{3\epsilon/\eta})s$. Starting at $s = D$, iteration and $1 - u \le e^{-u}$ give the stated number of translates.

Lean. The formal proof estimates the dimension of the inverse image of the chosen span. Here the Hilbert–Schmidt bound is tested on the orthogonal complement of the chosen span. See `isSpread_of_second_moment`, `IsSpread.exists_few_le_finrank_iSup_range`, `finrank_biSup_range_le`. □

Put $v = \dim X$. Let P be Π multiplied by the inverse square root of the product of the principal-level volume and the unipotent covolume. The identity class and a Bessel coordinate give $v > 0$. The first-moment bounds below give $\|eP\|_{\mathrm{HS}}^2 \ge (1 - o(1))v$, hence $D > 0$. Let $R : X \to X$ be the orthogonal projection onto X_{ram}, whose codimension is $v[1-(1-1/(N-1))^f] \le fv/(N-1) \le vb^{-8}$ by (11.6). Apply the lemma to $F = ePR : X \to V$. Since $I - F^*F = (I-R) + R(I-P^*P)R + RP^*(I-e)PR$, the first-moment bounds give

$$\operatorname{tr}(I - F^*F)_+ \le \operatorname{codim}_X X_{\mathrm{ram}} + \sqrt{v}\,\|I - P^*P\|_{\mathrm{HS}} + \|(I-e)P\|_{\mathrm{HS}}^2 \le (b^{-8} + b^{-6} + 4b^{-4})v.$$

Indeed one may take the trace against the spectral projection of $I - F^*F$ on its positive eigenspace and apply Cauchy–Schwarz to the middle term. Put $C_P = \mathbb{E}_g \rho(g)PP^*\rho(g)^*$. Since $0 \le R \le I$ and e commutes with ρ, the average of FF^* under conjugation is bounded by eC_Pe. Its Hilbert–Schmidt norm is at most $\|C_P\|_{\mathrm{HS}}$. By cyclicity of trace and invariance of the group average,

$\operatorname{tr} C_P^2 = \mathbb{E}_g \|P^* \rho(g) P\|_{\mathrm{HS}}^2$. The second-moment and dimension estimates give $D \operatorname{tr} C_P^2 \le (1+9b^{-4})v^2$ (Sections 8–10). Green's character formulas enter the estimate of the rejected finite types [Gre55].

Lean. `HasAmplitudeBlocks.isNearIsometry_piTensorElem_comp`, `targetDim_mul_sum_hilbertSchmidtNorm_sq_le`.

Apply Lemma 6.3 with $\epsilon = 9b^{-4}$ and $\eta = x^{-204}$. Then $3\epsilon = o(\eta)$, and $v/D \le (1-\epsilon)^{-1}$. The maps B_g on X_{ram} have the same images as $\rho(g)F$. From the translates in the lemma choose independent columns $B_g u$, $u \in \mathcal{S}$, spanning their sum. If T is the set of translates and n is the number of columns, then

$$|T|v \ll D \log x, \qquad D - n \le D x^{-204}. \tag{6.3}$$

Indeed the ceiling in the lemma adds at most v/D to $|T|v/D$. Corollary 24.44 bounds the cost of completing these columns to a basis by Db.

Lean. Lean restricts the source after selecting translates. Applying the moment bounds to ePR includes that restriction in the first moment. The auxiliary level in Section 24 allows the single cutoff x^{-204}. `sourceFormCoords`, `finrank_targetForms_ne_zero`, `le_finrank_ramifiedSourceForms`, `enough_forms`.

7 The two bounds for one determinant

Let $y_1, \dots, y_n$ be the columns chosen in (6.3). Hadamard's inequality and Proposition 6.2 give an upper bound e^c for the Gram determinant of the conjugated columns of $\Theta_1 y$, at every archimedean realization, where

$$c = n(\mathcal{A} - mS + 2\sum_p g_p \log p + C_1 faL). \tag{7.1}$$

We describe the finite-place information about precisely these columns.

For an isomorphism $\iota : \mathbb{C} \to \overline{\mathbb{Q}}_p$, write $\mathcal{O}_\iota$ for the pullback of the valuation ring and normalize $v_\iota(p) = 1$. An m-fold twist is allowed at p when $p \nmid \mathrm{N}Q_\xi$. Given real h_p of finite support, we say that y has index at most δ relative to these valuation bounds if there are $e_p \ge 0$ with $\sum_p e_p \log p \le \delta$ such that for each ι an integral matrix $u_\iota \in M_n(\mathcal{O}_\iota)$ has $v_\iota(\det u_\iota) \le e_p$, and each row of $u_\iota y$ has lifted valuation at least h_p at every allowed twist. Taking $h_p = 0$ gives the usual index for integral lifts.

For each translate use the local row operations on all its selected columns together, with h_p as in (6.2). Equation (11.12) bounds the total logarithmic index by $O(Dfa(1+\log x_1)\log x) = o(DfaL)$. In particular we may use

$$\delta = DfaL \tag{7.2}$$

at sufficiently large x. The resulting matrices at each embedding work simultaneously for all allowed twists.

Lean. `IndexLe`, `indexLe_normalizedPoincare`, `covolumeLe_of_normSqAt_le`.

The determinant as an arithmetic degree. The slope of a Hermitian lattice over a number field is the Arakelov degree of its determinant line divided by the product of the rank and the degree of the field over $\mathbb{Q}$. The metric on this line assigns to a wedge of columns the square root of their Gram determinant. For a map between fixed Hermitian lattices, Bost derives his slope inequality by summing the logarithms of the norms on the determinant line [Bos01, §4.1.1 and proof of Proposition 4.5]. Here the lift can change with the finite embedding. The squared norm of the untwisted exterior product is fixed. Lemma 7.2 carries out the degree calculation with the local changes of twist retained.

7.1 The comparison of single lifts

The lower bound comes from comparing twists. To state it we need a height for a family $w = (w_\iota)$, in which the number at a finite place may vary with ι. Write $h(w) \le C$ if there are a single $u \in \mathbb{C}^\times$, a real α, and $\lambda_p \ge 0$ such that

$$|\sigma u| \le e^{\alpha}, \qquad |\iota(uw_\iota)|_p \le p^{\lambda_p}, \qquad \alpha + \sum_{p \in F} \lambda_p \log p \le C$$

for all $\sigma \in \mathrm{Aut}(\mathbb{C})$, all ι above each p, and all finite prime sets F. Multiplication of the witnesses adds the bounds. They can be negative. For a fixed $z \neq 0$ we shall use

$$h((z)_\iota) \le C, \quad |\sigma z| \le e^{c} \ \ (\text{all } \sigma) \quad \Longrightarrow \quad 0 \le C + c. \tag{7.3}$$

Indeed u and z are algebraic, since a transcendental complex number has unbounded images under $\mathrm{Aut}(\mathbb{C})$. In a number field containing both, take the normalized sum of the logarithms of the absolute values of uz over all places. The product formula makes this sum zero. The bounds at infinity and at finite places give (7.3). For the classical product formula see Bombieri and Gubler [BG06, §§1.3–1.4].

Lean. `HeightLe`, `neg_le_of_norm_map_le`.

Let $Z(\xi) = \det(\Theta_\xi^* \Theta_\xi)$ on V. The chain is injective for each permitted m-fold twist. In the chain proof the first return map is a positive Gram operator. Exchanging adjacent lifts transports the inner-product identities across the triangular array of intermediate spaces to the final return. This is the injectivity used below. The Rallis inner-product formula is classical background for the theta pairings [GQT14]. Section 12 derives the exchanges and their scalars on the spaces used here.

Lean. `lowerChainAtLevel_injective`, `injective_lift_targetLattice`.

At a step $1 \le j \le m$, let u_ϑ be the lift on the selected local types with twist ϑ. Write P_ϑ for the orthogonal projection onto its image and put $Z_j(\vartheta) = \det(u_\vartheta^* u_\vartheta)$. Its source has dimension D by the packet comparison of Section 10. Gluing to the common theta kernel gives an operator T_ϑ on the target. Let ρ_ϑ be the ratio of the prescribed products of Hecke L-values at ϑ and at 1, as in (21.58). This scalar is nonzero, and $\rho_1 = 1$. Taking the adjoint of the gluing map on the lift image gives a single positive definite operator U_0 on the source such that, for every allowed twist at this step,

$$u_\vartheta^* T_\vartheta u_\vartheta = \rho_\vartheta U_0.$$

Write D_ϑ for the determinant of $P_\vartheta T_\vartheta$ on that image.

Lemma 7.1. *The twist quotient for a single lift is*

$$\frac{Z_j(1)}{Z_j(\vartheta)} = \frac{D_\vartheta}{D_1} \rho_\vartheta^{-D}.$$

Proof. Identify the source with $\operatorname{im} u_\vartheta$ by u_ϑ, and write C_ϑ for the compressed operator in this identification. Then $u_\vartheta C_\vartheta = P_\vartheta T_\vartheta u_\vartheta$. Multiplication by u_ϑ^* gives $(u_\vartheta^* u_\vartheta) C_\vartheta = \rho_\vartheta U_0$. Taking determinants at ϑ and 1 cancels $\det U_0$.

Lean. The code first identifies the common determinant $\det U_0$ for all twists. The quotient used here eliminates it as soon as the two determinant identities are written. See `inner_goodLift_comparisonOp`, `IsComparisonOperator.exists_normDet_sq_mul_det_eq`. □

The congruence theorem selects a twist ξ_ι allowed at p whose powers $\vartheta = \xi_\iota^j$ work simultaneously for $1 \le j \le m$. For each step its denominator bound supplies $\beta_p \ge 0$, with $\sum_p \beta_p \ll fmbL/k$, such that for some $C_{Q,d} > 0$ independent of the step,

$$|\iota D_{\xi_\iota^j}|_p \le e^{Dr\beta_p}, \qquad \log |\sigma D_1| \le C_{Q,d}\, D f m b L.$$

For the first estimate a Hecke operator clears the denominator of the projection to the lift image on the forms orthogonal to the unwanted lift spaces. The comparison operator preserves integrality and has this orthogonality on the lift image. Their composite thus has integral eigenvalues. For the second estimate the archimedean contraction is proved on tested forms after conjugation. The choice of covector of unit norm is made in the conjugated space.

Let O be a discrete valuation ring and $e_1 = (1, 0) \in O^2$. For distinct $\alpha, \beta \in O$, consider the endomorphism $T = \left(\begin{smallmatrix} \alpha & 1 \\ 0 & \beta \end{smallmatrix}\right)$ of O^2. The projector onto its α-eigenline is $P = (T - \beta I)/(\alpha - \beta)$. Its image on O^2 is $(\alpha - \beta)^{-1} O e_1$, whereas the intersection of that eigenline with O^2 is $O e_1$. The length of the quotient is exactly the valuation of $\alpha - \beta$. Thus the congruence of two eigenvalues already measures the cost of separating their eigenspaces in rank two.

In higher rank one bounds the denominators by Selmer groups. A congruence between the desired Hecke system and another one gives an extension class in a Galois lattice. Its local conditions place it in a Selmer group. Bounding the length of that group bounds the valuation lost when the Hecke operator separates the two systems. Ribet's construction is a classical instance of the passage from congruences to extensions [Rib76]. Here the estimates must hold for the ambient Galois representations as well as the chosen summand, for all coefficient primes and all m powers of the twist. The theorem providing these bounds also controls the inverse ratios of special values. In the notation above its consequence for the scalar in Lemma 7.1 is

$$h((\rho_{\xi_\iota^j}^{-1})_\iota) \le -\mathcal{A}/m + O(fmbL).$$

The construction of the separator is Theorem 25.7; the compression estimate is (25.30). For the test pair $(\chi_M \upsilon^{-2}, 2)$, the inverse ratio of adjusted norms has height at most $C_{\mathrm{crit}} < 1$ by (13.58). The main-pair bound is therefore $4\mathcal{B} + 2C_{\mathrm{crit}} - \mathcal{D}$ in (25.31). Counting the other ramified factors gives (25.32).

Lean. `New.Selmer.exists_mFoldTwist_selmerLengthLe_inv_ratio_le`, `New.exists_isRegularFamily`, `heightLe_liftDeterminant_ratio`.

Use D_1 as the common multiplier in $h((D_{\xi_\iota^j}/D_1)_\iota)$. The two determinant estimates bound this height by $O(Dfmb L)$, since $r \le 2k$. Adding D times the height of the inverse scalar and using $m^2 \le 4x$, $a = xb$, we obtain a constant $C_2 = C_2(Q, d)$ such that

$$h((Z_j(1)/Z_j(\xi_\iota^j))_\iota) \le D(C_2 faL - \mathcal{A})/m. \tag{7.4}$$

The determinant of the full chain factors as the product of the single-lift determinants times a scalar to the power D. In the quotient $Z(1)/Z(\xi_\iota)$, the additional scalar before this power is an inverse product of ratios of Hecke values and a rational power of the norm of the twist level. The latter is a p-adic unit at every allowed ι. For the simultaneous choice of twists, each of the at most $m^2 t$ factors has height less than one by (13.58). Their total cost is at most $m^2 t \le b$. Summing (7.4) yields

$$h((Z(1)/Z(\xi_\iota))_\iota) \le D(b + C_2 faL - \mathcal{A}). \tag{7.5}$$

This is the source of the leading term with the opposite sign.

Lean. `exists_hasSmallRatios_heightLe_liftDeterminant_div`, `heightLe_chainDeterminant_one_div`.

7.2 The product formula on the completed basis

Lemma 7.2 (The arithmetic degree of the untwisted determinant)**.** *Let $W \neq 0$ have algebraic coordinates in a positive definite rational symmetric metric, and write $\Delta = \langle W, W \rangle$. For each finite embedding let $q_\iota \neq 0$ be a local factor with $h((q_\iota)_\iota) \le A$. Suppose there are real numbers d_ι,*

nonnegative outside a finite set of rational primes, such that, with $\widehat{\iota}(z) = \iota(\bar{z})$,

$$|\iota(\Delta/q_\iota)|_p \le p^{d_\iota + d_{\widehat{\iota}}}.$$

Choose a conjugation-stable number field $F \subset \mathbb{C}$ *containing the coordinates of* W*, a multiplier* u *for the height bound, and* $\sqrt{-1}$*. For each prime* p*, extend the embeddings* $F \hookrightarrow \overline{\mathbb{Q}}_p$ *to isomorphisms* $\mathbb{C} \to \overline{\mathbb{Q}}_p$*, one per embedding, so that the resulting set* I_p *is closed under* $\iota \mapsto \widehat{\iota}$*. Then*

$$0 \le A + \frac{2}{[F:\mathbb{Q}]} \sum_{\sigma : F \hookrightarrow \mathbb{C}} \log \|\sigma W\| + \frac{2}{[F:\mathbb{Q}]} \sum_p \sum_{\iota \in I_p} d_\iota \log p. \tag{7.6}$$

Proof. The multiplier u is algebraic by its bound at infinity. On the finite embeddings of F, composition with complex conjugation pairs distinct embeddings since $\sqrt{-1} \in F$. Extend one of each pair arbitrarily and take its conjugate for the other. At infinity put $\sigma^\natural(z) = \overline{\sigma(\bar{z})}$. Rationality of the metric and Cauchy–Schwarz give

$$|\sigma\Delta| = |\langle \sigma^\natural W, \sigma W\rangle| \le \|\sigma^\natural W\| \, \|\sigma W\|.$$

Apply the number field product formula to the one element $u\Delta \in F^\times$ [BG06, §§1.3–1.4]. Each place occurs with its local degree when we sum over embeddings. At infinity $\sigma \mapsto \sigma^\natural$ permutes the embeddings, and at p the two index sums coincide by our choice of I_p. The remaining bounds for $|\sigma u|$ and $|\iota(uq_\iota)|_p$ sum, after division by $[F:\mathbb{Q}]$, to at most A. This proves (7.6).

Lean. The formal proof bounds every conjugate Gram determinant and uses one index allowance per rational prime. Summing first keeps the mean of the conjugate wedge norms and the individual index losses. See `neg_le_of_norm_map_le`, `CovolumeLe.norm_map_det_gram_le`, `covolume_add_index_nonneg`. □

Proposition 7.3. *If an independent family of* n *target forms has index at most* δ *relative to* h_p*, and its untwisted conjugate Gram determinants are at most* e^c*, then*

$$D(\mathcal{A} - C_2 faL - b) \le c + 2\delta - 2n \sum_p h_p \log p + (D-n)(bx^{203} + b). \tag{7.7}$$

Proof. The auxiliary spanning theorem supplies a single set from which we complete the family y by $n' = D - n$ forms z. It supplies scalars $\gamma_\xi \ne 0$, $\gamma_1 = 1$, such that $\gamma_\xi \Theta_\xi z_i$ is integral wherever ξ is allowed, $\|\Theta_1 z_i\|_\sigma^2 \le e^{bx^{203}}$, and $h((|\gamma_{\xi_\iota}|^{-2})_\iota) \le b$. To obtain these scalars, exchange adjacent lifts in two paths of theta integrals. At a diagonal exchange the identity between the column and row *kernels* supplies a factor λ. If a' is the square-exchange factor, the backward period comparison has denominator a'/λ. Multiplying over the squares fixes one scalar family before choosing the completion forms. The ratios for this family are derived in the comparison of the two period normalizations in Section 24.

Take the exterior product W_ξ of the completed lifted basis and put $\Delta_\xi = \langle W_\xi, W_\xi\rangle$. Injectivity gives $\Delta_\xi > 0$. Changing the source basis multiplies every Gram determinant by the same factor, so $\Delta_1/\Delta_\xi = Z(1)/Z(\xi)$. For $\xi = \xi_\iota$ set

$$q_\iota = |\gamma_\xi|^{-2n'} Z(1)/Z(\xi).$$

Equation (7.5) gives $h((q_\iota)_\iota) \le n'b + D(b + C_2 faL - \mathcal{A})$. Fix this twist in both arguments of the pairing. It is allowed at $\widehat{\iota}$ as well as at ι. The respective row operations give valuation at least h_p on each of the n short columns, and the γ_ξ-scaled completion columns are integral. Writing $j_\iota = v_\iota(\det u_\iota)$, we obtain

$$|\iota(\Delta_1/q_\iota)|_p \le p^{j_\iota + j_{\widehat{\iota}} - 2nh_p}, \qquad \Delta_1/q_\iota = |\gamma_\xi|^{2n'} \Delta_\xi. \tag{7.8}$$

Indeed, for rational $t < h_p$, multiply the transformed short columns on each side by a local scalar of valuation $-t$ at the corresponding embedding. Both families now have integral gauges. Their

mixed Gram determinant is integral, since the measured pairing is integral on these gauges and the final class stabilizers are trivial. Removing the scalars gives the bound with t in place of h_p; let $t \uparrow h_p$. Thus in Lemma 7.2 take $d_\iota = j_\iota - nh_p$. At infinity the exterior-product norm gives $\|\sigma W_1\|^2 \le e^{c+n'bx^{203}}$ by Hadamard's inequality. This uniform bound implies that each coordinate of W_1 is algebraic: a transcendental coordinate t could be sent to $t+N$ for every integer N by extending a transcendence basis. Substituting these two bounds into (7.6) proves (7.7).

Lean. The general Lean lattice inequality keeps $n' = D - n$ explicit. It uses integral lifts after a common rescaling. Here the signed valuations of the short columns contribute directly to the product formula. See `exists_span_isAuxiliaryForm_eq_top`, `covolume_add_index_nonneg`, `integral_forms_long`. □

Theorem 7.4 (The prime-sum inequality)**.** *There is a constant $C_{Q,d} > 0$ such that the data of Proposition 5.2 at every sufficiently large x satisfy*

$$mS \le C_{Q,d}\, fa \log b. \tag{7.9}$$

Proof. By (6.3), $(D-n)(bx^{203}+b) \le Db$. Substitute (7.1) and (7.2) into (7.7). The gain $2n\sum_p g_p \log p$ in c and the finite contribution $-2n\sum_p h_p \log p$ leave only $2n\sum_p \epsilon_p \log p = O(nfaL)$ by (11.9). Hence

$$nmS \le (n-D)\mathcal{A} + O(DfaL) + 2Db.$$

Since $n \le D$ and $\mathcal{A} \ge 0$, the first term on the right is nonpositive. Now $b \le faL$, $n/D \to 1$, and $L = \log b$. Division by n gives (7.9).

Lean. The formal proof uses one index bound per translate and enlarges the individual height errors before summation. We apply the block index to every selected translate at the cutoff in (6.3) and retain the individual height estimates in (25.33). See `star_inequality`, using `mainAmount_nonneg`. □

7.3 The one-sided estimate at every scale

Proposition 7.5 (The one-sided estimate)**.** *For each fixed prime $d \equiv Q/2 - 1 \pmod Q$ there is a constant $C_{Q,d} > 0$ such that, for all sufficiently large real y,*

$$R_{d,Q}(y) \le y^{-1/2} \exp\bigl(C_{Q,d}\sqrt{\log y \log\log y}\bigr).$$

In particular (1.2) holds for every $\varepsilon > 0$.

Proof. Fix $X = 16^k$. Among the integers $2^k \le n \le 2^{k+1}$, choose n_* maximizing

$$2 \sum_{\substack{p \le n^4,\ p \text{ prime} \\ p \equiv 1 \pmod Q,\ \chi_d(p) = -1}} (\log p)(p^{-1} - n^{-4}) - 4\mu_Q \log n, \qquad x = n_*^4.$$

This expression differs from $\widetilde{R}_Q(n^4)$ in Lemma 2.1 by a constant independent of n. Thus at every fourth power in the interval, $R_Q(n^4) \le R_Q(x) + O_{Q,d}(X^{-2/3})$. Between consecutive fourth powers u, v, the chord bound (3.3) costs at most $\mu_Q(v-u)^2/(8u^2)$. Since $v - u \ll X^{3/4}$ and $u \ge X$,

$$\sup_{X \le y \le 16X} R_Q(y) \le R_Q(x) + O_{Q,d}(X^{-1/2}). \tag{7.10}$$

At this integer x, Proposition 5.2 gives one choice of the field, marked places, weights and second cutoff with $|R_Q(x_1) - (A/B)R_1(x_1)| \le x^{-2}$. By (4.2), dividing (7.9) by $maB > 0$ gives $R_Q(x) \le C_{Q,d} fL/(mB) + x^{-2}$. Since $f = A\varphi(Q)$, $m \ge \sqrt{x}$, and $L \ll_{Q,d} (\log x)^6$, there is a constant $C'_{Q,d} > 0$ such that

$$R_Q(x) \le C'_{Q,d} \frac{A}{B} \frac{(\log x)^6}{\sqrt{x}} + x^{-2}. \tag{7.11}$$

Apply the bound on A/B in (5.1). Every sufficiently large real y belongs to such an interval, where $x \asymp y \asymp X$. Equation (7.10) therefore gives the asserted bound after increasing $C_{Q,d}$.

Lean. Lean estimates the remainder at every sufficiently large integer and compares a real cutoff with its floor. Here the prime-only sum and the chord bound reduce the determinant calculation to one fourth power per interval. See `Data.primeRemainder_le`, `primeRemainder_sub_primeRemainder_floor_le`, `eventually_primeRemainder_le_rpow`. □

7.4 From the determinant bound to the critical line

Proof of Theorem 1.1. For each integer $j \geq 3$, set $Q_j = 18480j!$ and $\beta_j = 1/2 + 1/j$. By Proposition 3.1 choose one prime $d_j \equiv Q_j/2 - 1 \pmod{Q_j}$ so that every $L(s, \psi\chi_{d_j})$, $\psi \bmod Q_j$, is nonzero for $\Re s \geq \beta_j$, $|\Im s| \leq j$. Fix j and write $R = R_{d_j,Q_j}$. For the cutoff selected in each sufficiently large interval $[16^k, 16^{k+1}]$, the short columns of Proposition 6.2 and (6.3) have Gram bound (7.1) and index bound (7.2). Combining both bounds with Proposition 7.3 gives $mS \leq C_{Q_j,d_j}\, fa \log b$ by Theorem 7.4. Equations (7.11) and (7.10) in Proposition 7.5 give $R(y) \leq C_j y^{-1/2+1/j}$ for all sufficiently large real y. The threshold may depend on j. Integrating gives $P(Y) = Y^{-1}\int_1^Y R(y)\,dy \leq C'_j Y^{\beta_j - 1}$, so the positive part of P has convergent Mellin integral for every real $\sigma > \beta_j$. Equation (3.4) excludes zeros of all numerator characters modulo Q_j for $\Re s > \beta_j$, $|\Im s| \leq j$.

For a putative zero ρ of $L(s,\chi)$, $\chi \bmod q$, with $\Re\rho > 1/2$, Lemma 2.4 gives $\Re\rho < 1$. Take j so large that $q \mid j!$, $\beta_j < \Re\rho$ and $|\Im\rho| \leq j$. Inducing χ to modulus Q_j preserves ρ, since the additional Euler factors are nonzero in $\Re s > 0$. This contradicts the exclusion just proved. Applying the functional equation in Lemma 2.4 to the conjugate character excludes zeros in $0 < \Re s < 1/2$. Finally, for the principal character $\chi_0 \bmod q$,

$$L(s,\chi_0) = \zeta(s)\prod_{p\mid q}(1 - p^{-s}).$$

The finite product at $s = 1$ is $\varphi(q)/q > 0$, also for $q = 1$ by the empty-product convention. The simple pole of ζ at 1 thus gives the asserted simple pole.

Lean. Lean uses the remainder bound for each prime in the progression. Here one twist suffices at each factorial modulus and height. See `star_inequality`, `eventually_primeRemainder_le_rpow`, `New.LFunction_ne_zero_of_primeRemainder_le`, `grh`, `New.zeros_on_line`, `LFunctionTrivChar_residue_one`. □

Proposition 7.6 (A signed mean and a prime sum on a fixed window)**.** *For a fixed prime d in Proposition 3.2, either of the following estimates, for every $\varepsilon > 0$, implies its hypothesis on P. For the first, take an integer x_0 large enough for Proposition 5.2. For each N divide $[x_0, N] \cap \mathbb{Z}$ into dyadic intervals and then into the pieces of that proposition, counting common endpoints once. On each piece use its common data for the signed sum. It suffices that*

$$\sum_{x_0 \leq x \leq N} \frac{S_x}{a_x B_x} \leq C_{Q,d,\varepsilon} N^{1/2+\varepsilon} \qquad (N \text{ a sufficiently large integer}).$$

For the second estimate fix $\lambda > 1$ and put

$$k_y(n) = \begin{cases} \lambda^{-1}(y^{-1} - n^{-1}), & y < n \leq \lambda y, \\ n^{-1} - (\lambda^2 y)^{-1}, & \lambda y < n \leq \lambda^2 y, \\ 0, & \text{otherwise.} \end{cases}$$

It suffices that for all sufficiently large real y,

$$2 \sum_{\substack{p \text{ prime},\ p \equiv 1 \pmod{Q} \\ \chi_d(p) = -1}} (\log p) k_y(p) \ \geq\ \mu_Q(1 - \lambda^{-1}) \log\lambda - C_{Q,d,\lambda,\varepsilon} y^{-1/2+\varepsilon}.$$

Proof. Write $R = R_{d,Q}$. Sum (4.2) over the pieces. Each dyadic interval $[X, 2X)$ has at most $2X$ nonempty pieces, each with total error at most X^{-3}, including the piece truncated at N. Thus $\sum_{x_0 \le x \le N}(R(x) - S_x/(a_x B_x)) = O_{Q,d}(1)$. The same conclusion follows from the singleton choices and $\sum_{x \ge x_0} x^{-2} < \infty$. Since $R(y) + \mu_Q \log y$ is nondecreasing, on $n - 1 < y \le n$ we have $R(y) \le R(n) + O_Q(1/n)$ for $n \ge 2$. Integrating and summing adds $O_Q(\log(2N))$, and the incomplete unit interval contributes $O_{Q,d}(1)$ by Lemma 2.1. The first estimate therefore gives $P(Y) \le C'_{Q,d,\varepsilon} Y^{-1/2+\varepsilon}$ for every $\varepsilon > 0$. Its positive part has the required Mellin integral.

The full sum $\sum_n \lambda_Q(n) k_y(n)$ exceeds the prime sum by $O_{d,Q,\lambda}(y^{-2/3})$. Indeed $0 \le k_y(n) \le 1/y$, and the discarded terms through $\lambda^2 y$ have total weight $O_{d,Q,\lambda}(y^{1/3})$ by the proof of Lemma 2.1. Put $Tf(y) = f(\lambda y)$. Then $(T-1)(T-\lambda^{-1})$ sends $(1/n - 1/y)_+$ to $k_y(n)$ and $\mu_Q \log y + c_{d,Q}$ to the displayed main term. For $D(y) = R(y) - R(\lambda y)$ the estimate gives $D(\lambda y) \le \lambda^{-1} D(y) + C y^{-\alpha}$ for each $0 < \alpha < 1/2$. Iterating from a fixed interval $[Y_0, \lambda Y_0]$ gives $D(y) \le C' y^{-\alpha}$, since $\lambda^{\alpha - 1} < 1$ and D is bounded on that interval. Lemma 2.1 gives $R(\lambda^j y) \to 0$, so telescoping yields $R(y) \le C' y^{-\alpha}/(1 - \lambda^{-\alpha})$. Its mean satisfies the required upper bound as well.

Lean. The pointwise argument is `eventually_primeRemainder_le_rpow` and `New.LFunction_ne_zero_of_primeRemainder_le`. The first alternative takes a mean before applying Landau's theorem, and the second uses two dilations to eliminate the centering constant. □

Part III
Auxiliary levels, automorphic periods and independent lifts

The short forms of Proposition 6.2 arise from Poincaré series on a unitary group. Their source is definite. Unfolding the inner product twice expresses it as a sum over rational elements. The first two moments of this sum give a lower bound for the dimension of the image. We then pass to integral forms and prove that the ensuing theta lifts are injective.

8 The Bessel period and the first moment

We express the Poincaré map of Section 6 as a Bessel period. The identity term in the unfolded inner product gives the source norm. The other terms bound the change in norm under projection onto the prescribed finite types.

On a definite unitary group, Gross's algebraic modular forms are functions on a finite adelic double quotient with the prescribed coefficient representation [Gro99]. Choosing representatives expresses their inner product as a sum with reciprocal stabilizer orders. The Bessel period below has this normalization on its definite factor. Section 10 computes the dimension of the image of this finite projector by comparing the same automorphic packets on the definite and indefinite unitary groups.

Fix the modulus parameter Q, its auxiliary prime and an admissible datum D as in Proposition 5.2. For the zeta function, $Q = 18480$. We use the levels, Haar measures and Bessel functions of D. Its CM field E contains a primitive 5040-th root of unity. The prescribed lattice frame determines a definite Hermitian space X_0. Let X be the space of source forms on $U(X_0)$ and put $v = \dim X$. For each selected place w over either level prime, write κ_w for its residue field. The group $G = \prod_w \mathrm{GL}_r(\kappa_w)$ acts unitarily by ρ on the principal-level target forms of the prescribed archimedean type. Let e be the central orthogonal projection onto the finite types specified in Section 6, let H be its image, and put $d_0 = \dim H$. Fitting and admissibility.

The Bessel period. Let R be the Bessel subgroup of $U(Y_0)$ and let $[R] = R(E^+)\backslash R(\mathbb{A}_{E^+})$. There are a real point g_0 and a linear functional t such that for every source form u and target form y of the prescribed principal type the Poincaré map Π satisfies

$$\langle y, \Pi u\rangle = \int_{[R]} \mathcal{W}_t(u; r)\,\overline{y(rg_0)}\,dr.$$

Here $\mathcal{W}_t(u;\cdot)$ is the Bessel function associated to u and t, and dr is the Bessel measure of D. Unfolding gives the same pairing with the Poincaré series in the centrally transported class coordinates of u. Uniqueness of the Riesz representative identifies the two maps. In these coordinates put $L = (\mathrm{vol}(J)\nu)^{-1/2}\Pi$, where J is the principal level and $\nu > 0$ is the unipotent covolume. Let $e_{c,i}$ be the coordinate vector at the definite class c and coefficient index i. For $s \in G$, unfolding twice gives

$$\langle \rho(s)Le_{c',j}, Le_{c,i}\rangle = \nu^{-1} \sum_{\gamma \in \mathcal{T}(c,s,c')} A_\gamma(c, i; c', j).$$

The indices are the rational returns with the prescribed integral reduction s. Each amplitude uses both class cutoffs and the full Bessel functions $W_i(y)\overline{W_j(\gamma^{-1}y)}$. Period identity; Petersson unfolding; positive scale and normalized blocks.

The identification, including its central translation and Haar factors, is proved on p. 118. The source Hilbert norm is the Euclidean norm in those coordinates.

The first-moment estimate. Write $P_s = L^*\rho(s)L$ and $r_\psi(s)$ for its contribution from the reduced Bessel subgroup R_0. For $i = 0, 1$ let Ω_i consist of the reductions agreeing with R_0 at the indicated prime. We shall prove

$$\sum_{s \in \Omega_i} \|P_s - r_\psi(s)\|_{\mathrm{HS}}^2 \le b^{-12}|R_0|v.$$

On R_0, equivariance identifies each summand with $\|L^*L - I\|_{\mathrm{HS}}^2$. The character calculation at the end of this section bounds the contributions from the excluded types at the two level primes. For each subspace $U \subset X$, with orthogonal projection P_U, it gives $\|eLP_U\|_{\mathrm{HS}}^2 \ge \dim U - 5b^{-4}v$. Return and identity estimates; rejected energy; the actual projector. Taking $U = X$ shows that the good target has positive dimension, since the source is nonzero and $b \ge 1000$.

8.1 The weighted count for two returns with the same rational relation

Let A denote the isotropic subspace spanned by the flag. To each rational return γ associate the linear relation in $(A^\perp/A)^2$ formed by the pairs $([y], [\gamma^{-1}y])$ with $y \perp A + \gamma A$. Here brackets denote reduction modulo A. Fix a set Ω of reductions. For $s \in \Omega$ let $\mathrm{meetNorm}(s)$ be the product of the residue norms over all places above $M = p_0p_1$ where the frame and its translate have a nonzero intersection after reduction. Both places in each conjugate pair occur in this test. Choose a positive integer z with $z \le \mathrm{meetNorm}(s)$ for every $s \in \Omega$. A return γ is called near if y and $\gamma^{-1}y$ belong to the prescribed archimedean box for some y. A reduction is called agreeing if conjugate reductions of integral Bessel unipotents have the same character. For classes c, c', let $\mathcal{P}_z(c, c')$ consist of the ordered pairs of chosen near returns over a common agreeing $s \in \Omega$, with the same rational relation and with both discriminants a_i below nonzero. Put

$$C = \frac{v^2}{\deg_{\mathrm{Bess}} \mathrm{mass}_r}, \qquad \tau = b^{kf/4}, \qquad n_p = \frac{|\bar{R}_0|\,\#\mathcal{C}}{N^{kf}},$$

where $k = \dim_E X_0$ is the definite rank, $f = [E : \mathbb{Q}]/2$, $\mathcal{C}$ is the original class set, $\bar{R}_0$ is the reduced Bessel subgroup, and N is the datum's auxiliary split level. $\deg_{\mathrm{Bess}}$ is the product of the local Bessel degrees, and mass_r is the genus mass of the standard rank-r Hermitian lattice. For each return γ_i, the level factor G_i and the weight w_i bounding its orbit contribution are given in the formulas on p. 28 below. We shall prove

$$\sum_{c,c'} \sum_{\mathcal{P}_z(c,c')} w_1 w_2 \;\le\; \frac{\tau^7 C}{z^k} \;\le\; \left(\frac{b^{2f}}{z}\right)^k C. \tag{8.1}$$

The second inequality uses $\tau^7 = b^{7kf/4} \le b^{2kf}$. The covolume factor ν^2 enters when we integrate the periods. Pinned pair theorem.

The two frames. Write $A = \langle e_1, \dots, e_\varrho \rangle_E$ for the isotropic flag space and $F = (e_1, \dots, e_\varrho, z_0)$ for the frame, with $h(z_0, z_0) = -N$. Put $n = \varrho + 1$. The rational ambient space has dimension $2\varrho + k + 1$, and $\langle F \rangle^\perp = A \oplus X_0$. For each return put

$$Z_i = F^* h \gamma_i F, \quad \Delta_i = \det Z_i, \quad \Delta_i' = \det Z_i[1:\varrho, 1:\varrho], \quad a_i = \Delta_i \bar{\Delta}_i - N^2 \Delta_i' \bar{\Delta}_i'.$$

Here $1:\varrho$ includes both endpoints. Each near representative under consideration has a cell rank. Since $a_i \neq 0$, that rank is n, and its leading minor Δ_i is nonzero. Writing $\mathfrak{d}_i = \prod_{\sigma \bmod \mathrm{conj}} |\sigma(a_i)|$, the two isotropic row indices in the level factor are one, since all ϱ rows are selected. Hence on these terms

$$G_i = M^{2\varrho f} \max\left\{ \frac{|\operatorname{Nm}_{E/\mathbb{Q}} \Delta_i|}{N^f}, \mathfrak{d}_i \right\}, \qquad w_i = \tau^2 \sqrt{\frac{C}{n_p G_i^k}}.$$

Indeed both top-cell second moments equal C. Cell and discriminant, minor and top row indices, gauge and size, cell moment.

The common space has corank one. Restrict both coordinates of the rational relation to X_0. The resulting pairs are precisely the images of the vectors $y \in B_i^\perp$, where $B_i = \langle F, \gamma_i F \rangle_E$. Indeed the two additional equations are orthogonality to z_0 and to $\gamma_i z_0$. With $J_F = \mathrm{diag}(0, \dots, 0, -N)$ the full Gram determinant is

$$\det \begin{pmatrix} J_F & Z_i \\ Z_i^* & J_F \end{pmatrix} = (-1)^n a_i.$$

In the determinant expansion a term uses both terminal diagonal entries or neither. The latter terms give $(-1)^n |\Delta_i|^2$, and the former give $(-1)^{n-1} N^2 |\Delta_i'|^2$, where $|u|^2 = u\bar{u}$. Thus B_i is nondegenerate of dimension $2n$. The projection onto the first middle coordinate preserves the form, since every lift has the form $x + a$ with $x \in X_0$, $a \in A$. It is injective because $a \in A$ orthogonal to $\gamma_i F$ gives a relation among the first ϱ rows of the invertible Z_i. In particular the second middle coordinate is uniquely determined by the first. Since the relations agree, both projected spaces are the same space W, and

$$\dim_E W = 2\varrho + k + 1 - 2n = k - 1.$$

Thus the common lattice has corank one in X_0. The two rational relations coincide, so their intersection has codimension zero. Common space, rank calculation.

Meeting places after saturation. Work in the self-dual finite lattice Λ in class coordinates. Write g_i for γ_i in the source and target class bases, so $g_i\Lambda = \Lambda$ and both matrices reduce to s. These bases fix the frame columns and preserve their products. The middle coordinates of a vector in X_0 retain the prescribed π_N-divisibility at the distinguished coordinate, with π_N the finite idele denoted `S.piN` in the inclusion matrix. Class transport, integral Gram matrix.

Fix one $g = g_i$. Write A_Λ for the integral isotropic columns, $S_0 = A_\Lambda + gA_\Lambda$ and $S = \langle F, gF\rangle_{\widehat{\mathcal{O}}_E}$. Saturation means intersection of the ambient lattice with the span over the local fields, place by place. Put $t = [S_0^{\mathrm{sat}} : S_0]$. Independence of the full set of columns gives $S \cap S_0^{\mathrm{sat}} = S_0$. Consequently

$$[S^{\mathrm{sat}} : S] = t\,[S^{\mathrm{sat}} : S + S_0^{\mathrm{sat}}].$$

The second index supplies the factor from the primes of nonzero intersection.

At a meeting place let $\bar{g}$ denote the action on the actual residue frame. A common nonzero vector has an expression $Fx = \bar{g}Fy$. Let j be the last position at which its two coefficient lists are not both zero. If $j < n$, take any covector λ annihilating the first j columns of both frames. The rank-one unipotent $1 + r(Fx)\lambda$ has, in the other frame, the conjugate expression $1 + r(Fy)(\lambda\bar{g})$. These are admissible tests of character agreement for every residue scalar r. Indeed the integral Eichler transformation is

$$Y \longmapsto Y + h(v,Y)u - h(u,Y)\big(v + \xi h(v,v)u\big), \qquad \xi + \bar{\xi} = 1.$$

Take u in the isotropic prefix lifting rFx and choose v lifting the covector by unimodularity and annihilating the prefix. At a split place choose the coefficients of u with zero conjugate component. This gives the displayed rank-one reduction, with its unitary companion at the conjugate place. The other split pairs can be kept at identity. One may take $\xi = \zeta_6$ in E. The simple-root sum of the transformation in each frame equals the covector applied to the coefficient list shifted one position up. Only position j contributes, since λ kills the prefixes. Thus agreement and nontriviality of the residue character as r varies give $\lambda(x_jF_{j+1} - y_j\bar{g}F_{j+1}) = 0$. Annihilator duality puts this difference in the sum of the two prefix spaces. Adjusting the coefficients there produces a relation one position higher with the same nonzero coefficient pair. Induction gives a relation with terminal coefficients not both zero. Integral test and its character, split-place lifts, flag induction.

In $\Lambda/S_0^{\mathrm{sat}}$ this relation makes the images of the two terminal columns dependent modulo the place. The quotient is locally free and these two columns have rank two over the fraction field. Smith normal form therefore puts at least one uniformizer in the index of their span in its saturation. Multiplying the additive residue sizes over all meeting places, including the conjugate places at which the test succeeds, gives

$$[S^{\mathrm{sat}} : S] \;\geq\; t\ \mathrm{meetNorm}(s) \;\geq\; tz.$$

This gives the entire factor at each such prime even if S_0 already has nontrivial saturation index there.

The complementary discriminant. The positive Hermitian discriminant $d(P)$ is the square root of the absolute index of the Gram map. Thus an additive sublattice index multiplies $d(P)$ linearly. Set $K = \Lambda \cap B_i^{\perp}$ in finite class coordinates. For $P = S^{\mathrm{sat}}$, restriction of the self-dual pairing gives $P^{\#}/P \simeq \Lambda/(P+K) \simeq K^{\#}/K$. Locally every functional on either primitive sublattice extends to Λ, with conjugate places paired under the involution. This proves the displayed identifications and $d(K) = d(P)$. The full Gram determinant now yields

$$[S^{\mathrm{sat}} : S]\,d(K) = d(S) = \sqrt{|\,\mathrm{Nm}_{E/\mathbb{Q}}\,a_i|} = \mathfrak{d}_i.$$

The common lattice in the prescribed level. For $x \in W(\mathbb{A}_f)$ write $y_i(x) = x + \alpha_i(x)$ for the unique lift into $B_i^{\perp}(\mathbb{A}_f)$, using the original class coordinates for x. Here $\alpha_i(x) \in A(\mathbb{A}_f)$ denotes its isotropic part. The term lattice over W, denoted L_i^0, is exactly the set of x with $y_i(x) \in \Lambda$ and $\alpha_i(x) \in MA_\Lambda$. Indeed its defining condition is that x belong to the definite class lattice and admit such a level lift orthogonal to g_iF. The first-middle decomposition of an integral lift has integral isotropic part and middle part in the class lattice. Uniqueness of the lift gives the assertion. Integral coordinates and level-lift criterion, common part and pair lattice.

The subgroup of K on which the isotropic part vanishes modulo M has index at most $\#(A_\Lambda/MA_\Lambda) = M^{2\varrho f}$. Its projection is isometric onto L_i^0. For each term i, write $t_i = [S_{0,i}^{\mathrm{sat}} : S_{0,i}]$ and $d_i = d(L_i^0)$. We obtain the one-term bound

$$t_i z\, d_i \;\le\; M^{2\varrho f}\mathfrak{d}_i \;\le\; G_i. \tag{8.2}$$

There is no change of the N-level lattice in this comparison.

Now let $L_{12} = L_1^0 \cap L_2^0$ and $d_{12} = d(L_{12})$. The congruence $g_2 g_1^{-1} = q \equiv 1 \bmod M$ defines a homomorphism

$$L_1^0 \longrightarrow S_{0,2}^{\mathrm{sat}}/S_{0,2}, \qquad x \longmapsto [(q-1)y_1(x)/M].$$

Indeed $y_2 - y_1 \in A$ since the first coordinates agree, and $qy_1 - y_2 \in g_2 A$ since the second coordinates of the common relation agree. The numerator is M-divisible in Λ. If the class is zero, write $(q-1)y_1 = Ma + Mg_2a'$ with $a, a' \in A_\Lambda$. Then $y_2' = y_1 + Ma$ is integral, has first middle coordinate x, and has isotropic part in MA_Λ. Since $g_2^{-1}y_2' = g_1^{-1}y_1 - Ma'$ is orthogonal to F, it is an admissible lift for the second term. The kernel therefore lies in L_{12}, which proves $[L_1^0 : L_{12}] \mid t_2$. Exchanging the two terms gives $[L_2^0 : L_{12}] \mid t_1$. In particular L_{12} has rank $k-1$. Multiplying $d_{12} \le t_2 d_1$ and $d_{12} \le t_1 d_2$ and applying (8.2) twice proves the required gain

$$(z d_{12})^2 \le (M^{2\varrho f}\mathfrak{d}_1)(M^{2\varrho f}\mathfrak{d}_2) \le G_1 G_2. \tag{8.3}$$

This argument applies when the rational relations coincide and the discriminants are nonzero. The formal argument also treats distinct rational relations. Crossed indices, general gain.

The sum over pairs of returns. Set $H = b^{n^3 f}$. The archimedean box bound for near terms gives $G_i \le H$, so (8.3) and $z \ge 1$ give $d_{12} \le H$. For $x > 0$ write $\mathcal{W}(x) = \tau^2\sqrt{C/(n_p(M^{2\varrho f}x)^k)}$. The weight $w_i^{\mathrm{disc}} = \mathcal{W}(\mathfrak{d}_i)$ depends only on the absolute discriminant and will also bound the truncated integral. Since $G_i \ge M^{2\varrho f}\mathfrak{d}_i$, (8.3) gives

$$w_1 w_2 \le w_1^{\mathrm{disc}} w_2^{\mathrm{disc}} \le \frac{\tau^4 C}{n_p z^k}\, d_{12}^{-k}.$$

Near gauge bound, product of the weights.

Multiplicity over a fixed lattice. Fix the left class c. Assign each ordered pair its common lattice $L \subseteq \Lambda_c \cap X_0$ and the position $t \in \bar{R}_0$ of its common reduction in its coset, using the fixed coset representatives. Fix L, t and put $\ell = k - \operatorname{rank} L$. For $\ell = 1$, the number of possibilities for a single marked term (c', γ) is at most $b^{2000n^6(\ell+1)f}$. Here is how the finite frame count gives this estimate. The integral n-by-n frame-product matrix Z has at most $b^{270n^3 f}$ choices by archimedean packing. In the top cell put $\delta = \det Z \ne 0$, $a = M\delta$, $a' = M\det(ZZ^*)$, where M is the congruence level. Write $[x] = |\widehat{\mathcal{O}}_E/x\widehat{\mathcal{O}}_E|$ for these integral moduli. For fixed Z the number is at most

$$(\operatorname{disc} L)^{2\varrho}[a]^{n\ell}[a']^{n\varrho}.$$

Indeed, record the ϱ lift rows modulo the Gram rows of an adelic basis of L, whose index is $(\operatorname{disc} L)^2$. For L with basis yu, where y is rational and u is an invertible finite adelic matrix, let w be the difference of two rational lift matrices. If the rows of wu lie in the Gram row lattice, then $\eta = y(y^*y)^{-1}w^*$ has the prescribed pairings. Its columns lie in L since it equals yu times the conjugate transpose of the integral Gram coefficients. The allowed level shear with parameter η thus equalizes the pairings with L. The differences of the n definite columns then lie in the rank-ℓ lattice orthogonal to L. Record these columns modulo a. The adjugate of Z clears the denominators in the column difference Y multiplied by Z^{-1}. The columns of YZ^{-1}/M lie in the definite class lattice. The last column of YZ^{-1} has length at most $2/\sqrt{5} < 1$ at each embedding by the cell estimate. Its squared Hermitian length is a totally real algebraic integer of absolute norm less than one, hence zero. A level shear therefore removes the difference using the first ϱ rows of Z. Finally record the $n\varrho$ isotropic entries modulo a'. Write their difference as $a'U$ and let P be the first ϱ rows of Z. Preservation of the Gram matrix gives $U^*P + P^*U = 0$. As P is surjective, $\ker P \subseteq \ker U$, hence $U = K'P$ with $K'^* = -K'$. The first ϱ columns of Z^{-1} give a right inverse of P, so $\det Z\, K'$ is integral. Thus $a'K'/M$ is an integral skew-Hermitian parameter for the final level shear. Equal frames modulo these moves determine the marked term, including c'. The top cell has no shared columns left to record. The cited modulus bounds give $(\operatorname{disc} L)^2, [a], [a'] \le b^{2n^3 f}$, proving the claim. Matrix packing, Gram rows and level shear, adjugate and short column, skew-Hermitian shear, recovery of the marked term, moduli and marked-frame bound.

The terminal column and the finite labels. Keep $F = (e_1, \dots, e_{n-1}, z_0)$ in the rational Witt frame, where $h(z_0, z_0) = -N$, and put $x = Z^{-1}e_n$ in the top cell. If $y = \gamma Fx$, then $F^*hy = e_n$. Hence every dual isotropic coordinate of y is zero and $y_{z_0} = -1/N$. Evaluating $h(y, y) = h(Fx, Fx) = -N\bar{x}_n x_n$ at any embedding σ gives

$$\|\sigma(y_{X_0})\|^2 = N^{-1} - N|\sigma(x_n)|^2.$$

For two frames with the same Z, the last column of their definite difference times Z^{-1} therefore has norm at most $2/\sqrt{N} \le 2/\sqrt{5} < 1$ at every embedding. After recording the definite columns modulo $M \det Z$, the adjugate puts this difference column in the integral class lattice. Its squared length is integral and totally nonnegative, so the product formula forces it to vanish. Rational frame identity, integral short column.

Let δ_E be the chosen generator of the different of $E/\mathbb{Q}$, with $\bar{\delta}_E = -\delta_E$. For the chosen embedding $\Phi_v : E \to \mathbb{C}$ at each infinite place v, set

$$M_v = \operatorname{Re}\left(\frac{M\Phi_v(\delta_E)}{2\pi i}\right).$$

The quotient is real since $\Phi_v(\delta_E)$ is purely imaginary. Replacing Φ_v by its conjugate reverses the sign of M_v. For a near term, a single archimedean point puts both frames in boxes of radius b^{37n}. For either embedding σ at v, the rational-to-real frame change multiplies $|\sigma Z_{ij}|$ by $|M_v|^{(n-1-i)+(n-1-j)}/N$, with $0 \le i, j < n$. Since $|M_v| \ge 1$, this gives $|\sigma Z_{ij}| \le Nb^{74n} \le b^{134n}$. Each entry is integral, by the integral form on the two transported lattices. Distinct such matrices are separated by at least one in the maximum of the complex coordinate norms, since a nonzero integral difference has nonzero integral absolute norm. Pack disjoint half-radius balls in this norm in dimension $2fn^2$. This gives at most $(1 + 2b^{134n})^{2fn^2} \le b^{270n^3 f}$ matrices. Writing $\delta = \det Z$, the determinant expansion and $n \le b$ give $|\sigma\delta|^2 \le (n!b^{134n^2})^2 \le b^{270n^2}$. As $M \le b^{23}$, the squared embedding bounds for $a = M\delta$ and $a' = M \det(ZZ^*) = M\delta\bar{\delta}$ are $b^{46+270n^2}$ and $b^{46+540n^2}$, respectively. Both exponents are at most $2n^3$ when $n \ge 10^5$. Multiplying over the f embeddings

proves $[a], [a'] \le b^{2n^3 f}$. Real scale and its square, Scale of a frame vector, Size of the rational Gram entries, moduli.

Finally move two marked terms on the left by the allowed frame-fixing unipotents and suppose their frames agree. These moves reduce to the identity. The quotient of the moved frames' isometries fixes F, so the corresponding finite reductions s, s' lie in one coset of the reduced Bessel group. Their common recorded position t in that coset gives $s = s'$. The rational quotient now induces on X_0 an isometry carrying the source representative for c'' to that for c' modulo the source level. Hence $c'' = c'$, and that isometry is trivial by freeness of the source level. Indeed its rational stabilizers are finite by definiteness and embed in principal congruence groups at the two distinct residue characteristics p_0, p_1, which are pro-p_0 and pro-p_1, respectively. The quotient is consequently a level unipotent fixing F on the right. The two chosen double-coset representatives coincide. Definite Levi and level, fixed-label recovery.

The marked lattice sum. Fix a finite adelic unitary representative u and put $\Lambda = E^k \cap u\widehat{\mathcal{O}}_E^k$. A mark specifies a row modulo the level ideal determined by π_N, of norm N^f. Distinct marked cosets specify distinct rows.

Lemma 8.1. *For $0 \le h \le k$ and $H \ge 1$, sum $d(L)^{-k}$ over the rank $k-h$ sublattices $L \subseteq \Lambda$ of Hermitian discriminant $d(L) \le H$, counting each with the number of marked rows that vanish on it. For $h > 0$ this sum is at most*

$$e^{2f} H N^{fh} (25h \operatorname{rd}_E)^{fh(k+1)}.$$

For $h = 0$ it is at most $e^{2f} H^2$.

Proof. For an E-subspace V of codimension $h > 0$, let $S = \Lambda \cap V$, $R = \Lambda \cap V^\perp$, $d = d(S)$. The discriminant satisfies $d^2 = \#(S^\#/S)$, with the Hermitian dual taken in the span. Since Λ is unimodular and S, R are saturated over the Dedekind domain $\mathcal{O}_E$, integral functionals on either extend to Λ. Orthogonal projection identifies $S^\#/S$ and $R^\#/R$ with $\Lambda/(S+R)$. Hence $d(R) = d$ and

$$\operatorname{covol}(R) = 2^{-fh} \operatorname{rd}_E^{fh} d.$$

Indeed $\mathcal{O}_E$ has covolume $2^{-f}|D_E|^{1/2}$. For a free sublattice on columns U, the additional volume factor is $\prod_{v|\infty} \det(U_v^* U_v)$, its Hermitian discriminant. Passing to R divides both volumes and discriminants by the index.

In dimension $m = 2fh$ the Euclidean unit ball contains a cube of half-side $m^{-1/2}$. Minkowski's second theorem on successive minima [Cas97, Ch. VIII, §4.3, Theorem V] gives $\prod_i \lambda_i(R) \le m^{m/2} \operatorname{covol}(R)$. Choose successive E-independent $x_j \in R$ with $\|x_j\| \le \lambda_{2f(j-1)+1}(R)$, since the preceding E-span has real dimension $2f(j-1)$. Grouping the minima in blocks of $2f$ gives

$$\prod_{j=1}^{h} \|x_j\|^{2f} \le (fh \operatorname{rd}_E)^{fh} d.$$

Every nonzero vector of Λ has squared Euclidean norm at least f, by the arithmetic–geometric mean inequality for its totally positive integral Hermitian length. Thus $f^{-1/2}\Lambda$ is 1-separated. Disjoint half-unit balls bound the number of its vectors of length at most $t \ge 1$ by $(3t)^{2fk}$. With $p = 2f(k+1)$, layer integration gives

$$\sum_{0 \ne x \in \Lambda} \|x\|^{-p} \le f^{-p/2} p \int_1^\infty (3t)^{2fk} t^{-p-1}\, dt = f^{-p/2}(k+1)3^{2fk} \le (5/\sqrt{f})^p.$$

The tuple (x_j) determines $V = \bigcap_j x_j^{\perp}$. Summation over all tuples consequently gives $\sum_{\operatorname{codim} V=h} d^{-k-1} \le (25h \operatorname{rd}_E)^{fh(k+1)}$.

Now take $V = EL$ and $i = [S : L]$, so $d(L) = id$. In a local Smith basis the h free row entries give N^{fh} choices globally. A constrained entry belonging to an elementary divisor has at most its local index many choices. Thus at most iN^{fh} rows vanish on L. For $h > 0$, the cutoff gives $i(id)^{-k} \le Hd^{-k-1}i^{-k}$. The series for sublattices of a rank r projective $\mathcal{O}_E$-module is

$$\sum_{L \subseteq S} [S : L]^{-s} = \prod_{j=0}^{r-1} \zeta_E(s-j) \qquad (s > r),$$

where the sum is over finite indices. This is the usual submodule zeta formula [Ros17, §1, the zero endomorphism]. Indeed S is locally free, and local Hermite normal forms with diagonal valuations a_j have $q^{\sum_j (j-1)a_j}$ choices over a residue field of order q. Here $s = k,\ r = k - h$, so the arguments are distinct integers at least 2. Their product is at most e^{2f}, since $\zeta_E(j) \le \zeta(j)^{2f}$ and

$$\sum_{j \ge 2} \log \zeta(j) \le \sum_{n \ge 2} \sum_{j \ge 2} n^{-j} = \sum_{n \ge 2} \frac{1}{n(n-1)} = 1.$$

This proves the first bound. For $h = 0$ one has $d = 1$ and $i^{1-k} \le H^2 i^{-(k+1)}$, yielding the same Euler bound with $r = k,\ s = k + 1$.

Lean, comparison. `orthogonaldiscriminants` (L350-L405), `shortnormals` (L495-L529), `heightsum` (L630-L678), `markedrows` (L417-L468), `indexsum` (L566-L610). The formal count uses a cutoff power $H^{2\varrho+6}$. Here Minkowski's second theorem treats all positive coranks together, and one factor $H/(di)$ makes the index series converge. □

For fixed E, k the cutoff dependence in positive corank is even polylogarithmic. Put $\delta = 1/(1 + \log H)$ and use $i(id)^{-k} \le H^{\delta} d^{-k-\delta} i^{-(k-1+\delta)}$. In the packing integral with $p = 2f(k + \delta)$, each normal costs $O_{E,k}(\delta^{-1})$. The index series begins at $h + \delta$; only $h = 1$ can introduce the simple pole of ζ_E at 1 [Neu99, Ch. VII, Corollary 5.11]. Since $H^{\delta} \le e$, the sum is $O_{E,k}\big(N^{fh}(1 + \log H)^{h + \mathbf{1}_{h=1}}\big)$. The explicit bounds of the lemma will be used as E, k vary.

For the prescribed $H = b^{n^3 f}$, the bounds of the lemma are at most

$$H^{2\varrho+6} 2^k \big[N^f (200 f k \operatorname{rd}_E)^{f(k+1)}\big]^h \le \tau^{h+1}/b^f$$

by admissibility, since $e^{2f} \le H$ and $h \le k$. Average over the at least $(N^k/b)^f$ marked cosets and integrate over the source quotient. Right invariance of Haar measure and freeness of the source level give the uniform mean on $\mathcal{C}$. Thus the sum over the classes and lattices is at most $\#\mathcal{C}\, \tau^{h+1}/N^{kf}$, in particular $\#\mathcal{C}\, \tau^2/N^{kf}$ for the pairs with the same relation. Class mean, admissible parameters.

With c, L, t fixed, the pair injects into two choices of marked term. The square of the multiplicity above is at most τ by admissibility. Summing first over L, c and then over t yields $\sum d_{12}^{-k} \le \tau^3 n_p$ on our corank-one stratum. The weighted contribution on the same datum is consequently

$$\sum w_1 w_2 \le \sum w_1^{\mathrm{disc}} w_2^{\mathrm{disc}} \le \frac{\tau^7 C}{z^k} \le \left(\frac{b^{2f}}{z}\right)^k C.$$

The last inequality uses $\tau \ge 1$ and $\tau^8 = b^{2kf}$. The first bound uses exact corank one. Lean's general summation reserves one further factor τ for the possible coranks. This proves (8.1) on the fixed admissible datum. Fixed-corank count, same-link estimate.

Unfolding the contribution of one rational relation. Fix c, s, c' and a rational relation $\mathcal{L}$. Write Γ for the returns in that block with this relation and nonzero term discriminant. Let U_c be the rational Bessel unipotents at the principal level in class c, and U_c^0 its subgroup fixing the frame pointwise. For $y \in G_\infty$ put $K_\gamma(y) = W^{\mathrm{cut}}(y) \otimes W^{\mathrm{cut}}(\gamma^{-1}y)^*$. With the original class cutoffs and Haar measure, the signed block is

$$A^{\mathrm{cut}}_{c,s,c',\mathcal{L}} = \sum_{\gamma \in \Gamma} \int_{G_\infty} \beta_c(y)\beta_{c'}(\gamma^{-1}y) K_\gamma(y)\, dy.$$

The integral majorant for the two cutoffs makes this sum absolutely convergent. The relation and the discriminant condition are stable under rational Bessel unipotents. Moving a block by integral unipotents with reductions $\bar{u}, \bar{u}'$ multiplies it by $\psi(u)^{-1}\psi(u')$ and takes s to $\bar{u}s\bar{u}'^{-1}$. Indeed rational approximation gives lifts with the same reductions and characters, and translation in the integral gives this scalar. If s fails agreement, some such move fixes s with scalar different from one, so $A^{\mathrm{cut}}_{c,s,c',\mathcal{L}} = 0$. Measured amplitude, absolute convergence, character action.

Suppose s agrees. In each double coset $U_c^0 \gamma U_{c'}$ choose one element δ, preferring an element that is near and is the prescribed representative of its smaller coset $U_c^0 \delta U_{c'}^0$. The action of $U_c^0 \times U_{c'}$ on this double coset is free. Indeed, if $p\gamma = \gamma w$, then w fixes $\gamma^{-1}F$ pointwise. The invertible cross Gram matrix yields fixed vectors $b_1, \dots, b_\varrho$ in its span dual to a basis of A. As $wA = A$, pairing with the b_j shows that w fixes A pointwise. The space $A + \langle b_j \rangle$ is nondegenerate and fixed pointwise. Its orthogonal complement injects into $A^\perp/A$, where the middle block of w acts trivially. Thus $w = p = 1$. Free action.

Let ω be the indicator of the reduced points. For each y, the $a \in U_c$ with $\omega(ay) = 1$ form one coset $U_c^0 a_0$. Freeness gives exactly one pair $a \in U_c^0 a_0,\ b \in U_{c'}$ for which $a\gamma b^{-1}$ is the chosen representative of its double coset. Thus $\Phi(\gamma, y) = \mathbf{1}_{\{\gamma \text{ chosen}\}}\omega(y)$ has sum one on the action $(\gamma, y) \mapsto (a\gamma b^{-1}, ay)$ of $U_c \times U_{c'}$, as does $\beta_c(y)\beta_{c'}(\gamma^{-1}y)$. This action preserves the kernel and counting measure times dy. Insert the sum of the translates of Φ in the original integral, change variables in each term, and sum the translates of the original cutoffs. Tonelli applied first to the norm justifies the exchange. If the resulting integrand is nonzero, y is reduced and in the cut. Some $b \in U_{c'}$ takes $\gamma^{-1}y$ to a reduced point in the cut. Hence γb^{-1} is near, and the frame-fixing moves to its small-coset representative preserve nearness. Writing $\mathcal{R}$ for the selected near representatives gives the exact identity

$$A^{\mathrm{cut}}_{c,s,c',\mathcal{L}} = \sum_{\delta \in \mathcal{R}} \int_{G_\infty} \omega(y) K_\delta(y)\, dy.$$

Reduced coset, change of cutoff, near representatives.

One real place and the Haar factor. Fix v, suppress its index on $W = W_v$, and put $Z = F^* h \gamma_v F$ in the normalized real frame. Write $g = g_v = |\det Z|^2 - |\det Z_0|^2 > 0$, where Z_0 omits the last row and column. Let d_v be the number of coordinates of W, e_v its Bessel degree, and R the frame-box radius. Set

$$T = R^{9n}/g, \qquad P = T^{45n^2}, \qquad c = c_G(g/2)^{k-1}, \qquad c_G = \frac{\operatorname{vol} U(k)}{\operatorname{vol} U(r)} \frac{2\pi^k}{(k-1)!}.$$

The common box witness bounds each entry of Z by R^2. Hadamard gives $g \le |\det Z|^2 \le (nR^4)^n \le R^{5n}$, hence $T \ge R$ and $g \ge T^{-1}$ since $Tg = R^{9n}$. The section calculation below gives discriminant at least $g/2$ on the ball of radius P^{-6} about Z. Let γ_Z be the unitary representative with that Gram matrix. For $x = (h, u) \in U(k) \times U^0$, write $a_x = \gamma_Z(uh)^{-1}$, embedding h in the frame fixer. The ordered pairs of frames $(F, \gamma_Z F)$ and $(F, \gamma_v F)$ have the same nonsingular joint Gram matrix.

Witt extension gives an isometry s fixing F with $s\gamma_Z F = \gamma_v F$. Thus $t_0 = \gamma_v^{-1} s\gamma_Z$ fixes F and the continuous shifts $t_x = t_0(uh)^{-1}$ fix F and satisfy $\gamma_v t_x = s a_x$. Frame-fixing shifts, section at the chosen scale.

We use the following point evaluation on the Gram section. On the compact tube A of the source, for every bounded measurable β supported in A,

$$\int_{U(k)\times B_0} \Big\| \int_A \beta(y) W(a_x y)\, dy \Big\|^2 c\, dh\, du \le K \int_A |\beta(y)|^2\, dy, \qquad K = P^{2000n^2} d_v/e_v.$$

Here dh is probability Haar, du is Bessel Haar on U^0, and $B_0 \subset U^0$ is the tube's parameter set, of finite positive measure. The point-evaluation argument below combines the Gram-coordinate density with Schur bounds for derivatives through order $2n^2$. The tube constructed below contains the shifted cut support by the frame-length estimate. Each U^0-slice has Haar measure at most $3|B_0|$. Two points of the tube differing by $u \in U$ have $\|u\| \le P^{40}$, since both points and their inverses have norm at most P^{20}. Integrate first on U^0, then over the frame entries of U/U^0, then on $U\backslash G$, where the full squared norm integrates to d_v. This gives $\int_A \|W\|^2\, dy \le T^{45n}|B_0|(TP^{40})^{2n^2} d_v$. Here $3 \le T^{45n}$ and $2 \le T$ absorb the slice and box constants. Gram-coordinate integral, Sobolev estimate, tube geometry, tube mass.

Take $\beta_j = \mathbf{1}_A \phi\, \overline{W_j}$ and sum the estimate over j. The Bessel actions of s and t_x are unitary on the two indices, so the left side is the integrated squared Hilbert–Schmidt norm of the shifted matrix pairing. Dividing by $c|B_0|$ puts probability measure on the shifts and gives the bounding measure

$$d\mu_v(y) = \frac{K}{c|B_0|} \mathbf{1}_A(y) \|W(y)\|^2 dy, \qquad \mu_v(A) \le \frac{T^{10^5 n^4} d_v^2}{e_v c_G (g/2)^{k-1}} = p_v.$$

Indeed $|B_0|$ cancels, and $P^{2000n^2} T^{45n} (TP^{40})^{2n^2} = T^{93600n^4 + 2n^2 + 45n} \le T^{10^5 n^4}$. Column argument, measure and cancellation.

The basis measure in the prescribed Haar normalization. Let $J = J^* = J^{-1}$ have size h. On Hermitian and skew-Hermitian matrices use one real coordinate per diagonal entry and ordinary complex Lebesgue measure per entry above the diagonal. These are the chosen coordinates up to orthogonal changes on each pair. Write $V_h = \operatorname{vol} U(h) = \prod_{j=1}^h 2\pi^j/(j-1)!$. For $g \in U(J)$, the skew-Hermitian differential $S = g^* J\, dg$ is left invariant. Lebesgue density in these differential coordinates is $V_h\, dg$ for the datum's Haar measure dg. Indeed the Cayley chart $S_0 \mapsto (1 + JS_0/2)(1 - JS_0/2)^{-1}$ has derivative JS_0 at zero and Haar density $1/V_h$ there. Left invariance propagates this calibration to every point. Haar definition.

For an invertible basis Y put $T = Y^* J Y$, and write locally $Y = g\sigma(T)$ with $\sigma^* J \sigma = T$. Such smooth sections exist by the submersion theorem: at Y the variation $\dot{Y} = (Y^*J)^{-1} H/2$ realizes every Hermitian $\dot{T} = H$. Consider $B = Y^* J\, dY$. In the linear coordinates $B = K + H/2$, $K^* = -K$, $H = H^*$, ordinary matrix Lebesgue density is $2^{-h} dK\, dH$. A diagonal entry contributes $1/2$. For an off-diagonal pair with inputs $s = K_{ij}$ and $t = H_{ij}$, the outputs, conjugating the second, are $s + t/2, -s + t/2$. Their complex determinant is one. Coordinate Jacobian.

Here $H = dT$, while at fixed T the vertical derivative is $K = \sigma^* S \sigma$. Congruence by σ on skew-Hermitian matrices has real Jacobian $|\det \sigma|^{2h}$. In singular-value coordinates each diagonal coordinate contributes s_i^2, and each complex pair contributes $(s_i s_j)^2$. Unitary changes preserve volume. The derivative in (dT, S) is triangular, so any skew part of $\sigma^* J\, d\sigma$ drops out of the determinant. Finally left multiplication $dY \mapsto B$ on h complex columns has Jacobian $|\det(Y^*J)|^{2h} = |\det \sigma|^{2h}$. Cancellation gives

$$dY = 2^{-h} V_h\, dg\, dT.$$

This computes the smooth density at every invertible basis, hence the fibre at the specified T. For $J = 1_h$, project $U(h)$ to its last column on the unit sphere. At the identity, upper-entry density splits as the density on $\mathfrak{u}(h-1)$ times Euclidean surface density in this last column. Invariance propagates the splitting. Induction gives compact mass $\prod_{j=1}^{h} |S^{2j-1}| = V_h$ before division, so dg is probability. Taking $h = r$ and $h = k$ gives the two basis factors used in completing the frame. Vertical determinant, basis polar formula.

Completing a frame at its prescribed Gram value. Write $q(Y) = Y^*JY$, with J the Hermitian involution of $G = U(J)$, and $\mathcal{O} = GF$. This is the full-column-rank locus of $q^{-1}(q(F))$ by Witt's extension theorem. At each such Y and each Hermitian B, the equation $Y^*JE = B/2$ has a solution and $dq_Y(E) = B$. In submersion coordinates (t, x) with $t = q(Y)$, write $dY = a(t, x)\, dt\, dx$. The smooth fibre measure λ on $\mathcal{O}$ has density $a(q(F), x)\, dx$. It is G-invariant because $|\det g| = 1$ and $q(gY) = q(Y)$.

The stated fibre hypothesis implies $(g \mapsto gF)_* m \le \lambda$ on this orbit. Indeed, extend a nonnegative continuous test of compact support in a fibre chart to one of compact support in the full-rank chart. Its fibre integral varies continuously in t. Thus its integral over the preimage of any sufficiently small open Gram set is at most the volume of that set times its fibre integral at $q(F)$ plus ε times the same volume. Apply the hypothesis, let $\varepsilon \downarrow 0$, and use a partition of unity. The law of gF is supported on $\mathcal{O}$, since g is invertible. This handles the singular Gram value without assigning mass to lower-rank tuples. Pinned fibre hypothesis, frame law.

Complete $F = (e, z)$ first by the ϱ dual columns D, then by the k columns of its definite complement. The Gram coordinates split into the old Gram, the complex cross block counted once, and the new Gram, with product Lebesgue measure. In a submersion chart the variables first parametrize the old columns and then, with these fixed, the new columns. The Jacobian is block triangular, so Fubini gives the product of the two fibre densities at the specified values. The conditional computations can be made at the standard columns: transporting by g preserves both cross products and Lebesgue measure on the new columns.

At the standard frame the cross products select coordinate rows up to sign and permutation. With them fixed, $D = e^* + eA + xC$, where x is the standard definite basis. Its Gram is $T = A + A^* + C^*C$. Writing $A = S + (T - C^*C)/2$, $S^* = -S$, gives $dA\, dC = 2^{-\varrho} dT\, dS\, dC$. Each diagonal contributes $1/2$, and on each complex off-diagonal pair $(s, t) \mapsto (s + t/2, -s + t/2)$ has absolute determinant one. At $T = 0$ these columns are the unipotent translates of e^* with parameters $(-C^*, S)$; their specified Haar is $du = dC\, dS$. At the standard augmented frame, the remaining cross products fix the coordinate rows outside the definite complement with unit Jacobian. The square Gram formula in that complement gives relative density $(\operatorname{vol} U(k)/2^k)dh$ with dh probability Haar. Successive fibres, dual-column Jacobian.

For every $H \ge 0$, the integral $\int H(gs)\, ds$ with $ds = du\, dh$ depends only on gF by left invariance of fixer Haar. Replacing λ by the dominated frame law in the successive integrations and comparing with the full basis of size r gives

$$2^{-\varrho} \frac{\operatorname{vol} U(k)}{2^k} (m * ds) \;\le\; \frac{\operatorname{vol} U(r)}{2^r}\, dg.$$

Since $r - k - \varrho = n$, the comparison has the required normalization. Pinned comparison.

The normalized Gram-coordinate mean. Write $R^\circ = U^0 U(k)$ for the frame fixer, with Haar $ds = du\, dh$ and probability Haar dh. Put $g(Y) = |\det Y|^2 - |\det Y_0|^2$ in the open cell $g(Y) > 0$. We compute the lower density in the group integral. In a frame tuple with cross Gram Y, divide the columns on the right by Y. The new tuple is dual to F and its self Gram T' is $Y^{-*} T Y^{-1}$, where T

was the original self Gram. Right multiplication by Y on the $\varrho+k$ free complex rows scales volume by $|\det Y|^{2(\varrho+k)}$. Hermitian congruence scales dT' by $|\det Y|^{2n}$ to dT, as is seen by diagonalizing the singular values. The resulting factor is $|\det Y|^{2(k-1)}$. Dual-frame coordinate identity.

In the dual coordinates the first ϱ columns give $2^{-\varrho}$ on separating their Hermitian and skew-Hermitian parts. The contribution is $1/2$ per real diagonal entry. In the last column polar integration in $\mathbb{C}^k$ gives $\pi^k t^{k-1}dt/(k-1)!$ for squared radius $t>0$, with angular probability measure. At $T=F^*hF=\operatorname{diag}(0,\dots,0,-1)$, $t=1+(T')_{nn}=1-|(Y^{-1})_{nn}|^2=g(Y)/|\det Y|^2$. Thus the frame measure in these coordinates has density $2^{-\varrho}\pi^k g(Y)^{k-1}/(k-1)!$ against $dY\,ds$. Restriction to the open cell gives a measure dominated by the Gram fibre at F^*hF. The coordinate laws disintegrate Lebesgue measure, and their positive-radius densities and sections are continuous at this Gram matrix, so Fatou's lemma applies on shrinking Gram neighbourhoods. First columns, radial factor, fibre at the specified Gram matrix.

Apply the frame-fibre comparison just proved to the open-cell measure. Convolution with fixer Haar introduces the second stabilizer integral. Dividing its density $2^{-\varrho}\pi^k g(Y)^{k-1}/(k-1)!$ by $\operatorname{vol}U(r)/(2^n\operatorname{vol}U(k))$ gives $c_G g(Y)^{k-1}$, since $n=\varrho+1$. For every nonnegative measurable test H we have therefore proved

$$\int_{g(Y)>0} c_G g(Y)^{k-1}\iint_{R^\circ\times R^\circ} H(s'\gamma_Y s)\,ds'\,ds\,dY \;\le\; \int_G H\,dg.$$

group mean.

To pass to $U\backslash G$, take a nonnegative cutoff χ of integral one on each left U^0-orbit and restrict it to cross Gram in Q, obtaining χ_Q. Use $H=\chi_Q w$ for a nonnegative left Bessel-invariant w. Integration over s' removes the cutoff on Q, and inversion preserves ds. On the group side unfold first over U. On any contributing orbit choose one contributing point. Relative to it the possible frame entries for U/U^0 are bounded by P^2, since $\|Y\|\|Y'^{-1}\|\le P^2$ for $Y,Y'\in Q$. The upper triangular entries thus have volume at most $(2P^2)^{2n^2}$, and each U^0-integral is at most one. Since $g(Y)\ge g(Z)/2$ on Q, this gives the cube mean with measure $c_G(g(Z)/2)^{k-1}dh\,du$ used in the point evaluation. Triangular coordinates, quotient mean.

The Gram section on the evaluation cube. Write $n=\varrho+1$, $\Pi=e_ne_n^*$, $H=I-\Pi$, and let $F=(e_1,\dots,e_\varrho,z)$, $F'=(e_1^*,\dots,e_\varrho^*,x_{k_0})$ be the two prescribed frames. Their columns together are Euclidean orthonormal. In the Hermitian form J, their three pairings F^*JF, F^*JF', F'^*JF' are $-\Pi, H, \Pi$, respectively. For $X=Y^{-1}$ put

$$\nu=(1-|X_{nn}|^2)^{1/2},\quad D=H+\nu\Pi,\quad C(Y)=-\Pi-\tfrac12 HX^*\Pi X(I+\Pi),\quad W(Y)=FC(Y)+F'D.$$

On $\nu>0$, the section in the formula acts by

$$\gamma_Y F=S:=W(Y)Y,\qquad \gamma_Y F'=LD^{-1},\qquad L=FX^*-SC(Y^*),$$

and fixes the Euclidean orthogonal complement of the two frames. To check the section, observe $F^*JW(Y)=I$ and $W(Y)^*JW(Y)=-X^*\Pi X$. For the latter identity set $B=X^*\Pi X$. The mixed terms are $C(Y)^*H+HC(Y)=-B+\Pi B\Pi$, and the remaining terms are $(\nu^2-1)\Pi=-\Pi B\Pi$. Therefore the source and target bases $(FX,W(Y^*))$ and $(W(Y),FX^*)$ have identical Gram matrices. They are bases of the frame span since X,D are invertible. Sending the former to the latter gives precisely the two displayed columns. The complement is J-orthogonal to this span. Thus $\gamma_Y^*J\gamma_Y=J$, $\|\gamma_Y^{-1}\|=\|\gamma_Y\|$, and $F^*J\gamma_YF=Y$. Here and below all matrix norms are Euclidean operator norms. Pinned section, isometry.

Take $M=T=R^{9n}/g(Z)$ from the local pairing. The following bounds hold whenever $M\ge 8n$, $P\ge M^{9n}$, $|Z_{ij}|\le M^2$ and $g(Z)=|\det Z|^2-|\det Z_0|^2\ge M^{-1}$, with Z_0 the leading

$(n-1)$-minor matrix. By Cramer's rule $g(Z) = |\det Z|^2\nu(Z)^2$. Hence $\|Z\| \le M^3$, $|\det Z|^{-1} \le M^{1/2}$, $\|Z^{-1}\| \le \|Z\|^{n-1}/|\det Z|$, and $\nu(Z)^{-1} = |\det Z|/\sqrt{g(Z)} \le M^{3n+1/2}$. In particular $T_0 = P/2$ bounds $\|Z\|, \|Z^{-1}\|, \nu(Z)^{-1}$. If $\|Y - Z\| < P^{-6}$, then $\|Z^{-1}(Y - Z)\| < T_0^{-5}$. The inverse series gives $\|Y^{-1}\| \le 2T_0$ and the resolvent identity gives $\|Y^{-1} - Z^{-1}\| \le 2T_0^{-4}$. Thus

$$\nu(Y)^2 \ge \nu(Z)^2 - 8T_0^{-4} \ge (1 - 8T_0^{-2})\nu(Z)^2 \ge \tfrac{7}{8}\nu(Z)^2.$$

Every singular value of $I + Z^{-1}(Y - Z)$ is at least $1 - T_0^{-5}$, so $|\det Y|/|\det Z| \ge 1 - nT_0^{-5} \ge 7/8$. It follows that $g(Y) \ge g(Z)/2$, and $\|Y\|, \|Y^{-1}\|, \nu(Y)^{-1} \le P$. The cube adds real and imaginary entries in $[0, P^{-8}]$ to Z. Its operator radius is at most its Frobenius radius $\sqrt{2}\, nP^{-8} < P^{-6}$, and each coordinate direction has norm one. Fitted bounds on the ball, evaluation cube.

For a unit real direction denote differentiation by a dot. The inverse rule gives $\|\dot{X}\| \le P^2$, and $\dot{\nu} = -\Re(\overline{X_{nn}}\dot{X}_{nn})/\nu$. Thus $\|\dot{D}\| \le P^3$ and $\|\dot{D}^{-1}\| \le P^5$. The same bounds apply with Y^* in place of Y. Successive product rules, using $\|D\| \le 1$, $\|D^{-1}\| \le P$, give

	$C(Y)$	$W(Y)$	S	L
norm	$2P^2$	$3P^2$	$3P^3$	$7P^5$
derivative norm	$2P^3$	$3P^3$	$4P^4$	$15P^6$.

In the last column one uses $\|\dot{L}\| \le P^2 + (4P^4)2P^2 + (3P^3)2P^3$. Consequently $\|LD^{-1}\| \le 7P^6$ and $\|\dot{L}D^{-1} + L(\dot{D^{-1}})\| \le 15P^7 + 7P^{10} \le 8P^{10}$. The column formula above gives $\|\gamma_Y\| \le P^8$ and $\|D_{\mathbb{R}}\gamma_Y\| \le P^{12}$, uniformly on this cube. For $a = \gamma_Y k$, $k = (uh)^{-1}$ on the specified fixer ball, $\|k\| = \|k^{-1}\| \le P^{20}$ since h is compact and $J = J^* = J^{-1}$. On every coordinate line

$$\|a^{-1}\partial_j a\| \le \|k^{-1}\|\,\|\gamma_Y^{-1}\|\,\|\partial_j\gamma_Y\|\,\|k\| \le P^{60} \le P^{85}.$$

This proves the differential estimate needed for successive first derivatives on the same cube and fixer ball. Section and derivative bounds, logarithmic derivative.

The volume bound for a unipotent slice. The same dual-column coordinates give the tube and its slice measure for the test on one real group. Write an element of U^0 as (w, m), where w is k-by-ϱ, $m + m^* = w^*w$, and its action on e^* is $e^* - xw - em$. Here x denotes the columns of the definite basis. Its action on x is $x + ew^*$, and it fixes e, z. Matrix multiplication gives

$$(w, m)(w', m') = (w + w', m + m' + w^*w').$$

Haar measure is Lebesgue measure in w and the skew part of m: both translations are triangular with diagonal one. Its scale is the one used in completing the dual columns. Write $B_a = \{\|w\| \le a,\ \|m\| \le a^2\}$. The group law gives $B_aB_b \subset B_{a+b}$. The dilation $(w, m) \mapsto (aw, a^2m)$ maps B_1 onto B_a, so

$$|B_a| = a^D|B_1|, \qquad D = 2k\varrho + 2\varrho^2 \le r^2.$$

Indeed w has $2k\varrho$ real entries and the skew part of m has ϱ^2 real coordinates of degree two. These balls are compact and have positive volume for $a > 0$. Unipotent coordinates, ball volume.

At scale $S \ge 8r$ set $a = r^2S^2$, $B_0 = B_a$ (the parameter set in the preceding estimate), $K_S = \{g \in G : \|g\| \le S\}$ and $A = B_aK_S$. Since $\|g^{-1}\| = \|g\|$ on $U(J)$, K_S is compact. For $u \in B_a$ the displayed action gives $\|u\| \le (1 + a)^2$; hence $\sup_{y\in A}\|y\| \le (1 + a)^2S \le S^{10}$.

Fix a nonempty slice $\{u \in U^0 : uy \in A\}$. There is $h \in U^0$ with $hy \in K_S$. For any u in the slice write $uy = vz$ with $v \in B_a$, $z \in K_S$. Then $v^{-1}uh^{-1} = z(hy)^{-1}$ has norm at most S^2, so its two matrix blocks put it in B_{S^2}. Thus the slice times h^{-1} is contained in B_{a+S^2}. Right invariance gives

$$\frac{|\{u : uy \in A\}|}{|B_0|} \le (1 + r^{-2})^D \le e < 3.$$

This holds for every orbit with the same B_0, A. In the one-place estimate $S = T^{45n}$ and $P = S^n$. The remaining support implication follows from the Gram section. For $v \in B_0$, z in the prescribed box, and $h \in U(k)$, the bound BesselLength$(s\gamma_Z(vh)^{-1}z) \le T$ gives $\|v^{-1}z\| \le S$, hence $z \in A$. The next argument proves this estimate for the stated section. Tube construction, frame-length implication.

Support in the tube at $S = T^{45n}$. Write $F = (e_1, \dots, e_{n-1}, z_0)$ for the Witt frame and $M_0 = T^{9n}$, where $T \ge 8n$. The open-cell hypotheses give $\mathfrak{g}(Z) \ge T^{-1}$ and $|Z_{ij}| \le T^2$, with $\mathfrak{g}(Z) = |\det Z|^2 - |\det Z_{[1,n-1]}|^2$. For a shift (h, v), put $q = (vh)^{-1}z$. The fixer vh fixes F, so each column of $q^{-1}F$ has norm at most T whenever z is in the frame box. Assume BesselLength$(s\gamma_Z q) \le T$. Left translation by s preserves this length. By its definition as an infimum of $\|s'\gamma_Z q\| + 1$, we can choose s' with $\|s'\gamma_Z q\| \le T$. Removing its unitary left factor gives w in the Bessel unipotent group with $\|y\| \le T$ for $y = w\gamma_Z q$.

Set $Z' = F^* J w \gamma_Z F$. If $wF = FU$, then U is upper unitriangular and $Z = U^* Z'$. Thus the leading minors and the discriminant are preserved. The entries of Z' are the pairings between $y^{-1}F$ and $q^{-1}F$. Both have columns of norm at most T, since $\|y^{-1}\| = \|y\|$ and $\|J\| = 1$. For either Z or Z' the determinant has modulus at least $T^{-1/2}$ and each cofactor has modulus at most $(\sqrt{n}\,T^2)^{n-1}$ by Hadamard. Cramer's rule therefore bounds both inverse norms by $n\sqrt{T}(\sqrt{n}\,T^2)^{n-1} \le M_0$.

Let $Z^{-*} = (Z^*)^{-1}$. For $a < n$ the dual column of the prescribed section satisfies

$$q^{-1}e_a^* = \sum_{i=1}^{n} (Z'^{-*})_{ia}\, y^{-1} f_i + \tfrac{1}{2} \sum_{i<n} (Z^{-1})_{in} (Z^{-*})_{na}\, q^{-1} f_i.$$

The section gives $\gamma_Z e_a^* = (FZ^{-*})_a + \frac{1}{2}\sum_{i<n}(Z^{-1})_{in}(Z^{-*})_{na}\gamma_Z f_i$. Multiply by $q^{-1}\gamma_Z^{-1}$ and use $UZ^{-*} = Z'^{-*}$. This gives $\|q^{-1}e_a^*\| \le nTM_0 + (n-1)TM_0^2/2 \le M_0^4$. Section columns, dual-column estimate.

The negative space of J has orthonormal basis z_0 and $(e_a - e_a^*)/\sqrt{2}$, $a < n$. Its projection $P_- = (1 - J)/2$ therefore satisfies $\|q^{-1}P_-\|_{\mathrm{F}}^2 \le (2n-1)M_0^8$. Now $q^*q = J + 2q^*P_-q$ and $\|P_-q\| = \|q^*P_-\| = \|q^{-1}P_-\|$, as $q^* = Jq^{-1}J$. Thus $\|q\|^2 \le 1 + 2(2n-1)M_0^8$ and $\|v^{-1}z\| = \|hq\| \le 8nM_0^4 \le M_0^5 = S$. For every $v \in B_0$ this puts $z = v(v^{-1}z)$ in $B_0K_S = A$, as required by the conditioned measure on the same tube. Bessel length, shifted support, application to the tube.

Evaluation at one Gram matrix. Here is a proof of the single-point Sobolev step in the preceding column estimate. Put $m = 2n^2$, $D_v = d_v/e_v$, $\epsilon = P^{-8}$, and $\mathcal{B} = \{(h, u) : \|u\| \le P^{20}\}$, with measure $c\,dh\,du$. Write Q for the translate by Z of the real coordinate cube $[0, \epsilon]^m$ in the complex n-by-n matrices. These coordinates preserve Lebesgue measure, and Q lies in the ball of radius P^{-6}. Write $a_x(Y) = \gamma_Y(uh)^{-1}$. For $x \in \mathcal{B}$ every coordinate curve in Q has $\|a_x(Y)^{-1}\partial_j a_x(Y)\| \le P^{85}$. The section calculation above gives the stronger bound P^{60} in these real coordinates. Section bounds, coordinates and derivative.

Let $V_\emptyset(g) = \int \beta(y) W(gy)\, dy$. The derivative along $g' = gX$ acts on $W(gy)$ by $y^{-1}Xy$. On the support of β, both y and y^{-1} have norm at most P^{20}. Expand the entries of $y^{-1}Xy$, absorbing a normalized product of two entries of y^{-1}, y into β at each differentiation. This gives functions V_w indexed by words on $s = r^4$ letters, with the new weights bounded by $|\beta|$. With $F_w(q, x) = V_w(a_x(Z + q))$, the derivative inequality and the datum's Schur estimate for derivative words give, for $|w| \le m$ in the second inequality,

$$\|\partial_j F_w\| \le P^{40+85} \sum_l \|F_{lw}\|, \qquad \int_{U \setminus G} \|V_w\|^2 \le K_0 := P^{2m} D_v \|\beta\|_2^2.$$

Here U is the full Bessel unipotent. More precisely the input for each derivative $D_w W$ and integrable test ψ is

$$\int_{U\backslash G} \left\| \int \psi(y)(D_w W)(gy)\, dy \right\|^2 \le P^{2|w|}(d_v/e_v)\|\psi\|_2^2, \qquad |w| \le 2n^2.$$

The Gram-coordinate mean gives $\int_{[0,\epsilon]^m \times \mathcal{B}} \|F_w\|^2 \le (2P^2)^m K_0$. Its density dominates c on Q. After unfolding along U^0, two Gram matrices $Y, Y' \in Q$ allow only upper triangular frame entries bounded by $\|Y\|\|Y'^{-1}\| \le P^2$, giving the stated volume factor. Gram mean, application to derivative words, Schur input on the datum.

For a Hilbert-valued F on $[0,\epsilon]$, the fundamental theorem of calculus and Cauchy–Schwarz give $\|F(0)\|^2 \le 2\epsilon^{-1}\int \|F\|^2 + 2\epsilon \int \|F'\|^2$. Apply this on one coordinate, integrate on the remaining face and on $\mathcal{B}$, and bound the sum of squared derivative norms by s^2 times the largest resulting word integral. Thus releasing each of the m coordinates contributes at most $A = 2\epsilon^{-1}(1 + (sP^{125})^2)$. After m steps the words have length at most m, where the Gram mean applies. Since $s \le P^4$ and $P \ge 8$,

$$A(2P^2) \le 8P^{268} \le P^{275}, \qquad \int_{\mathcal{B}} \|F_\emptyset(0,x)\|^2 c\, dh\, du \le P^{550n^2} K_0 \le P^{2000n^2} D_v \|\beta\|_2^2.$$

This proves point evaluation using the normalized Gram-coordinate mean and the quantitative section estimates. We now derive the Schur bound for the derivatives on the full normalized Bessel function. Fixed-point estimate.

Schur orthogonality for the Bessel function. Let R be right translation on the U-equivariant L^2-space on $U\backslash G$, with the quotient measure fixed above. The Bessel vector W has squared norm d_v, its number of coordinates. For the unit lowest-K-type highest vector u of the discrete series π, put $c(g) = \langle u, \pi(g)u\rangle$, with inner product linear in its second argument. With formal degree e_v for the chosen Haar measure, the reproducing kernel is $h = e_v \bar{c} \in L^1(G)$ and $R(h)W = W$. Schur orthogonality for this irreducible square-integrable representation of the unimodular group G gives $h * \delta_g * h = c(g)h$ [DM76]. As $h = h^*$, insertion on both sides gives

$$\langle R(a)W, R(b)W\rangle = d_v\, c(a^{-1}b).$$

Equality of Gram matrices makes $\pi(a)u \mapsto R(a)W/\sqrt{d_v}$ an isometric intertwiner onto the closed span of the translates of W. Integrable coefficient projector, quotient coefficient, identities for the actual function.

For a word w of complexified matrix units, let $u_w = d\pi(w)u$. The isometry identifies $D_w W/\sqrt{d_v}$ with u_w. To see that these derivatives are classical and bounded, apply a smooth compactly supported approximate identity to u and project back to its line by the compact matrix coefficient of the multiplicity-one lowest K-type. The scalar tends to one. Rescaling one of the kernels gives $\beta \in C_c^\infty(G)$ with $R(\beta)W = W$, pointwise by continuity and full support. Moving derivatives to β under convolution gives the assertion and preserves U-equivariance. Reproducing bump, classical derivatives.

For $\psi \in L^1(G) \cap L^2(G)$, test $\int \psi(y)\pi(y)u_w\, dy$ against any unit vector. Cauchy–Schwarz and Schur orthogonality give the squared bound $e_v^{-1}\|\psi\|_2^2\|u_w\|^2$. Transporting by the isometry proves

$$\int_{U\backslash G} \left\| \int_G \psi(y)(D_w W)(gy)\, dy \right\|^2 \le (d_v/e_v)\|u_w\|^2\|\psi\|_2^2.$$

Lean. `IsBesselFunction.hasSchurBound_of_convolve_eq`. The word norm is estimated on the tube scale P below.

A derivative bound. Let $\ell = 2\lambda$ be the doubled Harish-Chandra parameter in the unitary sign frame, with $|\ell_i| \le L = 3a$ on this datum. The highest weight ν of the lowest compact type satisfies $|\nu_i| \le L + r$. For the j-th step M_j generated from that type by at most j applications of the complexified Lie algebra, put $N_j = L + r + j$. The compact group is a product of two unitary groups. Each of its weights in the lowest type is in the convex hull of the permutations of ν within the two blocks. Each weight of the adjoint representation is zero or a difference of two coordinate characters. Since M_j is a quotient of the sum of the tensor products of the lowest type with up to j adjoint factors, the entries of every compact highest weight in M_j have absolute value at most N_j.

In the sign-diagonal frame put $E_{ab} = d\pi(e_{ab})$ and $\Omega = \sum E_{ab}E_{ba}$. Let Ω_K restrict the sum to indices in the same sign block. The unitary adjoint relation $E_{ab}^* = \epsilon_a\epsilon_b E_{ba}$ gives, on M_j,

$$\sum_{a,b} \|E_{ab}z\|^2 = \langle z, (2\Omega_K - \Omega)z\rangle.$$

The compact Casimir on a type of highest weight μ has eigenvalue $\sum_i \mu_i(\mu_i + 2\rho_{K,i}) \le rN_j(N_j + r)$, since $|2\rho_{K,i}| \le r$. The full Casimir on the discrete series has eigenvalue $(\|\ell\|^2 - \|2\rho_G\|^2)/4 \ge -r^3/4$. These give an upper bound for the sum of $(2rN_j(N_j + r) + r^3/4)\|z\|^2$. For a matrix X, each $|X_{ab}| \le \|X\|_{\mathrm{op}}$. Cauchy–Schwarz over r^2 entries now gives

$$\|d\pi(X)z\| \le r\sqrt{2rN_j(N_j + r) + r^3/4}\,\|X\|_{\mathrm{op}}\|z\| \le (r + 1)^4(N_j + 1)\|X\|_{\mathrm{op}}\|z\|.$$

The last inequality follows by squaring and using $N_j + r \le r(N_j + 1)$. Casimir estimate, parameter and unitary norm.

The frame carrying the original generators to this sign frame is unitary. Hence every conjugated matrix unit has operator norm one. For $s = |w| \le 2n^2$, induction from $\|u\| = 1$ gives

$$\|u_w\| \le \prod_{j=0}^{s-1}(r + 1)^4(L + r + j + 1) \le P^s,$$

since on the admitted datum $(r + 1)^4(L + r + 1 + 2n^2) \le b^5 \le P$. The preceding Schur inequality therefore gives exactly $P^{2s}d_v/e_v$ in the prescribed quotient norm. Action in the sign frame, word estimate, actual parameter, choice of P.

Normalization of the vector by its local period. Work at the given embedding in the datum's unitary sign frame. Write $c_\lambda(g) = \langle u_0, \pi_\lambda(g)u_0\rangle$ for the chosen unit discrete-series coefficient, e_v for its formal degree with the above dg, and d_v for the dimension of the compact value representation ρ_H. Inner products are linear in the second entry. On the Bessel group $R = U \rtimes U(k)$ use $ds = du\,dh$, where dh has mass one. For $s \in R$, let h_s be its compact component and $\sigma(s)$ the sum of its simple-root entries in the frame. Put $\rho_R(s) = e^{\sigma(s)-\overline{\sigma(s)}}\rho_H(h_s)$. For a prescribed unit vector v_0 and $x \in G$ set

$$W_x(y) = \int_R c_\lambda(x^{-1}sy)\rho_R(s)^{-1}v_0\,ds, \qquad I_x = \int_R c_\lambda(x^{-1}sx)\overline{\langle v_0, \rho_H(h_s)v_0\rangle}e^{\overline{\sigma(s)}-\sigma(s)}\,ds.$$

Thus $I_x = \langle v_0, W_x(x)\rangle$, with its original phase. Write $p(g) = e_v c_\lambda(g)$, with convolution $(W * p)(y) = \int_G p(a^{-1})W(ya)\,da$. Schur orthogonality gives $c_\lambda * p = c_\lambda$, since $\|u_0\| = 1$. Inserting the integral for W_x gives $W_x * p = W_x$. Fubini here uses $p \in L^1(G)$ and the uniform bound on $\int_R |c_\lambda(x^{-1}sy)|\,ds$ for the large-gap parameter. The latter also bounds W_x, while unfolding $\int_{U\backslash G}\int_R |c_\lambda(x^{-1}sy)|\,ds\,dy = \|c_\lambda\|_1$ shows that W_x is square integrable. In this unfolding the invariant quotient measure is used

on left U-invariant norms; on general integrands the corresponding group integral is taken with the datum's quotient cutoff. Bessel average, absolute convergence and reproduction.

Set $\Phi(g) = \overline{p(x^{-1}g)}\langle v_0, W_x(g)\rangle$. The function is integrable. The relation $W_x(sy) = \rho_R(s)W_x(y)$ gives $\int_R \Phi(sy)\,ds = e_v\|W_x(y)\|^2$. Unfold and put $g = xa$. Since $p(a^{-1}) = \overline{p(a)}$,

$$e_v\|W_x\|^2_{L^2(U\backslash G)} = \int_G \Phi(g)\,dg = \langle v_0, (W_x * p)(x)\rangle = I_x.$$

The needed quantitative input is that for the compact type and parameter at this embedding there exists x such that

$$\mathrm{BesselLength}(x) \le b^{5n}, \qquad V_x := \mathrm{vol}_R\{s : \|x^{-1}sx - 1\| \le 1\} \le b^{60(r+k)^2}|I_x|. \tag{8.4}$$

The ball contains a neighbourhood of the identity and is compact, so $0 < V_x < \infty$. Equation (8.4) gives both $\|W_x\|_2 > 0$ and the ball-to-norm estimate used in the decay bound. One may choose $W = (\sqrt{d_v}/\|W_x\|_2)W_x$; a unit scalar in this choice cancels in its squared norms and self-pairings. The next argument reduces (8.4) to a pointwise estimate on the signed phase at one position. Norm and period, stationary point, specialization with $\beta = b^3$.

The stationary frame on the prescribed datum. At an infinite place v put $n = \varrho + 1$, $r = k + 2n - 1$, $B = \mathrm{liftWeight}(0, v)$, and $a_c = \mathrm{columnCount}(v, 1)$. The large-group parameter λ has $k - 1$ positive-sign entries $k/2 - 1 - i$ on X, followed by $2n$ entries $-B - (j + \frac{1}{2})\Delta$ of sign $\epsilon_j = (-1)^j$. The lowest weight is $\mu = \lambda - C/2$, where C_i counts the signs below minus those above i, multiplied by its sign. The alternating tail sums to zero, so $C_i = k - 2 - 2i$ on X. On the tail, X contributes $-(k-1)\epsilon_j$ and the other tail entries contribute $-\epsilon_j$. Thus $\mu|_X = 0$ and

$$\xi_j = -B - (j + \tfrac{1}{2})\Delta + (k/2)(-1)^j.$$

Since $\Delta = 2r^8 > k$, $2n \le r \le \Delta$, and $\Delta^2 \le B$, the ξ_j are strictly decreasing integers in $[-3B, -B]$. Row parameter, lowest-weight formula, ordered weights. The compact lowest weight μ_H vanishes on X and equals $a_c \ge 0$ on its remaining coordinate. Here $B \ge 1$, $D_0 = 3B + a_c \le b^2$ and $8nr \le b$. Compact weight, size, specialization.

Set $P(t) = \prod_i (t - \xi_i)$. To a real polynomial F associate the vector $[F]$, zero on X, with coordinates $F(\xi_i)/\sqrt{|P'(\xi_i)|(a_c - \xi_i)}$ elsewhere. Since $\operatorname{sgn} P'(\xi_i) = (-1)^i$, its signed Hermitian pairing on these real vectors is

$$h([F],[G]) = \sum_i \frac{F(\xi_i)G(\xi_i)}{P'(\xi_i)(a_c - \xi_i)}.$$

Write $w_j = [t^j(t - a_c)]$ for $0 \le j \le 2\varrho$ and $y = \sqrt{P(a_c)}[1]$. Lagrange interpolation gives $\sum_i F(\xi_i)/P'(\xi_i) = [t^{2n-1}]F$ for $\deg F \le 2n - 1$. Apply it to $-t^{i+j}(t - a_c)$ and $-t^j$, respectively. Partial fractions at $t = a_c$ give $\sum_i ((a_c - \xi_i)P'(\xi_i))^{-1} = 1/P(a_c)$. Hence

$$h(w_i, w_j) = \begin{cases} 0 & i + j < 2\varrho, \\ -1 & i + j = 2\varrho, \end{cases} \qquad h(y, w_j) = 0, \qquad h(y, y) = 1.$$

Multiplication by μ sends w_j to w_{j+1} for $j < 2\varrho$, and $\mu y = a_c y + \sqrt{P(a_c)}w_0$. Define i_0 to be the identity on X and to send the extra compact basis vector to y. It is an isometry and $i_0^\dagger \mu i_0 = \mu_H$. The vectors $w_0, \dots, w_\varrho$ are independent by evaluation at the $2n$ distinct nodes. Together with i_0 they therefore have exactly the signed Gram matrix of the standard compact basis and the Bessel frame $e_1, \dots, e_\varrho, z$. Witt extension, after the fixed Euclidean unitary sign change p, gives x carrying pi_0 and $p(w_0, \dots, w_\varrho)$ to those prescribed columns. Stationary-position theorem, Witt extension.

Bessel length of the stationary frame. The integral gaps imply $|P'(\xi_i)| \geq 1$. Thus $|(w_j)_i| \leq D_0^{j+1/2}$, so every pw_j has Euclidean norm at most $M = \sqrt{r}\, D_0^{2n}$. Let J be the standard Hermitian involution in the split frame and Gram–Schmidt the vectors $pw_0, \dots, pw_\varrho$ in their given order, without rescaling, to get $u_0, \dots, u_\varrho$. Each u_i pairs to -1 with $pw_{2\varrho-i}$, because all its earlier correction terms pair to zero. Hence $M^{-1} \leq \|u_i\| \leq M$. The isotropic vectors u_i, $i < \varrho$, have isotropic duals $Ju_i/\|u_i\|^2$. These 2ϱ columns are mutually Euclidean orthogonal and their lengths are at most M. Their Euclidean orthogonal complement is J-stable with signature $(k, 1)$. It contains u_ϱ, which has signed norm -1.

In that complement choose orthonormal positive and negative directions along the components of u_ϱ, of lengths s, c, with $c^2 - s^2 = 1$. The isometry sending the unit negative vector to u_ϱ is a two-dimensional boost with norm $c + s \leq 2\|u_\varrho\|$, extended by the identity in the other positive directions. For $s = 0$ only a compact change is needed. Combining the dual columns and this middle isometry yields an isometry g from the standard frame with $\|g\| = \|g^{-1}\| \leq 2M$. The product xg changes the standard Bessel frame by an upper unitriangular matrix, since Gram–Schmidt only adds earlier vectors. It therefore belongs to the Bessel subgroup R, and $g^{-1} = (xg)^{-1}x$ measures the required left-coset length. Thus

$$\inf_{s \in R}(1 + \|sx\|) \leq 1 + 2M \leq 8n\sqrt{r}\, D_0^{2n} \leq b^{4n+1} \leq b^{5n}.$$

The period uses the original x carrying the prescribed compact columns. Frame criterion for R, dual-frame length bound.

For the scale used below put $\beta = b^3$, $t = (r + k)^2 \log \beta$. The datum has $k \leq r$, $r \geq x_{\mathrm{sc}}^5$ and $(\log b)^{100} \leq x_{\mathrm{sc}}$, where x_{sc} denotes the admissible scale. Thus $t \leq 12r^2 x_{\mathrm{sc}}^{1/100}$. At the admissible scales $x_{\mathrm{sc}} \geq 10^8$, this gives $10^{30}t^3 \leq r^7$. Both doubled parameters have absolute values bounded by $L = 3a$, where $r + k + L \leq \beta$, and the opposite-sign gaps of the large-group parameter allow excess $H = r^7$. Admissible scales, magnitude bounds, parameter gaps.

The Bessel character of the stationary frame. Let $v_1, \dots, v_n$ be the stationary chain, $A = \langle v_1, \dots, v_{n-1}\rangle$, $z = v_n$, and H_0 the image of the compact space under its stationary isometry. Write $[\,,\,]$ for the ambient Hermitian form and $\dagger$ for its adjoint. The Gram relations give $A^\perp = A \oplus H_0 \oplus \mathbb{C}z$ and $[z, z] = -1$. The weight $\mu = \mu^\dagger$ sends v_j to v_{j+1} for $j < n$ and compresses on H_0 to μ_H.

For $s \in R$, in this frame $Y = g - I$ sends each v_j into $\langle v_1, \dots, v_{j-1}\rangle$ and acts on H_0 modulo A as $h - I$. These are exactly the defining triangular action and compact block of the Bessel group in the frame carried by x. If $\sigma(g)$ denotes the sum of the coefficients of v_j in Yv_{j+1}, it is the prescribed simple-root sum $\sigma(s)$. On the chain a product of two strictly flag-lowering maps lowers by two positions. Thus σ adds under multiplication and $\sigma(g^{-1}) = -\sigma(g)$. Both Y and $Y^\dagger = g^{-1} - I$ lower the flag. Flag and compact action in the actual frame.

Complete A by a dual isotropic subspace orthogonal to $H_0 \oplus \mathbb{C}z$, and write ξ_j for the coordinate of v_j in this Witt decomposition. Define

$$L = \sum_{j<n} v_{j+1}\xi_j, \qquad M = \mu - L - L^\dagger.$$

The image of L is orthogonal to A, so $L^\dagger A = 0$. It follows that $MA = 0$ and $\operatorname{im} M \subset A^\perp$. The trace of MY can be computed on $A^\perp/A$, since Y preserves the filtration $A \subset A^\perp$ and the two other diagonal blocks of MY vanish. On that quotient Y has blocks $(h - I) \oplus 0$. The H_0-compression of M is μ_H, as L vanishes on H_0. Thus

$$\operatorname{tr}(MY) = \operatorname{tr}(\mu_H(h - I)), \qquad \operatorname{tr}(LY) = \sum_{j<n} \xi_j(Yv_{j+1}) = \sigma(g), \qquad \operatorname{tr}(L^\dagger Y) = \overline{\sigma(g^{-1})}.$$

Adding gives the weighted trace identity with exactly the exponent $\sigma(s) - \overline{\sigma(s)}$ used in I_x. The unweighted trace of Y is $\operatorname{tr}(h - I)$, since its actions on A and on the dual quotient are strictly triangular. The latter is adjoint to the action of $g^{-1} - I$ on A.

Finally, a unitary on the common zero-weight coordinates X, extended by identity, fixes the chain pointwise. In the frame carried by x it therefore belongs to the Bessel group and its compact action is the same unitary on X, extended by identity there as well. This supplies the simultaneous translation used in localization. The two trace identities, Common unitary subgroup, Transport to the Bessel group.

How much control of the signed phase suffices. Set $\beta = b^3$, $D = (r+k)^2$ and $\delta = (200\beta^5)^{-1}$. Fix a candidate x at this place with the length bound in (8.4). Write $\theta(s)$ for the integrand of I_x, $J_s = (x^{-1}sx) \oplus h_s$ in the unitary sign frames, $G(s) = (2/\|J_s\|)^{9D}$, and $v_* = \operatorname{vol}_R\{s : \|J_s - 1\| \le \delta\}$. We use operator norm here. The first block is proper on R, and $J_1 = 1$, so $0 < v_* < \infty$. The large-gap coefficient decay gives $|\theta| \le G^2$: the second block is unitary, the first has norm at least one, and the gap permits the power $18D$. Coefficient decay.

The phase input for this test is the following pointwise estimate on that same R for the chosen x:

$$\big(|\theta(s)| - 2\Re\theta(s)\big)_+ \le 12\beta^{-100D} \qquad (s \in R). \tag{8.5}$$

It permits a negative real part where the coefficient has very small modulus. The deficit is also at most $3|\theta|$. Taking the geometric mean of the two bounds gives an integrable error $6\beta^{-50D}G$. On the small ball $|\theta| \ge 1/4$. Indeed both matrix logarithms have norm at most 2δ and belong to the respective Lie algebras, by the convergent logarithm series and the unitary defining equation. The first derivative estimate proved above, at word length one, bounds the change of each unit coefficient along $\exp(tX)$ by $\beta^5\|X\|$. Here the two ranks and the parameter bound total at most β. The twist has modulus one. Thus each coefficient has modulus at least $1 - 2\beta^5\delta > 1/2$. Small-ball bound, phase estimate.

A single covering argument controls the integrated error. In the set of matrices J_s of norm at most $M \ge 1$, take a maximal family separated by more than δ/M. Disjoint half-radius balls in real dimension $2D$ bound its cardinality by $(1 + 2M^2/\delta)^{2D}$. Since $\|J_t^{-1}\| = \|J_t\|$, each J_s is within δ/M of some J_t with $\|J_s J_t^{-1} - 1\| \le \delta$. The preimages are thus covered by right translates of the small ball. With $K = (1 + 8/\delta)^{2D} \le \beta^{12D}$, right invariance gives

$$V_x \le \operatorname{vol}_R\{\|J_s\| \le 2\} \le Kv_*, \qquad \int_R G(s)\,ds \le 2^{9D}Kv_* \sum_{j \ge 0} 2^{-5Dj} \le 2^{9D+1}Kv_*.$$

Indeed the j-th annulus $2^j \le \|J_s\| \le 2^{j+1}$ contributes at most $2^{4Dj}Kv_*$ in volume. The same count proves G^2 integrable. Since $\beta \ge 10^{12}$, $6\beta^{-50D}\int G \le v_*/8$. Integrating the phase inequality therefore gives

$$v_*/4 \le \int_R |\theta| \le 2\Re I_x + v_*/8, \qquad V_x \le 16\beta^{12D}|I_x| \le \beta^{13D}|I_x|.$$

This is within the required allowance β^{20D}. For the bounded-length position it therefore suffices to prove (8.5) for the prescribed coefficients and character. We start with the compact coefficient. Position on the datum, phase control.

Removing the compact coefficient from the phase estimate. Let $t = D\log\beta$, $\eta = e^{-100t}$, and take a neighbourhood of the identity in both blocks of Frobenius radius d with $dt \le 10^{-12}$. At this embedding the highest weight of $U(k)$ is $a_c = \operatorname{columnCount}(v, 1)$ in one coordinate j and

zero elsewhere. The highest vector of $\operatorname{Sym}^{a_c}\mathbb{C}^k$ is $e_j^{\otimes a_c}$ in the corresponding ordering. Hence the compact coefficient at h is $c_H(h) = h_{jj}^{a_c}$. Highest-vector coefficient, compact weight.

For unitary h with $\|h-I\|_{\mathrm{F}} \le d$, put $A = -i\log h$. It is Hermitian and $\|A\|_{\mathrm{op}} \le 2d$. Let X have its spectral distribution in e_j, so $m = \mathbb{E}X = A_{jj}$ and $h_{jj} = \mathbb{E}e^{iX}$. Write $Y = X - m$, $v_2 = \mathbb{E}Y^2$. Then $|Y| \le 4d$ and Taylor's formula with integral remainder gives

$$\mathbb{E}e^{iY} = 1 - v_2/2 + r_0, \qquad |r_0| \le (4d/6)v_2.$$

Here $v_2 \le 16d^2$. Using $|\log(1+z) - z| \le |z|^2/(1-|z|)$ at $z = -v_2/2 + r_0$ gives $\log \mathbb{E}e^{iY} = -v_2/2 + r_1$, $|r_1| \le 3dv_2$. Consequently $c_H(h)e^{-ia_c m} = \exp(-q+E)$, with $q = a_c v_2/2$, $|E| \le 6dq$. If $q \le 200t$, this differs from e^{-q} by at most $e^{-q}(e^{1200dt} - 1) \le e^{-q}/10$. If $q > 200t$, its modulus is at most η. In both cases it is within $A_0/10 + \eta$ of some $A_0 \in [0,1]$, including when $a_c v_2 = 0$.

Write μ, μ_H for the two diagonal lowest weights. At a stationary position the linear functional $\mathcal{T}(B,C) = \operatorname{tr}(\mu B) - \operatorname{tr}(\mu_H C)$ satisfies $\mathcal{T}(g_s - I, h_s - I) = \phi(s) = \sigma(s) - \overline{\sigma(s)}$, an additive character with values in $i\mathbb{R}$, where $g_s = x^{-1}sx$ in the sign frame. Apply this identity to s^i. The binomial expansion of $(g_s - I)^j, (h_s - I)^j$ now gives zero under $\mathcal{T}$ for $j \ge 2$, since the j-th finite difference of $i\phi(s)$ is zero. The logarithm series proves $\mathcal{T}(\log g_s, \log h_s) = \phi(s)$. Thus $\theta(s)$ is the product of the two phase-corrected coefficients, with the compact one conjugated. The required estimate on the noncompact coefficient is

$$\exists\, a \in [0,1] \quad \left| c_\lambda(g) e^{-\operatorname{tr}(\mu \log g)} - a \right| \le a/10 + \eta.$$

Indeed if the two approximants are $a, c \in [0,1]$, the product is within $0.21ac + 2.2\eta + \eta^2 \le ac/3 + 4\eta$ of ac. This implies (8.5) there. To pass to all s, the source uses localization: either $|\theta(s)| \le \eta$, or the same signed value occurs in the small neighbourhood above. We verify the dichotomy below at the position constructed for this datum. Logarithmic phase, localization, noncompact estimate.

The compact measure in the noncompact coefficient. Let $K = U(r_+) \times U(r_-)$, $r_+ + r_- = r$, with probability Haar du. For an integral weight w, let M_w denote its product of principal minors. With $a = \min_i w_i$, $b' = \max_i w_i$ and P_j the coordinate projection onto $\{i : w_i \ge j\}$, it is

$$M_w(u) = (\det u)^a \prod_{a < j \le b'} \det(P_j u P_j|_{\operatorname{im} P_j}).$$

Write $Q_w(B) = \sum_{i,j} |w_i - w_j|\,|B_{ij}|^2$. For any such projection P, let $A = PuP|_{\operatorname{im} P}$ and $C = (1-P)uP|_{\operatorname{im} P}$. Unitarity gives $A^*A = 1 - C^*C$. Multiplying $1 - x \le e^{-x}$ over the eigenvalues of C^*C,

$$|\det A|^2 \le e^{-\|C\|_{\mathrm{F}}^2} = e^{-\|[P,u]\|_{\mathrm{F}}^2/2}.$$

Indeed $Pu(1-P)$ and C have the same squared Frobenius norm, as $\operatorname{tr}(AA^*) = \operatorname{tr}(A^*A)$. The modulus of the determinant character is one. Each ordered pair of indices is separated by exactly $|w_i - w_j|$ of the projections, so multiplication proves

$$|M_w(u)|^2 \le e^{-Q_w(u)/2}.$$

Counting cuts, pinned minor bound.

For $w = \mu$, the highest weight of the lowest K-type, let $m(u) = |M_\mu(u)|^2$ and $N_\mu = \int_K m(u)\,du$. In each sign block order the coordinates by decreasing μ_i. A principal minor is the coefficient of the corresponding unit wedge of the initial basis vectors. The tensor of these wedges, tensored with the determinant character to the power a, is a unit highest vector of weight μ. It generates

the irreducible Cartan component V_μ, since the highest weight line in this tensor product is one-dimensional. Thus $M_\mu(u)$ is its diagonal coefficient. Number the coordinates within each ordered sign block. For $i<j$ in a block put $b_{ij}=(\mu_i-\mu_j)/(j-i)\geq 0$. Then, for real $s\geq 0$,

$$\int_K m(u)^s\,du=\prod_{i<j \text{ in the same block}}(1+sb_{ij})^{-1}. \tag{8.6}$$

For integral s, the s-fold tensor power of the highest vector generates $V_{s\mu}$. Schur orthogonality gives $(\dim V_{s\mu})^{-1}$, and Weyl's dimension formula gives the product [Sug03, eqs. (18)–(21)]. In particular $N_\mu=1/\dim V_\mu$. For independent uniform variables U_{ij} on $[0,1]$, the variable $\prod U_{ij}^{b_{ij}}$ has the same integer moments as $m(u)$. Polynomial approximation on $[0,1]$ gives equality in distribution and hence equality of all real moments $s\geq 0$. The formal estimate uses the dimension at $s=1$. Lowest weight, normalization estimate.

The normalized coefficient and its splittings. Write $K=U(r_+)\times U(r_-)$, $\ell=2\lambda$, $\nu=\lambda+\rho_G$ and $\mu=\nu-2\rho_c$, with roots ordered by decreasing ℓ. Write M_γ for the weight product of principal minors used above. Fix an invertible matrix Y. Let G_c be the product of the two complex general linear groups on the sign blocks and B_c its upper triangular subgroup in decreasing ℓ-order. For $p,s\in G_c$ with $sY^{-1}p$ accretive, define

$$I(p,s)=\int_K \Psi_{p,s}(k)\,dk,\qquad \Psi_{p,s}(y)=\frac{M_\mu(py)M_\mu(y^*s)}{M_\nu(y^*sY^{-1}py)}\quad (y\in G_c).$$

Initial-column and initial-row determinants under $b\in B_c$ give $\Psi_{p,s}(yb)=|b^{\mu-\nu}|^2\Psi_{p,s}(y)$, where $b^\gamma=\prod_i b_{ii}^{\gamma_i}$. Here both ν and μ decrease weakly in each block. This factor, $|b^{-2\rho_c}|^2$, is the real Jacobian of the action of $x\in G_c$ at the flag kB_c when $xk=k'b$ by block QR. Indeed the two unitary frames are isometries for a K-invariant flag metric, while the complex tangent determinant of b is $\prod_{i>j} b_{ii}/b_{jj}$ over pairs in the same block, by its action on matrices modulo the upper triangular algebra. Probability Haar pushes to invariant probability volume on G_c/B_c. Thus changing variables on this flag manifold gives

$$I(px,x^*s)=\int_K \Psi_{p,s}(xk)\,dk=I(p,s).$$

The integral is holomorphic in p,s because accretivity keeps the denominator uniformly nonzero locally for compact K. Differentiate the last identity for $x=\exp(tA)$, then $x=\exp(itA)$, t real. By complex linearity the differentials in $(pA,0)$ and $(0,A^*s)$ both vanish. Take $x=(s^*)^{-1}$ in the invariance identity to reduce to $s=I$. The allowed p's then form a convex set in the block diagonal algebra, since $Y^{-1}p$ must have positive Hermitian part. Thus $I(p,s)$ is independent of the splitting.

For $Y=g\in U(J)$, set $a=(g+JgJ)/2$. The matrix $g^{-1}a=(I+g^{-1}(g^*)^{-1})/2$ is positive definite. By Jacobi's complementary-minor identity $M_\nu(H^{-1})=M_{-\nu}(H)$ on the accretive domain. Consequently $I(a,I)/N_\mu$, $N_\mu=\int_K|M_\mu|^2$, is exactly the definition of $c_\lambda(g)$. For any splitting $g=pzq^*$ with $p,q\in G_c$ and z accretive, evaluation at (p,q^*) gives the normalized integral with numerator $M_\mu(pk)\overline{M_\mu(qk)}$ and denominator $M_{-\nu}(k^*zk)$. The same independence applies to any accretive Y, by starting at (I,I). The discrete-series realization of this coefficient as a unit-vector coefficient with the prescribed formal degree is used in the Schur argument above. Definition of the coefficient, Flag density and invariance, descent and splitting.

A dimension-free remainder for one compressed determinant. Let P be an orthogonal projection and $S=I-P$. Suppose $\|A_i\|_{\mathrm{op}}\leq h\leq 1/20$ for $i=1,2,3$, and put $\zeta=\max_i\|[P,A_i]\|_{\mathrm{F}}$.

The determinant of the compression of $e^{A_1}e^{A_2}e^{A_3}$ to $P\mathbb{C}^r$ is

$$\exp\left(\sum_i \operatorname{tr}(PA_i) + \frac{1}{2}\sum_i \operatorname{tr}(PA_iSA_i) + \sum_{i<j} \operatorname{tr}(PA_iSA_j) + R_P\right), \qquad |R_P| \le 40h\zeta^2.$$

The two powers of the commutator allow summation over all weight cuts even when many of the weights are large.

On $P\mathbb{C}^r$, set $C(z) = P\prod_i e^{zA_i}P$ and $B(z) = \prod_i e^{zPA_iP}$. In a word of length j, compare its compression with the product of its compressed letters. Insert $I = P + S$ between the letters, and group terms according to their first passage from P to S and their first return. Once those two positions are specified, sum over the rest of the word without inserting projections. The intervening P- and S-blocks have operator norm at most h per letter. The two crossing blocks have Hilbert–Schmidt norm at most ζ. The trace norm of the difference is therefore at most $\binom{j}{2}h^{j-2}\zeta^2$ for $j \ge 2$, and is zero for $j < 2$.

The exponential coefficients for words of length j sum to $3^j/j!$. Thus the series of trace norms of the coefficients of $C - B$ is bounded coefficient by coefficient by $d(t) = \frac{9}{2}t^2\zeta^2e^{3ht}$. For $C - I$ and $B - I$ the corresponding operator-norm majorant is $p(t) = e^{3ht} - 1$, with $p(1) < 1$. Telescoping the difference of the jth powers in the logarithm series gives the majorant $\sum_{j\ge1} d(t)p(t)^{j-1} = d(t)/(1 - p(t))$ for $\operatorname{tr}\log C - \operatorname{tr}\log B$. Here $\operatorname{tr}\log B = z\sum_i \operatorname{tr}(PA_i)$ by determinants and continuity from zero. The quadratic term of the difference comes from a single exit and return, which contributes $\operatorname{tr}(PA_iSA_j)$ with coefficient one for $i < j$ and one half for $i = j$. The sum of the terms of degree at least three at $z = 1$ is bounded in absolute value by

$$\frac{d(1)}{1 - p(1)} - \tfrac{9}{2}\zeta^2 = 9\zeta^2\frac{e^{3h} - 1}{2 - e^{3h}} \le 40h\zeta^2,$$

using $e^{3h} \le 7/6$ and $e^{3h} - 1 \le 3he^{3h}$.

Comparison with Lean. Lean scales the off-diagonal blocks by a complex parameter and uses evenness to gain its square. Counting the first exit and return gives the two Hilbert–Schmidt factors directly. The other letters need only an operator-norm bound. The formal result assumes $\|A_i\|_{\mathrm{F}} \le h$ and gives the constant 10^4.

Lean. `norm_logMinorRemainder_le`, `exists_det_one_add_proj_mul_eq_exp`.

The signed quadratic term in the noncompact coefficient. Let $g = e^Z \in U(r_+, r_-)$, $h = \|Z\|_{\mathrm{F}} \le 2d$, $h \le 1/20$, $r = r_+ + r_-$. Use the parameter $\ell = 2\lambda$ of the datum, its compact signs σ_i, $\nu = \lambda + \rho_G$, $w = -\nu$, and $\mu = \nu - 2\rho_c$. Order each sign block by decreasing ℓ_i. Let U, V be the strict lower and upper parts of the sign-block diagonal of Z, respectively, and put

$$(A_1, A_2, A_3) = (-U, Z, -V), \qquad Y = e^{A_1}e^{A_2}e^{A_3}.$$

Write M_b for the weight product of principal minors. With dk probability Haar on $K = U(r_+) \times U(r_-)$, set

$$F(H) = \frac{\int_K |M_\mu(k)|^2/M_w(k^*Hk)\,dk}{\int_K |M_\mu(k)|^2\,dk}.$$

Both $H = e^Z, Y$ have positive Hermitian part, since their operator distance from 1 is at most $e^{3h} - 1 < 1$. Thus the compressed matrices occurring in the denominator are invertible. The normalized Flensted–Jensen formula gives $c_\lambda(g) = F(e^Z) = F(Y)$. For the second equality use its splitting formula for $Y = pe^Zq^*$, with $p = e^{-U}$ and $q = (e^{-V})^*$. It replaces the numerator at e^Z by

$M_\mu(pk)\overline{M_\mu(qk)}$. Both p, q are lower unitriangular in each sign block. The entries of μ are weakly decreasing there. Between consecutive coordinates of a block the drops in λ, ρ_G are each at least one, and the drop in $2\rho_c$ equals two. Each row set defining a μ-minor therefore consists of initial segments of the two blocks. Left multiplication by p or q effects a determinant-one row operation on these rows. The numerator stays $|M_\mu(k)|^2$. Normalized coefficient and splitting formula, triangular reduction.

For a threshold $P = \mathbf{1}_{\{w_i \ge j\}}$ let $D = kPk^*$ and

$$x = \|[P, Z]\|_{\mathrm{F}}, \quad e = \|[P, k]\|_{\mathrm{F}} = \|D - P\|_{\mathrm{F}}, \qquad z = x + 2he.$$

Masking entries gives $\|[P, A_a]\|_{\mathrm{F}} \le x$, whence $\|[D, A_a]\|_{\mathrm{F}} \le z$. For an orthogonal projection S write

$$H_S = \sum_{a \le b}{}' \operatorname{tr}\left(SA_a(1-S)A_bS\right),$$

where the diagonal terms have weight $1/2$. The second-order expansion of the compressed determinant at D is

$$\log\det(DYD|_{\operatorname{im} D}) = \operatorname{tr}\left(D\sum_a A_a\right) + H_D + R_D, \qquad |R_D| \le 10^4 hz^2.$$

Here the logarithm continues from the identity. In each sign block $\sum A_a$ is diagonal. Since D preserves the sign blocks and $\sum_i |D_{ii} - P_{ii}| = \sum_{i,j:P_{ii}\ne P_{jj}} |k_{ij}|^2 = e^2$, the linear term differs from $\operatorname{tr}(PZ)$ by at most he^2. Each off-diagonal block $DA_a(1-D)$ differs from $PA_a(1-P)$ in Frobenius norm by at most $2he$, by replacing its two projections one at a time. Both blocks have norm at most z, as do their reverse-direction counterparts. Cauchy–Schwarz for each trace gives $|H_D - H_P| \le (9/2)4hez = 18hez$.

For $i \in P,\ j \notin P$ we have $\ell_i < \ell_j$, because $\lambda + \rho_G$ increases strictly with ℓ. Write $s = Z_{ij}$, $s' = Z_{ji} = -\sigma_i\sigma_j\bar{s}$. At equal signs the entries of (A_1, A_2, A_3) in the two directions are $(-s, s, 0)$ and $(0, s', -s')$. Their contribution to H_P is $-ss'/2 = |s|^2/2$. At opposite signs the triples are $(0, s, 0)$ and $(0, s', 0)$, giving $ss'/2 = |s|^2/2$. Thus $H_P = x^2/4$. Conjugation by k identifies the compression at D with the P-minor of k^*Yk. Its logarithm differs from $\operatorname{tr}(PZ) + x^2/4$ by at most $10^5 h(x^2 + e^2)$, by the three bounds above and $2h \le 1/10$. Linear term, quadratic sign, one-minor bound.

Put $a = \min w_i$, $b = \max w_i$, and $P_j = \mathbf{1}_{\{w_i \ge j\}}$. In $M_w(k^*Yk) = (\det Y)^a \prod_{a<j\le b} \det_{P_j}(k^*Yk)$ the full determinant is exactly $e^{\operatorname{tr} Z}$. Since $\sum P_j = \operatorname{diag}(w) - aI$, the linear terms add to $\operatorname{tr}(wZ)$, writing a weight in a trace for its diagonal matrix. Each ordered pair i, j crosses exactly $|w_i - w_j|$ cuts. With $Q_w(C) = \sum_{ij} |w_i - w_j|\,|C_{ij}|^2$, we obtain

$$M_w(k^*Yk) = \exp\{\operatorname{tr}(wZ) + q + R_0(k)\}, \quad q = \tfrac{1}{4}Q_w(Z) \ge 0, \quad |R_0(k)| \le 10^5 h\{4q + Q_w(k)\}.$$

This includes all determinant twists $a \in \mathbb{Z}$. The entries of $2\rho_c = \nu - \mu$ have absolute value at most r. Hence $|\operatorname{tr}((\nu-\mu)Z)| \le r^2 h$ and $Q_w(k) \le Q_\mu(k) + 2r\|k\|_{\mathrm{F}}^2 = Q_\mu(k) + 2r^2$. Setting $R(k) = \operatorname{tr}((\nu-\mu)Z) - R_0(k)$ gives

$$M_w(k^*Yk) = \exp\{q - \operatorname{tr}(\mu Z) - R(k)\}, \qquad |R(k)| \le 10^6 d\,(q + Q_\mu(k) + r^2).$$

The continuous logarithms for the individual compressions give a measurable choice of R. Taking the reciprocal gives the Gaussian e^{-q} in the normalized integral for $e^{-\operatorname{tr}(\mu Z)}c_\lambda(g)$. Product of minors, comparison of the two weights.

Averaging the Gaussian on the compact group. Write r for the size of the noncompact unitary group, K for its maximal compact subgroup and dk for probability Haar. Put $Z = \log g$, $w = -(\lambda + \rho_G)$, $Q_\nu(A) = \sum_{i,j} |\nu_i - \nu_j|\,|A_{ij}|^2$, and $q = Q_w(Z)/4$. Let μ be the highest weight of the lowest K-type and $M_\mu(k)$ its unit highest-vector coefficient. Give K the probability measure $d\mathbb{P} = m(k)\,dk/N_\mu$. The normalized Flensted–Jensen integral and the logarithmic denominator estimate give

$$c_\lambda(g)e^{-\operatorname{tr}(\mu Z)} = e^{-q}\int_K e^{R(k)}\,d\mathbb{P}(k), \qquad |R(k)| \le \alpha(q + Q_\mu(k) + r^2), \quad \alpha = 10^6 d,$$

where $\|g - I\|_{\mathrm{F}} \le d$ and $\|Z\|_{\mathrm{F}} \le 2d$. Only $e^{R(k)}$, determined by the nonvanishing denominator, enters the integral. Denominator and integral, application at $\log g$.

Since $m(k) \le e^{-Q_\mu(k)/2}$, formula (8.6) gives, for $0 \le \alpha \le 1/4$,

$$\int_K e^{\alpha Q_\mu(k)}\,d\mathbb{P}(k) \le \frac{\int_K m(k)^{1-2\alpha}\,dk}{N_\mu} \le (1-2\alpha)^{-r(r-1)/2} \le e^{2\alpha r^2}.$$

Indeed each factor of the ratio is at most $(1-2\alpha)^{-1}$, and $-\log(1-2\alpha) \le 4\alpha$. With $t = (r+k)^2 \log\beta$ we have $t \ge \max(1, r^2)$ and $dt \le 10^{-12}$. The absolute error from e^{-q} is at most $e^{-q}(\exp(\alpha(q+3r^2)) - 1)$, by $|e^z - 1| \le e^{|z|} - 1$. For $q \le 200t$ this is at most $e^{-q}/10$ since $203\alpha t \le 203 \cdot 10^{-6}$. For $q > 200t$, bounding it by $\exp(-(1-\alpha)q + 3\alpha t)$ gives at most e^{-100t}. Thus the required $a \in [0,1]$ is $e^{-Q_w(\log g)/4}$. The formal proof uses a pointwise estimate and the compact normalization bound. The estimate for the moment ratio is uniform in μ. Compact dimension, pinned error bound.

Determinant decay outside the Gaussian region. Put $J = \operatorname{diag}(\varepsilon_i)$, $a(g) = (g + JgJ)/2$ and $Y(g) = g - a(g)$. Let H be any nonnegative integer at most the gap between opposite-sign entries of the undoubled parameter λ. The hypothesis HasLargeGap(H) gives gap $500r^3 + H$ and in particular allows the prescribed $H = r^7$. For the normalized coefficient the exact estimate is

$$|c_\lambda(g)|^2\,|\det a(g)|^H \le 1.$$

To see this, first take $y = y^* > 0$ in $U(J)$ and write $a = a(y)$. Then $y^{-1} = JyJ$, $\det y = 1$, and $a = (y + y^{-1})/2 > 0$. Jacobi's complementary-minor identity gives $\Delta_{I^c}(y) = \Delta_I(y^{-1}) = \Delta_I(y)$ for each coordinate set I. Put $\nu = \lambda + \rho$, where positive roots are ordered by λ. In this ordering ρ is decreasing along with λ, so all opposite-sign gaps of ν are at least H. At integer threshold j, set $A = \{\nu_i \ge j\}$, $B = \{\nu_i + H \ge j\}$ and $S = B \setminus A$. The weights on S lie in $[j - H, j)$, so its indices have one sign if $S \ne \varnothing$. Thus $\Delta_S(a) = \Delta_S(y)$. Hadamard–Fischer on a and on y gives

$$\Delta_B(a) \le \Delta_A(a)\Delta_S(y) = \Delta_A(a)\Delta_{A \sqcup B^c}(y) \le \Delta_A(a)\Delta_A(y)\Delta_B(y).$$

Multiply over the thresholds between a common lower and upper bound of $\nu, \nu + H$, retaining the common determinant prefactor for a. This gives $M_{\nu+H}(a) \le M_\nu(a)M_\nu(y)M_{\nu+H}(y)$. Since $M_{\nu+H}(x) = (\det x)^H M_\nu(x)$ and $M_\nu(a) > 0$, $(\det a)^H \le M_\nu(y)^2 = M_{-\nu}(y)^2$, the last equality following by complementing the cuts and changing j to $1 - j$. Principal-minor bound.

Write $g = k_0 y$ in polar form, $k_0 \in K$. For $u \in K$, $a(u^* y u) = u^* a(y) u$ and $a(g) = k_0 a(y)$. The normalized polar formula at this same place is

$$c_\lambda(g) = N_\mu^{-1}\int_K \frac{M_\mu(k_0 u)\overline{M_\mu(u)}}{M_{-\nu}(u^* y u)}\,du, \qquad N_\mu = \int_K |M_\mu(u)|^2\,du.$$

Every denominator is positive and at least $|\det a(g)|^{H/2}$. Cauchy–Schwarz and left invariance of du give an upper bound N_μ for the integral of the absolute numerator, proving the claim. Finally $a(g)^* a(g) = 1 + Y(g)^* Y(g)$, by expanding $g^* J g = J$. Taking determinants and using $\prod_i (1 + s_i^2) \geq 1 + \sum_i s_i^2$ for the singular values of $Y(g)$ gives the far-region bound $|c_\lambda(g)|^4 \leq (1 + \|Y(g)\|_{\mathrm{F}}^2)^{-H}$. It also gives the operator-norm decay used in the covering argument. If $s = \|g\|_{\mathrm{op}} > 1$, the eigenvalues of y include s, s^{-1} because $JyJ = y^{-1}$. Both contribute $(s + s^{-1})/2$ to $\det a(y)$, with all other factors at least one. Thus $|\det a(g)| \geq (s/2)^2$. For $s \geq 2$ this yields $|c_\lambda(g)| \leq (2/s)^j$ for each $0 \leq j \leq H$. For $1 \leq s \leq 2$ the same inequality follows from $|c_\lambda(g)| \leq 1$. Normalized polar formula, determinant decay, fourth-power bound.

Localization at the same stationary position. Write $(g, h) = (x^{-1} s x, h_s)$ in the sign frames, $J = \mathrm{diag}(\epsilon_i)$, and $Y = (g - JgJ)/2$, $F = \|Y\|_{\mathrm{F}}$. Set $u = 10^{-13}/t$. The first parameter has excess gap $H = r^7 \geq 10^{30} t^3$. The determinant decay proved above gives $|c_\lambda(g)|^4 \leq (1 + F^2)^{-H}$. If $F \geq u$, use $\log(1 + u^2) \geq u^2/2$ and $Hu^2/8 \geq 1250t$. The compact coefficient and character have modulus at most one, so $|\theta(s)| \leq e^{-100t}$, which implies (8.5). Determinant and Frobenius norm, Gap on the chosen parameter.

Suppose $F \leq u$. The stationary frame above has the following properties. The two factors contain a common positive subspace X of weight zero, and the image of the Bessel group contains the diagonal $U(X)$. The weights satisfy $-3B \leq \mu_i \leq -B$ off X, $B \geq 1$, and $\mu_H \geq 0$. On the group we have

$$\mathrm{tr}(g - I) = \mathrm{tr}(h - I), \qquad \mathrm{tr}(\mu(g - I)) - \mathrm{tr}(\mu_H (h - I)) = \phi(s) \in i\mathbb{R}.$$

The i-th diagonal entry of $gJg^* = J$ gives $\|(g - I)_{i,*}\|_2^2 = 2 \sum_j |Y_{ij}|^2 - 2\Re(g_{ii} - 1)$. Multiply by $-\mu_i$ and sum. Since h is unitary, $\Re\,\mathrm{tr}(\mu(g - I)) = \Re\,\mathrm{tr}(\mu_H (h - I)) \leq 0$. Thus the sum of the squared row norms off X is at most $6F^2$. The same argument with $g^* J g = J$ bounds the columns off X by $6F^2$.

Set $a = g_{XX}$, $c = g_{X^\perp X}$. Then $a^* a - I = -c^* J|_{X^\perp} c$ has Frobenius norm at most $6u^2$. In the polar decomposition $a = U|a|$ one has $\||a| - I\|_{\mathrm{F}} \leq \|a^* a - I\|_{\mathrm{F}} \leq u$. Translate on the left by the diagonal copy of U^*. Each highest vector is fixed by this copy. Indeed its compact matrix coefficient on $U(X)$ is the determinant to weight zero, as every proper weight cut contains all of X or none of it. The weighted trace identity also gives zero character there. Thus the translated point s' has the same signed value and the same F. Its X-block is $|a|$. The row and column bounds at s' give $\|g' - I\|_{\mathrm{F}}^2 \leq u^2 + 12u^2$. The unweighted trace identity and the row formula summed over all rows give $\|h' - I\|_{\mathrm{F}}^2 = \|g' - I\|_{\mathrm{F}}^2 - 2F^2$. Both factors are within $4u$ of identity. Since $4ut < 10^{-12}$, the two coefficient estimates and logarithmic phase cancellation give (8.5) at s', hence at s. This proves the defect bound throughout the group at the chosen x. Inserting it in the covering estimate yields

$$0 < V_x \leq 16 \beta^{12(r+k)^2} |I_x| \leq b^{60(r+k)^2} |I_x|, \qquad \mathrm{BesselLength}(x) \leq b^{5n}.$$

In particular the same local W_x can be normalized in the prescribed quotient norm. Weighted row and column bounds, Polar translation, Phase defect.

A decay majorant for the normalized local pair. At this place write $M = 9r^2$, $i = 3r^3 f$, $m = 60(r + k)^2$, and $L = \max_j |2\lambda_j|$. The datum gives $i \leq r^7$, $r + L + 1 \leq b^4$, $b \geq 2000$. Write $\phi(g) = (2/\|g\|)^M$, $\ell = \mathrm{BesselLength}$ and $u(y) = \max(1, \ell(y)/b^{6n})$. The determinant proof works with every exponent up to the full opposite-sign gap $500r^3 + r^7$. Its operator-norm consequence gives $|c_\lambda(g)| \leq (2/\|g\|)^{M+j}$ for $j = 0, i$. Coefficient decay, datum bounds.

For $g = x^{-1} s y$, $s \in R$, submultiplicativity and $\|g\| \geq 1$ give $\ell(y) = \ell(xg) \leq \ell(x)\|g\|$, so $\ell(y)/b^{6n} \leq \|g\|/2$. The two choices of j give $u(y)^i |c_\lambda(g)| \leq \phi(g)$. Let $C = \sqrt{d_v}/\|W_x\|_2$,

$h(y) = C\int_R \phi(x^{-1}sy)\,ds$. The compact type and twist are unitary and the cutoff lies in $[0,1]$, so $u^i\|W^{\mathrm{full,cut}}\| \le h$. Right invariance gives $h(sy) = h(y)$ for $s \in R$, and unfolding with the prescribed $ds = du\,dh_K$, $\int dh_K = 1$, gives $\int_{U\backslash G} h = C\int_G \phi$. Unfolding, length weight.

Here are both Haar estimates in the same operator norm. For a homomorphism $j : S \to G$ with right Haar ds, put $V(\delta) = \mathrm{vol}\{s : \|j(s) - 1\| \le \delta\}$, $0 < \delta \le 1$. For $t \ge 1$, pack the set $\{j(s)z : \|j(s)z\| \le t\}$ at separation δ/t. Disjoint half-radius balls in real dimension $2r^2$ give at most $(3t^2/\delta)^{2r^2}$ centres. Since $\|(j(s)z)^{-1}\| = \|j(s)z\|$, their right translates of the δ-ball cover its preimage. For any $M \ge 8r^2$ the same argument with $\phi(g) = (2/\|g\|)^M$ gives

$$\int_S \phi(j(s)z)\,ds \le 2^M(3/\delta)^{2r^2}V(\delta)\int_1^\infty Mt^{4r^2-M-1}\,dt \le 2^{M+1}(3/\delta)^{2r^2}V(\delta). \tag{8.7}$$

For $j(s) = x^{-1}sx$, $z = x^{-1}y$, $\delta = 1$, this gives $h \le a := C16^M V_x$. For $j = \mathrm{id}_G$, $z = 1$, take $\delta = (200(r+L+1)^5)^{-1}$. On this ball $\|\log g\| \le 2\delta$, so the first derivative bound $(r+L+1)^5$ gives $|c_\lambda(g)| \ge 1/2$. Thus $e_v V(\delta) \le 4$ by $e_v\int_G |c_\lambda|^2 = 1$. As $3/\delta \le b^{25}$, $16^M e_v\int_G \phi \le 8\cdot 32^M b^{50r^2} \le b^{99r^2}$. By (8.4), $C^2V_x \le b^m e_v d_v$. Hence

$$a\int_{U\backslash G} h = C^2V_x16^M\int_G \phi \le b^{m+99r^2}d_v \le b^{400r^2}d_v,$$

using $k \le r$. The factors of e_v cancel because both integrations use the same dg. Packing, formal-degree integral, normalized majorant.

A cutoff shared by all infinite places. The reduced indicator ω can couple the embeddings. To combine the one-place estimates let m_v be the probability law of the shift and μ_v the bounding measure just constructed. With A_v the compact tube and $t_v = t_v(x_v)$, the local contraction sends a test ϕ to the matrix

$$(T_v\phi)(x_v) = \int_{A_v} \phi(y)W_v(\gamma_v t_v y)\,W_v(t_v y)^*\,dy, \qquad \|T_v\phi\|_{L^2(m_v;\mathrm{HS})} \le \|\phi\|_{L^2(\mu_v)}.$$

All the shifts fix the frame. Write $m = \bigotimes m_v$, $\mu = \bigotimes \mu_v$. The tensor product of the T_v is a contraction on Hilbert tensor products. For a product test, Fubini identifies it with the integral of the product kernel on $\prod A_v$. The same identification holds for bounded measurable tests. Indeed on each compact tube use the finite measure $\mu_v + dy|_{A_v}$. Conditional expectations on increasing finite rectangular partitions approximate the test by uniformly bounded rectangular simple functions in the product control measure and, along a subsequence, almost everywhere. This controls both μ and product Haar. For each fixed shift the full kernel is bounded on the compact product, so dominated convergence identifies the integral there. Fatou gives the contraction for this test. In particular

$$\int \|(\textstyle\bigotimes_v T_v)\omega\|_{\mathrm{HS}}^2\,dm \le \int |\omega|^2\,d\mu \le \prod_v p_v.$$

Choose one simultaneous shift with squared norm at most $\prod_v p_v$ in the full-measure set on which the tubes contain the shifted cut support. For the latter assertion, a reduced point in the cut is in the frame box, and the other vector in the cut has Bessel length at most $b^{6n+2} \le R$. The frame-fixing shift preserves the box and ω. Thus on the product tube compare the full kernel to the cut kernel. Left invariance of Haar restores the original cut integral, including its signs. To put its indices in the order of the measured block apply this argument to γ^{-1} and take adjoints; the cross Gram matrix becomes Z^*, with the same discriminant. This proves

$$\Big\|\int \omega(y)K_\gamma^{\mathrm{cut}}(y)\,dy\Big\|_{\mathrm{HS}} \le \sqrt{\prod_v p_v} + E_\gamma, \qquad E_\gamma = \int \omega(y)\|K_\gamma^{\mathrm{full}}(y) - K_\gamma^{\mathrm{cut}}(y)\|_{\mathrm{HS}}\,dy.$$

The common near witness supplies the two box conditions in each local estimate. No independence assumption on ω occurs in the proof. Product of contractions, shifted cut support, adjoint, invariance and boxes.

Putting the product on the arithmetic scale. Let $x = \mathfrak{d}_\gamma$. At each infinite place $B = \langle F, \gamma F\rangle$ has negative index n. Indeed it contains z_0 of negative norm, and $B \cap z_0^\perp$ is nondegenerate and contains the totally isotropic space A of dimension $n-1$. The ambient negative index is n. The sign of the Gram determinant $(-1)^n\sigma_v(a_\gamma)$ is therefore $(-1)^n$. The integral class frames have integral cross products, so a_γ is an algebraic integer and $x = \sqrt{|\operatorname{Nm} a_\gamma|} \geq 1$. The scale of the a-th frame vector is $M_v^{\varrho-a}/\sqrt{N}$. Write $s_{v,a} = M_v^{2(\varrho-a)}/N$ for its square, $0 \leq a \leq \varrho$, $s_v = \prod_a s_{v,a}$, and $L_* = \prod_{v,a} s_{v,a}^{k+2(\varrho-a)}$. The normalized determinant acquires the factor s_v, and its penultimate minor the factor Ns_v. Thus $g_v = s_v^2|\sigma_v(a_\gamma)|$. Set $\alpha = 10^5n^4$, $d_B = \prod_v d_v$, $e_B = \prod_v e_v$. For P_* the product of the p_v evaluated at $g_v = s_v^2$, this gives

$$\prod_v p_v = P_*\, x^{-(\alpha+k-1)}, \qquad P_* \leq 2^{kf}\tau\, \frac{d_B^2}{e_B c_G^f L_*^2}.$$

Indeed $1 \leq |M_v| \leq b^{23}$, $1 \leq N \leq b^{60n}$ give $b^{-60n^2} \leq s_v \leq b^{46n^2}$ and $\prod_a s_{v,a}^{k+2(\varrho-a)} \leq s_v^k b^{92n^3}$. Hence $R^{9n}/s_v^2 \leq b^{453n^2}$. The exponent of b spent per place is at most $453n^2\alpha + 92n^2 + 184n^3 \leq 10^5n^9 \leq k/4$ by admissibility. Multiplication over the f places gives the bound. Frame scaling, integrality, product and size.

Write $V = (2/\mathrm{rd}_E)^f$, with rd_E the root discriminant, and put

$$c_S = \left(\frac{\pi^k}{(k-1)!\,N^k}\right)^f |\mathrm{disc}(E)|^{1/4}V^k, \qquad A_* = |\bar{R}_0|\#\mathcal{C}\, M^{f\varrho(2k+2\varrho+1)}c_S.$$

For $X_* = 200fk\,\mathrm{rd}_E$, admissibility gives $X_*^{1000} \leq b$. Since $(k-1)! \leq X_*^k$, $V \geq X_*^{-f}$, and $n \leq k$, we have $c_S V^n \geq N^{-kf}X_*^{-3kf}$. As $2^{kf}X_*^{3kf} \leq \tau$, this implies $2^{kf}n_pM^{2\varrho fk} \leq \tau A_* V^n$. The following genus-mass and covolume calculations give

$$A_* V^n \leq (1+b^{-9})Ce_B c_G^f(\nu L_*/d_B)^2.$$

Multiplying gives $n_pM^{2\varrho fk}P_* \leq (1+b^{-9})\tau^2\nu^2C \leq (\nu\tau^2/2)^2C$, since $\tau^2 \geq 8$. As $\alpha+k-1 \geq k$ and $x \geq 1$, we obtain $\sqrt{\prod_v p_v} \leq \nu\mathcal{W}(x)/2$. Arithmetic comparison, calibration on the datum.

One step of the genus mass. Let $F = E^+$, $f = [F:\mathbb{Q}]$, $d_F = |\operatorname{disc} F|$, η the quadratic character of E/F, and $M_j = \sum_{[\Lambda]\in\mathrm{gen}(\mathcal{O}_E^j)} |\operatorname{Aut}(\Lambda)|^{-1}$ for the standard definite form. These are the class masses in the unfolding because the adelic double cosets are precisely these lattice classes and the stabilizer is their full unitary automorphism group. The prescribed E contains ζ_{5040}, so $E = F(\sqrt{-1}) = F(\zeta_3)$. The defining polynomials have discriminants $-4, -3$, one of which is a unit at every finite place. Thus E/F is unramified at all finite places and $d_F = \mathrm{rd}_E^f$. Adelic genus mass, roots of unity.

For $j \geq 1$, the definite Hermitian mass formula gives

$$\frac{M_{j+1}}{M_j} = \left(\frac{\mathrm{rd}_E^{j+1/2}\,j!}{(2\pi)^{j+1}}\right)^f L_F(j+1, \eta^{j+1}). \tag{8.8}$$

In rank m the discriminant contributes $d_F^{m^2/2}$, the archimedean factor is $\prod_{a=1}^m((a-1)!/(2\pi)^a)^f$, and the Tamagawa number 2 cancels. The standard lattice is self-dual everywhere. At a place

of norm q, the relative finite factor for rank m over rank $m-1$ is $(1-\eta(\mathfrak{p})^m q^{-m})^{-1}$. Indeed the split and inert reductive fibres are $\mathrm{GL}_m(\mathbb{F}_q)$ and $\mathrm{U}_m(\mathbb{F}_q)$, with orders $q^{m^2}\prod_{a=1}^{m}(1-\eta(\mathfrak{p})^a q^{-a})$. The lattices are unimodular over a quadratic étale algebra, also at $\mathfrak{p} \mid 2$. Reduction lifts smoothly since for a unitary basis the linear correction is $X + X^*$, onto Hermitian matrices, by surjectivity of the residue trace on the diagonal and free choice in one entry of each off-diagonal pair. Gan–Yu, Thm. 10.20, Kirschmer, Thm. 4.2.3, Prop. 4.2.7(3), Lemma 4.2.9.

For $s = j+1 \geq 3$, use $|\log(1-z)| \leq 2|z|$ for real $|z| \leq 1/2$. There are at most f places above each rational prime, each with norm at least that prime. The full Euler product thus gives

$$|\log L_F(s,\eta^s)| \leq 2f \sum_{m\geq 2} m^{-s} \leq 2f\, 2^{-s}(1+2/(s-1)) \leq 4f\, 2^{-s}.$$

Here the middle bound follows by integrating from 2 to infinity. In the prescribed Haar norms, $\prod_v \mathrm{sph}_{j+1,v} = ((2\pi)^{j+1}/(j!\,\mathrm{rd}_E^{j+1/2}))^f$, by the product of the chosen generators of the different. Consequently the logarithm of this product minus $\log(M_j/M_{j+1})$ is exactly $\log L_F(j+1,\eta^{j+1})$. For $j \geq k \geq 4\varrho + 12$ and $8f \leq 2^\varrho$, its absolute value is at most $4f\,2^{-(j+1)} \leq 2/2^\varrho$. The formal proof obtains the same estimate from the norm-one Siegel–Weil coefficient and an adelic Gauss-sum bound. Pinned mass ratio, sphere normalization.

Genus mass at the source level. Write m_j for the genus mass of the standard rank-j Hermitian lattice over E, $n_X = \#\mathcal{C}$, $H = |\bar{H}|$ for the order of the reduced compact subgroup, $Q_N = N^{kf}$, and $Y = (N^k-1)^f$. Then

$$n_X = HY m_k, \qquad C = \frac{(n_X d_B)^2}{e_B m_r}.$$

Indeed the embedded integral compact group fixes the prescribed last row modulo the chosen place on each of the f split factors at N. In each $\mathrm{GL}_k(\mathbb{F}_N)$ the orbit of that row consists of the N^k-1 nonzero rows. Reduction is onto there (lift on the chosen half and take the inverse adjoint on its conjugate). The further level at $M = p_0 p_1$ is the kernel of reduction with image $\bar{H}$, giving total index YH. The level is free. For a rational stabilizer g, $k - \operatorname{tr} g$ is integral and vanishes modulo the f selected primes over p_0, so a nonzero norm would be at least p_0^f. Definiteness at all infinite places bounds the norm by $(2k)^{2f} < p_0^f$. Thus $\operatorname{tr} g = k$, and unitarity forces $g = 1$. Above a standard genus class with stabilizer Γ, the number of refined classes is consequently $YH/|\Gamma|$. Summing proves the count. Lattice condition and class count, free level, main second moment.

Put $w_v = |\Phi_v(\delta_E)|$ and write

$$S_{j,v} = \frac{(2\pi)^j}{w_v^{j-1/2}(j-1)!}, \qquad F_v = N^k \prod_{j=k+1}^{r} S_{j,v}, \qquad F_* = \prod_v F_v.$$

The sphere factor is radial integration on $\mathbb{C}^j$ for the prescribed quotient measure. The consecutive-rank estimate gives $|\log \prod_v S_{j+1,v} - \log(m_j/m_{j+1})| \leq 2^{1-\varrho}$. Telescoping yields

$$\log \frac{F_* m_r}{Q_N m_k} \leq (r-k)2^{1-\varrho}.$$

Since $\log(N^k/(N^k-1)) \leq 1/(N^k-1) \leq 2/N$, the source even allows two copies of Q_N/Y within the factor $1+b^{-9}$. Indeed write $t = \log b \geq 2$, $m = 2\varrho+2 = r-k+1$. Admissibility gives $m \geq \sqrt{x} \geq t^{50}$, $f \leq x^{12} \leq m^{24}$ and $N \geq b^m$. For $m \geq 2^{50}$, $\log(16m) + 9m^{1/50} \leq m/4$ and $\log(16m^{24}) \leq 2(m-9)$. Hence

$$(r-k)2^{1-\varrho} + 4f/N \leq 4m\,2^{-m/2} + 4m^{24}e^{-mt} \leq b^{-9}/2 \leq \log(1+b^{-9}).$$

Sphere products and error budget, admissible ranges, lower level bound.

Cancellation with the unipotent covolume and the real fibre. Let $D_E = |\operatorname{disc} E|$, $R_E = D_E^{1/(2f)}$, $V = (2/R_E)^f$, $a = \varrho(k+\varrho)$, $n = \varrho+1$. The split reduction of the unipotent group is upper block unitriangular with block sizes $1,\dots,1,k+1,1,\dots,1$; it has $\binom{2\varrho}{2} + 2\varrho(k+1) = 2a+\varrho$ free entries. Thus its order at the f selected places over each p_i is, in total, $u_M = M^{f(2a+\varrho)}$. The block diagonal identifies the compact factor after reduction, so $|\bar{R}_0| = Hu_M$ and $A_* = Hu_M^2 n_X c_S$. Split unipotent reduction, compact factor.

With the prescribed unipotent Haar measure, the class covolume has

$$\nu L_* = \frac{u_M D_E^{\varrho/4}}{Q_N V^a}.$$

In detail, row $h < \varrho$ has $q_h = k+1+2(\varrho-1-h)$ free complex entries and one imaginary entry. For the defined squared frame scales $s_{v,h}$, its free lattice has covolume $\big(V^{-1}\prod_v M^2/s_{v,h}\big)^{q_h}$. Indeed the free block contains the entire middle block of the given class basis and has determinant $\det(h_c)/\bar{\pi}_N$. Unitarity gives index one for $\det(h_c)$, and $\pi_N\bar{\pi}_N = N$ gives index N^{-f} for the block. This cancels the product N^{-f} of the squared column scales (each pair of dual scales cancels). Its imaginary lattice has covolume $D_E^{1/4}\prod_v M/s_{v,h}$. Indeed the skew integers are $i\mathcal{O}_{E^+}$, and the discriminant of E/E^+ divides both (4) and (3) using $i,\zeta_3 \in E$. To lift an integral free row l, use its hyperbolic pair and the remaining block with form J'. The upper unitriangular three-block matrix with upper entries $Ml, -M^2\theta l(J')^{-1}l^*, -M(J')^{-1}l^*$ in positions $(1,2),(1,3),(2,3)$ is unitary since $\theta = -\zeta_3$ has trace one. It has level M since the given lattice is unimodular. Adding Ms in position $(1,3)$ realizes every skew integral translation. Successive reduction of the rows, last first, gives a product fundamental domain. Now $\sum_h q_h = a$, $q_h + 1 = k + 2(\varrho - h)$, and $s_{v,\varrho} = N^{-1}$, so the last index in L_* contributes Q_N^{-1}. The split class basis and its unimodularity are used here. Class basis, unimodular lattice, Class determinant and free lattice, integral row lifts, Haar covolume.

Write $B_j = 2\pi^j/(j-1)!$, so $c_G = B_k/\prod_{j=k+1}^r B_j$ and $c_S = (B_k/2)^f D_E^{1/4} V^k/Q_N$. In $F_* c_G^f$ the factorials cancel termwise. Since $\prod_v w_v = R_E^f$ and $\sum_{j=k+1}^r (j-1) = (2\varrho+1)(k+\varrho) = k+n+2a-1$, the remaining powers of 2 and R_E give the exact identity

$$F_* c_G^f D_E^{\varrho/2} = c_S Q_N^2 V^{n+2a}.$$

Substitution, including the free class count, gives

$$\frac{A_* V^n}{C e_B c_G^f (\nu L_*/d_B)^2} = \frac{HF_* m_r}{n_X} = \frac{F_* m_r}{Q_N m_k}\frac{Q_N}{Y} \le 1 + b^{-9}.$$

Real and finite constants, scaled Siegel constant.

Separation of distinct frames in the truncation estimate. Here the norm of a tuple of frames is the maximum over its columns and archimedean places of the Euclidean norm of a column. The cutoff uses a domain of representatives modulo $M\mathcal{O}_E$ inside the box $\max_v |z_v| \le b^{26}$. Its size has a short proof. In $\mathbb{C}^f$ give the embedding lattice $\Lambda = \iota(\delta_E^{-1}\mathcal{O}_E)$ the usual real Euclidean inner product. The trace pairing gives $\Lambda^* = 2\overline{\iota(\mathcal{O}_E)}$. The product formula and AM–GM give $\lambda_1(\Lambda^*) \ge 2\sqrt{f}$. Banaszczyk's Euclidean transference theorem in dimension $2f$ therefore bounds the covering radius of Λ by $\sqrt{f}/2$. After multiplication by $M\delta_E$, any coset has a representative with $|z_v| \le M|\Phi_v(\delta_E)|\sqrt{f}/2 \le b^{26}$ simultaneously. Here $M \le b^{23}$, $|\Phi_v(\delta_E)| \le b$ and $2f \le b$. Each coset meets the compact box finitely, so its lexicographically least representative gives a Borel section. The argument below applies to the chosen domain, using only these same bounds. Banaszczyk, Math. Ann. 296 (1993), Theorem 2.2, Bounded residue domain.

On the support of the class cutoff the triangular coordinates of Gram–Schmidt above the diagonal belong to that domain in rational coordinates. Passing to the real frame multiplies them by scale ratios of modulus at most one. The orthogonalized a-th column has norm at most $\mathrm{BesselLength}(z_v)$. Indeed its norm is the distance to the preceding span, invariant under adding earlier columns. For every Bessel left translate the frame changes by such unitriangular operations, and the translated column length is bounded by the operator norm of the inverse, equal to that of the translate. Adding at most $n \le b$ columns bounds the original frame vectors by $b^{27}\,\mathrm{BesselLength}(z_v)$. At length $\le b^{6n+2}t'$ this is at most $\rho = b^{37n}t'$ for $t' \ge 1$. Comparison in the actual bases.

Compare two returns by setting $z = \gamma_0^{-1}y$ and $g = \gamma_0^{-1}\gamma_1$, with the rational elements embedded in the real groups at each place. The finite lattice conditions for g are those for the two right classes. Put

$$\delta_{va} = \|z_v^{-1}(g_v - I)f_a\|, \qquad F = (f_0, \dots, f_\varrho) = (e_1, \dots, e_\varrho, z_0).$$

Suppose all $\delta_{va} < 1/(N\rho)$. The rational frame Gram differences $h(f_j, (g-I)f_a)$ are algebraic integers, by pairing the integral frame columns in the finite class bases. Their normalized archimedean readings have absolute value at most $\rho\delta_{va}$ by Cauchy–Schwarz against $z_v^{-1}f_j$. The factor relating the normalized reading to the rational one is $M_v^{2\varrho-a-j}/N$, with $|M_v| \ge 1$. Thus every conjugate has absolute value less than one and all these differences vanish. They fix the dual isotropic and z_0 coordinates of gf_a. Preservation of its signed norm then sets its remaining positive definite component equal to zero. Consequently $(g-I)f_a$ belongs to the isotropic span for every a. Integrality in the class bases, Rigid Gram readings.

If an isotropic coefficient in such a column is nonzero, take the largest row index i with this property, and call its coefficient d_i. This is still an algebraic integer, since the definite class bases preserve the indicated entries. At each v choose a real unipotent w with $\|wz_v\| \le b^{6n+2}t'$, using the slack of one in Bessel length and discarding the compact unitary factor. On the isotropic columns w is upper unitriangular. Thus row i of $w(g_v - I)f_a$ reads precisely d_i multiplied at v by the frame scale ratio, whose inverse has modulus at most $b^{23\varrho}$. It follows that

$$|(d_i)_v| \le b^{23\varrho}\|wz_v\|\delta_{va} \le \rho\delta_{va} < 1.$$

This contradicts the product formula. Hence $gF = F$. Distinct rational frames are $1/(N\rho)$-separated in the archimedean box.

At a fixed reduction label the ratio of terms with identical frames is a frame fixer at principal level. Its component in the definite group identifies the two right classes. Freeness of the source level then makes that component the identity. Taking ratios with any fixed member thus injects the fibre into its rational class unipotent group. Its cutoff weights sum to at most one by the partition of unity. This proves the separation and multiplicity used in the weighted volume count below. Integral isotropic entries and separation, Freeness at the prescribed level, Cutoff weight of a frame.

Truncation of both Bessel vectors. Fix c, s, sum over the right class c', and write $x = \gamma^{-1}y$ for each return in the stable set under consideration. Let $i = r^3 f$, $d_B = \prod_v d_v$, and set

$$u_v(y_v) = \max\{1, \mathrm{BesselLength}(y_v)/b^{6n}\}, \qquad U(y) = \prod_v u_v(y_v).$$

Use the decay majorants h_v for the chosen normalized cut pair. They are invariant under the real Bessel subgroup and satisfy

$$u_v^{3i}\|W_v^{\mathrm{full,cut}}\| \le h_v \le a_v, \qquad a_v \int_{U_v^{\mathrm{unip}}\backslash G_v} h_v \le b^{400r^2} d_v.$$

The quotient measure here is the one induced by the prescribed unipotent Haar measure. The construction above proves these local bounds. See the construction of the decay majorant. Where the two kernels differ, a local length at x or y exceeds $b^{6n+2}/2$, so $U(x)U(y) \geq b$. The norm of a pure tensor is the product of the norms. Apply the triangle inequality and use $2i$ powers of the two lengths to obtain

$$\|K_\gamma^{\mathrm{full}}(y) - K_\gamma^{\mathrm{cut}}(y)\| \leq 2b^{-2i} \prod_v h_v(y_v) \prod_v a_v\, U(x)^{-i}.$$

For fixed y, let $F_y(t)$ be the sum of $\beta_{c'}(x)$ over the terms with $U(x) \leq t$, where $t \geq 1$. The frame separation estimate puts the distinct frames in a ball of radius $2Rt$, with separation at least $1/(NRt)$, in real dimension $m_0 = 2nrf$, where $R = b^{37n}$. Equal frames have equal right class and differ by a frame-fixing unipotent of that class, so their weights sum to at most one. Disjoint balls of half the separation give $F_y(t) \leq (1 + 4NR^2t^2)^{m_0} \leq (b^{6n^2}t^2)^{m_0}$. Since $i > 4m_0$, integration in t gives

$$\sum_{c',\gamma} \beta_{c'}(x)U(x)^{-i} = i \int_1^\infty F_y(t)t^{-i-1}\, dt \leq 2b^{12n^3rf}.$$

Frame separation, class weights.

Unfolding the left cutoff integrates it along the real unipotent group to ν, for every c and every quotient point. Thus $\int \beta_c(y) \prod_v h_v(y_v)\, dy = \nu \prod_v \int h_v$. Multiplication of the three bounds yields

$$\sum_{c',\gamma} \int \beta_c(y)\beta_{c'}(x)\|K_\gamma^{\mathrm{full}}(y) - K_\gamma^{\mathrm{cut}}(y)\|\, dy \leq 4\nu d_B b^{-2r^3f+12n^3rf+400r^2f} \leq \tfrac{\nu}{2} b^{-k^3f}.$$

Indeed $d_B \leq b^{2kf}$, $8 \leq b$, and $12n^3r + 400r^2 + 2k + 1 \leq 402r^2 \leq r^3$ by $12n^3 + 500 \leq k < r$. The same calculation without the saving b^{-2i} gives absolute convergence. The cutoff-change identity applies also to this nonnegative difference norm, so its bound carries over to the selected representatives in each fixed right class. Class covolume, product quotient integral, majorant integral and exponents, summed cut error.

The squared norm for a fixed rational relation. The preceding estimates give the cut-block bound needed for the pair count. If the selected set $\mathcal{R}$ is empty, the block vanishes. Otherwise every selected representative has $1 \leq x \leq b^{270n^2f}$, and $M \leq b^{23}$, $C \geq n_p$. Consequently

$$w_\gamma^{\mathrm{disc}} = \mathcal{W}(x) \geq b^{-(23\varrho+135n^2)fk} \geq b^{-k^3f}.$$

The last inequality follows from $k^2 \geq 23\varrho + 135n^2$. Thus a single representative bounds the total truncation error $\sum_{\mathcal{R}} E_\gamma \leq \nu b^{-k^3f}/2$. Summing $\|\int \omega K_\gamma^{\mathrm{cut}}\|_{\mathrm{HS}} \leq \nu w_\gamma^{\mathrm{disc}}/2 + E_\gamma$ and using the triangle inequality in the unfolded block yields

$$\|A_{c,s,c',\mathcal{L}}^{\mathrm{cut}}\|_{\mathrm{HS}} \leq \nu \sum_{\mathcal{R}} w_\gamma^{\mathrm{disc}} \leq \nu \sum_{\mathcal{N}_{c,s,c',\mathcal{L}}} w_\gamma^{\mathrm{disc}}.$$

Here $\mathcal{N}$ contains all near representatives of the smaller frame-fixing double cosets in this block; each selected representative occurs once. In expanding the square, each ordered pair determines its reduction and its relation for fixed c, c'. The discriminant-weight form of (8.1) now gives

$$\mathcal{E}(\Omega) := \sum_{c,c'} \sum_{s \in \Omega,\, \mathcal{L}} \|A_{c,s,c',\mathcal{L}}^{\mathrm{cut}}\|_{\mathrm{HS}}^2 \leq \nu^2\, \tau^7 C/z^k.$$

The unfolding divides by ν^2 in the normalized square. On Ω_i, take $z = p_i^{2f} \geq (M/b)^f$. Since $C \leq (M/b^6)^{kf} J_0$, where $J_0 = |\bar{R}_0|v$, even the allowance $\tau^8 = b^{2kf}$ gives $\mathcal{E}(\Omega_i)/\nu^2 \leq b^{-3kf} J_0$. The

sharper individual weight w_γ in the Lean source requires a separate Schur estimate when the frame minor dominates. The preceding pair count absorbs the larger w_γ^{disc} throughout this contribution from pairs with the same relation. Open-cell block, two branches for the individual gauge, sum over blocks, admissible ratio C/J_0, level-prime budget.

The complementary pair estimate. Partition the cut sum on the remaining rational terms Γ_* by the rational relation when $a_\gamma \neq 0$, assigning all $a_\gamma = 0$ to one extra part. Retain the squared norm of each nonzero-discriminant sum over the relation. In the expansion, all other inner products are bounded in modulus by the product of the corresponding block norms. This includes the square of the extra part. Put $m = kf$. With the full level gauge of each term we claim

$$\sum_{c,c'} \sum_{\substack{\text{near pairs over an agreeing block} \\ a_1 a_2 = 0 \text{ or } \mathcal{L}_1 \neq \mathcal{L}_2}} w_1 w_2 \leq b^{-m} J_0. \tag{$*$}$$

A pair determines its reduction, so summing this majorant charges it once. Here $w_\gamma = \tau^2 \sqrt{D_j/(n_p G_\gamma^k)}$ at cell j, where $D_j = C$ at the top and $D_j = J_0/\tau^{16}$ below. Other-pair estimate, all-cell orbital bound, expansion by pieces.

Let i be the codimension of the intersection in the first relation, ℓ the corank of the common definite lattice, and d its discriminant. The lattice count below gives, for each $\ell \leq n^3$, $\sum_{\ell \text{ fixed}} d^{-k} \leq n_p \tau^{\ell+2}$. If $(\eta d)^2 \leq G_1 G_2$ and the two cell moments are at most D, then $w_1 w_2 \leq \tau^4 D\, \eta^{-k} d^{-k}/n_p$ directly from their definitions. Thus an upper bound for $\tau^{\ell+6} D \eta^{-k}$, uniform on each corank, also bounds that corank's contribution. The integral splitting and congruence determinant below give

$$\left[d(M/2)^{if}\right]^2 \leq G_1 G_2.$$

For pairs with the same rational relation in $(*)$, the cell classification gives two vanishing discriminants, $\ell = 0$, and $D = J_0/\tau^{16}$. Take $\eta = 1$, with bound J_0/τ^{10}. For distinct rational relations it gives $i \geq 1$, $\ell \leq i + 1$, $D = C$. Take $\eta = (M/2)^{if}$ and $u = \tau(2/M)^m \leq 1$, since $M \geq 2b$ and $\tau^4 = b^m$. Then, using the admissible bound $C \leq (M/b^6)^m J_0$,

$$\tau^{\ell+6} C \eta^{-k} \leq \tau^7 u^i C \leq \tau^8 (2/b^6)^m J_0 \leq \tau^{-12} J_0.$$

Thus at fixed ℓ each product is at most $d^{-k} J_0/(n_p \tau^{\ell+12})$, including when both kinds of pairs occur at that corank. Summation and $n^3 + 1 \leq \tau$ give $\tau^{-9} J_0 \leq b^{-m} J_0$ as claimed. Below we prove the geometric gain, the lattice count and the comparison with the measured cut blocks on this same datum. General discriminant gain, link dimensions and cell ranks, all-corank count, corank summation.

Which cells enter the pair count. Use zero-based flag indices in this paragraph. Write $Z = F^* h \gamma F$ and $A = \langle e_0, \ldots, e_{\varrho-1} \rangle$. The two frame spans meet exactly when $a_\gamma = 0$. Indeed if $Zy = 0$, the nonpositive vector $\gamma F y$ belongs to $F^\perp = A \oplus X_0$. Positivity on X_0 puts it in A. If Z is invertible, the only possible line of shared coefficients is $y = Z^{-1} e_{\mathrm{last}}$, up to a scalar, where e_{last} is the terminal coordinate vector. Its definite part has squared norm $a_\gamma/(N|\det Z|^2)$ at every embedding, as in the terminal column calculation. This vanishes precisely when the vector is shared. In the singular case $a_\gamma = 0$: the same norm calculation on the last adjugate column gives $a_\gamma \geq 0$, whereas $\det Z = 0$ gives $a_\gamma = -N^2 |\delta_{n-1}|^2 \leq 0$. Discriminant and common vectors.

If $a_\gamma \neq 0$ the cell is $j = n$. Suppose $a_\gamma = 0$ and the rational characters agree. Let j be the smallest top index of a nonzero shared vector, taking the larger of the last indices of its two coordinate lists. The flag calculation below carries its leading pair to every higher index. At the

terminal index the two norms agree, so after scaling the pair is $(\zeta, 1)$ with $\zeta\bar{\zeta} = 1$. It follows that the shared vectors at indices $j, \dots, n-1$ span all tail coordinates triangularly in each frame. The leading j-minor is nonzero. Otherwise take a nonzero y supported below j in the kernel of that minor. The vector $\gamma F y$ is orthogonal to the first j columns of F and to all shared vectors, since it belongs to γA. These span F together, so $Zy = 0$. The first paragraph gives a shared vector. Carry its own leading coefficient pair to the terminal index. Equality of the terminal norms then forces both original leading coefficients to be nonzero. Since y is supported below j, this contradicts minimality. Finally $j = 0$ implies $\gamma A = A$ by triangularity, excluded in Γ_*. Propagation of a leading pair, leading minor and cell rank.

In a lower cell the terminal shared vector identifies the two terminal functionals up to a norm-one factor, so the graph over X_0 returning to X_0 has dimension k. In the top cell it has dimension $k-1$: the two disjoint frame spans have total dimension $2n$, and the projection of their orthogonal complement to the graph has kernel $A \cap \gamma A = 0$. Intersecting this subspace with the second relation reduces its dimension by at most i, the codimension of the intersection of the two relations in the first. Thus $\ell \le i + 1$, and if the rational relations coincide and one discriminant vanishes, both vanish and $\ell = 0$. Dimension in each cell.

For a near term, each rational Gram entry has modulus $\le b^{134n}$ at every embedding in the same box. The determinant expansion and $n \le b$ bound every leading minor by b^{135n^2}. Hence $D, \mu \le b^{270n^2 f}$, where $D = \prod_v |a_{\gamma,v}|$ and $\mu = |\mathrm{Nm}\, \delta_j| / N^{2f(j-\varrho)_+}$. Both row indices are at least one. From $M \le b^{23}$ and $N \le b^{60n}$ the level gauge satisfies $G \le b^{(46n+60n+270n^2)f} \le H = b^{n^3 f}$. The gain proved below, $d \ge 1$, and $M/2 \ge b^{18}$ now give $d \le H$ and $18i \le n^3$. In particular $\ell \le n^3$, in the range of the lattice count. Near gauge, corank cutoff.

The factor $M/2$ for distinct rational relations. In each pair of middle coordinates interchange the two z_0-coordinates. The difference of the original two signed forms then becomes the difference of two copies of the positive form $H = \mathrm{diag}(I_k, N)$. Each relation is the graph of an H-isometry U_j. Indeed it is isotropic by preservation of the form. Its dimension is $\dim(A_0 + \gamma A_0)^\perp - \dim(A_0 \cap \gamma A_0) = r - 2\varrho = k + 1$, where A_0 is the isotropic flag space and the intersection is the kernel of the two middle-coordinate maps taken together. Positivity makes each coordinate projection injective. The induced metric on the graph is the H-metric on the first coordinate. Let $K = \ker(U_1 - U_2)$, and choose rational complements to the shared graph in the two relations, of dimension i. Write A, B for bases of their source coordinates and $\mathfrak{a}, \mathfrak{b}$ for the determinant ideals of lattices in these bases. For example the positive discriminant of the first lattice is $\Delta_1 = \mathrm{N}(\mathfrak{a}) \prod_v \det(A_v^* H_v A_v)$. Metric and discriminant, positive graph coordinates.

The cross pairing in these bases is $P = A^* H (I - U_1^{-1} U_2) B$. The image of $I - U_1^{-1} U_2$ is $K^\perp$ and its kernel is K. Thus P is nonsingular. At each embedding the operator in parentheses has norm at most two. Applying its i-th exterior power and Cauchy–Schwarz gives

$$|\det P_v| \le 2^i \sqrt{\det(A_v^* H_v A_v) \det(B_v^* H_v B_v)}.$$

Suppose the two complement lattices admit integral lifts after adding vectors of the common graph. Such additions leave the cross pairing unchanged. For integral lifts t, t' in the class basis, the difference of the two signed pairings is $t^* J (1 - g_1 g_2^{-1}) t'$, where J is the integral lattice Gram matrix and the two class matrices reduce to the same s. It lies in $M\widehat{\mathcal{O}}_E$. This follows by subtracting the two Gram products before projection to the middle: all removed isotropic parts pair to zero with their respective lifts. Consequently $(\det P)\bar{\mathfrak{a}}\mathfrak{b} \subseteq M^i \mathcal{O}_E$. Taking ideal norms and using the archimedean inequality gives

$$M^{2if} \le |\mathrm{Nm}_{E/\mathbb{Q}} \det P|\, \mathrm{N}(\mathfrak{a})\, \mathrm{N}(\mathfrak{b}) \le 2^{2if} \Delta_1 \Delta_2.$$

Integral lifts over one reduction, determinant bound.

The integral splitting below chooses these lifted lattices with $t_j d_j^{\mathrm{com}} \Delta_j \le G_j$ for each term. Here t_j is its torsion index and d_j^{com} is the discriminant of its lattice on the common definite space. Given this construction and the crossed-index bounds $d \le t_2 d_1^{\mathrm{com}}$, $d \le t_1 d_2^{\mathrm{com}}$, multiplication immediately yields $[d(M/2)^{if}]^2 \le G_1 G_2$. The crossed-index argument extends to lower cells as follows. In the first term's lattice on the common definite space, choose a linear section $x \mapsto y_1$ of the integral level lifts locally, by lifting a basis over each discrete valuation ring. The common relation gives a lift y_2 with the same two middle coordinates over the fraction field. Hence $(g_2 g_1^{-1} - 1) y_1 \in A_0 + g_2 A_0$. Congruence puts its quotient by M in the saturation of $A_\Lambda + g_2 A_\Lambda$. The zero-class calculation above produces the integral second lift $y_2' = y_1 + Ma$, even when that lift is nonunique. Thus the index divides t_2, and interchange gives the other index. This uses only the level-lift descriptions of the two common parts. Required splittings, construction in the source, general crossed indices.

The discriminant of the rational relation. Fix one term γ in cell j, in its given class lattice Λ. Put $s = \min(j, \varrho)$, $I = A_0 \cap \gamma A_0$ and $H = (I^\perp \cap \Lambda)/(I \cap \Lambda)$. The shared flags give $\varrho - s$ independent vectors in I, one at each isotropic index beyond s. The first s vectors of each flag, together with its terminal vector, have nonsingular cross-pairing matrix of determinant δ_*. In fact at the top $\delta_* = \delta_j$. Below the top, eliminate the first j entries in the terminal column using the invertible leading block. The cell relation identifies the resulting vector on the second side with $\zeta z_0 + a$ on the first, $a \in A_0$, $\zeta\bar{\zeta} = 1$. Thus the final Schur entry is $-N\zeta$, and

$$|\operatorname{Nm} \delta_*| = N^{2f} \mu, \qquad \mu = |\operatorname{Nm} \delta_j| / N^{2f(j-\varrho)_+}.$$

The same nonsingularity bounds $\dim I$ by $\varrho - s$, since every vector of I pairs to zero with the opposite frame. Cell and terminal minor.

Work at the finite places, taking conjugate places together. Even above N the lattice basis has terminal block $\left(\begin{smallmatrix} 0 & -1 \\ -1 & -N \end{smallmatrix}\right)$ of determinant -1, so the full form is unimodular. The quotient H is self-dual. Indeed the primitive summand $I \cap \Lambda$ has integral dual vectors by self-duality of Λ. The Gram matrix of these two lists has blocks $\left(\begin{smallmatrix} 0 & 1 \\ 1 & * \end{smallmatrix}\right)$, hence their orthogonal complement identifies with H. Write J for its Gram matrix, F for the classes of the first s isotropic vectors of each flag, and $p = (z_0, \gamma z_0)$ for the two terminal classes. Adjoin two independent vectors of Gram NI_2 and put

$$\widehat{J} = J \oplus NI_2, \qquad B = \begin{pmatrix} F & p \\ 0 & I_2 \end{pmatrix}.$$

Ordering each flag with its own enlarged terminal vector gives the Gram matrix of B zero diagonal blocks and cross determinant δ_*. In particular F has rank $2s$ and spans the classes of $A_0 + \gamma A_0$, by dimension.

Let F_0 generate its saturation, of index h over the lattice generated by F, and replace F by F_0 in B to obtain B_0. Choose an integral basis (Q, U) of H with $F_0^* J (Q, U) = (1, 0)$, possible by primitivity and self-duality. The columns U are the classes of integral lifts of the whole relation. Their Gram matrix in its positive metric equals $Y^* \widehat{J} Y$, where

$$Y = \begin{pmatrix} U \\ -p^* JU/N \end{pmatrix}.$$

Indeed both signed middle Gram matrices equal $U^* JU$, and correcting their mean to the induced metric adds $N^{-1}(p^* JU)^*(p^* JU)$. The field-valued change of basis $S = ((Q, 0), (0, I_2), Y)$ has

determinant $\pm\det(Q,U)$, a unit even above N, since its bottom row block in Y does not affect that determinant. Put $K = S^*\widehat{J}S$. One has

$$B_0^*\widehat{J}S = (C,0), \qquad C = \begin{pmatrix} 1 & 0 \\ p^*JQ & NI_2 \end{pmatrix}.$$

Thus $\det(B_0^*\widehat{J}B_0)$ is N^4 times the leading principal minor of K^{-1} of order $2s+2$. Jacobi's complementary-minor identity makes this $N^4\det(Y^*\widehat{J}Y)/\det K$, where $\det K$ is N^2 times a unit. Passing back from F_0 to F multiplies the absolute ideal norm of the Gram determinant by h^2. Taking its square root gives the exact identity for the lattice P^1 of integral lifts in the relation,

$$h\,d(P^1) = |\operatorname{Nm}\delta_*|/N^{2f} = \mu.$$

Bordered Jacobi identity, unimodular lattice basis, metric on lifts.

Starting from the columns of F, adjoin the remaining isotropic columns on the first side, then those on the second, then saturate. Pairing with the opposite full frame in the first two steps surjects onto the corresponding quotients of row lattices of orders R, R'. In the last step quotienting by $I\cap\Lambda$ preserves the saturation quotient, since $I\cap\Lambda$ already lies in the first full flag lattice. Thus $tRR' \mid h$, where t is the torsion index. The A_0-coordinate of an integral lift modulo $(I\cap\Lambda)+M(A_0\cap\Lambda)$ is independent of its choice. The target has order M^{2sf}, since $I\cap\Lambda$ is primitive of corank s there. On the kernel P^+ one can subtract an integral vector of I to obtain a level lift. Consequently

$$t\,d(P^+) \le M^{2sf}\mu/(RR').$$

Three indices, level-lattice estimate.

Restriction to the common definite lattice. In the first relation V, let D_0 be its intersection with the second relation, and U_0 the simultaneous kernel of the two terminal forms in D_0. Use the orthogonal decomposition $V = U_0 \perp (D_0\ominus U_0)\perp(V\ominus D_0)$ for the positive induced metric. Put $P_0 = P^+\cap U_0$, let C be the orthogonal projection of $P^+\cap D_0$ on the second summand, and let P_t be the projection of P^+ on the third. Over each local ring an integral basis can be adapted to the two primitive intersections. Its matrix in this decomposition is block triangular, so

$$d(P^+) = d(P_0)d(C)d(P_t).$$

Elements of P_t have level lifts after adding elements of D_0. On U_0 the definite part of the first coordinate is an isometry: both terminal entries vanish and the two signed middle forms agree. Its image of P_0 lies in the term lattice by the level-lift criterion and spans the common definite space. Thus the lattice defining d^{com}, obtained by intersecting that space with the term lattice, contains this image and $d^{\mathrm{com}} \le d(P_0)$. In particular the terminal bound $d(C)G_* \ge \mu$, with $G_* = \max\{\prod_v |a_{\gamma,v}|, N^f\mu\}$, gives

$$t\,d^{\mathrm{com}}\,d(P_t) \le M^{2sf}G_*/(RR') = G_\gamma.$$

Here $R = R' = 1$ on the top cell and $a_\gamma = 0$ below it, which identifies the right side with the original level gauge. Orthogonal slicing, definite lattice, link splitting.

Index bound for the kernels of the two terminal forms. In the shared graph let C be the lattice obtained by projecting orthogonally off the simultaneous kernel of the two terminal forms. Write h for its rank. The forms are injective taken together on its span. A lift of a vector

of C differs from its projection by a vector killed by both forms. In the class coordinates their values on an integral lift t are the pairings with the last columns of the two integral frames, so they are integral at every finite place. In positive graph coordinates the two forms are represented, up to sign and order, by z^+ and $U^{-1}z^+$. Indeed their values before the interchange of terminal coordinates are $-Nt_z$ and $-N(\gamma^{-1}t)_z$. Transport of the terminal rows, integrality on lifts.

Let δ_j be the leading frame minor in cell j, and put

$$\mu = \frac{|\operatorname{Nm}_{E/\mathbb{Q}} \delta_j|}{N^{2f(j-\varrho)_+}}, \qquad G_* = \max\{D, N^f\mu\}, \qquad D = \prod_{v|\infty} |a_{\gamma,v}|.$$

For $j \le \varrho$ the terminal flag relation gives $a + \zeta z_0 = \gamma(a' + z_0)$ with $a, a' \in A_0$ and $\zeta\bar{\zeta} = 1$. Pairing with $t \perp A_0 + \gamma A_0$ makes the two rows proportional, so $h \le 1$. If $h = 2$, then $j = n = \varrho + 1$. Write F for the frame, $Z = F^*J\gamma F$, and $q = Z^{-1}e_n$. The vector $t = \gamma Fq$ has $F^*Jt = e_n$, hence terminal coordinate $-1/N$; $\gamma^{-1}t = Fq$ has zero definite part and terminal coordinate $q_n = \delta_{n-1}/\delta_n$. Interchanging the terminal coordinates gives $U(t_{X_0}, q_n) = (0, -1/N)$. Thus the terminal coordinate of $U^{-1}z^+$ is $u = -N\delta_{n-1}/\delta_n$. The Gram determinant of the two rows is $N^2(1 - |u_v|^2)$ at v, and its product over the f complex places is D/μ, since $a_\gamma = \delta_n\bar{\delta}_n - N^2\delta_{n-1}\bar{\delta}_{n-1}$. For one row the corresponding product is N^f, and for no rows it is 1. Each applicable product is at most G_*/μ. Terminal flag relation, terminal Gram determinant, normalized minor.

Choose h independent rows on C, with representing columns V. For a rational basis B of its span and coefficient determinant ideal $\mathfrak{a}$ of C, put $P = V^*HB$. Then $(\det P)\mathfrak{a} \subseteq \mathcal{O}_E$ is a nonzero integral ideal. By exterior Cauchy–Schwarz at each complex place,

$$1 \le \mathrm{N}(\mathfrak{a})|\operatorname{Nm}_{E/\mathbb{Q}} \det P| \le d(C) \prod_{v|\infty} \det(V_v^* H_v V_v) \le d(C)\, G_*/\mu.$$

This gives the required cut $d(C)G_* \ge \mu$ when $x = 1$, including the rank-zero case with empty determinants. Terminal cut inequality.

For each $0 \le h \le k$, apply the class average following Lemma 8.1 to the lattice obtained by the terminal cuts. Its total marked weight is at most $\#\mathcal{C}\,\tau^{h+1}/N^{kf}$. We next count the frames giving one such lattice.

Counting the shared columns in every lower cell. Fix the left class c, the reduction label t, and an integral lattice $L \subset \Lambda_c \cap X_0$ of corank h and discriminant $d \le b^{n^3 f}$. Only count near terms with $L \subseteq L_\gamma^M$. Write $F = (e_0, \dots, e_{\varrho-1}, z_0)$, $n = \varrho + 1$, and use zero-based indices. For a lower cell $0 \le j < n$, form Y from the first j rows of the frame-product matrix Z followed by rows $j, \dots, n-1$ of the identity. Its determinant δ is the nonzero leading minor defining the cell.

For each $a \ge j$ the defining common vector has coefficients on γF supported through a, with coefficient one at a. Backward elimination therefore gives a common vector $\gamma Fx = Fw$ with any prescribed tail $x_{j:n}$. Take the tail of e_l, $l \ge j$. Pairing with $e_0, \dots, e_{j-1}$ gives zero since these vectors are orthogonal to F. Hence $Yx = e_l$. It follows that each column $W_l = \gamma F \operatorname{adj}(Y)e_l$, $l \ge j$, lies in $\langle F \rangle$. Record its first ϱ coordinates, and use zero columns for $l < j$. The recorded entries are integral since the class representatives fix the isotropic coordinates, the transported frame is integral, and Y is integral. All level moves fixing F pointwise preserve these entries. Triangular elimination, shared columns, integrality and count.

Here is the cost of the extra coordinates. At an infinite place, let F_y denote the normalized frame at the witnessing point y. Its columns, and those of $F_{\gamma^{-1}y}$, have norm at most b^{37n}. The cut at y gives a Bessel unipotent u with $\|uy\| \le b^{7n}$, by the definition of Bessel length as the infimum of $\|sy\| + 1$ and removal of the compact factor of s. Its action T on the standard normalized frame

is unitriangular. Taking coordinates in uyF_yx shows that $\|Tx\|_\infty \le b^{7n}\|F_yx\|$, hence $|T_{ai}| \le b^{44n}$. The cofactor expansion and $n \le b$ give

$$\|x\|_\infty \le b^{1+(2+44n)\varrho+7n}\|F_yx\| \le b^{46n^2}\|F_yx\|.$$

Write $v = \operatorname{adj}(Y)e_l$, $Fw = \gamma Fv$. The corresponding normalized coefficient lists are $(\sigma(w_i)/s_i)_i$ at y and $(\sigma(v_i)/s_i)_i$ at $\gamma^{-1}y$, with $|s_a/s_i| \le b^{23\varrho}$ from the basis scales. The resulting vectors coincide. Apply the last display at y and the triangle inequality at $\gamma^{-1}y$. Since $|\sigma(v_i)| \le b^{(2+134n)\varrho}$ by cofactors and the bound on Z, each recorded w_a satisfies

$$|\sigma(w_a)| \le b^{46n^2+1+23\varrho+37n+(2+134n)\varrho} \le b^{182n^2}.$$

Distinct integral records are one-separated in the maximum embedding norm by the product formula. Packing half-radius balls gives at most $(1+2b^{182n^2})^{2fn\varrho} \le b^{366n^4f}$ records. Cut representative, coordinate bound, basis and cofactor bounds, two-box estimate.

Recovering the frame in a lower cell. Write P for the first ϱ rows of Z. The lift sorting over L contributes $d^{2\varrho}$ as before. Indeed its equation is $D_\gamma^* p + MP^*w = 0$, where D_γ is the X_0-block and pa is an adelic basis of L with p rational. Containment in L_γ^M gives finite adelic solutions with wa integral. Rational row reduction and additive approximation in the kernel give a rational w satisfying the same open integrality condition. The Gram-row shear equalizes these pairings within each lift label. Thus the n definite difference columns lie in the rank-h lattice perpendicular to L. Sorting relative to a fixed origin modulo $M\delta$ costs $[M\delta]^{nh}$. For equal residues, let X be the definite difference. Then XY^{-1}/M is integral in the class lattice by the adjugate formula. Its columns of index $\ge j$ vanish since the W_l lie in $\langle F\rangle$, which has zero X_0-projection. Consequently $X = M\eta P$, with η the first ϱ columns of XY^{-1}/M. The level shear of parameter η makes the definite blocks agree. Integral lifts and shears, denominator clearing.

The dual isotropic and z_0 coordinates are already fixed by Z. Sort the remaining isotropic coordinates modulo $a' = M\delta\bar\delta$, at cost $[a']^{n\varrho}$. Write the difference as $a'U$. Preservation of the Hermitian form gives $U^*P = -P^*U$. The equal shared labels imply $(U \operatorname{adj} Y)e_l = 0$ for $l \ge j$. For α the first ϱ columns of $\operatorname{adj} Y$, this implies $U\alpha P = U\operatorname{adj}(Y)Y = \delta U$. The integral matrix

$$K = \bar\delta U\alpha - \delta(U\alpha)^* - \alpha^*P^*U\alpha$$

is skew-Hermitian, and $KP = \delta\bar\delta U$. Indeed, in KP the last two terms cancel by the two preceding identities. The central level shear adds MKP, so the frames agree. Fixed-label recovery then identifies the right class and the chosen double-coset term. This argument includes $j = 0$, for which $Y = I$. Skew extension and central shear, marked-term recovery.

The determinant estimate for the top cell applies to Y, whose new entries are zero or one. Hence $[M\delta], [a'], d^2 \le b^{2n^3f}$. Include the top-cell count already proved, at most b^{2f} cell labels, and b^{270n^3f} matrices Z. The logarithm to base b of the total number of terms over c, t, L is at most

$$f\{2 + 270n^3 + 366n^4 + 2n^3(\varrho + nh + n\varrho)\} \le 2000n^6(h+1)f.$$

For $h \le n^3$, the square of this multiplicity bound is at most $\tau = b^{kf/4}$, since $k \ge 1000(2n)^9$. Multiply the preceding class sum for the lattices by τ and by $|\bar R_0|$ for the common reduction label. On this same datum the ordered-pair sum of d^{-k} at corank h is at most $n_p\tau^{h+2}$. Multiplicity over one lattice.

The projected-volume determinant at one real place. Use the real-place normalization $h(u,v) = u^*Jv$, $J^2 = 1$ and $h(z,z) = -1$. Write $F = (e_1, \dots, e_\varrho, z)$, $A_0 = \langle e_1, \dots, e_\varrho \rangle$, $F^\perp = A_0 \oplus X_0$, and $Y = yV_-$. Projection onto Y along its positive orthogonal complement is an isomorphism on each of $\langle F \rangle$ and $\langle \gamma F \rangle$. Indeed both have dimension $\dim Y$ and are nonpositive. Let ν be the ratio of the projected complex volumes of γF and F, in the metric $-h$ on Y. For a cell j put $i = \min(j, \varrho)$, $m = i + 1$, $F_b = (e_1, \dots, e_i, z)$, $Z = F^*J\gamma F$ and $B = F_b^*J\gamma F_b$.

In a lower cell $i = j$, eliminate the skipped isotropic coefficients of the terminal shared flag vector by subtracting the earlier shared vectors in descending order. This gives $w = \gamma F_b c = Fx$, with $c_m = 1$, $|x_z| = 1$. Thus $Bc = -x_z e_m$. Replacing the last column of B by Bc leaves its determinant unchanged, giving $|\det B| = |\delta_j(Z)|$ for the leading minor $\delta_j(Z)$. At the top $B = Z$, so the same formula holds. The cell minor is nonzero. The shared vectors at the skipped isotropic indices span a space $W \subseteq A_0 \cap \gamma A_0$ of dimension $\varrho - i$ by their triangular leading coefficients.

The map $u \mapsto F^*Ju$ on Y is invertible by the projection just proved. There is therefore a unique matrix ω_0 in Y having the same frame products as $\gamma F_b B^{-1}$. Its subframe products form the identity. Apply Gram–Schmidt for $-h$ starting at the last column. This gives $\omega = \omega_0 L$ with L invertible lower triangular and $\omega^*J\omega = -I_m$. Both ω, L may be fixed using only y, Z. The columns of ω are orthogonal to W, since $W \subset \gamma A_0$ annihilates γF_b and $W \subset \langle F \rangle$. In both projected volumes replace the skipped columns by the same ordered basis of shared flag vectors. The triangular column changes have determinant of modulus one. The common projected columns cancel, and ω is an orthonormal basis of their orthogonal complement in Y. As $F_b^*J\omega = L$, this gives

$$\nu = \frac{|\det(\omega^*J\gamma F_b)|}{|\det L|}.$$

Projected-volume ratio, triangular basis, bordered minor.

Set $\omega' = \gamma F_b B^{-1} L$, $K = \omega^*J\omega'$. The differences $\omega' - \omega$ annihilate F, so their Gram is v^*v for their X_0-components v. Expanding and using $\omega'^*J\omega' \le 0$ gives $-K - K^* \ge I_m + v^*v$. For $H = (-K - K^*)/2 > 0$, $-K = H^{1/2}(I + \mathrm{i}S)H^{1/2}$ with $S = S^*$; each eigenvalue of $I + \mathrm{i}S$ has modulus at least one. Consequently $|\det K| \ge \det H \ge 2^{-m} \det(I_m + v^*v)$. The volume identity gives $|\det K| = \nu |\det L|^2 / |\det B|$. Sylvester's determinant identity yields

$$|\det B| \det(I_k + vv^*) \le 2^m |\det L|^2 \nu.$$

Geometric determinant lower bound.

If $\|y^{-1} f_a\| \le \rho$ for all frame vectors, pairing with the Euclidean unit vectors $y^{-1}\omega_a \in V_-$ gives $|L_{aa}| \le \rho$, whence $|\det L| \le \rho^m$. In the lower cell the last column of L is $L_{mm} e_m$, so $\omega'_m = -(L_{mm}/x_z)w$. It lies in $\langle F \rangle$ and is orthogonal to $\omega'_m - \omega_m$. Its norm for h is $-|L_{mm}|^2$, giving $|L_{mm}|^2 = 1 + \|v_m\|^2$. At the top $\omega_m \perp A_0$. A Bessel unipotent u has $u^{-1}z - z \in A_0$, so pairing $u^{-1}z$ with ω_m gives the same L_{mm}. This pairing is bounded by $\|(uy)^{-1}z\| \le \|uy\|$, since J is a unitary involution. The Bessel length is $\inf_{s \in S}(1 + \|sy\|)$. Its compact factor acts unitarily, so length at most θ gives some u with $\|uy\| \le \theta$. In both cases

$$\frac{|L_{mm}|^2}{\det(I_k + v_m v_m^*)} \le \theta^{2(j-i)}.$$

Frame-box bound, Bessel length, terminal-column bounds.

The Schur weight in the prescribed coordinates. Put $e = \min(j, \varrho)$, $s = k + e$. In the local unit frame let B be the $(e+1)$-row subframe block and L the lower triangular product matrix of the preceding calculation. An Eichler translation with parameter $X \in \mathbb{C}^{e \times k}$ adds X^* times the

first e rows of B to the definite part of the moved frame. Multiplication by $B^{-1}L$ thus adds X^* times the first e rows of L to the difference matrix. Its terminal column v_z stays fixed. With L_1 the initial e-square block, $H_z = I + v_z v_z^*$, and c the adjoint of the other initial difference columns, the local estimate gives the majorant

$$\left(\frac{2^{e+1}|\det L|^2}{|\det B|}\right)^s \det(H_z + (L_1^*X + c)^*(L_1^*X + c))^{-s}$$

for the inverse s-th power of the projected-volume ratio. All volumes of complex coordinates here are ordinary Lebesgue volumes. Affine frame change, local weight.

For positive H, the determinant lemma and the change of variable $u = wH^{1/2}$ give, for $t > k$,

$$\int_{\mathbb{C}^{1\times k}} \det(H + u^*u)^{-t}\, du = \det(H)^{1-t}\, \frac{\pi^k \Gamma(t-k)}{\Gamma(t)}.$$

The scalar integral follows by polar coordinates. Integrate the rows of $L_1^*X + c$ successively at exponents $k+e, \dots, k+1$. Each scalar factor is at most $\pi^k/k! \le 1$ since $k \ge 7$. The affine row change contributes $|\det L_1|^{-2k}$; the remaining determinant is $\det(H_z)^{-k}$. Using $|\det L| = |\det L_1|\,|L_{zz}|$, $|\det L| \le R^{e+1}$ and $|L_{zz}|^2/\det H_z \le \theta^{2(j-e)}$, the local integral is at most

$$(2R^2)^{(e+1)e}\big(2^{e+1}\theta^{2(j-e)}\big)^k |\det B|^{-(k+e)}.$$

Affine matrix integral.

The shift has rows scaled by $M/\mathrm{basisScale}_v(e_a)$. Put $w_v = |M_v|^{\sum_{a<j}(\varrho-a)_+}$, $d'_v = |\delta_{j,v}|$. Its determinant has modulus $(M\sqrt{N})^e/w_v$, while $|\det B| = w_v^2 d'_v/N^j$. For the latter identity the common terminal flag in a lower cell adds a last diagonal factor of modulus one to the leading minor after subtracting the earlier rows and columns; at the top it is the whole minor. The inverse real Jacobian is $\big((M\sqrt{N})^e/w_v\big)^{-2k}$. It cancels w_v in the k-th power, leaving a factor $w_v^{-2e} \le 1$. Now $R = b^{37n}$, $\theta = b^{6n+2}$. For $e \ge 1$, $2^{e+1}\theta^{2(j-e)} \le b^{13e}$: below the top $j = e$, and at the top $e = n-1 \ge 10^5$. Since $\prod_v d'_v \ge 1$ by integrality and $(2R^2)^{e+1}N^j \le b^{135n^2}$, the combined e-th power contributes at most $b^{135n^3 f} \le \tau$ by admissibility. Thus the product of the integrals after the row scaling is at most

$$\tau\,(b^{13}/M^2)^{ekf}\,\mu^{-k/2}, \qquad \sqrt{\mu} = \frac{\prod_v d'_v}{N^{(j-e)f}}. \tag{8.9}$$

Unit subframe determinant, row Jacobian and product.

Summing over a single double coset. Fix c, s, c' and a representative γ of cell $j > 0$. Denote its frame-fixing double coset by $\mathcal{O}$, set $e = \min(j, \varrho)$, and let I_1, I_2 be the same two row indices used in the lattice discriminant. Write B_c for the original class cutoff restricted to the box, $W^\flat$ for the product of the cut Bessel functions, and $r_v(g,y) > 0$ for the projected-volume ratio. The directed Schur integral is

$$S_c(g) = \int B_c(y)\|W^\flat(y)\|^2 B_{c'}(g^{-1}y) \prod_v r_v(g,y)^{-(k+e)}\, dy.$$

On the support of the integrand, y and $g^{-1}y$ both belong to the prescribed box. The ratios for (g,y) and $(g^{-1}, g^{-1}y)$ are reciprocals. Cauchy–Schwarz, first in y, then over $\mathcal{O}$, bounds the square of the sum of absolute cut amplitude majorants by $(\sum_{g\in\mathcal{O}} S_c(g))(\sum_{g\in\mathcal{O}} S_{c'}(g^{-1}))$. Boxed weights, reciprocal weights.

Here is the multiplicity in the first sum. A right frame fixer leaves γF fixed. A left frame fixer at level M changes its middle coordinates by integral combinations of M times the isotropic rows of Z, as is seen by multiplying its middle row block into the dual isotropic coordinates of γF. The terminal coordinate is fixed by Z. In a local basis of the primitive definite lattice there are k rows to record modulo the span of the first e isotropic rows, so there are at most I_1^k residue lists. Choose one base point in each nonempty list. Within one list the difference of definite parts is $M\,q\,\xi$, with ξ the first e isotropic rows of the rational Z. A rational right inverse of ξ gives the unique rational q; its columns are integral in the class coordinates by the defining residue condition. Thus the shifts range in the conjugate archimedean embedding of $(\Lambda_c \cap X_0)^e$. Integral difference, recovery from the row classes.

At a fixed q, all remaining differences of frames lie in A. In the box they therefore occupy a translate of a real subspace of dimension at most $2fn\varrho$, one isotropic vector for each frame column and place. The frame separation and class-cutoff packing proved above give a total cutoff weight at most $b^{6n^2(2fn\varrho)} \le \tau$ on this fibre. This weights repeated frames by their own class cutoff. A fundamental parallelepiped D_c of the column lattice has volume at least one. Indeed its real Gram entries are $\frac{1}{2}\mathrm{Tr}_{E/\mathbb{Q}}(a)$, with a an integral Hermitian pairing. Since $\mathbb{Q}(i) \subset E$, this is the real part of $\mathrm{Tr}_{E/\mathbb{Q}(i)}(a) \in \mathbb{Z}[i]$. Hence its squared covolume is a positive integer. Packing one fibre, Column covolume.

For every y in this box and every real u fixing the frame, take $u^{-1}(p_0)_\infty$ as reference, where p_0 is the base point in the residue list. It has the same frame products since $uF = F$. The shift by x replaces the conjugate embedding of q by $\bar{\iota}(q) + x$. The majorant may depend on u, y, but its integral bound (8.9) is uniform in both. Summing its translates and integrating over D_c gives the full integral by tiling. Division by $\mathrm{vol}\, D_c$ contributes at most one. This average is valid inside the original integral: disintegrate along the real frame-fixing unipotent group. Its left arithmetic subgroup permutes the double coset, and the remaining varying cutoff on the orbit has translates summing to one. The quotient integral is invariant under right shifts by unimodularity. The indicator of the box and the norm of $W^\flat$ are constant on the orbit. The integral of the original cutoff against $\|W^\flat\|^2$ is at most νd_B, by the local quotient norms, where d_B is the number of Bessel coordinates. With (8.9) these facts give

$$\sum_{g \in \mathcal{O}} S_c(g) \le \nu d_B\, \tau^2 (b^{13}/M^2)^{ekf} (I_1/\sqrt{\mu})^k.$$

Inverting the orbit exchanges c, c' and replaces I_1 by I_2 with the same μ. translated majorant, quotient averaging, One directed sum.

Matching the level gauge. Put $m = kf$ and $D_j = C$ at the top, $D_j = J_0/\tau^{16}$ below. When $D = \prod_v |a_{\gamma,v}| \le N^f \mu$,

$$G_\gamma = M^{2ef} N^f \mu/(I_1 I_2).$$

After the two applications of Cauchy–Schwarz the bound for the square of the orbit sum, divided by $\nu^2\tau^4$, is

$$d_B^2 (b^{13}/M^2)^{2em} (I_1 I_2/\mu)^k \le \frac{D_j}{n_p\, G_\gamma^k}.$$

To see the last inequality multiply by $n_p G_\gamma^k$. Use $n_p N^{kf} d_B = J_0$, $d_B \le \tau^8$, and $b^{13}/M \le b^{-4}$. The result is at most $J_0 \tau^{8-32e} \le D_j$, since $e \ge 1$ and $C \ge J_0$. The measured orbit bound is thus νw_γ on this branch. When $D > N^f \mu$ the discriminant is nonzero, $j = n$, and $I_1 = I_2 = 1$. The open-cell bound already proved for the signed kernel uses D there and gives the required νw_γ after grouping the terms by the chosen representatives. Both branches are stable under the rational unipotents on

both sides: their triangular frame actions preserve the leading minors and the term discriminant. A nonzero cut contribution has a near representative. Summing by cell and branch bounds the block norm by $\nu \sum_\gamma w_\gamma$ on its agreeing reduction. On a disagreeing reduction translation by a character witness annihilates the stable block. The orbit bound can now be applied to the cross products in $(*)$ at precisely the gauge used to count the pairs. Schur product, gauge conversion, two branches, stable block.

Which rational terms survive on Ω_i. Write $F = (f_1, \dots, f_n) = (e_1, \dots, e_\varrho, z_0)$ with $h(z_0, z_0) = -N$, $A = \langle e_1, \dots, e_\varrho \rangle$, and R_{rat} for the rational Bessel group. Its elements send each f_j to itself modulo the earlier columns. Write Γ_* for the terms with nonzero term discriminant together with the agreeing zero-discriminant terms with $\gamma A \neq A$. Here agreement means $\psi_U(u_f) = \psi_U(u'_f)$ for the finite character whenever $u\gamma = \gamma u'$ with u, u' rational Bessel unipotents. Every $\gamma \in R_{\mathrm{rat}}$ preserves A and fixes e_1. The first column of its frame Gram matrix and of its leading ϱ-minor is zero, so $a_\gamma = 0$ and $\gamma \notin \Gamma_*$. The Bessel contribution in the second moment of Section 9 is therefore disjoint from the sum over the remaining rational terms. Remaining terms.

Suppose $\gamma \notin \Gamma_*$ agrees. Then $\gamma A = A$. In a relation $Fx = \gamma Fy$ supported on the first j columns, the coefficient pair (x_j, y_j) can be carried unchanged to position $j+1$ when $j < n$. Indeed take any E-linear form l killing both prefixes. Set $w = tFx = t\gamma Fy$, $l = h(v, \cdot)$. The rational isometry $Y \mapsto Y + l(Y)w - h(w, Y)(v + \frac{1}{2}h(v, v)w)$ belongs to both frame unipotents since w lies in each isotropic prefix and v is orthogonal to both. On each frame column its correction is just the covector value times the common isotropic vector. Thus the two simple-root sums are $tx_j l(f_{j+1})$ and $ty_j l(\gamma f_{j+1})$. Agreement for every $t \in E$ forces their coefficients to agree, since scaling absorbs $-M\delta_E$ and the finite additive character is nontrivial on E. Annihilator duality puts $x_j f_{j+1} - y_j \gamma f_{j+1}$ in the sum of the prefixes, giving the next relation. At the terminal position equality of the norms gives $x_j \bar{x}_j = y_j \bar{y}_j$. Eichler transformation, rational character and terminal norm.

Now expand γe_1 in A. If its last nonzero coefficient were at $j > 1$, the pair there would be $(x_j, 0)$, contradicting the norm equality. Hence $\gamma e_1 = \zeta e_1$, $\zeta\bar{\zeta} = 1$. Carry $(\zeta, 1)$ to each successive position. By induction the two earlier spans coincide and $\gamma f_j \equiv \zeta f_j$ modulo that span. The frame criterion gives $\zeta^{-1}\gamma \in R_{\mathrm{rat}}$. Over an integral class block, the basis and both class transports fix the first column line with its generator. The eigenvalue ζ is therefore integral, and $s \in \zeta \bar{R}_0$ since $\zeta^{-1} = \bar{\zeta}$ is integral. If $s \in \Omega_i$, it fixes e_1 at all f selected places above p_i, so $\zeta \equiv 1$ there. A nonzero $\mathrm{Nm}_{E/\mathbb{Q}}(\zeta - 1)$ has absolute value at least p_i^f and at most 2^{2f}, impossible since $p_i > 4$. Flag criterion, integral scalar, level test.

The remaining terms outside $R_{\mathrm{rat}} \cup \Gamma_*$ disagree. For each such orbit unfold its kernel through the closed unipotent group $L = U_\infty \cap \gamma U_\infty \gamma^{-1}$. Its discrete stabilizer Λ lies in L, and the integrand restricted to $\Lambda\backslash L$ transforms by the quotient of the two characters. A rational witness gives a right translation with multiplier other than one. Since L is unimodular, the integral vanishes. Absolute convergence and unfolding in stages therefore identify the full and cut amplitude sums outside R_{rat}, respectively, with those of Γ_* on every fixed (c, s, c') with $s \in \Omega_i$. Orbit integrals, block identity.

Passing to the full return on the two level-prime sets. For either i the cut contribution from nonzero-discriminant equal relations on Ω_i, divided by ν^2, is at most $b^{-3kf} J_0$. With $(*)$ and its orbital majorant the cut second moment on Γ_* is consequently at most $(b^{-3kf} + b^{-kf})J_0$. We verify the change of this cut square to the full return. For any fixed c, s, the cut-tail integral already proved bounds the sum over c' of the norms of the differences of the full and cut blocks by $\nu b^{-k^3 f}/2$. Their squared norms therefore sum to at most $\nu^2 b^{-2k^3 f}/4$. This holds for every subset of rational

terms, by absolute majorization. In the finite group there are

$$|G| \le (p_0 p_1)^{r^2 f} = M^{r^2 f} \le b^{23 r^2 f}$$

reductions. Since $r \le 2k$, $k \ge 1000$, we have $23r^2 f + 40 \le 2k^3 f$. Also $\#\mathcal{C} \le |\bar{R}_0| \#\mathcal{C}\, d_B = J_0$. Summing the row bound and dividing by ν^2 gives a full-minus-cut second moment at most $b^{-40} J_0$. Total cut error.

On the prescribed Ω_i the full block outside the Bessel subgroup is the full block of Γ_*, by the flag and character calculation. The measured ℓ^2 triangle inequality now gives

$$\left(\sum_{s \in \Omega_i} \|P_s - r_\psi(s)\|_{\mathrm{HS}}^2 \right)^{1/2} \le \left(b^{-20} + b^{-3kf/2} + b^{-kf/2} \right) \sqrt{J_0} \le b^{-6} \sqrt{J_0}.$$

Here $kf \ge 40$ and $b \ge 1000$. This completes the passage from the complementary-pair bounds to the full return bound on each of the two sets. For the good projection it remains to bound the contribution of the two rejected finite types. Full and cut squares, two prime sets.

The dimension discarded at either level prime. Fix $q = p_i$, $r = \#\mathrm{RowIndex}$, $f = \#\{w \text{ chosen above } p_i\}$, and abbreviate the order of the residue character at the tower exponent s by $m_0 = q_2(s) = (q_0 d)^{s+6}$. For $G = \mathrm{GL}_r(\mathbb{F}_q)$, a squarefree monic label $c = \prod P$ of degree r with $c(0) \ne 0$ selects the Green character with partition (1) at each Frobenius orbit corresponding to P. The fixed faithful multiplicative character gives this correspondence. Green's character parametrization and degree formula give distinct irreducibles for distinct labels, with degrees

$$\chi_c(1) = \frac{\prod_{i=1}^{r} (q^i - 1)}{\prod_{P|c} (q^{\deg P} - 1)}.$$

The identity coefficient of its central idempotent is $\chi_c(1)^2/|G|$, its share of the regular representation. Since $|G| = q^{r(r-1)/2} \prod_{i=1}^{r} (q^i - 1)$ and the denominator in the degree formula is at most q^r, this share is at least $q^{-r} \prod_{i=1}^{r} (1 - q^{-i}) \ge q^{-r}(1 - r/q)$. Green, *The characters of the finite general linear groups*, §8, Theorem 14, label degrees, irreducibility.

Draw f independent uniform monic degree-r polynomials. A polynomial is allowed if it is a label and avoids every m_0-th root of unity in $\mathbb{F}_q$. The probability of a forbidden root, including zero, is at most $(1 + m_0)/q$. For each $1 \le d' \le r/2$ there are at most $q^{d'}$ monic degree-d' choices for a repeated factor and at most $q^{r-2d'}$ multiples of its square. Thus the probability of a repeated factor is at most r/q. There are exactly $\varphi(q^r - 1)/r$ primitive polynomials, since each generator of $\mathbb{F}_{q^r}^{\times}$ has a Frobenius orbit of size r. By Rosser and Schoenfeld's totient estimate $n/\varphi(n) \ll \log\log(3n)$ [RS62, Theorem 15], the primitive fraction $u = \varphi(q^r - 1)/(rq^r)$ satisfies $u \gg 1/[r \log(r \log q)]$ at these levels.

A tuple is good when all entries are allowed and some entry is primitive. Its probability of failure is at most $f(1 + m_0 + r)/q + e^{-fu}$. Each good tuple has regular-representation share at least $(1 - r/q)^f$ times its uniform probability, by the tensor product formula. As $1 - (1 - r/q)^f \le fr/q$, the identity coefficient of $x_i = 1 - \sum_{c \text{ good}} e_c$, the rejected projector in the group algebra of the i-th prime factor, is at most

$$\frac{f(1 + m_0 + 2r)}{q} + \exp\left(-\frac{cf}{r \log(r \log q)} \right)$$

for an absolute $c > 0$.

Lean. The selected tuples are the same in Lean. The totient estimate improves the probability of missing all primitive labels. `Tuplebound` (L220-L272).

On the same datum write $x = \exp(\log X)$ for its first scale. The size bounds give $m_0 \le b^{1/9}$, $f \le b/1000$, $r = k + 2\varrho + 1 \le 2b$, $b^{10} \le q \le b^{11}$, $r \le 3x^5$, $f \ge x^{11}$, and $x \ge (\log b)^{100}$. The first term is at most $6b^{-8}/1000$. Since $r \le 3x^5$, $\log q \le 11 \log b \le 11x^{1/100}$ and $f \ge x^{11}$, we have $9r \log(r \log q) \log b = o(f)$. The second term is at most b^{-9}, so their sum is at most $b^{-8}/2$ for each of the two projectors. Scales, row rank, degree and level prime, residue-character order, the actual two projectors.

Other irreducible labels. Let $u_0 = q^{r/3}$ and choose a prime $p_* \ne 3$ dividing $u_0^2 + u_0 + 1 > 3$, which is 3 mod 9. Then $\operatorname{ord}_{p_*}(q) = r$, since $p_* \nmid u_0 - 1$. For a generator g take $\xi = g^e$ with $(e, 3p_*) = 1$ and $p^2 \nmid e$ for every other $p \mid m_0$. For every prime power ℓ^a, ξ^{ℓ^a} retains the full 3- or p_*-part of $q^r - 1$ and hence lies outside the maximal proper subfield $\mathbb{F}_{q^{r/3}}$. At $p \mid m_0$, the order of the norm has valuation at least $v_p(q-1) - 1 > v_p(m_0)$, since $m_0^2 \mid q - 1$ and $v_p(m_0) \ge 2$. Multiplying by an m_0-th root in $\mathbb{F}_q^\times$ preserves both conclusions, since Frobenius fixes the multiplier and its norm has order dividing m_0. As $p_* \ge 7$, the Chinese remainder theorem gives a fraction $\ge (2/3)(6/7)\prod_{n\ge 2}(1 - n^{-2}) = 2/7$ of the exponents, hence $\ge 2(1 - q^{-r})/(7r) \ge 1/(5r)$ of all monic degree-r polynomials. With these labels the cuspidality test in (10.2) uses irreducibility. The two order properties above would replace the primitive-place hypothesis of Theorem 24.8 in the proof of Theorem 17.7.

The finite Bessel character mean. Extend the reduced unipotent character ψ by zero to the residue group G and let $\psi^\natural(s) = |G|^{-1} \sum_{g \in G} \psi(gsg^{-1})$. We need $\sum_s |\psi^\natural(s)|^2 \le 2$. Use the local group at the chosen split places and their conjugates; write K for its integral points, $J = K(M)$ for the principal congruence subgroup and $V = \operatorname{vol}(K)$. Reduction identifies K/J with G: each matrix at a chosen place lifts integrally and determines the inverse adjoint at its conjugate. Let χ_M be the extended local Bessel character. On K it is the pullback of ψ, since unipotent membership and the simple-root character modulo M both factor through reduction. Consequently

$$a(x) := \int_K \chi_M(yxy^{-1})\, dy = V\psi^\natural(\bar{x}) \qquad (x \in K).$$

Each residue class has measure $\operatorname{vol}(J)$. The local mass $\mathcal{M}_T(K)$ integrates $|a|^2$ over the entire local group. The finite shell estimate on p. 81 is obtained from the local lattice and character alone and is independent of the first-moment estimate. With $n = \varrho + 1$ it gives

$$V^2 \operatorname{vol}(J) \sum_{s \in G} |\psi^\natural(s)|^2 = \int_K |a(x)|^2\, dx \le \mathcal{M}_T(K) \le V \operatorname{vol}(J)(V + 2^{-n}). \tag{8.10}$$

The lower volume bound (9.2) gives $2^{-n} \le V$; cancellation proves the claim.

Comparison with Lean. The formal proof allows any family of distinct residue classes. Taking every class turns its coset sum into the single integral over K.

Lean. `integral_extendedBesselChar_conj_eq`, `sum_sq_norm_integral_extendedBesselChar_conjugate_le`, `sum_norm_reducedCharMean_sq_le`.

Trace on the rejected finite type. Write $v = \dim X$. For s supported on one prime factor, $\operatorname{tr} r_\psi(s) = v\psi(s)$. In a diagonal class block, the definite unitary part δ of a rational Bessel return is integral up to conjugacy and reduces to the identity at the other prime p. Thus $\operatorname{tr}(\delta - 1)$ is divisible by the product of its f chosen places. Its absolute norm is at most $(2k)^{2f} < p^f$, so $\operatorname{tr}(\delta - 1) = 0$. Unitarity at infinity gives $\delta = 1$. A unipotent return translates the identity block by $\psi(s)$, and away from its reduced group there is no diagonal return. Embed the single prime factor by $\iota_i(s)$, the identity at the other prime. For its central rejected idempotent x the trace sum is unchanged

on replacing $\psi(\iota_i(s))$ by $\psi^\natural(\iota_i(s))$. The squared sum on this slice is bounded by the full sum over G. Cauchy–Schwarz gives

$$|\operatorname{tr} r_\psi(x)| \le v\sqrt{2\sum_s |x(s)|^2} = v\sqrt{2x(1)} \le b^{-4}v.$$

The middle equality uses $x = x^* = x^2$ at the identity in the group algebra. For the error $A(s) = P_s - r_\psi(s)$, cut it to the set $\bar{R}_0 G_i$ from the return estimate. Right convolution by x is an orthogonal projection on square-summable operator-valued functions. On each $h \in \bar{R}_0$ the resulting value has the same Hilbert–Schmidt norm as $A_x = \sum_s x(s)A(s)$, by left equivariance and unitarity on $\bar{R}_0$. The return estimate therefore gives $|\bar{R}_0|\|A_x\|_{\mathrm{HS}}^2 \le b^{-12}|\bar{R}_0|v$. Consequently $\|xL\|_{\mathrm{HS}}^2 \le (b^{-4}+b^{-6})v$ by taking the trace of $L^*xL = r_\psi(x) + A_x$. Convolution on the cut, diagonal trace, central trace, rejected energy.

Let x_0, x_1 be these commuting rejected projectors. Then $e = (1-x_0)(1-x_1)$ and $1-e \le x_0+x_1$. For every source subspace U, write $\pi = \mathrm{pr}_U$. The identity $\|L\pi\|_{\mathrm{HS}}^2 = \dim U + \operatorname{tr}((L^*L - I)\pi)$ and $\|\pi\|_{\mathrm{HS}} = \sqrt{\dim U} \le \sqrt{v}$ give $\|L\pi\|_{\mathrm{HS}}^2 \ge \dim U - b^{-6}v$ by the full return estimate. Orthogonality and $\|(1-e)L\pi\|_{\mathrm{HS}} \le \|(1-e)L\|_{\mathrm{HS}}$ now give

$$\|eL\pi\|_{\mathrm{HS}}^2 \ge \dim U - (2b^{-4} + 3b^{-6})v \ge \dim U - 5b^{-4}v.$$

Every term is measured on the same admissible datum.

9 The second moment at finite and infinite places

Pairs of returns may have the same reduction without being equal. We integrate first on a fixed reduction class and then sum over classes. Keeping the two returns in the same lattice fixes the power of the level in the resulting bound.

The local integral for a prescribed vector. For j chosen places over p_0, fix a nonzero frame vector at each selected place or its conjugate, as prescribed. Let Ω consist of the open reductions in the common stabilizer. The required nonzero-discriminant estimate is

$$\mathcal{E}(\Omega) \le (1 + b^{-4})b^{-6kj}\nu^2 C.$$

Here $\mathcal{E}$ is the sum, over rational relations, of the squared norms of the corresponding truncated sums. Consider first the local integral at one place where a vector is prescribed. Fixed-vector link energy.

Take one selected conjugate pair of places, with integral group K, level kernel K_M, and Z the preimage of the vector stabilizer. Use the filling measure vol, $V = \mathrm{vol}(K)$, $z = \mathrm{vol}(Z)$, $v_M = \mathrm{vol}(K_M)$, and put

$$\mathcal{M}(Z) = \int_{G_{\mathrm{loc}}} \left| \int_Z \chi_M(yxy^{-1})\, dy \right|^2 dx.$$

Here χ_M extends the local Bessel character from the unipotent group to its product with K_M, by right K_M-invariance and by zero off this product. Since K_M is normal in K and χ_M equals one on K_M, the inner integral equals z there. For the complementary integral we shall prove

$$\int_{G_{\mathrm{loc}} \setminus K_M} \left| \int_Z \chi_M(yxy^{-1})\, dy \right|^2 dx \le z v_M B^{-(t+1+n)}, \qquad B = b^9. \tag{9.1}$$

The left side includes the integral positions $K \setminus K_M$. We estimate it by summing over the two denominator orders at the split pair. Shell bound and normalization.

Take $t = k-10^{10}n^4$. Admissibility gives $t+1+n+2\cdot 10^9 n^4 \le k$, $6k \le 9t$, $t+1 < r$. Since $\mathrm{GL}_r(\mathbf{F}_{p_0})$ acts transitively on nonzero vectors, also for the inverse adjoint action, $V/z = p_0^r - 1 \ge B^{t+1}$. The isotropic flag gives n successive coordinate charts. In each chart, restricting the scalar to $R^\times$ and the remaining parameters to their integral boxes gives integral factors. The integral Heisenberg coordinates have volume one, each scalar has mass $(1-p_0^{-1})^2$, and the terminal stabilizer has integral mass one. The chart Jacobian at a unit has modulus one. Counting this part of the convolution gives $V \ge (1-p_0^{-1})^{2n} \ge 2^{-n} \ge B^{-n}$. Integral part of the floor charts. We deduce the one-place estimate

$$\frac{\mathcal{M}(Z)}{z} \le b^{-6k}\frac{\mathcal{M}(K)}{V}.$$

Proof. The contribution of K_M to $\mathcal{M}(Z)$ is exactly $z^2 v_M$. On its complement apply (9.1). Using $z \le VB^{-(t+1)}$ and $B^{-n} \le V$, we obtain

$$\frac{\mathcal{M}(Z)}{z} \le v_M\bigl(z + B^{-(t+1+n)}\bigr) \le 2Vv_MB^{-(t+1)} \le Vv_MB^{-t}.$$

The last inequality uses $B \ge 2$. For the full integral group, the same computation on K_M gives $\mathcal{M}(K) \ge V^2 v_M$. Since $B = b^9$ and $9t \ge 6k$, the asserted inequality follows.

Lean. `massIntegral_le_main_add_error`, `index_comap_vectorFixer`, `IsFillingData.massIntegral_mul_le_of_relIndex_eq`. □

Conjugate pairs of marked places are disjoint. Taking the product of these local comparisons gives the factor b^{-6kj}, with the full integral group at the other places. Applying this factor to $\mathcal{E}(\Omega)$ also uses the source's comparison with the top orbital densities, the full-group mass normalization, and its sieve for the other frequencies. One-place reduction, product of local groups, finite sieve system, upper-moment comparison.

Exact denominator ideals at one conjugate pair. We prove (9.1). At these two places the integer ring R is a product of two discrete valuation rings of residue size p_0. Write g for the level element and use $|c|$ for the additive Haar modulus of multiplication by c on this product, with $\mathrm{vol}(R) = 1$. For x put $D_x = (x-1)/g$. The ideal clearing all entries of D_x is $J = cR$ of index $I = |c|^{-1} = p_0^{a+b}$, where $a, b \ge 0$ are the maximum denominator orders in the two components. Here $I = 1$ if and only if $x - 1 \in gM_r(R)$, the defining congruence of K_M in G_{loc}. Indeed such an x is integral, and its inverse is integral by the unitary identity for the unimodular form. Consequently the sets $H_J = \{x : J(x) = J\}$ with $a + b > 0$ partition $G_{\mathrm{loc}} \setminus K_M$, including every integral x whose reduction is different from the identity.

Here are the local inputs to this summation. For an active y, meaning $\langle W_Z(y), W_Z(yx)\rangle \ne 0$, let $\mathfrak{d}_l(y,x)$ be the set of pairings with e_l of stable vectors on stage l (the l-th step of the isotropic flag); its inverse volume is the depth. Use the unit-scale gauge Γ. The stable-coordinate and matrix calculations below give, at each $l < \varrho$, an integral $d_l \in \mathfrak{d}_l(y,x)$, nonzero in both components, with $|d_l| = \mathrm{vol}(\mathfrak{d}_l(y,x))$, such that

$$C_yD_x \text{ is integral}, \qquad C_y = \prod_{l<\varrho} d_l\bar{d}_l.$$

In particular the uniform denominator C_y^{99n} also works. If $I > 1$, some d_l is a nonunit of R. Writing $u = \min_l \mathrm{vol}(\mathfrak{d}_l)$ therefore gives $u \le p_0^{-1} \le B^{-1}$. Also $C_y \in J$, so $u^{2\varrho} \le |C_y| \le I^{-1}$. At each

active y there is consequently a stage of depth at least $\max(B, I^{1/(2\varrho)})$. Stable products and depth, integral readings, matrix denominator estimate.

Let $\mathcal{W}(x)$ denote the square root of the weighted product of squared correlations for W_Z at the initial stage of the flag. The tuples of preceding Weyl parameters are empty, so both weights are one and the two correlations coincide. Write s for the volume exponent of the free coordinates and U for the unipotent mass. We use two moment estimates. If each active y has a stage of depth at least d, the signed integrand in (9.1) is at most $U^2 d^{-s}\mathcal{W}(x)$. Away from the contributing positions the majorant vanishes. On the contributing positions of H_J the one-frame bound gives

$$U^2 \int_{H_J \cap \text{positions}} \mathcal{W}(x)\, dx \le z v_M\, 2^{1500000n^3+25000n^2+1} I^{50000n^2}.$$

Indeed $c\bar{c}$ clears D_x, so the auxiliary frame gauge has loss $2I^2$; the unit gauge has loss two. Their respective powers in the one-frame bound are $25000n^2$ and $1500000n^3+1$. The shell normalization times U^2 is at most zv_M. These are statements in the filling Haar measure. Weighted square product, floor-zero depth gain, one-frame integral, shell normalization, signed integrand.

Put $h = t+1+n$, $h' = h + 10^9 n^4$. The prescribed dimension bound gives $s - h' \ge 9\cdot 10^7 n^4$, since $h' + 10^8 n^4 \le k$. For $I > 1$ the depth gain just obtained is at most

$$B^{-h'} I^{-(s-h')/(2\varrho)} \le B^{-h'} I^{-50000n^2-2}.$$

Including the integral for one frame, the sum over all denominator ideals in the complement is at most $\sum_{a,b\ge 0,\ a+b>0} p_0^{-2(a+b)} = (1-p_0^{-2})^{-2} - 1 < 2$. Finally $2^{1500000n^3+25000n^2+2} \le B^{10^9 n^4}$, which proves (9.1) from the displayed local inputs. The following paragraphs verify the geometric and measure estimates at the selected split pair. Free dimension margin.

The same congruence coset and the stable-coordinate ideals. Work at the selected split pair. Set $A = F\times F$, $R = \mathcal{O}_F \times \mathcal{O}_F$, and let $\mathcal{U}$ be the local unipotent group. The local level g is integral and invertible in A. The finite Bessel vector has one coordinate for each right coset $K_M c$ in Z:

$$W_Z(y)_c = \chi_M(yc^{-1}), \qquad \operatorname{supp}\chi_M = \mathcal{U}K_M.$$

If $\langle W_Z(y), W_Z(yx)\rangle \ne 0$, at least one coordinate is nonzero at both arguments. Choose its representative $c \in Z$. There are $u, v \in \mathcal{U}$ with $P = uy \in K_M c$ and $P' = vyx \in K_M c$. Both matrices and their inverses are integral. Moreover $P^{-1}P' \in K_M$, since $c \in K$ normalizes K_M. In particular

$$(P'-P)/g = P\,((P^{-1}P'-1)/g) \ \in M_r(R).$$

The choice of P, P' works simultaneously for the entire flag. Coset coordinates and common support, common integral representatives.

Put $E_l = \langle e_0, \dots, e_{l-1}\rangle_A$. To obtain a stable reading at index l, take $w \in E_l^{\perp}$ and $w' = w + a$ with $a \in E_l$. Require that $y^{-1}w$ and $(yx)^{-1}w'$ are integral and congruent modulo g. The possible coordinates $w(e_l^*)$ form the set $\mathfrak{d}_l$ used in the depth estimate. Here $w(e_l^*)$ denotes the coefficient on the dual basis vector. These conditions are linear. Indeed, on the subspace $S = E_l^{\perp}\times E_l$, take the inverse image of R^{2r} under the map

$$(w,a) \longmapsto \left(y^{-1}w,\ \frac{y^{-1}w - (yx)^{-1}(w+a)}{g} \right).$$

This map on the ambient A^{2r} is invertible by solving the two displayed equations in order. Its inverse image intersected with S is a full R-lattice in S, since componentwise it lies between two

powers of the uniformizer times the coordinate lattice of S. Projection onto $w(e_l^*)$ is surjective on S, since $e_l^* \in E_l^\perp$. On the lattice its image is therefore a product of two nonzero fractional ideals of $\mathcal{O}_F$.

At an active pair this image lies in R. In fact $Py^{-1}w = uw$ is integral. A block upper unitriangular u preserves the coefficient at e_l^* on $E_l^\perp$: the only other dual coefficients its row can read are the e_b^* coefficients with $b < l$, which vanish. Consequently $\mathfrak{d}_l = d_l R$ for a generator $d_l \in R$ nonzero in both components. The Haar normalization $\mathrm{vol}(R) = 1$ gives the exact equality $\mathrm{vol}(\mathfrak{d}_l) = |d_l|$. Stable coordinates, principal ideals and depth.

Clearing the matrix one isotropic line at a time. In the integer coordinates at the selected pair let A be the product of the two local fields, R its integer ring, $L = R^r$, and $h(v, w) = v^* J w$ the unimodular integral form. Write $g = \bar{g} \in R \cap A^\times$ for the level element, $y_1 = y$, $y_2 = yx$, $P_i = y_i^{-1}$ and $E(w, w') = (P_1 w - P_2 w')/g$. Take lifts $U_1 = uy_1$, $U_2 = vy_2 \in K$ of the active pair, with u, v Bessel unipotents and $(U_2 - U_1)/g$ integral. Let $A_a = \langle e_0, \dots, e_{a-1} \rangle_A$. A chosen reading $d_a \in R \cap A^\times$ has stable lifts with pairing d_a against e_a. Divide them by d_a to get ζ, ζ' in $A_a^\perp$, congruent modulo A_a, with

$$h(e_a, \zeta) = 1, \qquad P_1\zeta,\ E(\zeta, \zeta'),\ P_2\zeta' \ \in d_a^{-1} L.$$

Indeed the first stable lift reads integrally and the difference of the two readings lies in gL before division. Stable dual and upper quotient, Integral boxed lifts.

Put $T_a = \prod_{a \le l < \varrho} d_l \bar{d}_l$. We show that for every fractional ideal $\mathfrak{b}$ and every $w \in A_a^\perp$ with $P_1 w \in \mathfrak{b}L$, there is a $w' \equiv w \bmod A_a$ with $E(w, w') \in T_a^{-1}\mathfrak{b}L$. At $a = \varrho$ take $w' = v^{-1}uw$; a Bessel unipotent acts trivially on $A_\varrho^\perp / A_\varrho$. Then $E(w, w') = U_2^{-1}(U_2 - U_1)P_1 w/g$ belongs to $\mathfrak{b}L$.

For the step from $a+1$ to a, write $d = d_a$, $T = T_{a+1}$, $f = u^{-1}e_a$, $f' = v^{-1}e_a$. Both lift e_a mod A_a; $P_1 f = U_1^{-1}e_a$ and $P_2 f' = U_2^{-1}e_a$ are integral. Thus $\lambda = h(e_a, w) = h(P_1 f, P_1 w) \in \mathfrak{b}$. The vector $w_0 = w - \lambda\zeta$ lies in $A_{a+1}^\perp$ with $P_1 w_0 \in d^{-1}\mathfrak{b}L$. The induction hypothesis gives $w_0' \equiv w_0 \bmod A_{a+1}$ and $E(w_0, w_0') \in T^{-1}d^{-1}\mathfrak{b}L$. Let μ be the e_a-coefficient of $w_0' - w_0$. Only this coefficient survives in $\mu = h(\zeta', w_0') - h(\zeta, w_0)$. Hence

$$\frac{\mu}{g} = -h(E(\zeta, \zeta'), P_1 w_0) - h(P_2\zeta', E(w_0, w_0')) \ \in (Td\bar{d})^{-1}\mathfrak{b}.$$

Here conjugate linearity in the first argument supplies $\bar{d}^{-1}$. The corrected lift $w' = \lambda\zeta' + w_0' - \mu f'$ is congruent to w mod A_a. Its discrepancy is $\lambda E(\zeta, \zeta') + E(w_0, w_0') + (\mu/g)P_2 f'$; each term lies in $T_a^{-1}\mathfrak{b}L$. This proves the step. Coefficient of the removed line, Descent over the flag.

At $a = 0$ take $\mathfrak{b} = R$ and $w = y_1 e_j$ for each standard basis vector. Then $w' = w$, so $T_0(1 - x^{-1})/g$ is integral. Taking its J-adjoint gives $T_0(1 - x)/g$ integral as well, since $\bar{T}_0 = T_0$ and J, J^{-1} are integral. Thus $C_y = T_0$ already clears D_x, which implies the stated C_y^{99n} bound. The sharper exponent follows by working with fractional ideals in the finite lattice. In the uniform estimate of the formal proof, the exponents are 9 for the top quotient, 32 per line and 4 for the final adjoint. Their sum is $13 + 32\varrho \le 99n$. With these lifts and the specified coordinates this completes the matrix estimate. Columns and adjoint, Uniform exponent.

The Haar factor in the squared pairing. Write η for Haar measure on $\mathcal{U}$, normalized by $\eta(\mathcal{U} \cap K) = 1$, and $U = \eta(\mathcal{U} \cap K_M)$. These are precisely the datum's measures: in the triangular coordinates of the successive Heisenberg groups the integral subgroups have volume one; at level g their product is $U = \prod_{l < \varrho} \mathrm{floorModulus}(l)$. Write θ for the induced quotient measure on $\mathcal{U}\backslash G_{\mathrm{loc}}$, using the original Haar measure on G_{loc}. The following two calculations fix the normalization

independently of the estimates for the individual flag steps. Congruence volumes in the prescribed measures.

For any y, x, integrate $\mathbf{1}_Z(uy)\chi_M(uyxy^{-1}u^{-1})$ in $u \in \mathcal{U}$. On $uy \in K_M c$ its value is $\overline{\chi_M(yc^{-1})}\chi_M(yxc^{-1})$. Indeed $\chi_M(uyc^{-1}) = 1$ there, left translation by u multiplies χ_M by the unipotent character, and the final $(uyc^{-1})^{-1}$ can be removed by right K_M-invariance. The fibre $uy \in K_M c$ has measure U if nonempty and is nonempty precisely when $\chi_M(yc^{-1}) \neq 0$. Summing in c and integrating in the quotient gives the signed identity

$$\int_Z \chi_M(z'xz'^{-1})\,dz' = U \int_{\mathcal{U}\backslash G_{\mathrm{loc}}} \langle W_Z(y), W_Z(yx)\rangle\,d\theta(y)$$

with the inner product linear in the second entry. Unfolding a correlation, quotient integral.

There is also an exact calculation of the two-variable norm. Let H be the image of $\mathcal{U} \cap Z$ in Z/K_M and put $h = |H| = \eta(\mathcal{U} \cap Z)/U$. The support of W_Z in the quotient is the disjoint union of the images of $K_M c$, one for each left H-orbit in Z/K_M. Each image has measure v_M/U by disintegrating $\mathbf{1}_{K_M c}$ along $\mathcal{U}$. At $c \in Z$ precisely the h coordinates belonging to its orbit are nonzero and have modulus one. On its whole orbit in the quotient the vector agrees with $W_Z(c)$ up to a unit phase: right K_M fixes every coordinate and left $\mathcal{U}$ scales all coordinates by the same character. Different orbits have disjoint coordinate supports. There are $z/(hv_M)$ orbits. Hence

$$\begin{aligned}\iint |\langle W_Z(y), W_Z(y')\rangle|^2\,d\theta(y)d\theta(y') &= \frac{z}{hv_M}\left(\frac{v_M}{U}\right)^2 h^2 = \frac{zv_M\,\eta(\mathcal{U}\cap Z)}{U^3}\\ &\leq \frac{zv_M}{U^3}.\end{aligned}$$

This gives the zv_M in the shell estimate after multiplying by the product U of the flag-step moduli and the U^2 from the signed identity. Pinned pairing norm inequality.

A common sublattice and its exact depth moment. Let $D = \dim_A X'$ be the number of free orthonormal coordinates, s their volume exponent, and $d = D - 3\varrho = s + m + 5$, where the local datum has $m+1$ additional hyperbolic pairs. Let $\iota : X' \to A^r$ denote the isometric coordinate inclusion. At a point $b_0 \in K$ in the prescribed box and a contributing position x, write $x = t^{-1}utk_0$ with $u \in \mathcal{U},\ k_0 \in K_M$. Prescribe also up to ϱ linear equations $Vw = 0$ on X'. The common vectors $\mathcal{C}$ satisfy these equations and have integral, level-congruent readings of ιw at $b_0, b_0 x$.

On each factor of A, $\operatorname{rank}(u^{-1} - 1) \leq 2\varrho$. Indeed $u^{-1} - 1$ maps $E_\varrho^\perp$ into E_ϱ, and both the dimension of the latter and the codimension of the former are ϱ. Thus the simultaneous equations

$$Vw = 0, \qquad (u^{-1} - 1)tb_0^{-1}\iota w = 0$$

have nullity at least d in each factor. In each factor the intersection of the nullspace with $\mathcal{O}_F^D$ is saturated, hence a direct summand by Smith normal form over the discrete valuation ring. Choose d columns per factor from a basis of the summand and combine them in a matrix Y which extends to $\mathrm{GL}_D(R)$. For $w \in YR^d$ the second reading equals $k_0^{-1}b_0^{-1}\iota w$. It agrees modulo g with the first, so $YR^d \subset \mathcal{C}$. Clearing the three sets of rows.

For $v \in A^D$ let $J(v)$ be the fractional ideal generated by the entries of Y^*v, and put $\phi(v) = 1/\operatorname{vol}(J(v))$, allowing $+\infty$ on the null set where one component vanishes. The products $\langle v, \mathcal{C}\rangle$ contain $\overline{J(v)}$; their inverse volume is at most $\phi(v)$. Also, for any two integral congruent pairs, the difference of their pairings divided by $g = \bar{g}$ is integral, by bilinearity. Thus if all products with cv belong to the set of such differences, then $cJ(v) \subset R$. The volume of possible c is at most $\phi(v)$. These are the two estimates for which the direct and inverse depth weights use the same function.

For any integral self-conjugate $\alpha \in A^\times$ and $j \geq 0$, the split calculation yields

$$\int_{\alpha^{-j} R^D} \phi(v)^s \, dv = |\alpha|^{-j(D-s)} \left(\frac{1 - q^{-d}}{1 - q^{-(d-s)}} \right)^2 \;\leq\; (2/|\alpha|)^{99n(j+1)}.$$

To see this, complete Y to an integral basis. The adjoint basis change preserves the box and its Haar measure. For d uniform coordinates over one integer ring, their minimum valuation equals $i \geq 0$ with probability $(1 - q^{-d})q^{-di}$. Multiplication by q^{si} and summation give the displayed one-factor ratio. The two components are independent. Scaling the box multiplies volume by $|\alpha|^{-jD}$ and the integrand by $|\alpha|^{js}$. Finally $d - s = m + 5$, $m < 9n$, $D - s = 3\varrho + m + 5 \leq 99n$, and $q \geq 4$. This computation is uniform in the position and in the prescribed rows. Common bound and fitted dimension, two depth inequalities.

The density of a dot product. Let F be a nonarchimedean local field, with integer ring $\mathcal{O}$, uniformizer π, residue cardinality q, and $\mathrm{vol}(\mathcal{O}) = 1$. For $d \geq 2$, the dot product of two independent uniform vectors in $\mathcal{O}^d$ has continuous density

$$\rho(\beta) = (1 - q^{-d}) \sum_{j \geq 0} q^{-j(d-1)} \mathbf{1}_{\pi^j \mathcal{O}}(\beta), \qquad \rho(0) = \frac{1 - q^{-d}}{1 - q^{1-d}}.$$

The same upper bound $\rho(0)$ holds on any translate of either or both boxes. Indeed, conditional on the first vector $x \neq 0$, the dot product with the second vector is uniform on a coset of the ideal generated by the coordinates of x. If the first box is $\mathcal{O}^d$, this ideal is $\pi^j \mathcal{O}$ with probability $(1 - q^{-d})q^{-jd}$. This gives the formula for centered boxes and the bound for a translated second box. If the first box is disjoint from $\mathcal{O}^d$, the least coordinate valuation is constant and negative, so the conditional density is at most one. In both cases its supremum is integrable in x. Since its coset indicator is locally constant in β, dominated convergence gives continuity. The prescribed continuous fibre measure on each box has the same density almost everywhere by disintegration, hence at every β.

At a split pair the rank-$d = 2m + 4$ block form is $x \cdot y$ up to an integral unit. Scaling x, y by u_+^{-1}, u_-^{-1}, write $|u| = |u_+ u_-|_F$. The fibre bound scales by $|u|^{1-d}$, the ratio of the ambient and value moduli. As $\rho(0) \leq (1 - q^{-m})^{-2}$, this gives the required split bound.

Lean. `QuadricDisintegration` (L249-L267), `StandardLevel` (L540-L557). The maximum of the dot-product density also bounds translated boxes.

For the one-frame moment the two Fourier charts solve the quadric in disjoint blocks, leaving common free coordinates. The local argument uses a common center for the supports in the Fourier charts after passage modulo the earlier isotropic coordinates. At a step $l < \varrho$ the required comparison integrates the weighted squared correlation over the l-th Heisenberg group and bounds it by its next-step integral times $\mathrm{floorModulus}(l)\, \mathrm{kernelBound}^{\#\mathrm{FreeIndex}}\, \Lambda^{4900n}$, with Λ the loss of the chosen gauge. At the top the factor is one. Two-block Fourier estimate, common center, weighted correlation inequality.

The depth bound at the first stage. Fix the split conjugate pair, its level g, an admissible position $x = t^{-1} u t k'$ with $u \in U$, $k' \in K(g)$, and the cut vector $W = W_Z$. Write $R = \mathcal{O} \times \mathcal{O}$, $\mathrm{vol}(R) = 1$, $\varrho = n - 1$, and $s = \mathrm{volumeExponent}$. We use the gauge of scale one. At step a, θ_a is the prescribed orbit measure, and ν_a is the law of $\sigma_a \cdots \sigma_0$ for the product measure $\prod_{b=0}^{a} d\theta_b(\sigma_b)$. Write N_a for the set of points near at step a, meaning they can be moved into the integral box by

the prescribed partial group. Let $d_a(y) = \mathrm{vol}(\mathfrak{d}_a(y,x))^{-1}$ for $a < \varrho$, with $\mathfrak{d}_a$ the products with e_a of the stable vectors of that stage. Put

$$w_a(y) = \prod_{b<a} d_b(y)^{-s}, \qquad j_a(y) = \mathbf{1}_{N_a}(y) w_a(y)^{-1}.$$

Stable products and weights, floor measures and correlations.

For every h and $a < \varrho$ the split pair satisfies

$$\int \mathbf{1}_{N_{a+1}}(\sigma h) d_a(\sigma h)^s \, d\theta_a(\sigma) \le 4. \tag{P}$$

If $h \notin N_a$ the integrand vanishes. Otherwise translate h into the integral group by the partial group at a. This keeps the integral since its integrand is left invariant by the partial group at $a+1$ and the orbit measure is right invariant on such tests. For integral h and $\sigma h \in N_{a+1}$, write $\xi \sigma h \in K$ with ξ unitriangular on the first $a+1$ frame vectors. The vector $v = \sigma^{-1} e_a$ is the projection of $(\xi\sigma)^{-1} e_a$ off the earlier hyperbolic pairs, since $\xi^{-1} e_a - e_a$ belongs to the earlier isotropic span and σ fixes those pairs. Hence v is integral. The orbit measure pushes forward under $\sigma \mapsto v$ to the prescribed quadric at that stage measure. In the basis adapted to the flag the free coordinates are $z = E^* H v$, where H is the Gram matrix and E embeds the free block R^{D_X}. The change to the basis adapted to the flag has integral columns and integral inverse since both Gram matrices are unimodular. Disintegrate in a standard orthonormal block disjoint from z. For fixed outside coordinates its required form value is the total value minus their form. At every such value the block unit-box mass is at most 2 by the split dot-product density proved above. The other coordinates off the free block have unit boxes of mass one. Thus a nonnegative weight bounded by $\Phi(z)$ on these points has integral at most $2 \int_{R^{D_X}} \Phi(z)\, dz$. Orbit chart, orbit normalization, block disintegration, free coordinates.

Use the primitive common summand Y of rank $d = D_X - 3\varrho = s+m+5$ from the common-vector estimate, where the datum has $m+1$ further hyperbolic pairs. Transporting its vectors by σ gives stable vectors since the free block is orthogonal to the earlier flag. Their products with e_a contain the conjugate of the content ideal of $Y^* z$. The depth is therefore at most its index. As $Y^* z$ has uniform distribution on R^d, the content bound is

$$\int_{R^{D_X}} [R : \mathrm{cont}(Y^* z)]^s \, dz \le \sum_I [R : I]^{-(d-s)} \le \zeta_R(m) \le 2.$$

Combined with the preceding fibre bound this proves (P). Ideal zeta and block bound, common summand, one-floor estimate.

Let $C_b(y)$ denote the partial correlation of $\langle W(y), W(yx) \rangle$ integrated on the left against the prescribed quotient measure from step b. Set

$$P = \int |C_1(\sigma)| \, d\theta_0(\sigma), \quad S_a = \int w_a(y) |C_{a+1}(y)|^2 \, d\nu_a(y), \quad S = \sum_{a=0}^{\varrho} S_a.$$

Suppose every y with $\langle W(y), W(yx) \rangle \neq 0$ has some $a < \varrho$ with $d_a(y) \ge D > 0$. Integrate successive orbit coordinates, retaining the signed tail until the first such stage. At a shallow stage the triangle inequality bounds its tail by the next integral of the absolute tail. Later successive flag subgroups preserve the depths already read, and a branch shallow throughout has zero terminal correlation by hypothesis. Enlarging each first-deep piece to its full deep set therefore gives

$$P \le \sum_{a<\varrho} \int_{\{d_a \ge D\}} |C_{a+1}(y)| \, d\nu_a(y).$$

Nonzero C_{a+1} implies $y \in N_{a+1}$ by the support of W. On the deep piece this gives

$$|C_{a+1}(y)|^2 \le D^{-s} j_{a+1}(y)\, w_a(y) |C_{a+1}(y)|^2.$$

Earlier depths are invariant under the current flag subgroup, and $\sigma h \in N_{a+1}$ implies $h \in N_a$. Applying (P) thus gives $\int j_{a+1}(\sigma h)\, d\theta_a(\sigma) \le 4 j_a(h)$. Iteration bounds $\int j_{a+1}\, d\nu_a \le 4^{a+1}$. Cauchy–Schwarz first in each integral and then in the sum gives

$$P^2 \le D^{-s} \left(\sum_{a<\varrho} 4^{a+1} \right) S \le D^{-s} K S, \qquad K = 2^{10^6 n^3}.$$

At the initial stage the two pivots coincide, so the form majorant is P^2, and the weighted product of squared correlations is $K^3 S^2$. It follows that the square of the majorant is at most $D^{-2s} K^3 S^2$. This proves the depth gain pointwise in x. Integrating $K^{3/2} S$ in x is the separate one-frame estimate. First deep floor, weighted Cauchy–Schwarz, coincident pivots.

A common center for the two Fourier supports. Fix the split local datum, write $M = g$ for its level element, and use its gauge and $K(M) \le Z \le K$. For $a < \varrho$ write $A_a = \langle e_i : i < a \rangle$, $U_a = U_{(<a)}$ for the earlier unipotent group, N_a for the current Heisenberg group, G_a for the group fixing the earlier hyperbolic pairs, and Q_a for the group unitriangular on the earlier flag. The position ball $\mathcal{B}$ requires $(x-1)/M$ to have gauge size 1. Nearness at a means $Q_a g \cap K \ne \varnothing$. The coordinates of $W_Z(g)$ are $\chi_M(gc^{-1})$ for $K(M)c \subset Z$, with χ_M supported on $UK(M)$. Consequently its support is contained in UK, and translation on the left by U gives its prescribed character. A nonzero partial correlation above stage a therefore forces g near at $a+1$: some simultaneous translate contributing to its integral has $W_Z(\sigma g) \ne 0$, with $\sigma \in G_{a+1} \subseteq Q_{a+1}$. This uses precisely the same vector with all its coset coordinates. Coset coordinates, support of the correlation.

Fix the two arguments g, y. A nonzero cut square at (g, ny) requires $n \in N_a$ in the Heisenberg integral, nearness of g at $a+1$, and some $\mu \in U_a$ with $g^{-1}\mu n y \in \mathcal{B}$. Indeed the cut includes the indicator of positive U_a-mass of this last condition. If the cut integral is positive, take one n_0 in its support where the integrand is nonzero and choose $x_0 = g^{-1}\nu_0 n_0 y \in \mathcal{B}$, $\nu_0 \in U_a$. Choose also $\xi \in Q_{a+1}$ with $B = \xi g \in K$. Eliminating the earlier isotropic blocks writes $\xi = t\nu$ with $\nu \in U_a$ and $t \in G_a \cap Q_{a+1} = G_{a+1} N_a$. Here the factorization at each preceding stage removes the Heisenberg column of the corresponding hyperbolic pair and leaves the unitary transformation on its orthogonal complement. Flag factorization, Support and basepoint.

For any other n in this support put $x = g^{-1}\mu n y \in \mathcal{B}$. With $D = t n_0 n^{-1} t^{-1}$, the identity

$$(Bx_0)(Bx)^{-1} = t\nu\nu_0 (n_0 n^{-1}) \mu^{-1} \nu^{-1} t^{-1} = uD, \qquad u \in U_a,$$

follows because G_a normalizes U_a. Also $D \in N_a$ because G_{a+1} normalizes N_a. Thus $De_a^* \in A_a^\perp$; on $A_a^\perp / A_a$ every $u \in U_a$ acts as the identity. The column $((Bx_0)(Bx)^{-1} - 1)e_a^*$ is therefore a lift of $(D-1)e_a^*$ modulo A_a. Divided by M, it has size at most 14, hence at most the required 17. Indeed its matrix is

$$B((x_0 - 1)/M - (x-1)/M)(Bx)^{-1}.$$

The first two factors together have size 4. The position bound gives x size 3, Bx size 5, and its unitary inverse size 9; the product rule gives $4 + 9 + 1 = 14$. Matrix and column estimates, Centering.

The same B also bounds the variable on the quadric. Factor $t = sr$, $s \in G_{a+1}$, $r \in N_a$, and write $b = a+1$. Then $s^{-1}\xi = r\nu \in U_b$. For almost every contributing $\sigma \in G_b$, the next correlation forces σg near at $b+1$. The point $\sigma s^{-1} B$ is also near there, since it differs on the left

by $\sigma(s^{-1}\xi)\sigma^{-1} \in U_b \subset Q_{b+1}$. Choose $\eta \in Q_{b+1}$ with $K_0 = \eta\sigma s^{-1}B \in K$. The vector $s\sigma^{-1}f_b$ is the column of BK_0^{-1} at the frame vector f_b, with the earlier-stage coordinates removed: $\eta^{-1}f_b - f_b \in A_b$ and σs^{-1} fixes the earlier pairs. Hence its size is at most $1 + 5 + 1 = 7$. Both bounds now hold in the chart from the same n_0, ξ. For the untwisted one-frame step the frequency is M^{-1} and the coordinate rescaling is M, so their product has modulus one. Orbit column, quadric support, shared coordinates.

The Fourier inequality at the nonterminal stages. Keep the center just constructed and put $b_0 = a + 1$. Write θ_{b_0} for the next measure at that stage, $h(u, w) = u^*Jw$, and C_j for the partial correlation from stage j. In Heisenberg coordinates $n(\omega, \beta)$, β is skew and

$$C_{b_0}(g, n(\omega, \beta)y) = \int \psi(-M^{-1}h(\omega, \sigma^{-1}f_{b_0}))C_{b_0+1}(\sigma g, \sigma y)\, d\theta_{b_0}(\sigma).$$

Indeed, unfold one stage and move $\sigma n\sigma^{-1}$ to the left in the second argument. The sole adjacent entry of this Heisenberg element on the frame is $-h(\omega, \sigma^{-1}f_{b_0})$; its central entry is off the adjacent diagonal. This gives the displayed character with the original level. One-floor character.

Write the center $t = sr$ as above. In the integral over N_a substitute $n = r^{-1}s^{-1}n(Mv, \beta)srn_0$. Both translations preserve Heisenberg Haar measure. Conjugation by $s \in G_{b_0}$ acts on its vector part by an isometry of absolute determinant one and fixes the skew center. If ω_0 is the vector part of n_0, the phase becomes $\psi(-h(v + M^{-1}s\omega_0, w))$, where $w = s\sigma^{-1}f_{b_0}$. The shift gives a scalar of modulus one depending only on w. The cut is bounded by the squared modulus of this integral.

The vector part has rank $r_a = k + 1 + 2(\varrho - a - 1)$. Its scaling has Jacobian $|M|^{r_a}$, with $|\cdot|$ the additive modulus on the split algebra. On the centered support v has size 17. For fixed v the e_a-entry of $(n(Mv, \beta)^{-1} - 1)e_a^*$ is $c(v) - \beta$. Thus the differences of allowed β's divided by M have size 18. In a split pair the skew line is $\{(t, -t)\}$ with its integral lattice of mass one. Its scaling by M has modulus $|M|^{1/2}$; its size-18 ball has mass at most Λ^{19}. This bounds the integral in β by $|M|^{1/2}\Lambda^{19}$ times the squared transform at v. Skew coordinate, two Haar factors.

The image of $d\theta_{b_0}$ under $\sigma \mapsto w$ is the prescribed quadric measure $d\mu$. Below the terminal orbit, the flag coordinate chart has density $|t'|^{r_{b_0}}\, dt'\, du\, d\beta'$; the quadric chart uses $(t', t'u, c(u) - \beta')$, giving the same Jacobian. On the terminal orbit the measure at that stage is defined by pushing the quadric chart through its isometry section. The isometry s preserves both ambient volume and the form and hence the continuous disintegration at this orbit value. Push the correlation onto the quadric by conditional expectation, including the unit phase. Its resulting density $F(w)$ has size-7 support, and

$$\int |F|^2\, d\mu \le \int |C_{b_0+1}(\sigma g, \sigma y)|^2\, d\theta_{b_0}(\sigma).$$

This inequality also follows by testing the pushforward against an arbitrary $L^2(\mu)$ function and applying Cauchy–Schwarz. Floor chart Jacobian, terminal orbit chart, correlation on the orbit.

It remains to bound $\int_{v \text{ of size } 17} |\int \psi(-h(v, w))F(w)d\mu(w)|^2\, dv$. Use the prescribed basis adapted to the flag on both sides. It carries product volume to stage volume. Its Gram matrix swaps the two columns of each hyperbolic pair and fixes the orthonormal coordinates. Applying the Gram dual to a size-j vector gives coordinates of size $j + 4$. Reserve two disjoint orthonormal blocks of rank $d = 2m + 4$ and write D_X for the number of remaining free orthonormal coordinates X'. On the input side use one reserved block z, free variables $u \in X'$ and outer variables y'. In these coordinates the quadric measure is

$$dy'\, du\, d\mu_{\beta_0 - Q(y', u)}(z),$$

where β_0 is the orbit value and Q denotes the form on the complementary coordinates in the given value-line normalization. This identity follows by Fubini disintegration of a sum of orthogonal forms. It holds at the specified value: for a compactly supported locally constant test the block density is continuous. Away from the origin this follows by solving in a coordinate with nonzero derivative. At the origin the test is constant on a small ball, whose continuous density is the unit-box series above after scaling. Products of such tests suffice on a split product. Integrating over the compact outer support preserves continuity, so agreement almost everywhere gives agreement at β_0. Disintegration in a block, Coordinate sizes and blocks.

For output stage volume the other reserved block has ordinary product measure. Fix all output coordinates outside X'. The only phase involving the free output variable u' is $\psi(-\bar{u}' \cdot u)$; absorb the remaining phase into F. Apply Cauchy–Schwarz in the outer input variables, Plancherel in u, then Cauchy–Schwarz in the conditional z-measure. Finally integrate the outer output variables, including their reserved block. On each side, for original size j, an ordinary coordinate box contributes at most Λ^{j+5} per coordinate. The reserved input block contributes at most $\Lambda^{d(j+5)}$ by the uniform split fibre bound above, and the reserved output block obeys the same bound by volume. There are at most $99n$ coordinates outside X'. The two square-norm losses are therefore bounded by $\Lambda^{99n(7+6)}$ on the input and $\Lambda^{99n(17+6)}$ on the output. Plancherel contributes c^{D_X}, with $c = \mathrm{kernelBound} = 1$ on this finite datum: the frequency after rescaling is $-M^{-1}\bar{M} = -1$. Multiplying by the two Haar factors, the full squared-norm cost is at most

$$|M|^{r_a+1/2} c^{D_X} \Lambda^{19+99n(13+23)} \;\le\; \mathrm{floorModulus}(a)\, c^{D_X} \Lambda^{4900n}.$$

All integrals on the right are for the original θ_{a+1}. Outer coordinate count, finite gauge, finite Plancherel normalization, one-floor bound.

The direct depth weight over the earlier unipotents. Work at the selected split pair, with local algebra $A = K_p \times K_p$, integer ring R, product modulus $|\cdot|$, integral hermitian Gram matrix $J \in \mathrm{GL}(R)$, level $M = \bar{M} \in R \cap A^\times$ and gauge $\alpha = \bar{\alpha} \in R \cap A^\times$. Put $\Lambda = 2/|\alpha|$, $s =$ volume exponent. Let $\mathcal{B}$ be the prescribed positions with $(x - I)/M \in \alpha^{-1}\,\mathrm{Mat}(R)$. Write $d_a(g, z)$ for the volume of the stable pivot readings at the two absolute points g, z, so $D_j(g, z) = \prod_{a<j} d_a(g, z)^s$. Write $U_{<j}$ for the earlier unipotents with their ordered Haar convolution. We prove, for $j \le \varrho$ and g near at j,

$$\int_{U_{<j}} \mathbf{1}_{\mathcal{B}}(g^{-1}\nu y) D_j(g, \nu y)\, d\nu \;\le\; \prod_{a<j} (\Delta_a \Lambda^{496n}), \qquad \Delta_a = |M|^{r_a+1/2}, \quad r_a = k + 1 + 2(\varrho - a - 1).$$

First suppose g is an integral isometry and $x_0 = g^{-1}y \in \mathcal{B}$. For a stage-$a$ shear $h = h(\omega, \beta)$, let S be the condition that $x_1 = g^{-1}\nu h y \in \mathcal{B}$ for some $\nu \in U_{<a}$. Then $H = y(\nu h y)^{-1} = g x_0 x_1^{-1} g^{-1} = h^{-1}\nu^{-1}$ satisfies

$$(H - I)/M = g\big[(x_0 - I)/M - (x_1 - I)/M\big] x_1^{-1} g^{-1} \in \alpha^{-2}\,\mathrm{Mat}(R).$$

Indeed $x_1^{-1} = J^{-1} x_1^* J \in \alpha^{-1}\,\mathrm{Mat}(R)$. On $A_a^\perp / A_a$, with $A_a = \langle e_i : i < a \rangle$, ν^{-1} is the identity. The column $(h^{-1} - I)e_a^*$ there is $-\omega + (b(\omega) - \beta)e_a$, with b independent of β. Thus $v = \omega/M \in \alpha^{-2} L_{a+1}$, the scaled lattice at that stage. At fixed ω the allowed skew coordinates lie in one translate of $M\alpha^{-2}R^-$, of measure $|M|^{1/2}|\alpha|^{-1}$ since R^- has mass one. The change $\omega = Mv$ contributes $|M|^{r_a}$. Two positions, shear column, skew Haar.

Let $W \subset X'$ be the common vectors for the original g, y, with zero extra rows. A stable reading $c = \langle e_a, \zeta \rangle$ at g, hy has lifts ζ, ζ' congruent modulo A_a. For $o \in W$, the integral congruent readings of these lifts and of o give, by unitarity,

$$MR \ni \langle \zeta, o \rangle - \langle h^{-1}\zeta', o \rangle = \bar{c}\, \langle \omega, o \rangle.$$

Here $ho = o - \langle \omega, o\rangle e_a$ and $o \perp A_a$. Thus $\langle c \operatorname{pr}_{X'} v, W\rangle \subset R$. The common-vector content majorant ϕ therefore bounds $d_a(g, hy)$ at $\operatorname{pr}_{X'} v$, and its s-moment on $\alpha^{-2} R^{\dim X'}$ is at most Λ^{297n}. The integral orthonormal columns O spanning X' have integral retraction $O^* J$. Thus the lattice at that stage splits as their lattice plus its orthogonal complement, with product Haar of unit lattice mass. There are $r_a - \dim X' \le 99n$ complementary coordinates. Their scaled volume is $|\alpha|^{-2(r_a - \dim X')}$. Consequently

$$\int \mathbf{1}_S(h) d_a(g, hy)^s \, dh \le \Delta_a |\alpha|^{-1-2(r_a-\dim X')} \Lambda^{297n} \le \Delta_a \Lambda^{496n}.$$

Pivot identity, content moment, complement size, orthonormal projection.

For arbitrary y, if the next integral is positive, right translate by a contributing $\tau \in U_{<a+1}$ so $g^{-1}\tau y \in \mathcal{B}$. In the convolution write νh with $\nu \in U_{<a}$ and h on stage a. Replacing the second lift by $\nu\zeta'$ preserves its reading and its class modulo A_a, so $d_a(g, \nu z) = d_a(g, z)$. The inner integral is therefore bounded by the previous uniform bound times $\mathbf{1}_S(h) d_a(g, h\tau y)^s$. The one-stage inequality iterates from $D_0 = 1$. Finally choose $\gamma n_0 g$ integral, where $n_0 \in U_{<j}$ and γ fixes the first j pairs. Simultaneous transport by γn_0 preserves the stable readings. In the boxed integral at $\gamma n_0 g, \gamma y$ substitute $\nu \mapsto n_0^{-1}\gamma^{-1}\nu\gamma$ to recover the original integral. This preserves Haar since translations do, and conjugation by γ acts unitarily on each earlier vector coordinate and fixes its skew coordinate. The resulting exponent is within the stated $4600n$ allowance at every stage. Transport of readings, ascending convolution.

How the local bounds give a moment for one frame. Use the same Z and write $K_W(g, y) = \langle W_Z(g), W_Z(y)\rangle$. Let P_{l+1} be its simultaneous correlation over the stages strictly above l, and $\Theta_{\le l}$ the descending product of the measures at the successive stages through l. Put $h_l(g, x) = \prod_{a<l} \operatorname{depth}_a(g, x)^{-s}$. The sum entering the unit-gauge majorant at the initial stage is

$$\mathcal{N}(x) = \sum_{l=0}^{\varrho} \int h_l(g, x) |P_{l+1}(g, gx)|^2 \, d\Theta_{\le l}(g), \qquad \mathcal{W}(x) = 2^{1500000 n^3} \mathcal{N}(x).$$

Indeed both factors in the weighted square product are identical and its prefactor is $(2^{10^6 n^3})^3$ before taking the square root. Weighted square and its integral, product for the two heads.

Enclose the contributing part of H_J in the position ball $\mathcal{B}$ of the gauge $\alpha = c\bar{c}$, where $J = cR$. Its loss is $\Lambda = 2I^2$. Let $U_{<l}$ be the earlier unipotent group with the product measure $d\nu$. Define

$$F_l(g, y) = |P_{l+1}(g, y)|^2 \mathbf{1}_{\{\operatorname{vol}\{\nu \in U_{<l} : g^{-1}\nu y \in \mathcal{B}\} > 0\}}.$$

The two estimates needed in the passage through the flag are the weighted integral in ν over this ball, bounded by $\prod_{a<l} c_a$ at a near point g, and the inequality

$$\int_{N_l} F_l(g, ny) \, dn \le c_l \int F_{l+1}(\sigma g, \sigma y) \, d\theta_{l+1}(\sigma).$$

Here N_l is the current Heisenberg group, $c_l = \operatorname{floorModulus}(l) \Lambda^{4900n}$ for $l < \varrho$, and $c_\varrho = 1$. The finite-place self-dual Fourier normalization has kernel bound one.

To insert the cut on the right, observe that its defining positive-mass condition is invariant under simultaneous translation by σ on stage $l+1$. If the mass is zero, Fubini for $U_{<l+1} = U_{<l} N_l$ makes $F_l(g, ny) = 0$ for almost every n. Otherwise the right cut is one. At $l = \varrho$ there is no Heisenberg integration. The remaining integral is over the stabilizer H. The h with $hg \in \mathcal{U}K$ occupy at most one right coset of $H \cap K$: for two such h's, the ratio of their integral representatives has Levi component $h_1 h_0^{-1} \in H \cap K$. Its mass is at most one, and Cauchy–Schwarz gives $c_\varrho = 1$. Cut at the following floor, terminal stabilizer measure.

For the l-th term of $\int_{\mathcal{B}} \mathcal{N}(x)\,dx$, set $x = g^{-1}\nu y$ and disintegrate Haar measure over $U_{<l}\backslash G_{\mathrm{loc}}$. The absolute value of P_{l+1} is invariant under earlier unipotents in its second argument by character equivariance. Where it is nonzero g is near. The first estimate leaves $\prod_{a<l} c_a$ times the integral of F_l. Disintegrate the quotient measure again along N_l and apply the displayed inequality. In the quotient by $U_{<l+1}$, the substitution $y \mapsto \sigma y$ preserves measure: σ fixes the earlier hyperbolic pairs, and conjugation preserves their unipotent Haar measure (the vector actions have determinant of norm one and the central coordinates are fixed). Continue to the last stage. The last cut square is at most $|K_W|^2$, integrated in both variables against the original quotient measure. This proves the reduction

$$\int_{\mathcal{B}} \mathcal{N}(x)\,dx \le nU\Lambda^{4900n\varrho} \iint |K_W|^2\,d\theta\,d\theta.$$

The factor $n = \varrho + 1$ counts the initial terms, and U is the product of the flag moduli. Inserting the exact pairing norm above and using $n\Lambda^{4900n\varrho} \le \Lambda^{25000n^2}$ gives the bound for a single frame on a shell on page 70. Together with the pointwise depth gain this proves (9.1) and the factor b^{-6k} for the selected split pair. Quotient measures at consecutive steps, summation of the weighted squares.

The orthonormal block at every finite place in T. Let T be a finite set of places stable under conjugation, $A = \prod_{w\in T} E_w$, $R = \prod_{w\in T} \mathcal{O}_w$, $A^- = \{x : \bar{x} = -x\}$, $R^- = R \cap A^-$. Use the filling's Haar measures, for which R and R^- have mass one. Write $|a|$ for the additive modulus on A, and retain the datum's skew unit $\delta \in R^\times$ and integral trace-one element τ. For its block of rank $D = 2m + 4$ let μ_β be the prescribed fibre measure at $\beta = \delta z^* z \in A^-$. We prove

$$\mu_\beta(a^{-1}R^D) \le \zeta_R(m)|a|^{-(D-1)} \qquad (\beta \in A^-,\ a \in A^\times).$$

Prescribed data, block estimate.

The field contains i and an integral trace-one element, for instance $-\zeta_3$. For the given τ, $i\tau - \overline{i\tau} = i$ is a unit, so conjugation acts nontrivially on the residue field at each fixed place. Thus over each place of the fixed field $F = E^+$ the local integer algebra is split or unramified quadratic. In either case there is a unit c of norm -1: in the split case take $(-1, 1)$; in the unramified case lift a finite-field norm solution one power of the uniformizer at a time using surjectivity of the residue trace. The columns $(1, c), (\bar{\tau}, -\tau c)$ have determinant $-c$ and hyperbolic Gram matrix. In the resulting $h = m + 2$ hyperbolic pairs the block value is $\delta\,\mathrm{Tr}(\sum_i \bar{u}_i v_i)$.

At each place of F the trace pairing is unimodular: its reduction is the nondegenerate trace pairing of a separable residue algebra [Neu99, Ch. III, §2]. In mutually dual integral bases the form divided by the skew unit δ is consequently an ordinary dot product in $2h = D$ coordinates over the fixed field. Both basis changes preserve Haar measure and all translates of the unit box. For residue cardinality q of the fixed field, the dot-product bound is at most $(1 - q^{1-D})^{-1}$. This is bounded by the local factor of $\zeta_R(m)$, namely $(1 - q^{-2m})^{-1}$ in the unramified case and $(1 - q^{-m})^{-2}$ in the split case. Multiply over the places. Via δ^{-1} the skew line identifies with the fixed algebra, so a self-conjugate multiplier b acts on it with modulus $|b|^{1/2}$. Scaling by a^{-1} multiplies ambient volume by $|a|^{-D}$ and skew-line volume by $|a|^{-1}$, since the value scales by $(a\bar{a})^{-1}$. This proves the asserted fibre bound for every translate of $a^{-1}R^D$.

Lean. `SemilocalProducts` (L414-L431), `StandardQuadrics` (L96-L149). Trace-dual bases reduce all the local blocks to the same scalar density.

In the prescribed dimensions $m = n + 2\varrho + 4$, $n[E : \mathbb{Q}] \le 2^{2\varrho}$. The finite Euler product gives $\zeta_R(m) \le \zeta(m)^{[E:\mathbb{Q}]}$. The integral test gives $\zeta(m) - 1 \le (1 + 2/(m-1))2^{-m} \le 2^{1-m}$. Thus $n \log \zeta_R(m) \le 2^{-n-3} \le \log(1 + 2^{-n})$, uniformly in T. Degree hypothesis, fitted ranks and Euler bound.

The volume of the integral group at all places in T. Use the original finite product $A = \prod_{w\in T} E_w$, its integer ring R of additive mass one, and the filling Haar measure on $G(A)$. Write K for the integral isometries, $n = \varrho + 1$ and $m = 3\varrho + 5$. With $q_w = \#(\mathcal{O}_w/\mathfrak{p}_w)$, put

$$Z_m = \zeta_R(m) = \prod_{w\in T} (1 - q_w^{-m})^{-1}. \qquad \text{Then} \qquad V_T = \operatorname{vol}(K) \le Z_m^n \le 1 + 2^{-n}.$$

For the first inequality use the unit-box block bound Z_m on every fibre of the standard rank $2m+4$ block, in the filling measure. Write Q_a for the group unitriangular on the first a frame vectors, $N_a = \{g : Q_a g \cap K \neq \varnothing\}$, and θ_a for the prescribed measure at that stage. For $0 \le a < n$ and every g,

$$\int \mathbf{1}_{N_{a+1}}(\sigma g)\, d\theta_a(\sigma) \le Z_m.$$

Indeed there is no contribution unless $g \in N_a$. Right translation on functions left invariant by Q_{a+1} allows one to replace g by some $g_0 \in Q_a g \cap K$. If $\xi\sigma g_0 \in K$ with $\xi \in Q_{a+1}$, the current vector $\sigma^{-1} f_a$ is obtained from the integral column $(\xi\sigma)^{-1} f_a$ by omitting the earlier hyperbolic pairs, since $\xi^{-1} f_a - f_a$ is in the earlier isotropic span. Its coordinates in the basis adapted to the flag are integral by the integral Gram dual. The pushforward of θ_a is the prescribed quadric measure, also for the last vector f_ϱ. Integrate in the reserved standard block. At each fixed value of the outside coordinates the fibre has mass at most Z_m, while the other integral coordinates range over a box of mass one. Floor translation and box, orbit pushforward, reserved block.

Let $\mathcal{U}$ be the full flag unipotent group. Unfolding $\mathbf{1}_K$ along $\mathcal{U}$ gives the measure of the near set $\mathcal{U}K$ in $\mathcal{U}\backslash G(A)$, because every nonempty unipotent fibre is a translate of $\mathcal{U} \cap K$ of mass one in triangular coordinates. The quotient uses the n successive orbit measures followed by the Haar measure of the terminal stabilizer H. Its normalization gives $H \cap K$ mass one. At any fixed y, all $h \in H$ with $hy \in \mathcal{U}K$ lie in one right coset of $H \cap K$ if there are any. Indeed the ratio of two corresponding integral representatives lies in K and is block triangular with Levi factor $h_1 h_0^{-1} \in H \cap K$. Hence the terminal mass is at most one. If any later stages can bring σg into $\mathcal{U}K$, then $\sigma g \in N_{a+1}$ because those stages fix the earlier flag. Iterating the preceding bound from the terminal measure gives $V_T \le Z_m^n$. Stabilizer coset, unfolded integral volume, successive near sets.

For T the chosen places and their conjugates we also need a lower bound in this Haar measure. The product integer ring R has additive volume one. Write $u = \operatorname{vol}(R^\times)$. At a nonterminal stage, restrict both Heisenberg coordinates to their integral boxes of mass one and take a unit Weyl parameter. Its density $|c|^d$ is then one and the Weyl matrix is integral. At the terminal stage the normalization of the quadric measure in the lattice datum gives its integral chart mass at least u, and the integral stabilizer has mass one. Convolution multiplies these lower bounds, since K is a subgroup. There are n stages in all. As T contains at most $4f \le b$ places of residue size at least b^{10}, we obtain

$$V_T \ge u^n \ge \Big(1 - \sum_{w\in T} q_w^{-1}\Big)^n \ge 2^{-n}. \tag{9.2}$$

The upper bound $V_T \le 1 + 2^{-n}$ follows from the Euler product above.

Lean. `le_partialGroupMeasure_boxSet`, `inv_two_pow_le_groupMeasure`.

Absolute mass at the full finite set. Let $R = \prod_{v\in T} \mathcal{O}_{E_v}$, with $\operatorname{vol}(R) = 1$, and retain the filling Haar measure on the unitary group over this product. Set $Z = K$, $V_T = \operatorname{vol}(K)$, and $v_g = \operatorname{vol}(K(g))$. We check the modifications to the one-frame estimate before summing the shells.

The lattice and denominator arguments are componentwise over discrete valuation rings. The Hermitian descent still uses $d_a \bar{d}_a$ and the same common level $g = \bar{g}$; this also covers places fixed by

conjugation. The common summand has the same rank in every component. Its content function has unit-box moment

$$\prod_{v\in T}\frac{1-q_v^{-d}}{1-q_v^{-(d-s)}}\le \zeta_R(d-s)\le \zeta_R(m)\le 2,\qquad d-s=m+5.$$

The scaled-box formula still contributes $|\alpha|^{-j(D-s)}$ in the product modulus. Use the preceding dot-product density bound for the solved block in the quadric charts. Its unit-box bound is $\zeta_R(m)$ over the entire product, so the reciprocal-depth moment per stage is at most $\zeta_R(m)\zeta_R(m+5)\le 4$.

In the direct depth and Fourier estimates, a self-conjugate scalar on the skew line has Jacobian $|\alpha|^{1/2}$, its integral lattice has volume one, and the free-coordinate Fourier transform is unitary in the prescribed measure. Thus the vector Jacobian $|g|^{r_a}$, skew Jacobian $|g|^{1/2}$, and all gauge powers in those estimates are unchanged. For instance the reserved input block at size $j+4$ contributes at most $\zeta_R(m)|\alpha|^{-(j+4)(2m+3)}\le \Lambda^{(2m+4)(j+5)}$. All ordinary boxes are measured in product Haar, giving one power of the product modulus per coordinate. The two centered matrix identities hold over R, and the orbit formula holds factorwise, including on the terminal chart, since the specified section uses the integral trace-one and skew units. The leading chart coordinate is nonzero at every factor almost everywhere, since T is finite and each coordinate hyperplane is null. The filling character is trivial on traces and self-dual for these measures. Consequently the same bound for a single frame on a shell applies with $\Lambda=2I^2$ and $I=[R:J]$. The mass calculation for W_Z used only cosets, so it also holds with $Z=K$. Character and measure normalizations, large lattice over all of T.

For $I>1$ some stable generator is a nonunit, so the maximum depth is at least $\max(2,I^{1/(2\varrho)})$. Put $h'=n+10^9n^4$. The same dimension margin gives $s-h'\ge 9\cdot 10^7n^4\ge 2\varrho(50000n^2+m)$. The depth factor in the squared pairing is therefore at most $2^{-h'}I^{-50000n^2-m}$. After the one-frame estimate the remaining sum over denominator ideals contributes at most $\zeta_R(m)\le 2$. The power $2^{1500000n^3+25000n^2+2}$ is at most $2^{10^9n^4}$. The squared integral outside $K(g)$ is thus at most $2^{-n}V_Tv_g$. On $K(g)$ it equals $V_T^2v_g$. With the compact volume estimate and the prescribed $2^{-\varrho}\le b^{-6}$ this gives

$$\mathcal{M}_T(K)\le V_Tv_g(V_T+2^{-n})\le (1+b^{-6})V_Tv_g.$$

Full-group mass.

The zero frequency on the same datum. Let j distinct places above p_0 be marked. Partition T into their j conjugate pairs and the remaining star-stable set. On each pair take its marked-vector fixer, and on the remainder take the full integral group. Their product Z contains every integral element satisfying the marks. The local characters multiply: the unipotent and congruence membership conditions are coordinatewise, the simple-root sum is evaluated in the product algebra, and the additive character is the product of the prescribed characters. Under this decomposition the original group Haar is $c>0$ times product Haar by invariance. Each mass therefore scales by c^3, and each group volume by c. In the relative comparison $\mathcal{M}_T(Z)\operatorname{vol}(K)$ and $\mathcal{M}_T(K)\operatorname{vol}(Z)$ acquire the same factor c^4. Tonelli after factoring the inner integrals gives

$$\mathcal{M}_T(Z)\operatorname{vol}(K)\le b^{-6kj}\mathcal{M}_T(K)\operatorname{vol}(Z).$$

Consequently the product bound for the prescribed vectors on T is independent of the normalizations of the local chart measures. Haar factors in the mass, choice of the local subgroups.

At the initial stage the two parameter tuples are empty. By Fubini, the product of the two copies of the correlation integral is $\left|\int\langle W_Z(y),W_Z(yx)\rangle\,d\theta(y)\right|^2$. Integration over x and the signed identity

above give $\mathcal{I}_0 = \mathcal{M}_T(Z)/U^2$. Write $\mathrm{mass}_{\mathrm{rel}}(Z) = \mathrm{vol}(Z)v_g/U^2$ for the relative mass (`massRatio` in Lean). Then

$$0 \le \frac{\mathcal{I}_0}{\mathrm{mass}_{\mathrm{rel}}(Z)} \le (1 + b^{-6})b^{-6kj}.$$

Empty heads and unit weight, normalized bottom of the sieve.

The finite-place frequency estimate. Assume the inverse level is integral outside T, and let f be the number of complex places. Put $I(\xi) = |\xi|^{-1}$ in the product Haar modulus and $D(\xi) = [R : R \cap \xi^{-1}R]$, for ξ nonzero in every component. For a sequence p of invertible Weyl parameters put $P_i = p_0 \cdots p_i$. The weight $H_l(p)$ below uses only the tuple of its first l entries. The density of two such tuples and the associated form are `headDensity` and `floorForm` in Lean. Set $H_l(p) = \prod_{i<l} I(P_i)^{n^6/2}$ if every such product is integral, zero otherwise. The two frequency cases use

$$\frac{|\mathcal{I}_l(p, q; \omega)|}{\mathrm{mass}_{\mathrm{rel}}(Z)} \le b^{n^9 f} \frac{I(\xi)^{(k-n^7)/2}}{D(\xi)^3} H_l(p)H_l(q) \qquad (0 \le l \le \varrho). \tag{9.3}$$

Here $\omega(x) = \psi(\xi\,\mathrm{Gram}(x)_{ll})$; the other needed case is $\omega = 1,\ l = i + 1,\ \xi = g^{-1}(p_i^{-1} - q_i^{-1})$. In the second case the difference must be nonzero in every component. We prove the first case at $l = 0$ and both cases for $1 \le l \le \varrho$ on this same datum. Two frequencies and their required decay, decay normalization, head weights, global energy and sieve.

Fourier decay at the first stage. Fix the full T, the same subgroup Z, and a frequency $\xi \in A^\times$. Let $J = R \cap \xi^{-1}R$ and $D = [R : J]$. Write r for the ambient rank of the first quadric, and $h(v, w) = v^*Hw$ for its integral unimodular form. Its orbit measure is the prescribed fibre measure. We first bound the measure of each residue class.

For any $a \in A^\times$, center $x_0 \in A^r$ and fibre value β,

$$\mu_\beta(x_0 + aR^r) \le \zeta_R(m)|a|^{r-1}.$$

Indeed, scaling by a contributes $|a|^{r-1}$ to the fibre measure. In a translate of R^r, fix the coordinates outside the reserved orthonormal block. They range over a box of mass one and change only the prescribed value on that block. The translated-block estimate above is uniform in the value and the center, which proves the claim.

At the initial stage the lower integration variable is 1 and the tuples are empty. At a fixed position x both factors use the same $F_x(\sigma) = P_1(\sigma, \sigma x)$. For $F_x(\sigma) \ne 0$, some $\gamma \in Q_1$ has $\gamma\sigma \in K$. This group fixes e_0, so $v = \sigma^{-1}e_0 = (\gamma\sigma)^{-1}e_0 \in R^r$. The Fourier kernel on these orbit vectors v, w is $\psi(\xi h(v, w))$. It only sees the classes of v mod $\bar{J}R^r$ and w mod JR^r. Self-duality of R and $H \in \mathrm{GL}_r(R)$ make this a perfect pairing of finite groups of order D^r. Consequently its unnormalized character matrix has operator norm $D^{r/2}$, by orthogonality of its rows. Orbit pairing at the two arguments, annihilator of the integral lattice.

Integrate F_x over the preimage of each class under the orbit map. For either partition the vector of these integrals has squared norm at most $\zeta_R(m)D^{1-r}\int |F_x|^2 d\theta_0$, by Cauchy–Schwarz on each class and the class-mass bound. Multiplication by the character matrix gives the pointwise estimate

$$|\mathcal{A}_x(\xi)| \le \zeta_R(m)D^{\,1-r/2} \int |F_x|^2 \, d\theta_0.$$

Here $\mathcal{A}_x$ is the twisted double integral at x. For $\xi \in R$ its phase is identically one on its support, so $\mathcal{I}_0(p, q; \psi(\xi\,\mathrm{Gram}_{00})) = \mathcal{I}_0(p, q; 1)$.

The two denominator factors. Keep $D = [R : R \cap \xi^{-1} R]$ for the frequency, and I for the index of the ideal clearing $(x-1)/g$. The two indices concern different variables. Let $\mathcal{W}(x)$ be the initial weighted square from page 70. Its term at $a = 0$ contains the entire squared norm $\int |F_x|^2 d\theta_0$. The free coordinates are a subspace of A^r, so $s \le r - 2$ by $D_X - 3\varrho = s + m + 5$. The residue-class estimate and the prefactor in $\mathcal{W}$ give $|\mathcal{A}_x(\xi)| \le D^{-s/2} \mathcal{W}(x)$.

Independently, taking absolute values of the two orbit integrals bounds $|\mathcal{A}_x(\xi)|$ by the same pair of absolute pivots as at the empty tuple. On an active shell of index I, the first-deep-stage estimate gives $|\mathcal{A}_x(\xi)| \le I^{-s/(2\varrho)} \mathcal{W}(x)$. For $I = 1$ use the pivot bound at depth one, since integral stable readings give depth at least one on the support. Away from the contributing positions the pivots vanish.

Put $t = k - n^7$, $L = 50000n^2 + m$. Admissibility gives $t + 6 + 10^8 n^4 \le k$, and the free-dimension calculation gives $s \ge t + 6 + 9 \cdot 10^7 n^4 \ge t + 6 + 2\varrho L$. Taking the geometric mean of the two estimates with weights $(t+6)/s$ and $1 - (t+6)/s$ yields

$$|\mathcal{A}_x(\xi)| \le D^{-(t+6)/2} I^{-L} \mathcal{W}(x).$$

The one-frame shell integral contributes at most $\mathrm{mass}_{\mathrm{rel}}(Z)\, 2^{1500000n^3 + 25000n^2 + 1} I^{50000n^2}$ in the original measures. This also holds for $I = 1$ by the unit position ball. Summation over the ideals contributes $\zeta_R(m) \le 2$. Finally $I(\xi) = |\xi|^{-1} \ge D^{-1}$; in valuations this just drops the positive numerator factors. Thus for every $\xi \in A^\times$,

$$\frac{|\mathcal{I}_0(p, q; \psi(\xi\, \mathrm{Gram}_{00}))|}{\mathrm{mass}_{\mathrm{rel}}(Z)} \le 2^{1500000n^3 + 25000n^2 + 2} D^{-(t+6)/2} \le b^{n^9 f} \frac{I(\xi)^{t/2}}{D^3}.$$

This proves (9.3) at the initial stage with the test at the original level. At the first stage the second orbit is translated by the Weyl element of the tuple. The two supports then have different sizes in a common coordinate system. Shell dimension margin, two heads in a common gauge, required frequency loss.

Distinct parameters at the first stage with external weight 1. Fix the admissible filling at the conjugation-stable set T, with $A = \prod_{w \in T} E_w$, $R = \prod_{w \in T} \mathcal{O}_w$, and $g = M|_T$. Assume M^{-1} is integral off T. Use the same $K(g) \subseteq Z \subseteq K$, its vector W_Z, and its Haar measures. We first take $\omega_{\mathrm{ext}} = 1$ with $\xi = g^{-1}(p_0^{-1} - q_0^{-1}) \in A^\times$, so the difference is nonzero at every place of T. All oscillation in this term comes from the two Bessel characters. Write $e = e_0$, $e^* = e_0^*$, $P = p_0$, $Q = q_0 \in A^\times$ and $w_P e = P e^*$. Nearness at the first stage means that some element fixing e carries the point into K. In particular, if y and $w_P^{-1} y$ are near, the columns $y^{-1} e$, $P y^{-1} e^*$ are integral. Their Hermitian pairing is P. A nonzero absolute majorant gives such a pair for P, y and for Q, yx: each nonzero pivot supplies a nonzero correlation, its two arguments are near, and later stages fix e. Thus $P, Q \in R$ whenever the majorant is nonzero anywhere. Otherwise the form vanishes at all frequencies. This proves the zero-weight case of (9.3). Weyl action, support of both heads.

On this support take the integral self-adjoint scale

$$\alpha = g\, \overline{PQ}\, PQ, \qquad \Lambda = 2/|\alpha|.$$

The elements $\alpha/P, \alpha/Q, \alpha/g$ and $\alpha(P^{-1} - Q^{-1})$ are integral. In this gauge, size u means that multiplying by α^u gives integral entries. One common change of coordinates suffices for the two orbits. Indeed, write a boxing element for y as $s n_0$, with s fixing e, e^* and n_0 in the first Heisenberg group N_0. Put $k_0 = s n_0 y \in K$, $n' = s n_0 s^{-1}$. Then

$$P n' e^* = k_0 (P y^{-1} e^*) \in R^r, \qquad s y = (n')^{-1} k_0.$$

The last column of n' is $(c, v, 1)$ in the decomposition $Ae \oplus V_1 \oplus Ae^*$; the other nonconstant entries are those of $-v^* H_1$, for the integral middle Gram matrix H_1. As $\alpha/P, \alpha/\bar{P} \in R$, both $\alpha n'$ and its Hermitian adjoint $\alpha(n')^{-1}$ are integral. Hence sy has size one. Since s commutes with w_P and αw_P^{-1} is integral, $s w_P^{-1} y$ has size two. The corresponding supported orbit vectors have these same bounds. For example, when $\gamma\sigma y \in K$ with γ unitriangular on the first two vectors and σ on the first stage, project $(sy)(\gamma\sigma y)^{-1} e_1$ onto V_1. This gives $s\sigma^{-1} e_1$ because $\gamma^{-1} e_1 - e_1 \in Ae$. Simultaneous change of coordinates, two orbit supports.

The depth estimate for the remaining lines is proved for arbitrary tuple length on p. 87.

The measure and the phase on the first Heisenberg section. Write $P = p_0,\ Q = q_0,\ u = P^{-1}$. Let w_P, w_Q be the length-one Weyl elements. Write C_j for the partial correlation of the measured vector W_Z with the stated Weyl ratio $d = w_Q^{-1} w_P$ on the second argument (equivalently, d has entry P/Q on e_0). Write $h(z, y) = z^* H y$. For fixed arguments a, a', put $F(\sigma) = C_2(\sigma a, \sigma a')$ and $y = \sigma^{-1} f_1$. The first Heisenberg group N has coordinates $\nu(\omega, \gamma)$, with ω in the middle space of rank $r_0 = k + 1 + 2(\varrho - 1)$ and $\gamma \in A^-$. For $e = e_0,\ e^* = e_0^*$, the section point is

$$y' = \bar{P}\, \nu^{-1} e^* = \bar{P}(e^* - \omega + c\, e), \qquad h(e, y') = \bar{P}.$$

Here c is the e-entry of $\nu^{-1} e^*$. The first-stage isometries fix both e, e^*, so y is orthogonal to both. Unfolding one stage in $C_1(\nu a, \nu a')$ conjugates ν by σ. The only adjacent entry in its frame block is $-h(\omega, y)$. The ratio d multiplies that entry by P/Q, and the first argument contributes its negative. The raw discrepancy is $\lambda = g^{-1}(P/Q - 1) = -\xi P$. Its character is $\psi(-\lambda h(\omega, y))$. Since $h(y', y) = -P h(\omega, y)$, the signed identity is

$$C_1(\nu a, \nu a') = \int \psi(-\xi h(y', y)) F(\sigma)\, d\theta_1(\sigma).$$

Conjugation and the adjacent root, phase on the section.

Let $S = \{\nu \in N : w_P \nu a \text{ is near at } 1\}$. The estimate is immediate if $C_1(\nu a, \nu a')$ vanishes on S. Otherwise its active subset also has νa near, by support. There the preceding common-center calculation, applied to $w_P \nu a, \nu a$, gives s_ν fixing e, e^* with $s_\nu \nu a$ of size $i = 99n^2$, and P of size at most $10n$. Choose one contributing ν_0 and put $s = s_{\nu_0}$. For every contributing ν, the column

$$(s\nu_0 \nu^{-1} s^{-1} - 1) e^* = [(s\nu_0 a)(s_\nu \nu a)^{-1} - 1] e^*$$

has size at most $2i + 6$ by the unitary inverse formula. Multiplying by $u^{-1} = P$ leaves it of size at most $j = 999n^2$. At the initial stage there are no earlier isotropic directions to quotient out. To bound the input orbit using the same s, for $F(\sigma) \neq 0$ take $l \in Q_2$ with $l\sigma a \in K$. Project the f_1-column of $(s\nu_0 a)(l\sigma a)^{-1}$ to the middle space. Since $l^{-1} f_1 - f_1 \in Ae$ and ν_0 acts trivially on $e^\perp / Ae$, the result is sy, of size at most $i + 6$. Column comparison, support from a single center.

Use $d\nu = d\omega\, d\mu_-(\gamma)$, the original product Haar measure with both integral coordinate lattices of mass one. Make the substitution $\nu = s^{-1} \nu(uv, \gamma) s \nu_0$. Right translation preserves Haar measure. Conjugation by s fixes the skew centre and has product determinant modulus one on the vector coordinates, since $(\det s)\overline{\det s} = 1$ there. The vector change $\omega = uv$ contributes $|u|^{r_0}$. In these coordinates the bounded column is $(\nu(uv, \gamma)^{-1} - 1) e^* / u$. It forces v into the box of size j. Its e-entry before division is $c(v) - \gamma$, with $c(v)$ independent of γ. Thus for fixed v, the active skew slice S_v, if nonempty, lies in $\gamma_0 + D_v$, where $\gamma_0 \in S_v$ and $D_v = A^- \cap b_j R,\ b_j = u\alpha^{-(j+1)}$.

Let $\tau + \bar{\tau} = 1$ and $\bar{\delta} = -\delta$ be the prescribed integral trace-one element and skew unit. The additive isomorphism

$$(A^-)^2 \longrightarrow A, \quad (d_1, d_2) \longmapsto \tau \delta^{-1} d_1 + d_2$$

has inverse $z \mapsto (\delta(z+\bar{z}),\, z-\tau(z+\bar{z}))$. It maps $(R^-)^2$ onto R, so it sends $\mu_- \otimes \mu_-$ to the specified Haar measure on A. Since $\tau\delta^{-1} \in R$, it sends D_v^2 into $b_j R$. Consequently

$$\mu_-(S_v)^2 \le \mu_-(D_v)^2 \le |b_j|, \qquad \mu_-(S_v) \le |u|^{1/2}|\alpha|^{-(j+1)/2} \le |u|^{1/2}\Lambda^{j+2}.$$

This is a bound on the skew slice for arbitrary $u \in A^\times$. At a split pair the single skew coordinate has to satisfy both component bounds. Its allowed mass is at most their minimum. Combining the slice bound with the vector Jacobian gives the stage modulus $\Delta_0(u) = |u|^{r_0+1/2}$ times the gauge loss. Floor modulus, integral elements and normalized Haar, skew mass, Heisenberg Haar integration.

The vector part after recentering is $s^{-1}uv + \omega_0$. The translate ω_0 contributes a unit phase depending only on the input. The Fourier scalar on the variable v is $\vartheta = -\lambda\bar{u} = \xi P/\bar{P}$, hence $|\vartheta| = |\xi|$ in the product modulus. Under $\sigma \mapsto s\sigma^{-1}f_1$, the input measure pushes to the first-stage quadric measure μ. With the unit phase included, write F_* for the conditional mean of F on these fibres. It satisfies $\int |F_*|^2 d\mu \le \int |F|^2 d\theta_1$ and is supported in the box of size $i+6$. The output has ambient vector Haar measure.

Split the basis adapted to the flag as $X' \oplus (X')^\perp$, and write $y = (a, z)$, $v = (b, c)$, $D_X = \dim_A X'$. Passing to this basis adds four to each box exponent. Disintegrate the boxed input as $da\, d\mu_a(z)$, with da the original Haar measure. With a fixed, solve the norm equation in one reserved orthonormal block of rank $d_B = 6\varrho + 14$. At coordinate size $l = i + 10$ its mass is at most $\Lambda^{d_B(l+1)}$ at every shifted norm. The other $r_0 - D_X - d_B$ coordinates each contribute at most Λ^{l+1}. As $r_0 - D_X = 20\varrho + 40 < 99n$, this gives $\sup_a \mu_a(1) \le M_{\mathrm{in}} = \Lambda^{99n(i+12)}$. On the output, box the disjoint reserved block and the other coordinates of c at size $j+4$. They have ordinary Haar volume at most $M_{\mathrm{out}} = \Lambda^{99n(j+6)}$. Floor coordinates and the two reserved blocks.

For fixed c put $G_c(a) = \int \psi(\vartheta h(c,z)) F_*(a,z)\, d\mu_a(z)$. Cauchy–Schwarz gives $\|G_c\|_2^2 \le M_{\mathrm{in}} \|F_*\|_2^2$. The transform at (b,c) is now the ordinary Fourier transform of G_c in the orthonormal coordinates a, with kernel $\psi(\vartheta\bar{b}\cdot a)$. Integrating its square over all $b \in A^{D_X}$ gives exactly $I(\xi)^{D_X}\|G_c\|_2^2$ by self-dual Plancherel. Integrating over the allowed c gives the squared-norm bound $M_{\mathrm{out}} M_{\mathrm{in}} I(\xi)^{D_X} \|F\|_2^2$. With the skew-slice loss included the gauge exponent is at most $j + 2 + 99n(i+j+18) \le 100n(i+j+31) \le 10^6 n^3$. Finite Plancherel normalization, Fourier estimate, its ambient output on the section.

Consequently, for $C_\Gamma = \Lambda^{10^6 n^3}$, the section estimate is

$$\int_S |C_1(\nu a, \nu a')|^2 d\nu \le I(\xi)^{D_X}\, \Delta_0(P^{-1})\, C_\Gamma \int |C_2(\sigma a, \sigma a')|^2 d\theta_1(\sigma). \tag{9.4}$$

Free and reserved ranks, fitted dimension, translated frequency, section bound.

The Fourier operator on two quadric fibres. Let the two orbit vectors lie in $a_1^{-1}R^r$ and $a_2^{-1}R^r$, where $a_1, a_2 \in R \cap A^\times$ are self-adjoint. Equip both fibres with their original orbit measures. For $\eta \in A^\times$ put

$$B_0 = |a_1 a_2|^{-1}, \qquad D' = [R : R \cap (\eta/(a_1 a_2))^{-1} R].$$

In the scaled variables $a_1 v, a_2 w$ the character $\psi(\eta h(v,w))$ factors through the paired residue classes of frequency $\eta/(a_1 a_2)$. Their character matrix has norm $(D')^{r/2}$ by orthogonality. In the original variables each class on the i-th fibre has mass at most $\zeta_R(m)(|a_i|D')^{1-r}$ by the translated-box estimate on p. 83. Cauchy–Schwarz on the classes then bounds the operator norm by

$$\zeta_R(m)\, B_0^{1/2} (B_0/D')^{(r-2)/2}.$$

Since $D' \ge |\eta/(a_1 a_2)|$, one has $B_0/D' \le I(\eta)$. The other Fourier estimate, for unequal last entries, uses (9.4). We now apply both estimates for all nonempty tuples, including those with one entry. Gram character on two orbits, supports of the two orbit vectors.

Integral minors for tuples of arbitrary length. Fix $1 \le j \le \varrho$. Write $P_a = \prod_{i=0}^{a} p_i$, $Q_a = \prod_{i=0}^{a} q_i$, and $P_{-1} = Q_{-1} = 1$. If $z, w_p^{-1}z$ are near at j, choose $s, u \in Q_j$ with $sz, uw_p^{-1}z \in K$. In the integral matrix $S = sw_pu^{-1} = (sz)(uw_p^{-1}z)^{-1}$, take columns $e_0, \dots, e_a$ and rows $e_0^*, \dots, e_{a-1}^*, i$, $a < j$. Their determinant is $P_a(se_a^*)_i$. Indeed u^{-1} acts on these columns by an upper unitriangular matrix; $w_pe_b = p_be_b^*$; and the first a selected rows of the resulting dual columns of s are lower unitriangular followed by a zero last column. Choosing $i = e_a^*$ gives $P_a \in R$; allowing every i gives $P_az^{-1}e_a^* \in R^r$ since $(sz)^{-1}$ is integral. Nonzero pivots give these two nearness conditions at y for p and at yx for q, since all later successive flag subgroups fix the first j pairs and commute with the two Weyl elements. Thus all cumulative products are integral on the nonzero support, even when individual entries have poles. Integral minor, both head supports.

Set $b_p = \prod_{a<j} P_a$, $b_q = \prod_{a<j} Q_a$ and $\alpha = g\overline{b_pb_q}\, b_pb_q$, $\Lambda = 2/|\alpha|$. Each P_a divides b_p in R, so α clears p_a and p_a^{-1}, ratios of successive cumulative products. The same holds for q, their conjugates and the inverse level. To center the two supports, factor the boxing element $s = tn_0$ with t fixing the first j hyperbolic pairs and n_0 in their flag unipotent group. This factors off the induced isometry on the middle quotient. Put $U = tn_0t^{-1}$, so $U(tz) = sz$. Apply the minor just proved to $tz, w_p^{-1}tz$, using U as the boxing element. In the decomposition into the first isotropic span, its middle space and its dual, write

$$U = \begin{pmatrix} A & B & C \\ 0 & 1 & D \\ 0 & 0 & E \end{pmatrix}.$$

The last j columns Ue_a^* are integral after multiplication by α. The block E is lower unitriangular, hence $\alpha^{j-1}E^{-1}$ is integral by the finite geometric sum. Unitarity gives $A = (E^{-1})^*$ and $B = -AD^*H_{\mathrm{mid}}$. Consequently α^jU and its Hermitian adjoint α^jU^{-1} are integral. Thus tz has size j, and $tw_p^{-1}z = w_p^{-1}tz$ has size $j+1$. Projecting a boxed next-frame column onto the middle space gives the same two sizes for the supported orbit vectors: the next flag unipotent changes that column only by the earlier isotropic span. Common translate, projection of the orbit column.

At every active pair with the translated nearness conditions, $c_a = gP_aQ_a$ is a stable reading for $a < j$: the identical lifts $c_ae_a^*$ have both readings in gR^r by the minor formula. Thus $|d_a| \ge |c_a|$ for the stable generators. The descent on p. 72 bounds the position index by

$$I_x \le \prod_{a<\varrho} |d_a|^{-2} \le (\Lambda/2)^{2j} \prod_{j\le a<\varrho} |d_a|^{-2}.$$

Here $\prod_{a<j} |c_a| = |g|^j|b_pb_q| \ge (2/\Lambda)^j$. If $j < \varrho$ and $I_x > \Lambda^{4j}$, a remaining stage has depth at least $I_x^{1/(4(\varrho-j))}$. For $j = \varrho$ the empty product gives $I_x \le (\Lambda/2)^{2\varrho}$ throughout the contributing support. Depth after the head.

Transport of all earlier depth weights. Fix the position x and an integral self-adjoint scale α adapted to p: $\alpha/\prod_{b<l} p_b \in R$ for $0 \le l \le j$. Set $\Lambda = 2/|\alpha|$, $w_l = \mathrm{weylElem}(l, p)$, $h_l(z) = \prod_{b<l} \mathrm{depth}_b(z, x)^{-s}$, and $m_l(z) = \mathbf{1}_{\mathrm{near}_l}(z)/h_l(z)$. Write η_a for the original Haar measure on N_a, $\eta_{<l}$ for Haar on $U_{<l}$ in the successive Heisenberg coordinates, and ϑ_l for the descending product of the orbit measures of stages $0, \dots, l-1$. The lower point has measure $\mu_l = \eta_{<l} * \vartheta_l$. Put $C = \Lambda^{10^6n^3}$ and $D_l = \prod_{a<l} \Delta_a(p_a^{-1})$, where $\Delta_a(t) = |t|^{r_a+1/2}$, $r_a = k + 1 + 2(\varrho - a - 1)$. For $l \le b$, $b_0 \ge 0$ measurable, left Q_b-invariant and supported on points near at b, and any σ-finite measure ν',

$$\int b_0(z) m_l(w_l^{\pm 1}z)\, d(\eta_{<l} * \nu')(z) \le CD_l \int b_0(z)\, d\nu'(z). \tag{9.5}$$

Here is the calculation with an earlier unipotent $\nu \in U_{<a}$ still present. Let $k_0 \in K$, $u \in N_a$ and suppose $\gamma w_{a+1} u\nu k_0 \in K$ for $\gamma \in Q_{a+1}$. Write $T = (\gamma w_{a+1} u\nu)^{-1}$, which is integral. Its

columns on $e_0,\dots,e_a$ are those of $((u\nu)^{-1}e_b^*\bar{p}_b)_{b\le a}$ times an upper unitriangular matrix, since $\gamma^{-1}e_b-e_b\in\langle e_c:c<b\rangle$. Take rows $e_0^*,\dots,e_{a-1}^*,t$. The dual block of $(u\nu)^{-1}$ is lower unitriangular in this ordering. The minor is therefore $(\prod_{b\le a}\bar{p}_b)((u\nu)^{-1}e_a^*)_t$, also for a repeated row, when both sides vanish. The tuple condition and projection off the earlier pairs give

$$z'=\bar{p}_a u^{-1}e_a^*\in\alpha^{-1}R^r,$$

since ν^{-1} acts trivially on $\langle e_b:b<a\rangle^\perp/\langle e_b:b<a\rangle$. Leading-column minor, projection to the section.

In the common summand at k_0,k_0x, impose the a extra equations $(\nu\iota v)_{e_b}=0$ for $b<a$. These are among the allowed ϱ rows in the common-content calculation. Transport the resulting vectors by $w_{a+1}u\nu$. Their two readings are integral and level-congruent. Their earlier dual coordinates vanish by the extra equations, so they are stable at stage a. Their products with e_a give the common-content majorant $\mathrm{depth}_a\le\phi(z'_{X'})$, by the same projection. In $u=(\omega,\gamma_0)$, change the middle variable to $-\bar{p}_a\omega$, which contributes $|p_a|^{-r_a}$. On its box $\alpha^{-1}R^{r_a}$, the integral of ϕ^s is at most $2|\alpha|^{-(r_a-s)}$ by the scaled content moment on X' and ordinary volume on the complement. At fixed middle coordinate the skew parameters have differences in $\bar{p}_a^{-1}\alpha^{-1}R$, so their mass is at most $|p_a\alpha|^{-1/2}$ by the trace-one decomposition used for (9.4). The resulting integral of $\mathbf{1}_{\mathrm{near}_{a+1}}\mathrm{depth}_a^s$ contributes at most $\Delta_a(p_a^{-1})\Lambda^{200n}$ since $r_a-s+1<200n$. Common rows and section Haar.

The already read factor $m_a(w_a\nu k_0)$ is constant in this integration: the Weyl element at the new stage and u fix the earlier pairs, commute with w_a, and preserve the weight and the near cut of length a. On the new near set $m_{a+1}=m_a\,\mathrm{depth}_a^s$. Iterating the preceding estimate gives $D_l\Lambda^{200nl}\le CD_l$. For an outer point near at l, rebox by $s_0u_0\in Q_l$, with $u_0\in U_{<l}$ and s_0 fixing the first l pairs. Right translation by u_0 preserves $\eta_{<l}$; conjugation by s_0 has vector determinant of modulus one and fixes each skew coordinate. It also commutes with w_l. This proves the same bound at every such point. Pulling b_0 through the unipotent integral proves (9.5). For w_l^{-1} conjugate every parameter, which keeps its modulus. Iteration and reboxing.

In particular at the second stage the full modulus is $D_2=\Delta_0(p_0^{-1})\Delta_1(p_1^{-1})$. The earlier factor occurs in both uses of the transport inequality below. Partial measure, Weyl transport, earlier inverse weight, crossed sections.

The signed integral and both squared norms at each stage. Fix $1\le j=i+1\le\varrho$, a position x, and a scale α adapted to the parameters with loss Λ. Write $w_a=w_{p,<a}$ and $d=w_{q,<j}^{-1}w_j$. Set $P=p_i$, $Q=q_i$, $r_i=k+1+2(\varrho-i-1)$, $X=\dim_A X'$, $\Delta_a(u)=|u|^{r_a+1/2}$, $D_a=\prod_{l<a}\Delta_l(p_l^{-1})$, and $C=\Lambda^{10^6n^3}$. Use the untwisted and d-twisted correlations C_l^1,C_l^2 of the same W_Z. For kernel $b=1,2$ let E_a^b be the sum defining $\mathcal{N}(x)$ above restricted to $l\ge a$, with C_{l+1}^b in place of P_{l+1}. Write $B_b(u)$ for the sum from j with the first j orbit variables held fixed at their product u. Integrating B_b over those variables gives E_j^b.

On the Heisenberg group N_i, all earlier hyperbolic pairs are fixed. In its middle space of rank r_i, the section and the next orbit vector are $y'=\bar{P}\nu^{-1}e_i^*=\bar{P}(e_i^*-\omega+ce_i)$ and $y=\sigma^{-1}f_j$. The latter is orthogonal to e_i,e_i^*. Only the adjacent entry $-h(\omega,y)$ of the conjugated Heisenberg element contributes to the difference of simple-root sums. The ratio d multiplies this entry by P/Q; the earlier entries give zero. Thus at $\xi=g^{-1}(P^{-1}-Q^{-1})\in A^\times$,

$$C_j^2(\nu a,\nu a')=\int\psi(-\xi h(y',y))\,C_{j+1}^2(\sigma a,\sigma a')\,d\theta_j(\sigma).$$

The superscript 2 labels the twisted kernel. The phase is integrated before taking absolute values. Section identity.

Here the proof of (9.4) works on N_i with its original Haar measure. To check the support, only ν with both $w_j\nu a$ and νa near at j contribute. The common change of coordinates gives $s_\nu \in G_j$ with $s_\nu\nu a$ of size $99n^2$, and the parameter P has size at most $10n$. Choose one such ν_0 and $s = s_{\nu_0}$. The column $(s\nu_0\nu^{-1}s^{-1} - 1)e_i^*$ equals the e_i^*-column of $(s\nu_0 a)(s_\nu\nu a)^{-1} - 1$, since s, s_ν fix that column. It has size at most $198n^2 + 6$. After multiplying by P it has size at most $999n^2$. Projecting the supported next vector off the first j pairs in these same coordinates bounds sy by size $99n^2 + 6$. In $\nu = s^{-1}\nu(P^{-1}v, \gamma)s\nu_0$, the vector Jacobian and skew-volume bound of (9.4) give $\Delta_i(P^{-1})$; the effective Fourier scalar $\xi P/\bar{P}$ has product modulus $|\xi|$. The same free-coordinate Plancherel estimate applies since $X + 1 \le r_i \le r_0,\ r_i - X < 99n$. With $\phi_2(a) = |C_j^2(a, ax)|$, and $B_2(a)$ the sum of weighted squares from stage j, this proves

$$\int_{\{w_j\nu a \text{ near}_j\}} \phi_2(\nu a)^2\, d\nu \le I(\xi)^X \Delta_i(P^{-1}) C\, B_2(a)/h_j(a).$$

Here h_a denotes the positive depth weight of length a. At $j = \varrho$, $f_j = z$ has norm $-N$. The orbit measure is the prescribed measure at this value of the middle form. Its lattice is unimodular since the first j hyperbolic planes split integrally from the unimodular full lattice. The translated-box estimate uses the reserved block and holds at every fibre value, including $-N$. Section support and Jacobian, integral lattice isometry, last orbit vector.

Write $v_a = \mathbf{1}_{\text{near}_a}/h_a$, μ_j for the full measure below the tuple, and Θ_j for the product of the first j orbit measures in descending order. The squared correlations with their depth weights, and hence their sums B_b, are invariant under Q_j on the indicated supports. Its unipotent part changes the correlation by a unit character, and its isometry part preserves the orbit integral. For a left G_j-invariant test, changing u to $w_j^{\pm 1}u$ preserves the lower integral: disintegrate Haar measure over G_j, which commutes with w_j. Thus (9.5) applies on both sides, with cost CD_a at length a. Invariance of the weighted squares, Weyl change of variables. For the Gram character choose η as on p. 90, and first suppose $I(\eta) < 1$. On the two supported orbits the Gram entry has size at most $200n^2$. The two-box estimate applied with box exponents $60n^2 + 6, 99n^2 + 6$ bounds the squared signed pivot by $CI(\eta)^X[B_1v_j](u)[B_2v_j](w_j^{-1}u)$. Indeed the orbit rank minus two is at least X and the box prefactor is at most $4\Lambda^{159n^2+12} \le C$. Cross the inverse weights between the two factors in Cauchy–Schwarz. Both integrals incur CD_j, and $\int B_1 d\Theta_j \le E_0^1,\ \int B_2 d\Theta_j = E_j^2$, by the definitions of the three sums. Crossed Gram integrals, Gram phase on the support.

For unequal last entries put $\phi_1(u) = |C_j^1(u, ux)|$. The integrand has modulus $\phi_1(u)\phi_2(w_j^{-1}u)$. On common support use the squared Cauchy–Schwarz factors

$$h_i(u)\phi_1(u)^2\, v_j(w_j^{-1}u), \qquad \mathbf{1}_{\text{near}_j}(u)\,[h_j\phi_2^2](w_j^{-1}u)v_i(w_iw_j^{-1}u).$$

Their product is the square of the integrand: $w_iw_j^{-1}$ is the single last Weyl inverse, which fixes the earlier depth readings. The first integral is bounded by $CD_j \int h_i\phi_1^2\, d\Theta_j$. In the second translate by w_j, then integrate N_i first. Both h_j and $v_i(w_i\,\cdot)$ are constant on its fibres. The displayed section bound cancels h_j and leaves $I(\xi)^X\Delta_i(P^{-1})C\, B_2(a)v_i(w_ia)$. The i earlier Heisenberg integrations cost CD_i; the remaining integral is E_j^2. Finally $\int h_i\phi_1^2\, d\Theta_j$ is one summand of E_0^1. Because $\Delta_i(P^{-1})D_i = D_j$, the squared position integrand in both branches is bounded, whenever $I(\eta) < 1$, by

$$I(\eta)^X\, C^3 D_j^2 E_0^1 E_j^2.$$

In the unequal-entry branch $\eta = \xi$. The bound by the absolute pivot majorant applies for all frequencies. The comparison at the last *entry*, of length $\varrho + 1$, is a separate statement. Earlier weights after section integration, two modulus products, normalized position bound.

Uniform summation over positions. Fix $1 \le j \le \varrho$, put $t = k - n^7$, $r = t + 12$, $L = 50000n^2 + m$, and write $\mathcal{P}_x = C^3 D_j^2 E_0^1 E_j^2$ for the product obtained above. The density of the tuples is $\kappa_j = \prod_{a<j} |p_a q_a|^{r_a/2}$. For the exact denominator ideal $J = (c)$ of $(x-1)/g$, let S_J be its contributing shell and $I = [R : J]$. The shell lies in the position ball of scale $c\bar{c}$, with loss $\Lambda' = 2I^2$. We claim

$$\kappa_j \int_{S_J} \sqrt{\mathcal{P}_x}\, dx \le \mathrm{mass}_{\mathrm{rel}}(Z)\, \Lambda^{1500000n^3+1} (\Lambda')^{25000n^2}. \tag{S}$$

Let Π be the squared pairing norm on the two quotient variables, and U the unipotent level volume. The one-frame bound, starting at zero for the first kernel and at j for the second, gives

$$\int_{S_J} E_0^1 \le U(\Lambda')^{25000n^2}\Pi, \qquad \int_{S_J} E_j^2 \le U(\Lambda')^{25000n^2}\Pi \prod_{a<j} |p_a/q_a|^{r_a+1}.$$

Indeed $d = w_q^{-1} w_p$ commutes with all stage groups from j onwards. The orbit measures, the simple-root characters and the depth weights are therefore the same in both iterations. The earlier unipotents contribute unit characters to the second kernel. In its terminal integral, translation by d on the second quotient variable replaces the a-th Weyl parameter by $c_a = (p_a/q_a)c'_a$. Its measure $|c_a|^{r_a} dc_a$ gives the factor displayed above. Cauchy–Schwarz in x now leaves

$$\kappa_j^2 D_j^2 \prod_{a<j} |p_a/q_a|^{r_a+1} = \prod_{a<j} |q_a|^{-1} = |Q_{j-1}|^{-1} \le \Lambda^2,$$

since $\alpha/Q_{j-1} \in R$. As $U\Pi \le \mathrm{mass}_{\mathrm{rel}}(Z)$ and $C^{3/2} = \Lambda^{1500000n^3}$, this proves (S). Weyl quotient measures, character above the prescribed tuple, integrated energies and density.

Let $F(x)$ be the integral over the lower point of the modulus of the signed double orbit integral, and put $V = \zeta_R(m)$. Let $A(x)$ place the absolute values inside that double integral as well. The two branches give $F(x) \le e^s \sqrt{\mathcal{P}_x}$ for $e = \min(1, \sqrt{I(\eta)}) < 1$; for $e = 1$ use $F \le A \le V\sqrt{\mathcal{P}_x}$, $1 \le V \le 2$. Indeed in the unit gauge the integral orbit lies in the unit-coordinate box after reboxing. Fix its coordinates outside the reserved orthonormal block. The remaining norm-fibre mass is at most $\zeta_R(m)$ at every shifted norm, and the fixed coordinates range over boxes of volume one. This includes the final norm $-N$. Floor mass from the coordinate box. For $j < \varrho$ and $I > \Lambda^{4j}$, put $d_* = I^{1/(4(\varrho-j))}$. The minor calculation forces a depth $\ge d_*$ on some stage $a \ge j$ at every active pair with both translated frames near. A nonzero second pivot forces this nearness throughout the first pivot's flag subgroup orbit. Stop at the first stage $a \ge j$ where the depth is at least d_*, and apply Cauchy–Schwarz there. The inverse weight at step $a+1$, integrated from step j, is at most $4^{a-j+1}/h_j(u)$ by (P). The squared first pivot is therefore bounded by $C d_*^{-s} B_1(u)/h_j(u)$, with $\varrho^2 4^\varrho \le C$. Bound the second squared pivot by $V B_2(w_j^{-1}u)/h_j(w_j^{-1}u)$. Applying (9.5) in the two orientations gives

$$A(x) \le V I^{-s/(8(\varrho-j))} \sqrt{\mathcal{P}_x}.$$

For $j = \varrho$, the majorant vanishes when $I > \Lambda^{4j}$ by the empty product of remaining depths.

Since $s - r \ge 9 \cdot 10^7 n^4 \ge 8\varrho L$, interpolation with weights r/s and $1 - r/s$ leaves I^{-L} on the large shells. On the other shells insert $1 \le \Lambda^{4jL} I^{-L}$. After (S) the sum contributes at most $\sum_J [R : J]^{-m} \le 2$; $4jL + 1500000n^3 + 25000n^2 + 3 \le 10^9 n^4$. Thus $\kappa_j \int F \le \mathrm{mass}_{\mathrm{rel}}(Z) e^{t+12} \Lambda^{10^9 n^4}$. The same argument with $e = 1$ gives absolute integrability of the majorant.

The frequency denominator. Write $D = D(\xi) = [R : R \cap \xi^{-1}R]$, $\beta = \alpha^{200n^2}$, and $B = |\beta|^{-1}$. In the Gram case $\beta h(v, w) \in R$ on the support. At each place take $\eta_v = \xi_v$ if $v(\xi_v) \le v(\beta_v)$, and $\eta_v = \beta_v$ otherwise. Then $\eta - \xi \in \beta R$, so the character is unchanged. With q_v the residue cardinality,

$$I(\eta) = D^{-1} \prod_v q_v^{\min(\max(v(\xi_v),0), v(\beta_v))} \le \min(I(\xi), B/D).$$

For unequal last entries take $\eta = \xi$. Here $\alpha^2\xi = \alpha^2 g^{-1}(p_{j-1}^{-1} - q_{j-1}^{-1}) \in R$, so $D \le B$. In either case $e^{t+12} = \min(1, I(\eta))^{(t+12)/2} \le I(\xi)^{t/2}(B/D)^6$. The frequency on the support. Finally $H_j(p) = I(b_p)^{n^6/2}$ on the nonzero support, with the same formula for q, and $\Lambda = (2/|g|)(I(b_p)I(b_q))^2$. Thus $\Lambda^{10^9 n^4} B^6 \le (2/|g|)^{n^6} H_j(p) H_j(q)$, since $4(10^9 n^4 + 1200 n^2) \le n^6$. For the prescribed level, $|g|^{-1} = M^{2f}$ and $M \le b^{23}$, whence $(2/|g|)^{n^6} \le b^{(1+46f)n^6} \le b^{n^9 f}$. If a cumulative product is nonintegral, the corresponding support is empty by the minor calculation on p. 87. This proves (9.3) with D^{-6} for $1 \le j \le \varrho$. The same shell summation covers $j = 1$. Shell interpolation and integrability, frequency and tuple factors.

Recursion at a finite place. Write $p[c]$ for the tuple with entry j set to c, using the prescribed unit extension on the Haar-null complement of $A^\times$. For $j < \varrho$, and any measurable $|\phi| \le 1$,

$$\int_A \phi(c)\mathcal{I}_{j+1}(p[c], q[c]; 1)\, dc = \mathcal{I}_j(p, q; \phi(\mathrm{Gram}_{jj})).$$

To see the measure, expand both partial correlations at stage j with orbit variables σ, τ and lower point u. In the second orbit make the substitution $\tau = w_j(c)^{-1} n \sigma$, where here $w_j(c)$ acts on the single j-th plane. In this chart the measure is $|c|^{r_j} dc\, d\eta_j(n)$. Right translation by σ is valid on this integrand: as a test of τ it is left invariant by $Q_{j+1} \cap G_j$, and the orbit measure on those tests is right G_j-invariant. Indeed the Gram reading is unchanged under that left action, and the simultaneous unipotent phases in the second correlation cancel there since the earlier Weyl ratio commutes with G_j. The new lower point $n\sigma u$ has precisely measure $\eta_j * \theta_j * \mu_j = \mu_{j+1}$. The weight reads $\phi(h(f_j, \sigma\tau^{-1} f_j)) = \phi(c)$; the common entry cancels out of the Weyl ratio and multiplies κ_j by $|c|^{r_j}$. These identities give the displayed formula. The integral of the absolute double-pivot integrand is bounded by the integrable majorant just proved, so Fubini applies. Orbit substitution and invariance, descent with common entry.

The Gram matrix of the last parameter. Let G_ϱ fix all the hyperbolic planes, $z = f_\varrho$ have norm $-N$, and $H \subset G_\varrho$ fix z. Use the prescribed Haar measure $d\lambda$ on G_ϱ and the factorization $d\lambda = d\eta_H * d\theta_\varrho$. For $W = W_Z$ and $\ell \in G_\varrho$ put

$$T_p(\ell)_{ab} = \kappa_\varrho(p, 1) \int_G W(y)_a \, \overline{W(\ell w_p^{-1} y)_b}\, dy.$$

Here a, b range over the original level cosets. For $e = \ell^{-1} z$ let $\Phi_{pq}(e) = \int_H \langle T_p(h\ell), T_q(h\ell)\rangle\, dh$, with value zero off the orbit. The definition is independent of the lift by right invariance of dh. The compactness and continuity details appear below. The last stage identity with $\omega_\xi(y) = \psi(\xi h(z, yz))$ is

$$\mathcal{I}_\varrho(p, q; \omega_\xi) = \int_{G_\varrho} \omega_\xi(\ell^{-1}) \langle T_p(\ell), T_q(\ell)\rangle\, d\lambda(\ell) = \int \psi(\xi h(z, e)) \Phi_{pq}(e)\, d\theta_{\mathrm{quad}}(e).$$

For the first equality expand the two tensors and contract their coset indices. In the two integration variables put $y_2 = y_1 x$, then disintegrate $y_1 = su$, with $s = h_1\sigma$ in the top group and u in the lower measure μ_ϱ. In the ℓ integral substitute $\ell = as^{-1}$, $a = h_2\tau$; this uses right invariance of top-group Haar. The four vectors have arguments $su, sux, aw_p^{-1}u, aw_q^{-1}ux$. Their two contracted inner products have opposite order, as in the original signed pivot. The phase is $\omega_\xi(\sigma\tau^{-1})$ because both h_i fix z. The product of the two scalar densities is exactly $\kappa_\varrho(p, q)$. This is the expanded definition of $\mathcal{I}_\varrho$. The second equality disintegrates $h\ell$ over $e = \ell^{-1} z$ in these same measures. Normalized tensor, changes of variables, terminal orbit.

Take the specified $\pi = \pi_N|_T$ with $\pi\bar{\pi} = N$. Put $t = h(z, e)$; then $e = x - tz/N$, $x \perp z$, and $h(x, x) = t\bar{t}/N - N$. Write $d\mu_t$ for the prescribed norm-fibre measure at this value (its skew

index includes δ). In the coordinates $X_0 \oplus A(z/\pi)$ the final line has norm -1; the coordinate along it is $-t/\bar{\pi}$. This basis has the same volume as the basis adapted to the flag: their Gram determinants have unit modulus, so their change of basis has modulus one. The change from t therefore contributes $|\pi|^{-1}$. The remaining measure is $d\mu_t$. Indeed, solve the norm equation in a hyperbolic pair of X_0; after fixing the last coordinate the same pair, chart and skew measure define the sphere at the shifted norm. Equivalently, disintegrate ambient Haar in the norm variable in these coordinates and use continuity of compact-test integrals to evaluate at $-N$. This proves

$$J_{\mathrm{last}}(p[t], q[t]) = |\pi|^{-1} \int \Phi_{pq}(x - tz/N)\, d\mu_t(x), \qquad \int_A \psi(\xi t) J_{\mathrm{last}}(p[t], q[t])\, dt = \mathcal{I}_\varrho(p, q; \omega_\xi).$$

At a common t any finite linear combination of tuples has quadratic form

$$|\pi|^{-1} \iint \Big\| \sum_p c_p T_p(h\ell_x) \Big\|^2 dh\, d\mu_t(x), \qquad \ell_x^{-1} z = x - tz/N$$

with zero integrand off the orbit. This proves positivity for the last-entry matrix; division by $\mathrm{mass}_{\mathrm{rel}}(Z) > 0$ preserves it. Last-coordinate Haar factor, terminal Fourier identity, positivity at a common entry.

Support and continuity of the terminal density. Write G_ϱ for the fixer of the hyperbolic pairs, z for the last vector, and $H = (G_\varrho)_z$. Use the tensor $T_p(l)$ defined above for $l \in G_\varrho$. For p' near p, the finitely many prefix valuations and $\kappa_\varrho(p', 1)$ are constant. Choose one scale clearing all the inverse prefixes and their conjugates in this neighborhood. If the two factors are nonzero, $y \in UK$ and $w_{p'}^{-1} y$ is near at ϱ, since $l \in G_\varrho$. In the common-box argument on p. 87 take the boxing element in U itself. Thus y belongs to a fixed compact matrix box C in G. The same bound holds for $y' = l w_{p'}^{-1} y$ by interchanging the two points and conjugating the tuple, since l commutes with $w_{p'}$. Consequently $l = y' y^{-1} w_{p'}$ stays in a compact set for nearby tuples. This bounds the support of the individual tensor integrands before integration. Their boundedness on these compact sets also justifies Fubini for the expanded tensor pairing above.

There is a single neighborhood of $(p, 1)$ such that $T_{p'}(ls) = T_p(l)$ for every $l \in G_\varrho$, $s \in G_\varrho$ in that neighborhood. Indeed compactness of C and openness of $K(g)$ imply $y^{-1} w_p s w_{p'}^{-1} y \in K(g)$ for all $y \in C$. Right $K(g)$-invariance identifies the two integrands there. Outside C both vanish. This gives local constancy jointly in the tuple and relation with uniformity in the relation. Tensor and congruence invariance, uniform supports and translations.

The fibre pairing at $e = l^{-1} z$ integrates $\langle T_p(hl), T_q(hl) \rangle$ over H, and is zero off this orbit. Its support lies in the compact image of the support of the relation. The orbit map has local sections: over each local fixed field $h(e, e) = -N \neq 0$; a tangent vector $ce + v$, $v \perp e$, $c + \bar{c} = 0$, is the value at e of the skew-adjoint map sending e to $ce + v$ and $w \perp e$ to $-e\, h(v, w)/h(e, e)$. The local-field implicit-function theorem applies, also on each split pair. Uniform right translations therefore make the pairing locally constant on the quadric. Extend it to a compactly supported step function Φ on the ambient space: cover its compact support by finitely many lattice cosets on which the restriction is constant, take a common refinement and extend by zero. If invariance under $H \cap K(g)$ is needed, average the extension over this compact group; $W(ky) = W(y)$ there, and in the tensor integral the substitution $y \mapsto ky$ gives $T_p(lk^{-1}) = T_p(l)$. Support, orbit section and step extension.

This same Φ works for all nearby lower tuples by the uniform tensor identity. In the coordinates $e = x - tz/N$, $t = p_\varrho$, nearby t' translate the argument by $-(t' - t)z/N$, an additive period of Φ. Only the norm-fibre parameter $\delta(t'\bar{t}'/N - N)$ varies. Its measure is weakly continuous on compactly supported step tests by the hyperbolic-block disintegration, including at zero. Since $x \mapsto x - tz/N$ is a closed affine embedding, its pullback of Φ is such a test. Multiplication by the fixed factor

$|\pi|^{-1}$ proves joint continuity of the last-entry density; the definition uses only the lower entries of q. Norm fibres, joint continuity.

For the subgroup Z chosen for the marked-place bottom estimate, put

$$J_j = \frac{\mathcal{I}_j(p,q;1)}{\mathrm{mass}_{\mathrm{rel}}(Z)} \quad (j \le \varrho), \qquad J_{\varrho+1} = \frac{J_{\mathrm{last}}}{\mathrm{mass}_{\mathrm{rel}}(Z)}.$$

Integration over a common entry gives J_j at zero frequency. Equation (9.3) controls its Fourier transform and the unequal-entry term. At the top, continuity and positivity hold for these same tuples. These are the finite sieve hypotheses. Assembly on the chosen subgroup.

The squared norm over a fixed rational column. Use $Y = Z$ for the subgroup just constructed. Fix a left class c, the value $t = \langle z, \gamma z\rangle$, and the rational X_0-part of γz. Put $\lambda = (c', \gamma)$ and $l_\lambda = (h_c^{-1}\gamma h_{c'})_T^{-1}$. All $l_\lambda^{-1} z = e$ are equal. Here γ fixes the hyperbolic pairs, the class translates are integral off T, the tuples are units off T, and M^{-1} is integral there. For each λ use distinct reductions $\bar{n}$ with $n_T \in Y$. Write $\mathcal{C} = Y/K(g)$, $v = \mu_G(K(g))$, $v_H = \mu_H(H \cap K(g))$, where $H = (G_\varrho)_z$. Use the measures of the filling datum.

For an open reduction the leading flag minors stay nonzero over E because their reductions are nonzero. The open Bruhat decomposition writes a contributing element $uw_p\gamma u'^{-1}$; its relation determines γ. The norm of its definite column gives $|\Phi_v(t)|^2 - N^2 = N\|(\gamma z)_{X_0,v}\|^2$. The discriminant condition gives $t\bar{t} - N^2 \neq 0$, so this is positive at every embedding. We use the classification of contributing tuples in the finite entry set, with their integral representatives off T. Distinct blocks give distinct tuples $(c, c', \bar{n}, \gamma)$ because these entries recover the relation and reduction. Open-link representatives, application to finite subsums.

Let $V_p(l) \in \ell^2(\mathcal{C}^2)$ be the finite orbital tensor with its density for the prescribed parameters κ_p, $U_{p,\lambda}$ the orbital tensor of the cut Bessel function at $(w_p\gamma)_\infty$, and $A_{\lambda,\bar{n},p}$ the corresponding global coset amplitude. Suppose $w = \rho_p\kappa_p$ is independent of p and $R_{pq} = \langle \rho_p U_{p,\lambda}, \rho_q U_{q,\lambda}\rangle$ is common to all λ. Then

$$v_H v^2 |\mathcal{C}|\, w^2 \sum_{\lambda,\bar{n}} \Big\| \sum_p A_{\lambda,\bar{n},p} \Big\|^2 \;\le\; \Re \sum_{p,q} R_{pq}\, \mathrm{fiberPairing}_Y(p,q;e). \tag{9.6}$$

The measured entry used here, for $b = [n_T^{-1}a]$, is

$$V_p(l_\lambda)_{ab}\, U_{p,\lambda} = \kappa_p v\, A_{\lambda,\bar{n},p}.$$

Indeed, put $x = (h_c^{-1} w_p \gamma h_{c'})_T$. In the finite orbital integral the supporting $K(g)$-cosets, after right translation by b, are indexed by pairs $u, u' \in U/(U \cap K(g))$ with $u^{-1}xu' \in n_T K(g)$. Each contributes $v\chi(u)\overline{\chi(u')}$. The global unfolding uses the same pairs: unipotent approximation in the two class lattices supplies representatives integral off T, unique modulo the respective class unipotents. For tuples integral with integral inverse off T, the unfolded contributing representatives are integral there. The congruence at T is then the entire reduction condition. Translation in the infinite orbital integral contributes the same phase by the product character, trivial on the rational unipotent group. This gives the displayed identity with one factor of v. Finite orbital entry, class representatives, phases, and measured unfolding.

Write $d_{ab}(l) = (V_p(l)_{ab})_p$ and $\mathcal{J}(l) = \sum_{ab} d_{ab}(l)^* R\, d_{ab}(l)$. Every summand is nonnegative. For fixed λ, the $(a, [n_T^{-1}a])$ are distinct as $(a, \bar{n})$ vary by normality of $K(g)$ in Y and the definition of reduction. At such an entry the quadratic form is $\|\sum_p V_p(l_\lambda)_{ab}\rho_p U_{p,\lambda}\|^2 = v^2 w^2 \|\sum_p A_{\lambda,\bar{n},p}\|^2$. Summing the selected entries gives the inner bound by $\mathcal{J}(l_\lambda)$, with the factor $|\mathcal{C}|$.

Choose l_0 in the nonempty fibre. In H, the cosets $(H \cap K(g))l_\lambda l_0^{-1}$ are disjoint. Indeed coincidence puts the quotient of the two class translates in the principal level, by integrality off T.

The fixed column and t determine γz; the rational quotient therefore lies in the definite stabilizer. The double cosets identify the right classes, and freeness of the principal level identifies the two rational elements. On each such coset $\mathcal{J}(hl_0) = \mathcal{J}(l_\lambda)$ by left $H \cap K(g)$-invariance of the tensor. Right Haar measure gives each coset mass v_H. Positivity bounds the sum of their integrals by $\int_H \mathcal{J}(hl_0)\,dh$, the right side of (9.6) by definition. The empty family contributes zero. This proves the fixed-fibre inequality given the displayed common archimedean Gram. Summation over rational columns requires the separate weighted Siegel estimate. Principal-level separation, fixed-fibre comparison, Haar integral on the fibre.

The common Gram on the fixed fibre. Suppose p_a is a unit off T for $a < \varrho$ and $|\Phi_v(t)| > N$ at every archimedean place, as in the top-stage sum. Put $p_\varrho = t$, $s_{va} = M_v^{2(\varrho-a)}/N$, $x_{va} = s_{va}\Phi_v(p_a)$, $c_v = x_{v\varrho} = \Phi_v(t)/N$, and $d_a = k - 1 + 2(\varrho - a)$. Conjugation by the frame scales takes $(w_p\gamma)_v$ to $w_{x_v}\gamma_v$, with last frame entry c_v for γ_v. Rescaled Weyl element, rescaled last entry.

At v use the truncated vector F, with unitary value representation τ on $\mathbb{C}^{I_v}$. Write $O(g) = \int F(y)F(g^{-1}y)^*\,dy$ in the prescribed Haar measure. Its entries are absolutely integrable: for the tuples here, simultaneous nonvanishing puts y in a bounded closed matrix ball, and F is bounded. Let $a(c_v)$ be the standard boost with last frame entry c_v. In $X_0 \oplus \mathbb{C}z$ the z-column of γ_v has last coordinate $-c_v$ and X_0-part of squared length $|c_v|^2 - 1$. An $h \in U(X_0)$ moves the boost column to this column, and the remaining isometry fixes z. Thus $\gamma_v = h\,a(c_v)h'$ with $h' \in U(X_0)$. The Weyl elements commute with h. Substituting $y = hy'$ and using $F(hy') = \tau(h)F(y')$ gives, for every tuple on this relation,

$$O(w_{x_v}\gamma_v) = \tau(h)\,O(w_{x_v}a(c_v))\,\tau(h').$$

Both multipliers are unitary and independent of p, so they preserve all Hilbert–Schmidt inner products. This proves equality of the cross terms as well as the diagonal terms. Boost decomposition, compact overlap, local Gram transport.

Set

$$r_{vp} = \frac{\sqrt{c_G}}{|I_v|}(|c_v|^2 - 1)^{(k-1)/2}\prod_{a<\varrho}|x_{va}|^{d_a}, \qquad \rho_p = \prod_v r_{vp}.$$

The top local form $J_{v,n}(x_v(p), x_v(q))$ is the Gram of $r_{vp}O(w_{x_v}a(c_v))$ in these coordinates. The cut vector and Haar measure at infinity are products, so Fubini identifies $U_{p,\lambda}$ with the tensor product of the actual $O(w_{x_v}\gamma_v)$. Hence $R_{pq} = \prod_v J_{v,n}(x_v(p), x_v(q)) = \langle \rho_p U_{p,\lambda}, \rho_q U_{q,\lambda}\rangle$ for each λ. Finally, $\kappa_p = \prod_{a<\varrho}|p_a|_T^{d_a/2}$. The product formula gives $|p_a|_T^{1/2}\prod_v|\Phi_v(p_a)| = 1$, so the common weight is

$$w = \rho_p\kappa_p = \underbrace{\left(\prod_v \frac{\sqrt{c_G}}{|I_v|}\prod_{a=0}^{\varrho} s_{va}^{d_a}\right)}_{r=\text{normalizingConst}(k-1)}\prod_v(|\Phi_v(t)|^2 - N^2)^{(k-1)/2} > 0.$$

The hypotheses on the local model used here are the unitary equivariance, cut support, and the choice of top form just given. Construction and lower-stage bounds of that system are separate inputs. Choice of top form, filling's cut system, weight cancellation, product Gram.

Summing the rational columns. Keep the filling datum of (9.6), with T stable under conjugation, N^{-1} integral off T, and the lattice basis invertible there over the integers. Set

$$\beta = (t\bar{t} - N^2)/N, \qquad G_t = \prod_{\sigma\in\Phi}(|\sigma t|^2 - N^2)^{k-1}, \qquad e_t(y) = y - (t/N)z$$

in the prescribed lattice basis. Assume $|\sigma t| > N$ for all σ. For a rational column $u = (\gamma z)_{X_0}$ the norm identity is $u^* u = \beta$. The matrix $h_c^{-1} \gamma h_{c'}$ is integral off T. Since $h_{c'}$ fixes z and the lattice basis and its inverse are integral there, its z-column has integral definite part $x = h_c^{-1} u$ there. Its local point is $e_t(x_T)$. Count each pair (c, u) once. Let $\Psi \geq 0$ be the compactly supported, locally constant extension of the weighted fibre pairing, invariant under $H \cap K(g)$, as on p. 92. Such an extension can be chosen nonnegative since the orbit value is an integral of squared norms: take the positive part of the real extension before averaging over $H \cap K(g)$.

The test on finite adelic vectors is $f_t(y) = \mathbf{1}_{(\widehat{\mathcal{O}}_E^T)^k}(y^T) \Psi(e_t(y_T))$, where the indicator tests each coordinate outside T. It has compact support because e_t is a closed affine embedding. The source level preserves integrality off T and acts through $H \cap K(g)$ at T, hence preserves f_t. The source level is free, so its mass is its class number n_X. Positivity bounds the sum of the selected columns by the full weighted representation sum. For $k \geq 3$, the definite one-vector Siegel formula gives the latter as

$$n_X |d_E|^{1/4} \prod_{\sigma \in \Phi} \frac{\pi^k (\sigma\beta)^{k-1}}{(k-1)!} \operatorname{dens}(f_t, \beta).$$

Here the finite phase is $\psi_f(s(y^* y - \beta)/2)$, the self-adjoint codifferent has volume one, and $V = \operatorname{vol}(\widehat{\mathcal{O}}_E) = 2^f / \sqrt{|d_E|}$, $f = |\Phi|$. Weil, Acta Math. 113, Thm. 5, one-vector coefficient in these measures.

For the density bound retain the entire T-test. The two factors in the Fourier integral are obtained by splitting vectors at T and sending a self-adjoint frequency s_T to the skew frequency $-\delta_E s_T$. The integral vector and skew frequency boxes at T both have volume one. With the prescribed character, integration over increasing frequency boxes is the average of the norm-fibre integral over shrinking slabs. Continuity of that integral at $\delta\beta_T$, with δ the prescribed skew unit, gives the T-factor $I_t = \int \Psi(e_t(y))\, d\nu_{\delta\beta_T}(y)$. Phase and Haar factorization, slab inversion under integrability.

The factor away from T is bounded in absolute value by the integral of the absolute standard-lattice Gauss sum over all self-adjoint finite frequencies. For frequency s, let I be the conductor ideal of $a \mapsto \psi_f(sa)$ on $\widehat{\mathcal{O}}_E$ and $q = [\widehat{\mathcal{O}}_E : I]$. Squaring and translating the inner integral kills every difference outside I^k by character orthogonality. Thus the Gauss sum is at most $V^k q^{-k/2}$, uniformly in the constant phase. Frequencies of conductor I occupy at most q codifferent cosets, one per additive character of the quotient. Hence this factor is at most $V^k \zeta_E(m)$, $m = 3\varrho + 4$, since $k/2 \geq m + 1$. In the prescribed range $[E : \mathbb{Q}] \leq 2^{2\varrho}$, the Euler product and $\zeta(m) - 1 \leq 2^{1-m}$ give $\log \zeta_E(m) \leq 2^{-\varrho-3} \leq \log(1 + 2^{-\varrho})$. The datum gives $2^{-\varrho} \leq b^{-6}$. Refine a compact step test at T into finitely many cosets $a + mR^k$ with one rational scalar $m \neq 0$. Writing $y = a + mu$ changes the quadratic frequency to $m^2 s$. The cross term is linear in u, so differencing adds only a scalar of modulus one at each difference. The same conductor bound is integrable after this fixed rescaling, as required for slab inversion. Gauss estimate by conductor, off-T and sphere bounds.

Write $\pi_T \bar{\pi}_T = N$ for the chosen level idele at T. Since $|\pi_T|^2 = |N|_T$ and N is a unit off T, the product formula for N gives $|\pi_T| = N^{-f}$, and $\prod_\sigma (\sigma\beta)^{k-1} = N^{-f(k-1)} G_t$. Thus

$$\sum_{(c,u)} \Psi(e_t((h_c^{-1} u)_T)) \leq (1 + b^{-6})\, n_X C_{\mathrm{S}}\, G_t\, |\pi_T|^{-1} I_t, \qquad C_{\mathrm{S}} = |d_E|^{1/4} V^k \left(\frac{\pi^k}{(k-1)!\, N^k} \right)^f.$$

All tuples have last entry t, so $|\pi_T|^{-1} I_t = \Re \sum_{p,q} R_{pq} J_{\mathrm{last}}(p, q)$ for the unnormalized terminal density by its sphere definition. Product formula, weighted column sum.

For the mass comparison put $D = |\bar{H}||\bar{U}|^2$, with bars denoting the integral reductions at M. Integral H has mass one and reduction kernel $H \cap K(g)$, so $|\bar{H}| v_H = 1$. The stage volumes

multiply to $m_U = (M^f)^{-\varrho(2k+2\varrho+1)} = |\bar{U}|^{-1}$. The cosets in Y have mass v, giving $\mathrm{mass}_{\mathrm{rel}}(Y) = \mu_G(Y)v/m_U^2 = D\,v_H v^2|\mathcal{C}|$. Finally $C_{\mathrm{scaled}} = Dn_X C_{\mathrm{S}}$, since multiplication of the reduced unipotent and definite groups is bijective onto the reduced Bessel subgroup. The common Gram calculation gives $w = r\sqrt{G_t}$. Summing (9.6) and cancelling $G_t > 0$ gives

$$r^2 \sum_i \Big\| \sum_p A_{i,p} \Big\|^2 \le (1 + b^{-6}) C_{\mathrm{scaled}}\, \Re \sum_{p,q} R_{pq} \frac{J_{\mathrm{last}}(p,q)}{\mathrm{mass}_{\mathrm{rel}}(Y)}.$$

The top sum for several t's follows by addition. Each inner sum still pairs tuples with the same last entry. Floor volumes, mass comparison, terminal cancellation.

The truncated integral at a single infinite place. Write $m = \varrho = n - 1$, d for the number of value coordinates, $e = e_v$ for the formal degree, W for the full vector and $F = W^\flat$. Put $\rho = b^{6n+2}$, $t = b^{6r}$, and

$$C = \{x : \max_{ij} |(x - I)_{ij}| \le (t\rho^{11n})^{99n}\}.$$

Let $c_F = (\mathrm{vol}\, U(k)/\mathrm{vol}\, U(r))/(\lambda d)^2$, where $d\mu = \lambda\, dg$ is the successive Haar measure. The same quotient measure comparison gives the factor λ in each copy of $K_F(x) = \langle F, R(x)F\rangle_{U\backslash G}$; the two bottom correlations are conjugate. Thus the normalized absolute integrand is $c_F\lambda^3|K_F|^2 dg$. The stated Haar comparison gives $c_F\lambda^3 d^2 \le 1$. Haar and quotient measures, bottom correlations.

The Schur identity already proved gives $\|K_W\|_2 = d/\sqrt{e}$. The existing decay majorant gives $\|F - W\|_2 \le b^{-r^3 f}\sqrt{d}$: on the support of the difference the length ratio exceeds b. One can therefore extract $3r^3 f$ powers from the majorant, whose squared integral is at most $b^{400r^2} d$. Since $\|F\|_2 \le \|W\|_2 = \sqrt{d}$, unitarity gives $|K_F - K_W| \le 2d\, b^{-r^3 f}$. On C, the operator norm is at most b^{700nr}: bound it by $1 + r^2(t\rho^{11n})^{99n}$ and use $\rho \le b^{7n}$, $r \ge 10^9 n^3$. The preceding Haar packing bound gives $e\,\mathrm{vol}(C) \le b^{6300nr^3 + 99r^2}$. As $f \ge 4000n$, the triangle inequality in $L^2(C)$ bounds the normalized contribution by $(1 + b^{-12})^2/e$. matrix ball, exponent budget.

A lower bound for depth outside the archimedean box. Suppose $\langle F(y), F(yx)\rangle \ne 0$. Choose unipotents u, v with $P = uy$, $Q = vyx$ of operator norm at most ρ. Their inverses have the same norms, since the Gram matrix J is a Hermitian involution. Write $X_a = \langle e_0, \dots, e_{a-1}\rangle$. The real depth h_a is the infimum of $\|y^{-1}z\|^2 + \|(yx)^{-1}z'\|^2$ subject to $z' - z \in X_a$, $z \perp_J X_a$, and $\langle e_a, z\rangle_J = 1$. The possible readings of vectors of squared length sum less than one form a disc of area π/h_a by homogeneity, so the depth in the stage weight is h_a/π. two lifts and disc area.

If every $h_a < T^2$ with $T \ge \rho$, choose witnesses z_a, z_a' of cost less than T^2. In the Witt decomposition $X \oplus V \oplus X^\vee$, let R, S have columns uz_a, vz_a'. Their norms are at most $\sqrt{m}\,\rho T \le T^3$. Their dual blocks R_3, S_3 are lower unitriangular, since the unipotents preserve the flag and act trivially on successive quotients. Write

$$w = uv^{-1} = \begin{pmatrix} A & B & C' \\ 0 & I & D \\ 0 & 0 & E \end{pmatrix}, \qquad wS = R + (K, 0, 0)^t.$$

Here K is strictly upper triangular, column a belonging to X_a. Isometry gives

$$R_3^* K = \mathrm{strictUpper}(S^* J S - R^* J R), \qquad E = R_3 S_3^{-1},$$
$$D = (R_2 - S_2)S_3^{-1}, \qquad A = E^{-*}, \qquad B = -AD^* J_V.$$

Indeed $R_3^* K$ is strictly upper, so it is recovered from its sum with its adjoint. Finally $C' = (R_1 + K - AS_1 - BS_2)S_3^{-1}$. Both triangular inverses have norm at most T^{4m} by the finite geometric

series. Strict upper truncation increases the operator norm by at most $\sqrt{m}$, by the Frobenius norm. The displayed products then give $\|K\| \le T^{4m+8}$ and $\max(\|A\|, \|B\|, \|C'\|, \|D\|, \|E\|) \le T^{12m+16}$. Consequently $\|x - I\| = \|P^{-1} w Q - I\| \le T^{99n}$. This proves the required contrapositive at the initial stage. real depth criterion.

The initial sum outside the archimedean box. Write $\mathcal{M}_0$ for the product of the two absolute pivots and $\mathcal{P}_0$ for their weighted square product in the gauge of loss ρ^2. They use the kernel $\langle F(y), F(y') \rangle$ twice. Let H be the least dyadic number ≥ 1 bounding every entry of $x - I$, and S_H its shell. For $x \in S_H \setminus C$ the preceding argument applied with $T = t\rho^{11n}$ and $T = (H/4)^{1/(99n)}$ shows that at every active y

$$\max_a h_a/\pi \ge D_H := \max(t^2, H^{1/(99n)}).$$

Indeed $H/2 < \max_{ij} |(x - I)_{ij}|$ and $H^{1/(99n)} > t\rho^{11n} \ge \pi\, 4^{2/(99n)}$. The deep vector gives the first bound below. The second follows by integrating one frame in the original measures at the successive stages.

$$\begin{aligned} \mathcal{M}_0(x) &\le D_H^{-s} \sqrt{\mathcal{P}_0(x)} && (x \in S_H \setminus C), \\ \int_{S_H} \sqrt{\mathcal{P}_0(x)}\, d\mu(x) &\le U_F\, \rho^{2A_0} H^{50000n^2}, && A_0 = 1500000n^3 + 25000n^2 + 1. \end{aligned} \tag{9.7}$$

Here $U_F = \pi^{2(k-5)m} \iint |\langle F(y), F(z) \rangle|^2$ with the stage quotient measures in both variables. In the first inequality the squared pivot carries D_H^{-s}, and the majorant is exactly this squared pivot. In the second, S_H lies in the position ball of the gauge of loss $(\rho H)^2$. identical pivots, one-frame Haar estimate.

The depth integral for the pointwise bound uses positive powers. For every $a < m, x, g$,

$$\int_{\{\sigma:\, \sigma g \text{ near at } a+1\}} \mathrm{depth}_a(\sigma g, x)^s\, d\theta_a(\sigma) \le (\rho^2)^{10000n}$$

Indeed the stage quotient identity for the stabilizer of the preceding flag reduces to g in the box and sends θ_a under $\sigma \mapsto v = \sigma^{-1} e_a$ to the prescribed quadric measure. A contributing v has coordinates bounded by ρ^7, or ρ^{11} in the basis adapted to the flag. Write u for its free positive coordinates. In their space of dimension $s + 1 + 3n$ choose a subspace of dimension $s + 1$ on which both g^{-1} and $(gx)^{-1}$ take values in the $+1$ eigenspace of J. Each negative projection imposes at most n equations. Let u_1 be the orthogonal projection of u on this subspace. For $u_1 \ne 0$, $w = u_1/\|u_1\|^2$ pairs to one with v. The vector σw is orthogonal to the earlier frame, and the two lengths of this lift are both $\|w\|$; hence the stage depth is at most $2/(\pi \|u_1\|^2)$. floor quotient and box, common positive space.

Put $\Lambda = \rho^2$. In a unitary basis adapted to this subspace, $\|u\|^2 \le r\rho^{22} \le \rho^{23}$; the other $3n$ coordinates lie in discs of radius ρ^{12}. Polar coordinates in $s + 1$ complex dimensions give

$$\int_{\|u_1\|^2 \le \rho^{23}} (36/\|u_1\|^2)^s\, du_1 = \pi (36\pi)^s\, \rho^{23}/s! \le 4e^{144} \rho^{23}.$$

With the complementary discs this contributes at most $\Lambda^{99n \cdot 12}$. There are at most $99n$ remaining coordinates in the basis adapted to the flag, including an orthogonal block of two positive coordinates. Condition on the others and solve the quadric in that block. Its squared-radius density for the imaginary coordinate $\beta = i\|z\|^2/2$ is $2\pi^2 \|z\|^2$; in the box its fibre mass is at most $4\pi^2 \rho^{22}$. Each other disc contributes $\pi\rho^{22}$. Together these factors contribute at most $\Lambda^{99n \cdot 13}$, and $99(12 + 13) < 10000$. Iterated Cauchy–Schwarz against the inverse depth weights now gives the squared-pivot bound by summing over the first deep stage. Its factors are at most $\Lambda^{10000nm}$ times the square of the number of stages, absorbed in $\Lambda^{10^6 n^3}$. two-coordinate disintegration, weighted pivot estimate.

The depth weight in the shell integral. Put $L = \rho H$, $\Lambda = L^2$, $\mathcal{B} = \{x : \max_{ij} |(x-I)_{ij}| \le L\}$, and let $v_a(g,z) = \mathrm{depth}_a(g, g^{-1}z)^{-1}$ be the area of the stable readings at the absolute points g, z. For g near at $j \le m$, with the ordered successive Haar measure,

$$\int_{U_{<j}} \mathbf{1}_{\mathcal{B}}(g^{-1}\nu y) \prod_{a<j} v_a(g, \nu y)^s \, d\nu \le \Lambda^{100nj}.$$

Initially take $\|g\| \le \rho$, $x_0 = g^{-1}y \in \mathcal{B}$. For a shear $h = h(\omega, it)$ on stage a, suppose $x_1 = g^{-1}\nu h y \in \mathcal{B}$ for some $\nu \in U_{<a}$. Since $J = J^{-1} = J^*$, inverses have equal operator norm. Hence

$$\|h^{-1}\nu^{-1} - I\| = \|g(x_0 - x_1)x_1^{-1}g^{-1}\| \le \rho^2(2rL)(1+rL) \le L^8 \qquad (4 \le L,\ 8r \le \rho \le L).$$

Modulo $X_a = \langle e_i : i < a\rangle$, the e_a^*-column of this difference is $-\omega + (b(\omega) - it)e_a$, since ν acts trivially on $X_a^{\perp}/X_a$. Thus $\|\omega\| \le L^8$, and the allowed t's at fixed ω occupy length at most $2L^8$. The original measure is $d\omega\, dt$ in these coordinates. shear measure, skew measure, two positions.

Choose an $(s+1)$-plane S in the free positive coordinates X' on which the negative projections of g^{-1} and y^{-1} vanish. This contributes at most $2n$ linear conditions and $\dim X' = s + 1 + 3n$. For $o \in S$, J-isometry gives $\|g^{-1}o\| = \|y^{-1}o\| = \|o\|$. If $c = \langle e_a, \zeta\rangle_J$ is a stable reading at g, hy, its lifts ζ, ζ' agree modulo X_a and satisfy $\|g^{-1}\zeta\|^2 + \|(hy)^{-1}\zeta'\|^2 < 1$. Since $ho = o - \langle \omega, o\rangle_J e_a$,

$$|\bar{c}\langle \omega, o\rangle_J| = |\langle \zeta, o\rangle_J - \langle h^{-1}\zeta', o\rangle_J| \le \sqrt{2}\,\|o\|.$$

Here Cauchy–Schwarz uses the two lifts together. With $p = \mathrm{pr}_S \omega$, the area of possible c is at most $2\pi/\|p\|^2$ for $p \ne 0$. The free coordinates are standard positive axes, so the orthogonal splitting at S preserves Lebesgue measure. Writing $R = L^{16}$, polar integration gives

$$\int_{\|p\|^2 \le R} (2\pi/\|p\|^2)^s \, dp = \frac{\pi(2\pi^2)^s}{s!} R \le \pi e^{2\pi^2} R.$$

At most $r - (s+1) = 5n + 4 \le 9n$ complex coordinates remain, each costing at most πR. Thus the one-shear cost is at most $2L^8\, \pi e^{2\pi^2} R(\pi R)^{9n} \le L^{153n+41} \le \Lambda^{100n}$, using $\pi e^{2\pi^2} \le 4^{16}$. reading identity, axes and radial measure, free dimension.

For arbitrary y, right translation by a contributing element of $U_{<a+1}$ arranges $g^{-1}y \in \mathcal{B}$ whenever the integral is positive. In the ordered convolution νh, $\nu \in U_{<a}$ preserves the second lift modulo X_a; thus $v_a(g, \nu h y) = v_a(g, hy)$. Integrate the earlier weights first, uniformly in hy, then apply the one-shear estimate. Induction starts at $j = 0$. For g near at j, move it into the box by γn_0, with $n_0 \in U_{<j}$ and γ fixing the first j hyperbolic pairs. Transport both lifts; translation by n_0 and conjugation by γ preserve the ordered Haar measure (the latter acts by isometries of complex determinant of modulus one on each earlier vector part and fixes the centre). Here the stage modulus is 1. iteration and transport.

The Heisenberg integral for the cut Bessel vector. Here $F = W^{\flat} = \mathrm{besselCut}_b(W)$, where W is the normalized Bessel average at the chosen embedding. The cut retains the level-one unipotent character and is supported on $U\{g : \|g\| \le \rho\}$. Define $K(g,y) = \langle F(g), F(y)\rangle$. Write $P_j(g,y)$ for its simultaneous left average over the stages from j upwards, and $P_{m+2} = K$. the chosen cut, character and support. Set $D = k - 5$. Size j means coordinate modulus at most L^j. Put

$$B_a(g,y) = \mathbf{1}_{\{\int_{U_{<a}} \mathbf{1}_{\mathcal{B}}(g^{-1}vy)\, dv > 0\}} |P_{a+1}(g,y)|^2,$$

and let N_a be the current Heisenberg group. For $a < m$,

$$\int_{N_a} B_a(g, ny)\, dn \ \le\ \pi^{2D}\Lambda^{4900n} \int B_{a+1}(\sigma g, \sigma y)\, d\theta_{a+1}(\sigma).$$

Indeed choose one contributing n_0 and put $x_n = g^{-1}v_n n y \in \mathcal{B},\ v_n \in U_{<a}$, for each contributing n. Because $P_{a+1}(g, n_0 y) \neq 0$, some simultaneous translate of g from the higher stages belongs to the support of F. Thus $B = t\nu g \in \mathrm{box}(\rho)$, where $\nu \in U_{<a}$ and $t = s_0 r_0$ with $s_0 \in G_{a+1},\ r_0 \in N_a$, by removing the preceding Heisenberg columns. As in the common-center calculation, $(Bx_{n_0})(Bx_n)^{-1} = v\, t n_0 n^{-1} t^{-1}$ for some $v \in U_{<a}$. The e_a^*-columns agree modulo the preceding isotropic space. The difference from the identity on the left is $B(x_{n_0} - x_n)x_n^{-1}B^{-1}$. Since $\|B^{\pm 1}\| \leq \rho,\ \|x_n^{\pm 1}\| \leq 1 + rL,\ L \geq 8r$, its entries have size at most 17. On the support of $P_{a+2}(\sigma g, \sigma y)$, nearness at $a+2$ likewise gives $B' = \eta\sigma s_0^{-1} B \in \mathrm{box}(\rho)$ with η unitriangular on the flag through f_{a+1}. Removing the preceding coordinates from the f_{a+1}-column of $B(B')^{-1}$ gives $w = s_0 \sigma^{-1} f_{a+1}$, of size at most 7. center of the cut, orbit support.

Write $n = r_0^{-1} s_0^{-1} n(\omega, i\tau) s_0 r_0 n_0$. Translations preserve Heisenberg Haar measure, and s_0 acts on the vector coordinates with determinant of modulus one, fixing the center. Hence $dn = d\omega\, d\tau$, with planar Lebesgue measure on each complex coordinate. The column just bounded gives $|\omega_i| \leq L^{17}$ and its central entry is $c(\omega) - i\tau$. At fixed ω the permitted τ's have length at most $2L^{17} \leq \Lambda^{19}$. Unfolding the next stage and using the unipotent character of F gives the phase $\psi(-h(\omega, w))$ times a unit phase in w from the translation. Indeed the only adjacent entry is the pairing of the vector coordinate with the next frame vector and the level is 1. Under $\sigma \mapsto w$ the measure at that stage pushes forward to the prescribed quadric measure $d\mu_Q(w)$. Indeed, below the last orbit the chart substitutes $u \mapsto zu$ on $\mathbb{C}^{r_{a+1}}$. Its real Jacobian $|z|^{2r_{a+1}}$ is the stage density. Translation and reflection of the skew coordinate preserve its length measure. On the last orbit, a vector with negative coordinate z has positive part $\sqrt{|z|^2 - 1}\, he$, where e is a fixed unit vector and h has probability Haar measure on $U(k)$. In the value parameter $\langle v, v\rangle_J/2$, polar coordinates give density $2\pi^k(|z|^2 - 1)_+^{k-1}/(k-1)!$ against $d^2 z$, exactly the last stage density. The isometry s_0 preserves the form and ambient volume, hence the continuous disintegration. flat chart, last orbit. The phased amplitude pushes to $f\, d\mu_Q$, with $\int |f|^2 d\mu_Q \leq \int |P_{a+2}|^2 d\theta_{a+1}$ by Cauchy–Schwarz against functions of w. orbit and character, Haar substitution.

Use the volume-preserving basis adapted to the flag, which enlarges either size bound by at most 4. Its $k-1$ orthonormal axes include two reserved positive pairs, leaving $D = (k-1) - 4$ free axes. Split off the free coordinates $A, \xi \in \mathbb{C}^D$ on the input and output, with complements z, d'. There are at most $99n$ complementary complex coordinates. For fixed A the restricted input fibre has mass $M_i \leq \Lambda^{99n(7+6)}$. In fact each ordinary coordinate of size $\ell = 7 + 4$ contributes at most $\pi L^{2\ell} \leq \Lambda^{\ell+1}$. On a reserved positive $\mathbb{C}^2$ the value parameter is $t' = \|z'\|^2/2$ with Haar dt'. Differentiating $\pi^2(2t')^2/2$ gives the full fibre density $4\pi^2 t'$, at most $4\pi^2 L^{2\ell} \leq \Lambda^{2(\ell+1)}$ on the restricted block. This bound includes the zero fibre since the density tends to zero. For continuous compactly supported tests, polar integration on the sphere gives a continuous density at every value. Integrating the other variables on compact sets therefore gives the prescribed disintegration at the orbit value. The complementary output coordinates of size $17+4$ have volume $M_o \leq \Lambda^{99n(17+6)}$. For fixed d', set $G_{d'}(A) = \int \psi(-h(d', z)) f(A, z)\, d\mu_A(z)$, where $d\mu_Q = dA\, d\mu_A$. Then $\|G_{d'}\|_2^2 \leq M_i \|f\|_2^2$. Since $\psi(z) = \exp(2i \operatorname{Im} z)$, the ordinary Fourier frequency for the free variables is a real orthogonal transform of ξ/π. Plancherel on the signed section gives

$$\int_{\mathbb{C}^D} \left| \int_{\mathbb{C}^D} \psi(-\xi^* A) G_{d'}(A)\, dA \right|^2 d\xi = \pi^{2D} \|G_{d'}\|_2^2.$$

Integrating d' and the skew slice contributes at most $M_o \Lambda^{19}$; the total loss exponent is $19 + 99n(13 + 23) \leq 4900n$. Finally, if the next close-position mass is zero, the present cut vanishes almost everywhere by $U_{<a+1} = U_{<a} N_a$ and Fubini. If it is positive, simultaneous translation by $\sigma \in G_{a+1}$ preserves that mass. This restores the next cut indicator and proves the assertion. original measures, sphere density, free coordinates.

The integral over a shell. Put $c_D = \Lambda^{100n}$, $c_P = \pi^{2D}\Lambda^{4900n}$, where $\Lambda = (\rho H)^2$, and recall $S_H \subset \mathcal{B}$. Write η_a for the Heisenberg measure, θ_a for the measure at that stage, $\eta_{<l} = \eta_0 * \cdots * \eta_{l-1}$, and $\Theta_l = \theta_l * \cdots * \theta_0$. The terminal measures are $\eta_m = \delta_1$ and θ_{m+1} the probability Haar measure of $U(k)$. Thus $Q = \Theta_{m+1}$ represents the Bessel quotient and $\mu = \eta_{<l} * R_l$, with $R_{m+1} = Q$ and $R_l = \eta_l * R_{l+1}$. These factorizations follow by moving each earlier unipotent Haar measure past the later successive flag subgroups, which normalize it with modulus one. floor measures.

For $l < m$ the preceding section estimate contributes c_P. At $l = m$ the same inequality contributes one by Cauchy–Schwarz in θ_{m+1}. After integrating against $R_{l+1}(dy)$, one can remove σ from the second argument. Indeed $B_{l+1}(\sigma g, \cdot)$ is invariant under $U_{<l+1}$ by unitarity of the Bessel character, and σ preserves the quotient integral by left invariance of μ and modulus-one conjugation on $U_{<l+1}$. Integrating in g against Θ_l now produces Θ_{l+1}. At the end $B_{m+1} \le |K|^2$. Consequently

$$\iint B_l(g,y)\, dR_l(y)\, d\Theta_l(g) \le c_P^{m-l}\Pi_F, \qquad \Pi_F = \iint |K(g,y)|^2\, dQ(y)\, dQ(g).$$

cutoff and quotient induction.

Expanding the weighted square integral at the empty tuple gives

$$N(x) = \sum_{l=0}^{m} \int \left(\prod_{a<l} v_a(g, gx)^s \right) |P_{l+1}(g, gx)|^2\, d\Theta_l(g).$$

In its l-th summand, substitute $x = g^{-1}\nu y$ using $\mu = \eta_{<l} * R_l$. The factor $|P_{l+1}(g, \nu y)|^2$ is independent of ν. Where it is positive, g is near at l, so the direct depth-weight integral on $\mathcal{B}$ contributes at most c_D^l. Where the ball has zero mass the summand integrates to zero. Thus the cost is $c_D^l c_P^{m-l}\Pi_F$ for the l-th summand. Since $c_D \le c_P$, the sum contributes at most $n c_P^m \Pi_F$. At the empty tuple both kernels equal K. The tuple modulus is one, and the prefactor of the product of the two moments is $((\rho^2)^{10^6 n^3})^3$. Hence $\sqrt{\mathcal{P}_0} = (\rho^2)^{1500000n^3} N$. Therefore

$$\int_{S_H} \sqrt{\mathcal{P}_0}\, d\mu \le n(\rho^2)^{1500000n^3} (\rho H)^{9800nm} \pi^{2(k-5)m} \Pi_F \le \rho^{2A_0} H^{50000n^2} U_F.$$

Here $n \le \Lambda^n$ and $n + 4900nm \le 25000n^2$. Both factors of Q belong to this same kernel. In the original Bessel quotient measure each contributes precisely the Haar ratio λ when evaluating the invariant integrand $|K|^2$. empty-head product, sum over floors, weighted summands, quotient normalization.

Summing the remote shells. Schur's lemma for the compact type gives $\int F \otimes F^* \le I$ in the original quotient measure (its trace is at most d); hence $c_F U_F \le \Theta = \pi^{(2m+1)k} \operatorname{vol} U(k) / \operatorname{vol} U(r)$. The preceding inequalities now give

$$c_F \int_{C^c} \mathcal{M}_0\, d\mu \le \Theta \rho^{2A_0} t^{-s} \sum_{H=1,2,4,\dots} H^{50000n^2 - s/(198n)} \le 2\Theta \rho^{2A_0} t^{-s} \le b^{-12}/e.$$

Indeed $s \ge 10^{10} n^4$, $r = s + 5n + 5$, and the Weyl-dimension and compact-volume products give $e\Theta \le b^{4r^2}$. These imply $1 + 4r^2 + 14nA_0 + 12 \le 6rs$. Adding the near contribution gives $|J_{v,0}| \le (1 + b^{-10})/e$, since $(1 + b^{-12})^2 + b^{-12} \le 1 + b^{-10}$ for $b \ge 2$. degree and volume, bottom mass.

Fourier decay for the empty tuple. We retain the truncated vector, the measures at the successive stages, and the weighted moments of (9.7). Write

$$m = n - 1, \qquad D = k - 5, \qquad k = s + 3n + 6, \qquad \rho = b^{6n+2}, \qquad \Theta = \pi^{(2m+1)k} \frac{\operatorname{vol} U(k)}{\operatorname{vol} U(r)}.$$

The prescribed parameter bounds used below are

$$b \geq 2, \quad n \geq 10^8, \quad \rho \geq 8r, \quad s \geq 10^{10} n^4, \quad k \geq n^7.$$

As before, d is the dimension of the value representation, $d\mu = \lambda\, dg$, and

$$c_F = \frac{\operatorname{vol} U(k)}{\operatorname{vol} U(r)(\lambda d)^2}.$$

Let $\mathcal{J}_0(\nu)$ be the normalized initial stage form with weight $\psi(\nu \operatorname{Gram}_{00})$, where $\psi(z) = \exp(2i \operatorname{Im} z)$. Its Fourier estimate is

$$d\,|\mathcal{J}_0(\nu)| \leq \Theta b^{2000 n^9} |\nu|^{-(k-n^7)} \min(1, |\nu|^{-1})^6, \qquad \nu \neq 0. \tag{9.8}$$

This is the case $j = 0$ of (9.13), proved below for every tuple length. The following Plancherel estimate and Haar normalization enter that proof.

The Fourier inequality on two quadrics. We first record the elementary measure argument, including the reason for reserving two different positive blocks. Suppose a Hermitian coordinate space has D orthonormal coordinates x, two disjoint orthonormal blocks z, w, each of complex dimension two, and the remaining coordinates. All these positive coordinates are orthogonal to the remaining ones. Let the input measure be disintegrated in the block z:

$$d\mu_{\mathrm{in}}(y) = da\, dx\, d\kappa_{a,x}(z).$$

Here a denotes every coordinate outside x, z, including the coordinates in the other reserved block. Disintegrate the output measure in that other block:

$$d\mu_{\mathrm{out}}(v) = db\, d\xi\, d\kappa'_{b,\xi}(w).$$

On the prescribed rectangular boxes suppose the a- and b-coordinate volumes are at most A, A', uniformly in the free coordinates, and the two conditional block masses are at most M, M'. Then the operator

$$(T_\nu f)(v) = \int \psi(\nu h(v, y)) f(y)\, d\mu_{\mathrm{in}}(y)$$

from functions supported in the input box, restricted to the output box, satisfies

$$\int_{\mathrm{out}} |T_\nu f|^2\, d\mu_{\mathrm{out}} \leq AA'MM' \left(\frac{\pi^2}{|\nu|^2}\right)^D \int_{\mathrm{in}} |f|^2\, d\mu_{\mathrm{in}}. \tag{9.9}$$

Indeed the phase has the form

$$h(v, y) = \overline{\xi} \cdot x + \overline{w} \cdot a_w + \Phi_b(a, x, z),$$

where a_w is the part of a in the output block. For fixed b, put

$$G_b(a, x) = \int \psi(\nu \Phi_b(a, x, z)) f(a, x, z)\, d\kappa_{a,x}(z).$$

Cauchy–Schwarz in a bounds $|T_\nu f(v)|^2$ by

$$A \int \left| \int_{\mathbb{C}^D} \psi(\nu\overline{\xi} \cdot x) G_b(a,x)\, dx \right|^2 da.$$

This bound is independent of w; integration in its conditional measure contributes M'. Enlarge the remaining ξ-integral to all of $\mathbb{C}^D$ and apply Plancherel:

$$\int_{\mathbb{C}^D} \left| \int_{\mathbb{C}^D} \psi(\nu\overline{\xi} \cdot x) g(x)\, dx \right|^2 d\xi = \left(\frac{\pi^2}{|\nu|^2} \right)^D \int |g|^2.$$

The factor follows from ordinary real Fourier transform with the character $\exp(2\pi i\, u \cdot v)$: the real linear frequency change has determinant $(|\nu|/\pi)^{2D}$. Cauchy–Schwarz in z contributes M, and integration over b contributes A'. This proves (9.9). Truncation proves the same assertion for measurable square-integrable amplitudes. The two reserved blocks are disjoint precisely so that the w-term depends only on a; otherwise the first Cauchy–Schwarz step would not remove the output fibre variable.

These disintegrations apply to the quadric measure in the basis adapted to the flag. In a positive $\mathbb{C}^2$-block the value variable is $t = \|z\|^2/2$, and its fibre density is

$$\frac{d}{dt} \left(\frac{\pi^2 (2t)^2}{2} \right) = 4\pi^2 t \quad (t \geq 0),$$

with density zero for $t < 0$. On the box $|z_i| \leq L^\ell$ this is at most $4\pi^2 L^{2\ell}$. The continuous density is zero at $t = 0$, so no exceptional norm value is discarded. The basis adapted to the flag preserves volume and enlarges a size exponent by at most four. We take $L \geq \rho \geq 8r$. With $\Lambda = L^2$, each ordinary complementary coordinate contributes at most $\pi L^{2\ell} \leq \Lambda^{\ell+1}$, and a reserved pair contributes at most $\Lambda^{2(\ell+1)}$. There are at most $99n$ complementary coordinates. Thus input and output size exponents j, i, in the original basis, give the bound

$$AA'MM' \leq \Lambda^{99n(i+6)} \Lambda^{99n(j+6)}. \tag{9.10}$$

The same computation applies with ordinary volume on the output: one then disintegrates a product measure instead of a quadric.

Normalization of the quotient measure. Let

$$A_F = \int_{U \backslash G} F(y) F(y)^*\, dy.$$

Compact equivariance makes A_F a scalar on the irreducible value representation, and the cut norm gives $\operatorname{tr} A_F \leq d$. Thus $0 \leq A_F \leq I$ and $\operatorname{tr}(A_F^2) \leq d$. Both quotient measures in the definition of U_F are λ times the original measure. Hence

$$U_F = \pi^{2Dm} \lambda^2 \operatorname{tr}(A_F^2), \qquad d\, c_F U_F \leq \pi^{2Dm} \frac{\operatorname{vol} U(k)}{\operatorname{vol} U(r)}.$$

In particular,

$$d\, c_F U_F\, \pi^k \leq \Theta. \tag{9.11}$$

Archimedean Fourier decay with the earlier parameters fixed. Retain the normalization and the parameters on p. 101, in particular

$$m = n - 1, \quad D = k - 5 = s + 3n + 1, \quad K = k - n^7, \quad \rho = b^{6n+2} \geq 8r, \quad n \geq 10^8.$$

For a tuple $p = (p_0, p_1, \ldots)$ of nonzero complex numbers, write $P_l = \prod_{a<l} p_a$, and define

$$H_j(p) = \prod_{l=1}^{j} \left(\mathbf{1}_{\{|P_l| \leq b^{60n^2}\}} \max(1, |P_l|^{-1})^{n^6} \right), \qquad H_0 = 1. \tag{9.12}$$

Let $\mathcal{I}_j(p, q; \omega)$ be the stage form in the original measures at the successive stages, including its density for the prescribed parameters, and put $\mathcal{J}_j = c_F \mathcal{I}_j$. We prove both estimates

$$d\,|\mathcal{J}_j(p, q; \psi(\nu \operatorname{Gram}_{jj}))| \leq \Theta b^{2000n^9} |\nu|^{-K} \min(1, |\nu|^{-1})^6 H_j(p) H_j(q), \tag{9.13}$$

$$d\,|\mathcal{J}_{i+1}(p, q; 1)| \leq \Theta b^{2000n^9} |\nu|^{-K} \min(1, |\nu|^{-1})^6 H_{i+1}(p) H_{i+1}(q), \tag{9.14}$$

The wave bound holds for $0 \leq j \leq m$ and $\nu \neq 0$. The distinct-parameter bound holds for $i + 1 \leq m$, with $\nu = p_i^{-1} - q_i^{-1} \neq 0$. Here and below absolute values on complex scalars are their usual moduli. Thus the modulus of multiplication on complex additive Haar measure is $|z|^2$. This convention matters in all the parameter factors.

The lower measure and the two weighted moments. Choose $L \geq \rho$ such that

$$|P_l|^{-1},\ |Q_l|^{-1} \leq L \quad (0 \leq l \leq j), \qquad Q_l = \prod_{a<l} q_a,$$

and put $\Lambda = L^2$, $\mathcal{C} = \Lambda^{10^6 n^3}$. Size t means coordinate modulus at most L^t. Write $\mathcal{Q}_j$ for the flag group denoted by partialUnitriangularGroup(j), G_j for the group fixing the first j hyperbolic pairs, and $U_{<j}$ for the unipotent group of those pairs. Retain the measures η_a, θ_a, and put

$$\vartheta_j = \theta_{j-1} * \cdots * \theta_0, \qquad \mathfrak{m}_j = \eta_{<j} * \vartheta_j, \qquad \vartheta_0 = \delta_1.$$

These are the lower-point measures. The measure $\mathfrak{m}_j$ need not be replaced by Haar measure on G. For a G_j-left-invariant integrand its integral is invariant under left multiplication by $w_j(p)$. Indeed the remaining stages give Haar measure on G_j, their convolution with $\mathfrak{m}_j$ gives the full group Haar measure, and $w_j(p)$ centralizes G_j. Haar invariance and the quotient factorization therefore give precisely this restricted invariance. This is also the Weyl transport used in the earlier-weight argument on p. 87.

Set $w = w_j(p)$, $\delta = w_j(q)^{-1} w_j(p)$, and

$$K_1(g, y) = \langle F(g), F(y) \rangle, \qquad K_\delta(g, y) = \langle F(g), F(\delta y) \rangle.$$

For either kernel write P_l^δ for its correlation over the stages from l upwards; the superscript 1 denotes K_1. At a fixed position x, let

$$h_l(g) = \prod_{a<l} \operatorname{depth}_a(g, x)^{-s}, \qquad m_l(g) = \frac{\mathbf{1}_{\mathrm{near}_l}(g)}{h_l(g)}.$$

The depths are positive and finite; the stable readings form a nondegenerate bounded disc. Define the nonnegative sums

$$B_l^\delta(g) = \sum_{t=l}^{m} \int h_t(\sigma g) |P_{t+1}^\delta(\sigma g, \sigma g x)|^2 \, d(\theta_t * \cdots * \theta_l)(\sigma), \qquad N_l^\delta(x) = \int B_l^\delta(g) \, d\vartheta_l(g).$$

In particular

$$N_l^\delta(x) = \sum_{t=l}^{m} \int h_t(g) |P_{t+1}^\delta(g, gx)|^2 \, d\vartheta_{t+1}(g).$$

The functions B_j^1, B_j^δ are $\mathcal{Q}_j$-left-invariant and vanish off near$_j$. This follows by moving the flag group through the measure at that stage: its unipotent factors multiply the correlation by a unit complex number, while its stage factors are absorbed by quotient invariance. Also

$$\int |P_{j+1}^\delta(\sigma g, \sigma g x)|^2 \, d\theta_j(\sigma) \le B_j^\delta(g)/h_j(g), \qquad N_j^1 \le N_0^1.$$

For $a < m$ put

$$d_a = k + 1 + 2(m - a - 1), \qquad \Delta_a(z) = |z|^{2d_a+1}, \qquad A_l(p) = \prod_{a<l} \Delta_a(p_a^{-1}).$$

The two-frame weighted product, including all earlier moduli, is

$$\mathcal{P}_j(x; L) = A_j(p)^2 \mathcal{C}^3 N_0^1(x) N_j^\delta(x). \tag{9.15}$$

All uses of Tonelli and Cauchy–Schwarz below can first be made with nonnegative truncations. The shell estimates subsequently prove finiteness of the needed integrals.

Head minors and a common frame. The leading-column determinant calculation on p. 87 is an identity over $\mathbb{C}$. If $g, w^{-1}g$ are near at j, choose $s, u \in \mathcal{Q}_j$ with $sg, uw^{-1}g$ in the operator-norm box of radius ρ. The matrix

$$S = swu^{-1} = (sg)(uw^{-1}g)^{-1}$$

has entries at most ρ^2. The minor with columns $e_0, \dots, e_a$ and rows $e_0^*, \dots, e_{a-1}^*, v$ is $P_{a+1}(se_a^*)_v$. With $v = e_a^*$ it is P_{a+1}. Since its order is at most n, the determinant bound gives, with ample slack,

$$|P_l| \le \rho^{9n}, \qquad |(se_a^*)_v| \le L^{10n}, \qquad |p_a|, |p_a|^{-1} \le L^{10n} \quad (a < j). \tag{9.16}$$

The second assertion divides the minor by P_{a+1}; the last uses ratios of consecutive products. The estimates use $n, r \le L$, so that the factorials and sums in the minors are included in the displayed powers.

Factor $s = tn_0$ with $t \in G_j$, $n_0 \in U_{<j}$, and put $U = tn_0t^{-1}$. Then $U(tg) = sg$ and $Ue_a^* = se_a^*$. In the Witt decomposition through stage j,

$$U = \begin{pmatrix} A & B & C_0 \\ 0 & I & D_0 \\ 0 & 0 & E \end{pmatrix}, \qquad A = E^{-*}, \qquad B = -AD_0^* J_{\mathrm{mid}}.$$

The columns just bounded bound C_0, D_0, E by L^{10n}. As E is lower unitriangular, its inverse is the finite geometric sum of at most j powers of $E - I$; its entries are at most L^{12n^2}. The displayed identities then bound all entries of U, and of $U^{-1} = JU^*J$, by L^{24n^2}. Multiplication by the boxed matrix, and then by w^{-1}, yields

$$|(tg)_{ab}| \le L^{60n^2}, \qquad |(tw^{-1}g)_{ab}| \le L^{99n^2}. \tag{9.17}$$

The same t is essential: it preserves the Hermitian pairing between the two orbit vectors.

On a contributing pair the minor calculation also applies at gx with the tuple q. Nonzero partial correlations imply the requisite nearness by a contributing higher-stage translate, and those successive flag subgroups commute with $w_j(p), w_j(q)$. Consequently a nonzero form or absolute

pivot majorant $\mathcal{M}_j$ has $|P_l|, |Q_l| \le \rho^{9n} \le b^{60n^2}$. If any of these inequalities fails, the form and $\mathcal{M}_j$ vanish. No such assertion is needed for the weighted product $\mathcal{P}_j$. This proves the support indicator in (9.12). head minors and common size, common translate.

Transport of inverse depth weights at the complex place. We require the complex version of (9.5): for $l \le b_0$, and a measurable nonnegative function B which is $\mathcal{Q}_{b_0}$-left-invariant and supported near at b_0,

$$\int B(g) m_l(w_l(p)^{\pm 1} g)\, d(\eta_{<l} * \zeta)(g) \le A_l(p) \mathcal{C} \int B(g)\, d\zeta(g). \tag{9.18}$$

Here ζ is any sigma-finite measure. We give the analytic calculation, since the complex Haar exponents differ from the nonarchimedean exponents.

Fix a point y in the prescribed box, an earlier $\nu_0 \in U_{<a}$, and $\gamma = \mathbf{n}(\omega, it) \in N_a$. On the set where $w_{a+1}(p)\gamma\nu_0 y$ is near at $a+1$, the leading minors applied with the conjugate tuple and boxing element $(\gamma\nu_0)^{-1}$ imply

$$z' = \bar{p}_a \gamma^{-1} e_a^* = \bar{p}_a (e_a^* - \omega + c(\omega, t) e_a) \quad \text{has size } 20n.$$

In fact $\bar{p}_a (\gamma\nu_0)^{-1} e_a^*$ has that size by (9.16); it is congruent to z' modulo X_a, and z' has zero coordinates on the earlier pairs, so projection gives the assertion. The coefficient of it in $c(\omega, t)$ has modulus one.

In the free positive space X', of dimension $D = s + 1 + 3n$, impose the at most m equations $(\nu_0 \iota v)_{e_b} = 0$, $b < a$, together with the negative projections of $y^{-1}\iota v$ and $(yx)^{-1}\iota v$. There are at most $3n$ linear conditions. Their common kernel contains an orthonormal $(s+1)$-plane S. For $v \in S$, both transformed vectors are in the positive eigenspace of J; hence their Euclidean lengths equal $\|v\|$. The ball of radius $1/2$ in S consists of admissible common vectors. Transport by $w_{a+1}\gamma\nu_0$ gives stable lifts at stage a; the additional equations give the earlier orthogonality conditions. Their readings are the products with the free component $z'_{X'}$. Therefore the stable-reading disc contains the disc of radius $\| \operatorname{pr}_S z'_{X'} \|/2$, and

$$\operatorname{depth}_a(w_{a+1}\gamma\nu_0 y, x) \le \phi(z'_{X'}), \qquad \phi(v) = \frac{36}{\| \operatorname{pr}_S v \|^2}.$$

At zero projection the bound is interpreted as $+\infty$, a null set in the ensuing Lebesgue integral.

For every $t_0 \ge 0$, polar coordinates prove

$$\int_{|v_i| \le L^{t_0}} \phi(v)^s\, dv \le \Lambda^{99n(t_0+1)}. \tag{9.19}$$

Indeed a unitary decomposition into $S \oplus S^\perp$ preserves volume and puts both components in the ball of squared radius $R = DL^{2t_0} \le L^{2t_0+1}$. The integral in S is

$$\frac{36^s \pi^{s+1} R}{s!} \le \pi e^{36\pi} R.$$

The remaining $3n$ coordinates contribute at most $(\pi R)^{3n}$. Since $L \ge \rho \ge 8r$ and $n \ge 10^8$, this is less than the right side of (9.19). Bounds on the projected complements are uniform in the free coordinates; enlarging the projections to these fixed balls is legitimate for nonnegative integrands.

Now change the vector part of the Heisenberg coordinate to $-\bar{p}_a\omega$. There are d_a complex coordinates, so this contributes exactly $|p_a|^{-2d_a}$. At a fixed vector coordinate, the condition $|\bar{p}_a c(\omega, t)| \le L^{20n}$ bounds the allowed real t's by length $2L^{20n}/|p_a|$. Change to the volume-preserving basis adapted to the flag, increasing a size exponent by at most four; use (9.19) on the

free coordinates and disc volumes on the at most $99n$ complementary coordinates. The resulting loss is bounded, very generously, by

$$\Lambda^{20n+3+99n(20n+7)+99n(20n+6)} \le \Lambda^{10^6 n^2}.$$

Thus the one-stage estimate is

$$\int_{N_a} \mathbf{1}_{\mathrm{near}_{a+1}} \operatorname{depth}_a(w_{a+1}\gamma\nu_0 y, x)^s \, d\gamma \le \Delta_a(p_a^{-1})\Lambda^{10^6 n^2}.$$

The earlier inverse weight is constant in this integration: the new Weyl and Heisenberg factors preserve the earlier pairs and their stable readings. Iteration over $a < l$ gives $A_l(p)\Lambda^{10^6 n^2 l} \le A_l(p)\mathcal{C}$. For an arbitrary point near at l, rebox using an element $t\nu \in G_l U_{<l}$. Right translation by ν preserves unipotent Haar measure, and conjugation by t has complex vector determinant of modulus one and fixes each real centre. It also commutes with w_l. This proves the estimate uniformly at all near points. Pulling the invariant B out of the unipotent integral proves (9.18). For w_l^{-1}, replace the tuple by its conjugate; all displayed moduli remain the same. section transport, common positive subspace.

The twisted one-frame shell bound and the density cancellation. The one-frame proof on pp. 98–100 also applies to N_j^δ. Here is why its hypotheses and normalizations persist. The element δ normalizes the Bessel unipotent group and commutes with G_j. Its character on each Heisenberg stage $a \ge j$ is therefore unchanged. The direct depth-weight integral uses only the two absolute points g, y, so its proof is unchanged. The squared correlation is invariant under unipotent translation in its second variable, since the possibly changed character still has modulus one. In the Heisenberg descent from any stage $a \ge j$, the character is exactly the one used on p. 98; its cost remains $\pi^{2D}(L'^2)^{4900n}$. The terminal compact averaging has mass one. The same quotient induction consequently gives, on the position ball of radius $L' \ge \rho$,

$$\int N_j^\delta(x)\, d\mu(x) \le \pi^{2Dm}(L'^2)^{25000n^2}\Pi_\delta, \qquad \Pi_\delta = \iint |\langle F(g), F(\delta y)\rangle|^2 \, dQ(y)\, dQ(g). \tag{9.20}$$

The integral on the left is restricted to that ball. More explicitly, the summand with index $l \ge j$ contributes at most $(L'^2)^{100nl}[\pi^{2D}(L'^2)^{4900n}]^{m-l}\Pi_\delta$. Summing at most n terms and using $n + 4900nm \le 25000n^2$ proves the stated bound.

To compute Π_δ, put $t_a = p_a/q_a$. On the a-th Heisenberg group, conjugation by δ has vector Jacobian $|t_a|^{2d_a}$ and centre Jacobian $|t_a|^2$. The later diagonal pairs have mutually cancelling determinants of absolute value one. Thus its modulus on the full Bessel unipotent group is

$$J_j = \prod_{a<j} |p_a/q_a|^{2(d_a+1)}.$$

Disintegrating group Haar measure into unipotent Haar measure and Q shows that $\int \Phi(\delta y)\, dQ(y) = J_j \int \Phi(y)\, dQ(y)$ for unipotent-invariant nonnegative Φ. The sign of this Jacobian can also be read directly: the substitution $n \mapsto \delta n \delta^{-1}$ contributes J_j^{-1}, while left translation on the full group contributes one. Hence $\Pi_\delta = J_j \Pi_1$.

The density for the prescribed parameters in the complex convention is $\kappa_j = \prod_{a<j} |p_a q_a|^{d_a}$. The exact cancellation is

$$\kappa_j^2 A_j(p)^2 J_j = \prod_{a<j} |p_a q_a|^{2d_a} |p_a|^{-(4d_a+2)} |p_a/q_a|^{2d_a+2} = |Q_j|^{-2} \le L^2. \tag{9.21}$$

Applying Cauchy–Schwarz in x to (9.15), then using (9.20) twice with $L' = LH$, gives

$$\kappa_j \int_{S_H} \sqrt{\mathcal{P}_j(x; L)}\, d\mu(x) \le U_F L^{2A_0} H^\alpha, \quad A_0 = 1500000n^3 + 25000n^2 + 1, \quad \alpha = 50000n^2. \tag{9.22}$$

Indeed the square root of (9.21) is at most $L \le \Lambda$, the factor $\mathcal{C}^{3/2}$ is $\Lambda^{1500000n^3}$, and $U_F = \pi^{2Dm}\Pi_1$. This explicitly accounts for every tuple modulus. The normalization from (9.11) remains

$$dc_F U_F \pi^k \le \Theta. \tag{9.23}$$

twisted moments and density cancellation.

Separation of two frames. We spell out the Cauchy–Schwarz step so that the earlier tuple moduli are not hidden. If B, B' are nonnegative, $\mathcal{Q}_j$-left-invariant and supported near at j, and

$$f(g)^2 \le R\,\frac{B(g)}{h_j(g)}\,\frac{B'(w^{-1}g)}{h_j(w^{-1}g)},$$

cross the two inverse weights and apply Cauchy–Schwarz in $d\mathfrak{m}_j(g)$. The two resulting integrals are

$$\int B(g) m_j(w^{-1}g)\, d\mathfrak{m}_j(g), \qquad \int B'(w^{-1}g) m_j(g)\, d\mathfrak{m}_j(g).$$

The first is bounded by (9.18). In the second, the restricted Weyl invariance described above changes g to wg; then the same transport applies. Thus

$$\left(\int f\, d\mathfrak{m}_j\right)^2 \le R A_j(p)^2 \mathcal{C}^2 \left(\int B\, d\vartheta_j\right)\left(\int B'\, d\vartheta_j\right). \tag{9.24}$$

The transformed integrand is G_j-left-invariant because w commutes with G_j, and the weights and B' have that invariance; hence no unjustified invariance of $\mathfrak{m}_j$ is used.

The wave estimate at a fixed position. For fixed x, g unfold the next stage and put

$$f(\sigma) = P^1_{j+1}(\sigma g, \sigma g x), \qquad f'(\tau) = P^\delta_{j+1}(\tau w^{-1} g, \tau w^{-1} g x), \qquad v_\sigma = \sigma^{-1} f_j.$$

Here $f_j = e_j$ for $j < m$, and f_m is the negative terminal frame vector. The signed inner integral is exactly

$$I_\nu(x,g) = \iint \psi(\nu h(v_\sigma, v_\tau)) f(\sigma)\overline{f'(\tau)}\, d\theta_j(\tau)\, d\theta_j(\sigma).$$

This is the identity $\mathrm{Gram}_{jj}(\sigma\tau^{-1}) = h(\sigma^{-1}f_j, \tau^{-1}f_j)$, with no change of frequency. When the integral contributes, (9.17) provides one $t \in G_j$. On the supports of f, f', the orbit vectors tv_σ, tv_τ have sizes $60n^2+6, 99n^2+6$, respectively. To see this, box the point σg, compare its next-frame column with that of tg, and project off the earlier isotropic span. The flag unipotent changes the column only in that span. This is the same projection argument used for the quadric pushforward in p. 98; it also applies on the terminal negative orbit.

The pushforward measures are exactly the quadric measures proved there. Apply (9.9)–(9.10) to one amplitude and Cauchy–Schwarz against the other. The loss exponent is at most

$$99n(60n^2+12) + 99n(99n^2+12) \le 10^6 n^3.$$

It follows that

$$|I_\nu(x,g)|^2 \le \left(\frac{\pi^2}{|\nu|^2}\right)^D \mathcal{C}\,\frac{B^1_j(g)}{h_j(g)}\,\frac{B^\delta_j(w^{-1}g)}{h_j(w^{-1}g)}.$$

Using (9.24) and $N^1_j \le N^1_0$, we obtain

$$\phi_\nu(x)^2 \le \left(\frac{\pi^2}{|\nu|^2}\right)^D \mathcal{P}_j(x;L), \qquad \phi_\nu(x) = \int |I_\nu(x,g)|\, d\mathfrak{m}_j(g). \tag{9.25}$$

crossed weights, wave and common frame.

Unequal last tuples: the exact section phase. Let $j = i + 1 \le m$ and $\nu = p_i^{-1} - q_i^{-1} \neq 0$. For $\gamma = \mathbf{n}(\omega, it) \in N_i$, conjugation by δ multiplies its only relevant adjacent root entry by p_i/q_i. The two simultaneous Bessel characters therefore differ by $p_i/q_i - 1$. Moving this factor through the next stage is the same block calculation as on p. 88; the entry read there is $-h(\omega, \sigma^{-1} f_j)$. Unfolding that stage gives

$$\begin{aligned} P_j^\delta(\gamma g, \gamma g') &= \int \psi(-(p_i/q_i - 1)h(\omega, v_\sigma)) P_{j+1}^\delta(\sigma g, \sigma g')\, d\theta_j(\sigma), \\ z'(\gamma) &= \bar{p}_i \gamma^{-1} e_i^* = \bar{p}_i(e_i^* - \omega + c(\omega, t)e_i), \\ h(z'(\gamma), v_\sigma) &= -p_i h(\omega, v_\sigma). \end{aligned}$$

The last identity holds because v_σ is orthogonal to e_i, e_i^*. Since $\nu p_i = 1 - p_i/q_i$, the phase in the first line is exactly

$$\psi(-\nu h(z'(\gamma), v_\sigma)). \tag{9.26}$$

The skew coordinate drops out of this phase.

We next prove the section estimate

$$\int_{\{\gamma : w_j(p)\gamma g \text{ near}_j\}} |P_j^\delta(\gamma g, \gamma g')|^2\, d\eta_i(\gamma) \le \left(\frac{\pi^2}{|\nu|^2}\right)^D \Delta_i(p_i^{-1}) \mathcal{C} \int |P_{j+1}^\delta(\sigma g, \sigma g')|^2\, d\theta_j(\sigma). \tag{9.27}$$

Only points with γg also near at j can contribute. Choose one such γ_0. By (9.17), there is $t_0 \in G_j$ with $t_0\gamma_0 g$ of size $99n^2$. For each contributing γ, choose $t_\gamma \in G_j$ with $t_\gamma \gamma g$ of that size. Both t_0, t_γ fix e_i^*. Consequently the e_i^*-column of

$$t_0\gamma_0(t_0\gamma)^{-1} - I$$

equals that of $(t_0\gamma_0 g)(t_\gamma \gamma g)^{-1} - I$. It has size $198n^2 + 6$. The parameter bound in (9.16) gives $|p_i| \le L^{10n}$, so the column multiplied by $\bar{p}_i$ has size at most $999n^2$.

Use the Haar-preserving substitution

$$\gamma = t_0^{-1} \mathbf{n}(\omega, it) t_0 \gamma_0.$$

Then $t_0\gamma_0(t_0\gamma)^{-1} = \mathbf{n}(\omega, it)^{-1}$. Thus the scaled vector coordinate $\zeta = \bar{p}_i \omega$ lies in a box of size $999n^2$, and for fixed ζ the allowed skew coordinate has length at most $2L^{999n^2}/|p_i|$. The vector change costs $|p_i|^{-2d_i}$. Together the exact tuple cost is $\Delta_i(p_i^{-1})$.

Under this substitution, the phase is the Fourier phase of frequency ν in ζ against $t_0 v_\sigma$, times a unit phase depending only on v_σ. Indeed the vector part of γ is $t_0^{-1}\omega + \omega_0$; multiplying it by $\bar{p}_i$ and using preservation of h gives the assertion from (9.26). No extra factor $|p_i|$ remains in the frequency. The input vectors $t_0 v_\sigma$ have size at most $99n^2 + 6$: compare with the boxed centre $t_0\gamma_0 g$ and project away e_i and the earlier isotropic coordinates, on which γ_0 alone changes the next vector.

Apply the flat-output version of (9.9), using the quadric input and ordinary volume in ζ. Integrate the skew slice afterwards. The total loss exponent is at most

$$99n(99n^2 + 12) + 99n(999n^2 + 6) + (999n^2 + 1) \le 10^6 n^3.$$

This proves (9.27), including its precise Haar factor. The argument applies to all measurable amplitudes by the same truncation and conditional Cauchy–Schwarz used for (9.9).

The earlier weights in the section estimate. Put $\phi_1(g) = |P_j^1(g, gx)|$, $\phi_2(g) = |P_j^\delta(g, gx)|$, and $w_l = w_l(p)$. For weight one, the double stage integral factors, and its absolute value is $\phi_1(g)\phi_2(w_j^{-1}g)$. The function $h_i\phi_1^2$ is $\mathcal{Q}_j$-left-invariant and supported near at j; ϕ_2 is G_j-left-invariant. Define

$$u(g) = \mathbf{1}_{\mathrm{near}_j}(w_j g)h_j(g)m_i(w_i g)\phi_2(g)^2.$$

On the contributing set, $m_i(w_i w_j^{-1}g) = h_i(g)^{-1}$, since the last Weyl factor does not affect an earlier stable reading. Hence

$$(\phi_1(g)\phi_2(w_j^{-1}g))^2 = h_i(g)\phi_1(g)^2\, m_j(w_j^{-1}g)\, u(w_j^{-1}g).$$

Cauchy–Schwarz produces two integrals. By (9.18), the first is at most

$$A_j(p)\mathcal{C}\int h_i(g)\phi_1(g)^2\, d\vartheta_j(g) \le A_j(p)\mathcal{C}N_0^1(x).$$

For the second, restricted Weyl invariance replaces $u(w_j^{-1}g)$ by $u(g)$. The function is indeed G_j-left-invariant. Factor the unipotent measure into η_i and $\eta_{<i}$, which may be interchanged because the former normalizes the latter with modulus one. Both $h_j(\gamma g)$ and $m_i(w_i\gamma g)$ are constant as $\gamma \in N_i$ varies. Apply (9.27) first; $\int |P_{j+1}^\delta|^2 d\theta_j \le B_j^\delta/h_j$ cancels h_j. Apply (9.18) to the remaining i earlier stages. Writing $R_\nu = (\pi^2/|\nu|^2)^D$, the second integral is at most

$$R_\nu\Delta_i(p_i^{-1})\mathcal{C}\, A_i(p)\mathcal{C}\int B_j^\delta\, d\vartheta_j = R_\nu A_j(p)\mathcal{C}^2 N_j^\delta(x).$$

Multiplication of these two bounds proves

$$\left(\int \phi_1(g)\phi_2(w_j^{-1}g)\, d\mathfrak{m}_j(g)\right)^2 \le \left(\frac{\pi^2}{|\nu|^2}\right)^D \mathcal{P}_j(x; L). \tag{9.28}$$

In particular the modulus of every earlier tuple occurs twice, as required by (9.15). section centring, earlier weights, exact section phase.

The absolute bound and remote positions. Let $\mathcal{M}_j(x)$ be the integral in the lower point of the product of the two absolute pivots. For either wave or distinct-parameter integrand just considered, denote its lower-point absolute integral by $\phi(x)$. Then

$$\phi(x) \le \mathcal{M}_j(x), \qquad \phi(x) \le (\pi/|\nu|)^D\sqrt{\mathcal{P}_j(x; L)}.$$

The mass bound, uniformly at every stage $j \le m$, is

$$V = (\pi\rho^{24})^k\rho^{9999n} \ge 1.$$

After the stage quotient reduction a supported orbit has size at most seven, or eleven in the basis adapted to the flag. Its D free coordinates contribute at most ρ^{24D}; the remaining coordinates, including the positive pair used to disintegrate the quadric, contribute at most ρ^{2574n}. These quantities are bounded by V. This also covers the terminal negative quadric. Cauchy–Schwarz in each pivot, followed by (9.24), therefore gives

$$\mathcal{M}_j \le V\sqrt{\mathcal{P}_j}. \tag{9.29}$$

If every active point has a stage $a \ge j$ of depth at least T_1, the argument at the first stage exceeding the depth threshold already proved before the depth-weight calculation on p. 98 gives

$$\mathcal{M}_j \le VT_1^{-s/2}\sqrt{\mathcal{P}_j}. \tag{9.30}$$

Here one pivot uses the deep estimate and the other uses its mass bound. More explicitly, the square of the first pivot is at most $T_1^{-s}\mathcal{C}B_j^1/h_j$. The positive-depth integral on each intervening stage is at most $(\rho^2)^{10000n}$, by the common-positive-plane calculation already supplied; summing at most m first-deep pieces costs $m^2\Lambda^{10000nm} \le \mathcal{C}$. The second pivot has square at most VB_j^δ/h_j. Crossed separation gives $T_1^{-s}V\mathcal{P}_j$, which is at most $T_1^{-s}V^2\mathcal{P}_j$. A nonzero second pivot ensures the translated nearness conditions for both tuples along the entire G_j-orbit of the first pivot. Indeed a contributing translate gives $w_j(p)^{-1}g$ and $w_j(q)^{-1}gx$ near at j, and all these translates at that stage commute with the Weyl elements. Thus the deep condition applies to every kernel point used in the restriction to the first stage exceeding the depth threshold.

We verify that remote positions have this property after a tuple. At an active y, the minor estimate and the boxed inverse give

$$\|y^{-1}e_a^*\|,\ \|(yx)^{-1}e_a^*\| \le L^{10n+2} \qquad (a<j).$$

Taking the identical dual lifts shows that their real depth costs are less than L^{22n}. If $T \ge L^{11n}$ and all the remaining real depth bounds are less than T^2, every stage has witnesses of cost less than T^2. The explicit triangular matrix calculation preceding (9.7) then gives $\|x-I\| \le T^{99n}$. For a least-dyadic shell S_H, set

$$H_* = L^{2400n^2}, \qquad \beta = s/(198n).$$

If $H > H_*$, use $T = (H/4)^{1/(99n)}$. Then $T \ge L^{11n}$, but the shell has an entry of $x-I$ larger than $H/2 > T^{99n}$. Some remaining real depth is therefore at least T^2, and its inverse-area depth is at least $T^2/\pi \ge H^{1/(99n)}$. This proves, including the case $j=m$ when the remote active support is empty,

$$\mathcal{M}_j(x) \le VH^{-\beta}\sqrt{\mathcal{P}_j(x;L)} \qquad (x \in S_H, H > H_*). \tag{9.31}$$

first deep floor and two pivots, remote support and floor mass.

A common shell summation for both gains. Write

$$u = |\nu|, \qquad g = (\pi/u)^D, \qquad a = \alpha/\beta = 9900000n^3/s.$$

We have $0 < a < 1$, $\alpha \ge 1$, and $\beta - \alpha \ge 1$. Equations (9.22), (9.29), and (9.31) imply the useful interpolation bound

$$\kappa_j \int \phi(x)\,d\mu(x) \le 4U_F L^{2A_0} H_*^\alpha V^a \min(g,V)^{1-a}. \tag{9.32}$$

To check it directly, first suppose $g \le V$, and split the dyadic shells at $X = \max(H_*, (V/g)^{1/\beta})$. Below X use $\phi \le g\sqrt{\mathcal{P}_j}$; the geometric sum is at most $2gX^\alpha$. Above X use $\phi \le VH^{-\beta}\sqrt{\mathcal{P}_j}$; the sum is at most $2VX^{\alpha-\beta}$. Since $X^\alpha \le H_*^\alpha (V/g)^a$ and $X^{\alpha-\beta} \le (V/g)^{a-1}$, the claimed bound follows. If $g \ge V$, split at H_*, use the absolute bound on the first part and the remote bound on the second, obtaining $4VH_*^\alpha$. This is the same formula.

There is a large margin in the frequency exponent:

$$Da \le 19800000n^3 < 2\cdot 10^7 n^3, \qquad E := D(1-a) \ge k - n^7 + 6 = K + 6.$$

Indeed $D/s \le 2$, and $n^7 \ge 2\cdot 10^7 n^3 + 11$. Moreover

$$V^a g^{1-a} = \pi^{ka+D(1-a)}\rho^{(24k+9999n)a}u^{-E}, \qquad ka + D(1-a) \le k, \qquad (24k+9999n)a \le 10^9 n^3.$$

The last inequalities use $k/s \le 2$ and $9999n/s \le 1$. For $u \ge 1$, (9.32) therefore gives the factor u^{-K-6}, with a remaining ρ-cost at most $\rho^{10^9 n^3}$. For $\rho^{-24} \le u \le 1$, one has $g \le V$ and

$$u^{-E} \le u^{-D} \le u^{-K}\rho^{24(n^7-5)}.$$

For $0 < u \le \rho^{-24}$, simply bound $V^a \min(g, V)^{1-a} \le V$; because $K \ge 0$,

$$\rho^{24k+9999n} \le u^{-K} \rho^{24n^7+9999n}.$$

In all three cases the π-power is at most π^k and is absorbed by (9.23).

Finally choose

$$L = \max(\rho,\ |P_l|^{-1},\ |Q_l|^{-1} : 0 \le l \le j).$$

The parameter-dependent exponent is

$$B_L = 2A_0 + 2400n^2\alpha = 120000000n^4 + 3000000n^3 + 50000n^2 + 2 \le n^6.$$

On the nonzero support, all indicators for the tuples equal one, and

$$L^{B_L} \le L^{n^6} \le \rho^{n^6} H_j(p) H_j(q).$$

The latter follows by bounding a maximum by the product of the factors $\max(1, |P_l|^{-1})$, $\max(1, |Q_l|^{-1})$, times ρ. All remaining factors are bounded by

$$4\rho^{n^6+10^9n^3+24n^7+9999n} \le 4\rho^{30n^7} \le b^{2000n^9}.$$

Multiplying (9.32) by dc_F proves (9.13) and (9.14), with the parameter weights and the sixth-power damping. If an indicator in a tuple weight vanishes, (9.16) gives a zero stage form. At $j = 0$ there are no prescribed entries, the earlier density is the empty product, and $H_0(p) = H_0(q) = 1$. The first estimate therefore gives (9.8) with the same constant and frequency exponent. In the formal proof the shell bound also applies to every tuple length.

Lean. `norm_floorForm_le`.

Removing the entries in the product over places. Fix a finite set $A_0 \subset E^\times$ of allowed entries, all units off T. Write $J_i = \prod_v J_{v,i} \cdot J_{T,i}$ on the scaled tuples used above. Use the real stage recursions and the decay estimates (9.13)–(9.14) for the truncated vectors. Products preserve the common-entry recursion by Fubini and positivity at the last entry by the Schur product theorem. Recursion carries that positivity to each lower common-entry matrix. For varying unit tuples the sum of all its entries is lower semicontinuous: start with continuity at the top and use Fatou on the nonnegative sums of the common extensions at each integration.

For $i \ge 1$ let S_i be the real sum of J_i over pairs of i-entry tuples in A_0 with equal last entry, and S_0 the real bottom mass. Put

$$c_i = \frac{\mathrm{vol}(\widehat{\mathcal{O}}_E)}{\prod_v s_{vi}^2}, \qquad W_i = \sum_{p \in A_0^i} H_i^{\mathrm{glob}}(p), \qquad s_{vi} = M_v^{2(\varrho - i)}/N.$$

Suppose ε bounds the sum of the decay factors for nonzero rational frequencies integral off T at both sets of frequency scales below. Suppose also $W_i^2 \le \Gamma \prod_{a<i} c_a$. Then

$$S_n \le \Big(\prod_{a<n} c_a\Big)(S_0 + 2n\Gamma\varepsilon), \qquad n = \varrho + 1.$$

Indeed, sum over two i-entry tuples first, with a variable common next entry. Extend that nonnegative density to the adeles by the indicator of integrality off T and the changes $z_v \mapsto s_{vi} z_v$. Poisson summation bounds its sum on A_0 by c_i times its integral in the local variables plus $c_i \varepsilon W_i^2$. To justify the one-sided bound, periodize on $\mathbb{A}_E/E$. The hypothesis gives an absolutely convergent

Fourier series for the periodization, agreeing with it almost everywhere. A positive approximate identity at zero bounds the finite rational subsum from below in the limit by lower semicontinuity at its points. The determinant of the complex dilations and the integral volume away from T give c_i. The dual index y/δ_E has archimedean coordinates $2\pi i\Phi_v(y)/(\Phi_v(\delta_E)s_{vi})$ in the local tests and coordinates y_T at T.

At zero frequency the integral is the sum over all pairs of earlier tuples by recursion. Their equal last entries give S_i. For distinct last entries u, u', the difference $y = (u^{-1} - u'^{-1})/M$ is a nonzero T-integer. Its finite frequency is y_T; its infinite frequencies are $M\Phi_v(y)/s_{v,i-1}$. The second decay estimate therefore bounds each cross term by $\varepsilon H_i^{\mathrm{glob}}(p)H_i^{\mathrm{glob}}(q)$. Thus $S_{i+1} \le c_i(S_i + 2\varepsilon W_i^2)$, also for $i = 0$ when there are no distinct last entries. Divide by $\prod_{a\le i} c_a$ and sum. Restricting the terminal entries to those occurring in the energy can only decrease the answer. One-sided Poisson, two errors and iteration.

The numerical frequency sum. Here and in the next two paragraphs all quantities belong to the same admissible datum. Put $n = \varrho + 1$, $f = [E : \mathbb{Q}]/2$, $K = k - n^7$, and

$$s_{va} = \frac{M_v^{2(\varrho-a)}}{N}, \qquad x_v = \frac{|M_v|^{2\varrho+1}}{MN}, \qquad \mathcal{D} = \Theta^f \prod_v x_v^K.$$

Only indices $0 \le a \le \varrho$ occur. Thus $\mathcal{D}$ is the quantity denoted decayConst above. The size hypotheses used below are

$$b \ge 1000, \quad \varrho \ge 10^5, \quad 512000n^9 \le k \le b, \quad f \le b/1000, \quad \mathrm{rd}_E \le b, \tag{9.33}$$

$$1 \le |M_v| \le b^{23}, \qquad 1 \le M \le b^{23}, \qquad 1 \le N \le b^{60n}. \tag{9.34}$$

These are properties of the datum, not estimates obtained by summing frequencies.

For $y \in E$, let $\mathfrak{b}(y) = \{a \in \mathcal{O}_E : ay \in \mathcal{O}_E\}$ and put

$$\mathcal{H}(y) = \mathrm{N}\,\mathfrak{b}(y) \prod_v \max(1, |\Phi_v(y)|)^2.$$

For every finite $X \subset E$,

$$\sum_{y\in X} \mathcal{H}(y)^{-3} \le (59/5)^f \zeta_E(2) < 32^f. \tag{9.35}$$

In any coset of $\mathcal{O}_E$, the polydisks of radius $1/2$ about the images $(\Phi_v(y))_v$ have disjoint interiors. Otherwise the difference of two centers would come from a nonzero algebraic integer with all conjugates of modulus less than one. In the disk $|u - z| < 1/2$, one has $\max(1, |z|)^{-6} \le \max(1, |u| - 1/2)^{-6}$. The integral of this majorant divided by the disk area is

$$8\left(\int_0^{3/2} r\,dr + \int_{3/2}^{\infty} r(r - 1/2)^{-6}\,dr\right) = \frac{59}{5}.$$

Integrating over the disjoint polydisks bounds the sum of the archimedean weights in each coset by $(59/5)^f$. The elements with denominator $\mathfrak{b}$ lie in $\mathfrak{b}^{-1}$, which contains $\mathrm{N}\,\mathfrak{b}$ cosets of $\mathcal{O}_E$. Multiplying by $(\mathrm{N}\,\mathfrak{b})^{-3}$ and summing gives the factor $\zeta_E(2)$. Its Euler product is bounded by $\zeta(2)^{2f} = (\pi^2/6)^{2f}$, proving (9.35). Lean sums over square-grid corners in `sum_height_inv_cube_le`.

Let $J_T(y)$ be the generalized index $[\mathcal{O}_T : y\mathcal{O}_T]$, and let $D_T(y) = [\mathcal{O}_T : \mathcal{O}_T \cap y^{-1}\mathcal{O}_T]$. For a nonzero T-integer y, the product formula and the denominator ideal give

$$J_T(y)^{1/2} \le \prod_v |\Phi_v(y)|, \qquad \mathrm{N}\,\mathfrak{b}(y) \le D_T(y). \tag{9.36}$$

Indeed the generalized index is the product of $\mathrm{N}\,\mathfrak{p}^{\operatorname{ord}_{\mathfrak{p}} y}$ over $\mathfrak{p} \in T$; the omitted factors are at least one. The second assertion follows by restricting the denominator conditions to the places in T.

Suppose $|c_v|^{-1} \le x_v \le B$ and $B \ge 1$. The elementary inequality

$$\min(1, |c_v \Phi_v(y)|^{-1}) \le B \max(1, |\Phi_v(y)|)^{-1}$$

and (9.36) imply

$$\prod_v \left(|c_v \Phi_v(y)|^{-K} \min(1, |c_v \Phi_v(y)|^{-1})^6 \right) \frac{J_T(y)^{K/2}}{D_T(y)^3} \le \left(\prod_v x_v^K B^6 \right) \mathcal{H}(y)^{-3}. \tag{9.37}$$

The powers $|\Phi_v(y)|^K$ supplied by the product formula cancel the inverse powers in the archimedean factors. Therefore, for every finite set of nonzero T-integers, its sum in (9.37) is at most $\prod_v x_v^K (bB^2)^3$, by (9.35) and $32 \le b^3$.

Apply this with frequency coefficient c_v/s_{va}, where $|c_v|^{-1} \le |M_v|/M$. Since $|M_v| \ge 1$,

$$|c_v/s_{va}|^{-1} \le x_v, \qquad x_v \le B := b^{23(2\varrho+1)}.$$

Moreover

$$(bB^2)^3 = b^{3+138(2\varrho+1)} \le b^{n^9}.$$

The global frequency majorant contains the additional factors $\Theta^f b^{2000 n^9 f}$ from the complex places and $b^{n^9 f}$ from the finite places. Its sum is consequently bounded by

$$\varepsilon = b^{2002 n^9 f} \mathcal{D}. \tag{9.38}$$

This holds at both frequency scales in the sieve. For Poisson summation, $c_v = 2\pi i/\Phi_v(\delta_E)$ and $|c_v|^{-1} = |M_v|/M$; for the unequal-last-entry terms, $c_v = M$ and $|c_v|^{-1} \le |M_v|/M$. The bound is uniform in the finite set of frequencies and in the stage. It sums the stated local decay majorants; it does not prove the local oscillatory estimates which supply those majorants.

The weighted count of parameter tuples. Let $\mathcal{A} \subset E^\times$ be finite, with all its elements integral off T, and let $0 \le i \le \varrho$. For $p = (p_0, \dots, p_{i-1}) \in \mathcal{A}^i$, put

$$\delta_j = p_0 \cdots p_j, \qquad c_{jv} = \prod_{a=0}^{j} s_{va} \qquad (0 \le j < i).$$

Set $R_0 = b^{60n^2}$. The global parameter weight is the product, over $j < i$, of

$$\left\{ \prod_v \mathbf{1}_{|c_{jv}\Phi_v(\delta_j)| \le R_0} \max(1, |c_{jv}\Phi_v(\delta_j)|^{-1})^{n^6} \right\} \mathbf{1}_{\delta_j \in \mathcal{O}_T} J_T(\delta_j)^{n^6/2}.$$

This is the weight $H_i^{\mathrm{glob}}(p)$ used in the preceding recursion. Its support and size can both be bounded independently of the number of elements of $\mathcal{A}$.

First,

$$c_{jv}^{-1} \le N^{j+1} \le b^{60n^2} = R_0.$$

If the weight is nonzero, every δ_j is integral at T and off T, hence lies in $\mathcal{O}_E$, and

$$|\Phi_v(\delta_j)| \le R_0^2 =: R = b^{120n^2}.$$

Using (9.36), one factor before its power n^6 is bounded by

$$J_T(\delta_j)^{1/2}\prod_v \max(1,|c_{jv}\Phi_v(\delta_j)|^{-1}) \le \prod_v \max(|\Phi_v(\delta_j)|, c_{jv}^{-1}) \le R^f.$$

Thus every tuple has weight at most b^{120n^8fi}.

Second, the map $p \mapsto (\delta_0,\dots,\delta_{i-1})$ is injective: recover $p_0=\delta_0$ and $p_j=\delta_j/\delta_{j-1}$. Distinct tuples of algebraic integers are separated by at least one in the maximum of their complex embedding norms, because some nonzero integral difference has an embedding of modulus at least one. Balls of radius $1/2$ in this norm are disjoint and lie in the ball of radius $R+1/2$. Volume comparison in real dimension $2fi$ bounds the number of possible tuples by $(1+2R)^{2fi}$. Since $1+2R \le bR$ and $i \le n$,

$$\begin{aligned} W_i := \sum_{p\in\mathcal{A}^i} H_i^{\mathrm{glob}}(p) &\le b^{120n^8fi}(1+2R)^{2fi} \\ &\le b^{(120n^9+240n^3+2n)f} \le b^{130n^9f}. \end{aligned} \tag{9.39}$$

The empty tuple satisfies the same estimate.

Write $V=\mathrm{vol}(\widehat{\mathcal{O}}_E)=(2/\operatorname{rd}_E)^f$. Since $V \ge b^{-f}$ and $s_{va} \le b^{46\varrho}$, the Poisson constants satisfy

$$\prod_{a<i} c_a = \prod_{a<i}\frac{V}{\prod_v s_{va}^2} \ge b^{-i(1+92\varrho)f} \ge b^{-100n^2f}.$$

Combining this with (9.39) gives precisely

$$W_i^2 \le b^{400n^9f}\prod_{a<i} c_a. \tag{9.40}$$

Indeed $260n^9+100n^2 \le 400n^9$. Taking $\Gamma=b^{400n^9f}$ and using $2n \le k \le b \le b^{n^9f}$, the error in the sieve is

$$2n\Gamma\varepsilon \le b^{2403n^9f}\mathcal{D}. \tag{9.41}$$

Comparison with the second-moment bound. Let d_{B} be the number of tensor-product Bessel coordinates, e_{B} their product formal degree, $h=\#\mathcal{C}$, and $R_{\mathrm{B}}=|\bar{R}_0|$. Write C and I for the main and identity second moments, so $I=R_{\mathrm{B}}hd_{\mathrm{B}}$, and retain ν for the unipotent covolume. The scalar in Siegel's step is denoted a_*. Put

$$\begin{gathered} c_G=\frac{\mathrm{vol}\,U(k)^2}{\mathrm{vol}\,U(k+2\varrho+1)\,\mathrm{vol}\,U(k-1)}, \qquad \Theta=\pi^{(2\varrho+1)k}\frac{\mathrm{vol}\,U(k)}{\mathrm{vol}\,U(k+2\varrho+1)}, \\ \Lambda=\prod_v\prod_{a=0}^{\varrho} s_{va}^{k+2(\varrho-a)}, \qquad r_0=\frac{1}{d_{\mathrm{B}}}\prod_v\left(\sqrt{c_G}\prod_{a=0}^{\varrho} s_{va}^{k-1+2(\varrho-a)}\right), \\ w=V^n(d_{\mathrm{B}}/\Lambda)^2/c_G^f. \end{gathered}$$

Direct cancellation of one power of each s_{va} gives

$$\prod_{a<n} c_a = r_0^2 w. \tag{9.42}$$

The arithmetic normalization and the two size estimates needed here are

$$a_* w \le (1+b^{-9})Ce_{\mathrm{B}}\nu^2, \qquad d_{\mathrm{B}} \le b^{2kf}, \qquad b^{11kf}I \le C. \tag{9.43}$$

The first is the prescribed mass estimate; the second is the Bessel-coordinate dimension estimate; the third is the prescribed main-to-identity moment separation. The following calculation uses exactly these assertions.

We first prove the remaining scalar estimate

$$a_* w\mathcal{D} \le b^{n^9 f} d_{\mathrm{B}} I\nu^2. \tag{9.44}$$

This follows from the definitions, not from the first inequality in (9.43). To display the cancellation, put $s = \sqrt{\mathrm{rd}_E} \ge 1$, $Q_0 = Ms^2/(2\pi)$, $x = Q_0^{2\varrho+1}/(MN)$, and $D = \varrho(2k + 2\varrho + 1)$. The norm of the different gives $\prod_v |M_v| = Q_0^f$; in particular $Q_0 \ge 1$. In these terms the defining constants are

$$a_* = R_{\mathrm{B}} h \left[M^D \frac{\pi^k}{(k-1)!N^k} s(2/s^2)^k \right]^f, \qquad \nu = \frac{1}{\Lambda} \left[\frac{M^D}{N^k} (s^2/2)^{\varrho(k+\varrho)} s^{\varrho} \right]^f.$$

Set $c_* = R_{\mathrm{B}} h (d_{\mathrm{B}}/\Lambda)^2$. Then $a_* w\mathcal{D} = c_* A^f$ and $d_{\mathrm{B}} I\nu^2 = c_* B^f$, where

$$A = M^D \frac{\pi^k}{(k-1)!N^k} s(2/s^2)^k \frac{(2/s^2)^n}{c_G} \Theta x^{k-n^7},$$
$$B = \left[\frac{M^D}{N^k} (s^2/2)^{\varrho(k+\varrho)} s^{\varrho} \right]^2.$$

The quotient formula $\operatorname{vol} U(k)/\operatorname{vol} U(k-1) = 2\pi^k/(k-1)!$, which is the sphere volume in the prescribed normalization, gives $\Theta/c_G = (k-1)!\pi^{2\varrho k}/2$. Substitution yields the exact identity

$$\frac{A}{B} = (2/s^2)^{n+2\varrho^2} \frac{s^{1-2\varrho}}{2M^{\varrho(2\varrho+1)}} x^{-n^7}.$$

Here $2/s^2 \le b$, $s^{1-2\varrho} \le 1$, and $x^{-1} \le MN \le b^{23+60n}$. Therefore

$$\frac{A}{B} \le b^{n+2\varrho^2+(23+60n)n^7} \le b^{n^9}.$$

The last elementary exponent comparison follows from $n \ge 100001$. Raising it to the power f proves (9.44).

Let $\ell \le f$ be the number of marked places and put $Y = b^{-6k\ell} C\nu^2$. The mass term in the sieve is bounded using (9.43) by

$$a_* w \left(\prod_v \frac{1 + b^{-10}}{e_v} \right) (1 + b^{-6}) b^{-6k\ell} \le (1 + b^{-9})^2 (1 + b^{-6}) Y.$$

Indeed $\prod_v e_v = e_{\mathrm{B}}$, and $(1 + b^{-10})^f \le \exp(b^{-9}/1000) \le 1 + b^{-9}$. The error term satisfies

$$\begin{aligned} a_* w\, b^{2403n^9 f} \mathcal{D} &\le b^{(2404n^9+2k)f} I\nu^2 \\ &\le b^{(2404n^9-9k)f} C\nu^2 \le b^{-10} Y. \end{aligned}$$

For the last inequality it is enough that $2404n^9 f + 2kf + 10 + 6k\ell \le 11kf$; this follows at once from $\ell \le f$, $f \ge 1$, and $k \ge 512000n^9$.

Finally, for $u = b^{-1} \le 1/1000$,

$$(1 + u^6)((1 + u^9)^2(1 + u^6) + u^{10}) \le 1 + u^4.$$

For example, set $t = u^4/100$. Each of u^6, u^9, u^{10} is at most t, and the left side is at most $1 + t(5 + 7t + 4t^2 + t^3) \le 1 + 17t \le 1 + u^4$. Multiplying by the outer Siegel factor $1 + b^{-6}$, and using (9.42), therefore gives the required upper bound

$$(1 + b^{-6})a_*\Big(\prod_{a<n} c_a\Big)\left\{\left(\prod_v \frac{1 + b^{-10}}{e_v}\right)(1 + b^{-6})b^{-6k\ell} + b^{2403n^9 f}\mathcal{D}\right\} \le r_0^2(1 + b^{-4})b^{-6k\ell}C\nu^2.$$

The archimedean bounds in this estimate are (9.13) and (9.14). It remains to verify the three inequalities in (9.43) for the data of Section 5.

Lean references. `Frequencies.lean` proves the denominator and height estimates; `GlobalSieve.lean` contains `boundsDecay_prod`, `sum_globalWeight_le`, and `sq_sum_globalWeight_le`; `EnergyEstimate.lean` contains `prod_poissonConst_eq`, `scaledSiegelConst_mul_decayConst_le`, and `sieve_constant_le`. The prescribed inputs are established for the constructed data in `FillingAdmissible.lean`; their scalar and representation-theoretic reconstruction follows below.

Three inequalities for the chosen datum. We verify (9.43) for the field, source level, compact type and Haar measures of the preceding mass and covolume calculations. Here $L = \log b$, and $m = 2\varrho + 2 = 2n$ denotes the number of arithmetic lifts; thus $r = k + m - 1$. This local use of m differs from the abbreviation $m = n - 1$ in the preceding complex-place proof. This paragraph concerns dimensions and scalar normalizations; it does not use a bound for a measured Poincaré block.

The mass normalization is already available. The calculation in "Cancellation with the unipotent covolume and the real fibre" gave

$$\frac{A_* V^n}{Ce_B c_G^f (\nu L_*/d_B)^2} = \frac{F_* m_r}{Q_N m_k}\frac{Q_N}{(N^k - 1)^f} \le 1 + b^{-9}.$$

In the notation of the numerical bound, $a_* = A_*$, $\Lambda = L_*$, $d_{\mathrm{B}} = d_B$, $e_{\mathrm{B}} = e_B$, and $w = V^n(d_B/L_*)^2/c_G^f$. Multiplying by the positive denominator therefore gives, on this very datum,

$$a_* w \le (1 + b^{-9})Ce_B\nu^2. \tag{9.45}$$

The source is `FillingAdmissible.lean`, declaration `scaledSiegelConst_le`.

The number of compact coordinates. At v the compact type is $\mathrm{Sym}^{a_v}\mathbb{C}^k$, where $a_v = \mathrm{columnCount}(v, 1) \le a = xb$. Its dimension $d_v = \binom{k+a_v-1}{k-1}$ gives a single product for both bounds:

$$d_B = \prod_v d_v \le b^{2kf}, \tag{9.46}$$

$$1 \le d_B((k-1)!)^f = \prod_v \prod_{i=1}^{k-1}(a_v + i) \le b^{5kf/4}. \tag{9.47}$$

Indeed $k + a \le 2xb \le b^{5/4}$ eventually, since $k \le 3x^5$ and $b \ge x^{36}$. The factorials in the second line are the ones left by the formal degree and the genus mass below.

Lean. `card_besselCoord_mul_factorial_le`.

The formal degree on the chosen parameter. Write κ_v for the integral weight at the place v in this paragraph, and put $S = \sum_v \log \kappa_v$. Thus $0 \le S \le fL$. The rank k is even. With $\Delta = 2r^8$, the ordered row parameter is

$$\varsigma_{v,i} = k/2 - 1 - i \quad (0 \le i < k - 1), \qquad \varsigma_{v,k-2+j} = -\kappa_v - j\Delta \quad (1 \le j \le m).$$

In the prescribed Haar normalization the Harish–Chandra degree formula is

$$e_v = \prod_{l=k-1}^{r-1} \frac{\prod_{0\le i<l}(\varsigma_{v,i}-\varsigma_{v,l})}{l!}. \tag{9.48}$$

For $l < k-1$, the differences in the full formula are $l-i$, whose product is $l!$; those factors have cancelled. Thus no additional compact dimension or volume factor is omitted in (9.48).

In the last m rows, the differences between tail entries are $(j-i)\Delta$ for $1 \le i < j \le m$. Their product at each place is $\Delta^{\binom{m}{2}} \prod_{j=1}^{m-1} j!$. Each of the $m(k-1)$ differences between a tail entry and an initial entry lies in $[\kappa_v, \kappa_v + 2m\Delta]$ and hence has logarithm in $[\log\kappa_v, \log\kappa_v + 1/x]$. Indeed $2m\Delta \le x^{42} \le \kappa_v/x$ eventually, by $m \le 2\sqrt{x}$, $r \le 3x^5$, $\kappa_v \ge xb/x_1$ and $b \ge x_1^{10^6}$, $x_1 \ge x^{10^6}$. Thus (9.48) gives

$$\begin{aligned} \log e_B + f\sum_{l=k-1}^{r-1} \log(l!) &= m(k-1)S + E_{\deg}, \\ 0 \le E_{\deg} - f\log\left(\Delta^{\binom{m}{2}} \prod_{j=1}^{m-1} j!\right) &\le fm(k-1)/x. \end{aligned} \tag{9.49}$$

Lean. Lean bounds all the tail differences by the largest difference. Here their product is a Vandermonde determinant. `log_filling_besselDegree_mem_Icc`.

The ratio of the two moments. The free source-level count proved above is $n_X = H(N^k-1)^f m_k$, and $|\bar{R}_0| = Hu_M$, where $u_M = M^{f\varrho(2k+2\varrho+1)}$. Thus the definitions $C = (n_X d_B)^2/(e_B m_r)$ and $I = |\bar{R}_0| n_X d_B$ give the exact identity

$$\frac{C}{I} = \frac{d_B(N^k-1)^f m_k}{e_B m_r u_M}. \tag{9.50}$$

All factors are positive. This is `mainSecondMoment_div_identitySecondMoment_eq`.

Let $R_E = \mathrm{rd}_E$. The genus-mass calculation already proved in the note gives

$$\log\frac{m_r}{m_k} = f\sum_{j=k}^{r-1}\Big((j+\tfrac12)\log R_E + \log(j!) - (j+1)\log(2\pi)\Big) + E_{\mathrm{mass}}, \quad |E_{\mathrm{mass}}| \le 4f\,2^{-k}. \tag{9.51}$$

Indeed its error at step j is $\log L_{E^+}(j+1, \eta^{j+1})$, of modulus at most $4f2^{-(j+1)}$; sum the geometric series. This uses the same standard lattice and Haar normalization as (9.50).

Put

$$G = \log\frac{MR_E}{2\pi}, \qquad h_0 = \log\frac{\sqrt{R_E}}{2\pi}.$$

Proposition 5.2 chooses the fine prime in a dyadic interval, so

$$\log N = mS/f + (m-1)G - \tfrac{15}{2}L + E_N, \qquad 0 \le E_N \le \log 2.$$

Also write $e_N = \log(1-N^{-k})$, so $-2/N \le e_N \le 0$. Substituting these formulas, (9.49), and (9.51) into (9.50), all factorials cancel except $(k-1)!$. The result is

$$\log(C/I) = fk(\log M - \tfrac{15}{2}L) + \log(d_B((k-1)!)^f) + T, \tag{9.52}$$

$$T = mS - f(m-1)(\varrho G + h_0) + fkE_N - E_{\deg} - E_{\mathrm{mass}} + fe_N. \tag{9.53}$$

Here $\sum_{j=k}^{r-1} j = (m-1)(k+\varrho)$, since $r = k+m-1$ and $m = 2\varrho+2$. The power u_M cancels the remaining unipotent factor. As $S \le fL$, $G, |h_0| \ll L$, and $\log(m\Delta) \ll \log x \ll L$, equations (9.49)–(9.53) give

$$\frac{T}{fk} = E_N + O\left(\frac{m^2 L}{k} + \frac{m}{x} + \frac{1}{k}\right) = E_N + O(x^{-1/2}). \tag{9.54}$$

We used $m \le 2\sqrt{x}$, $k \ge x^5/2$ and $L^{100} \le x$. In particular $|T| \le fkL/4$ for sufficiently large x. Inserting this and (9.47) into (9.52) gives

$$b^{11kf} \le \frac{C}{I} \le (M/b^6)^{kf}. \tag{9.55}$$

For the lower bound use $M \ge b^{20}$; for the upper bound the exponent of b is $-15/2+5/4+1/4 = -6$.

Lean. The formal comparison allows $E_N \le \sqrt{L}$. The dyadic interval and the product in (9.49) give the error in (9.54). `mainSecondMoment_div_identitySecondMoment_mem_Icc`.

Equations (9.45), (9.46), and (9.55) discharge the three scalar hypotheses in (9.43). They use the compact type and formal degree of the specified representation and the earlier genus-mass and source lattice identities. They do not assert that an arbitrary chosen Poincaré map has those coordinates, or prove any local oscillatory, shortness, denominator, or height estimate for that map.

Further Lean references. The explicit weights, row parameter, and chosen level are in `LevelAndTwists.lean`, declarations `targetParameter_eq`, `log_liftWeight_le`, and `log_N_mem_Icc_logLevelTarget`. The sphere-product comparison is in `FrameMass.lean`, declarations `log_prod_unitSphereMass`, `abs_log_prod_unitSphereMass_sub_le`, and `prod_frameMass_mul_le`.

There is a shorter summation once these normalized estimates are available. At a fixed stage $i < \varrho$, let t, u range over the finite alphabet in $E^\times$ and w_t be the sum of the actual nonnegative global weights of prefixes ending in t. For $t \ne u$ put $y_{tu} = (t^{-1} - u^{-1})/M$ and form the matrix $B_{tu} = \beta(((M/r_v)\Phi_v(y_{tu}))_v, (y_{tu})_T)$, with zero diagonal. Here r_v is the prescribed frame scale squared at i and $\beta \ge 0$ is the global frequency majorant. The inverses of the entries and of M are integral off T. For fixed t, the nonzero y_{tu} are distinct, and the same holds for fixed u. The stated bound $\sum_y \beta \le \varepsilon$ on finite sets of such frequencies at this scale therefore bounds each row and each column sum. Using $2w_t w_u \le w_t^2 + w_u^2$, the total absolute unequal-entry contribution is at most $\varepsilon \sum_t w_t^2$. Equal last entries give precisely the original density sum. Admissible scales and frequency budget, reciprocal heads and the equal-entry sum, global head weights.

9.1 The period map and its matrix coefficients

We identify the operators in the moment estimates with the blocks of the Poincaré map by using the normalized Bessel functions, the free source level and the absolute convergence estimates proved above. Write $\mathcal{G}$ for the adelic unitary group of Y_0 and G for its finite residue group.

The finite frame and the two projectors. Let $F = E^+$, let $H = U(X_0)$, and write $R = U \rtimes H$ for the Bessel subgroup of the algebraic group of Y_0. Let $\Gamma = U(Y_0)(F)$, let K be the full integral finite level, and let $J \lhd K$ be the principal level at the chosen places. Then $K/J = G$. The target before imposing its good types is the space T of J-invariant automorphic forms of the prescribed archimedean type. Here a form means a bounded continuous left Γ-invariant function fixed by right convolution with the kernel e_∞ defined below. Thus each $y \in T$ has a bound $|y(g)| \le C_y$ on all of $\mathcal{G}$; no bound uniform in y is required. This is the definition of the form space used for the principal-level target. Bounded forms of a coefficient type. Its inner product is

$$\langle y, z\rangle_T = \int_{\Gamma\backslash\mathcal{G}} \overline{y(g)} z(g)\, dg.$$

The quotient measure is induced by the specified product Haar measure on $\mathcal{G} = G_\infty G_{\mathrm{fin}}$. Right translation by a lift $k_s \in K$ defines $\rho(s)y(g) = y(gk_s)$. It is independent of the lift and unitary.

There are two different projectors in this construction. The archimedean coefficient kernel is

$$e_\infty(x) = \prod_{v|\infty} e_v \overline{c_v(\iota_{\infty,v}(x_v))}, \qquad e_B = \prod_{v|\infty} e_v, \tag{9.56}$$

where ι_∞ is the product of the fixed sign-frame group isomorphisms to the groups G_v. Right convolution with e_∞ selects the prescribed coefficient type. Schur orthogonality gives its self-adjoint idempotence, and the large-gap estimates give $e_\infty \in L^1(G_\infty)$. The finite group projector e selecting the good residue types will be identified below. These projectors act in different variables.

The fit between the lattice coordinates used in the amplitudes and the orthonormal finite coordinates of Y_0 is exact. Put $u = \mathrm{latticeScale}(0)$ and let $\eta = u^{-1}$ act centrally on X_0. Comparing the middle and string blocks of the displayed graded lattice basis gives

$$L_{\mathrm{graded}} = u\,\iota(\eta) L_{\mathrm{filling}}. \tag{F}$$

The finite frame of Y_0 is $L_{\mathrm{graded}} k_0$, where $k_0 \in \mathrm{GL}_r(\widehat{\mathcal{O}}_E)$ reduces to 1 at every chosen place. Consequently the finite coordinate map Ψ is conjugation by k_0 on integral matrices. Thus

$$k \in K \Longleftrightarrow \Psi(k) \in K_{\mathrm{filling}}, \quad k \in J \Longleftrightarrow \Psi(k) \in J_{\mathrm{filling}}, \quad \mathrm{red}\,\Psi(k) = \mathrm{red}\,k. \tag{9.57}$$

Indeed reduction commutes with multiplication, and $\mathrm{red}\,k_0 = 1$. The scalar u in (F) disappears under conjugation; its middle-block correction is precisely η.

the frame k_0 can be obtained by first orthonormalizing the unimodular graded lattice at the finite places, then multiplying this orthonormal frame by an integral unitary matrix with inverse reductions at the chosen split places. At a split place, a matrix on the chosen factor lifts arbitrarily and determines the inverse adjoint on the conjugate factor. This proves the required surjectivity of reduction. The only classification used here is the usual local classification of unimodular Hermitian lattices over the split or unramified quadratic local extension present in this datum. These are the only cases: for any finite rational prime p, the cyclotomic part of E contains a root of unity of odd order prime to p, on which complex conjugation is nontrivial. Inertia at p acts trivially on that root, so it cannot contain complex conjugation. Hence E/E^+ is unramified at every finite place.

Let $\chi : G_\infty \longrightarrow \prod_v G_v$ be the real coordinate map to the groups used for the local Bessel functions. More explicitly, put $q_\infty = \chi(1)$. Then

$$\chi(x) = q_\infty \iota_\infty(x), \qquad \vartheta_\infty(s) = q_\infty \iota_\infty(s) q_\infty^{-1}, \qquad \chi(sx) = \vartheta_\infty(s)\chi(x).$$

The variable in the product of the local Bessel groups corresponding to $s \in R(F_\infty)$ is $\vartheta_\infty(s)$. Thus χ need not be a homomorphism. The exact identity needed for coefficient arguments is

$$\chi(g)^{-1}\vartheta_\infty(s)\chi(x) = \iota_\infty(g^{-1}sx). \tag{9.58}$$

Both factors $q_\infty^{-1} q_\infty$ cancel upon substitution. The sign-frame isomorphism carries the specified Haar measures to each other, and left translation by q_∞ preserves Haar, so

$$\chi_*(dx) = \prod_v dg_v. \tag{M}$$

For a rational point γ, these coordinates satisfy $\chi(\gamma_\infty x) = \gamma_{\mathrm{fill},\infty}\chi(x)$. They are the same coordinates that occur in the amplitude integrals.

Source coordinates and the central translation. Let J_H be the source level and $\mathcal{C} = H(F)\backslash H(\mathbb{A}_f)/J_H$. The free-level proof above shows that every stabilizer is trivial. Evaluation at representatives c therefore gives a linear isomorphism between source forms at this level and arbitrary vectors of their compact coefficient space, one for each $c \in \mathcal{C}$.

Let V_H be that coefficient space, let I index the tensor product of the Bessel value spaces, and write $\sigma = \bigotimes_v \sigma_v$ for the compact representation on the Bessel values. The equality of the prescribed compact types gives an invertible map $A : V_H \longrightarrow \mathbb{C}^I$, with values interpreted as row vectors, such that

$$A(\tau(h)\zeta) = (A\zeta)\sigma(h)^{-1}. \tag{D}$$

Here τ is the action on source values. Take A to be the fixed intertwiner used in `sourceFormCoords`, called `weylBesselEquiv` in the source. It is obtained by choosing local intertwiners between the representations of the same highest weight and taking their tensor product. This is the dual-coordinate convention; the inverse in (D) is essential.

For an original source form ϕ, set $\psi = T_{\eta^{-1}}\phi$, where T denotes the central finite translation, and put

$$C\phi(c,i) = (A\psi(c))_i, \qquad C : X_{\mathrm{alg}} \xrightarrow{\sim} X_{\mathrm{coord}} = \mathbb{C}^{\mathcal{C}\times I}. \tag{9.59}$$

For this datum $T_{\eta^{-1}}$ is the translation by latticeScale(0). This is precisely the source-coordinate map used by the filling construction. It is a linear equivalence. The Hilbert space in the second-moment argument is X_{coord} with its ordinary Euclidean inner product; equivalently one can transport that inner product to X_{alg} using C. No assertion that arbitrary algebraic coordinates are orthonormal is needed.

Let h_c be the class representative in filling coordinates and set

$$\widetilde{h}_c = \Psi^{-1}(\iota(\eta)^{-1}h_c).$$

In the Bessel group this is the finite point $\iota(c\eta^{-1})$. A rational point has finite coordinate image

$$\Psi(\gamma_f) = \iota(\eta)^{-1}\gamma_{\mathrm{fill},f}\iota(\eta).$$

The central factors therefore cancel in

$$\Psi(\widetilde{h}_c^{-1}\gamma_f\widetilde{h}_{c'}) = h_c^{-1}\gamma_{\mathrm{fill},f}h_{c'}. \tag{R}$$

It follows from (R) and the residue-preserving property of Ψ that the rational-return set used below is exactly $\mathcal{T}(c,s,c')$ of the amplitude definition.

Convergence of the Poincaré series. Write $W(y) = \bigotimes_v W_v(y_v)$ for the full normalized Bessel function, and β_c for the class cutoff. For $a = (c,i)$ define

$$f_{c,i}(x,k) = \beta_c(\chi(x))\, W_i(\chi(x))\, \mathbf{1}_{\widetilde{h}_c J}(k), \qquad \mathcal{P}_{c,i}(g) = \sum_{\gamma\in\Gamma} f_{c,i}(\gamma g). \tag{9.60}$$

The cutoff here is the class cutoff. The function W has not been replaced by its archimedean truncation.

Here is a direct convergence check using the estimates already proved. Let $i_0 = r^3 f$, let $U(y)$ be the product of the scaled Bessel lengths, and let a_v be the finite constants in the local positive majorants. The weighted frame-packing argument gives, uniformly in y, c, s,

$$\sum_{c',\,\gamma\in\mathcal{T}(c,s,c')} \beta_{c'}(\gamma^{-1}y)U(\gamma^{-1}y)^{-i_0} \le 2b^{12n^3rf}.$$

The local majorants give $\|W(z)\| \le (\prod_v a_v) U(z)^{-i_0}$. Thus the corresponding sum with $\|W(\gamma^{-1}y)\|$ is uniformly bounded. To apply this to an arbitrary point g in (9.60), either every finite-level indicator vanishes, or choose one surviving rational translate δ. Every other surviving translate is uniquely $\gamma^{-1}\delta$, with $\gamma \in \mathcal{T}(c,1,c)$. The preceding bound at the real point of δg proves

$$\sup_g \sum_{\delta\in\Gamma} |f_{c,i}(\delta g)| \le 2b^{12n^3rf} \prod_v a_v < \infty. \tag{C}$$

The quotient $\Gamma\backslash\mathcal{G}$ has finite volume. Tonelli and (C) show that every seed is integrable and every series is bounded and square integrable. Multiplying by any bounded automorphic form is consequently compatible with unfolding.

Let U_c be the rational unipotents satisfying $h_c^{-1}u_f h_c \in J_{\text{filling}}$. The class cutoff satisfies $\sum_{u\in U_c} \beta_c(u_\infty y) = 1$. The full Bessel vector is invariant under these rational translates: its finite character is trivial on the principal level, and its adelic character is trivial on rational points. Hence grouping the series first over U_c removes the class cutoff and leaves a translate of W_i, multiplied by its finite-level indicator. The reproduction identity for W shows that each such grouped term is fixed by right convolution with e_∞. Bound (C) and $e_\infty \in L^1$ allow convolution to pass through the sum. The result is therefore fixed by the prescribed kernel, left Γ-invariant and right J-invariant. Convolution with this continuous integrable kernel also gives its continuous representative. Thus it belongs to the same principal target T. Define the linear map $\mathcal{P} : X_{\text{coord}} \longrightarrow T$ by the columns $\mathcal{P}_{c,i}$.

Unfolding a single series and then integrating over $\widetilde{h}_c J$ proves, for a bounded automorphic function z of principal level,

$$\int_{[\mathcal{G}]} z(g)\mathcal{P}(v)(g)\,dg = \operatorname{vol}(J) \sum_c \int_{G_\infty} \beta_c(\chi(x)) \left(\sum_i v_{c,i} W_i(\chi(x)) \right) z(x, \widetilde{h}_c)\,dx. \tag{9.61}$$

In the finite integral $z(x, \widetilde{h}_c j)$ is constant in j. The coefficient is therefore $\operatorname{vol}(J)$.

Evaluation of the period at a single point. We next identify $\mathcal{P}C$ with the map defined by the Bessel period. This step matters because the subsequent determinant construction uses that period-defined map.

At each infinite place the local vector has the construction already established in the local-period argument:

$$W_v(y) = a'_v \int_{R_v} c_v(x_v^{-1} s y)\, \chi_v(s)^{-1} \sigma_v(h_s)^{-1} u_v\, ds_v. \tag{A}$$

Here a'_v is its fixed normalization, x_v its chosen stationary point, and the integral is absolutely convergent. Choose $g_0 \in G_\infty$ with $\chi(g_0) = (x_v)_v$. In (A), the local variable s_v corresponding to a real Bessel point s is $\vartheta_\infty(s)_v$. In particular

$$\prod_v c_v\Big(\chi(g_0)_v^{-1} \vartheta_\infty(s^{-1})_v \chi(x)_v\Big) = e_B^{-1} \overline{e_\infty(g_0^{-1} s^{-1} x)}$$

by (9.58); this verifies the coefficient argument without treating χ as a group homomorphism. The product of the local Haar measures on R_v, transported to $R(F_\infty)$, is $c_\infty\, ds$ for a fixed $c_\infty > 0$. This follows from uniqueness of Haar measure. Put $w = \bigotimes_v u_v$ and define the fixed functional

$$t_0(\zeta) = \frac{c_\infty \prod_v a'_v}{e_B} (A\zeta) \cdot w. \tag{9.62}$$

Tensoring (A), applying Fubini, using (D), and then replacing s by s^{-1} gives

$$(A\psi(c)) \cdot W(\chi(x)) = \int_{R(F_\infty)} \overline{e_\infty(g_0^{-1}s^{-1}x)}\, \mathcal{W}_{t_0}(T_\eta\psi; s, \widetilde{h}_c)\, ds. \tag{9.63}$$

The inverse-variable substitution preserves measure because $R(F_\infty)$ is unimodular. Explicitly its character becomes $\chi_R(s)$, and its coefficient becomes $t_0(\tau(h_s)^{-1}\psi(c))$, exactly the definition of $\mathcal{W}_{t_0}$ on this class. The character agrees with the datum's generic character because

$$\jmath\,\mathrm{genericConst} = -(M\delta_E)^{-1}.$$

This equality is part of the displayed grading; it checks the sign and conductor of the period. In particular g_0, t_0 are fixed before either ψ or a target test form is selected.

We record the quotient-measure computation. Set $J' = R(\mathbb{A}_f) \cap J$. The natural decomposition is

$$R(F)\backslash R(\mathbb{A})/J' = \coprod_{c\in\mathcal{C}} U_c\backslash R(F_\infty). \tag{Q}$$

To see surjectivity, decompose a point of R into its unipotent and H parts, select the class of the latter, and approximate the finite unipotent part by a rational unipotent. Such approximation follows by applying the Chinese remainder theorem successively in additive coordinates of the unipotent group. For disjointness, a rational Bessel element connecting two class cosets first identifies their H-classes. Its remaining rational H-component is a source-class stabilizer and is therefore 1. The remaining element is precisely in U_c.

The fixed Haar measure on $R(\mathbb{A})$ is a positive multiple of the product of ds and a Haar measure on $R(\mathbb{A}_f)$. Integrating that finite factor over J' gives a number $v_{J'} > 0$, independent of c. Thus (Q), or equivalently its cutoff partition of unity, gives

$$\int_{[R]} F(r)\, dr = v_{J'} \sum_c \int_{R(F_\infty)} \beta_c(\chi(s)) F(s, \widetilde{h}_c)\, ds \tag{9.64}$$

for every integrable left rational-invariant, right J'-invariant function F. This formula uses the Bessel measure dr; no normalization of it has been assumed.

Insert (9.63) into (9.61), taking $z = \overline{y}$ for $y \in T$. The class-cutoff quotient can be moved from the x variable to the s variable. Indeed the integrand is invariant under simultaneous left multiplication of (s, x) by U_c; insert the sum of the translates of either cutoff, which is 1, and change variables term by term. Absolute integrability follows from boundedness of y, $e_\infty \in L^1$, and integrability of $\mathcal{W}_{t_0}$ on $U_c\backslash R(F_\infty)$. The last property follows also from (Q): the source period function is continuous on the compact quotient $[R]$. After moving the cutoff, the inner integral is

$$\int_{G_\infty} \overline{e_\infty(g_0^{-1}s^{-1}x)}\, \overline{y(x, \widetilde{h}_c)}\, dx = \overline{y(sg_0, \widetilde{h}_c)}.$$

This is the conjugate coefficient-projector identity, with the substitution $x = sg_0z$. Now apply (9.64) and put

$$t = \frac{\mathrm{vol}(J)}{v_{J'}} t_0.$$

We obtain, simultaneously for every source form ϕ and every $y \in T$,

$$\langle y, \mathcal{P}C\phi\rangle_T = \int_{[R]} \mathcal{W}_t(\phi; r)\, \overline{y(rg_0)}\, dr. \tag{9.65}$$

Here the central translations cancel: $T_\eta\psi = T_\eta T_{\eta^{-1}}\phi = \phi$. Both Haar ratios have been absorbed into the one fixed functional t. No factor depending on a source vector, class, target vector, or translate has been introduced.

By the defining property of the period map Π, the right side of (9.65) is $\langle y, \Pi\phi\rangle_T$. Nondegeneracy of the Petersson inner product gives

$$\Pi = \mathcal{P}C. \tag{9.66}$$

The Petersson block and its scale. For $s \in G$, unfold the second factor of $\langle \rho(s)\mathcal{P}_{c',j}, \mathcal{P}_{c,i}\rangle_T$ by (9.61). The translated first factor is still right J-invariant because $J \triangleleft K$. At the point $(x, \widetilde{h}_c k_s)$, its series is

$$\mathcal{P}_{c',j}(x, \widetilde{h}_c k_s) = \sum_{\gamma \in \mathcal{T}(c,s,c')} \beta_{c'}(\gamma^{-1}\chi(x)) W_j(\gamma^{-1}\chi(x)).$$

To check the indexing, its γ^{-1}-term survives exactly when $\widetilde{h}_{c'}^{-1}\gamma_f^{-1}\widetilde{h}_c k_s \in J$, or equivalently $\widetilde{h}_c^{-1}\gamma_f\widetilde{h}_{c'} \in k_s J$. Equation (R) turns this into the integral return with reduction s.

Using (M) now proves the exact formula

$$\langle \rho(s)\mathcal{P}_{c',j}, \mathcal{P}_{c,i}\rangle_T = \mathrm{vol}(J) \sum_{\gamma \in \mathcal{T}(c,s,c')} \int \beta_c(y)\beta_{c'}(\gamma^{-1}y) W_i(y)\overline{W_j(\gamma^{-1}y)}\, dy. \tag{9.67}$$

The positive amplitude majorant proved above justifies exchanging this sum and integral. In particular these are the full amplitude blocks, with both class cutoffs and the original unipotent measure.

The common unipotent covolume is the same $\nu > 0$ that appears in the class-cutoff identity $\int_{U(F_\infty)} \beta_c(uy)\, du = \nu$. Set

$$\alpha = (\mathrm{vol}(J)\nu)^{-1/2} > 0, \qquad L = \alpha\, \Pi C^{-1} = \alpha\, \mathcal{P}.$$

Since the inner product is conjugate linear in its first variable, the scalar in its two arguments contributes α^2. Equation (9.67) therefore becomes

$$\langle \rho(s) L e_{c',j}, L e_{c,i}\rangle_T = \nu^{-1} A(c,s,c')_{ij}. \tag{9.68}$$

With $P_s = L^*\rho(s)L$, the matrix entry in this basis is

$$(P_s)_{(c,i),(c',j)} = \langle e_{c,i}, P_s e_{c',j}\rangle = \nu^{-1}\overline{A(c,s,c')_{ij}}.$$

Thus $\|P_s\|_{\mathrm{HS}}^2 = \nu^{-2}\sum_{c,c',i,j} |A(c,s,c')_{ij}|^2$, exactly the second moment used throughout this section.

The same good projector and the same arithmetic maps. At a chosen place w, let $z_{w,\lambda}$ be the central orthogonal idempotent for the finite type labelled by λ. These idempotents are pairwise orthogonal. For $i = 0, 1$, define in the group algebra above p_i

$$z_i = \sum_{\lambda \in \mathrm{Good}_i} \bigotimes_{w|p_i} z_{w,\lambda_w}.$$

The families Good_i are exactly the good tuples from the finite character calculation. Expanding the product shows

$$z_0 \otimes z_1 = \sum_{\lambda_0 \in \mathrm{Good}_0,\, \lambda_1 \in \mathrm{Good}_1} \bigotimes_{i,w|p_i} z_{w,\lambda_{i,w}} = e_0, \tag{9.69}$$

where the last expression is the group-algebra idempotent defining the target of prescribed finite type. The equality holds coefficient by coefficient. Reindexing the product residue group introduces no factor of $|G|$. Since ρ is unitary, the operator $e = \rho(e_0) = \rho(z_0)\rho(z_1)$ is precisely the central orthogonal projector used in the lower-mass argument.

Finally, for the original ramified source X_{ram}, put $X'_{\mathrm{ram}} = C(X_{\mathrm{ram}})$. The arithmetic maps of the next section satisfy

$$B_s = c_{\mathrm{ss}}^{-1}|G|\, e\rho(s)\Pi|_{X_{\mathrm{ram}}} = c_{\mathrm{ss}}^{-1}|G|\alpha^{-1}(e\rho(s)L|_{X'_{\mathrm{ram}}})\, C|_{X_{\mathrm{ram}}}. \tag{9.70}$$

The prefactor is nonzero and $C|_{X_{\mathrm{ram}}}$ is an isomorphism onto X'_{ram}. Hence the individual ranges, the ranges of every selected family of translates, and the source codimensions are the ones used by the moment and spanning arguments.

Dimension of the image of the finite projector. Equations (9.66), (9.68) and (9.69) identify the period map, its matrix coefficients and the projector onto the prescribed finite types. To normalize the second moment we still need the bound

$$d_0 \le |G|\, e_B\ \mathrm{mass}_r\,.$$

Each finite-type summand has a supercuspidal component above both split primes. Section 10 uses these components to identify the finite Hecke modules at each W_j with those at W_m. Counting definite classes then gives the bound.

Lean references. All declarations here refer to commit `72fead102f663d5c7a407964a13dc0cf9d977207`, under `grh/src/`. The finite-frame construction is in `Rows.lean`, `exists_space_isStd_of_le_and_gram_eq_of_lt`; the fit and central coordinates are in `FillingAdmissible.lean`, `filling_fits` and `sourceFormCoords`. The seed, convergence, target membership and Petersson computation are in `PoincareSeries.lean`, `cutBessel`, `exists_tsum_enorm_cutBessel_le`, `exists_levelSpace_tsum_cutBessel_eq`, and `petersson_poincare`. The local-to-global functional is `BesselFormula.lean`, `exists_besselFn_eq_integral`. The period disintegration and Riesz identification are `SeriesPeriod.lean`, `exists_integral_besselMeasure_eq_sum`, `exists_integral_poincareSeries_eq_period`, and `poincare_eq_poincareMap_comp`. Finally, `SpanningTheorem.lean`, `hasAmplitudeBlocks_scaledPoincare` and `levelRepresentation_goodIdempotentAbove_eq`, are precisely the two identities proved above.

10 Automorphic packets and the dimension of the target

The dimension of the target normalizes the second moment. We compare the image of the finite projector on each unitary group with its image on the definite group, where a class count bounds the dimension.

For a Hermitian space W in the sequence $Y_0, W_1, \dots, W_m$, let goodForms(W) denote the principal-level automorphic forms of the prescribed archimedean type in the image of the finite projector e. Write $e_B = \prod_{v|\infty} e_v$ for the product of the archimedean formal degrees. We prove, for every $1 \le j \le m$,

$$\dim_{\mathbb{C}} \text{goodForms}(W_j) = \dim_{\mathbb{C}} \text{goodForms}(Y_0) = d_0, \qquad d_0 \le |G| e_B \,\text{mass}_r\,. \tag{10.1}$$

The spaces $Y_0, W_1, \dots, W_m$ have the same finite local Hermitian spaces over E/F, where $F = E^+$. Fix the specified finite isometries, which identify their finite adelic group U_f, maximal level K_f, and principal level J. Their common rank r is a power of three by Proposition 5.2. At a real place v the chosen discrete series has regular Harish–Chandra parameter λ_v. Its unordered entries are the same for every space, and its sign pattern prescribes which entries are assigned to each part of the signature.

Let $e_{v,s}$ be the minimal coefficient idempotent of the chosen discrete series at v, for the sign pattern s. Thus $e_{v,s} = d_{v,s}\overline{\beta_{v,s}}$ with the coefficient convention of this note, $e_{v,s} * e_{v,s} = e_{v,s} = e_{v,s}^*$, and convolution by $e_{v,s}$ selects one line in each copy of that discrete series. Haar measure is the compact-dual measure already used for e_B; the compact group $U(r)$ and each compact Cartan have probability measure. Put $e_{\infty,s} = \prod_v e_{v,s}$.

The two cuspidal factors. Write $e_0 = \sum_{c\in\mathcal{C}} A_c$, where $A_c = \bigotimes_w e_{c_w}$ runs over the orthogonal good-label projectors. Each good family contains a primitive label above p_0 and a primitive label above p_1. These give distinct split places for each individual summand. At either place put $K = \mathrm{GL}_r(\mathcal{O}_w)$ and $K_1 = \ker(K \to \mathrm{GL}_r(\kappa_w))$. Green's parametrization assigns the primitive degree-r label a cuspidal representation τ of K/K_1 [Gre55]. The isotypic projector of this finite group [Ser77, §2.6, Theorem 8(ii)], expressed in Haar measure with $\mathrm{vol}\, K_1 = \mathrm{vol}\, K/|K/K_1|$, is

$$\phi_\tau(k) = \frac{\dim \tau}{\mathrm{vol}\, K}\overline{\chi_\tau(\bar{k})}\, 1_K(k). \tag{10.2}$$

The inflated τ is a type for the unramified twists of a level-zero supercuspidal representation of $\mathrm{GL}_r(F_w)$ [BK93, Theorem 8.4.1]. Thus a constituent selected by A_c is supercuspidal at the two indicated places. It is cuspidal globally: a summand in the continuous or residual spectrum comes from induction from a proper Levi, whose local subquotients at a split place cannot be supercuspidal.

Changing the archimedean packet member. Let $\pi = \pi_\infty \otimes \pi_f$ be such a constituent on one of the spaces. Since r is odd, the two supercuspidal components give a cuspidal weak base change Π to $\mathrm{GL}_r(\mathbb{A}_E)$ [HL04, Theorem 2.2.2]. Use the same standard base-change normalization for all the inner forms. Corollary 3.3.2 of [KMSW14] assigns a parameter for the unitary group to the nonzero discrete spectral summand containing π. Its standard transfer agrees with Π at almost all places. Strong multiplicity one for isobaric representations of the general linear group [JS81] therefore identifies its entire isobaric expansion with the single cuspidal summand Π. Thus the parameter is $\Pi[1]$, with trivial Arthur SL_2.

The isometries from the quasi-split Hermitian space over an algebraic closure form a torsor under its unitary group. Hence the groups under consideration are pure inner forms. For a generic

parameter on a pure inner form the multiplicity formula of Kaletha, Mínguez, Shin and White applies [KMSW14, Theorem 1.7.1 and the scope statement following it]. The centralizer of the simple parameter consists of the invariant central scalars $\{\pm 1\}$. Dividing by the invariant center therefore gives the trivial global component group. The character condition in the multiplicity formula is automatic: every product of locally relevant packet members, unramified almost everywhere, occurs with multiplicity one.

Transport π_f by the fixed finite isometries. At each real place the discrete packet is determined by the unordered regular parameter. The prescribed sign pattern gives its specified member on the new signature [Ato20, Theorem A.4]. The product with this new archimedean part is automorphic by the multiplicity formula. Reversing the change recovers π. The correspondence is independent of the summand A_c that detects π. Count each constituent hit by e_0 once. Since the real coefficient selects one line, a matched pair has the same full space π_f^J with its Hecke action, and hence the same projected space $e_0(\pi_f^J)$. Every finite Hecke operator at level J commuting with e_0 acts identically under this identification. The direct sum over the matched pairs proves the dimension equality in (10.1).

Comparison with Lean. The formal proof compares positive Hecke traces by orbital integrals and recovers the spherical multiplicities from a parahoric identity. The packet argument identifies the finite modules themselves.

Lean. `LineDeterminant` (L282-L348), `EqualMultiplicities` (L232-L292).

Counting on the definite terminal form. On W_m, the infinite group is compact. Let $V_\lambda = \bigotimes_v V^c_{\lambda_v}$ be its algebraic weight. A principal-level form is a choice of a vector of V_λ at each finite double coset, invariant under its rational stabilizer. Those stabilizers are trivial. Indeed they are finite, since the Hermitian space is definite. Their elements reduce to 1 at a chosen place over each of the distinct primes p_0, p_1. The kernel of reduction in $\mathrm{GL}_r(\mathcal{O}_w)$ is pro-p_i; a finite-order element in it has p_i-power order. Its order is therefore both a p_0-power and a p_1-power, hence is 1. This argument also works after conjugating the finite frame.

The mass at maximal level is $\mathrm{mass}_r = \sum_x |\operatorname{Aut}(x)|^{-1}$. Over a maximal-level class with automorphism group Γ_x, the principal-level classes are its orbits on K_f/J. The action is free by the preceding argument, so their number is $|G|/|\Gamma_x|$. Summing gives exactly

$$\#\, U(W_m)(F)\backslash U_f/J = |G|\,\mathrm{mass}_r\,.$$

Thus

$$\dim \mathrm{forms}^J_{\lambda,W_m} = |G|\,\mathrm{mass}_r \prod_v \dim V^c_{\lambda_v}.$$

Harish–Chandra's formal-degree formula in compact-dual measure is

$$d_{v,s} = \dim V^c_{\lambda_v} = \frac{\displaystyle\prod_{i<j}(\lambda^{\downarrow}_{v,i} - \lambda^{\downarrow}_{v,j})}{\displaystyle\prod_{j=1}^{r-1} j!}.$$

It is independent of s, so $\prod_v \dim V^c_{\lambda_v} = e_B$ for the Bessel systems used above. The good forms are a subspace of these principal-level forms. The first part of (10.1), applied to W_m, therefore proves $d_0 \le |G| e_B\,\mathrm{mass}_r$.

Lean. The compact-dual formal degree, The definite mass and dimension estimate..

The second moment and the choice of translates. Set $\beta(s) = \|P_s\|_{\mathrm{HS}}^2$ and let $C = v^2/(\deg_{\mathrm{Bess}} \operatorname{mass}_r)$, where $\deg_{\mathrm{Bess}}$ is the product of the local Bessel degrees and mass_r is the genus mass of the standard rank-r lattice. Then $\beta(a^{-1}b)$ is the Hilbert–Schmidt inner product of $\rho(a)LL^*\rho(a)^*$ and $\rho(b)LL^*\rho(b)^*$. For a stabilizer of frame vectors at j places, maximize its total β-mass multiplied by b^{6kj} over the places and vectors. Denote this maximum by M_0. Apply the global and complex-place estimates just proved to the open-stabilizer sum. At the maximizing stabilizer the open elements contribute at most $(1 + 4b^{-4})C$ after weighting. The complement contributes at most $b^{-4}M_0$ when another place is available. For the full set of places the bound is C. Thus $\sum_s \beta(s) \le M_0 \le (1+9b^{-4})C$ since $b \ge 1000$. Equation (10.1) gives $d_0 \le |G| \deg_{\mathrm{Bess}} \operatorname{mass}_r$ on this datum. Combining it with this estimate gives the required second moment on the same L:

$$d_0 \, \mathbb{E}_{s\in G} \|L^*\rho(s)L\|_{\mathrm{HS}}^2 \le (1 + 9b^{-4})v^2.$$

Maximal stabilizer argument; dimension and moment.

Together with the first moment, this is the input to Lemma 6.3 used in (6.3). We use those translates throughout. The formal proof chooses enough translates to leave codimension at most $4d_0/b^2$. Ramified rank and cost.

11 Short forms and their integral index

We apply the norm and divisibility bounds to the same translates of definite forms as in the dimension estimate. Let X denote the full source space and $V \subset X$ its ramified subspace. Retain d_0 for the dimension of the target. In the notation of the datum D, the archimedean term A and the remainder R are the quantities $\mathcal{A}$ of Section 6 and S of (4.1), respectively. Thus

$$d_0 = \dim_{\mathbb{C}} D.\text{targetForms}, \qquad A = D.\text{mainAmount}, \qquad R = D.\text{totalRemainder}.$$

Here m is the number of lifts and $a = xb$. The map on the ramified source V is

$$B_s = \#G\, e_0 \rho(s) c_{\mathrm{ss}}^{-1} \Pi, \qquad c_{\mathrm{ss}} \neq 0.$$

The map Θ_ξ is the chain of m lifts on the target forms for an allowed m-fold twist ξ; at a rational prime ℓ, allowed means that its modulus has norm prime to ℓ. The same normalization of B_s is used in the norm, index and rank estimates. Normalized map.

11.1 Short integral source forms and the local index

Put $D_X = \dim_{\mathbb{C}} X$ and $L = \log b$. We first construct short integral vectors in V. The local theta estimates then bound the norms of their lifts and the index of the lattice they generate.

A uniform short frame in a definite adelic lattice. Let E be the CM field, with f infinite places, and put $C_{\mathrm{diff}} = q_1 d$. This positive integer belongs to the different of $E/\mathbb{Q}$. For a finite adelic unitary matrix g of rank k, let

$$\Lambda_g = E^k \cap g\widehat{\mathcal{O}}_E^k, \qquad \|y\|_\infty^2 = \sum_{\tau : E^+ \hookrightarrow \mathbb{R}} \sum_{i=1}^{k} |\tau(y_i)|^2.$$

Choose one complex extension at each infinite place in this formula. There is a matrix $Y_g \in \mathrm{GL}_k(E)$ such that

$$g^{-1} Y_g \in M_k(\widehat{\mathcal{O}}_E), \qquad |\sigma((Y_g)_{ij})| \leq C_0, \qquad C_0 = 6 C_{\mathrm{diff}} (fk)^2 \tag{11.1}$$

for every embedding $\sigma : E \hookrightarrow \mathbb{C}$. The columns need only be an E-basis contained in Λ_g; no assertion that they are an $\mathcal{O}_E$-basis is required.

Give the Minkowski space the real inner product $\frac{1}{2}\operatorname{Tr}_{E/\mathbb{Q}} h$. Local Hermitian self-duality and trace duality for the different give $\Lambda_g^* = 2\mathfrak{D}_E^{-1}\Lambda_g$, so $(C_{\mathrm{diff}}/2)\Lambda_g^* \subseteq \Lambda_g$. For $0 \neq y \in \Lambda_g$, $h(y,y)$ is a totally positive algebraic integer. Its trace is at least f by the arithmetic–geometric mean inequality. Writing λ_j for the Euclidean successive minima, we obtain $\lambda_1(\Lambda_g^*) \geq 2\sqrt{f}/C_{\mathrm{diff}}$. Banaszczyk's transference theorem [Ban93, Theorem 2.1] gives

$$\lambda_{2fk}(\Lambda_g) \leq \frac{2fk}{\lambda_1(\Lambda_g^*)} \leq R_k := C_{\mathrm{diff}} k \sqrt{f} \leq C_0.$$

The lattice vectors in this ball span the whole real space; a proper E-subspace has smaller real dimension after Minkowski embedding. Hence k of the vectors give the required E-basis.

Lean. `Weaktransference` (L71-L211), `Tracedualityandshortframes` (L673-L918). The successive-minima bound gives the column norm R_k directly; the formal proof uses the entry bound C_0.

The number of definite classes. For $n \geq 1$, set $C_n = 6C_{\mathrm{diff}}(fn)^2$ and let $h(K) = \#U(E^+)\backslash U(\mathbb{A}_f)/K$ for the standard definite form of rank n. If K lies between the principal level $K(Q)$ and the standard maximal compact $K_{\max}$, then

$$h(K) \leq (3Q)^{2fn^2} M_n \leq Q^{2fn^2}(1 + 2C_n^3)^{4fn^2}. \tag{11.2}$$

Indeed $K(3Q)$ has trivial class stabilizers. Each stabilizer is finite and acts trivially modulo 3 on the underlying $\mathbb{Z}$-lattice of rank $2fn$. The kernel of reduction modulo 3 in $\mathrm{GL}_{2fn}(\mathbb{Z})$ is torsion-free [Ser06, Lecture I, Lemma 1]. Above a maximal-level class with stabilizer Γ there are thus $[K_{\max} : K(3Q)]/|\Gamma|$ classes at level $K(3Q)$. The reduction modulo $3Q$ has at most $(3Q)^{2fn^2}$ possible matrices. Summing over Γ proves the first inequality.

In rank one, Hilbert 90 maps the ideal class group of E onto the classes of the norm-one torus via $a \mapsto a/\bar{a}$ on finite ideles. Local lifts may be taken to be units almost everywhere: at an unramified place multiply a lift by a power of the base uniformizer, and at a split place use the first factor. Hence $M_1 \leq h_E$. Put $R = \mathrm{rd}_E \leq C_{\mathrm{diff}}$. The bound $L_F(j, \eta^j) \leq \zeta(j)^f$ holds for $j \geq 2$, and $\sum_{j\geq 2} \log \zeta(j) \leq \sum_{m\geq 2} 1/(m(m-1)) = 1$. Multiplying (8.8) therefore gives $M_n \leq h_E e^f (nR)^{fn^2/2}$. Minkowski's bound represents each ideal class by an integral ideal of norm at most R^f. Summing their inverse squared norms gives $h_E \leq R^{2f}\zeta_E(2) \leq (R\zeta(2))^{2f}$. Since $9e(2C_{\mathrm{diff}})^2\sqrt{nC_{\mathrm{diff}}} \leq C_n^4$, these bounds give the second inequality in (11.2).

Lean. `ClassCount` (L122-L190). Lean counts integral matrices for short frames. Here the class number at a torsion-free level is its genus mass.

For the upper-chain and untwisted final levels, take $Q = MN$. For every admissible record, $x \geq 10^4$, $f, C_{\mathrm{diff}} \leq x^{13}$, $n \leq x^6$, $\log(MN) \leq 22L + 25mL$, $m \leq 2\sqrt{x}$, and $L \leq x^{1/100}$. Thus $C_n \leq 6x^{51}$, $\log(1 + 2C_n^3) \leq 160 \log x$ and $\log(MN) \leq 72x^{51/100}$. Since $\log x \leq \sqrt{x}$, (11.2) gives

$$\log h(K) \leq x^{25}(640 \log x + 144x^{51/100}) \leq x^{30},$$

also for maximal level $Q = 1$.

Integral evaluation vectors and delta forms.

Lemma 11.1. *On E^n take the standard Hermitian form $h(u, v) = \sum_i \bar{u}_i v_i$ and its self-dual lattice $\mathcal{O}_E^n$. Let K be a free compact open subgroup of $U(h)(\mathbb{A}_{E^+,f})$ stabilizing $\widehat{\mathcal{O}}_E^n$. Write the algebraic weight as polynomial diagrams with H boxes in total and determinant powers e_τ, counting any columns assigned to the latter only there. Put $s = \#\{\tau : e_\tau \neq 0\}$. The level space is spanned by forms with integral transported values at every finite embedding and squared Weyl norm at most*

$$R_n^{2ns}(nR_n)^{2H}$$

at every point and every complex embedding.

Proof. Fix a class representative g. For independent matrices of indeterminates Z_τ, form the vector polynomial

$$P_g(Z) = \prod_{e_\tau \neq 0} \det(\tau Y_{g^{e_\tau}}) \ \mathrm{ev}_\kappa((\tau(Y_g)Z_\tau)_\tau).$$

In the tableau basis the evaluation coordinates are products of flag minors [Lax16, §3]. Their linear independence and the invertibility of Y_g imply that the coefficient vectors of P_g span, since a functional annihilating them annihilates the polynomial. Their entries lie in the normal closure of E. A finite embedding composed with each τ gives the corresponding finite embedding of E. In the gauge at g, the polynomial part uses the integral matrix $g^{-1}Y_g$, and each determinant becomes

$\det((g^{e_\tau})^{-1}Y_{g^{e_\tau}})$ at the corresponding place. The minors have integer coefficients. Thus every coefficient vector is integral in this gauge, including for negative e_τ.

At any complex embedding the conjugates of the labeled matrices are again evaluated at embeddings of E. Each determinant has modulus at most R_n^n by Hadamard. On the unit torus in the variables, each column of $\tau(Y_g)Z_\tau$ has norm at most nR_n. In the exterior-tensor realization of the Weyl norm an evaluation vector is the tensor of the column wedges prescribed by the diagrams. Hadamard gives norm at most $(nR_n)^H$. Each coefficient of the conjugated polynomial is a Fourier coefficient on this torus, hence has norm at most $R_n^{ns}(nR_n)^H$. At a free level each coefficient defines a delta form ϕ supported on the class of g. For $g' = \gamma g k$, with $\gamma \in U(h)(E^+)$ and $k \in K$, one has $\phi(g') = \rho(\gamma)\phi(g)$ and, at each finite embedding, $\rho(g'^{-1})\phi(g') = \rho(k^{-1})\rho(g^{-1})\phi(g)$. Here k^{-1} preserves the integral weight lattice. At infinity every conjugate of γ is unitary for h and preserves the Weyl norm. Taking the direct sum over the class set proves the claim.

Lean. `ShortColumns` (L217-L328), `ShortColumns` (L339-L440), `WeylKernel` (L403). Lean spans by evaluations on roots of unity. Here the polynomial coefficients give a finite family; integrality is checked on the coefficients before applying the complex Fourier bound. □

For the source X_0 all $e_\tau = 0$. Its diagrams have only columns of height one, $a_\tau = a\,1_{\tau\in\Phi_a}$ at τ, so $B_0 = \sum_\tau a_\tau \le fa$. In degree q the evaluation coordinates are the monomials z^α, $|\alpha| = q$, with squared Weyl norm $\sum_\alpha \binom{q}{\alpha}|z^\alpha|^2 = \|z\|_2^{2q}$. This is the normalization in the lemma. The resulting bound is

$$(kR_k)^{2B_0} \le (k^3C_0^2)^{B_0} \le C_0^{6(kf+fa)}. \tag{11.3}$$

Lean. `Sourcecolumnsanddeterminantexponent` (L48-L71).

To see that K_0 is free, let a rational unitary element γ stabilize gK_0. Its characteristic coefficients are integral at all finite primes. At each of the f chosen places above p_0, $g^{-1}\gamma g$ reduces to the identity. Hence $\alpha = \operatorname{tr}\gamma - k$ is an algebraic integer divisible by their product, of norm p_0^f. Every complex conjugate of α has absolute value at most $2k$. Since $p_0 > (2k)^2$, the norm inequality forces $\alpha = 0$. A unitary matrix of rank k and trace k is the identity. The same argument applies to every level between K_0 and the chosen level. Freeness of the chosen levels.

Lemma 11.1 now applies to X_0.

For the present scales, $C_{\mathrm{diff}}, f, k \le b$, $k \le a$, and $L \ge 10$. Thus $\log C_0 \le 6L$ and

$$\log C_0^{6(kf+fa)} \le 72faL. \tag{11.4}$$

Thus X is spanned by integral delta forms whose values have squared norm at most $\exp(72faL)$ at every complex embedding.

Ramification without denominators. For each $w \mid \pi_N$, choose a local unit whose residue generates $\mathbb{F}_N^\times$, put its conjugate inverse at $\bar w$, and put one at all other primes. Let T_w be right translation by the corresponding norm-one scalar idele. Both that idele and its inverse are integral. The operators T_w commute and satisfy $T_w^{N-1} = 1$ on X. There are f such places. The source level imposes that the last row be $(0,\dots,0,1)$ modulo π_N, so the order of this scalar modulo K_0 is exactly $N-1$.

Put

$$P_{\mathrm{ram}} = \prod_{w\mid\pi_N}(1-T_w), \qquad V = \bigcap_{w\mid\pi_N}\ker(1+T_w+\cdots+T_w^{N-2}).$$

The translations generate $H = \prod_{w\mid\pi_N}\mathbb{F}_N^\times$ modulo K_0, since their residue scalars occupy independent coordinates. Let K' be the group generated by their representatives and K_0. The same trace

argument at p_0 proves that K' is free, since the added representatives are the identity there. Each fibre over a K'-class is consequently a free transitive H-set.

Choose one representative of each fibre and translate it by H. In these coordinates the forms on the fibre are the regular representation with coefficients in the value space. The same description holds for integral forms: the scalar representatives and their inverses preserve the transported coefficient lattice. For any coefficient ring A, the kernel of the norm sum in $A[\mathbb{F}_N^\times]$ is the augmentation ideal $I_w = (1 - T_w)A[\mathbb{F}_N^\times]$, since the norm sum replaces each coefficient by their sum. Augmentation splits as an A-module map by the identity basis vector. Tensoring these splittings shows that the simultaneous kernel on $A[H]$ is $\bigotimes_w I_w$, the image of P_{ram}. This calculation works with any value module, including at primes dividing $N - 1$.

Apply P_{ram} to the integral delta forms. Its 2^f translates of a given delta form have disjoint class supports because $N > 2$. At a point with nonzero value the norm is the norm of one translate. Thus

$$V = \operatorname{span}_{\mathbb{C}} S, \qquad S \subseteq X_{\mathrm{int}}, \qquad \|\sigma\phi(g)\|_{\mathrm{Weyl}}^2 \le \exp(72faL) \quad (\phi \in S,\ \sigma,\ g). \tag{11.5}$$

Each I_w is free of rank $N - 2$, so the same orbit calculation gives

$$\dim V = \left(\frac{N-2}{N-1}\right)^f D_X \ge \left(1 - \frac{f}{N-1}\right) D_X \ge D_X/2. \tag{11.6}$$

The last inequality uses $f \le b$ and $N - 1 \ge b^9$.

Lean. Lean bounds the kernels for one scalar at a time and uses a triangle inequality for their differences. Freeness of their joint action gives the exact dimension and disjoint supports. `Centralscalings` (L597-L761), `Dimensionandnorm` (L818-L1011).

The local assertion for theta lifts. For a prime p, write

$$g_p = D.\mathrm{gain}(p), \qquad \lambda_p = \frac{16fa}{p} \begin{cases} m, & p \equiv 1 \text{ or } -1 \pmod m, \\ 1, & \text{otherwise,} \end{cases}$$

and define

$$\varepsilon_p = \begin{cases} 2\lambda_p, & p < x_1,\ \chi_K(p) \ne 1, \\ 2mfb, & p = N, \\ 0, & \text{otherwise.} \end{cases} \tag{11.7}$$

Here $N \ge x_1$, so the two nonzero cases are disjoint. These losses give the finite correction in the determinant estimate of Section 7. Normalize $v_p(p) = 1$. For a chosen finite embedding over p, let $\mathcal{O}$ be its valuation ring and $\mathcal{M}_p = X_{\mathrm{int,p}}$. Write $B_{\xi,\Psi}$ for the composite upper theta lift, followed by restriction to Y_m, with twist ξ and chain datum Ψ. The required local theta statement is the existence of an integer $j'_p \ge 0$ and an $\mathcal{O}$-submodule $\mathcal{S}_p \subset X$ such that

$$\begin{aligned} &j'_p \le \varepsilon_p D_X, \qquad \mathrm{ColengthDvd}(\mathcal{M}_p, \mathcal{S}_p, p^{j'_p}), \\ v_p(B_{\xi,\Psi}\phi) \ge g_p - \varepsilon_p \quad &\text{for every } \phi \in \mathcal{S}_p \cap V, \text{ every allowed } \xi, \text{ and every chain datum } \Psi. \end{aligned} \tag{11.8}$$

The valuation inequality means the inequality for the integral gauge at every finite adelic point. The same $\mathcal{S}_p$ must work for all the indicated twists and chain data. Merely constructing one module separately for each twist would not establish this statement. Exact simultaneous local statement.

We use a definition valid even when $\mathcal{O}$ is not a discrete valuation ring: $\mathrm{ColengthDvd}(M, M', z)$ means that every finite family $\phi_1, \dots, \phi_n \in M$ has an n-by-n matrix u over $\mathcal{O}$ with $\det u \mid z$ and

$\sum_j u_{ij}\phi_j \in M'$ for every i. It neither requires a basis of M nor requires $M' \subseteq M$. Intersecting M' with M changes none of these conclusions.

These colength bounds multiply under composition of such changes of family, by multiplying their matrices. They pull back along linear maps by applying the same matrix to a preimage family. They push forward because a linear map commutes with the row combinations. They also add over disjoint lists of families: use a block diagonal matrix after permuting the list. These assertions prove the functorial operations needed below directly over $\mathcal{O}$. Colength and its functorial properties.

The error sum is elementary. Put $W = \log x_1$. For $p = mc \pm 1$ one has $m/p \le 2/c$. For all other primes put $c = p$. At most three primes receive the same integer c. Since $\log p \le W$ for $p < x_1$, summing over c gives, for any finite set F of primes,

$$\sum_{p\in F} \varepsilon_p \log p \le 192faW(1+W) + 2mfb\log N \ll faL. \tag{11.9}$$

Here $W^6 \le L$, $\log N \le 25mL$, $m^2 \le 4x$ and $a = xb$. The formal estimate uses the larger nonsplit allowance faL^2/p with the same congruence multiplier. Allowed-loss summation.

Transport of the local valuation bounds. Let $\mathcal{P} = \{p < x_1 : \ p \text{ prime}, \ \chi_K(p) \ne 1\}$. The gains and losses vanish off $\mathcal{P} \cup \{N\}$, and $g_N = 0$. In the notation $\varepsilon_p = \epsilon_p$ of this section,

$$h_p = g_p - \varepsilon_p, \qquad \sum_p (g_p - h_p)\log p \ll faL \tag{11.10}$$

by (11.9). The rational normalization in Lean.

Let $\mathcal{L}_\xi$ denote the lower chain on the unprojected principal-form space; its restriction to the target is Θ_ξ. The seesaw identity, with its common nonzero constant c_{ss}, says that for every allowed twist ξ there is a chain datum Ψ_ξ with

$$\mathcal{L}_\xi(c_{\mathrm{ss}}^{-1}\Pi\phi) = B_{\xi,\Psi_\xi}\phi \qquad (\phi \in V), \tag{11.11}$$

Theorem 23.38 proves this identity with one scalar for all ξ, using the finite tensor of p. 565. Corollary 23.39 supplies its exact original normalization and coefficient bound. By (11.8) the lower lift of each $\phi \in \mathcal{S}_p \cap V$ has valuation at least h_p. The operator $\#Ge_0\rho(s)$ preserves this bound. To check it, a central idempotent is a sum of primitive central idempotents, so

$$|G|e_0(t) = \sum_{\pi \text{ selected}} \dim(\pi)\chi_\pi(t^{-1})$$

is an algebraic integer. The lower lift intertwines the finite action with integral right translations. Its projected value is therefore a sum of values of valuation at least h_p with integral coefficients. Intertwining and integral projection.

Now take any finite family $y_i = B_s\phi_i$ with $\phi_i \in V \cap X_{\mathrm{int}}$, for one fixed s. Apply the matrix supplied by ColengthDvd$(\mathcal{M}_p, \mathcal{S}_p, p^{j'_p})$ to its preimage family. The row combinations remain in V; their images have lifted valuation at least h_p for every twist allowed at p. Since $\det u_p \mid p^{j'_p}$,

$$|\iota_p(\det u_p)|_p \ge p^{-j'_p}, \qquad j'_p \le D_X\varepsilon_p.$$

For the sharper index, only the colength of the module is charged. At a split or ordinary nonsplit prime $p \ne N$ the whole lattice suffices. At an exceptional prime the construction below costs at most $15faD_X/p$. At N, (11.13) and (11.14) give a block cost at most $(m/2+1)faD_X/k$. Since

$\sum_{p<x_1}(\log p)/p \le 2W$ and $\log N \le 25mL$, one translate has logarithmic index $O(D_X fa(W + m^2L/k)) = O(D_X fa(1+W))$. Use a block for every translate selected in (6.3). Their block diagonal matrix has logarithmic index at most

$$O(Dfa(1+W)\log x) = o(DfaL). \tag{11.12}$$

Here $D_X = v$, $|T|v = O(D\log x)$, $m^2L/k = o(1)$, and $(1+W)\log x = o(L)$ by (5.3) and (5.4). Each block works at every embedding for all twists allowed at its prime. The source's index deduction.

The colength at the level prime N. At $p = N$ we can prove the colength estimate in (11.8) directly. Let $\mathcal{A}_0, \mathcal{A}_L, \mathcal{A}_\vee$ be the modules of source forms integral at $1, \delta, \delta^\vee$, respectively, where $\delta = D.\mathrm{diagPointN}(0)$ and $\delta^\vee = D.\mathrm{dualPointN}(0)$. In the local matrix coordinates used below, put $D_N = D.\mathrm{diagPoint}(0, \Sigma_N)$; the defining relation is $\delta^\vee = \delta D_N^{-1}$. For every value coordinate α, let d_α be its total degree in the last variable, summed over the infinite places, and let

$$d_{\mathrm{tot}} = \sum_{c,\alpha} d_\alpha,$$

where c runs over source classes. Then

$$k\, d_{\mathrm{tot}} \le D_X fa. \tag{11.13}$$

For the present symmetric powers, permutation of the k variables permutes the monomial basis and its degrees. Thus their summed degree is equal for every variable. Summing over the k variables gives the total degree $B_0 \le fa$ for each coordinate and class, proving (11.13). Equivalently this follows by conjugating the one-coordinate diagonal action by permutation matrices and comparing determinants.

The explicit diagonal matrices defining these lattices give

$$\mathcal{A}_0 \subseteq \mathcal{A}_\vee, \qquad \mathrm{ColengthDvd}(\mathcal{A}_\vee, \mathcal{A}_L, N^{d_{\mathrm{tot}}}). \tag{$*$}$$

Here are the coordinate details of the second assertion. At each source class choose a rationally adjusted representative r_c for which $r_c\delta^\vee$ is integral above N and $r_c\delta$ is $\mathrm{diag}(1, \dots, 1, N)$ at the relevant embeddings, up to integral factors. Such representatives exist by weak approximation in the rational unitary group: its Cayley chart is a dense open chart in a vector space of skew-hermitian matrices, so ordinary weak approximation meets the required open local conditions. For simultaneous adaptation, choose the integral factor in the intersection of the maximal compact and its conjugate by that diagonal matrix.

Evaluation at these r_c makes the coordinates of $\mathcal{A}_\vee$ integral. Requiring the coordinate (c, α) to be divisible by N^{d_α} suffices for $\mathcal{A}_L$. This condition propagates from the representatives to the class because $\delta^{-1}K_0\delta$ is integral above N, the direct last-row congruence defining K_0. The required coordinate operator is the diagonal matrix with entries N^{d_α}, whose determinant is $N^{d_{\mathrm{tot}}}$.

To justify its colength interpretation over a nondiscrete valuation ring, one may impose the coordinate congruences in turn. For a single linear coordinate f and a finite family, choose a family member whose f-value divides all the others, as is possible in a valuation ring. Subtract its multiples to zero the other f-values and multiply that member by the required divisor q. The resulting change-of-family matrix has determinant q. Iterating the coordinates multiplies the determinants, proving $(*)$ without a freeness assumption on the full integral-form module. Mean degree and the two lattices; Coordinate congruences over a valuation ring.

For the finite Fourier operator U at N, Lemma 20.37 proves directly from its residue-matrix formula that U preserves the source space and $U\mathcal{A}_L \subseteq \mathcal{A}_\vee$. Starting with a finite family in $\mathcal{A}_0$,

impose $(*)$ successively on its images under $1, U, \dots, U^h$. At each step the image lies in $\mathcal{A}_\vee$; the new integral change of columns preserves the earlier congruences. The composite has determinant $N^{(h+1)d_{\mathrm{tot}}}$ and sends the family into $(U^h)^{-1}\mathcal{A}_L$. Taking $h = m/2$, inequality (11.13) gives

$$(m/2+1)d_{\mathrm{tot}} \le mfbD_X, \tag{11.14}$$

since $a = xb,\ k \ge x,\ m \ge 2$. For this same preimage and the original vacuum scalar, Corollary 20.38 gives the valuation bound $-2mfb$ for the final lift of every ramified form. Level-prime colength, Valuation after the lifts.

Norms of the lifts of the short source forms. Here $t = 5r$, $n_j = k + j(t+1)$ is the rank of X_j, and $n_m = k + m(t+1) \le x^6$ eventually for the constructed data. We use

$$\overline{\log \kappa} = \frac{1}{f}\sum_{\tau} \log \kappa_\tau, \qquad c_1 = \operatorname{Re} G_{d,1}(1), \qquad s_{\mathrm{main}} = a(\log a - \overline{\log \kappa} + c_1).$$

Thus $s_{\mathrm{main}} = D.\mathrm{mainPrimeSum}$; $G_{d,1}$ is the corrected average of logarithmic derivatives defined in the prime-sum section below. The required archimedean estimate is the following; it is proved in Theorem 19.14. For the trivial twist, a single translated normalized-vacuum kernel at stage j has, after every $\sigma \in \mathrm{Aut}(\mathbb{C})$, operator norm at most

$$\exp(x^{50} + r_j/2), \qquad r_j = \sum_{\tau,i} s_\infty(\text{row length}_{j,\tau,i}). \tag{11.15}$$

The expansion of a glued chain into translated kernels has coefficient sum at most $e^{x^{60}}$, proved in Theorem 18.10; the class bound gives at most $e^{x^{30}}$ intermediate classes at each stage. Its restriction to Y_m multiplies squared norms by at most

$$\exp\bigl(x^{70} - (|\Phi_a|a - fm\Delta_0)\log N\bigr), \qquad \Delta_0 = r^8.$$

These are the exact normalized estimates used by the source. The finite gluing expansion is now proved with its original vacuum normalization. The normalized one-kernel and restriction bounds are proved separately in Theorems 19.14 and 19.16, with the same vacuum normalization. Normalized-vacuum jet estimate; Its one-kernel specialization; Expansion and restriction.

Here is the deduction, including its losses. If the conjugate values of ϕ have squared norm at most e^B, multiply the operator norms, sum the coefficients and count the intermediate classes. Before restriction the norm is at most

$$\exp\left(mx^{30} + x^{60} + mx^{50} + \frac{1}{2}\sum_{j<m} r_j + \frac{B}{2}\right).$$

The row lengths give

$$\sum_{j<m} r_j \le C_{\mathrm{row}} + O(faL), \qquad C_{\mathrm{row}} = m\left(|\Phi_a| s_\infty(a) + k\sum_{\tau} s_\infty(\kappa_\tau)\right).$$

Indeed the main rows have lengths $a, 0, \dots$ at the a-places and $k-1$ further lengths κ_τ. Their excess lengths sum, at each place and stage, to at most $b + a/x_1 + n_m\Delta_0^2$. Each excess box contributes at most $25L$ in s_∞, since $\log(2a+1) + \log(M/\Omega) \le 25L$. Thus the total excess is at most $25mfL(b + a/x_1 + n_m\Delta_0^2) = O(faL)$, since $m \ll \sqrt{x}$, $m \le x_1$, and $n_m\Delta_0^2$ is polynomial in x. Replacing $k-1$ by k only increases the upper bound.

Put $\Gamma = \sum_{p\in\mathcal{P}} g_p \log p$. Grouping the von Mangoldt sums prime by prime gives the exact identity

$$A - mR + 2\Gamma = mk \sum_{\tau} s_\infty(\kappa_\tau) + m|\Phi_a| s_{\mathrm{main}}.$$

Stirling's elementary upper bound and the period normalization

$$\log(M/\Omega) = 1 + c_1 + \log(M \,\mathrm{rd}_E\,/(2\pi))$$

give

$$s_\infty(a) - s_{\mathrm{main}} \le a\big(\overline{\log\kappa} + \log(M\,\mathrm{rd}_E\,/(2\pi))\big) + 1 + \tfrac{1}{2}\log a.$$

The defining lower bound for $\log N$ cancels the mean term:

$$m(s_\infty(a) - s_{\mathrm{main}}) - a\log N \le a\big(23L + \tfrac{15}{2}L\big) + m(1 + \tfrac{1}{2}\log a) \le 32aL.$$

It follows that

$$C_{\mathrm{row}} - |\Phi_a| a \log N \le A - mR + 2\Gamma + 32faL.$$

All these equalities and inequalities concern the displayed parameters, not unknown zeros of an L-function. Stirling, the level window and the gain identity.

The remaining terms $x^{70} + fm\Delta_0 \log N + 2(mx^{30} + x^{60} + mx^{50})$ are polynomial in x and hence $o(b)$ on the constructed scales. Including the Stirling error $32faL$ therefore gives

$$\|\sigma B_{1,\Psi}\phi(g)\|^2_{\mathrm{Weyl}} \le \exp\big(B + A - mR + 2\Gamma + O(faL)\big). \tag{11.16}$$

For the good projection the integral-character formula above also gives $|\sigma e_0(t)| \le 1$: the selected irreducible dimensions have squared sum at most $|G|$. Thus $|G| e_0\rho(s)$ multiplies the pointwise norm by at most $|G|^2$. Passing to the squared class norm contributes at most

$$e^{x^{30}}|G|^4 \le e^{x^{99}}.$$

Indeed $\log|G| \le fr^2 \log M$, by counting all matrices over the chosen residue fields, and the same polynomial scale bounds give $x^{30} + 4\log|G| \le x^{99}$. Combining (11.5), (11.11) and (11.16), and using $x^{99} \le b$, gives

$$\|\Theta_1 B_s \phi\|^2_\sigma \le \exp(A - mR + 2\Gamma + C_1 faL)$$

for the same spanning set at every s, σ, as asserted in Proposition 6.2. The short-lift deduction.

The four local lattices. The module in (11.8) is chosen separately at each prime. At a prime $p \ne N$ which splits in K or satisfies $p \ge x_1$, it uses the entire integral source and the integrality of the single theta kernel. At a nonexceptional nonsplit prime it again uses the entire source, with a jet valuation and a collision estimate. At N, the colength is (11.14); the finite gluing and operator identities are proved in §18.1, and the valuation follows from Corollary 18.16, its stated source-form hypotheses, and the original integral-vacuum clause. The ordinary CM argument deriving (J_N) from that clause is supplied in Corollary 18.32.

At an exceptional nonsplit prime, the decisive additional input is particularly explicit. A full mark Λ selects one prime from each conjugate pair induced by the a-places. For $\mathfrak{p} \in \Lambda$, let

$$A_{\mathfrak{p}} = a\,|\{\tau \in \Phi_a : \tau \text{ induces } \mathfrak{p}\}|, \qquad Y_{\mathfrak{p}} = p^{A_{\mathfrak{p}}} T_{\mathfrak{p}}, \qquad r_p = \lceil \log_p m \rceil,$$

where $T_{\mathfrak{p}}$ is the Hecke operator of index r_p. The required elementary-divisor assertion is that $Y_{\mathfrak{p}}\mathcal{M}_p \subseteq \mathcal{M}_p$ and, for every integer $j \geq 0$, there is an integer $e \geq 0$ with

$$\begin{gathered} e \leq \left(\frac{12A_{\mathfrak{p}}}{mp} + \frac{j}{m^2 f}\right) D_X, \\ \operatorname{ColengthDvd}(\mathcal{M}_p, Y_{\mathfrak{p}}\mathcal{M}_p + p^j\mathcal{M}_p, p^e). \end{gathered} \tag{11.17}$$

It is proved in the source by constructing the Frobenius cycle of the embeddings inducing $\mathfrak{p}$ or $\bar{\mathfrak{p}}$, identifying the Hecke action with its edge operator, and applying the Frobenius-cycle elementary-divisor theorem. That theorem is more specific than Smith normal form: it bounds the elementary divisors of these particular large Hecke matrices. Its proof is not supplied by the coordinate argument above. Exact exceptional estimate; Actual Hecke action and the edge operator; Frobenius-cycle theorem.

The passage from (11.17) to the local colength is elementary. Set $j = \lceil 2mfa/p \rceil$ and $Y = \prod_{\mathfrak{p}\in\Lambda} Y_{\mathfrak{p}}$. Colength bounds modulo p^j add under products of integral endomorphisms, by the pullback and composition argument above. Since $\sum_{\mathfrak{p}} A_{\mathfrak{p}} \leq af$ and $|\Lambda| \leq f$, the cost for Y^m is at most

$$\left(\frac{12fa}{p} + \frac{2mfa/p+1}{m}\right) D_X = \frac{14fa + p/m}{p} D_X \leq \lambda_p D_X.$$

Indeed $14fa + p/m \leq 15fa$, since $p < x_1 \leq a$. Thus one may take $\mathcal{S}_p = Y^m\mathcal{M}_p + p^j\mathcal{M}_p$. The local theta valuation proves $v_p(B_{\xi,\Psi}\phi) \geq g_p - 2\lambda_p$ on this module, uniformly in the allowed twist and chain datum; it uses the marked Hecke-transfer identity as well as the single-jet divisibility estimate. Valuation and product of marked operators.

Take the basis $y_1, \ldots, y_n$ selected in (6.3) from the translates of the short integral source forms. The preceding norm bound and Hadamard's inequality give

$$\det \operatorname{Gram}(\sigma(\Theta_1 y_i))_{i=1}^n \leq e^c, \qquad c = n(A - mR + 2\Gamma + C_1 faL), \tag{11.18}$$

where σ acts on the value coordinates before the inner products are taken. These coordinates have a positive definite rational metric; the final class stabilizers are trivial. Rational metric; The common family.

For the same basis, (11.12) gives the logarithmic index bound $\delta = d_0 faL$ relative to $(h_p)_p$ in (7.2). Thus there are $e_\ell \geq 0$ with

$$\sum_{\ell\in F} e_\ell \log \ell \leq \delta \quad \text{for every finite set of primes } F, \tag{11.19}$$

and, for every $\iota : \mathbb{C} \simeq \overline{\mathbb{Q}}_\ell$, an ι-integral matrix u_ι such that

$$|\iota(\det u_\iota)|_\ell \geq \ell^{-e_\ell}. \tag{11.20}$$

Every row combination $\sum_k (u_\iota)_{ik} y_k$ has lifted valuation at least h_ℓ under every twist allowed at ℓ. In this readable edition the matrix is chosen before the twist. The compact edition permits dependence on the allowed twist; it still permits one to use that same chosen twist at each of the two conjugate embeddings below. Index definition.

For the completed family, Proposition 7.3 applies the exterior-power metric to (11.18) and the mixed integral pairing to the rows in (11.20). The conjugate embedding in each pairing uses the same twist.

Lean. The formal proof uses mixed Gram matrices and Hadamard's inequality to adjoin columns. Here both estimates are applied to exterior products. See `CovolumeLe.norm_map_det_gram_le` and `CovolumeLe.sumElim`.

12 Unitary theta lifts and the exchange of adjacent lifts

We prove injectivity for every allowed chain of lifts Θ_ξ, as required for the determinant quotient. The return of a single lift to its source gives an injective operator in the range considered here. To deduce injectivity of a chain we move the return map past each adjacent lift. The linear algebra is as follows.

Lemma (chains in a triangle). For $i, n \geq 0$ with $i + n \leq m$, let $F_{i,n}$ be vector spaces. Let $j_i : V_0 \to F_{i,0}$ be injective and let

$$A_{i,n} : F_{i,n} \to F_{i,n+1}, \qquad D_{i,n} : F_{i,n+1} \to F_{i+1,n}$$

be linear. Suppose that, whenever the indicated vertices exist,

$$A_{i+1,n} D_{i,n} = c_{i,n} D_{i,n+1} A_{i,n+1}$$

for some scalars $c_{i,n}$. Suppose also that

$$D_{i,0} A_{i,0} j_i = j_{i+1} K_i$$

for an injective endomorphism K_i of V_0. Then every composite $P_{i,n} = A_{i,n-1} \cdots A_{i,0} j_i$ is injective.

Proof. Commuting the descent through the successive arrows gives

$$P_{i+1,n} K_i = \left(\prod_{a=0}^{n-1} c_{i,a} \right) D_{i,n} P_{i,n+1}.$$

For $n = 0$ this is the assumed first-step identity; multiplication by $A_{i+1,n}$ and the square relation prove the next case. Induct on the chain length. The length-zero map is j_i. If $P_{i,n+1} v = 0$, the displayed identity gives $P_{i+1,n} K_i v = 0$. The induction hypothesis and injectivity of K_i imply $v = 0$. No division by a square scalar occurs.

Lean, formal context. `TriangleLifts` (L221-L380), `ChainNorms` (L794). □

Theta lifts in the lower triangle. Fix an allowed m-fold twist ξ. Let $S_{i,n}$ denote the Hermitian space at position (i, n) of the lower triangle, denoted by $D.\mathrm{gridSpace}(i, \mathrm{false}, 0, n)$ in Lean. Its rank is $r_n = r + nt$, where $t = 5r$. The $S_{i,0}$ are successive opposites of Y_0. The identification $U(S) = U(-S)$, with the same quotient measure, gives isometries j_i from the space V_0 of principal-level forms on Y_0 into the forms on $S_{i,0}$. The established cusp-kernel property of the selected archimedean type at the fixed finite level makes these forms rapidly decreasing, and hence bounded; it supplies the decay hypothesis in the doubling identity below, for every allowed finite twist. Let $\mathfrak{m} = \mathfrak{m}(\xi)$ be the triangle modulus and put

$$R_\xi = \mathcal{O}_E/\mathfrak{m}, \qquad N_n = |\mathrm{GL}_{r_n}(R_\xi)| > 0.$$

For the prescribed splitting characters and joint harmonic $v_{i,n}$, let $L_{i,n}$ be the theta lift with finite datum $\mathbf{1}_{X_0}$, $X_0 = (I_{r_n}; 0)$, at $\mathfrak{m}$, and the vacuum elsewhere. The ascending arrow of the abstract lemma is $A_{i,n} = N_n L_{i,n}$. This factor changes the measure giving the integral finite group volume one to the measure giving its principal subgroup at $\mathfrak{m}$ volume one. The first step has ranks r and $6r$, independently of i, so $2r < 6r$. The auxiliary triangle has first ranks $r+1$ and $6r$; the parameter inequalities also give $2(r+1) < 6r$. Nodes, arrows and point datum; Weighted chains.

Siegel–Weil and the Rallis inner-product identity. The convergent-range identities used below are `siegel_weil` and `inner_product_formula`. The hypotheses on support and ranks can be checked directly on the data for the first arrow of the chain.

Let a, b be the ranks of S, T and w the Witt index of T. For a unitary pair the convergent range of the Rallis formula includes $b - w > a$ [GQT14, §1.7]. Here $w \le b/2$, so $2a < b$ implies this inequality on both first arrows. Use the convergent Siegel–Weil formula [Wei65] with the doubled source $S \oplus (-S)$. Suppose that S is anisotropic over F, or that the Schwartz datum f is *clean* at a finite place, meaning that its Schrödinger support consists of injective maps $S \to T$. For rapidly decreasing automorphic forms y, y' one obtains

$$\langle \Theta_f y', \Theta_f y \rangle_T = \mathrm{vol}[U(T)] \int_{U(S)(\mathbb{A})} \langle f, \omega(g) f \rangle \langle y', R(g) y \rangle_S \, dg. \tag{12.1}$$

The volume is for the quotient and data conventions of this note. One can equivalently normalize $[U(T)]$ to probability measure and omit this factor.

Indeed, the product of the kernel and its conjugate is the theta kernel of the doubled space. In the Siegel–Weil sum a maximal isotropic space with noninjective projection contributes zero at a clean place. If S is anisotropic, the kernel of either projection would be an isotropic subspace of S, so both projections are injective to begin with. The contributing spaces have dimension a and are graphs of isometries $\gamma \in U(S)(F)$, with terms $\langle f, \omega(g'^{-1}\gamma g) f \rangle$. Integrating against $\overline{y'(g')} y(g)$ on $[U(S)]^2$ unfolds the graph sum to (12.1). In the convergent range the theta majorant grows at most polynomially in the source heights, and rapid decay gives absolute integrability. The finite support condition and the displayed volume fix the normalization needed in (12.4). Unfolding with the chosen measures. Second, we use the local joint-harmonic theorem for real unitary groups with the prescribed, sufficiently separated discrete-series parameters. Index the discrete series π_l by the doubled Harish–Chandra parameter l, as in Lean. For the chosen unit vector u_l and formal degree d_l, the normalized coefficient is

$$e_l(g) = d_l \, \overline{\langle u_l, \pi_l(g) u_l \rangle}.$$

At each real place τ, let l_τ, l'_τ be the row and column parameters of the joint harmonic v_τ. The lift has source parameter $-l_\tau$ and target parameter l'_τ. Its source coefficient is therefore $e(x) = \prod_\tau e_{-l_\tau}(x_\tau)$. The contragredient identity $\overline{e_{-l_\tau}(x_\tau)} = e_{l_\tau}(x_\tau)$ gives the oscillator row coefficient. For $v = \bigotimes_\tau v_\tau$ one has

$$Z_\infty(v, e) := \int_{U(S)(F_\infty)} \langle v, \omega_\infty(x) v \rangle \frac{\overline{e(x)}}{e(1)} \, dx \neq 0. \tag{12.2}$$

Interchanging the groups and conjugating the types gives the corresponding assertion in the other direction. The parameters in the grid satisfy the local correspondence at every real place, and finite twists leave them unchanged.

Indeed, integration over the product of the real groups gives

$$e(1) Z_\infty(v, e) = \prod_\tau \langle v_\tau, \omega_\tau(e_{l_\tau}) v_\tau \rangle,$$

where $\omega_\tau(e_{l_\tau})$ is the oscillator representation integrated against e_{l_τ}. Each factor has positive real part. The local proof averages the discrete-series kernel against the highest-K-type coefficient on the left and right. Polar decomposition and the Fock matrix-coefficient formula then write the average as an integral of positive highest-weight minors times $\langle v_\tau, \omega_\tau(z) v_\tau \rangle > 0$ for positive hermitian z. The parameter gaps justify the averaging. Division by $e(1) = \prod_\tau d_{-l_\tau} > 0$ gives (12.2). We use this local joint-harmonic result as a classical representation theorem; see J.-S. Li, *Theta lifting for unitary representations with nonzero cohomology*, Duke Math. J. **61** (1990), 913–937,

doi:10.1215/S0012-7094-90-06135-6. For the explicit lowest-K-type rule see also Ichino, *Theta lifting for discrete series representations of real unitary groups*, arXiv:2002.09148, Lemma 7.8 and the proof of Lemma 7.7. The Lean calculation uses `conj_dsKernel_neg`, the polar positivity integral, joint-harmonic positivity, the product over real places, and the source and target types.

Support and stabilizer of the finite theta datum. Let $a = \dim S$, $b = \dim T$, and

$$X_0 = \begin{pmatrix} I_a \\ 0 \end{pmatrix} \in \mathrm{Mat}_{b\times a}(R_\xi), \qquad F = \mathbf{1}_{X_0} \otimes \mathrm{vac}.$$

Write $K_0 = U(S)(\mathbb{A}_f) \cap \mathrm{GL}_a(\widehat{\mathcal{O}}_E)$, and $J = J(\mathfrak{m})$. The splitting character is tame at $\mathfrak{m}$ and unramified elsewhere, so J fixes F. Every matrix X in its Schrödinger support has an integral left inverse at the places of $\mathfrak{m}$, since its top square block is I_a modulo $\mathfrak{m}$.

Suppose g is integral outside $\mathfrak{m}\overline{\mathfrak{m}}$ and the coefficient of F at g is nonzero. Then some X and Xg both belong to the support. Their integral left inverses imply that g and g^{-1} are integral at $\mathfrak{m}$, and unitarity supplies its conjugate places. Reduction gives $X_0\bar{g} = X_0$, whence $\bar{g} = I_a$. Thus $g \in J$. Conversely every $g \in J$ fixes F, so its coefficient is $\|F\|^2$. Normalized counting measure on matrices modulo $\mathfrak{m}$ therefore gives

$$\|F\|^2 = \frac{1}{|\mathrm{Mat}_{b\times a}(R_\xi)|}, \qquad c_f := \|F\|^2 \,\mathrm{vol}(J) = \frac{1}{|\mathrm{Mat}_{b\times a}(R_\xi)|\,|\mathrm{GL}_a(R_\xi)|} > 0. \tag{12.3}$$

The modulus contains a chosen place. Injectivity of matrices on the support supplies the clean place required in (12.1). This proves sharpness for the specified datum. Sharpness from the left inverse.

An injective unramified convolution. The required nonvanishing can be proved without invoking a global automorphic L-value. Let A be a conjugation-stable set of finite places away from the level, let G_A be the finite adelic group supported there, and let K_A be its integral subgroup. On bounded K_A-fixed functions put

$$\mathcal{R}_A y = \sum_{g \in G_A/K_A} \omega_b(g) R(g) y, \qquad |\omega_b(g)| = I(g)^{-b/2}, \qquad I(g) = [gL : gL \cap L], \quad L = \widehat{\mathcal{O}}_E^a.$$

The coefficient is the inverse splitting-character value on $\det g$, times the inverse b-th power of the vacuum multiplier. Its identity coefficient is one and it is bi-invariant under K_A.

We give the lattice count proving convergence and injectivity. A neighbour vector x at a prime w, supported at w, has $\varpi x \in L$, $x \notin L$, and integral norm. It determines

$$L_x = \{z \in L : \langle x, z\rangle \in \widehat{\mathcal{O}}_E\} + \widehat{\mathcal{O}}_E x.$$

A neighbour is specified by the line of ϖx in κ_w^a and one polar residue. To define the latter choose a reference vector x_0 on the line, rescale x by a unit so that $x - x_0 \in L$, and take the residue of $\varpi\langle x_0, x - x_0\rangle$. The possible residues come from elements of pole order at most one with integral trace. At a split prime this residue is zero. At an inert prime take ϖ from the base field; the residues then form the trace-zero subspace of the quadratic residue field, of size $\sqrt{q}$, where $q = |\kappa_w|$. The Weil field is unramified over its maximal real subfield at finite places; this feature of the fixed field is being used. There are therefore at most

$$\frac{q^a - 1}{q - 1}\sqrt{q} \quad \text{steps at } w, \qquad \frac{q^a - 1}{q - 1} \quad \text{if } w \neq \bar{w}.$$

We count paths of neighbours from L to a prescribed $K = gL$. Let $J \neq K$ be the current self-dual lattice. Self-duality rules out a strict inclusion $K \subset J$, so at some w there is an $x \in K_w$ with a simple pole relative to J_w. Extend x by zero at the other places and put

$$H = \{z \in J : \langle x, z\rangle \in \widehat{\mathcal{O}}_E\}, \qquad J' = H + \widehat{\mathcal{O}}_E x.$$

Perfect duality gives $[J : H] = q_w$: pairing with the primitive residue of $\varpi_w x$ gives a nonzero residue-field functional on $J_{\bar{w}}$. Since K is integral, so is $\langle x, x\rangle$. Thus J' is integral and $\varpi_w x \in H$, whereas $x \notin J$. It follows that $[J' : H] = q_w$. Hence J' has the same covolume as J; the inclusion of J' in its dual is an equality. This is the one-prime neighbour construction of [GV14, Propositions 5.3–5.4].

Since $J \cap K \subset H$ and $x \in K$, the intersection $J' \cap K$ contains $(J \cap K) + \widehat{\mathcal{O}}_E x$, which has index q_w over $J \cap K$. Consequently

$$[K : K \cap J'] \leq [K : K \cap J]/q_w.$$

Iteration reaches K, since an index of one gives $K = J$ by self-duality. The line and residue count above applies at every step: locally a self-dual Hermitian lattice over the unramified quadratic extension has an orthonormal basis, and at a split pair its two components are dual. Choose one path for each endpoint. Distinct cosets gK_A give distinct endpoints gL, and the product of q_w along a path is at most $I(g)$. The product of the step weights $q_w^{-b/2}$ thus bounds $|\omega_b(g)|$.

Define

$$s_w = \frac{q_w^a - 1}{q_w - 1}\, u_w q_w^{-b/2}, \qquad u_w = \begin{cases} 1, & w \neq \bar{w}, \\ \sqrt{q_w}, & w = \bar{w}. \end{cases}$$

Since $b > 2a$,

$$s_w \leq (q_w - 1)^{-1}, \qquad s_w \leq ((q_w - 1)\sqrt{q_w})^{-1} \quad (w \neq \bar{w}).$$

The field contains primitive third and fourth roots of unity. Thus $q_w \geq 4$, because $X^2 + X + 1$ has no root over $\mathbb{F}_2$ and $X^2 + 1$ has none over $\mathbb{F}_3$. The sum over one conjugate pair is at most $1/3$. Moreover $\sum_w s_w < \infty$: split terms are $O(q_w^{-3/2})$; at inert places q_w is a square, so the terms are $O(p_w^{-2})$. There are at most $[E : \mathbb{Q}]$ primes over each rational prime.

If $s_A = \sum_{w \in A} s_w \leq 1/3$, the word count gives

$$\sum_{\substack{g \in G_A/K_A \\ g \neq K_A}} |\omega_b(g)| \leq \sum_{j \geq 1} s_A^j \leq \frac{1}{2}.$$

Thus $\mathcal{R}_A = 1 + T_A$, with $\|T_A\|_{\infty \to \infty} \leq 1/2$, and is injective. For arbitrary A, remove finitely many conjugate pairs so that the remaining weight sum is at most $1/3$. Cosets and coefficients factor over disjoint place sets, so $\mathcal{R}_{A_1 \cup A_2} = \mathcal{R}_{A_1}\mathcal{R}_{A_2}$. Every removed pair and the remaining tail are injective. Their finite product proves injectivity and absolute convergence for all A. Only absolute values entered, so the conclusion holds for all allowed twist phases. Lean decreases the same index by explicit unitary transformations; the lattice argument uses only duality and the two indices at a neighbour. Neighbour construction and index decrease; Weights and word bound; Injective convolution.

Injectivity of the specified first lift. Apply (12.1) with $f = v_{i,0} \otimes F$. Sharpness and right J-invariance reduce the finite integral to $c_f \mathcal{R}_A$, where $A = (\mathfrak{m}\overline{\mathfrak{m}})^c$. For forms y', z of the selected archimedean type, Schur orthogonality gives

$$\langle y', R(x)z\rangle = \frac{\overline{e(x)}}{e(1)}\langle y', z\rangle \qquad (x \in U(S)(F_\infty)).$$

Finite translations commute with this type projector. The series $z = \mathcal{R}_A y$ has the same type and fixed finite level: its coefficient is bi-invariant at A, and the operators commute with the level outside A. To verify that it is a form, average each partial sum over the fixed compact finite level. These averages are forms in one finite-dimensional space, and converge uniformly to z, because the full series is level invariant. That space is closed in the uniform norm, so z belongs to it. We obtain

$$\begin{aligned} \langle L_{i,0}y', L_{i,0}y\rangle &= C_{i,\xi}\langle y', \mathcal{R}_A y\rangle, \\ C_{i,\xi} &= \frac{\mathrm{vol}[U(S_{i,1})]\, Z_\infty(v_{i,0}, e)}{|\mathrm{Mat}_{6r\times r}(R_\xi)|\,|\mathrm{GL}_r(R_\xi)|} \neq 0. \end{aligned} \tag{12.4}$$

If $L_{i,0}y = 0$, take $y' = \mathcal{R}_A y$. The equality gives $C_{i,\xi}\|\mathcal{R}_A y\|^2 = 0$. Thus $\mathcal{R}_A y = 0$, and injectivity of $\mathcal{R}_A$ gives $y = 0$. Hence $a_i = L_{i,0}j_i$ is injective on V_0 for every ξ. A sign assertion about the complex scalar $C_{i,\xi}$ is unnecessary. Specified first-step formula.

The first return is an adjoint Gram operator. The descent at $n = 0$ can be chosen among the transposed theta lifts. Conjugating a theta kernel replaces its source by the opposite space and replaces α_M by $\alpha_M^{-1}\upsilon^{\dim M}$. Its finite datum becomes

$$\Psi_0'(X) = c\,\overline{\kappa}\,\overline{\Psi_0(\varepsilon^{-1}X)},$$

where ε is the fixed finite norm--1 unit, c is the theta-conjugation scalar, and κ is the scalar obtained by projecting the conjugated real polynomial onto the prescribed partner highest-type line. The local joint-harmonic theorem gives this scalar projection. For this identity extend j_i to the full source-type space by the same opposite-space isometry. Kernel conjugation and Fubini then prove, for all source-type forms y and target-type forms x,

$$\langle x, L_{i,0}j_i y\rangle = \langle D_{i,0}x, j_{i+1}y\rangle. \tag{12.5}$$

Thus a theta descent has the required adjointness. Opposite-space adjoint datum.

Its return preserves the chosen principal level. For an integral source element u, choose an integral target element g reducing to $\mathrm{diag}(\bar{u}, I)$ modulo $\mathfrak{m}$; reduction is surjective at the selected split primes. This pair fixes X_0. Weil equivariance gives

$$R_T(g)a_i = c(u)a_i R_S(u), \qquad |c(u)| = 1.$$

Consequently the Gram pairing of a_i is invariant under simultaneous $R_S(u)$ on its two arguments. The adjoint return commutes with these integral elements and preserves the principal-level subspace V_0. Transporting (12.5) by j_{i+1} gives

$$D_{i,0}A_{i,0}j_i = j_{i+1}K_i, \qquad K_i = N_0 a_i^* a_i\,|_{V_0}\,.$$

For $y \neq 0$, $\langle K_i y, y\rangle = N_0\|a_i y\|^2 > 0$. Thus K_i is injective for every allowed twist. Integral covariance; Level and injectivity.

The terminal normalization is elementary and positive. Let $J_{\mathrm{last}}(\xi) \subset K_0$ be the final finite level, and let W_{alg} be the final Weyl module of dimension $d_W > 0$. The final unitary group is definite. The scalar realization of a W_{alg}-valued algebraic form is its matrix coefficient against a fixed unit vector of W_{alg}. Schur orthogonality for the compact real group, with Haar probability measure, contributes $1/d_W$. A finite coset contributes $\mathrm{vol}(J_{\mathrm{last}}) = [K_0 : J_{\mathrm{last}}]^{-1}$. Summing with the same stabilizer weights as the algebraic inner product proves

$$\langle F_1, F_2\rangle_{\mathrm{alg}} = [K_0 : J_{\mathrm{last}}(\xi)]\, d_W \langle \mathrm{ofAlg}(F_1), \mathrm{ofAlg}(F_2)\rangle_{\mathrm{Pet}}.$$

The algebraic chain Θ_ξ, denoted by lowerChainAtLevel(ξ) in Lean, has scalar realization $P_{0,m}$. Therefore

$$\|\Theta_\xi y\|_{\mathrm{alg}}^2 = [K_0 : J_{\mathrm{last}}(\xi)]\, d_W\, \|P_{0,m} y\|_{\mathrm{Pet}}^2, \qquad [K_0 : J_{\mathrm{last}}(\xi)]\, d_W > 0. \tag{12.6}$$

The group orders in the ascending arrows are essential to this identification; positivity itself requires no nonvanishing theorem. Terminal normalization.

The complementary-rank square identity. We state the complementary-rank exchange for the chosen Hermitian spaces before reconstructing its fixed-character proof. For a square indexed by i, n, set

$$V = S_{i,n+2}, \quad V' = S_{i+1,n}, \quad W = S_{i,n+1} \oplus S_{i+1,n+1}.$$

Then

$$\dim V = \dim V' + 2t, \qquad \dim W = \dim V + \dim V' = 2(r + (n+1)t).$$

Its negative signatures satisfy $q_W = q_V + q_{V'}$, by comparing the alternating base signs; added blocks are positive. Also $q_W \le 2r < r + (n+1)t$, hence $2q_W + 2 \le \dim W$. For

$$\nu_{n,\xi} = (\nu_{\mathrm{quot,n+1}} \xi^{\,n+1})^{-1}, \qquad \mu_\tau = (n+1)\,\mathrm{gap},$$

the datum ensures tame ramification exactly at $\mathfrak{m}(\xi)$ with residue character χ. Its weight is μ_τ, and $r + (n+1)t \le \mu_\tau$, so $\dim W \le 2\mu_\tau + 2q_W$. These hypotheses hold for every m-fold twist.

Here is the exact meaning of matched data. Choose the index identification $\sigma : V' \oplus V \simeq W$. For $k \in \mathrm{GL}_{\dim W}(R_\xi)$, let $C(k)$ be its first $\dim V'$ columns and $R(k)$ the last $\dim V$ rows of k^{-1}, in this ordering. Functions $\Phi : \mathrm{Mat}_{\dim V \times \dim W}(R_\xi) \to \mathbb{C}$ and $\Psi : \mathrm{Mat}_{\dim W \times \dim V'}(R_\xi) \to \mathbb{C}$ are matched if they vanish off these sets and

$$\Phi(R(k)) = \chi(\det k)\Psi(C(k)) \quad (k \in \mathrm{GL}_{\dim W}(R_\xi)).$$

For the real Schwartz vector v from the lower edges, the required exchange is the existence of a real Schwartz vector w and $c \neq 0$ such that every matched pair satisfies

$$\int_{[U(V)]} \theta^{W,V}_{\gamma,\alpha}(w \otimes \Phi)(g,h)\, dh = c \int_{[U(V')]} \theta^{V',W}_{\alpha',\gamma}(v \otimes \Psi)(h,g)\, dh, \tag{12.7}$$

where $\gamma = \nu_{n,\xi}^{-1} \upsilon^{\dim W/2}$ and $\alpha\alpha' = \gamma$. These are the rank-normalized norm-one splitting characters; the latter equality is equivalent to the source's dimension-dependent splitting-character product condition. The assertion includes convergence of the indicated integrals.

The formal source proves the stronger statement that w is chosen before the finite character and

$$|c|^2 = C \left(\prod_{w|\mathfrak{m}} N w \right)^{t^2} \prod_{a=1}^{t} |L^{\mathfrak{m}}(a, \nu_{n,\xi})|^2, \qquad C > 0.$$

Neither this size formula nor the character-independent choice of w is required for chain injectivity. The nonzero scalar and the exchange identity are required. They do not follow from the elementary seesaw interchange alone, and the ordinary first-step Rallis formula does not supply them. Complementary-rank exchange; Its grid hypotheses.

12.1 Full-rank theta integrals and the top configuration

We give the intermediate exchange from which hyperbolic-plane removal starts. In particular, the extension from a self-dual Fourier coefficient to an identity of automorphic functions is proved here. No multiplicity-one statement for the theta lifts is being assumed.

Let Y be a Hermitian space of even rank $2j$ with an adapted hyperbolic plane, let B have rank b, and put

$$X = H^b \oplus Y, \qquad A = B \oplus Y.$$

All spaces carry the finite self-dual lattices and adelic frames used elsewhere in this note. We retain the hypothesis $\zeta_{5040} \in E$ of the Weil datum. In particular E/E^+ is unramified at every finite place: at a rational prime p, choose $\ell \in \{3, 5\}$ with $\ell \neq p$. Inertia acts trivially on $\mathbb{Q}(\zeta_\ell)$, whereas complex conjugation acts nontrivially there, so conjugation cannot be a nontrivial inertia element of the quadratic extension. This hypothesis is used in the integral orthonormal-basis arguments. Write D for the indicated splitting of X over Y, with isotropic frames K, L, and D^{op} for the splitting with those frames exchanged. Fix a character ν with positive integral infinite weights μ_τ, tamely ramified on a nonempty set S of split places, with $S \cap \overline{S} = \varnothing$. Put $F_S = \prod_{w \in S} E_w$, $O_S = \prod_{w \in S} \mathcal{O}_w$, and let χ be its tame residue character, extended by zero off the units when necessary.

For an integral b-by-$\dim X$ matrix z, use the standard matched functions

$$R_z(x) = \mathbf{1}_{\mathrm{GL}_{\dim X}(O_S)} \begin{pmatrix} z \\ x \end{pmatrix} \chi^{-1}\left(\det \begin{pmatrix} z \\ x \end{pmatrix} \right), \qquad x \in \mathrm{Mat}_{\dim A \times \dim X}(F_S), \tag{12.8}$$

$$C_z(y) = \mathbf{1}_{\mathrm{Mat}(O_S)}(y) \chi^{-1}(\det zy), \qquad y \in \mathrm{Mat}_{\dim X \times b}(F_S). \tag{12.9}$$

The chosen ordering of the rows is fixed throughout. At the conjugate places the small datum is read by transpose, as in the definition of thetaSmall. For real Schwartz vectors P, Q, abbreviate

$$\mathcal{T}_+(P, R_z) = \mathrm{thetaLarge}(X, A, \nu; P, R_z), \qquad \mathcal{T}_-(Q, C_z) = \mathrm{thetaSmall}(X, B, \nu; Q, C_z).$$

Thus the large kernel has both splitting characters $\nu^{-1}\upsilon^{\dim X/2}$; the small kernel has characters $\nu\upsilon^{\dim X/2}$ and υ^b. The later adjoint flip converts this convention to the displayed complementary-rank identity.

Lemma 12.1 (Full-rank convergence and termwise unfolding)**.** *Let U, Z be Hermitian spaces and let a datum have a real Schwartz vector and a compactly supported, locally constant component at a nonempty collection of split places disjoint from its conjugate. Suppose that this finite component is supported on surjective maps $U \longrightarrow Z$. Its theta kernel, for the first group in a fixed compact set, is bounded on $U(Z)(\mathbb{A})$, and in fact decays faster than every inverse power of the first-minimum height of Z. The theta integral over $[U(Z)]$ is absolutely convergent and continuous in the first variable.*

For the mixed model, fix an isotropic frame with span K and a first-group point g. At a selected split place w, write $K_w^\perp$ in the adapted coordinates at the identity. Suppose that every y in the support of the finite datum satisfies

$$y_w(g_w^{-1} K_w^\perp) = Z_w.$$

Here it suffices to fix one such w for the support. Then the sum of the absolute values of the rational-frame terms at g is bounded in the second-group variable and integrable over $[U(Z)]$. The sum may also be integrated term by term in a central Fourier coefficient at g. Ordinary convergence applies to every standard pair (12.8)–(12.9), with the latter read at $\overline{S}$. The assertion about the absolute sum of mixed-model terms applies to the frame-adapted pair $z = L^*$ at the parabolic points used below.

Proof. Here are the decay argument and the support checks. They are useful because unrestricted theta integrals in these ranks need not converge. Let $H \geq 1$ be the reciprocal of the shortest nonzero vector of the moved lattice of Z, cut off at one. Ordinary lattice counting gives a bound $CH^a p_M(v)$ for the theta distribution on a fixed Schwartz seminorm p_M, with the finite datum fixed, for some a, M. The same estimate holds for a finite sum of mixed-model terms with arbitrary coefficients of absolute value one. One can obtain it by counting vectors in successive balls: $\#(\Lambda \cap B_R) \leq C(1 + RH)^d$, followed by a convergent sum against $(1 + R)^{-d-2}$. Reduction of the moved self-dual lattice changes the relevant Schwartz seminorm by at most another power of H. For the ordinary kernel the constants are uniform on a compact set of first-group points. For the mixed estimate we fix g.

For $H > 2$, choose a nonzero rational lattice vector d of length at most $2/H$. It is isotropic: its Hermitian norm is integral at every finite place and has absolute value less than one at every infinite embedding, so its field norm is an integer of absolute value less than one. Choose a rational frame c, with finite integral coordinates and bounded infinite coordinates, spanning U for the full kernel, or $K^\perp$ for the mixed-model terms. If $x_{U,w}$ carries standard coordinates to the given local space, then $C_w = x_{U,w}^{-1} c$ is the frame at the identity. At g its columns are $C_{g,w} = g_w^{-1} C_w$. A single integer clears their finite denominators. Multiplication by this integer preserves their span over E_w. The rank-one maps

$$d\langle c_i, \cdot \rangle$$

span an isotropic rational subspace of $\mathrm{Hom}(U, Z)$, and their moved infinite lengths are $O(H^{-1})$. The theta distribution is invariant under their rational Heisenberg translations. In the mixed model the same is true term by term, because these translations kill the fixed isotropic frame and act on the remaining theta distribution by rational translations.

We recall explicitly the elementary Fourier estimate used at this point. Let a continuous linear functional be invariant under the rational translations of an isotropic space, and suppose its adelic translation average annihilates a fixed finite Schwartz datum. If this space is spanned by rational lattice vectors of infinite length at most δ, then for every L

$$|\ell(\omega(g, h)f)| \leq C_L H^a \delta^L p_{M+L+d+2}(f_\infty). \tag{12.10}$$

Here the hypothesis on the functional is the preceding order-M bound $CH^a p_M$, including its differentiated versions; d bounds the real dimension of the isotropic averaging space. There is no norm on the whole Schwartz space in this assertion. To see the estimate, multiply a fixed integral basis of the coefficient field by the spanning vectors and by one integer depending on the finite datum. These give periods of length $O(\delta)$ for the translated functional at infinity. Additive approximation identifies its adelic mean with its mean on the resulting real torus. That mean is zero. Expand on this torus and use integration by parts of order exceeding its dimension plus L. Each derivative in a period direction contributes its length; the nonzero Fourier modes have a summable reciprocal-power majorant. This gives (12.10), with a possibly larger Schwartz seminorm. Linear dependence in a spanning family causes no problem: choose a basis from it. The dimension is bounded by the fixed dimension of $\mathrm{Hom}(U, Z)$.

The required zero mean follows directly in the split Schrödinger model. At a second-group point h, let $\lambda_{h,w}$ be the w-component of the adjoint row of d in the moved target frame. It is nonzero because d is rational and nonzero and the change of frame at the conjugate place is invertible. The frequencies of the translation characters are the entries of

$$\lambda_{h,w}\, y_w C_{g,w}.$$

In the ordinary model $C_{g,w}$ spans the full source and y_w is onto. In the mixed model its columns span $g_w^{-1} K_w^\perp$, so the stated support hypothesis gives a right inverse of $y_w C_{g,w}$. In either case a zero

frequency row would, on multiplication by a right inverse, give $\lambda_{h,w} = 0$. Character orthogonality therefore kills the constant term on the finite support, so its translation average is zero. Applying (12.10) with $\delta = C/H$ and $L > a + N$ gives $O(H^{-N})$. For a finite sum of mixed terms the initial polynomial bound and the derivative estimates are uniform in their phases. Choose their phases to turn the sum into a sum of absolute values, and then let the finite set increase.

For the standard data with $z = L^*$, take $C_w = (K \mid I)_w$, where I includes the complementary summand. On the large side the stacked matrix of (12.8) in the basis (L, K, I) is block triangular with diagonal blocks I_b and $y_w C_w$. On the small side transposition at the conjugate places gives the first b columns of $y_w C_w$ in (12.9). Thus $M_w = y_w C_w N$ has unit determinant on the relevant support: take N to be the identity on the large side and $\binom{I_b}{0}$ on the small side. At a parabolic point p stabilizing K, unitarity gives $p_w K_w^{\perp} = K_w^{\perp}$. Consequently there is an invertible change of basis A_w with $C_{p,w} A_w = C_w$, and

$$(y_w C_{p,w})(A_w N M_w^{-1}) = I_{\dim Z}.$$

This verifies the mixed support hypothesis at every such p.

For the ordinary integral, the compact-uniform bound and the finite volume of $[U(Z)]$ give continuity by dominated convergence. In the mixed model, at the fixed first point the bounded absolute sum is integrable on the same quotient. Central translation multiplies each rational-frame term by a character of modulus one. The same absolute sum therefore controls the integral over the compact central quotient, and Fubini gives the assertion about Fourier coefficients.

Lean. `Ordinarykernel` (L364-L410), `Transportedframe` (L222-L238), `hasIntegrableTerms_integrated_schrodingerOp`, `hasIntegrableTerms_localDatum_of_eq_stdDatum`, `centreCoeff_integral_eq_tsum`. □

Short-period decay and surjective support; The parabolic change of frame; The small datum at the conjugate places.

12.2 Matched data and fixed-character normalization of the exchange

We give the remaining finite representation argument and the nonvanishing argument for the exchange scalar. The character ν is fixed throughout. The construction may depend on ν; the theta chain requires neither a common real vector for different characters nor a formula for the absolute value of the exchange scalar.

Put $F_S = \prod_{w \in S} E_w$, $O_S = \prod_{w \in S} \mathcal{O}_w$, where S is the nonempty set of split ramified places, disjoint from its conjugate. Write χ for the unitary character on $O_S^{\times}$ obtained from ν. Its extension to $F_S^{\times}$ in the formulas below is the one specified by ν, not an arbitrary extension. For an even integer $N = a + b$, set

$$\eta(t) = \nu(t/\bar{t})|t|_S^{N/2}.$$

Here $t/\bar{t}$ denotes the norm-one idele supported at $S \cup \overline{S}$. Thus $\eta|_{O_S^{\times}} = \chi$. Every general linear group over F_S has Haar measure giving its integral subgroup volume one.

A column matrix is *primitive* if it has an integral left inverse and integral entries; a row matrix is primitive if it has an integral right inverse and integral entries. If $k \in \mathrm{GL}_N(F_S)$, let $C(k)$ be its first b columns and $R(k)$ the last a rows of k^{-1}. The row order is fixed. A matched pair (Φ, Ψ) consists of locally constant functions supported on primitive a-by-N rows and N-by-b columns, respectively, such that

$$\Phi(R(k)) = \chi(\det k)\Psi(C(k)) \qquad (k \in \mathrm{GL}_N(O_S)). \tag{12.11}$$

The supports are compact. Completing a primitive frame to an integral basis in (12.11) gives

$$\Phi(hX) = \chi(\det h)^{-1}\Phi(X), \qquad \Psi(Yj) = \chi(\det j)^{-1}\Psi(Y) \tag{12.12}$$

for $h \in \mathrm{GL}_a(O_S)$ and $j \in \mathrm{GL}_b(O_S)$. For example, replacing k by $k\,\mathrm{diag}(j, I_a)$ proves the column identity; replacing it by $k\,\mathrm{diag}(I_b, h^{-1})$ proves the row identity.

Lemma 12.2 (Generation by one standard column datum)**.** *For $s \in \mathrm{Mat}_{b\times N}(O_S)$, put*

$$p_s(Y) = \mathbf{1}_{\mathrm{Mat}(O_S)}(Y)\mathbf{1}_{\mathrm{GL}_b(O_S)}(sY)\chi(\det(sY))^{-1}.$$

Suppose $sx = I_b$, with x integral, and $p = r\,p_s$ is a locally constant compactly supported function, where $r \geq 0$ and $r(x) > 0$. Every complex subspace P of the full-rank Schwartz–Bruhat functions which contains p and is stable under $f(Y) \mapsto f(AY)$, $A \in \mathrm{GL}_N(F_S)$, contains every column member Ψ of a matched pair. No topological closure of P is required.

Proof. Choose a uniformizer ϖ of the product O_S. Since Ψ is locally constant of compact support, there is an integer $c \geq 1$ such that $\Psi(Y + \varpi^c Z) = \Psi(Y)$ for every integral Z and every Y. Set

$$e = xs, \qquad d = e + \varpi^{-c}(I - e), \qquad K = \mathrm{GL}_N(O_S).$$

The matrix e is idempotent, $dx = x$, and d is invertible. If $z = dky$ lies in the support of p, then $g = sz \in \mathrm{GL}_b(O_S)$. Since $sd = s$,

$$y = k^{-1}xg + \varpi^c k^{-1}(z - xg).$$

The second coefficient is integral. By translation invariance and (12.12), $\Psi(y) = \chi(\det g)^{-1}\Psi(k^{-1}x)$. It follows, including the case $p(dky) = 0$, that

$$\Psi(k^{-1}x)p(dky) = |p(dky)|\Psi(y).$$

Let $I(y) = \int_K |p(dky)|\,dk$. The group K acts transitively on primitive column frames: complete each such frame to an integral basis. Thus $I(y) = I(x)$ on the support of Ψ. Moreover $I(x) > 0$, since $p(dx) = p(x) = r(x) > 0$ and the integrand is locally constant. Hence

$$\int_K \Psi(k^{-1}x)p(dk\,\cdot)\,dk = I(x)\Psi.$$

This integral is a finite linear combination in P. Indeed a compact group has a finite orbit on a smooth Schwartz–Bruhat vector, and the coefficient is locally constant. Dividing by $I(x)$ proves the assertion.

Lean. `StableSpacesStandardData` (L80-L236). □

Sections on the full-rank locus. For arbitrary full-rank Schwartz–Bruhat functions define

$$S_a\Phi(X) = \int_{\mathrm{GL}_a(F_S)} \eta(\det h)\Phi(hX)\,dh, \qquad S_b\Psi(Y) = \int_{\mathrm{GL}_b(F_S)} \eta(\det j)\Psi(Yj)\,dj.$$

For each fixed full-rank matrix the integrating variable lies in a compact set: a left or right inverse bounds its entries, and the full-rank compact support bounds its determinant away from zero. Call (Φ, Ψ) a same-section pair if

$$S_a\Phi(R(k)) = \eta(\det k)S_b\Psi(C(k)) \qquad (k \in \mathrm{GL}_N(F_S)). \tag{12.13}$$

This Haar-integral definition works for arbitrary full-rank Schwartz–Bruhat functions. On the unit-covariant functions (12.12) it is the usual finite coset sum definition of the section.

Every matched pair is a same-section pair. To check this, choose h, j such that $X_0 = hR(k)$ and $Y_0 = C(k)j$ are primitive. Then $X_0Y_0 = 0$, and primitivity implies $\ker X_0 = \operatorname{im} Y_0$ over O_S. Choose an integral right inverse Z_0 of X_0; the matrix $k_0 = (Y_0, Z_0)$ is integral invertible and $R(k_0) = X_0$. Comparing with k, its determinant is $\det k_0 = \det k \det j / \det h$. In each section exactly one coset of the integral general linear group contributes. Unit covariance makes the integrand constant on that coset, so the sections are $\eta(\det h)\Phi(X_0)$ and $\eta(\det j)\Psi(Y_0)$. Equation (12.11) at k_0 is precisely (12.13).

Lemma 12.3 (The large theta map factors through its column section)**.** *Fix real Schwartz data and the rank-normalized splitting characters of the exchange. The large theta integral of a same-section pair depends linearly only on its column member Ψ. In particular, if $(\Phi, 0)$ is a same-section pair, then its large theta integral is zero.*

Proof. Fix an adelic point g of the first unitary group and let $\Lambda(\Phi)$ be the large theta integral at g. Full-rank convergence makes this a linear functional. The split Schrödinger action on the target group and right translation of its integration variable give

$$\eta(\det h)\Lambda(\Phi(h\,\cdot)) = \Lambda(\Phi) \qquad (h \in \mathrm{GL}_a(F_S)). \tag{12.14}$$

Indeed, the target point with component h^{-1} acts by

$$\Phi(X) \longmapsto \nu(\det h/\overline{\det h})\,|\det h|_S^{N/2}\Phi(hX).$$

The scalar is $\eta(\det h)$, and the Haar measure on the unitary quotient is right invariant.

Let $u(X)$ be the indicator of the primitive row matrices and put

$$\Pi\Phi(X) = u(X)\int_{\mathrm{GL}_a(F_S)} \eta(\det h)\Phi(hX)\,dh.$$

This is a full-rank Schwartz–Bruhat function. For a fixed full-rank X, the set of h for which $h^{-1}X$ is primitive is a single coset of $\mathrm{GL}_a(O_S)$, and therefore has volume one. Apply (12.14) to $X \mapsto u(h^{-1}X)\Phi(X)$, then integrate in h. It gives

$$\Lambda(\Pi\Phi) = \Lambda\left(\Phi(X)\int u(h^{-1}X)\,dh\right) = \Lambda(\Phi).$$

These exchanges need no completion of a vector space. On the compact full-rank support of Φ, the possible lattices hO_S^a lie between two fixed scalar lattices. There are only finitely many such lattices, and local constancy refines each compact coset into finitely many pieces on which the Schwartz–Bruhat function being averaged is constant.

For primitive X, choose $k \in \mathrm{GL}_N(O_S)$ with $R(k) = X$. The same-section relation identifies $\Pi\Phi(X) = S_a\Phi(X)$ with $\eta(\det k)S_b\Psi(C(k))$. Thus $\Pi\Phi$, and hence $\Lambda(\Phi)$, is determined by Ψ. This holds at every g, and all constructions are linear.

Lean. `MatchingPartners` (L428-L502), `EquivariantDifference` (L235-L268). □

Lemma 12.4 (Equivariance of the difference)**.** *Let M be the column projection of the space of same-section pairs. For fixed real large and small data and any $c \in \mathbb{C}$, let*

$$\mathcal{D}(\Psi) = T_{\mathrm{large}}(\Psi) - c\,T_{\mathrm{small}}(\Psi), \qquad \Psi \in M,$$

where T_{large} is the map of Lemma 12.3. Then M and $\ker \mathcal{D}$ are stable under $\Psi(Y) \mapsto \Psi(AY)$, $A \in \mathrm{GL}_N(F_S)$. More precisely, if $\widehat{A}$ is the unitary adelic point supported at $S \cup \overline{S}$ with S-component A, then

$$\mathcal{D}\big(\Psi(A\,\cdot)\big)(g\widehat{A}) = \kappa(A)^{-1}\mathcal{D}(\Psi)(g), \qquad \kappa(A) = \big(\upsilon_S(\det A)|\det A|_S^{1/2}\big)^b \neq 0. \tag{12.15}$$

The functions are right invariant under the standard integral level away from $S \cup \overline{S}$.

Proof. The same-section relation at Ak says that

$$\Phi_A(X) = \eta(\det A^{-1})\Phi(XA^{-1}), \qquad \Psi_A(Y) = \Psi(AY)$$

is again a same-section pair. This proves stability of M. The split Weil formula on $\mathrm{Hom}(W, V)$ is the change of variables $X \mapsto h^{-1}Xg$, multiplied by

$$\frac{\alpha_W(\det h)}{\beta_V(\det g)} \frac{|\det g|_S^{a/2}}{|\det h|_S^{N/2}},$$

where α_W, β_V are the derived Hecke characters on $\mathbb{A}_E^\times$ of the two spaces. For the large kernel $\beta_V = \widetilde{\nu}^{-1}\upsilon^b$, with $\widetilde{\nu}(t) = \nu(t/\bar{t})$; the original two norm-one splitting characters are both $\nu^{-1}\upsilon_{U(1)}^{N/2}$. Combining this formula with the scalar in Φ_A cancels ν and leaves $(\upsilon_S(\det A)|\det A|_S^{1/2})^{-b}$. For the small kernel the datum is read at $\overline{S}$ by transpose; the determinant there is $\overline{\det A}^{-1}$. The identities $\upsilon(\bar{t}) = \upsilon(t)^{-1}$ and $|\bar{t}^{-1}| = |t|^{-1}$ give exactly the same scalar. Right translation of g now proves (12.15). The level invariance away from these places is the integral invariance of the vacuum; the characters are unramified there.

Lean. `EquivariantDifference` (L81-L220). □

12.3 The ramified line calculation and removal of planes

The following argument supplies the descent part of (12.7). Throughout this subsection the character is fixed. No choice uniform over finite characters is needed. Let E/F be the CM extension already used in the datum, let S be the nonempty set of split ramified places, with $S \cap \overline{S} = \varnothing$, and let ν be the unitary norm-one character of positive integral weights μ_τ. Write $\widetilde{\nu}(a) = \nu(a/\overline{a})$, $Q_S = \prod_{w \in S} q_w$, and χ for its tame character on $\prod_{w \in S} \mathcal{O}_{E,w}^\times$. At every $w \in S$, the residue character is nontrivial. Here $L^S(s, \nu)$ means the finite Hecke L-function of $\widetilde{\nu}$, with the ramified factors omitted, as in the preceding exchange statement.

Take adapted splittings

$$W^+ = W \perp H, \qquad V^+ = V \perp H,$$
$$n = \dim W \in 2\mathbb{Z}, \quad r = \dim V \leq n, \quad b = n - r, \quad e = r - n/2.$$

An adapted frame is integral with integral inverse at the finite places and orthonormal in the chosen positive majorant at infinity. For an integral $b \times n$ matrix y, the standard row datum is

$$\Phi_y(X) = \mathbf{1}_{\binom{y}{X} \in \mathrm{GL}_n(\mathcal{O}_S)} \chi\left(\det \begin{pmatrix} y \\ X \end{pmatrix}\right)^{-1}, \qquad \mathcal{O}_S = \prod_{w \in S} \mathcal{O}_{E,w}.$$

The fixed ordering of rows is understood. Let $I_{W,V}(v, y)(g)$ denote its theta integral over $[U(V)]$, with real Schwartz vector v, the vacuum away from S, and the rank-normalized splitting characters of the exchange statement. The column datum paired with Φ_y is

$$\Psi_y(Y) = \mathbf{1}_{Y \text{ integral},\ yY \in \mathrm{GL}_b(\mathcal{O}_S)} \chi(\det(yY))^{-1}.$$

Write $J_{W,U}(w,y)$ for its small-side theta integral, using the transpose at the conjugate places and $\dim U = b$. Lemma 12.1 applies to these data. In particular these are ordinary, absolutely convergent theta integrals, and their dependence on a real Schwartz vector is continuous.

Real cone operators and their convergence. Let A and B be the isotropic lines in the two added planes, and choose $x_B \neq 0$ spanning B. In the mixed model for A, let slice_x denote evaluation at the vector x of V_∞^+. Let Z_B be the target zero term along B: in a Schrödinger model it is obtained by a partial Fourier transform and evaluation at zero in the corresponding isotropic variables. Put

$$\ell_+(a, v^+) = Z_B(\mathrm{slice}_{x_B a}\, v^+).$$

The operator ℓ_- is obtained from the other mixed model of the same hyperbolic plane; one-dimensional Poisson summation interchanges the two line terms. For $a = (a_\tau) \in (E \otimes \mathbb{R})^\times$, use $|a|_\infty = \prod_\tau |a_\tau|^2$. Define, for $x > 0$,

$$\mathcal{C}_x^\pm v^+ = \int_{K(V^+)\times(E\otimes\mathbb{R})^\times} \left(\prod_\tau (a_\tau/|a_\tau|)^{2\mu_\tau}\right) |a|_\infty^x\, \ell_\pm(a, \omega(1,\kappa)v^+)\, d\kappa\, d^\times a. \tag{12.16}$$

The Weil action in this formula has target weight $(n+2)/2 - \mu_\tau$; after taking a slice, the weight is $n/2 - \mu_\tau$.

These integrals converge in the Schwartz topology. Indeed restriction, partial Fourier transform and integration in an isotropic variable are continuous maps of Schwartz spaces. Consequently, for every Schwartz seminorm p_N and every M, there are M' and C such that, uniformly for κ in the compact group,

$$p_N\big(\ell_\pm(a, \omega(1,\kappa)v^+)\big) \leq C(1 + \|a\|)^{-M} p_{M'}(v^+).$$

For every $x > 0$ a suitable M also gives

$$\prod_\tau |a_\tau|^{2x} \leq C_x(1 + \|a\|)^M \prod_\tau (1 + \log^2 |a_\tau|)^{-1}.$$

To see this, bound $t^{2x}(1 + \log^2 t)$ by a polynomial in $1 + t$: it tends to zero as $t \downarrow 0$, and has polynomial growth as $t \to \infty$. Multiply this bound over the places. The displayed logarithmic majorant is integrable for $d^\times a$, by $u_\tau = \log|a_\tau|$. It also controls all Schwartz seminorms of the integral. Combining it with the uniform target bound for full-rank data justifies interchanging the cone integral and $I_{W,V}$. The real cone estimate.

Theorem 12.5 (The one-plane constant terms)**.** *For every real Schwartz vector v^+, integral y, and $g \in U(W)(\mathbb{A})$, there is a real-analytic function $\mathcal{Z}_{v^+,y,g}$ on $\mathbb{R}$ such that*

$$\mathrm{CT}_A\, I_{W^+,V^+}(v^+, y \circ \mathrm{pr}_W)(1,g) = \mathcal{Z}_{v^+,y,g}(e+1), \tag{12.17}$$

$$\mathcal{Z}_{v^+,y,g}(x) = \epsilon_+ A^x Q_S^{x-1/2} L^S(x,\nu) I_{W,V}(\mathcal{C}_x^+ v^+, y)(g), \tag{12.18}$$

$$\mathcal{Z}_{v^+,y,g}(1-x) = \epsilon_- A^x Q_S^{x-1/2} L^S(x,\nu) I_{W,V}(\mathcal{C}_x^- v^+, y)(g) \tag{12.19}$$

for every real $x \geq 1$. Here $A > 0$ is the absolute value of the fixed scale idele, and $\epsilon_+, \epsilon_- \neq 0$ depend on the fixed character, frames and measures, but not on v^+, y, g, x. If $r = n$, the constant term at (u,g) is the same as at $(1,g)$ for every $u \in \mathbb{A}_E^\times$.

On the small side, for a real Schwartz vector w^+,

$$\mathrm{CT}_A\, J_{W^+,U}(w^+, y \circ \mathrm{pr}_W)(1,g) = \kappa_0 J_{W,U}(\mathrm{slice}_0\, w^+, y)(g), \qquad \kappa_0 \neq 0. \tag{12.20}$$

Proof. We give the local calculation, the global unfolding, and the convergence justification separately.

At one ramified place choose an additive character of conductor $\mathcal{O}$ and give $\mathcal{O}$ additive volume one. For $f(t) = \mathbf{1}_{\mathcal{O}^\times}(t)\chi(t)^{-1}$,

$$\widehat{f}(a) = \begin{cases} \gamma_w \chi(\varpi a), & \operatorname{ord}(a) = -1, \\ 0, & \operatorname{ord}(a) \neq -1, \end{cases} \qquad |\gamma_w|^2 = q_w^{-1}. \tag{12.21}$$

For $\operatorname{ord}(a) \geq 0$, the integral is zero because the nontrivial residue character sums to zero. For $\operatorname{ord}(a) < -1$, integration on a coset of $1 + \varpi\mathcal{O}$ gives zero. For valuation -1, it is q_w^{-1} times a residue-field Gauss sum. The identity $|\sum_{u \neq 0} \chi(u)\psi(u)|^2 = q_w$ follows by writing the square as a sum over u/v and summing first over v. This proves the stated modulus and its nonvanishing.

Here is why the matrix datum produces this one-dimensional factor. In the adapted frames its matrix, after integral invertible changes of coordinates, has the block triangular form

$$\begin{pmatrix} a & * & * \\ 0 & \binom{y}{X} & * \\ 0 & 0 & z \end{pmatrix}.$$

The other mixed model reverses the two outside blocks. Its determinant is a fixed unit times $az \det \binom{y}{X}$. The off-diagonal coordinates have the factor $\mathbf{1}_{\mathcal{O}}$, whose normalized integral is one. The last coordinate has $\mathbf{1}_{\mathcal{O}^\times}(z)\chi(z)^{-1}$, and its partial Fourier transform is (12.21). All integral frame determinants contribute nonzero unit phases. Thus the finite part of the line term, apart from nonzero vacuum constants, is

$$\gamma\, \mathbf{1}_{a \text{ integral off } S \cup \overline{S}} \mathbf{1}_{a_S \in \mathcal{O}_S^\times} \mathbf{1}_{\varpi \overline{a}_S \in \mathcal{O}_S^\times} \chi(a_S)^{-1} \chi(\varpi \overline{a}_S), \qquad |\gamma| = Q_S^{-1/2}.$$

It multiplies precisely the lower datum Φ_y. Choose a finite idele c, supported at $\overline{S}$, whose conjugate component at S is ϖ. Then $|c^{-1}|_f = Q_S$, and the change $a_{\mathrm{old}} = c^{-1} a_{\mathrm{new}}$ removes the ϖ in the indicator and character. The resulting finite Tate integral, for $x > 1$, is $L^S(x, \nu)$ times the positive volume of finite integral units. This follows directly by partitioning the finite ideles into their valuation classes: at an unramified prime the sum is $\sum_{j \geq 0} \widetilde{\nu}(\varpi)^j q_w^{-jx}$, and at the ramified primes the unit characters cancel against $\widetilde{\nu}$. The translate contributes Q_S^x, the Gauss sum contributes $Q_S^{-1/2}$, and the fixed scale idele contributes A^x. This proves the exact power $A^x Q_S^{x-1/2}$; every remaining scalar is nonzero. Block determinant and Gauss calculation; The finite Tate factor.

For the global calculation, expand the theta kernel in the mixed model for A and integrate over its unipotent radical modulo rational points. Fourier orthogonality leaves rational isotropic vectors of V^+. The vector zero contributes nothing: on that slice the rows $y \circ \mathrm{pr}_W$ and the remaining rows all annihilate the nonzero line A, whereas their stacked matrix must be invertible on the support of the standard datum. The nonzero isotropic vectors form one rational unitary orbit, by Witt's extension theorem. Group them first by their rational isotropic lines, leaving a sum over the nonzero scalar on each line. Unfold the line orbit to the stabilizer of B, and use its Levi coordinates (a, h), with $a \in \mathbb{A}_E^\times$, $h \in U(V)(\mathbb{A})$, and a maximal compact variable κ. The quotient measure has the factor $|a|^{r+1}$, and the Weil line term contributes $|a|^{-n/2}$. The remaining exponent is exactly $e + 1 = r + 1 - n/2$.

More explicitly, up to a fixed nonzero measure scalar, the unfolded expression at a general exponent s is

$$\int_{E^\times \backslash \mathbb{A}_E^\times} \int_{[U(V)]} \int_{K(V^+)} \widetilde{\nu}(a)|a|^s \sum_{\xi \in E^\times} \theta^{W,V}\Big(\ell_+(\xi a, \omega(\kappa)v^+) \otimes \Phi_{y,\mathrm{line}}(\xi a)\Big)(g, h)\, d\kappa\, dh\, d^\times a.$$

Here $\Phi_{y,\mathrm{line}}$ is the finite line factor just computed, including the vacuum off the specified places. Equivalently this is the Tate integral of the full adelic line term; writing the factors separately only displays its real and finite parts. At $s = e + 1$ it is the constant term. For $s > 1$, unfold the sum over $E^\times$, use the finite calculation, and integrate the real line term. This gives (12.18).

Poisson summation on the line has no zero term on either side: $f(0) = \widehat{f}(0) = 0$ at a ramified place by (12.21). With the fixed self-dual normalization, its exact form is

$$|a| \sum_{\xi \in E} F_+(\xi a) = \sum_{\xi \in E} F_-(\xi \overline{a}^{-1}),$$

where $F_\pm$ are the two full adelic line functions; their zero summands both vanish. Changing a to $\overline{a}^{-1}$ in the dual expression preserves $\widetilde{\nu}$, since $\widetilde{\nu}(\overline{a}^{-1}) = \widetilde{\nu}(a)$, and changes the exponent to $1 - s$. It gives (12.19) for $x > 1$. The real cone estimate and the full-rank theta bound justify absolute convergence on each of these two half-lines. For any bounded interval of exponents, choose $R > 1$ with the interval inside $(1 - R, R)$. The inequality $t^s \le t^{1-R} + t^R$, and its version with powers of $|\log t|$ using slightly wider endpoints, gives an integrable majorant for the Mellin integral and its derivatives. Thus $\mathcal{Z}$ is real analytic on all of $\mathbb{R}$, and the unfolding at $e + 1$ is valid. Continuity extends the two factorizations to $x = 1$; Hecke's theorem supplies continuity of $L^S(x, \nu)$ there. When $r = n$, the source Levi factor involving u cancels the corresponding Weil factor, and the only occurrence of u in the unfolded integral is through au. Translation in the idele-class variable removes it. This proves the stated independence of u. It is an equal-rank statement $r = n$, not the condition $e = 0$. Isotropic-orbit unfolding; The exponent and two factorizations.

For (12.20), expand the same source constant term on the small side. A rational nonisotropic vector is killed by central Fourier orthogonality. For a nonzero isotropic vector $x \in U$, take the target zero term along Ex. The resulting linear functional on Schwartz data is invariant under every Heisenberg translation along x. In the Schrödinger model at a conjugate ramified place these translations multiply the datum by the additive characters of X^*x. On its support, X^* is injective, so $X^*x \neq 0$. An invariant functional therefore kills this datum: average over a sufficiently large compact additive subgroup to annihilate all these characters. Such a subgroup exists uniformly because the support is compact and avoids $X^*x = 0$. This argument works after every adelic translate of the rational vector; invertibility preserves its nonzero local component. Only the vector zero remains. Its mixed slice is the lower column datum times the slice $\mathrm{slice}_0\, w^+$, with the nonzero vacuum normalization κ_0. This proves (12.20).

Lean. `LineGaussSums` (L353-L503), `FiniteTateIntegral` (L216-L322), `RemovingPlanes` (L306-L475). □

Lemma 12.6 (The image of a cone operator)**.** *There is a function $B : \mathbb{R} \to \mathbb{R}_{>0}$, real analytic on the whole real line, such that for every real Schwartz vector v on (W, V) there is one real Schwartz vector v^+ on (W^+, V^+) satisfying*

$$I_{W,V}(\mathcal{C}_x^+ v^+, y)(g) = B(x) I_{W,V}(v, y)(g) \qquad (x > 0) \tag{12.22}$$

for every standard datum and every g. In particular, after rescaling v^+, the cone operator at any prescribed $x > 0$ has the same theta-integral image as all real Schwartz vectors.

Proof. Choose v_B with target zero term $Z_B v_B = v$. This is a surjective Schwartz-space map: in the mixed model tensor v with a Schwartz function whose required value or integral is one. Near the nonzero isotropic vector x_B, the orbit map $U(V^+)_\infty \to U(V^+)_\infty x_B$ is a submersion. It has a smooth local section $k(x)$ with $k(x)x_B = x$ on the isotropic cone. Composing with a smooth local retraction onto that cone extends the section to a neighborhood in the ambient real vector space.

Choose $\beta \geq 0$ smooth of compact support there, with $\beta(x_B) > 0$. Shrink its support so that at each real place

$$\tfrac{1}{2}\|x_B\|_\tau^2 \leq \|x\|_\tau^2 \leq 2\|x_B\|_\tau^2.$$

Prescribe the slices of v^+ by

$$\operatorname{slice}_x v^+ = \beta(x)\,\omega_{n/2-\mu,0}(1, k(x))v_B.$$

A compactly supported smooth family of Schwartz vectors is a Schwartz function in the mixed model, so these slices define the required v^+.

For $x = \kappa^{-1}x_B a$ in this support, $\kappa k(x)$ preserves B and scales its chosen vector by a. The target zero-term formula for the Levi action is

$$Z_B\omega_{n/2-\mu,0}(1, \kappa k(x))v_B = \left(\prod_\tau (a_\tau/|a_\tau|)^{-2\mu_\tau}\right)|a|_\infty^{n/2}\,\omega_{n/2-\mu,0}(1, h_{\kappa,a})v$$

for some $h_{\kappa,a} \in U(V)_\infty$. The power $|a|_\infty^{n/2}$ is the half-density of the n complex isotropic variables; the determinant twist gives the displayed phase. Substitution in (12.16) cancels that phase exactly. The theta integral over $[U(V)]$ is invariant under the remaining action of $h_{\kappa,a}$, by right translation of its integration variable. Hence the scalar in (12.22) is

$$B(x) = \int \beta(\kappa^{-1}x_B a)|a|_\infty^{n/2+x}\,d\kappa\,d^\times a.$$

Since the compact group preserves the chosen Euclidean lengths, the support lies in $1/2 \leq |a_\tau|^2 \leq 2$ at every place. It is a compact annulus. The integrand is nonnegative and positive on a neighborhood of $(1, 1)$. Thus $B(x) > 0$ for every real x. Compactness of the annulus proves analyticity for all real x, indeed an entire Mellin extension. The previously proved seminorm bound justifies the interchange with the theta integral.

Lean. `ConeImage` (L221-L449). □

Lemma 12.7 (Integer nonvanishing and vanishing descent)**.** *Let $k \geq 0$ be an integer with $k \leq \mu_\tau$ at every real place. The function $L^S(x, \nu)$, initially for $x \geq 1$, has a real-analytic continuation $\mathcal{L}_\nu(x)$ to $x > -k$, and*

$$\mathcal{L}_\nu(j) \neq 0 \qquad (j \in \mathbb{Z},\ j > -k). \tag{12.23}$$

If $e = r - n/2 \geq -k$ and I_{W^+,V^+} vanishes on every real Schwartz vector and every standard datum, then so does $I_{W,V}$.

Proof. The nontrivial positive infinity type makes $\widetilde{\nu}$ nontrivial. Tate's functional equation gives its analytic continuation. Its archimedean gamma factors, in the current normalization and up to nonzero elementary factors, are $\prod_\tau \Gamma(x + \mu_\tau)$. They are finite and nonzero on $x > -k$. For an integer $j \geq 1$, nonvanishing follows from the Euler product when $j > 1$, and from the classical Hecke nonvanishing theorem at $j = 1$. If $-k < j \leq 0$, the functional equation relates the completed value at j to the completed value of the inverse character at $1 - j \geq 1$. The epsilon factor, conductor powers, and both gamma factors are nonzero, so the value at j is nonzero. All omitted finite factors are genuinely ramified factors, equal to one; there are no extra omitted unramified factors that could introduce trivial zeros here. This proves (12.23). It asserts nonvanishing at these integers only, not on an interval inside the critical strip.

For descent fix v, y, g, and set $I = I_{W,V}(v, y)(g)$. Use Lemma 12.6 to choose v^+ that works for every $x > 0$. Equations (12.18) and (12.22) give, for $x > 1$,

$$\mathcal{Z}_{v^+,y,g}(x) = \epsilon_+ A^x Q_S^{x-1/2}\mathcal{L}_\nu(x)B(x)I.$$

Both sides are real analytic on the connected interval $(-k,\infty)$; therefore they agree throughout that interval. Universal vanishing upstairs makes the constant term zero, so at $x=e+1$ the left side is zero. But $e+1$ is an integer strictly greater than $-k$, and every displayed factor other than I is nonzero. Thus $I=0$. Since v,y,g were arbitrary, universal vanishing descends. Adding one plane raises e by one. Consequently the same proof removes any finite number of planes if the inequality $e\ge -k$ holds at the bottom of the tower. The required integer values of Tate's functional equation; One-plane and iterated vanishing descent. For a classical account of the analytic theorem used here see S. S. Kudla, *Tate's thesis*, in *An Introduction to the Langlands Program*, Birkhäuser (2004), 109–131, author's lecture notes.

Lean. `FunctionalEquation` (L355-L440), `VanishingDescends` (L93-L185). □

Corollary 12.8 (Descent of an exchange on standard data)**.** *Suppose* $\dim W=\dim V+\dim U$, $\dim W$ *is even, and for real Schwartz vectors* v^+,w^+ *one has*

$$I_{W^+,V^+}(v^+,y^+)=c^+J_{W^+,U}(w^+,y^+)$$

on all standard data y^+*. Then there is a real Schwartz vector* v *with*

$$I_{W,V}(v,y)=c\,J_{W,U}(\mathrm{slice}_0\,w^+,y),\qquad c=c^+\kappa_0/\gamma_e,\quad \gamma_e\neq 0. \tag{12.24}$$

In particular $c\neq 0$ *whenever* $c^+\neq 0$*. The statement iterates along any adapted sequence of planes.*

Proof. Apply the same source constant term to the assumed identity on $y^+=y\circ \mathrm{pr}_W$. If $e\ge 0$, use $v=\mathcal{C}^+_{e+1}v^+$ and

$$\gamma_e=\epsilon_+A^{e+1}Q_S^{e+1/2}L^S(e+1,\nu).$$

If $e<0$, then $-e\ge 1$ and $e+1=1-(-e)$; use $v=\mathcal{C}^-_{-e}v^+$ and

$$\gamma_e=\epsilon_-A^{-e}Q_S^{-e-1/2}L^S(-e,\nu).$$

In both cases the exponent of the cone operator is positive and the Hecke value occurs at an integer at least one. Thus $\gamma_e\neq 0$. The small constant term is (12.20); dividing gives (12.24). Successive zero slices compose to the zero slice for the direct sum of the removed planes. No division by a theta value occurs in this descent.

Lean. `RemovingPlanes` (L508-L684). □

Lemma 12.9 (Constancy in equal ranks)**.** *Let* Y *have even rank and contain a hyperbolic plane. For every real Schwartz vector* z*, the equal-rank integral*

$$F_z(g)=\mathrm{thetaLarge}(Y,Y,\nu;z,p_\varnothing)(g),\qquad p_\varnothing(X)=\mathbf{1}_{\mathrm{GL}_{\dim Y}(O_S)}(X)\chi(\det X)^{-1},$$

is constant on $U(Y)(\mathbb{A})$*. Denote its value by* $I_Y(z)$*.*

Proof. Choose a split place v_0 below S. In Lemma 12.4 take $b=0$ and the small map zero. The column space is a point and $\kappa(A)=1$. Thus F_z is right invariant under $U(Y)(F_{v_0})$. It is continuous and left $U(Y)(F)$-invariant by full-rank convergence and rational invariance of the theta kernel.

Put $G=\mathrm{SU}(Y)$. It is simply connected and absolutely almost simple, and $G(F_{v_0})\simeq \mathrm{SL}_{\dim Y}(F_{v_0})$ is noncompact. Strong approximation makes $G(F)G(F_{v_0})$ dense in $G(\mathbb{A})$ [Rap13, Theorem 2.3]. For fixed g, the function $s\mapsto F_z(sg)$ equals $F_z(g)$ on this dense set, since g normalizes the subgroup supported at v_0. Hence $F_z(g)$ depends only on $\det g$.

The unipotent radical of the parabolic stabilizing the hyperbolic line has determinant one. The constant term at its Levi point $m(u,1)$ is therefore $F_z(m(u,1))$, with quotient volume one. The

equal-rank formula of Theorem 12.5 makes this value independent of u. The determinants $u/\bar{u}$ run through the norm-one ideles by local Hilbert 90 [Mil20]: at almost all places a solution can be chosen to be a unit, since the extension is unramified there. This proves constancy.

Lean, comparison. Strong approximation gives invariance on each determinant fiber directly. Lean proves the adelic invariance through its strong-approximation lemmas. `LargeTriple` (L383-L425). □

12.4 Fourier recognition and the top exchange

The precise Fourier recognition statement. For the following lemma, an equivariant map means a linear map $\mathcal{D}$ from a $\mathrm{GL}_{\dim X}(F_S)$-stable space M of Schwartz functions on full-column-rank matrices to continuous automorphic functions on $U(X)$, with the transformation law

$$\mathcal{D}(\Psi(k\,\cdot))(gk) = \zeta(\det k)^{-b}\mathcal{D}(\Psi)(g), \qquad \zeta = v|\cdot|^{1/2}, \tag{12.25}$$

at the split places S, and invariance under the standard integral level away from $S \cup \overline{S}$. Such a map is supplied by the difference of the two theta integrals, by 12.3 and 12.4, using Lemma 12.1. In particular, its kernel is stable under all these local general linear transformations.

Write $f_T(g)$ for the central Fourier coefficient of an automorphic function along D, with rational skew-Hermitian index T. Equivalently, write $T = \mathfrak{j}\beta$ with β Hermitian, where $\mathfrak{j} = -\bar{\mathfrak{j}}$ is the fixed Weil scalar. A *self-dual test* consists of $\beta = J_B$ and a parabolic point whose Levi frame, after division by the fixed scale idele, spans a self-dual lattice for J_B at every finite place. We also test every invertible β whose infinite signatures differ from those of B, at every parabolic point.

Lemma 12.10 (Recognition from self-dual tests)**.** *Let $\mathcal{D}$ satisfy (12.25), and suppose $C_{L^*} \in M$, where L^* is the dual row block of the adapted frame of D. If $f = \mathcal{D}(C_{L^*})$ has zero central Fourier coefficients on both classes of tests just specified, then $\mathcal{D}$ vanishes on every locally constant datum admitting a matched partner.*

Proof. We separate the global lattice argument from its finite Fourier selector. All Fourier means used below are integrals of continuous functions over compact additive adelic quotients, so their linearity, continuity and Fourier uniqueness require no convergence assumption beyond those already stated.

First consider an extra hyperbolic plane in Y, with dual isotropic vectors u, u'. Let $n_+(a)$ and $n_-(b)$ be the elementary shifts of these vectors along the frame K of D, for adelic rows a, b. Matrix multiplication gives

$$n_+(a)n_-(b) = z(-a^*b)n_-(b)n_+(a),$$

where z is the central subgroup for D. Define

$$f^+(g) = \int_{E^b\backslash\mathbb{A}_E^b} f_T(n_-(b)g)\,db, \qquad \phi_g^+(a) = f^+(n_+(a)g),$$

and define the minus versions by interchanging the two lines. For rational a_0, the value $\phi_g^+(a+a_0)$ is the Fourier coefficient with index $-Ta_0^*$ of $b \mapsto f_T(n_-(b)n_+(a)g)$. When T is invertible these are all its Fourier coefficients. Consequently ϕ_g^+ determines $f_T(g)$ and ϕ_g^-, and a scalar relation between two plus functions is equivalent to the same relation between the corresponding minus functions. Also

$$\phi^+_{n_+(a)g}(y) = \phi_g^+(y+a), \qquad \phi^-_{n_+(a)g}(b) = \psi(-\operatorname{tr} Ta^*b)\phi_g^-(b). \tag{12.26}$$

The partial means of the standard datum are invariant under the local shears along either extra line. Here is the finite argument, including why the standard datum is needed. Write

$S_s(y) = \mathbf{1}_{\mathrm{Mat}(O_S)}(y)\chi^{-1}(\det sy)$. If a linear functional Λ is invariant under $y \mapsto y + uzsy$, and $su = ru = 0$, then

$$\Lambda(S_s(y + ury)) = \Lambda(S_s(y)).$$

Indeed average both functions under $y \mapsto y + urtsy$, with t ranging over integral matrices. On any nonzero slice choose an integral left inverse g of sy and an integral point $y + u\rho$ in that slice. The change of variables $t \mapsto t + (y + u\rho)g$ identifies the two averages; otherwise both averages are zero. Each locally constant compactly supported datum has a finite orbit under this compact integral shear group. Its average is therefore a finite linear combination, with which an arbitrary linear functional commutes. Apply this observation to the partial-mean functional. Its invariance under uzs is the defining average over n_-, transported by (12.25); its other shears have determinant one. The frame condition in this application must be retained. Let H_K be the subgroup fixing every vector of K at $S \cup \overline{S}$. If $x \in H_K$, unitarity $x^*Jx = J$, together with the frame equality at the conjugate place, gives

$$K^*Jx = K^*J$$

at the place where the determinant datum is read. Up to the fixed form normalization, this is exactly its dual row. A local shear v has matrix $1 + uc$ on the read half, with $cu = 0$; this dual row kills u, so it is also unchanged by v. Thus the transported partial-mean functional at vx reads the same dual row as the original standard datum. The finite averaging calculation applies at every $x \in H_K$. It proves

$$f^+(vx) = f^+(x) \qquad (x \in H_K)$$

for the split-supported radical shears. No condition on $vx \in H_K$ is needed here. The same argument applies to the other extra line.

We now justify the passage from these shears to the full adelic radical. Let R_u be the unipotent radical of the stabilizer of the line spanned by u, and write $z_u(s)$ for its one-line central subgroup in the matrix parametrization radicalCentre(s). This parametrization may have a kernel. All means below use probability Haar measure on the additive parameter quotients. Put

$$B_0 = \begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix}$$

and let F_{B_0} be the central Fourier coefficient for the composite frame (u, K) at B_0. We will prove

$$\begin{aligned} F_{B_0}(hg) &= F_{B_0}(g) \quad (g \in H_K,\ h \in R_u(\mathbb{A})), \\ f^+(hg) &= f^+(g) \quad (h \in R_u(\mathbb{A}),\ g, hg \in H_K). \end{aligned} \tag{12.27}$$

The second assertion requires the frame condition at both endpoints.

First, at every argument x, including those outside H_K,

$$F_{B_0}(x) = \int_{E\backslash\mathbb{A}_E} f^+(z_u(s)x)\, ds. \tag{12.28}$$

Indeed, write the Fourier variable for (u, K) in blocks

$$M = \begin{pmatrix} s & a \\ c & N \end{pmatrix}.$$

Matrix multiplication factors its central element, in order, as the center along K at N, the shift $n_-(a - c^*)$, and $z_u(s)$. The Fourier character at B_0 reads only N. Integrating first in N gives f_T. The map $(a, c) \mapsto a - c^*$ is a surjective homomorphism of compact additive quotients, since $c = 0$

gives a right inverse. It therefore carries probability Haar measure to probability Haar measure. The two remaining off-diagonal means give f^+, and the s-mean proves (12.28). No index or volume factor is introduced.

For rational s_0, the element $z_u(s_0)$ commutes with $n_-(b)$ and fixes K. Move it through the shift and the Fourier center of K, then remove it by rational left invariance of f. This proves

$$f^+(z_u(s_0)x) = f^+(x)$$

at every x. Set $e = \mathbf{1}_{\overline{S}}$. The element $z_u(es_1)$ is one of the allowed local shears; its unitary matrix can have components on both conjugate halves. Therefore, for $x \in H_K$, the same equality holds for $s = s_0 + es_1$. Such parameters are dense by additive strong approximation away from any one place of the nonempty set $\overline{S}$. Continuity extends the equality to all s, and (12.28) yields

$$F_{B_0}(x) = f^+(x) \qquad (x \in H_K).$$

The local shear identity holds at every central translate of the base point: $z_u(s)$ fixes K pointwise, so $z_u(s)g \in H_K$ whenever $g \in H_K$. Every radical shear commutes with this center. Hence

$$\begin{aligned} F_{B_0}(vg) &= \int f^+(z_u(s)vg)\,ds \\ &= \int f^+(vz_u(s)g)\,ds = \int f^+(z_u(s)g)\,ds = F_{B_0}(g). \end{aligned}$$

Here (12.28) is used at vg; the argument does not assume $vg \in H_K$.

For the rational part of the radical, use the rational splitting with basis consisting of u, the middle Hermitian space, and the opposite vector. Write $r(q,d)$ for the radical element with middle column q and scalar parameter d, and let J_0 be the rational middle form. It fixes u and sends a middle vector w to $w + uq^*J_0w$. Thus a rational $r(q_0,0)$ acts on the composite frame by

$$U = \begin{pmatrix} 1 & q_0^*J_0K \\ 0 & I_b \end{pmatrix}, \qquad U^*B_0U = B_0.$$

In this display K denotes its middle-space coordinates. In the defining mean for $F_{B_0}(r(q_0,0)x)$, substitute $M = UNU^*$. This is an automorphism of the compact additive quotient preserving probability Haar measure. It moves the rational radical element to the left of the Fourier element, where automorphy removes it; the character has index $U^*B_0U = B_0$. Consequently

$$F_{B_0}(r(q_0,0)x) = F_{B_0}(x)$$

at every x. Likewise $F_{B_0}(z_u(s)x) = F_{B_0}(x)$ at every x, since a translation in the upper-left Fourier block pairs trivially with B_0.

The radical group law in these coordinates is

$$r(q,d)r(q',d') = r(q+q', d+d'+\mathrm{j}q^*J_0q').$$

For the dense set of columns $q = q_0 + eq_1$, with q_0 rational, it gives

$$r(q,d) = r(q_0,0)\,r(0,d-\mathrm{j}q_0^*J_0eq_1)\,r(eq_1,0).$$

The middle factor is central; in these parametrizations, $r(0,d') = z_u(-d'/(2\mathrm{j}))$. The last is an allowed local shear: its quadratic term vanishes because $e\overline{e} = 0$, and on the read half its matrix is $1 + uc$, with $cu = 0$. The rational and central invariances just proved hold at every argument,

so they remove the first two factors. The local shear identity for F_{B_0} removes the last at $g \in H_K$. Additive strong approximation in q and continuity, with d fixed, prove the first assertion of (12.27). For the second, identify F_{B_0} with f^+ at both endpoints in H_K. Interchanging u, u' proves the minus version.

The transported dual row and the pointwise frame condition; The bordered mean, central translates and full radical; The radical coordinates and group law.

We now show that nonzero coefficients at invertible indices can be detected on self-dual lattices. Let Q denote the finite places outside $S \cup \overline{S}$. Put $\delta = \mathfrak{j}\kappa\overline{\kappa}$, with κ the fixed scale idele. At a point acting on the frame K by γ, its normalized Gram matrix is

$$C = \delta^{-1}\gamma^* T \gamma.$$

Standard integral-level invariance implies that C is integral on Q whenever the coefficient is nonzero: translate by an integral central element and cancel the nonzero coefficient; the resulting character must be trivial, and the integral lattice is self-dual for the normalized additive character.

The endpoint conditions in (12.27) are preserved in the straightening step as follows. For a unitary adelic point a, let $c_Q(a)$ keep its finite components on Q and replace all other components by the identity. The set Q is conjugation-stable, so this operation is a group homomorphism. Keep a point $X \in H_K$ distinct from an auxiliary point g, with $X_Q = g_Q$. Adelic Iwasawa decomposition for the flag (u, K) gives a right factor κ, integral at every finite place and chosen at infinity by real Iwasawa decomposition, such that $g\kappa$ acts by a block upper-triangular matrix on that flag. Its radical–Levi decomposition is

$$g\kappa = hg_0, \qquad h \in R_u(\mathbb{A}),$$

where g_0 scales u, u' by t, t', $\overline{t}'t = 1$, and acts on K by γ. Define the modified points in the original group

$$X_\kappa = Xc_Q(\kappa), \qquad X' = c_Q(h^{-1})X_\kappa. \tag{12.29}$$

Both points lie in H_K, since all modifications are supported on Q. The level of f gives $f^+(X_\kappa) = f^+(X)$. The point $c_Q(h^{-1})$ is still in $R_u(\mathbb{A})$, so the two-endpoint assertion in (12.27) gives $f^+(X') = f^+(X_\kappa)$. Moreover $X'_Q = (g_0)_Q$. Thus a nonzero partial mean gives a nonzero shifted mean at the straight point consisting of the shifted point X' and the auxiliary block point g_0. On $S \cup \overline{S}$, the original point fixes every vector of K; no agreement with g_0 there is asserted.

If the starting shifted mean is nonzero, choose a for which the partial mean at $n_+(a)X$ is nonzero. Every line shift $n_\pm(a)$ fixes K pointwise. Consequently the same construction applies after each shift used below: the shifted point remains in H_K, its Q-components agree with those of the correspondingly shifted auxiliary point, and both endpoints of each radical removal satisfy the required frame condition. The compact right factor at all places; Frame preservation under line shifts; Straightening the original and auxiliary points. Among straight points with $\phi^+ \neq 0$, choose one minimizing the positive integer

$$d(C) = \prod_{w \in Q} q_w^{v_w(\det C)}.$$

The product is finite and its exponents are nonnegative by integrality. Fourier uniqueness in (12.26) gives the corresponding minimality for the opposite straight point, with the two line scalars exchanged and the same γ. Indeed any opposite straight point with nonzero minus function, viewed with its two lines interchanged, has a nonzero plus function and exactly the same denominator. Thus the minimum value applies to both families without making a new choice.

At this minimum, the supports satisfy

$$\phi^+(a) \neq 0 \implies \gamma^{-1}a^*t \text{ is integral on } Q, \tag{12.30}$$

$$\phi^-(b) \neq 0 \implies \gamma^{-1}b^*t' \text{ is integral on } Q. \tag{12.31}$$

To verify the first assertion, apply finite Iwasawa decomposition to $n_+(a)$ times the straight point. Before decomposition its flag matrix has lower blocks $(-a^*t, \gamma)$; afterwards it is upper triangular with last block γ'. The inverse integral change-of-basis matrix has blocks a_{21}, a_{22} satisfying

$$-a^*t = \gamma' a_{21}, \qquad \gamma = \gamma' a_{22}.$$

If $\gamma^{-1}a^*t$ is not integral, $\det a_{22}$ is not a unit at some place of Q, since otherwise its inverse is integral. The new Gram matrix C' is integral, and $C = a_{22}^* C' a_{22}$. Hence $d(C') < d(C)$, contrary to minimality. This proves (12.30); exchanging the lines proves (12.31).

If Cx is integral on Q, (12.31) makes the character in (12.26) trivial on the support of ϕ^- for the shift $a = \mathbf{1}_Q(\gamma x\overline{t'})^*$. Fourier uniqueness therefore makes a a period of ϕ^+. Choose a point of its nonempty support and subtract (12.30) for this point and its translate by a. Since $\overline{t'}t = 1$, this says that x is integral on Q. Apply it to every column of C^{-1}: C^{-1} is integral on Q. Thus a nonzero coefficient occurs at a point with a unimodular normalized Gram matrix there.

Now suppose that $\delta^{-1}T$ is unimodular at S, and start at a finite point trivial at $S \cup \overline{S}$. The preceding procedure leaves that frame unchanged. The normalized Gram matrix is then unimodular at every finite place, by gluing its Q-part with $\delta^{-1}T$ at $S \cup \overline{S}$. A self-dual Hermitian lattice over an unramified or split quadratic local algebra has an integral orthonormal basis. If its infinite signatures are those of B, Landherr's theorem [Lan35] gives a rational isometry to B; moving this isometry from the Fourier index to the argument turns it into the first self-dual test. If its signatures differ, it is the second test. The assumed vanishing therefore contradicts the nonzero coefficient. We have proved

$$f_T(1, g_f) = 0 \quad \text{if } g_f \text{ is trivial at } S \cup \overline{S} \text{ and } (\delta^{-1}T)_S \in \mathrm{GL}_b(O_S). \tag{12.32}$$

It remains to recover all data from these primitive indices. Use the opposite integral frame, so that its standard datum $S_0 = C_{K^*}$ is the original C_{L^*}, and write

$$S_z(y) = S_0(y + LzK^*y).$$

For every Fourier-coefficient functional Λ of $\mathcal{D}$, at a point $(1, g_f)$ with g_f trivial at S, equivariance gives $\Lambda(S_z) = \psi_S(\operatorname{tr}\tau z)\Lambda(S_0)$, where $\tau = (\delta^{-1}T)_S$. If $\Lambda(S_0) \neq 0$, invariance under integral z forces τ to be integral. Let $R = O_S/\varpi O_S$, let η be the residue character giving χ^{-1}, and choose lifts $\widetilde{r}$ of its matrices with $\widetilde{I} = I$. Form the finite sum

$$P = \sum_{r \in \mathrm{Mat}_b(R)} \eta(\det r) S_{\varpi^{-1}\widetilde{r}}.$$

Every such Λ kills P. For unit $\det\tau$ this is (12.32); otherwise the finite matrix Gauss sum is zero. The latter fact follows by choosing, at a residue-field factor where τ is singular, an invertible matrix fixing τ and having determinant on which η is nontrivial, and changing the summation variable by that matrix. Primitivity of the tame character means nontriviality at each such factor. If $\Lambda(S_0) = 0$, the conclusion is immediate.

Fourier uniqueness along the compact center gives $\mathcal{D}(P)(1, g_f) = 0$ when g_f is trivial at S. Here is the precise use of weak approximation to obtain all points. Put $F = \mathcal{D}(P)$. Local constancy of P and the local equivariance give a compact open subgroup K_P at S under which F is right invariant;

shrink it so that the determinant multiplier is one. Fix (g_∞, g_f). By weak approximation choose rational unitary points γ_i with $\gamma_{i,\infty} \to g_\infty$ and $\gamma_{i,S}^{-1} g_S \in K_P$. Rational invariance gives

$$F(\gamma_{i,\infty}, g_f) = F(1, \gamma_{i,f}^{-1} g_f) = 0,$$

because the finite component at S can be removed by K_P. Continuity, with g_f fixed, proves $F(g_\infty, g_f) = 0$. Weak approximation for the unitary group here follows from its Cayley parametrization by skew-Hermitian matrices on a dense open set and ordinary approximation in the number field. Translate P by $y \mapsto -\varpi^{-1} y$. Direct substitution in the two standard data shows that each nonzero summand has value $C_{L^*}(y)$. Their number is an integer-valued function $r(y) \geq 0$, and the summand indexed by I is nonzero at $y = L - \varpi K$, where $L^* y = I$. Thus the stable kernel of $\mathcal{D}$ contains $r C_{L^*}$, with $r(L - \varpi K) > 0$. The positive compact-average generator 12.2 implies that this kernel contains every locally constant matched datum. This proves the lemma.

Lean. `StableSpacesStandardData` (L686-L751), `CentreCoefficients` (L623-L839), `UnimodularCoefficients` (L628-L668). □

The lattice-minimum argument above is the adelic version of the singular-form argument underlying the source; it is not being cited as a holomorphic theorem in a nonholomorphic situation. Fourier pairs of partial means; Local shears and singularity; The minimal denominator argument; The primitive-index Gauss selector; Self-dual recognition.

A real section with a prescribed opposite slice. Use adapted real sign coordinates, so $J_B^2 = J_Y^2 = I$, at an infinite place and put

$$J = J_B \oplus J_Y, \qquad x_0 = \begin{pmatrix} I_b \\ 0 \end{pmatrix}.$$

We use the mixed Schrödinger–Fock realization attached to D. Its slices are Schwartz functions of a frame, with values in the smaller Fock space. Interchanging the two isotropic frames Fourier transforms this variable; in particular the opposite slice at zero is its integral, up to the fixed nonzero vacuum normalization.

There is a smooth local choice of unitary matrices $k(y)$ with $y = k(y) x_0 c(y)$ for frames y near x_0. Here is a direct construction, so that no global smooth choice in Witt's theorem is required. Put $w = (x_0^* J x_0)^{-1} x_0^*$, and for a square matrix a set

$$L(a) = (a x_0 w - x_0 w a^*) J, \qquad T(a) = \exp(L(a))(a - L(a)).$$

The matrix $L(a)$ is in the unitary Lie algebra, $L(I) = 0$, and $(a - L(a)) x_0 = x_0 w a^* J x_0$. The derivative of T at I is the identity. By the inverse function theorem, T has a smooth local inverse near I. Apply it to $I - (x_0 - y) w J$, whose product with x_0 is y, and take the exponential factor. Extending its Lie-algebra argument with a bump gives a smooth k on all frames that has the asserted property near x_0.

Choose also a globally smooth function γ agreeing with matrix inversion near J_B, and a nonnegative smooth bump σ, on families of frames at all infinite places, equal to one at x_0. Its compact support is sufficiently small that $y^* J y$ is invertible, $\gamma(y^* J y) = (y^* J y)^{-1}$, and y lies in $k(y) \operatorname{span} x_0$ wherever $\sigma(y) \neq 0$. Define the smooth, globally defined projection expression

$$p(x) = x \gamma(x^* J x) x^* J x_0.$$

Where the relevant bump factors are nonzero this is the orthogonal projection of x_0 onto $\operatorname{span} x$, and is unchanged by replacing x with xC, C invertible. More explicitly, if a bump factor forces γ to equal the inverse Gram matrix, then

$$xC((xC)^*JxC)^{-1}(xC)^*Jx_0 = x(x^*Jx)^{-1}x^*Jx_0.$$

If it does not, the section constructed next is zero. This is why a global inverse of every Gram matrix is not needed.

Lemma 12.11 (The real bump and its period)**.** *Let v be any real Schwartz vector for (Y,B), and z one for (Y,Y). Suppose that the equal-rank theta integral of z with finite datum $\chi^{-1}\circ\det$ is the constant I_0 on $U(Y)$. Then there are real Schwartz vectors P for (X,A), Q for (X,B), and a nonnegative function ρ, positive near the identity on $(U(B)\times U(Y))\backslash U(A)$ at infinity, with these properties:*

Q has opposite zero slice v; the real slices of P vanish unless their Gram matrices have the signatures of B; and their theta periods satisfy

$$\text{large slice period at } h, B_0 = \frac{\rho(h)I_0}{\nu^{\mathrm{bc}}(\det B_{0,\infty})|\det B_{0,\infty}|_{\mathbb{A}}^{j}}\,\text{small slice functional at } B_0. \tag{12.33}$$

In this formula the large slice is translated by a suitable $p_h \in U(B)\times U(Y)$ before integration over $[U(Y)]$. The same p_h works for every invertible adelic B_0, every remaining Heisenberg translation, and every remaining group argument.

Proof. We specify both slices. All formulas below are products over the infinite places, and x'_1 denotes the upper square block of a frame x'. Choose a nonzero scalar κ_0 so that

$$Q[B_0] = \kappa_0\sigma(x_0B_0)\left|\prod_\tau(\det B_{0,\tau})^{j+\mu_\tau}\right|^2 v \tag{12.34}$$

has opposite zero slice v. Such a scalar exists because the integral of its real scalar weight is strictly positive: the weight is nonnegative, compactly supported, and positive near $B_0 = I$. Multiply the reciprocal of that integral by the fixed nonzero Fourier/vacuum normalization. The mixed-model Fourier inversion then gives exactly the claimed opposite slice.

For $x \in \mathrm{Mat}_{(b+2j)\times b}(\mathbb{C})$, put $y = p(x)$, $x' = k(y)^{-1}x$, and prescribe

$$P[x] = \kappa_0\sigma(y)\sigma(x')\prod_\tau \overline{\det x'_{1,\tau}}^{2\mu_\tau}\omega_{\infty,j-\mu}(1,k(y))(v\boxtimes z). \tag{12.35}$$

The subscript means that the real Weil operator includes the determinant power $j-\mu_\tau$, exactly as dictated by $\nu^{-1}\upsilon^j$. This section is smooth. Its support is compact: both y and x' lie in the compact support of σ, and $x = k(y)x'$. Every Schwartz seminorm is bounded there, since the Weil action on Schwartz vectors is smooth and the group parameters range in a compact set. Thus the inverse mixed-model map gives Schwartz vectors P, Q. On the support of (12.35), the frame x is $k(y)x_0c$ for an invertible c; hence $x^*Jx = c^*J_Bc$, which proves the asserted signature support.

For a real point h, set

$$x_1 = h^{-1}x_0, \quad y = p(x_1), \quad \rho(h) = \sigma(y).$$

If $\rho(h) \neq 0$, local Witt decomposition gives unitary $u \in U(B)$, $q \in U(Y)$ with

$$k(y)x_0u = h^{-1}x_0, \qquad \operatorname{diag}(u, q^{-1})hk(y) = I.$$

Take $p_h = \operatorname{diag}(u, q^{-1})$. The frame at which the large slice is evaluated is $h^{-1}x_0u^{-1}B_0$. Its span agrees with that of x_1, so the projection factor is y; after multiplication by $k(y)^{-1}$ it is x_0B_0. The Weil operators for p_hh and $k(y)$ cancel. If $\rho(h) = 0$, the projection identity and the second bump factor show directly that both sides of the asserted period identity vanish; choose $p_h = I$.

We are therefore left with

$$\kappa_0\rho(h)\sigma(x_0B_0)\prod_\tau \overline{\det B_{0,\tau}}^{2\mu_\tau}(v \boxtimes z).$$

The theta functional factors across $A = B \oplus Y$. Its second factor integrates to I_0, independently of the remaining group argument by the assumed equal-rank constancy. Finally, for the idele d having infinite components $\det B_{0,\tau}$ and finite components one,

$$\prod_\tau \overline{\det B_{0,\tau}}^{2\mu_\tau} = \frac{\left|\prod_\tau(\det B_{0,\tau})^{j+\mu_\tau}\right|^2}{\nu^{\mathrm{bc}}(d)|d|_{\mathbb{A}}^j}.$$

This follows from $\nu^{\mathrm{bc}}(d) = \prod_\tau(d_\tau/\bar{d}_\tau)^{\mu_\tau}$ and $|d|_{\mathbb{A}} = \prod_\tau |d_\tau|^2$. Comparison with (12.34) proves (12.33), including its uniformity in all the remaining variables. The projection of x_0 onto its own span is x_0, so $\rho = 1$ at the identity and is positive near it. Its support in the quotient is compact: if $\rho(h) \neq 0$, the coset of h has the representative $k(y)^{-1}$ with $y \in \operatorname{supp}\sigma$.

Lean. `RealVectors` (L50-L177), `RealVectors` (L237-L460), `RealVectors` (L524-L734). □

Opposite slices and smooth Witt decomposition; The bump and the compact section; The determinant factor and period identity.

Theorem 12.12 (Top exchange on the standard finite data)**.** *In the setting above, fix real Schwartz vectors v for (Y, B) and z for (Y, Y), and suppose the latter equal-rank theta integral is the constant I_0. There are real Schwartz vectors P, Q, with $Q[D^{\mathrm{op}}, 0] = v$, and a scalar c_{top} such that, for every integral z_0,*

$$\mathcal{T}_+(P, R_{z_0}) = c_{\mathrm{top}}\mathcal{T}_-(Q, C_{z_0}), \qquad |c_{\mathrm{top}}| = K|I_0|, \qquad K > 0. \tag{12.36}$$

The construction works when $I_0 = 0$; then $c_{\mathrm{top}} = 0$. The factor K depends on the spaces, their measures and the chosen real bump, and is independent of the finite part of a tame character having the specified infinite weights.

Proof. Take P, Q from Lemma 12.11 and first use $z_0 = L^*$, the dual row block of D. Expand the two theta kernels along the center of the parabolic of D. The central coefficient at $\mathrm{j}J_B$ is the sum of terms whose rational frame has Gram matrix J_B. Witt's theorem makes these frames one rational orbit. On the large side use the representative $x_0 = (I_b; 0)$, with stabilizer $U(Y)$; on the small side use I_b, with trivial stabilizer. At each parabolic test point p, the frame stabilizer preserves $K^\perp$. For R_{L^*} and the transpose of C_{L^*} at the conjugate places, the right inverse in Lemma 12.1 therefore applies to the transported frame. The integrable absolute sum at this p justifies the central Fourier average and the unfolding of the frame orbit.

the scalar computation can be made before the last quotient integral. Write a parabolic point as $n(c, Z)m(q, g)$, where e_B is the fixed square adelic frame carrying the rational Gram matrix of B to the adapted standard forms, $q = \kappa e_B B_0$, and $B_{0,f}$ spans a self-dual lattice. The term of a rational frame x is

$$\psi\Big(\tfrac{1}{2}\operatorname{tr} Zx^*J_Ax\Big)\,\chi_A(\det q)^{-1}|\det q|_{\mathbb{A}}^{\dim A/2} \times \text{finite slice} \times \text{real theta slice}. \tag{12.37}$$

The corresponding formula with B in place of A holds on the small side. It follows directly from the Schrödinger action of the parabolic: its center multiplies by the displayed quadratic character,

its Levi changes variables and supplies the square-root modulus, and its remaining unipotent coordinates act by the indicated Heisenberg translation.

Multiplying the stacked row matrix by the adapted integral basis (L, K, I) gives a block triangular matrix with diagonal blocks I_b and $x(K, I)$. Thus R_{L^*} is the adapted standard determinant datum times a scalar ε with $|\varepsilon| = 1$. At a self-dual finite frame, the integral Witt theorem says that a nonzero term lies in $(U(B) \times U(Y))K_f$. Otherwise both finite integrands vanish. Inside this set, integral-level invariance removes K_f, and the finite slices in (12.37) are

$$\varepsilon\chi^{-1}(\det x_S)(\mathrm{vac}_B \boxtimes (\chi^{-1} \circ \det)) \quad \text{and} \quad \chi^{-1}(\det \overline{x}_S)\mathrm{vac}_B,$$

respectively. Here x is the finite frame in B, integral with unit determinant. The relation between them is

$$\chi^{-1}(\det x_S)\nu_f^{\mathrm{bc}}(\det x) = \chi^{-1}(\det \overline{x}_S).$$

This uses precisely the prescribed tame character at S, its inverse at $\overline{S}$, and unramifiedness elsewhere.

Apply (12.33) to the real slices. The powers of $|\det B_0|$, the infinite character of its determinant and the finite character just displayed cancel against the quotient of the two parabolic factors in (12.37). The remaining multiplier is

$$\kappa_1\varepsilon\nu^{\mathrm{bc}}(d_0)I_0, \qquad d_0 = \det(\kappa e_B), \qquad \kappa_1 = \frac{\text{vacuum factor for } (X, A)}{\text{vacuum factor for } (X, B)}|d_0|_{\mathbb{A}}^{j}.$$

Both vacuum factors are fixed nonzero Gaussian normalization constants; in particular $\kappa_1 \neq 0$.

Disintegrate $U(Y)(F)\backslash U(A)(\mathbb{A})$ through $U(B)(\mathbb{A}) \times U(Y)(\mathbb{A})$. The remaining quotient weight is ρ at infinity and the indicator of $(U(B)_f \times U(Y)_f)K_f$ at the finite places. Its integral, including the positive Haar-disintegration constant, is a finite positive real number R: its support in the quotient is compact and it is positive on a neighborhood of the identity. We obtain equality of the two self-dual Fourier coefficients with

$$c_{\mathrm{top}} = R\kappa_1\varepsilon\nu^{\mathrm{bc}}(d_0)I_0, \qquad K = R|\kappa_1| > 0.$$

All these calculations remain valid if the period I_0 is zero. At an invertible index outside the real signature class of B, both Fourier coefficients vanish: for P this is its slice support, and for Q it follows because a frame in B cannot have that Gram matrix.

The difference of the two theta maps is an equivariant map $\mathcal{D}$, by the same-section factorization and (12.25). We have proved that its value at C_{L^*} vanishes on all self-dual tests, with the same scalar c_{top} for every test point. Apply Lemma 12.10. It gives the identity for every matched datum in its domain, in particular for every standard pair (R_{z_0}, C_{z_0}). The opposite slice was fixed in the construction of Q, and the absolute-value formula follows because the remaining two character factors have modulus one.

Lean. `Transportedrightinverse` (L149-L172), `Integrabilityattheparabolictests` (L953-L995), `LargeTriple` (L517-L741), `RealVectors` (L736-L786). □

Parabolic terms and convergence; The two finite slices; Comparison of the unfolded terms; Fourier comparison and the top identity; The prescribed opposite slice.

The zero-period argument for arbitrary real data. The preceding bump construction alone does not show that every real datum vanishes when an equal-rank period is zero. We now prove the additional sign argument needed for that assertion. It is a statement about the compact stabilizer of a Fourier frame, and applies to all Schwartz vectors.

Lemma 12.13 (Vanishing outside the negative cone)**.** *Suppose that B is negative definite at every infinite place, and that $q_\tau(Y) \le j$, where q_τ denotes the negative index and* $\dim Y = 2j$*. Let P be any real Schwartz vector for (X, A). For every invertible Hermitian matrix β which is not negative definite at some infinite place, the central Fourier coefficient of $\mathcal{T}_+(P, R_{L^*})$ at $\mathfrak{j}\beta$ vanishes at every parabolic point of D.*

Proof. If no rational b-frame in A has Gram matrix β, the unfolded coefficient is an empty sum. Otherwise Witt's theorem makes its frames one rational orbit, and unfolding gives the integral of one mixed-model term over its rational stabilizer quotient. The test point p belongs to the stabilizer of K, so $p_w^{-1}K_w^\perp = K_w^\perp$. The unit determinant on the support of R_{L^*} gives surjectivity there, and Lemma 12.1 gives absolute integrability. Fix a place τ where β_τ is not negative definite. Since β is invertible, it has a positive vector at this place. An isometry of A_τ takes this vector to a positive coordinate vector e_i. After that change of coordinates write x for the Fourier frame, so that $xv = e_i$ for some $v \neq 0$.

Put $q = q_\tau(Y)$, $p = 2j - q$, and $\lambda_\tau = j - \mu_\tau$. The smaller Weil representation on the slice is the one for (Y, A), with character $\nu^{-1}\upsilon^j$ acting on $U(A)$; hence its real weight is λ_τ. The positive and negative dimensions of A_τ are p and $b + q$, respectively, and

$$N = p - \lambda_\tau = j - q + \mu_\tau > 0. \tag{12.38}$$

The identity relating the large and slice characters is $\nu^{-1}\upsilon^{b+j} = (\nu^{-1}\upsilon^j)\upsilon^b$. It is precisely the determinant correction in the mixed-model slice formula. Thus an element fixing x acts on the slice by this smaller Weil representation.

Let C be the subgroup of the maximal compact group $U(p) \times U(b + q)$ fixing x. We show that its only fixed polynomial Fock vector is zero. Let F be its polynomial and fix an arbitrary value $\xi = (\xi_{kj})$ for the polynomial variables at all infinite places. A row r supported on the negative coordinates of A_τ can be chosen so that

$$rx = 0, \qquad \sum_k \overline{r}_k \xi_{kj} = 0 \quad (j \text{ a negative coordinate of } Y_\tau), \qquad \sum_k |r_k|^2 = 1.$$

Here is the dimension count. The negative projection of x has rank at most $b - 1$, because it annihilates the nonzero vector v with $xv = e_i$. The first constraint therefore imposes at most $b - 1$ equations on a space of dimension $b + q$. After complex conjugation the other constraints are q linear equations in r. Their common kernel has dimension at least one; normalize any nonzero vector in it.

For $|\eta| = 1$, set

$$c_\eta = I + (\overline{\eta} - 1)r^* r.$$

Since $rr^* = 1$, the matrix r^*r, with entries $\bar{r}_i r_j$, is the orthogonal projection onto $\mathbb{C}r^*$. Thus c_η acts by $\bar{\eta}$ on this negative line and fixes its orthogonal complement. The identity $rx = 0$ gives $c_\eta x = x$, so $c_\eta \in C$. The polynomial Fock formula for its action is a linear substitution times η^N. Indeed its positive determinant is one and its negative determinant is $\overline{\eta}$. The vacuum contributes η^p and the character contributes $\eta^{-\lambda_\tau}$, which gives (12.38). For a column belonging to a negative coordinate of Y_τ, the substitution is

$$\xi_{kj} \longmapsto \xi_{kj} + (\overline{\eta} - 1) r_k \sum_i \overline{r}_i \xi_{ij};$$

for a positive column it is

$$\xi_{kj} \longmapsto \xi_{kj} + (\eta - 1) \overline{r}_k \sum_i r_i \xi_{ij}.$$

Other places and all remaining combinations of variables are fixed. The negative-column correction vanishes at the chosen ξ. Consequently invariance of F says on the unit circle that

$$F(\xi) = \eta^N F\big(\xi + (\eta - 1)\xi'\big),$$

where ξ' is independent of η, supported in the positive columns at τ. The right side is a polynomial in η with zero constant coefficient, since $N > 0$. Its circle average is zero. Therefore $F(\xi) = 0$. Since ξ was arbitrary, $F = 0$.

Each fixed homogeneous degree in the polynomial Fock model is a finite-dimensional C-stable space. Averaging over the probability Haar measure of C projects it onto its fixed vectors, which we have just shown to be zero. Polynomial Gaussian vectors span a dense subspace in the Schwartz topology: the Hermite expansion of a Schwartz function converges with every Schwartz seminorm. The Weil action is continuous on this topology and uniformly continuous as a family of operators over the compact group C. It follows that compact averaging annihilates every Schwartz vector. In particular every continuous C-invariant linear functional on the real Schwartz space is zero. This last conclusion concerns the Schwartz topology, and does not assert that the theta distribution is bounded on an L^2 completion.

To apply this to the unfolded integral, let S_x be the adelic stabilizer of the original frame, let Γ_x be its rational stabilizer, and undo the isometry above, replacing C by its compact conjugate inside S_x. Disintegrate $\Gamma_x\backslash U(A)(\mathbb{A})$ through S_x: fix h in the outer quotient $S_x\backslash U(A)(\mathbb{A})$ and integrate k over $\Gamma_x\backslash S_x$. The average to insert is the one with argument kch, $c \in C$, rather than a right translate of h in the larger quotient. The slice formula is

$$\operatorname{slice}_x \omega(g, kch) f = \omega_{\mathrm{smaller}}(1, kc)\big(\operatorname{slice}_x \omega(g, h) f\big).$$

The compact c-average of the real Schwartz vector on the right is zero by the preceding argument. Theta evaluation is continuous on the real Schwartz space by lattice counting, so its average is zero as well. Finite adelic Schwartz data are finite sums of tensors after their compact support and open stabilizer have been fixed; hence this applies to every finite part. Right translation by C preserves Haar measure on S_x: its modular character is trivial on a compact group. Thus insertion of the average does not change the inner stabilizer integral. Absolute integrability justifies this disintegration and Fubini. The Fourier coefficient is consequently zero.

Lean. `NegativeIndices` (L50-L142), `NegativeIndices` (L146-L329), `NegativeIndices` (L369-L512). □

The frame and the stabilizing circle; The positive rotation weight and absence of invariant functionals; Compact averaging in the unfolded coefficient.

To state the required zero-period hypothesis without a hidden restriction on the real datum, put $\alpha = \nu^{-1}\upsilon^j$. Let d_Y be the equal-rank finite determinant datum $\chi^{-1} \circ \det$, with the fixed vacuum elsewhere. For a Schwartz datum f on $\mathrm{Hom}(Y, B)(\mathbb{A})$, a matrix $a \in \mathrm{Hom}(Y, B)(\mathbb{A})$, an auxiliary splitting character β_0, and $g \in U(Y)(\mathbb{A})$, $y \in U(B)(\mathbb{A})$, denote by

$$\mathcal{H}_{\alpha,\alpha\beta_0}(f \boxtimes d_Y; (a, 0); (g, y \oplus u); w)$$

the theta distribution of the pair $(Y, B \oplus Y)$, with those finite data, Heisenberg argument $(a, 0)$, group argument $(g, y \oplus u)$, and real Schwartz vector w. Equivalently one may fix the finite part of f and absorb its real factor into w. This is the ordinary theta functional in the mixed-model expansion, with no period integral in its definition.

Corollary 12.14 (Zero periods force all top data to vanish)**.** *Assume the signs of Lemma 12.13. Let P be any real Schwartz vector for (X, A). Suppose that for every $h \in U(A)_\infty$ there is a*

$p_h \in U(B)_\infty \times U(Y)_\infty$ *such that*

$$\int_{[U(Y)]} \mathcal{H}_{\alpha,\alpha\beta_0}\left(f \boxtimes d_Y; (a,0); (g, y \oplus u); \omega_{\infty,j-\mu}(1, p_h h) P[(p_h h)^{-1} x_0 B_0]\right) du = 0 \tag{12.39}$$

for every invertible adelic B_0, every auxiliary character β_0, every f, a, g, y just specified. Then

$$\mathcal{T}_+(P, R_z) = 0 \qquad \text{for every integral } z.$$

In particular, if (12.39) holds for every real Schwartz vector P, all such large theta integrals vanish.

Proof. At a self-dual Fourier test use the same rational frame x_0 and the same finite slice calculation as in the proof of Theorem 12.12. Outside $(U(B)_f \times U(Y)_f)K_f$ the integrand is zero. Inside it, its integral over the stabilizer quotient $[U(Y)]$, after the permitted translate p_h, is a fixed scalar times the left side of (12.39). It is therefore zero. The full-rank integrability lemma justifies disintegration through $U(B) \times U(Y)$ and Fubini, so the entire self-dual coefficient is zero. No integrable auxiliary weight on this quotient is needed: the original unfolded integrand is absolutely integrable and its inner integral itself is zero.

At the other invertible indices the coefficient vanishes by Lemma 12.13. The same-section factorization of the large theta map gives the equivariant map $\mathcal{D}$ on column data, with the small map identically zero. Lemma 12.10 now proves $\mathcal{D}(C_z) = \mathcal{T}_+(P, R_z) = 0$ for all standard data. All quantifiers in (12.39) precede this argument, so the conclusion applies independently to every P for which the zero periods have been established.

Lean. `LargeTriple` (L762-L792), `LargeTriple` (L1077-L1147). □

The full period identity; The negative-sign alternative for arbitrary real data.

Application to equal-rank periods. The period identity just proved assumes that the equal-rank integral is a constant I_0 and includes the case $I_0 = 0$. The equal-rank argument below proves its constancy and constructs a real Schwartz vector for which $I_0 \neq 0$. To prove nonvanishing one needs Corollary 12.14 for an arbitrary real Schwartz vector. Its sign hypotheses hold for $B = J^-$ and $Y = H \oplus J$ with J positive definite: if $\dim J = 2k$, then $j = k + 1$, $q(Y) = 1$, and the positive rotation weight is $k + \mu_\tau$. The plane constants and the construction of the image on the cone then permit descent to the ranks of the exchange identity by Corollary 12.8.

12.5 Nonvanishing of equal-rank periods and the exchange identity

Lemma 12.15 (Vanishing of every equal-rank period)**.** *Suppose $I_Y(z) = 0$ for every real Schwartz vector z. Let B be any Hermitian space. In the theta kernel for $(Y, B \perp Y)$, use the determinant datum $p_\varnothing$ in the Y-factor and any fixed finite datum in the B-factor. The integral over the second $[U(Y)]$ is zero for every real Schwartz vector, every fixed point of the other groups, and every Heisenberg translation whose Y-component is zero. The assertion also holds after taking a real slice and applying a real Weil operator.*

Proof. Fix the other variables and let T be the theta distribution on the real Schwartz space for (Y, B), including its finite datum, splitting characters and Heisenberg translation. It is a continuous linear functional on that Schwartz space. On product real vectors the orthogonal-sum formula for the Weil representation and the theta sum gives the product of this value and the equal-rank theta kernel for (Y, Y). The extra splitting character, if present, enters only the first factor and is already included in T.

This factorization extends to every Schwartz vector w. Here is an explicit way to take the partial distribution. In Schrödinger coordinates choose Hermite bases (h_p), (h'_q) on the two real vector spaces and write

$$w = \sum_{p,q} a_{pq} h_p \otimes h'_q.$$

The coefficients decrease faster than every power of $(1 + |p| + |q|)$; each Schwartz seminorm of a Hermite function grows at most polynomially in its index. Continuity of T therefore gives polynomial growth of $T(h_p)$, and

$$z_w = \sum_{p,q} a_{pq} T(h_p) h'_q$$

converges in every Schwartz seminorm. The corresponding statement in the Fock realization follows by the real model isomorphism. At every point $\ell \in U(Y)(\mathbb{A})$ the product formula, extended by continuity, identifies the original kernel with the equal-rank kernel of z_w. Its ℓ-integral is consequently $I_Y(z_w) = 0$. This proves the identity before integrating: no exchange with an unbounded sum of period integrals is being assumed. Real slices, partial Fourier transforms and real Weil operators preserve the Schwartz space, so the same argument applies to the slices occurring in the top Fourier calculation.

Lean. `TheoremForPartners` (L410-L484). □

The arbitrary-period implication used for nonvanishing. The constructed-vector statement Theorem 12.12 at $I_0 = 0$ does not by itself imply universal vanishing. The following application uses Corollary 12.14: if J is positive definite of rank $2k$, $Y_0 = H \perp J$, and every $U(Y_0)$-period of a real Schwartz vector P for

$$(X, A) = (H^{2k} \perp Y_0,\ J^- \perp Y_0)$$

is zero, then the large theta integral of P vanishes on every standard finite datum. Its proof includes the wrong-sign invertible Fourier indices for an arbitrary P, using Lemma 12.13, with $q(Y_0) = 1$, $\dim Y_0 = 2k + 2$, and positive character weights. The period-nullness lemma supplies the complete hypothesis (12.39), with $p_h = 1$: every sliced and translated real vector occurring there is Schwartz. This verifies the universal quantifier required below.

Lemma 12.16 (Fixed-character equal-rank nonvanishing)**.** *Let W have even rank $N = 2k + 2q$, with q negative signs and $k \geq 1$. Suppose $k \leq \mu_\tau$ at every infinite place. Then there is a real Schwartz vector z on $(H \perp W, H \perp W)$ such that*

$$I_{H \perp W}(z) \neq 0.$$

Proof. First let J be the standard positive definite space of rank $2k$ and $Y_0 = H \perp J$. Suppose, to the contrary, that I_{Y_0} vanishes on every real Schwartz vector. Lemma 12.15 makes every $U(Y_0)$-period in the top pair (X, A) zero, for every real vector P. Corollary 12.14 therefore makes $I_{X,A}$ identically zero for every real Schwartz vector and every standard finite datum.

Both X and A have $2k + 1$ hyperbolic planes to remove; their ranks are $6k + 2$ and $4k + 2$, respectively. After removing those planes they become $(J, 0)$. At the top the parameter $e = \dim A - \dim X/2$ is $k + 1$; at the bottom it is $-k$. Lemma 12.7 therefore applies at every step, because $k \leq \mu_\tau$. In particular it gives universal vanishing for $(J, 0)$. But its vacuum at $g = 1$, with the standard datum $s = I_{2k}$, has value $\mathrm{vol}[U(0)] > 0$: the only theta summand and the finite determinant datum both equal one. This contradiction proves $I_{Y_0}(z_0) \neq 0$ for some z_0.

We can add a hyperbolic plane while keeping this value nonzero. If Y has rank $2h$ and $I_Y(z) \neq 0$, use Lemma 12.6 at exponent $h + 1$ to choose z^+ on $Y^+ = Y \perp H$ with

$$I_Y(\mathcal{C}^+_{h+1} z^+) = B(h + 1) I_Y(z) \neq 0.$$

By equal-rank constancy, the value of the upper integral equals its constant term. The plane formula consequently gives

$$I_{Y^+}(z^+) = \epsilon_+ A^{h+1} Q_S^{h+1/2} L^S(h+1,\nu) B(h+1) I_Y(z) \neq 0.$$

All scalar factors are nonzero. Notice that in this application the lower plane parameter is $e = h$, because the ranks are equal; it is not $e = 0$.

Adding q planes to Y_0 gives $J \perp H^{q+1}$, with the same signatures as $H \perp W$. At each finite place the two spaces have self-dual lattices by the hypotheses of Section 12.1. The local quadratic algebra is unramified or split, so these lattices admit orthonormal bases of the same size, as in the proof of Lemma 12.10. Thus the spaces are isometric at every place. The Hasse principle for Hermitian forms gives a rational isometry between them. Under such an isometry the rational theta lattice is unchanged; comparing the chosen adelic frames translates the group variable, transforms the real Schwartz vector by a real Weil operator, and multiplies the kernel by nonzero unitary splitting factors. The theta integral is constant in its group variable by Lemma 12.9, so its nonzero value is preserved. This gives the asserted vector on the actual space $H \perp W$.

Lean. `TheoremForPartners` (L504-L578), `TheoremForPartners` (L736-L846). □

Theorem 12.17 (Complementary ranks for a fixed character)**.** *Let V, V', W have ranks $a, b, N = a + b$, with*

$$a = b + 2e, \quad e \geq 0, \qquad q_W = q_V + q_{V'}, \qquad 2q_W + 2 \leq N.$$

Suppose $N/2 - q_W \leq \mu_\tau$ for every infinite place. For each prescribed real Schwartz vector v for (V', W), there exist a real Schwartz vector w for (W, V) and $c \neq 0$ such that every matched finite pair (Φ, Ψ) satisfies

$$\int_{[U(V)]} \theta^{W,V}_{\gamma,\alpha}(w \otimes \Phi)(g,h)\, dh = c \int_{[U(V')]} \theta^{V',W}_{\alpha',\gamma}(v \otimes \Psi)(h,g)\, dh, \qquad \gamma = \nu^{-1} \upsilon_{U(1)}^{N/2}, \quad \alpha\alpha' = \gamma. \tag{12.40}$$

The real vector and scalar may depend on the fixed ν. They work for all matched pairs, all indicated splitting characters, and all g; the integrals converge absolutely.

Proof. Put $B = (V')^-$, the Hermitian space with the opposite form, and let y be the adjoint real datum associated with v. This is a real linear change of oscillator coordinates and preserves the Schwartz space. Set $Y = H \perp W$. The preceding lemma provides z with $I_Y(z) \neq 0$. Choose a Schwartz vector y^+ for (Y, B) with zero slice y along the added plane. Such a choice is elementary in the mixed model: tensor y with a Schwartz function whose required value, or partial Fourier integral, equals one.

Apply Theorem 12.12 to

$$X = H^b \perp Y, \qquad A = B \perp Y.$$

It gives real vectors P, Q, with the prescribed opposite zero slice of Q equal to y^+, and an exchange on all standard finite data with scalar $c_{\mathrm{top}} \neq 0$. The nonvanishing follows directly from $|c_{\mathrm{top}}| = K|I_Y(z)|$, $K > 0$; no normalization formula for a product of L-values is needed here.

Remove $b + 1$ hyperbolic planes on the source and the large target, using Corollary 12.8. The source becomes W, and the large target has rank

$$(b + N + 2) - 2(b + 1) = a$$

and negative count

$$(b - q_{V'} + q_W + 1) - (b + 1) = q_V.$$

The fixed finite local forms also agree with those of V. Thus the adapted splittings, followed by the same rational isometry and adelic-frame change as above, identify the result with the pair (W, V). The small real vector becomes y, since successive zero slices compose to the zero slice along the direct sum of the removed planes. The plane parameter decreases from $N/2+1$ to $e \geq 0$. Each descent divides by a nonzero factor and multiplies by the nonzero small vacuum factor, so the resulting standard-data identity has a scalar $c_{\mathrm{std}} \neq 0$.

This identity extends from standard data to every matched pair. Form the difference map of Lemma 12.4. Its kernel is stable under $\mathrm{GL}_N(F_S)$ and contains, for example, p_s with $s = (I_b, 0)$ and $x = (I_b, 0)^t$. Apply Lemma 12.2 with $r = 1$. The kernel contains every matched Ψ, and section factorization identifies the corresponding large value with the original Φ. We have proved

$$\mathrm{thetaLarge}(W, V; w, \Phi) = c_{\mathrm{std}}\, \mathrm{thetaSmall}(W, B; y, \Psi)$$

for every matched pair.

We spell out the final adjoint and character changes. In the finite model the adjoint coordinate map is $X \mapsto \varepsilon \overline{X}^{\,t}$, where the fixed finite adelic unit $\varepsilon \in \widehat{\mathcal{O}}_E^{\times}$ satisfies $\varepsilon\overline{\varepsilon} = -1$. The accompanying change of the framed opposite space ensures that the rational adjoint is $X \mapsto J_{V'}^{-1} X^* J_W$. It carries rational theta coordinates bijectively. The real coordinate map is the signed adjoint and is unitary. Change of variables in the Schrödinger theta distribution (or Poisson summation when the polarization is changed) therefore intertwines the theta functionals with a nonzero unitary scalar. Its compatibility with the Weil action is obtained by the identity $(hXg^{-1})^* = (g^{-1})^* X^* h^*$. The characters of a flipped kernel with original characters (a', b') are

$$((b')^{-1}\upsilon_{U(1)}^{N},\ (a')^{-1}\upsilon_{U(1)}^{b}).$$

By unit covariance, $\Psi(\varepsilon X) = \chi(\varepsilon)^{-b}\Psi(X)$. Thus the adjoint moves its datum from S to $\overline{S}$, exactly as required in thetaSmall, with another unitary scalar independent of the matched pair. Identifying $U((V')^-)$ with $U(V')$, the quotient measures agree. Consequently there is one d, $|d| = 1$, such that

$$\gamma(\det g)\, \mathrm{thetaSmall}(W, (V')^-; y, \Psi)(g) = d \int_{[U(V')]} \theta_{\gamma,\gamma}^{V',W}(v \otimes \Psi)(h, g)\, dh.$$

Also, directly from the determinant character in the kernel,

$$\int_{[U(V)]} \theta_{\gamma,1}^{W,V}(w \otimes \Phi)(g, h)\, dh = \gamma(\det g)\, \mathrm{thetaLarge}(W, V; w, \Phi)(g).$$

These two formulas prove (12.40) for $\alpha = 1, \alpha' = \gamma$, with $c = c_{\mathrm{std}} d \neq 0$. If $\alpha\alpha' = \gamma$, both sides acquire the same factor $\alpha(\det g)^{-1}$, so the same w, c work.

Lean. `Flip` (L195-L290), `TheoremForPartners` (L85-L232), `TheoremForPartners` (L1048-L1105). □

Reduction modulo the twist modulus and the exchange identity. Let $R_\xi = \mathcal{O}_E/\mathfrak{m}(\xi)$, and let $\pi : O_S \to R_\xi$ be reduction. It is onto and detects units. In particular $\mathrm{GL}_N(O_S) \to \mathrm{GL}_N(R_\xi)$ is onto: lift the entries and use the invertibility of the reduced determinant in each local factor. Primitive frames over R_ξ lift to primitive frames over O_S. Extend the pullbacks of Φ, Ψ by zero off the integral matrices. Unit detection and the equality of the tame unit characters show directly that these are matched semilocal functions. Their finite adelic data are exactly the original data modulo $\mathfrak{m}(\xi)$, pointwise, so no volume factor is inserted in this passage. This remains true if the modulus has higher prime powers; no assumption that R_ξ is a product of fields is used. Reduction and equality of the two finite data.

For the square of (12.7), take

$$V = S_{i,n+2}, \qquad V' = S_{i+1,n}, \qquad W = S_{i,n+1} \perp S_{i+1,n+1}.$$

The ranks give $a = b+2t$ and $N = 2(r+(n+1)t)$. The alternating base signs give $q_W = q_V + q_{V'} \le 2r$, and $t = 5r$ ensures $2q_W + 2 \le N$. The allowed character has $\mu_\tau = (n+1)\,\mathrm{gap} \ge N/2$, hence certainly $N/2 - q_W \le \mu_\tau$. Its tame conductor is the required nonempty split modulus at every allowed m-fold twist. Thus all rank, signature, finite-datum and weight hypotheses of Theorem 12.17 hold. This proves (12.7) for every allowed twist. The already proved finite square calculation, first returns and positive terminal normalization then apply without any uniform-in-character choice of w.

12.6 The square law on the grid

The finite algebra connecting the exchange to the square law is explicit. Start with the adjoint datum $\Psi_{i,0}$ and put

$$\Psi_{i,n+1}\begin{pmatrix} \eta & s \\ 0 & I_t \end{pmatrix} = \frac{\Psi_{i,n}(-\eta)}{|\mathrm{Mat}_{r_{n+1}\times t}(R_\xi)|}, \qquad \Psi_{i,n+1} = 0 \quad \text{on other blocks.}$$

Let $D_{i,n}$ be the transposed lift with this datum and a prescribed partner joint harmonic. Seesaw expresses both square compositions as pairings of a theta integral on W with the exterior forms. Insert (12.7) and project w onto the prescribed highest-type line with scalar $\kappa_{i,n}$. The resulting finite functions obey the matched-data identity with coefficient $\eta_{i,n} = \kappa_{i,n}/c$.

Summing over bases of the complementary column and row subspaces gives group orders N_n and N_{n+2} on the two sides. Averaging the determinant-one shear $\left(\begin{smallmatrix} I & s \\ 0 & I \end{smallmatrix}\right)$ over s gives the denominator in $\Psi_{i,n+1}$. Multiplying by $\Psi_{i,n}(-\eta)$ and summing over η therefore gives

$$N_n L_{i+1,n} D_{i,n} = \eta_{i,n} N_{n+2} D_{i,n+1} L_{i,n+1}.$$

In terms of $A_{i,n} = N_n L_{i,n}$, this is

$$A_{i+1,n} D_{i,n} = \left(\eta_{i,n} \frac{N_{n+2}}{N_{n+1}}\right) D_{i,n+1} A_{i,n+1}. \tag{12.41}$$

The coefficient may vanish: the abstract triangle lemma does not divide by it. Recursive datum; Finite shear calculation; Seesaw and exchange; Weighted square.

With (12.7) proved above, the abstract triangle lemma, the first returns above, and (12.6) imply injectivity of lowerChainAtLevel(ξ) for every allowed ξ. Its restriction Θ_ξ to the good target is injective, so

$$Z(\xi) = \det(\Theta_\xi^* \Theta_\xi) > 0. \tag{12.42}$$

Part IV
Congruences and integral Galois representations

A congruence between Hecke eigensystems can enlarge the denominator of an integral spectral projector. We bound this denominator by passing to Galois extensions with prescribed local conditions. At the coefficient prime the bound involves the length of a quotient of lattices, so the rational dimension of the corresponding Selmer group alone would lose the term needed in (7.2).

13 Congruences, auxiliary primes and common twists

At each coefficient prime the Gram determinant requires a single twisting character whose first m powers satisfy the Selmer and special-value bounds simultaneously. Theorem 13.10 constructs auxiliary primes and polynomials whose valuations bound these quantities. We first show how to specialize all the polynomials at one tuple of roots of unity, with arbitrarily small error for every power.

Fix a rational prime ℓ. On $K = \overline{\mathbb{Q}}_\ell$ write v for the valuation normalized by $v(\ell) = 1$, with $v(0) = +\infty$. Its valuation ring is $\mathcal{O} = \{x \in K : v(x) \geq 0\}$. The polynomials below have coefficients in $\mathcal{O}$. For a nonzero polynomial $g(T) = \sum_i a_i T^i$ and rational $\rho \geq 0$, set

$$w_\rho(g) = \min_i \{v(a_i) + i\rho\}.$$

Gauss valuations and roots. For $g, h \in \mathcal{O}[T] \setminus \{0\}$, let $R(g)$ be the multiset of roots of g in $\mathcal{O}$, counted with multiplicity. Then

$$w_\rho(gh) = w_\rho(g) + w_\rho(h), \qquad w_\rho(g) \leq v(g(y)) \quad \text{if } v(y) \geq \rho, \tag{13.1}$$

$$w_\rho(g) = w_0(g) + \sum_{\alpha \in R(g)} \min\{v(\alpha), \rho\}, \qquad v(g(y)) = w_0(g) + \sum_{\alpha \in R(g)} v(y - \alpha) \quad (y \in \mathcal{O}), \tag{13.2}$$

Here $w_0(g) \geq 0$ by integrality of the coefficients. For multiplicativity, choose $\pi \in K$ with $v(\pi) = \rho$, replace T by πT, and divide each polynomial by a coefficient of smallest valuation. The two resulting reductions modulo the maximal ideal are nonzero polynomials over a field; their product is nonzero. This proves the first identity. The evaluation inequality is the ultrametric inequality applied to the monomials. Factoring over K gives

$$w_\rho(g) = v(a_{\deg g}) + \sum_{\alpha:\, g(\alpha) = 0} \min\{v(\alpha), \rho\}.$$

The terms with $v(\alpha) < 0$ are independent of ρ. For integral y they also satisfy $v(y - \alpha) = v(\alpha)$. Removing these terms proves (13.2). The root 0, if present, contributes ρ in the first formula.

For $h \geq 1$, put

$$\rho_h = \frac{1}{\varphi(\ell^h)}.$$

If ζ has exact order ℓ^h, then

$$v(\zeta - 1) = \rho_h. \tag{13.3}$$

Indeed $\Phi_{\ell^h}(1) = \ell$, and the primitive roots are conjugate over $\mathbb{Q}_\ell$, so the valuations of their differences from 1 are equal and add to 1. The positive numbers ρ_h are distinct and tend to zero. Consequently, for every $\rho > 0$, $B \geq 0$ and $\delta > 0$, there is an $H \geq 1$ such that

$$B < \delta \sum_{\substack{1 \leq h \leq H \\ \rho_h \leq \rho}} \frac{1}{h+1}. \tag{13.4}$$

This follows from divergence of the harmonic series: the condition $\rho_h \leq \rho$ omits only finitely many positive integers.

A uniform one-variable specialization. Suppose $0 < \delta < 1$, and choose H satisfying (13.4). For every $g \in \mathcal{O}[T]$ with $w_\rho(g) \leq B$, there are $1 \leq h \leq H$, $\rho_h \leq \rho$, and a primitive ℓ^h-th root ζ such that

$$v(g(\zeta - 1)) \leq w_{\rho_h}(g) + \delta. \tag{13.5}$$

In particular H is chosen before g, with no restriction on its degree.

Let R_h be the multiset of roots of g of valuation exactly ρ_h, and put $n_h = |R_h|$. Each root belongs to at most one such multiset, and (13.2) gives

$$\sum_{\substack{1 \leq h \leq H \\ \rho_h \leq \rho}} n_h \rho_h \leq w_\rho(g) \leq B.$$

Thus (13.4) supplies an h in this sum with

$$n_h \rho_h (h+1) \leq \delta. \tag{13.6}$$

For any α with $v(\alpha) > 0$, consider the monic integral polynomial

$$F_\alpha(T) = (T + 1 + \alpha)^{\ell^h} - 1.$$

Its coefficient of T has valuation h, since $1 + \alpha$ is a unit. Its roots include $\zeta - 1 - \alpha$ for every primitive ℓ^h-th root ζ. All its roots are integral. Therefore (13.2), with radius 1, gives

$$\sum_{\zeta \text{ primitive of order } \ell^h} \min\{v(\zeta - 1 - \alpha), 1\} \leq w_1(F_\alpha) \leq h + 1. \tag{13.7}$$

Sum over $\alpha \in R_h$ and average over the $\varphi(\ell^h) = 1/\rho_h$ primitive roots. For some ζ,

$$\sum_{\alpha \in R_h} \min\{v(\zeta - 1 - \alpha), 1\} \leq n_h \rho_h (h+1) \leq \delta.$$

Because $\delta < 1$, none of these summands can equal 1. The minima can therefore be removed. For the other integral roots α of g, the unequal-valuation rule gives

$$v(\zeta - 1 - \alpha) = \min\{\rho_h, v(\alpha)\}.$$

The two identities in (13.2) now yield

$$v(g(\zeta - 1)) = w_{\rho_h}(g) + \sum_{\alpha \in R_h} \big(v(\zeta - 1 - \alpha) - \rho_h\big) \leq w_{\rho_h}(g) + \delta,$$

as required. This also covers $R_h = \varnothing$.

Several polynomials and several variables. Fix an integer $s_0 \geq 0$, a rational $\rho > 0$, and numbers $B \geq 0$, $\varepsilon > 0$. There is an $H = H(\ell, s_0, \rho, B, \varepsilon)$ with the following property. Let P_i, $i \in I$, be any finite family of integral polynomials in $d \leq s_0$ variables. Suppose that points $x_i \in \mathcal{O}^d$ satisfy

$$v(x_{i,u}) \geq \rho, \qquad \sum_{i \in I} v(P_i(x_i)) \leq B.$$

Then there are roots of unity $\zeta_1, \dots, \zeta_d$ such that

$$\begin{aligned} \zeta_u^{\ell^H} &= 1, & v(\zeta_u - 1) &\leq \rho, \\ v(P_i(\zeta_1 - 1, \dots, \zeta_d - 1)) &\leq v(P_i(x_i)) + \varepsilon & (i \in I). \end{aligned} \tag{13.8}$$

Neither H nor its choice uses the degrees or coefficients of the polynomials, their individual points, or $|I|$ beyond the displayed bound on the sum of valuations.

To prove this, take

$$\delta = \frac{\min\{\varepsilon, 1\}}{s_0 + 2}$$

and choose H in (13.4) with $B + 1$ in place of B. Move the coordinates one at a time. At a given step, write y_i for the current points, and replace the next coordinate by a common variable T, obtaining $g_i(T) \in \mathcal{O}[T]$. The coordinate being replaced is still an original coordinate, so its valuation is at least ρ. Hence

$$w_\rho\Big(\prod_i g_i\Big) = \sum_i w_\rho(g_i) \leq \sum_i v(P_i(y_i)) \leq B + 1.$$

Apply (13.5) to $\prod_i g_i$. It gives one root ζ with $v(\zeta - 1) = \rho_h \leq \rho$ and

$$\sum_i \big(v(g_i(\zeta - 1)) - w_{\rho_h}(g_i)\big) \leq \delta.$$

Every difference in this sum is nonnegative by (13.1). Thus each difference is at most δ. Also

$$w_{\rho_h}(g_i) \leq w_\rho(g_i) \leq v(P_i(y_i)).$$

It follows both that each individual valuation increases by at most δ, and that their total increases by at most δ. After d steps the total increase is at most $d\delta \leq \min\{\varepsilon, 1\}$; this proves the induction bound $B + 1$ and (13.8). When $d = 0$ the assertion is immediate.

The same tuple for every power $1, \dots, m$. Fix $m \geq 1$, a rational $\rho > 0$, finitely many finite bounds $\beta_i \geq 0$, and $\varepsilon > 0$. There is an H, chosen before P_i and x_i, such that

$$v(x_{i,u}) \geq \rho, \qquad v(P_i(x_i)) \leq \beta_i$$

imply the existence of $\zeta_u^{\ell^H} = 1$ satisfying

$$\begin{aligned} v(\zeta_u - 1) &\leq \rho/m, \\ v(P_i((\zeta_u^j - 1)_u)) &\leq v(P_i(x_i)) + \varepsilon \qquad (i \in I,\ 1 \leq j \leq m). \end{aligned} \tag{13.9}$$

The number of variables is again at most s_0.

For the proof, introduce the integral polynomials

$$P_{i,j}((X_u)_u) = P_i(((1 + X_u)^j - 1)_u).$$

For each integral x with $v(x) \geq \rho$, the monic polynomial $(1+Y)^j - (1+x)$ has j integral roots with product $\pm x$. If $x \neq 0$, the sum of their valuations is $v(x)$; one root has valuation at least $\rho/j \geq \rho/m$. If $x = 0$, take the root 0. Thus there is a point $x'_{i,j}$ with all coordinates of valuation at least ρ/m and

$$P_{i,j}(x'_{i,j}) = P_i(x_i).$$

Apply (13.8) to the family indexed by (i,j), radius ρ/m, and sum bound $m\sum_i \beta_i$. This proves (13.9).

If additionally $0 < \rho < 1$, all these powers are nontrivial and

$$v(\zeta_u^j - 1) \leq \rho \qquad (1 \leq j \leq m). \tag{13.10}$$

Indeed let ζ_u have exact order ℓ^h, $h \geq 1$. The initial radius bound says $m/\varphi(\ell^h) \leq \rho < 1$, so $j \leq m < \varphi(\ell^h) \leq \ell^h$. Writing $a = v_\ell(j) < h$, formula (13.3) applied to ζ_u^j gives

$$v(\zeta_u^j - 1) = \frac{\ell^a}{\varphi(\ell^h)} \leq \frac{m}{\varphi(\ell^h)} \leq \rho.$$

The corresponding Lean proofs are in `RootsOfUnity`, in particular `exists_rootsOfUnity_ellVal_eval_le_add`, and the all-powers specialization in `GeneralTheorem`, `exists_rootsOfUnity_ellVal_eval_le`. Only elementary polynomial algebra, the ultrametric inequality and divergence of the harmonic series have been used.

From a finite torsion level to divisible coefficients. The strict threshold in the Selmer application has an elementary explanation. Let A be a discrete valuation ring with uniformizer π, fraction field F_A, and let $\mathcal{M}$ be any torsion A-module. If

$$\operatorname{length}_A \mathcal{M}[\pi^e] < e, \tag{13.11}$$

then $\mathcal{M} = \mathcal{M}[\pi^e]$. Indeed, otherwise some element has exact order π^a with $a > e$. Multiplying it by π^{a-e} produces an element of exact order π^e in $\mathcal{M}[\pi^e]$, generating a submodule of length e, a contradiction. In particular the length of the whole module then equals the length of its π^e-torsion.

For the application, take a continuous integral representation $\rho : G_E \to \mathrm{GL}_n(A)$. The module of continuous cocycles with values in $(F_A/A)^n$, subject to the specified homogeneous linear relations and unramified conditions, is torsion. Each cocycle has finite image, since G_E is compact and the target is discrete; a common denominator of its finitely many coordinate values kills it. For $a \neq 0$, the coefficient injection

$$A/aA \longrightarrow (F_A/A)[a], \qquad x \longmapsto x/a \bmod A,$$

is an isomorphism. Applied to cocycles, it identifies the module with coefficients in $(A/aA)^n$ with the a-torsion of the divisible-coefficient module. The cocycle equation, the homogeneous relations and vanishing on the designated inertia groups are preserved and reflected by this injection.

We use normalized cocycles to represent cohomology classes. If a distinguished element γ acts as a scalar λ with $\lambda - 1 \in A^\times$, each cohomology class has exactly one representative z with $z(\gamma) = 0$: subtract the coboundary of $(\lambda-1)^{-1}z(\gamma)$. Two such representatives differing by the coboundary of b give $(\lambda-1)b = 0$, hence $b = 0$. The same argument applies to finite and divisible coefficients. Thus the coefficient injection compares the normalized Selmer modules without an additional H^0 term. The local Selmer conditions are the fixed homogeneous relations on these representatives.

Write $e_\ell = v_A(\ell)$, so the normalized length is $\operatorname{length}_A / e_\ell$ and $v = v_A/e_\ell$. Suppose a polynomial value $P(x)$ bounds the finite-level kernel by

$$\operatorname{length}_A \mathrm{Sel}(T\chi \bmod \ell^{M_{\mathrm{tor}}}) \leq v_A(P(x)).$$

If $v(P(x)) < M_{\mathrm{tor}}$, the right side is less than $v_A(\ell^{M_{\mathrm{tor}}}) = e_\ell M_{\mathrm{tor}}$. Applying (13.11) proves

$$\mathrm{length}_{\mathrm{norm}}\,\mathrm{Sel}(T\chi \otimes F_A/A) \le v(P(x)). \tag{13.12}$$

The argument applies to every finite coefficient ring containing the original data and the character values. The value $v(P(x))$ and the threshold M_{tor} do not change under enlarging that ring. The bound therefore holds over every sufficiently large coefficient ring. The corresponding Lean theorems are `length_selmerModule_divisibleQuot_le`, `divisibleLength_pointChar_le`, and `twistedLengthLe_pointChar` in `Selmer/Line`, once the finite-level polynomial inequality is supplied.

Auxiliary primes and specialization polynomials. Let $G_E = \mathrm{Gal}(\overline{\mathbb{Q}}/E)$, let c be complex conjugation, and write $g^{-c} = cg^{-1}c^{-1}$. Fix a finite set S_0 of places of the CM field E containing the places above ℓ. Take finitely many Selmer data T_i as in §13.1 and Stickelberger systems Y_k as in §13.3. Assume that all these data have a common distinguished element $g_0 \in G_E$ in an inertia group above a place in S_0 of residue characteristic different from ℓ. Let A be the initial coefficient ring, π_A a uniformizer and $k_A = A/\pi_A A$. Let $\beta(T_i)$ denote the integer length bound of the Selmer datum and $\beta(Y_k)$ the bound of the Stickelberger system. Set

$$b_i = v(\pi_A)\,\beta(T_i), \qquad c_k = \beta(Y_k) < \infty.$$

For T_i, this bound includes its global and original-place terms and its determinant terms at g_0 and g_0^{-c}. For Y_k, it includes the valuation of the untwisted system value and its prescribed contribution at g_0. For the ring of integers $\mathcal{O}'$ of a finite extension of $\mathbb{Q}_\ell$, module lengths are normalized by $\mathrm{length}(\mathcal{O}'/\ell) = 1$. Fix an integer $M_{\mathrm{tor}} > b_i + 1$ for every i. The initial coefficient ring A for the T_i is also required to have residue field satisfying

$$|k_A| > \max_i \left\{ 2\,\mathrm{rank}(T_i)\,\mathrm{length}_A(A/\ell^{M_{\mathrm{tor}}}A) + 1 \right\}.$$

It can be enlarged by a finite unramified extension to satisfy this inequality: the residue-field cardinality grows without bound, while $\mathrm{length}_A(A/\ell^{M_{\mathrm{tor}}}A)$ and the normalized bounds are unchanged under unramified extension. Under the hypotheses of Theorem 13.10, the following primes and polynomials exist.

There is an integer $s_0 \ge 1$ such that, for every positive rational ρ with $s_0\rho < 1$, and every sufficiently large N, there are a set Q of auxiliary primes, characters $\lambda_u : G_E \to \mathbb{Z}/\ell^N\mathbb{Z}$, and integral polynomials P_i, V_k in variables indexed by Q, with these properties:

1. The primes in Q have rational prime norm, $Q, \bar{Q}, S_0$ are pairwise disjoint, and $2|Q| \le s_0$. Each λ_u is continuous, comes from the norm-one torus, is trivial on decomposition groups above ℓ, is unramified outside $\{u, \bar{u}\}$, and maps inertia at u onto $\mathbb{Z}/\ell^N\mathbb{Z}$.

2. There are points x_i, y_k, with every coordinate of valuation at least ρ, such that

$$v(P_i(x_i)) \le b_i + s_0\rho, \qquad v(V_k(y_k)) \le c_k + s_0\rho.$$

3. For any tuple $\zeta_u^{\ell^N} = 1$, put

$$\chi(g) = \prod_{u \in Q} \zeta_u^{\lambda_u(g)}.$$

Then

$$\chi(\Theta_{k,Q}) = V_k((\zeta_u - 1)_u).$$

The finite-level Selmer kernel of $T_i\chi$ mod $\ell^{M_{\mathrm{tor}}}$, with the specified normalized cocycles and local relations, has normalized length at most $v(P_i((\zeta_u-1)_u))$. This assertion holds over every sufficiently large finite coefficient ring containing the data and the character values.

In the third assertion the Selmer conditions are unramifiedness outside the original places and $Q\cup\bar{Q}$, the crystalline relations at the places above ℓ, and the relation at the distinguished element acting by a scalar. The Lean declarations are `TwistPolynomials` and `exists_twistPolynomials` in `Selmer/Line`.

A common twisting character. Fix $m\ge 1$ and $0<\epsilon<1$. Choose a positive rational ρ with $s_0\rho\le\epsilon/2$. Apply (13.9), with error $\epsilon/2$, to the combined finite list of bounds b_i+1, c_k+1. This chooses an H before any auxiliary-prime polynomials are selected. Only now choose $N\ge H$ large enough for the construction, and obtain $Q,\lambda_u,P_i,V_k,x_i,y_k$. The specialization produces roots $\zeta_u^{\ell^H}=1$, which also satisfy $\zeta_u^{\ell^N}=1$, with

$$\begin{aligned} v(P_i((\zeta_u^j-1)_u))&\le b_i+\epsilon<b_i+1,\\ v(V_k((\zeta_u^j-1)_u))&\le c_k+\epsilon \qquad (1\le j\le m).\end{aligned}$$

Put

$$\psi(g)=\prod_{u\in Q}\zeta_u^{\lambda_u(g)}.$$

This is one continuous character of ℓ-power order. Its j-th power is obtained from the same λ_u by replacing ζ_u with ζ_u^j. Clause 3 and (13.12), using $b_i+\epsilon<b_i+1<M_{\mathrm{tor}}$, therefore bound every divisible-coefficient Selmer length by $b_i+\epsilon$ and every system value by $c_k+\epsilon$, simultaneously for all i,k,j. The construction does not choose a new character when j changes.

We also check ramification, which is needed for an m-fold twist. At $u\in Q$, every λ_v, $v\ne u$, vanishes on inertia: the auxiliary primes and their conjugates are disjoint. Since λ_u is onto on inertia, the image of ψ^j there contains $\zeta_u^j\ne 1$. Thus every power $1\le j\le m$ is ramified at every $u\in Q$. The norm-one property gives the same assertion at $\bar{u}$. For a generator γ_w of tame inertia at $w=u$ or $\bar{u}$, the finite character has prime-to-residue-characteristic order, so its inertial image is generated by $\psi^j(\gamma_w)$. Some power of that value is ζ_u^j. The factorization $z^a-1=(z-1)(1+\cdots+z^{a-1})$, with z a unit, gives

$$v(\psi^j(\gamma_w)-1)\le v(\zeta_u^j-1)\le\rho.$$

Consequently

$$\sum_{w\in Q\cup\bar{Q}} v(\psi^j(\gamma_w)-1)\le 2|Q|\rho\le s_0\rho\le\epsilon/2. \tag{13.13}$$

The characters are trivial on the decomposition groups above ℓ, and remain unramified outside $Q\cup\bar{Q}$.

For completeness, write the norm-one condition explicitly. For each u, take the character η_u of $\mathrm{U}(1)(\mathbb{A}_{E^+})/\mathrm{U}(1)(E^+)$ supplied by that condition, so that $\lambda_u(\mathrm{Art}\,a)=\eta_u(a/\bar{a})$. If it is initially presented as a character of the finite idele class group of E, η_u here denotes its restriction to this norm-one quotient. Then

$$\eta=\prod_{u\in Q}\zeta_u^{\eta_u},\qquad \psi(\mathrm{Art}\,a)=\eta(a/\bar{a})$$

is a character of the same quotient, trivial on its rational points and with the required tame level. For the fixed field isomorphism $\iota:\mathbb{C}\simeq\overline{\mathbb{Q}}_\ell$, set $\tau=\iota^{-1}\circ\eta^{-1}$, using the inverse-avatar convention. Its inverse avatar is precisely ψ. The ramification just proved says that the local tame part of τ^j is nontrivial at every $u\in Q$, for $1\le j\le m$. The conductor is prime to ℓ because Q avoids S_0.

This proves the conclusion of the general simultaneous twisting argument using the auxiliary primes and polynomials, whose construction is proved in Theorem 13.10. For an arbitrary $\epsilon > 0$, applying the result with $\min\{\epsilon/2, 1/2\}$ gives the same conclusion with allowance ϵ. Applying it to the union of all main representations, all lift indices, and all rank-one main-pair representations gives simultaneity across those finite families as well. Their finiteness, lattices, local relations and bounds b_i are arithmetic requirements separate from polynomial specialization. For the families used here these hypotheses follow from Theorems 17.87, 17.94 and 17.105, together with Corollary 17.106. The Stickelberger interpretation of the critical-value ratios is supplied at every coefficient prime for $z \geq 2$ by Corollary 18.36. Theorem 18.45 proves the endpoint comparison (EP), and Corollary 18.47 supplies $z = 1$ at every prime as well.

A finite algebraic step in the Selmer descent. The following finite-module lemma is used in the auxiliary-prime descent below. Let A be a discrete valuation ring with residue field k, let $I \neq 0$ be an ideal, and let C be a finite-length A-module. Suppose

$$f, g : C \longrightarrow (A/I)^d$$

are A-linear. Outside at most d residue classes of $a \in A$,

$$\operatorname{length} \ker(af - g) \leq \operatorname{length}\big(\ker f \cap g^{-1}(f(C))\big). \tag{13.14}$$

To prove this, put $C' = \sum_{a \in A} \ker(af - g) \subset C$. For a generator $z \in \ker(af - g)$, one has $g(z) = f(az)$; thus $g(C') \subset f(C')$. Let

$$\Lambda = \{x \in A^d : x \bmod I \in f(C')\}.$$

This is free of rank at most d, and its reduction $\pi : \Lambda \to f(C')$ is onto. Projectivity of Λ gives an A-linear map $\sigma : \Lambda \to C'$ with $f\sigma = \pi$, and then an endomorphism U of Λ with $\pi U = g\sigma$. The characteristic polynomial of U mod $\mathfrak{m}$ has degree at most d; exclude its roots in k. For every remaining a, the map $a - U$ is invertible over A. The identity

$$(af - g)\sigma = \pi(a - U)$$

then implies $f(C') \subset (af - g)(C')$. Comparing image and kernel lengths on C' yields

$$\operatorname{length} \ker((af - g)|_{C'}) \leq \operatorname{length} \ker(f|_{C'}).$$

The left kernel is $\ker(af - g)$, by the definition of C'; the right kernel is contained in $\ker f \cap g^{-1}(f(C))$. This proves (13.14). For finitely many pairs (f, g), the union of exceptional residue classes has size at most the sum of their target ranks. If $|k|$ exceeds this sum by more than 1, a common slope can be chosen outside this union and outside the residue 0, so it is a unit. For a rank-n datum at e torsion levels, evaluation at a Frobenius element and its conjugate inverse has target rank $2n$; thus $|k| > 2ne + 1$ suffices.

This proves the finite-length version of `tilting` used in `Selmer/Descent`. It explains the large-residue-field hypothesis in that construction. The next subsection supplies primes realizing the evaluation maps and proves the stopping bound using the stated classical duality, reciprocity and Chebotarev theorems.

13.1 Descent with local conditions at auxiliary primes

Global duality bounds the number of auxiliary primes. Lean expresses duality through a module of relations at split primes. Here we apply the finite-module Poitou–Tate formula to the dual Selmer groups for the local conditions in (13.20).

Let E be the CM field, $G = G_E$, and write $g^c = cgc^{-1}$ and $g^{-c} = (g^c)^{-1}$. In this subsection A is the ring of integers of a finite extension of $\mathbb{Q}_\ell$, with uniformizer π, residue field k_A, and integer valuation $v_A(\pi) = 1$. Write λ_A for length over A. Let $\rho : G \to \mathrm{GL}_n(A)$ be the continuous action on the datum $T = A^n$, and let χ_ℓ be the ℓ-adic cyclotomic character. For $a \in A \setminus \{0\}$, put $V_a = T/aT$. Its Cartier dual V_a^* has underlying group $\mathrm{Hom}(V_a, \mathbb{Q}/\mathbb{Z})$ and actions

$$(gy)(u) = y(\chi_\ell(g)\rho(g)^{-1}u), \qquad (ty)(u) = y(tu) \quad (t \in A).$$

The superscript $*c$ means that this action is composed with $g \mapsto g^c$.

ρ is unramified outside a finite set $S \subset S_0$ containing the places above ℓ. We assume that

$$\mathrm{span}_A \rho(G) = \mathrm{Mat}_n(A).$$

At each $w \mid \ell$, a finite free module Z_w of integral local cocycles contains every coboundary and satisfies

$$\sum_{w|\ell} \mathrm{rank}_A Z_w \le (\#\{w \mid \ell\} + f)n, \qquad f = [E : \mathbb{Q}]/2. \tag{13.15}$$

The finite-coefficient local condition is to obey every finite linear relation among evaluations which holds on Z_w. Finally there is an element γ_*, inertial at a place of S prime to ℓ, with

$$\rho(\gamma_*) = \rho(\gamma_*^{-c}) = \zeta_* I_n, \qquad \zeta_*^2 - 1 \in A^\times. \tag{13.16}$$

In particular $\chi_\ell(\gamma_*) = 1$, and both $\zeta_* - 1$ and $\zeta_*^{-1} - 1$ are units. Each cohomology class, for either V_a or V_a^*, therefore has a unique cocycle representative vanishing at γ_*.

The two terms to be preserved are

$$\begin{aligned} G_T &= \sup_{a \neq 0} \lambda_A \,\mathrm{Hom}_{A[G]}(V_a, V_a^{*c}), \\ L_T &= \sum_{v \in S,\ v \nmid \ell} \sup_{a \neq 0} \max\{\lambda_A C_v(a) - \lambda_A V_a, 0\}. \end{aligned} \tag{13.17}$$

Here $C_v(a)$ is the image, under restriction to the chosen decomposition group at v, of all continuous global cocycles with values in V_a. These are exactly `globalTerm` and `placesTerm` in the source. Their finiteness is part of the datum, and their quantitative estimates remain the estimates specified for that datum.

The Greenberg–Wiles formula. For a finite Galois module V and local cohomology conditions $\mathcal{L}_v \subset H^1(E_v, V)$, unramified at almost every place, write $\mathcal{L}_v^\perp$ for their annihilators under local Tate duality. We use the finite-module Greenberg–Wiles formula

$$\frac{\#H^1_{\mathcal{L}}(E, V)}{\#H^1_{\mathcal{L}^\perp}(E, V^*)} = \frac{\#H^0(E, V)}{\#H^0(E, V^*)} \prod_v \frac{\#\mathcal{L}_v}{\#H^0(E_v, V)}. \tag{13.18}$$

The product includes infinity. Taking orders in Poitou–Tate and using the global Euler characteristic formula gives (13.18) for finite ℓ-primary modules [Mil06, Ch. I, Corollary 2.3 and Theorems 4.10, 5.1]. The local cup products of two global classes have sum of invariants zero by global reciprocity.

Local conditions at the auxiliary primes. Fix integers $N > M \geq 0$. An auxiliary set J consists of primes of prime absolute norm, disjoint from their conjugates and from $S_0 \cup \overline{S_0}$, with a distinguished prime $P_0 \in J$. At $v \in J$, fix a tame generator γ_v, an arithmetic Frobenius ϕ_v, and a continuous character $h_v : G \to \mathbb{Z}/\ell^N\mathbb{Z}$. The character comes from a finite idele class character by $a \mapsto a/\bar{a}$, is tamely ramified precisely at $v, \bar{v}$, is trivial on decomposition groups above ℓ, and satisfies

$$\begin{aligned} &h_w(\gamma_v) = \delta_{wv}, \\ &h_w(\phi_v) = \delta_{wP_0} \quad (v \neq P_0), \qquad\qquad h_v(\phi_v) = 0. \end{aligned} \tag{13.19}$$

All these values are in $\mathbb{Z}/\ell^N\mathbb{Z}$. The character is anticyclotomic, so its values at g and g^{-c} are equal. No prime in S_0 or $J \setminus \{P_0\}$ divides the rational prime $\mathrm{N}\, P_0$.

For each datum T_i, choose slopes $s_{v,i} \in A$, with $s_{P_0,i} = 0$. Whenever $s_{v,i} \neq 0$, require ϕ_v and ϕ_v^{-c} to act trivially on $T_i/\ell^M T_i$. For $a \mid \ell^M$, define $\mathcal{S}_i(J; a)$ to be the module of continuous cocycles with values in T_i/aT_i, normalized by $z(\gamma_{*,i}) = 0$, unramified outside $S_i \cup J \cup \bar{J}$, obeying the specified local relations, and satisfying

$$z(\gamma_v) = s_{v,i} z(\phi_v), \qquad z(\gamma_v^{-c}) = s_{v,i} z(\phi_v^{-c}) \quad (v \in J). \tag{13.20}$$

Write $\mathcal{S}_i^P(J; a)$ when ramification is additionally allowed at $P, \bar{P}$, without imposing (13.20) at either place. The Lean definition is `tiedSelmer`.

Lemma 13.1 (Auxiliary characters with prescribed Frobenius)**.** *Let B and R be finite sets of places of E, and let $z \in G$ lie in the normal subgroup generated by inertia at the places of $R \setminus (B \cup \bar{B} \cup \{v \mid \ell\})$. For every neighborhood X of z and every $N \geq 0$, there is a prime P of prime norm, with $P, \bar{P} \notin R$, $P \neq \bar{P}$, and data (γ_P, ϕ_P, h_P) such that $\phi_P \in X$, $h_P(\gamma_P) = 1$, $h_P(\phi_P) = 0$, and h_P comes from the norm-one torus with tame level P. It is trivial on the decomposition groups at $B \cup \bar{B} \cup \{v \mid \ell\}$.*

Proof. Choose a conjugation-stable set T containing $B \cup \bar{B} \cup \{v \mid \ell\}$ whose ideal classes generate $\mathrm{Cl}(E)$. The additional representatives can be chosen outside $R \cup \bar{R}$ by Chebotarev in the Hilbert class field [Neu99, Ch. VII, Theorem 13.4]. The ideles which are units off T therefore map onto $\mathbb{A}_{E,f}^\times / E^\times$.

Set $n = \ell^N$. Adjoin μ_n and the n-th roots of a finite generating set of T-units. This finite Galois extension is unramified off T, since the defining Kummer equations are étale there. Hence z fixes it. Choose a finite Galois extension containing this field such that the coset at z modulo its absolute Galois group lies in X. Chebotarev gives primes P with arithmetic Frobenius conjugate to the image of z in this finite quotient. Choosing a place above P selects a Frobenius lift σ with that image. We may exclude $T \cup R \cup \bar{R}$ and the ramified primes of E/E^+, and take P of residue degree one over $\mathbb{Q}$, since primes of higher residue degree have Dirichlet density zero. Thus $P \neq \bar{P}$. Splitting in the Kummer extension gives $\mathrm{N}P \equiv 1 \bmod n$ and makes each T-unit an n-th power in the residue field.

The residue logarithm at P, normalized to be 1 mod n on the chosen tame generator under local reciprocity, now kills the T-units. It descends from the ideles which are units off T to a continuous idele class character θ, since their intersection with $E^\times$ is precisely the T-units. By global reciprocity [Neu99, Ch. VI, §§5–6], $a \mapsto \theta(a/\bar{a})$ gives h_P with the required norm-one origin. It is trivial on the multiplicative groups at T and on the units away from $P, \bar{P}$, and has $h_P(\gamma_P) = 1 \bmod n$. Finally $\phi_P = \sigma \gamma_P^{-h_P(\sigma)}$ is a Frobenius with $h_P(\phi_P) = 0$. Inertia fixes the finite quotient at the chosen unramified prime, so this correction preserves $\phi_P \in X$.

Lean. `New.Selmer.exists_isAuxPrimeDataAt`. □

For $N = 0$, the character is trivial and the generator condition means $1 = 0$ in $\mathbb{Z}/1\mathbb{Z}$; the applications have $N > M$, so $N > 0$. The lemma gives existence and the stated character conditions, with no bound on the norms of the new primes.

The first prime and subsequent choices. Apply Lemma 13.1 with $B = \varnothing$ and the prescribed g_0, which is inertial at a place of S_0 prime to ℓ. Include in R every prime dividing the norm of a member of S_0. This gives P_0 with ϕ_{P_0} in any prescribed neighborhood of g_0, and with the required avoidance of its rational norm. Give it slope zero for every datum.

For a fixed datum T, let H be the subgroup generated by $[g, \gamma_*]$ and $[g, \gamma_*^c]$, for $g \in G$. It is normal and stable under c. It acts trivially on T, and every abelian character of G is trivial on it. Set

$$U = \ker\bigl(G \to \mathrm{GL}(T/\ell^M T)\bigr) \cap \ker \chi_{\ell^N}, \qquad \gamma_0 = \gamma_{P_0}.$$

Both H and γ_0 lie in $U \cap U^c$. For every $h \in H$ and every neighborhood of $\gamma_0 h$, Lemma 13.1, applied with $B = J \setminus \{P_0\}$, supplies a new prime whose Frobenius lies in that neighborhood and in $U \cap U^c$. To verify its hypothesis, enlarge R by $S_0, \bar{S}_0, J$, the desired finite avoidance set, and the primes dividing $\mathrm{N}\, P_0$. The element γ_0 is inertial at P_0, and H lies in the normal subgroup generated by inertia at $S \cup \bar{S}$, away from ℓ. These places are outside $B \cup \bar{B}$.

Shrink the neighborhood so that each old character is constant there. Its value at $\gamma_0 h$ is δ_{wP_0}, since it kills H. The new character is trivial on the decomposition groups at the old later primes; hence all compatibilities in (13.19) are preserved. The resulting prime can be added with any slope for this datum and with slope zero for all other data. Thus every neighborhood in G_E of every element of $\gamma_0 H$ contains the Frobenius of such a prime.

Lean. `exists_isNewAuxiliaryPrime`.

Lemma 13.2 (The local count with its coboundary terms)**.** *Fix a datum T, a tied set J, and $a \mid \ell^M$. There is an ordinary dual Selmer group $\mathcal{D}(J; a)$ such that*

$$\lambda_A \mathcal{S}(J; a) \le \lambda_A \mathcal{D}(J; a) + L_T. \tag{13.21}$$

Every member of $\mathcal{D}(J; a)$, restricted to U, is a G-equivariant homomorphism $U \to V_a^*$, and vanishes on inertia at $P_0, \bar{P}_0$. If P is a new auxiliary prime with $\phi_P, \phi_P^c \in U$, then for $z' \in \mathcal{S}^P(J; a)$, $y \in \mathcal{D}(J; a)$, and $t \in A$, reciprocity gives

$$y(\phi_P)\bigl(tz'(\gamma_P)\bigr) + y(\phi_P^c)\bigl(tz'(\gamma_P^{-c})\bigr) = 0. \tag{13.22}$$

Proof. Put $S' = S \cup (J \setminus \{P_0\}) \cup \overline{J \setminus \{P_0\}}$. For $v \in S'$, let L_v^Z be the module of restrictions to a chosen decomposition group of global cocycles obeying the local relations and ties, allowing arbitrary ramification and omitting the normalization at γ_*. This module contains all local coboundaries: each is the restriction of a global coboundary, and the ties are obeyed because inertia is trivial on V_a and a nonzero slope occurs only at a Frobenius acting trivially on V_a. Let $\mathcal{L}_v$ be its image in $H^1(E_v, V_a)$, and impose the unramified condition outside S'. Define $\mathcal{D}(J; a) = H^1_{\mathcal{L}^\perp}(E, V_a^*)$.

The slope at P_0 is zero. A cocycle satisfying its two ties is unramified at $P_0, \bar{P}_0$: at a prime away from ℓ, inertia acts trivially, its wild part has no nontrivial continuous homomorphism to the finite ℓ-group V_a, and its remaining ℓ-part is detected by the chosen tame generator. Thus the normalized tied module injects into $H^1_{\mathcal{L}}(E, V_a)$. The uniqueness of normalization uses $\zeta_* - 1 \in A^\times$. Both global invariant groups in (13.18) vanish, because γ_* acts on V_a by ζ_* and on its Cartier dual by ζ_*^{-1}.

At a finite place, the exact sequence of coboundaries gives

$$\lambda_A \mathcal{L}_v - \lambda_A H^0(E_v, V_a) = \lambda_A L_v^Z - \lambda_A V_a.$$

Each of the f complex places contributes $-\lambda_A V_a$. Taking logarithms to base $\#k_A$ in (13.18), we obtain

$$\lambda_A \mathcal{S}(J;a) \le \lambda_A \mathcal{D}(J;a) + \sum_{v\in S'} \lambda_A L_v^Z - (\#S' + f)\lambda_A V_a. \tag{13.23}$$

At $w \mid \ell$, the equations defining the local condition imply $\lambda_A L_w^Z \le \operatorname{rank}_A Z_w \, v_A(a)$. Here is the finite linear algebra even when the decomposition group is infinite. Let F be the free module on all coordinate-evaluation symbols, and map it to Z_w^* by evaluation. Its image Λ is a submodule of the finite free module Z_w^*, hence is free of rank at most $\operatorname{rank}_A Z_w$. A function obeying every relation in the kernel is exactly a member of $\operatorname{Hom}_A(\Lambda, A/a)$. Its length is at most the asserted bound. Matrix-valued relations give the same scalar relations by selecting one output coordinate at a time. Equation (13.15) consequently accounts for the extra $f\lambda_A V_a$ in (13.23).

At $v \in S$, $v \nmid \ell$, the definition of L_T gives the allowance beyond $\lambda_A V_a$. At a tied prime outside S, evaluation at its Frobenius injects L_v^Z into V_a: if that value is zero, the tie makes the tame value zero, and these two values determine the local cocycle. At the conjugate prime use the inverse conjugate pair of generators. Therefore every new local term is at most $\lambda_A V_a$. Substitution proves (13.21) with exactly the original L_T.

Represent a dual class by its unique cocycle vanishing at γ_*. Its restriction to U is a homomorphism, since U acts trivially on the Cartier dual; it is equivariant by the cocycle conjugation identity. The dual unramified condition outside S' implies that it is zero on inertia at $P_0, \bar{P}_0$. For $z' \in \mathcal{S}^P(J;a)$, all local pairings with this dual class vanish except possibly at $P, \bar{P}$: at the old places this is the annihilator condition, and elsewhere both classes are unramified. At a split auxiliary prime, fix compatible ℓ-power roots of unity and write t_ℓ for the tame character in these coordinates. Let $u = t_\ell(\gamma_P) \in \mathbb{Z}_\ell^\times$. The fact that u is a unit follows either from the tame-generator choice or already from $h_P(\gamma_P) = 1$ modulo ℓ^N. Transport the local uniformizer by complex conjugation when defining the tame character at $\bar{P}$. Because $\chi_\ell(c) = -1$, conjugation changes its value by a minus sign, and inversion changes it back. Thus

$$t_\ell(\gamma_P^{-c}) = u$$

at $\bar{P}$, in the same root-of-unity coordinates. The local cup-product formula for an unramified dual class consequently gives the two remaining invariants as

$$\epsilon u^{-1} y(\phi_P)\big(t z'(\gamma_P)\big), \qquad \epsilon u^{-1} y(\phi_P^c)\big(t z'(\gamma_P^{-c})\big),$$

where $\epsilon \in \{1, -1\}$ is the one common sign fixed by the cup-product and Artin conventions. Both dual Frobenius values are well-defined independently of the inertial lifts, and the Frobenius actions are trivial on both finite coefficient modules here. Multiplication of the sum-of-invariants equality by ϵu, acting through its residue modulo ℓ^M, gives exactly (13.22). There is no division by a possibly nonunit integer. This proves all assertions.

Lean. `exists_length_tiedSelmer_le`, `apply_first_tameGenerator_eq_zero_of_mem_dualSelmer`, `dualSelmer_frob_add_conj_frob_eq_zero`. □

Lemma 13.3 (A conjugate pairing and the global length bound)**.** *Suppose $z \in \mathcal{S}(J;a)$ has a value with a unit coordinate. Assume that for every new auxiliary prime P admissible for this datum, there is $z' \in \mathcal{S}^P(J;a)$ with*

$$\big(z'(\gamma_P), z'(\gamma_P^{-c})\big) = \big(z(\phi_P), z(\phi_P^{-c})\big).$$

Then $\lambda_A \mathcal{D}(J;a) \le G_T$, and consequently $\lambda_A \mathcal{S}(J;a) \le G_T + L_T$.

Proof. First $z(H)$ spans V_a. Indeed the cocycle identity and $z(\gamma_*) = 0$ give Sah's identity

$$z([g, \gamma_*]) = (1 - \zeta_*)z(g).$$

Thus some value on H has a unit coordinate. Its span is stable under $\rho(G)$, since H is normal and acts trivially, so $z(ghg^{-1}) = \rho(g)z(h)$. It is stable under every matrix because $\rho(G)$ spans $\mathrm{Mat}_n(A)$. Elementary matrices then carry the vector with a unit coordinate to every vector of V_a.

Fix $y \in \mathcal{D}(J; a)$, restricted to U, and define

$$\Psi_t(u, v) = y(v)\big(tz(u)\big) - y(u^c)\big(tz(v^c)\big) \qquad (u, v \in U \cap U^c,\ t \in A).$$

We first prove $\Psi_t(h, h) = 0$ for $h \in H$. Both z and y vanish at γ_0 and at γ_0^c, so replacing h by $g = \gamma_0 h$ leaves this diagonal value unchanged. The function $u \mapsto \Psi_t(u, u)$ is locally constant on the open subgroup $U \cap U^c$, since the cocycles have finite values and conjugation is continuous. The preceding approximation gives an admissible Frobenius $\phi \in U \cap U^c$ with $\Psi_t(\phi, \phi) = \Psi_t(g, g)$. By hypothesis there is z' with tame values $z(\phi)$ and $z(\phi^{-c}) = -z(\phi^c)$. Equation (13.22) says precisely that $\Psi_t(\phi, \phi) = 0$, proving the assertion.

The restrictions of z and y to H are homomorphisms. Expanding the diagonal identity at hh' therefore gives

$$\Psi_t(h, h') + \Psi_t(h', h) = 0. \tag{13.24}$$

Conjugation by γ_*, together with (13.16) and $\chi_\ell(\gamma_*) = 1$, gives

$$\Psi_t(\gamma_* h \gamma_*^{-1}, h') = \Psi_{\zeta_* t}(h, h'),$$
$$\Psi_{\zeta_* t}(h', \gamma_* h \gamma_*^{-1}) = \Psi_t(h', h).$$

For example, $z(\gamma_* h \gamma_*^{-1}) = \zeta_* z(h)$, whereas $y(\gamma_* h \gamma_*^{-1})$ is $y(h)$ precomposed with multiplication by ζ_*^{-1}; the conjugate terms transform with the reciprocal factors. Apply (13.24) to the conjugated pair with parameter $\zeta_* t$, and subtract it for the original pair with parameter t. The result is $\Psi_{(\zeta_*^2 - 1)t}(h, h') = 0$. Since $\zeta_*^2 - 1$ is a unit, $\Psi_t(h, h') = 0$ for all parameters. This argument does not divide by 2.

There is consequently a unique A-linear map

$$\eta_y : V_a \longrightarrow V_a^*, \qquad \eta_y(z(h)) = y(h^c) \quad (h \in H).$$

To check that it is well defined, take a relation $\sum_r a_r z(h_r) = 0$. The vanishing of Ψ, with parameters ta_r, shows that the character $\sum_r a_r y(h_r^c)$ kills every vector $tz(h'^c)$. These vectors span V_a, so that character is zero. Equivariance follows on the spanning vectors from

$$z(ghg^{-1}) = \rho(g)z(h), \qquad y(g^c h^c (g^c)^{-1}) = \rho_{V_a^*}(g^c) y(h^c).$$

Thus $\eta_y \in \mathrm{Hom}_{A[G]}(V_a, V_a^{*c})$.

The assignment $y \mapsto \eta_y$ is linear and injective on the dual cohomology classes. Indeed if $\eta_y = 0$, then the normalized global cocycle representing y vanishes on H. Sah's identity for that cocycle is

$$y([g, \gamma_*]) = (1 - \zeta_*^{-1}) y(g).$$

Its scalar is a unit, so the cocycle is zero. Therefore

$$\lambda_A \mathcal{D}(J; a) \le \lambda_A \,\mathrm{Hom}_{A[G]}(V_a, V_a^{*c}) \le G_T.$$

Lemma 13.2 completes the proof.

Lean, formal context. `mirrorForm_self_eq_zero`, `length_dualSelmer_le`, `length_tiedSelmer_le_of_isUnit`. The code bounds the dual module defined by its relations at the auxiliary primes. The proof here applies Poitou–Tate to ordinary dual Selmer classes and uses the mirror pairing to obtain the same bound G_T. □

Passing to the exponent. The unit-coordinate hypothesis is removed by using every torsion level. Let $e = v_A(\ell^M)$, and let π^k, $0 \le k \le e$, be the exponent of $\mathcal{S}(J;\pi^e)$. Multiplication by π^{e-k} identifies A/π^k with the π^k-torsion of A/π^e. Applied pointwise to cocycles, it preserves and reflects every homogeneous local relation, tie and unramified condition. Consequently

$$\mathcal{S}(J;\pi^k) \simeq \mathcal{S}(J;\pi^e)[\pi^k] = \mathcal{S}(J;\pi^e).$$

If $k = 0$, this group is zero. If $k > 0$, it has an element of exact order π^k; some value of that element has a unit coordinate, since otherwise π^{k-1} would kill it. Hence if the hypothesis about Frobenius values in Lemma 13.3 holds for every member at every level $1 \le k \le e$, then

$$\lambda_A \mathcal{S}(J;\ell^M) \le G_T + L_T. \tag{13.25}$$

The ideals (ℓ^M) and (π^e) are equal.

Theorem 13.4 (A uniform bound for the number of tied primes)**.** *Let T_i, $i \in I$, be finitely many of the above data, of ranks n_i, and let g_0 be inertial at a place of S_0 prime to ℓ. Fix $M \ge 0$, a neighborhood W of g_0, and put $e = v_A(\ell^M)$. Suppose*

$$\#k_A > 2n_i e + 1 \qquad (i \in I).$$

There is an integer s, independent of N, such that for every $N > M$ there are tied primes and characters satisfying (13.19), with $\phi_{P_0} \in W$, $\#J \le s$, and

$$\lambda_A \mathcal{S}_i(J;\ell^M) \le G_{T_i} + L_{T_i} \qquad (i \in I). \tag{13.26}$$

One may take

$$s = 1 + \sum_{i\in I} \sum_{k=1}^{e} \lambda_A \mathcal{S}_i(\varnothing;\pi^k). \tag{13.27}$$

Proof. The groups in (13.27) are finite. A continuous cocycle into a finite module defines a homomorphism into the semidirect product of that module with the finite image of the representation. Its kernel therefore gives an extension of bounded degree, unramified outside the fixed finite set. Hermite–Minkowski finiteness supplies one finite quotient through which all such cocycles factor. This also proves finiteness at every later finite set of primes.

For a tied configuration define the integer potential

$$F(J) = \sum_{i\in I} \sum_{k=1}^{e} \lambda_A \mathcal{S}_i(J;\pi^k).$$

Start with the first prime P_0 and all slopes zero. Killing the tame values removes its allowed ramification, so these initial groups are contained in $\mathcal{S}_i(\varnothing;\pi^k)$. Thus $1 + F(\{P_0\}) \le s$.

Suppose that for a datum i, a level $k \le e$, and an admissible new prime P, some member of $\mathcal{S}_i(J;\pi^k)$ has Frobenius pair outside the tame image of $\mathcal{S}_i^P(J;\pi^k)$. For each level $r = 1, \dots, e$, apply the finite tilting lemma (13.14) to

$$C_r = \mathcal{S}_i^P(J;\pi^r), \quad f_r(z) = \big(z(\gamma_P), z(\gamma_P^{-c})\big), \quad g_r(z) = \big(z(\phi_P), z(\phi_P^{-c})\big).$$

The target has rank $2n_i$ over A/π^r. For each level at most $2n_i$ residue classes of a coefficient u are excluded. The displayed residue-field hypothesis permits one unit $u \in A^\times$ avoiding all exclusions for all e levels. Set the new slope for datum i equal to u^{-1}, and set the slopes for every other datum equal to zero.

The old group at level r is $C_r \cap \ker f_r$: zero tame values remove the additional ramification at both primes. The new group is $C_r \cap \ker(uf_r - g_r)$. Tilting therefore gives

$$\lambda_A \mathcal{S}_i(J \cup \{P\}; \pi^r) \le \lambda_A\big(\mathcal{S}_i(J; \pi^r) \cap g_r^{-1}(f_r(C_r))\big).$$

No level grows, and the assumed failure makes the inequality strict against the old length at level k. For all other data the zero slope removes the newly allowed ramification, so their groups do not grow either. Thus F decreases by at least one. All character compatibilities are preserved by the auxiliary-prime construction, and $\#J + F(J)$ does not increase.

Repeat while such a failure exists. This process terminates because F is a nonnegative integer. At termination every datum, every level and every admissible new prime satisfies the inclusion required for (13.25). That equation proves (13.26), and the invariant $\#J + F(J) \le s$ proves the cardinal bound. The initial groups in (13.27) use only T_i, M and their fixed conditions; they do not use N, P_0, or the later characters.

Lean. `exists_isTiedPrimes_card_le_and_length_le`. □

The tied length is bounded by $G_T + L_T$. The two determinant terms at g_0 and g_0^{-c} enter when the first prime is released.

13.2 Polynomial bounds for the lengths of Selmer modules

We construct the polynomials that bound the Selmer lengths. There are three steps: place every specialization in one finite space of functions, express its length by augmented maximal minors, and find a small value of one such minor by restricting to a line. The line argument also gives the exact cost of removing the auxiliary local relations. The required tied-prime bound is Theorem 13.4.

Throughout this subsection, ℓ is a rational prime, O is the integer ring of a finite extension of $\mathbb{Q}_\ell$, ϖ is a uniformizer, $v_O = \mathrm{ord}_\varpi$, and

$$v(\ell) = 1, \qquad \mathrm{len}_v C = v(\varpi)\,\mathrm{length}_O C.$$

For a finite coefficient extension O'/O, normalized lengths use its own uniformizer and the same valuation v. Write $G_E = \mathrm{Gal}(\overline{\mathbb{Q}}/E)$. All cocycles below are locally constant, and all extra relations are homogeneous linear equations with finite support in G_E. These include the prescribed local relations and the scalar-element normalization of the Selmer datum; they are never replaced by a dimension condition.

One finite quotient for every specialization. The following ray class quotient works simultaneously for all coefficient groups of a given exponent. For cyclic coefficients it also gives finiteness of the characters unramified outside the specified places.

Lemma 13.5 (A quotient independent of the coefficient module)**.** *Let H be an open subgroup of G_E, let S be a finite set of places of E, and let $m \ge 1$. There is an open normal subgroup K of G_E, contained in H, with the following property. For every abelian group B killed by m, every locally constant function $z : G_E \to B$ that is additive on H and zero on inertia above every place outside S vanishes on K.*

Proof. Let F be the fixed field of H. Form a modulus $\mathfrak{m}$ of F containing all real places and each finite w over S with exponent $2v_w(m) + 1$, where v_w is the normalized valuation. For $u \in 1 + \mathfrak{p}_w^{2v_w(m)+1}$, Hensel's lemma at 1 applied to $X^m - u$ gives $u \in (F_w^\times)^m$. The restriction of z to $H = G_F$ is a continuous homomorphism with finite image. Under local reciprocity it kills the units away from S

and the m-th powers at each w over S. Global reciprocity therefore factors $z|_H$ through the finite ray class quotient $\mathrm{Cl}_{\mathfrak{m}}(F)/m\,\mathrm{Cl}_{\mathfrak{m}}(F)$ [Mil20, Ch. V, §3]. The kernel of the map from H to this quotient is open. Intersecting its finitely many G_E-conjugates gives the required K.

Lean. Lean separates the finite images by characters in `exists_openNormalSubgroup_forall_eq_zero`. The modulus here works for all the images at once. □

Fix a continuous representation $\rho : G_E \to \mathrm{GL}_n(O)$, a positive integer q, a finite family of continuous characters $(\lambda_u : G_E \to \mathbb{Z}/q\mathbb{Z})_{u\in J}$, and a torsion level $M \geq 1$. Fix also a finite set S of finite places of E and a set of the homogeneous local relations described above. For the representative $[\lambda_u(g)] \in \{0, \dots, q-1\}$, put

$$\Psi_X(g) = \prod_u (1 + X_u)^{[\lambda_u(g)]}, \qquad R_X(g) = \Psi_X(g)\rho(g).$$

The polynomial function Ψ_X need not be a character. For every coefficient map $e : O \to O'$ and point $x = (x_u) \in (O')^J$, define $\mathcal{S}(x)$ by the equations

$$\begin{aligned} z(gh) &= z(g) + R_x(g)z(h), \\ z(g) &= 0 \quad \text{on inertia outside } S, \\ \sum_g e(L_g)z(g) &= 0 \quad \text{for each prescribed relation } L, \end{aligned} \qquad z : G_E \longrightarrow (O'/\ell^M)^n.$$

The symbol $R_x(g)$ means the evaluated matrix acting modulo ℓ^M. These equations define a module even at a point where R_x is not a representation. At $x_u = \zeta_u - 1$ with $\zeta_u^q = 1$, they are exactly the cocycle equations for the character twist $\prod_u \zeta_u^{\lambda_u}$.

Lemma 13.6 (One polynomial matrix and its sharp minors)**.** *There is a finite matrix $A(X)$ over $O[X_u : u \in J]$, with d columns, such that for every O' and x as above,*

$$\mathcal{S}(x) \simeq \ker\Big(A(x) : (O'/\ell^M)^d \longrightarrow (O'/\ell^M)^a\Big).$$

Let $\mathcal{M}$ be the finite family of d-row minors of

$$\begin{pmatrix} A(X) \\ \ell^M I_d \end{pmatrix}.$$

For every discrete valuation ring O' of characteristic zero over O, with integer-valued valuation $v_{O'}$, one has

$$\mathrm{length}_{O'}\,\mathcal{S}(x) = \min_{P\in\mathcal{M}} v_{O'}(P(x)). \tag{13.28}$$

Consequently, at any one such point x_0, one can choose $P \in O[X_u]$ such that its valuation equals the length at x_0, while its valuation bounds the length from above at every other point over every such coefficient ring.

Proof. Put $H = \ker(\rho \bmod \ell^M) \cap \bigcap_u \ker\lambda_u$. Since $R_x|_H = 1$ for every x, each cocycle restricts to an additive map on H. Lemma 13.5 gives one open normal subgroup $K \subset H$ on which all these maps vanish. Then $z(gk) = z(g)$, so all the cocycles are functions on G_E/K. Use their $d = n|G_E/K|$ coordinates as unknowns. Summing coefficients over each fibre of $G_E \to G_E/K$ writes every defining equation as a row over $\Lambda = O[X_u : u \in J]$. These rows generate a submodule of the noetherian module Λ^d, so finitely many of them generate it. Take these as A. Their specializations imply every original equation, and every solution on the finite quotient pulls back to a locally constant solution on G_E.

Put $b = \ell^M$ and $Q = \Lambda^d/(\mathrm{rows}(A) + b\Lambda^d)$. At x, evaluation on the standard basis identifies $\mathcal{S}(x)$ with $\mathrm{Hom}_{O'}(Q_x, O'/b)$, where $Q_x = Q \otimes_{\Lambda,x} O'$. Every cyclic summand O'/c of Q_x has $c \mid b$, and its dual $(O'/b)[c]$ has the same length. The elementary divisors thus give $\mathrm{length}_{O'} \mathcal{S}(x) = v_{O'}(\mathrm{Fitt}_0 Q_x)$. Zeroth Fitting ideals commute with arbitrary base change [Eis95, Corollary 20.5], since maximal presentation minors specialize entrywise. Here those minors are precisely $\mathcal{M}$. Their least valuation gives (13.28). A minor attaining the minimum at x_0 bounds the length everywhere. For $d = 0$ take the empty minor 1.

Lean. The simultaneous matrix and sharp polynomial appear in `exists_matrix_selmerAtPoint_equiv` and `exists_sharp_polynomial`. Lean obtains the minimum by its elementary-divisor and row-lattice argument in `isLeast_length_matrixKernel`. □

This reconstructs `OneSystem`, including `exists_matrix_selmerAtPoint_equiv` and `exists_sharp_polynomial`, and the augmented-minor argument in `SharpMinors`. The finite-row reduction uses that the present coefficient ring is noetherian; the source also treats arbitrary coefficient rings by allowing an infinite row family.

The geometric sum and the tame cocycle identity. For $0 < r \le 1$ and $N \ge 0$, the Gauss valuation of the cyclic sum is

$$w_r\left(\frac{(1+X)^{\ell^N} - 1}{X}\right) = \sum_{j=1}^{N} \min\{1, \varphi(\ell^j) r\} =: D_N(r). \tag{13.29}$$

Indeed there are $\varphi(\ell^j)$ roots $\zeta - 1$ with ζ of order ℓ^j. Each has valuation $1/\varphi(\ell^j)$ by (13.3), so (13.2) gives the identity. Hence $Nr \le D_N(r) \le N$, and $v(\sum_{i<\ell^N} \alpha^i) \ge D_N(r)$ whenever $v(\alpha - 1) \ge r$, the sum starting at $i = 0$. In particular $D_N(r) \ge M$ kills the sum modulo ℓ^M and gives $\alpha^{\ell^N} \equiv 1 \bmod \ell^M$. It suffices to take

$$N \ge M + \max\left(0, \left\lceil \log_\ell \frac{1}{(\ell-1)r} \right\rceil\right),$$

since all summands after the indicated number of initial terms equal one.

Here is the exact local consequence. Suppose a finite coefficient module carries a continuous representation R and a locally constant cocycle z. At a place of residue characteristic different from ℓ, suppose inertia acts by scalars α, and $\sum_{j<q} \alpha^j = 0$ on the module, where $q = \ell^N$. The affine action

$$u \longmapsto z(g) + R(g)u$$

then has q-th power equal to the identity for every inertial element g: its linear part is $\alpha^q = 1$, and its translation part is $\sum_{j<q} \alpha^j z(g) = 0$. The affine action on inertia is continuous with finite image. It kills wild inertia, since its image has exponent a power of ℓ, whereas wild inertia is a pro-p group with $p \ne \ell$. In the tame quotient, conjugation by Frobenius is the norm-power map. If the residue norm is $1 \pmod q$, the affine maps of a Frobenius ϕ and an inertial element γ commute. Evaluation at $u = 0$ gives

$$(R(\phi) - I)z(\gamma) = (R(\gamma) - I)z(\phi). \tag{13.30}$$

The same identity holds if ϕ^{-1} is Frobenius: multiply the identity for ϕ^{-1} by $-R(\phi)$, use $z(\phi^{-1}) = -R(\phi^{-1})z(\phi)$, and use the scalar inertia action to commute the matrices. This is the only local Galois-group fact used below, namely the classical description of tame inertia and its Frobenius action.

Local hypotheses at the auxiliary primes. Let J be a finite set of auxiliary primes and $v_0 \in J$. The primes in J and their conjugates are distinct, lie outside S_0, and have norms $1 \pmod{\ell^N}$. At each $w \in J$ choose a Frobenius ϕ_w, a tame generator γ_w, and an integral slope s_w, with $s_{v_0} = 0$. For $b = +$ use these elements; for $b = -$ use their conjugate inverses:

$$\phi_{w,-} = (c\phi_w c)^{-1}, \qquad \gamma_{w,-} = (c\gamma_w c)^{-1}.$$

Here c is the chosen complex conjugation.

The characters $\lambda_u : G_E \to \mathbb{Z}/\ell^N\mathbb{Z}$ are anticyclotomic, unramified outside $u, \bar{u}$, and satisfy

$$\lambda_u(\gamma_{w,b}) = \delta_{u,w}, \qquad \lambda_u(\phi_{w,b}) = \delta_{u,v_0} \quad (w \neq v_0).$$

For the application, these characters come from the prescribed norm-one torus construction and are trivial on the decomposition groups above ℓ. Take $S = S_T \cup J \cup \bar{J}$, where $S_T \subset S_0$ is the datum's own ramification set, and retain exactly its relation set $\mathcal{L}_T$. Since the normalizing scalar element lies in inertia above S_T away from ℓ, each λ_u is trivial on it. The prescribed local relations and scalar-element normalization therefore apply to the twists in question. Assume that ρ is unramified outside S_0, and that

$$s_w \neq 0 \implies \rho(\phi_{w,b}) = I \pmod{\ell^M} \quad (b = +, -).$$

The extra relations defining the datum are fixed throughout. For a function z, its tie at (w, b) is the equation $z(\gamma_{w,b}) = s_w z(\phi_{w,b})$. The adjective *tied* below means imposing these displayed equations in addition to the original local relations.

Put $a_{v_0} = 1$, $a_w = s_w$ for $w \neq v_0$, and specialize on the line

$$x_w = a_w y.$$

Take y in the integer ring O' of a finite coefficient extension, with $v(y) \geq r$, $0 < r \leq 1$, and $D_N(r) \geq M$. For every g, the scalar $\Psi_x(g) - 1$ is divisible by y. Equation (13.29) therefore applies to $\Psi_x(g)$. In particular, all $1 + x_w$ are ℓ^N-th roots of unity modulo ℓ^M, so the evaluated twist does act through a representation on $(O'/\ell^M)^n$. This justifies applying (13.30); no multiplicativity over O' itself is asserted.

At $w \neq v_0$, the prescribed character values give

$$R_x(\phi_{w,b}) = (1 + y)\rho(\phi_{w,b}), \qquad R_x(\gamma_{w,b}) = (1 + s_w y)I \quad \text{on } (O'/\ell^M)^n.$$

Thus (13.30) and the slope hypothesis imply

$$\big((1 + y)\rho(\phi_{w,b}) - I\big)\big(z(\gamma_{w,b}) - s_w z(\phi_{w,b})\big) = 0. \tag{13.31}$$

Indeed the tame identity makes the first term $s_w y\, z(\phi_{w,b})$. If $s_w = 0$, this is already zero; otherwise $\rho(\phi_{w,b}) = I \pmod{\ell^M}$, so subtracting the second term cancels it.

Let $\mathcal{S}_{\mathrm{tied}}(x)$ denote the submodule obtained by adding the ties at $J \setminus \{v_0\}$. The joint tie map embeds $\mathcal{S}(x)/\mathcal{S}_{\mathrm{tied}}(x)$ into the product of the kernels in (13.31). For any integral n-by-n matrix D, Smith normal form gives

$$\operatorname{length}_{O'} \ker(D \bmod \ell^M) \leq v_{O'}(\det D);$$

the right side is $+\infty$ if the determinant is zero. Consequently, if

$$P_1(Y) = \prod_{\substack{w \in J \setminus \{v_0\} \\ b \in \{+,-\}}} \det\big((1 + Y)\rho(\phi_{w,b}) - I\big) \in O[Y], \tag{13.32}$$

then

$$\operatorname{len}_v \mathcal{S}(x) \le \operatorname{len}_v \mathcal{S}_{\mathrm{tied}}(x) + v(P_1(y)). \tag{13.33}$$

Every factor in P_1 has degree n and unit leading coefficient $\det \rho(\phi_{w,b})$. Thus, with the Gauss valuation already used above,

$$w_r(P_1) \le 2n(|J|-1)r \le 2n|J|r. \tag{13.34}$$

Equations (13.30)–(13.33) reconstruct the argument in `Untied`, including the needed control of inertia for specializations that are characters only modulo ℓ^M.

Releasing the first prime. At $y = 0$, the evaluated twist is the original representation. The same tame argument applies at v_0, because $s_{v_0} = 0$, $M \le N$, and $\sum_{j<\ell^N} 1 = \ell^N = 0 \pmod{\ell^M}$. Suppose the two Frobenius actions at v_0 agree modulo ℓ^M with those of g_0 and $g_0^{-c} = (cg_0c)^{-1}$. Then the two tie values at v_0 lie in the kernels of $\rho(g_0) - I$ and $\rho(g_0^{-c}) - I$. Dropping the first-prime ties therefore costs at most

$$\delta_0 = v_O \det(\rho(g_0) - I) + v_O \det(\rho(g_0^{-c}) - I). \tag{13.35}$$

In particular, a bound

$$\operatorname{length}_O \mathcal{S}_{\text{all ties}}(0) \le b_{\mathrm{glob}} + b_{\mathrm{loc}}$$

implies

$$\operatorname{length}_O \mathcal{S}_{\mathrm{tied}}(0) \le \beta_{\mathrm{int}} := b_{\mathrm{glob}} + b_{\mathrm{loc}} + \delta_0. \tag{13.36}$$

These are the three terms defining β_T. The datum hypotheses make δ_0 finite. The cost is derived from the two evaluation maps, rather than introduced as an unspecified loss.

Theorem 13.7 (A polynomial with a small value)**.** *Assume the tied-prime hypotheses above and (13.36). Let r be a positive rational number with $r \le 1$ and $D_N(r) \ge M$, and let $s_0 \ge 2n|J|$. There exist a polynomial $P \in O[X_w : w \in J]$ and a point x^0 in the integers of a finite extension of $\mathbb{Q}_\ell$ such that*

$$\begin{aligned} v(x_w^0) &\ge r \quad (w \in J), \\ v(P(x^0)) &\le v(\varpi)\beta_{\mathrm{int}} + s_0 r, \end{aligned} \tag{13.37}$$

and, for every finite coefficient extension and every integral point x,

$$\operatorname{len}_v \mathcal{S}(x) \le v(P(x)). \tag{13.38}$$

Here $\mathcal{S}(x)$ is the module with the original local relations and without any auxiliary ties.

Proof. Apply Lemma 13.6 to the module with the ties at $J \setminus \{v_0\}$, and select a polynomial $G(X)$ sharp at the point 0. Thus

$$v_O(G(0)) = \operatorname{length}_O \mathcal{S}_{\mathrm{tied}}(0) \le \beta_{\mathrm{int}}.$$

Restrict it to the specified line:

$$P_0(Y) = G((a_w Y)_w) \in O[Y].$$

Then $v(P_0(0)) \le v(\varpi)\beta_{\mathrm{int}}$, and P_0 is nonzero. It remains a length bound for the tied module at every point of this line.

There is a $y \in \overline{\mathbb{Q}}_\ell$ with $v(y) = r$ such that both P_0 and P_1 attain their Gauss valuations:

$$v(P_i(y)) = w_r(P_i) \quad (i = 0, 1).$$

For completeness, choose t with $v(t) = r$, substitute $Y = tU$, and divide each polynomial by a coefficient of least valuation. Its reduction is a nonzero polynomial over $\overline{\mathbb{F}}_\ell$. Choose a nonzero residue u outside the finite union of their root sets and lift it to a unit. Then $y = tu$ has the asserted property. All coefficients and the chosen algebraic elements lie in one finite coefficient extension.

Put $x^0_w = a_w y$. Choose, again by Lemma 13.6, a polynomial P sharp for the untied module $\mathcal{S}(x^0)$. This P has coefficients in the original ring O and gives (13.38) at every specialization. Using (13.33), the bound furnished by G, and (13.34), we obtain

$$\begin{aligned} v(P(x^0)) &= \operatorname{len}_v \mathcal{S}(x^0) \\ &\le v(P_0(y)) + v(P_1(y)) \\ &= w_r(P_0) + w_r(P_1) \\ &\le v(P_0(0)) + s_0 r \\ &\le v(\varpi)\beta_{\mathrm{int}} + s_0 r. \end{aligned}$$

Finally $a_w \in O$, so every coordinate of x^0 has valuation at least r.

Lean, comparison. Lean uses $Nr \ge M$. The exact Gauss valuation in (13.29) gives the same tame relations at smaller depth. `exists_boundsSelmer_ellVal_eval_le`. □

At a tuple of ℓ^N-th roots of unity, the matrix module is exactly the normalized Selmer module of the corresponding character twist. Equation (13.38) therefore supplies the finite-level polynomial inequality used in (13.12). The strict threshold in that equation then passes the bound to the divisible coefficients.

Dependence on the auxiliary exponent. The common quotient and the matrix may depend on the characters, on N, and on the auxiliary primes. Specialization at roots of unity is uniform in the sizes of the matrices and the degrees of their minors.

For finitely many data, suppose a preceding construction gives, for every sufficiently large N, a common tied-prime set with $|J| \le s$, the short tied-length bounds, and the first Frobenius congruences used in (13.35). Choose $c \ge 1$ bounding all ranks and set $s_0 = 2c(1 + s)$. This choice is independent of N and of the radius r. For each $0 < r \le 1$, choose N large enough that $D_N(r) \ge M$. Theorem 13.7 then gives the polynomials and the small-value points for all these data, with that same s_0, and with exactly the original local relations.

This proves the polynomial bound for Selmer lengths from Theorem 13.4. To control the special values we use the norm relations of a Stickelberger system.

13.3 The special-value polynomial from its norm relations

The norm relations of a Stickelberger system give a polynomial bound for its special values. The existence of the systems, their integrality, and their identification with the required special values are hypotheses of this argument; they are not consequences of polynomial specialization.

Continue to write $\mathcal{O}$ for the valuation ring of $K = \overline{\mathbb{Q}}_\ell$, with $v(\ell) = 1$. For each auxiliary set Q, let Cl^1_Q be the group of norm-one finite ideles modulo principal norm-one ideles and the integral units congruent to 1 at the places in Q. The only group-ring properties used below are finite support and evaluation under characters factoring through this quotient. Suppose that

$$\Theta_Q \in \mathcal{O}[\mathrm{Cl}^1_Q]$$

are given, together with a continuous unit-valued character $\vartheta : G_E \longrightarrow \mathcal{O}^\times$ and a list $e_1, \dots, e_r \in \mathbb{Z}$. For an auxiliary prime u of norm q_u, choose a Frobenius element ϕ_u, and define

$$E_u(a) = \prod_{k=1}^{r} (1 - a q_u^{-e_k})(1 - a^{-1} q_u^{-e_k}).$$

The required norm relation is

$$\psi(\Theta_{Q \cup \{u\}}) = \psi(\Theta_Q) E_u\big(\vartheta(\phi_u)\psi(\phi_u)\big), \tag{13.39}$$

whenever $u \notin Q$, the enlarged set is auxiliary, and ψ comes from the norm-one torus with modulus Q and is trivial above ℓ. This is the exact relation in the definition of a Stickelberger system. Put

$$b = \vartheta(g_0), \qquad A_0 = 1(\Theta_\varnothing), \qquad \beta = v(A_0) + 2r\, v(b-1),$$

and assume $b \neq 1$ and $A_0 \neq 0$. Thus $\beta < \infty$.

Specializing the group-ring element. Let J be a nonempty tied auxiliary set, with first prime u_0, and let $h_u : G_E \longrightarrow \mathbb{Z}/\ell^N\mathbb{Z}$ be its characters. For each class c occurring in Θ_J, choose a finite idele a'_c such that $a'_c/\bar{a}'_c$ represents c, using Hilbert 90 for finite ideles, and choose a Galois lift g_c of $\mathrm{Art}(a'_c)$. Write $[h_u(g_c)]$ for its representative in $\{0, \dots, \ell^N - 1\}$. If $\Theta_J = \sum_c a_c[c]$, define

$$V_J((X_u)_u) = \sum_c a_c \prod_{u \in J} (1 + X_u)^{[h_u(g_c)]}. \tag{13.40}$$

This is an integral polynomial. Before specialization the expression $\prod_u (1 + X_u)^{[h_u(g)]}$ need not be a character of G_E. At every tuple $\zeta_u^{\ell^N} = 1$, however, it is exactly the character $\psi(g) = \prod_u \zeta_u^{h_u(g)}$. Consequently

$$V_J((\zeta_u - 1)_u) = \psi(\Theta_J). \tag{13.41}$$

The right side is independent of the chosen representatives, since ψ factors through Cl_J^1. This proves the specialization formula for the group-ring element.

Lemma 13.8 (A cyclic congruence gives a small-value point)**.** *Let $0 < \rho \leq 1$ be rational, and let $P, R \in \mathcal{O}[X]$. Suppose j, N are nonnegative integers and*

$$(1+X)^{\ell^N} - 1 \;\mid\; (1+X)^j P - R, \qquad w_\rho(R) < \rho + D_N(\rho).$$

Then there is $y \in \mathcal{O}$, with $v(y) = \rho$, for which

$$v(P(y)) = w_\rho(R).$$

Proof. Take t with $v(t) = \rho$. Divide $R(tX)$ by a coefficient of least valuation and choose a unit z whose residue avoids the roots of this nonzero reduction in $\overline{\mathbb{F}}_\ell$. Then $y = tz$ satisfies $v(R(y)) = w_\rho(R)$. By (13.29), $v((1+y)^{\ell^N} - 1) \geq \rho + D_N(\rho) > v(R(y))$. The congruence and the fact that $1 + y$ is a unit give $v(P(y)) = v(R(y))$.

Lean, comparison. The cyclic Gauss valuation replaces the iterated lower bound in `StickelbergerLine` (L240-L270). □

The first-prime value. Assume that $q_{u_0} \equiv 1 \pmod{\ell^N}$ and that an integer M satisfies

$$\beta + 1 < M \le N, \qquad v(\vartheta(\phi_{u_0}) - b) \ge M. \tag{13.42}$$

For every integer e, one has $v(q_{u_0}^{-e} - 1) \ge N$. If $r > 0$, then $v(b-1) < M$, and the ultrametric equality gives

$$v(1 - \vartheta(\phi_{u_0}) q_{u_0}^{-e}) = v(b-1), \qquad v(1 - \vartheta(\phi_{u_0})^{-1} q_{u_0}^{-e}) = v(b-1).$$

For the inverse use $v(\vartheta(\phi_{u_0})^{-1} - b^{-1}) = v(\vartheta(\phi_{u_0}) - b)$ and $v(b^{-1} - 1) = v(b-1)$, since both numbers are units. Relation (13.39), at the trivial character, now gives

$$v(1(\Theta_{\{u_0\}})) \le \beta. \tag{13.43}$$

The assertion is also immediate when $r = 0$, as the Euler product is then 1.

The one-variable norm relation. Assume the tied-prime identities

$$h_{u_0}(\phi_u) = 1 \quad (u \in J \setminus \{u_0\}), \qquad q_u \equiv 1 \pmod{\ell^N} \quad (u \in J).$$

Let $P(X)$ be the restriction of V_J to the coordinate line $X_{u_0} = X$, with all other coordinates zero. Let $R_0(X)$ be the analogous polynomial for $\Theta_{\{u_0\}}$ and the character h_{u_0}. Then $R_0(0) = 1(\Theta_{\{u_0\}})$. For $u \ne u_0$, put $\alpha_u = \vartheta(\phi_u)$ and

$$R_u(X) = \prod_{a=1}^{r} (1 - \alpha_u q_u^{-e_a}(1+X))(1 + X - \alpha_u^{-1} q_u^{-e_a}).$$

Both coefficients of X in the paired linear factors are units. Multiplicativity of the Gauss valuation gives

$$w_\rho(R_u) \le 2r\rho.$$

Moreover, for every nonzero integral ζ,

$$R_u(\zeta - 1) = \zeta^r E_u(\alpha_u \zeta).$$

Set $R_1 = \prod_{u \in J \setminus \{u_0\}} R_u$ and $j = r(|J| - 1)$. For any ℓ^N-th root ζ, the character $\zeta^{h_{u_0}}$ is supported at u_0 and its conjugate. Applying (13.39) successively at the other primes, where its Frobenius value is ζ, gives

$$\zeta^j P(\zeta - 1) = R_0(\zeta - 1) R_1(\zeta - 1).$$

The ℓ^N numbers $\zeta - 1$ are the distinct roots of the monic polynomial $(1+X)^{\ell^N} - 1$. Monic polynomial division over $\mathcal{O}$, followed by the root bound over K for the remainder, proves

$$(1+X)^{\ell^N} - 1 \;\mid\; (1+X)^j P - R_0 R_1. \tag{13.44}$$

This argument is valid over the valuation ring $\mathcal{O}$, although it is not Noetherian; only division by a monic polynomial has been used.

Theorem 13.9 (A valuation bound for the special-value polynomial)**.** *Suppose the preceding system and tied-prime data are given. Let $s_0 \ge 1$, $\rho > 0$ rational, and M, N be integers such that*

$$2r|J| \le s_0, \qquad s_0\rho < 1, \qquad \beta + 1 < M \le D_N(\rho), \qquad v(\vartheta(\phi_{u_0}) - b) \ge M.$$

There is a point $y = (y_u)_{u \in J}$ satisfying

$$v(y_u) \ge \rho \quad (u \in J), \qquad v(V_J(y)) \le \beta + s_0\rho. \tag{13.45}$$

At every tuple of ℓ^N-th roots of unity, (13.41) gives the exact value of the same Stickelberger element under the specialized character.

Proof. The assumptions imply $0 < \rho < 1$ and $M \le N$, so (13.43) applies. Thus

$$w_\rho(R_0) \le v(R_0(0)) \le \beta, \qquad w_\rho(R_1) \le 2r(|J|-1)\rho \le s_0\rho.$$

Consequently

$$w_\rho(R_0R_1) \le \beta + s_0\rho < \beta + 1 < M \le D_N(\rho).$$

Apply Lemma 13.8 to (13.44). Take the resulting scalar as the u_0-coordinate of y, and put every other coordinate equal to zero. Their valuations are $+\infty$, and evaluation on this line is exactly evaluation of P. This proves (13.45).

Lean, formal context. `StickelbergerLine` (L372-L535), `FitTwistsExist` (L377). □

For a finite family of systems the same set of tied primes works. Choose $M > \max_k(\beta(Y_k) + 1)$ first. Continuity of the finitely many characters gives one neighborhood of g_0 on which all the required congruences modulo valuation M hold. Choose s_0 using the uniform bound for $|J|$ and the finitely many lengths r_k, then take N large enough that $M \le D_N(\rho) \le N$. The points y_k may differ from one system to another; the subsequent simultaneous specialization theorem already allows this. No common small-value point is assumed at this intermediate stage.

This proves the system-polynomial construction in `StickelbergerLine`, including its norm congruence, the point of small valuation, and evaluation at every torsion character. The remaining application-specific input is the construction of the integral systems Θ_Q with (13.39) and the prescribed special-value interpretation.

13.4 The finite primes and their common polynomial

Theorem 13.10 (Simultaneous twists for finitely many Selmer data)**.** *Fix the finite families and the common element g_0 of the construction on p. 174. Assume all the Selmer-datum hypotheses stated at the start of the subsection on p. 177, including (13.15)–(13.17), the finiteness of the two determinants in (13.35), and the integral system hypotheses in (13.39), with their stated nonzero base values. Assume also the displayed residue-field hypothesis at the fixed torsion level M_{tor}. Then the three assertions about the auxiliary primes and polynomials hold. Consequently the simultaneous character construction above gives, for every $m \ge 1$ and every $\epsilon > 0$, one twist whose first m powers satisfy all the stated Selmer, system-value and tame-ramification bounds.*

Proof. Keep $M_{\mathrm{tor}} > b_i + 1$ and the initial coefficient ring fixed. Choose a second integer $M_1 \ge M_{\mathrm{tor}}$, with $M_1 > c_k + 1$ for every system. Only M_{tor} enters the residue-field condition and the Selmer descent potential; M_1 is an additional precision for the system values. By continuity, choose one neighborhood W of g_0 such that for every $g \in W$, every datum i, and every system k,

$$\rho_i(g) \equiv \rho_i(g_0) \pmod{\ell^{M_{\mathrm{tor}}}}, \qquad \rho_i(g^{-c}) \equiv \rho_i(g_0^{-c}) \pmod{\ell^{M_{\mathrm{tor}}}},$$
$$v(\vartheta_k(g) - \vartheta_k(g_0)) \ge M_1.$$

Neither M_1 nor W depends on the later radius or character depth.

Apply Theorem 13.4 at level M_{tor}. It gives an integer s, independent of N, and for every $N > M_{\mathrm{tor}}$ a common tied-prime set J with $|J| \le s$, first Frobenius in W, and

$$\operatorname{length}_A \mathcal{S}_i(J; \ell^{M_{\mathrm{tor}}}) \le G_{T_i} + L_{T_i}.$$

The two first-Frobenius congruences and (13.35) show that the module with ties only at $J \setminus \{P_0\}$ has length at most

$$\beta(T_i) = G_{T_i} + L_{T_i} + v_A \det(\rho_i(g_0) - I) + v_A \det(\rho_i(g_0^{-c}) - I).$$

Thus its normalized bound is exactly b_i.

Choose an integer $c \geq 1$ at least every representation rank and every number of Euler-factor exponents in a system, and put

$$s_0 = 2c(1+s).$$

This integer is fixed before the radius ρ and before N. For any positive rational ρ with $s_0\rho < 1$, and every integer $N > M_{\mathrm{tor}}$ satisfying $D_N(\rho) \geq M_1$, proceed as follows. Construct the tied primes at this depth. The preceding inequalities give

$$2|J| \leq s_0, \qquad 2n_i|J| \leq s_0, \qquad 2r_k|J| \leq s_0.$$

Theorem 13.7, applied to each datum, therefore gives P_i and a point x_i with

$$v(P_i(x_i)) \leq b_i + s_0\rho, \qquad v((x_i)_u) \geq \rho,$$

and bounds the length of the finite Selmer module at every specialization. Theorem 13.9, using $M = M_1$, gives the corresponding V_k, y_k, with

$$v(V_k(y_k)) \leq c_k + s_0\rho, \qquad v((y_k)_u) \geq \rho.$$

Equation (13.41) identifies every torsion evaluation with the value of the system at that character. The auxiliary-character construction supplies all the ramification and norm-one conditions of clause 1.

Finally fix $m \geq 1$ and choose $0 < \epsilon' < \min\{\epsilon, 1\}$. Choose a positive rational ρ with $s_0\rho \leq \epsilon'/2$. Apply (13.9) with error $\epsilon'/2$ and the fixed finite list of upper bounds $b_i + 1, c_k + 1$. This chooses the character-order exponent H before N, without using matrix sizes or polynomial degrees. Now choose $N \geq H$, $N > M_{\mathrm{tor}}$, and $D_N(\rho) \geq M_1$, and take the primes and polynomials constructed above. The specialized bounds are at most $b_i + \epsilon'$ and $c_k + \epsilon'$, hence at most the requested bounds with ϵ. The strict inequality $b_i + \epsilon' < M_{\mathrm{tor}}$ permits the torsion-to-divisible comparison (13.12). The ramification calculation (13.13) applies to this same character and all its first m powers.

Lean. `exists_twistPolynomials`, `exists_anticyclotomic_twist`. □

To apply the theorem we need Galois lattices satisfying the stated local conditions and integral measures interpolating the special values. We turn first to the measures.

13.5 Integral measures and special values of twists

We construct the systems of Theorem 13.9 from twist measures. The construction assumes integrality, a nonzero trivial-character value and special-value interpolation, and proves the integral group-ring relations and ratio bound. Corollary 18.36 constructs the systems at every coefficient prime for $z \geq 2$, and Corollary 18.47 proves the endpoint case $z = 1$ at every prime. The geometric families also admit one common complex normalization integral at all embeddings, by Corollaries 18.34 and 18.47. These normalizations do not change the vacuum scalar in the theta-chain height argument.

Interpolation at a fixed modulus. Let E be the CM field, put $F = E^+$, and write c for complex conjugation. The norm-one torus has adelic points

$$T(\mathbb{A}) = \{u \in \mathbb{A}_E^\times : u\bar{u} = 1\}.$$

Fix the isomorphism $\iota : \mathbb{C} \xrightarrow{\sim} K = \overline{\mathbb{Q}}_\ell$ used to define the avatars. It is a field isomorphism; no compatibility with the complex absolute value is asserted. Write $\mathcal{O}$ for the valuation ring of K, normalize $v(\ell) = 1$, and put

$$v_+(a) = \max\{0, v(a)\}, \qquad v(0) = v_+(0) = +\infty.$$

Thus v_+ is the source's `ellVal`; on integral elements it agrees with v.

Let S_{ch} be the chosen finite places, disjoint from their conjugates. A *critical pair* $p = (\nu, z)$ consists of an integer $1 \le z \le t = 5r$ and an automorphic character ν of $T(\mathbb{A})/T(F)$. Its weight at each infinite place σ is $-c_\sigma$, with c_σ an integer in $[z, 4b]$, and its tame residue characters are trivial off S_{ch}. The pair is *ramified* if its tame character is nontrivial at every chosen place. In this subsection fix a ramified critical pair.

Lean. `CriticalPair` (L268-L290), `ramificationatthechosenplaces` (L112-L116).

The finite bad set S_0 contains the chosen places, their conjugates, the places above ℓ, and the other fixed level places. A set Q is auxiliary if its places have prime norm, $Q \cap \bar{Q} = \varnothing$, and $(Q \cup \bar{Q}) \cap S_0 = \varnothing$. Let $\mathcal{T}(Q)$ denote the weight-zero tame characters of $T(\mathbb{A})/T(F)$ with residue characters supported on Q. The modulus Q is part of the datum: it need not be the minimal conductor. Write $\mathcal{T}^{\mathrm{prim}}(Q)$ for those with nontrivial residue character at every place of Q, and put $NQ = \prod_{u \in Q} q_u$.

For $\xi \in \mathcal{T}^{\mathrm{prim}}(Q)$ and $\delta \in \{1, -1\}$, let

$$R_{p,\delta}(\xi) = \frac{\left| L^{S_{\mathrm{ch}} \cup Q}(z, (\nu \xi^\delta)^{-1}) \right|}{|L^{S_{\mathrm{ch}}}(z, \nu^{-1})|}. \tag{13.46}$$

Here L is the Hecke L-function after base change to E, and superscript S means omission at $S \cup \bar{S}$. The untwisted denominator is nonzero. In the source's totalized division convention, this nonvanishing also follows by applying the interpolation identity below to the trivial twist and using $F_\varnothing(1) \neq 0$. The absolute values in (13.46) are complex absolute values; their nonnegative real ratio is subsequently viewed as a complex number and mapped by ι.

The hypothesis `CriticalPair.IsTwistMeasure` supplies a finite set $\mathcal{I} \subset \mathbb{Z}_{\ge 0}$, an integer $m \ge 0$, and values $F_Q(\tau)$ for modulus-labelled tame characters, with the following four properties:

1. $F_\varnothing(1) \neq 0$.

2. For each Q, there are finitely many points $u_{Q,j} \in T(\mathbb{A})$ and complex coefficients $\alpha_{Q,j}$, independent of τ, such that

$$F_Q(\tau) = \sum_j \alpha_{Q,j} \tau(u_{Q,j}) \qquad (\tau \in \mathcal{T}(Q)). \tag{13.47}$$

   The coefficients are integral at every prime away from NQ. In particular, at an auxiliary Q, $\iota(\alpha_{Q,j}) \in \mathcal{O}$.

3. If $u \notin Q$ and $Q \cup \{u\}$ is auxiliary, then, for the same underlying character τ at the two labelled moduli,

$$F_{Q \cup \{u\}}(\tau) = F_Q(\tau) \prod_{e \in \mathcal{I}} (1 - a q_u^{-e})(1 - a^{-1} q_u^{-e}), \qquad a = (\nu\tau)^E(\varpi_u). \tag{13.48}$$

   Here $\chi^E(y) = \chi(y/\bar{y})$ denotes base change, and ϖ_u is an idele supported at u with uniformizer component there.

4. At every primitive twist,

$$F_Q(\xi) = F_\varnothing(1)(NQ)^m R_{p,1}(\xi)^{m+1}. \tag{13.49}$$

It is essential that (13.47) holds for all characters at one modulus using the same coefficients and points. Separate integral expressions depending on the character would not give the required group-ring element. For the critical pairs in the application the supplied parameters are

$$(\mathcal{I}, m, i) := \begin{cases} (\{0,1\}, 1, 2), & z = 1, \\ (\{z\}, 0, 1), & z \geq 2, \end{cases} \qquad i = m + 1. \tag{13.50}$$

The two constructions are given in the case $z \geq 2$ and the case $z = 1$ and the combined existence theorem.

Avatars and the characters quantified in the norm relation. Fix $\delta \in \{1, -1\}$. Let ϑ_δ be the ℓ-adic avatar of the algebraic Hecke character $(\nu^{-\delta})^E$. It is continuous and unramified outside S_0. Its values are units: the norm of its compact image is a compact subgroup of $\mathbb{R}_{>0}$, hence is trivial. For a weight-zero character τ, write ψ_τ for the avatar of $(\tau^{-1})^E$. Use the reciprocity convention in which a local uniformizer maps to geometric Frobenius. The zero infinity type gives the exact formula, for every finite idele y,

$$\psi_\tau(\mathrm{Art}(y)) = \iota\big(\tau^E(y)\big)^{-1}. \tag{13.51}$$

Characters of G_E in this formula are read on its abelianization. The usual avatar correction at the places above ℓ is 1 because the infinity type is zero. For ϑ_δ, whose infinity type need not be zero, the corresponding uncorrected formula is used only on local ideles away from ℓ.

A Galois character ψ comes from T with modulus Q and is trivial above ℓ in the following sense. The character ψ is locally constant and there is a finite-idele character η, trivial on $E^\times$, such that

$$\psi(\mathrm{Art}(y)) = \eta(y/\bar{y}).$$

Moreover $\eta(a) = 1$ whenever a is a unit at all places away from ℓ and $a_u \equiv 1 \pmod{u}$ for $u \in Q$; the components above ℓ are unrestricted. This is the `IsFromU1` condition used in the source. In particular the restriction of η to norm-one finite ideles factors through Cl^1_Q. Local constancy on the compact group G_E gives finite image, so these character values are roots of unity. Consequently their evaluations on integral group-ring elements are integral, and $v_+ = v$ on every system value used in the polynomial construction.

Lemma 13.11 (Realizing the auxiliary character by a tame avatar)**.** *Every ψ satisfying these conditions is ψ_τ for a character $\tau \in \mathcal{T}(Q)$.*

Proof. Define $\mu(u) = \iota^{-1}(\eta(u_f))$ for $u \in T(\mathbb{A})$, where u_f is its finite component. This is trivial on $T(F)$, on the infinite torus, and on the integral norm-one ideles congruent to 1 at Q. It therefore defines a weight-zero tame automorphic character with modulus dividing Q. Its kernel contains the open tame-level subgroup, so it is continuous. Finite-idelic Hilbert 90 writes every u_f as $y/\bar{y}$; thus $\mu(u) = \iota^{-1}(\psi(\mathrm{Art}(y)))$, and μ has finite image because ψ does. Put $\tau = \mu^{-1}$, retaining the labelled modulus Q. For every finite idele y,

$$\iota\big(\tau^E(y)\big)^{-1} = \eta(y/\bar{y}) = \psi(\mathrm{Art}(y)).$$

Formula (13.51) identifies these two Galois characters on Artin images; density and continuity identify them on G_E. This is the ordinary reciprocity argument in the inverse-avatar formula and its converse.

Lean. `ClassSums` (L420-L476). □

Lemma 13.12 (The integral group-ring elements)**.** *The integral-measure property gives elements* $\Theta_Q \in \mathcal{O}[\mathrm{Cl}_Q^1]$ *such that*

$$A_0 := 1(\Theta_\varnothing) = \iota(F_\varnothing(1)) \neq 0, \qquad \psi_\tau(\Theta_Q) = \iota(F_Q(\tau^\delta)) \tag{13.52}$$

whenever Q *is auxiliary and* ψ_τ *satisfies the preceding conditions at modulus* Q*.*

Proof. For each $u \in T(\mathbb{A})$, finite-idelic Hilbert 90 gives $y \in \mathbb{A}_{E,f}^\times$ with $y/\bar{y} = u_f$. One may use local Hilbert 90, choosing unit solutions at almost all places, to obtain this restricted-product idele. Then $u/\operatorname{divStar}(y)$ has finite component 1, so every weight-zero tame character satisfies

$$\tau(u) = \tau^E(y).$$

The choice of y works simultaneously for all such characters, since they are trivial on the infinite torus. Choose $y_{Q,j}$ in this way for the points in (13.47). Define

$$\Theta_Q = \sum_j \iota(\alpha_{Q,j})[\mathrm{divStarClass}_Q(y_{Q,j}^{-\delta})]. \tag{13.53}$$

All coefficients are integral, and the sum is finite. For nonauxiliary Q set $\Theta_Q = 0$, since neither the asserted evaluation nor the norm relations concern those moduli.

To evaluate a class represented by $y/\bar{y}$, choose any Galois lift of $\mathrm{Art}(y)$. The `IsFromU1` condition proves independence of that choice and of the representative: the value is precisely the induced character of Cl_Q^1. Using (13.51) gives

$$\begin{aligned}\psi_\tau\Big(\mathrm{divStarClass}_Q(y_{Q,j}^{-\delta})\Big) &= \iota(\tau^E(y_{Q,j}^{-\delta}))^{-1} \\ &= \iota(\tau(u_{Q,j})^\delta).\end{aligned}$$

Thus evaluation of (13.53) is $\iota(F_Q(\tau^\delta))$. The minus sign in $y_{Q,j}^{-\delta}$ is forced by the inverse in the avatar formula. At the trivial character the same calculation gives $A_0 = \iota(F_\varnothing(1))$; the measure's nonvanishing property proves $A_0 \neq 0$. In particular A_0 is integral. This reconstructs the finite-idele and group-ring construction.

Lean. `LValueSystems` (L155-L230). □

Lemma 13.13 (Euler relations and the first inertia element)**.** *The data* $\vartheta_\delta, \mathcal{I}, \Theta_Q$ *satisfy the norm relation* (13.39)*. If* w *is a chosen place away from* ℓ *and* g_0 *is its tame generator, then*

$$b := \vartheta_\delta(g_0) \neq 1, \qquad b^{q_w - 1} = 1, \qquad v(b-1) \leq v(q_w - 1). \tag{13.54}$$

Proof. Let $u \notin Q$ with $Q \cup \{u\}$ auxiliary, and let ϕ_u be arithmetic Frobenius. For a character ψ in the norm relation, use Lemma 13.11 to write $\psi = \psi_\tau$ with $\tau \in \mathcal{T}(Q)$. Both base-change characters are unramified at u, and $u \nmid \ell$. Since arithmetic Frobenius is the inverse of the geometric Artin image of a uniformizer,

$$\vartheta_\delta(\phi_u) = \iota(\nu^E(\varpi_u))^\delta, \qquad \psi_\tau(\phi_u) = \iota(\tau^E(\varpi_u)).$$

Consequently, with $a = (\nu\tau^\delta)^E(\varpi_u)$,

$$\vartheta_\delta(\phi_u)\psi_\tau(\phi_u) = \iota(a)^\delta.$$

The paired Euler factor $\prod_{e\in\mathcal{I}}(1 - tq_u^{-e})(1 - t^{-1}q_u^{-e})$ is unchanged by $t \mapsto t^{-1}$. Applying ι to (13.48) for τ^δ, and then (13.52), therefore gives exactly (13.39). Enlarging the labelled modulus preserves `IsFromU1`, so both evaluations are legitimate.

At w, take a local unit whose residue γ generates $\mathbb{F}_{q_w}^{\times}$ and whose local Artin image agrees with g_0 in the abelianization, in the convention used for the tame generator. If κ_w is the tame residue character of ν, then

$$b = \iota(\kappa_w(\gamma))^{-\delta}.$$

There is no ℓ-adic infinity-type correction because $w \nmid \ell$, and no contribution from a chosen conjugate place because $S_{\mathrm{ch}} \cap \bar{S}_{\mathrm{ch}} = \varnothing$. The character κ_w is nontrivial. As γ generates the residue units, its image is nontrivial and its order divides $q_w - 1$. This proves the first two assertions of (13.54).

For completeness, if $b \neq 1$ and $b^n = 1$, then $\sum_{k=0}^{n-1} b^k = 0$, whence

$$n = \sum_{k=0}^{n-1}(1 - b^k) = (1 - b)\sum_{k=1}^{n-1}\sum_{h=0}^{k-1} b^h.$$

The second factor is integral since b is a root of unity. Therefore $v(b-1) \leq v(n)$. Apply this with $n = q_w - 1$. The source counterparts are the tame-generator and Frobenius identities and the root-of-unity bound and the norm relation.

Lean. `LValueSystems` (L93-L153), `LValueSystems` (L232-L273). □

Specialization and cancellation of the base value. For a primitive ξ, the character ξ^{δ} remains primitive with the same modulus. Combining (13.52) with (13.49) gives the exact identity

$$\psi_{\xi}(\Theta_Q) = A_0(NQ)^m \iota(R_{p,\delta}(\xi))^i, \qquad i = m + 1. \tag{13.55}$$

Since Q is auxiliary, NQ is an ℓ-adic unit. It follows that

$$|\psi_{\xi}(\Theta_Q)|_{\ell} = |A_0|_{\ell}\,|\iota(R_{p,\delta}(\xi))|_{\ell}^{i}. \tag{13.56}$$

For $z = 1$ the exponent is 2, and the factor NQ is removed only from this norm identity; it is present in the exact identity (13.55). This is the interpolation and construction of the system.

Lemma 13.14 (Cancellation with the truncated valuation)**.** *Suppose $A \neq 0$ is integral, $i \geq 1$, $C \geq 0$, and $|x|_{\ell} = |A|_{\ell}|r|_{\ell}^{i}$. If*

$$v_+(x) \leq v(A) + C,$$

then $i\,v_+(r) \leq C$, and hence $v_+(r) \leq C$.

Proof. If $C = +\infty$ the assertion is immediate, so suppose it is finite. The hypothesis forces $x \neq 0$ and then $r \neq 0$. If $v(r) < 0$, the conclusion is immediate because $v_+(r) = 0$. If $v(r) \geq 0$, the norm identity and integrality of A give

$$v_+(x) = v(x) = v(A) + i\,v(r) = v(A) + i\,v_+(r).$$

Subtract the finite number $v(A)$ from the given bound. Finally $i \geq 1$. Thus no assumption that the normalized ratio itself is integral is needed.

Lean, formal context. `LValueSystems` (L275-L363), `FitTwistsExist` (L377). □

Theorem 13.15 (The special-value bound from the interpolation formula)**.** *Assume the twist-measure hypotheses above with the parameters (13.50), and let w be a chosen place away from ℓ. For each sign δ, the construction gives an integral Stickelberger system with first element g_0, base value $A_0 \neq 0$, and*

$$\beta = v(A_0) + 2|\mathcal{I}|\,v(b-1) \leq v(A_0) + 4v(q_w - 1).$$

For every primitive auxiliary twist ξ whose inverse avatar satisfies `IsFromU1`*, and every $c \geq 0$, one has*

$$v_+(\psi_{\xi}(\Theta_Q)) \leq \beta + c \quad \Longrightarrow \quad v_+\big(\iota(R_{p,\delta}(\xi))\big) \leq 2v(1-b) + c/i \leq 4v(q_w - 1) + c. \tag{13.57}$$

Proof. The preceding lemmas give each defining property of the system: integrality, continuity and unramifiedness of ϑ_δ, $b \neq 1$, nonzero A_0, and the norm relation for every quantified character. By (13.50), $i = |\mathcal{I}|$ is 1 or 2. The local norm identity (13.56) and Lemma 13.14 give $i\, v_+(\iota R_{p,\delta}(\xi)) \leq 2i\, v(1-b) + c$: the factor A_0 contributes the same valuation on both sides. Now use (13.54).

Lean. `LValueSystems` (L365-L407). The code bounds $v(1-b)$ by $v(q_w - 1)$ and drops the divisor i. Keeping them gives the first inequality in (13.57). □

Theorem 13.15, followed by Theorem 13.9 and simultaneous specialization, gives the special-value bound for this measure. Theorem 18.5 identifies the exact condition at a fixed coefficient embedding: a Fourier-coefficient lower bound uniform in the auxiliary modulus. Corollary 18.36 supplies this condition at every prime for $z \geq 2$, using the theta-lift distribution and its convergent norm identity. Corollary 18.47 supplies $z = 1$ at every prime, using the determinant-period comparison proved in Theorem 18.45.

The ramified critical values used in the determinant. Write $N = q_2(s_0)$ for the order of the tame character $\chi_{M,w}$ at a chosen old place w (Lemma 17.92). The parameters of Theorem 17.7 satisfy $2^{s_0}5^{s_0} \mid N$ and $2m < 2^{s_0}$. For $1 \leq |D| \leq 2m$, the order of $\chi_{M,w}^D$ is $N/\gcd(N, |D|)$, still divisible by both 2 and 5. For each ℓ, choose w over an old prime different from ℓ. If the tame character at w is $\chi_{M,w}^D$, then $b = \iota(\chi_{M,w}^D(\gamma))^{-\delta}$ has this same order for a generator γ of the residue units, since $\delta = \pm 1$. Hence $v_\ell(1-b) = 0$ for every coefficient prime by the cyclotomic unit formula [Was97, Ch. 2], proved in Lemma 17.37.

The powers D in the chain are $j - i$ for $1 \leq i < j \leq m$; the squares of the comparison grid have $D = i + j + 2$ for $0 \leq i, j < m$ by Proposition 24.26. The three characters in (21.75) have $D = j, j, -j$, respectively, for $1 \leq j \leq m$, since the carrier and the powers of υ are trivial on old residue units, and the twisting modulus is disjoint from the old level. In that exchange product we use this estimate for the $5m$ ramified pairs counted in (25.32). The $k/2$ main pairs use (25.31). Include also the endpoint-2 pair $(\chi_M \upsilon^{-2}, 2)$ used in testing that estimate. It has old power $D = 1$, so the same mixed-order argument applies.

In Theorem 13.10 include all these ramified critical pairs in the same finite list as the Selmer data and use error $2^{-\ell}$ at $\iota : \mathbb{C} \simeq \overline{\mathbb{Q}}_\ell$. The resulting one twist θ_ι meets each system's own threshold β with that error at all its required powers. Equation (13.57) therefore gives $v_+(\iota R_{p,\delta}(\theta_\iota^j)) \leq 2^{-\ell}/i$, where $i = |\mathcal{I}|$. With global multiplier 1, the inverse of each such ratio family has height at most

$$C_{\mathrm{crit}} = \sum_{\ell \text{ prime}} 2^{-\ell} \log \ell < 1. \tag{13.58}$$

Indeed $\log n < n - 1$ for $n \geq 2$, and $\sum_{n \geq 2}(n-1)2^{-n} = 1$.

For any fixed critical pair ramified at all the chosen old places, the same specialization gives height at most $4 \log 2 + C_{\mathrm{crit}} < 4$ when the pair is included in its finite list. Fix $w_0 \mid p_0$ and $w_1 \mid p_1$ among these places. Use w_0 when $\ell \neq p_0$, and w_1 when $\ell = p_0$. The order of the tame character at w_0 is independent of the embedding, so it can be a power of at most one coefficient prime. The use of w_1 contributes at most one more. At either exception of order ℓ^a, the local term $2v_\ell(1-b)\log \ell$ equals $2 \log \ell/\varphi(\ell^a) \leq 2 \log 2$, since $\log x/(x-1)$ decreases for $x > 1$.

Lean. `LValueSystems` (L385-L407), `FitTwistsExist` (L377-L399). The mixed order of the old tame character makes its first inertia factor a unit before the simultaneous specialization.

13.6 From relation-preserving cocycles to bounded determinants

We bound the determinants used to construct the common Ribet polynomial. Throughout this subsection, O is a discrete valuation ring, K its fraction field, ϖ a uniformizer, and $v_O(\varpi) = 1$.

Length always means O-module length. In particular, the length bounds below are not dimensions after reduction modulo ϖ. The proofs apply to the modules of relations and cocycles, including their torsion.

A block determinant from its divisible annihilator.

Lemma 13.16 (The divisible block determinant bound)**.** *Let A be an associative unital O-algebra, let $\psi : A \twoheadrightarrow \mathrm{Mat}_r(O)$ be a surjective O-algebra homomorphism, and let $N \subseteq A^d$ be a left A-submodule. Define*

$$Z_N = \left\{ (w_j)_{j=1}^d \in ((K/O)^r)^d : \sum_j \psi(t_j) w_j = 0 \quad (t \in N) \right\}.$$

If $\lambda_O(Z_N) < \infty$, there is a matrix $B \in \mathrm{Mat}_d(A)$ whose rows belong to N such that its dr by dr block image satisfies

$$v_O \det \psi(B) = \lambda_O(Z_N).$$

The empty block case has determinant 1 and length zero.

Proof. Let $L \subseteq O^{dr}$ be the span of all individual rows of the horizontal block matrices $(\psi(t_1) \ \cdots \ \psi(t_d))$, for $t \in N$. Every matrix with r rows in L is the image of an element of N. Indeed, to place a particular row of such an image in row position a, with all other rows zero, multiply t on the left by a lift under ψ of a suitable matrix unit. Linear combinations produce any chosen row of L, and summing these constructions produces any ordered r-tuple of rows of L.

The ordinary dot product identifies

$$Z_N = \mathrm{Hom}_O(O^{dr}/L, K/O).$$

By the elementary-divisor theorem, a free summand of O^{dr}/L would give a copy of K/O in this dual, of infinite length. Thus L has full rank. If its elementary divisors are $\varpi^{a_1}, \dots, \varpi^{a_{dr}}$, the quotient and its dual both have length $\sum_i a_i$. Choose an O-basis of L, and group its dr rows into d groups of r. The preceding matrix-unit argument lifts each group to a row of a matrix B with rows in N. Its block determinant has valuation $\sum_i a_i$, as required.

Lean, formal context. `BlockFittingBound` (L157), `FitTwistsExist` (L377). □

This proves the required specialization of `exists_rows_addVal_det_le_of_divisible` directly by lattice duality. In particular there is no extra factor of r in the bound.

The cocycle module as an annihilator. Let G be a group and let $u_x : G \to \mathrm{GL}_n(O)$ be a finite family. Write $U(g) = (u_x(g) - 1)_x$. Choose elements $\gamma_1, \dots, \gamma_d$ such that the $U(\gamma_i)$ generate the O-span of all $U(g)$, and choose coordinates

$$U(g) = \sum_i z_i(g) U(\gamma_i), \qquad z_i(g) \in O.$$

Such a finite generating subfamily exists because the ambient module of matrix families is finite over the noetherian ring O. For a finite set J, put $R = O[X_j : j \in J]$ and $A = R[G]$. For $t \in A^d$ write

$$\mathrm{rv}(t) = \sum_i t_i(\gamma_i - 1).$$

Let $\mathcal{I}$ be a left ideal containing the common kernel of the actions u_x, the elements $p(g-1)$, the prescribed ties

$$\tau_j - 1 - X_j \epsilon_j(\phi_j)(\phi_j - 1)$$

(and any second family of such ties), and every $(g-1)\rho_0$. Here $p \neq 0$, and the ϵ_j have coefficients in O. Define the tied-row module $N = \mathrm{rv}^{-1}(\mathcal{I})$. All assertions in the following lemma concern this module itself.

Lemma 13.17 (Main tied rows and normalized Ribet cocycles)**.** *Let $\rho : G \to \mathrm{GL}_r(O)$ have full matrix span, and suppose*

$$\rho(\gamma_*) = \zeta\, 1, \qquad c_1(\zeta - 1) = 1 \quad (c_1 \in O).$$

For any $a \in G$, take

$$n_* = c_1(\gamma_* - 1), \qquad \rho_0 = n_* + (1 - n_*)(a - 1).$$

Let $\psi : A \to \mathrm{Mat}_r(O)$ act by ρ and set all X_j to zero. The annihilator Z_N in Lemma 13.16 embeds into the module of ρ-cocycles $f : G \to (K/O)^r$ that preserve all O-linear relations among the $U(g)$ and vanish at γ_* and all the tied τ_j. Consequently a length bound b_0 for that actual cocycle module gives a square matrix of tied rows with

$$v_O \det \psi(B_0) \leq b_0.$$

Proof. Let I be the augmentation ideal of $A = R[G]$, the kernel of $\sum a_g g \mapsto \sum a_g$. Modulo $\mathcal{I}$, the coordinate identity gives $g - 1 = \sum_i z_i(g)(\gamma_i - 1)$, since the difference acts by zero under every u_x. The $g - 1$ generate I over R, so the row map induces $A^d/N \simeq I/(I \cap \mathcal{I})$. Equip $W = (K/O)^r$ with the A-action through ψ. By definition $Z_N = \mathrm{Hom}_A(A^d/N, W)$. For such a homomorphism λ put $f(g) = \lambda(g - 1)$, using the quotient just identified. The identity $gh - 1 = (g - 1) + g(h - 1)$ proves the cocycle law, and the values at γ_i recover the original row coordinates. This is the augmentation-ideal description of derivations [Bro82, Ch. IV, §2, Exercise 2].

Each relation $\sum_g a_g U(g) = 0$ gives $\sum_g a_g(g-1) \in I \cap \mathcal{I}$. The ties give $f(\tau_j) = 0$ since $\psi(X_j) = 0$. Also $\lambda(\rho_0) = c_1 f(\gamma_*)$ since $\psi(n_*) = 1$. Applying λ to $(\gamma_* - 1)\rho_0 \in \mathcal{I}$ gives $(\zeta - 1)c_1 f(\gamma_*) = 0$, so $f(\gamma_*) = 0$. The same unit $\zeta - 1$ makes normalized cocycles inject into cohomology. Finally ψ is onto by the full matrix span, and Lemma 13.16 gives the determinant bound.

Lean. `exists_tiedRows_addVal_det_le`. The code checks the coordinate rows separately. The augmentation quotient imposes the same relations on $g \mapsto g - 1$. □

These are the ordinary algebraic arguments in `exists_tiedRows_addVal_det_le`. Theorem 17.7 verifies the scalar and full-matrix-span hypotheses for every stable lattice whose rational representation has the selected tame type at a level prime different from the coefficient prime. The representation on the main summand and these structural properties are supplied in Theorem 17.87. The rank-one rows and full ambient family are supplied in Theorems 17.94 and 17.105.

The lattice generated by the matrix differences.

Lemma 13.18 (Ribet cocycles as equivariant maps)**.** *Let $M = \prod_x \mathrm{Mat}_n(O)$ with the action by left multiplication by the u_x, and let $I = \sum_{g \in G} O\, U(g)$. For any $O[G]$-module A, evaluation on $U(g)$ identifies $\mathrm{Hom}_{O[G]}(I, A)$ with the cocycles $G \to A$ preserving the O-linear relations among the $U(g)$. If O is the ring of integers of a finite extension of $\mathbb{Q}_\ell$, G is profinite and the u_x are continuous, then every such cocycle with values in $(K/O)^r$ has finite image and is continuous. It vanishes wherever all $u_x(g) = 1$.*

Proof. Since $hU(g) = U(hg) - U(h)$, the lattice I is G-stable. Preservation of relations says precisely that the assignment $U(g) \mapsto f(g)$ defines an O-linear map on I. The same identity makes equivariance equivalent to the cocycle identity. For $A = (K/O)^r$ the image of the finitely generated

I is finite and killed by a power of the uniformizer. The lattice topology on I is induced from M, by elementary divisors. Thus the linear map is continuous on I, and its composite with U is continuous and zero on $U^{-1}(0)$.

Lean, comparison. Lean extends the map to a graph at each depth in `ribetCocycles.exists_graph_subrepresentation`. Here equivariant maps on I describe the cocycles at every depth. □

At an auxiliary place of residue characteristic different from ℓ, if ρ is unramified, the restriction of f to inertia is a homomorphism to a finite ℓ-group. It kills wild inertia and the prime-to-ℓ tame factors. Vanishing at a generator of the tame pro-ℓ factor therefore makes it unramified at that place. The same argument applies at the conjugate place. Thus the tied tame conditions remove the auxiliary ramification from these actual continuous cocycles.

Lemma 13.19 (The main cocycle length from local defects)**.** *Let O be the ring of integers of a finite extension of $\mathbb{Q}_\ell$, and use the preceding normalization and tame conditions for continuous u_x on the profinite Galois group. Use the crystalline local conditions defined by the fixed homogeneous matrix relations in (15.13), as in the Selmer conditions. For each $m \geq 1$ let Z_m be the resulting relation-preserving cocycles with values in $(O/\ell^m)^r$. Suppose the local conditions above ℓ give a map*

$$Z_m \longrightarrow \bigoplus_{v|\ell} D_{v,m},$$

whose kernel consists of cocycles satisfying the original crystalline Selmer condition, and suppose

$$\lambda_O(D_{v,m}) \leq \Delta_v, \qquad \sum_{v|\ell} \Delta_v < \infty,$$

uniformly in m. Suppose that normalized crystalline Selmer classes in $(K/O)^r$ have length at most $s_T < \infty$. For tied rows with $p = \ell^\nu$, $\nu \geq 1$, it suffices to assume the finite-level local hypotheses at $m = \nu$ to bound $\lambda_O(Z_N)$. Under the uniform hypotheses the whole divisible-coefficient cocycle module of Lemma 13.17 has the same upper bound

$$b_0 = \sum_{v|\ell} \Delta_v + s_T.$$

Proof. Lemma 13.18 gives continuous cocycles, and the tame vanishing conditions remove the added primes. The kernel of the displayed local map therefore lies in the original finite-level Selmer module. Its map to the divisible Selmer group is injective on normalized cocycles: if the image is a coboundary, evaluating at γ_* and using the unit $\zeta - 1$ makes that coboundary zero. Corollary 15.13 proves that these finite and divisible local relation conditions are preserved and reflected by the natural coefficient inclusions. Hence

$$\lambda_O(Z_m) \leq \sum_{v|\ell} \Delta_v + s_T.$$

For each standard row vector e_i, $\mathrm{rv}(pe_i) = p(\gamma_i - 1) \in \mathcal{I}$, hence $pe_i \in N$. When $p = \ell^\nu$, every coordinate of Z_N lies in $(K/O)^r[\ell^\nu]$. The injection of Lemma 13.17 thus factors through Z_ν, using $O/\ell^\nu \simeq (K/O)[\ell^\nu]$. Its bound at this single level suffices for the determinant. For the general divisible module, Lemma 13.18 puts each finite set of cocycles in some Z_m; the same uniform length bound passes to their union.

Lean. `RibetElement` (L200-L223), `RibetMatrices` (L471-L486). The divisible bound treats every depth. In the determinant calculation the tied ideal annihilates the row coordinates at the prescribed depth. □

This is the length comparison in `exists_length_le_loss_add_crystallineSelmerLength`. For specified local classes containing the restrictions of the actual representations, Theorem 15.11 supplies these comparison maps by taking the quotients by the local relation conditions. Corollary 15.12 bounds their total colength by the sum of the dual cup-product defects. Theorem 17.40 verifies the chart, window, sign and sparsity conditions for the constructed data and proves the one-place bounds from the supplied rational representations and their Hodge–Tate and local type compatibilities. Corollary 17.88 applies these estimates to the rows from the main summand. Corollary 17.106 supplies the required class membership and the same allowance for the full ambient family. Continuity and unramifiedness away from ℓ alone do not imply the crystalline local condition.

13.7 The triangular Fitting argument for the character blocks

Here $R = O[X_j : j \in J]$, where J is finite and ordered, and S is a finite set of elements of G. Fix an O-linear map $\Phi : O^n \to R^S$ and vectors $z(g) \in O^n$ for $g \in G$. For an R-submodule $N \subseteq R^S$ with $\Phi(pO^n) \subseteq N$, $p \neq 0$, put

$$\mathcal{Z}_N = \{\Psi \circ z : \Psi \in \mathrm{Hom}_O(O^n, O/p),\ \Psi(\Phi^{-1}N) = 0\}.$$

This is a finite-length module of functions even when G is infinite: it is an image of a submodule of $(O/p)^n$. Write $\mathrm{Fit}(N)$ for the ideal generated by determinants of square matrices whose rows belong to N, equivalently $\mathrm{Fitt}_0(R^S/N)$.

The change in length on adding a relation. If $qv \in N$ for $q \in R$ and $v \in R^S$, then

$$q\,\mathrm{Fit}(N + Rv) \subseteq \mathrm{Fit}(N). \tag{13.59}$$

Indeed, expand a determinant of rows $n_i + a_i v$ multilinearly. Terms with two rows proportional to v vanish. In a term with one such row, multiplying by q puts that row into N; in a term with none, multiplying any row by q does the same. When S is empty both Fitting ideals are R, and the assertion is immediate.

Lemma 13.20 (Relative duality and the Fitting ideal)**.** *For any O-submodule V of the functions $G \to O/p$, there are $e \in O \setminus \{0\}$ and an R-submodule $K \supseteq N$ such that*

$$\begin{aligned} v_O(e) &\le \lambda_O(\mathcal{Z}_N/(\mathcal{Z}_N \cap V)), \\ e\,\mathrm{Fit}(K) &\subseteq \mathrm{Fit}(N), \end{aligned}$$

and K contains $\Phi(v)$ whenever every linear $\Psi : O^n \to O/p$ with $\Psi \circ z \in V$ kills v. The last condition ranges over all such Ψ, not just those annihilating $\Phi^{-1}N$.

Proof. Write $D = O^n/\Phi^{-1}N$ and $D^* = \mathrm{Hom}_O(D, O/p)$. Evaluation identifies $D = (D^*)^*$, and this dual preserves length. On a cyclic summand O/ϖ^a the dual is $(O/p)[\varpi^a] \simeq O/\varpi^a$, with a generator pairing of annihilator (ϖ^a). Take direct sums of these pairings. The map $D^* \to \mathcal{Z}_N/(\mathcal{Z}_N \cap V)$, $\Psi \mapsto [\Psi \circ z]$, is onto. Write J for its kernel. The annihilator $H \subseteq D$ of J is then the dual of D^*/J, so the sum of the valuations equals its length.

Choose lifts $x_i \in O^n$ of generators of the cyclic summands O/ϖ^{a_i} of H and set $K = N + \sum_i R\Phi(x_i)$. Every v in the assertion is killed by all the functionals in J, so its class belongs to H and $\Phi(v) \in K$. Since $\varpi^{a_i}\Phi(x_i) \in N$, successive applications of (13.59) give $\varpi^{\sum_i a_i}\,\mathrm{Fit}(K) \subseteq \mathrm{Fit}(N)$. The scalar has valuation $\sum_i a_i = \lambda_O(H)$. An empty sum gives $K = N$ and the scalar 1.

Lean. The determinant inclusion is `fittingIdeal.mul_mem_of_smul_mem`. The formal proof of `fittingIdeal.exists_addVal_le_relLength` adjoins one row at a time by Noetherian induction. □

The proof also explains why no independence assumption on the functions $z(g)$ is required. It is the relative length of their image, not the length of the entire space of linear forms, that bounds the contribution of the extra relations. This reconstructs `fittingIdeal.exists_addVal_le_relLength`.

Lemma 13.21 (Constant term of a Fitting element)**.** *Let $T \subseteq R^S$ contain $\Phi(pO^n)$ and satisfy*

$$\Phi(z(g)) - e_g \in T \quad (g \in S).$$

Put $\mathfrak{m}_X = (X_j : j \in J)$ and $T_0 = T + \mathfrak{m}_X R^S$. For $C \in O \setminus \{0\}$ there is $d \in \mathrm{Fit}(T)$ with

$$v_O(d(0)) \le \lambda_O(\mathcal{Z}_{T_0}/\mathcal{Z}_{T_0}[C]) + |S| v_O(C).$$

Proof. At $X = 0$ let $A = (R^S/T) \otimes_R O$. The map $O^n \to A$ induced by Φ sends $z(g)$ to e_g for $g \in S$, so it is onto with kernel $\Phi^{-1}T_0$. In particular $pA = 0$. Composition with $g \mapsto \Phi(z(g)) \bmod T_0$ identifies $\mathrm{Hom}_O(A, O/p)$ with $\mathcal{Z}_{T_0}$. Surjectivity follows from the definition of $\mathcal{Z}_{T_0}$, and the values on S determine the functional.

As in Lemma 13.6, specializing the presentation minors gives the zeroth Fitting ideal of A [Stacks, Lemma 15.8.4]. Over the discrete valuation ring O its valuation is $\lambda_O(A) = \lambda_O(\mathcal{Z}_{T_0})$, by the elementary divisors and $pA = 0$. Evaluation at zero is surjective on the coefficient ring, so a generator of this ideal is $d(0)$ for some $d \in \mathrm{Fit}(T)$. There are at most $|S|$ cyclic summands. Each contributes at most $v_O(C)$ to the length of $\mathcal{Z}_{T_0}[C]$, giving the inequality.

Lean. Lean adjoins the relations Ce_g before taking constant terms in `exists_mem_fittingIdeal_addVal_constantCoeff_le`. Here specialization gives the least possible valuation directly. □

Triangular relations on the character summands. Fix maps $\tau, \tau', \phi, \phi' : J \to G$, a subset $I \subseteq J$, $C \neq 0$, and a nonnegative integer b. Say that $\mathcal{Z}_N$ has triangular cost (I, C, b) if there is a set $L \subseteq G$ such that, for $Z_0 = \{f \in \mathcal{Z}_N : f|_L = 0\}$, the following hold. For $j \in I$, if $f \in Z_0$ satisfies

$$f(\tau_i) = f(\tau'_i)\ (i < j), \qquad f(\phi_i) = f(\phi'_i)\ (i < j,\ i \in I),$$

then $C(f(\tau_j) - f(\tau'_j)) = 0$. Moreover, for every submodule $Z_1 \subseteq Z_0$ whose functions vanish at every τ_j, τ'_j and agree at ϕ_j, ϕ'_j for $j \in I$, one has

$$\lambda_O(\mathcal{Z}_N/Z_0) + \lambda_O(Z_1/Z_1[C]) \le b. \tag{13.60}$$

This is the condition `HasTriangleCost` in Lean. Theorem 13.44 chooses auxiliary primes and inertia and Frobenius elements successively. Together with the local and odd-class bounds, they give the length inequality above.

Theorem 13.22 (The triangular Fitting bound)**.** *Suppose $\Phi(pO^n) \subseteq N$, $p \neq 0$, and*

$$\Phi(z(g)) - e_g \in N \quad (g \in S),$$

and suppose the two ties hold for every $j \in J$:

$$\Phi(z(\tau_j)) - X_j \Phi(z(\phi_j)) \in N, \qquad \Phi(z(\tau'_j)) - X_j \Phi(z(\phi'_j)) \in N.$$

Assume that for every $j \in I$ and $i < j$, either $i \in I$ or $X_i R^S \subseteq N$. If $\mathcal{Z}_N$ has triangular cost (I, C, b), then there is $F \in R$ with

$$F \prod_{j \in I} X_j \in \mathrm{Fit}(N), \qquad v_O(F(0)) \le b + (|I| + |S|) v_O(C). \tag{13.61}$$

Proof. Apply Lemma 13.20 with V the functions vanishing on L. It gives $K \supseteq N$ and e with

$$e\,\mathrm{Fit}(K) \subseteq \mathrm{Fit}(N), \qquad v_O(e) \le \lambda_O(\mathcal{Z}_N/Z_0).$$

For $g \in L$, the membership conclusion gives $\Phi(z(g)) \in K$. Consequently every $\mathcal{Z}_{K'}$, for $K' \supseteq K$, lies in Z_0.

For $j \in I$ in increasing order, adjoin the vector $v_j = \Phi(z(\phi_j) - z(\phi'_j))$. Let T' be the module before the jth addition. For each $i < j$, either v_i has already been adjoined, or $X_i R^S \subseteq N$. Hence $X_i v_i \in T'$ in both cases. Subtracting the two ties gives $\Phi(z(\tau_i) - z(\tau'_i)) \in T'$. Every linear form killing $\Phi^{-1}T'$ therefore gives a function in Z_0 satisfying all the preceding equalities in the triangular hypothesis. That hypothesis and finite-module duality imply

$$C\Phi(z(\tau_j) - z(\tau'_j)) \in T'.$$

Subtracting C times the difference of the ties shows $CX_j v_j \in T'$. Equation (13.59) proves that adjoining v_j costs exactly the factor CX_j. After all additions, the resulting module $T \supseteq K$ satisfies

$$C^{|I|}\left(\prod_{j\in I} X_j\right)\mathrm{Fit}(T) \subseteq \mathrm{Fit}(K).$$

Set $T_0 = T + \mathfrak{m}_X R^S$. The ties imply that every function in $\mathcal{Z}_{T_0}$ vanishes at every τ_j, τ'_j, and the adjoined v_j imply agreement at ϕ_j, ϕ'_j for $j \in I$. This module is contained in Z_0, so (13.60) applies to $Z_1 = \mathcal{Z}_{T_0}$. Lemma 13.21 gives $d \in \mathrm{Fit}(T)$ with

$$v_O(d(0)) \le \lambda_O(Z_1/Z_1[C]) + |S| v_O(C).$$

Take $F = eC^{|I|}d$. The two Fitting inclusions give the first assertion of (13.61), and the two relative lengths add to at most b, giving the second. This proves the theorem without changing the finite coefficient module O/p.

Lean, formal context. `FittingBlock` (L137), `FitTwistsExist` (L377). □

This proves `exists_mul_prod_X_mem_fittingIdeal`. The factors $|I|v_O(C)$ and $|S|v_O(C)$ have different origins: the first bounds the loss from the triangular relations, and the second bounds the loss from the final constant-term determinant. They will both be retained in the common Ribet valuation bound below.

13.8 Constructing the local first-order operator

We now construct the operator in (25.13). The spherical chain pairing follows from the classification of the spherical unitary dual and the Jacquet module at the line parahoric. The other local input, parahoric local–global compatibility for the automorphic representations in this construction, is supplied by Theorem 17.105.

Fix one auxiliary place, of residue field k and order q, and let $n \ge 1$ be the representation rank. Put

$$M = \ell^\nu, \qquad M \mid q - 1.$$

Here $\mathcal{O}$ is the ring of integers in $\overline{\mathbb{Q}}_\ell$, viewed through ι^{-1} as scalars on the complex automorphic forms. Let $\mathcal{E}$ be their ring of integral endomorphisms at the fixed principal auxiliary level. Congruences in this subsection are modulo $M\mathcal{E}$. The ring $\mathcal{E}$ is torsion free as an abelian group, since it is a subring of a complex endomorphism algebra.

Write $\Gamma = \mathrm{GL}_n(k)$, let P be the stabilizer of the first coordinate line, and write ρ for its action by integral operators. Choose a generator z of $k^\times$ and the homomorphism $\log_z : k^\times \to \mathbb{Z}/M\mathbb{Z}$ taking z to 1. It exists because $M \mid q-1$. Put

$$\alpha(p) = \log_z(p_{11}), \qquad \beta(g) = \log_z \det g, \qquad d = \mathrm{diag}(z, 1, \dots, 1).$$

Then $\beta(d) = 1$. Fix representatives r for the left cosets Γ/P, and put

$$C = \sum_r \rho(r), \quad C^\vee = \sum_r \rho(r^{-1}), \quad N_P = \sum_{p \in P} \rho(p), \quad N_\Gamma = \sum_{g \in \Gamma} \rho(g).$$

For an integral weight $f : \mathbb{Z}/M\mathbb{Z} \to \mathbb{Z}$, define

$$L_P(f) = \sum_{p \in P} f(\alpha(p))\rho(p), \qquad L_\Gamma(f) = \sum_{g \in \Gamma} f(\beta(g))\rho(g).$$

Let $\lambda : \mathbb{Z}/M\mathbb{Z} \to \mathbb{Z}$ be any lift of residue classes. Taking representatives of least absolute value gives the source's `valMinAbs`; no special property of that choice is needed in the congruences below.

Lemma 13.23 (Transfer of the first entry)**.** *The transfer to Γ of the character $p \mapsto p_{11}$ of P is* det*. Consequently the transfer of α is β.*

Proof. Identify Γ/P with the lines in k^n. The transfer formula [Neu99, Ch. VI, §7, proof of Lemma 7.7] groups its factors by orbits of g on these lines. An orbit of length a contributes the eigenvalue of g^a on its representative line. For $g = \mathrm{diag}(x, 1, \dots, 1)$ the first line contributes x. Every other line has a nonzero fixed coordinate outside the first, so its orbit contributes 1. Conjugacy gives the answer at every one-entry diagonal matrix. Both characters kill the transvections, whose orders are powers of the residue characteristic whereas $|k^\times| = q-1$. Diagonals and transvections generate Γ by row reduction, proving the equality. Compose with $\log_z$ to obtain the assertion for α.

Lean. `TiedTraces` (L130), `transfer_comp_firstEntry`. The specialization to the tame character follows from the transfer of the first entry. Here the orbit formula computes that transfer on the lines of k^n. □

Lemma 13.24 (Logarithms of finite-group norms)**.** *The following congruences hold modulo $M\mathcal{E}$:*

$$C L_P(\lambda) C^\vee \equiv L_\Gamma(\lambda), \qquad (\rho(d) - 1) L_\Gamma(\lambda) \equiv -N_\Gamma. \tag{13.62}$$

If an additive map $B : \mathbb{Z}^{\mathbb{Z}/M\mathbb{Z}} \to \mathcal{E}$ satisfies

$$B(f)\rho(p) = B\big(f(\,\cdot - \alpha(p))\big) \qquad (p \in P), \tag{13.63}$$

then for each $g \in \Gamma$ there is $R_g \in \mathcal{E}$ such that

$$B(\lambda) C^\vee (\rho(g) - 1) \equiv B(1) R_g \pmod{M\mathcal{E}}. \tag{13.64}$$

Proof. For each representative r, write $gr = r' p_{g,r}$ with $p_{g,r} \in P$. In the formal group-algebra expansion of $C L_P(\lambda) C^\vee$, the coefficient of g is $\sum_r \lambda(\alpha(p_{g,r}))$. Modulo M this is the transfer of α at g, hence $\beta(g)$, by Lemma 13.23. This proves the first congruence. Multiplication of a weighted norm by $\rho(g)$, on either side, replaces its weight $f(x)$ by $f(x - \beta(g))$. Since $\lambda(x-1) - \lambda(x) \equiv -1 \pmod M$, the second congruence follows.

Right multiplication of $C^\vee$ by $\rho(g)$ gives

$$C^\vee \rho(g) = \sum_r \rho(p_{g,r}) \rho(r^{-1}),$$

because r' runs through the same representatives. The shift identity and $\lambda(x-y)-\lambda(x)\equiv-\lambda(y)$ (mod M) therefore give (13.64) with

$$R_g=-\sum_r \lambda(\alpha(p_{g,r}))\rho(r^{-1}).$$

All coefficients are integers. For every additive map on integral weights, weights congruent modulo M have images congruent modulo $M\mathcal{E}$, since their difference is M times an integral weight. This justifies each reduction used above.

Lean, formal context. `TiedTraces` (L70-L112), `FitTwistsExist` (L377). □

Local compatibility and the resulting matrix relation. Let ϕ,γ be the geometric Frobenius and tame-inertia elements at this place, with the chosen local Artin convention. Use the matching local choice: a uniformizer ϖ and a unit u whose reciprocity symbols are the classes of ϕ and γ in the abelianized decomposition group, with residue z and $\log_z(z)=1$. The residue is primitive for a tame generator; the logarithm then exists because $M\mid q-1$. The uniformizer can be adjusted by a unit to match the chosen Frobenius class. These are precisely the local reciprocity choices in `LocalChoice`. This fixes the sign of both the tame character and the Hecke normalization. On a Hecke system x, write $F_x=N_x(\phi)$, $T_x=N_x(\gamma)$, and $M_x(g)=N_x(g)$; in formulas on forms these matrices and scalars are transported by ι^{-1}. The representation N_x is the prescribed twist of the attached representation, by a character trivial at γ. Write $F=E_v$ for the local field, and also write ρ for the full local translation representation on the ambient forms. Let U be the pro-unipotent Iwahori Hecke operator of $\mathrm{diag}(\varpi,1,\dots,1)$, with counting normalization, and let $\widetilde{U}=a(\phi)U$ be its prescribed scalar normalization.

We use two local compatibility statements on the actual projections. If the system x contains a determinant type $\psi\circ\beta$, then $T_x=\psi(1)I$, including the trivial character. If $\psi\neq1$ and its line type $\psi\circ\alpha$ occurs, decompose that projection into automorphic constituents over characteristic zero. On each such constituent there is a scalar u with

$$\widetilde{U}y=uy,\qquad (F_x-uI)(T_x-I)=0. \tag{13.65}$$

Indeed, after transport by ι, the unique ramified inertia line has the same normalized Frobenius eigenvalue as the Hecke operator on the line type (Lemmas 17.83 and 17.86). The twist is unramified there. For each constituent of the projection, Theorem 17.105 identifies its semisimple attached representation with N_x after the prescribed normalization. This uses local compatibility only on a nonzero projection. The actual ranks are at least two; in rank one the line and determinant types coincide and local reciprocity gives the same identity.

For $p\in\mathcal{O}[X]$, regard the following scalar functions as integral Hecke operators, as in the existing Hecke–Galois setup:

$$s_{p,g}(x)=\mathrm{tr}(p(F_x)M_x(g)),\qquad t_{p,g}(x)=\mathrm{tr}(p(F_x)(T_x-I)M_x(g)),\qquad t_g=t_{1,g}.$$

The membership in the integral Hecke algebra follows by expanding the polynomials into traces of group elements. Theorem 17.58 and Corollary 17.60 prove the required characteristic-coefficient and tied-trace membership from the finite integral Hecke lattice and rational Frobenius attachment. No matrix representation over the Hecke algebra is inferred from those traces.

Lemma 13.25 (The trace identities on the two types)**.** *On the line-type space for $\psi\circ\alpha$, if $\psi\neq1$, then*

$$t_{p,g}(x)y=t_g(x)p(\widetilde{U})y. \tag{13.66}$$

For every integral weight f, on the full level space one has

$$L_\Gamma(f)t_{p,g}=(\rho(d)-1)L_\Gamma(f)s_{p,g},\qquad N_\Gamma t_g=0. \tag{13.67}$$

Proof. On each constituent in the line-type projection, (13.65) gives $p(F_x)(T_x - I) = p(u)(T_x - I)$, and $p(\widetilde{U})$ acts by $p(u)$. Multiplying the matrix identity on the right by $M_x(g)$ and taking traces gives (13.66).

For the determinant-type assertion, finite Fourier expansion of weights on $\mathbb{Z}/M\mathbb{Z}$ suffices. If a weight is a character χ, then $L_\Gamma(\chi)y$ has determinant type $\chi^{-1}\circ\beta$. On such vectors the compatibility input gives $T_x = \chi^{-1}(1)I$; the operator $\rho(d)$ acts by the same scalar. This proves the first identity in (13.67) for character weights. Characters span all complex weights, so it holds for integral weights as well. Take $f = 1$ and $p = 1$; the norm is fixed by $\rho(d)$, giving the second identity.

Lean, formal context. `TiedTraces` (L638-L675), `FitTwistsExist` (L377). □

The integral weighted Hecke operator. For every polynomial $p \in \mathcal{O}[X]$, define

$$B(f) = p(\widetilde{U})L_P(f).$$

This is an additive map from integral weights to $\mathcal{E}$. Indeed the image of $L_P(f)$ is fixed by the inverse image of $\ker\alpha$, which contains the pro-unipotent Iwahori subgroup I_1. On I_1-fixed vectors the counting-normalized Hecke operator is a finite sum of translations. Its powers preserve the I_1-fixed space, hence the principal-level condition; all other local level and type conditions commute with these translations. The coefficients of p and the powers of the prescribed integral twist value $a(\phi)$ are integral. Thus $B(f)$ is an integral operator supported at the chosen place and its conjugate. Its integral realization on the finite level space does not require dividing by $|P|$.

Reindexing the weighted sum gives (13.63). Moreover P fixes $B(1)$ from the left. Here $L_P(1) = N_P$ has values fixed by the line parahoric. The operator U preserves that fixed space: the double cosets of $\mathrm{diag}(\varpi, 1, \dots, 1)$ for the line parahoric and for I_1 have the same left representatives $1 + \sum_{j>1} \widetilde{z}_j E_{1j}$, $z_j \in k$, when applied to parahoric-fixed vectors. Thus both give the same finite coset sum, which is parahoric invariant. For the first-entry character, write $hu(z)b = u(z')bh'$, with h, h' in the line parahoric. Conjugation by b preserves the top left entry, and multiplication by these upper row unipotents changes it only by an element of $\varpi\mathcal{O}_F$. Hence $h'_{11} \equiv h_{11} \pmod{\varpi}$. Reindexing the coset sum therefore preserves that character type. This is the explicit operator realization in `exists_aevalHecke_weightedSum`.

Lemma 13.26 (The weighted trace identity after transfer)**.** *For every integral weight f on $\mathbb{Z}/M\mathbb{Z}$,*

$$CB(f)t_g = CL_P(f)t_{p,g}. \tag{13.68}$$

Proof. Work first over $\mathbb{C}$ on one Hecke system. For a nontrivial character weight χ, $L_P(\chi)$ takes values in the $\chi^{-1}\circ\alpha$ type. Thus (13.66) already gives $B(\chi)t_g = L_P(\chi)t_{p,g}$.

For the constant character both $B(1)$ and $L_P(1)$ take values in the P-invariants. The coset sum C takes these to the Γ-invariants. If either output after applying C is nonzero, the system has trivial determinant type. Then $T_x = I$ and both trace scalars vanish. This proves the identity for the constant character as well. Fourier expansion gives the claim for all weights, and the equality holds on the integral lattice since both sides preserve it.

Lean, comparison. Lean uses the line identity only for characters nontrivial on $n!$, with a resulting divisor $2n!$ in the modulus. Here the ramified-line identity applies to every nontrivial character, and the trivial character is killed after the coset sum. See `TiedTraces` (L638-L675). □

The spherical Hecke polynomial and the line parahoric. Write F for the local field, $\mathcal{O}_F$ for its integers, ϖ for a uniformizer, and $K = \mathrm{GL}_n(\mathcal{O}_F)$. Let ρ be a smooth complex representation

with a positive definite invariant Hermitian form $\langle\ ,\ \rangle$, conjugate linear in its first variable. Let $w \neq 0$ be fixed by K and satisfy

$$\mathcal{H}_j w = \eta_j w, \qquad P_w(X) = \sum_{j=0}^{n} (-1)^j q^{\binom{j}{2}} \eta_j X^{n-j},$$

where $\mathcal{H}_j$ is the counting-normalized spherical Hecke operator for lattices of colength j between $\varpi\mathcal{O}_F^n$ and $\mathcal{O}_F^n$. In particular $\eta_0 = 1$, and $\eta_n \neq 0$, since $\mathcal{H}_n w = \rho(\varpi I)w$.

Let A be averaging over I_1. Smoothness makes this a finite average on each vector; unitarity makes it an orthogonal projection. Put $b = \mathrm{diag}(\varpi, 1, \dots, 1)$ and define operators on the whole representation by

$$\mathcal{U} = q^{n-1} A\rho(b)A, \qquad \mathcal{U}^* = q^{n-1} A\rho(b^{-1})A.$$

They are adjoint. The matrices $u(z) = I + \sum_{j>1} \widetilde{z}_j E_{1j}$, $z \in k^{n-1}$, give the cosets $u(z)bI_1$ of $I_1 b I_1$. Indeed conjugation by b makes the first-row entries above the diagonal divisible by ϖ; these are the only extra conditions in $I_1 \cap bI_1 b^{-1}$. Thus this intersection has index q^{n-1}, and on I_1-fixed vectors $\mathcal{U}$ is exactly the operator U used above.

Let $J \subset K$ be the inverse image of the stabilizer of the first coordinate line modulo ϖ, and let $V_w = \mathrm{span}\{\rho(g)w : g \in \mathrm{GL}_n(F)\}$. Both $\mathcal{U}$ and $\mathcal{U}^*$ preserve V_w^J. Indeed the forward cosets for J have the same upper-row representatives as for I_1. For b^{-1}, both groups have the lower-column representatives $1 + \varpi \sum_{i>1} \widetilde{z}_i E_{i1}$, $z \in k^{n-1}$: the extra condition on that column in the intersection with its b^{-1}-conjugate is divisibility by ϖ^2. On J-fixed vectors the two operators are therefore the counting operators for JbJ and $Jb^{-1}J$, respectively.

Lemma 13.27 (The polynomial on line-parahoric invariants)**.** *The roots of P_w, with multiplicity, are a union of blocks*

$$q^{(n-1)/2} z\, q^{-(d-1)/2+i} \quad (0 \le i < d), \qquad -\tfrac{1}{2} < \log_q |z| < \tfrac{1}{2}.$$

For each block let β be its last member ($i = d-1$). The product of $X - \beta$, one factor for each block, annihilates $\mathcal{U}$ on V_w^J.

Proof. The spherical Hecke algebra is generated by the $\mathcal{H}_j$ and $\mathcal{H}_n^{-1}$ by the Satake isomorphism [Sat63]. The K-average of every translate of w is consequently a multiple of w: it is the normalized double-coset average at w. Thus $V_w^K = \mathbb{C}w$. For each nonzero $v \in V_w$ some translate $\rho(g)w$ pairs nontrivially with v. The K-average of $\rho(g^{-1})v$ is then a nonzero multiple of w. This finite average belongs to the subrepresentation generated by v, proving that V_w is irreducible. Its Hermitian form is positive definite.

By Tadić's classification [Tad86, Theorem 3.2], this spherical representation is an irreducible normalized parabolic induction of unramified determinant characters $\chi \circ \det$ on blocks GL_d. Unitary characters and complementary pairs give $z = \chi(\varpi)$ with the stated strict bounds. The normalized Satake parameters of such a block are $zq^{-(d-1)/2+i}$, $0 \le i < d$. If e_j denotes the elementary symmetric function on the full parameter, Tamagawa's formula for the Satake transform of the j-th double coset gives $\eta_j = q^{j(n-j)/2} e_j$ [Gro98, (3.14)]. Since $\binom{j}{2} + j(n-j)/2 = j(n-1)/2$, the roots in the counting normalization are exactly those displayed.

Let $P = MN$ be the upper $(1, n-1)$ parabolic, with the evident interpretation $P = G = \mathrm{GL}_1$ for $n = 1$. The parahoric J contains the standard Iwahori and has an Iwahori decomposition for P by block elimination. The projection to N-coinvariants identifies V_w^J with $r_P(V_w)^{J \cap M}$, where r_P is the normalized Jacquet module [MY14, Theorem 2.1 and Proposition 2.3]. The q^{n-1} upper-row cosets have the same image there, so $\mathcal{U}$ acts as $q^{n-1}\delta_P(b)^{1/2} r_P(b) = q^{(n-1)/2} r_P(b)$, with $\delta_P(b) = q^{-(n-1)}$.

The geometric lemma [BZ77, Lemma 2.12] filters this Jacquet module by the block from which the first coordinate is taken: the permutation double cosets are indexed by that choice. On the corresponding graded piece the first GL_1 acts by $\chi|\cdot|^{-(d-1)/2}$: restricting $\chi\circ\det$ to the upper $(1,d-1)$ Levi gives χ on that coordinate, and normalized coinvariants multiply it by the inverse square root of the modulus character. Consequently $q^{(n-1)/2}r_P(b)$ is the scalar $q^{(n-1)/2}zq^{(d-1)/2}=\beta$ on this piece. Multiplying the linear annihilators of the successive quotients proves the assertion, including the factors for repeated values of β.

Lean, comparison. Lean derives forward and adjoint recurrences for finite lattice sums. The Jacquet filtration supplies the annihilator on the parahoric invariants directly. `HeckeDegrees` (L433-L580), `SphericalChainLemma` (L97-L207). □

Lemma 13.28 (A root and its next multiple)**.** *If $P_w(c)=0$ and $P_w(qc)\neq 0$, then $|c|^2>q^{n-2}$. If c and qc are both roots, then $P_w/(X-c)$ annihilates $\mathcal{U}$ on V_w^J.*

Proof. The logarithms to base q of the moduli of the block centers $q^{(n-1)/2}z$ all lie in the open interval $((n-2)/2,n/2)$. If qc is absent, c ends every block containing it, hence $\log_q|c|>(n-2)/2$. If qc is present and every occurrence of c were last, every occurrence of qc would be first. A block ending at c has center of modulus at most $|c|$, and one starting at qc has center of modulus at least $q|c|$. Their logarithms cannot both lie in that open interval of length one. Thus at least one occurrence of c is not last. Removing it leaves every last-member factor of Lemma 13.27, with its multiplicity.

Lean, comparison. The block centers replace the adjoint identities and the positivity calculation in `SphericalChainLemma` (L209-L261) and `SphericalChainLemma` (L523-L536). □

Theorem 13.29 (Spherical chain pairing)**.** *Suppose $a\neq 0$, $t\geq 1$, and*

$$P_w(X/a)=\prod_{i=0}^{t-1}(X-q^i)S(X),\qquad S\mid H. \tag{13.69}$$

Then, for every I_1-fixed vector y,

$$\langle w,((X-q^{t-1})H)(aU)y\rangle=0. \tag{13.70}$$

Proof. For $0\leq i<t-1$, both $c_i=q^i/a$ and qc_i are roots. The proof of Lemma 13.28 removes one occurrence of each c_i without removing a last member. The c_i are distinct, so we can make all these removals at once. What remains is, up to scalar, $(aX-q^{t-1})S(aX)$ by (13.69). This includes $t=1$, when no factor is removed. Lemma 13.27 and $S\mid H$ now give $(a\mathcal{U}-q^{t-1}I)H(a\mathcal{U})=0$ on V_w^J. That space contains w and is invariant under $\mathcal{U}$ and $\mathcal{U}^*$, so the adjoint of this polynomial operator kills w. Pairing with y gives (13.70), as $U=\mathcal{U}$ on I_1-fixed vectors.

Lean. Lean cancels successive roots using the adjoint lattice sums. Here the final member of each inducing block is retained. `SphericalChainLemma` (L207-L261), `SphericalChainLemma` (L619-L627). □

Lemma 13.30 (The unitary root string)**.** *If $P_w(c)=0$ and $|c|^2q^t=q^n$ for an integer $t\geq 1$, then*

$$P_w(cq^i)=0\qquad(0\leq i<t).$$

Proof. Take a block containing c, and write $s=\log_q|z|$. At its index i the norm condition reads

$$s-\tfrac{d-1}{2}+i=-\tfrac{t-1}{2}.$$

Thus s is a half-integer. Since $|s|<1/2$, $s=0$ and $2i=d-t$. The same block therefore contains the entries from index i through $i+t-1=d-1-i$, as asserted.

Lean, comparison. Lean proves symmetry of the roots and uses the endpoint inequality. The classification gives the entire string inside one block. `SphericalChainLemma` (L537-L613). □

The global norm vanishing. Let N be the sum over the product of all auxiliary residue groups. For $\delta' \in \mathbb{T}$, assume (13.69) with $a = a(\phi)$ for every Hecke system x that occurs at base level and satisfies $\delta'(x) \neq 0$. Put $p = (X - q^{t-1})H$ in B. Then

$$\delta' NB(1) = 0. \tag{13.71}$$

Indeed, take y in the auxiliary-level space of a single system x. The local operators preserve x, and $z = NB(1)y$ is fixed by every auxiliary residue group. Thus z lies in the base-level space of x. If this space is zero, then $z = 0$; if $\delta'(x) = 0$, then $\delta' z = 0$. In the remaining case the root-chain hypothesis applies. For each spherical Hecke eigenvector w in this base-level space, (13.70) gives $\langle w, B(1)y\rangle = 0$, since $B(1)y = p(\widetilde{U})N_P y$ and $N_P y$ is I_1-fixed. Each summand of N is unitary and fixes w, so $\langle w, z\rangle = 0$. Such vectors span the base-level space of x. Hence $z = 0$, proving (13.71) on every system.

Lean. `fullnormandbase-levelinvariance` (L756-L789).

Theorem 13.31 (The explicit one-place operator)**.** *Assume the determinant-type identity $T_x = \psi(1)I$ and (13.65) on each nontrivial line type, together with the integral automorphic realization above. For each Hecke system x with a nonzero base-level subspace and $\delta'(x) \neq 0$, assume the root-chain factorization (13.69) at the chosen place, with $a = a(\phi)$ and the same polynomial H for all such x. Suppose the common stabilizer multiple c supplies an integral operator Ψ_P such that*

$$\sum_{p \in P} \rho(p^{-1})\Psi_P \rho(p) = c\,\mathrm{id}.$$

Then

$$\partial = -CB(\lambda)C^\vee \in \mathcal{E} \tag{13.72}$$

is supported at the chosen place and its conjugate, and satisfies

$$\begin{gathered} N_\Gamma t_g = 0, \qquad \partial t_g \equiv N_\Gamma s_{(X-1)H,g}, \\ \text{for each } h \in \Gamma \text{ there is } Z_h \in \mathcal{E} \text{ with} \quad c\partial(\rho(h) - 1) \equiv N_\Gamma Z_h, \qquad \delta' N Z_h = 0. \end{gathered} \tag{13.73}$$

Both congruences hold modulo $\ell^\nu \mathcal{E}$. If the line trace identity (13.66) is known only for $\psi(n!) \neq 1$, the conclusions still hold modulo $(M/(2n!))\mathcal{E}$ whenever $M/(2n!)$ is integral.

Proof. Integrality and support follow directly from the construction. The norm identity is (13.67). Integral Hecke operators commute with the local operators, since the latter preserve each Hecke system. Therefore (13.68), (13.62) and (13.67) give

$$\begin{aligned} \partial t_g &= -CL_P(\lambda)C^\vee t_{p,g} \equiv -L_\Gamma(\lambda)t_{p,g} \\ &= -(\rho(d) - 1)L_\Gamma(\lambda)s_{p,g} \equiv N_\Gamma s_{p,g}. \end{aligned}$$

Since $M \mid q - 1$, the polynomials $p = (X - q^{t-1})H$ and $(X - 1)H$ differ by a multiple of M. Their trace elements are consequently congruent modulo M, proving the first line of (13.73).

For the second line take R_h from (13.64) and put

$$Z_h = -\Psi_P B(1) R_h.$$

Because P fixes $B(1)$ from the left, the norm-section identity implies $cB(1) = N_P \Psi_P B(1)$. As $CN_P = N_\Gamma$,

$$c\partial(\rho(h) - 1) \equiv -C\,cB(1)R_h = N_\Gamma Z_h.$$

Put $L = \delta' N$. It is fixed by right multiplication by $\rho(h)$ for every $h \in \Gamma$, and $LB(1) = 0$ by (13.71). Hence

$$|P|\, L\Psi_P B(1) = LN_P \Psi_P B(1) = cLB(1) = 0.$$

Torsion freeness of $\mathcal{E}$ gives $L\Psi_P B(1) = 0$, so $LZ_h = 0$ exactly. This cancellation proves an equality of operators; the constructed Z_h was already integral, and no division by $|P|$ is used to construct it.

For the weaker hypothesis put $k = n!$ and $D(f) = B(f)t_g - L_P(f)t_{p,g}$. The weight $f(\cdot - k) - f$ has Fourier coefficient zero at all characters trivial on k, so its image under D is zero. Taking $f(x) = \lambda(x)(\lambda(x) + k)$ gives $f(\cdot - k) - f + 2k\lambda = Mh$ for some integral weight h. Thus $2kD(\lambda) = MD(h)$, whence $D(\lambda) \in (M/(2k))\mathcal{E}$ by torsion-freeness. Substituting this congruence for the weighted trace identity in the first line of the proof gives the asserted weaker modulus.

Lean, comparison. `AuxiliaryPrimeTie` (L115-L210), `FitTwistsExist` (L377). □

The required Ψ_P follows from the orbit-stabilizer construction of Lemma 25.3: stabilizers for the subgroup P have orders dividing the corresponding full-group stabilizer orders, hence divide c. Different auxiliary places have disjoint supports, so their local operators commute. Theorem 13.31 therefore supplies precisely (25.13), and Lemma 25.4 supplies the deformed norm family. The source counterpart is `exists_mul_traceElement_tameGenerator_eq_frob`.

The line-parahoric identity is supplied by Lemmas 17.83 and 17.86 for the projections identified by Theorem 17.105. For $\delta' = A\Delta$, the common polynomial in (13.116) supplies the root-chain hypothesis; the verification at the base level is given after Lemma 25.4.

13.9 Relations over the real subfield and the Kummer maps

We construct the maps of the auxiliary-prime argument by comparing dual Selmer cocycles at consecutive powers of ℓ. Their local conditions give the reciprocity identity over the real field. Transfer to the CM field contributes a factor of two.

The real character and its local conditions

Let E be a CM number field containing μ_{5040}, let $F = E^+$, and fix an algebraic closure Ω. Put

$$\Gamma = G_F, \qquad G = G_E, \qquad c \in \Gamma, \quad c^2 = 1,$$

where c is the chosen complex conjugation. The group G is an open normal subgroup of index two, and each $\sigma \in \Gamma$ has a unique expression gc^ϵ, $g \in G$ and $\epsilon \in \{0, 1\}$. Write

$$r(gc^\epsilon) = g, \qquad \varepsilon(gc^\epsilon) = (-1)^\epsilon, \qquad \kappa = \chi_\ell \varepsilon : \Gamma \longrightarrow \mathbb{Z}_\ell^\times.$$

Here χ_ℓ denotes the ordinary cyclotomic character on Γ, and its restriction to G will also be denoted by χ_ℓ. Then

$$\kappa(\sigma) = \chi_\ell(r(\sigma)), \qquad \kappa|_G = \chi_\ell|_G, \qquad \kappa(c) = 1. \tag{13.74}$$

Both cosets of G are clopen, so r is continuous. Its multiplication rule is

$$r((gc^\epsilon)(hc^\eta)) = g(c^\epsilon h c^{-\epsilon}). \tag{13.75}$$

The character κ is the source's `cyclotomicPlus`.

Lemma 13.32 (Finite inertia and the real cyclotomic character)**.** *The extension E/F is unramified at every finite place. In particular every finite inertia group in Γ lies in G. The character κ is continuous, is unramified outside ℓ, is 1 at every complex conjugation in Γ, and satisfies $4 \mid \kappa(\sigma) - 1$ in $\mathbb{Z}_\ell$ for every σ. Also $[F : \mathbb{Q}] \geq 4$.*

Proof. Since $12 \mid 5040$, choose a primitive twelfth root of unity $\zeta \in E$. It satisfies

$$\zeta - \bar{\zeta} = \zeta - \zeta^{-1} = \zeta^3,$$

a unit of O_E. If a finite inertia element of Γ acted nontrivially on E, it would act as the nontrivial automorphism of the quadratic extension E/F, namely complex conjugation. Its action on the residue of the integral element ζ would then force $\zeta - \bar{\zeta}$ to lie in the corresponding prime, contradicting its being a unit. This proves the inertia assertion and finite unramifiedness.

Continuity follows from the definitions. Outside ℓ the cyclotomic character is trivial on inertia, and the quadratic character ε is trivial there by the preceding paragraph. Thus κ is unramified outside ℓ. Every complex conjugation in G_F acts nontrivially on E, since E is totally imaginary, and has cyclotomic value -1. Both factors in $\kappa = \chi_\ell \varepsilon$ are therefore -1 at such an element. Formula (13.74) shows that $\kappa(\sigma)$ is a cyclotomic value of an element fixing μ_4. For $\ell = 2$ this is congruent to 1 modulo 4, and for odd ℓ the asserted divisibility by 4 is automatic because 4 is a unit. Finally $16 \mid 5040$, so F contains the degree-four real field $\mathbb{Q}(\mu_{16})^+$. Hence $[F : \mathbb{Q}] \geq 4$.

Lean. `inertiaGroup_le_range_toGalPlus`, `continuous_cyclotomicPlus`, `cyclotomicPlus_eq_one_of_mem_inertiaAbove`, `cyclotomicPlus_eq_one_of_isComplexConj`. □

Fix $\nu > 0$, $m = \ell^\nu$, $R = \mathbb{Z}/m\mathbb{Z}$, and an exponent $e \geq 0$. Reduction modulo m is denoted by a subscript m. Define

$$U = G \cap \ker \chi_m \subset \Gamma, \qquad M = R(\kappa_m^{-e}). \tag{13.76}$$

Thus M has underlying additive group R, with $\sigma a = \kappa_m(\sigma)^{-e} a$. The subgroup U is open and normal in Γ, and its fixed field is $E(\mu_m)$.

Let $x_1, \dots, x_k$ be the earlier ν-split Frobenius data: each consists of a place w_i of E of prime rational norm $q_i \neq \ell$, a place A_i of Ω above it, and a geometric Frobenius $\phi_i \in G$ with $\chi_m(\phi_i) = 1$. Assume their underlying primes v_i of F are distinct. Put

$$S = \{v \text{ of } F : v \mid \ell\} \cup \{v_1, \dots, v_k\}. \tag{13.77}$$

Choose the place A_v above v to be A_i when $v = v_i$, and choose it arbitrarily otherwise. Write $D_v \subset \Gamma$ for its decomposition group. Define local conditions on actual functions $D_v \to M$ by

$$\mathcal{L}_v = \begin{cases} \{f : f|_{D_v \cap \ker \kappa} = 0\}, & v \mid \ell, \\ \{0\}, & v \nmid \ell. \end{cases} \tag{13.78}$$

We impose these conditions at the places in S.

Lemma 13.33 (Verification of the relation setting)**.** *The data (13.76)–(13.78) satisfy the following properties. We have $mM = 0$, U fixes M and μ_m, no complex conjugation belongs to U, every complex conjugation acts trivially on M, and M is unramified outside S. The extension $E(\mu_m)/F$ fixed by U is unramified outside ℓ. Every local coboundary belongs to $\mathcal{L}_v$ for $v \in S$. The further count hypotheses are*

$$\begin{gathered} |M[\ell]| = \ell, \qquad \ker \kappa \subset U \cup Uc, \qquad 4 \mid \kappa(\sigma) - 1, \\ |S \setminus \{v : v \mid \ell\}| = k, \qquad [F : \mathbb{Q}] \geq 4. \end{gathered} \tag{13.79}$$

Proof. The assertions about m, U, and its action follow immediately from the definitions. A complex conjugation acts on E by its nontrivial CM involution, so it is outside G and hence outside U; its κ-value is 1 by Lemma 13.32. The same lemma proves that M is unramified away from ℓ, hence away from S. In fact every inertia group away from ℓ lies in G and has cyclotomic value 1,

so it lies in U. This proves the stronger unramifiedness assertion for the field of U; in particular $E(\mu_m)/F$ is unramified at the primes outside S in the reciprocity identity below.

At a prime $v \mid \ell$, the action on M is trivial on $D_v \cap \ker\kappa$. Thus a coboundary $\sigma \mapsto \sigma a - a$ vanishes there, exactly as required. At $v = v_i$, the residue field of F embeds in the prime field of the degree-one place w_i, so its norm is also q_i. The element ϕ_i^{-1} is consequently an arithmetic Frobenius for F at A_i. It belongs to U. Inertia at v_i lies in G by Lemma 13.32, and its cyclotomic value is 1 since $q_i \neq \ell$; thus it too lies in U. Inertia together with arithmetic Frobenius topologically generates D_{v_i}, giving $D_{v_i} \subset U$. Hence the action of the entire decomposition group on M is trivial. Its coboundaries are zero functions and belong to $\mathcal{L}_{v_i} = 0$.

As $\nu > 0$, the subgroup $R[\ell] = \ell^{\nu-1}R$ has order ℓ. If $\sigma = gc^{\epsilon}$ satisfies $\kappa(\sigma) = 1$, then $\chi_\ell(g) = 1$, so $g \in U$. Thus either $\sigma \in U$ or $\sigma c \in U$, proving the kernel inclusion. The remaining conditions follow from Lemma 13.32, distinctness of the v_i, and $q_i \neq \ell$. This checks all hypotheses, including the containment of local coboundaries as actual functions, rather than only a condition on their classes.

Lean. `kummerSetting`, `exists_card_lt_finrank_kummerReductions`. □

For a finite set P of primes of F disjoint from S, let $\mathcal{Z}(P)$ be the module of continuous cocycles $Z^1(\Gamma, M)$ unramified outside $S \cup P$, whose restrictions at $v \in S$ belong to $\mathcal{L}_v$, and which vanish at *every* complex conjugation in Γ. This is the source's `kummerSetting.cocycles`; the last condition concerns actual values, including at the different real places of F.

The relation module and its reduction

Let

$$D = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}).$$

Its Cartier-dual action is

$$(\sigma f)(a) = \chi_m(\sigma) f(\kappa_m(\sigma)^e a). \tag{13.80}$$

Multiplication by an element of R on a character is well defined, since its values are killed by m. This describes explicitly the cyclotomic factor in the dual action.

For a finite $T \supset S$, the relation module $\mathcal{R}(T)$ consists of locally constant functions $Y : \Gamma \to D$ with the following properties:

1. $Y(\sigma) = 0$ for $\sigma \notin U$, and $Y|_U$ is an additive homomorphism.
2. For $\sigma \in \Gamma$ and $u \in U$,
$$Y(\sigma u \sigma^{-1})(a) = \chi_m(\sigma) Y(u)(\kappa_m(\sigma)^e a). \tag{13.81}$$
3. Y vanishes on inertia outside T.
4. Suppose p_j are finitely many distinct primes of F outside T, with chosen places above them, arithmetic Frobenii $\sigma_j \in U$, and tame generators t_j normalized modulo m using the fixed primitive m-th root of unity. Then
$$\sum_j Y(\sigma_j)(Z(t_j)) = 0 \quad \text{in } \mathbb{Q}/\mathbb{Z} \qquad \big(Z \in \mathcal{Z}(\{p_j\})\big). \tag{13.82}$$

These functions form an R-module. Its pointwise reduction is

$$\mathrm{Red}_\ell\, \mathcal{R}(T) = \mathrm{im}\big(\mathcal{R}(T) \longrightarrow \mathrm{Fun}(\Gamma, D/\ell D)\big). \tag{13.83}$$

Theorem 13.34 (The count at two consecutive depths)**.** *For the data* (13.76)–(13.78)*, put* $\delta = 1$ *if* $\ell = 2$ *and* $\delta = 0$ *otherwise. Then*

$$\dim_{\mathbb{F}_\ell} \mathrm{Red}_\ell \mathcal{R}(S) \geq k + [F:\mathbb{Q}] - 1 - \delta \geq k + 1. \tag{13.84}$$

Proof. Keep F, S, U fixed. For $1 \leq i \leq \nu$, write M_i for $(\mathbb{Z}/\ell^i)(\kappa^{-e})$ and D_i for its Cartier dual. Let Z_i be the primal Selmer cocycles at depth i: they satisfy (13.78) at S, are unramified at the other finite places and vanish at every real conjugation. Let Y_i consist of all cocycles in D_i whose local classes annihilate these primal conditions under local Tate duality. At depth zero take both groups trivial. These groups are finite by finiteness of Selmer groups for finite modules and of the coboundary groups.

Apply the Greenberg–Wiles formula (13.18) over F; at real places its denominator is ordinary H^0, also for $\ell = 2$ [MP22, Theorem 3.4]. At each old place v_j the action is trivial by Lemma 13.33. Thus each old or real place contributes ℓ^{-i}. The unramified factor away from S is one, by equality of the kernel and cokernel orders of Frobenius minus one. At $v \mid \ell$ an admissible cocycle factors through $\kappa(D_v)$. This group contains $\chi_\ell(G_{E_w})$ for $w \mid v$, an open subgroup of $\mathbb{Z}_\ell^\times$. It is procyclic, including at 2, where it lies in $1 + 4\mathbb{Z}_2$. Every value at a topological generator gives a continuous cocycle: if the generator acts on M_i with order r, take its cyclic quotient of order $r\ell^i$. The norm of that quotient on M_i is zero. Thus there are ℓ^i local cocycles; dividing by the local coboundary order $\ell^i/|H^0(F_v, M_i)|$ gives factor one. Multiplying the two global Selmer orders by the respective coboundary orders cancels the global invariant factors. We obtain

$$|Y_i| = |Z_i|\, \ell^{i(k+[F:\mathbb{Q}])}.$$

All primal local conditions reduce onto those at depth $i-1$. At $v \mid \ell$ lift the generator value, and at any other unramified place lift the Frobenius value. At the old and real places both conditions are zero. Dual to $M_i \to M_{i-1}$ is the inclusion $D_{i-1} \simeq \ell D_i \hookrightarrow D_i$. By functoriality of the local pairing and the surjectivity just proved, a cocycle taking values in ℓD_i lies in Y_i precisely when its preimage belongs to Y_{i-1}. This assertion concerns cocycles as functions, so the preimage is unique. Multiplication by ℓ embeds Z_{i-1} in Z_i. Consequently the pointwise image of Y_ν in $\mathrm{Fun}(\Gamma, D_\nu/\ell D_\nu)$ has order

$$|Y_\nu|/|Y_{\nu-1}| \;\geq\; \ell^{k+[F:\mathbb{Q}]}.$$

A residual cocycle zero on U factors through $\Gamma/U \simeq C_2 \times \chi_m(G)$, by (13.75). For odd ℓ, the generator c of C_2 acts by -1 on $D_\nu/\ell D_\nu$. Commuting c past g gives $(c-1)b(g) = (g-1)b(c)$, so b is determined by $b(c)$. There are at most ℓ such cocycles. For $\ell = 2$ the action is trivial and the quotient is generated by at most two elements, giving at most four cocycles. Restriction to U therefore loses at most $1 + \delta$ dimensions.

For $b \in Y_\nu$, let $Y = b$ on U and $Y = 0$ elsewhere. The cocycle identity gives additivity and (13.81) because U fixes D. At finite places outside S the dual class is unramified: the two unramified conditions are mutual annihilators, and coboundaries vanish on inertia [Mil06, Ch. I, §2]. For Z in (13.82) the local invariants of $[Z] \cup [b]$ sum to zero. Orthogonality kills the terms at S, the primal class at infinity is zero, and at the other places off the p_j both classes are unramified. At p_j both modules are trivial locally, because inertia acts trivially and $\sigma_j \in U$. The unramified–tame local pairing with the specified normalized generator is $b(\sigma_j)(Z(t_j))$, up to the common Frobenius sign. This proves (13.82) with $T = S$. These relations realize all the residual restrictions just counted. Finally $[F:\mathbb{Q}] \geq 4$ by Lemma 13.32.

Lean, comparison. Lean chooses auxiliary primes detecting roots of units in `exists_splitPlaces_forall_exists_mem_relations` and estimates the reduction space in `exists_pow_le_card_reductions`. For these scalar modules the count at consecutive depths suffices with $T = S$. □

Relations as scalar Kummer maps

Let $\iota_m : R \hookrightarrow \mathbb{Q}/\mathbb{Z}$ be the injection $\bar{a} \mapsto a/m \bmod \mathbb{Z}$. The map

$$R \xrightarrow{\sim} D, \qquad \beta \longmapsto \lambda_\beta, \qquad \lambda_\beta(a) = \iota_m(a\beta) \tag{13.85}$$

is an R-linear isomorphism. Indeed an additive character of R is determined by its value at 1, that value is killed by m, and every such value has a unique form $\iota_m(\beta)$. For $Y \in \mathcal{R}(T)$ define $\psi_Y : G \to R$ by

$$Y(g)(a) = \iota_m(a\psi_Y(g)) \qquad (g \in G, a \in R). \tag{13.86}$$

This is a unique function; no choice of a division in R is used.

Lemma 13.35 (The Kummer type of a relation)**.** *The function ψ_Y is locally constant and vanishes outside $U \subset G$. On U it is additive, and*

$$\psi_Y(u^c) = -\psi_Y(u), \qquad \psi_Y(gug^{-1}) = \chi_m(g)^{e+1}\psi_Y(u) \quad (u \in U, g \in G). \tag{13.87}$$

It vanishes on inertia at the primes of E whose underlying primes of F lie outside T.

Proof. Local constancy, support, additivity and inertia vanishing follow by applying the inverse of (13.85) to the corresponding properties of Y. Equivariance gives, for $\sigma \in \Gamma$ and $u \in U$,

$$Y(\sigma u\sigma^{-1})(a) = \iota_m(a\chi_m(\sigma)\kappa_m(\sigma)^e\psi_Y(u)).$$

At $\sigma = c$ the multiplier is -1, because $\chi_m(c) = -1$ and $\kappa_m(c) = 1$. At $\sigma = g \in G$ it is $\chi_m(g)^{e+1}$, because $\kappa_m(g) = \chi_m(g)$. Injectivity of ι_m proves both identities in (13.87), including when $\ell = 2$.
Lean. `IsKummerMap.hasKummerType`. □

Transfer of even cocycles

Lemma 13.36 (Extension of an even cocycle)**.** *Let $z : G \to R(\chi_m^{-e})$ be a continuous cocycle satisfying $z(g^c) = z(g)$ for all $g \in G$. Then*

$$X(gc^\epsilon) = z(g) \tag{13.88}$$

defines a continuous cocycle $X : \Gamma \to M$. It restricts to z on G and satisfies $X(c) = 0$.

Proof. Continuity follows from continuity of r and z. If $\sigma = gc^\epsilon$ and $\tau = hc^\eta$, then (13.75), evenness, and the cocycle identity for z give

$$\begin{aligned} X(\sigma\tau) &= z(g) + \chi_m(g)^{-e}z(c^\epsilon hc^{-\epsilon}) \\ &= X(\sigma) + \kappa_m(\sigma)^{-e}X(\tau). \end{aligned}$$

This proves the cocycle identity without averaging or division by two. The remaining assertions follow from the definition.
Lean. `comp_ofGalPlus_mem_Z1`. □

Lemma 13.37 (The doubled extension satisfies all real local conditions)**.** *Let P be a finite set of primes of F disjoint from S. In addition to the hypotheses of Lemma 13.36, assume z is unramified away from the primes of E above $\ell \cup P$, vanishes on $D_w \cap \ker \chi_\ell$ at every $w \mid \ell$, and satisfies $z(\phi_i) = 0$ for the earlier Frobenii. Then $2X \in \mathcal{Z}(P)$.*

Proof. For $\sigma \in \ker \kappa$, the action on M is trivial, so

$$X(\sigma^2) = 2X(\sigma). \tag{13.89}$$

If $d \in \Gamma$ is any complex conjugation, then $\kappa(d) = 1$ and $d^2 = 1$. Hence $2X(d) = X(1) = 0$. Thus the vanishing holds at every real place.

Let $v \mid \ell$ and $\sigma \in D_v \cap \ker \kappa$. Since G has index two, $\sigma^2 \in G$. It is still in the decomposition group of the chosen place above v, and

$$\chi_\ell(\sigma^2) = \kappa(\sigma^2) = 1.$$

It therefore lies in the local subgroup on which z vanishes. By restriction, $X(\sigma^2) = z(\sigma^2) = 0$, so (13.89) gives $2X(\sigma) = 0$. This is exactly the local condition above ℓ. No assumption that v splits in E/F is needed.

At inertia outside $\ell \cup P$, the inertia group of F lies in G by Lemma 13.32; thus X equals z there and vanishes by the unramified hypothesis. In particular $2X$ is unramified outside $S \cup P$. At $v = v_i$, this gives zero on inertia, because v_i lies in neither ℓ nor P. Also $X(\phi_i^{-1}) = z(\phi_i^{-1}) = 0$, by the cocycle inverse formula and the assumed value at ϕ_i. The zero set of a continuous cocycle is a closed subgroup. Inertia and the arithmetic Frobenius ϕ_i^{-1} topologically generate D_{v_i}, so X vanishes on all of D_{v_i}; in particular $2X$ belongs to $\mathcal{L}_{v_i} = 0$.

One and the same multiplication by 2 has imposed the conditions both at infinity and above ℓ. No further factor is introduced by these two checks. At $\ell = 2$ we retain $2X$ throughout; the proof does not infer $X = 0$ from a doubled vanishing statement.

Lean. `two_nsmul_comp_ofGalPlus_mem_cocycles`. □

Theorem 13.38 (Exact reciprocity for the Kummer maps)**.** *Let $Y \in \mathcal{R}(T)$ and $\psi = \psi_Y$. Let w_j be finitely many ν-split Frobenius data of E above distinct primes p_j of F outside T, with geometric Frobenii ϕ_j. Choose $t_j \in G$ in inertia at the corresponding place of Ω, normalized modulo m using a uniformizer of F. For every commutative $\mathbb{Z}_\ell$-algebra O finite as a $\mathbb{Z}_\ell$-module and every continuous even cocycle*

$$z : G \longrightarrow (O/\ell^\nu O)(\chi_m^{-e})$$

unramified outside $\ell, w_j, \bar{w}_j$, vanishing on $D_w \cap \ker \chi_\ell$ above ℓ, and satisfying $z(\phi_i) = 0$ for the earlier data x_i, one has

$$\sum_j 2\psi(\phi_j) z(t_j) = 0 \quad \text{in } O/\ell^\nu O. \tag{13.90}$$

Proof. Normalized t_j exist in the inertia group of F: choose an element multiplying an m-th root of a local uniformizer by the fixed primitive m-th root of unity. They belong to G by Lemma 13.32. As the place w_j has prime norm, the underlying place p_j of F has the same norm, so ϕ_j^{-1} is an arithmetic Frobenius at that place and belongs to U. Thus these are precisely split-prime data to which (13.82) applies.

First suppose $O/\ell^\nu O = R$. Set $P = \{p_j\}$. Its inverse image in E consists of the chosen places and their conjugates: finite unramifiedness and degree one make each p_j split in the quadratic extension E/F. Lemma 13.37 therefore gives $2X \in \mathcal{Z}(P)$ for the extension of z. Applying (13.82) with arithmetic Frobenii ϕ_j^{-1} yields

$$0 = \sum_j Y(\phi_j^{-1})(2X(t_j)) = \iota_m \left(\sum_j 2\psi(\phi_j^{-1}) z(t_j) \right).$$

On U, ψ is additive and hence $\psi(\phi_j^{-1}) = -\psi(\phi_j)$. Injectivity of ι_m and multiplication by -1 give (13.90). This explains the Frobenius sign as well as the factor 2.

For general O, its quotient $O/\ell^\nu O$ is a finite R-module. Each R-linear map $f : O/\ell^\nu O \to R$ is continuous for the discrete topologies and commutes with the scalar Galois action. Thus $f \circ z$ satisfies the same local conditions and is even. By the scalar case, f kills the sum in (13.90). To see that these maps separate points, write the finite abelian group $O/\ell^\nu O$ as a direct sum of cyclic groups $\mathbb{Z}/\ell^j\mathbb{Z}$ with $1 \le j \le \nu$. On each summand $1 \mapsto \ell^{\nu-j}$ defines an R-linear injection into R. Composing with the projections proves that the sum is zero.

Lean. `IsKummerMap.sum_eq_zero` uses the same separation by additive maps for the rings of integers in `CoeffRing.exists_mem_iff`. □

Corollary 13.39 (The one-term annihilator for the chosen tame generator)**.** *In Theorem 13.38, suppose $\psi(\phi_i) = 0$ for $i \neq j$. If $n \in \mathbb{Z}$ satisfies $2\psi(\phi_j) \mid n$ in R, then*

$$nz(\tau_j) = 0 \tag{13.91}$$

for the chosen tame generator τ_j, or indeed for any element τ_j of the same inertia group.

Proof. The reciprocity identity gives $2\psi(\phi_j)z(t_j) = 0$. At this prime the cyclotomic action is unramified, so the restriction of z to inertia is a continuous homomorphism to an ℓ-primary group. The pro-p_j wild-inertia image is zero. The remaining ℓ-primary tame quotient, modulo ℓ^ν, is cyclic of order ℓ^ν, generated by the image of the normalized t_j. Consequently $z(\tau_j) = az(t_j)$ for some $a \in R$. Write $n = 2\psi(\phi_j)b$ in R and multiply the vanishing identity by ab. No unit comparison between the two tame generators, and no division by 2, is required.

Lean. `IsKummerMap.nsmul_apply_tameGenerator_eq_zero`. □

The dimension of the space of reductions

Let $\mathcal{K}(T) = \{\psi_Y : Y \in \mathcal{R}(T)\}$, an R-submodule of the functions $G \to R$. Define

$$\overline{\mathcal{K}}(T) = \{\psi|_U \bmod \ell : \psi \in \mathcal{K}(T)\} \subset \mathrm{Fun}(U, \mathbb{F}_\ell).$$

This image is already an $\mathbb{F}_\ell$-subspace: scalar multiplication by a residue class is induced by any lift of that scalar to R. Thus it equals the span called `kummerReductions` in the source.

Lemma 13.40 (The reductions are canonically identified)**.** *There is a canonical $\mathbb{F}_\ell$-linear isomorphism*

$$\mathrm{Red}_\ell \mathcal{R}(T) \xrightarrow{\sim} \overline{\mathcal{K}}(T). \tag{13.92}$$

Both spaces are finite. In particular this identification loses no factor at $\ell = 2$ and does not require lifting a modulo-ℓ function to a cocycle modulo ℓ^ν.

Proof. The identification (13.85) induces

$$D/\ell D \simeq R/\ell R = \mathbb{F}_\ell.$$

Restrict a function in $\mathrm{Red}_\ell \mathcal{R}(T)$ to U and apply this identification pointwise. For a representative Y the result is precisely $\psi_Y|_U \bmod \ell$, so the map is well defined and surjective. Every relation is zero outside U. Consequently its pointwise reduction on the whole of Γ is determined by its reduction on U. The pointwise identification is injective, proving injectivity and linearity of (13.92).

For finiteness, put $L = E(\mu_m)$, so $U = G_L$. The restrictions of the relations to U are homomorphisms to the finite group D, killed by m and unramified away from the primes over T.

By the ray class quotient in the proof of Lemma 13.5, all factor through one finite group. There are finitely many such homomorphisms. The relations are determined by their restrictions, so both reduction spaces are finite.

Lean, formal context. Lean proves finiteness in `finite_kummerReductions` and the cardinal inequality in `card_reductions_le`. Here the identification of the two pointwise images also gives the inverse linear map. □

Theorem 13.41 (Abundance of the admissible Kummer maps)**.** *For every $\nu > 0$, every $e \geq 0$, and every finite family $x_1, \dots, x_k$ of ν-split data above distinct primes of F, take $T = S$. Then*

$$\dim_{\mathbb{F}_\ell} \overline{\mathcal{K}}(T) \geq k + 1. \tag{13.93}$$

Every map in $\mathcal{K}(T)$ has the Kummer type (13.87) and the even-class reciprocity (13.90), for every finite coefficient ring O as above. In particular, given any k functions $v_i : U \to \mathbb{F}_\ell$, some such map ψ has reduction outside their span.

Proof. By Theorem 13.34 and Lemma 13.40,

$$\ell^{\dim_{\mathbb{F}_\ell} \overline{\mathcal{K}}(T)} = |\operatorname{Red}_\ell \mathcal{R}(T)| \geq \ell^{k+1}.$$

Since $\ell > 1$, this proves (13.93). The type and reciprocity statements were proved in Lemma 13.35 and Theorem 13.38. Finally the span of k functions has dimension at most k, so it cannot contain $\overline{\mathcal{K}}(T)$. By its definition as an image, a point outside that span is the reduction of an actual map $\psi \in \mathcal{K}(T)$.

Lean. `exists_card_lt_finrank_kummerReductions`. □

Theorem 13.41 proves the existence assertion previously stated in the Kummer-abundance paragraph. The corresponding admissible map carries the avoidance set T, the precise factor 2 in reciprocity, and the chosen-tame-generator annihilator of Corollary 13.39. Its definition and proof distinguish cocycles, cohomology classes and pointwise reductions. These Kummer maps will be used in the induction of Theorem 13.44.

13.10 Uniform auxiliary-prime triangles and the Tate-twist cost

Theorem 13.41 now supplies the arithmetic existence assertion for the prime triangle. The uniform index set, the triangular induction, the passage to the even and odd parts, and the construction of common integral root-chain factors are proved below. The relation count is a separate argument from the earlier ordinary dual-Selmer descent. The local conditions and reciprocity were proved in §13.9. The numerical odd-cocycle bound is proved in Theorem 14.5.

The source locations are TrianglePrimes, TateTwistCocycles, and the simultaneous root-chain application. Their real-subfield relation dependency is now supplied by Theorems 13.34 and 13.41, including the precise reduction space.

A finite set of evaluations independent of the exponent

Let $E \subset \mathbb{Q}(\mu_N)$ be a CM field containing μ_{5040}, with $N > 0$, and put $G = G_E$ and $\chi = \chi_\ell$. Conjugation by the fixed complex conjugation is denoted by $g \mapsto g^c$. Choose $a \in G$ whose image generates $\chi(G)$ topologically. The coefficient ring O is the ring of integers of a finite extension of $\mathbb{Q}_\ell$; replacing it by a finite Galois extension when needed does not change the following assertions. Write

$$A_\nu = O/\ell^\nu O, \qquad Z_e(S) = Z^1_{S,\mathrm{cont}}(G, A_\nu(\chi^{-e})),$$

where S is the allowed set of finite ramified places. These are modules of cocycles, not cohomology classes modulo coboundaries.

Lemma 13.42 (Uniform detecting points)**.** *There is a finite set $s \subset \ker\chi$, depending on E, N, ℓ, a but independent of O, ν, e, with the following property:*

$$z \in Z_e(\{v : v \mid \ell\}), \quad z(a) = 0, \quad z(g) = 0 \ (g \in s) \quad \Longrightarrow \quad z = 0. \tag{13.94}$$

Whenever $N \le y$ and $\ell \le y^2$, one may take $|s| \le 6y^3$.

Proof. Put $m = N\ell$, $L = \mathbb{Q}(\mu_m)$, $d = [L : \mathbb{Q}]$, and let S be the places of L above ℓ. The last nonzero ℓ-multiple of a cocycle restricts to a homomorphism on G_L: the cyclotomic twist acts trivially on $A_\nu[\ell]$ there. Representatives of G/G_L and lifts of a basis of the maximal elementary abelian ℓ-quotient of G_L unramified outside S therefore give a detecting set s_0 in G.

Since $\mu_\ell \subset L$, Kummer theory represents the dual of this quotient by classes in $L^\times/L^{\times\ell}$ with valuations divisible by ℓ off S. Dividing the divisor there by ℓ maps these classes to $\mathrm{Cl}(\mathcal{O}_{L,S})[\ell]$. The kernel is contained in the S-units modulo powers, of dimension at most $r_1(L) + r_2(L) + |S|$ by the unit theorem. By Minkowski each ideal class has a representative of norm at most $B = 2^d|\operatorname{disc} L|^{1/2}$ [Neu99, Ch. I, §§5–7]. At most $B^2\zeta_L(2) \le B^2\zeta(2)^d$ ideals have this norm or less, and $|\operatorname{disc} L| \le m^d$. Thus $h_L \le (8m)^d$. Including at most d coset representatives gives $|s_0| \le d(3 + \log_\ell(8m))$, with $d \le N\ell$. Let $C = \overline{\langle a \rangle}$. Since $\chi(C) = \chi(G)$, each $g \in s_0$ can be written $g = h_g c_g$ with $h_g \in \ker\chi$ and $c_g \in C$. If $z(a) = 0$, its closed zero subgroup contains C. Thus $s = \{h_g : g \in s_0\}$ has the required property.

Here $N \ge [\mathbb{Q}(\mu_{35}) : \mathbb{Q}] = 24$ because $\mu_{35} \subset E$. If $y \le \ell \le y^2$, then $8N \le \ell^2$ and the bound is at most $6N\ell \le 6y^3$. If $\ell < y$, it is at most $y^2(7 + \log_2 y) \le y^3$ for $y \ge 24$.

Lean. `New.OddPart.exists_detecting_points_mem_ker`. Lean projects detectors for the exponent-$N\ell^2$ class-field quotient into $\ker\chi$. The same projection here reuses the mod-ℓ Kummer detector. □

Reciprocity for the auxiliary Kummer maps

Put $F = E^+$, $G_\nu = \ker(\chi \bmod \ell^\nu)$ and $R_\nu = \mathbb{Z}/\ell^\nu\mathbb{Z}$. A Frobenius datum x consists of a degree-one finite place w_x of E, a place of the algebraic closure above it, and a geometric Frobenius ϕ_x. It is ν-split if $w_x \nmid \ell$ and $\phi_x \in G_\nu$. Its chosen tame generator is τ_x. We exclude the finitely many places ramified over F whenever required; a degree-one unramified place of E then lies over a split place of F.

For $e > 0$, define $\mathcal{E}_e(S)$ to consist of those cocycles in $Z_e(\{v : v \mid \ell\} \cup S \cup \bar{S})$ which satisfy

$$z(g^c) = z(g), \qquad z(g) = 0 \quad (g \in D_v \cap \ker\chi,\ v \mid \ell). \tag{13.95}$$

This is exactly the even-class condition used below, uniformly for all O.

An admissible Kummer map of exponent e, attached to a finite family $x_1, \dots, x_k$ of ν-split data above distinct primes of F, is a locally constant map $\psi : G \to R_\nu$, together with a finite set T of primes of F containing those above ℓ and below the x_i, with the following properties. Its restriction to G_ν is additive, and on that subgroup

$$\psi(g^c) = -\psi(g), \qquad \psi(bgb^{-1}) = \chi(b)^{e+1}\psi(g) \quad (b \in G). \tag{13.96}$$

It also has the following reciprocity property. Suppose w_j is a finite family of ν-split data above distinct primes of F outside T, and $z \in \mathcal{E}_e(\{w_j\})$ satisfies $z(\phi_{x_i}) = 0$ for all i. For tame generators γ_j normalized using uniformizers over F,

$$\sum_j 2\psi(\phi_{w_j})z(\gamma_j) = 0. \tag{13.97}$$

In particular, if $\psi(\phi_{w_i}) = 0$ for $i \neq j$, then

$$2\psi(\phi_{w_j}) \mid n \text{ in } R_\nu \quad \Longrightarrow \quad nz(\tau_{w_j}) = 0. \tag{13.98}$$

Indeed, inertia acts trivially on the Tate twist away from ℓ. Its pro-p wild subgroup has zero image in an ℓ-primary module when $p \neq \ell$, and the remaining image is generated by the value on γ_j. Thus $z(\tau_{w_j})$ is an integer multiple of $z(\gamma_j)$, and (13.98) follows from (13.97). No division by 2 or by a tame generator comparison factor is involved.

A lower bound for the dimension of the relations. For every $\nu > 0$, every $e > 0$, and every family $x_1, \dots, x_k$ as above, there is a finite T for which the restrictions $\psi|_{G_\nu} \bmod \ell$ of admissible Kummer maps span an $\mathbb{F}_\ell$-space of dimension at least $k+1$. This is Theorem 13.41: its subgroup U is the present G_ν, and its maps have exactly the type and reciprocity (13.96)–(13.97). The condition of at least four real places of F is automatic from $\mu_{16} \subset E$, by Lemma 13.32.

Theorem 13.34 counts the restrictions of dual Selmer cocycles and proves reciprocity off S. Theorem 13.38 transfers the identity to E with a factor of two. The corresponding formal statements are `exists_card_lt_finrank_kummerReductions` and its relation count.

The finite construction of the triangle

Let $\gamma = a^{-1}$, put $u = \chi(\gamma)$, and suppose

$$N + t < y, \qquad u^j - 1 \mid \ell^h \quad (0 < j < t), \qquad \ell \leq y^2 \text{ or } h = 0. \tag{13.99}$$

Let W be a set of allowed Frobenius data. We use the following precise approximation condition on W: for every $\nu > 0$ and $g \in G$, some $\delta \in \ker \chi$ has the property that, in every neighborhood of $\delta^{\ell^\nu} g$, one can choose a datum in W avoiding any prescribed finite set of places. The places can be required to have degree one. The source's condition $W \in \mathrm{mostFrobeniiAt}(\delta^{\ell^\nu} g)$ is stronger than this formulation and implies it by Chebotarev.

Lemma 13.43 (Separation without dividing by a determinant)**.** *Let $\psi_i : G_\nu \to R_\nu$ be additive maps. If their reductions are linearly independent as $\mathbb{F}_\ell$-valued functions, their joint map to R_ν^r is surjective. Suppose moreover that each ψ_i has Kummer type $e_i < t$ and ψ_0 has type $e < t$, with the reduction of ψ_0 independent of the earlier maps of type e. There is $v \in G_\nu$ such that*

$$\psi_i(v) = 0 \quad \textit{for all earlier } i, \qquad \psi_0(v) = D_e := \prod_{\substack{0 \leq k < t \\ k \neq e}} (u^e - u^k) \pmod{\ell^\nu}. \tag{13.100}$$

Proof. The reduction of the joint image is all of $\mathbb{F}_\ell^r$: otherwise a nonzero linear functional annihilating that image would give a relation among the reduced functions. The image is an R_ν-submodule. If M is the quotient of R_ν^r by this image, then $M = \ell M$, hence $M = \ell^\nu M = 0$. This proves surjectivity, including when the domain group is noncommutative.

Choose $v_0 \in G_\nu$ with $\psi_0(v_0) = 1$ and all earlier type-e values zero. For an exponent $k \neq e$, replace any $b \in G_\nu$ by

$$b' = (\gamma b \gamma^{-1})^A b^B, \qquad A \equiv u^{-1}, \quad B \equiv -u^k \pmod{\ell^\nu},$$

using integer representatives for A, B. A map of type d takes at b' the value $(u^d - u^k)\psi(b)$, by (13.96). Apply this operation once for every $k < t$, $k \neq e$. Maps of type different from e acquire a zero factor; the type-e maps preserve their zero values and ψ_0 acquires the product D_e. This proves (13.100).

Lean. `New.Selmer.surjective_of_linearIndependent_cast`, `New.Selmer.exists_separating_element`. □

Theorem 13.44 (A fixed finite index set at every depth)**.** *Assume the approximation condition on W and (13.99), in the CM setting stated above. There is a finite linearly ordered index set*

$$J = \{1, \dots, t-1\} \times s, \qquad m = |J| = \max(t-1, 0)|s|, \tag{13.101}$$

ordered first by the exponent, with an increasing type map $j \mapsto e_j$, such that for every $\nu > 0$ one can choose ν-split data $(w_j, \tau_j, \phi_j) \in W$, indexed by this same J, with distinct underlying primes of F. Set $C = 6y^5h$. If $t \le 1$ put $b = 0$; otherwise take

$$b = v_\ell\left(4\prod_{j=1}^{t-1}(1-u^j)\right).$$

The following assertions hold for all coefficient rings O:

1. *$q_{w_j} \equiv 1 \pmod{\ell^\nu}$; the places and their conjugates are pairwise distinct and avoid the prescribed fixed bad places.*
2. *For every $0 < e < t$ and $g \in s$, the index $j = (e, g)$ satisfies $z(\phi_j) = 2z(g)$ for every $z \in \mathcal{E}_e(\varnothing)$.*
3. *$D_e \mid \ell^b$ for every $0 \le e < t$, and $4(u^e - 1) \mid \ell^b$ for $0 < e < t$.*
4. *If $z \in \mathcal{E}_{e_j}(\{w_i : i \in J\})$ satisfies $z(\tau_i) = 0$ for $i < j$ and $z(\phi_i) = 0$ for $i < j$ of type e_j, then $\ell^b z(\tau_j) = 0$.*
5. *$(|J| + |s| + 1)b \le C$, and a, s detect the cocycles unramified away from ℓ as in (13.94).*

The choices of s, J, m, b precede the choice of ν.

Proof. Use Lemma 13.42 for $a = \gamma^{-1}$. Fix a depth $\nu > 0$ only after s, J have been chosen. At an index $j = (e, g)$ suppose all earlier data and their Kummer maps ψ_i have been chosen. Apply Theorem 13.41 to the earlier data of type e. It gives a map ψ_j, with avoidance set T_j, whose reduction is outside the span of the preceding type-e maps. The earlier maps of each fixed type are linearly independent by induction. Lemma 13.43 supplies $v \in G_\nu$ for which all earlier maps vanish and $\psi_j(v) = D_e$.

Apply the approximation hypothesis for W to $gg^c v(v^c)^{-1}$. For the resulting $\delta \in \ker\chi$, put

$$\sigma = \delta^{\ell^\nu} gg^c v(v^c)^{-1}. \tag{13.102}$$

This element lies in G_ν. Additivity and oddness give, for every earlier map and the new map,

$$\psi(\sigma) = 2\psi(v). \tag{13.103}$$

The δ^{ℓ^ν} contribution vanishes, the g, g^c contributions cancel, and the $v, (v^c)^{-1}$ contributions add. For an even cocycle $z \in \mathcal{E}_e(\varnothing)$ the same calculation gives

$$z(\sigma) = 2z(g). \tag{13.104}$$

All factors act trivially on A_ν whenever this is needed in the cocycle identity. Alternatively, $v(v^c)^{-1}$ already lies in $\ker\chi$, and comparison with its product with v^c gives the same cancellation directly.

There is one open neighborhood on which (13.103), (13.104), and $\chi \equiv 1 \pmod{\ell^\nu}$ hold, uniformly over O in the even-cocycle assertion. To justify this last uniformity, fix ν, e and first use R_ν coefficients. Evaluation on $s \cup \{a\}$ is injective on all cocycles unramified outside ℓ, since the difference of two with the same evaluations vanishes by Lemma 13.42. There are consequently

at most $|R_\nu|^{|s|+1}$ such cocycles. Intersect their neighborhoods of constancy. Every additive homomorphism $O/\ell^\nu O \to R_\nu$ is R_ν-linear and preserves the scalar Tate-twist action. These homomorphisms separate points, since $O/\ell^\nu O$ is finite free over R_ν. The same neighborhood therefore works for all O.

Now choose a datum in W with Frobenius in this neighborhood, excluding T_j, every earlier T_i, and the primes below all earlier places. The result satisfies

$$\psi_i(\phi_j) = 0 \quad (i < j), \qquad \psi_j(\phi_j) = 2D_{e_j} \pmod{\ell^\nu}, \qquad z(\phi_j) = 2z(g). \tag{13.105}$$

This completes the finite induction. At every later step, the avoidance set T_j remains excluded.

For $t \geq 2$, put $\Pi_n(X) = \prod_{j=1}^{n}(1 - X^j)$, with $\Pi_0 = 1$. Up to a unit, $D_e = \Pi_e(u)\Pi_{t-1-e}(u)$. The quotient $\Pi_{t-1}(X)/(\Pi_e(X)\Pi_{t-1-e}(X))$ is the Gaussian binomial polynomial in $\mathbb{Z}[X]$, by its Pascal recurrence [DLMF, (17.2.29), (17.2.32)]. Thus every D_e divides $\Pi_{t-1}(u) = D_0$. For $0 < e < t$, $u^e - 1$ also divides this product. This proves assertion 3, and (13.105) gives $2\psi_j(\phi_j) = 4D_{e_j} \mid \ell^b$ in R_ν. By (13.99), each factor of $\Pi_{t-1}(u)$ divides ℓ^h; also $4 \mid u - 1$ since E contains μ_4. Hence $b \leq th \leq yh$. For $t \leq 1$ the empty products give assertion 3 directly.

Let z be as in assertion 4. Its vanishing on an earlier tame generator, and its evenness, remove ramification at the earlier place and its conjugate: on inertia the cocycle is a homomorphism and its ℓ-primary image is generated by that tame value. Apply the reciprocity condition for ψ_j to the remaining places w_i, $i \geq j$. They lie outside T_j; the required local conditions at the earlier type-e_j places hold by the assumed Frobenius vanishing. Moreover $\psi_j(\phi_i) = 0$ for $i > j$, by (13.105). The one-term implication (13.98) now gives $\ell^b z(\tau_j) = 0$.

Geometric Frobenius has cyclotomic value $q_{w_j}^{-1}$, so $\phi_j \in G_\nu$ gives $q_{w_j} \equiv 1 \pmod{\ell^\nu}$. Distinct underlying primes of F give all the required pairwise distinctness, including the conjugates. Finally, when $\ell \leq y^2$, Lemma 13.42 gives $|s| \leq 6y^3$. Since $t + 2 \leq y$, the construction has $|J| + |s| + 1 \leq (t+2)6y^3 \leq 6y^4$. Since $b \leq yh$, this gives the asserted cost. When $h = 0$ the cost is zero for every finite detecting set.

Lean. `New.Selmer.exists_isPrimeTriangle`. Lean takes $b = yh$ and requires $4(u^e - 1)^2 \mid \ell^b$. Here the common Gaussian-binomial denominator suffices because Lemma 13.45 uses only one power of $u^e - 1$. □

The length bound from the even and odd parts

This part assumes only the conclusions defining a prime triangle; it does not assume or reprove the abundance input. For $e \geq 0$ let

$$L_e = \begin{cases} \bigcup_{v \mid \ell} I_v, & e = 0, \\ \bigcup_{v \mid \ell} (D_v \cap \ker \chi), & e > 0. \end{cases}$$

Include all conjugate decomposition and inertia groups in these unions, so the definitions do not depend on a choice of places above v. Let $\mathrm{Sha}_e^-(A_\nu)$ denote the cocycles unramified outside ℓ which vanish on L_e, with the additional condition $z(g^c) = -z(g)$ when $e > 0$. For $e = 0$ impose no sign condition: this module consists of everywhere-unramified homomorphisms. This module is `sha0dd` in Lean.

Let Z_f be the module of functions obeying the integral linear relations annihilating all characteristic-zero crystalline cocycles at the places above ℓ. The usual local crystalline calculation gives

$$z \in Z_f \cap Z^1(G, A_\nu(\chi^{-e})) \quad \Longrightarrow \quad z|_{L_e} = 0. \tag{13.106}$$

Here is the precise characteristic-zero fact used. For $e > 0$, $H^1_f(K, V(-e)) = 0$ for a finite extension $K/\mathbb{Q}_\ell$ and trivial coefficient space V; hence each crystalline cocycle is a coboundary and vanishes where $\chi = 1$. For $e = 0$ the crystalline classes are unramified and the cocycles vanish on inertia. The first assertion follows from the crystalline local exact sequence: the de Rham tangent quotient is zero and $1 - \varphi$ is invertible on $D_{\mathrm{cris}}(V(-e))$. The exponent-zero case gives the usual unramified one-dimensional factor. Evaluation at any specified element of L_e is therefore one of the integral annihilating linear relations. It also annihilates every function obeying those relations modulo ℓ^ν. If the specified element is conjugated into the chosen local decomposition group, the cocycle formula transports its zero value back, because the element acts trivially on the twist. This proves (13.106); it uses the characteristic-zero local calculation, not a claim that arbitrary torsion crystalline classes are torsion coboundaries. Compare the integral-relation argument in the source.

Lemma 13.45 (Residual even kernel)**.** *Let the data form a prime triangle with step exponent b. Fix $0 < e < t$ and let $S \subset Z_e(\{v : v \mid \ell\})$ be an O-submodule whose members vanish on L_e. Set $d = u^e - 1$ and assume*

$$dz(a) = 0, \qquad (z + z \circ c)(\phi_j) = 0 \quad (e_j = e), \qquad z \in S. \tag{13.107}$$

Then

$$\mathrm{length}_O(S/S[\ell^b]) \le \mathrm{length}_O \mathrm{Sha}^-_e(A_\nu). \tag{13.108}$$

Proof. Conjugation preserves the cyclotomic character, the set of allowed ramified places, and L_e. Thus

$$\mathcal{A} : S \longrightarrow \mathrm{Sha}^-_e(A_\nu), \qquad \mathcal{A}(z) = z \circ c - z,$$

is an O-linear map. Its kernel consists of even z. For such z, put $w = z + z \circ c = 2z$. This is an even class unramified outside ℓ. For each $g \in s$, the triangle provides an index j of type e with

$$4z(g) = 2w(g) = w(\phi_j) = 0.$$

The cocycle $4dz$ vanishes on s and at a, by (13.107). Detection gives $4dz = 0$. Since $4d \mid \ell^b$, it follows that $\ell^b z = 0$. Thus $\ker \mathcal{A} \subset S[\ell^b]$. The quotient $S/S[\ell^b]$ is a quotient of $S/\ker \mathcal{A}$, and the latter embeds in $\mathrm{Sha}^-_e(A_\nu)$. Taking lengths proves the result. In particular this proof remains valid at $\ell = 2$.

Lean, comparison. `relLength_torsionBy_le_length_shaOdd`. The detector above already kills $4dz$, so a single factor d suffices. □

Theorem 13.46 (Tate-twist triangle cost)**.** *Assume a prime triangle as in Theorem 13.44, fix $0 \le e < t$, and let*

$$Z \subset Z_e(\{v : v \mid \ell\} \cup \{w_j, \bar{w}_j : j \in J\})$$

be an O-submodule. Suppose every member of Z unramified outside ℓ satisfies $(u^e - 1)z(a) = 0$. Define

$$\tau'_j = (\tau_j^c)^{-1}, \quad \phi'_j = (\phi_j^c)^{-1}, \quad I_e = \{j : e_j = e\}, \quad Z_0 = \{z \in Z : z|_{L_e} = 0\},$$

and let

$$B_e = \mathrm{length}_O\big(Z/(Z \cap Z_f)\big) + \mathrm{length}_O \mathrm{Sha}^-_e(A_\nu). \tag{13.109}$$

Then $Z, \tau, \tau', \phi, \phi', I_e$ have HasTriangleCost *with multiplier ℓ^b and cost B_e. Explicitly:*

1. *If $j \in I_e$, $z \in Z_0$, and*

$$z(\tau_i) = z(\tau'_i) \quad (i < j), \qquad z(\phi_i) = z(\phi'_i) \quad (i < j,\ i \in I_e),$$

then $\ell^b z(\tau_j) = \ell^b z(\tau'_j)$.

2. If $S \subset Z_0$ and every $z \in S$ vanishes on all τ_j, τ_j' and satisfies $z(\phi_j) = z(\phi_j')$ for $j \in I_e$, then

$$\operatorname{length}_O(Z/Z_0) + \operatorname{length}_O(S/S[\ell^b]) \le B_e. \tag{13.110}$$

Proof. Because $\chi(\tau_j) = 1$ and $\chi(\phi_j) \equiv 1$ modulo ℓ^ν, the cocycle formula for inverses gives

$$z(\tau_j') = -z(\tau_j^c), \qquad z(\phi_j') = -z(\phi_j^c).$$

For $e > 0$ the even part $z + z \circ c$ lies in $\mathcal{E}_e(\{w_j\})$. Agreement of the paired values in assertion 1 says exactly that this even part vanishes on the earlier tame generators and the earlier same-type Frobenii. The triangular vanishing assertion therefore proves assertion 1. For $e = 0$, I_e is empty and assertion 1 has no instances.

By (13.106), $Z \cap Z_f \subset Z_0$, so

$$\operatorname{length}_O(Z/Z_0) \le \operatorname{length}_O(Z/(Z \cap Z_f)).$$

For S as in assertion 2, vanishing at both tame generators removes ramification at every w_j and $\bar{w}_j$. Thus its members are unramified away from ℓ. If $e = 0$, they belong directly to $\mathrm{Sha}_0^-(A_\nu)$, giving the desired second length bound. If $e > 0$, paired Frobenius agreement is equivalent to $(z + z \circ c)(\phi_j) = 0$ for $j \in I_e$. The other hypotheses of Lemma 13.45 follow from the assumptions on Z and Z_0. Apply that lemma and add the two length bounds.

Lean, formal context. `OddSha` (L645-L679), `FitTwistsExist` (L377). □

No cardinality of an even Selmer group appears in (13.109). The odd-cocycle term is essential. The next subsections construct the integer and prove the finite length estimates used in OddSha: for all $e < t$ and all depths its length is bounded by $v_O(X_-)$ for $X_- = \mathcal{X}$ of (14.3), with $\mathcal{X} \le h_E T^{24T^5}$ and $T = N + t + 3$. The coefficient ring must contain the values of all Dirichlet characters modulo N. Theorem 14.5 proves the bound by the refined Coates–Sinnott theorem at odd primes and by Stickelberger's theorem and Kummer theory at 2.

Corollary 13.47 (Application to the tied character cocycles)**.** *In the finite common-Ribet construction, take $\chi_e = \chi^{-e}$ and $p = \ell^\nu$. Assume the family u_x is continuous and unramified outside ℓ, a fixed finite set of old places away from ℓ, and the new places $w_j, \bar{w}_j$. The old elements δ_w are tame generators covering all those old places, including their conjugates. Assume the scalar normalizing element γ_* of (14.14) belongs to an inertia group at a place away from ℓ. This γ_* is the element denoted by $\gamma = \gamma_\pi$ in that equation; it is distinct in role from the cyclotomic generator a^{-1} used above.*

Let Z be any O-submodule of the character cocycles read from the tied rows, so that its members preserve the O-linear relations among $u_x(g) - 1$, vanish at the δ_w, and obey (14.20). Then evaluation on

$$s' = \{a\} \cup s \cup \{\tau_j, \tau_j' : j \in J\} \quad \text{is injective on } Z, \qquad |s'| \le 2(|J| + |s| + 1). \tag{13.111}$$

Moreover Z has the triangle cost of Theorem 13.46. Its local term is bounded by the crystalline colength of the ambient module of relation-preserving character cocycles. These are precisely the character detection and cost hypotheses in (14.21).

Proof. Lemma 13.18 makes each relation-preserving cocycle continuous and zero wherever all $u_x(g)$ equal 1; apply its finite-coefficient version, or embed $O/\ell^\nu O$ into K/O by $x \mapsto \ell^{-\nu} x$. Thus the cocycle is initially unramified outside the indicated old and new places and ℓ. On inertia away from ℓ, $\chi_e = 1$, so the restriction is a homomorphism to an ℓ-primary group. Its wild-inertia image is zero, and its tame image is generated by the value at a tame generator. The equalities

$z(\delta_w) = 0$ therefore remove all old ramification. We have obtained the ramification hypothesis of Theorem 13.46.

Whenever such a z is unramified outside ℓ, the stated inertial hypothesis implies $z(\gamma_*) = 0$. The tied-normalizer row gives

$$(\chi_e(g) - 1)(c_1 z(\gamma_*) + z(a)) = 0 \qquad (g \in G),$$

as proved in Lemma 14.8. Taking $g = a$ now gives $(u^e - 1)z(a) = 0$, exactly the prerequisite of Theorem 13.46. In the application the scalar normalizer is chosen in inertia at the original level, away from ℓ, so the inertial hypothesis is the one actually available.

If z vanishes on s', the two new tame values remove all new ramification. Lemma 13.42 then gives $z = 0$ from its values at a, s. Applying this to differences proves injectivity. The displayed cardinality bound follows from $|s'| \le |s| + 1 + 2|J|$. If $\mathcal{R}_e$ denotes the ambient module of relation-preserving character cocycles, the inclusion

$$Z/(Z \cap Z_f) \hookrightarrow \mathcal{R}_e/(\mathcal{R}_e \cap Z_f)$$

bounds its local length by crystallineColength(χ_e, u, ν). Finally the polynomials H_j constructed below satisfy $H_j(1) = 1$, and $\chi_e(\phi_j), \chi_e(\phi'_j) \equiv 1$ modulo ℓ^ν. Their evaluations therefore satisfy the congruence (14.17). The character separation required in (14.8) is exactly $D_e \mid \ell^b$ from the prime triangle. This checks the remaining character hypotheses with the stated uniform bound on the number of detecting elements.

Lean, formal context. `OddSha` (L645-L679), `FitTwistsExist` (L377). □

In the notation of the numerical allowance, one may consequently take $k = |J|$, $n_{\mathrm{test}} = |s| + 1$, $n_{\mathrm{det}} = |s'|$, and step exponent $b \le yh$. Assertion 5 of Theorem 13.44 and (13.111) give exactly (25.1). The factor two for the conjugate tame generators is already included in that allowance.

Common integral root-chain factors at the chosen primes

We record the finite algebra which turns a root chain for each weak system into a single integral polynomial for each auxiliary place. All integrality in this subsection is ℓ-adic, transported through the fixed isomorphism $\iota : \mathbb{C} \simeq \overline{\mathbb{Q}}_\ell$ when a complex polynomial occurs.

Lemma 13.48 (An integral multiple with value one)**.** *Let $A, A', B, C \in \overline{\mathbb{Q}}_\ell[X]$, with B monic and C integral. Suppose coefficientwise modulo the maximal ideal*

$$AB \equiv C \equiv A'B, \qquad |A'(1)|_\ell \ge 1.$$

Then A is integral, $A(1)$ is a unit, and $H = A/A(1)$ is integral with $H(1) = 1$ and $A \mid H$.

Proof. The Gauss norm on polynomials is multiplicative, and a monic polynomial has Gauss norm at least one. From $AB \equiv C$ and the integrality of C one obtains $\|AB\|_{\mathrm{G}} \le 1$, hence $\|A\|_{\mathrm{G}} \le 1$. Also $\|(A - A')B\|_{\mathrm{G}} < 1$, so $\|A - A'\|_{\mathrm{G}} < 1$ and $|A(1) - A'(1)|_\ell < 1$. If $|A(1)|_\ell < 1$, the ultrametric inequality contradicts $|A'(1)|_\ell \ge 1$. Thus $|A(1)|_\ell = 1$, and the asserted formula for H proves everything. The strict Gauss bounds are legitimate because each polynomial has only finitely many coefficients. Compare the monic-cancellation calculation.

Lean. `HeckeStrings` (L155-L211). □

Let $\mathcal{X}$ be the finite set of systems at the fixed base level. Write ρ_x for their Galois representations, α for the normalizing character, and $N_x = \alpha\rho_x$. The representation-theoretic inputs needed here are the following ones from the preceding local discussion:

1. The characteristic polynomials of $N_x(g)$ are integral. A weak lift means that 1 is a root of those polynomials for every $g \in G$.

2. At a good split place $w \nmid \ell$, spherical compatibility makes the Hecke polynomial annihilate $\rho_x(\phi_w)$. The normalizing scalar $z_0 = \iota^{-1}(\alpha(\phi_w))^{-1}$ satisfies $|z_0|^2 q_w^t = q_w^n$ in the complex absolute value.

3. The spherical unitary root-string implication proved in Lemma 13.30 says that this root z_0 yields all roots $z_0 q_w^i$, $0 \le i < t$. Thus, for a weak lift,

$$P_{x,w}(z_0 X) = T_{x,w}(X) \prod_{0 \le i < t} (X - q_w^i). \tag{13.112}$$

   The factors are distinct because $q_w > 1$.

4. For most Frobenius data, the attached-representation condition gives coefficientwise congruence of the Hecke polynomial and $\mathrm{char}(\rho_x(\phi_w))$ modulo the ℓ-adic maximal ideal. This can be imposed simultaneously for the finitely many $x \in \mathcal{X}$.

The step from item 2 to the initial root is elementary: an annihilating polynomial vanishes at every eigenvalue of the matrix, and $N_x(\phi_w)$ has eigenvalue 1. Items 2 and 4 state the precise compatibility input; the present argument does not deduce them from a characteristic-polynomial congruence.

For such a datum put

$$B_\phi(X) = \prod_{0 \le i < t} (X - \chi(\phi)^{-i}), \qquad C_{x,\phi}(X) = \mathrm{char}(N_x(\phi)).$$

Transporting (13.112) through ι and normalizing a degree-n characteristic polynomial gives

$$\big(\alpha(\phi)^n \iota(T_{x,w})\big) B_\phi \equiv C_{x,\phi}. \tag{13.113}$$

The scalar $\alpha(\phi)$ has ℓ-adic norm one, since it is the value of a continuous character of the compact group G. If $C_{x,\phi} \equiv S B_\phi$ with $|S(1)|_\ell \ge 1$, apply Lemma 13.48 to $A = \alpha(\phi)^n \iota(T_{x,w})$ and $A' = S$. It gives an integral polynomial $H_{x,w}$ with $H_{x,w}(1) = 1$, divisible by $\iota(T_{x,w})$. Thus (13.112) is precisely the required $\mathrm{HasRootChain}(P_{x,w}, H_{x,w}, q_w, z_0, t)$. This proves the finite algebra in eventually_exists_hasRootChain from the listed representation inputs and the local root-string lemma.

Lemma 13.49 (The scalar shift preserves all triangle tests)**.** *Let $T : G \to \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell)$ be the continuous main representation after the auxiliary character twist. Suppose there is $\eta \in \ker \chi$ with $T(\eta) = z I_r$ and*

$$|(z^{\ell^\nu})^j - 1|_\ell = 1 \quad (\nu \ge 0,\ 1 \le j \le 4r+2).$$

For every ν and $g \in G$ some $\delta = \eta^i$ has $\chi(\delta) = 1$ and

$$\det(1 - T(\phi)) \text{ is a unit} \quad \text{for all } \phi \text{ sufficiently near } \delta^{\ell^\nu} g.$$

Consequently the condition $|\mathrm{char}(T(\phi))(1)|_\ell \ge 1$ can be included in W in Theorem 13.44, together with the finitely many root-chain conditions just described.

Proof. The characteristic coefficients of $T(g)$ are integral. One can see this directly from compactness: the powers of $T(g)$ and their inverses lie in a bounded set, so each eigenvalue has absolute value one. Consider the integral polynomial $P(Z) = \det(1 - ZT(g))$, of degree at most r and constant term 1. The $r+1$ residue classes $1, z^{\ell^\nu}, \dots, (z^{\ell^\nu})^r$ are distinct. Its nonzero reduction cannot vanish

at all of them, so $P((z^{\ell^\nu})^i)$ is a unit for some $0 \le i \le r$. Since $T((\eta^i)^{\ell^\nu} g) = (z^{\ell^\nu})^i T(g)$, this is the desired unit determinant at the center point. Continuity gives the neighborhood assertion.

Intersect this open condition with the root-chain conditions for the finite family $\mathcal{X}$ and the cofinite conditions excluding bad places and the base level. The most-Frobenii condition and Chebotarev allow a datum in every sufficiently restricted neighborhood, after any further finite exclusions. This is the approximation property required of W. In the application η comes from scalar inertia at the original level, where the auxiliary twist is trivial; hence the scalar identity for the twisted T is indeed the displayed one. The use of δ^{ℓ^ν} is essential: it contributes zero to all R_ν-valued triangle tests in (13.102).

Lean, formal context. `HeckeStrings` (L345-L416), `FitTwistsExist` (L377). □

Finally, suppose a fixed base system x_0 has the lift trace

$$\operatorname{tr} N_{x_0}(g) = \operatorname{tr} T(g) + \sum_{0 \le i < t} \chi(g)^{-i} \quad (g \in G). \tag{13.114}$$

Apply this identity to g^k for all $k \ge 1$. Newton identities in characteristic zero give

$$C_{x_0,g}(X) = \operatorname{char}(T(g))(X) B_g(X). \tag{13.115}$$

There are only finitely many weak base systems x whose characteristic coefficients are congruent to those of x_0 at every g. Call this set $\mathcal{X}_{\mathrm{near}}$. For the primes supplied by Theorem 13.44 with the W of Lemma 13.49, the preceding polynomial argument applies to every $x \in \mathcal{X}_{\mathrm{near}}$ with $S = \operatorname{char}(T(\phi_j))$. Define

$$H_j = \prod_{x \in \mathcal{X}_{\mathrm{near}}} H_{x,w_j}. \tag{13.116}$$

Then H_j is integral and $H_j(1) = 1$. Each T_{x,w_j} divides its corresponding factor and hence divides H_j, so the same H_j gives (13.69) at w_j for every nearby weak base-level system. All these finitely many polynomials lie in one finite coefficient extension if such an extension is needed for the subsequent integral algebra.

Choice of the exponent ν

The separator n_0 of the earlier denominator argument depends only on the finite base-level systems. The fixed stabilizer integer c_* is also base data. In the denominator application, the choice of u, h for the denominator estimate verifies (13.99) before this step. Equation (13.101) determines m independently of ν. We may therefore choose $\nu > 0$ large enough that

$$(n_0 c_*^2)^{m+1} \mid \ell^\nu \tag{13.117}$$

in the coefficient valuation ring, and only then choose the auxiliary primes, the larger-level systems, and their common polynomials H_j. For example it suffices that

$$\nu \ge (m+1) v_\ell(n_0 c_*^2),$$

where $v_\ell(\ell) = 1$; take an integer ceiling and increase it to at least one. The divisibility is in the local coefficient ring, not an assertion that a general integer has no prime factors other than ℓ.

Theorem 13.41 bounds the number of auxiliary primes in the construction leading to (13.109). The congruences between roots and the compatibility of the spherical Hecke operators give the common factors H_j. Equation (13.117) specifies the modulus for polynomial cancellation. The uniform length bound of Theorem 14.5 follows from Lemma 14.2 at odd coefficient primes and

Lemma 14.4 at 2. The scalar-inertia property for the chosen main rows is supplied by Theorems 17.87 and 17.94; Theorem 17.105 supplies the two stated compatibility properties. Together with the preceding prime-triangle construction, these give the required arithmetic and local representation setup, with the stipulated lift trace (13.114) retained. These dependencies remain separate from the finite induction and the even/odd length estimate.

14 Cyclotomic cocycles and the common congruence polynomial

Theorem 14.5 bounds the length of the odd part of a module of cyclotomic cocycles. This bound enters the construction of the common congruence polynomial in Theorem 14.11.

A common multiple of generalized Bernoulli values controls the length. For odd coefficient primes the refined Coates–Sinnott theorem bounds the length of a Galois cohomology group. For the prime 2, Stickelberger's theorem gives an annihilator and Kummer theory bounds the number of evaluations needed to detect a cocycle. The formal finite-level estimate is `exists_length_shaOdd_le`.

Let ℓ be a prime and $N \geq 1$ an integer. Let μ_N denote the group of N-th roots of unity, $E \subseteq \mathbb{Q}(\mu_N)$ a CM field, $G_E = \mathrm{Gal}(\overline{\mathbb{Q}}/E)$, and $D_v \subset G_E$ a decomposition group at v. Let $\mathcal{O}$ be the ring of integers of a finite extension of $\mathbb{Q}_\ell$, with normalized valuation $v_{\mathcal{O}}$. Assume that $\mathcal{O}$ contains the values of all characters of $A = (\mathbb{Z}/N\mathbb{Z})^\times$. For an integer $\nu \geq 0$, put $R_\nu = \mathcal{O}/\ell^\nu\mathcal{O}$. Write ϵ for the ℓ-adic cyclotomic character and c for complex conjugation. For $e > 0$, the module $S_{e,\nu}$ consists of locally constant functions $x : G_E \to R_\nu$ such that

$$x(gh) = x(g) + \epsilon(g)^{-e}x(h), \qquad x(cgc) = -x(g),$$

which vanish on inertia away from ℓ, and on $D_v \cap \ker\epsilon$ for every place above ℓ. For $e = 0$, let $S_{0,\nu}$ be the everywhere unramified continuous homomorphisms $G_E \to R_\nu$, with no parity restriction. In both cases the elements are cocycles themselves, including those that are coboundaries. This is the Lean definition of `shaOdd`.

Generalized Bernoulli values. Fix an integer $t \geq 0$. For $0 < e < t$ put $n = e + 1$, $C_e = (e + 2)!(e + 1)N$, and

$$v_\psi = -2C_eL(-e, \psi), \qquad d_e = \left| \prod_{\psi(-1)=(-1)^n} v_\psi \right|, \qquad X_e = d_e \prod_{a\in\{N+1,2N+1\}} (a^e - 1)(a^n - 1).$$

Here ψ runs over Dirichlet characters modulo N satisfying $\psi(-1) = (-1)^n$, including imprimitive characters of this parity. Write $T = N + t + 3$.

Lemma 14.1 (A common multiple of generalized Bernoulli values)**.** *The v_ψ are nonzero algebraic integers and d_e, X_e are positive integers, with $X_e \leq T^{T^2}$. For $a \equiv 1 \bmod N$, $a > 0$, and $m \geq 1$ prime to a,*

$$(a^n - 1)v_\psi \equiv 2C_ea^e \sum_{j=0}^{Nm-1} \left\lfloor \frac{aj}{Nm} \right\rfloor j^e\psi(j) \pmod{m\mathbb{Z}[\psi]}. \tag{14.1}$$

Here $\mathbb{Z}[\psi]$ is the ring generated by the values of ψ.

Proof. Write $B_n(X)$ for the Bernoulli polynomial with $B_1(X) = X - \frac{1}{2}$. For each $0 \leq j < N$, the difference formula for Bernoulli polynomials gives

$$F_j(X) = \frac{N^n}{n+1}\left(B_{n+1}\left(X + \frac{j}{N}\right) - B_{n+1}\left(\frac{j}{N}\right)\right), \qquad F_j(m) = \sum_{i=0}^{m-1}(j + iN)^n \quad (m \geq 0).$$

The polynomial F_j is integer-valued on nonnegative integers. Its Newton expansion in $\binom{X}{k}$, $0 \leq k \leq n + 1$, has integral coefficients [Fri13, Remark 3(i)–(ii)]. Indeed, successive evaluation at $X = 0, 1, \ldots, n + 1$ gives a unit lower triangular integer matrix $(\binom{i}{k})$. Thus $(n + 1)!F_j \in \mathbb{Z}[X]$. Its

constant term is zero and its linear coefficient is $\beta_j = (n+1)!N^n B_n(j/N)$, by $B'_{n+1} = (n+1)B_n$. Evaluating modulo m^2 gives

$$(n+1)! \sum_{i=0}^{m-1} (j+iN)^n \equiv m\beta_j \pmod{m^2\mathbb{Z}}. \tag{14.2}$$

Set $f = Nm$, and assume $(a,m) = 1$, $a \equiv 1 \pmod{N}$. Multiplication by a permutes residues modulo f. For $k_j = \lfloor aj/f \rfloor$, expand $(aj - fk_j)^{e+1}$ modulo f^2; its character value is $\psi(j)$. Summing gives

$$(a^{e+1} - 1) \sum_{j<f} \psi(j) j^{e+1} \equiv (e+1)a^e f \sum_{j<f} k_j j^e \psi(j) \pmod{f^2\mathbb{Z}[\psi]}.$$

Multiply by $(e+2)!$, partition the first sum into residue classes modulo N, apply (14.2), and cancel the integer m in $\mathbb{Z}[\psi]$. The resulting congruence is (14.1), since

$$v_\chi = 2 \sum_{j=0}^{N-1} \chi(j)\beta_j = -2C_e L(-e, \chi).$$

The equality follows by grouping $L(s,\chi)$ into residue classes and using $\zeta(1-k,u) = -B_k(u)/k$ at $k = e+1$ [Apo76, p. 255, (8), and p. 264, (17)]. For $N = 1$ use $B_{e+1}(1) = B_{e+1}(0)$. If χ^* induces χ primitively, the parity condition makes the gamma factor at $-e$ finite; the functional equation and the Euler product at $e+1 > 1$ give $L(-e,\chi^*) \neq 0$. To recover $L(-e,\chi)$, multiply by $1 - \chi^*(p)p^e$ for each $p \mid N$ with $p \nmid \operatorname{cond}(\chi^*)$. These factors are nonzero since $p^e > 1$. Thus every v_χ is a nonzero algebraic integer.

Galois conjugation permutes these nonzero algebraic integers, so their product is a nonzero rational integer. In particular each v_χ divides d_e in the ring of integers containing the character values. The two choices $N+1, 2N+1$ are relatively prime, so at least one is an ℓ-adic unit for each ℓ.

For $k \geq 1$, integrating $B_k = B'_{k+1}/(k+1)$ on $[0,1]$ gives zero since $B_{k+1}(1) = B_{k+1}(0)$. Thus B_k has a zero there. Now $B'_k = kB_{k-1}$ and $B_0 = 1$ give $\|B_k\|_{\infty,[0,1]} \leq k!$ by induction. Thus $|v_\chi| \leq 2\varphi(N)(e+2)!N^{e+1}(e+1)! \leq T^{3e+6}$. There are at most N factors in d_e. The four extra factors in X_e have product at most T^{8e+4}, since $2N+1 \leq T^2$. Hence

$$\log_T X_e \leq N(3e+6) + 8e + 4 \leq 4(N+2)(e+2) \leq (N+e+4)^2 \leq T^2.$$

Lean. The special-value identity, parity test and finite congruence are `sum_mul_bernoulli_eq_neg_mul_LFunction`, `sum_mul_bernoulli_ne_zero` and `voronoi_congruence_bernoulli`. The code constructs its multiple by a parity-corrected determinant in `exists_isStickelbergerBound`. The Newton expansion gives the progression congruence directly, and d_e is the product over the characters of the required parity. □

14.1 The length bound at finite cyclotomic level

Lemma 14.2 (Odd coefficient primes)**.** *Let $0 < e < t$ and ℓ be odd. Then*

$$\operatorname{length}_{\mathcal{O}} S_{e,\nu} \leq [E : \mathbb{Q}]\, v_{\mathcal{O}}(X_e).$$

If $\ell > T$, the right hand side may be replaced by $v_{\mathcal{O}}(d_e)$.

Proof. First take coefficients $\mathbb{Z}/\ell^\nu$. A coboundary for the twist $-e$ takes the same value at g and *cgc*. Since 2 is a unit, the odd cocycles inject into cohomology. Their classes lie in the image of $P = (1 + (-1)^n c)/2$, since c acts on the coefficients by $(-1)^e$.

Choose $a \in \{N+1, 2N+1\}$ prime to ℓ. At each $v \mid \ell$ the cyclotomic image of inertia contains a. Indeed, local inertia surjects onto the Galois group of the totally ramified extension $\mathbb{Q}_\ell(\mu_{\ell^\infty})/\mathbb{Q}_\ell$. A lift of a fixes the prime-to-ℓ roots of unity, and fixes the ℓ-part of μ_N since $a \equiv 1 \bmod N$. It therefore fixes E. By the local condition every cocycle on D_v descends to $\epsilon(D_v)$. This closed subgroup of $\mathbb{Z}_\ell^\times$ is procyclic. Evaluation at a topological generator γ embeds its first cohomology in $(\mathbb{Z}/\ell^\nu)/(\gamma^{-e}-1)$: a continuous cocycle is determined by its value there, and those of coboundaries form the displayed ideal. Since a is a limit of powers of γ, the length is at most $v_\ell(a^e-1)$ at each such place.

Put $U = \operatorname{Spec} \mathcal{O}_E[1/\ell]$ and $V = \operatorname{Spec} \mathcal{O}_E[1/N\ell]$. The classes with zero local images belong to the kernel of localization in $H^1(U, \mathbb{Z}/\ell^\nu(-e))$. The archimedean local groups vanish. Poitou–Tate duality pairs this kernel perfectly with the corresponding kernel in $H^2(U, \mathbb{Z}/\ell^\nu(n))$ [Mil06, Ch. I, Theorem 4.10 and Ch. II, §2]. The pairing is invariant under c, so it respects P. Integral H^2 surjects onto finite-coefficient H^2 here: the Galois group unramified outside ℓ has ℓ-cohomological dimension at most two, and finiteness at finite levels gives vanishing of integral H^3 [GP15, §6.2]. Localization embeds $H^2(U, \mathbb{Z}_\ell(n))$ into $H^2(V, \mathbb{Z}_\ell(n))$: its kernel comes from the $H^0(k_w, \mathbb{Z}_\ell(e))$ at $w \mid N,\ w \nmid \ell$, which vanish because $|k_w|^e \neq 1$.

Write $G = \operatorname{Gal}(E/\mathbb{Q})$. Apply the refined Coates–Sinnott theorem [GP15, Theorem 6.11] with Euler factors removed at the primes dividing N. The required Iwasawa μ-invariant vanishes by Ferrero–Washington since $E(\mu_\ell)/\mathbb{Q}$ is abelian [FW79]. The torsion in $H^1(V, \mathbb{Z}_\ell(n))$ is $H^0(E, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))$ [GP15, Lemma 6.9] and is killed by a^n-1. Consequently $(a^n-1)\Theta(1-n)$ belongs to the zeroth Fitting ideal of $PH^2(V, \mathbb{Z}_\ell(n))$ over $P\mathbb{Z}_\ell[G]$, where Θ is the equivariant incomplete L-function.

Extend scalars to $\mathcal{O}$. The cocycle conditions extend by flatness, or directly by expanding their values in a $\mathbb{Z}_\ell$-basis of $\mathcal{O}$. Set $A = P\mathcal{O}[G]$ and $M = PH^2(V, \mathbb{Z}_\ell(n)) \otimes \mathcal{O}$. The fraction field of $\mathcal{O}$ contains the character values. Intersecting A with successive sums of its character lines over that field gives a filtration of A by ideals. Each quotient is a torsion-free rank-one $\mathcal{O}$-lattice on which G acts by one character ψ with $\psi(c) = (-1)^n$. Denote this module by $\mathcal{O}_\psi$. By right exactness of tensor product,

$$\operatorname{length}_{\mathcal{O}} M \leq \sum_{\psi} \operatorname{length}_{\mathcal{O}}(\mathcal{O}_\psi \otimes_A M).$$

Fitting ideals commute with specialization [Stacks, Lemma 15.8.4]. Over $\mathcal{O}$ each term is bounded by $v_{\mathcal{O}}((a^n-1)L(-e, \psi^{-1}))$. Here the characters of G are inflated to modulus N, so their Euler factors are those of Lemma 14.1. Inversion permutes the characters of this parity. Since the $v_\psi = -2C_eL(-e, \psi)$ are integral and their product over all characters of this parity has absolute value d_e, the sum is at most $[E:\mathbb{Q}]v_{\mathcal{O}}(a^n-1) + v_{\mathcal{O}}(d_e)$. There are at most $[E:\mathbb{Q}]$ places above ℓ. Adding the local contributions proves the first bound by the definition of X_e.

If $\ell > T$, $E_v/\mathbb{Q}_\ell$ is unramified, and $\epsilon(D_v)$ surjects onto $\mathbb{F}_\ell^\times$. For a topological generator γ of this image, both $\gamma^{-e}-1$ and γ^n-1 are units since $0 < e, n < \ell-1$. The first gives $H^1(\epsilon(D_v), (\mathbb{Z}/\ell^\nu)(-e)) = 0$ by evaluation at γ. The second gives $H^0(E, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) = 0$, so the integral $H^1(V, \mathbb{Z}_\ell(n))$ has no torsion. Thus $\Theta(1-n)$ itself belongs to the Fitting ideal. The same sum gives the bound $v_{\mathcal{O}}(d_e)$.

Lean, comparison. The finite-level proof uses character reconstruction and Jacobi-sum derivatives in `exists_detecting_points_component`. Here the refined Coates–Sinnott theorem and the integral character filtration also cover primes dividing $[E:\mathbb{Q}]$. □

Lemma 14.3 (The dyadic annihilator)**.** *For $0 < e < t$ and $\ell = 2$, the integer $2\varphi(N)X_e$ annihilates $S_{e,\nu}$.*

Proof. For $\nu \leq 1$, $2R_\nu = 0$. Suppose $\nu \geq 2$, put $f = N2^\nu$, $F = \mathbb{Q}(\mu_f)$, and choose an odd $a \in \{N+1, 2N+1\}$. For $x \in S_{e,\nu}$, its restriction to G_F is additive, since the twist is trivial there

modulo 2^ν. As in the preceding proof, $a \in \epsilon(D_v)$ for $v \mid 2$. The restriction of x to D_v descends to the abelian image $\epsilon(D_v)$. Commuting a lift of a past $h \in D_v \cap G_F$ in the cocycle identity gives $(a^e - 1)x(h) = 0$. Thus $w = (a^e - 1)x|_{G_F}$ is unramified at 2 and, by the defining conditions, at all other finite places. By global reciprocity w is a character of $\mathrm{Cl}(F)$ [Mil20, Ch. V, §3].

For $u \in (\mathbb{Z}/f)^\times$, lift $\sigma_u : \zeta \mapsto \zeta^u$ to $s_u \in G_\mathbb{Q}$ and put $(T_u w)(h) = u^{-e} w(s_u^{-1} h s_u)$. This is independent of the lift, since w is a homomorphism. These operators give a group action. If $u \equiv 1 \bmod N$, then s_u fixes E, so the cocycle rule gives $w(s_u^{-1} h s_u) = u^e w(h)$. Thus the action on the translates of w factors through $A = (\mathbb{Z}/N)^\times$. Moreover $T_{-1} w = (-1)^n w$ by raw oddness.

The classical Stickelberger element is $\theta_F = \sum_u \{u/f\}\sigma_u^{-1}$, where braces denote fractional part. The integral element $(1 - c)(a - \sigma_a)\theta_F$ annihilates $\mathrm{Cl}(F)$ [Was97, Theorem 6.10]; its coefficient at σ_u^{-1} is $F_a(u/f)$, where $F_a(y) = \lfloor a\{y\} \rfloor - \lfloor a\{-y\} \rfloor$. Applying w to its action on an ideal class gives $\sum_u F_a(u/f) u^e T_u w = 0$.

For each character of A with $\psi(-1) = (-1)^n$, put $B_\psi = \sum_{v \in A} \psi(v)^{-1} T_v$. Since $B_\psi T_u = \psi(u) B_\psi$, the last equality gives $\Sigma B_\psi w = 0$, where $\Sigma = \sum_u F_a(u/f) u^e \psi(u)$. To compare this scalar with v_ψ, let S_r be the floor sum in (14.1) at $m = 2^r$. Removing the even terms from S_ν leaves $S_\nu - 2^e \psi(2) S_{\nu-1}$; the remaining nonunits contribute zero. The change $u \mapsto -u$ and the character parity therefore give

$$\Sigma = 2(S_\nu - 2^e \psi(2) S_{\nu-1}) \quad \text{in } R_\nu.$$

Use (14.1) at the two depths. Since $e \geq 1$, the lower-depth congruence multiplied by 2^e holds modulo 2^ν. It follows that $C_e a^e \Sigma = (1 - 2^e \psi(2))(a^n - 1) v_\psi$. The Euler factor is a unit in R_ν, and $v_\psi \mid d_e$. Hence $(a^n - 1) d_e B_\psi w = 0$.

The sum of these B_ψ acts on w as $\varphi(N)$. Indeed the characters of $A/\langle -1 \rangle$ have sum zero off $\{1, -1\}$ by orthogonality. In the specified parity the coefficients at $1, -1$ are $\varphi(N)/2, (-1)^n \varphi(N)/2$, respectively. These are integer identities before reduction, and $-1 \neq 1$ in A since E is CM. Thus $\varphi(N) X_e x$ vanishes on G_F and descends to $\mathrm{Gal}(F/E)$. Complex conjugation acts trivially on this abelian quotient. Oddness gives $2\varphi(N) X_e x = 0$.

Lean, comparison. Lean constructs Jacobi quotients and reconstructs the cocycle from character components in `exists_isJacobiFamily` and `ReciprocityComponentBounds` (L281-L390). Here Stickelberger's theorem is applied to the restricted cocycle before taking character sums. □

Lemma 14.4 (Dyadic square classes and length)**.** *For $0 < e < t$, let $K = \lceil N(5 + \log_2 N) \rceil$. At $\ell = 2$ one has*

$$\mathrm{length}_\mathcal{O} S_{e,\nu} \leq K v_\mathcal{O}(2\varphi(N) X_e).$$

Proof. The last nonzero power-of-two multiple of any nonzero x is a homomorphism to $R_\nu[2]$, since the cyclotomic twist acts trivially there. It factors through the maximal elementary abelian 2-quotient of G_E unramified outside the set S of dyadic places. Lifts of a basis of this quotient detect every x.

The Kummer count in the proof of Lemma 13.42, now with $\ell = 2$ and $L = E$, bounds this dimension by $r_2(E) + |S| + \log_2 h_E$. No coset representatives are needed, since $\mu_2 \subset E$. Put $D = [E : \mathbb{Q}]$. The tower discriminant formula gives $|\operatorname{disc} E|^{1/D} \leq |\operatorname{disc} \mathbb{Q}(\mu_N)|^{1/\varphi(N)} \leq N$. The same ideal count therefore gives $h_E \leq (8N)^D$. At most $D(5 + \log_2 N) \leq K$ evaluations suffice. By Lemma 14.3 each evaluation belongs to $R_\nu[2\varphi(N) X_e]$, whose length is at most $v_\mathcal{O}(2\varphi(N) X_e)$. The evaluation map is injective, giving the claim.

Lean, comparison. Lean passes to a larger cyclotomic field in `New.OddPart.exists_detecting_points_cocycles` and bounds length in `OddPart.length_le_of_detecting_points`. For the last nonzero dyadic multiple, Kummer theory over E suffices. □

Theorem 14.5 (A uniform bound at finite cyclotomic level)**.** *Let the CM field $E \subseteq \mathbb{Q}(\mu_N)$, the coefficient ring $\mathcal{O}$, and the odd cocycle modules $S_{e,\nu}$ be as in the definition of the odd local conditions.*

Let h_E be the class number of E. For $t \geq 0$, put $T = N + t + 3$, let $K = \lceil N(5 + \log_2 N) \rceil$, and use the integers X_e of Lemma 14.1. Then

$$\mathcal{X} = h_E \left(\prod_{0<e<t} 2\varphi(N) X_e \right)^K \tag{14.3}$$

satisfies

$$1 \leq \mathcal{X} \leq h_E T^{2KT^3} \leq h_E T^{24T^5}, \qquad \operatorname{length}_{\mathcal{O}} S_{e,\nu} \leq v_{\mathcal{O}}(\mathcal{X}) \quad (0 \leq e < t)$$

simultaneously for every coefficient prime ℓ, every allowed $\mathcal{O}$, and every ν.

Proof. For $e = 0$, class field theory identifies the group with $\operatorname{Hom}(\mathrm{Cl}(E), R_\nu)$. Decomposition of the ℓ-part of the class group into cyclic factors bounds its length by $v_{\mathcal{O}}(h_E)$. This includes the only twist that occurs when $t = 1$.

For $0 < e < t$, Lemmas 14.2 and 14.4 give the length bound, since $[E : \mathbb{Q}] \leq N \leq K$. By Lemma 14.1, $X_e \leq T^{T^2}$, and $2\varphi(N) \leq T^2$. There are fewer than T factors. This gives the first size bound in the statement, including the empty product; the second follows from $K \leq 6T^2$.

Lean, comparison. `New.exists_length_shaOdd_le` gives $h_E 2^{4T^{27}}$. The two prime estimates above give the smaller common integer in (14.3). □

14.2 Constructing one common Ribet polynomial

We now prove the finite algebra behind the common polynomial in (25.16). The construction uses one matrix for the main summand, character cocycles satisfying a stated triangular length bound, and the trace relations defining the tied Hecke ideal. The same polynomial works for every determinant β_g. For the corresponding Lean constructions see `RibetMethod`, the character rows, and the common-element theorem.

Coefficient rings and trace relations. Let O be a discrete valuation ring, with fraction field K and additive valuation v_O, normalized to give a uniformizer value 1. Let G be a group. Let Y and J be finite sets, with J linearly ordered, and put

$$R = O[X_j : j \in J], \qquad \mathcal{L} = \prod_{x \in Y} K[X_j : j \in J], \qquad \mathcal{A} = R[G].$$

Fix a common rank n and representations $u_x : G \longrightarrow \mathrm{GL}_n(O)$ for $x \in Y$. Their extensions to $\mathcal{A}$ are also denoted u_x. Let $\mathbb{S} \subseteq \mathcal{L}$ be an O-subalgebra containing the constant families X_j and every family of characteristic coefficients of $u_x(g)$, for $g \in G$. In particular the natural map $R \longrightarrow \mathcal{L}$ factors through $\mathbb{S}$. Let $\mathcal{J} \subseteq \mathcal{L}$ be an $\mathbb{S}$-submodule and write

$$\mathfrak{a} = \mathcal{J} \cap \mathbb{S}.$$

This is an ideal of $\mathbb{S}$. In the application it is the tied Hecke ideal. Fix $p \in O \setminus \{0\}$, group elements $\tau_{j,\pm}, \phi_{j,\pm}$, and polynomials $\varepsilon_j(T) \in O[T]$. Put

$$t_{j,\pm} = \tau_{j,\pm} - 1 - X_j \varepsilon_j(\phi_{j,\pm})(\phi_{j,\pm} - 1) \quad \text{in } \mathcal{A}. \tag{14.4}$$

The required trace assumptions are, for every $g \in G$,

$$(x \mapsto p \operatorname{tr} u_x(g)) \in \mathcal{J}, \qquad (x \mapsto \operatorname{tr} u_x(t_{j,\pm} g)) \in \mathcal{J}. \tag{14.5}$$

No trace identity beyond these and characteristic-coefficient membership is implicit in the argument.

Let $\mathcal{X} \subseteq Y$ be nonempty, with a distinguished $\pi \in \mathcal{X}$. Suppose there are integral decompositions

$$u_x = \rho_x \oplus \chi_0 \oplus \cdots \oplus \chi_{t-1} \quad (x \in \mathcal{X}), \qquad t \geq 1, \quad \chi_0 = 1. \tag{14.6}$$

The main summands have a common rank $r = n - t$. Integral changes of basis in (14.6) do not affect any determinant or trace below. For every $x \in \mathcal{X}$, assume an element $\gamma_x \in G$ and $z_x \in O$ satisfy

$$\rho_x(\gamma_x) = z_x I_r, \qquad z_x - 1 \in O^\times, \qquad \chi_e(\gamma_x) = 1 \quad (0 \leq e < t). \tag{14.7}$$

Choose $a \in G$ and $C \in O \setminus \{0\}$ such that

$$\prod_{f \neq e} (\chi_e(a) - \chi_f(a)) \mid C \quad (0 \leq e < t). \tag{14.8}$$

Integral characteristic coefficients and adjugates. The coefficient extension in the next argument matters. The characteristic coefficients belong to $\mathbb{S}$, but need not belong to R. Accordingly the Cayley–Hamilton construction is performed over $\mathbb{S}[G]$, after extending $R[G]$ to this algebra.

Lemma 14.6 (Characteristic coefficients of sums and blocks)**.** *Let S be a commutative subring of a commutative ring L, and let $\varrho : \Lambda \longrightarrow M_n(L)$ be a representation of an S-algebra. If the characteristic coefficients of all $\varrho(w)$ lie in S, then the characteristic coefficients of every flattened block matrix with entries $\varrho(w_{ij})$ lie in S. This remains true over $S[T]$ for matrices with entries in $\Lambda[T]$, where T is central. For a group algebra $S[G]$, it suffices to check the characteristic coefficients on G.*

Consequently, for a square matrix $D(T)$ over $\Lambda[T]$, there are $P(T) \in S[T]$ and a matrix $D'(T)$ over $\Lambda[T]$ such that

$$\varrho_{\mathrm{block}}(D'D) = P(T)I, \qquad P(0) = \det \varrho_{\mathrm{block}}(D(0)). \tag{14.9}$$

Here a scalar in $S[T]$ is read inside $L[T]$ in the first identity.

Proof. Amitsur's integral identity [Ami80] gives

$$\det\left(1 - T\sum_i s_i A_i\right) = \prod_{[w]\ \text{primitive}} \det\left(1 - T^{|w|} s_w A_w\right). \tag{14.10}$$

Here w runs through primitive cyclic words, A_w is the product of the matrices in that order, and s_w is the product of their scalar weights. The identity holds over every commutative ring. Only words of length at most d contribute in degree d, so each coefficient belongs to the ring generated by the characteristic coefficients of the words and the s_i.

For a group algebra, words in the group elements are group elements. For a block matrix take $A_{ij} = E_{ij} \otimes \varrho(w_{ij})$. A nonzero word in these elementary blocks has a single block. An off-diagonal block contributes determinant 1 to the product, while a diagonal block contributes the determinant for $\varrho(w_1 \cdots w_h)$. This proves the coefficient assertions over S, and over $S[T]$ by taking the powers of T as scalar weights. The characteristic polynomial of the flattened $D(T)$ therefore has coefficients in $S[T]$. Evaluate its adjugate polynomial at $D(T)$ in the block algebra to obtain $D'(T)$. Cayley–Hamilton gives (14.9) with $P(T) = \det \varrho_{\mathrm{block}}(D(T))$. For the empty matrix $P = 1$.

Lean, formal context. `RibetLemma` (L519-L685), `FitTwistsExist` (L377), `amitsur_charpoly_mem_of_span`, `amitsur_charpoly_comp_coeff_mem_range`. Lean proves coefficient descent by induction in a truncated polynomial ring. The primitive-word product gives the same descent coefficient by coefficient here. □

Lemma 14.7 (The noncommutative multiplier)**.** *In Lemma 14.6, suppose $\mathcal{M} \subseteq L$ is an S-submodule and put*

$$\mathfrak{n} = \{w \in \Lambda : \operatorname{tr} \varrho(wb) \in \mathcal{M} \text{ for every } b \in \Lambda\}.$$

Let $y = (y_1, \dots, y_d)^{\mathsf{T}}$, let $I = \sum_i \Lambda y_i$, and suppose a square matrix B over Λ and $\rho_0 \in \Lambda$ satisfy

$$(By)_i \in \mathfrak{n} + I\rho_0 \quad \text{for every } i.$$

There exists $P \in S[T]$ with

$$P(0) = \det \varrho_{\mathrm{block}}(B), \qquad IP(\rho_0) \subseteq \mathfrak{n}. \tag{14.11}$$

For any $\rho_1 \in \Lambda$, the element $f = \operatorname{tr} \varrho(\rho_1 P(\rho_0))$ belongs to S and satisfies

$$\det \varrho(w)\, f \in \mathcal{M} \quad (w \in I). \tag{14.12}$$

Proof. Cyclicity of trace makes $\mathfrak{n}$ a two-sided ideal containing $\ker \varrho$. Write $By = b + B_1 y \rho_0$ with $b_i \in \mathfrak{n}$ and B_1 a matrix over Λ. Apply Lemma 14.6 to $D(T) = B - TB_1$. On $Q = \Lambda/\mathfrak{n}$, let Λ act by left multiplication and let T act by right multiplication by $\overline{\rho_0}$. These actions commute, even though ρ_0 need not be central. The image of $D(T)$ kills $\overline{y}$. The coefficientwise identity (14.9) also holds in Q, since its error lies in $\ker \varrho \subseteq \mathfrak{n}$. It follows that every $y_i P(\rho_0)$ lies in $\mathfrak{n}$, proving (14.11).

The trace defining f is a characteristic coefficient, so is in S. For $w \in I$, the same adjugate construction gives $v \in \Lambda$ with $\varrho(vw) = \det(\varrho(w)) I_n$. Since $vw \in I$, the element $\rho_1 vw P(\rho_0)$ lies in $\mathfrak{n}$. Its trace is $\det(\varrho(w)) f$, and the definition of $\mathfrak{n}$ with $b = 1$ gives (14.12).

Lean, formal context. `RibetLemma` (L519-L685), `FitTwistsExist` (L377). □

Coordinates and tied rows. Fix a finite list of old elements δ_w, indexed by W, and a finite detecting set $S \subseteq G$. There is a finite list of additional generators indexed by I_0 such that, on $\Omega = W \sqcup I_0 \sqcup S$, the elements g_α are the δ_w, these additional generators, and the members of S, and there exist $z_\alpha(g) \in O$ satisfying

$$u_x(g) - 1 = \sum_{\alpha \in \Omega} z_\alpha(g)(u_x(g_\alpha) - 1) \quad (g \in G,\ x \in Y). \tag{14.13}$$

Indeed the O-span of the families $(u_x(g) - 1)_{x \in Y}$ is a submodule of the finite free module $\prod_x M_n(O)$, hence is finitely generated. Choose a finite subset of the given families that spans it and append the prescribed elements. Redundant coordinates cause no problem.

Put $\gamma = \gamma_\pi$, choose $c_1 \in O$ with $c_1(z_\pi - 1) = 1$, and define

$$e_0 = c_1(\gamma - 1), \qquad \rho_0 = e_0 + (1 - e_0)(a - 1) \quad \text{in } \mathcal{A}. \tag{14.14}$$

Then $\rho_\pi(\rho_0) = I_r$ and $\chi_0(\rho_0) = 0$. Let I_{aug} be the augmentation ideal of $\mathcal{A}$, and let $K_u = \bigcap_{x \in Y} \ker u_x$. The tie module is the left ideal

$$\mathcal{T} = K_u + pI_{\mathrm{aug}} + \sum_{j,\pm} \mathcal{A} t_{j,\pm} + I_{\mathrm{aug}} \rho_0.$$

For a row $r \in \mathcal{A}^\Omega$ put $V(r) = \sum_\alpha r_\alpha (g_\alpha - 1)$ and define the left $\mathcal{A}$-module of tied rows by

$$N = V^{-1}(\mathcal{T}). \tag{14.15}$$

Here $pI_{\mathrm{aug}} \subseteq \mathcal{T}$, so p kills the row annihilator. At $p = \ell^{\nu}$, Lemma 13.19 bounds the main determinant using the local quotient lengths at depth ν alone. If $\widehat{\mathcal{A}} = \mathbb{S}[G]$ and $\mathcal{U} : \widehat{\mathcal{A}} \to M_n(\mathcal{L})$ is the family representation, the two-sided trace ideal

$$\mathfrak{n} = \{w : \operatorname{tr}\mathcal{U}(wb) \in \mathcal{J} \text{ for every } b \in \widehat{\mathcal{A}}\}$$

contains the images of K_u, pI_{aug}, and every $t_{j,\pm}$. For the latter assertions expand in group elements and use (14.5), $\mathbb{S}$-linearity, and cyclicity of trace. Thus every tied row satisfies the hypothesis of Lemma 14.7 after extension to $\mathbb{S}[G]$. To check this last point with the finite ideal $I = \sum_{\alpha} \mathcal{A}(g_{\alpha} - 1)$, replace every $g - 1$ in an augmentation element by $\sum_{\alpha} z_{\alpha}(g)(g_{\alpha} - 1)$. The error lies in K_u by (14.13), and its product with ρ_0 also lies in K_u. Hence $I_{\mathrm{aug}}\rho_0$ can be replaced by $I\rho_0$ modulo the trace ideal.

The character hypotheses. Write $\eta_e : \mathcal{A} \to R$ for extension by χ_e. Assume $\chi_e(\delta_w) = 1$ and choose $o_w(T) \in O[T]$ with

$$o_w(u_x(\delta_w))(u_x(\delta_w) - 1) = 0 \quad (x \in Y), \qquad o_w(1) \neq 0. \tag{14.16}$$

Also assume

$$\varepsilon_j(\chi_e(\phi_{j,\pm})) \equiv 1 \pmod{p}. \tag{14.17}$$

A character cocycle for χ_e means a function $c : G \to O/pO$ satisfying all of the following conditions:

$$c(gh) = c(g) + \chi_e(g)c(h), \tag{14.18}$$

$$\sum_g \theta_g(u_x(g) - 1) = 0 \text{ (all } x) \implies \sum_g \theta_g c(g) = 0, \tag{14.19}$$

$$c(\delta_w) = 0, \qquad c(\gamma) = 0 \implies (\chi_e(g) - 1)c(a) = 0 \text{ (all } g). \tag{14.20}$$

The sums in (14.19) have finite support and $\theta_g \in O$. The implication in (14.20) is imposed on each cocycle under consideration; we do not assert that this implication by itself defines a submodule.

Let $\mathrm{ty} : J \to \{0, \dots, t-1\}$ be increasing, and put $J_e = \mathrm{ty}^{-1}(e)$. For every O-submodule Z consisting of these character cocycles, suppose that evaluation on S is injective and that Z has triangle cost b_e with factor C along J_e. Explicitly, there is a set $A \subseteq G$ such that, for $Z_0 = \{c \in Z : c|_A = 0\}$, the following hold. For $j \in J_e$ and $c \in Z_0$, the conditions

$$c(\tau_{i,+}) = c(\tau_{i,-}) \quad (i < j), \qquad c(\phi_{i,+}) = c(\phi_{i,-}) \quad (i < j,\ i \in J_e)$$

imply $C(c(\tau_{j,+}) - c(\tau_{j,-})) = 0$. For every submodule $Z_1 \subseteq Z_0$ whose elements vanish at all $\tau_{j,\pm}$ and agree at $\phi_{j,+}, \phi_{j,-}$ for $j \in J_e$, one has

$$\mathrm{length}_O(Z/Z_0) + \mathrm{length}_O(Z_1/Z_1[C]) \leq b_e, \tag{14.21}$$

where $Z_1[C] = \{c \in Z_1 : Cc = 0\}$. These are explicit inputs; their arithmetic verification is a separate step.

Lemma 14.8 (Reading tied rows on a character)**.** *Let $F : R^{I_0 \sqcup S} \to R^d$ be an R-linear map. Read a row on χ_e, forget its W coordinates, and apply F. Suppose $H \subseteq R^d$ contains the images of all tied rows. Let $\Phi : O^{\Omega} \to R^d$ be the resulting map on constant coordinate rows. Every O-linear map $\lambda : O^{\Omega} \to O/pO$ vanishing on $\Phi^{-1}(H)$ gives a character cocycle $c(g) = \lambda(z(g))$ satisfying* (14.18)–(14.20).

Proof. Write $[v]$ for the row in $\mathcal{A}^\Omega$ obtained from $v \in O^\Omega$. The following rows have value in K_u, by (14.13):

$$[z(gh)] - [z(g)] - g[z(h)], \qquad [z(g_\alpha)] - \mathbf{e}_\alpha, \qquad \left[\sum_g \theta_g z(g)\right]$$

in the last case whenever the family relation in (14.19) holds. They therefore belong to N. Reading the first and third rows proves the cocycle and family-relation conditions. Reading the second for a W coordinate proves $c(\delta_w) = 0$, since the W coordinates are forgotten.

The row

$$r_0 = c_1[z(\gamma)] + (1 - c_1(\gamma - 1))[z(a)]$$

has $V(r_0) - \rho_0 \in K_u$. Hence $(g-1)r_0 \in N$. Since $\chi_e(\gamma) = 1$, its reading on χ_e is $(\chi_e(g)-1)(c_1 z(\gamma) + z(a))$. Applying λ gives $(\chi_e(g) - 1)(c_1 c(\gamma) + c(a)) = 0$; this proves the last condition when $c(\gamma) = 0$.
Lean, formal context. `RibetLemma` (L519-L685), `FitTwistsExist` (L377). □

Lemma 14.9 (Character determinants and the valuation of the constant term)**.** *Under the character hypotheses, put*

$$X_{<e} = \prod_{\mathrm{ty}(j)<e} X_j, \quad X_e = \prod_{j \in J_e} X_j, \quad \mathfrak{q} = (X_j^2 : j \in J), \quad \mathfrak{q}_e = (\mathfrak{q} : X_{<e}).$$

There exist polynomials $d_e, \zeta_e, \Gamma_e \in R$ such that

$$d_e + \zeta_e = \Gamma_e X_e, \quad \zeta_e \in \mathfrak{q}_e, \quad v_O(\Gamma_e(0)) \le b_e + (|J_e| + |S|)v_O(C). \tag{14.22}$$

For each e, there are finitely many matrices $B_{e,h}$ of size $|S|$ over R and weights $w_{e,h} \in R$ with $d_e = \sum_h w_{e,h} \det B_{e,h}$. For every choice $h = (h_e)_{e<t}$ there is a square matrix M_h indexed by Ω, with every row in N, such that

$$\det \eta_e(M_h) = c_0 \det B_{e,h_e}, \qquad c_0 = \left(\prod_{w \in W} o_w(1)\right) C^{|S|}. \tag{14.23}$$

Proof. First we construct rows $\nu_i \in N$, for $i \in I_0$, whose I_0 part is the ith unit row under every η_e. Fix e, and let $H \subseteq R^{I_0}$ be the image of N under character evaluation followed by projection to I_0. The associated Φ in Lemma 14.8 sends pO^Ω into H, since every row $p[v]$ is tied. For each $s \in S$, the generator relation in that lemma gives $\Phi(z(s)) \in H$. Thus all resulting cocycles vanish on S and are zero by detection. In particular every form annihilating $\Phi^{-1}(H)$ vanishes on $z(g_i)$ for $i \in I_0$.

Forms with values in O/pO separate the elements of a finitely generated O-module killed by p: by the structure theorem it is a sum of modules $O/(\varpi^a)$ with $a \le v_O(p)$, each of which embeds in O/pO. Apply this to $O^\Omega/\Phi^{-1}(H)$. We obtain $\Phi(z(g_i)) \in H$. The generator relation then puts the ith unit row in H. This gives rows $m_{e,i} \in N$ with the required unit row on this one character.

Form the matrix M_e with rows $m_{e,i}$ and let U_e be its square I_0-column block. Take ν_i as the rows of

$$\sum_e \left(\prod_{f<e} (I - U_f)\right) M_e,$$

with products ordered by increasing index. The rows belong to the left module N. Their I_0-block telescopes to $I - \prod_e (I - U_e)$. Under η_e its e-th factor in the product vanishes, since $\eta_e(U_e) = I$. This provides the simultaneous unit rows integrally.

For each e, eliminate the I_0 coordinates by defining

$$F_e(r)_s = \eta_e(r_s) - \sum_{i \in I_0} \eta_e(r_i)\eta_e((\nu_i)_s), \qquad K_e = F_e(N) \subseteq R^S.$$

The W coordinates are omitted. Put $H_e = K_e + \mathfrak{q}_e R^S$ and let $\Phi_e : O^\Omega \to R^S$ be F_e on constant rows. We have

$$\begin{aligned} \Phi_e(pO^\Omega) &\subseteq K_e, \\ \Phi_e(z(s)) - \mathbf{e}_s &\in K_e \quad (s \in S), \\ \Phi_e(z(\tau_{j,\pm})) - X_j \Phi_e(z(\phi_{j,\pm})) &\in K_e. \end{aligned}$$

The first two assertions follow from $p[v] \in N$ and the generator relations. For the last, the row $[z(\tau_{j,\pm})] - X_j \varepsilon_j(\phi_{j,\pm})[z(\phi_{j,\pm})]$ has value $t_{j,\pm}$ modulo K_u. Its character reading gives the assertion using (14.17) and the first assertion.

Every form defining the function module for H_e is a character cocycle by Lemma 14.8. It therefore has the triangle cost (14.21). Moreover, if $i < j$ and $j \in J_e$, monotonicity of the type implies either $i \in J_e$ or $X_i \in \mathfrak{q}_e$: in the latter case $X_i X_{<e}$ has the square factor X_i^2. All the hypotheses of the finite triangle–Fitting theorem 13.22 are now verified. It gives

$$\Gamma_e X_e \in \mathrm{Fitt}_0(R^S/H_e), \qquad v_O(\Gamma_e(0)) \le b_e + (|J_e| + |S|) v_O(C). \tag{14.24}$$

The modules R^S/H_e and R^S/K_e have the same reduction modulo $\mathfrak{q}_e$. By base change for Fitting ideals [Eis95, Corollary 20.5], their zeroth Fitting ideals have the same image in $R/\mathfrak{q}_e$. Lifting (14.24) gives d_e, ζ_e in (14.22). The determinant definition of the Fitting ideal expresses d_e as a finite sum $\sum_h w_{e,h} \det B_{e,h}$ with every row of $B_{e,h}$ in K_e. Lift each such row to $r \in N$ and replace it by $r - \sum_{i \in I_0} r_i \nu_i$. Its I_0 part under η_e is zero, and its S part is the original row of $B_{e,h}$. Denote these lifted rows by $\widetilde{B}_{e,h,s}$.

Write $a_e = \chi_e(a)$, choose $u_e \in O$ with $u_e \prod_{f \ne e}(a_e - a_f) = C$, and set

$$q_e = u_e \prod_{f \ne e} (a - a_f) \in O[G]. \tag{14.25}$$

Then $\eta_f(q_e) = C\delta_{ef}$. In the order generated by $(a_f)_f$, the ideal of possible values at e of elements vanishing at every $f \ne e$ is exactly $\prod_{f \ne e}(a_e - a_f)O$. Indeed, their representing polynomials are divisible by each $X - a_f$ in succession, since the a_f are distinct and O is a domain. The product itself attains the generator, also for the empty product. Thus only indices in the residual class of a_e contribute positive valuation. For a general comparison of congruence ideals and differentials in local complete intersections, see [HT20, §8].

For M_h take the W-rows with single entries $o_w(\delta_w)$, the I_0-rows ν_i, and the S-rows $\sum_e q_e \widetilde{B}_{e,h_e,s}$. They are tied by (14.16) and left linearity. After evaluation at e, expand along the W-rows. The remaining matrix is block upper triangular with diagonal blocks I and CB_{e,h_e}. This gives (14.23).

Lean. The formal unit rows, determinant decomposition, and common character matrices are `exists_unit_rows_mem_tiedRows`, `exists_sum_mul_det_of_mem_fittingIdeal_evalRowAtChar`, and `nonempty_characterRows`. The product formula for the simultaneous unit rows and the optimality in the order generated by a are the arguments given here. □

Lemma 14.10 (The coefficient that survives all characters)**.** *For the polynomials in* (14.22)*, one has*

$$\prod_e d_e \equiv X_J \prod_e \Gamma_e \pmod{\mathfrak{q}}, \qquad X_J = \prod_{j \in J} X_j. \tag{14.26}$$

Consequently, for every $F \in R$, all coefficients of $F \prod_e d_e$ at proper divisors of X_J are zero, and its coefficient at X_J is $F(0) \prod_e \Gamma_e(0)$.

Proof. Use the ordered telescoping expansion

$$\prod_e (d_e+\zeta_e)-\prod_e d_e=\sum_e \zeta_e \prod_{f<e}(d_f+\zeta_f)\prod_{f>e} d_f.$$

The product of earlier factors is $X_{<e}\prod_{f<e}\Gamma_f$. Thus the eth term lies in $\mathfrak{q}$ because $\zeta_e X_{<e}\in\mathfrak{q}$. This proves (14.26). The ideal $\mathfrak{q}$ is spanned by monomials having some exponent at least 2, so none contributes to a monomial dividing X_J. In $FX_J\prod_e\Gamma_e$, the coefficient of X_J is the constant coefficient of $F\prod_e\Gamma_e$, and every proper divisor has coefficient zero.

Lean, formal context. `RibetLemma` (L519-L685), `FitTwistsExist` (L377). □

Theorem 14.11 (One common polynomial and the full algebraic valuation bound)**.** *Assume the preceding coefficient, trace, decomposition, scalar, separation, old-element, and character hypotheses. Suppose there is a square matrix B_0 indexed by Ω, all of whose rows are tied, for which*

$$D_x=\det \rho_{x,\mathrm{block}}(B_0)\in R,\qquad v_O(D_\pi(0))\le b_0. \tag{14.27}$$

Here b_0 and all b_e are finite. Then there are $f\in\mathbb{S}$ and $d\in O\setminus\{0\}$ such that, for every finitely supported $\omega: G\to O$,

$$\beta_\omega=\left(x\mapsto \det\left(\sum_g \omega(g)(u_x(g)-1)\right)\right)\in\mathbb{S},\qquad \beta_\omega|_{\mathcal{X}}=0, \tag{14.28}$$

$$\beta_\omega f\in\mathfrak{a}, \tag{14.29}$$

$$[X_A]f(x)=0\quad (x\in\mathcal{X},\ A\subsetneq J),\qquad [X_J]f(\pi)=d, \tag{14.30}$$

$$v_O(d)\le b_0+\sum_{e<t} b_e + t\left(\sum_{w\in W} v_O(o_w(1))+2|S|v_O(C)\right)+(|J|+1)v_O(C). \tag{14.31}$$

In (14.28), membership means the family of constant polynomials. In particular the conclusion holds for $\beta_g(x)=\det(1-u_x(g))$, by taking $\omega=-\mathbf{1}_{\{g\}}$.

Proof. For every $x\in\mathcal{X}$ define

$$k_x=1-(z_x-1)^{-1}(\gamma_x-1)\in O[G],\qquad k=\prod_{x\in\mathcal{X}} k_x,$$

in any fixed order. Each k_x is 1 on every character and is zero on the main block ρ_x. Therefore k is 1 on all characters and zero on every main block: in the latter evaluation its product has a zero factor. No commutativity of the factors is needed.

For a choice $h=(h_e)$ from Lemma 14.9, form, by left multiplication of entries,

$$B_h=(1-k)B_0+kM_h.$$

Its rows are tied. Its main block at every lift is B_0, and its character matrices are those of M_h. Use $\rho_1=kq_0$, where q_0 is (14.25). At every lift, ρ_1 acts by C on the trivial character and by zero on every other summand. Since $\chi_0(\rho_0)=0$, the matrix $u_x(\rho_1\rho_0)$ is zero.

Apply Lemma 14.7 over $\mathbb{S}[G]$ to B_h, the coordinate vector $(g_\alpha-1)_\alpha$, and the trace ideal defined above. It gives $f_h\in\mathbb{S}$ which multiplies every determinant from the coordinate ideal into $\mathcal{J}$. At a lift x, write $P(T)=P(0)+TQ(T)$. Since $u_x(\rho_1\rho_0)=0$,

$$f_h(x)=P(0)(x)\operatorname{tr} u_x(\rho_1)=C\det u_{x,\mathrm{block}}(B_h)=CD_x\prod_e \det\eta_e(M_h).$$

The final determinant factorization follows by reordering the rows and columns of the flattened block matrix together, so there is no permutation sign.

For arbitrary ω, the element

$$w_\omega = \sum_g \omega(g) \sum_\alpha z_\alpha(g)(g_\alpha - 1)$$

lies in the coordinate ideal and has image $\sum_g \omega(g)(u_x(g) - 1)$. Thus every f_h serves every β_ω. Lemma 14.6 places these determinants in $\mathbb{S}$. Their trivial-character block is zero on $\mathcal{X}$, proving their vanishing there.

Set

$$f = \sum_h \left(\prod_e w_{e,h_e} \right) f_h. \tag{14.32}$$

All the weights are in $R \subseteq \mathbb{S}$, so $f \in \mathbb{S}$ and all $\beta_\omega f$ lie in $\mathcal{J} \cap \mathbb{S}$. Expanding the independent finite sums and using (14.23) gives, at every lift,

$$f(x) = C c_0^t D_x \prod_e d_e. \tag{14.33}$$

Lemma 14.10 now proves the proper-coefficient vanishing and gives the integral leading coefficient

$$d = C \left(\left(\prod_w o_w(1) \right) C^{|S|} \right)^t D_\pi(0) \prod_e \Gamma_e(0). \tag{14.34}$$

All its factors are nonzero by the finite valuation bounds and the stated nonvanishing assumptions. Taking valuations, the selector q_0 contributes one $v_O(C)$, c_0^t contributes $t \sum_w v_O(o_w(1)) + t|S| v_O(C)$, and the Γ_e contribute at most $\sum_e b_e + (|J| + t|S|) v_O(C)$. Adding the main bound b_0 gives exactly (14.31).

Lean, formal context. `RibetLemma` (L519-L685), `FitTwistsExist` (L377). □

The determinant hypothesis and the factor c^2. The matrix hypothesis (14.27) follows from Lemmas 13.17 and 13.16 when the image of ρ_π spans $M_r(O)$ over O and the row annihilator has length at most b_0. For $p = \ell^\nu$, the local-condition map at depth ν gives this bound by Lemma 13.19. Thus the present theorem does not assume a common polynomial, a small determinant on the characters, or any cancellation of valuation losses.

For the arithmetic application, let $v_\ell = v_O / v_O(\ell)$ on K. Rename the number t of character blocks in this subsection as q_0; it is bounded by the ambient parameter denoted t in the denominator theorem. Let $c \neq 0$ be the level coefficient from the local operator construction. The exact numerical condition needed after (14.31) is

$$\frac{b_0 + \sum_e b_e}{v_O(\ell)} + q_0 \sum_w v_\ell(o_w(1)) + (2q_0|S| + |J| + 1) v_\ell(C) + 2v_\ell(c) \leq \mathrm{sel} + 2q_0 \mathrm{ram} + r_0(4fD_\ell) + E_\ell. \tag{14.35}$$

Corollary 25.1 proves this conversion from the stated local, odd-class, tame, and prime-count estimates. With those arithmetic inputs, the theorem gives

$$v_\ell(c^2 f_J(\pi)) \leq \mathrm{sel} + 2q_0 \mathrm{ram} + r_0(4fD_\ell) + E_\ell,$$

together with the other two assertions of (25.16). The two powers of c are exactly the two added valuation terms in this last normalization; the common-polynomial construction itself introduces no additional power of c. Its separation cost is the explicit $(2q_0|S| + |J| + 1) v_\ell(C)$ already displayed.

Section 17 verifies the trace relations for the Hecke systems under consideration. Corollary 25.1 bounds the valuation of the resulting normalized polynomial.

14.3 Old-inertia polynomials and the tame valuation allowance

We now supply the old-inertia polynomials in (14.16) and the tame estimate in (25.4), from explicit local hypotheses on the representations. The integer used below is the same group-order integer that clears the type projections; no larger integer is introduced. The source statements are the power relation on old inertia and `exists_tame_relations`.

Let E be a CM field with $[E:\mathbb{Q}] = 2f$, and let j be a fixed complex conjugation on $\overline{\mathbb{Q}}$. Write $\bar{v}$ for the conjugate of a finite place v of E. Fix a prime ℓ, the integers $\mathcal{O}$ of a finite extension of $\mathbb{Q}_\ell$, and an embedding $\mathcal{O} \hookrightarrow \overline{\mathbb{Q}}_\ell$. Normalize $v_\mathcal{O}$ by its value 1 on a uniformizer, and set

$$v_\ell(z) = \frac{v_\mathcal{O}(z)}{v_\mathcal{O}(\ell)} \quad (z \in \mathrm{Frac}(\mathcal{O})^\times), \qquad v_\mathcal{O}(0) = v_\ell(0) = +\infty.$$

On rational integers this is the usual ℓ-adic valuation. The two old level primes are denoted by p_0, p_1.

The chosen integer and the matrix power relation

Let $h \geq 1$, and let $\mathcal{P}$ be a finite family of chosen places above p_0 and p_1, containing a place above each. For $w \in \mathcal{P}$, write k_w for its residue field. Put

$$c = \prod_{w \in \mathcal{P}} |\,\mathrm{GL}_h(k_w)|. \tag{14.36}$$

This is a positive integer. In the source application $h = D.r\,D.t$ and (14.36) is exactly D.chosenGroupOrder; see its definition.

Lemma 14.12 (The required orders divide the chosen group order)**.** *For $i \in \{0,1\}$ and $1 \leq d \leq h$, one has*

$$\prod_{j=1}^{d} (p_i^j - 1) \mid c. \tag{14.37}$$

In particular $p_i^d - 1 \mid c$ and $v_\ell(p_i^d - 1) \leq v_\ell(c)$.

Proof. Choose $w \in \mathcal{P}$ above p_i and put $q = |k_w|$. Counting ordered bases gives

$$|\mathrm{GL}_h(k_w)| = \prod_{a=0}^{h-1} (q^h - q^a) = q^{h(h-1)/2} \prod_{j=1}^{h} (q^j - 1).$$

Since q is a power of p_i, each $p_i^j - 1$ divides $q^j - 1$. The group order is a factor of c by (14.36).

Lean. `AttachedRepresentations` (L198-L220), `levelPrime_pow_sub_one_dvd_chosenGroupOrder`. The code extracts a single factor from the group order. The full product enters the choice of auxiliary levels. □

Lemma 14.13 (Finite inertia and the prescribed spectrum)**.** *Let T be an invertible matrix over $\overline{\mathbb{Q}}_\ell$ of finite order. Suppose that, for some $i \in \{0,1\}$, each eigenvalue α of T satisfies $\alpha^{p_i^d - 1} = 1$ for an integer d with $1 \leq d \leq h$. Then*

$$T^c = 1. \tag{14.38}$$

Consequently, if a characteristic-zero representation has finite inertia image at a place above p_i and has this spectrum on each inertia element, (14.38) holds on its entire inertia group.

Proof. A finite-order matrix over a field of characteristic zero is semisimple, since its minimal polynomial divides a separable $X^b - 1$. By Lemma 14.12 each eigenvalue of T is a c-th root of unity. Thus $T^c = 1$, elementwise also on the finite inertia image.

Lean, formal context. `AttachedRepresentations` (L198-L220), `FitTwistsExist` (L377), `levelPrime_pow_sub_one_dvd_chosenGroupOrder`. The cited formal divisibility controls each eigenvalue order. In the present finite-image case the minimal polynomial divides a separable cyclotomic product, which gives the matrix identity before reduction. □

The spectral condition is the exact information extracted from the scaled type polynomials in the source. The elementary origin of the exponent is that a nonzero root of degree d over $\mathbb{F}_{p_i}$ lies in $\mathbb{F}_{p_i^d}^\times$, so its order divides $p_i^d - 1$. Theorem 17.76 and Corollary 17.77 identify these spectra for the good rows of the main summand. The mixed type theorem and Theorem 17.105 supply the corresponding assertion for every required ambient system.

The finite-image hypothesis is also needed. For example, the matrix $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ has only the eigenvalue 1, but $T^c = \left(\begin{smallmatrix} 1 & c \\ 0 & 1 \end{smallmatrix}\right) \neq 1$ for every positive integer c in characteristic zero. The argument uses semisimplicity before applying the spectral orders, and does not divide by the size of the inertia image.

Lemma 14.14 (Polynomial identities on the integral models)**.** *Let L be a finite free $\mathcal{O}$-module, let $u : G_E \to \mathrm{GL}_{\mathcal{O}}(L)$ be an integral model of a representation ρ on a vector space V over $\overline{\mathbb{Q}}_\ell$, and let $P \in \mathcal{O}[X]$. If $P(\rho(g)) = 0$, then $P(u(g)) = 0$.*

Proof. An identification of $L \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_\ell$ with the space of ρ gives an injective $\mathcal{O}$-algebra map

$$F : \mathrm{End}_{\mathcal{O}}(L) \longrightarrow \mathrm{End}_{\overline{\mathbb{Q}}_\ell}(V), \qquad F(u(g)) = \rho(g).$$

In matrices, F is scalar extension followed by one change of basis, so it is injective. Polynomial evaluation commutes with an algebra map; hence $F(P(u(g))) = P(\rho(g)) = 0$. Injectivity proves the assertion. There is no denominator in this descent, even if the chosen change of basis is not integral. This is the relevant consequence of `IsLatticeIn.exists_algHom`.

Lean. `AttachedRepresentations` (L268-L287). □

The two conjugate copies have the same scalar cost

For $g \in G_E$ write $g^* = jg^{-1}j^{-1}$. If g is in inertia above v, then g^* is in inertia above $\bar{v}$. We use the auxiliary character $\psi : G_E \to \mathcal{O}^\times$ from the twisting argument. Its norm-one-idele property is the condition imposed before Lemma 13.11: there is an idele character η such that

$$\psi(\mathrm{Art}(y)) = \eta(y/\bar{y}) \tag{14.39}$$

for finite ideles y. The choice of arithmetic or geometric Artin convention does not affect the calculation below, provided the same convention is used throughout.

Lemma 14.15 (The conjugate-inverse values coincide)**.** *The character in* (14.39) *satisfies*

$$\psi(g^*) = \psi(g) \qquad (g \in G_E). \tag{14.40}$$

Proof. Functoriality of Artin reciprocity under conjugation sends $\mathrm{Art}(y)^*$ to $\mathrm{Art}(\bar{y}^{-1})$ in the abelianization. But

$$\frac{\bar{y}^{-1}}{\overline{\bar{y}^{-1}}} = \frac{\bar{y}^{-1}}{y^{-1}} = \frac{y}{\bar{y}}.$$

Formula (14.39) gives equal ψ-values on these two Artin images. Artin images are dense in the abelianized Galois group, and ψ is continuous and abelian, so equality holds on all of G_E.

Lean. `IsFromU1.apply_idOrConjInv`. □

The polynomials, covering, and exact valuation sum

Let Q be the support of the auxiliary twist, before the new triangle primes are introduced. Its places are away from ℓ, p_0, p_1; write $\bar{Q} = \{\bar{v} : v \in Q\}$. For each $v \in Q$, retain the tame generator τ_v supplied by the twist construction, and assume its finite ramification allowance is

$$\sum_{v \in Q} v_\ell(\psi(\tau_v) - 1) \le \mathrm{ram} < \infty. \tag{14.41}$$

This is the same scalar sum as in (13.13). In particular $\psi(\tau_v) \neq 1$ for every $v \in Q$.

Put

$$M = \{v : v \mid p_0 p_1,\ v \nmid \ell\}, \qquad W = Q \sqcup M.$$

The set M is stable under conjugation. The symbol $\sqcup$ records a disjoint union of index sets. For $w \in W$, let q_w be its underlying place. On the Q-part put $\delta_w = \tau_w$; on the M-part choose a place of $\overline{\mathbb{Q}}$ above w and take its fixed tame generator δ_w. Finally put

$$\delta_{w,0} = \delta_w, \qquad \delta_{w,1} = \delta_w^*. \tag{14.42}$$

Theorem 14.16 (Old-inertia relations with their full cost)**.** *Let u_x be integral models of a finite family of characteristic-zero representations ρ_x of G_E. Assume the following two local properties for every x:*

1. *At every $v \in M$ above p_i, the image of inertia is finite, and every eigenvalue of every inertia element has order dividing $p_i^d - 1$ for some $1 \le d \le h$.*
2. *For every inertia element g above a place in $Q \cup \bar{Q}$, one has*

$$(\rho_x(g) - \psi(g)\,1)(\rho_x(g) - 1) = 0. \tag{14.43}$$

Use the character, generators and allowance just specified. For $w \in M$, let $A_{w,b}$ be the set of distinct eigenvalues different from 1 of the operators $\rho_x(\delta_{w,b})$ for all x. Then the polynomials

$$o_{w,b}(X) = \begin{cases} \psi(\delta_{w,b}) - X, & w \in Q, \\ \prod_{\xi \in A_{w,b}} (X - \xi), & w \in M \end{cases} \qquad (b = 0, 1) \tag{14.44}$$

belong to $\mathcal{O}[X]$ and satisfy

$$o_{w,b}(u_x(\delta_{w,b}))(u_x(\delta_{w,b}) - 1) = 0, \qquad o_{w,b}(1) \neq 0 \tag{14.45}$$

for all w, b, x. The underlying places are prime to ℓ and cover all required old and auxiliary places up to conjugation:

$$\{v : v \mid p_0 p_1\} \cup Q \cup \bar{Q} \subseteq \{v : v \mid \ell\} \cup \{q_w : w \in W\} \cup \{\bar{q}_w : w \in W\}. \tag{14.46}$$

The exact normalized cost is

$$T_{\mathrm{tame}} := \sum_{w \in W} \sum_{b=0}^{1} v_{\mathcal{O}}(o_{w,b}(1)), \tag{14.47}$$

$$\begin{aligned} \frac{T_{\mathrm{tame}}}{v_{\mathcal{O}}(\ell)} &= 2 \sum_{v \in Q} v_\ell(\psi(\tau_v) - 1) + \sum_{w \in M} \sum_{b=0}^{1} v_\ell(o_{w,b}(1)) \\ &\le 2\,\mathrm{ram} + 8 f\, v_\ell(c). \end{aligned} \tag{14.48}$$

The polynomials are common to the whole family of representations.

Proof. The tame quotient is procyclic, and conjugation and inversion preserve its generators. The old places omitted from M lie above ℓ, which gives (14.46). For $w \in Q$, (14.43) gives the annihilator, and Lemma 14.15 gives its value $\psi(\tau_w) - 1 \neq 0$ at 1 for either generator. For $w \in M$, Lemma 14.13 gives $\rho_x(\delta_{w,b})^c = 1$. The operator is therefore semisimple. Its nontrivial eigenvalues must be roots of every polynomial satisfying (14.45), and their product of linear factors suffices. The set of these roots is Galois stable over the coefficient field, so its monic polynomial has integral coefficients there. It divides $(X^c - 1)/(X - 1)$ over the same ring. All the annihilating identities restrict to the stable lattices.

At 1 the divisor relation gives $v_\ell(o_{w,b}(1)) \leq v_\ell(c)$. More precisely,

$$v_\ell(o_{w,b}(1)) = \sum_{j \geq 1} \frac{\#\{\xi \in A_{w,b} : \operatorname{ord}(\xi) = \ell^j\}}{\varphi(\ell^j)}.$$

A root with nontrivial prime-to-ℓ part differs from 1 by a unit, and (13.3) applies to the remaining roots. Now $|M| \leq 2[E : \mathbb{Q}] = 4f$, so (14.41) gives (14.48).

Lean, comparison. Lean uses $(X^c - 1)/(X - 1)$ at every old place. Here only the distinct eigenvalues that occur contribute. `exists_tame_relations`. □

Corollary 14.17 (The tame contribution to the common congruence polynomial)**.** *For character summands $\chi_e = \epsilon_\ell^{-e}$, the doubled family* (14.42) *and polynomials* (14.44) *supply the old-element hypotheses* (14.16) *and the tame inequality in* (25.4)*, under the two representation hypotheses of Theorem 14.16 and the auxiliary twist allowance* (14.41)*.*

Proof. At a place of residue characteristic different from ℓ, inertia fixes all ℓ-power roots of unity. Therefore $\chi_e(\delta_{w,b}) = 1$ for every e, w, b. Theorem 14.16 supplies the polynomial relations, their nonzero values at 1, the required covering, and exactly $T_{\mathrm{tame}}/v_{\mathcal{O}}(\ell) \leq 2\,\mathrm{ram} + 8f\, v_\ell(c)$. Reindexing (w, b) as the old-element index set in the Ribet construction proves every asserted input. No further valuation factor is incurred by the integral models or the conjugate copies.

Lean. The cost used in the congruence polynomial is the last assertion of `exists_tame_relations`. The uniform old-place bound and the auxiliary evaluations are computed in `TraceRelations` (L969-L978). □

The ordinary polynomial construction and its valuation bound are thus proved from the specified representation data. Finite old inertia and the precise main-row spectra are supplied in §17.13. The auxiliary quadratic follows from the determinant-block type and a supplied attached representation with the stated compatibility by Lemma 17.82. Theorem 17.105 supplies this attachment and compatibility for the full ambient family. The crystalline-defect problem at places above ℓ, including the case $\ell = p_0$ or p_1, is separate from this argument.

15 Local duality and crystalline period lattices

At places above the coefficient prime, the local condition on extensions is defined by crystalline periods. Passing to integral lattices introduces a cokernel. We compute its normalized length, keeping track of the coefficient functional in the cup pairing.

We first compute the rank of the module of crystalline cocycles. An exponent bound for a quotient of this module then gives a bound for its length. The corresponding Lean calculations are the crystalline-cocycle rank and exponent bounds and the sum over places and summands.

Let $K/\mathbb{Q}_\ell$ be a finite extension of degree d, let $\mathcal{O}$ be the integers of a finite coefficient extension $F/\mathbb{Q}_\ell$, and normalize $v_{\mathcal{O}}$ by assigning value 1 to a uniformizer. Write

$$e_{\mathcal{O}} = v_{\mathcal{O}}(\ell), \qquad \widehat{\lambda}_{\mathcal{O}}(M) = \frac{\operatorname{length}_{\mathcal{O}} M}{e_{\mathcal{O}}}.$$

Thus $\widehat{\lambda}_{\mathcal{O}}(\mathcal{O}/\ell^n\mathcal{O}) = n$. This is $\mathcal{O}$.`normLength` applied to the module length: Lean uses the valuation $v_\ell(\ell) = 1$, for which a uniformizer of $\mathcal{O}$ has value $1/e_{\mathcal{O}}$ (definition of normalized length). Write $G_K = \mathrm{Gal}(\overline{K}/K)$ and (1) for the cyclotomic Tate twist. We use Fontaine's period rings B_{cris} and B_{dR}. The module $D_{\mathrm{dR}}(V) = (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_\ell} V)^{G_K}$ has the filtration induced by that on B_{dR}. For a continuous G_K-stable free $\mathcal{O}$-lattice L of rank r, put $V = L \otimes_{\mathcal{O}} F$ and

$$Z_f(L) = \{z \in Z^1_{\mathrm{cont}}(G_K, L) : [z] \in H^1_f(K, V)\}, \qquad L^\dagger = \mathrm{Hom}_{\mathcal{O}}(L, \mathcal{O})(1).$$

Here $H^1_f(K,V)$ is the kernel of $H^1(K,V) \to H^1(K, B_{\mathrm{cris}} \otimes_{\mathbb{Q}_\ell} V)$. Thus $Z_f(L)$ consists of the integral cocycles that become coboundaries over B_{cris}.

For a de Rham representation, Bloch and Kato identify the dimension of the finite local condition and its annihilator under Tate duality [BK90, Proposition 3.8 and Corollary 3.8.4]. We first pass from their cohomology group to the lattice of cocycles that occurs in the determinant estimate.

Lemma 15.1 (The rank of the module of crystalline cocycles)**.** *Suppose V is de Rham. Then $Z_f(L)$ is a finite free $\mathcal{O}$-module and*

$$\begin{aligned} \operatorname{rank}_{\mathcal{O}} Z_f(L) &= r + \dim_F(D_{\mathrm{dR}}(V)/\operatorname{Fil}^0 D_{\mathrm{dR}}(V)), \\ r \le \operatorname{rank}_{\mathcal{O}} Z_f(L) &\le (d+1)r \le 2dr. \end{aligned} \tag{15.1}$$

Moreover,

$$\operatorname{rank}_{\mathcal{O}} Z_f(L) + \operatorname{rank}_{\mathcal{O}} Z_f(L^\dagger) = (d+2)r. \tag{15.2}$$

The submodule $Z_f(L)$ is saturated in $Z^1_{\mathrm{cont}}(G_K, L)$.

Proof. Put $t_V = D_{\mathrm{dR}}(V)/\operatorname{Fil}^0$. Bloch–Kato's dimension formula gives $\dim_F H^1_f(K,V) = \dim_F H^0(K,V) + \dim_F t_V$ [BK90, Corollary 3.8.4]. The kernel of $Z_f(V) \to H^1_f(K,V)$ is $V/H^0(K,V)$, hence $\dim_F Z_f(V) = r + \dim_F t_V$.

Evaluation at finitely many elements of G_K embeds this finite-dimensional space in a finite product of copies of V. Indeed, as long as the common kernel is nonzero, choose a cocycle in it and an element where that cocycle is nonzero. The next evaluation strictly lowers the dimension of the kernel. Thus $Z_f(L)$ embeds in a finite product of copies of L and is finite free over $\mathcal{O}$. Every continuous cocycle has compact image, which lies in $\varpi^{-a}L$ for some a. Clearing this denominator shows that $Z_f(L)$ spans $Z_f(V)$. It is saturated among integral cocycles since $H^1_f(K,V)$ is an F-subspace.

Finally $0 \le \dim_F t_V \le dr$. For the Tate dual each filtration jump u is paired with $-1-u$, with the same multiplicity. Exactly one of the two is negative, so $\dim_F t_V + \dim_F t_{V^\dagger} = dr$. Both displayed formulas follow.

Lean, formal context. `CrystallineCocycles` (L76-L131), `TraceRelations` (L717-L755), `finrank_crysCocycles_eq_add_neg_jumps`. The formal rank formula counts the negative jumps over the coefficient embeddings. The Bloch–Kato dimension formula and the coboundary space $V/H^0(K,V)$ give the same integer here. □

Lemma 15.2 (An exponent bounds the normalized defect)**.** *Keep the preceding hypotheses, and let $B \subseteq Z_f(L)$ be an $\mathcal{O}$-submodule. If $k \in \mathbb{N}$ and*

$$\ell^k Z_f(L) \subseteq B,$$

then $Z_f(L)/B$ has finite length and

$$\widehat{\lambda}_{\mathcal{O}}(Z_f(L)/B) \le k \operatorname{rank}_{\mathcal{O}} Z_f(L) \le (d+1)rk \le 2drk. \tag{15.3}$$

For a family $\mathcal{Q}$ of test lattices, let $\mathrm{Ann}_\smile(\mathcal{Q}, L)$ consist of the functions $c : G_K \to L$ such that, for each $Y \in \mathcal{Q}$, each $a \in Z_f(Y)$, and each admissible pairing of the form (15.5) with second argument in L, the cup product of a and c is a coboundary. This is `cupOrthogonal` in Lean. Write $H_{\mathcal{Q}}(L) = Z_f(L) \cap \mathrm{Ann}_\smile(\mathcal{Q}, L)$. The bound applies to $B = H_{\mathcal{Q}}(L)$ when $\ell^k Z_f(L) \subseteq H_{\mathcal{Q}}(L)$.

Proof. The inclusion gives a surjection

$$Z_f(L)/\ell^k Z_f(L) \longrightarrow Z_f(L)/B.$$

The module on the left has length $ke_{\mathcal{O}} \operatorname{rank}_{\mathcal{O}} Z_f(L)$. Length decreases under a quotient. Divide by $e_{\mathcal{O}}$ and apply Lemma 15.1. For $k = 0$ the inclusion says $B = Z_f(L)$ and both sides are zero.
Lean, formal context. `TraceRelations` (L717-L755), `FitTwistsExist` (L377). □

Lemma 15.3 (Coefficient extension preserves normalized length)**.** *Let $\mathcal{O} \subseteq \mathcal{O}'$ be the rings of integers in a finite extension of coefficient fields, and let M be a finite-length $\mathcal{O}$-module. Then*

$$\widehat{\lambda}_{\mathcal{O}'}(M \otimes_{\mathcal{O}} \mathcal{O}') = \widehat{\lambda}_{\mathcal{O}}(M). \tag{15.4}$$

Proof. Write $e' = v_{\mathcal{O}'}(\pi)$ for a uniformizer π of $\mathcal{O}$. The structure theorem over a DVR expresses M as a direct sum of modules $\mathcal{O}/\pi^{a_j}\mathcal{O}$. After tensoring, the corresponding lengths are $e'a_j$. Thus

$$\operatorname{length}_{\mathcal{O}'}(M \otimes \mathcal{O}') = e' \operatorname{length}_{\mathcal{O}}(M), \qquad v_{\mathcal{O}'}(\ell) = e' v_{\mathcal{O}}(\ell).$$

Dividing proves the assertion. This statement concerns extension of a fixed module; it does not identify different classes of allowed local lattices after extending coefficients.
Lean, formal context. `TraceRelations` (L717-L755), `FitTwistsExist` (L377). □

The use of the rational prime ℓ in (15.3) is essential to its normalization: the length of $\mathcal{O}/\ell\mathcal{O}$ is $e_{\mathcal{O}}$, while the length of the residue field is 1. The preceding division accounts for all coefficient ramification.

15.1 From local cup-product defects to crystalline relation colength

We prove the structural comparison that turns a local cup-product defect into the crystalline colength used by the determinant argument. The local relations cut out exactly the reductions of integral crystalline cocycles. The defect bounds the excess contributed by the other lattices in the local class. The source statements are `relLength_le_of_comesFromClass`, the graph construction and its local comparison, and `crystallineColength_le_sum_dualDefect`.

Let $K/\mathbb{Q}_\ell$ be finite, $\Gamma = G_K$ and $d = [K : \mathbb{Q}_\ell]$. Let O be the integers of a finite extension of $\mathbb{Q}_\ell$, with fraction field F. A lattice means a finite free O-module with a continuous O-linear Γ-action. Write

$$L^\dagger = \mathrm{Hom}_O(L, O)(1), \qquad M_m = L/\ell^m L.$$

All cocycles below are continuous. Write $Z_f(L)$ for the crystalline cocycles: the functions $z : \Gamma \to L$ such that

$$z(gh) = z(g) + gz(h), \qquad 1 \otimes z(g) = gb - b \quad \text{for some } b \in B_{\max} \otimes_{\mathbb{Z}_\ell} L.$$

Here $B_{\max}$ is the crystalline period ring used in the source. Its primitive condition agrees with the B_{cris} condition in Lemma 15.1. Indeed the classical period rings satisfy

$$\varphi(B_{\max}) \subseteq B_{\mathrm{cris}} \subseteq B_{\max},$$

and φ fixes $\mathbb{Q}_\ell$ and commutes with the Galois action; see Colmez, *Théorie d'Iwasawa des représentations de de Rham d'un corps local*, §III.2, p. 24. If $1 \otimes z(g) = gb - b$ with b in $B_{\max} \otimes L$, applying $\varphi \otimes 1$ supplies the B_{cris} primitive $(\varphi \otimes 1)b$. The converse follows from the second inclusion. Thus the two definitions give the same actual cocycles. There is also no continuity condition lost in passing from the source's primitive definition to this one. Give $B_{\max}$ its usual Hausdorff period topology, for which the Galois action and the inclusion $\mathbb{Q}_\ell \hookrightarrow B_{\max}$ are continuous. In a $\mathbb{Z}_\ell$-basis of L, the map $L \hookrightarrow B_{\max} \otimes_{\mathbb{Z}_\ell} L$ is a continuous injection from a compact space into a Hausdorff space, and hence a topological embedding. Thus $g \mapsto gb - b$, when it takes values in L, is continuous as an L-valued map. Bloch–Kato gives that $Z_f(L)$ is finite free over O when L is potentially crystalline, with the rank in Lemma 15.1.

Fix a class $\mathcal{Q}$ of potentially crystalline lattices, closed under finite direct sums and under equivariant isomorphisms of rational representations. In the application it is exactly $\mathrm{InClass}(\mathcal{F}, \mathcal{C}, \mathcal{U})$: the prescribed period admissibility along the given filter, the allowed Hodge–Tate jumps $\mathcal{U}$, and the inertia-polynomial identities $\mathcal{C}$ on periods. These conditions are unchanged by an equivariant isomorphism after tensoring with F, and are preserved by finite direct sums. For the first property the period spaces are identified by the rational isomorphism; for the second they are direct sums. The jump multisets are correspondingly unchanged or added, and the operator identities hold on each summand. We will use only these two elementary stability properties of the class.

The exact cup-product defect. The pairings tested in the source have a finite modulus ℓ^n, an O-linear form

$$\Phi : Y \otimes_O L^\dagger \longrightarrow O(1),$$

and a $\mathbb{Z}_\ell$-linear map $\lambda : O \to \mathbb{Z}_\ell$. The form is required to satisfy

$$g\Phi - \Phi = \ell^n b_\Phi(g), \qquad b_\Phi \in Z_f\big((Y \otimes_O L^\dagger)^\dagger\big). \tag{15.5}$$

Thus $\lambda\Phi$ modulo ℓ^n is an equivariant pairing into $\mathbb{Z}/\ell^n\mathbb{Z}(1)$. For $z \in Z_f(L^\dagger)$, the condition $z \in H_{\mathcal{Q}}(L^\dagger)$ of Lemma 15.2 means that its cup pairing with every $a \in Z_f(Y)$ vanishes in cohomology, for all $Y \in \mathcal{Q}$ and all the pairings just described. The test can be made either in $H^2(K, \mu_{\ell^n})$ or in $H^2(K, \overline{K}^\times)$: the map between these groups is injective by Hilbert 90 and the Kummer sequence. Set

$$\Delta_{\mathcal{Q}}(L^\dagger) = \mathrm{length}_O\big(Z_f(L^\dagger)/H_{\mathcal{Q}}(L^\dagger)\big) \in \mathbb{N} \cup \{\infty\}. \tag{15.6}$$

This is precisely the source's unnormalized $\mathrm{dualDefect}(\mathcal{Q}, L^\dagger)$. It is an O-module quotient: a scalar in a dual cup product can be moved to the crystalline test cocycle of Y.

Finite local duality for actual cocycles

Lemma 15.4 (An integral coefficient functional with no valuation loss)**.** *There is a $\mathbb{Z}_\ell$-linear map $\lambda : O \to \mathbb{Z}_\ell$ such that*

$$\mathrm{Hom}_O(L, O) \longrightarrow \mathrm{Hom}_{\mathbb{Z}_\ell}(L, \mathbb{Z}_\ell), \qquad \varphi \longmapsto \lambda \circ \varphi$$

is an isomorphism. Consequently it identifies $L^\dagger/\ell^m L^\dagger$ with the Cartier dual $M_m^D = \mathrm{Hom}(M_m, \mu_{\ell^m})$.

Proof. The module $D = \mathrm{Hom}_{\mathbb{Z}_\ell}(O, \mathbb{Z}_\ell)$ is a finite torsion-free O-module under $(a\lambda)(x) = \lambda(ax)$. To see torsion-freeness, for $a \neq 0$ choose $c \in O$ and $0 \neq b \in \mathbb{Z}_\ell$ with $ac = b$, by clearing the denominator of a^{-1} in F. If $a\lambda = 0$, then $b\lambda = 0$, and $\mathbb{Z}_\ell$-torsion-freeness gives $\lambda = 0$. Comparing ranks over $\mathbb{Z}_\ell$ shows that D has rank one over O. It is free because O is a DVR. Choose an O-generator λ of D. The assertion for a free lattice L follows coordinate by coordinate from $O \simeq D$. The induced pairing

is equivariant into $\mathbb{Z}_\ell(1)$, since the coefficient ring has trivial Galois action. Reducing a perfect integral pairing modulo ℓ^m remains perfect. No discriminant or different factor is introduced.

Lean. `exists_perfect_pairing`, `TateDualBound` (L122-L140). □

Lemma 15.5 (Orthogonal subgroups of finite cocycles)**.** *Let M be a finite ℓ-primary Γ-module with Cartier dual $M^D = \mathrm{Hom}(M, \overline{K}^\times)$. For $Y \subseteq Z^1(K, M^D)$ a subgroup, let $Y^\perp \subseteq Z^1(K, M)$ be its annihilator under the cup product. Then*

$$|Y^\perp|\,|Y| \le |M|^{d+2}, \tag{15.7}$$

with equality if Y contains $B^1(K, M^D)$.

Proof. Local Tate duality gives a perfect pairing on H^1. Passing from classes to cocycles gives

$$|Y^\perp|\,|Y| = |B^1(K, M)|\,|H^1(K, M)|\,|Y \cap B^1(K, M^D)|.$$

The local Euler characteristic formula and duality in degrees zero and two give $|H^1(K, M)| = |H^0(K, M)|\,|H^0(K, M^D)|\,|M|^d$ [Mil06, I, Corollary 2.3 and Theorem 2.8]. Now $|B^1(K, M)| = |M|/|H^0(K, M)|$, and the same identity holds for M^D, which has order $|M|$. Substitution proves the assertion.

Lean, comparison. The formal bound uses a prime-to-ℓ index subgroup. Finite local duality also gives the equality case above. See `DualityInequality` (L304-L335). □

Lemma 15.6 (Equivariant lifts and the local cup product)**.** *Put $W = \mathrm{Hom}_O(Y, L)$. An equivariant map $t : Y \to L/\ell^m L$ and any O-linear lift $q : Y \to L$ give the admissible pairing $\Phi(y, \varphi) = \varphi(qy)$ of (15.5). The obstruction to lifting $\ell^j t$ equivariantly is $\ell^j[b_\Phi] \in H^1(K, W)[\ell^m]$. If it vanishes, ℓ^j kills every test cup for this pairing. In particular, if $\ell^j H^1(K, \mathrm{Hom}_O(Y, L))_{\mathrm{tors}} = 0$ for every $Y \in \mathcal{Q}$, then*

$$\widehat{\lambda}_O(Z_f(L^\dagger)/H_{\mathcal{Q}}(L^\dagger)) \le j \operatorname{rank}_O Z_f(L^\dagger).$$

Proof. Freeness of Y gives q and identifies $W/\ell^m W = \mathrm{Hom}_O(Y, L/\ell^m L)$. Under $W = (Y \otimes_O L^\dagger)^\dagger$, the connecting homomorphism for multiplication by ℓ^m sends t to the class of $b_\Phi(g) = (gq - q)/\ell^m$. This cocycle is the coboundary of q/ℓ^m over F, so it belongs to $Z_f(W)$. Exactness gives the asserted lifting obstruction. If q_j is an equivariant lift of $\ell^j t$, the finite pairing multiplied by ℓ^j is the reduction of $(y, \varphi) \mapsto \lambda(\varphi(q_j y))$. For $a \in Z_f(Y)$ and $z \in Z_f(L^\dagger)$, the integral cup under this pairing maps to zero over $\mathbb{Q}_\ell$. Indeed, the pairing induces an equivariant map to the rational Tate dual of $L^\dagger$ over $\mathbb{Q}_\ell$, the finite local condition is functorial, and the two finite local conditions are orthogonal [BK90, Proposition 3.8]. Passing to the inverse limit in finite local Tate duality and inverting ℓ, the local invariant identifies $H^2(K, \mathbb{Z}_\ell(1)) \to H^2(K, \mathbb{Q}_\ell(1))$ with $\mathbb{Z}_\ell \hookrightarrow \mathbb{Q}_\ell$ [Mil06, I, Corollary 2.3]. Thus the integral cup and its reduction both vanish. Every admissible form reduces to such a map t. The uniform hypothesis gives $\ell^j Z_f(L^\dagger) \subseteq H_{\mathcal{Q}}(L^\dagger)$, and Lemma 15.2 gives the length bound.

Lean, formal context. `TateDualBound` (L147-L173), `Pairing`, `HasCupExponent`, `normLength_dualDefect_le_of_hasCupExponent`. The connecting class and Bloch–Kato duality verify the formal exponent hypothesis for the pairings with the stated lifts. □

Suppose $W^{G_K} \to (W/\varpi W)^{G_K}$ is surjective, where ϖ is a uniformizer of O. The invariants sequence gives $H^1(K, W)[\varpi] = 0$, hence $H^1(K, W)_{\mathrm{tors}} = 0$. This applies in particular when the residual representations of Y and L admit no nonzero equivariant map.

Corollary 15.7 (The lifting exponent for two Tate twists)**.** *Suppose $K/\mathbb{Q}_\ell$ is unramified, $Y = \mathbb{Z}_\ell(u)$, $L = \mathbb{Z}_\ell(v)$, and $r = v - u$. The least $j \geq 0$ for which every $\ell^j t$ in Lemma 15.6 lifts equivariantly at every modulus is*

$$\begin{cases} 0 & r = 0 \text{ or } (\ell > 2 \text{ and } \ell - 1 \nmid r), \\ 1 + v_\ell(r) & \ell > 2,\ r \neq 0,\ \ell - 1 \mid r, \\ 1 & \ell = 2,\ r \text{ odd}, \\ 2 + v_2(r) & \ell = 2,\ r \neq 0 \text{ even}. \end{cases}$$

Proof. For $r = 0$ every map lifts. Otherwise $W = \mathbb{Z}_\ell(r)$ has no integral invariants. The cyclotomic character has image $\mathbb{Z}_\ell^\times$ on G_K, since the shifted cyclotomic polynomial $\Phi_{\ell^m}(1+T)$ is Eisenstein over the unramified base for every $m \geq 1$. An invariant scalar modulo ℓ^m is therefore killed by the ideal generated by $x^r - 1$ for all $x \in \mathbb{Z}_\ell^\times$. If this ideal is (ℓ^j), then j suffices and, for $j > 0$, the invariant scalar $1 \bmod \ell^j$ attains it. For odd ℓ, cyclicity of $\mathbb{F}_\ell^\times$ gives $j = 0$ unless $\ell - 1 \mid r$. In the latter case Fermat's theorem gives $v_\ell(x^r - 1) \geq 1 + v_\ell(r)$, with equality for $x = 1 + \ell$. For $\ell = 2$ and odd r take $x = -1$. For even r, every odd square is $1 \bmod 8$, and $x = 5$ gives equality in $v_2(x^r - 1) \geq 2 + v_2(r)$. These valuations follow by the binomial expansion. Whenever $v_\ell(y - 1) > 1/(\ell - 1)$, raising y to the ℓ-th power increases this valuation by one and raising it to an integer prime to ℓ preserves it, including after inversion.

Lean, formal context. `InClass.of_cyclotomic_pow`. Lean places each Tate power in its one-jump period class. The valuation computation here gives a test-specific lifting exponent. □

Corollary 15.8 (The unramified local test)**.** *Let O be the integers of a finite extension of $\mathbb{Q}_\ell$. Suppose Y and L become unramified over a finite extension K'/K of degree d'. Then $\ell^{v_\ell(d')}$ kills every test cup for this pair. In particular $\Delta_{\mathcal{Q}}(L^\dagger) = 0$ if, for every $Y \in \mathcal{Q}$, one can take d' prime to ℓ. This includes crystalline representations of arbitrary rank with all filtration jumps zero on Y and L, with $K' = K$.*

Proof. First work over K'. Unramified rational representations are precisely the crystalline representations with trivial filtration [FO, Proposition 7.14]. A finite cocycle $a \in Z_f(Y)$ defines a crystalline extension of the trivial representation by Y_F. Its jumps are zero by strict exactness of the de Rham filtration, so it is unramified. Thus a itself vanishes on inertia, as does $s = t \circ a$. Choose $x \in L$ lifting the value of s at arithmetic Frobenius. Writing A for its action on L, the powers of $\left(\begin{smallmatrix} A & x \\ 0 & 1 \end{smallmatrix}\right)$ extend continuously from $\mathbb{Z}$ to $\widehat{\mathbb{Z}}$, since $\mathrm{GL}_{1+\operatorname{rank} L}(O)$ is the inverse limit of its finite congruence quotients. The upper-right entry gives an unramified cocycle in L reducing to s, since the two reductions agree at a topological generator. Its extension is unramified and hence crystalline. Its evaluation cup with $z \in Z_f(L^\dagger)$ is zero by the integral duality argument in Lemma 15.6, proving vanishing over K'. Restriction preserves the finite local condition. Applying corestriction to a restricted cup multiplies the original class by d'. It is ℓ-primary, giving the asserted power.

Lean, formal context. `dualDefect_eq_zero_iff`, `Pairing.isCupCoboundary_of_eq_coboundary`. The general formal argument descends cup vanishing from a chart subgroup. Here lifting a Frobenius value proves it over the unramified representation field. □

Lifting the cocycle can be much easier than lifting the morphism. Over $\mathbb{Q}_\ell$, take $Y = \mathbb{Z}_\ell$ and let L have unramified character sending Frobenius to $1 + \ell^h$, for any $h \geq 1$ and odd ℓ. Then $W^{G_K} = 0$, while $1 \bmod \ell^h$ is invariant, so its connecting class has order ℓ^h, although the cup vanishes by Corollary 15.8. Fontaine–Laffaille full faithfulness in the small-weight interval $[0, \ell - 2]$ identifies morphisms between torsion objects [FL82, Theorem 6.1]. The Bhatt–Morrow–Scholze comparison bounds torsion in the geometric étale cohomology of a proper smooth formal scheme by torsion in the crystalline cohomology of its special fibre [BMS18, Theorem 1.1(ii)]. Here the obstruction lies

in the local Galois cohomology of W. For tests without controlled equivariant lifts, Corollary 16.20 gives the uniform cup-product estimate for the crystalline extensions specified there, under its common tame chart, opposite jump window and size hypotheses.

In the automorphic application, Corollary 17.88 uses the characters $C_c = \chi_\ell^{-c}$ for $0 \le c < t$. All maps on a diagonal pair $Y = L = C_c$ lift, since the action on its endomorphisms is trivial. For two distinct Tate characters over an unramified base, Corollary 15.7 gives the exponent over $\mathbb{Z}_\ell$ with $r = a - c$ for $Y = C_a, L = C_c$. In particular, if $\ell > t$ and the local base is unramified, every test between two characters in the string has zero loss, also over an extended coefficient ring. Indeed for $a \neq c$ one has $0 < |a - c| < \ell - 1$, so some cyclotomic unit separates the characters already in the residue field. The same local condition also tests the main Galois representation. Its nonzero Hodge–Tate weights persist under finite extension. Corollary 17.31 gives crystallinity over a common extension, and the integral estimate is still needed for this family.

Theorem 15.9 (Finite tests for the local cup-product defect)**.** *Assume $L \in \mathcal{Q}$. For $m \ge 1$, let A_m be the reduction of $Z_f(L)$ in $Z^1(K, M_m)$, and let C_m be generated by the cocycles*

$$s = t \circ a, \quad a \in Z_f(Y), \quad Y \in \mathcal{Q}, \quad t : Y \longrightarrow M_m \quad \text{equivariant and } O\text{-linear.} \tag{15.8}$$

Taking $Y = L$ gives $A_m \subseteq C_m$. Put $A_0 = C_0 = 0$. Then

$$\Delta_{\mathcal{Q}}(L^\dagger) = \sup_{m \ge 1} \operatorname{length}_O(C_m / A_m).$$

If the defect is finite, some modulus and finitely many maps (15.8) attain it. In particular every O-submodule $N \subseteq C_m$ satisfies

$$\operatorname{length}_O(N/(N \cap A_m)) \le \Delta_{\mathcal{Q}}(L^\dagger). \tag{15.9}$$

Proof. Use the coefficient functional from Lemma 15.4 to identify $L^\dagger/\ell^m L^\dagger = M_m^D$, and let B_m be the reduction of $Z_f(L^\dagger)$. Bloch–Kato orthogonality [BK90, Proposition 3.8] gives $B_m \subseteq A_m^\perp$: the integral cup vanishes over $\mathbb{Q}_\ell$, hence integrally by the injection of local invariant groups used in Lemma 15.6. Every finite coboundary lifts to an integral coboundary by lifting its defining vector. In particular A_m, B_m contain all coboundaries on their respective sides. The kernel of reduction on $Z_f(L)$ is $\ell^m Z_f(L)$, since a period primitive for a divisible cocycle can be divided by ℓ^m. The same holds on the dual. Thus (15.2) gives

$$|A_m|\,|B_m| = |M_m|^{d+2}. \tag{15.10}$$

The equality case of Lemma 15.5, applied with M and M^D interchanged, now gives $B_m = A_m^\perp$. Finite local duality on cohomology therefore induces a perfect pairing

$$C_m/A_m \;\times\; B_m/C_m^\perp \;\longrightarrow\; \mathbb{Q}_\ell/\mathbb{Z}_\ell. \tag{15.11}$$

These two finite O-modules have equal cardinality and hence equal length.

Let U_m be the inverse image of $C_m^\perp$ in $Z_f(L^\dagger)$. Then $Z_f(L^\dagger)/U_m = B_m/C_m^\perp$. By Lemma 15.6, the maps (15.8) give exactly the admissible forms modulo ℓ^m. Every coefficient functional is an O-multiple of the one chosen above, and that scalar can be absorbed into the test cocycle. Consequently $\bigcap_m U_m = H_{\mathcal{Q}}(L^\dagger)$. To see $U_{m+1} \subseteq U_m$, compose a test with $M_m \hookrightarrow M_{m+1}$, $v \mapsto \ell v$. The new cup is the image of the old one under $\mu_{\ell^m} \hookrightarrow \mu_{\ell^{m+1}}$. Both have the same local invariant in $\mathbb{Q}/\mathbb{Z}$.

The lengths of $Z_f(L^\dagger)/U_m$ thus increase. If they are bounded, they stabilize and the U_m then equal their intersection. If they are unbounded, the quotient by the intersection has infinite length.

This proves the formula. At each m the group C_m is finite, so finitely many maps generate it. They can be combined into one map from their direct sum, which belongs to $\mathcal{Q}$. Finally $N/(N \cap A_m)$ embeds in C_m/A_m.

Lean, comparison. Lean proves the colength inequality by a cardinality bound. The perfect pairing identifies the quotient that attains the defect. See `relLength_le_of_comesFromClass`. □

A lattice realizing the local Ribet cocycles

Let G be the profinite global group, $\eta : \Gamma \to G$ the continuous local restriction homomorphism, and let $u_x : G \to \mathrm{GL}_n(O)$, $x \in X$, be a finite family of continuous representations. Let $\rho : G \to \mathrm{GL}_r(O)$ be continuous and denote the local lattice of $\rho \circ \eta$ by L. Write $\mathcal{R}_m$ for the actual ρ-cocycles $w : G \to (O/\ell^m O)^r$ satisfying

$$\sum_g \theta_g (u_x(g) - 1) = 0 \quad \text{for every } x \quad \Longrightarrow \quad \sum_g \theta_g w(g) = 0 \tag{15.12}$$

for every finitely supported $\theta : G \to O$. This is the source's ribetCocycles.

Lemma 15.10 (A Ribet cocycle comes from a crystalline cocycle of the class)**.** *Suppose the local lattices of all $u_x \circ \eta$ belong to $\mathcal{Q}$. Then the restriction of each $w \in \mathcal{R}_m$ satisfies (15.8). If the u_x are semisimple over F, one may use the same lattice and integral cocycle at every depth.*

Proof. Use M, I, U from Lemma 13.18. The cocycle w is given by an equivariant map $I \to M_m$. Suppose first that the u_x are semisimple over F, as in Theorem 17.105. The columns of $M_F = M \otimes_O F$ are copies of these representations, so I_F admits an equivariant projection $P : M_F \to I_F$. Use the full lattice

$$Y = I \oplus (M \cap \ker P).$$

Extend the map for w by zero on the second summand. With $\mathbf{1}$ the tuple of identity matrices, $U(g) = (g-1)\mathbf{1}$ lies in I, so $U(g) = (g-1)P(\mathbf{1})$. Thus $U \circ \eta$ is an integral cocycle in Y that becomes a coboundary over F, hence belongs to $Z_f(Y)$. Its image is $w \circ \eta$. Since $Y_F = M_F$, closure under direct sums and rational isomorphism puts the local lattice Y in $\mathcal{Q}$.

For a general family, elementary divisors give k such that $I \cap \ell^k M \subseteq \ell^m I$. Take $Y = I + \ell^k M$ and extend the map for w by zero on $\ell^k M$; the displayed inclusion makes this well-defined. Again $Y_F = M_F$, and $U \circ \eta$ is the rational coboundary of $\mathbf{1}$. Both lattices carry the induced ℓ-adic topology, so the cocycles are continuous.

Lean, comparison. Lean uses a graph lattice depending on the depth in `exists_graph_subrepresentation` and `comesFromClass_of_mem_ribetCocycles`. The semisimple family admits a fixed rational complement. □

The relation-defined local and global conditions

For the local lattice $L = O^r$, define $\mathcal{B}_m(L)$ to be the module of functions $s : \Gamma \to M_m$ obeying every matrix-valued finite relation of $Z_f(L)$. Explicitly, for every finitely supported $A : \Gamma \to \mathrm{End}_O(L)$,

$$\left(\sum_g A(g) z(g) = 0 \quad \text{for all } z \in Z_f(L) \right) \quad \Longrightarrow \quad \sum_g A(g) s(g) = 0. \tag{15.13}$$

This is precisely obeying$(\mathrm{relationsOn}(Z_f(L)), O/\ell^m O)$ in the source. For an arbitrary O-module A, use the same relations for functions $\Gamma \to L \otimes_O A$.

Theorem 15.11 (The local crystalline colength comparison)**.** *The functions with coefficients in A obeying (15.13) are precisely the image of $Z_f(L) \otimes_O A$, and its evaluation map is injective. In particular $\mathcal{B}_m(L) = A_m$, the reduction of $Z_f(L)$. If the local lattices of $\rho \circ \eta$ and all $u_x \circ \eta$ belong to $\mathcal{Q}$, then*

$$\operatorname{length}_O(\mathcal{R}_m/\{w \in \mathcal{R}_m : w \circ \eta \in \mathcal{B}_m(L)\}) \le \Delta_{\mathcal{Q}}(L^\dagger). \tag{15.14}$$

Proof. Put $U = Z_f(L)$ and choose an O-basis of L. By Lemma 15.1, U is finite free. If all the values of $z \in U$ are divisible by a uniformizer ϖ, then z/ϖ is a continuous integral cocycle and its rational class is still crystalline. Thus the coordinate evaluations separate $U/\varpi U$. Choose evaluations whose reductions form a basis of its dual. By Nakayama's lemma they form a basis of $U^\vee$. Every other coordinate evaluation is an O-linear combination of these. Each such identity is a relation in (15.13), by placing its coefficients in one matrix row. Consequently any solution over A is determined by the selected values. Those values are arbitrary: the dual basis in U tensored with them gives the unique solution. For $A = O/\ell^m O$ this gives the asserted equality.

Let N be the restricted image of $\mathcal{R}_m$. Lemma 15.10 supplies its presentations (15.8). The quotient in (15.14) is $N/(N \cap A_m)$, which Theorem 15.9 bounds by the dual defect.

Lean, comparison. The proof of `relLength_ribetCocycles_le` uses the inclusion of the reduction in the solution module. Coordinate evaluations give equality for the integral lattice $Z_f(L)$. □

Corollary 15.12 (The sum of the local defects bounds the global colength)**.** *Let E be the global field and, at every $v \mid \ell$, let $\mathcal{Q}_v$ be the specified local class containing the restrictions of ρ and all u_x. Let L_v be the local lattice of ρ. For every $m \ge 0$,*

$$\operatorname{crystallineColength}(\rho, u, m) \le \sum_{v \mid \ell} \Delta_{\mathcal{Q}_v}(L_v^\dagger). \tag{15.15}$$

The right side is independent of m.

Proof. Let $\mathcal{B}_{v,m}$ be the inverse image in $\mathcal{R}_m$ of the relation module $\mathcal{B}_m(L_v)$ under restriction at v. By the definition of the global crystalline relations, their solution module in $\mathcal{R}_m$ is $\bigcap_{v\mid\ell} \mathcal{B}_{v,m}$. There is an injective diagonal map

$$\mathcal{R}_m/\bigcap_{v\mid\ell} \mathcal{B}_{v,m} \longrightarrow \bigoplus_{v\mid\ell} \mathcal{R}_m/\mathcal{B}_{v,m}.$$

Length is monotone under injection and additive on a finite direct sum. Theorem 15.11 bounds the summand at v by $\Delta_{\mathcal{Q}_v}(L_v^\dagger)$. This proves (15.15) for the exact source definition of crystalline colength.

Lean. `crystallineColength_le_sum_dualDefect`. □

Corollary 15.13 (Compatibility with finite and divisible coefficients)**.** *The local conditions defined by (15.13), and the Ribet cocycle relations (15.12), are preserved and reflected by injective O-linear coefficient maps. In particular, the inclusions*

$$O/\ell^m O \longrightarrow F/O, \qquad x \longmapsto \ell^{-m} x$$

give the finite-to-divisible compatibility used in Lemma 13.19, for the original local conditions specified in the definition of the Selmer group.

Proof. Under a coefficient injection a finite matrix sum vanishes precisely when its image vanishes. This applies to both sets of relations and to the cocycle identities. The same holds for vanishing on inertia or at the normalizing element. The indicated map to F/O is injective, which gives the asserted compatibility.

Lean. `comp_mem_obeying_iff`, `mem_obeying_localRelations_iff`. □

The comparison (15.15) is now proved from the specified local classes, finite local Tate duality, the local Euler characteristic and the crystalline cocycle rank formula. Theorem 17.40 applies the quantitative estimates to the constructed data, and Corollary 17.41 gives the original summed allowance. The actual main-row application, including its local types, is Corollary 17.88; the full ambient-family application is Corollary 17.106. No such construction is included as a premise under the name *crystalline colength* in the proof above.

15.2 Polynomial degree and the tame local period loss

The tame branch of the crystalline-defect estimate concerns a place whose residue characteristic is the coefficient prime. It is separate from the old-inertia relations at places away from that prime. Here we prove the polynomial and ideal calculations that account for the tame exponent. The three period statements required to apply them are then reconstructed in the following subsections, with the integral frame constructed in Theorem 16.19. The relevant source files are TameCharts and the tame cup exponent.

Throughout this subsection p is the coefficient prime. Thus the local field has residue characteristic p, whereas the old-inertia calculation uses places of residue characteristic different from the coefficient prime.

Lemma 15.14 (A monic polynomial for a tensor operator)**.** *Let R be a commutative ring, let A and B be endomorphisms of R-modules V and W, and suppose that monic polynomials $P, Q \in R[X]$ of degrees d, d' satisfy $P(A) = 0$ and $Q(B) = 0$. There is a monic $H \in R[X]$ of degree dd' such that*

$$H(A \otimes B) = 0.$$

If the modules are finite free, the same polynomial annihilates the transpose of $A \otimes B$.

Proof. For $d, d' > 0$, the algebra

$$C = R[X, Y]/(P(X), Q(Y))$$

is free over R with basis $X^i Y^j$, $0 \le i < d$ and $0 \le j < d'$. Let H be the characteristic polynomial of multiplication by XY on this free module. It is monic of degree dd'. The adjugate identity, hence Cayley–Hamilton over a commutative ring, gives $H(XY) = 0$ in C.

The commuting operators $A \otimes 1$ and $1 \otimes B$ make $V \otimes_R W$ a C-module. Applying the last identity to that module proves the assertion. Transposing a polynomial identity preserves it. If one degree is zero, its monic polynomial is 1; the corresponding module is zero, and $H = 1$ proves the statement. No diagonalizability or splitting of P and Q is used.

Lean, formal context. `CupOrthogonalityBounds` (L532-L600), `FitTwistsExist` (L377). □

One other elementary fact will be used below. If an endomorphism β preserves an exact sequence $0 \to V \to E \to C \to 0$, acts as 1 on C, and is killed by q on V, then

$$q(\beta)(\beta - 1) = 0 \quad \text{on } E. \tag{15.16}$$

Indeed $(\beta - 1)E \subseteq V$, where $q(\beta)$ is zero. Consequently a polynomial of degree d for the original lattice gives one of degree $d + 1$ for its extension by a trivial parameter lattice. This is the origin of the extra 1 in the tame cost.

A polynomial makes most coordinate-lowering steps free

Let Λ be the ring of integers of a finite unramified extension of $\mathbb{Q}_p$, with residue field k. Suppose that a commutative Λ-algebra A contains $R = \Lambda[[z]]$, and that an automorphism s of A fixes Λ and satisfies $s(z) = \xi z$, for $\xi \in \Lambda^\times$. Let C_A be a finite free A-submodule of a semilinear A-module T_A, with a basis fixed by s. Let E_A be another semilinear module and let $\pi : E_A \to T_A$ be an equivariant A-linear map. Assume multiplication by z is injective on T_A. For an A-submodule $M_A \subseteq E_A$, put

$$J = \{u \in A : uC_A \subseteq \pi(M_A)\}.$$

This is an ideal of A.

Lemma 15.15 (The polynomial lowering identity)**.** *Let $M \subseteq E_A$, and let $Q(X) = \sum_{a=0}^{D} q_a X^a \in \Lambda[X]$ satisfy*

$$Q(1) = 0, \qquad Q(s)M \subseteq zM_A. \tag{15.17}$$

Fix $r \in \Lambda$, $t \geq 0$ and $j \geq 1$. Suppose there is one $u \in R$ such that, for every vector c in the fixed basis of C_A, an element $x_c \in M$ satisfies

$$\pi(x_c) = (p^t z^j - ru)c.$$

Then

$$Q(\xi^j)p^t z^{j-1} \in J + (r). \tag{15.18}$$

If $\overline{Q}(\overline{\xi}^j) \neq 0$ in k, then $p^t z^{j-1} \in J + (r)$.

Proof. Write $u = u_0 + zv$, with $u_0 \in \Lambda$ and $v \in R$. Since $Q(1) = 0$ and s fixes Λ,

$$Q(s)u = \sum_a q_a(u_0 + \xi^a z\, s^a(v)) = zv', \qquad v' = \sum_a q_a \xi^a s^a(v) \in A.$$

Also $Q(s)(p^t z^j) = Q(\xi^j)p^t z^j$. By (15.17), write $Q(s)x_c = zy_c$, with $y_c \in M_A$. Equivariance of π and invariance of c give

$$z\pi(y_c) = z(Q(\xi^j)p^t z^{j-1} - rv')c.$$

Cancel z on T_A. The resulting identity holds for every basis vector, with the same v', so its scalar belongs to J. This proves (15.18). In the last assertion, $Q(\xi^j)$ is a unit of the discrete valuation ring Λ, hence of A, and can be removed from ideal membership.

Lean, formal context. `CupProductBound` (L994-L1022), `FitTwistsExist` (L377). □

In the intended period construction, M is the integral Kisin module, M_A its A_{inf} version, and π is the projection from the extension associated with a crystalline cocycle to its trivial parameter lattice. The strong lifting condition and (15.17) are specific period-module assertions; the lemma proves their exact algebraic consequence. It does not assume that s acts as a scalar on the Galois lattice.

Theorem 15.16 (Tame ideal descent controlled by polynomial degree)**.** *Let A be a commutative Λ-algebra with $p \in \mathrm{Jac}(A)$, let $z \in A$, and suppose $\overline{\xi} \in k^\times$ has order $e \geq 1$. Let $J_i \subseteq A$ be a family of ideals. Fix nonnegative integers n, κ, ρ and a monic polynomial $Q \in \Lambda[X]$ of degree D, with*

$$e \leq p^\rho, \qquad \rho D \leq \kappa, \qquad n + \kappa + 2 < p. \tag{15.19}$$

Write $J_i^{(\kappa)} = J_i + (p^{\kappa+1})$ and assume:

1. *The set* $\mathcal{A} = \{a \geq 0 : z^a \in J_i^{(\kappa)} \text{ for every } i\}$ *is nonempty, and if* $a \in \mathcal{A}$, *then*

$$\left\lfloor \frac{a + e(n+\kappa)}{p} \right\rfloor \in \mathcal{A}.$$

2. *For* $j \geq 1$ *and* $t \geq 0$, *membership* $p^t z^j \in J_i^{(\kappa)}$ *implies*

$$p^{t+v_p(j)+1} z^{j-1} \in J_i^{(\kappa)}.$$

3. *The same membership also implies*

$$Q(\xi^j) p^t z^{j-1} \in J_i^{(\kappa)}.$$

Then $p^\kappa \in J_i$ *for every* i.

Proof. Take the least $\nu \in \mathcal{A}$. The recurrence and minimality give

$$\nu \leq \left\lfloor \frac{\nu + e(n+\kappa)}{p} \right\rfloor, \qquad (p-1)\nu \leq e(n+\kappa).$$

Since $n + \kappa < p - 1$, we obtain $\nu < e$.

Call a step j, $1 \leq j \leq \nu$, exceptional if $\overline{Q}(\overline{\xi}^j) = 0$. The elements $\overline{\xi}^j$ in this range are pairwise distinct. Reduction preserves the degree of a monic polynomial, so the elementary root bound over k shows that there are at most D exceptional steps.

Start from $z^\nu \in J_i^{(\kappa)}$ and descend in j. At a nonexceptional step, condition 3 and the unit $Q(\xi^j)$ lower the power of z without increasing the power of p. At an exceptional step, condition 2 costs $v_p(j) + 1$. Because $0 < j < e \leq p^\rho$,

$$v_p(j) + 1 \leq \rho.$$

Thus, after all the steps, $p^t \in J_i^{(\kappa)}$ for some $t \leq \rho D \leq \kappa$. Multiplying by $p^{\kappa - t}$ gives

$$p^\kappa \in J_i + (p^{\kappa+1}).$$

Write $p^\kappa = y + p^{\kappa+1} b$, $y \in J_i$. Then $p^\kappa(1 - pb) = y$. Since p is in the Jacobson radical, $1 - pb$ is a unit, proving $p^\kappa \in J_i$. This last step needs no finite-generation assumption on J_i. If $\nu = 0$, the same argument starts directly with 1 in the ideal modulo $p^{\kappa+1}$.

Lean, formal context. `CupProductBound` (L994-L1022), `FitTwistsExist` (L377). □

For a monic annihilator q of degree d on the original lattice, (15.16) supplies $Q = q(X)(X - 1)$ of degree $d + 1$. The theorem therefore gives the precise loss

$$\kappa = \rho(d+1), \qquad e \leq p^\rho, \tag{15.20}$$

provided the displayed height condition holds. The number of coordinate steps can be close to e, but only at most $d + 1$ of them cost a power of p.

The first two ideal hypotheses are the precise specializations of ordinary coordinate lowering and the frame and Frobenius height estimates. The third follows from Lemma 15.15 once the period-module lifting and divisibility statements in the tame polynomial relation and its application to lifting ideals are supplied.

Removing tame inertial action and assembling the exponent

Lemma 15.17 (A common tame extension for the inertial periods)**.** *Let $1 \le r < p$, and put*

$$e = \mathrm{lcm}_{1\le j\le r}(p^j - 1), \qquad f = r!.$$

Then $p \nmid ef$, $e \mid p^f - 1$, and $e \le p^{r^2}$. There is a finite Galois extension

$$K = \mathbb{Q}_{p^f}(p^{1/e})/\mathbb{Q}_p$$

of degree ef whose absolute inertia subgroup $I_K \subseteq I_{\mathbb{Q}_p}$ is the kernel of the tame character modulo e. If β is a continuous representation of inertia with open kernel and $\beta(g)^e = 1$ for every inertia element g, then β is trivial on the inertia of K.

Proof. Each $p^j - 1$ is prime to p, and $r < p$ makes $r!$ prime to p. Since $j \mid r!$, each $p^j - 1$ divides $p^f - 1$. Also

$$e \le \prod_{j=1}^{r}(p^j - 1) \le p^{r(r+1)/2} \le p^{r^2}.$$

The unramified degree-f extension contains μ_e. The polynomial $X^e - p$ is Eisenstein, so adjoining a root gives a totally ramified extension of degree e; its roots are all present. The resulting extension of $\mathbb{Q}_p$ is Galois, and its inertia quotient is the usual tame cyclic quotient of order e.

The image of β is finite because inertia is compact and the kernel is open. Wild inertia has a finite p-group image. Every element of that image has order dividing e, which is prime to p, so the wild image is trivial. The remaining representation factors through the procyclic tame inertia group. Its image has exponent dividing e, hence it factors through the tame quotient of order e. This quotient is trivial on inertia over K.

Lean, formal context. `DeepPoints` (L378-L409), `FitTwistsExist` (L377). □

Applied to the inertial action on potentially crystalline periods, this lemma gives the common tame extension used in the source. The usual descent criterion for potentially crystalline periods then makes the lattices crystalline over its Galois group: trivial inertial action on the period space permits descent across the remaining unramified extension. The quantitative integral comparison below is an additional assertion.

Three integral period estimates. For two factors that become unramified over an extension of degree prime to ℓ, Corollary 15.8 already gives zero cup loss. For the general case we use the following estimates for the modules defined by crystalline forms. The integral frame is constructed in Theorem 16.19. Corollary 16.20 combines the three estimates under the comparison and representation hypotheses below.

1. For the extension attached to each crystalline cocycle, its standard-exponent A_{inf} period module has lifting ideals with the nonempty set of initial exponents, Frobenius height recurrence and ordinary coordinate-lowering relation in Theorem 15.16. Here $n \ge 0$ is an upper bound for the Hodge–Tate jumps and e is the ramification index of the tame chart. These assertions are required for the actual standard-exponent modules when the jump sets contain negative integers as well: no positivity of the jumps and no unrecorded Tate shift is allowed. In the source, the standard exponents at level zero prescribe order $-u$ for a jump u, so negative jumps are included in the module definition.

2. If $q \in \mathbb{Z}_p[X]$ annihilates the linear inertial operator on the potentially crystalline period space, the semilinear sum $q(s)$ on the integral Kisin module has image in zM_A. If the multiplier f belongs to the chart ring $S = \Lambda[[z]]$ and to a lifting ideal modulo $p^{\kappa+1}$, it admits the common S-valued correction required by Lemma 15.15. The restriction $f \in S$ is essential: it is false for general $f \in A_{\mathrm{inf}}$, as the counterexample following Theorem 15.37 shows.

3. For an admissible finite-coefficient pairing as in (15.5), take the two crystalline cocycle extensions and the extension of the Tate dual tensor associated with the pairing's crystalline error cocycle. Suppose p^κ kills the three standard-exponent period cokernels. Suppose their first two jump sets lie in

$$[-\eta, \eta] \cup -\varepsilon[b, b+W], \qquad [-\eta, \eta] \cup \varepsilon[b, b+W], \qquad \varepsilon \in \{1, -1\},$$

and $p > b + 6(\eta + W)$. Then the cup product of the two cocycles, after multiplication by p^H, is a continuous coboundary for

$$H \geq 3\big(\kappa + (48(\eta + W) + 4)d\big), \tag{15.21}$$

where $d = 0$ for an unramified chart and $d = 1$ for a nontrivial tame chart.

The first statement is proved in Lemmas 15.40 and 15.42 and Theorems 15.47 and 15.49. The second is Theorems 15.35 and 15.37. Theorem 15.65 proves the third, including the stronger sufficient exponent $3\kappa + 2(48(\eta + W) + 4)d$.

The third statement is the $a = 0$ specialization of `isScaledCupCoboundary_of_killsCokernel`. In that specialization the excluded interval $[-a, 0)$ is empty. The condition on p makes every nonzero integer within distance $6(\eta + W)$ of 0 or $\pm b$ prime to p, which is the source's exact gap hypothesis. Its proof uses integral period points, their allowed changes, and a continuous cup-coboundary construction; no assertion of this strength is being inferred solely from the eigenvalues of an inertia matrix.

Theorem 15.18 (The tame cup exponent from the period estimates)**.** *Assume the three period estimates just stated, and let $r \geq 1$ and $N, b, W, \eta \geq 0$ be integers. Let a class of potentially crystalline $\mathcal{O}$-lattices over $G_{\mathbb{Q}_p}$ have the following type data. At every inertia element, its period operator is annihilated by $X^e - 1$, with $e = \mathrm{lcm}_{1 \leq j \leq r}(p^j - 1)$, and by a monic polynomial over $\mathbb{Z}_p$ of degree at most N. Let the paired lattice L also be potentially crystalline and have these two annihilator bounds at every inertia element, including inverses. In the source it belongs to the class with inverse inertial type; that hypothesis supplies these bounds. Suppose the jump sets of the class and of L lie in the opposite windows in the period estimates. If*

$$b + 6(W + \eta) + r^2(N+1)^2 + 2 < p, \tag{15.22}$$

then the class has cup exponent

$$H = 3r^2(N+1)^2 + 150(\eta + W + 1). \tag{15.23}$$

In other words, multiplying any crystalline cocycle of the paired lattice by p^H makes its cup product with every crystalline cocycle of every lattice of the class a continuous coboundary, for every allowed finite-coefficient pairing.

Proof. Set $\kappa = r^2(N+1)^2$ and $n = b + 6(W + \eta)$. The size hypothesis implies $r < p$ when $r \geq 1$. Lemma 15.17 gives a common tame chart with ramification index $e \leq p^{r^2}$, so take $\rho = r^2$ in Theorem 15.16.

Choose the degree-at-most-N inertial polynomials for each lattice at the generator and its inverse. Lemma 15.14 gives a monic polynomial of degree at most N^2 for the operator on the Tate dual of their tensor product. Concretely one applies the lemma to the two inverse operators and transposes; the Tate factor has trivial inertial type on the crystalline period space.

Each of the three extensions by its trivial parameter lattice therefore has a polynomial of degree at most N^2+1 or $N+1$, by (15.16). In either case

$$\rho \deg Q \le r^2(N+1)^2 = \kappa.$$

Every jump of either original lattice is at most $b+W+\eta$. A jump of the Tate dual tensor is $-u-u'-1$, where u, u' lie in opposite windows. At most one of u, u' can lie below $-\eta$, so $-u-u'-1 \le b+W+2\eta$. The parameter extensions add only the jump 0. Thus all jumps relevant to the three lifting ideals are at most n. The first two period estimates, Lemma 15.15 and Theorem 15.16 show that p^κ kills all three period cokernels.

The third period estimate now applies. For a tame chart $d \le 1$, and

$$3(\kappa + (48(\eta+W)+4)d) \le 3\kappa + 144(\eta+W) + 12 \le 3\kappa + 150(\eta+W+1) = H.$$

Thus the required cup class vanishes after multiplication by p^H on the Galois group on which the period construction is performed. If the third period estimate is formulated on G_K, descend this conclusion to $G_{\mathbb{Q}_p}$ by restriction and corestriction. For every finite p-primary coefficient module, their composite is multiplication by $[K:\mathbb{Q}_p] = ef$. This integer is prime to p by Lemma 15.17, so it acts invertibly on the cohomology group. The original cup class multiplied by p^H is therefore zero as well. Vanishing in continuous cohomology means that the continuous two-cocycle is the coboundary of a continuous one-cochain. No additional power of p is used in this descent.

Lean, formal context. `DeepPoints` (L378-L409), `FitTwistsExist` (L377). □

The normalization of the resulting defect is supplied by Lemma 15.2. Theorem 15.18 explains the polynomial-degree contribution and its precise numerical constant. Corollary 16.20 now supplies its three period statements, including the construction of the explicit frame matrices, and gives a smaller sufficient exponent. The following sections prove the nonzero-pole bridge and the few-low-jump estimate. Corollary 16.56 supplies the small-loss one-place allowance. The quantitative frame, long-gap and cut arguments culminate in Corollary 16.93, which covers the other prime ranges under their stated local hypotheses.

15.3 Summing the local crystalline defects

The structural comparison in Corollary 15.12 reduces the crystalline colengths to defects of the actual Tate-dual lattices. We record the exact arithmetic needed to combine the local estimates. It proves the factor $4f$ and the character normalization appearing in (25.3).

Theorem 15.19 (Summing the local crystalline defects)**.** *Let E be a CM field of degree $2f$, let $r_0 \ge 1$, and let $0 \le q_0 \le t_*$ with $t_* > 0$. At every $v \mid \ell$ let $L_{v,0}$ have rank r_0, and let $L_{v,j}$ have rank 1 for $1 \le j \le q_0$. Let $\mathcal{Q}_v$ be a local lattice class containing the restrictions of the representations used in the relation construction and the indicated summands. Write*

$$d_v = [E_v : \mathbb{Q}_\ell], \qquad \delta_{v,j} = \widehat{\lambda}_{\mathcal{O}}\left(\frac{Z_f(L_{v,j}^\dagger)}{H_{\mathcal{Q}_v}(L_{v,j}^\dagger)}\right).$$

Suppose $D_\ell \ge 0$ and the local estimates are

$$\delta_{v,0} \le d_v r_0 D_\ell, \qquad \delta_{v,j} \le d_v D_\ell / t_* \quad (1 \le j \le q_0). \tag{15.24}$$

Let $L_T, L_1, \dots, L_{q_0}$ be the crystalline colengths of the main and character summands. Under the hypotheses of Corollary 15.12,

$$\frac{L_T + \sum_{j=1}^{q_0} L_j}{v_{\mathcal{O}}(\ell)} \le 2f\left(r_0 + \frac{q_0}{t_*}\right) D_\ell \le 4f r_0 D_\ell. \tag{15.25}$$

The coefficient ring and congruence depth introduce no additional factor.

Proof. The decomposition of the finite $\mathbb{Q}_\ell$-algebra

$$E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \simeq \prod_{v|\ell} E_v$$

gives $\sum_{v|\ell} d_v = [E : \mathbb{Q}] = 2f$ by taking dimensions. The structural comparison for each summand gives

$$\frac{L_T + \sum_j L_j}{v_{\mathcal{O}}(\ell)} \le \sum_{v|\ell} \sum_{j=0}^{q_0} \delta_{v,j}.$$

Apply (15.24) and then sum the local degrees to obtain the first inequality in (15.25). Finally $q_0/t_* \le 1 \le r_0$, so $r_0 + q_0/t_* \le 2r_0$. This proves the second inequality.

Lean, formal context. `CrystallineDefect` (L344-L377), `FitTwistsExist` (L377). □

For the source application, $t_* = D.t$, $q_0 \le D.t$, and $D_\ell = \mathrm{defectBound}(D, \ell)$. Its definition is

$$D_\ell = 10^6 \frac{m}{k}\left(b\, v_\ell(M) + \lambda_\ell\right), \qquad \lambda_\ell = v_\ell\left((2b)! \left(\prod_{\substack{b-30h^3 < n \le b+30h^3 \\ n>0}} n\right)^{h^3}\right), \tag{15.26}$$

where $m = D.m$, $k = D.k$, $b = D.\mathrm{natB}$, $M = D.M$ and $h = D.\mathrm{halfGap}$ are the data parameters. Here the lower bound of the product is interpreted in the integers, with only positive n included; this agrees with the source's truncated natural-number interval.

Lemma 15.20 (The sum of the bounds involving integer parts)**.** *Let m, k, t_*, x be positive integers with $x \le t_*$, let $\rho, \lambda \in \mathbb{N}$, and let $b, v \ge 0$. Put $D = 10^6(m/k)(bv + \lambda)$. Then*

$$10^5\left(\rho m \left\lfloor \frac{2\lambda}{k t_*} \right\rfloor + \lambda \left\lfloor \frac{\rho m}{kx} \right\rfloor\right) \le \frac{\rho D}{x}. \tag{15.27}$$

Proof. Since $x \le t_*$, the two floor terms satisfy

$$\left\lfloor \frac{2\lambda}{k t_*} \right\rfloor \le \frac{2\lambda}{kx}, \qquad \left\lfloor \frac{\rho m}{kx} \right\rfloor \le \frac{\rho m}{kx}.$$

The left side of (15.27) is therefore at most $3 \cdot 10^5 \rho m \lambda/(kx)$. As $bv \ge 0$ and $3 \cdot 10^5 \le 10^6$, this is at most $10^6 \rho m(bv + \lambda)/(kx) = \rho D/x$. This proves the assertion, including $\rho = 0$ or $\lambda = 0$.

Lean, formal context. `CrystallineDefect` (L344-L377), `FitTwistsExist` (L377). □

This is the finite numerical calculation in `crystalline_bound_le_defectBound`. With $\lambda = \lambda_\ell$ and $v = v_\ell(M)$, it converts the source's floor-valued crystalline estimate to $\rho D_\ell/x$. Theorem 16.92 proves the preceding estimate of the actual local defect by those floor terms under its displayed chart, window, coefficient and sparsity hypotheses.

Theorem 15.19 proves the summation and normalization. Corollary 16.93 supplies the crystalline local estimates (15.24) when the specified local classes satisfy its hypotheses; Corollary 16.20 supplies the tame cup exponent under its type and window hypotheses. Theorem 17.40 verifies the numerical, field and window requirements for the constructed data and gives both applications from the stated rational representation and local compatibility inputs. In the source the one-place application is `summand_defect_le_at_place`. Its crystalline branch uses the window and small-jump estimates of `normLength_dualDefect_le_of_crystallineSizes`; its potentially crystalline branch at a level prime uses a tame period-module comparison and a quantitative cup exponent. The old-inertia polynomials away from ℓ do not establish either local assertion at the coefficient prime.

15.4 Scalar periods, coefficient retractions, and integral period matrices

The tame cup-product estimate uses the intersection of two scalar period lattices, the lattice for the Tate twist, and coefficient retractions that preserve the de Rham filtration. We prove these facts here. The matrix identities needed to detect denominators follow from crystalline comparison in Theorem 16.19.

Let $C = \widehat{\overline{\mathbb{Q}}_p}$ and let $C^\flat$ denote its tilt. Let $A = W(\mathcal{O}_{C^\flat})$ be the ring of p-typical Witt vectors, with Frobenius φ and Fontaine's surjection $\theta : A \to \mathcal{O}_C$. Choose a compatible system $\epsilon = (1, \zeta_p, \zeta_{p^2}, \dots)$ of primitive p-power roots of unity, regarded as an element of $\mathcal{O}_{C^\flat}$. Writing $[\,\cdot\,]$ for the Teichmüller lift, put

$$\mu = [\epsilon] - 1, \qquad \tau = [\epsilon^{1/p}] - 1, \qquad \xi = \sum_{j=0}^{p-1} [\epsilon^{1/p}]^j.$$

Thus $\varphi\tau = \mu = \xi\tau$ and $\ker\theta = (\xi)$. In Lean, ϵ and $\epsilon^{1/p}$ are `epsTilt` and `epsRoot`, and τ, μ, ξ are `tau`, `mu`, and `xi` (cyclotomic periods). These basic period-ring constructions and the definition of B_{dR}^+ as the $\ker\theta$-adic completion of $A[1/p]$ are classical; see Bhatt–Morrow–Scholze, §3.3. We use the same conventions as the scalar intersection lemmas.

Lemma 15.21 (The intersection of all Frobenius kernels)**.** *In A one has*

$$\bigcap_{j\ge 0} \ker(\theta\varphi^j) = \mu A, \qquad \bigcap_{j\ge 1} \ker(\theta\varphi^j) = \tau A. \tag{15.28}$$

Proof. Let I denote the first intersection. It contains μ, and it is saturated for multiplication by p, since $\mathcal{O}_C$ is p-torsion-free. Write a bar for reduction modulo p, and let v be the rank-one valuation on $\mathcal{O}_{C^\flat}$. Set $\alpha = v(\bar\xi) > 0$. The identities above give

$$v(\bar\mu) = p\, v(\bar\tau), \qquad \alpha = (p-1)v(\bar\tau), \qquad v(\bar\mu) = \frac{p\alpha}{p-1}.$$

If $x \in I$, then $x = \xi b$. For $j \ge 1$ the cyclotomic sum gives $\theta(\varphi^j\xi) = p$. Consequently $\varphi b \in I$. Suppose $\bar x \neq 0$ and put $s = v(\bar x)$. Repeating $x \mapsto \varphi(x/\xi)$ gives elements of I with nonzero reductions and valuations

$$s_r = p^r\left(s - \frac{p\alpha}{p-1}\right) + \frac{p\alpha}{p-1} \quad (r \ge 0).$$

All these valuations are nonnegative. Since $p > 1$, this forces $s \ge p\alpha/(p-1) = v(\bar\mu)$. The valuation ring property therefore gives $\bar\mu \mid \bar x$. The conclusion also holds when $\bar x = 0$.

Choose $b_0 \in A$ lifting $\bar x/\bar\mu$; if $\bar x = 0$ take $b_0 = 0$. Then $x - \mu b_0 = px_1$ with $x_1 \in I$. Repeat this construction. For every N,

$$x = \mu \sum_{r=0}^{N-1} p^r b_r + p^N x_N, \qquad b_r \in A, \quad x_N \in I.$$

The p-adic completeness and separation of A give $x = \mu \sum_{r\ge 0} p^r b_r$. Hence $I = \mu A$. For the second intersection, apply the first assertion to φx and then use the automorphism φ^{-1} and $\varphi^{-1}\mu = \tau$.
Lean. `Ainf` (L862), `Ainf` (L891). □

For a crystalline $\mathbb{Z}_p$-lattice T, write $N_A(T)$ for the following module in $W(C^\flat) \otimes T$:

$$\left\{ x \in A[1/\tau] \otimes T : \lambda(\varphi^j x) \in B_{\mathrm{dR}}^+ \text{ for every } j \ge 1 \text{ and every invariant de Rham form } \lambda \right\}. \quad (15.29)$$

Here a form is linear in the period coefficient and is equivariant on $1 \otimes T$. Its restriction to the displayed localization is determined by its values on T. This is the actual filtered definition of the source module.

Lemma 15.22 (The exact rank-one Tate lattice)**.** *Let e_c generate $\mathbb{Z}_p(c)$, where $g e_c = \chi(g)^c e_c$, for any integer c. Then*

$$N_A(\mathbb{Z}_p(c)) = \tau^{-c} A e_c. \quad (15.30)$$

In particular $\tau N_A(\mathbb{Z}_p(1)) = A e_1$, and $N_A(\mathbb{Z}_p) = A$.

Proof. The invariant forms are the constant-field multiples of $e_c \mapsto t^c$. For every $j \ge 1$, $\varphi^j \tau$ has de Rham order exactly one: it is $[\epsilon]^{p^{j-1}} - 1$, whose expansion in $t = \log[\epsilon]$ has nonzero linear coefficient p^{j-1}. Thus $\tau^{-c} A e_c$ satisfies (15.29) for every integer c.

Conversely write $x = \tau^{-N} a e_c$, with $a \in A$ and $N \ge 0$. Its membership implies $\varphi^j a \in \mathrm{Fil}^{N-c} B_{\mathrm{dR}}$ for every $j \ge 1$. If $N - c > 0$, then $\theta(\varphi^j a) = 0$ for all such j, so Lemma 15.21 gives $a = \tau a_1$. The same condition for a_1 has exponent $N - c - 1$. After $N - c$ iterations $a = \tau^{N-c} b$ with $b \in A$. If $N - c \le 0$, simply put $b = \tau^{c-N} a$. In both cases $x = \tau^{-c} b e_c$ as required.
Lean, formal context. `Modules` (L232-L246), `FitTwistsExist` (L377). □

Extending a coefficient retraction through de Rham orders

Fix a Kummer chart

$$S = \Lambda[[z]] \subset A, \qquad E = p + z^e V \in S, \quad V \in S^\times, \qquad (E) = (\xi) \text{ in } A,$$

where $\Lambda = W(\mathbb{F}_{p^f})$, $\varphi S \subset S$, and the chart field is $K = \mathrm{Frac}(\Lambda)(\theta z)$. Use the embedded rings

$$A_{\max} = \widehat{A[E/p]}_p, \qquad B^+ = A_{\max}[1/p], \qquad S_B = \widehat{S[E/p]}_p[1/p] \subset B^+.$$

The completion notation for S_B means its image in B^+. At an integer $a \ge 0$ put

$$\iota_a(r) = \iota_0(\varphi^{-a} r), \qquad \mathcal{K}_a = \mathrm{Frac}\big(K[S_B, \iota_a(S)]\big) \subset B_{\mathrm{dR}}.$$

In particular $\mathcal{K}_0 = \mathrm{Frac}(K[S_B])$. These are the value fields in the Kummer chart construction.

Lemma 15.23 (Scalar maps preserve every de Rham order)**.** *Let $f : A \to A$ be S-linear. For $a \ge 0$, $r_1, \dots, r_q \in A$ and $v_1, \dots, v_q \in \mathcal{K}_a$, one has*

$$\sum_i \iota_a(r_i) v_i \in \mathrm{Fil}^n B_{\mathrm{dR}} \quad \Longrightarrow \quad \sum_i \iota_a(f(r_i)) v_i \in \mathrm{Fil}^n B_{\mathrm{dR}} \qquad (n \in \mathbb{Z}). \quad (15.31)$$

The same assertion holds with $\{0\}$ in place of Fil^n. If $f(1) = 1$, it also preserves congruences of these sums to any fixed element of $\mathcal{K}_a$ modulo Fil^n.

Proof. First take $a = 0$. The map f is p-adically bounded, since $f(p^r A) \subset p^r A$. Extend it to $A[1/p]$. Its S-linearity shows that it preserves $A[E/p]$ and on this ring satisfies

$$f\left(\sum_i a_i (E/p)^i\right) = \sum_i f(a_i)(E/p)^i.$$

This is well-defined by the extension to $A[1/p]$, rather than by a choice of polynomial expression. It therefore extends to an S_B-linear map $F : B^+ \to B^+$ carrying $A_{\max}$ into itself.

Here is the necessary completion check. Every element of $A_{\max}$ has a presentation $\sum_{i\geq 0} a_i (E/p)^i$ with $a_i \to 0$ p-adically. Indeed, lift it successively by polynomials modulo p^r and collect their coefficients. The map θ on $A[E/p]$ takes integral values and is p-adically continuous, so it extends to $A_{\max} \to \mathcal{O}_C$. On the displayed series its value is $\theta(a_0)$. If this is zero, write $a_0 = Eb_0$; then division of the series by E gives

$$b_0 + \sum_{i\geq 1} (a_i/p)(E/p)^{i-1} \in B^+.$$

Thus $\ker(\theta : B^+ \to C) = EB^+$. By induction, an element of $A[1/p]$ which lies in $E^r B^+$ already lies in $E^r A[1/p]$: cancel one E using $\ker(\theta : A[1/p] \to C) = EA[1/p]$, and repeat. Truncating the series gives surjectivity, so

$$A[1/p]/E^r A[1/p] \;\simeq\; B^+/E^r B^+ \qquad (r \geq 1). \tag{15.32}$$

The E-adic completion of B^+ is consequently B^+_{dR}. Since F is E-linear, it extends to this completion and, after inverting E, to B_{dR}. This extension preserves $E^n B^+_{\mathrm{dR}} = \mathrm{Fil}^n B_{\mathrm{dR}}$ for every integer n. Only the E-adic topology is needed in this step; no assertion about a bounded inverse for the de Rham period t is being used.

We next verify linearity over K. Let P be the separable minimal polynomial of θz over $K_0 = \mathrm{Frac}(\Lambda)$. Its coefficients belong to $S[1/p]$, $P(z) \in EB^+_{\mathrm{dR}}$ and $P'(z)$ is a unit of B^+_{dR}. Newton iteration starting at z gives elements of $\mathrm{Frac}(S) \cap B^+_{\mathrm{dR}}$ converging E-adically to the embedded constant $\theta z \in K$. The limit is that constant because it is the unique root of P with residue θz. Thus K lies in the E-adic closure of $\mathrm{Frac}(S)$ in B_{dR}. The extension of F commutes with multiplication by $\mathrm{Frac}(S)$: it commutes with nonzero elements of S, and those elements are invertible in B_{dR}. Continuity gives commutation with K. It already commutes with S_B. Commuting with multiplication by a nonzero element also implies commuting with its inverse, so F is $\mathcal{K}_0$-linear. Applying F to the sum proves (15.31) at $a = 0$.

For general a apply this construction to $f_a = \varphi^{-a} f \varphi^a : A \to A$. This map is S-linear because $\varphi^a S \subset S$; it is also linear over $\varphi^{-a} S$. Its extension therefore commutes with K, S_B, and $\iota_a(S)$, and hence is $\mathcal{K}_a$-linear. Since $f_a(\varphi^{-a} r) = \varphi^{-a} f(r)$, it gives the stated formula. A linear map carries zero to zero. If $f(1) = 1$, the extension fixes $\mathcal{K}_a$ pointwise, which proves the congruence assertion.
Lean, formal context. `KummerChartDeRham` (L252-L355), `FitTwistsExist` (L377). □

Corollary 15.24 (Integral intersection with the value field)**.** *Suppose $\lambda_0 : A \to S$ is S-linear and $\lambda_0(1) = 1$. Then, inside B_{dR},*

$$A \cap \mathcal{K}_0 = S. \tag{15.33}$$

For a trivial finite free lattice C, its module defined by (15.29) is $A \otimes C$, and its value-field submodule is $S \otimes C$.

Proof. If $x \in A \cap \mathcal{K}_0$, apply Lemma 15.23 to the zero relation $x \cdot 1 - 1 \cdot x = 0$. It gives $\lambda_0(x) = x$, so $x \in S$. The converse inclusion is part of the definition of $\mathcal{K}_0$. The final assertion follows

coordinatewise from Lemma 15.22 with $c = 0$. The scalar retraction itself is constructed in Lemma 15.30 below.

Lean, formal context. `KummerChartDeRham` (L252-L355), `FitTwistsExist` (L377). □

Lemma 15.25 (Vector retraction by descent of relations)**.** *Let M be an S-module in a period vector space and suppose a family of linear forms separating its A-span takes every element of M into $\mathcal{K}_0$. For an S-linear $\lambda_0 : A \to S$ with $\lambda_0(1) = 1$, the rule*

$$r\left(\sum_i a_i v_i\right) = \sum_i \lambda_0(a_i) v_i, \qquad a_i \in A, v_i \in M, \tag{15.34}$$

defines an S-linear retraction $AM \to M$. It commutes with S-linear maps of such modules induced by maps on their generators, in particular projections onto the parameters and onto the eigenspaces of the coefficient root of unity. In the notation of the eigenspace decomposition, i ranges over the units of the finite field used in the Kummer chart (`Letter`). Its Teichmüller lift $[i]$ is the eigenvalue, and `letterProj` is the projection onto that eigenspace. At every de Rham level a, the retraction preserves any order or fixed-value congruence defined by forms whose values on M lie in $\mathcal{K}_a$. The same statements hold after localizing at finitely many nonzero elements of S.

Proof. If $\sum_i a_i v_i = 0$, apply a separating form. The zero case of Lemma 15.23 gives $\sum_i \lambda_0(a_i)\lambda(v_i) = 0$ for every such form; separation gives $\sum_i \lambda_0(a_i) v_i = 0$. Thus (15.34) is independent of the expression. It is S-linear and fixes M because $\lambda_0(1) = 1$. Naturality follows by applying the formula to the images of the same generators.

At level a, the value of a form on $\sum_i a_i v_i$ is $\sum_i \iota_a(a_i) v_i'$ with $v_i' \in \mathcal{K}_a$. Its value on the retracted vector replaces a_i by $\lambda_0(a_i)$. Lemma 15.23, including its congruence assertion, proves exactly the required preservation. For a localization, clear a common scalar denominator in S. Its image belongs to $\mathcal{K}_a$; the extended scalar map commutes with that image and its inverse. The same calculation therefore applies without changing the asserted de Rham order. No freeness of M or exactness of a Kisin functor is used here.

Lean. `ExponentLattices` (L725), `ExponentLattices` (L792). □

Integral period matrices and divisibility

Matrix identities. Use the period rings $B_{\max}$, B^+ and S_B and the generator E specified in the definition of the Kummer chart below; here $B_{\max} = B^+[1/\mu]$. Let T be a lattice of rank d whose filtration jumps lie in $[-h, h]$. Choose a crystalline period matrix $P \in \mathrm{GL}_d(B_{\max})$ invariant under the chart subgroup, so $\rho(g)g(P) = P$ for every element g of that subgroup. Theorem 16.19 constructs matrices

$$Q, Q' \in \mathrm{Mat}_d(A), \quad Z, Y \in \mathrm{Mat}_d(B^+), \quad Z \in \mathrm{Mat}_d(S_B),$$

and $U = (E\tau)^h P \in \mathrm{Mat}_d(B^+)$, satisfying

$$QQ' = \tau^{2h} I_d, \qquad UZ = E^{2h} Q, \qquad YZ = E^{2h} I_d. \tag{15.35}$$

No Frobenius matrix or quantitative radius estimate is needed as an additional hypothesis on the matrices. The quantitative annihilators in Theorems 16.27 and 16.30 are proved from the actual lifting ideals below; no additional radius estimate is used in those proofs. We do not assume the module spanning, common denominator, or divisibility conclusions below as part of this input. The specific construction is the integral period-matrix frame; the construction of the lattice frame from these matrices. For a G_K-stable lattice in a crystalline representation, the Breuil–Kisin functor

produces a finite free module over $W(k)[[u]]$ and identifies its positive crystalline realization with that of the rational representation [BMS18, Theorem 4.4 and Proposition 4.34]. The bounds used below concern the powers of E in the entries of particular matrices. Theorem 16.19 constructs these matrices and checks their denominators.

Put $n_j = \tau^{-h} Q e_j$ in the period ambient module and $M = A\langle n_1, \dots, n_d\rangle$. Let $N(T)$ denote the submodule of $N_A(T)$ on which every invariant form takes values in $\mathcal{K}_0$.

Lemma 15.26 (Coefficients detected by the frame forms)**.** *Under* (15.35)*, write $x = \sum_j a_j n_j$ with $a_j \in A$. For an invariant form λ, let b_λ be the row of its values on the chosen basis of T, multiplied by P. Then*

$$E^h \lambda(x) = b_\lambda Z a. \tag{15.36}$$

There are crystalline forms $\sigma_1, \dots, \sigma_d$ for which b_{σ_i} are the coordinate rows. Consequently, for any B^+-submodule $W \subset B_{\mathrm{dR}}$,

$$\big[\lambda(x) \in W \text{ for every crystalline } \lambda\big] \quad \Longrightarrow \quad E^{2h} a_j \in W \quad (1 \le j \le d). \tag{15.37}$$

All these statements hold after applying φ^m to the frame, with E replaced by $\varphi^m E$.

Proof. Cancel $E^h \tau^h$ in $(E\tau)^h P Z = E^{2h} Q$ inside B_{dR}. This gives $PZ = E^h \tau^{-h} Q$, hence (15.36). Take the forms σ_i to have rows equal to the rows of P^{-1}. They are equivariant because $\rho(g) g(P) = P$, and their values come from $B_{\max}$, so they are crystalline. The coordinate identities and $YZ = E^{2h} I$ give

$$E^{2h} a = Y \big(E^h \sigma_i(x)\big)_i.$$

Multiplication by E^h and by the entries of Y preserves W. This proves (15.37). Applying φ^m gives the last assertion; the rows of $(\varphi^m P)^{-1}$ are again equivariant crystalline forms.

Lean. `FrameTheorem` (L295), `FrameTheorem` (L371). □

Theorem 15.27 (Identification of the filtered modules)**.** *Assume* (15.35) *and a scalar retraction $\lambda_0 : A \to S$ as above. Then*

$$N_A(T) = \bigoplus_{j=1}^{d} A n_j, \qquad N(T) = \bigoplus_{j=1}^{d} S n_j. \tag{15.38}$$

The value-field part of $\varphi^l N_A(T)$ is $\bigoplus_j S \varphi^l n_j$ for every $l \ge 0$.

Proof. For an invariant form λ, the entries of $b_{\lambda,m} = \lambda(e) \varphi^m P$ are fixed in B_{dR}, hence belong to the finite chart field K. For $m \ge 1$, $\varphi^m E$ is a unit of B_{dR}^+: E is associated to ξ, while $\theta(\varphi^m \xi) = p \ne 0$. The Frobenius version of (15.36) therefore shows that every n_j satisfies (15.29). Here φ preserves B^+ and S_B. Indeed $E = p + z^e V$ gives

$$\varphi(E)/p = 1 + z^{e(p-1)} V^{-1} \varphi(V) (E/p - 1),$$

so φ preserves $S[E/p]$ and its p-adic completion; the same argument applies to $A[E/p]$. At $m = 0$ and at every nonnegative Frobenius level the same identity shows that the values of all invariant forms on $\varphi^m n_j$ belong to $\mathcal{K}_0$.

The matrix Q is invertible after inverting τ by $QQ' = \tau^{2h} I$. Thus the n_j are linearly independent. They span a module containing $\tau^h (A \otimes T)$. For any $x \in N_A(T)$, some $\tau^N x$ is consequently in M. It remains to show that one can remove a factor τ. Suppose $\tau x = \sum_j a_j n_j$ with $x \in N_A(T)$. For

every $m \geq 1$ and every invariant form, the value on $\varphi^m(\tau x)$ has de Rham order at least one. Apply Lemma 15.26 to the mth Frobenius frame with $W = \mathrm{Fil}^1$. It gives

$$(\varphi^m E)^{2h}\varphi^m(a_j) \in \mathrm{Fil}^1 .$$

The first factor is a de Rham unit, so $\theta(\varphi^m a_j) = 0$ for all $m \geq 1$. Lemma 15.21 gives $a_j = \tau a_j'$. Cancel τ in the torsion-free ambient module to obtain $x = \sum_j a_j' n_j$. Repeating removes τ^N and proves the first equality in (15.38).

Let now $x \in N(T)$, and write $x = \sum_j a_j n_j$. The special forms σ_i show $Za = (E^h \sigma_i(x))_i \in \mathcal{K}_0^d$. Since Z has entries in S_B and nonzero determinant, its inverse is over $\mathcal{K}_0$. Thus every $a_j \in \mathcal{K}_0$. It already lies in A, so Corollary 15.24 gives $a_j \in S$. The reverse inclusion was proved above. The identical argument with $\varphi^l P$, $\varphi^l Z$ and $\varphi^l n_j$ proves the assertion at level l.

Lean. `FrameTheorem` (L505), `FrameTheorem` (L517), `FrameTheorem` (L554). □

Lemma 15.28 (A common denominator for crystalline forms)**.** *Under the matrix-frame identities, for every crystalline form λ one has*

$$E^h \lambda(N(T)) \subset S_B. \tag{15.39}$$

The exponent h is independent of both the vector and the form.

Proof. For a crystalline form the row b_λ in (15.36) comes from $B_{\max}$ and is fixed by the chart subgroup. The classical equality $B_{\max}^{G_K} = K_0$ puts its entries in $K_0 \subset S_B$. The chart condition gives the same inclusion if the subgroup under consideration contains G_K. For $x \in N(T)$ its coefficient vector belongs to S^d by Theorem 15.27. Since Z has entries in S_B, the identity $E^h\lambda(x) = b_\lambda Za$ proves the claim. No denominator depending on λ is chosen.

Lean. `FrameTheorem` (L603). □

Lemma 15.29 (Divisibility detected by all crystalline forms)**.** *Use the matrix-frame identities. Let $d_0 \in A$ satisfy the scalar cancellation property*

$$d_0 \mid E^b r \text{ in } B^+, \quad r \in A, b \geq 0 \quad \Longrightarrow \quad d_0 \mid r \text{ in } A. \tag{15.40}$$

If $x \in N_A(T)$ and, for one $n \geq 0$, $E^n\lambda(x) \in d_0 B^+$ for every crystalline form λ, then $x \in d_0 N_A(T)$.

Proof. Write $x = \sum_j a_j n_j$ using Theorem 15.27. Apply Lemma 15.26 to $E^n x$ and $W = d_0 B^+$. This gives $E^{2h+n} a_j \in d_0 B^+$. Property (15.40) implies $a_j = d_0 b_j$ with $b_j \in A$. Hence $x = d_0 \sum_j b_j n_j \in d_0 N_A(T)$. Lemma 15.31 below proves the scalar property for the actual elements z, τ, and $z\tau$.

Lean. `FrameTheorem` (L627). □

The consequences just proved use only the displayed matrix identities, basic period-ring comparisons, and the constructed scalar retraction. Theorem 16.19 constructs those matrices for every crystalline lattice under the stated classical period background. The scalar intersection, Tate integrality, retraction, spanning deduction and form-divisibility arguments above are therefore available for the lattices. The additional quantitative approximation is proved in Theorem 16.63.

15.5 The integral tame polynomial relation and the common correction

This subsection proves the polynomial-divisibility and common-lift steps used in the second period input. The source is the crystalline-form argument and its application to the lifting ideals. We use the period frame described above, whose existence is proved in Theorem 16.19. Its consequences for identification with the form-defined modules, common denominators, and detection of divisibility are proved in Theorem 15.27 and Lemmas 15.28 and 15.29. The argument below does not assume the tame polynomial relation as a hypothesis on the frame.

The period rings and the integral matrices. Write $A = A_{\mathrm{inf}} = W(\mathcal{O}^{\flat}_{\mathbf{C}_p})$, $\tau = [\varepsilon^{1/p}] - 1$ and $\xi = ([\varepsilon] - 1)/\tau$. The Kummer chart has

$$\Lambda = W(\mathbf{F}_{p^f}), \quad S = \Lambda[[z]] \subset A, \quad E = z^e V + p \quad (V \in S^{\times}), \quad E = u\xi \quad (u \in A^{\times}).$$

The element z is a nonzero nonunit Teichmüller lift. Let $A_{\max}$ be the p-adic completion of $A[E/p] \subset A[1/p]$, and put $B^+ = A_{\max}[1/p] = B^+_{\max}$. This is the source's positive period ring, which is distinct from B^+_{cris}; replacing E by its associated generator ξ gives the same ring. Let $S_B \subset B^+$ be the p-adic closure of $S[E/p]$ with p inverted, the source's *bicommutant*.

For a lattice T crystalline over the chart subgroup H, write $N(T)$ and $N_A(T)$ for its form-defined Kisin module and its A-module, respectively. Theorem 15.27 gives a finite S-basis $n_1, \dots, n_d$ of $N(T)$ which is an A-basis of $N_A(T)$. For a crystalline form μ, one common nonnegative integer h has

$$E^h \mu(N(T)) \subset S_B \subset B_{\mathrm{dR}}.$$

It also gives the divisibility test of Lemma 15.29. For $l \geq 0$ put

$$N_l(T) = \sum_j S\,\varphi^l n_j, \qquad M_l(T) = \sum_j A\,\varphi^l n_j = \varphi^l N_A(T).$$

By Lemma 15.40, the actual standard-order module of level l is $M_l(T)$; its letter-i part is $M_{l,i}(T)$. The corresponding integral module $N_{l,i}(T)$ is obtained by the same letter projector. Only these standard levels will be used for the common correction; no assertion about retraction of an arbitrary Hodge-order module is needed.

Scalar retractions and cancellation of period denominators

Lemma 15.30 (A scalar coordinate which fixes the chart ring)**.** *There is an S-linear map*

$$\lambda_0 : A \longrightarrow S, \qquad \lambda_0|_S = \mathrm{id}_S\,.$$

Proof. Put $I = (p, z)A$ and $\mathfrak{m} = (p, z)S$. The sequence (p, z) is regular on A: multiplication by p is injective on Witt vectors, and $A/pA = \mathcal{O}^{\flat}_{\mathbf{C}_p}$ is a domain in which the residue of z is nonzero. The ring A is complete and separated for I. Indeed, on every finite Witt length, multiplication by the Teichmüller lift z^m multiplies Witt coordinate number r by $\bar{z}^{mp^r}$. The z-adic topology there is therefore the componentwise valuation topology, which is complete. The ideals (p^m, z^m) are cofinal with the powers of I. Taking the inverse limit over the Witt length consequently gives $A = \varprojlim_m A/I^m$.

Choose a basis $(\bar{b}_j)_{j \in J}$ of the $\mathbf{F}_{p^f}$-vector space A/I, including $\bar{b}_0 = 1$, and choose lifts $b_j \in A$, with $b_0 = 1$. The associated-graded formula in this two-generator case gives

$$\mathrm{gr}_I A \simeq (A/I)[P, Z], \qquad \mathrm{gr}_{\mathfrak{m}} S \simeq \mathbf{F}_{p^f}[P, Z].$$

For completeness, this formula needs no Noetherian hypothesis. Reduction modulo p proves $(I^{n+1} : p) = I^n$: in an expression for an element of I^{n+1}, the coefficient of the sole term z^{n+1} must be divisible by p, and one then cancels p. Reducing a homogeneous relation first modulo p and then using this colon identity successively shows that all its coefficients lie in I, which proves the displayed formula. Consequently the map sending the jth basis vector to b_j induces an isomorphism at every finite quotient by a power of (p, z). One proves this successively in the degree using the displayed graded isomorphism. Completeness then identifies A with the $\mathfrak{m}$-adic completion of $\bigoplus_{j \in J} S$. Its elements are coefficient families with finite support modulo every $\mathfrak{m}^n$. The coordinate at $j = 0$ is an S-linear map to S and fixes S, because $b_0 = 1$. This is λ_0. This is the scalar coordinate denoted by `coord`$_0$ in the source.

Lean. `Charts` (L430), `Charts` (L470). □

Lemma 15.31 (Cancellation of powers of E from scalar divisibility)**.** *Let $d \in A$ have nonzero image in A/pA, and suppose*

$$E = pc + dh, \qquad c \in A^\times,\ h \in A. \tag{15.41}$$

Then, for $a \in A$ and $m \geq 0$,

$$E^m a \in dB^+ \quad \Longrightarrow \quad a \in dA. \tag{15.42}$$

Moreover dB^+ is closed in B^+, and division by d on this image is continuous. These conclusions apply to $d = z$ and $d = \tau$. The implication (15.42) and the closed-image assertion also apply to $d = z\tau$.

Proof. First A/dA has no p-torsion. If $pa = db$, reduction modulo p forces $b = pb_1$, since $\overline{d} \neq 0$ in the domain A/pA. Cancelling p gives $a = db_1$. The same argument by induction shows that d is a nonzero divisor modulo every p^n. It also shows that dA is p-adically closed: if $a = dc_n$ modulo p^n, the c_n are compatible modulo p^n, hence converge to c with $a = dc$. Adjusting representatives of a compatible sequence in $A/(d, p^n)$ now proves that A/dA is p-adically complete.

The map $A[1/p] \to (A/dA)[1/p]$ sends E/p to $\overline{c}$. Its restriction to $A[E/p]$ therefore takes values in A/dA and extends by p-adic continuity. After inverting p it gives

$$q_d : B^+ \longrightarrow (A/dA)[1/p].$$

Its kernel is exactly dB^+. To check the nontrivial inclusion, write an element of $A_{\max}$ as $x = \sum_{j\geq 0} a_j (E/p)^j$, with $a_j \in A$ tending p-adically to zero. Put $a = \sum_{j\geq 0} a_j c^j \in A$. The identity for a difference of powers gives

$$x - a = \frac{dh}{p} \sum_{j\geq 1} a_j \sum_{v=0}^{j-1} (E/p)^{j-1-v} c^v. \tag{15.43}$$

The rightmost series converges in $A_{\max}$, since each inner polynomial is integral and $a_j \to 0$. If $q_d(x) = 0$, then $a \in dA$, because A/dA has no p-torsion. Equation (15.43) gives $x \in dB^+$. Multiplying a general element of B^+ by a power of p reduces to this case.

Now $q_d(E) = p\overline{c}$ is a unit. If $E^m a \in dB^+$, applying q_d gives $q_d(a) = 0$, and p-torsion-freeness of A/dA gives $a \in dA$. The same series computation gives the useful boundedness statement

$$x \in dB^+ \cap p^n A_{\max} \quad \Longrightarrow \quad x/d \in p^{n-1} A_{\max}.$$

Thus division is continuous, and a convergent sequence of multiples of d has a convergent sequence of quotients. Completeness of B^+ shows that its limit is again a multiple of d.

For $d = z$, use $E = p + z(z^{e-1}V)$. For $d = \tau$, write $t = [\varepsilon^{1/p}] = 1 + \tau$. Since $\xi = 1 + t + \cdots + t^{p-1}$, one has $\xi = p + \tau h_0$, and $E = u\xi$ gives (15.41) with $c = u$. Both z and τ have nonzero reduction modulo p. For $d = z\tau$, first cancel z to write $a = za_1$; then cancel z in B^+ and apply the τ case to a_1. Continuous division by $z\tau$ is the composite of the two continuous divisions, and its image is closed by the same Cauchy-sequence argument.

Lean, comparison. Lean proves the intersection with A_{inf} and cancels E for the chart divisors. The quotient q_d above treats these divisors together and also gives continuity of division. See `CrystallineRings` (L588-L636), `RadiusLemma` (L60-L104). □

Lemma 15.32 (The scalar Galois differences used by the chart)**.** *Put $\delta = z\tau$. For $g \in H$, the chart satisfies*

$$(g-1)S \subseteq \delta EA, \qquad (g-1)S_B \subseteq \delta B^+. \tag{15.44}$$

If s is the tame generator, fixes Λ, and $s(z) = \zeta z$ with $\zeta \in \Lambda^\times$, then every power s^k satisfies

$$(s^k - 1)S_B \subseteq zB^+. \tag{15.45}$$

Proof. An element of H fixes the chosen root $\theta(z)$ and the unramified constants. On its compatible system of p-power roots it has

$$g(z) = z[\varepsilon]^{a_g} \quad (a_g \in \mathbf{Z}_p).$$

The binomial series for $(1+X)^{a_g} - 1$ has integral coefficients and is divisible by X. As $[\varepsilon] - 1 = \xi\tau = E\tau u^{-1}$, this gives $g(z) - z \in \delta EA$. For a series $F \in \Lambda[[X]]$, the formal divided difference

$$\frac{F(gz) - F(z)}{gz - z} = \sum_{j \geq 1} F_j \sum_{v=0}^{j-1} (gz)^v z^{j-1-v}$$

converges in the weak, equivalently (p,z)-adic, period topology. Thus the first inclusion in (15.44) follows. It also gives $g(E/p) - E/p \in \delta A_{\max}$, since $E \in S$ and $(g(E) - E)/p$ is a multiple of $\delta E/p$. Therefore $g - 1$ sends $S[E/p]$ into δB^+. The action is continuous on B^+, and Lemma 15.31 makes δB^+ closed. Taking the closure and then inverting p proves the second inclusion.

For s^k, write $F(z) = F(0) + zG(z)$. It fixes $F(0)$ and multiplies z by a unit, so $(s^k - 1)S \subset zA$. It follows that $(s^k - 1)(E/p) \in zB^+$. Every Galois element takes E to an associated element, since E generates $\ker\theta$; hence it acts continuously on B^+. The same polynomial, closure and inversion argument, using closedness of zB^+, proves (15.45).

Lean, comparison. The source proves the two divisibilities on the chart and its bicommutant. The power-series argument above passes to S_B by closure. See `zTau_mul_E_dvd_smul_sub`, `tau_mul_z_dvd_smul_sub`, `z_dvd_sub`. □

Lemma 15.33 (Clearing a denominator in a Galois difference)**.** *Let g act on the period rings, let $d \in A$, and assume $(g-1)S_B \subset dB^+$ and $g(E)$ is associated to E. If $E^h b \in S_B$ for $b \in B_{\mathrm{dR}}$, then*

$$E^{2h}(g(b) - b) \in dB^+. \tag{15.46}$$

Proof. Put $F = E^h$, $a = Fb \in S_B$, and choose $v \in A^\times$ with $F = v\,g(F)$. Write $g(a) - a = dt$ and $g(F) - F = dr$ with $r, t \in B^+$; the latter is allowed since $F \in S_B$. In B_{dR} one has

$$F^2(g(b) - b) = v\{F(g(a) - a) - (g(F) - F)a\} = dv(Ft - ra).$$

This proves the assertion. The calculation accounts for both powers of the original denominator.

Lean. `pow_smul_gal_sub_mem_span_of_dvd`. □

The inertial polynomial on actual integral periods

Lemma 15.34 (The polynomial relation among conjugate crystalline forms)**.** *Suppose H is normal, T is crystalline over H, and s lies in inertia and normalizes H. Let β be the linear inertial operator on $B_{\max} \otimes_{\mathbf{Z}_p} T$ and let $q = \sum_k q_k X^k \in \mathbf{Z}_p[X]$ satisfy $q(\beta) = 0$. For a crystalline form μ put*

$$\mu_k(x) = s^{-k}\mu(s^k x).$$

Then μ_k is crystalline and, for $x \in N_A(T)$,

$$\sum_k q_k \mu_k(x) = 0. \tag{15.47}$$

Proof. Normality of H gives the equivariance of the conjugate form on T; its values still come from $B_{\max}$. Let λ_k be the $B_{\max}$-linear form on $B_{\max} \otimes T$ extending $\mu_k|_T$. The H-fixed crystalline periods span this $B_{\max}$-module by crystalline comparison. On such a period y, the vector $s^k y$ is again fixed by H, and $\lambda_0(s^k y)$ lies in the fixed subfield of $B_{\max}$, the maximal unramified subfield of

the field fixed by H. An inertia element fixes that subfield. By the definition of the linear inertial operator, $\beta^k y = s^k y$ on these periods, and consequently

$$\lambda_k(y) = \lambda_0(\beta^k y).$$

It follows that $\sum_k q_k \lambda_k = \lambda_0 q(\beta) = 0$, first on the spanning periods and then everywhere. Thus the corresponding form relation holds on $1 \otimes T$. Every element of $N_A(T)$ is in $A[1/\tau] \otimes T$; multiply by a power of τ and use A-linearity to extend the relation to $N_A(T)$. The target B_{dR} has no τ-torsion, so cancellation proves (15.47).

Lean. `TameCharts` (L91), `TameCharts` (L117). □

Theorem 15.35 (The tame polynomial maps the Kisin module into zN_A)**.** *In the tame chart, let $q \in \mathbf{Z}_p[X]$ annihilate the linear inertial operator β of s on the potentially crystalline representation T. Under the explicit period-frame hypothesis,*

$$q(s)N(T) \subseteq zN_A(T). \tag{15.48}$$

Here $q(s)$ means $\sum_k q_k s^k$ for the semilinear Galois action on the ambient module.

Proof. The module $N_A(T)$ is stable under an element normalizing H. Indeed, conjugation preserves the equivariant forms; Galois commutes with Frobenius and preserves B_{dR}^+, so all positive-Frobenius tests remain true. The condition of lying in $A[1/\tau] \otimes T$ is preserved because a Galois conjugate of τ is associated to τ.

Fix $x \in N(T)$ and put $w = \sum_k q_k s^k x \in N_A(T)$. For every crystalline form μ, the preceding lemma gives

$$\mu(w) = \sum_k q_k (s^k - 1)\mu_k(x).$$

Lemma 15.28 supplies the same h for all the crystalline forms μ_k: $E^h \mu_k(x) \in S_B$. Lemmas 15.32 and 15.33 therefore give

$$E^{2h}\mu(w) \in zB^+.$$

The scalar z satisfies the cancellation condition of Lemma 15.29, by Lemma 15.31. Applying the frame divisibility test to w proves $w \in zN_A(T)$, as required.

Lean. `TameCharts` (L187). □

Corollary 15.36 (The ordinary period difference is divisible by $z\tau$)**.** *Under the same period-frame hypothesis, for $g \in H$,*

$$(g-1)N(T) \subseteq z\tau N_A(T). \tag{15.49}$$

Proof. For $x \in N(T)$ and a crystalline form μ, equivariance on T extends to its fractional A-span and gives $\mu(gx - x) = g\mu(x) - \mu(x)$. Apply Lemma 15.28, the ordinary case of Lemma 15.32, and Lemma 15.33. The result is $E^{2h}\mu(gx - x) \in z\tau B^+$. Stability gives $gx - x \in N_A(T)$, and Lemmas 15.31 and 15.29 conclude the proof.

Lean. `ExponentLattices` (L627). □

One correction in S for all parameter constants

Let $0 \to T \to \widetilde{T} \xrightarrow{\pi} C \to 0$ be the extension belonging to a crystalline cocycle family, with C carrying trivial action. Denote by $C_{A,i}$ the letter-i part of $A \otimes_{\mathbf{Z}_p} C$ and put

$$J_{r,l,i} = (r) + \{a \in A : aC_{A,i} \subseteq \pi(M_{l,i}(\widetilde{T}))\}.$$

This is the exact source lifting ideal, after the standard-order module has been identified with M_l in Lemma 15.40. The elements $c_i = e_i(1 \otimes c)$ generate $C_{A,i}$ over A; this is why lifting all such constants is sufficient for its colon ideal, without choosing a basis of the letter part. The map π sends $N(T_1)$ into $N(T_2)$ for any equivariant lattice map $T_1 \to T_2$: precomposition preserves the defining equivariant forms and all their Frobenius and value-field conditions. It therefore sends $N_l(T_1)$ into $N_l(T_2)$ as well.

Theorem 15.37 (A common power-series correction for all constants)**.** *Let $r \in \mathbf{Z}_p$, $l \geq 0$, $f \in S$, and $f \in J_{r,l,i}$. There is a single $u \in S$ such that every parameter constant c_i, the letter-i projection of $1 \otimes c$, has a lift $x_c \in N_{l,i}(\widetilde{T})$ satisfying*

$$\pi(x_c) = (f - ru)c_i. \tag{15.50}$$

The modulus is unchanged and the correction is independent of c.

The restriction $f \in S$ is essential. For the identity map on the trivial rank-one lattice one has $J_{r,l} = A$ and $N_l = S$. If $f \in A \setminus S$, then no $u \in S$ can make $f - ru$ an element of S, so the unrestricted assertion for every element of the lifting ideal would be false.

Proof. The basis $\varphi^l n_j$ identifies $M_l(T)$ with $A \otimes_S N_l(T)$. The scalar retraction in Lemma 15.30 therefore induces

$$R_l : M_l(T) \longrightarrow N_l(T), \qquad R_l\left(\sum_j a_j \varphi^l n_j\right) = \sum_j \lambda_0(a_j)\varphi^l n_j.$$

It fixes $N_l(T)$ and satisfies $R_l(ay) = \lambda_0(a)y$ for $y \in N_l(T)$. This description is independent of the chosen S-basis, because λ_0 is S-linear. For an equivariant lattice map, its matrix on the bases of N_l has entries in S, so R_l commutes with the map. The same observation applies to the letter projectors: they are polynomials in the coefficient action, with coefficients in Λ and denominator $p^f - 1$, a unit. Thus R_l preserves each letter part.

Write $f = rt + a$ with $t \in A$ and $aC_{A,i} \subseteq \pi(M_{l,i}(\widetilde{T}))$. For each c, choose $y_c \in M_{l,i}(\widetilde{T})$ with $\pi(y_c) = ac_i$. By Corollary 15.24, $N(C) = S \otimes C$. The constant c_i therefore lies in $N_{l,i}(C)$: Frobenius permutes the letters, fixes $1 \otimes c$, and the S-span of $\varphi^l(S \otimes C)$ is $S \otimes C$. Consequently

$$\pi(R_l y_c) = R_l(\pi y_c) = \lambda_0(a)c_i = (f - r\lambda_0(t))c_i.$$

Take $u = \lambda_0(t)$ and $x_c = R_l y_c$. The same t, and hence the same u, works for every constant. This proves (15.50).

Lean, comparison. Lean takes a natural-number modulus. For a nonzero $r \in \mathbf{Z}_p$, replacing r by its associated power of p only rescales u by a unit. See `exists_lift_of_mem_liftIdeal`. □

Corollary 15.38 (Lowering the degree in the lifting ideals)**.** *Let $Q \in \mathbf{Z}_p[X]$, $Q(1) = 0$, annihilate the linear inertial operator on $\widetilde{T}$. For $j \geq 1$ and $t \geq 0$, under the period-frame hypothesis,*

$$p^t z^j \in J_{r,0,i} \quad \Longrightarrow \quad Q(\zeta^j)p^t z^{j-1} \in J_{r,0,i}, \tag{15.51}$$

where $s(z) = \zeta z$.

Proof. Theorem 15.37 gives the common correction $u \in S$ and lifts in $N(\widetilde{T})$. Theorem 15.35 gives $Q(s)x_c \in zN_A(\widetilde{T})$. Write $u = u_0 + zv$ with $u_0 \in \Lambda$ and $v \in S$. The element s fixes c_i, since it fixes Λ and acts trivially on C. As $Q(1) = 0$, applying $Q(s)$ to $(p^t z^j - ru)c_i$ gives

$$z\left(Q(\zeta^j)p^t z^{j-1} - r\sum_k Q_k \zeta^k s^k(v)\right)c_i.$$

The same correction in parentheses works for every c. Project the divided lift to the letter i and cancel z in the ambient module. This is legitimate because z is a unit in $W(\mathbf{C}_p^\flat)$. The resulting lifts show membership in $J_{r,0,i}$. This is the proof of Lemma 15.15 applied to the generating constants c_i; it does not require the chosen constants to form a free basis of the letter part.

Lean. `aeval_tameMultiplier_pow_mul_mem_liftIdeal`. □

Theorems 15.27 and 16.19 identify the integral modules of the chosen forms and construct their period matrices. The tame relation on these modules uses the same S-valued correction for every parameter and preserves the power of p.

15.6 Standard period lattices and the Frobenius height recurrence

We now reconstruct the first input in 15.2 for the actual period modules. The seed and the identification of standard exponents below use the usual de Rham comparison theorem, but do not use the existence of an integral Kisin frame. The coordinate-lowering estimates additionally use the explicit frame and chart conclusions stated in 15.4, 15.28, 15.29, 15.36, and 15.37. These dependencies will remain visible. The corresponding source is the period-module definitions, standard exponents, and the lifting ideals.

The period lattice and the sign of a jump

Put $A = A_{\mathrm{inf}} = W(\mathcal{O}_C^\flat)$, and write φ for its Frobenius automorphism. Fix a compatible system of p-power roots of unity ϵ, and set

$$q = [\epsilon], \qquad \tau = [\epsilon^{1/p}] - 1, \qquad \mu_0 = q - 1 = \varphi(\tau).$$

Let E be the chart generator of $\ker\theta$, put $E_a = \varphi^a(E)$, and write B_{dR}^+ for the usual complete discrete valuation ring with filtration $\mathrm{Fil}^m B_{\mathrm{dR}} = t^m B_{\mathrm{dR}}^+$, $m \in \mathbb{Z}$. There is a unit $v \in A^\times$ such that

$$\varphi(\tau) = vE\tau, \qquad \varphi^\ell(\tau) \sim \tau \prod_{0 \le a < \ell} E_a. \tag{15.52}$$

Here $\sim$ means equality up to an A-unit. For $a, b \ge 0$, the image of $\varphi^{-a}(E_b)$ in B_{dR}^+ has order 1 if $a = b$ and order 0 otherwise; the image of $\varphi^{-a}(\tau)$ has order 0. In particular, for $r \in A$ and $m \ge 0$,

$$\varphi^{-a}(r) \in t^m B_{\mathrm{dR}}^+ \quad \Longleftrightarrow \quad E_a^m \mid r. \tag{15.53}$$

Indeed the case $m = 1$ is $\ker(\theta\varphi^{-a}) = E_a A$, and induction proves the general case. The assertions about the orders follow from $E \sim ([\epsilon] - 1)/([\epsilon^{1/p}] - 1)$ by evaluation under θ. Thus the E_a are pairwise nonassociated prime elements. This is also the scalar calculation in the distinct Frobenius primes.

Let L be a finite free $\mathbb{Z}_p$-lattice whose representation $V = L[1/p]$ is crystalline over the finite chart field F. An invariant form is an element of

$$D^*(V) = \mathrm{Hom}_{\mathbb{Q}_p[G_F]}(V, B_{\mathrm{dR}}) = D_{\mathrm{dR},F}(V^\vee).$$

Extend such forms to the common period ambient space. All instances of φ below act on the scalar factor and fix L. Define

$$\begin{aligned} N_A(L) &= \left\{ x \in A[1/\tau] \otimes_{\mathbb{Z}_p} L : \lambda(\varphi^m x) \in B_{\mathrm{dR}}^+ \text{ for all } m \ge 1,\ \lambda \in D^*(V) \right\}, \\ N_{\mathrm{loc},A}(L) &= N_A(L)[E_0^{-1}, E_1^{-1}, \ldots]. \end{aligned} \tag{15.54}$$

These are the modules defined by `kisinAinf` and `kisinLocAinf`, rather than abstract replacements for them.

We use the source convention for Hodge–Tate jumps: the representation $\mathbb{Q}_p(c)$ has jump $-c$. Consequently an invariant form of type u on a letter sends that letter of $1 \otimes L$ into $t^{-u}B_{\mathrm{dR}}^+$. Letter projectors e_i are the usual finite idempotents obtained from the coefficient root of unity. They are defined over the unramified chart constants, commute with the Galois action, and satisfy $\varphi e_i = e_{\sigma i}\varphi$ for the permutation σ of letters. The argument can be read without coefficients by using a single letter.

Lemma 15.39 (Filtered forms detect the integral de Rham lattice)**.** *On each letter there is an F-basis λ_j of its invariant forms, with actual jumps u_j, such that*

$$y \in e_i(B_{\mathrm{dR}}^+ \otimes L) \quad \Longleftrightarrow \quad \lambda_j(y) \in t^{-u_j}B_{\mathrm{dR}}^+ \quad \text{for every } j \tag{15.55}$$

for y in that letter of $B_{\mathrm{dR}} \otimes V$. In addition, if $u_j \le n$ for every jump and $n \ge 0$, then every invariant form sends $B_{\mathrm{dR}}^+ \otimes L$ into $t^{-n}B_{\mathrm{dR}}^+$.

Proof. Choose a filtration-adapted basis of $D^*(V)$, with λ_j of filtration degree $-u_j$, and respect the letter idempotents. The classical filtered de Rham comparison theorem says that $t^{u_j}\lambda_j$ are a B_{dR}^+-basis of the dual of $B_{\mathrm{dR}}^+ \otimes L$. One can see the integrality and the basis assertion separately: integrality follows from the filtration degree; reduction modulo t is a basis by the Hodge–Tate comparison and $\operatorname{gr} D_{\mathrm{dR}}(V^\vee) = D_{\mathrm{HT}}(V^\vee)$, so the determinant is a unit of B_{dR}^+. Evaluation against this dual basis proves (15.55). This use of comparison is classical foundational input; see Brinon–Conrad, *CMI Summer School Notes on p-adic Hodge Theory*, §6.1 and Proposition 6.3.2.

Finally every invariant form is an F-linear combination of the λ_j. Nonzero elements of F have t-order zero, and $-u_j \ge -n$, proving the last assertion. No nonnegativity of an individual u_j was used.

Lean, formal context. `KummerChartDeRham` (L50-L170), `FitTwistsExist` (L377). □

Let U_i be any finite jump set containing the actual jumps on letter i. The standard orders at level ℓ are

$$c_\ell(a,u) = \begin{cases} 0, & a < \ell, \\ -u, & a \ge \ell. \end{cases}$$

Write $X_{\ell,i}(L)$ for the elements $x \in e_i N_{\mathrm{loc},A}(L)$ such that, for every $a \ge 0$, every $u \in U_{\sigma^{-a}i}$, and every invariant form of type u at $\sigma^{-a}i$,

$$\lambda(\varphi^{-a}x) \in t^{c_\ell(a,u)}B_{\mathrm{dR}}^+.$$

Thus $X_{\ell,i}$ is exactly the source module `withOrders i (stdOrders l)`.

Lemma 15.40 (Modules of standard exponents)**.** *For every $\ell \ge 0$ and every letter i,*

$$X_{\ell,i}(L) = e_i\varphi^\ell N_A(L). \tag{15.56}$$

This equality holds for arbitrary integral jumps, including negative jumps.

Proof. First note two direct consequences of (15.54). If $x \in N_{\mathrm{loc},A}(L)$, then

$$\lambda(\varphi^m x) \in B_{\mathrm{dR}}^+ \quad (m \ge 1,\ \lambda \in D^*(V)). \tag{15.57}$$

Indeed, clear a finite product of E_b to put x in $N_A(L)$. After applying φ^m, all the resulting denominators are E_{b+m}, which are units in B_{dR}^+. Also $\varphi^\ell N_A(L) \subseteq N_{\mathrm{loc},A}(L)$. To check this,

write $y \in N_A(L)$ with denominator τ^h. By (15.52), multiplying $\varphi^\ell y$ by $(\prod_{a<\ell} E_a)^h$ clears all its denominators except τ^h. Its positive Frobenius values are integral by the defining condition on y, so that multiple belongs to $N_A(L)$.

Suppose now $x = \varphi^\ell y$ with $y \in N_A(L)$ and x in letter i. For $a < \ell$, the expression $\lambda(\varphi^{-a}x) = \lambda(\varphi^{\ell-a}y)$ is integral by definition. For $a \geq \ell$, put $b = a - \ell$. The vector $\varphi^{-b}y$ belongs to $A[1/\varphi^{-b}(\tau)] \otimes L$, whose coefficients map into B^+_{dR} because $\varphi^{-b}(\tau)$ is a unit there. Every form of type u therefore takes its value in $t^{-u}B^+_{\mathrm{dR}}$. This proves one inclusion.

Conversely let $x \in X_{\ell,i}(L)$. Choose one expression for its coordinates in a $\mathbb{Z}_p$-basis of L with common denominator

$$\tau^h \prod_{a \in S} E_a^{m_a},$$

where S is finite and the numerators belong to A. For any $a \geq \ell$, the standard bounds and Lemma 15.39 imply $\varphi^{-a}x \in B^+_{\mathrm{dR}} \otimes L$. All denominator factors except $E_a^{m_a}$ are units at this point. Equation (15.53) therefore shows that $E_a^{m_a}$ divides every numerator. All such factors with $a \geq \ell$ can be cancelled together: distinct E_a are prime and do not divide one another, so cancellation of one does not destroy divisibility by another. It follows that

$$x \in A[1/\tau, 1/E_0, \ldots, 1/E_{\ell-1}] \otimes L = A[1/\varphi^\ell(\tau)] \otimes L.$$

Thus $y = \varphi^{-\ell}x \in A[1/\tau] \otimes L$.

It remains to check its positive Frobenius values. If $1 \leq m \leq \ell$, set $a = \ell - m < \ell$. The standard order at a is zero. Apply this condition to the adapted invariant basis on letter $\sigma^{-a}i$; every invariant form is an F-linear combination of that basis, so $\lambda(\varphi^m y) = \lambda(\varphi^{-a}x)$ is integral for all λ. If $m > \ell$, use (15.57) on $\varphi^m y = \varphi^{m-\ell}x$. Hence $y \in N_A(L)$, as required. The bounds used at $a \geq \ell$ are exactly $-u$, without replacing them by 0 when $u < 0$.

Lean, formal context. `KummerChartDeRham` (L50-L170), `FitTwistsExist` (L377). □

Lemma 15.41 (The two height inclusions)**.** *Suppose every jump of L is at most the nonnegative integer n. Then*

$$\tau^n(A \otimes L) \subseteq N_A(L), \qquad E^n N_A(L) \subseteq \varphi N_A(L). \tag{15.58}$$

Proof. Every invariant form is at least of order $-n$ on $A \otimes L$ by Lemma 15.39. For every $m \geq 1$,

$$\varphi^m(\tau) = (1+\mu_0)^{p^{m-1}} - 1$$

has t-order 1: division by μ_0 gives a series with nonzero constant term p^{m-1} in B^+_{dR}. Thus the positive Frobenius values of $\tau^n(A \otimes L)$ have order at least zero. This proves the first inclusion directly from the definition.

For the second, take $x \in N_A(L)$ and put $y = \varphi^{-1}(E^n x)$. Since $\varphi^{-1}(\tau) \mid \tau$, the vector y lies in $A[1/\tau] \otimes L$. At the first positive Frobenius iterate,

$$\lambda(\varphi y) = E^n \lambda(x) \in B^+_{\mathrm{dR}} :$$

the denominator τ is a de Rham unit, so $\lambda(x) \in t^{-n}B^+_{\mathrm{dR}}$, whereas E has order 1. For $m \geq 2$,

$$\lambda(\varphi^m y) = \varphi^{m-1}(E)^n \lambda(\varphi^{m-1}x) \in B^+_{\mathrm{dR}}.$$

Hence $y \in N_A(L)$. Negative jumps cause no additional factor: their values have higher, rather than lower, filtration order.

Lean, formal context. `KummerChartDeRham` (L50-L170), `FitTwistsExist` (L377). □

Actual lifting ideals and the common correction

Consider the crystalline extension associated with a cocycle,

$$0 \longrightarrow Y \longrightarrow T \xrightarrow{\pi} C_0 \longrightarrow 0,$$

where C_0 is a trivial finite free parameter lattice. Its underlying lattice is split, so we may use the vector $(0, c) \in T$ to lift $c \in C_0$. Put

$$M_{\ell,i} = e_i \varphi^\ell N_A(T), \qquad C_{A,i} = e_i(A \otimes C_0), \qquad c_i = e_i(1 \otimes c).$$

Define the ideals

$$I_{\ell,i} = \{s \in A : sC_{A,i} \subseteq \pi(M_{\ell,i})\}, \qquad J_{\ell,i}(r) = I_{\ell,i} + (r), \quad r \in \mathbb{Z}_p. \tag{15.59}$$

By Lemma 15.40, these are the actual standard-exponent lifting ideals of the source. In particular,

$$s \in J_{\ell,i}(r) \iff \exists u \in A\ \forall c \in C_0\ \exists x_c \in M_{\ell,i} : \pi(x_c) = (s - ru)c_i. \tag{15.60}$$

The scalar u is common to all c.

Lemma 15.42 (Initial exponents and Frobenius transport)**.** *If the jumps of T are at most $n \geq 0$, then $\tau^n \in I_{0,i}$ for every i. For every $m \geq 1$ there is an $a \geq 0$ such that $z^a \in J_{0,i}(p^m)$ for every i. Furthermore,*

$$\varphi^\ell(s) \in J_{\ell,\sigma^\ell i}(r) \quad \iff \quad s \in J_{0,i}(r), \qquad E^n J_{0,i}(r) \subseteq J_{1,i}(r). \tag{15.61}$$

Proof. Lift $\tau^n c_i$ by $e_i(\tau^n(1 \otimes (0, c)))$ and use Lemma 15.41. This proves the first assertion. If $n = 0$, take $a = 0$. Otherwise $A/p = \mathcal{O}_C^\flat$ is a rank-one valuation ring, and both $\overline{z}$ and $\overline{\tau}$ are nonzero elements of positive valuation. Choose b large enough that $(\overline{\tau})^n$ divides $(\overline{z})^b$. Then $z^b \in (p, \tau^n)$, hence $z^{bm} \in (p^m, \tau^n) \subseteq J_{0,i}(p^m)$ for every i.

The Frobenius assertion follows by applying φ^ℓ and its inverse to (15.60): it fixes r, commutes with π, maps c_i to $c_{\sigma^\ell i}$, and maps $N_A(T)$ onto $\varphi^\ell N_A(T)$. Finally multiply each lift and its common correction in (15.60) by E^n and apply the second inclusion in (15.58).

Lean. `exists_forall_z_pow_mem_liftIdeal`, `phi_pow_smul_mem_liftIdeal_iff`, `mul_E_pow_mem_liftIdeal_one`. □

For the lowering argument, let $\mathfrak{S} = \Lambda[[z]] \subset A$ be the chart ring, and let $N(T) \subset N_A(T)$ be the integral period module of the given frame. Set $N_\ell(T) = \mathfrak{S} \cdot \varphi^\ell N(T)$. The frame identifies $A \cdot N(T) = N_A(T)$, and thus $A \cdot N_\ell(T) = \varphi^\ell N_A(T)$. The strong-lifting conclusion 15.37 says that if $f \in \mathfrak{S} \cap J_{\ell,i}(r)$, the common u in (15.60) can be chosen in $\mathfrak{S}$, and all the x_c can be chosen in $e_i N_\ell(T)$. This statement concerns the actual modules just identified.

Lemma 15.43 (The extra factor at level one)**.** *Let $\delta = z\tau$, and let g belong to the chart subgroup. The conclusions 15.32 and 15.36 imply*

$$(g - 1)N(T) \subseteq \delta N_A(T), \qquad (g - 1)N_1(T) \subseteq \delta E\, \varphi N_A(T). \tag{15.62}$$

The respective quotients preserve the letter of an input vector.

Proof. The first inclusion is precisely the ordinary period difference proved from the frame matrices and scalar cancellation in the cited result. For the second, write $x = \sum_j s_j \varphi(n_j)$, with $s_j \in \mathfrak{S}$ and $n_j \in N(T)$. Since g commutes with φ,

$$(g - 1)x = \sum_j g(s_j)\varphi((g - 1)n_j) + \sum_j (g(s_j) - s_j)\varphi(n_j).$$

The first sum belongs to $\varphi(\delta)\varphi N_A(T)$, and

$$\varphi(\delta) = z^p v E\tau = \delta E\, z^{p-1} v.$$

The second sum belongs to $\delta E\varphi N_A(T)$ by $(g-1)\mathfrak{S} \subseteq \delta EA$. Applying the letter projector to the quotient gives the assertion for each letter. This proves the extra E directly, without changing or discarding any negative standard order.

Lean. `exists_zTau_smul_eq_smul_sub`, `CupProductBound` (L867-L891). □

Lemma 15.44 (A divided difference preserves a lifting ideal)**.** *Fix $\ell = 0$ or 1, and put $d = \delta$ or $d = \delta E$, respectively. Let $f \in \mathfrak{S} \cap J_{\ell,i}(r)$, and let g belong to the chart subgroup. If $s, \alpha, \beta \in A$ satisfy*

$$d(s + \beta f) = \alpha(g(f) - f), \tag{15.63}$$

then $s \in J_{\ell,i}(r)$.

Proof. Use the strong lift with a common $u \in \mathfrak{S}$. The chart scalar difference gives $g(u) - u = dv$ for one $v \in A$, and Lemma 15.43 gives $g(x_c) - x_c = dy_c$ with $y_c \in M_{\ell,i}$. Each c_i is fixed by g. Apply π, multiply by α, and use (15.63) to obtain

$$d\pi(\alpha y_c) = d(s + \beta f - r\alpha v)c_i.$$

Cancel the nonzero scalar d in the torsion-free period ambient space. The same αv works for every c, proving $s + \beta f \in J_{\ell,i}(r)$. Since βf already belongs to the ideal, the conclusion follows.

Lean. `mem_liftIdeal_of_eq_zTau_mul_smul_sub`, `mem_liftIdeal_of_eq_mul_smul_sub`. □

The ordinary coordinate step, with its exact cost

Lemma 15.45 (The Kummer shift on a tame chart)**.** *For the tame chart built from $\varpi^e = p$ and a compatible tower $\varpi_r^{p^r} = \varpi$, there is an element g of the chart subgroup such that $g(z) = qz$ and g fixes Λ.*

Proof. Over each finite unramified extension of $\mathbb{Q}_p$, the polynomial $X^{ep^r} - p$ is Eisenstein. Hence inertia is transitive on its roots: it is still irreducible over the maximal unramified algebraic extension. In particular the closed subset of inertia consisting of elements that send ϖ_r to $\epsilon_r\varpi_r$ is nonempty. These subsets are nested as r increases. Compactness of inertia gives an element lying in all of them. It fixes ϖ and the unramified constants and therefore belongs to the chart subgroup. Its action on the compatible tower is multiplication by ϵ, so its action on $z = [\varpi^\flat]$ is $g(z) = [\epsilon]z$. If the local group is presented with a surjection onto $G_{\mathbb{Q}_p}$, take any preimage of this element. This is the ordinary construction underlying the tame chart.

Lean. `TameCharts` (L520-L636). □

Lemma 15.46 (The scalar cyclotomic identity)**.** *For every integer $j \geq 1$, with $v_j = v_p(j)$, there are $\alpha \in A^\times$ and $\beta \in A$ such that*

$$\tau E(p^{v_j} + \beta z) = \alpha(q^j - 1). \tag{15.64}$$

Proof. Write $j = p^{v_j} m$ with $p \nmid m$. The geometric factor $1 + q^{p^{v_j}} + \cdots + q^{(m-1)p^{v_j}}$ is an A-unit: its residue in the residue field of the local ring A is $m \neq 0$. It follows from (15.52) that

$$q^j - 1 \sim q^{p^{v_j}} - 1 \sim \tau \prod_{a=0}^{v_j} E_a.$$

The chart identity $E \equiv p \pmod z$, together with $\varphi(z) = z^p$, gives $E_a \equiv p \pmod z$ for every a. Consequently $\prod_{a=1}^{v_j} E_a = p^{v_j} + \beta z$. Absorbing the unit from the preceding associatedness proves the formula, also when $v_j = 0$ and the product is empty. Compare the scalar lifting calculation.
Lean. `PeriodMatrixBasis` (L134-L194). □

Theorem 15.47 (Ordinary lowering at levels zero and one)**.** *On the tame chart of Lemma 15.45, under the stated frame and strong-lifting hypotheses, for all $c \ge 0$, $j \ge 1$, $r \in \mathbb{Z}_p$, and letters i,*

$$p^c z^j \in J_{0,i}(r) \implies p^{c+v_p(j)+1} z^{j-1} \in J_{0,i}(r), \tag{15.65}$$

$$p^c z^j \in J_{1,i}(r) \implies p^{c+v_p(j)} z^{j-1} \in J_{1,i}(r). \tag{15.66}$$

Proof. Take g from Lemma 15.45 and put $f = p^c z^j$. Then $g(f) - f = p^c z^j (q^j - 1)$. Multiplying (15.64) by $p^c z^j$ gives

$$\delta E (p^{c+v_p(j)} z^{j-1} + \beta f) = \alpha(g(f) - f).$$

Apply Lemma 15.44 with $\ell = 1$. For $\ell = 0$, write $E = p + zb$ and set $\beta' = p^{v_p(j)} b + \beta E$. The same scalar identity becomes

$$\tau(p^{v_p(j)+1} + z\beta') = \alpha(q^j - 1),$$

hence

$$\delta(p^{c+v_p(j)+1} z^{j-1} + \beta' f) = \alpha(g(f) - f).$$

The level-zero case of the operator lemma finishes the proof.
Lean, formal context. `PeriodMatrixBasis` (L134-L194), `FitTwistsExist` (L377). □

Lemma 15.48 (Frobenius descent and the factorial cost)**.** *Let $pa \le b$. If*

$$p^c z^b \in J_{1,\sigma i}(r), \qquad c + v_p(b!) \le k + v_p((pa)!),$$

then $p^k z^a \in J_{0,i}(r)$. In particular,

$$z^b \in J_{1,\sigma i}(r) \implies z^{\lfloor b/p \rfloor} \in J_{0,i}(r). \tag{15.67}$$

Proof. Apply (15.66) successively from exponent b to exponent pa. The total increase of the p-exponent is

$$\sum_{j=pa+1}^{b} v_p(j) = v_p(b!) - v_p((pa)!).$$

Increase the exponent further to k if necessary, and use $\varphi(p^k z^a) = p^k z^{pa}$ and (15.61). When $a = \lfloor b/p \rfloor$, none of $pa + 1, \dots, b$ is divisible by p, so the displayed cost is zero. This proves (15.67).
Lean, formal context. `PeriodMatrixBasis` (L134-L194), `FitTwistsExist` (L377). □

Theorem 15.49 (The height recurrence for the lifting ideals)**.** *Work on the same tame chart, with the frame hypotheses of Theorem 15.47. Assume every jump of the extension T is at most $n \ge 0$, and $z^e \in (E,p)\mathfrak{S}$. For $\kappa \ge 0$, put*

$$\mathcal{A}_\kappa = \{a \ge 0 : z^a \in J_{0,i}(p^{\kappa+1}) \text{ for every letter } i\}.$$

This set is nonempty, and

$$a \in \mathcal{A}_\kappa \implies \left\lfloor \frac{a + e(n+\kappa)}{p} \right\rfloor \in \mathcal{A}_\kappa. \tag{15.68}$$

Its least member ν satisfies

$$(p-1)\nu \le e(n+\kappa). \tag{15.69}$$

In particular $\nu < e$ whenever $n + \kappa < p - 1$.

Proof. Nonemptiness is Lemma 15.42. Choose $u, v \in \mathfrak{S}$ with $z^e = uE + vp$. For every letter, height raising gives $z^a E^n \in J_{1,i}(p^{\kappa+1})$. Expand

$$z^{a+e(n+\kappa)} = z^a(uE + vp)^{n+\kappa}.$$

A term with E-exponent at least n is in the ideal by the previous membership; every other term has p-exponent at least $\kappa + 1$, so it too is in the ideal. Apply (15.67) and use that σ permutes the letters. This proves the recurrence.

For its least member ν, minimality gives $\nu \le \lfloor(\nu + e(n + \kappa))/p\rfloor$. Therefore $p\nu \le \nu + e(n + \kappa)$, proving (15.69). Its final consequence follows by comparison with $e(p - 1)$.

Lean, formal context. `PeriodMatrixBasis` (L134-L194), `FitTwistsExist` (L377). □

Corollary 15.50 (The small-jump cokernel)**.** *Under the same frame hypotheses, if $en \le p - 2$, then $I_{0,i} = A$ for every letter. Equivalently, the map from the actual level-zero standard period module onto the trivial parameter module is surjective at each letter.*

Proof. Set $\kappa = 0$. The height bound forces $\nu = 0$, so $1 \in I_{0,i} + (p)$. Write $1 = s + pb$ with $s \in I_{0,i}$. The element $s = 1 - pb$ is a unit because A is p-adically complete. Thus $I_{0,i} = A$.

Lean, formal context. `PeriodMatrixBasis` (L134-L194), `FitTwistsExist` (L377). □

The ideal at level zero. The initial exponent, height raising and ordinary lowering give the first two hypotheses of Theorem 15.16 for $J_i = I_{0,i}$, including negative jumps. The frame is constructed in Theorem 16.19 and identified with the module defined by forms in Theorem 15.27. To obtain the third hypothesis one needs a depth estimate and a continuous cochain for the cup product. These enter Corollary 16.20.

15.7 Localization of the period module and preservation of depth orders

The depth argument uses the localized integral module, not only its standard-exponent parts. Its required closure properties follow from the matrix consequences and scalar differences already proved. We use A, S, $E_a = \varphi^a E$, $N_A(T)$, $N(T)$, and the value field $\mathcal{K}_0$ as in 15.27. Let $\mathcal{D}$ be the multiplicative set generated by all E_a, $a \ge 0$, and put

$$M_A = N_A(T)[\mathcal{D}^{-1}], \qquad M = N(T)[\mathcal{D}^{-1}].$$

Every denominator in $\mathcal{D}$ belongs to S and is nonzero. The source definitions are the value-field and localized modules.

Lemma 15.51 (Localization of the module over the value field)**.** *Inside the common period ambient space,*

$$M = \{x \in M_A : \lambda(x) \in \mathcal{K}_0 \text{ for every invariant form } \lambda\}. \tag{15.70}$$

Thus M and M_A are exactly the modules denoted by ***kisinLoc*** *and* ***kisinLocAinf*** *in the source.*

Proof. If $x = n/D$ with $n \in N(T)$ and $D \in \mathcal{D}$, then $\lambda(x) = \lambda(n)/D \in \mathcal{K}_0$, because $S \subset \mathcal{K}_0$ and this is a field. Conversely, for an x on the right side choose $D \in \mathcal{D}$ with $Dx \in N_A(T)$. Every invariant form takes Dx into $\mathcal{K}_0$, so $Dx \in N(T)$ by the actual form definition of $N(T)$. Hence $x \in M$. This argument does not require a new existence or exactness theorem.

Lean. `Modules` (L274), `Modules` (L279). □

Lemma 15.52 (Positive Frobenius preserves the localized modules)**.** *One has*

$$\varphi M_A \subseteq M_A, \qquad \varphi M \subseteq M. \tag{15.71}$$

The first assertion follows directly from the filtered definition; the second follows from the properties of the period matrices.

Proof. If $x \in N_A(T)$ and $\tau^b x \in A \otimes T$, then $E^b \varphi x \in A[1/\tau] \otimes T$ because $\varphi\tau$ is a unit times $E\tau$. For every $j \geq 1$ and invariant form λ,

$$\lambda(\varphi^j(E^b \varphi x)) = E_j^b \lambda(\varphi^{j+1} x) \in B_{\mathrm{dR}}^+.$$

Thus $E^b \varphi x \in N_A(T)$ and $\varphi x \in M_A$. Since $\varphi(E_a) = E_{a+1}$, this proves $\varphi M_A \subseteq M_A$.

We record explicitly why $\varphi S_B \subseteq S_B$. Write the chart equation as $E = p + z^e V$, $V \in S^\times$. Because $\varphi S \subseteq S$ and $\varphi z = z^p$,

$$\frac{\varphi E}{p} = 1 + p^{p-1}(E/p - 1)^p V^{-p} \varphi(V) \in S[E/p].$$

The Frobenius map fixes p and is p-adically bounded, so this inclusion extends to the completion and after inverting p.

Let $n_1, \dots, n_d$ be the actual S-basis of $N(T)$ obtained in 15.27. In the notation of the matrix frame, for every invariant form λ,

$$E_1^h \lambda(\varphi n_j) = \big(\lambda(e)\varphi P\big)(\varphi Z)_{*,j}. \tag{15.72}$$

The row $\lambda(e)\varphi P$ is fixed by the chart subgroup: $\rho(g)g(P) = P$, $g(\lambda(e)) = \lambda(e)\rho(g)$, and φ commutes with g and fixes the entries of $\rho(g)$. Its entries therefore belong to the finite chart field K. The entries of φZ belong to S_B, and $E_1 \in S$ is nonzero. Formula (15.72) proves $\lambda(\varphi n_j) \in \mathcal{K}_0$. The first part of the proof already puts φn_j in M_A; Lemma 15.51 now puts it in M. Finally use $\varphi S \subseteq S$ and $\varphi \mathcal{D} \subseteq \mathcal{D}$ to conclude $\varphi M \subseteq M$.

Lean. `ExponentLattices` (L685), `ExponentLattices` (L695). □

Lemma 15.53 (Values at every inverse-Frobenius level)**.** *For $a \geq 0$, let*

$$\mathcal{K}_a = \mathrm{Frac}\big(K[S_B, \iota_a(S)]\big) \subset B_{\mathrm{dR}}, \qquad \iota_a(s) = \varphi^{-a} s.$$

For every $x \in M$ and every invariant form λ,

$$\lambda(\varphi^{-a} x) \in \mathcal{K}_a. \tag{15.73}$$

This is the source's `valueAt_mem_valueField`. It follows from the present matrix input and positive-Frobenius stability, without adding a Frobenius transition matrix to the existence hypothesis.

Proof. Put $R = S[\mathcal{D}^{-1}]$ and $F_S = \mathrm{Frac}(S)$. The actual frame $n_1, \dots, n_d$ is an R-basis of M. By Lemma 15.52, each $\varphi^a n_j$ belongs to M. There is consequently a matrix $C_a \in \mathrm{Mat}_d(R)$ such that, in frame-matrix notation,

$$\varphi^a(n) = n C_a.$$

The original frame and its Frobenius translate are both bases over $\mathrm{Frac}(A)$ in the common fraction ambient space: the frame is a basis there by $QQ' = \tau^{2h} I$, and φ is an automorphism of that scalar

field. Hence $\det C_a \neq 0$. Since $R \subset F_S$, the inverse matrix has entries in F_S. It follows that every $x \in M$ admits an expression

$$x = \sum_{j=1}^{d} r_j \varphi^a n_j, \qquad r_j \in F_S.$$

Apply φ^{-a} and then λ to this identity. The form extends uniquely to this fraction ambient space, since its target B_{dR} is a field and every nonzero element of A has nonzero image there. Thus

$$\lambda(\varphi^{-a}x) = \sum_j \iota_a(r_j)\lambda(n_j).$$

Here $\iota_a(r_j) \in \mathrm{Frac}(\iota_a S)$, while $\lambda(n_j) \in \mathcal{K}_0 \subseteq \mathcal{K}_a$ by the definition of $N(T)$. Both factors lie in $\mathcal{K}_a$, proving (15.73).

Lean. `ExponentLattices` (L747-L787). □

Lemma 15.54 (The localized ordinary divided difference)**.** *Put $\delta = z\tau$, and let g belong to the chart subgroup. Then*

$$\frac{gx - x}{\delta} \in M_A \quad (x \in M). \tag{15.74}$$

This follows from the already proved ordinary difference 15.36; it is not a further period-module hypothesis.

Proof. Write $x = n/D$ with $n \in N(T)$ and $D \in \mathcal{D}$. By 15.36, $gn - n = \delta n'$ for some $n' \in N_A(T)$. Since $D \in S$, the scalar chart difference 15.32 gives $gD - D = \delta E b$ for some $b \in A$. Every gE_a is associated to E_a in A: g preserves $\ker\theta = (E)$ and commutes with φ. Thus $gD = uD$ for some $u \in A^\times$, so all denominators below are allowed in M_A. Direct subtraction gives

$$\frac{gx - x}{\delta} = \frac{n'}{gD} - \frac{Ebn}{DgD} \in M_A.$$

This also verifies the required denominator support after localization.

Lean. `ExponentLattices` (L651). □

Corollary 15.55 (Dividing by $z\tau$ preserves every depth order)**.** *For every $a \geq 0$, the image $\iota_a(\delta) = \varphi^{-a}(z\tau)$ is a unit in B_{dR}^+. Consequently, for every integer m, every invariant form λ, and every vector w for which the expressions are defined,*

$$\lambda(\varphi^{-a}(w/\delta)) \in \mathrm{Fil}^m \quad \Longleftrightarrow \quad \lambda(\varphi^{-a}w) \in \mathrm{Fil}^m\,. \tag{15.75}$$

In particular, if $x \in M$ has any prescribed integer order function at a letter, then $(gx - x)/\delta \in M_A$ has those same lower bounds at that letter. More generally, every stronger order bound proved for $gx - x$ survives the division.

Proof. The elements $\theta(\varphi^{-a}z)$ and $\theta(\varphi^{-a}\tau) = \epsilon_{a+1} - 1$ are both nonzero; the first is a nonzero root of the chart uniformizer and the second uses a primitive p^{a+1}-st root of unity. Their product is nonzero, so $\iota_a(\delta)$ has de Rham order zero. Linearity of a form gives

$$\lambda(\varphi^{-a}(w/\delta)) = \iota_a(\delta)^{-1}\lambda(\varphi^{-a}w),$$

which proves (15.75) for all integers m. For the asserted membership, use Lemma 15.54. Equivariance of forms and commutation of g with φ show that $gx - x$ preserves every lower filtration bound of x, because the Galois action preserves the filtration of B_{dR}. It preserves the letter as well. The letter projector commutes with scalar multiplication by δ, so applying it to the quotient preserves the identity. The equivalence just proved then preserves all asserted depth bounds, including positive bounds arising from negative jumps.

Lean. `KummerChartDeRham` (L194), `KummerChartDeRham` (L297), `ExponentLattices` (L892). □

Together these results supply localization, positive-Frobenius closure, and value-field membership at every inverse-Frobenius level used by the depth construction from the existing matrix input. No additional statement about exactness of Kisin modules, preservation of an arbitrary filtration by an unspecified map, or existence of a depth operator has been inserted as an assumption.

15.8 The Witt cochain behind the local cup-product bound

We reconstruct the cohomological part of the three-cokernel argument. In particular, a path of integral period points will produce an explicit continuous cochain whose coboundary is the required multiple of the cup product. The source is the pairing lemma and its corrected cochain. The later application derives its period-depth estimates from the explicit integral frame, constructed in Theorem 16.19.

Frobenius modulo the small Witt subgroup

Let $C^\flat$ be the complete algebraically closed tilted field of characteristic p, with maximal ideal $\mathfrak{m}^\flat$ in its valuation ring. For $m \geq 1$ put

$$R_m = W_m(C^\flat), \qquad \mathfrak{s}_m = W_m(\mathfrak{m}^\flat).$$

The second notation means the additive subgroup of truncated Witt vectors all of whose Witt coordinates have strictly positive valuation. It is an ideal in $W_m(\mathcal{O}_{C^\flat})$, but is not asserted to be an ideal in R_m. Give these finite Witt vectors the topology of their m coordinates. Then $\mathfrak{s}_m$ is open and is stable under Frobenius and Galois. The notation $R_m(1)$ and $\mathfrak{s}_m(1)$ includes the usual cyclotomic action on the coefficient factor; Frobenius acts only on the Witt factor.

Lemma 15.56 (Constants and Frobenius in the small Witt quotient)**.** *The map $\varphi - 1 : R_m \to R_m$ is surjective, and*

$$(R_m/\mathfrak{s}_m)^{\varphi=1} = \mathbb{Z}/p^m\mathbb{Z}, \qquad (\mathbb{Z}/p^m\mathbb{Z}) \cap \mathfrak{s}_m = 0. \tag{15.76}$$

Consequently a vector $v \in R_m(1)$ with $(\varphi - 1)v \in \mathfrak{s}_m(1)$ is congruent to a unique element of $\mathbb{Z}/p^m\mathbb{Z}(1)$ modulo $\mathfrak{s}_m(1)$. This extraction of the constant is continuous on its domain.

Proof. For $m = 1$, surjectivity is the existence of a root of $X^p - X - a$ in $C^\flat$. Inductively solve first modulo p, lift the solution, and divide the remaining error by p. The identification $pR_m \simeq R_{m-1}$ as additive groups commutes with Frobenius, so induction corrects the error. Here multiplication by p identifies the length-$(m-1)$ Witt vectors with pR_m because $C^\flat$ is perfect.

For the fixed quotient, first suppose $m = 1$ and $v^p - v \in \mathfrak{m}^\flat$. If v had negative valuation, $v^p - v$ would have valuation $p\,\mathrm{val}(v) < 0$. Thus v is integral, and its residue satisfies $\bar{v}^p = \bar{v}$. It is therefore an element of $\mathbb{F}_p$. For the induction step subtract the Teichmüller lift of this residue, then subtract a vector in $\mathfrak{s}_m$ with the remaining zeroth coordinate. The difference is pw. Its Frobenius difference belongs to $\mathfrak{s}_m$, and

$$pw \in \mathfrak{s}_m \quad \Longleftrightarrow \quad w \in \mathfrak{s}_{m-1}.$$

This follows directly from the Witt coordinates of multiplication by p and the preservation of positive valuation by Frobenius. Induction now supplies a constant modulo p^{m-1} for w. Together with the first residue this gives a constant modulo p^m for the original vector.

A nonzero constant has a first nonzero Witt coordinate in $\mathbb{F}_p^\times$, of valuation zero. It cannot belong to $\mathfrak{s}_m$, proving uniqueness. Finally $R_m/\mathfrak{s}_m$ is discrete because $\mathfrak{s}_m$ is open. The inverse of its inclusion of the finite constant group is therefore continuous on that subgroup. Tensoring with the rank-one Tate module gives the asserted twisted statements.

Lean, comparison. The source states Frobenius surjectivity, extraction of a constant and uniqueness for the Tate twist. Here these follow by induction on the Witt length. See `ArtinSchreierSequence` (L430-L464). □

A cochain for a sequence of changes in the two variables

Let G be a local Galois group and $H \subseteq G$ an open subgroup of index prime to p. Let L, L' be finite free $\mathcal{O}$-lattices, with continuous cocycles a, a'. They can belong to families of cocycles; fix their parameter vectors c, c'. Write E, E' for the corresponding extensions of the trivial parameter lattices by L, L'. The chosen module splittings have first projections pr and pr'. If x projects to $p^{e_1}c$, then

$$\mathrm{pr}(gx) = g\,\mathrm{pr}(x) + p^{e_1}a(g), \tag{15.77}$$

and similarly on the second side.

Let $P : L \times L' \to \mathbb{Z}/p^m\mathbb{Z}(1)$ be the equivariant pairing obtained from an admissible pairing in (15.5). Extend it bilinearly to the Witt ambient modules, reducing modulo p^m in the target. Frobenius commutes with G and fixes all vectors $1 \otimes t$. For a nonnegative integer κ abbreviate

$$B(v, v') = p^{\kappa} P(\mathrm{pr}\, v, \mathrm{pr}'\, v').$$

Consider a change in one coordinate that fixes its image in the parameter lattice. Changing the first coordinate by v while keeping x' fixed satisfies the pairing condition if

$$B(v, gx' - x') \in \mathfrak{s}_m(1) \quad (g \in H),$$

and changing the second coordinate by v' while keeping x fixed satisfies it if

$$B(gx - x, gv') \in \mathfrak{s}_m(1) \quad (g \in H).$$

The factor p^{κ} is included in B. These two conditions are `IsSafeMoveFst` and `IsSafeMoveSnd` in the Lean definition of an admissible path.

Theorem 15.57 (The corrected Witt cochain)**.** *Let x, w project to $p^{e_1}c$, and let x', w' project to $p^{e_2}c'$. Suppose the four successive changes*

$$(\varphi x, \varphi x') \longrightarrow (w, \varphi x') \longrightarrow (w, w') \longrightarrow (x, w') \longrightarrow (x, x')$$

satisfy the pairing conditions. Suppose also

$$B(gx - x, g(hx' - x')) \in \mathfrak{s}_m(1) \quad (g, h \in H). \tag{15.78}$$

Then $p^{e_1+e_2+\kappa}(a \cup_P a')$ is the coboundary of a continuous cochain on H. Its class on G is zero as well. Any larger power of p has the same property.

Proof. Put $X = \mathrm{pr}\, x$ and $X' = \mathrm{pr}'\, x'$. By Lemma 15.56, choose $y \in R_m(1)$ satisfying

$$(\varphi - 1)y = B(w, \varphi x' - w') + B(x, w' - x'). \tag{15.79}$$

Define, for $g \in H$,

$$\begin{aligned} f(g) = {} & p^{\kappa} P(\mathrm{pr}(gx), gX') - g(p^{\kappa} P(X, X')) \\ & - B(x, gx' - x') + gy - y. \end{aligned} \tag{15.80}$$

Every term is continuous in the finite Witt-coordinate topology.

We record the two identities that perform the calculation. Expanding (15.77), equivariance of P, and the cocycle equations gives

$$\begin{aligned} p^{e_1+e_2+\kappa} P(a(g), ga'(h)) = {} & B(gx - x, g(hx' - x')) \\ & + gf(h) - f(gh) + f(g). \end{aligned} \tag{15.81}$$

For completeness, write $A_g = p^{e_1}a(g)$, $A'_g = p^{e_2}a'(g)$, $dX_g = gX - X$ and $dX'_g = gX' - X'$. The part of $f(g)$ before $gy - y$ is

$$p^{\kappa}(P(A_g, gX') - P(X, dX'_g) - P(X, A'_g)).$$

Its coboundary is

$$-p^{\kappa}P(dX_g + A_g, gdX'_h) - p^{\kappa}P(dX_g, gA'_h).$$

This is exactly the difference between the left side of (15.81) and its first right-hand term; the correction $gy - y$ has zero coboundary.

Next, commuting g with φ and using (15.79) gives

$$\begin{aligned}(\varphi - 1)f(g) = {} & B(gw - w, g(\varphi x' - w')) - B(\varphi x - w, g\varphi x' - \varphi x') \\ & - B(w - x, gw' - w') + B(gx - x, g(w' - x')).\end{aligned} \tag{15.82}$$

One obtains this by substituting $(\varphi - 1)(gy - y) = (g - 1)[B(w, \varphi x' - w') + B(x, w' - x')]$ into (15.80); the remaining terms group into the four displayed bilinear terms. Each is small by one of the four pairing conditions.

Lemma 15.56 therefore gives a unique $k(g) \in \mathbb{Z}/p^m\mathbb{Z}(1)$ with $f(g) - k(g) \in \mathfrak{s}_m(1)$. The map k is continuous. Reduce (15.81) modulo the small subgroup. Its first right-hand term vanishes by (15.78). Both remaining sides are constants, and their difference is small, so uniqueness gives

$$p^{e_1+e_2+\kappa}P(a(g), ga'(h)) = gk(h) - k(gh) + k(g).$$

This is the required continuous coboundary on H. The class on G is killed by restriction followed by corestriction, hence by $[G : H]$, and is also killed by p^m. These integers are coprime, so its class on G is zero. Multiplying the cochain by an additional power of p gives the last assertion. The case $m = 0$ has zero target and is immediate.

Lean. `nsmul_modPairing_eq_coboundary`, `of_isSafePath`. □

Lemma 15.58 (The pairing conditions on the sum of the eigenspaces)**.** *Suppose the coefficient root of unity decomposes the ambient modules into eigenspaces indexed by i. At each i, suppose that the vectors in the path satisfying the pairing conditions lie in the i-th eigenspace. If Frobenius permutes the eigenspaces and the path at i runs from $(\varphi x_{i^-}, \varphi x'_{i^-})$ to (x_i, x'_i), the sums of these paths satisfy Theorem 15.57, provided* (15.78) *holds for every index.*

Proof. Different letter parts pair to zero. In fact, multiplication by the coefficient root of unity can be moved from one argument of the original $\mathcal{O}$-bilinear pairing to the other. On two different parts its two eigenvalues have distinct residues, so their difference is a unit. The pairing is zero. This remains true after the coefficient functional and reduction. Galois preserves each part. Thus each pairing for the sums is the sum of its diagonal letter expressions, and is small. Finally

$$\varphi \sum_i x_i = \sum_i \varphi x_{i^-}, \qquad \varphi \sum_i x'_i = \sum_i \varphi x'_{i^-}$$

by reindexing the finite permutation of letters. The projectors sum to one, so the sums lie over the same constants required by Theorem 15.57. No factor equal to the number of letters is introduced.

Lean. `wittPairing_wittFst_eq_zero_of_ne`, `SafeMoves` (L490-L547). □

15.9 Period depth, opposite windows, and the three-cokernel exponent

We next supply the passage from the period estimates to the safe paths just used. This makes explicit the smaller period statements on which the application depends. It does not assert that their analytic construction follows from the Witt-cochain calculation.

At a letter i, write $\mathrm{Ord}_i(o)$ for the source's period vectors with orders $o(a,u)$ at the level $\varphi^a E$ on a form of jump u at the letter $i^{-a} = \sigma^{-a} i$. Thus its value $\nu(\varphi^{-a}x)$ belongs to $t^{o(a,u)} B_{\mathrm{dR}}^+$. A point has depth d if $gx - x \in \mathrm{Ord}_i(d)$ for every $g \in H$. All points used below belong to the actual localized Kisin module of the relevant cocycle extension. Write $\delta = z\tau$ as in the source. At a letter, a constant c means its projected vector $e_i(1 \otimes c)$.

The proved period facts used in the tame application. For the remainder of this subsection work on the tame chart of Lemma 15.45, and assume the matrix-frame input in 15.4 for the lattices and cocycle extensions used below. Let $M = N[E_a^{-1} : a \geq 0]$ and let $M_A = N_A[E_a^{-1} : a \geq 0]$. These are the actual source modules by Lemma 15.51. We use the following established consequences.

1. Lemma 15.40 identifies standard orders $o(a,u) = -u$ with the integral module N_A. For the Tate representation the jump is -1, so its standard order is 1. Combining that identification with Lemma 15.22 gives $\tau \,\mathrm{Ord}_i(1) \subseteq A_{\mathrm{inf}} \otimes \mathcal{O}(1)$.

2. Lemmas 15.54 and 15.55 show that $\delta^{-1}(g-1)x \in M_A$ for $x \in M$, without changing any stated order of $(g-1)x$. Lemma 15.52 gives $\varphi M \subseteq M$ and $\varphi M_A \subseteq M_A$.

3. The coordinate retraction of Lemma 15.25 is
$$r\left(\sum_\ell a_\ell v_\ell\right) = \sum_\ell \lambda_0(a_\ell) v_\ell, \qquad a_\ell \in A_{\mathrm{inf}}, \quad v_\ell \in M.$$
It preserves all orders, all congruences to fixed-field constants, letter parts and zero parameter projection. This formula is independent of the chosen expansion; no free basis for M is assumed.

The scalar depth step will now be proved from the defining chart data. Thus no separate depth-operator theorem remains as an input to this subsection.

At level a, form values on M belong to the chart's value field V_a by Lemma 15.53; the scalar maps preserve orders over this field by Lemma 15.23. We use the classical fixed-part statement for B_{dR}: a value invariant modulo $t^n B_{\mathrm{dR}}^+$ is congruent modulo that submodule to a scalar in the fixed local field. A nonzero fixed scalar has t-adic order zero. Indeed, if its leading order r is below n, that leading term belongs to $(\mathrm{gr}^r B_{\mathrm{dR}})^H = (C_p(r))^H$. The Tate–Sen fixed-part theorem makes this zero for $r \neq 0$ and equal to the fixed local field for $r = 0$. In the latter case subtract that fixed residue; any remaining leading term below n would have positive order and is again impossible. This proves the statement for every integer n.

Lemma 15.59 (The level-zero value field has fixed residue)**.** *Every $v \in V_0 \cap B_{\mathrm{dR}}^+$ satisfies $gv - v \in tB_{\mathrm{dR}}^+$ for every $g \in H$.*

Proof. Choose $N \geq 0$ such that $p^N \theta(v) \in \mathcal{O}_C$, and choose $a \in A_{\mathrm{inf}}$ with $\theta(a) = p^N \theta(v)$. Then $p^N v - a \in tB_{\mathrm{dR}}^+$. Apply the scalar extension of λ_0 from Lemma 15.23. It fixes V_0 and preserves this order, so
$$p^N v - \lambda_0(a) \in tB_{\mathrm{dR}}^+.$$
The chart subgroup fixes $\theta(S)$. Consequently $p^N(gv - v) \in tB_{\mathrm{dR}}^+$. Since p is a unit of B_{dR}^+, the claim follows.

Lean, formal context. `ExponentLattices` (L80-L232), `FitTwistsExist` (L377). □

Lemma 15.60 (The exact scalar cost of a nonzero depth step)**.** *Let $d_0 = 0$ if the chart ramification index e is 1, and $d_0 = v_p(e) + 1$ otherwise. For every nonzero integer j there are $\gamma \in H$, $\omega \in A_{\mathrm{inf}}^{\times}$ and $v_j, X_j \in A_{\mathrm{inf}}$ such that*

$$\gamma E = \omega E, \qquad \delta X_j = \omega^j - 1, \qquad v_j X_j - p^{v_p(j)+d_0} \in EA_{\mathrm{inf}}. \tag{15.83}$$

The element γ can be chosen independently of j.

Proof. Use $\gamma z = [\epsilon]z$ from Lemma 15.45. The generator E of $\ker\theta$ satisfies $\gamma E = \omega E$ for an A_{inf}-unit ω. Choose $u \in A_{\mathrm{inf}}^{\times}$ with $[\epsilon] - 1 = E\tau u$, and put $\pi = \theta(z)$. Then

$$\gamma z - z = E\delta u.$$

The scalar chart difference in Lemma 15.32, applied to E, also gives $\delta \mid \omega - 1$.

We give the derivative estimate used in the calculation. Weierstrass preparation of $E = p + z^e V$, $V \in S^{\times}$, gives a monic degree-e Eisenstein polynomial $F \in \Lambda[X]$ and $G \in S^{\times}$ with $F(z) = EG$. Normalize $v_p(p) = 1$; then $v_p(\pi) = 1/e$. If $e = 1$, $F' = 1$. For $e > 1$, write $F = X^e + \sum_{r<e} a_r X^r$ and $a_e = 1$. The nonzero summands $ra_r\pi^{r-1}$ of $F'(\pi)$ have pairwise distinct fractional parts of their valuations:

$$v_p(ra_r\pi^{r-1}) = v_p(ra_r) + \frac{r-1}{e}, \qquad 1 \le r \le e.$$

The first term on the right is an integer because Λ is unramified. Hence the least valuation occurs once and

$$v_p(F'(\pi)) \le v_p(e) + \frac{e-1}{e} < v_p(e) + 1 = d_0. \tag{15.84}$$

In particular $\theta(F'(z)) \mid p^{d_0}$ in $\mathcal{O}_C$, including the case $e = 1$. This proves the derivative bound directly.

Taylor expansion at z, with increment $E\delta u$, and cancellation of E give

$$\omega\,\gamma G = G + F'(z)\delta u + ER$$

for an integral remainder R. Apply θ, using that γ fixes $\theta(G)$:

$$\theta(\omega - 1)\theta(G) = F'(\pi)\theta(\delta)\theta(u).$$

The right side is nonzero. Thus $a = \theta(\omega - 1)$ is nonzero, and

$$v_p(\theta\delta) \le v_p(a) \le v_p(\theta\delta) + d_0. \tag{15.85}$$

Here the first inequality follows from $\delta \mid \omega - 1$, and the second uses $v_p(\theta G) \ge 0$, the fact that $\theta(u)$ is a unit of $\mathcal{O}_C$, and the derivative bound. Moreover

$$v_p(\theta\delta) = \frac{1}{e} + \frac{1}{p-1} > \frac{1}{p-1}.$$

In this range the lifting-the-exponent formula gives

$$v_p((1+a)^j - 1) = v_p(a) + v_p(j) \qquad (j \in \mathbb{Z} \setminus \{0\}).$$

For a positive integer prime to p, the first binomial term has strictly least valuation. For the pth power, the first term has valuation $1 + v_p(a)$, the intermediate terms have larger valuation, and the last term has valuation $pv_p(a) > 1 + v_p(a)$. Induction proves the formula for positive j; a negative power differs by a unit from the corresponding positive power.

Because ω is a unit and $\delta \mid \omega - 1$, the element $X_j = (\omega^j - 1)/\delta$ belongs to A_{inf} even for negative j. The preceding identity and (15.85) imply

$$0 \le v_p(\theta X_j) \le v_p(j) + d_0.$$

Thus $p^{v_p(j)+d_0}/\theta X_j \in \mathcal{O}_C$. Lift this quotient through the surjection θ to $v_j \in A_{\mathrm{inf}}$. The last equality in (15.83) follows from $\ker\theta = EA_{\mathrm{inf}}$. The source performs the same Taylor and valuation calculation in the nonzero scalar depth step.

Lean. `ExponentLattices` (L80-L232). □

Lemma 15.61 (The depth operator from the scalar calculation)**.** *Put $c(j) = v_p(j) + d_0$, with $v_p(0) := 0$. For each integer j there is a single additive map $A_j : M \to M$, valid for all letters and forms, with the following properties. If x lies over a constant and has depth o, then $A_j x - p^{c(j)}x$ lies over zero and has orders o. If a level-zero form value of x is constant modulo t^j, the corresponding value of $A_j x$ is constant modulo t^{j+1}. Consequently A_j raises depth j to $j+1$ for that form.*

Proof. First take $j \neq 0$ and the data in (15.83). Define

$$D_j x = p^{c(j)}x - v_j \frac{\gamma x - x}{\delta}, \qquad A_j x = r(D_j x).$$

The divided difference belongs to M_A by Lemma 15.54. Lemmas 15.55 and 15.25 preserve its orders and zero parameter projection, proving the cost assertion.

For a form value $v = c + y$, with c fixed and $y \in t^j B_{\mathrm{dR}}^+$, write $y = E^j q$ with $q \in V_0 \cap B_{\mathrm{dR}}^+$. Lemma 15.59 and $\gamma E = \omega E$ give

$$\gamma y - \omega^j y \in t^{j+1} B_{\mathrm{dR}}^+.$$

The image of $\delta = z\tau$ is a unit of B_{dR}^+ because its image under θ is nonzero. Dividing this congruence by δ yields

$$\nu\left(\frac{\gamma x - x}{\delta}\right) - X_j y \in t^{j+1} B_{\mathrm{dR}}^+.$$

Together with $v_j X_j \equiv p^{c(j)} \pmod{E}$, this proves

$$\nu(D_j x) - p^{c(j)} c \in t^{j+1} B_{\mathrm{dR}}^+.$$

The vector retraction retains this congruence.

For $j = 0$ take simply $A_0 x = p^{d_0} x$. A form value constant modulo t^0 is integral. By Lemma 15.59 it is invariant modulo t, and hence is a fixed-field constant modulo t. Thus A_0 has the required raising property; its cost difference is zero. This argument is specific to level zero and does not divide by j or assume the higher-level averaging theorem. Taking Galois differences of a congruence to a fixed constant proves the asserted increase in depth.

Lean, formal context. `ExponentLattices` (L80-L232), `FitTwistsExist` (L377). □

Lemma 15.62 (Depth improves positive orders and transports by Frobenius)**.** *If x has orders o and depth d, then its orders are at least*

$$o^{\mathrm{new}}(a,u) = \begin{cases} d(a,u), & o(a,u) > 0, \\ \min\{d(a,u), 0\}, & o(a,u) \le 0. \end{cases} \tag{15.86}$$

Together with its original orders it has the maximum of these two lower bounds. φx has orders 0 at level zero and orders $o(a-1,u)$ at levels $a \ge 1$. Its depth is at least 1 at level zero and $d(a-1,u)$ at levels $a \ge 1$.

Proof. For each form value v, write $v = c + r$, where c is fixed and r has order at least $d(a,u)$. If both the original order and the depth are positive, then c is a fixed scalar of positive order, hence is zero. In all other cases a fixed scalar has order at least zero. These observations give (15.86), including nonpositive depths.

For Frobenius, the value at level $a \geq 1$ becomes the old value at level $a-1$. At level zero, clear a product of the factors $\varphi^b E$ from a vector in M_A. By the filtered definition of the integral period module, the value of its positive Frobenius iterate is integral. After one Frobenius the cleared denominator is a product of $\varphi^{b+1}E$, whose de Rham orders are zero. It can therefore be cancelled without changing integrality. This proves nonnegative order at the new level zero directly from that filtered definition. For depth write

$$g\varphi x - \varphi x = \varphi(\delta)\,\varphi(\delta^{-1}(gx - x)).$$

The divided difference has the old depth orders by Corollary 15.55. Its Frobenius has the shifted orders just described. The factor $\varphi(\delta)$ has order at least one at E and nonnegative order at every other level: $\varphi(\tau) = \mu$ is associated to $E\tau$, and $\varphi(z)$ is integral. This supplies the asserted extra one at level zero.

Lean, formal context. `ExponentLattices` (L80-L232), `FitTwistsExist` (L377). □

Lemma 15.63 (The pairing criterion from the Tate-dual cokernel)**.** *Suppose a point of the Tate-dual side at the letter i lies over p^κ and has standard orders $-u$. Its Frobenius pullback from the letter l steps before i supplies the rule*

$$R_l(a,s) = \begin{cases} 1, & a < l, \\ -s, & a \geq l. \end{cases} \tag{15.87}$$

Specifically, a first-side move with orders $o(a,u)$ is safe at a second-side point of depth $d'(a,u')$ whenever $o(a,u) + d'(a,u') \geq R_l(a, u+u')$ at every pair of jumps. The same statement holds with the sides exchanged.

Proof. Let $\Phi : L \otimes L' \to \mathcal{O}(1)$ be the integral form underlying the finite pairing, and write $g\Phi - \Phi = p^m b(g)$. The third extension has underlying lattice $(L \otimes L')^\dagger \oplus \mathcal{O}$ and cocycle b. There is an integral equivariant map

$$F : (L \otimes L')^\dagger \oplus \mathcal{O} \longrightarrow (L \otimes L')^\dagger, \qquad F(v,c) = p^m v + c\Phi.$$

Indeed $F(g(v,c)) = gF(v,c)$ follows by substituting $p^m b(g) = g\Phi - \Phi$. Its evaluation form is $\Psi((v,c),w) = p^m v(w) + c\Phi(w)$. Modulo p^m, evaluation at a third point over p^κ is therefore p^κ times the original finite pairing. More precisely its constant is $p^\kappa e_{i-l}(1)$ before the Frobenius pullback and $p^\kappa e_i(1)$ afterwards. Balanced $\mathcal{O}$-bilinearity makes e_i act as the identity on the evaluation of two letter-i vectors. For general coefficients, compose this evaluation with the fixed integral $\mathbb{Z}_p$-linear coefficient functional and work with the underlying $\mathbb{Z}_p$-lattices. This preserves the same formula and its equivariance.

Here is the order calculation, with no blanket tensor-order estimate assumed. At each level and letter take the adapted invariant form bases λ_r on L and λ'_s on L' from Lemma 15.39, of jumps u_r, u'_s. Their matrices on integral lattice bases have the form

$$\operatorname{diag}(t^{-u_r})A, \qquad \operatorname{diag}(t^{-u'_s})A', \qquad A, A' \in \mathrm{GL}(B^+_{\mathrm{dR}}).$$

The invariant forms η_{rs} on the Tate dual, dual to $\lambda_r \otimes \lambda'_s$ through the Tate form $e_1 \mapsto t$, satisfy

$$t\ \mathrm{ev}(q, v \otimes v') = \sum_{r,s} \eta_{rs}(q)\lambda_r(v)\lambda'_s(v'). \tag{15.88}$$

Their entries on the dual integral lattice are products of entries of A^{-1} and $(A')^{-1}$ multiplied by $t^{u_r+u'_s+1}$. Consequently η_{rs} is of type $-u_r-u'_s-1$. A move or a Galois difference lies in the kernel lattice of its cocycle extension. Strict exactness of filtered de Rham comparison extends each form on that kernel to the extension with the same type. Thus the given orders and depths apply to the λ_r and λ'_s just chosen.

Let ν be the standard third point at the letter l steps before i, and put $q=\varphi^l F(\nu)$. Functoriality of N_A gives $F(\nu)\in N_A((L\otimes L')^\dagger)$. Thus at level a the value of η_{rs} on q has order at least zero if $a<l$, and at least $u_r+u'_s+1$ if $a\ge l$. For the latter assertion use Lemma 15.40; for the former use the positive-Frobenius condition in the definition of N_A. This tests the image under F directly, so the extra trivial parameter jump of the third extension introduces no additional case.

Apply (15.88) to a first-side move and to $\delta^{-1}(gx'-x')$. By Corollary 15.55 the second vector has the stated depth orders. At every summand the assumed inequality $o(a,u_r)+d'(a,u'_s)\ge R_l(a,u_r+u'_s)$ therefore gives order at least one on the right side. Hence the resulting Tate value ξ has standard order one at every level.

We also verify its membership in the localized Tate module, as required when using these order tests. Clear the finitely many E_a denominators of the three inputs. The positive-Frobenius values of the equivariant trilinear evaluation are finite fixed-field linear combinations of products of invariant-form values, by the tensor comparison theorem. Each product is integral, and each fixed-field coefficient has nonnegative de Rham order. The defining condition for N_A follows. The same reasoning for kernels identifies the vectors over zero with the localized modules of L,L'. Restoring the denominators proves $\xi\in N_{\mathrm{loc},A}(\mathbb{Z}_p(1))$.

By Lemma 15.40, ξ in fact belongs to $N_A(\mathbb{Z}_p(1))$. Lemma 15.22 makes $\tau\xi$ integral. Multiplication by $z\in W(\mathfrak{m}^\flat)$ makes its reduction small: the small subgroup is an ideal in the integral Witt ring. Since $z\tau=\delta$, that reduction is exactly p^κ times the pairing of the move with $gx'-x'$. This proves the pairing condition in the first variable. For the second variable divide $gx-x$ by δ and apply the same argument to g times the second move. Order modules are Galois-stable, so the same inequalities apply.

Lean, formal context. `DeepPoints` (L378-L409), `FitTwistsExist` (L377). □

Lemma 15.64 (A finite set of level-zero steps raises depth)**.** *Let $J\subset\mathbb{Z}$ be finite. If x lies over $p^e c$, has orders o and depth d, and J contains every integer between $d(0,u)$ and a proposed depth $d'(0,u)$, with the upper endpoint excluded, then, provided $d'(a,u)\le d(a,u)$ for $a>0$, there is a point over*

$$p^{e+\sum_{j\in J}c(j)}c$$

with depth d' and at least the original orders o. It also has the improved orders of Lemma 15.62.

Proof. Apply the operators A_j of Lemma 15.61 in increasing order of j. At a current step j, every form whose proposed final depth exceeds j and whose initial depth is at most j has already reached depth j. The raising property gives depth $j+1$ for that form. Forms already deeper retain their depth: the cost property says that $A_jx-p^{c(j)}x$ has those same depth orders, and taking Galois differences preserves their order module. The same argument keeps every other level unchanged. This induction is finite and works simultaneously for all forms and all jumps, without a factor for their number.

The cost property also preserves the original orders. Indeed a vector of orders o is automatically deep for o, since the order modules are Galois-stable. The difference from $p^{c(j)}x$ then lies over zero and has orders o. Projection multiplies the original constant by $p^{c(j)}$ at each step, giving the displayed total exponent. Apply Lemma 15.62 after the last step.

Lean, formal context. `DeepPoints` (L378-L409), `FitTwistsExist` (L377). □

Theorem 15.65 (The bound from the three cokernels at pole order zero)**.** *On the tame chart, use the period matrices specified above, and take κ, η, W, b_i to be nonnegative integers. Let d_0 be the chart bound of Lemma 15.60: it is 0 if $e = 1$ and $v_p(e) + 1$ otherwise, hence at most 1 on this tame chart. At every letter, suppose p^κ kills the level-zero period cokernels of the two cocycle extensions and of the Tate-dual extension associated with their admissible pairing. Suppose the two jump sets lie in opposite windows*

$$[-\eta, \eta] \cup -\varepsilon_i[b_i, b_i + W], \qquad [-\eta, \eta] \cup \varepsilon_i[b_i, b_i + W], \quad \varepsilon_i \in \{1, -1\},$$

and no nonzero multiple of p is within distance $n = 6(\eta + W)$ of 0 or $\pm b_i$. Put $C = (8n + 4)d_0$. Then every tested cup product is killed in continuous cohomology by

$$p^{3\kappa + 2C}. \tag{15.89}$$

In particular the exponent $3(\kappa + (48(\eta + W) + 4)d_0)$ used in (15.21) is sufficient. These are the $a = 0$ constants of the source's three-cokernel theorem.

Proof. Apply Corollary 16.35 with pole order $a = 0$. We have $p^\kappa E^0 = p^\kappa$, and the excluded interval $[-a, 0)$ is empty. The two window hypotheses are precisely (16.32). Thus $n = 6(\eta + W)$, $C = (8n + 4)d_0$, and (16.34) gives $3\kappa + 2C \le 3(\kappa + C)$. For a tame chart subgroup of prime-to-p index, restriction and corestriction give the same annihilator in the ambient group.

Lean, comparison. The formal three-cokernel theorem `isScaledCupCoboundary_of_killsCokernel` also covers a nonpositive set of jumps. Here the gap case alone suffices, using `isScaledCupCoboundary_of_gap`. □

The proof of the general gap estimate in Section 16.5 uses the depth operators, the pairing criteria and the continuous Witt cochain constructed above. Its integral frames are obtained independently in Theorem 16.19.

15.10 The tame local conclusion after the period reconstruction

The three period estimates listed in the preceding tame argument are first deduced from the explicit matrix identities. The following statement records that implication. Theorem 16.19 then constructs its matrices, and Corollary 16.20 removes the extra existence hypothesis while retaining the global coefficient.

Corollary 15.66 (The tame exponent from the explicit frame matrices)**.** *Retain the representation, type, opposite-window and size hypotheses of Theorem 15.18. Assume the matrix input (15.35) exists for the crystalline lattices and cocycle extensions occurring on the common tame chart, including the extension associated with each admissible pairing. Then all three period estimates needed in that theorem hold for the actual form-defined modules, with the correction multiplier restricted to S as in Theorem 15.37. In particular its asserted cup exponent*

$$3r^2(N + 1)^2 + 150(\eta + W + 1)$$

is valid under this matrix-existence hypothesis. More precisely, a sufficient exponent is

$$3r^2(N + 1)^2 + 96(\eta + W) + 8. \tag{15.90}$$

Proof. The standard-exponent module is identified in Lemma 15.40. The initial exponent, height raising, ordinary lowering and Frobenius recurrence are proved in Lemmas 15.41 and 15.42 and Theorems 15.47 and 15.49. The common correction and tame polynomial lowering are Theorem 15.37 and Corollary 15.38. All multipliers used in the descent are $p^t z^j \in S$, so the necessary restriction

on the correction does not weaken any step of Theorem 15.16. That theorem and the same three polynomial-degree estimates as before therefore kill the three cokernels with $\kappa = r^2(N+1)^2$.

The localized period and divided-difference properties are Lemmas 15.52 and 15.54, with all integer orders preserved by Corollary 15.55. The retractions are supplied by Lemmas 15.23 and 15.25. The depth and safe-path construction of Theorem 15.65 now applies. The size hypothesis implies the required gap around 0 and $\pm b$. On the common tame chart its different allowance d_0 is at most 1, so the cup exponent it supplies satisfies

$$3\kappa + 2(48(\eta + W) + 4)d_0 \le 3\kappa + 96(\eta + W) + 8.$$

The corrected Witt cochain in Theorem 15.57 establishes continuity of the primitive, and the common chart has degree prime to p by Lemma 15.17. Restriction and corestriction therefore descend the conclusion without a new p-power. Finally $96(\eta + W) + 8 \le 150(\eta + W + 1)$, which retains the previous global numerical allowance.

Lean, comparison. The three cokernels and the final enlargement of the exponent occur in `CupOrthogonalityBounds` (L532-L598). Here the estimate preceding that enlargement is recorded in (15.90). □

The six period matrices. It remains to construct the six matrices in (15.35). Section 16 constructs an integral basis, proves its determinant is nonzero modulo p, and gives the required factorizations.

16 Integral Frobenius frames and the local defect

We construct compatible period matrices for a lattice and its Tate dual by Frobenius contraction. Bounds on Taylor coefficients at successive filtration jumps then control the denominators in the local cup pairing.

The identities in (15.35) require bounds over the same coefficient ring for the lattice and its Tate dual. We obtain them by de Rham comparison and descent to the Kummer extension.

Keep $A, S, E, \tau, \xi, \mu, B^+, S_B$ as above. Write $A_{\max} = \widehat{A[\xi/p]}_p$; this is the same ring as $\widehat{A[E/p]}_p$ because $E/\xi \in A^\times$. Set $X = \xi/p \in A_{\max}$. We use $B_{\max} = B^+[1/\mu]$, with the convention for B^+ given in the definition of the Kummer chart.

Lemma 16.1 (A direct approximation modulo μ)**.** *The maps*

$$A[1/p] \longrightarrow B^+/\mu B^+, \qquad A[1/p] \longrightarrow B^+/\tau B^+$$

are surjective.

Proof. Since $\xi = \sum_{j=0}^{p-1}(1+\tau)^j$, there is $g \in A$ with $\xi - p = \tau g$. Consequently

$$X - 1 = \tau g/p, \qquad X^2 - X = \mu g/p^2.$$

Every $b \in A_{\max}$ can be written $b = \sum_{i\geq 0} a_i X^i$, where $a_i \in A$ and $a_i \to 0$ p-adically. The polynomial identities

$$X^i - 1 = (X-1)\sum_{j=0}^{i-1} X^j, \qquad X^i - X = (X^2 - X)\sum_{j=0}^{i-2} X^j \quad (i \geq 2)$$

may be summed: the coefficients of the finite sums are integral, so the terms multiplied by a_i tend to zero in $A_{\max}$. Thus, for $c_\tau, c_\mu \in A_{\max}$,

$$b = \sum_{i\geq 0} a_i + (X-1)c_\tau, \qquad b = a_0 + X\sum_{i\geq 1} a_i + (X^2 - X)c_\mu.$$

The displayed constant expressions belong to $A[1/p]$. After inverting p these identities give both surjections.

Lean. `CrystallineRings` (L686). □

Lemma 16.2 (Frobenius kernel intersections in B^+)**.** *One has*

$$\bigcap_{j\geq 0} \ker(\theta\varphi^j : B^+ \to C) = \mu B^+, \qquad \bigcap_{j\geq 1} \ker(\theta\varphi^j : B^+ \to C) = \tau B^+. \tag{16.1}$$

For $b \in B^+$ and $n \geq 0$,

$$(\varphi\tau)^n \mid \varphi b \quad \Longrightarrow \quad \tau^n \mid b \qquad \text{in } B^+.$$

Proof. For the first equality choose $a \in A[1/p]$ with $b - a \in \mu B^+$ by Lemma 16.1. All maps $\theta\varphi^j$, $j \geq 0$, kill μ. If they kill b, they therefore kill a. Clear a power of p and apply Lemma 15.21; it follows that $a \in \mu A[1/p]$, hence $b \in \mu B^+$. The reverse inclusion follows from $\theta\varphi^j\mu = 0$. The second equality is identical, using the approximation modulo τ, the positive values $j \geq 1$, and the second intersection in that lemma.

If $\varphi\tau \mid \varphi b$, applying $\theta\varphi^j$ for $j \geq 0$ shows $\theta\varphi^{j+1}b = 0$, since $\varphi\tau = \mu$. Hence $\tau \mid b$. If $(\varphi\tau)^n \mid \varphi b$ and $n > 0$, write $b = \tau b_1$ and cancel $\varphi\tau$ in the domain B^+. Induction proves the last assertion.

Lean. `CrystallineRings` (L726), `PeriodMatrixBasis` (L110). □

Lemma 16.3 (Uniformly bounded poles remove the localization)**.** *If $b \in B_{\max}$ and $a \geq 0$ satisfy*

$$\varphi^j b \in \mathrm{Fil}^{-a} B_{\mathrm{dR}} \qquad (j \geq 0),$$

then $\mu^a b \in B^+$. In particular $(E\tau)^a b \in B^+$.

Proof. Choose $N \geq 0$ with $c = \mu^{a+N} b \in B^+$. Every $\varphi^j \mu = [\epsilon]^{p^j} - 1$ has de Rham order one, as follows by expanding it in $t = \log[\epsilon]$. If $N > 0$, it follows that $\varphi^j c \in \mathrm{Fil}^N B_{\mathrm{dR}}$ for every $j \geq 0$. Thus all $\theta\varphi^j c$ vanish, and Lemma 16.2 gives $c = \mu c_1$ with $c_1 \in B^+$. Cancel μ and decrease N by one. After finitely many repetitions $\mu^a b \in B^+$. Finally $E\tau$ is associated to μ in A.
Lean, formal context. `FixedRatios` (L47-L85), `FitTwistsExist` (L377). □

This gives an ordinary proof of `pow_smul_mem_range_BPlus`. Only the scalar kernel calculation is used here.

Lemma 16.4 (Frobenius of the chart Eisenstein element)**.** *The element φE is a unit of S_B.*

Proof. Write $E = p + z^e V$, with $V \in S^\times$, and put $X_E = E/p$. Since $\varphi z = z^p$,

$$\frac{\varphi E}{p} = 1 + p^{p-1}(X_E - 1)^p \frac{\varphi V}{V^p}.$$

The second term belongs to $p\widehat{S[E/p]}_p$. The geometric series for its negative therefore gives the inverse of $\varphi E/p$ in that complete ring. Since p is inverted in S_B, the assertion follows, also for $p = 2$.
Lean. `Bicommutant` (L455). □

From crystalline periods to the matrix axioms

Let T be a crystalline lattice of rank d on the chart subgroup and suppose its jumps lie in $[-a, b]$, where $a, b \geq 0$. Choose $P \in \mathrm{GL}_d(B_{\max})$ with invariant columns, and put $k = a + b$. The existence of P is the defining crystalline comparison, not the integral construction to be proved.

We use two precise classical period facts. For the finite chart field $K = K_0(\theta z)$, with $K_0 = \mathrm{Frac}\,\Lambda$,

$$B_{\max}^{G_K} = K_0, \qquad B_{\mathrm{dR}}^{G_K} = K.$$

The positive-period convention is compatible with the crystalline one by Colmez, §III.2; in particular $\varphi(B_{\max}) \subset B_{\mathrm{cris}} \subset B_{\max}$. Indeed an element of $B_{\max}$ fixed by G_K has Frobenius in $B_{\mathrm{cris}}^{G_K} = K_0$, and bijectivity of Frobenius on K_0, together with its injectivity on $B_{\max}$, puts the element itself in K_0. The standard fixed-field calculation for B_{cris} is Brinon–Conrad, §9.1. The second fact is filtered de Rham comparison: a basis of $D_{\mathrm{dR}}(T[1/p])$ adapted to its filtration has a period matrix

$$PC = W\,\mathrm{diag}(t^{u_1}, \ldots, t^{u_d}), \qquad C \in \mathrm{GL}_d(K), \quad W \in \mathrm{GL}_d(B_{\mathrm{dR}}^+), \quad -a \leq u_i \leq b. \tag{16.2}$$

This is the filtration compatibility in Brinon–Conrad, Propositions 6.3.2 and 6.3.7. It is the same filtered comparison used in Lemma 15.39. All positive Frobenius twists of P have invariant columns and admit the same statement with the same bounds.

Theorem 16.5 (Integrality and Frobenius conditions on the period matrix)**.** *There are matrices U, W_0, F, F' over B^+ and a unit $v \in A^\times$, with the entries of F in S_B, such that*

$$U = (E\tau)^a P, \qquad UW_0 = (E\tau)^k I_d, \tag{16.3}$$

$$F'F = I_d, \qquad \varphi U = vUF. \tag{16.4}$$

For each g fixing Λ pointwise and θz, there is $R_g \in \mathrm{GL}_d(A)$ such that

$$R_g\, g(U) = U. \tag{16.5}$$

For every S-linear $f : A \to A$, let $\widehat{f} : B^+ \to B^+$ be the extension of Lemma 15.23, applied entrywise to vectors. Then

$$Uy \in E^k(B^+)^d \quad \Longrightarrow \quad U\widehat{f}(y) \in E^k(B^+)^d \qquad (y \in (B^+)^d). \tag{16.6}$$

Proof. By (16.2), every entry of every $\varphi^j P$ has de Rham order at least $-a$. Inverting the equality shows that every entry of $(\varphi^j P)^{-1}$ has order at least $-b$. Lemma 16.3 therefore puts

$$U = (E\tau)^a P, \qquad W_0 = (E\tau)^b P^{-1}$$

in $\mathrm{Mat}_d(B^+)$ and proves (16.3). In particular U is nonsingular over the fraction field of B^+.

Both P and φP have invariant columns. Thus $X_0 = P^{-1}\varphi P$ and X_0^{-1} are fixed by G_K; their entries lie in $K_0 \subset S_B$ by the fixed-field calculation. Put

$$\omega = \frac{\mu}{E\tau} \in A^\times, \qquad v = \omega^a, \qquad F = (\varphi E)^a X_0, \qquad F' = (\varphi E)^{-a} X_0^{-1}.$$

Lemma 16.4 shows that these matrices belong to the asserted rings. Since $\varphi(E\tau) = (\varphi E)\mu = \omega(\varphi E)(E\tau)$, the equations (16.4) follow.

An element g as in the theorem lies in G_K. The ideals (E) and (gE) are the same kernel of θ. Also (τ) is g-stable by the second intersection in Lemma 15.21, since g commutes with φ and θ is equivariant. Applying g^{-1} gives equality of the ideals; since A is a domain, $g(E\tau)$ and $E\tau$ are associates. Let $r_g = (E\tau)/g(E\tau) \in A^\times$. For the integral representation matrix $\rho(g)$ set $R_g = r_g^a \rho(g)$. Invariance $\rho(g)gP = P$ gives (16.5).

It remains to check (16.6); this is not a formal consequence of the Frobenius equation. Use (16.2) and write $q = C^{-1}y \in (B_{\mathrm{dR}})^d$. Since W is invertible over B_{dR}^+ and $E\tau$ has de Rham order one,

$$Uy \in \mathrm{Fil}^k(B_{\mathrm{dR}})^d \quad \Longleftrightarrow \quad q_i \in \mathrm{Fil}^{\,k-a-u_i} B_{\mathrm{dR}} \quad (1 \le i \le d).$$

The extension of $\widehat{f}$ to B_{dR} is K-linear and preserves every filtration step by Lemma 15.23. Hence $C^{-1}\widehat{f}(y) = \widehat{f}(q)$ satisfies the same inequalities. It follows that $U\widehat{f}(y) \in \mathrm{Fil}^k(B_{\mathrm{dR}})^d$. Its entries already lie in B^+, where (15.32) identifies this intersection with $E^k B^+$. This proves the final axiom.
Lean. `PeriodMatrixBasis` (L954). □

The theorem proves the assertion called `IsChartOf.exists_isPeriodMatrixOf` in the pinned source. We next construct the integral basis for these matrices and then prove its determinant is not divisible by p. The two tasks are kept separate because nonsingularity over a fraction field alone does not control integral denominators.

16.1 Constructing the period basis by Frobenius contraction

We construct the basis used in the determinant calculation. The notation A, S, E, τ, ξ, B^+ and S_B is that of the preceding period calculation. Put $R = A_{\max}$ and $X = \xi/p \in R$. Since E and ξ are associated in A, the two presentations $R = \widehat{A[E/p]}_p = \widehat{A[\xi/p]}_p$ agree. The construction follows the period-basis argument; all the boundedness and freeness steps required by that argument are proved here.

Lemma 16.6 (Integral bounds in the maximal period ring)**.** *Write $J = (p, \xi)A$. The following statements hold.*

1. *$R/pR \simeq (A/J)[X]$. Consequently a monic polynomial in X is a non-zero-divisor modulo every $p^m R$.*
2. *For $a \in A$, one has*
$$a \in p^m R \quad \Longleftrightarrow \quad a \in J^m.$$
3. *For $r \in R$ and $n \geq 0$ there are $a \in A$ and $w \in R$ such that $p^n r = a + \tau^n w$.*
4. *If $e \in A$ divides μ and $e^n b \in R$ for $b \in B^+$, then $p^{2n} b \in R$.*
5. *If $\bar{\delta} \neq 0$ in A/pA, then δA is both p-adically closed and J-adically closed, and $A/\delta A$ has no p-torsion.*

Proof. The presentation $A[X]/(pX - \xi)$ is p-torsion-free. Indeed, reducing $pf = (pX - \xi)g$ modulo p gives $\bar{\xi}\bar{g} = 0$ in $(A/pA)[X]$, so $g = pg_1$ and one cancels p. Its p-adic completion is again p-torsion-free: if $pr = 0$, reduction modulo p^m of this presentation shows $r \in p^{m-1}R$ for every m. Completing does not change the quotient modulo p, which proves the first assertion. A monic polynomial is regular in a polynomial ring over any commutative coefficient ring. Induction and cancellation of p then give its regularity modulo p^m.

Certainly $J^m \subset p^m R$, since $\xi = pX$. For the reverse inclusion, induct on m. If $a \in A \cap p^{m+1}R$, applying $\theta : R \to \mathcal{O}_C$ and lifting by the surjectivity of $\theta : A \to \mathcal{O}_C$ gives

$$a = \xi c + p^{m+1} d, \qquad c, d \in A.$$

Therefore $pXc \in p^{m+1}R$ and hence $Xc \in p^m R$. Regularity of the monic polynomial X modulo p^m gives $c \in p^m R$. The induction hypothesis gives $c \in J^m$, so $a \in J^{m+1}$.

The proof of Lemma 16.1 writes each $r \in R = A_{\max}$ as $a + (X-1)w$ with $a \in A$ and $w \in R$. Since $p(X-1) = \xi - p \in \tau A$, this gives $pR \subset A + \tau R$. Multiplying this inclusion by p successively gives $p^n R \subset A + \tau^n R$ for every $n \geq 0$.

For the fourth assertion, choose $t \in A$ with $et = \xi(\xi - p)$; this is possible because $e \mid \mu = \xi\tau$ and $\tau \mid \xi - p$. Set $G = X(X-1)$, so $p^2 G = et$. If $c = e^n b \in R$, choose m with $b' = p^m p^{2n} b \in R$. In R,

$$G^n b' = p^m t^n c.$$

The first assertion, applied to the monic polynomial $(X(X-1))^n$, gives $b' \in p^m R$. Cancel p^m in B^+ to obtain $p^{2n} b \in R$.

Finally, $\bar{\delta} \neq 0$ and the fact that A/pA is a domain give

$$\delta c \in p^m A \quad \Longrightarrow \quad c \in p^m A$$

by induction. If $a - \delta c_m \in p^m A$, then $c_{m+1} - c_m \in p^m A$, so completeness of A gives $a \in \delta A$. This proves p-adic closedness; reducing $pa = \delta b$ modulo p and cancelling p proves the absence of p-torsion in $A/\delta A$. The positive valuation of $\bar{\xi}$ in the rank-one valuation ring A/pA permits a choice of $q \geq 1$ such that $\bar{\delta} \mid \bar{\xi}^q$. Thus $J^q \subset (p, \delta)$ and $J^{qm} \subset (p, \delta)^m \subset (p^m, \delta)$ for $m \geq 1$. The p-adic closedness just proved therefore implies J-adic closedness. For the third assertion the source divides by $X - b$ for general b; here the case $b = 1$ already occurs in Lemma 16.1.

Lean, formal context. `CrystallineRings` (L423), `PeriodMatrixBasis` (L339-L670), `FitTwistsExist` (L377). □

Lemma 16.7 (Frobenius contraction with a uniform denominator)**.** *Let $\mathcal{V} \subset R$ and $u \in A$. Suppose that for every $b \in \mathcal{V}$ there is $c \in \mathcal{V}$ with*

$$\xi^k b = u\varphi(c).$$

Then $p^{k+1} b \in A$ for every $b \in \mathcal{V}$.

Proof. For every $m \geq 0$, uniformly for $b \in \mathcal{V}$, we construct $a_m \in A$ and $r_m \in R$ satisfying

$$p^{k+1}b = a_m + \tau^{k+1}p^m r_m. \tag{16.7}$$

For $m = 0$ this is the third assertion of Lemma 16.6. Suppose the assertion at m is known for all members of $\mathcal{V}$ and choose c as in the hypothesis. Apply it to c, apply Frobenius, and multiply by u. The identities $\varphi\tau = \xi\tau$ and $\xi = pX$ give

$$u\varphi(a_m(c)) = \xi^k \left(p^{k+1}b - \tau^{k+1}p^{m+1}uX\varphi(r_m(c))\right).$$

Here $\varphi R \subset R$; this follows directly from $\varphi\xi - p \in \mu A \subset pR$, which puts φX in R. Divisibility by ξ^k of an element of A in B^+ descends to A: for one factor apply θ and use $\ker(\theta|_A) = \xi A$, then cancel and repeat. Thus $u\varphi(a_m(c)) = \xi^k a_{m+1}$ for some $a_{m+1} \in A$. Cancellation proves (16.7) at $m + 1$.

For $l \geq m$, the difference $a_l - a_m$ lies in $A \cap p^m R = J^m$, by Lemma 16.6. Here $J = (p, E) = (p, z^e)$, so its powers are cofinal with those of (p, z); completeness of A from Lemma 15.30 gives a limit $a \in A$. Since $J^m R \subset p^m R$ and R is p-adically separated, (16.7) implies $a = p^{k+1}b$. This proves the assertion without a finiteness assumption on $\mathcal{V}$. The argument is the contraction lemma in the period-ring construction.

Lean. `CrystallineRings` (L1019-L1090). □

Lemma 16.8 (Finite freeness from regularity and a rank bound)**.** *Let $S = \Lambda[[z]]$, with maximal ideal $\mathfrak{m} = (p, z)$, and let M be an $\mathfrak{m}$-adically separated S-module. Suppose p acts injectively on M, z acts injectively on M/pM, and every S-linearly independent finite subset of M has at most d elements. Then M is finite free of rank at most d.*

Proof. Choose lifts $b_i \in M$ of a basis of $M/\mathfrak{m}M$. Their images in M/pM are independent over $S/pS = (\Lambda/p\Lambda)[[z]]$. Indeed, in a nonzero finite relation divide the coefficients by their largest common power of z. Injectivity of z on M/pM preserves the relation, whose reduction modulo z now contradicts the choice of the b_i. Hence every finite S-relation among the b_i has all coefficients in pS. Cancel p on M and iterate. Since $\bigcap_n p^n S = 0$, the b_i are independent and their number is some $r \leq d$.

Let $\beta : S^r \to M$ send the standard basis to the b_i. Surjectivity modulo $\mathfrak{m}$ gives $\mathfrak{m}^n M = \beta(\mathfrak{m}^n S^r) + \mathfrak{m}^{n+1}M$ for every n. For each $x \in M$, successively subtract these lifts to obtain a series in S^r whose partial sums map to x modulo successive powers of $\mathfrak{m}$. The series converges by completeness of S^r, and its sum maps to x by separation of M. Thus β is an isomorphism. The source constructs continuous coordinate maps for a general complete coefficient ring; here S/pS is a discrete valuation ring, so reduction modulo p already gives independence.

Lean. `AdicCompletion` (L583-L604). □

Lemma 16.9 (Coordinate expansion in the period ring)**.** *Take the topological S-basis (b_i) of A constructed in Lemma 15.30, and write $\lambda_i : A \to S$ for its coordinates. Each extends to an S_B-linear map $\widehat{\lambda}_i : B^+ \to S_B$. For every $y \in B^+$ and every integer $L \geq 0$, some finite set I satisfies*

$$y - \sum_{i \in I} b_i \widehat{\lambda}_i(y) \in p^L R.$$

One finite set can be chosen for finitely many elements y. The coordinates λ_i separate elements of A. Moreover S_B equals the bicommutant used in the source: the elements of B^+ whose multiplication operators commute with every S-linear endomorphism $B^+ \to B^+$ carrying R into itself.

Proof. Lemma 15.23 extends λ_i through $A[E/p]$ and its completion, and its values lie in the completion of $S[E/p]$, with p inverted. On A, the finite coordinate sums converge for (p,z). In R the identity $E = p + z^e V$ gives $z^e \in pR$, so $(p,z)^{eL}R \subset p^L R$. The same sums thus converge p-adically after mapping into R. For a polynomial in E/p the assertion follows term by term from the assertion for its finitely many coefficients. For a general element of R, first approximate by such a polynomial modulo $p^L R$. Every finite sum of the operators $b_i\widehat{\lambda}_i$ preserves $p^L R$, so the approximation remains valid. Finally clear one power of p for an element of B^+. Enlarging the finite set does not worsen any of these error estimates, and therefore handles finitely many elements simultaneously. Separation by the coordinates is part of the completed direct-sum description in the proof of Lemma 15.30. For the last assertion, every such endomorphism commutes with multiplication by p^{-1}, because p is a unit in B^+, and therefore with $S[E/p]$. Its preservation of R makes it p-adically continuous there, so it commutes with the completion of $S[E/p]$ and with S_B. Conversely the extension of the distinguished coordinate $\lambda_0 : A \to S$, with $\lambda_0(1) = 1$, has image in S_B and preserves R, so it belongs to this family of endomorphisms. If multiplication by $b \in B^+$ commutes with this extension, then $\widehat{\lambda}_0(b) = b\widehat{\lambda}_0(1) = b$, whence $b \in S_B$. This proves both inclusions.

Lean. `Bicommutant` (L108), `Bicommutant` (L411). □

For the remaining lemmas let $U \in \mathrm{Mat}_d(B^+)$, $k \geq 0$, and $v \in A^\times$ satisfy the following three properties:

$$\begin{aligned} &UW = (E\tau)^k I_d \quad \text{for some } W \in \mathrm{Mat}_d(B^+), \\ &\varphi U = vUF, \qquad F \in \mathrm{Mat}_d(S_B), \qquad F'F = I_d \quad \text{for some } F' \in \mathrm{Mat}_d(B^+), \\ &Uy \in E^k(B^+)^d \quad \Longrightarrow \quad U\widehat{f}(y) \in E^k(B^+)^d \quad \text{for every } S\text{-linear } f : A \to A. \end{aligned} \tag{16.8}$$

In the last line f acts coordinatewise and $\widehat{f}$ is its extension from Lemma 15.23. These are precisely the three fields of a period matrix used in its basis construction; the Galois field is needed only in the subsequent determinant argument. The first identity makes U injective over B^+, and also gives $WU = (E\tau)^k I_d$ by cancellation over its fraction field.

Lemma 16.10 (Uniform integral coefficients after all scalar maps)**.** *Under* (16.8)*, there exists* $N \geq 0$ *such that, whenever*

$$Uy = E^k x, \qquad U\widehat{f}(y) = E^k w, \qquad x \in A^d,\ y, w \in (B^+)^d,$$

one has $p^N w \in A^d$*. The integer* N *is independent of* x, y, w *and of the* S*-linear map* f*.*

Proof. First choose $r_U, r_W \geq 0$ with $p^{r_U}U$ and $p^{r_W}W$ integral over R. From $WUy = (E\tau)^k y$ and $Uy = E^k x \in R^d$, the fourth assertion of Lemma 16.6, with $e = E\tau$ (which is associated to μ), gives $p^{2k+r_W} y \in R^d$. The map $\widehat{f}$ preserves R. Apply the same assertion with $e = E$ to the second displayed equation to obtain

$$p^m w \in R^d, \qquad m = 4k + r_U + r_W. \tag{16.9}$$

This bound is already uniform in all the displayed choices.

We next show that this collection of vectors admits Frobenius preimages with the same properties. Put

$$x' = \varphi^{-1}(vE^k x) \in A^d.$$

If $Ua = rx'$ with $r \in A$ and $a \in (B^+)^d$, applying Frobenius and cancelling the injective matrix U gives

$$F\varphi(a) = \varphi(r)y. \tag{16.10}$$

Take $a = Wx'$ and $r = (E\tau)^k$, and multiply by F'. Then $\varphi(\tau)^k$ divides every coordinate of $\varphi(Wx')$ in B^+. The scalar kernel calculation of Lemma 16.2 gives

$$\varphi(\tau) \mid \varphi(b) \quad \Longrightarrow \quad \tau \mid b \qquad (b \in B^+) :$$

indeed, the hypothesis makes $\theta\varphi^j(b) = 0$ for every $j \geq 1$. Apply this one-factor assertion repeatedly, cancelling $\varphi\tau$ at each step, to obtain $Wx' = \tau^k y'$ with $y' \in (B^+)^d$. Consequently $Uy' = E^k x'$.

Set $f' = \varphi^{-1} f \varphi : A \to A$; it is S-linear because $\varphi S \subset S$. Its extension satisfies

$$\varphi \widehat{f'} = \widehat{f} \varphi.$$

This is first checked on a term $a(E/p)^j$ using $\varphi f' = f\varphi$ and S_B-linearity; p-adic continuity proves it on B^+. Let $U\widehat{f'}(y') = E^k w'$, which exists by (16.8). Formula (16.10) for $a = y'$ is $F\varphi(y') = \varphi(E)^k y$. Apply $\widehat{f}$ to it. Its S_B-linearity, including commutation with the entries of F and with φE, gives

$$\varphi(w') = vE^k w.$$

This proves the required uniform Frobenius descent.

Write $\xi = \eta E$, with $\eta \in A^\times$. Let $\mathcal{V}$ consist of all coordinates $p^m w_j$ arising in (16.9). It lies in R, and the preceding descent gives, for every $b \in \mathcal{V}$, some $c \in \mathcal{V}$ with

$$\xi^k b = \eta^k v^{-1} \varphi(c).$$

Lemma 16.7 gives $p^{k+1}\mathcal{V} \subset A$. Thus one can take $N = m + k + 1 = 5k + r_U + r_W + 1$.

Lean. `PeriodMatrixBasis` (L444). □

Lemma 16.11 (A finite free module of integral chart solutions)**.** *Under* (16.8)*, define*

$$P = \{x \in A^d : \ Uy = E^k x \text{ for some } y \in S_B^d\}.$$

Then P is finite free over S of rank at most d. There are $Q \in \mathrm{Mat}_d(A)$ and $Z \in \mathrm{Mat}_d(S_B)$ such that

$$UZ = E^k Q, \qquad P = QS^d.$$

At this stage the matrices are allowed to have zero columns.

Proof. For $x \in P$, its solution $y(x) \in S_B^d$ is unique by injectivity of U. A finite family of vectors $x_j \in P$ and the family $y(x_j)$ have exactly the same A-linear relations: apply the injective map U and cancel E^k. If the x_j are independent over S, an A-linear relation $\sum_j a_j x_j = 0$ therefore gives $\sum_j a_j y(x_j) = 0$. Apply each extended coordinate $\widehat{\lambda}_i$ from Lemma 16.9. Because $y(x_j) \in S_B^d$, this gives $\sum_j \lambda_i(a_j) y(x_j) = 0$, hence $\sum_j \lambda_i(a_j) x_j = 0$. Independence over S gives $\lambda_i(a_j) = 0$ for all i, j, and coordinate separation gives $a_j = 0$. The x_j are thus independent over A, so there can be at most d of them.

The module P is p-saturated in A^d: if $px \in P$ and $x \in A^d$, divide its solution by p in S_B. If $zx = px'$ for $x, x' \in P$, reduction modulo p in the domain A/pA shows $x \in pA^d$, since $\bar{z} \neq 0$. Saturation then gives $x \in pP$. Thus z is regular on P/pP, and p is regular on P. Moreover P is (p, z)-adically separated since it embeds in A^d. Lemma 16.8 now makes P finite free of rank $r \leq d$.

Choose a basis $q_1, \ldots, q_r$ of P, take their unique solutions $z_j = y(q_j)$, and add zero columns until both families have d columns. The resulting matrices Q, Z have all the asserted properties.

Lean. `PeriodMatrixBasis` (L475). □

Lemma 16.12 (Divisibility detected by the chart solutions)**.** *Let Q, Z be as in Lemma 16.11. Suppose $x \in A^d$, $y \in (B^+)^d$, and $Uy = E^k x$. If $\delta \in A$ has $\bar{\delta} \neq 0$, and row vectors $l, g \in A^{1\times d}$ satisfy $lQ = \delta g$, then*

$$\delta \mid lx \quad \text{in } A.$$

Proof. For every coordinate i, write

$$U\widehat{\lambda}_i(y) = E^k w_i, \qquad x_i = p^N w_i \in A^d,$$

using Lemma 16.10 with one common N. Since $p^N\widehat{\lambda}_i(y) \in S_B^d$, we have $x_i \in P$, and hence $x_i = Qs_i$ for some $s_i \in S^d$.

Choose $c \geq 0$ such that

$$Ub = E^k t, b \in R^d \quad \Longrightarrow \quad p^c t \in R^d;$$

one can take $c = r_U + 2k$ by Lemma 16.6. For each $m \geq 0$, choose a finite coordinate set I with

$$y - \sum_{i\in I} b_i \widehat{\lambda}_i(y) \in p^{m+c} R^d.$$

Then the integral vector

$$t_m = p^N x - \sum_{i\in I} b_i x_i \in A^d$$

lies in $p^m R^d$, by applying the displayed bound to the error and the factor p^N. By Lemma 16.6, it therefore lies in $J^m A^d$. Using $x_i = Qs_i$ and $lQ = \delta g$ gives

$$p^N lx - \delta \sum_{i\in I} b_i g s_i \in J^m.$$

The ideal δA is J-adically closed by the same lemma. Thus $p^N lx \in \delta A$. The absence of p-torsion in $A/\delta A$ cancels p^N and proves the assertion. This completes the approximation step without assuming an integral change of basis in advance.

Lean. `PeriodMatrixBasis` (L624). □

Theorem 16.13 (An integral basis of crystalline periods)**.** *Under (16.8), there exist $Q \in \mathrm{Mat}_d(A)$ and $Z \in \mathrm{Mat}_d(S_B)$ with the following properties:*

$$\begin{gathered} \det Q \neq 0, \qquad UZ = E^k Q, \\ \{x \in A^d : \ Uy = E^k x \text{ for some } y \in S_B^d\} = QS^d, \\ Uy = E^k x, \quad x \in A^d, \ y \in (B^+)^d, \quad \bar{\delta} \neq 0, \quad lQ = \delta g \quad \Longrightarrow \quad \delta \mid lx. \end{gathered} \tag{16.11}$$

In particular the module in the middle line has rank exactly d over S, and the columns of Q are a basis.

Proof. Lemmas 16.11 and 16.12 give all assertions except $\det Q \neq 0$. If Q were singular, clearing denominators in a left kernel over $\mathrm{Frac}(A)$ would give a nonzero row $l \in A^{1\times d}$ with $lQ = 0$. For each $a \in A^d$, the vector Wa solves

$$U(Wa) = E^k(\tau^k a).$$

For every $n \geq 0$, apply Lemma 16.12 with $x = \tau^k a$, $\delta = z^n$, and $g = 0$. It gives $z^n \mid \tau^k la$. Since $\bigcap_n z^n A \subset \bigcap_n (p, z)^n A = 0$, we obtain $\tau^k la = 0$. The element τ is nonzero in the domain A, so $la = 0$ for every $a \in A^d$, contradicting $l \neq 0$. The rank assertion follows, and the padded zero columns in Lemma 16.11 are consequently absent.

Lean. `PeriodMatrixBasis` (L641). □

Theorem 16.13 deduces the conditions defining a period basis (`IsPeriodBasis`) from the three stated properties of the period matrices. No determinant claim modulo p has been used in its proof. That stronger determinant claim, and the construction of the two inverse matrices required by (15.35), belong to the next argument.

16.2 The determinant obstruction and construction of the integral frame

Let $k = \mathbb{F}_{p^f}$ and let $K = \mathrm{Frac}(W(k))(\varpi)$ be the chart field, with uniformizer $\varpi = \theta(z)$. Indeed, Weierstrass preparation of $E = p + z^e V$, $V \in S^\times$, gives an Eisenstein polynomial of degree e having ϖ as a root. Let $\pi : A \to R = \mathcal{O}_{C^\flat}$ be reduction modulo p, and put $\bar{z} = \pi(z)$. Since $z = [\bar{z}]$, the elements $\varpi_n = \theta\varphi^{-n}(z)$ satisfy $\varpi_0 = \varpi$ and $\varpi_{n+1}^p = \varpi_n$. An element of the absolute Galois group fixes $S = W(k)[[z]]$ pointwise exactly when it fixes $W(k)$ and all the ϖ_n. Its pointwise stabilizer is therefore $H_\infty = G_{K_\infty}$, where $K_\infty = \bigcup_{n\ge 0} K(\varpi_n)$.

Under the embedding $F = k((\bar{z})) \subset C^\flat$, the field-of-norms theorem identifies the action of H_∞ on the separable closure of F in $C^\flat$ with the full absolute Galois group of F [Win83]. For this particular embedding, see Brinon–Conrad, diagram (11.1.3) and Theorem 11.1.2.

Theorem 16.14 (Frobenius prevents a singular reduction of the basis)**.** *Let $Q \in \mathrm{Mat}_d(A)$ satisfy*

$$t \in S^d, \quad Qt \in pA^d \quad \Longrightarrow \quad t \in pS^d. \tag{16.12}$$

Suppose that

$$QB = \rho\varphi(Q), \qquad B \in \mathrm{Mat}_d(S), \quad \pi(\det B) \neq 0, \quad \rho \in \mathrm{Mat}_d(A), \tag{16.13}$$

and that for every $g \in H_\infty$ there is $\rho_g \in \mathrm{Mat}_d(A)$ with $Q = \rho_g g(Q)$. Then $\pi(\det Q) \neq 0$.

Proof. Write bars for reduction modulo p, and let $W = \ker(\bar{Q} : (C^\flat)^d \to (C^\flat)^d)$. If W contained a nonzero vector in F^d, clearing denominators would give one in $k[[\bar{z}]]^d$. Lift it to $t \in S^d$. Then $Qt \in pA^d$, contradicting (16.12). Thus $W \cap F^d = 0$.

Frobenius on $C^\flat$ takes p-th powers. Denote its coordinatewise action by $[p]$. Reduction of (16.13) gives $\bar{B}W^{[p]} \subset W$. Since $\bar{B}$ is invertible over F and $C^\flat$ is perfect, the dimensions agree. Likewise, $Q = \rho_g g(Q)$ gives $gW \subset W$, hence $gW = W$.

Suppose $j = \dim W > 0$. Choose a nonzero vector x in the line $\bigwedge^j W$, expressed in the exterior basis, and set $D = \bigwedge^j \bar{B}$. Then $Dx^{[p]} = \lambda x$ for some $\lambda \neq 0$. As $C^\flat$ is algebraically closed, scaling x by a $(p-1)$-st root of λ^{-1} gives $Dx^{[p]} = x$. Its coordinates are separable algebraic over F. Indeed, $X_i^p = (D^{-1}X)_i$ define a finite F-algebra, since reduction by these relations lowers total degree until every exponent is less than p. The Jacobian matrix is $-D^{-1}$, so this algebra is étale.

The ratios of the coordinates of x to any one nonzero coordinate are fixed by H_∞, which preserves the line $\bigwedge^j W$. The field-of-norms theorem therefore puts the ratios in F. Choose a basis matrix for W and normalize a nonzero j-row minor to the identity. By Cramer's rule the entries of this matrix are signed ratios of the same Plücker coordinates, so its columns belong to $W \cap F^d$. This contradicts $j > 0$.

Lean, comparison. The formal proof uses a kernel vector of minimal support and approximates its coordinates in the tilt. Here Frobenius makes the Plücker coordinates separable after rescaling, and the field of norms gives descent. For the chart and determinant, see `z_eq_teichmuller` and `not_dvd_det_of_frobenius`. □

The basis and its Frobenius transform. Let $k \ge 0$, $U \in \mathrm{Mat}_d(B^+)$, and suppose the period-matrix construction supplies

$$UW = (E\tau)^k I, \qquad \varphi(U) = vUF, \qquad v \in A^\times, \quad F \in \mathrm{Mat}_d(S_B). \tag{16.14}$$

For every element g fixing Λ and $\theta(z)$, suppose $\rho_g g(U) = U$ for some integral matrix ρ_g. Assume there are matrices $Q \in \mathrm{Mat}_d(A)$ and $Z \in \mathrm{Mat}_d(S_B)$ such that $\det Q \neq 0$, $UZ = E^k Q$, and the following two properties:

$$x \in A^d, y \in S_B^d, Uy = E^k x \implies x \in QS^d, \tag{16.15}$$

$$x \in A^d, y \in (B^+)^d, Uy = E^k x,\ \pi(\delta) \neq 0, lQ = \delta g \implies \delta \mid lx. \tag{16.16}$$

In the second line $\delta \in A$ and l, g are integral row vectors. These are precisely the span and denominator outputs of Theorem 16.13. They do not already assert that Q spans all integral solutions over A.

Lemma 16.15 (Transport of a period basis by Frobenius and Galois)**.** *Under* (16.14)–(16.16)*, the matrix Q is p-saturated over S, and there is $B \in \mathrm{Mat}_d(S)$ with*

$$\varphi(Q) = vQB, \qquad \pi(\det B) \neq 0, \qquad \pi(\det Q) \neq 0. \tag{16.17}$$

Proof. If $Qs = pw$ with $s \in S^d$ and $w \in A^d$, then $U(Zs/p) = E^k w$, with $Zs/p \in S_B^d$. By (16.15), $w = Qs'$ for some $s' \in S^d$. Since $\det Q \neq 0$ and A is a domain, cancellation gives $s = ps'$. This proves (16.12).

The elements p and $\varphi(E)$ are units in S_B, by its definition and Lemma 16.4. The formula in that lemma also shows that Frobenius preserves S_B. Applying Frobenius to $UZ = E^k Q$ gives

$$U\left((E/\varphi(E))^k F\varphi(Z)\right) = E^k v^{-1}\varphi(Q).$$

The parenthesized matrix has entries in S_B, so applying (16.15) column by column gives $B \in \mathrm{Mat}_d(S)$ with $QB = v^{-1}\varphi(Q)$.

If $p \mid \det B$, determinants in this identity show that $p^n \mid \det Q$ implies $p^{n+1} \mid \varphi(\det Q)$, and hence $p^{n+1} \mid \det Q$, since Frobenius is an automorphism fixing p. Induction and p-adic separation would give $\det Q = 0$. Thus $\pi(\det B) \neq 0$.

For $g \in H_\infty$, continuity shows that g fixes S_B pointwise, so $g(Z) = Z$ and $g(E) = E$. Consequently

$$E^k Q = UZ = \rho_g g(U) g(Z) = E^k \rho_g g(Q),$$

and cancellation gives $Q = \rho_g g(Q)$. Theorem 16.14, with $\rho = v^{-1}I$, now gives $\pi(\det Q) \neq 0$.

Lean, formal context. `PeriodMatrixFrame` (L51-L247), `FitTwistsExist` (L377). □

Lemma 16.16 (All integral solutions are spanned over A)**.** *Under the preceding hypotheses, if $X \in \mathrm{Mat}_d(A)$ and $Y \in \mathrm{Mat}_d(B^+)$ satisfy $UY = E^k X$, there is $L \in \mathrm{Mat}_d(A)$ with $X = QL$.*

Proof. Apply (16.16) to each column of X, with $\delta = \det Q$, and to each row of $\mathrm{adj}(Q)$. The required row identity is $\mathrm{adj}(Q)Q = (\det Q)I$; the condition $\pi(\det Q) \neq 0$ was proved in Lemma 16.15. Thus every entry of $\mathrm{adj}(Q)X$ is divisible by $\det Q$ in A. Write $\mathrm{adj}(Q)X = (\det Q)L$, multiply by Q, and cancel the nonzero scalar $\det Q$. This gives $X = QL$.

Lean, formal context. `PeriodMatrixFrame` (L51-L247), `FitTwistsExist` (L377). □

Lemma 16.17 (Approximation modulo the product of the two periods)**.** *For every $k \geq 0$ the natural map induces an isomorphism*

$$A[1/p]/(E\tau)^k A[1/p] \ \simeq\ B^+/(E\tau)^k B^+. \tag{16.18}$$

In particular every $b \in B^+$ has an expression $b = a + (E\tau)^k w$ with $a \in A[1/p]$ and $w \in B^+$.

Proof. For E^k this is (15.32). Lemma 15.31, including its proof of the surjective map q_τ, gives $B^+/\tau B^+ \simeq A[1/p]/\tau A[1/p]$. It also gives $A[1/p] \cap \tau B^+ = \tau A[1/p]$. Induction now gives the same assertion for τ^k: lift the constant term modulo τ, divide the difference by τ, and repeat; injectivity follows by the same successive cancellation.

Since $E = u\xi$ with $u \in A^\times$ and $\xi = p + \tau h_0$, one has $p \in (E, \tau)A$. Thus E and τ, and hence E^k and τ^k, are comaximal in $A[1/p]$ and in B^+. The Chinese remainder isomorphisms identify the two sides of (16.18) with the corresponding products of the E^k- and τ^k-quotients. The isomorphisms just proved identify those products. For $k = 0$ both quotients are zero and the final assertion is immediate.

Lean, formal context. `PeriodMatrixFrame` (L51-L247), `FitTwistsExist` (L377). □

Theorem 16.18 (Construction of the integral period matrices)**.** *Under* (16.14)–(16.16)*, there are* $Q' \in \mathrm{Mat}_d(A)$ *and* $Y \in \mathrm{Mat}_d(B^+)$ *such that*

$$QQ' = \tau^k I, \qquad UZ = E^k Q, \qquad YZ = E^k I. \tag{16.19}$$

The same Q *satisfies* (16.17)*. For every* g *fixing* Λ *and* $\theta(z)$ *there is* $G_g \in \mathrm{Mat}_d(A)$ *with* $g(Z) = ZG_g$*.*

Proof. Apply Lemma 16.16 to $UW = (E\tau)^k I = E^k(\tau^k I)$ to obtain $Q' \in \mathrm{Mat}_d(A)$ with $QQ' = \tau^k I$.

Apply Lemma 16.17 entrywise to U, writing

$$U = V + (E\tau)^k R, \qquad V \in \mathrm{Mat}_d(A[1/p]), \quad R \in \mathrm{Mat}_d(B^+).$$

Choose n with $X = p^n V$ integral. Since $(E\tau)^k I = UW$, this says $X = Up^n(I - WR)$, and therefore $U(E^k p^n(I - WR)) = E^k X$. Lemma 16.16 gives $X = QL_0$ with $L_0 \in \mathrm{Mat}_d(A)$. Hence

$$U = Q\left(p^{-n}L_0 + E^k Q' R\right) = QY \quad \text{for } Y \in \mathrm{Mat}_d(B^+).$$

The equation $UZ = E^k Q$ becomes $QYZ = QE^k I$. Because $\det Q \neq 0$ in the domain B^+, cancellation gives $YZ = E^k I$.

For the final assertion, $g(E) = u_g E$ with $u_g \in A^\times$, since g preserves $\ker\theta$ and E generates it. The period-matrix Galois relation gives

$$Ug(Z) = \rho_g g(UZ) = E^k u_g^k \rho_g g(Q).$$

The rightmost matrix $u_g^k \rho_g g(Q)$ is integral. Lemma 16.16 writes it as QG_g. Thus $Ug(Z) = UZG_g$. The identity $UW = (E\tau)^k I$ implies $\det U \neq 0$, so cancellation gives $g(Z) = ZG_g$.

Lean, formal context. `PeriodMatrixFrame` (L51-L247), `FitTwistsExist` (L377). □

With $k = 2h$, Theorem 16.18 supplies exactly the three identities in (15.35). There is no remaining determinant or passage-from-S-span-to-A-span hypothesis in this step. The hypotheses to check are the period-matrix axioms for $U = (E\tau)^h P$ and the explicit basis conditions (16.15)–(16.16); their constructions must be used together with this theorem. The nonsingularity of the reduction follows from the field-of-norms argument in Theorem 16.14.

16.3 Integral period matrices in the tamely ramified case

Theorem 16.19 (The explicit frame exists for every crystalline lattice)**.** *Let* T *be a crystalline lattice of rank* d *on a Kummer chart, with all Hodge–Tate jumps in* $[-h, h]$*,* $h \geq 0$*. There are an invariant crystalline period matrix* $P \in \mathrm{GL}_d(B_{\max})$ *and matrices*

$$Q, Q' \in \mathrm{Mat}_d(A), \qquad Z \in \mathrm{Mat}_d(S_B), \qquad Y \in \mathrm{Mat}_d(B^+)$$

such that $U = (E\tau)^h P \in \mathrm{Mat}_d(B^+)$ *and*

$$QQ' = \tau^{2h} I_d, \qquad UZ = E^{2h} Q, \qquad YZ = E^{2h} I_d, \qquad \det Q \notin pA. \tag{16.20}$$

Thus the matrices satisfy (15.35)*, by the classical period comparisons specified in the period-matrix construction and the determinant criterion of Theorem 16.14.*

Proof. Apply Theorem 16.5 with $a = b = h$ and $k = 2h$. It verifies all the period-matrix properties, including scalar-map stability and the Galois relation. Theorem 16.13 then constructs Q, Z and proves the basis identities (16.11). These are exactly (16.15) and (16.16). Lemma 16.15 gives $\det Q \notin pA$; Theorem 16.18 gives Q', Y and the three identities. Thus no matrix-existence assumption remains in the statement.

Lean. `FrameTheorem` (L219), `PeriodMatrixFrame` (L452). □

Corollary 16.20 (The exponent in the tame cup-product bound)**.** *Retain the representation, type, opposite-window and size hypotheses of Theorem 15.18, including crystallinity of the specified lattices and cocycle extensions on its common tame chart. No additional frame-existence or period-depth hypothesis is required. The three period estimates hold for the actual form-defined modules, with the common correction multiplier in S as in Theorem 15.37. Writing*

$$\kappa = r^2(N+1)^2,$$

the resulting cup products are killed by

$$p^{\,3\kappa+96(\eta+W)+8}. \tag{16.21}$$

In particular the previous allowance $p^{\,3\kappa+150(\eta+W+1)}$ remains valid.

Proof. Every crystalline lattice in the three extensions and every extension attached to an admissible pairing has finitely many jumps. Choose an integer h bounding their absolute values for each lattice. Theorem 16.19 supplies all matrices assumed in Corollary 15.66. That corollary and its proved continuous cup primitive give (16.21). Its numerical exponent involves the stated type and window bounds, not the auxiliary denominator in Lemma 16.10; the latter denominator was cancelled in Lemma 16.12. Finally $96(\eta + W) + 8 \le 150(\eta + W + 1)$.

Lean, comparison. With $a = 0$, the gap bound below gives $3\kappa + 2C$, where $C \le 48(\eta + W) + 4$. The formal tame-type theorem uses $3(\kappa + C)$. `isScaledCupCoboundary_of_gap`, `hasCupExponent_of_isTameType`. □

From the frame to the local loss. Theorem 16.18 supplies the integral frame and its transport under Frobenius and Galois. For a nonzero pole, the new issue is to annihilate the cokernels with a controlled exponent. Theorems 16.27 and 16.30 give the annihilators and Theorem 16.34 converts them into a cup-product bound. The small-loss estimate is Corollary 16.56, and the estimate for all prime ranges is Corollary 16.93.

16.4 Quantitative annihilators beyond the zero-pole case

The integral frame now allows the lifting-ideal calculation to continue with a nonzero power of E. We first record the chart calculation at every Frobenius level, including its zero step. We then prove a family of actual cokernel annihilators and a contraction estimate along several successive letters. These arguments use the constructed period frame; they require no additional analytic-radius assertion.

Keep A, S, E, z, τ and the actual ideals $I_{\ell,i}$ and $J_{\ell,i}(r) = I_{\ell,i} + (r)$ from (15.59). Write

$$E_a = \varphi^a(E), \qquad \theta_a = \theta\varphi^{-a}, \qquad \delta = z\tau, \qquad d = \begin{cases} 0, & e = 1, \\ v_p(e) + 1, & e > 1. \end{cases}$$

The integer e is the ramification index of the chart. Valuations are normalized by $v_p(p) = 1$, and $v_p(0) = 0$ only when it occurs in the definition of a step cost.

Lemma 16.21 (Kummer shifts on an Eisenstein chart)**.** *Let $K_0 = \mathrm{Frac}(\Lambda)$ be unramified over $\mathbb{Q}_p$, let $K = K_0(\pi)$ be totally ramified with uniformizer π, and form the chart from compatible roots $\pi_r^{p^r} = \pi$, with $z = [\pi^\flat]$. For every integer m there is $g_m \in G_K$ fixing Λ such that*

$$g_m(z) = [\epsilon]^m z.$$

The same statement holds in the inverse image of G_K under a surjection onto the ambient Galois group. In particular the ordinary lowering and height recurrence of Theorem 15.47 and Theorem 15.49 hold on these charts, including ramified charts with $p \mid e$.

Proof. The polynomial $X^{p^r} - \pi$ is Eisenstein over K and remains Eisenstein over every finite unramified extension of K. Consequently inertia in G_K acts transitively on its roots. There is therefore an inertia element sending π_r to $\epsilon_r^m \pi_r$. The sets of such elements are nonempty closed subsets of the compact inertia group and are nested as r increases. Their intersection supplies g_m. It fixes K, hence Λ, and the action on the compatible tower gives the displayed identity.

The earlier ordinary-lowering proof used precisely the shift g_1, the scalar identities $E \equiv p \pmod z$ and $\varphi z = z^p$, and the frame, common-correction and divided- difference statements. Those scalar identities and period statements hold on the present chart. The proof of (15.64) and both lowering formulas therefore applies verbatim. Existence of an initial exponent, height raising, and $z^e \in (E, p)S$ give the same least-exponent recurrence. No prime-to-p assumption on e enters these steps.

Lean. `KummerChartExistence` (L306). □

The following statements use these local charts. More generally their proofs apply whenever the chart subgroup has the shifts just constructed and fixes Λ and $\theta(z)$. These are the defining conditions on the Kummer chart used here.

Lemma 16.22 (Scalar steps at every Frobenius level)**.** *For every $a \geq 0$ and every nonzero integer j, put*

$$c_a(j) = v_p(j) + d + 3a.$$

There are $g \in H$, $\Omega \in A^\times$ and $\upsilon, X \in A$ such that

$$\begin{aligned} gE_a &= \Omega E_a, \qquad E_a \mid gs - s \quad (s \in S), \\ \delta X &= \Omega^j - 1, \qquad \upsilon X - p^{c_a(j)} \in E_a A. \end{aligned} \tag{16.22}$$

Proof. Take $g = g_{p^a}$ from Lemma 16.21 and write $gE = \omega E$ with $\omega \in A^\times$. Since g commutes with Frobenius, take $\Omega = \varphi^a\omega$. The difference $gz - z = ([\epsilon]^{p^a} - 1)z$ is divisible by E_a; continuity and the fact that g fixes Λ give $E_a \mid gs - s$ for all $s \in S$.

The scalar chart difference gives $\delta \mid \omega - 1$. Also $\delta \mid \varphi^a\delta$, since $\varphi^a z = z^{p^a}$ and $\tau \mid \varphi^a\tau$. Thus $\delta \mid \Omega^j - 1$, for positive or negative j; define X by the third equation in (16.22).

We give the quantitative part explicitly. As in Lemma 15.60, write $F(z) = EG$ with F monic Eisenstein of degree e and $G \in S^\times$. The derivative calculation there proves $v_p(F'(\pi)) \le d$, where $\pi = \theta z$. Taylor expansion now uses the increment

$$gz - z = E\delta u \sum_{r=0}^{p^a-1} [\epsilon]^r, \qquad u \in A^\times.$$

After division by E and application of θ, it gives

$$\theta(\omega - 1)\theta G = F'(\pi)\theta\delta\, \theta u\, p^a.$$

The images of G and u are units. Hence

$$v_p(\theta\delta) + a \le v_p(\theta(\omega - 1)) \le v_p(\theta\delta) + a + d.$$

Here $v_p(\theta\delta) = 1/e + 1/(p-1) > 1/(p-1)$. The strict lifting-the-exponent calculation in Lemma 15.60 therefore gives

$$v_p(\theta(\omega^j - 1)) \le v_p(j) + d + a + v_p(\theta\delta).$$

Applying θ_a to $\delta X = \varphi^a(\omega^j - 1)$ yields

$$v_p(\theta_a X) \le v_p(j) + d + a + v_p(\theta\delta) - v_p(\theta_a\delta) \le v_p(j) + d + 3a.$$

For the last inequality, it is equality of the extra terms when $a = 0$; if $a \ge 1$, use $v_p(\theta\delta) \le 2 \le 2a$ and $v_p(\theta_a\delta) > 0$. The denominator is nonzero, so $p^{c_a(j)}/\theta_a X \in \mathcal{O}_C$. Lift this element through the surjection $\theta_a : A \to \mathcal{O}_C$. Its lift v gives the final congruence, since $\ker\theta_a = E_a A$. This reconstructs the nonzero depth step at every level.

Lean. `ExponentLattices` (L185-L250). □

Lemma 16.23 (The zero step by finite averaging)**.** *For $a \ge 0$ put $c_a(0) = d + 3a$ and $N = p^{c_a(0)}$. There are $t_0, \dots, t_{N-1} \in H$ such that*

$$E_a \mid g\left(\sum_{l=0}^{N-1} t_l s\right) - \sum_{l=0}^{N-1} t_l s \quad (g \in H,\ s \in S). \tag{16.23}$$

If $v \in \mathcal{K}_a \cap B_{\mathrm{dR}}^+$, then $\sum_l t_l v$ has residue in $K = \mathbb{Q}_p^{\mathrm{alg,H}}$.

Proof. Choose $t_l = g_l$ in Lemma 16.21. Let $\pi_a = \theta_a z$ and $\zeta = \theta_a[\epsilon]$. Then $\pi_a^{p^a} = \pi$, and ζ is a primitive p^a-th root of unity. Every $\theta_a s$, $s \in S$, is a polynomial of degree less than ep^a in π_a with coefficients in the unramified ring Λ. Indeed π_a is a uniformizer with Eisenstein polynomial $F(X^{p^a})$, and convergent power series reduce by this polynomial to the displayed finite module. On a monomial $w\pi_a^m$, the sum over the chosen shifts is

$$w\pi_a^m \sum_{l=0}^{N-1} \zeta^{lm} = \begin{cases} Nw\pi^{m/p^a}, & p^a \mid m, \\ 0, & p^a \nmid m. \end{cases}$$

The geometric sum vanishes in the second case because $p^a \mid N$. The first expression is fixed by H. This proves (16.23).

For the final assertion we justify passage to the entire value field, rather than just S. Choose $b \in A$ and $M \ge 0$ with $p^M v - \varphi^{-a} b \in \mathrm{Fil}^1$. The scalar extension of $\varphi^{-a}\lambda_0\varphi^a$, as in Lemma 15.23, fixes $\mathcal{K}_a$ and preserves filtration. It therefore gives

$$p^M v - \varphi^{-a}\lambda_0(b) \in \mathrm{Fil}^1.$$

Thus the residue of v lies in $K(\pi_a)$ and is a p^{-M}-multiple of an element of $\theta_a(S)$. The polynomial calculation just made shows that its sum over the shifts has residue in K. This is the averaging argument of the source's zero depth step.

Lean. `ExponentLattices` (L270-L309). □

Theorem 16.24 (An operator for every level and every depth)**.** *For $a \geq 0$, $j \in \mathbb{Z}$, put $c_a(j) = v_p(j) + d + 3a$, with $v_p(0) = 0$. There is a single additive map $A_{a,j} : M \to M$, valid simultaneously for all letters and invariant forms, with these properties:*

1. *if x lies over a constant and has depth o, then $A_{a,j}x - p^{c_a(j)}x$ lies over zero and has orders o;*
2. *if the value of a form at level a on x is a fixed-field constant modulo Fil^j, its value on $A_{a,j}x$ is a fixed-field constant modulo Fil^{j+1}.*

In particular, on an unramified chart a level-zero step that is zero or prime to p is free, and the corresponding level-one step costs at most 3.

Proof. Let $r : M_A \to M$ be the retraction of Lemma 15.25. For $j \neq 0$ take the scalar data from Lemma 16.22, and set

$$A_{a,j}x = r\left(p^{c_a(j)}x - \upsilon\frac{gx - x}{\delta}\right).$$

The localized divided difference lies in M_A and preserves all its asserted orders by Lemmas 15.54 and 15.55. Retraction preserves these orders and the zero parameter projection, proving the first property.

For the second property write the level-a form value as $v = c + y$, with $c \in K$ fixed and $y \in \mathcal{K}_a \cap \mathrm{Fil}^j$. The element $\varphi^{-a}E_a$ is a de Rham uniformizer, and $y = (\varphi^{-a}E_a)^j q$ with $q \in \mathcal{K}_a \cap B_{\mathrm{dR}}^+$. The residue argument in the preceding lemma, now using the single g which fixes S modulo E_a, shows $gq - q \in \mathrm{Fil}^1$. Consequently

$$gy - (\varphi^{-a}\Omega)^j y \in \mathrm{Fil}^{j+1}.$$

The image of δ under φ^{-a} is a de Rham unit. Divide the last congruence by it and use $\upsilon X \equiv p^{c_a(j)} \pmod{E_a}$. The level-$a$ value before retraction is then congruent to $p^{c_a(j)}c$ modulo Fil^{j+1}. The retraction preserves that congruence by Lemma 15.25.

For $j = 0$ use the averaging elements of Lemma 16.23, and put

$$A_{a,0}x = r\left(\sum_{l=0}^{N-1} t_l x\right).$$

The difference from Nx is a sum of Galois differences, so the first property again holds. A value which is constant modulo Fil^0 is integral, since every element of the embedded fixed field belongs to B_{dR}^+. The averaging lemma makes its residue a fixed-field constant. Retraction retains this congruence. This proves the second property also at zero.

Lean. `ExponentLattices` (L1152), `SidePoints` (L647). □

Corollary 16.25 (The exponent in a composition of depth operators)**.** *Let $\Sigma \subset \mathbb{N} \times \mathbb{Z}$ be finite. Suppose it contains every pair (a, j) occurring in an increase from a depth function o to a larger depth function o' on the prescribed jumps. A point over $p^\kappa c$, with orders ℓ and depth o, can be replaced by a point over*

$$p^{\kappa + C}c, \qquad C = \sum_{(a,j)\in\Sigma} c_a(j),$$

with the original orders ℓ and depth o'. It consequently has the improved orders of Lemma 15.62.

Proof. At each level use the operators of the preceding theorem in increasing order of j, and process the finitely many levels in any order. The correction term has the orders of the current depth function. Since these order modules are Galois-stable, every depth already attained remains attained. A single operator raises all forms at its level that require the same step, so each pair (a, j) is charged once. The original orders are preserved because a vector having those orders is already invariant modulo the same order module. Parameter projection multiplies by $p^{c_a(j)}$ at each step. This is the argument of Lemma 15.64 with a finite set of levels.

Lean. `SidePoints` (L733). □

Lemma 16.26 (Removing the modulus and lowering a monomial)**.** *For $m \le n$, if $p^c z^n \in J_{0,i}(r)$ and*

$$c + n + v_p(n!) \le b + m + v_p(m!),$$

then $p^b z^m \in J_{0,i}(r)$. Also, if $p^b \in J_{\ell,i}(p^{b+1})$, then

$$J_{\ell,i}(p^{b+1}) = I_{\ell,i}. \tag{16.24}$$

Proof. Apply the level-zero ordinary step from exponent n down to m. Its total cost is $n - m + v_p(n!) - v_p(m!)$. For the second statement write $p^b = s + p^{b+1}u$ with $s \in I_{\ell,i}$. Then $s = p^b(1 - pu)$, and $1 - pu$ is a unit of the p-adically complete ring A. Thus $p^b \in I_{\ell,i}$, and hence $(p^{b+1}) \subseteq I_{\ell,i}$.

Lean. `CupProductBound` (L671), `CupProductBound` (L1028). □

Theorem 16.27 (The unramified trade between p and E)**.** *Let the chart be unramified, normalized by $E = z + p$, and let $p \ge 5$. Let T be any of the crystalline cocycle extensions under consideration. If all its jumps at all letters are at most $(p-3)n$, where $n \ge 0$, then*

$$(p, E)^n \subseteq I_{0,i} \quad \text{for every letter } i. \tag{16.25}$$

In particular $p^s E^t$ kills the actual standard-period cokernel for every index whenever $s, t \ge 0$, $s + t = n$. There is no lower bound on the jumps in this statement.

Proof. The least-exponent bound of Theorem 15.49, now justified by Lemma 16.21, supplies $b \ge 0$ with

$$z^b \in J_{0,i}(p^{n+1}) \quad \text{for every } i, \qquad (p-1)b \le (p-3)n + n = (p-2)n.$$

Legendre's formula gives $(p-1)v_p(b!) \le b$. Therefore

$$(p-2)(b + v_p(b!)) \le (p-2)b + (p-1)v_p(b!) \le (p-1)b \le (p-2)n,$$

and $b + v_p(b!) \le n$.

For each $0 \le m \le n$ we claim $p^{n-m} z^m \in J_{0,i}(p^{n+1})$. If $m \ge b$, this is an ideal multiple of z^b. If $m < b$, Lemma 16.26 applies because

$$b - m + v_p(b!) - v_p(m!) \le n - m.$$

The case $m = 0$ gives $p^n \in J_{0,i}(p^{n+1})$. Equation (16.24) drops the modulus, so all the displayed monomials belong to $I_{0,i}$. They generate $(p, z)^n$, which equals $(p, E)^n$ since $E = z + p$. This proves (16.25) and includes $n = 0$. The only jump condition used in the proof of the two height inclusions is an upper bound, so negative jumps incur no added shift. This gives the conclusion of `killsCokernel_of_isUnramified` for the actual modules.

Lean. `CupProductBound` (L1140-L1191). □

Lemma 16.28 (Trading one E for one z at level one)**.** *If $p^c z^a E^n \in J_{1,i}(r)$, then*

$$p^{c+dn+v_p(n!)} z^{a+n} \in J_{1,i}(r). \tag{16.26}$$

Proof. It suffices to trade a single factor when $n \geq 1$. Use Lemma 16.22 at level zero and step n, writing its data as

$$gE = \omega E, \qquad \delta X = \omega^n - 1, \qquad \upsilon X = p^{v_p(n)+d} + Eb_1.$$

The scalar chart difference gives $gz = z(1+\tau E\upsilon)$ with $\upsilon \in A$; choose $b_2 \in A$ such that $(1+\tau E\upsilon)^a = 1 + \tau E\upsilon b_2$. For $f = p^c z^a E^n$ set

$$s = p^{c+v_p(n)+d} z^{a+1} E^{n-1}, \qquad \beta = zb_1 + \upsilon\upsilon b_2 \omega^n.$$

Direct expansion gives

$$\delta E(s + \beta f) = z\upsilon(gf - f).$$

The multiplier f belongs to S, so the common-correction condition of Lemma 15.44 applies and puts s in the same level-one ideal. Repeating for $n, n-1, \ldots, 1$ has total cost $dn + v_p(n!)$ and proves the assertion. This is the E-to-z trade.

Lean. `CupProductBound` (L931-L989). □

Lemma 16.29 (Frobenius contraction on successive eigenspaces)**.** *Suppose $z^a \in J_{0,j}(r)$ for every letter j. Fix a letter i and an integer $J \geq 0$. If all jumps at the letters $\sigma^t i$, $1 \leq t \leq J$, are at most $\omega' \geq 0$, then there are $c, n \geq 0$ with*

$$\begin{gathered} p^c z^n \in J_{0,i}(r), \qquad c \leq J\big(d\omega' + v_p(\omega'!)\big), \\ n \leq \frac{a}{p^J} + \frac{\omega'}{p-1}. \end{gathered} \tag{16.27}$$

The last inequality is an inequality of real numbers.

Proof. For one step, raise $p^c z^n$ from level zero to level one by multiplying by $E^{\omega'}$, and apply Lemma 16.28. Here height raising uses only the jump bound at the current letter: for x in that letter of $N_A(T)$, the first positive Frobenius test on $\varphi^{-1}(E^{\omega'}x)$ is $E^{\omega'}\lambda(x)$, whose order is nonnegative by the current-letter bound. Every later positive Frobenius test is already integral by $x \in N_A(T)$. This is the letterwise version of the proof of Lemma 15.41. This gives $p^{c+d\omega'+v_p(\omega'!)} z^{n+\omega'}$ at level one. The factorial descent of Lemma 15.48 then gives at the preceding letter

$$p^{c+d\omega'+v_p(\omega'!)} z^{\lfloor (n+\omega')/p \rfloor}.$$

The last descent has zero extra cost, because the integers between $p\lfloor (n+\omega')/p \rfloor + 1$ and $n + \omega'$ are prime to p. Apply this step J times, starting at $\sigma^J i$. The costs add, and the exponents obey $n_{r+1} \leq (n_r + \omega')/p$. Thus

$$n_J \leq \frac{a}{p^J} + \omega' \sum_{r=1}^{J} p^{-r} \leq \frac{a}{p^J} + \frac{\omega'}{p-1}.$$

Lean. `CupProductBound` (L1191). □

Theorem 16.30 (A cokernel bound from J successive eigenspaces)**.** *Suppose every jump for every index is at most $\omega \geq 0$, and every jump at $\sigma^t i$, $1 \leq t \leq J$, is at most $\omega' \geq 0$. If*

$$8e \leq p^{J+1}, \qquad 2Jd\omega' + \frac{(2J+4)\omega'}{p-1} + \frac{8e\omega}{p^{J+1}} \leq \kappa, \tag{16.28}$$

where κ is a nonnegative integer, then

$$p^{\kappa} \in I_{0,i}. \tag{16.29}$$

Thus p^{κ} kills the actual standard-period cokernel at that letter.

Proof. Use the height recurrence modulo $p^{\kappa+1}$ to find $a \geq 0$ with

$$z^a \in J_{0,j}(p^{\kappa+1}) \quad \text{for every } j, \qquad (p-1)a \leq e(\omega+\kappa).$$

Lemma 16.29 supplies c, n with the membership there and

$$c \leq Jd\omega' + \frac{J\omega'}{p-1}, \qquad n \leq \frac{e(\omega+\kappa)}{(p-1)p^J} + \frac{\omega'}{p-1}.$$

Using $v_p(n!) \leq n/(p-1)$ and $p/(p-1) \leq 2$, we obtain

$$\begin{aligned} c + n + v_p(n!) &\leq Jd\omega' + \frac{(J+2)\omega'}{p-1} + \frac{4e(\omega+\kappa)}{p^{J+1}} \\ &\leq Jd\omega' + \frac{(J+2)\omega'}{p-1} + \frac{4e\omega}{p^{J+1}} + \frac{\kappa}{2} \leq \kappa. \end{aligned}$$

For the first inequality we also used $p^2/(p-1)^2 \leq 4$; the second uses $8e \leq p^{J+1}$; the last is exactly the second inequality in (16.28), divided by two. Lemma 16.26 now removes z^n, giving $p^{\kappa} \in J_{0,i}(p^{\kappa+1})$. The same lemma drops the modulus and proves (16.29). This reconstructs the successive-letter cokernel estimate, including the absorption of the initial κ in its own seed bound.
Lean. `CupProductBound` (L1193-L1243). □

Application to the local cup product. Theorems 16.27 and 16.30 concern the actual standard-exponent modules and may be applied separately to each of the two cocycle extensions and to the extension associated with their admissible pairing. They supply precisely $p^{\kappa}E^a$ or p^{κ} annihilators; their proofs impose upper bounds on jumps and do not change any negative jump. Additional jump-window, pole-gap, or multiplicity conditions belong to the subsequent cup and colength arguments and are not included implicitly here.

16.5 Pole order and the cup-product bound

We now allow the period cokernel to be killed by $p^{\kappa}E^a$ instead of p^{κ}. The exponent a is a nonnegative integer; we use m for the inverse-Frobenius level. All parameters $\kappa, a, t, \eta, W, b_i, C$ below are nonnegative integers. Work on one of the local Kummer charts of Lemma 16.21, with chart subgroup H. All cocycle extensions in this subsection are crystalline on H. The integral frames exist by Theorem 16.19. The localized modules, retractions, and level-zero depth operators are those of Lemmas 15.54, 15.25, and Theorem 16.24. Thus the finite composition argument of Lemma 15.64 applies on these charts. Cup-coboundary conclusions are first asserted on H. They descend to an ambient group G whenever H is open in G and $[G:H]$ is prime to p; no such index condition is needed for the conclusion on H itself.

Let U_i, U_i' be the jump sets of the two cocycle extensions at a letter i; these sets include the parameter jump 0. A third extension is attached to the crystalline error of the admissible pairing, as in Lemma 15.63. An assertion that $p^{\kappa}E^a$ kills a period cokernel means precisely $p^{\kappa}E^a \in I_{0,i}$ for the lifting ideal (15.59). Put

$$O_a(m,u) = \begin{cases} -a-u, & m=0, \\ -u, & m>0, \end{cases} \qquad D_{a,t}(m,u) = \begin{cases} a+t-u, & m=0, \\ -u, & m>0. \end{cases}$$

Thus O_0 is the standard order function. Throughout, d_0 is the different allowance of Lemma 16.22, and $c(j) = v_p(j) + d_0$, with $v_p(0) = 0$.

Lemma 16.31 (A cokernel annihilator gives a point with the stated pole)**.** *If $p^\kappa E^a \in I_{0,i}$, then every parameter constant c_i has a point in the localized integral period module lying over $p^\kappa c_i$, with orders and depth O_a. If every jump of the extension for every index is nonpositive, then $1 \in I_{0,i}$ for every i.*

Proof. Apply Theorem 15.37 with $f = p^\kappa E^a$, $r = 0$, and $l = 0$. It gives $x \in N_i$ over $p^\kappa E^a c_i$ with standard orders. The point $E^{-a}x$ belongs to M and lies over $p^\kappa c_i$. The scalar E has order one at level zero and order zero at every positive level, so its orders are O_a. Order modules are Galois-stable, and therefore these orders are also lower bounds for its depth.

For the last assertion, use Lemma 15.41 with $n = 0$. It gives $A \otimes \widetilde{T} \subseteq N_A(\widetilde{T})$. The underlying lattice vector $(0, c)$ lifts c, and the letter projector gives the required lift of c_i. Hence $1 \in I_{0,i}$.

Lean. `pointsOver_nonempty_of_killsCokernel`, `killsCokernel_zero_zero_of_small_jumps`. □

Lemma 16.32 (The pairing criterion when the third point has a pole)**.** *Suppose $p^\kappa E^a$ kills the Tate-dual period cokernel at the letter l steps before i. Define*

$$R_{l,a}(m,s) = \begin{cases} 1, & m < l, \\ a - s, & m = l, \\ -s, & m > l. \end{cases} \tag{16.30}$$

A first-side move over zero with orders $o(m,u)$ satisfies the pairing condition at a second-side point of depth $d'(m,u')$ whenever

$$o(m,u) + d'(m,u') \geq R_{l,a}(m, u+u')$$

for every level and pair of jumps. The same criterion holds with the sides exchanged.

More generally, let a third point lie over p^κ before the Frobenius shift, and suppose its shifted image under the equivariant evaluation map F has, on each adapted dual form, order at least $q_l(m, -u-u'-1)$. Then the sufficient condition is

$$q_l(m, -u-u'-1) + o(m,u) + d'(m,u') \geq 1. \tag{16.31}$$

All inputs must belong to their localized period modules, as in Lemma 15.63.

Proof. Lemma 16.31 supplies a Tate-dual point ν over p^κ with orders O_a. Let q be the finite coefficient exponent of the admissible pairing, and apply $F(v,c) = p^q v + c\Phi$ from the proof of Lemma 15.63. At a pair of jumps u, u', the corresponding Tate-dual form has jump $-u-u'-1$. After applying φ^l, its orders are at least

$$0 \quad (m < l), \qquad u + u' + 1 - a \quad (m = l), \qquad u + u' + 1 \quad (m > l).$$

Their sums with (16.30) are one in all three cases. The evaluation identity (15.88) therefore has order at least one after inserting the move and the divided Galois difference of the point at rest. The latter division by $\delta = z\tau$ preserves every integer order by Corollary 15.55.

The localized Tate-module membership, cancellation of its denominators by the standard-order tests, and multiplication by τ and then by z are exactly the operations justified in Lemma 15.63. They make the reduced value small and give the required pairing condition with factor p^κ. The same evaluation calculation proves (16.31). If the third point is specified by an order function

antitone in its jump variable, its bounds pass through F as follows. Precompose an adapted form with the integral equivariant map F; a form of type w remains of type w. Its expansion in an adapted basis uses only types $v \le w$, with fixed-field coefficients of de Rham order at least zero. Antitonicity bounds every term below by the prescribed order at w. This also justifies using an improved, antitone order function in (16.31). In the special case O_a, the extra pole changes only the middle entry of the order table.

Lean. `isSafe_of_orders`, `isSafeMoveFst_of_killsCokernel`, `isSafeMoveSnd_of_killsCokernel`. □

Lemma 16.33 (Depth removes a pole across the exact jump gap)**.** *Suppose $p^\kappa E^a$ kills the cokernel at i and $U_i \cap [-a, 0) = \varnothing$. Let $J_i \subset \mathbb{Z}$ be finite and contain every integer in*

$$[-u - a, a + t - u) \qquad (u \in U_i),$$

where $t \ge 0$. If $\sum_{j \in J_i} c(j) \le C$, then every constant c_i admits a point over $p^{\kappa+C} c_i$ with standard orders O_0 and depth $D_{a,t}$.

Proof. Start with Lemma 16.31 and apply Lemma 15.64 using J_i; supply unused cost by scalar multiplication. All higher-level depths remain $-u$. At level zero the improved-order formula (15.86) gives the required standard order as follows. If $u < -a$, the initial order $-a - u$ is positive and the improved order is $a + t - u \ge -u$. If $u \ge -a$, the jump gap gives $u \ge 0$, and the improved order is at least $\min(a + t - u, 0) \ge -u$.

Lean. `pointsOver_nonempty_of_gap`. □

The endpoint convention is necessary in this argument. If $-a \le u < 0$, a nonzero fixed scalar has order zero, satisfies the initial lower bound $-a - u \le 0$, and has arbitrarily large depth; it still fails the required standard bound $-u > 0$. The depth-improvement argument therefore cannot discard $[-a, 0)$ from its hypotheses. The jump 0 causes no such obstruction.

Theorem 16.34 (The cup-product bound at nonnegative pole order)**.** *Suppose $p^\kappa E^a$ kills, for every index, the period cokernels of the two cocycle extensions and the Tate-dual extension of their admissible pairing. Suppose*

$$\begin{aligned} U_i &\subseteq [-\eta, \eta] \cup -\varepsilon_i [b_i, b_i + W], \\ U_i' &\subseteq [-\eta, \eta] \cup \varepsilon_i [b_i, b_i + W], \qquad \varepsilon_i \in \{1, -1\}, \end{aligned} \tag{16.32}$$

and both U_i and U_i' avoid $[-a, 0)$. For each i let J_i contain all the integer intervals $[-u - a, a + \eta - u)$ with $u \in U_i \cup U_i'$, and suppose $\sum_{j \in J_i} c(j) \le C$. Then the tested cup products are continuous coboundaries after multiplication by

$$p^{3\kappa + 2C}. \tag{16.33}$$

Proof. Lemma 16.33 gives points x_i, x_i' over $p^{\kappa+C} c_i, p^{\kappa+C} c_i'$, with standard orders and depth $D_{a,\eta}$. Put $y_i = \varphi x_{i^-}$ and $y_i' = \varphi x_{i^-}'$. Their orders are zero at level zero and $-u$ above it; their depths are one at level zero, $a + \eta - u$ at level one, and $-u$ above one, by Lemma 15.62.

Suppose first that the negative far window is on the first side. Then $u \le \eta$ and $u' \ge -\eta$ at this letter. The improved-order formula shows that x_i has nonnegative order at level zero. The path

$$(y_i, y_i') \longrightarrow (x_i, y_i') \longrightarrow (x_i, x_i')$$

is safe. For the first move use $R_{1,a}$. At level zero the order-depth sum is at least $0 + 1$; at level one it is at least $-u + (a + \eta - u') \ge a - u - u'$; above one it is at least $-u - u'$. For the second move use $R_{0,a}$. At level zero its order-depth sum is at least

$$a + \eta - u + \min(0, -u') \ge a - u - u',$$

because $u' \geq -\eta$; at positive levels it is at least $-u-u'$. If the negative far window is on the second side, reverse the roles and the order of the two moves. Zero moves put either path in the four-move format of Theorem 15.57.

For the additional condition on the two cocycles, the sum of the two depths is $2(a+\eta)-u-u'$ at level zero and $-u-u'$ at every positive level. These dominate $R_{0,a}$, since $a, \eta \geq 0$. Lemma 16.32 proves (15.78). Sum over the letters using Lemma 15.58, then apply the corrected Witt cochain of Theorem 15.57. The exponent is $(\kappa+C)+(\kappa+C)+\kappa = 3\kappa+2C$. Applied with ambient group H itself, that theorem supplies continuity of the primitive. If $H \subseteq G$ has index prime to p, its restriction–corestriction argument also descends the vanishing to G.

Lean. `isScaledCupCoboundary_of_gap`. □

Corollary 16.35 (The numerical bound under the gap hypothesis)**.** *Retain the cokernel, opposite-window, and jump-gap hypotheses of Theorem 16.34, and put $n = 6(a+\eta+W)$. Suppose no nonzero multiple of p lies within distance n of 0 or $\pm b_i$, at any letter. Then one may take*

$$C = (8n+4)d_0 = (48(a+\eta+W)+4)d_0, \qquad H = 3\kappa+2C. \tag{16.34}$$

In particular the source allowance $H = 3(\kappa+C)$ is valid.

Proof. Take

$$J_i = ([-n,n] \cup [b_i-n, b_i+n] \cup [-b_i-n, -b_i+n]) \cap \mathbb{Z}.$$

If $u \in [-\eta, \eta]$, every required step lies between $-a-\eta$ and $a+2\eta$; if u belongs to a far window, it lies within $a+\eta+W$ of the opposite far centre. These intervals are contained in J_i. The gap condition gives $v_p(j) = 0$ for all $j \in J_i$, using $v_p(0) = 0$. Finally $|J_i| \leq 3(2n+1) \leq 8n+4$. Apply Theorem 16.34 and use $C \geq 0$.

Lean, comparison. The gap argument gives $3\kappa+2C$. The formal theorem covering both the gap and nonpositive cases uses $3(\kappa+C)$. `sum_cost_le_of_noMultipleNear`, `isScaledCupCoboundary_of_gap`, `isScaledCupCoboundary_of_killsCokernel`. □

An unramified chart with one nonpositive side

For the next result the chart is unramified, so $d_0 = 0$. The jump gap below zero will be replaced by a sign condition on the second side. Its pole-free initial lift comes from the last assertion of Lemma 16.31.

Lemma 16.36 (Free deepening for a nonpositive side)**.** *Suppose $U'_j \subseteq [-\eta, 0]$ for every index and the chart is unramified. Let $a, t, n \geq 0$, $t \geq \eta$, $a+t+\eta \leq n$, and assume no nonzero multiple of p is within distance n of zero. There are points x'_i over c'_i with standard orders and depth $D_{a,t}$. For $y'_i = \varphi x'_{i^-}$, there is also a point q'_i over c'_i whose orders are zero at level zero and $-u'$ above zero, and whose depth is at least*

$$\begin{cases} a+t-u', & m=0, \\ t-u', & m=1, \\ -u', & m>1. \end{cases} \tag{16.35}$$

Proof. Start from the standard point of Lemma 16.31 with $\kappa = a = 0$. Raise its level-zero depth from $-u'$ to $a+t-u'$. All steps lie in $[0, a+t+\eta] \subseteq [0, n]$. Each has cost zero, so Lemma 15.64 gives x'_i over the unchanged constant with its original standard orders.

The point y'_i has order zero and depth at least one at level zero. At level one it has standard order $-u'$ and depth $a+t-u' \geq t-u'$, while above one it has standard orders and depth. Raise only its level-zero depth to $a+t-u'$. The needed steps again belong to $[0, n]$ and cost zero. The depth-composition lemma preserves the other levels and all original orders, proving the result.

Lean, formal context. `DeepPoints` (L378-L409), `FitTwistsExist` (L377). □

Theorem 16.37 (The nonpositive-side cup estimate)**.** *Suppose the chart is unramified and $p^\kappa E^a$ kills, for every index, the cokernels of the first side and of the Tate-dual side of its admissible pairing. Suppose the first jump sets satisfy the first inclusion in (16.32), and $U_i' \subseteq [-\eta, 0]$ for every index. Put $n = 6(a+\eta+W)$ and assume the same no-multiple condition around $0, \pm b_i$ as in Corollary 16.35. Then the tested cup products are continuous coboundaries after multiplication by*

$$p^{2\kappa}. \tag{16.36}$$

Proof. Set $t = a+\eta$. Begin on the first side with the point of orders and depth O_a from Lemma 16.31. Raise its level-zero depth to $a+t-u$. The required steps lie within $a+t+\eta+W \le n$ of 0 or the opposite far centre, so all costs are zero. The resulting point x_i lies over $p^\kappa c_i$, has depth $D_{a,t}$, and retains orders $O_a \ge O_t$. Lemma 16.36 gives x_i' over c_i' with standard orders and depth $D_{a,t}$. Put $y_i = \varphi x_{i^-}$ and $y_i' = \varphi x_{i^-}'$.

If the far window of the first side is negative, every first-side jump at i satisfies $u \le \eta \le t$. Depth improvement makes the order of x_i at level zero nonnegative. Use the two moves $(y_i, y_i') \to (x_i, y_i') \to (x_i, x_i')$. For the first, the rule $R_{1,a}$ holds: at level zero the sum is at least one; at level one it is at least $(-t-u)+(a+t-u') = a-u-u'$; above it the sum is $-u-u'$. For the second use $R_{0,a}$. Its level-zero sum is at least $a+t-u+\min(0,-u') = a+t-u \ge a-u-u'$, since $-t \le u' \le 0$. At higher levels the sum is at least $-u-u'$.

If that far window is positive, then $u \ge -\eta \ge -t$. Use the point q_i' in Lemma 16.36 and the path

$$(y_i, y_i') \longrightarrow (y_i, q_i') \longrightarrow (x_i, q_i') \longrightarrow (x_i, x_i').$$

For the first move use $R_{1,a}$ with the sides exchanged. At level zero its sum is at least $0+1$; at level one it is at least $-u'+(a+t-u) \ge a-u-u'$; above one it is $-u-u'$. For the middle move use $R_{0,a}$. Its order at level zero is at least $\min(0,-t-u) = -t-u$, and its order at level one is at least $-t-u$. Adding (16.35) gives $a-u-u'$ at zero and $-u-u'$ at one, as required. All higher sums are again $-u-u'$. For the last move use $R_{0,a}$ with the sides exchanged. Its level-zero order is at least $\min(0,-u') = 0$, and the depth of x_i there is $a+t-u \ge a-u-u'$. At positive levels both relevant orders are standard.

Finally the two depths add to $2(a+t)-u-u'$ at level zero and $-u-u'$ elsewhere, which dominates $R_{0,a}$. This proves the condition on the two cocycles. Insert a zero move to obtain the four-move format, sum over the letters, and apply Theorem 15.57, first on H itself. Its three exponents are $\kappa, 0, \kappa$, giving (16.36). Prime-to-p descent applies whenever the stated ambient-group hypothesis holds.

Lean, formal context. `DeepPoints` (L378-L409), `FitTwistsExist` (L377). □

The cup-product bound from the three cokernels. Corollary 16.35 and Theorem 16.37 prove the two alternatives of `isScaledCupCoboundary_of_killsCokernel` on these charts. The first gives the sufficient exponent $3\kappa + 2(48(a+\eta+W)+4)d_0$; the second gives 2κ. The common upper bound $3(\kappa+(48(a+\eta+W)+4)d_0)$ follows in both cases. Only level-zero depth operators are used. The remaining local task for an application is to supply its stated $p^\kappa E^a$ cokernel annihilators and check its jump and arithmetic gap conditions. The frame-existence theorem alone does not give those numerical annihilators.

16.6 Multiplicity, depth, and the few-low-jump bound

We prove the estimate that charges the pole loss only to the multiplicities of the low jumps. Throughout this subsection the chart is unramified, $p \ge 5$, $S = \Lambda[[z]]$, $E = z+p$, and $f = [\Lambda[1/p] : \mathbb{Q}_p]$. The modules $M = N[E_j^{-1} : j \ge 0]$ and $M_A = N_A[E_j^{-1} : j \ge 0]$ are those already constructed.

We retain the chosen coefficient root of unity and its letter projectors from the local setup. When the coefficient ring is $\mathcal{O}$, this means a fixed $\zeta \in \mathcal{O}$ with $\zeta^{p^f-1} = 1$, and an embedding lies above i when its value on ζ is the Teichmüller value of i. Cup-coboundary assertions are on the chart subgroup H; they also hold on an ambient group G when H is open in G and $[G : H]$ is prime to p. We impose this index condition whenever a defect is computed on G rather than H. For a jump lattice write

$$M_i(o) = M \cap \mathrm{Ord}_i(o), \qquad R_t(j,u) = \begin{cases} t-u, & j = 0, \\ -u, & j > 0. \end{cases}$$

Write $o^{[j]}$ for o with every order at level j increased by one, and $o^{[j,u]}$ for o with only its value at (j,u) increased by one. Let $m_i(u)$ be the multiplicity of the jump u on the underlying $\mathbb{Z}_p$-lattice at letter i. The comparison basis in Lemma 15.39 identifies this number with the dimension of the quotient of type-u forms by type-$(u-1)$ forms. Indeed a basis adapted to that increasing filtration has exactly $m_i(u)$ basis members of type u; all the other members have smaller or larger type. This observation also applies when $m_i(u) = 0$.

The scalar retraction of Lemma 15.25 maps M_A to M, preserves all these orders, and fixes M. For a cocycle side, let π be its parameter projection, $C_i = \mathrm{im}(c \mapsto e_i(1 \otimes c))$, and put

$$D_i(t,c) = \pi\big(M_i(R_t) \cap \mathrm{Deep}_i(c)\big).$$

Here $\mathrm{Deep}_i(c)$ consists of points over a member of C_i whose Galois differences have orders c. For two submodules X, Y of a common $\mathbb{Z}_p$-module, write $\ell(X : Y) = \mathrm{length}_{\mathbb{Z}_p}(X/(X \cap Y))$.

Lemma 16.38 (A primitive relation with a multiplicity bound)**.** *Fix a level b, a letter i, and an integer u. Suppose $o(b,u) < o(b,v)$ for every allowed jump $v < u$ at i^{-b}, and $m_{i^{-b}}(u) \le n$. Any $fn+1$ points $x_j \in M_i(o)$ admit coefficients $c_j \in \mathbb{Z}_p$, one of which is 1, such that*

$$\sum_j c_j x_j \in M_i(o^{[0]}) + M_i(o^{[b,u]}). \tag{16.37}$$

Proof. Write $h = m_{i^{-b}}(u)$ and choose h adapted forms ν_k at jump u, as in Lemma 15.39. For each x_j take the vector y_j of residues of $E^{-o(b,u)}\nu_k(\varphi^{-b}x_j)$. These values have nonnegative de Rham order. They belong to the level-b value field by Lemma 15.53, since $E = p + (\varphi^{-b}z)^{p^b}$ belongs to that field and has order one. The residue calculation in Lemma 16.23 therefore puts y_j in K_b^h, where $\pi_b = \theta_b(z)$, $\pi_b^{p^b} = -p$, and $K_b = \Lambda[1/p](\pi_b)$.

For $b = 0$, $fh+1$ vectors in K_0^h are dependent over $\mathbb{Q}_p$. Divide by a coefficient of smallest valuation to obtain a relation over $\mathbb{Z}_p$ with one coefficient 1. For $b > 0$, $X^{p^b} + p$ is Eisenstein and $\theta_b(S) = \Lambda[\pi_b] = \mathcal{O}_{K_b}$. Indeed θ_b acts on Λ by inverse Frobenius to the b-th power, and the valuations of $a_k\pi_b^k$, $a_k \in \Lambda[1/p]^\times$, are distinct modulo $\mathbb{Z}$ for $0 \le k < p^b$. Also $\theta_b(E) = p + \pi_b$ is a uniformizer. The $\mathcal{O}_{K_b}$-span Y of the y_j is free of rank at most h, so $Y/\theta_b(E)Y$ has dimension at most fh over $\mathbb{F}_p$. Lift a dependence of the $fn+1$ vectors there to coefficients $c_j \in \mathbb{Z}_p$, one equal to 1. Then $\sum_j c_j y_j = \theta_b(E)\sum_j \theta_b(s_j)y_j$ with $s_j \in S$. For $b = 0$ use its exact relation and take $s_j = 0$.

Consequently $\sum_j (c_j - Es_j)x_j$ gains one order on the type-u forms: its indicated residues vanish and the smaller types already have strictly greater order. The other summand $E\sum_j s_j x_j$ gains one order at every level-zero jump. This proves (16.37), also when $h = 0$.

Lean. `exists_sum_smul_mem_locKisinPart_sup`. Lean first works over C_p and retracts its coefficients. For points in the localized module the residues already belong to K_b. □

Lemma 16.39 (Length from a bounded exponent and primitive relations)**.** *Let Q be a $\mathbb{Z}_p$-module killed by p^e. Suppose any $r+1$ elements of $Q[p]$ satisfy a relation over $\mathbb{Z}_p$ with one coefficient equal to 1. Then*

$$\mathrm{length}_{\mathbb{Z}_p} Q \le er.$$

Proof. The relation hypothesis gives $\dim_{\mathbb{F}_p} Q[p] \le r$. For $1 \le j \le e$, multiplication by p^{j-1} embeds $Q[p^j]/Q[p^{j-1}]$ into $Q[p]$. The filtration by these kernels ends at Q, so its length is at most er. This writes out the filtration used in the source's induction on the exponent.

Lean. `FittingIdealsAndLengths` (L428). □

Lemma 16.40 (Division by p for a single depth step)**.** *Let c be a depth function, and fix (b,u). Suppose*

$$p \nmid n \quad \text{for every allowed jump } v \text{ at } i \text{ and every integer } c(0,v) < n < c(0,v) + p^b. \tag{16.38}$$

Let $x, w \in M$ lie at the letter i. If x has depth c, w lies over zero with orders $c^{[0]}$, and $px - w$ has depth $c^{[b,u]}$, there exists a point m over zero with orders c such that $x - m$ has depth $c^{[b,u]}$.

Proof. On the unramified chart a level-zero step of nonzero index prime to p has cost zero. Compose the operators of Theorem 16.24 over the finite union of the intervals in (16.38), indexed by the allowed jumps. The resulting operator B has cost zero and raises the depth of w from $c^{[0]}$ to $c + p^b$ at level zero, preserving the other levels and the zero projection.

The same is true of its orders. Indeed Bw retains its original orders $c^{[0]}$. At level zero, the interval in (16.38) cannot contain 0. If $c(0,v) \ge 0$, the original order is positive, and the depth-improvement lemma gives the full raised depth as an order. If $c(0,v) + p^b \le 0$, the improved order is again at least the target. These two cases include the possible zero endpoints. Thus

$$w' = E^{-p^b} Bw$$

lies over zero with orders c. In the unramified chart $E^{p^b} - E_b \in pS$, so choose $r \in S$ with $E^{p^b} = E_b + pr$. Define

$$m = rw' - (Bx - x).$$

The cost property puts m over zero with orders c, and

$$p(x - m) = B(px - w) + E_b w'.$$

The first term has depth $c^{[b,u]}$: a cost-zero operator preserves any existing depth, since its difference from the identity has the corresponding orders and zero projection. The second has those orders and zero projection, hence that depth. Finally p has order zero at every E_j, so it can be cancelled from every Galois-difference order condition. This gives the required depth of $x - m$.

Lean, formal context. `HodgeExpansion` (L599-L774), `FitTwistsExist` (L377). □

Lemma 16.41 (The colength of a single depth step)**.** *Suppose (16.38) holds, $c(j,v) \ge -v$ on all allowed jumps, and*

$$c(b,u) < c(b,v) \quad \text{for every allowed } v < u \text{ at } i^{-b}.$$

For a cocycle side whose kernel lattice has multiplicities $m_i(v)$,

$$\ell\big(D_i(0,c) : D_i(0,c^{[b,u]})\big) \le \big(v_p(c(b,u)) + 3b\big) f m_{i^{-b}}(u), \tag{16.39}$$

with $v_p(0) = 0$ in this cost convention.

Proof. Put $X = M_i(R_0) \cap \mathrm{Deep}_i(c)$, $X^+ = M_i(R_0) \cap \mathrm{Deep}_i(c^{[b,u]})$, and let K be the points over zero with orders c. Since $c \ge R_0$, $K \subset X$. The quotient of the two constant modules is a quotient of $Q = X/(X^+ + K)$.

The one-step operator of Theorem 16.24 shows that $p^e Q = 0$, with $e = v_p(c(b,u)) + 3b$: its value is in X^+ and its difference from p^e times the argument lies in K. Take $f m_{i^{-b}}(u) + 1$ elements of $Q[p]$, represented by $x_j \in X$. Write

$$p x_j = x_j^+ + m_j, \qquad x_j^+ \in X^+, \quad m_j \in K.$$

Identify each m_j with a vector in the kernel lattice. This identification preserves the localized module and all orders, since equivariant forms on the kernel extend with the same filtration degree by exactness of filtered de Rham comparison [BC09, Proposition 6.3.3]. The value-field condition is preserved by the same extension of forms. Thus only the kernel of the period map is used here, rather than an exactness assertion for its cokernel.

Apply Lemma 16.38 to these kernel vectors. A primitive combination of the m_j is $w + z$, where w has orders $c^{[0]}$ and z has orders $c^{[b,u]}$, both over zero. For $x = \sum c_j x_j$ this says $px - w$ has depth $c^{[b,u]}$. Lemma 16.40 gives $x - m$ of that depth for some $m \in K$. It still has orders R_0, because x and m do. The original elements of $Q[p]$ therefore have a primitive relation. Apply Lemma 16.39 and then the quotient map to the constant modules.

Lean, formal context. `FewLowJumps` (L141-L413), `FitTwistsExist` (L377). □

Lemma 16.42 (Summing the multiplicity costs)**.** *Suppose $c \geq R_0$, (16.38) holds, and a target c' satisfies $c'(j,v) \leq c(j,v)$ for $j \neq b$. Suppose $u \mapsto c'(b,u)$ is antitone on the allowed jumps. Then*

$$\ell\big(D_i(0,c) : D_i(0,c')\big) \leq f \sum_{u \in U_{i^{-b}}} m_{i^{-b}}(u) \sum_{c(b,u) \leq n < c'(b,u)} \big(v_p(n) + 3b\big). \tag{16.40}$$

Proof. If a step remains, choose the least jump u whose current order is below its target. Every smaller jump v has already reached its target, so antitonicity gives $c(b,v) \geq c'(b,v) \geq c'(b,u) > c(b,u)$. Lemma 16.41 applies. Raising this one order removes exactly one summand from the right side. For $b > 0$ the level-zero free interval is unchanged; for $b = 0$ it has length 1 and contains no integer. Induct on the finite number of steps and add colengths. When no step remains the current depth implies c', which contributes zero colength. Decreases at other levels cost nothing. This also proves the assertion if some target orders at level b are below the original orders.

Lean, formal context. `FewLowJumps` (L141-L413), `FitTwistsExist` (L377). □

The preceding division and length arguments are the ordinary proof of the single-step multiplicity estimate.

Removing the pole and choosing the depths

Fix integers $a, W \geq 0$, $\eta \geq 1$, a positive integer b_i at each letter, and signs $\varepsilon_i \in \{1, -1\}$. Assume

$$\begin{aligned} b_i + W + \eta &\leq (p-3)a, \qquad 6(a + \eta + W) < b_i, \\ p \nmid n \quad &\text{if } n \neq 0 \text{ and } \min\{|n|, |n - b_i|, |n + b_i|\} \leq 6(a + \eta + W). \end{aligned} \tag{16.41}$$

Suppose the side under consideration has jumps in $[-\eta, \eta] \cup \varepsilon_i[b_i, b_i + W]$. Its extension has the same allowed sets, with 0 already in the low window. Define

$$n_i = f \sum_{|u| \leq \eta} m_i(u), \qquad r_i = f \sum_{u} m_i(u).$$

The multiplicities here are those of the kernel lattice, not those of its extension by the parameter lattice.

Lemma 16.43 (Deep points before the pole is removed)**.** *Every parameter has a period point of depth $R_{2a+\eta}$ and orders at least*

$$o(j,u)=\begin{cases}\alpha(u), & j=0,\\ -u, & j>0,\end{cases}\qquad \alpha(u)=\begin{cases}2a+\eta-u, & u<-a,\\ \min\{2a+\eta-u,0\}, & u\geq -a.\end{cases} \tag{16.42}$$

In particular $C_i\subset D_i(-\eta,R_{2a+\eta})$. Also $p^aC_i\subset D_i(0,R_{2a+\eta})$.

Proof. Every jump is at most $b_i+W+\eta$, hence at most $(p-3)a$. Theorem 16.27 gives both annihilators E^a and p^a for the actual period cokernel. The first gives a point over the original parameter with orders and depth R_{-a}; the second gives a point over p^a times the parameter with orders and depth R_0.

Raise level-zero depth to $R_{2a+\eta}$ in both cases. For a low jump the necessary indices lie between $-a-\eta$ and $2a+2\eta$. For a far jump they lie in the corresponding interval about b_i or $-b_i$, with deviation at most $3a+2\eta+W$. All these integers are within the range in (16.41), and the zero index is free as well. Corollary 16.25 therefore raises the depth at no cost. Apply Lemma 15.62 to the first point to get (16.42).

For low jumps, $\alpha(u)\geq -\eta-u$. A negative far jump is below $-a$, so the same inequality holds; a positive far jump gives $\alpha(u)=2a+\eta-u$, again sufficient. This proves the first inclusion. The second point retains its original orders R_0, giving the second inclusion.

Lean, formal context. `FewLowJumps` (L418-L759), `FitTwistsExist` (L377). □

Lemma 16.44 (The cost of removing a pole)**.** *Under* (16.41)*,*

$$\ell(C_i : D_i(0,R_{2a+\eta}))\leq af\sum_{-\min(a,\eta)\leq u<0} m_i(u)\leq an_i. \tag{16.43}$$

Proof. Put $s=2a+\eta$. Let X be the module of period points in M at letter i with depth R_s and orders (16.42). Lemma 16.43 gives $\pi(X)=C_i$ and $p^aC_i\subseteq D_i(0,R_s)$. On all allowed jumps the initial orders already dominate R_0, except possibly at level zero for $-\min(a,\eta)\leq u<0$, where $\alpha(u)=0$.

Use the adapted basis of invariant forms from Lemma 15.39 for the extension lattice. It has $h=\sum_{-\min(a,\eta)\leq u<0} m_i(u)$ forms at these jumps, since the parameter lattice adds jumps only at zero. Taking their residues on X defines a $\mathbb{Z}_p$-linear map $X\to\Lambda[1/p]^h$. Indeed each form value v is integral in B_{dR}^+ and belongs to the level-zero value field by Lemma 15.51. As in the proof of Lemma 15.59, scaling by a power of p and applying the scalar retraction gives $p^Nv\equiv w \bmod tB_{\mathrm{dR}}^+$ with $w\in S$. The residue of w is $w(-p)\in\Lambda$.

In the kernel the indicated basis forms have positive order. Each earlier basis form either occurs in the same residue map or has positive order by (16.42). Thus every form of each indicated type has positive order. Lemma 15.62 therefore raises the orders to $s-u$. Thus the projection of the kernel lies in $D_i(0,R_s)$, and the residue image surjects onto $C_i/D_i(0,R_s)$. Any $fh+1$ elements of the image are linearly dependent over $\mathbb{Q}_p$. Dividing a nonzero relation by a coefficient of least valuation makes it integral with one coefficient equal to 1. The quotient is killed by p^a, so Lemma 16.39 bounds its length by afh.

Lean. `relLength_constantAt_range_deepConstants_le`. Lean uses the one-step estimate at line 491 and sums over pole orders. The residue map treats the obstructing jumps together. □

For a letter with negative far window define the depth

$$d_i^-(j,u)=\begin{cases}b_i+W+a+\eta, & j=0,\\ W+1-u+2a\,\mathbf{1}_{\{|u|\leq\eta\}}, & j=1,\\ -u, & j\geq 2.\end{cases} \tag{16.44}$$

For a positive far window use $d_i^+ = R_{2a+\eta}$, and write $d_i = d_i^{\varepsilon_i}$.

Lemma 16.45 (The additional length for a negative far window)**.** *If $\varepsilon_i = -1$ and $i^- = \sigma^{-1}i$ denotes the preceding Frobenius letter, then*

$$\ell\big(D_i(0, R_{2a+\eta}) : D_i(0, d_i^-)\big) \le (a+1)n_i + 3(W+1)r_{i^-} + 6an_{i^-}. \tag{16.45}$$

Proof. Let q be the least integer with $T = qp \ge b_i + W + a + \eta$. Thus

$$b_i + W + a + \eta \le T < b_i + W + a + \eta + p.$$

First change the depth at level zero to the constant T, leaving the higher depths $-u$. The target is antitone, and the free-interval condition for a level-zero change is automatic.

For a low jump, the costs are bounded by

$$\sum_{0 \le n \le T} v_p(n) = v_p(T!) \le \frac{T}{p-1} \le a+1,$$

where the summand at 0 is 0. The last inequality uses $b_i + W + \eta \le (p-3)a$, which gives $T \le (p-1)(a+1)$. For a negative far jump, a step divisible by p and less than T would be at most $T - p$; the entire possible interval then lies within $6(a + \eta + W)$ of b_i. Such a nonzero multiple is excluded by (16.41). These far lines therefore cost zero. By Lemma 16.42, this change costs at most $(a+1)n_i$ in length.

Next raise level-one depth from $-u$ to $W + 1 - u + 2a\mathbf{1}_{\{|u| \le \eta\}}$, and lower level-zero depth from T to $b_i + W + a + \eta$ if necessary. The level-zero interval $(T, T+p)$ contains no multiple of p, as required for a level-one step. The target at level one is antitone on the allowed jumps: within each window it decreases with slope -1, and the separation $b_{i^-} > 6(a + \eta + W)$ exceeds the possible extra $2a$ at the low window.

There are $W + 1$ steps for a far jump and $W + 1 + 2a$ for a low jump. Every step is zero or prime to p, because it is within $6(a + \eta + W)$ of 0 or $\pm b_{i^-}$. Its cost is therefore at most 3 by Theorem 16.24. The starting depth is at least R_0: at level zero this uses $-u \le b_i + W + \eta \le T$ for the negative far window. Apply Lemma 16.42 again. This gives $3(W+1)r_{i^-} + 6an_{i^-}$, proving the result.

Lean, formal context. `FewLowJumps` (L418-L759), `FitTwistsExist` (L377). □

Theorem 16.46 (A submodule with all the required depths)**.** *Under (16.41) there is a $\mathbb{Z}_p$-submodule V of the parameter lattice C such that every $c \in V$, for every index i, has a point over $e_i(1 \otimes c)$ with orders R_0 and depth d_i. Moreover*

$$\operatorname{length}_{\mathbb{Z}_p}(C/V) \le (8a+1)\sum_i n_i + 3(W+1)\sum_i r_i \le 11\sum_i \big((a+1)n_i + (W+1)r_i\big). \tag{16.46}$$

Proof. Take

$$V = \bigcap_i \{c \in C : e_i(1 \otimes c) \in D_i(0, d_i)\}.$$

For a positive far window Lemma 16.44 gives colength at most an_i. For a negative one, add Lemma 16.45. Thus for every index the colength is at most

$$(2a+1)n_i + 3(W+1)r_{i^-} + 6an_{i^-}.$$

The colength of a finite intersection is at most the sum of the colengths, by the diagonal map to the respective quotients. The map $i \mapsto i^-$ permutes the letters, so the sum is the first expression in (16.46). The second inequality is numerical. The definition of V gives the points.

Lean, formal context. `FewLowJumps` (L418-L759), `FitTwistsExist` (L377). □

These results reconstruct pole removal and the two depth changes, followed by the intersection over letters. In particular the low-jump multiplicity is retained throughout the length estimate.

The two paths and the normalized defect

Lemma 16.47 (A deeper point on the Tate-dual side)**.** *For two sides with the opposite windows in (16.41), the Tate-dual side of every admissible pairing has, at each letter, a point over* 1 *whose orders are*

$$q(j,w)=\begin{cases}\zeta(w), & j=0,\\ -w, & j>0,\end{cases}\qquad \zeta(w)=\begin{cases}a-w, & w<-a,\\ \min\{a-w,0\}, & w\geq -a.\end{cases}\tag{16.47}$$

Its depth at level zero is at least $a-w$.

Proof. The jumps of this side are 0 and the integers $w=-u-v-1$, where u,v belong to opposite windows. Such a w is within $2\eta+W+1$ of 0, or within $\eta+W+1$ of one of $b_i,-b_i$. Also $w\leq b_i+W+\eta\leq(p-3)a$; when both jumps are low this uses $b_i>6(a+\eta+W)$.

Theorem 16.27 gives the annihilator E^a, hence a point with orders and depth R_{-a} over 1. Raise its level-zero depth to R_a. Every necessary index is within $6(a+\eta+W)$ of 0 or $\pm b_i$: the largest extra deviation is $a+2\eta+W+1$, which is within that range since $\eta\geq 1$. All these steps are free. The depth-improvement lemma gives (16.47). Its level-zero function ζ is antitone, as is the function $-w$ at the other levels, so the general evaluation criterion (16.31) applies to these orders.

Lean, formal context. `FewLowJumps` (L795-L894), `FitTwistsExist` (L377). □

Theorem 16.48 (The bounded-colength submodule cups to zero)**.** *Let the side S have windows of sign ε_i, and let V be the submodule from Theorem 16.46. For every crystalline side T with the opposite windows, every admissible pairing, and every pair of parameters c for T and $d\in V$ for S, the cup product is a continuous coboundary. There is no additional power of p in this assertion.*

Proof. Choose points x_i for T as in Lemma 16.43, with orders (16.42) and depth $R_{2a+\eta}$. Choose points y_i for S over d, with orders R_0 and depth d_i, using the definition of V. All these points project to the original parameters. The ordinary Tate-dual point with orders R_{-a} gives the rule $R_{l,a}$ of (16.30) at cost zero.

Write

$$M(u)=\min\{-u,\alpha(u)\}=\begin{cases}0, & -a\leq u<0,\\ -u, & \text{otherwise.}\end{cases}$$

Thus $M(u)+u\geq -a$ always. On an allowed window it also satisfies $M(u)+u\geq-\eta$, because a negative far jump is strictly below $-a$. This formula includes the endpoint $u=-a$.

Suppose first that $\varepsilon_i=1$. The far window of T at i is negative, and (16.42) gives nonnegative level-zero order for x_i. Use the path

$$(\varphi x_{i^-},\varphi y_{i^-})\longrightarrow(x_i,\varphi y_{i^-})\longrightarrow(x_i,y_i).$$

For the first move use $R_{1,a}$. At level zero the moving vector has nonnegative order and the resting point has depth at least 1. At level one the moving order is at least $M(u)$, with u a jump at i^-. If the sign of S at i^- is positive, the resting depth is $2a+\eta-v$, and

$$M(u)+2a+\eta-v\geq a-u-v.$$

If that sign is negative, the resting depth is $b_{i^-}+W+a+\eta$. Since $v\geq -b_{i^-}-W$ and $M(u)+u\geq-\eta$, one again has

$$M(u)+b_{i^-}+W+a+\eta\geq a-u-v.$$

At all higher levels the order-depth sum is at least $-u-v$; at level two this uses $d_{i^-}(1,v) \geq -v$. This checks every case of $R_{1,a}$, including a change of sign between the letters.

For the second move use $R_{0,a}$. At level zero a low jump v of S has y_i of nonnegative order by depth improvement. For a positive far jump it has order at least $2a+\eta-v$. Consequently the sum of the moving order and the depth $2a+\eta-u$ of x_i is at least $a-u-v$ in both cases: for low v use $v \geq -\eta$, and for far v add the two displayed $2a+\eta$ terms. At positive levels y_i and φy_{i^-} both have orders at least $-v$, and x_i has depth $-u$. The second change satisfies the pairing condition.

Now suppose $\varepsilon_i = -1$. The constant positive level-zero depth in (16.44) and the original orders R_0 imply that y_i has nonnegative level-zero order. Use the reverse path

$$(\varphi x_{i^-}, \varphi y_{i^-}) \longrightarrow (\varphi x_{i^-}, y_i) \longrightarrow (x_i, y_i).$$

The first change satisfies the pairing condition by $R_{1,a}$. At level zero its order is nonnegative and the resting depth is at least 1; at level one the sum is at least $-v+(2a+\eta-u) \geq a-u-v$; above it the sum is at least $-u-v$.

For the remaining move, use the Frobenius of the deeper Tate-dual point in Lemma 16.47 at i^-. At level zero its order is 0. A low jump of the moving side gives moving order at least 0. A positive far jump gives moving order at least $2a+\eta-u$, so adding the resting depth gives

$$(2a+\eta-u)+(b_i+W+a+\eta) \geq 3a+2\eta \geq 1.$$

The same positivity is immediate for a low jump.

At level one put $w=-u-v-1$, with the jumps now read at i^-. The required inequality is

$$\zeta(w)+M(u)+W+1-v+2a\mathbf{1}_{\{|v|\leq\eta\}} \geq 1. \tag{16.48}$$

There are three cases. If v is low, then $\zeta(w) \geq -a-w$ and $M(u) \geq -a-u$, so the left side is at least $W+2$. If v is far and u is low, separation gives $|w| > a$, hence $\zeta(w) = a-w$; together with $M(u) \geq -a-u$ this again gives at least $W+2$. If both are far, their signs are opposite, $|u+v| \leq W$, and $M(u) = -u$. In this case $\zeta(w) \geq -(W+1)-w$, and the left side is at least 1. This proves (16.48). At levels at least two the three orders are $-w, -u, -v$ and sum to 1. The general evaluation criterion (16.31) therefore makes this move safe as well.

Finally the Galois-difference pairing satisfies the pairing condition by $R_{0,a}$. At level zero, if $\varepsilon_i = 1$ the two depths add to $4a+2\eta-u-v$. If $\varepsilon_i = -1$ they add to $3a+2\eta+b_i+W-u$, which is at least $a-u-v$ because $v \geq -b_i-W$. At positive levels their sum is at least $-u-v$.

Insert zero moves to make the two displayed paths into the four-move paths of Theorem 15.57. Lemma 15.58 sums their safety conditions, and Theorem 15.57 gives the continuous cup coboundary on H, and on G under the index condition stated at the beginning of the subsection. Every parameter and every Tate-dual point used here has scaling exponent zero, so no power of p is added.

Lean, formal context. `FewLowJumps` (L795-L894), `FitTwistsExist` (L377). □

The level-one inequality is the calculation in the negative-window safe move. The two explicit paths above also verify the final cup assertion of the few-low-jump theorem, without leaving a separate stop-point construction as an input.

Theorem 16.49 (The normalized defect with few low jumps)**.** *Let $\mathcal{O}$ be a coefficient discrete valuation ring with the root of unity and letter projectors just specified, let L be a crystalline $\mathcal{O}$-lattice of rank r, and let $\mathcal{Q}$ be a class of crystalline lattices with the opposite windows to L on an unramified chart. Assume the letter-window conditions* (16.41)*, with $\eta \geq 1$ and $p \geq 5$. Suppose that*

at each coefficient embedding at most μ Hodge–Tate jumps of L, counted with multiplicity, lie in $[-\eta,\eta]$. Then the normalized defect satisfies

$$\widehat{\lambda}_{\mathcal{O}}\left(\frac{Z_f(L)}{H_{\mathcal{Q}}(L)}\right) \le 11f(\mu(a+1)+r(W+1)). \tag{16.49}$$

Here $\widehat{\lambda}_{\mathcal{O}}(\mathcal{O}/p\mathcal{O})=1$. The class may consist of all crystalline lattices with the prescribed opposite jump sets.

Proof. Take the universal side of L, with parameter lattice $C=Z_f(L)$ and the identity cocycle map. This is finite free over $\mathcal{O}$ by the previously stated crystalline cocycle finiteness and rank theorem. Let $D=[\mathrm{Frac}(\mathcal{O}):\mathbb{Q}_p]$, and let d_i be the number of coefficient embeddings lying over letter i. Different letters have distinct Teichmüller residues, so these sets of embeddings are disjoint and $\sum_i d_i \le D$.

The jump multiplicity on letter i is the sum of the multiplicities at its coefficient embeddings. Therefore the quantities in Theorem 16.46 satisfy

$$n_i \le f\mu d_i, \qquad r_i \le frd_i.$$

That theorem gives a $\mathbb{Z}_p$-submodule $V\subset C$ with

$$\mathrm{length}_{\mathbb{Z}_p}(C/V) \le 11fD\big(\mu(a+1)+r(W+1)\big).$$

Theorem 16.48, applied to every crystalline cocycle of every tested lattice and every admissible pairing, puts V in $C\cap \mathrm{Ann}_{\smile}(\mathcal{Q},L)$. The defect quotient is thus a quotient of C/V.

If $e_{\mathcal{O}}, f_{\mathcal{O}}$ are the ramification index and residue degree of the coefficient field, restriction of scalars gives

$$\mathrm{length}_{\mathbb{Z}_p} X = f_{\mathcal{O}}\,\mathrm{length}_{\mathcal{O}} X, \qquad D=e_{\mathcal{O}}f_{\mathcal{O}}$$

for every finite $\mathcal{O}$-module X. Consequently $\widehat{\lambda}_{\mathcal{O}}(X)=\mathrm{length}_{\mathbb{Z}_p}(X)/D$. Divide the preceding bound by D to obtain (16.49). No coefficient-field ramification factor remains.

Lean, formal context. `CupOrthogonalityBounds` (L287-L340), `FitTwistsExist` (L377). □

This is the ordinary reconstruction of the few-low-jump defect bound. Its period inputs are the frame constructed in Theorem 16.19, the actual annihilators of Theorem 16.27, and the proved depth operators of Theorem 16.24. The sparse-pattern and arbitrary-pattern cut estimates at the other primes are separate assertions.

16.7 Small crystalline loss and vanishing of the local defect

The nonzero-pole estimates have an immediate application to the valuation appearing in (15.26). We give the arithmetic explicitly, since a small valuation will force a genuine gap in the depth steps. Throughout this subsection h,G,b,η,W and δ_0 are nonnegative integers satisfying

$$\begin{gathered} h\ge 30, \qquad G\ge 99h^3, \qquad b\ge 99G(h^3+1),\\ 0\le \delta_0<\eta, \qquad 99(\eta+W)\le h^3. \end{gathered} \tag{16.50}$$

Here δ_0 is the shift of a far window; it is unrelated to the scalar $z\tau$ in the period construction. For a prime p put

$$\lambda = v_p\left((2b)!\left(\prod_{\substack{b-30h^3<n\le b+30h^3\\ n>0}} n\right)^{h^3}\right), \qquad a=2(\lambda+1). \tag{16.51}$$

All valuations here are normalized by $v_p(p)=1$.

Lemma 16.50 (What the factorial and window product detect)**.** *The valuation in* (16.51) *satisfies*

$$2b < p(\lambda + 1). \tag{16.52}$$

If $\lambda < h^3$, then p divides no positive integer in $(b - 30h^3, b + 30h^3]$ and $p > 198G$.

Proof. Among the factors of $(2b)!$ there are $\lfloor 2b/p \rfloor$ multiples of p. Hence $\lambda \geq v_p((2b)!) \geq \lfloor 2b/p \rfloor$, which proves (16.52). If a factor in the displayed window product were divisible by p, its h^3-th power would contribute at least h^3 to λ. This proves the exclusion. Finally, $\lambda < h^3$ gives $\lambda + 1 \leq h^3$, so

$$p > \frac{2b}{\lambda + 1} \geq \frac{198G(h^3 + 1)}{h^3} > 198G.$$

Lean, formal context. `CrystallineCocycles` (L186-L229), `FitTwistsExist` (L377). □

Lemma 16.51 (The precise small-loss gap and height inequalities)**.** *Suppose $\lambda < h^3$, set $n = 6(a + \eta + W)$, and take $\beta \in \{b + \delta_0, G + \delta_0\}$. Then $p \geq 7$,*

$$n < \beta, \qquad \beta + W + \eta \leq (p - 3)a, \tag{16.53}$$

and no nonzero multiple of p lies within distance n of 0, β or $-\beta$.

Proof. Since $a \leq 2h^3$ and $\eta + W \leq h^3/99$,

$$n \leq (12 + 6/99)h^3 < 13h^3 < G \leq \beta, \qquad \delta_0 + n < 14h^3 < 30h^3.$$

Also $G + \delta_0 + n < 2G < p$ by Lemma 16.50. Thus the neighborhoods of 0 and $\pm(G + \delta_0)$ contain no nonzero multiple of p. If a nonzero multiple u of p were within distance n of $\pm(b + \delta_0)$, its positive absolute value would satisfy

$$||u| - b| \leq \delta_0 + n < 30h^3.$$

It would therefore be a forbidden factor in the product of Lemma 16.50. This proves the gap assertion.

For the height bound, the size conditions give $\beta + W + \eta \leq b + 2\eta + W < 2b$. Since $p \geq 7$ and $a = 2(\lambda + 1)$,

$$(p - 3)a = 2(p - 3)(\lambda + 1) \geq p(\lambda + 1) > 2b.$$

This proves both assertions of (16.53).

Lean, formal context. `CrystallineCocycles` (L186-L229), `FitTwistsExist` (L377). □

The same estimates apply for every index even when the choice of β and the sign of the far window vary with the letter. They are the ordinary arithmetic behind the valuation detection lemmas and the small-loss gap lemma.

Lemma 16.52 (A common upper jump for the three extensions)**.** *Let $\eta \leq \beta$, let $\varepsilon \in \{1, -1\}$, and suppose*

$$U \subset [-\eta, \eta] \cup -\varepsilon[\beta, \beta + W], \qquad U' \subset [-\eta, \eta] \cup \varepsilon[\beta, \beta + W].$$

Every member of U, U', or $\{-u - u' - 1 : u \in U,\ u' \in U'\}$ is at most $\beta + W + \eta$. Adding the trivial parameter jump 0 preserves this upper bound.

Proof. The bounds for U and U' follow from their displayed windows. For the dual tensor jump, if both original jumps are low, then $-u - u' - 1 \leq 2\eta - 1 \leq \beta + W + \eta$. If one is low and the other is in a far window, their sum is at least $-\eta - \beta - W$. If both are far, their signs are opposite and their sum is at least $-W$. These three cases give the asserted bound. The number $\beta + W + \eta$ is nonnegative, which also handles the parameter jump.

Lean, formal context. `CrystallineCocycles` (L453-L470), `FitTwistsExist` (L377). □

Theorem 16.53 (Exact orthogonality in the small-loss range)**.** *Let $F/\mathbb{Q}_p$ be unramified of degree f, and work with the actual form-defined modules on its unramified Kummer chart. Let $\mathcal{C}$ be a class of crystalline $\mathcal{O}$-lattices and let L be a crystalline $\mathcal{O}$-lattice. Include all the crystalline cocycle extensions and all extensions of the Tate-dual tensor associated with the admissible pairings of (15.5). At each letter i suppose the class and L have allowed jump sets U_i, U_i' in the opposite windows*

$$\begin{aligned} &U_i \subset [-\eta,\eta] \cup -\varepsilon_i[\beta_i, \beta_i + W], \\ &U_i' \subset [-\eta,\eta] \cup \varepsilon_i[\beta_i, \beta_i + W], \qquad \beta_i \in \{b + \delta_0, G + \delta_0\}. \end{aligned} \tag{16.54}$$

Assume (16.50), (16.51) and $\lambda < h^3$. Either of the following conditions is sufficient for every tested cup product to be a continuous coboundary:

1. *every negative member of $U_i \cup U_i'$ is at most $-h/2$, and $\lambda < \lfloor h/5 \rfloor$;*
2. *$U_i' \subset [-\eta, 0]$ for every letter i.*

Consequently

$$Z_f(L) \subset \mathrm{Ann}_{\smile}(\mathcal{C}, L), \qquad \widehat{\lambda}_{\mathcal{O}}\Big(\frac{Z_f(L)}{H_{\mathcal{C}}(L)}\Big) = 0. \tag{16.55}$$

No additional power of p is used in this conclusion.

Proof. Lemmas 16.51 and 16.52 bound every jump of each of the three relevant extensions by $(p-3)a$. Theorem 16.27, with $s = 0$ and $t = a$, shows that E^a annihilates all their standard period cokernels, for every index. Thus the nonzero-pole cup theorem has $\kappa = 0$. The gap hypothesis is exactly Lemma 16.51. The chart is unramified, so its different allowance is zero.

In the first case,

$$2a = 4(\lambda + 1) \le 4\lfloor h/5 \rfloor < h.$$

Every negative allowed jump is therefore strictly less than $-a$, and neither jump set meets $[-a, 0)$. Theorem 16.34 and Corollary 16.35 give cup exponent zero. In the second case Theorem 16.37 gives the same conclusion. Apply the result to each crystalline cocycle of L, each crystalline cocycle of each member of $\mathcal{C}$, and every admissible finite-coefficient pairing. This is precisely the first inclusion in (16.55); the quotient in that formula is zero. Continuity of its primitives is supplied by the Witt-cochain construction used in the cited cup theorems.

Lean, formal context. `CrystallineCocycles` (L453-L470), `FitTwistsExist` (L377). □

Lemma 16.54 (The remaining length in the small-loss case)**.** *Let $Q, m \ge 1$, let $\rho, \mu \ge 0$, and in addition to (16.50) suppose*

$$99(Q^2 + (W+1)Q + Q + W + m) \le h. \tag{16.56}$$

If $\lfloor h/5 \rfloor \le \lambda$, and $K = \lfloor 2\lambda/Q \rfloor$, then

$$11\big(\mu(2\lambda + 3) + \rho(W+1)\big) \le 10^5\big(\rho K m + \mu\lambda\big). \tag{16.57}$$

Proof. For $h \ge 30$, $\lfloor h/5 \rfloor \ge h/5 - 1 \ge h/6$. Consequently $\lambda \ge 1$ and

$$2\lambda \ge h/3 \ge 33(W+1)Q, \qquad W + 1 \le K.$$

Also $2\lambda + 3 \le 6\lambda$ and $m \ge 1$. Therefore

$$11\big(\mu(2\lambda + 3) + \rho(W+1)\big) \le 66\mu\lambda + 11\rho K \le 10^5\big(\rho K m + \mu\lambda\big).$$

Lean, formal context. `CrystallineCocycles` (L453-L470), `FitTwistsExist` (L377). □

Theorem 16.55 (The complete small-loss branch of the local estimate)**.** *Retain the unramified local setting and the opposite windows of Theorem 16.53, with $\lambda < h^3$. Let $\rho = \operatorname{rank}_{\mathcal{O}} L$ and let $\mu \geq 0$ be an integer. Suppose either*

1. *every negative allowed jump is at most $-h/2$, and at every coefficient embedding the number of jumps of L in $[-\eta, \eta]$, counted with multiplicity, is at most μ;*
2. *the allowed jumps of L all belong to $[-\eta, 0]$.*

Assume also (16.56), with $Q, m \geq 1$. Then

$$\widehat{\lambda}_{\mathcal{O}}\left(\frac{Z_f(L)}{H_{\mathcal{C}}(L)}\right) \leq f\, 10^5 \left(\rho m \left\lfloor \frac{2\lambda}{Q} \right\rfloor + \mu\lambda\right). \tag{16.58}$$

In the second case the left side is zero. It is also zero in the first case when $\lambda < \lfloor h/5 \rfloor$.

Proof. The zero assertions are Theorem 16.53. It remains to consider the first case with $\lambda \geq \lfloor h/5 \rfloor$. Lemma 16.51 verifies all the height, far-window and prime-gap hypotheses of Theorem 16.49, with $a = 2(\lambda + 1)$. The condition $\delta_0 < \eta$ ensures $\eta \geq 1$. That theorem, including its conversion from $\mathbb{Z}_p$-length to normalized $\mathcal{O}$-length, gives the stronger bound

$$11 f\big(\mu(2\lambda + 3) + \rho(W + 1)\big).$$

Apply Lemma 16.54. This proves (16.58) with the exact floor in $\lfloor 2\lambda/Q \rfloor$.

Lean, formal context. `CrystallineCocycles` (L453-L470), `FitTwistsExist` (L377). □

Corollary 16.56 (The local defect in the small-loss case)**.** *In Theorem 16.55, take*

$$Q = kt_*, \qquad \mu = \left\lfloor \frac{\rho m}{kx} \right\rfloor, \qquad k, m, t_*, x \geq 1, \quad x \leq t_*.$$

Let D_p have the definition (15.26), with $\lambda_p = \lambda$. Then the normalized local defect is at most

$$f\, \frac{\rho D_p}{x}. \tag{16.59}$$

For $\rho = r_0, x = 1$ and for $\rho = 1, x = t_*$, these are the two inequalities in (15.24) at an unramified place whose data satisfy these small-loss hypotheses.

Proof. Substitute the displayed values in (16.58) and apply Lemma 15.20. Its nonnegative term $b\, v_p(M)$ can only enlarge the allowance. The local degree is f because the chart field is unramified. Thus no ramification or coefficient-degree factor is lost in (16.59).

Lean, formal context. `CrystallineCocycles` (L453-L470), `FitTwistsExist` (L377). □

The cut construction at small primes. When the prime is small or the crystalline loss is at least h^3, one needs closer control of the frame after Frobenius. Theorem 16.63 bounds its denominator, Theorem 16.73 constructs the points across a long gap, and Theorem 16.82 assembles the paths along each Frobenius orbit. Corollary 16.93 gives the resulting local bound.

Quantitative approximation of a frame after Frobenius

The long-gap construction needs a quantitative assertion in addition to the frame matrices. We prove it here with its denominator $p^{4\delta}$. Retain $A = A_{\mathrm{inf}}$, $R = A_{\mathrm{max}}$, $B^+ = R[1/p]$, $S = \Lambda[[z]]$ and $E = p + z^e v(z)$ with $v(z) \in S^\times$. Put $K_0 = \operatorname{Frac}\Lambda$ and $R_S = \widehat{S[E/p]}_p$, so $S_B = R_S[1/p]$ inside B^+. Let $\Pi : B^+ \to S_B$ be the extension of the scalar coordinate $\lambda_0 : A \to S$ in Lemma 16.9. It is S_B-linear, fixes S_B, and sends A into S.

Lemma 16.57 (The completed scalar chart and its approximations)**.** *There is a faithful description*

$$R_S \simeq \Lambda\langle T\rangle[z]/(z^e - pT) = \bigoplus_{r=0}^{e-1} z^r \Lambda\langle T\rangle, \qquad T = z^e/p. \tag{16.60}$$

Here $\Lambda\langle T\rangle$ consists of power series whose coefficients tend to zero p-adically. Substitution embeds this ring and S_B into $K_0[[z]]$. Consequently S_B is z-adically separated, evaluation at $z = 0$ is a homomorphism $S_B \to K_0$, and

$$f \in S_B,\ f(0) = 0 \quad \Longrightarrow \quad f/z \in S_B.$$

For every integer $a \geq 0$ one also has

$$p^a R_S \subset S + E^a R_S. \tag{16.61}$$

Proof. The identities $E/p = 1 + v(z)z^e/p$ and $z^e/p = v(z)^{-1}(E/p - 1)$ identify the two polynomial subrings $S[E/p] = S[z^e/p]$. The presentation $S[T]/(pT - z^e)$ is p-torsion-free: if $pF = (pT - z^e)G$, reduction modulo p shows that G is divisible by p, and cancellation gives the assertion. After inverting p it embeds in $K_0[[z]]$ by $T = z^e/p$, which proves this presentation for the subring.

Modulo p^m the relation gives $z^{em} = 0$. Every power series in z can therefore be truncated, and the quotient is a free module on $1, z, \dots, z^{e-1}$ over $(\Lambda/p^m)[T]$, with $z^e = pT$. Taking inverse limits proves (16.60). Explicitly the series $\sum_{n\geq 0} c_n z^n$ representing an element of R_S satisfy

$$p^{\lfloor n/e\rfloor} c_n \in \Lambda, \qquad p^{\lfloor n/e\rfloor} c_n \longrightarrow 0 \quad p\text{-adically as } n \to \infty.$$

These coefficients are unique. This also shows that the formal series map is continuous in each coefficient and that division by z, when the constant term is zero, loses at most one power of p.

For completeness this abstract completion embeds faithfully in the actual period ring. On $A[E/p]$ the formula

$$\sum_j a_j (E/p)^j \longmapsto \sum_j \lambda_0(a_j)(E/p)^j$$

is well-defined by extending λ_0 to $A[1/p]$. It takes values in $S[E/p]$, preserves powers of p, and is the identity on that subring. Its extension to completions is a left inverse to $R_S \to R$. Thus the formal description is valid for the embedded ring being used here.

Every element of R_S has a presentation $\sum_{j\geq 0} s_j (E/p)^j$ with $s_j \in S$ tending to zero p-adically: choose polynomial lifts at successive powers of p and collect their coefficients. After multiplication by p^a, the terms with $j < a$ lie in S, and the remaining terms are E^a times a convergent series in R_S. This proves (16.61).

Lean, formal context. `RadiusLemma` (L232-L542), `FitTwistsExist` (L377). □

Lemma 16.58 (A row step with its factorial cost)**.** *Choose γ fixing the chart constants and θz, with $\gamma z = qz$, $q = [\epsilon]$, as in Lemma 16.21. Suppose*

$$\gamma Y = YG, \qquad G \in \mathrm{Mat}_r(A), \qquad G - I \in z\tau\,\mathrm{Mat}_r(A).$$

Let a row $W \in S_B^r$ and a row $A_0 \in A^r$ satisfy

$$A_0 = mY_i - z^{n+1}W, \qquad m, n \geq 0.$$

Then, with $v_n = v_p(n+1)$, there are rows $s \in S^r$ and $W' \in S_B^r$ such that

$$p^{v_n+1}W = s + zW'. \tag{16.62}$$

Moreover

$$p^{v_n+1}A_0 + z^{n+1}s = (p^{v_n+1}m)Y_i - z^{n+2}W'$$

is an integral row of the same kind at the next index.

Proof. By Lemma 15.32, $\gamma W - W \in z\tau(B^+)^r$. Put

$$D = \frac{W(G-I) - q^{n+1}(\gamma W - W)}{z\tau} \in (B^+)^r.$$

Subtracting the two expressions for the transformed row gives

$$A_0G - \gamma A_0 = z^{n+1}((q^{n+1}-1)W - z\tau D).$$

The scalar identity (15.64) gives $\alpha \in A^\times$ and $\beta \in A$ with $\tau E(p^{v_n} + \beta z) = \alpha(q^{n+1}-1)$. Its proof applies to the present chart: it uses only the associated Frobenius factors of $q^{n+1}-1$ and $\varphi^a E \equiv p \pmod z$. Write $E = p + zr_E$, $r_E \in S$. Then

$$\alpha(A_0G - \gamma A_0) = \tau z^{n+1}\left(p^{v_n+1}W - zW''\right), \quad W'' = \alpha D - (p^{v_n}r_E + \beta E)W. \tag{16.63}$$

The left side is integral. Repeated use of Lemma 15.31, first for τ and then for z, shows that its quotient by τz^{n+1} lies in A^r. Applying Π to that quotient gives (16.62), with $W' = \Pi(W'')$. Here $\Pi(W) = W$ and $\Pi(zW'') = z\Pi(W'')$. The last displayed identity follows by substitution and has integral left side because $s \in S^r \subset A^r$.

Lean. `RadiusLemma` (L291). □

Lemma 16.59 (A convergent series after Frobenius)**.** *Suppose $Y = I + zW$ with $W \in \mathrm{Mat}_r(S_B)$, and retain the transformation hypotheses of Lemma 16.58. Let $N = p^\ell \geq 3e$ and put $t_N = z^N/p^2$. Evaluation $X \mapsto t_N$ defines an S-algebra homomorphism*

$$\mathrm{ev} : S[[X]] \longrightarrow R_S.$$

There is a matrix $F \in \mathrm{Mat}_r(S[[X]])$ with

$$p^2\varphi^\ell(W) = \mathrm{ev}(F). \tag{16.64}$$

In particular $\varphi^\ell(Y) = \mathrm{ev}(I + XF)$, where $I + XF$ is an invertible matrix over $S[[X]]$.

Proof. Since $z^e = pT$ in R_S, $z^N \in p^3R_S$, and hence $t_N \in pR_S$. Every evaluated power series therefore converges in the p-adically complete ring R_S. For a row W_0 of W, start the row recurrence with $m = 1$, $n = 0$, and A_0 the corresponding row of I. It produces rows $W_n \in S_B^r$ and $s_n \in S^r$ with

$$p^{v_p(n+1)+1}W_n = s_n + zW_{n+1}.$$

Put $a_n = n - v_p(n!) \geq 0$ and $H_n = p^{a_n+2}\varphi^\ell(W_n)$. The factorial identity $a_n + 2 = a_{n+1} + v_p(n+1) + 1$ gives

$$H_n = p^{a_{n+1}}\varphi^\ell(s_n) + t_N H_{n+1}.$$

Consequently

$$H_0 = \sum_{n<M} t_N^n p^{a_{n+1}} \varphi^\ell(s_n) + t_N^M H_M$$

for every M. In the formal embedding $S_B \subset K_0[[z]]$ the remainder is divisible by z^{NM}. The series with coefficients $p^{a_{n+1}}\varphi^\ell(s_n) \in S^r$ evaluates to an element of R_S, with the same formal series as H_0. Faithfulness of that embedding proves equality. This proves (16.64) entry by entry. Finally $I + XF$ has constant matrix I, so its inverse exists in the formal power-series matrix ring.

Lean. `RadiusLemma` (L347), `RadiusLemma` (L406). □

Theorem 16.60 (Factorization of the period matrix on the specified radius)**.** *Let $Z \in \mathrm{Mat}_r(S_B)$ and $Z' \in \mathrm{Mat}_r(B^+)$ satisfy*

$$Z'Z = E^k I, \qquad \gamma Z = ZG, \quad G \in \mathrm{Mat}_r(A).$$

For $p^\ell \geq 3e$ and any integer $h \geq 0$, there are $C \in \mathrm{GL}_r(K_0)$ and $\mathcal{V} \in \mathrm{GL}_r(S[[X]])$ such that

$$\varphi^\ell Z = E_\ell^h C \,\mathrm{ev}(\mathcal{V}), \qquad E_\ell = \varphi^\ell E, \quad X \mapsto z^{p^\ell}/p^2. \tag{16.65}$$

In particular $\mathrm{ev}(\mathcal{V})$ and its inverse have entries in $R_S \subset R$, and the chart subgroup fixes C.

Proof. Apply Π entrywise to $Z'Z = E^k I$ and use its S_B-linearity. Evaluation at $z = 0$ then shows that $Z(0)$ is invertible over K_0. Put $B = Z(0)^{-1}$ and write $BZ = I + zW$ with $W \in \mathrm{Mat}_r(S_B)$ using Lemma 16.57. The element γ fixes B, so $\gamma(BZ) = (BZ)G$.

The entries of $\gamma Z - Z$ lie in $z\tau B^+$ by Lemma 15.32. Multiplication by Z' shows $E^k(G - I) \in z\tau\,\mathrm{Mat}_r(B^+)$. Lemma 15.31 therefore gives $G - I \in z\tau\,\mathrm{Mat}_r(A)$. The preceding lemma supplies

$$\varphi^\ell(B)\varphi^\ell(Z) = \mathrm{ev}(I + XF).$$

Write $E = p + zr_E$. Then

$$E_\ell = p\,\mathrm{ev}(\varepsilon), \qquad \varepsilon = 1 + pX\varphi^\ell(r_E) \in S[[X]]^\times.$$

Set $\mathcal{V} = \varepsilon^{-h}(I + XF)$ and $C = p^{-h}\varphi^\ell(B)^{-1}$. These choices give (16.65). Since C has entries in K_0, every element of the chart subgroup fixes it.

Lean. `RadiusLemma` (L417), `RadiusLemma` (L477). □

Lemma 16.61 (Approximation of a power series modulo E^a)**.** *Let $N \geq 3e$, let $\mathrm{ev}(X) = z^N/p^2$, and suppose $ea \leq N\delta$, with $a, \delta \geq 0$ integers. For every $F \in S[[X]]$ there is $s_F \in S$ with*

$$s_F - p^{2\delta}\,\mathrm{ev}(F) \in E^a S_B. \tag{16.66}$$

Proof. Truncate F after its first δ coefficients. Multiplication by $p^{2\delta}$ makes this truncation an element of S, and leaves the tail in $z^{N\delta}R_S$. Since $ea \leq N\delta$, this tail lies in $p^a R_S$. Equation (16.61) writes it as an element of S plus an element of $E^a R_S$. Adding the two S-terms proves the claim, also for $a = 0$.

Lean. `RadiusLemma` (L191), `RadiusLemma` (L232). □

Lemma 16.62 (The two approximations and the exponent 4δ)**.** *Under the hypotheses of Lemma 16.61, let $\mathcal{V} \in \mathrm{GL}_r(S[[X]])$ and put $V = \mathrm{ev}(\mathcal{V})$. The set*

$$\mathcal{G}_a(V) = \{y \in S^r : \ Vy - r_y \in E^a(S_B)^r \text{ for some } r_y \in \mathbb{Z}_{\geq 0}^r\}$$

has S-span containing $p^{4\delta}S^r$.

Proof. Approximate the entries of $\mathcal{V}^{-1}$ and $\mathcal{V}$ by the preceding lemma. This gives matrices $M, M' \in \mathrm{Mat}_r(S)$ with

$$M \equiv p^{2\delta} V^{-1}, \qquad M' \equiv p^{2\delta} V \pmod{E^a S_B}.$$

Thus $VM \equiv p^{2\delta} I$. For $R_0 = p^{4\delta} I - MM'$ one also has $VR_0 \equiv 0$. Each column of M and each column of R_0 belongs to $\mathcal{G}_a(V)$, with the indicated constant vectors $p^{2\delta} e_j$ and 0 respectively. For every $x \in S^r$,

$$p^{4\delta} x = R_0 x + M(M'x),$$

which proves the spanning assertion. The two matrix approximations account for both factors $p^{2\delta}$.
Lean. `RadiusLemma` (L232). □

Theorem 16.63 (Approximation of points over the value field)**.** *Let L be a crystalline lattice on the chart, and use its actual integral frame from Theorem 16.19. Let $N_\ell(L)$ denote the value-field submodule of $\varphi^\ell N_A(L)$, as in Theorem 15.27. If*

$$3e \le p^\ell, \qquad ea \le p^\ell \delta, \tag{16.67}$$

then $p^{4\delta} N_\ell(L)$ is contained in the S-span of the vectors $y \in N_\ell(L)$ such that

$$g\mu(y) - \mu(y) \in \mathrm{Fil}^a B_{\mathrm{dR}} \quad \text{for every invariant form } \mu \text{ and every } g \in H. \tag{16.68}$$

The same assertion holds after projecting to a letter.

Proof. Let the jumps of L lie in $[-h, h]$ and use the frame P, Q, Z, Y of Theorem 16.19. Its vectors $n_j = \tau^{-h} Q e_j$ give $N_\ell(L) = \bigoplus_j S\varphi^\ell n_j$ by Theorem 15.27. The matrix Z satisfies $YZ = E^{2h} I$ and $\gamma Z = ZG_\gamma$ with G_γ integral, by the final assertion of Theorem 16.18. Therefore Theorem 16.60 applies with exponent h.

For an invariant form μ, let $b_{\mu,\ell} = \mu(e)\varphi^\ell(P)$. Equivariance of μ and invariance of the crystalline period matrix show that this row is fixed by H. Its entries belong to the finite chart field $K \subset B_{\mathrm{dR}}^+$, as in the proof of Theorem 15.27. For $v \in S^r$, the Frobenius version of (15.36), followed by (16.65), gives

$$\mu\left(\sum_j v_j \varphi^\ell n_j\right) = b_{\mu,\ell} C V v, \qquad V = \mathrm{ev}(\mathcal{V}). \tag{16.69}$$

Both $b_{\mu,\ell}$ and C are fixed by H, and their entries have nonnegative de Rham order. If $v \in \mathcal{G}_a(V)$, write $Vv = r_v + E^a w$. The first term $b_{\mu,\ell} C r_v$ in (16.69) is fixed by H; the second has order at least a because $w \in (S_B)^r \subset (B^+)^r$. The filtration is Galois-stable, so this vector satisfies (16.68). Lemma 16.62 now gives the desired span after multiplication by $p^{4\delta}$.

Letter projectors commute with the chart subgroup and preserve $N_\ell(L)$. Composing an invariant form with a letter projector is again an invariant form. Applying a projector to the spanning relation therefore proves the final assertion.
Lean. `FrameTheorem` (L560). □

This proves the quantitative input used by the long-gap argument, including its exponent. It reconstructs the radius calculation and the approximation of value-field points. The formal scalar model in Lemma 16.57 supplies the separation needed for the series argument; no separation claim about all of B^+ is required for that step.

The cost of steps in two integer intervals

Write $\nu_p(j) = v_p(j)$ for $j \neq 0$, and set $\nu_p(0) = 0$ in the depth-step cost. Put $L_p(B) = \lfloor \log_p B \rfloor$ for $B \geq 1$, with $L_p(0) = 0$.

Lemma 16.64 (The cost of two intervals)**.** *Let p be prime and R, n, B be nonnegative integers. If $c \in \mathbb{Z}$, $R \leq pn$, and $|c| + R \leq B$, put $\mathcal{W}(R, c) = ([-R, R] \cup [c - R, c + R]) \cap \mathbb{Z}$. Then, for $D \geq 0$,*

$$\sum_{j \in \mathcal{W}(R,c)} (\nu_p(j) + D) \leq 8n + 2L_p(B) + (4R + 2)D. \tag{16.70}$$

In particular, this bounds the cost at level a with $D = d_0 + 3a$ in Theorem 16.24.

Proof. The union contains at most $4R + 2$ integers. For $B = 0$ it contains only zero. Otherwise write $\nu_p(j)$ for $j \neq 0$ as the number of i with $1 \leq i \leq L_p(B)$ and $p^i \mid j$. Consecutive multiples of p^i differ by p^i, so each interval contains at most $2R/p^i + 1$ of them. Summing over both intervals gives

$$\sum_{j \in \mathcal{W}(R,c)} \nu_p(j) \leq \frac{4R}{p - 1} + 2L_p(B) \leq 8n + 2L_p(B).$$

Adding the constant cost proves (16.70).

Lean, comparison. Lean first bounds one interval and then sums over the union. Here the powers of p are counted in both intervals at once. `sum_twoWindows_le`. □

16.8 Taylor coefficients across a long gap

We now use the quantitative frame approximation of Theorem 16.63 to construct the point needed at a long gap. The argument extracts a Taylor coefficient whose parameter projection is known, then removes its projection error by an actual cokernel annihilator. The source is the long-gap construction.

Throughout this subsection the chart is unramified, $S = \Lambda[[z]]$, $E = z + p$, and $\Lambda = W(\mathbf{F}_{p^f})$ is fixed by the chart subgroup H. Write $i^{-a} = \sigma^{-a} i$ for the letter a steps before i, and use the actual integral localized module $N_{\mathrm{loc}}(L)$ of a crystalline lattice L. Its integral standard-order part is

$$N_{\ell,i}(L) = e_i \sum_j S\varphi^\ell n_j, \qquad c_\ell(a, u) = \begin{cases} 0, & a < \ell, \\ -u, & a \geq \ell. \end{cases}$$

The constructed frame and Theorem 15.27 identify this module with the value-field vectors having the displayed orders. It is the integral S-module; its A_{inf}-span is $M_{\ell,i}(L) = e_i \varphi^\ell N_A(L)$.

For an integer-valued function o on jumps put

$$c_{\ell,o}(a, u) = c_\ell(a, u) + \mathbf{1}_{a=0} o(u).$$

Let $\mathcal{C}_{\ell,D,i}(L)$ consist of the vectors $\xi \in N_{\ell,i}(L)$ for which every $g\xi - \xi$, $g \in H$, has orders $c_{\ell,D}$, where the second subscript denotes the constant function with value D. Here $D \geq 0$ is an integer depth parameter. These are the constants modulo E^D used below. They form a $\mathbb{Z}_p$-module, stable under multiplication by Λ. The latter assertion uses that H fixes Λ; stability under arbitrary multiplication by S is not being asserted.

Lemma 16.65 (Extracting a constant with positive order)**.** *Let $\ell > 0$, $\xi \in \mathcal{C}_{\ell,D,i}(L)$, and let y have the standard orders c_ℓ. If $\xi + Ey$ has orders $c_{\ell,o}$, then ξ has orders*

$$c_\ell(a, u) + \mathbf{1}_{a=0} \begin{cases} D, & o(u) > 0, \\ 0, & o(u) \leq 0. \end{cases}$$

Proof. Multiplication by E adds one to the order at level zero and leaves the orders at positive levels unchanged. Consequently $\xi = (\xi + Ey) - Ey$ has order at least $\min\{o(u), 1\}$ at level zero. When $o(u) > 0$, this is at least one, because the orders are integers. Its depth there is at least D. The positive-order improvement of Lemma 15.62 therefore gives order at least D. When $o(u) \le 0$, use the original standard order, which is zero at level zero because $\ell > 0$. The standard orders at all other levels already hold for ξ.

Lean. `LongGap` (L123). □

Lemma 16.66 (Extracting a Taylor coefficient)**.** *Let $\ell > 0$, let every ξ_n belong to $\mathcal{C}_{\ell,D,i}(L)$, and let y have orders c_ℓ. Suppose $o(u) \le D$ for every jump and*

$$x = \sum_{n=0}^{N} E^n \xi_n + E^{N+1} y$$

has orders $c_{\ell,o}$. Then ξ_N has orders

$$c_\ell(a,u) + \mathbf{1}_{a=0} \begin{cases} D, & N < o(u), \\ 0, & o(u) \le N. \end{cases} \tag{16.71}$$

Proof. Induct on N. The case $N = 0$ is Lemma 16.65. For the induction step write $x = \xi_0 + Ey'$, where

$$y' = \sum_{n=0}^{N-1} E^n \xi_{n+1} + E^N y.$$

All terms of y' have the standard orders. The preceding lemma says that ξ_0 has order at least D where $o(u) > 0$, and at least zero otherwise. Since $o \le D$, it follows that ξ_0 has orders $c_{\ell,o}$. Subtract it from x and divide by E. The result y' has orders $c_{\ell,o-1}$, with $o - 1 \le D$. Apply the induction hypothesis to y'. The condition $N - 1 < o(u) - 1$ is exactly $N < o(u)$, giving (16.71). Division by E occurs in the already defined localized module, so it creates no additional existence assertion.

Lean. `LongGap` (L145). □

Lemma 16.67 (Integral digits in powers of E)**.** *For every $s \in S$ there are $w_n \in \Lambda$ such that, for each $M \ge 0$, one has*

$$s = \sum_{n=0}^{M-1} E^n w_n + E^M t_M, \qquad t_M \in S.$$

Proof. The ring $S = \Lambda[[z]]$ is complete for (p, z). Substitution $U \mapsto z + p$ gives an isomorphism $\Lambda[[U]] \to S$, with inverse $z \mapsto U - p$, since both substitutions preserve the maximal ideals. The coefficients of the inverse image of s are the w_n. Truncation at any degree M gives the claimed remainder.

Lean. `exists_expansion_pow_E`. The change of variable gives all digits at once. □

Theorem 16.68 (Taylor expansion with nearly invariant coefficients)**.** *Let $\ell > 0$, $3 \le p^\ell$, and $D \le p^\ell \delta$, where $D, \delta \ge 0$. For $x \in N_{\ell,i}(L)$, there are $\xi_n \in \mathcal{C}_{\ell,D,i}(L)$ such that, for each $M \ge 0$, there is $y_M \in N_{\ell,i}(L)$ with*

$$p^{4\delta} x = \sum_{n=0}^{M-1} E^n \xi_n + E^M y_M. \tag{16.72}$$

Proof. Theorem 16.63, with ramification index one, writes $p^{4\delta}x$ as a finite S-linear combination of standard-order value-field vectors whose invariant form values are fixed modulo Fil^{D} at level zero. Project every vector in that combination to letter i. The projection commutes with H; composing a form with this projection is again an invariant form. Thus the projected vectors $b_1, \dots, b_q$ have depth at least D at zero. At every other level their Galois differences retain the standard orders. Since $\ell > 0$, they therefore lie in $\mathcal{C}_{\ell,D,i}(L)$, and

$$p^{4\delta}x = \sum_{j=1}^{q} s_j b_j, \qquad s_j \in S.$$

Expand s_j by Lemma 16.67, using digits $w_{j,n} \in \Lambda$ and remainders $t_{j,M}$. Set

$$\xi_n = \sum_{j=1}^{q} w_{j,n} b_j, \qquad y_M = \sum_{j=1}^{q} t_{j,M} b_j.$$

The first vector is a constant modulo E^{D} because Λ is fixed by H. The second belongs to $N_{\ell,i}(L)$, which is an S-module. Substitution proves (16.72).

Lean. `LongGap` (L233). □

Lemma 16.69 (A monic polynomial which raises the early levels)**.** *Let $s, g, \ell \geq 0$. Suppose y has orders $-u - s$ at level zero and $-u$ at every positive level. Put*

$$N = s + g \sum_{a=0}^{\ell-1} p^{a}, \qquad f(X) = X^{s} \prod_{a=0}^{\ell-1} \left((X-p)^{p^{a}} + p\right)^{g}. \tag{16.73}$$

Then $f \in \mathbb{Z}[X]$ is monic of degree N, and $f(E)y$ has orders

$$g\mathbf{1}_{a<\ell} - u.$$

Proof. Every factor in (16.73) is monic of the stated degree. Since $E = z + p$ and $\varphi z = z^{p}$,

$$f(E) = E^{s} \prod_{a=0}^{\ell-1} (\varphi^{a} E)^{g}.$$

At the de Rham point of level a, the element $\varphi^{a}E$ has order one and the other $\varphi^{b}E$, $b \neq a$, have order zero. The factor E^{s} first cancels the pole $-s$ at zero; the remaining factors add g precisely at the levels $a < \ell$. This proves the assertion.

Lean. `LongGap` (L283). □

Consider a crystalline cocycle extension

$$0 \longrightarrow Y \longrightarrow T \xrightarrow{\pi} C \longrightarrow 0$$

with trivial parameter lattice C, as in the earlier lifting-ideal construction. The finite allowed jump sets of T contain zero. Write $c_i = e_i(1 \otimes c)$ for $c \in C$. The constant c_i lies in $\mathcal{C}_{\ell,D,i}(C)$ for every ℓ, D, and π preserves the order and constant conditions: the sole jump of C is zero, and composition with π is an invariant form of that type on T.

Lemma 16.70 (The parameter projection of the leading coefficient)**.** *Let $\ell > 0$, let $f \in \mathbb{Z}[X]$ be monic of degree $N < D$, let $q \geq 0$, and suppose $\xi_n \in \mathcal{C}_{\ell,D,i}(T)$ and y has orders c_ℓ. If*

$$\pi\left(\sum_{n=0}^{N} E^n \xi_n + E^{N+1} y\right) = p^q f(E) c_i,$$

then

$$\pi(\xi_N) - p^q c_i \text{ has orders } c_{\ell,D}. \tag{16.74}$$

Proof. Write f_n for the coefficient of X^n in f and put $\eta_n = \pi(\xi_n) - p^q f_n c_i$. Each η_n belongs to $\mathcal{C}_{\ell,D,i}(C)$. The hypothesis gives

$$\sum_{n=0}^{N} E^n \eta_n + E^{N+1} \pi(y) = 0.$$

Apply Lemma 16.66 to this zero vector, with o the constant function D. Because $N < D$, η_N has orders $c_{\ell,D}$. The leading coefficient $f_N = 1$ then gives (16.74).

Lean. `LongGap` (L313). □

Lemma 16.71 (A long-gap coefficient before exact correction)**.** *Suppose $3 \leq p^\ell$, $D \leq p^\ell \delta$, and $E^s \in I_{0,i}(T)$. Suppose also that every allowed jump at i^{-a}, $0 \leq a < \ell$, is at most $g \geq 0$. Put*

$$N = s + g \sum_{a=0}^{\ell-1} p^a$$

and assume $N < D$. For every $c \in C$ there is $\xi \in \mathcal{C}_{\ell,D,i}(T)$ with orders

$$o_{\ell,D,N}(a,u) = c_\ell(a,u) + D\mathbf{1}_{a=0}\mathbf{1}_{u<-N} \tag{16.75}$$

such that $\pi(\xi) - p^{4\delta} c_i$ has orders $c_{\ell,D}$.

Proof. The inequality $3 \leq p^\ell$ implies $\ell > 0$. The actual annihilator $E^s \in I_{0,i}$, the common integral lift, and Lemma 16.31 supply a point $y \in N_{\mathrm{loc},i}(T)$ over c_i with orders $-u - s$ at zero and $-u$ elsewhere. Take the polynomial f of Lemma 16.69. Its value on y has orders $g - u$ below level ℓ and $-u$ from level ℓ onward. The jump bound $u \leq g$ therefore puts $f(E)y$ in $N_{\ell,i}(T)$. At zero its order is also at least $o(u) = \min\{D, g - u\}$.

Use Theorem 16.68 and take its expansion through degree N:

$$p^{4\delta} f(E) y = \sum_{n=0}^{N} E^n \xi_n + E^{N+1} y_{N+1}.$$

Apply Lemma 16.66 with $o(u) = \min\{D, g - u\}$. If $u < -N$, then $N < g - u$, because $g \geq 0$, and also $N < D$. Hence $N < o(u)$, and the extracted coefficient $\xi = \xi_N$ has the orders (16.75). Project the expansion by π. Since $\pi(y) = c_i$, its projection is $p^{4\delta} f(E) c_i$. Lemma 16.70 proves the asserted error estimate.

Lean. `LongGap` (L375). □

Lemma 16.72 (Lifting an error at a later standard level)**.** *If $p^\kappa \in I_{0,i^{-\ell}}(T)$ and $w \in N_{\ell,i}(C)$, there is $v \in N_{\ell,i}(T)$ with $\pi(v) = p^\kappa w$.*

Proof. Frobenius transport of the lifting ideals gives $p^{\kappa} \in I_{\ell,i}(T)$, since Frobenius fixes p. Thus $p^{\kappa}w$ has a lift in the actual A_{inf}-module $M_{\ell,i}(T)$. Apply the vector retraction of Lemma 15.25 to this lift. The retraction preserves the standard orders, hence takes it to $N_{\ell,i}(T)$, and commutes with π. It fixes w, which is already in the integral value-field module. The retracted lift therefore has the required projection. No exactness of an integral Kisin functor is assumed.

Lean. `LongGap` (L355). □

Theorem 16.73 (The long-gap point with its exact exponent)**.** *On the unramified chart, suppose*

$$\begin{aligned} &3 \le p^{\ell}, \qquad D \le p^{\ell}\delta, \qquad E^{s} \in I_{0,i}(T), \qquad p^{\kappa} \in I_{0,i^{-\ell}}(T), \\ &u \le g \quad (u \in U_{i^{-a}},\ 0 \le a < \ell), \qquad N = s + g\sum_{a=0}^{\ell-1} p^{a} < D, \end{aligned} \tag{16.76}$$

where $D, \delta, s, \kappa, g, \ell$ are nonnegative integers. For every $c \in C$ there is a point $x \in N_{\mathrm{loc},i}(T)$ over $p^{\kappa+4\delta}c_i$, with orders $o_{\ell,D,N}$ from (16.75) and depth

$$d(a,u) = c_{\ell}(a,u) + D\mathbf{1}_{a=0}. \tag{16.77}$$

All annihilators in (16.76) refer to the actual period modules of the given extension.

Proof. Choose ξ from Lemma 16.71 and put $w = \pi(\xi) - p^{4\delta}c_i$. This vector belongs to the localized integral module of C, and has orders $c_{\ell,D}$. Consequently $E^{-D}w \in N_{\ell,i}(C)$. By Lemma 16.72, choose $v \in N_{\ell,i}(T)$ such that

$$\pi(v) = p^{\kappa}E^{-D}w.$$

Define

$$x = p^{\kappa}\xi - E^{D}v.$$

Its projection is exactly

$$\pi(x) = p^{\kappa}(w + p^{4\delta}c_i) - p^{\kappa}w = p^{\kappa+4\delta}c_i.$$

The correction $E^{D}v$ has orders $c_{\ell,D}$, which dominate $o_{\ell,D,N}$. Thus x retains (16.75). Every Galois difference of $E^{D}v$ has those same $c_{\ell,D}$ orders: the order conditions and the letter are stable under H. The differences of ξ have $c_{\ell,D}$ orders by its membership in $\mathcal{C}_{\ell,D,i}(T)$. Subtracting gives precisely (16.77). All operations take place in $N_{\mathrm{loc}}(T)$ or its Galois-stable order modules. In particular the error has been removed by an actual lift, rather than discarded after projection.

Lean. `LongGap` (L426). □

Corollary 16.74 (Transporting the long-gap point to its stop)**.** *Apply Theorem 16.73 at a letter $j = i^{-t}$. The point $\varphi^{t}x$ at i lies over $p^{\kappa+4\delta}c_i$. Its orders and depths are at least the following values:*

level a	*order*	*depth*
$a < t$	0	1
$a = t$	$D\mathbf{1}_{u<-N}$	D
$t < a < t+\ell$	0	0
$a \ge t+\ell$	$-u$	$-u$.

Thus the remaining depth corrections below $t+\ell$ are only zero-to-one steps at $t < a < t+\ell$.

Proof. Iterate the Frobenius order and depth transport in Lemma 15.62. Every newly introduced level has order at least zero and depth at least one; the old levels shift upward by t. Frobenius carries c_j to c_i and fixes p. The displayed table follows directly from (16.75) and (16.77).

Lean. `SidePoints` (L785). □

The strict inequalities $u < -N$ and $N < D$ were used at different steps: the first selects the coefficient whose order improves, and the second separates the leading parameter coefficient from the projection error. The factor $p^{4\delta}$ is paid exactly once, in the quantitative frame approximation. The additional p^{κ} is paid exactly once, to lift and remove the projection error at the preceding cut.

Cuts and the assembly of local cup bounds

We continue with the actual localized modules of the preceding sections, their parameter lattices, and the integral admissible pairing of two cocycle sides. The coefficient ring and the letter projectors are those already fixed above. Write H for the chart subgroup, i^{-a} for the letter a Frobenius steps before i, and $d_0 = 0$ when the ramification index e_K is 1, with $d_0 = v_p(e_K) + 1$ otherwise. All coboundaries below are continuous coboundaries on H. They descend to an ambient group when H is open of index prime to p, by the argument in Theorem 15.57; in particular there is no index factor when the chart group is the local group under consideration.

The allowed jump sets, including the trivial parameter jump, are

$$U_i \subseteq [-\eta, \eta] \cup -\varepsilon_i[\beta_i, \beta_i + V], \qquad U_i' \subseteq [-\eta, \eta] \cup \varepsilon_i[\beta_i, \beta_i + V], \qquad \varepsilon_i \in \{1, -1\}. \tag{16.78}$$

All interval restrictions on jumps concern integers. The nonnegative parameters satisfy $\eta \geq 1$ and $V + 4\eta \leq W$. The symbols Δ and B below denote a depth allowance and a height parameter; they are unrelated to the element $\delta = z\tau$.

The order functions used at a cut. For a nonnegative pole parameter s, put

$$P_s(a, u) = -u - s\,\mathbf{1}_{a=0}, \qquad D_{s,\eta,l}(a, u) = -u + s\,\mathbf{1}_{a=0} + \eta\,\mathbf{1}_{a<l}.$$

For any order function o define its Frobenius shift by

$$(\mathrm{sh}_{l,c}\, o)(a, u) = \begin{cases} c, & a < l, \\ o(a - l, u), & a \geq l. \end{cases}$$

Let $\mathcal{I}(o, d)$ denote the improved-order function in (15.86). We shall use

$$\begin{aligned} C_{s,\eta,l} &= \mathcal{I}(P_s, D_{s,\eta,l+1}), \\ T_{\eta,l}(a, u) &= \begin{cases} \max\{0, -u - \eta\}, & a < l, \\ -u, & a \geq l, \end{cases} \\ R_{l,s}(a, w) &= \begin{cases} 1, & a < l, \\ s - w, & a = l, \\ -w, & a > l. \end{cases} \end{aligned} \tag{16.79}$$

A standard point at a cut with previous-cut distance $l > 0$ has orders P_η and depth $D_{s,\eta,l}$. A stop is a point over the same parameter constant with orders $T_{\eta,l}$. These order modules are Galois-stable, so a stop also has depth $T_{\eta,l}$.

Lemma 16.75 (The pairing inequalities at the endpoints of an interval)**.** *Suppose $l, l_0 > 0$, and the Tate-dual cokernels at i and i^{-l} are killed by $p^{\kappa}E^{s}$ and $p^{\kappa}E^{s_0}$, respectively. Let x, x' be standard points at i with depth $D_{s,\eta,l}$, and let x_0, x_0' be standard points at i^{-l} with depth D_{s_0,η,l_0}. If a stop q on the first side makes the second-side move from $\varphi^l x_0'$ to x' safe, then the path*

$$(\varphi^l x_0, \varphi^l x_0') \longrightarrow (q, \varphi^l x_0') \longrightarrow (q, x') \longrightarrow (x, x')$$

satisfies the pairing condition with exponent κ. The analogous assertion holds with the sides interchanged. A zero move puts either path in the four-move form of Theorem 15.57.

Proof. Lemma 16.32 applies with the rule R_{l,s_0} at the preceding cut and $R_{0,s}$ at the current cut. Its proof applies to these antitone order functions: the Tate-dual point has orders $\mathrm{sh}_{l,0} P_{s_0}$, and, at its jump $-u-v-1$, this order plus $R_{l,s_0}(a, u+v)$ is exactly 1. Thus it suffices to check

$$R_{l,s_0}(a, u+v) \le \min\{T_{\eta,l}, \mathrm{sh}_{l,0} P_\eta\}(a,u) + (\mathrm{sh}_{l,1} D_{s_0,\eta,l_0})(a,v),$$
$$R_{0,s}(a, u+v) \le \min\{P_\eta, T_{\eta,l}\}(a,u) + D_{s,\eta,l}(a,v).$$

In the first inequality, below l the minimum is 0 and its companion depth is 1. At l the two terms are $-u-\eta$ and $s_0+\eta-v$, whose sum is the required s_0-u-v. Above l their sum is at least $-u-v$. In the second inequality, the sum at zero is $-u-\eta+s+\eta-v = s-u-v$. For $0<a<l$ the minimum is at least $-u-\eta$, and the companion depth is $\eta-v$. From l on the sum is $-u-v$. The difference of points over the same constant lies in the kernel and has at least the minimum of their orders. These calculations give the first and last moves; the middle move is the assumed safe move at the stop.

Lean. `exists_isSafePath_of_stop`. □

Theorem 16.76 (Summation over Frobenius orbits)**.** *Choose a set of cuts meeting every Frobenius orbit, and let l_i be the least positive integer for which i^{-l_i} is a cut. At every cut i, suppose there are standard points x_i, x_i' over $p^{e_1}c, p^{e_2}c'$, with pole allowance η, depths D_{s_i,η,l_i}, and a Tate-dual annihilator $p^\kappa E^{s_i}$. Suppose that one side has a stop at each cut, making the other side's move from the Frobenius translate of its preceding-cut point safe with exponent κ. Then every tested cup product is a continuous coboundary after multiplication by*

$$p^{e_1+e_2+\kappa}.$$

Proof. Set $d_i = 0$ at a cut and $d_i = l_i$ elsewhere, and define at every letter

$$X_i = \varphi^{d_i} x_{i^{-d_i}}, \qquad Y_i = \varphi^{l_i} x_{i^{-l_i}},$$

with the same definitions on the second side. The nearest-cut property gives $l_i = d_{i^{-1}}+1$. Reindexing the finite letter sum by Frobenius therefore gives

$$\varphi \sum_i X_i = \sum_i Y_i, \qquad \varphi \sum_i X_i' = \sum_i Y_i'.$$

At a noncut the two pairs agree. At a cut, the preceding lemma supplies a four-move safe path between them.

We also check the cocycle condition; it is not an additional assumption. At a point transported d steps from a cut, both Galois differences have orders at least $\mathrm{sh}_{d,1} D_{s,\eta,l}$. The sum of these depths, at jumps u, v, dominates $R_{d,s}(a, u+v)$: below d it is $2 \ge 1$, at d it is $2s+2\eta-u-v \ge s-u-v$, and above d it is at least $-u-v$. Apply the criterion for the Tate-dual pairing at that cut. This proves the condition on the two cocycles for X_i, X_i'. Lemma 15.58 sums the paths and these cocycle conditions. Theorem 15.57 now gives the continuous primitive with exponent $e_1+e_2+\kappa$.

Lean. `isScaledCupCoboundary_of_stops`. □

Lemma 16.77 (A stop when the signs are constant between cuts)**.** *Let $l > 0$. Suppose that, for $0 \le a < l$, one side has all its jumps at i^{-a} at most η, and the other side has all its jumps there at least $-\eta$. At i suppose both points have orders P_η and depth $D_{s,\eta,l+1}$, and their preceding-cut points have orders P_η. If $p^\kappa E^s$ kills the Tate-dual cokernel at i, then the first of these points is a stop for which the move of the other side satisfies the pairing condition with exponent κ.*

Proof. On the side whose jumps satisfy $v \le \eta$, depth improvement makes the orders nonnegative for $a < l$, and at least $-v-\eta$ there; above these levels the orders are at least $-v$. These are exactly the stop orders $T_{\eta,l}$. For the moving side use its jump $u \ge -\eta$ when $a < l$. The order of its move is at least $\min\{P_\eta, \mathrm{sh}_{l,0} P_\eta\}(a,u)$. At zero this minimum is $-u-\eta$, and its sum with the resting depth is $s-u-v$. At $0 < a < l$ it is $\min\{-u, 0\}$, whose sum with $\eta - v$ is at least $-u-v$. At $a = l$ it is $-u-\eta$, and the extra η in $D_{s,\eta,l+1}$ gives exactly $-u-v$. Above l the sum is again $-u-v$. These inequalities are the rule $R_{0,s}$, which proves the pairing condition. Exchanging the two sides proves the other orientation.

Lean. `exists_mem_stopPoints_isSafeMoveSnd`, `exists_mem_stopPoints_isSafeMoveFst`. □

The numerical conditions for the cuts. Take nonnegative integers $X, \Delta, \kappa, B, n, J, s_0, G$ and shifts s_i. We use the following explicit inequalities:

$$\begin{aligned} n + L_p(2B) + (J+1+d_0)W + d_0 G &\le X, \\ 12((J+1)X + \Delta) &< \kappa, \\ e_K(2B) \le p^{J+1}\Delta, \qquad 8e_K &\le p^{J+1}, \\ \beta_i + s_i + W &\le 2B \quad \text{for all } i, \\ G &\le (p-1)n, \\ \beta_i + W &\le (p-3)s_0 \quad \text{for all } i \quad (p \ge 5), \\ s_i + W &\le pn \quad \text{for all } i. \end{aligned} \tag{16.80}$$

Here L_p is the integer logarithm defined in Lemma 16.64. A cut is of an allowed kind if either

1. $s_i = 0$ and $\beta_{\varphi^t i} + W \le G$ for $1 \le t \le J$; or
2. the chart is unramified, $p \ge 5$, $s_i < \beta_i$, and $s_0 \le \kappa + s_i$.

The first kind will be used when the next J letters have small windows. The second uses the unramified trade between p and E.

Lemma 16.78 (Cokernels at the allowed cuts)**.** *Assume* (16.78) *and* (16.80)*. At any allowed cut i, the actual three cokernels of the paired sides and their Tate-dual side are annihilated by $p^\kappa E^{s_i}$.*

Proof. Each of the two original sides has all its jumps at i at most $\beta_i + W$. The Tate-dual jumps are 0 and $-u-u'-1$. If both u, u' are low, this is at most $2\eta - 1$; if exactly one is far, it is at most $\beta_i + V + \eta - 1$; if both are far their opposite signs give the upper bound $V - 1$. Each bound is at most $\beta_i + W$.

At a cut of the first kind use Theorem 16.30 with global upper jump $2B$ and successor upper jump G. By (16.80), $d_0 G \le X$, $G/(p-1) \le n \le X$, and $e_K(2B)/p^{J+1} \le \Delta$. Therefore its required exponent is at most

$$2JX + (2J+4)X + 8\Delta = 4(J+1)X + 8\Delta < \kappa.$$

This proves the annihilation with $s_i = 0$. At a cut of the second kind, every jump for every index is at most $(p-3)s_0 \le (p-3)(\kappa + s_i)$. Theorem 16.27, with $n = \kappa + s_i$, gives the required monomial $p^\kappa E^{s_i}$. The same argument applies to all three extensions just bounded.

Lean. `jumps_le_of_fitsWindows`, `killsCokernel_of_isKillingCut`. □

Lemma 16.79 (The cost of the cut points)**.** *Under* (16.80)*, let i satisfy $s_i = 0$ or $d_0 = 0$, and let $\sigma_j \in \{1, -1\}$ be the sign of a side's far window. For $0 \le a \le 2J+1$ take the union of the two*

integer windows of radius $R_a = s_i\mathbf{1}_{a=0} + W$*, centered at* 0 *and* $-\sigma_{i^{-a}}\beta_{i^{-a}}$*. Let* Σ_i *consist of the resulting pairs* (a, j)*. Then*

$$\kappa + 4\Delta + \sum_{(a,j)\in\Sigma_i} \big(\nu_p(j) + d_0 + 3a\big) \le 25\kappa. \tag{16.81}$$

Here $\nu_p(0) = 0$ *and* $\nu_p(j) = v_p(j)$ *for* $j \neq 0$*.*

Proof. At a given level the window radius is at most pn, and the absolute center plus its radius is at most $2B$. Lemma 16.64 bounds the cost by

$$8n + 2L_p(2B) + (4R_a + 2)(d_0 + 3a).$$

If the extra radius s_i is nonzero, it occurs at $a = 0$ and $d_0 = 0$, so its contribution to the last product is zero. Otherwise $R_a = W$ for that product. Since $W \ge 4$ and $a \le 2J + 1$,

$$(4W + 2)(d_0 + 3a) \le 30W(J + 1 + d_0).$$

Thus each level costs at most $30X$. There are $2J + 2$ levels, so the total costs at most $60(J + 1)X$. Finally

$$4\Delta + 60(J + 1)X \le 24\kappa$$

follows from $12((J + 1)X + \Delta) < \kappa$. This proves (16.81).

Lean. `sum_cost_windowSteps_le`. □

Lemma 16.80 (The standard points at an endpoint)**.** *Assume* (16.78) *and* (16.80)*. At an allowed cut* i*, for every* $l \le 2J + 1$ *and every parameter* c*, there is a point over* $p^{25\kappa}c$ *with orders* $C_{s_i,\eta,l}$ *and depth* $D_{s_i,\eta,l+1}$*. It also has orders* P_η*.*

Proof. Lemma 16.78 and Lemma 16.31 give an actual localized point over $p^\kappa c$ with orders and depth P_{s_i}. Apply the all-level composition Corollary 16.25 to raise its depth to $D_{s_i,\eta,l+1}$. At level zero the required integers lie between $-s_i - u$ and $s_i + \eta - u$, and at levels $1, \dots, l$ between $-u$ and $\eta - u$, with the upper endpoint excluded. A low jump lies in $[-\eta, \eta]$; a far jump has the form $\sigma(\beta + v)$, $0 \le v \le V$. These descriptions put every required step in the windows of Lemma 16.79, since $V + 4\eta \le W$. An allowed cut has $s_i = 0$ or $d_0 = 0$, so that lemma applies. Unused allowance is supplied by scalar multiplication. Depth improvement gives $C_{s_i,\eta,l}$.

For positive levels this improved bound is at least $-u$. At level zero, if $u < -s_i$, the original order is positive, so the new order is $s_i + \eta - u \ge -\eta - u$. Otherwise an allowed cut satisfies either $s_i = 0$ or $s_i < \beta_i$; the window condition then implies $u \ge -\eta$. The new order $\min\{s_i + \eta - u, 0\}$ is again at least $-\eta - u$. This proves the final assertion, including the boundary case $u = -s_i$.

Lean. `pointsOver_cutOrders_nonempty`, `mem_pointsOver_plainDepth_of_cutOrders`. □

Long gaps between cuts. For $0 \le t < l$, call the interval before i a long gap with pivot i^{-t} if the chart is unramified, $p \ge 5$, and

$$s_{i^{-l}} = 0, \qquad t + J + 1 = l,$$
$$s_0 + \omega\sum_{a=0}^{J} p^a < \beta_{i^{-t}}, \qquad \eta \le \omega, \tag{16.82}$$

and, for $0 \le a < l$, $a \neq t$, either $\varepsilon_{i^{-a}} \neq \varepsilon_{i^{-t}}$ or $\beta_{i^{-a}} + V \le \eta$. In addition require $\beta_{i^{-a}} + V \le \omega$ for $t < a < l$. Define the target depth of its stop by

$$D^{\mathrm{gap}}_{l,t,r}(a, u) = \begin{cases} r, & a = t, \\ 1, & a < l,\ a \neq t, \\ -u, & a \ge l. \end{cases}$$

Lemma 16.81 (The stop supplied by a long gap)**.** *Assume* (16.78)*,* (16.80)*, and* (16.82)*, with* $l \le 2J+1$ *and* i^{-l} *an allowed cut. Then the side whose far window is negative at the pivot has a point over* $p^{25\kappa}c$*, with orders* $T_{\eta,l}$ *and depth* $D^{\mathrm{gap}}_{l,t,2B}$*. If the other side moves between its cut point at* i *and the Frobenius translate of its cut point at* i^{-l}*, the move satisfies the pairing condition with exponent* κ*.*

Proof. Denote the resting side by R and its opposite by R'. At the pivot $j = i^{-t}$, the jumps of R are at most $\eta \le \omega$. At the J preceding letters they are at most ω, by the final condition in the definition of a long gap. The cut $i^{-l} = j^{-(J+1)}$ has shift zero; Lemma 16.78 therefore gives its annihilator p^{κ}. Theorem 16.27 and (16.80) give the pivot annihilator E^{s_0}. Put

$$N = s_0 + \omega \sum_{a=0}^{J} p^a.$$

Then $N < \beta_j \le 2B$, and $2B \le p^{J+1}\Delta$. Apply Theorem 16.73 with length $J+1$ and depth $2B$. It gives a point at j over $p^{\kappa+4\Delta}c$ with depth $2B$ at level zero, zero depth at levels $1, \dots, J$, and standard depth above those levels. Its order at level zero is $2B$ on jumps $u < -N$ and zero on the other jumps; it has zero orders at levels $1, \dots, J$ and standard orders afterwards.

Transport this point by φ^t to i. Its depth is already 1 below t, is $2B$ at t, is zero for $t < a < l$, and is $-u$ from l on. Raise only the zero depths between t and l to 1. The required pairs $(a, 0)$ lie in the windows of Lemma 16.79; its cost bound includes the initial 4Δ. Thus the resulting point lies over $p^{25\kappa}c$. For $a < l$, $a \ne t$, all resting-side jumps are at least $-\eta$, so the required stop order is zero. At the pivot, a low jump also requires only order zero; a negative far jump has $u \le -\beta_j < -N$, so its order is $2B$, at least $-u - \eta$ because $-u \le \beta_j + V \le 2B$. From level l on the standard orders are unchanged. This proves the asserted stop orders and depth.

The move of R' has orders at least

$$\min\{C_{s_i,\eta,l}, \mathrm{sh}_{l,0}\, C_{0,\eta,l_0}\}(a, u),$$

where l_0 is the preceding cut's own previous-cut distance. For $a < l$, $a \ne t$, the moving-side jumps satisfy $u \le \eta$. Its improved cut order is then nonnegative, so this minimum is zero; the resting depth is 1. At the pivot, the minimum is at least $\min\{\eta - u, 0\}$. Here $u \le \beta_j + V < \eta + 2B$, so this minimum plus $2B$ is at least 1. At and above l both order functions are at least $-u$, and the resting depth is $-v$. The sum therefore dominates $R_{l,0}(a, u+v)$ at every level. The Tate-dual annihilator at i^{-l} is p^{κ}, again by Lemma 16.78. The safety rule proves the final assertion. The same argument covers either orientation of the original two sides.

Lean. `pointsOver_longGapDepth_nonempty`, `exists_stop_of_isLongGap`. □

Theorem 16.82 (The cut cup bound)**.** *Assume* (16.78) *and* (16.80)*. Choose a set of allowed cuts such that every index has a cut between* 1 *and* $2J+1$ *steps before it. For the nearest cut before each cut* i*, write its distance as* l*. Suppose that either all the intervening far windows have a common sign, apart from windows ending at or below* η*, or there is a long gap as in* (16.82)*. Then every tested cup product is a continuous coboundary after multiplication by*

$$p^{51\kappa}. \tag{16.83}$$

Proof. Use Lemma 16.80 at every cut and on both sides, with its own nearest-cut distance. The points have exponent 25κ, orders P_η and depth $D_{s_i,\eta,l+1}$, which is at least the standard cut depth. The Tate-dual annihilators come from Lemma 16.78. A constant-sign interval has one side with jumps at most η and the other with jumps at least $-\eta$; Lemma 16.77 supplies its stop. At a long gap use Lemma 16.81. Theorem 16.76 gives the exponent $25\kappa + 25\kappa + \kappa = 51\kappa$.

Lean. `isScaledCupCoboundary_of_cut_windows`. □

Sparse large windows and arbitrary patterns

Lemma 16.83 (Cuts followed by small windows)**.** *Let T be a set of thick letters, and let $J \geq 1$. Suppose that for every letter j and every $1 \leq r, t \leq J$,*

$$T(j) \text{ and } T(\varphi^r j) \quad \Longrightarrow \quad \neg T(\varphi^{r+t} j).$$

Thus no three thick positions have both intervening forward distances in $[1, J]$. Call a letter a cut if none of its next J letters is thick. Then every letter has a cut between 1 and $2J+1$ steps before it. If i^{-l} is its nearest preceding cut, a thick letter i^{-r} with $0 \leq r < l$ must satisfy

$$r = 0 \quad \text{or} \quad r + J + 1 = l.$$

Proof. First observe that if a thick letter is not a cut, the letter $J+1$ steps before it is a cut. Otherwise a thick letter in the J positions before it, the given thick letter, and a thick letter in its next J positions form a forbidden triple. For a fixed i, first try $i^{-(J+1)}$. If it is not a cut, one of its next J letters is thick and equals i^{-r} for $1 \leq r \leq J$. Either i^{-r} is a cut, or the observation gives a cut at $i^{-(r+J+1)}$. Both possibilities lie in the required range.

Now let $0 < r < l$ and i^{-r} be thick. It is not a cut by minimality. Since the first J letters after i^{-l} are thin, $l - r \geq J + 1$. The observation makes $i^{-(r+J+1)}$ a cut. Minimality rules out $r + J + 1 < l$, so equality holds. These arguments apply on finite Frobenius cycles as well: they use positions along the orbit, and the sparsity hypothesis includes repetitions of a letter reached after a full cycle.

Lean, formal context. `CutExponentBounds` (L603-L1052), `FitTwistsExist` (L377). □

Theorem 16.84 (The sparse-window cup exponent)**.** *Let $\eta \geq 1$, $J \geq 1$, and $G + V \leq \eta$. In (16.78) take $\beta_i = b$ at thick letters and $\beta_i = G$ elsewhere, and assume the sparsity condition of Lemma 16.83. Suppose either*

1. *the chart is unramified, $p \geq 7$, and $10p^J\eta \leq b$; or*
2. *the signs ε_i are constant, and $8e_K \leq p^{J+1}$.*

If

$$2e_K(b + 5\eta) \leq p^{J+1}\Delta, \qquad 20(J + 1 + d_0)\eta + L_p(2(b + 5\eta)) \leq X, \tag{16.84}$$

then every tested cup product is a continuous coboundary with exponent

$$10^3(\Delta + (J + 1)X). \tag{16.85}$$

In the second case this result uses no long-gap construction.

Proof. Use zero shifts and the cuts of Lemma 16.83. In the general cut inequalities take

$$W = 5\eta, \quad B = b + 5\eta, \quad n = G_{\mathrm{cut}} = 10\eta, \quad \kappa = 12((J + 1)X + \Delta) + 1.$$

For $p \geq 5$ take $s_0 = \lfloor (b + 10\eta)/(p - 3) \rfloor + 1$; for $p = 2, 3$ put $s_0 = 0$, since the corresponding inequality is unused. The first inequality of (16.80) follows from

$$10\eta + 5(J + 1 + d_0)\eta + 10d_0\eta \leq 20(J + 1 + d_0)\eta.$$

The height and thin-window inequalities follow from $\beta_i \leq \max\{b, \eta\}$, $p \geq 2$, and the definition of s_0. The depth condition is (16.84). In the unramified case, $8e_K = 8 \leq p^2 \leq p^{J+1}$; the constant-sign

case assumes this inequality. All shifts satisfy the width condition. Since a cut has only thin successors, $\beta + W = G + 5\eta \le 6\eta \le G_{\mathrm{cut}}$ there, so every cut is allowed.

It remains to verify the sign-or-gap condition. With constant signs it is immediate. Otherwise consider a nearest-cut interval. The combinatorial lemma leaves at most two possibly large windows: at its current letter and at a pivot t with $t + J + 1 = l$. If the signs are not constant up to small windows, choose the pivot whose sign differs from the current sign. Every other interior letter is thin, so its far window ends at or below η; at the current letter the sign is opposite to the pivot. The long-gap conditions hold with $\omega = \eta$ once

$$s_0 + \eta \sum_{a=0}^{J} p^a < b$$

is checked. Put $A = p^J \eta$. The geometric sum is at most $2p^J$, $p - 3 \ge 4$, and $b \ge 10A \ge 10$. Hence

$$s_0 + \eta \sum_{a=0}^{J} p^a \le \frac{b}{4} + \frac{5}{2}\eta + 1 + 2A \le \frac{7}{10} b + 1 < b.$$

Thus Theorem 16.82 applies. Finally $A_0 = (J+1)X + \Delta$ is a positive integer, and

$$51\kappa = 612A_0 + 51 \le 1000A_0.$$

Increasing the exponent proves (16.85).

Lean, formal context. `CutExponentBounds` (L603-L1052), `FitTwistsExist` (L377). □

Lemma 16.85 (Two-size cut inequalities on an unramified chart)**.** *Suppose the chart is unramified, $p \ge 7$, and*

$$0 < G \le b, \qquad b < Kp^2, \qquad 2G < Kp, \qquad W \le K.$$

Put $q = \lfloor b/p \rfloor$, $L = L_p(2(b+W))$, $X = 3K + L$, $\Delta = 4K$, $\kappa = 12(2X + 4K) + 1$, and $s_0 = q + 12K$. If $\beta_i \le b$ and $0 \le s_i \le \max\{q - 7K, 0\}$, then (16.80) holds with $J = 1$, $n = K$, $B = b + W$, and $G + W$ in place of the general cut parameter G.

Proof. We have $K \ge 1$, $q < Kp$, and $d_0 = 0$. Thus the first cost inequality is $K + L + 2W \le 3K + L = X$. The depth bound follows from $2(b + W) \le 4Kp^2$, and $8 \le p^2$. Also $\beta_i + s_i + W \le b + q + W \le 2(b + W)$. The inequality $G + W \le (p-1)K$ follows from $2G < Kp$, $W \le K$, and $p \ge 7$. For the remaining height bound, use $b < pq + p$, $q < Kp$, and $p \le Kp$. They give

$$b + W < (p-3)q + 3Kp + p + K \le (p-3)q + (4p+1)K \le (p-3)(q + 12K),$$

where the last inequality is $4p + 1 \le 12(p-3)$. Finally, if $q \le 7K$, the shift plus W is at most $K \le pK$; otherwise it is at most $q - 6K < pK$. The strict cost inequality holds by the definition of κ.

Lean, formal context. `CutExponentBounds` (L603-L1052), `FitTwistsExist` (L377). □

Theorem 16.86 (The cup exponent for any pattern of the two sizes)**.** *On an unramified chart with $p \ge 7$, let the opposite windows in (16.78) have centers $\beta_i = b$ at an arbitrary set of thick letters and $\beta_i = G$ elsewhere. Suppose*

$$\eta \ge 1, \qquad V + 4\eta \le W \le K, \qquad 0 < G \le b, \qquad b < Kp^2, \qquad 2G < Kp.$$

Then every tested cup product is a continuous coboundary with exponent

$$10^4\big(K + L_p(2(b+W))\big). \tag{16.86}$$

Proof. Use the parameters of Lemma 16.85. Set $s_i = \max\{q - 7K, 0\}$ if the successor of i is thick, and $s_i = 0$ otherwise. Declare i to be a cut precisely when $s_i < \beta_i$. Every thick letter is a cut because $s_i \le q < b$. Every letter with a thin successor is also a cut, since $s_i = 0 < G \le \beta_i$. Consequently there is a cut one or two letters before every letter: if the first preceding letter is thick it is a cut; otherwise the second preceding letter has a thin successor.

Every cut is of an allowed kind. With a thin successor it has shift zero and its next window satisfies $\beta + W = G + W$. With a thick successor it satisfies

$$s_0 = q + 12K \le s_i + 19K \le s_i + \kappa,$$

so the unramified alternative applies.

Consider a nearest-cut interval. Its length is at most two. A length-one interval is plain, as is a length-two interval whose two signs agree. In the remaining case its current letter is thick, its preceding letter is thin and is not a cut, and the cut two steps back has shift zero. Indeed a thick preceding letter would be a cut, while a thin current letter would make its predecessor a cut. Since the intervening thin letter is not a cut,

$$G \le \max\{q - 7K, 0\}, \qquad \text{hence} \qquad G + 7K \le q.$$

This gives a long gap with $l = 2$, pivot $t = 0$, and $\omega = G + W$. The only nontrivial inequality is

$$q + 12K + (G + W)(1 + p) < b.$$

Since $G + W \le q - 6K$ and $q < Kp$, its left side minus $b \ge pq$ is at most

$$2q - 6Kp + 6K < -4Kp + 6K < 0.$$

The opposite signs supply the remaining long-gap condition, and $\eta \le W \le G + W$ bounds the low window. Thus Theorem 16.82 applies. With $L = L_p(2(b + W))$, the resulting exponent satisfies

$$51\kappa = 6120K + 1224L + 51 \le 10^4(K + L),$$

because $K \ge 1$. This proves the claimed allowance for an arbitrary pattern of thick and thin letters.
Lean, formal context. `CutExponentBounds` (L603-L1052), `FitTwistsExist` (L377). □

The preceding two theorems reconstruct the cut estimates in the cut, sparse-window, and arbitrary-pattern cup bounds. Their proofs include the stops, the order inequalities, and the Frobenius-orbit assembly. Their local hypotheses must still be verified for the actual Galois and weight data in any global application.

16.9 The remaining prime ranges and the full local defect bound

We apply the cut bounds to the prime ranges not covered by Theorem 16.55. The following inequalities strengthen (16.50) by specifying the cutoff Q for small primes and the logarithmic bound m. Let $Q, m, h, G, b, \eta, W, \delta_0$ be nonnegative integers with

$$\begin{gathered} Q, m \ge 1, \qquad h \ge 30, \qquad 99(Q^2 + (W + 1)Q + Q + W + m) \le h, \\ 99h^3 \le G, \qquad 99G(h^3 + 1) \le b, \qquad 2\log(4b) < m, \\ 0 \le \delta_0 < \eta, \qquad 99\big((\eta + W)Q + \eta + h\big) \le h^3. \end{gathered} \tag{16.87}$$

The logarithm in this display is natural. For a prime p, retain the valuation λ in (16.51) and put

$$K = \left\lfloor \frac{2\lambda}{Q} \right\rfloor, \qquad J = \lfloor \log_p(h^3) \rfloor. \tag{16.88}$$

The symbol K in this subsection is an integer allowance, not a local field.

Lemma 16.87 (The logarithmic bound for the Kummer chart)**.** *Assume* (16.87)*. Then*

$$\begin{aligned} Q \le h, \quad m \le h/99, \quad \eta + W \le h^3/(99Q), \quad \delta_0 + W \le G, \\ h^3 \le 4b, \qquad 2(b + \delta_0 + 10G) \le 4b, \qquad 2(b + \delta_0 + W + 4\eta) \le 4b. \end{aligned} \tag{16.89}$$

For every integer H with $1 \le H \le 4b$,

$$\lfloor \log_p H \rfloor < m. \tag{16.90}$$

If the chart has ramification index $1 \le e \le h$, and $d_0 = 0$ for $e = 1$, while $d_0 = v_p(e) + 1$ otherwise, then

$$J + 1 + d_0 \le h. \tag{16.91}$$

Proof. The first four assertions follow directly from (16.87), using $Q \ge 1$ and $\delta_0 < \eta$. In particular $\eta + W \le h^3/99$ and $G \ge 99h^3$. Thus $\delta_0 + 10G < 11G \le b$ and $\delta_0 + W + 4\eta \le 5(\eta + W) \le 5h^3/99 \le b$. These inequalities prove the last two bounds in (16.89); the bound for h^3 is immediate.

Since $p \ge 2$, integration of $1/t$ on $[1, 2]$ gives $\log p \ge \log 2 \ge 1/2$. Therefore

$$\lfloor \log_p H \rfloor \le \frac{\log H}{\log p} \le 2 \log(4b) < m.$$

In particular $J \le m - 1$. Also $v_p(e) \le \lfloor \log_p e \rfloor \le m - 1$ because $1 \le e \le h \le 4b$. Hence $d_0 \le m$ in both cases, and $J + 1 + d_0 \le 2m \le h$.

Lean, formal context. `CrystallineCocycles` (L333-L456), `FitTwistsExist` (L377). □

Lemma 16.88 (The two intervals on each eigenspace)**.** *Fix the coefficient root and letter projectors used in Theorem 16.49. At a coefficient embedding g, let t_g specify whether the embedding is thick, let $\epsilon_g \in \{1, -1\}$, and suppose the two allowed jump sets belong to*

$$[-\eta, \eta] \cup -\epsilon_g[\beta_g, \beta_g + W], \qquad [-\eta, \eta] \cup \epsilon_g[\beta_g, \beta_g + W], \qquad \beta_g = \begin{cases} b + \delta_0, & t_g, \\ G + \delta_0, & \neg t_g. \end{cases}$$

Let T be a predicate on the letters and let ϵ_i be their signs. Whenever g lies over i and either t_g holds or $p > Q$, assume

$$t_g \iff T(i), \qquad \epsilon_g = \epsilon_i.$$

At letter i take the union of the corresponding embedding jump sets and the parameter jump 0. Those unions satisfy

$$\begin{aligned} &U_i \subset [-R_p, R_p] \cup -\epsilon_i[\beta_i, \beta_i + W], \\ &U_i' \subset [-R_p, R_p] \cup \epsilon_i[\beta_i, \beta_i + W], \\ &R_p = \begin{cases} 2G, & p \le Q, \\ \eta, & p > Q, \end{cases} \qquad \beta_i = \begin{cases} b + \delta_0, & T(i), \\ G + \delta_0, & \neg T(i). \end{cases} \end{aligned} \tag{16.92}$$

Proof. The added jump 0 belongs to either low window. If $p > Q$, thickness and sign are read at every contributing embedding, so the assertion follows by taking unions. If $p \le Q$, a thick embedding is still read correctly. A thin embedding has all its allowed jumps in $[-2G, 2G]$, regardless of its sign, because $\eta \le G$ and $\delta_0 + W \le G$ by Lemma 16.87. Thus a thin embedding contributes only to the enlarged low window, even when its thickness or sign is not read by its letter.

Lean, formal context. `CrystallineCocycles` (L333-L456), `FitTwistsExist` (L377). □

Lemma 16.89 (Integer parts in the two remaining ranges)**.** *Assume* (16.87) *and* (16.88)*. If $p \le Q$, then*

$$K \ge 1, \qquad 2(b+\delta_0) < h^2 K. \tag{16.93}$$

If $p > Q$ and $\lambda \ge h^3$, then

$$\begin{gathered} K \ge 1, \qquad KQ \ge \lambda + 1, \qquad b + \delta_0 < Kp^2, \\ 2(G+\delta_0) < Kp, \qquad W + 4\eta \le K. \end{gathered} \tag{16.94}$$

Proof. The definition of K gives $2\lambda < (K+1)Q$. Lemma 16.50 gives $2b < p(\lambda+1)$. For $p \le Q$ these imply

$$4b < (K+1)Q^2 + 2Q.$$

The size conditions give $Q^2 + 2Q \le h \le b$ and $2\delta_0 < b$. Consequently

$$KQ^2 > 4b - Q^2 - 2Q \ge 3b > 2(b+\delta_0).$$

This proves $K \ge 1$ and (16.93), since $Q \le h$.

Now suppose $\lambda \ge h^3$ and $p > Q$. Since $h^3 \ge Q$, Euclidean division of 2λ by Q gives

$$KQ \ge 2\lambda - Q + 1 \ge \lambda + 1.$$

In particular $K \ge 1$. The low-window size inequality gives

$$\delta_0 < \frac{K}{99}, \qquad W + 4\eta \le 4(\eta + W) < \frac{4K}{99} \le K.$$

Indeed, $99Q(\eta + W) \le h^3 \le \lambda < KQ$. The factorial inequality now yields

$$b < \frac{p(\lambda+1)}{2} \le \frac{pKQ}{2} < \frac{Kp^2}{2}.$$

Adding $\delta_0 < K/99$ proves $b + \delta_0 < Kp^2$. Finally,

$$2G \le \frac{2b}{99(h^3+1)} < \frac{pKQ}{99(h^3+1)} \le \frac{Kp}{99}, \qquad 2\delta_0 < \frac{2K}{99} \le \frac{Kp}{99}.$$

Their sum is less than Kp. This proves (16.94).

Lean, formal context. `CrystallineCocycles` (L333-L456), `FitTwistsExist` (L377). □

For the next statements work on the canonical Kummer chart of G_F, where $F/\mathbb{Q}_p$ has ramification index e and residue degree f, so $[F:\mathbb{Q}_p] = ef$. The group on which the period points and their continuous primitives are constructed is G_F itself. Thus no descent from a larger local field is used. Let $\mathcal{C}$ be the crystalline lattice class specified by its allowed embedding jumps, and let L be a crystalline $\mathcal{O}$-lattice of rank ρ. Use the actual form-defined modules of their cocycle extensions, including the Tate-dual extensions attached to every admissible pairing. Retain the coefficient root and letter decomposition of the preceding lemma. These are representation hypotheses; the following estimates do not assert existence of such data for a global representation.

Theorem 16.90 (The cup exponent for small primes)**.** *Assume* (16.87), (16.88), *$p \le Q$, $e \le h$, and the letter windows* (16.92)*. Write Φ for Frobenius on the letters. Suppose no three thick letters occur with two successive gaps at most J: for every letter i and $1 \le r, s \le J$,*

$$T(i) \text{ and } T(\Phi^r i) \quad \Longrightarrow \quad \neg T(\Phi^{r+s} i). \tag{16.95}$$

Suppose either the chart is unramified with $p \ge 7$, or all the signs ϵ_i are equal. Then

$$p^{\,10^4 K(J+1)} Z_f(L) \subseteq \mathrm{Ann}_{\smile}(\mathcal{C}, L), \qquad J+1 \le m. \tag{16.96}$$

Proof. Apply Theorem 16.84 with low radius $2G$, far centres $b+\delta_0$ and $G+\delta_0$, width W, depth allowance $2K$, and step-cost allowance K. The thin far window lies in the enlarged low window because $G+\delta_0+W\le 2G$. Since $p\le Q\le h$, one has $J\ge 1$, and

$$p^J\le h^3<p^{J+1}.$$

For the unramified alternative,

$$10p^J(2G)\le 20Gh^3\le b+\delta_0.$$

For the constant-sign alternative, $8e\le 8h\le h^3<p^{J+1}$. Thus the respective sign or distance condition of that theorem holds.

Put $B=b+\delta_0$. By Lemma 16.89, $2B<h^2K$. Since $10G\le B$ and $e\le h$,

$$2e(B+10G)\le 4eB<2eh^2K\le 2h^3K<2Kp^{J+1}.$$

This proves its depth condition. Lemma 16.87 bounds its step cost by

$$\begin{aligned}20(J+1+d_0)(2G)+\lfloor\log_p(2(B+10G))\rfloor&\le 40Gh+hG\\&=41Gh<K.\end{aligned}$$

For the last inequality, $h^2K>2B\ge 2b\ge 198G(h^3+1)$ implies $K>198Gh$. All the hypotheses of the sparse-cut theorem are now verified. Its exponent is

$$10^3\bigl(2K+(J+1)K\bigr)\le 10^4K(J+1).$$

Apply the pairwise assertion to the identity-parametrized side $Z_f(L)$, to every crystalline cocycle of every member of $\mathcal{C}$, and to every admissible finite-coefficient pairing. This gives the inclusion in (16.96). The continuous primitives are those of the cut theorem on G_F. Finally (16.90), with $H=h^3$, gives $J<m$, equivalently $J+1\le m$.

Lean, formal context. `CrystallineCocycles` (L333-L456), `FitTwistsExist` (L377). □

Theorem 16.91 (The cup exponent for large loss)**.** *Assume* (16.87), (16.88), $p>Q$, $\lambda\ge h^3$, *and the letter windows* (16.92). *Suppose the chart is unramified and* $p\ge 7$. *No sparsity or constant-sign condition is required. Then*

$$p^{10^4Km}Z_f(L)\subseteq \mathrm{Ann}_{\smile}(\mathcal{C},L). \tag{16.97}$$

Proof. Apply Theorem 16.86 with low radius η, width W, centres $B=b+\delta_0$ and $G+\delta_0$, and $M=W+4\eta$. Here $\eta\ge 1$, $0<G+\delta_0\le B$, and Lemma 16.89 gives

$$B<Kp^2,\qquad 2(G+\delta_0)<Kp,\qquad M\le K.$$

The resulting exponent is $10^4(K+\lfloor\log_p(2(B+M))\rfloor)$. By Lemma 16.87, the logarithmic term is at most $m-1$. Since $K,m\ge 1$,

$$K+\lfloor\log_p(2(B+M))\rfloor\le K+m-1\le Km.$$

The last inequality is $(K-1)(m-1)\ge 0$. Using the universal side and all admissible pairings as in Theorem 16.90 proves (16.97).

Lean, formal context. `CrystallineCocycles` (L333-L456), `FitTwistsExist` (L377). □

The preceding arithmetic reconstructs the small-prime and large-loss applications of the cut bounds. Every logarithmic, ramification and floor allowance has been included in the displayed inequalities.

Theorem 16.92 (The normalized local defect in all prime ranges)**.** *Retain* (16.87)*, the canonical local setting above, and* (16.92)*. If $p \le Q$, assume $e \le h$, the sparsity condition* (16.95)*, and either an unramified chart with $p \ge 7$ or constant letter signs. If $p > Q$, assume an unramified chart with $p \ge 7$. Let $\mu \ge 0$ be an integer. For the range $p > Q$ and $\lambda < h^3$, suppose additionally that either*

1. *every negative allowed jump is at most $-h/2$, and at every coefficient embedding at most μ jumps of L, counted with multiplicity, lie in $[-\eta, \eta]$; or*
2. *all allowed jumps of L lie in $[-\eta, 0]$.*

Then

$$\widehat{\lambda}_{\mathcal{O}}\left(\frac{Z_f(L)}{H_{\mathcal{C}}(L)}\right) \le ef\,10^5\left(\rho m\left\lfloor\frac{2\lambda}{Q}\right\rfloor + \mu\lambda\right). \tag{16.98}$$

The coefficient and letter hypotheses may be supplied by Lemma 16.88. In the first alternative the defect is zero when $\lambda < \lfloor h/5\rfloor$; in the second it is zero throughout $\lambda < h^3$.

Proof. If $p \le Q$, Theorem 16.90 and $J + 1 \le m$ give cup exponent at most $10^4 Km$. If $p > Q$ and $\lambda \ge h^3$, Theorem 16.91 gives the same bound. In either case Lemma 15.2, with local degree ef, gives the stronger estimate

$$\widehat{\lambda}_{\mathcal{O}}\left(\frac{Z_f(L)}{H_{\mathcal{C}}(L)}\right) \le 2ef\,\rho\,10^4 Km.$$

Since $2 \cdot 10^4 \le 10^5$ and $\mu\lambda \ge 0$, this implies (16.98). This application uses the actual finite free crystalline-cocycle module, whose rank is at most $2ef\rho$; the exponent multiplies by the rational prime p, so coefficient ramification is already accounted for by the normalization.

It remains to consider $p > Q$ and $\lambda < h^3$. The chart is unramified, hence $e = 1$, and the low radius in (16.92) is η. Conditions (16.87) imply (16.50) and (16.56). Theorem 16.55 therefore gives exactly (16.98), including the two asserted zero-defect cases.

Lean, formal context. `CrystallineCocycles` (L333-L456), `FitTwistsExist` (L377). □

Corollary 16.93 (The local defect bound in all ranges)**.** *Under the hypotheses of Theorem 16.92, set*

$$Q = kt_*, \qquad \mu = \left\lfloor\frac{\rho m}{kx}\right\rfloor, \qquad k, m, t_*, x \ge 1, \quad x \le t_*.$$

With D_p defined by (15.26)*, the normalized local defect is at most*

$$[F : \mathbb{Q}_p]\,\frac{\rho D_p}{x}. \tag{16.99}$$

In particular $\rho = r_0, x = 1$ and $\rho = 1, x = t_*$ give the respective inequalities of (15.24)*, whenever the actual local data satisfy the hypotheses above.*

Proof. Substitute the displayed Q and μ in (16.98), then apply Lemma 15.20 with $\lambda_p = \lambda$. Its term $b\,v_p(M)$ is nonnegative. Since $ef = [F : \mathbb{Q}_p]$, the resulting factor is precisely the local degree in (16.99).

Lean, formal context. `CrystallineCocycles` (L333-L456), `FitTwistsExist` (L377). □

This corollary supplies all prime ranges of the crystalline local estimate as a local implication from its representation, chart, window and sparsity hypotheses. Theorem 17.26 below verifies the field, weight and numerical conditions for the constructed data, assuming the stated rational representations and their local compatibilities. The passage from the actual defect to relation colength and the sum over places are Corollary 15.12 and Theorem 15.19; they introduce no further coefficient-degree factor.

Passage to the prescribed local types. Theorem 16.92 covers the three prime ranges. To sum the local losses we must check its weight and field hypotheses for the prescribed representations. This is done in Corollary 17.106.

17 Galois representations of the Hecke systems

We determine the local Galois representations on the distinguished summands and then on every Hecke eigensystem in the ambient space. The latter result permits us to separate the eigensystems by an integral Hecke operator.

Lemma 13.17 requires the integral span of the Galois image to be the full matrix algebra. It also requires a scalar in the image whose difference from 1 is a unit. We deduce both properties from the primitive local type. After one finite extension of the coefficient field, they hold for every stable lattice in each distinguished summand. Theorem 17.76 identifies the local type of those summands, and Theorem 17.94 gives the corresponding rank-one conditions.

Throughout this subsection ℓ is the coefficient prime, O is the ring of integers of a finite extension $K/\mathbb{Q}_\ell$, and a root of unity is taken in $\overline{\mathbb{Q}}_\ell$ unless another field is specified.

Lemma 17.1 (The reduction test for roots of unity)**.** *For roots of unity x, y, the difference $x - y$ is a unit in any coefficient ring containing them if and only if x/y does not have ℓ-power order. Equivalently,*

$$x^{\ell^a} \neq y^{\ell^a} \quad \text{for every } a \geq 0 \quad \Longleftrightarrow \quad v_\ell(x - y) = 0.$$

Proof. Every root of unity is an integral unit. A root of unity of ℓ-power order reduces to 1 in residue characteristic ℓ. A root $z \neq 1$ of order d prime to ℓ cannot reduce to 1: the equality $1 + z + \cdots + z^{d-1} = 0$ would reduce to $d = 0$. Decompose the order of x/y into its ℓ-power and prime-to-ℓ parts. The preceding observations show that x/y reduces to 1 exactly when its prime-to-ℓ part is trivial. Since y is a unit, the same condition says that $x - y$ is a nonunit.

Lean, formal context. `LabelCharacters` (L561-L601), `FitTwistsExist` (L415). □

Lemma 17.2 (A primitive Frobenius orbit survives removal of one prime)**.** *Let $q > 1$ be an integer with $q \equiv 1 \pmod 3$, and let $r = 3^d$ with $d \geq 1$. Set $M = q^r - 1$. For every prime ℓ, the order of q modulo*

$$M_{\ell'} = M/\ell^{v_\ell(M)}$$

is exactly r. Consequently, if ξ has order M, then

$$\xi^{q^i \ell^a} \neq \xi^{q^j \ell^a} \qquad (0 \leq i < j < r,\ a \geq 0).$$

This statement holds in every field in which an element of order M exists.

Proof. Put $u = q^{r/3}$. Since $u \equiv 1 \pmod 3$, the integer $u^2 + u + 1 > 3$ is 3 mod 9, and hence has at least two distinct prime divisors. Choose one of them, $p \neq \ell$. The factorization $M = (u-1)(u^2+u+1)$ gives $v_p(M_{\ell'}) > v_p(u-1)$, so $M_{\ell'} \nmid q^{r/3} - 1$. The order of q mod $M_{\ell'}$ divides the power of three r and does not divide $r/3$. It is r. Equality of two of the indicated powers of ξ would give $M_{\ell'} \mid q^i(q^{j-i} - 1)$, forcing $r \mid j - i$.

Lean. The factorization handles all ℓ together. Compare `injOn_Iio_pow_card_pow_of_isPrimitiveRoot`. □

Lemma 17.3 (The selected character twist preserves primitive type)**.** *Let $\mathbb{F}_q$ be a finite field, let $N^2 \mid q - 1$, and let $\alpha \in \mathbb{F}_q^\times$ satisfy $\alpha^N = 1$. If $\xi \in \mathbb{F}_{q^r}^\times$ has order $q^r - 1$, then $\alpha\xi$ has the same order. In particular, multiplying all roots of a primitive degree-r polynomial by α gives another primitive degree-r polynomial.*

For the roots $\xi_i = (\alpha\xi)^{q^i}$, $0 \le i < r$, put

$$a_0 = \frac{q^r - 1}{q - 1}, \qquad y = (\alpha\xi)^{a_0}.$$

Then y has order $q-1$ and $\xi_i^{a_0} = y$ for every i.

Proof. Write $M = q^r - 1$. Since ξ generates $\mathbb{F}_{q^r}^\times$, one has $\alpha = \xi^{cM/N}$ for an integer c. Every prime divisor of M divides M/N: this is immediate for primes not dividing N, and follows from $N^2 \mid M$ for the remaining primes. Hence $1 + cM/N$ is prime to M, proving that $\alpha\xi = \xi^{1+cM/N}$ is primitive. Because $\alpha \in \mathbb{F}_q$, its Frobenius orbit is precisely the scaled orbit of ξ, so the degree remains r. The formula for the order of a power gives $\operatorname{ord}(y) = q - 1$. Thus $y \in \mathbb{F}_q^\times$ and $\xi_i^{a_0} = y^{q^i} = y$.

Lean, formal context. `LabelCharacters` (L561-L601), `FitTwistsExist` (L415). □

Theorem 17.4 (The integral endomorphism algebra of a tame pair)**.** *Let L be a free O-module of rank $r \ge 1$, and let $A, F \in \mathrm{GL}_O(L)$. Suppose that*

$$F^{-1}AF = A^q$$

for an integer $q \ge 1$, and that A has an eigenvalue $\zeta \in O^\times$ on $K \otimes_O L$. Put $\zeta_i = \zeta^{q^i}$. Assume

$$\zeta_r = \zeta_0, \qquad \zeta_i - \zeta_j \in O^\times \quad (0 \le i < j < r).$$

Then the O-algebra generated by A, F, F^{-1} is $\mathrm{End}_O(L)$, and $F^r = c\,1$ for some $c \in O^\times$.

Proof. Write $B = O[A, F, F^{-1}]$ and let k be the residue field of O. The first r translates by F of a ζ-eigenvector have distinct eigenvalues ζ_i, so they form a basis over K. The characteristic polynomial of A therefore has r distinct roots modulo the maximal ideal. Over $\bar{k}$, every $\bar{A}$-invariant subspace is a sum of its eigenlines. The relation $AF = FA^q$ makes $\bar{F}$ cycle through all these lines. The residual action is thus absolutely irreducible. Burnside's theorem gives the full endomorphism algebra over $\bar{k}$, and hence over k by dimension [EGH+11, §3.2]. Nakayama's lemma applied to $\mathrm{End}_O(L)/B$ gives $B = \mathrm{End}_O(L)$. Over K, F^r preserves each A-eigenline and therefore commutes with A as well as F. It lies in the centre of B and is invertible on L, so it is a unit scalar.

Lean. Lean proves residual spanning by interpolation on the same eigenvalues and lifts a basis by its unit determinant. `IntegralspanandscalarFrobenius` (L116-L194). □

Corollary 17.5 (Full span for every stable lattice)**.** *Let $R : G \to \mathrm{GL}(V)$ be a representation over K, and suppose $s, \phi \in G$ satisfy*

$$R(\phi)R(s)R(\phi)^{-1} = R(s)^q.$$

Suppose $R(s)$ has eigenvalue ζ, where $\zeta^{q^r} = \zeta$, and the r numbers ζ^{q^i}, $0 \le i < r = \dim_K V$, have pairwise unit differences. After adjoining ζ to K, every G-stable O-lattice $L \subset V$ satisfies

$$\mathrm{span}_O\{R(g)|_L : g \in G\} = \mathrm{End}_O(L), \qquad R(\phi^r)|_L \in O^\times 1.$$

Proof. For any such L, the O-span of $R(G)$ is an algebra and contains $A = R(s)$, $F = R(\phi^{-1})$ and F^{-1}. By Theorem 17.4 it is all of $\mathrm{End}_O(L)$. Inverting F^r gives the assertion for $R(\phi^r)$.

Lean. `Thespanofastablelattice` (L116-L194). □

Lemma 17.6 (A common power gives the required scalar inertia)**.** *Let s act on V with a squarefree annihilating polynomial whose roots are $\tau(\xi_1), \dots, \tau(\xi_a)$, where the ξ_i are roots of unity in a finite-field algebraic closure and τ is an injective multiplicative map on their group. Suppose $\xi_i^v = y$ for every i. Then $R(s^v) = \tau(y)\,1$.*

Suppose in addition that positive integers N, j, s_0, m, r satisfy

$$2^{s_0}5^{s_0} \mid N, \qquad 1 \le j \le m, \qquad m(4r+2) < 2^{s_0},$$

and that

$$y^a = 1 \implies N \mid ja.$$

Then, for every coefficient prime ℓ,

$$\tau(y)^{\ell^a t} - 1 \in O^\times \qquad (a \ge 0,\ 1 \le t \le 4r+2)$$

in every coefficient ring containing $\tau(y)$.

Proof. The annihilating polynomial divides $X^v - \tau(y)$, since it is squarefree and all its roots satisfy that equation. Evaluating at $R(s)$ proves the scalar identity.

Choose $p_* \in \{2, 5\}$ different from ℓ. If $\tau(y)^{\ell^a t}$ had ℓ-power order, then for some $b \ge 0$ injectivity of τ would give $y^{\ell^{a+b}t} = 1$. Hence

$$p_*^{s_0} \mid j\ell^{a+b}t.$$

Since $p_* \ne \ell$, this forces $p_*^{s_0} \mid jt$, although

$$0 < jt \le m(4r+2) < 2^{s_0} \le p_*^{s_0}.$$

This contradiction and Lemma 17.1 prove the unit assertion.

Lean, formal context. `LabelCharacters` (L561-L601), `FitTwistsExist` (L415). □

Theorem 17.7 (The matrix and scalar hypotheses from the selected labels)**.** *Fix the coefficient prime ℓ, the two distinct level primes p_0, p_1, the residue-character order N, the tower exponent s_0, and the ambient rank $r = 3^d$, $d \ge 1$. Suppose*

$$N^2 \mid p_i - 1, \qquad 9 \mid p_i - 1 \quad (i = 0, 1), \qquad 2^{s_0}5^{s_0} \mid N, \qquad m(4r+2) < 2^{s_0}.$$

Choose i with $p_i \ne \ell$ and a place $w \mid p_i$ with residue field $\kappa_w = \mathbb{F}_{p_i}$. Let R be one of the following representations, with the indicated local type at w.

1. *a rank-r representation whose type polynomial is a primitive degree-r polynomial with its roots multiplied by $\alpha_i \in \mathbb{F}_{p_i}^\times$, $\alpha_i^N = 1$;*
2. *a rank-one representation whose type polynomial is $X - \alpha_i$, where $\alpha_i = \omega_i^{-j}$, $\mathrm{ord}(\omega_i) = N$, and $1 \le j \le m$.*

Here the local type means that a tame generator s_i has the squarefree annihilator obtained from these roots by an injective multiplicative map into $\overline{\mathbb{Q}}_\ell^\times$. Assume that the representation kills wild inertia at this place; then arithmetic Frobenius ϕ_i satisfies the tame relation

$$R(\phi_i)R(s_i)R(\phi_i)^{-1} = R(s_i)^{p_i}.$$

Over a sufficiently large finite coefficient field, every stable lattice L in R has full integral matrix span. If its rank is $\rho \in \{r, 1\}$, then $R(\phi_i^\rho)$ is scalar, and

$$\chi_\ell(\phi_i^\rho) = p_i^\rho.$$

There is also an element γ_* in inertia at this place with

$$R(\gamma_*) = \zeta_* 1, \qquad \zeta_*^{\ell^a t} - 1 \in O^\times \quad (a \geq 0,\ 1 \leq t \leq 4r+2).$$

In particular both $\zeta_* - 1$ and $\zeta_*^2 - 1$ are units.

Proof. In the primitive case, Lemma 17.3 shows that the scaled polynomial remains primitive. Its roots are a complete Frobenius orbit $\xi, \xi^{p_i}, \dots, \xi^{p_i^{r-1}}$, with $\mathrm{ord}(\xi) = p_i^r - 1$. Lemma 17.2 and injectivity of the multiplicative map show that the images of distinct roots remain distinct after raising to any ℓ-power. Their differences are therefore units by Lemma 17.1. At least one of these images is an eigenvalue, because the annihilating polynomial kills $R(s_i)$ and the rank is positive. Its full orbit has r members, so Corollary 17.5 applies.

Take $v = (p_i^r - 1)/(p_i - 1)$. The common value $y = \xi^v$ has order $p_i - 1$, hence $y^a = 1$ implies $N \mid a$ and therefore $N \mid ja$, with $j = 1$ if desired. Lemma 17.6 gives the scalar inertia element $\gamma_* = s_i^v$ and all the unit differences.

In rank one the matrix-span and Frobenius-scalar statements are immediate. Take $v = 1$ and $y = \alpha_i = \omega_i^{-j}$. Then $y^a = 1$ implies $N \mid ja$, so the same scalar lemma applies. The cyclotomic formula in either case is the usual action of arithmetic Frobenius at a place of residue cardinality p_i, which is prime to ℓ.

Lean, formal context. `LabelCharacters` (L561-L601), `FitTwistsExist` (L415). □

For the data in Proposition 5.2, Lemma 17.29 gives $E_w = \mathbb{Q}_{p_i}$ at each chosen place $w \mid p_i$. Their level congruences give $N^2 \mid p_i - 1$, with $N = q_2(s_0) = (q_0 d)^{s_0+6}$. Here q_0 is the fixed modulus denoted by Q in that data construction, and d is its fixed auxiliary quadratic prime. Since $18480 \mid q_0$, the divisibility by 9 and by $2^{s_0}5^{s_0}$ follows. The explicit family has

$$\sqrt{x} \leq m \leq 2\sqrt{x}, \qquad r \leq 3x^5, \qquad 2^{s_0} = m^{22}.$$

For $x \geq 3$,

$$m(4r+2) \leq 28x^{11/2} < x^{11} \leq m^{22} = 2^{s_0}.$$

Thus the numerical hypotheses of Theorem 17.7 hold in this family.

Good label tuples supply a primitive polynomial at each level prime, as in the earlier type-selection argument; the rank-one label is $X - 1$ before the prescribed character scaling. For any coefficient prime ℓ, at least one of the two distinct level primes is different from ℓ, and the theorem uses its actual tame inertia action only at that place. An allowed twist is unramified at these old chosen places, so it does not change the selected type. Applying Theorem 17.7 therefore verifies the matrix and scalar clauses for any representation realizing those types. In particular the scalar and matrix hypotheses of Lemma 13.17 and the stronger scalar condition used by the root-chain argument are consequences of type compatibility.

This is the ordinary linear-algebra and finite-field argument behind the primitive orbit calculation, the integral tame-pair span, and the passage from selected labels to main-summand scalar elements. The integral projectors above also show directly that no lattice index or denominator is lost in the matrix-span assertion. Theorem 17.76 and Corollary 17.77 identify the actual main local types, so Theorem 17.87 applies the present matrix and scalar argument to those rows. Theorem 17.94 does the same for the original rank-one rows, and Theorem 17.105 supplies the full ambient realizations.

17.1 The jumps of the main summands and their Tate duals

We verify the weight hypotheses for the two multisets used by the descent pairs and the rank-one main pairs. This calculation is independent of the existence of their Galois representations.

Corollary 17.66 identifies these jumps for the realized main rows, in the convention of Lemma 15.1. The main local types are identified in §17.13; the rank-one and ambient comparisons are supplied in §17.16 and Theorem 17.105.

Write r for the target rank and retain the data

$$k = r + 1 - m, \qquad t_* = 5r, \qquad h = r^8, \qquad \Delta = 2h.$$

Here r is odd, m is even, $1 \le m < r$, and k is even. For the infinite place v of a complex embedding σ, put

$$\epsilon_\sigma = \begin{cases} 1, & \sigma \text{ belongs to the CM type,} \\ -1, & \text{otherwise,} \end{cases} \qquad w_j(v) = \kappa(v) + (2j+1)h.$$

The two values of κ are b on Φ_b and $G = xb/x_1$ elsewhere. We use the target parameters, numbered from zero,

$$\lambda_q(v) = \begin{cases} k/2 - 1 - q, & 0 \le q < k-1, \\ h - w_{q+2-k}(v), & k-1 \le q < r. \end{cases} \tag{17.1}$$

All of these are integers. The elementary size inequalities needed first are

$$r \ge 3, \qquad 0 < G \le b, \qquad 3t_* \le h, \qquad 4mh \le h^2, \qquad 2h^2 + t_* < G. \tag{17.2}$$

Lemma 17.14 verifies them, together with the quantitative bounds, for the data constructed in the existence argument.

Lemma 17.8 (The normalization of the row jumps)**.** *Let r and t_* be odd. For diagram entries $a_q(v)$ with Harish–Chandra parameters*

$$\lambda_q(v) = a_q(v) + (r-1)/2 - q,$$

the jumps in the CM convention are $q - a_q(v)$ at the chosen embedding and $r - 1 - q + a_q(v)$ at its conjugate. After subtracting $(r - t_*)/2 + \epsilon_\sigma w_j(v)$, the jumps are

$$u_{j,q}(\sigma) = \lfloor t_*/2 \rfloor - \epsilon_\sigma(\lambda_q(v) + w_j(v)). \tag{17.3}$$

Proof. The two unshifted expressions are both $(r-1)/2 - \epsilon_\sigma \lambda_q(v)$. Subtracting the displayed shift gives $(t_* - 1)/2 - \epsilon_\sigma(\lambda_q(v) + w_j(v))$. Since t_* is odd, $(t_* - 1)/2 = \lfloor t_*/2 \rfloor$. This proves the identity, including its integral normalization. It does not assert the Hodge–Tate comparison for a Galois representation; it identifies exactly the multiset that such a comparison must supply.

Lean, formal context. `RowsFromEigensystems` (L58-L258), `FitTwistsExist` (L415). □

Theorem 17.9 (The explicit descent-pair jump multiset)**.** *Assume (17.2), and let $1 \le j \le m$. Let $J_j(\sigma)$ be the multiset of the r integers in (17.3), with the parameters (17.1). Then the following hold.*

1. *For $q < k - 1$,*
$$|\epsilon_\sigma u_{j,q}(\sigma) + w_j(v)| \le t_*.$$
For $k - 1 \le q < r$,
$$h/2 \le |u_{j,q}(\sigma)| \le 4mh.$$
2. *Exactly m members of $J_j(\sigma)$ have absolute value at most $2h^2$, counted with multiplicity. In particular their number is at most $\lfloor rm/k \rfloor$.*

3. The total number of negative jumps at σ and $\bar\sigma$, counted with multiplicity, is at most r.

Proof. Put $\tau = \lfloor t_*/2 \rfloor$. On the first block, $|\lambda_q(v)| \le k/2 - 1$, and $k \le r \le t_*$. Therefore

$$|\epsilon_\sigma u_{j,q} + w_j| = |\epsilon_\sigma \tau - \lambda_q| \le \tau + k/2 - 1 \le t_*.$$

On the second block put $i = q + 2 - k$. As q runs through this block, i runs through $1, \dots, m$ exactly once. The large weight cancels:

$$\lambda_q(v) + w_j(v) = (2(j-i)+1)h.$$

Since $1 \le i, j \le m$, the absolute value of the odd integer $2(j-i)+1$ lies between 1 and $2m-1$. It follows that

$$h - \tau \le |u_{j,q}| \le (2m-1)h + \tau \le 2mh \le 4mh,$$

and $h - \tau \ge h/2$ by (17.2).

Every jump in the second block has absolute value at most h^2. Every jump in the first block has absolute value at least $w_j(v) - t_* > 2h^2$. Thus the small-jump count is exactly the length m of the second block. Since $k \le r$, $mk \le rm$, so $m \le \lfloor rm/k \rfloor$.

Conjugation preserves v and reverses ϵ_σ. The two jumps with the same index q consequently have sum $2\tau \ge 0$. At most one of them is negative. Sum this inequality over the r indices. Repeated values are still counted separately because the argument is indexed by q.

Lean, formal context. `RowsFromEigensystems` (L58-L258), `FitTwistsExist` (L415). □

Theorem 17.10 (The rank-one main-pair jump multiset)**.** *Assume* (17.2). *Let $1 \le j \le m$, let $m < z \le r$, and suppose an integral weight $c(v)$ satisfies*

$$w_j(v) + r - t_* \le c(v) \le w_j(v).$$

The one-element multiset

$$J(\sigma) = \{z - 1 - \epsilon_\sigma c(v)\}$$

has string length $s = 2z - 1 \le t_*$. Its member u satisfies

$$|\epsilon_\sigma u + w_j(v)| \le t_*.$$

It has no member with absolute value at most $2h^2$. At σ and $\bar\sigma$ together there is at most one negative member.

Proof. Write $d(v) = w_j(v) - c(v)$, so $0 \le d(v) \le t_* - r$. Then

$$\epsilon_\sigma u + w_j(v) = \epsilon_\sigma(z-1) + d(v).$$

Its absolute value is at most $(z-1) + t_* - r \le t_* - 1$. Hence $|u| \ge w_j(v) - t_* > 2h^2$. The conjugate jumps sum to $2(z-1) \ge 0$, proving the negative count. Finally $1 \le 2z - 1 \le 2r - 1 \le 5r = t_*$.

Lean, formal context. `RowsFromEigensystems` (L58-L258), `FitTwistsExist` (L415). □

Theorems 17.9 and 17.10 verify all the numerical clauses of the source's `IsMainSummandWeights` for the two displayed weight designs. The calculation reconstructs the descent-pair and main-pair weight lemmas.

Lemma 17.11 (Common opposite windows for a summand class)**.** *Assume* (17.2)*. Let* $0 \le j \le m$ *and let* $J(\sigma)$ *be any multiset such that each member* u *satisfies either*

$$h/2 \le |u| \le 4mh \quad \text{or} \quad |\epsilon_\sigma u + w_j(v)| \le t_*.$$

Let $s \le t_*$ *and suppose* $U_\sigma \subseteq \operatorname{supp} J(\sigma) \cup \{0, \dots, s-1\}$*. Set*

$$\eta = h^2 + 1, \qquad W = 4t_*, \qquad \delta_0 = (2j+1)h - 2t_*, \qquad \beta_\sigma = \kappa(v) + \delta_0. \tag{17.4}$$

Then

$$\begin{aligned} U_\sigma &\subset [-\eta, \eta] \cup -\epsilon_\sigma[\beta_\sigma, \beta_\sigma + W], \\ \{-u-1 : u \in U_\sigma\} &\subset [-\eta, \eta] \cup \epsilon_\sigma[\beta_\sigma, \beta_\sigma + W]. \end{aligned} \tag{17.5}$$

Proof. Every low member of $J(\sigma)$ has absolute value at most h^2. So do the added integers, since $s \le t_* \le h^2$. Their images under $u \mapsto -u-1$ have absolute value at most $h^2 + 1 = \eta$.

For a high member, $w_j(v) = \beta_\sigma + 2t_*$, so

$$\beta_\sigma + t_* \le -\epsilon_\sigma u \le \beta_\sigma + 3t_*.$$

This proves the first far-window inclusion. If $v' = -u - 1$, then $\epsilon_\sigma v' = -\epsilon_\sigma u - \epsilon_\sigma$. The preceding interval enlarged by one at either endpoint still lies in $[\beta_\sigma, \beta_\sigma + 4t_*]$, because $t_* \ge 1$. This proves the dual inclusion with the same window start. In particular the Tate shift by -1 has been included in the width.

Lean, formal context. `RowsFromEigensystems` (L58-L258), `FitTwistsExist` (L415). □

Lemma 17.12 (Low filtration jumps of the dual distinguished summand)**.** *Assume* (17.2)*, and let* L *have rank* ρ *and actual jump multiset* $J(\sigma)$ *at each coefficient embedding. Suppose each member is low or high as in Lemma 17.11, and suppose*

$$k\,\#\{u \in J(\sigma) : |u| \le 2h^2\} \le \rho m,$$

where the count includes multiplicity. Let U_σ *be as in that lemma, and let* U'_σ *contain only actual jumps of the Tate dual* $L^\dagger$*. Then every negative member of* $U_\sigma \cup U'_\sigma$ *is at most* $-h/2$*, and at most*

$$\mu = \lfloor \rho m/k \rfloor \tag{17.6}$$

jumps of $L^\dagger$ *lie in* $[-\eta, \eta]$ *at each embedding, where* $\eta = h^2 + 1$*.*

For the explicit descent-pair multiset, that dual low-jump count is exactly m*; for the rank-one main-pair multiset it is zero.*

Proof. Every member of $J(\sigma)$ has absolute value at least $h/2$: this is stated for the low members, and for the high members follows from $|u| \ge w_j(v) - t_* > 2h^2$. A negative member of U_σ cannot be one of its added nonnegative integers, so it is at most $-h/2$.

The Tate-dual jump corresponding to u is $v' = -u - 1$, with the same multiplicity. If $v' < 0$, then $u \ge 0$; the main multiset has no zero, so $u \ge h/2$ and $v' \le -h/2 - 1 \le -h/2$. If $|v'| \le h^2 + 1$, then

$$|u| = |-v' - 1| \le h^2 + 2 \le 2h^2.$$

The map $u \mapsto -u - 1$ is a bijection of the integer labels and preserves their multiplicities. The low-jump count of the dual is therefore at most the displayed count in J, which is at most (17.6) because $k > 0$.

For the explicit descent-pair multiset, each of the m low members has $|u| \le h^2$, so its dual lies in $[-h^2-1, h^2+1]$. A high member has $|u| > 2h^2$, whence $|-u-1| > 2h^2-1 \ge h^2+1$; thus there are exactly m. The rank-one main-pair multiset has only a high member, so its dual low-jump count is zero.

Lean, formal context. `RowsFromEigensystems` (L58-L258), `FitTwistsExist` (L415). □

The restriction to actual jumps in U'_σ is needed. Dualizing the whole allowed class would also add the jumps $-1, \dots, -s$ from the character summands. These are not jumps of the main dual and would destroy its negative-jump gap. The admissible class on the main-dual side is therefore intersected with the support of its actual jump multiset. This does not remove any vector or impose an extra condition on the main lattice.

Lemma 17.13 (The character dual has only a nonpositive low jump)**.** *Assume* (17.2). *If a rank-one character summand has the single jump c with $0 \le c < s \le t_*$, then its Tate dual has the single jump $-c-1$, and*

$$-\eta \le -c-1 < 0, \qquad \eta = h^2+1.$$

It therefore satisfies the nonpositive-side alternative of Theorem 16.92.

Proof. Filtered duality sends the jump c to $-c-1$. Since $c+1 \le s \le t_* \le h^2+1$, the stated interval follows. There is no requirement that this character satisfy the negative-jump gap of the main summand; it uses the other alternative of the local theorem.

Lean, formal context. `CrystallineDefect` (L49-L113), `FitTwistsExist` (L415). □

Lemma 17.14 (The constructed data meet every crystalline size condition)**.** *For the data of the existence construction, and all sufficiently large x, the choices*

$$Q = kt_*, \quad h = r^8, \quad G = xb/x_1, \quad \eta = h^2+1, \quad W = 4t_*, \quad \delta_0 = (2j+1)h - 2t_*$$

satisfy (17.2) *and all of* (16.87)*, simultaneously for $0 \le j \le m$. The threshold depends only on the fixed Dirichlet parameters.*

Proof. The construction in Proposition 5.2 gives $m \asymp \sqrt{x}$ and $r \asymp x^5$. By (5.3) and (5.4),

$$x^{10^6} \le x_1, \qquad x_1^{10^6} \le b, \qquad \log b = x^{o(1)}.$$

The construction gives $G = a/x_1 = b/C$, an integer between 1 and b. Also $h \asymp x^{40}$, $Q = O(x^{10})$, $\eta \asymp x^{80}$ and $W = 4t_* = O(x^5)$. Thus $Q, m \ge 1$, $r \ge 3$, $h \ge 30$, $3t_* \le h$ and $4mh \le h^2$ for all sufficiently large x. For $0 \le j \le m$ these estimates give

$$0 \le \frac{\delta_0}{\eta} \le \frac{2m+1}{h} = O(x^{-79/2}).$$

The two polynomial ratios in (16.87) satisfy

$$\frac{Q^2+(W+1)Q+Q+W+m}{h} = O(x^{-20}), \qquad \frac{(\eta+W)Q+\eta+h}{h^3} = O(x^{-30}).$$

For the other three inequalities, use $G = xb/x_1$ to obtain

$$\begin{aligned} \frac{h^3}{G} &\ll x^{119}\frac{x_1}{b} \le x^{119}x_1^{1-10^6} = o(1), \\ \frac{G(h^3+1)}{b} &= \frac{x(h^3+1)}{x_1} \ll \frac{x^{121}}{x_1} = o(1), \\ \frac{\log(4b)}{m} &= x^{-1/2+o(1)}. \end{aligned}$$

Since $2h^2 + t_* \le h^3$ for large x, the first line also gives $2h^2 + t_* < G$. All the bounds are uniform in j and the local prime. They imply (17.2) and (16.87).

Lean. Lean obtains the numerical comparisons from the logarithmic bounds on the data in `New.Selmer.crystallineSizes_data`. Here the ratios are evaluated at the scales of Proposition 5.2. □

Corollary 17.15 (The summed crystalline-cocycle rank for these weights)**.** *Let E be the CM field with f infinite places, and let a rank-ρ de Rham Galois lattice realize jump multisets $J(\sigma)$ for which*

$$\#\{u \in J(\sigma) : u < 0\} + \#\{u \in J(\bar{\sigma}) : u < 0\} \le \rho.$$

Then, for each rational prime p,

$$\sum_{v \mid p} \operatorname{rank}_{\mathcal{O}} Z_f(L|_{G_{E_v}}) \le (\#\{v : v \mid p\} + f)\rho.$$

In particular this holds for the descent-pair and rank-one main-pair multisets above whenever their actual Galois lattices realize the stated jumps.

Proof. After enlarging the coefficient field to split the embeddings, Lemma 15.1 says that the rank at v is ρ plus the number of negative jumps at the embeddings belonging to v. Each complex embedding, transported by the fixed complex–p-adic identification, belongs to exactly one place above p. Summing over v therefore counts each negative jump once. The embeddings form f conjugate pairs, each contributing at most ρ. This proves the bound. The coefficient extension changes neither rank nor the jump multiplicities.

Lean, formal context. `CrystallineDefect` (L49-L113), `FitTwistsExist` (L415). □

The two-window and dual-count arguments reconstruct the common-window comparison and the main-dual and character-dual alternatives; the final rank count is the conjugate-pair rank estimate. Thus the window widths, negative gap, low-jump multiplicity and numerical scales are proved for the actual weight formulas. Corollary 17.66 identifies these formulas with the jumps of the realized main rows and hence of their stable lattices. The corresponding local types are proved in Theorem 17.76 and Corollary 17.77.

17.2 From rational representations to the local classes

The local estimates concern the Tate duals of the actual summands. We first explain how a supplied rational representation gives the integral local objects to which those estimates apply. We then construct the letters, signs and sparse sets for the fields already constructed in the data-existence argument. This application retains its rational representation and local compatibility hypotheses. They are supplied for the actual main rows in Theorem 17.87; Corollary 17.88 applies the classes below to them. Theorem 17.105 and Corollary 17.106 supply the full ambient-family hypotheses and their application.

Lemma 17.16 (One coefficient ring for finitely many lattices and roots)**.** *Let G be a profinite group and let $V_1, \dots, V_a$ be continuous finite-dimensional representations of G, each defined over a finite extension of $\mathbb{Q}_p$. Given finitely many positive integers $n_1, \dots, n_b$, there is a finite Galois extension $K_c/\mathbb{Q}_p$, with ring of integers $\mathcal{O}$, such that all V_j are defined over K_c, every μ_{n_i} is contained in K_c, and each V_j has a G-stable free $\mathcal{O}$-lattice. If lattices have already been chosen over smaller coefficient rings, their scalar extensions can be used instead.*

Proof. Take the Galois closure of the compositum of the finitely many fields of definition and the splitting fields of $X^{n_i} - 1$. For one resulting representation choose a basis and the lattice $L_0 = \mathcal{O}^d$. Compactness of the image of G bounds all matrix entries, so $gL_0 \subseteq p^{-c}L_0$ for some c and every g. The $\mathcal{O}$-span $L = \sum_{g\in G} gL_0$ is a submodule of $p^{-c}L_0$ containing L_0. It is finite because $\mathcal{O}$ is noetherian, torsion-free and therefore free because $\mathcal{O}$ is a discrete valuation ring, and has full rank. Its definition makes it G-stable. Scalar extension of a previously chosen stable lattice has these same properties.

Lean, formal context. `LocalLattices` (L94-L272), `FitTwistsExist` (L415). □

The finite coefficient field in this lemma is automatic for a supplied continuous $\overline{\mathbb{Q}}_p$-valued representation: Theorem 17.43 proves it from compactness and Baire category. Boundedness alone would not suffice. The lemma itself does not construct the representation. For the letters below, the required additional roots have orders $p^{f_v} - 1$, one for each of the finitely many places $v \mid p$. We choose this coefficient ring before applying the local bounds. Thus no assertion that cup-orthogonal quotients commute with coefficient extension is needed.

Lemma 17.17 (Local conditions are independent of the stable lattice)**.** *Let V be a finite-dimensional K_c-representation of G_E and let L be a stable $\mathcal{O}$-lattice in V. At a place $v \mid p$, restriction of L has the same rational period admissibility, Hodge–Tate jump multiset at each coefficient embedding, and polynomial identities on its potentially crystalline periods as $V|_{G_{E_v}}$. In particular, if V is crystalline at v, so is L in the lattice sense; if it is only potentially crystalline, the same statement holds with that qualification.*

Proof. The inclusion gives $L[1/p] = V$ equivariantly. Since the period rings invert p, it identifies their tensor products with L and V, including the Galois actions, the Frobenius action and the de Rham filtrations. Fixed periods along any specified subgroup or filter are therefore identified, so their spanning property is unchanged. The same identification on the graded Hodge–Tate periods gives equality of the jump multiplicities at each coefficient embedding. Finally, the Weil operators on potentially crystalline periods are identified; polynomial evaluation commutes with this identification.

More concretely, if an integral matrix model u is identified with another rational model R by a matrix B, the map $x \mapsto Bx$ is injective on $\mathcal{O}^d$, extends to an isomorphism after inverting p, and intertwines all these operators. No integrality of B or B^{-1} is required for the period statements. An integral matrix identity descends without a denominator by Lemma 14.14.

Lean, formal context. `LocalLattices` (L94-L272), `FitTwistsExist` (L415). □

Fix an embedding $e_v : E \hookrightarrow \overline{\mathbb{Q}}_p$ that induces v, and the comparison isomorphism $\iota_p : \mathbb{C} \simeq \overline{\mathbb{Q}}_p$ used in the source. Put

$$\sigma_v(g) = \iota_p^{-1} g^{-1} e_v, \qquad \gamma_g = g|_{\mathcal{O}} : \mathcal{O} \longrightarrow \mathbb{C}_p.$$

Every $\mathbb{Z}_p$-algebra embedding of $\mathcal{O}$ in $\mathbb{C}_p$ is some γ_g: extend it to the finite Galois field K_c, and then to an automorphism of $\overline{\mathbb{Q}}_p$. Consequently a rational jump identity $\mathrm{HT}_{\gamma_g}(V) = J(\sigma_v(g))$ gives the same identity, with multiplicities, for L at every coefficient embedding. If two choices of g induce the same γ, these multisets agree because both equal the intrinsic jump multiset of L at γ. This is the choice-independence needed for the allowed jump sets.

Lemma 17.18 (The local test class)**.** *At a fixed $v \mid p$, let $\mathcal{R}$ be a supplied family of rational representations, and let U_γ be finite sets of integers. Suppose each member of $\mathcal{R}$ has the required period admissibility and has all its jumps at γ in U_γ. Prescribe, for each inertia element, any family of polynomial identities satisfied on the periods of every member of $\mathcal{R}$. Then every stable lattice*

in a member of $\mathcal{R}$ belongs to the class defined by those three conditions. The class is closed under finite direct sums and equivariant rational isomorphisms.

Away from the two level primes, if the supplied local compatibility asserts crystallinity, this is precisely the class of crystalline lattices with jumps in U_γ; there is no additional inertia-polynomial restriction. At a level prime, the same assertion uses the prescribed potentially crystalline period type.

Proof. Apply Lemma 17.17 to each member. Fixed period spaces, their spanning property and polynomial identities pass to a finite direct sum. The jump multisets add, so their supports stay in U_γ. An equivariant rational isomorphism identifies all three conditions. These are exactly the stability properties used in Theorem 15.9 and the lattices in Lemma 15.10.
Lean, formal context. `LocalLattices` (L94-L272), `FitTwistsExist` (L415). □

These arguments reconstruct the transfer of admissibility, jumps and inertia polynomials to the actual local lattices. They use the asserted local compatibility of the rational representation as an input; they do not replace it by a condition on a newly defined quotient.

The coefficient letters read the constructed field

In this subsection q_0 denotes the fixed modulus called Q in the data construction; the prime cutoff for the local estimates is $Q = kt_*$, where $t_* = 5r$. Thus these two moduli have different roles. Write

$$E = \mathbb{Q}(\sqrt{-d}, \zeta_{q_1}), \qquad q_1 = 3\ell_0\, 2^s q_0, \qquad N_E = q_1 d, \qquad h = r^8.$$

Here $18480 \mid q_0$, d is prime and $d \equiv q_0/2 - 1 \pmod{q_0}$, as in the fixed parameters. Let Φ_b be the constructed set of large-weight infinite places. Its elements have class -1 modulo q_0 and obey the forbidden configuration in (5.2). For an embedding $\sigma : E \to \mathbb{C}$, define $\varepsilon(\sigma) \in \{1, -1\}$ by

$$\sigma(\sqrt{-d}) = \varepsilon(\sigma)\sqrt{-d},$$

where the square root on the right is the fixed one defining the CM type. At the associated infinite place w, the chosen CM-type embedding sends ζ_{q_1} to $\sigma(\zeta_{q_1})^{\varepsilon(\sigma)}$.

Lemma 17.19 (The scalar sizes needed for the charts)**.** *For all sufficiently large scales in the data construction,*

$$[E : \mathbb{Q}] \le h, \qquad p \mid N_E \Longrightarrow p \le Q, \qquad h^3 \le 2^{12s}. \tag{17.7}$$

Moreover every prime $p < 7$ divides N_E.

Proof. The degree is $2\varphi(q_1) = O_{q_0}(x^{11} \log x)$, whereas $h = r^8 \ge x^{40}$. The prime divisors of N_E belong to the fixed finite set of prime divisors of $q_0 d$: multiplication by $3\ell_0 2^s$ introduces only 2 and 3, already dividing q_0. Since $Q = (r + 1 - m)5r$ tends to infinity, the second assertion follows. Finally $h^3 = r^{24} < 3^{24}x^{120}$ and $2^{12s} = m^{264} \ge x^{132}$. The last assertion follows from $2 \cdot 3 \cdot 5 \mid q_0$.
Lean, formal context. `KummerChartExistence` (L350-L452), `FitTwistsExist` (L415). □

Lemma 17.20 (The canonical chart uses the entire completion group)**.** *Let $F/\mathbb{Q}_p$ be finite, of ramification index e_F and residue degree f_F. The Kummer chart may be chosen with constant ring $\Lambda = W(\mathbb{F}_{p^{f_F}})$, chart field F, and subgroup $H = G_F$. Its parameter and kernel generator have the form*

$$S = \Lambda[[z]] \subset A_{\mathrm{inf}}, \qquad \mathcal{E} = p + z^{e_F} V, \quad V \in S^\times, \qquad \theta(z) = \pi_F,$$

where π_F is a uniformizer of F. In particular $e_F f_F = [F : \mathbb{Q}_p]$. If F is unramified one may take $\pi_F = -p$ and $\mathcal{E} = z + p$.

Proof. Choose compatible p-power roots of π_F and let $\pi_F^\flat$ be the corresponding element of $\mathcal{O}_{\mathbb{C}_p}^\flat$. Embed the finite residue field $\mathbb{F}_{p^{f_F}}$ in the tilt and use its Witt vectors as $\Lambda \subset A_{\mathrm{inf}}$. Put $z = [\pi_F^\flat]$. Evaluation defines an injective map $\Lambda[[X]] \to A_{\mathrm{inf}}$ with $X \mapsto z$. Indeed, modulo p a nonzero series has the valuation of its lowest nonzero term: the nonzero coefficients in the finite field have valuation zero and $\pi_F^\flat$ has positive valuation. This proves injectivity modulo p, and then p-adic separatedness proves injectivity before reduction. The series converge in the usual weak topology of A_{inf}. Frobenius preserves its image, since it preserves Λ and sends z to z^p.

The map θ sends Λ to the integers of the maximal unramified subfield F_0 of F. Successive expansion in the uniformizer π_F, with Teichmüller residue representatives, proves $\theta(S) = \mathcal{O}_F$. The element $-p/\pi_F^{e_F}$ is a unit of $\mathcal{O}_F$; lift it to a unit $V \in S$. Then $\theta(\mathcal{E}) = 0$ for $\mathcal{E} = p + z^{e_F}V$. The formal series $p + X^{e_F}V(X)$ is Eisenstein of degree e_F in the Weierstrass sense. Weierstrass preparation makes $S/(\mathcal{E})$ free of rank e_F over Λ. Its surjection onto $\mathcal{O}_F$, also free of that rank, is an isomorphism: the kernel has rank zero and lies in a torsion-free Λ-module.

The usual primitive-element criterion for A_{inf} also makes $\mathcal{E}$ a generator of $\ker\theta$ in A_{inf}. To check that criterion here, modulo p its first Witt coordinate has valuation $e_F v_p(\pi_F) = 1$, while modulo the ideal of positive- valuation tilt coordinates the element $\mathcal{E}$ equals p. Its second Witt coordinate is therefore a unit. These are exactly the primitive degree-one conditions used in the earlier period-ring setup.

Every element of G_F fixes Λ pointwise and fixes $\theta(z) = \pi_F$. Conversely an element of $G_{\mathbb{Q}_p}$ fixing both fixes $F_0(\pi_F) = F$. Thus the fixer of the chart is precisely G_F. The Kummer-shift elements can also be taken in G_F: for every $n > 0$, the polynomial $X^n - \pi_F$ is Eisenstein over F, so G_F acts transitively on its roots. Given an integer l, at level p^a choose an element taking the chosen root to its product with the l-th power of the chosen p^a-th root of unity. These conditions form nested nonempty closed subsets of the compact group G_F. Their intersection supplies one element with $gz = [\epsilon]^l z$ at all levels. All chart conditions therefore hold with subgroup $H = G_F$. For $e_F = 1$ choose $\pi_F = -p$ and $V = 1$ from the outset.

Lean, formal context. `KummerChartExistence` (L350-L452), `FitTwistsExist` (L415). □

This is the explicit construction in the canonical Kummer chart, including the choice of the whole completion group. It uses the same standard A_{inf} and Witt-vector properties as the earlier period calculations.

Fix p and $v \mid p$. Let $F = E_v$, and write e_F, f_F for its ramification index and residue degree. Use the chart of Lemma 17.20. The symbol $\mathcal{E}$ there is the kernel generator denoted by E in the preceding local period estimates; the field is still denoted by E in the global data.

Choose $\mathcal{O}$ as in Lemma 17.16 containing a primitive $(p^{f_F} - 1)$-st root ζ_v. The letter read by $g \in G_{\mathbb{Q}_p}$ is

$$a_v(g) = g\zeta_v.$$

The Frobenius permutation on letters is $a \mapsto a^p$. The coefficient-root projectors already used in Theorem 16.49 consequently exist over this $\mathcal{O}$. Their denominators divide $p^{f_F} - 1$ and are units, so this choice adds no power of p.

Lemma 17.21 (An unramified letter determines the embedding)**.** *If $p \nmid N_E$, the field F is unramified, $p \geq 7$, and*

$$a_v(g) = a_v(g') \quad \Longrightarrow \quad \sigma_v(g) = \sigma_v(g').$$

Proof. The field E is contained in $\mathbb{Q}(\zeta_{N_E})$. Prime-to-p roots of unity lie in unramified extensions of $\mathbb{Q}_p$, so $p \nmid N_E$ makes F unramified. It is the unique unramified extension of degree f_F inside $\overline{\mathbb{Q}}_p$ and equals $\mathbb{Q}_p(\zeta_v)$. Equality of the two letters says that $g'^{-1}g$ fixes ζ_v. Since $\mathbb{Q}_p(\zeta_v)/\mathbb{Q}_p$ is Galois, it follows that gg'^{-1} also fixes this field: the induced automorphisms of g and g' on the field are

equal. Hence gg'^{-1} fixes e_vE, or equivalently $g^{-1}e_v = g'^{-1}e_v$. This proves the embedding identity. The prime bound follows from the last assertion of Lemma 17.19.

Lean, formal context. KummerChartExistence (L350-L452), FitTwistsExist (L415). □

Lemma 17.22 (The place ratio read from a root)**.** *Let $g, g' \in G_{\mathbb{Q}_p}$, put $w_g = \mathrm{place}(\sigma_v(g))$, and let $m \mid q_1$ be prime to p. Suppose, for some $a, n \geq 0$,*

$$(g'g^{-1})(e_v(\zeta_{q_1})^{p^a}) = \left(e_v(\zeta_{q_1})^{p^a}\right)^{p^n}.$$

Then the ratio of the infinite places, read modulo m, is

$$\frac{w_g}{w_{g'}} \equiv \varepsilon(\sigma_v(g))\varepsilon(\sigma_v(g'))p^n \pmod{m}. \tag{17.8}$$

Proof. Let $z = e_v(\zeta_{q_1})$, let k be the exponent representing $w_g/w_{g'}$ in $(\mathbb{Z}/q_1\mathbb{Z})^\times$, and abbreviate the two signs by $\varepsilon, \varepsilon'$. The definition of the group law on infinite places gives

$$(\sigma_v(g)\zeta_{q_1})^{\varepsilon} = \left((\sigma_v(g')\zeta_{q_1})^{\varepsilon'}\right)^k.$$

Apply $g'\iota_p$ and then raise to p^a. The hypothesized action gives $z^{p^n\varepsilon p^a} = z^{\varepsilon' k p^a}$. Since z is primitive, q_1 divides $(p^n\varepsilon - \varepsilon' k)p^a$. Reduce modulo m and cancel p^a; then multiply by ε' to obtain the stated congruence.

Lean, formal context. KummerChartExistence (L350-L452), FitTwistsExist (L415). □

Lemma 17.23 (The sign is constant at ramified large-weight embeddings)**.** *If $p \mid N_E$, then all g with $w_g \in \Phi_b$ have the same value of $\varepsilon(\sigma_v(g))$.*

Proof. Suppose two such embeddings have opposite signs. First let $p \mid q_1$. Put $\beta = (1+\sqrt{-d})/2$ and $b_v = e_v\beta$. The sign assumption says that a local Galois automorphism carries b_v to $1 - b_v$. These two numbers consequently have the same p-adic absolute value. On the other hand,

$$b_v(1 - b_v) = \frac{d+1}{4}.$$

Every prime divisor of q_1 divides $(d+1)/4$. For odd primes this follows from $d+1 \equiv q_0/2 \pmod{q_0}$, and for $p = 2$ it follows from $16 \mid q_0$. Thus the displayed product has absolute value less than one. Both factors then have absolute value less than one, contradicting $b_v + (1 - b_v) = 1$.

The remaining case is $p = d$, since d is prime and $(d, q_1) = 1$. Every local Galois action on μ_{q_1} is a power of Frobenius, so for some $n \geq 0$ the relevant ratio of automorphisms raises $e_v(\zeta_{q_1})$ to its d^n-th power. Lemma 17.22, with $m = q_0$, makes the ratio of the two places congruent to $-d^n$. Both places have class -1 modulo q_0, so $d^n \equiv -1$. This is impossible modulo 2^a, where $a = v_2(q_0) \geq 4$: the congruence on d gives $d \equiv 2^{a-1} - 1 \pmod{2^a}$. This residue has square 1 and differs from both 1 and -1. Its powers are therefore never -1, giving the contradiction.

Lean, formal context. KummerChartExistence (L350-L452), FitTwistsExist (L415). □

Lemma 17.24 (The global sparsity passes to coefficient letters)**.** *Suppose $p \leq Q$ and put $J_p = \lfloor \log_p(h^3) \rfloor$. Define a letter a to be thick if $a = a_v(g)$ for some g with $w_g \in \Phi_b$. Then, for any $1 \leq u, v \leq J_p$, the three letters*

$$a, \qquad a^{p^u}, \qquad a^{p^{u+v}}$$

cannot all be thick. Repeated positions in a Frobenius cycle are included in this assertion.

Proof. The group G_F contains a lift F_v of residue Frobenius $x \mapsto x^{p^{f_F}}$. It fixes $e_v E$. Set $a_0 = v_p(q_1)$ and $\xi = e_v(\zeta_{q_1})^{p^{a_0}}$. This root has order prime to p. Frobenius acts on it by its p^{f_F}-th power, whereas F_v fixes it. Thus $\xi^{p^{f_F}-1} = 1$.

If $a_v(g') = a_v(g)^{p^n}$, then $g'g^{-1}$ raises every $(p^{f_F}-1)$-st root of unity to its p^n-th power: it does so on the generator $g\zeta_v$, and hence on all its powers. In particular it has this action on ξ. If the three indicated letters were thick, choose g_0, g_1, g_2 at large-weight places giving them. The automorphisms $g_2 g_0^{-1}$ and $g_2 g_1^{-1}$ raise ξ to its p^{u+v}-th and p^v-th powers. By Lemma 17.22, the ratios w_{g_0}/w_{g_2} and w_{g_1}/w_{g_2} are respectively $\pm p^{u+v}$ and $\pm p^v$ modulo $(q_1)_{p'}$.

Lemma 17.19 gives $J_p \le \log_2(h^3) \le 12s$, whence $0 < v < u+v \le 24s < \ell_0$. These are precisely the forbidden congruences of (5.2), with exponents v and $u+v$. That argument did not require the three places to be distinct; it excludes repeated cycle positions as well.

Lean, formal context. `CrystallineDefect` (L130-L295), `FitTwistsExist` (L415). □

Theorem 17.25 (All chart, sign and sparsity conditions for the constructed data)**.** *For every sufficiently large scale, every rational prime p and every $v \mid p$, one may choose the canonical chart and the coefficient root above, a thick predicate T on its letters, and signs $\varepsilon_a \in \{1,-1\}$ such that:*

1. *$e_F f_F = [E_v : \mathbb{Q}_p]$ and $e_F \le h$;*
2. *whenever $w_g \in \Phi_b$ or $p > Q$,*
$$w_g \in \Phi_b \quad \Longleftrightarrow \quad T(a_v(g)), \qquad \varepsilon(\sigma_v(g)) = \varepsilon_{a_v(g)};$$
3. *either the chart is unramified with $p \ge 7$, or $p \le Q$ and all letter signs are equal;*
4. *for $p \le Q$, the thick letters satisfy (16.95) with $J = J_p$.*

The coefficient ring can be chosen to work at every place above this fixed prime simultaneously.

Proof. Take $T(a)$ to mean that some g at a large-weight place has $a_v(g) = a$. If $p \nmid N_E$, Lemma 17.21 shows that the letter determines the embedding, so it determines both membership in Φ_b and the sign. Define these signs on the letters that occur and assign arbitrary signs to the unused letters. If $p \mid N_E$, use the constant sign from Lemma 17.23, choosing either sign if there is no large-weight embedding. Here $p \le Q$ by Lemma 17.19; clause 2 is required only for large-weight embeddings, where it holds by construction. The same lemma and $e_F \le [E_v : \mathbb{Q}_p] \le [E : \mathbb{Q}]$ give clause 1. The unramified alternative follows from Lemma 17.21, and clause 4 is Lemma 17.24. Finally there are finitely many $v \mid p$, so one finite coefficient extension contains their finitely many roots, by Lemma 17.16.

Lean, formal context. `CrystallineDefect` (L130-L295), `FitTwistsExist` (L415). □

This proves the content of the coefficient-letter, sparsity and chart construction, using the already proved sparse-place construction. At a small prime a thin embedding can share a letter with a thick one. Clause 2 deliberately makes no assertion for that thin embedding: Lemma 16.88 absorbs its window in $[-2G, 2G]$. This point is needed in the ramified case.

Application to the main summand and the character string

Let the supplied local summands be L_{main} of rank ρ and $L_c = \mathcal{O}(-c)$ for $0 \le c < t \le t_*$. Use the jump convention of the note, so L_c has the single jump c, while its Tate dual has the single jump $-c-1$. Let J_γ be the jump multiset of L_{main}. The test class allows

$$U_\gamma = \operatorname{supp} J_\gamma \cup \{0, 1, \dots, t-1\}. \tag{17.9}$$

Any smaller sets containing all the required test jumps may be used. For a specified summand L_j, indexed by the symbol main or by $0 \le j < t$, use on the dual side the actual support

$$U'_{j,\gamma} = \operatorname{supp} \mathrm{HT}_\gamma(L_j^\dagger) \subseteq \{-u - 1 : u \in U_\gamma\}. \tag{17.10}$$

One must retain this restriction for the main summand. Dualizing the entire test set would also insert $-1, \dots, -t$, which are not main-dual jumps and need not satisfy its negative-gap hypothesis.

Theorem 17.26 (The local crystalline estimate for the specified summands)**.** *Take any scale above the common eventual threshold for the data constructed above, and a finite family of supplied rational representations defined over finite coefficient fields. Suppose a specified main representation has the weight profile of Theorem 17.9 or Theorem 17.10, with rank ρ and character-string length $t \le t_*$, and has the asserted Hodge–Tate jump identities at every place above p. At a place $v \mid p$ away from the two level primes, suppose the main representation and all rational representations used in its relation construction are crystalline, and their jumps lie in (17.9). Let $\mathcal{C}_v$ be the class of crystalline lattices with those allowed jump sets. Then their stable lattices belong to $\mathcal{C}_v$, and*

$$\begin{aligned} \widehat{\lambda}_{\mathcal{O}} \left(\frac{Z_f(L^\dagger_{\mathrm{main}})}{H_{\mathcal{C}_v}(L^\dagger_{\mathrm{main}})} \right) &\le [E_v : \mathbb{Q}_p]\, \rho D_p, \\ \widehat{\lambda}_{\mathcal{O}} \left(\frac{Z_f(L^\dagger_c)}{H_{\mathcal{C}_v}(L^\dagger_c)} \right) &\le [E_v : \mathbb{Q}_p]\, D_p/t_* \qquad (0 \le c < t). \end{aligned} \tag{17.11}$$

Here D_p is the original number in (15.26). There are no further chart, letter, sign, sparsity or numerical size hypotheses in this application.

Proof. Lemmas 17.17 and 17.18 give the actual local lattices and their membership. Lemma 17.14 verifies (16.87) at

$$Q = kt_*, \quad h = r^8, \quad \eta = h^2 + 1, \quad W = 4t_*, \quad \delta_0 = i(2h) + h - 2t_*,$$

with the original G and b. The opposite embedding windows are Lemma 17.11. Theorem 17.25 and Lemma 16.88 pass them to coefficient letters and verify all chart, sign and small-prime sparsity conditions of Theorem 16.92.

For the main Tate dual, retain $U'_{\mathrm{main},\gamma}$ in (17.10). Lemma 17.12 gives the negative gap and the low-jump multiplicity $\mu = \lfloor \rho m/k \rfloor$. Corollary 16.93, with $x = 1$, proves the first bound. For a character Tate dual, Lemma 17.13 puts its actual jump in $[-\eta, 0]$. This is the second alternative of Theorem 16.92, so Corollary 16.93 applies with $x = t_*$ and proves the second bound. Its rank is one.

All admissible pairings in (15.5) are included in the tested class. Their crystalline error extensions are therefore the actual extensions used by the local cup theorems. The canonical chart is a chart of G_{E_v} itself, so there is no restriction–corestriction factor to add. The proof works with every already specified stable lattice; it does not change its integral structure in order to obtain the estimates.

Lean, formal context. `CrystallineDefect` (L532-L669), `FitTwistsExist` (L415). □

This is the crystalline part of the estimate for a main summand and its character string. The tame application at the two level primes is kept separate. The main-row realization, its jumps and all its local types are supplied in Theorem 17.87. The ambient realizations and their class membership are Theorem 17.105 and Corollary 17.106. The chart and sparse-letter statements for the constructed fields are proved independently of those representation inputs.

17.3 The tame level-prime application

The tame estimate can be applied to the specified local classes once their inertial period spectra are supplied. We prove here the root-polynomial construction, the common-chart assertion for all cocycle extensions, and the numerical comparison with the original one-place allowance. The inertia operators in this subsection are the linear operators on potentially crystalline periods, not the operators on the Galois lattice itself.

Lemma 17.27 (A polynomial from a bounded list of tame roots)**.** *Let p be prime, let $r \geq 1$, and let $\mathcal{M}$ be a multiset of at most a elements of $\overline{\mathbb{Q}}_p$. Suppose that for every $y \in \mathcal{M}$ there is an integer j, $1 \leq j \leq r$, with $y^{p^j-1} = 1$. Set*

$$e = \operatorname{lcm}_{1 \leq j \leq r}(p^j - 1).$$

There is a monic $q \in \mathbb{Z}_p[X]$ of degree at most ar which vanishes on $\mathcal{M}$. The polynomial $X^e - 1$ also vanishes on $\mathcal{M}$. If a finite-order linear operator in characteristic zero has its spectrum in $\mathcal{M}$, both polynomials annihilate it. The same assertions, with the same degree bound, hold for the inverse spectrum.

Proof. For $y^{p^j-1} = 1$, reduction identifies y with the Teichmüller lift of an element of $\mathbb{F}_{p^j}^{\times}$. Thus y lies in the unramified extension of $\mathbb{Q}_p$ of degree j. Its minimal polynomial m_y over $\mathbb{Q}_p$ is monic, has degree at most j, and belongs to $\mathbb{Z}_p[X]$: y is integral, and $\mathbb{Z}_p$ is integrally closed. Put

$$q = \prod_{y \in \mathcal{M}} m_y,$$

with multiplicity. Then $\deg q \leq ar$, and q vanishes on every member of $\mathcal{M}$. Since $p^j - 1 \mid e$, each member also satisfies $y^e = 1$.

A finite-order operator A satisfies $A^d = 1$ for some $d > 0$. The polynomial $X^d - 1$ has no repeated root in characteristic zero, so A is diagonalizable after extending scalars. A polynomial vanishing on its spectrum therefore annihilates it; faithful scalar extension gives the identity over the original field. Finally y^{-1} has the same order as y, so the same argument applies to the inverse multiset.

Lean, formal context. `LocalLattices` (L40-L82), `FitTwistsExist` (L415). □

This is the ordinary proof of `isTameType_of_roots`. The finite-order condition is essential when one infers an annihilating identity from a spectral list. For potentially crystalline representations it is supplied by the finite inertia action on their period space.

Corollary 17.28 (The main and character type degree)**.** *Suppose that, for each inertia element s, the period spectra of a class $\mathcal{Q}$ of potentially crystalline lattices over $G_{\mathbb{Q}_p}$ lie in one multiset $\mathcal{M}_s$, common to the entire class, of cardinality at most $r + t_*$. Assume that each member has order dividing $p^j - 1$ for some $1 \leq j \leq r$. Then, at every s, the class has the two annihilators required by Theorem 15.18, with*

$$N = (r + t_*)r.$$

If $L_0 \in \mathcal{Q}$, its Tate dual has the inverse type required for the paired lattice in that theorem. In particular, the hypotheses hold if $\mathcal{M}_s$ is obtained by adjoining at most t_* character values of orders dividing $p - 1$ to at most r main values satisfying the stated order bound.

Proof. Apply Lemma 17.27 with $a = r + t_*$. For a Tate dual the inertial period operator is the transpose of the inverse operator; the Tate factor has trivial crystalline inertial type. Polynomial annihilation is preserved by transpose, and the inverse-spectrum assertion gives the same degree

bound. More precisely, every polynomial that kills the period operator of L_0 at s^{-1} kills that of $L_0^\dagger$ at s. This is the inverse-type condition on the class.

Lean, formal context. `LocalLattices` (L40-L82), `FitTwistsExist` (L415). □

Lemma 17.29 (A split level prime gives one local embedding)**.** *Let $C \geq 1$ and suppose $E \subseteq \mathbb{Q}(\mu_C)$ and $p \equiv 1 \pmod{C}$. For every place $v \mid p$, the completion E_v is $\mathbb{Q}_p$. If $\jmath : E \hookrightarrow \overline{\mathbb{Q}}_p$ is the embedding associated with v, then every $g \in G_{\mathbb{Q}_p}$ fixes $\jmath(E)$. Hence, for a fixed $\iota : \mathbb{C} \simeq \overline{\mathbb{Q}}_p$,*

$$\iota^{-1} g^{-1} \jmath = \iota^{-1} \jmath.$$

In particular the CM sign and the high-window center read at this embedding are independent of the coefficient embedding.

Proof. The congruence gives $C \mid p-1$. The roots of $X^C - 1$ in $\mathbb{F}_p$ are simple and number C; Hensel lifting therefore places all of μ_C in $\mathbb{Q}_p$. The image of every embedding of E into $\overline{\mathbb{Q}}_p$ is contained in $\mathbb{Q}_p$. Its completion contains the closure of $\mathbb{Q}$, which is $\mathbb{Q}_p$, and is contained in $\mathbb{Q}_p$, proving the first assertion. The remaining claims follow directly.

Lean, formal context. `CrystallineDefect` (L303-L318), `FitTwistsExist` (L415). □

For the data, $C = D.\mathrm{cyclotomicConductor} = q_1 d$. The construction chooses its two level primes to be 1 modulo q_2^2, where

$$q_1 = 3^{\lfloor \log_3 s \rfloor + 5} 2^s q_0, \qquad q_2 = (q_0 d)^{s+6}, \qquad s \geq 1, \qquad 18480 \mid q_0.$$

Here $C \mid q_2$, and hence $C \mid q_2^2$. Indeed, at 2 the extra valuation s in q_1 is at most $(s+5)v_2(q_0 d)$, and at 3 its extra valuation $\lfloor \log_3 s \rfloor + 5$ is at most $s + 5 \leq (s+5)v_3(q_0 d)$. At every other prime, C has just the valuation of $q_0 d$, which is at most its valuation in q_2. Thus the chosen primes are 1 modulo C. The corresponding source assertion is `mem_range_of_modEq_one`. Thus the tame application at a level prime is over $G_{\mathbb{Q}_p}$; no larger local degree is concealed in that application.

Lemma 17.30 (Crystalline cocycles preserve the common chart)**.** *Let G be a local Galois group and let $H \leq G$ be an open subgroup on which a lattice T is crystalline, and let C_0 be a trivial finite free parameter lattice. Suppose $c : G \to \mathrm{Hom}_{\mathcal{O}}(C_0, T)$ is a continuous cocycle and there is an $\mathcal{O}$-linear map $b : C_0 \to B_{\max} \otimes_{\mathbb{Z}_p} T$ with*

$$c(g) = gb - b \qquad (g \in G).$$

The integral cocycle extension with underlying lattice $T \oplus C_0$ and action

$$g(t, a) = (gt + c(g)(a), a)$$

is crystalline on the same subgroup H. The assertion applies simultaneously to any finite family of crystalline cocycles. If Y and L are crystalline on H, it also applies on H to the extension associated with every admissible pairing error in (15.5).

Proof. For each $a \in C_0$, the vector $(-b(a), a)$ in the period extension is fixed by G, since

$$-gb(a) + c(g)(a) = -b(a).$$

A crystalline period basis for T on H, together with these vectors for a $\mathbb{Z}_p$-basis of C_0, spans the entire period extension. Indeed its projection to $B_{\max} \otimes C_0$ is the standard basis, and its kernel is the period module of T. This is the crystalline spanning condition on H. For a finite family, choose a period witness for each basis vector of its parameter lattice and combine the witnesses linearly.

Duals and tensor products of crystalline representations are crystalline on H, as is the Tate twist. The admissible pairing condition says exactly that its error cocycle belongs to $Z_f((Y \otimes L)^\dagger)$. Its period witness therefore gives the same construction with $T = (Y \otimes L)^\dagger$.

Lean, formal context. `CrystallineDefect` (L303-L318), `FitTwistsExist` (L415). □

Corollary 17.31 (One tame chart for all the tested extensions)**.** *Under the type assumptions of Corollary 17.28, suppose $r < p$. Put*

$$e = \operatorname{lcm}_{1 \le j \le r}(p^j - 1), \qquad f = r!, \qquad K = \mathbb{Q}_{p^f}(p^{1/e}).$$

Every member of $\mathcal{Q}$, every paired Tate dual, and all the cocycle extensions and pairing-error extensions used in the cup calculation are crystalline over G_K. The chart is independent of the chosen members and cocycles, and $[K : \mathbb{Q}_p] = ef$ is prime to p.

Proof. Lemma 17.27 shows that every inertial period operator has e-th power one. Apply Lemma 15.17. Its common tame extension kills the finite inertial action on the potentially crystalline periods; the crystalline descent criterion stated there makes each representation crystalline over G_K. Inverse spectra have the same exponent, so the conclusion includes the paired Tate duals. Lemma 17.30 supplies all the remaining extensions on this same subgroup. The degree and its prime-to-p property are those of Lemma 15.17.

Lean, formal context. `CrystallineDefect` (L303-L318), `FitTwistsExist` (L415). □

In particular, the common-chart crystallinity of the extensions listed in Corollary 16.20 is a consequence of the type conditions on the original lattices and the definition of the tested crystalline cocycles. It is not an additional arithmetic hypothesis for each cocycle or pairing.

Lemma 17.32 (Numerical bounds at a tame place)**.** *Let r, m be integers with $r \ge 2$, $1 \le m \le r$, and put*

$$k = r + 1 - m, \qquad t_* = 5r, \qquad h = r^8, \qquad N = (r + t_*)r, \qquad \kappa = r^2(N+1)^2.$$

Let b, G, i be nonnegative integers with $b \ge r^{18}$, $0 \le G \le b$, $i \le m$, and put

$$S = \beta + (2i+1)h - 2t_*, \qquad \beta \in \{b, G\}.$$

Set $\eta = h^2 + 1$, $W = 4t_*$, and

$$H = 3\kappa + 150(\eta + W + 1).$$

For every prime $p \ge b^{10}$,

$$S + 6(W + \eta) + \kappa + 2 < p. \tag{17.12}$$

If $p \mid M$, and D_p is the number in (15.26), then, for $1 \le x \le t_*$,

$$2H \le \frac{D_p}{x}. \tag{17.13}$$

In this prime range the crystalline valuation loss λ_p in (15.26) is zero.

Proof. We have $N = 6r^2$ and $N + 1 \le 7r^2$, so $\kappa \le 49r^6$. Also $h \ge 2t_*$ because $r^7 \ge 10$; thus $S \ge 0$, and

$$S \le b + (2m+1)h \le b + 3r^9.$$

Consequently

$$\begin{aligned} S + 6(W + \eta) + \kappa + 2 &\le b + 3r^9 + 120r + 6r^{16} + 49r^6 + 8 \\ &\le b + 186r^{16} \le 187b < b^{10} \le p. \end{aligned}$$

The strict inequality uses $b \geq 2$, for which $b^9 \geq 512 > 187$. The same elementary estimates give

$$H \leq 147r^6 + 150r^{16} + 3000r + 300 \leq 3597r^{16}.$$

Since $k \leq r$, $m \geq 1$, $x \leq 5r$, and $v_p(M) \geq 1$,

$$\frac{D_p}{x} \geq 10^6 \frac{m}{kx}\, b \geq 200000 \frac{b}{r^2} \geq 200000r^{16} \geq 7194r^{16} \geq 2H.$$

This proves (17.13).

For the final assertion, every factor of $(2b)!$ is less than p. Moreover $h^3 = r^{24} \leq b^2$, so every positive integer in the other product is at most

$$b + 30h^3 \leq b + 30b^2 \leq 31b^2 < b^{10} \leq p.$$

Here $b^8 \geq 256 > 31$. None of these factors is divisible by p, and hence $\lambda_p = 0$.

Lean, formal context. `Magnitudes` (L75-L85), `FitTwistsExist` (L415). □

These size comparisons follow from the scales of the admissible data. For completeness, if

$$x_0 \geq 3, \qquad r \leq 3x_0^5, \qquad x_1 \geq x_0^{10^6}, \qquad b \geq x_1^{10^6},$$

then $r \leq x_0^6$ and $r^{18} \leq x_0^{108} \leq x_0^{10^{12}} \leq b$. Also $G = x_0 b/x_1 \leq b$. The data have $m \leq r$ and $p \in [b^{10}, b^{11}]$. Thus the numerical hypotheses used above are consequences of the stated data scales, rather than a further estimate on the representations. The scale bounds and the level-prime interval are recorded in the scale inequalities and the rank scales, the level-prime and half-gap scales.

Theorem 17.33 (The tame local bound for the test class)**.** *Retain the parameters and inequalities of Lemma 17.32. Suppose $E \subseteq \mathbb{Q}(\mu_C)$, $p \equiv 1 \pmod{C}$, $p \geq b^{10}$ and $p \mid M$. At a place $v \mid p$ let $\mathcal{Q}$ be a class of potentially crystalline $\mathcal{O}$-lattices satisfying the spectral hypotheses of Corollary 17.28. Let L be the Tate dual of a member of that class, and put $\rho = \operatorname{rank}_{\mathcal{O}} L$. Suppose the class and L have their jumps in the opposite windows*

$$[-\eta, \eta] \cup -\varepsilon[S, S+W], \qquad [-\eta, \eta] \cup \varepsilon[S, S+W], \qquad \varepsilon \in \{1, -1\},$$

with the same S and ε at every coefficient embedding. Then, for every integer $1 \leq x \leq t_*$,

$$\widehat{\lambda}_{\mathcal{O}}\left(\frac{Z_f(L)}{H_{\mathcal{Q}}(L)}\right) \leq [E_v : \mathbb{Q}_p]\, \frac{\rho D_p}{x} = \frac{\rho D_p}{x}. \tag{17.14}$$

No separate common-chart assumption on the cocycle extensions, and no coefficient-degree factor, is needed.

Proof. Lemma 17.29 identifies E_v with $\mathbb{Q}_p$ and makes the complex embedding read by every coefficient embedding constant. The stated windows are therefore the windows to which the tame theorem applies. Since $p \geq b^{10} > r$, Corollary 17.31 supplies one tame chart for every lattice and every extension used in the calculation. Corollary 17.28 supplies the type degree $N = (r + t_*)r$.

The inequality (17.12) is exactly the size hypothesis of Theorem 15.18, with far-window start S. Corollary 16.20 consequently gives cup exponent at most H. Its construction applies to every pairing in (15.5), because Lemma 17.30 verifies the required crystallinity of the pairing-error extension. Restriction and corestriction from the common chart introduce no p-power, since its degree is prime to p.

Apply Lemma 15.2 with local degree one. The normalized defect is at most $2\rho H$, and (17.13) gives $2\rho H \leq \rho D_p/x$. This is (17.14). Normalization divides the integral length by $v_{\mathcal{O}}(p)$, as in that lemma; there is no extra factor from the coefficient field.

Lean, formal context. `CrystallineDefect` (L463-L514), `FitTwistsExist` (L415). □

The theorem reconstructs the tame application in the tame local bound. Its numerical and common-chart requirements have been proved. The theorem requires a specified global representation to have a potentially crystalline local realization with the asserted finite inertial spectra. Theorem 17.105 verifies these hypotheses for the actual full family, and Corollary 17.106 applies the bound. The opposite-window statement is supplied by the actual weight calculation; neither a spectrum nor a jump multiset is being deduced from the existence of an arbitrary stable lattice.

17.4 The Selmer conditions for the specified local types

We verify the datum used in the tied Selmer descent, including its original global term and the two first-prime determinants. The arithmetic assumptions in this subsection are the existence of a continuous representation over a finite coefficient field, its stated conjugate self-duality, its unramifiedness and Hodge–Tate comparison, and its local type at the chosen places away from the coefficient prime. The scalar, full-span and numerical length conditions will be deduced.

Let E be the CM field of degree $2f$, let O be a coefficient valuation ring with fraction field K, and fix a rank-ρ stable lattice L with representation $R : G_E \to \mathrm{GL}_O(L)$. We assume $1 \le \rho \le r$. Write $\chi = \chi_\ell$, $g^c = cgc^{-1}$ and $g^{-c} = (g^c)^{-1}$. The integer $v_O(\ell)$ is denoted by e_O, so normalized lengths are ordinary O-lengths divided by e_O. Put

$$g_* = \prod_{w \text{ chosen}} |\,\mathrm{GL}_{rt_*}(\kappa_w)|, \qquad t_* = 5r, \qquad b_\ell = v_\ell(g_*).$$

This is the integer in (14.36), with its matrix-size parameter equal to rt_*.

Lemma 17.34 (Trace self-duality transports scalar elements)**.** *Suppose $1 \le s \le t_*$, the image of R spans $\mathrm{End}_O(L)$, and*

$$\operatorname{tr} R(x^c) = \chi(x)^{1-s} \operatorname{tr} R(x^{-1}) \qquad (x \in G_E). \tag{17.15}$$

If $R(g) = \alpha I$, then $\alpha \in O^\times$ and

$$R(g^{-c}) = \chi(g)^{s-1}\alpha I. \tag{17.16}$$

If $\chi(g) = 1$, the matrices $R(g)$ and $R(g^{-c})$ have the same characteristic polynomial, whether or not they are scalar.

Proof. In an O-basis of L, the representations $x \mapsto R(x^{-c})^{\mathsf{T}}$ and $x \mapsto \chi(x)^{s-1}R(x)$ have equal traces by (17.15) at x^{-1}. Both span the full matrix algebra, since conjugation and inversion permute G_E, transpose preserves the matrix span, and the twists are units. They are conjugate by the trace criterion [Car94, Theorem 1].

For the full-span case take $A = O$ or $A = O/\pi^j$, where π is a uniformizer and $j \ge 1$. If P, Q have equal traces, the rule $P(x) \mapsto Q(x)$ respects every A-linear relation, as one sees by multiplying its image by $Q(y)$ and taking traces. Here one uses equality at xy and the perfect matrix trace pairing. Interchanging P, Q gives an inverse, and multiplication on group elements gives an algebra automorphism of $M_\rho(A)$. Such an automorphism is inner for local A. Indeed, the images of the diagonal matrix units split A^ρ into ρ isomorphic free rank-one summands. Transport a generator of the first to the others by the images of the matrix units e_{j1}. The resulting basis conjugates all the matrix units to their images.

If $R(g) = \alpha I$, invertibility gives $\alpha \in O^\times$, and the conjugacy gives (17.16). When $\chi(g) = 1$, the conjugacy and invariance under transpose give the asserted equality of characteristic polynomials.

Lean. `SelmerData` (L657-L721), `FitTwistsExist` (L415). The trace criterion gives conjugacy on the whole lattice. □

Lemma 17.35 (The global congruence term)**.** *Retain the preceding hypotheses and assume* $1 \le \rho \le r$*. For* $a \in O \setminus \{0\}$ *let* $V_a = L/aL$*, with the conjugate Cartier dual of* (13.17)*. Define the ideal*

$$\mathfrak{d} = \big((1-\chi(x)^s)\operatorname{tr} R(x) : x \in G_E\big) \subseteq O, \qquad v_O(0) = +\infty.$$

The O*-module* $\operatorname{Hom}_{O[G_E]}(V_a, V_a^{*c})$ *is cyclic of length* $\min\{v_O(a), v_O(\mathfrak{d})\}$*. If* $R(g) = \alpha I$ *and* $\chi(g) = p_i^{\rho}$ *for one of the two level primes, then*

$$\sup_{a\neq 0} \widehat{\lambda}_O \operatorname{Hom}_{O[G_E]}(V_a, V_a^{*c}) = v_O(\mathfrak{d})/e_O \le b_\ell. \tag{17.17}$$

Proof. Write $n = v_O(a)$. In dual coordinates the action is the reduction of $R'(x) = \chi(x)R(x^{-c})^{\mathsf{T}}$. Indeed, for a generator ξ of the inverse different of $O/\mathbb{Z}_\ell$, the perfect pairing on O/π^n is $(u,v) \mapsto \operatorname{Tr}_{K/\mathbb{Q}_\ell}(\pi^{-n}\xi uv) \bmod \mathbb{Z}_\ell$. By (17.15), $\operatorname{tr} R'(x) = \chi(x)^s \operatorname{tr} R(x)$. Both representations span all matrices at every level. By the trace criterion just proved, their reductions modulo π^j are isomorphic exactly when $j \le v_O(\mathfrak{d})$.

Put $d = \min\{n, v_O(\mathfrak{d})\}$. A nonzero intertwiner modulo π^n has the form $\pi^{n-j}B$, where $1 \le j \le n$ and B intertwines modulo π^j with nonzero reduction. The residual representations are irreducible by the full matrix span, so this reduction is an isomorphism. Thus B is invertible and $j \le d$. If $d = 0$ the Hom module is zero. Otherwise multiplication by π^{n-d} maps the Hom module at level d isomorphically onto it. Fixing an isomorphism at that level identifies the latter with O/π^d, since only scalars commute with the full matrix algebra.

At the specified element g, Lemma 17.34 gives $R'(g) = \chi(g)^s \alpha I$. Hence $\alpha(1 - p_i^{\rho s})$ kills this cyclic module. After division by e_O, its length is at most $v_\ell(p_i^{\rho s} - 1) \le b_\ell$ by Lemma 14.12, since $1 \le \rho s \le r t_*$.

Lean. `congruenceTerm_le`. Lean embeds the Hom module into the torsion of a rank-ρ dual. Identifying its isomorphisms at each level proves cyclicity and removes the factor ρ in the global term. □

Lemma 17.36 (The local excess is bounded by an inertial determinant)**.** *Let* $v \nmid \ell$ *be a finite place of* E*, and suppose* $\gamma_v \in I_v$ *satisfies* $\det(R(\gamma_v) - I) \neq 0$*. Let* $C_v(a)$ *be the restrictions to* G_{E_v} *of all continuous global cocycles with values in* V_a*. Then*

$$\sup_{a\neq 0} \max\{\lambda_O C_v(a) - \lambda_O V_a, 0\} \le v_O \det(R(\gamma_v) - I). \tag{17.18}$$

Proof. For a finite ℓ-primary local Galois module at a place prime to ℓ, the finite local Euler characteristic and local Tate duality give

$$\lambda_O H^1(E_v, V_a) = \lambda_O H^0(E_v, V_a) + \lambda_O H^0(E_v, V_a^*).$$

Adding the coboundaries, whose length is $\lambda_O V_a - \lambda_O H^0(E_v, V_a)$, gives

$$\lambda_O Z^1(G_{E_v}, V_a) = \lambda_O V_a + \lambda_O H^0(E_v, V_a^*).$$

These are the same classical finite local duality formulas used in the descent argument. Since $C_v(a)$ is a submodule of this cocycle module, its excess is at most the last invariant length.

Inertia fixes the ℓ-power roots of unity at $v \nmid \ell$, so $\chi(\gamma_v) = 1$. An invariant character therefore annihilates $(R(\gamma_v) - I)V_a$. By finite duality,

$$\lambda_O H^0(E_v, V_a^*) \le \lambda_O(V_a/(R(\gamma_v) - I)V_a).$$

If the Smith exponents of $R(\gamma_v) - I$ are $d_1, \dots, d_\rho$, the right side is $\sum_j \min\{d_j, v_O(a)\} \le \sum_j d_j = v_O \det(R(\gamma_v) - I)$. This proves (17.18), including its supremum over a. No enlargement of the original local condition $C_v(a)$ is used; the full local cocycle module is only an upper bound.

Lean, formal context. `SelmerData` (L657-L721), `FitTwistsExist` (L415). □

Lemma 17.37 (The determinant bound from the good labels)**.** *Assume* (17.15)*. At a chosen place w above $p_i \neq \ell$, suppose the local type of R is given by a degree-ρ label over $\mathbb{F}_{p_i}$, scaled by the selected residue character. In rank r the unscaled label is nonzero at 0 and avoids every N-th root of unity, where the scaling element ω^{-j} has order dividing N. In rank one the unscaled label is $X-1$, with $\mathrm{ord}(\omega) = N$ and $1 \leq j \leq m < N$.*

Then the prescribed tame generator γ_w satisfies

$$\begin{aligned} \det(R(\gamma_w) - I) \neq 0, \qquad \det(R(\gamma_w^{-c}) - I) \neq 0, \\ v_\ell \det(R(\gamma_w) - I) = v_\ell \det(R(\gamma_w^{-c}) - I) \leq \rho \min\{b_\ell, 1/(\ell-1)\}. \end{aligned} \tag{17.19}$$

At every conjugate unchosen place the same conclusion holds with generator γ_w^{-c}.

Proof. In rank r, a root of the scaled label could equal 1 only if ω^j were a root of the unscaled label. This is excluded by its avoidance of the N-th roots. In rank one the scaled root is $\omega^{-j} \neq 1$, because $0 < j < N$. Every scaled root ξ is nonzero and has degree $d \leq \rho \leq r$ over $\mathbb{F}_{p_i}$. Hence

$$\xi^{p_i^d - 1} = 1, \qquad \xi \neq 1.$$

The prescribed tame generator maps this finite group of roots injectively into the characteristic-zero roots of unity. Indeed a generator of tame inertia has exponent prime to the order of every finite quotient in question. Thus every eigenvalue z of $R(\gamma_w)$ is nontrivial and satisfies $z^{p_i^d-1} = 1$ for some $1 \leq d \leq r$. This is the exact use of the local type compatibility hypothesis.

For a root of unity $z \neq 1$, the cyclotomic valuation of $1 - z$ vanishes unless z has ℓ-power order, since prime-to-ℓ roots of unity retain their order in the residue field. If z has order ℓ^a, the identity $\Phi_{\ell^a}(1) = \ell$ gives

$$v_\ell(1 - z) = 1/\varphi(\ell^a) \leq 1/(\ell - 1).$$

Indeed the factors $1 - z^j$ with j prime to ℓ have the same valuation, since $(1 - z^j)/(1 - z)$ reduces to j for each positive such j [Was97, Ch. 2]. In this case $\ell \mid p_i^d - 1 \mid g_*$ by Lemma 14.12. Summing over the ρ eigenvalues proves the bound for γ_w. Since $\chi(\gamma_w) = 1$, Lemma 17.34 gives the same characteristic polynomial for γ_w^{-c}, a generator at the conjugate place.

Lean. `MainSummands` (L193-L237), `ellVal_levelPrime_pow_sub_one_le`. The order bound in Lean suffices for the uniform allowance. Here the cyclotomic polynomial gives the exact valuation for each eigenvalue. □

Lemma 17.38 (The place terms and the first-prime terms)**.** *Assume* (17.15)*. Let S consist of the places above ℓ, p_0, p_1 and suppose every place above $p_0 p_1$ is chosen or is the conjugate of a chosen place, with the types of Lemma 17.37. For a chosen $w \nmid \ell$, put $g_0 = \gamma_w$. Let L_T be the place term in* (13.17) *and put*

$$F_T = v_O \det(R(g_0) - I) + v_O \det(R(g_0^{-c}) - I).$$

Then

$$\frac{L_T + F_T}{e_O} \leq (4f + 2)\rho \min\{b_\ell, 1/(\ell - 1)\}. \tag{17.20}$$

Proof. Every $v \in S$ prime to ℓ lies above p_0 or p_1. Apply Lemma 17.36, using the generator at v supplied by Lemma 17.37. Each normalized local term is at most $\rho \min\{b_\ell, 1/(\ell-1)\}$. For either rational prime p_i, the equality $\sum_{v \mid p_i} e_v f_v = [E : \mathbb{Q}] = 2f$ shows that there are at most $2f$ places above it. Hence at most $4f$ local terms occur. The two specified first-prime determinants satisfy the same individual bound. Adding proves the assertion.

Lean, formal context. `SelmerData` (L657-L721), `FitTwistsExist` (L415). □

Theorem 17.39 (The Selmer conditions and their bound)**.** *Let a continuous rational Galois representation be defined over a finite extension of $\mathbb{Q}_\ell$, with rank $\rho \in \{r, 1\}$ and conjugate self-duality (17.15), for $1 \le s \le t_*$. Assume it is unramified outside the places above $\ell p_0 p_1$, is potentially crystalline above ℓ, and has the jump multiset of Theorem 17.9 or Theorem 17.10, respectively. Assume its good labels at the old chosen places give the local types used in Theorem 17.7 and Lemma 17.37, at places prime to ℓ only.*

After one finite enlargement of the coefficient field, every stable lattice L yields the original Selmer datum of the tied descent, with

$$S = \{v : v \mid \ell p_0 p_1\}, \qquad Z_v = Z_f(L|_{G_{E_v}}) \quad (v \mid \ell).$$

Its local relation modules are exactly the relations defining the original crystalline Selmer condition. For any chosen $w \nmid \ell$, using $g_0 = \gamma_w$ for the first-prime terms, its parameter

$$\beta_T = G_T + L_T + F_T$$

satisfies

$$\frac{\beta_T}{e_O} \le b_\ell + (4f+2)\rho \min\{b_\ell, 1/(\ell-1)\} \le (4f+3)\rho b_\ell. \tag{17.21}$$

Proof. Lemma 17.16 supplies a common finite coefficient extension and stable lattices, and allows any already specified lattices to be retained. Enlarge further to contain the finitely many roots used in Theorem 17.7. That theorem applies to every stable lattice and supplies the full integral matrix span, a scalar $R(g) = \alpha I$ with $\chi(g) = p_i^\rho$, and an inertial element γ_* above an old prime different from ℓ with

$$R(\gamma_*) = \zeta_* I, \qquad \zeta_*^2 - 1 \in O^\times.$$

Because $\chi(\gamma_*) = 1$, Lemma 17.34 also gives $R(\gamma_*^{-c}) = \zeta_* I$. Thus the exact scalar condition (13.16) holds on this lattice.

Continuity makes all finite reductions locally constant, and the assumed unramifiedness gives the required set S. The actual crystalline-cocycle modules are finite free and contain all coboundaries by Lemma 15.1. Theorem 17.9 or Theorem 17.10, followed by Corollary 17.15, proves exactly the rank allowance (13.15). Taking the finite evaluation relations of these actual modules therefore gives the original local relations, without changing the finite or divisible Selmer condition.

Lemma 17.35 bounds G_T/e_O by b_ℓ. Lemma 17.38 bounds $(L_T + F_T)/e_O$ by $(4f + 2)\rho \min\{b_\ell, 1/(\ell-1)\}$. In particular both terms are finite, which verifies the last finiteness requirements of the datum. Adding proves (17.21).

Lean, formal context. `SelmerData` (L657-L721), `FitTwistsExist` (L415). □

For data of fixed rank ρ at all coefficient primes, the contribution of $L_T + F_T$, after normalization and summation with weights $\log \ell$, is at most $(4f+2)\rho(4 \log Y + 1)$ for $Y = \max\{2, \lceil \log g_* \rceil\}$. In fact $\log(Y!) \ge \sum_{\ell \le Y} \lfloor Y/\ell \rfloor \log \ell \ge (Y/2) \sum_{\ell \le Y} (\log \ell)/\ell$. Thus $\sum_{\ell \le Y} (\log \ell)/(\ell-1) \le 4 \log Y$. Primes $\ell > Y$ dividing g_* contribute at most $(\log g_*)/Y \le 1$ to this last sum over the support. The global congruence term has the separate allowance $\sum_\ell b_\ell \log \ell = \log g_*$.

This reconstructs the original congruence-term estimate and the construction and allowance of the main Selmer datum. The two first-prime determinants are included in the coefficient $4f + 3$: $4f$ comes from the old places, 2 from those determinants and 1 from the global congruence term. The result verifies the numerical datum input for the prior twisting and descent arguments. Theorem 17.87 verifies all these hypotheses for the normalized good main rows. Theorem 17.94 supplies the original rank-one application, and Theorem 17.105 supplies the ambient representations.

17.5 Local estimates for the specified weights and types

The preceding arguments separate the arithmetic realization of the representations from the estimates required after such a realization. We record the resulting statement for the original main summand and its character string, keeping that realization hypothesis explicit.

Theorem 17.40 (Local bounds for the specified rational representations)**.** *Take a sufficiently large member of the family in the admissible-data construction, and put $t_* = 5r$. Fix a coefficient prime p and a finite family of continuous rational Galois representations, each defined over a finite extension of $\mathbb{Q}_p$. Let its main summand have rank $\rho \in \{r, 1\}$, with the descent-pair or main-pair jump multiset of Theorems 17.9 and 17.10, respectively. Include a string of characters with jumps c, $0 \le c < t \le t_*$.*

Assume that the supplied representations realize these jump identities, and that all the representations used in the local relation classes have jumps in

$$\operatorname{supp} J(\sigma) \cup \{0, \dots, t-1\}.$$

At places above p, assume they are crystalline if $p \notin \{p_0, p_1\}$. If $p \in \{p_0, p_1\}$, assume instead that they are potentially crystalline and that, for each inertia element s at v, there is one multiset $\mathcal{M}_{v,s}$, common to all the supplied local representations, which contains their period spectra and has at most $r + t_*$ members. Each member has order dividing $p^j - 1$ for some $1 \le j \le r$. The main and character summands themselves belong to these local classes.

Choose a common coefficient ring as in Lemma 17.16, and use any specified stable lattices L_{main} and L_c. For each $v \mid p$, let $\mathcal{C}_v$ be the crystalline class with the allowed jumps away from the two level primes. At a level prime, let it be the potentially crystalline class with those jumps and the identities

$$\prod_{y \in \operatorname{supp} \mathcal{M}_{v,s}} (\beta_T(s) - y) = 0 \qquad (s \in I_{E_v}),$$

where β_T is the inertial action on the periods of its member T. These identities hold for the supplied family because its inertial period operators have finite order. Then the actual defects satisfy

$$\begin{aligned} \delta_{v,\mathrm{main}} &:= \widehat{\lambda}_{\mathcal{O}} \left(\frac{Z_f(L^\dagger_{\mathrm{main}}|_{G_{E_v}})}{H_{\mathcal{C}_v}(L^\dagger_{\mathrm{main}}|_{G_{E_v}})} \right) \le [E_v : \mathbb{Q}_p]\, \rho D_p, \\ \delta_{v,c} &:= \widehat{\lambda}_{\mathcal{O}} \left(\frac{Z_f(L^\dagger_c|_{G_{E_v}})}{H_{\mathcal{C}_v}(L^\dagger_c|_{G_{E_v}})} \right) \le [E_v : \mathbb{Q}_p]\, D_p/t_*. \end{aligned} \tag{17.22}$$

Here D_p is exactly (15.26). Every bound holds for every congruence depth used in the original relation construction. There is no further chart, coefficient-letter, sparsity, sign, common-extension or scalar-size hypothesis.

Proof. Away from the two level primes, Theorem 17.26 gives both estimates. Its proof uses the actual support of the main dual for the few-low-jump alternative and the nonpositive jump of a character dual for the other alternative. Thus it applies to the two specified summands without inserting the character jumps into the main-dual support.

At a level prime, Lemma 17.29 and the subsequent divisibility calculation give $E_v = \mathbb{Q}_p$. Consequently every coefficient embedding reads the same σ, and hence the same CM sign and window start. Lemma 17.11 supplies the opposite windows. The assumed spectral list supplies Corollary 17.28 with $N = (r + t_*)r$. Corollary 17.31 and Lemma 17.32 give the common chart and the required numerical bounds. Apply Theorem 17.33 with $x = 1$ for $L^\dagger_{\mathrm{main}}$, and with $x = t_*$ for each rank-one $L^\dagger_c$. This proves (17.22).

All these estimates are statements about the integral cup-orthogonal quotients of the given lattices. The local theorems test every admissible finite-coefficient pairing, so their exponents and normalized lengths are uniform in the congruence depth. Lemma 17.16 chooses the coefficient ring before the comparisons; no quotient is replaced by one obtained through an unproved base-change identity.

Lean, formal context. `FitTwistsExist` (L415). □

Corollary 17.41 (The sum of the local bounds)**.** *Under the hypotheses of Theorem 17.40, suppose the supplied family is the family used in the original relation construction of Corollary 15.12. Write* $2f = [E : \mathbb{Q}]$*, and let* L_T, L_c *denote its main and character crystalline colengths. Then*

$$\frac{L_T + \sum_{0 \le c < t} L_c}{v_{\mathcal{O}}(p)} \le 2f \left(\rho + \frac{t}{t_*}\right) D_p \le 4f\rho D_p. \tag{17.23}$$

In particular the local term in (25.3) *has its original coefficient and normalization.*

Proof. Lemma 17.18 puts the restrictions of the supplied family and its indicated summands in the prescribed classes. Theorem 17.40 proves exactly (15.24). Apply Theorem 15.19; its proof sums $[E_v : \mathbb{Q}_p]$ to $2f$, and uses $t \le t_*$ and $\rho \ge 1$ for the last inequality.

Lean, formal context. `FitTwistsExist` (L415). □

The distinction between the two uses of a level prime is essential. If p is the coefficient prime, the tame estimates use its operators on potentially crystalline periods. The matrix-span argument of Theorem 17.7 instead selects one of the two level primes different from p, where the asserted local type concerns the actual tame inertia action. Neither assertion has been substituted for the other.

The Galois representations in the main rows. The local estimates apply once the main rows are realized by representations with the prescribed weights and local types. We first construct these rows, descend their coefficients to finite extensions, and compare their Hecke eigenvalues. The larger family needed for the denominator calculation is treated in Theorem 17.105.

17.6 Finite coefficients and exact identities from attachment

We first choose a finite coefficient field for the attached representations. We then use spherical Hecke duality to identify their conjugate traces and to compare two theta lifts before cancelling their common summands.

A finite coefficient field from compactness

Lemma 17.42 (Countably many finite coefficient fields)**.** *The finite extensions of* $\mathbb{Q}_\ell$ *inside a fixed algebraic closure form a countable set. Consequently there is an increasing sequence of finite extensions* $K_m/\mathbb{Q}_\ell$ *with* $\overline{\mathbb{Q}}_\ell = \bigcup_m K_m$*.*

Proof. Let $K = \mathbb{Q}_\ell(\alpha)$ be a finite extension and f the minimal polynomial of α. Because $f'(\alpha) \neq 0$, Hensel's lemma gives, for a sufficiently close monic rational approximation to f, a root $\beta \in K$ as close to α as desired. Krasner's lemma then gives $\mathbb{Q}_\ell(\beta) = K$. There are countably many roots of rational polynomials in the fixed algebraic closure. Take successive composita of the fields they generate.

Lean, formal context. `FitTwistsExist` (L415). □

Theorem 17.43 (Compact representations have finite coefficients)**.** *Let G be a compact topological group, and let*

$$R : G \longrightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$$

be continuous for the ℓ-adic topology on the target. There is a finite extension $K/\mathbb{Q}_\ell$ such that $R(G) \subseteq \mathrm{GL}_n(K)$ in the given basis. The resulting K-valued representation is continuous.

Proof. Put $H = R(G)$. For the fields of Lemma 17.42, the subgroups $H \cap \mathrm{GL}_n(K_m)$ are closed by completeness and cover H. Baire's theorem makes one of them open, hence of finite index by compactness. Adjoining the matrix entries of coset representatives to K_m gives K, whose topology is the induced one.

Lean, formal context. `FitTwistsExist` (L415), `eventually_exists_isLatticeIn`. The formal lattice theorem uses compactness, a choice of basis with bounded entries, and descent of the entries to a coefficient ring. Here Baire's theorem on the compact image gives the finite coefficient field directly. □

For a profinite group this is Brinon–Conrad, *CMI Summer School Notes on p-adic Hodge Theory*, Exercise 14.4.4(4). The target is the algebraic closure with its ℓ-adic topology. The conclusion can fail over $\mathbb{C}_\ell$.

Corollary 17.44 (One coefficient ring for a finite family)**.** *A finite family of continuous $\overline{\mathbb{Q}}_\ell$-valued representations of G_E, including any finitely many character twists used with it, is defined over a common finite extension $K/\mathbb{Q}_\ell$. After enlarging K if necessary, each has a stable lattice over O_K. Any finitely many already specified stable lattices over finite coefficient fields can be retained by extension of scalars.*

Proof. Take the compositum of the finitely many fields given by Theorem 17.43 and the fields of the specified lattices. Over its integers O_K, keep the scalar extensions of those lattices. For any remaining representation R, compactness bounds the matrix entries, so the stable module $\sum_g R(g)O_K^n$ lies between O_K^n and $\varpi^{-a}O_K^n$ for some a. It is therefore a full free lattice.

Lean, formal context. `FitTwistsExist` (L415), `eventually_exists_isLatticeIn`. The formal theorem gives a lattice over every sufficiently large coefficient ring. For finitely many representations, this proof forms a single compositum and spans the orbit of an integral basis. □

Trace forms and the attachment limit

Lemma 17.45 (The nondegenerate image trace form)**.** *Let F be an algebraically closed field of characteristic zero, and $R : G \to \mathrm{GL}(V)$ a finite-dimensional representation. The bilinear form*

$$(a, b) \longmapsto \mathrm{tr}_V(ab) \quad \text{on} \quad A = \mathrm{span}_F\, R(G)$$

is nondegenerate if and only if V is semisimple as a G-module.

Proof. Let J be the Jacobson radical of A. Since J is nilpotent, $\mathrm{tr}_V(xa) = 0$ for every $x \in J$, $a \in A$. Nondegeneracy therefore gives $J = 0$, hence semisimplicity. Conversely, if V is semisimple, then J annihilates V and faithfulness gives $J = 0$. In the Wedderburn decomposition $A = \prod_i M_{d_i}(F)$, the ith simple module occurs in V with multiplicity $m_i > 0$. On this factor the pairing is m_i times the nondegenerate matrix trace pairing, and $m_i \neq 0$ in F.

Lean, formal context. `FitTwistsExist` (L415). □

Lemma 17.46 (Semisimple representations are determined by traces)**.** *Over an algebraically closed field of characteristic zero, two finite-dimensional semisimple representations with equal traces on every group element are isomorphic. For continuous $\overline{\mathbb{Q}}_\ell$-valued representations of a compact group, the isomorphism can be defined over a common finite coefficient extension.*

Proof. Apply the preceding decomposition to the diagonal action on the direct sum. Equality of traces extends by linearity from G to its image algebra. At the identity of each matrix factor the trace is its size times its multiplicity, so the multiplicities agree (Brauer–Nesbitt). In the ℓ-adic case adjoin the entries of an intertwiner to the common coefficient field of Theorem 17.43.

Lean, formal context. `FitTwistsExist` (L415), `IsTraceNondeg.exists_conj_of_trace_eq`. Both arguments recover conjugacy from traces using Wedderburn decomposition. Here the trace of each central component records its multiplicity. □

The construction of the Galois representation uses the following limit. For each $g \in G_E$, use the proper Frobenius filter mostFrobeniiAt(g) from the earlier construction. A datum x has a place w_x and a geometric Frobenius ϕ_x, with $\phi_x \to g$. Finitely many bad places may be excluded. Conjugating the place and Frobenius carries this filter to the filter at g^c: conjugation is an open homeomorphism and preserves the cofinite place conditions. If R is attached to a monic Hecke-polynomial system H_w, then, for every coefficient,

$$[X^j]\iota(H_{w_x}) - [X^j]\det(X - R(\phi_x)) \longrightarrow 0. \tag{17.24}$$

This is the consequence of the source's arbitrarily small coefficient approximation that will be used. For finitely many attached systems the conditions hold simultaneously by finite intersection in the same proper filter. The fixed field isomorphism $\iota : \mathbb{C} \simeq \overline{\mathbb{Q}}_\ell$ is applied to the Hecke coefficients before taking these ℓ-adic limits; no continuity of ι is asserted.

Lemma 17.47 (Polynomial identities pass through attachment)**.** *Suppose finitely many continuous representations and their Hecke-polynomial systems satisfy* (17.24)*. Any polynomial identity among their Hecke coefficients and finitely many continuous functions of ϕ_x, valid eventually in the Frobenius filter, holds with those coefficients replaced by the characteristic coefficients at g and those functions evaluated at g.*

Proof. The characteristic coefficients and the given polynomial are continuous. Hence its values along the filter tend to the expression at g. They are eventually zero, and the limit in a Hausdorff field along a proper filter is unique.

Lean, formal context. `FitTwistsExist` (L415). □

Conjugate self-duality and normalization

Let $\chi = \chi_\ell$ be the cyclotomic character; thus $\chi(\phi_w) = q_w^{-1}$ for a geometric Frobenius away from ℓ. Write $g^c = cgc^{-1}$ for the CM involution.

Lemma 17.48 (The opposite-place Hecke coefficient identity)**.** *At a split unramified place of residue cardinality q, let $n \geq 1$ and write the degree-n Hecke polynomials of a common nonzero spherical eigenvector as*

$$H_w(X) = \sum_{i=0}^{n} (-1)^i q^{i(i-1)/2} s_i(w) X^{n-i}, \qquad s_0(w) = 1,$$

and similarly at $\bar{w}$. With the standard opposite-place identification of the unitary group, they satisfy

$$H_w(0)[X^{n-1}]H_{\bar{w}} = q^{n-1}[X]H_w. \tag{17.25}$$

Proof. Normalize Haar measure by giving the spherical compact subgroup volume one. In the split GL_n coordinates let T_i be the double coset of $\mathrm{diag}(\varpi I_i, I_{n-i})$ and let $Z = T_n$, the central double coset of ϖI_n. At the conjugate place the unitary relation gives the inverse transpose coordinates.

Hence its ith coset is that of $\mathrm{diag}(\varpi^{-1}I_i, I_{n-i})$, and multiplication by Z gives T_{n-i}. No coset multiplicity occurs in this product because Z is central and is a single compact coset. For a common eigenvector this gives

$$s_n(w)s_1(\bar{w}) = s_{n-1}(w).$$

The three coefficients are

$$\begin{aligned} H_w(0) &= (-1)^n q^{n(n-1)/2} s_n(w), \\ [X^{n-1}]H_{\bar{w}} &= -s_1(\bar{w}), \\ [X]H_w &= (-1)^{n-1} q^{(n-1)(n-2)/2} s_{n-1}(w). \end{aligned}$$

Their signs agree after multiplication, and $n(n-1)/2 = (n-1) + (n-1)(n-2)/2$. This proves (17.25).

Lean, formal context. `AttachedRepresentations` (L313-L413), `FitTwistsExist` (L415). □

Theorem 17.49 (Conjugate trace identity from an attached eigenform)**.** *Let $R : G_E \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ be continuous and attached to the Hecke-polynomial system of a nonzero eigenform. Suppose the eigenform is spherical at both w and $\bar{w}$ outside a finite set, with the normalization of Lemma 17.48. Then*

$$\operatorname{tr} R(g^c) = \chi(g)^{1-n} \operatorname{tr} R(g^{-1}) \qquad (g \in G_E). \tag{17.26}$$

If the image trace form of R is nondegenerate, then

$$R^c \simeq R^\vee \otimes \chi^{1-n}$$

over a finite coefficient extension.

Proof. For $n = 0$ both traces vanish. Otherwise conjugation carries a geometric Frobenius at w to one at $\bar{w}$, and is a homeomorphism of G_E. Thus the attachment approximation for $H_{\bar{w}_x}$ has limit $\det(X - R(g^c))$. Exclude the finitely many nonspherical places and those above ℓ. Then (17.25), $q_{w_x}^{n-1} = \chi(\phi_x)^{1-n}$ and Lemma 17.47 give

$$[X^0]\det(X - R(g))\,[X^{n-1}]\det(X - R(g^c)) = \chi(g)^{1-n}[X]\det(X - R(g)).$$

For every invertible matrix A of size n,

$$[X^0]\det(X - A) = (-1)^n \det A, \quad [X^{n-1}]\det(X - A) = -\operatorname{tr} A,$$

and

$$[X]\det(X - A) = (-1)^{n-1} \det A \operatorname{tr}(A^{-1}).$$

Substituting and cancelling the nonzero determinant proves (17.26). No division by the rank is involved. Nondegeneracy implies semisimplicity by Lemma 17.45; the two representations in the last assertion are semisimple and have equal traces. Apply Lemma 17.46.

Lean, formal context. `AttachedRepresentations` (L313-L413), `FitTwistsExist` (L415). □

The source comparison is the opposite-place coefficient identity and conjugate self-duality. Continuity and the nondegenerate image trace form are properties of the supplied attached representation. The theorem derives its self-duality from attachment; it does not derive attachment from a collection of formal Hecke polynomials.

Lemma 17.50 (The normalization changes the duality exponent)**.** *Suppose a rank-r representation R satisfies (17.26) with $n = r$. Let $a \in \mathbb{Z}$ satisfy $r - t = 2a$, and let θ be a continuous character with*

$$\theta(g^c)\theta(g) = \chi(g)^{2a}.$$

Then $T = \theta R$ satisfies

$$\operatorname{tr} T(g^c) = \chi(g)^{1-t} \operatorname{tr} T(g^{-1}). \tag{17.27}$$

If the Hodge–Tate jump of θ at σ is $-a + \epsilon_\sigma w(v)$, then the jump multiset of T is

$$\{u - a + \epsilon_\sigma w(v) : u \in J_R(\sigma)\},$$

with multiplicities. If $P_s(R(s)) = 0$ for an inertial element s, and P_s has degree d, then $\theta(s)^d P_s(X/\theta(s))$ annihilates $T(s)$.

Proof. The trace of $T(g^c)$ is

$$\theta(g^c)\chi(g)^{1-r} \operatorname{tr} R(g^{-1}) = \chi(g)^{2a+1-r}\theta(g)^{-1} \operatorname{tr} R(g^{-1}),$$

and $2a + 1 - r = 1 - t$. This proves (17.27). Hodge–Tate weights add under tensor product with a character, which gives the asserted multiset in the jump convention of the note. The last assertion is direct substitution in the polynomial, since $T(s) = \theta(s)R(s)$.

Lean, formal context. `SelmerHypotheses` (L143-L205), `FitTwistsExist` (L415). □

For the avatar of the character $\mu^{-1}(x/\bar{x})|x|^a$ used in the descent pair, the asserted character-product identity follows from its Frobenius values at w and $\bar{w}$: their product is q_w^{-2a}. Both sides are continuous characters, so Frobenius density extends the identity to G_E. The jump formula is the local avatar formula proved with the cited rank-one character dictionary in Lemmas 17.89 and 17.91. These are the normalizations in the avatar duality calculation and the main-summand twist.

17.7 Galois representations of the main eigensystems

The definite terminal main space W_m admits a shorter realization argument than the unrestricted all-rank construction: a primitive Green label at one split place forces a cuspidal base change. The elementary local argument below, followed by precise classical theorems of Labesse and Chenevier–Harris, constructs the raw rank-r representation for every coefficient prime. Prescribed ramified local types are a separate assertion. For theta lifts that need not have cuspidal base change, we also prove the two-lift cancellation and its local consequences.

Lemma 17.51 (A finite cuspidal type forces supercuspidality)**.** *Let F_v be a nonarchimedean local field, let $H = \mathrm{GL}_n(F_v)$, $K = \mathrm{GL}_n(\mathcal{O}_v)$, and $K_1 = \ker(K \to \mathrm{GL}_n(k_v))$. Let τ be an irreducible complex representation of $\mathrm{GL}_n(k_v)$ whose invariants under the unipotent radical of every proper parabolic subgroup are zero. If an irreducible smooth admissible representation π of H contains the inflation of τ to K, then π is supercuspidal.*

Proof. If π is a subquotient of induction from a proper standard parabolic $P = MN$, take K_1-invariants. This is exact by compact averaging. Iwasawa decomposition $H = PK$ identifies the invariants of the induction with

$$\operatorname{Ind}_{\overline{P}}^{\mathrm{GL}_n(k_v)} \sigma^{M \cap K_1}.$$

Indeed restriction to K makes an invariant function a function on K/K_1, with the indicated left equivariance. The modulus character is trivial on $P \cap K$. Frobenius reciprocity factors any map

from τ through its $\overline{N}$-coinvariants. These vanish because averaging identifies them with $\tau^{\overline{N}} = 0$. Semisimplicity over the finite group then rules out τ in every subquotient.

Lean, formal context. `FitTwistsExist` (L415), `New.GaloisRep.sphericalCompat_std`. This finite-type test is used in the classical base-change argument. The cited formal compatibility theorem is obtained through cohomology of unitary groups and theta lifting. □

The corresponding compact-induction construction is also stated in Prasad–Raghuram, *Representation theory of* $\mathrm{GL}(n)$ *over non-Archimedean local fields*, Theorem 6.2. The proof above gives the implication needed here for an arbitrary irreducible constituent, rather than merely constructing one representation containing the type.

Satake normalization. Fix $\iota_\ell : \mathbb{C} \simeq \overline{\mathbb{Q}}_\ell$. At a split spherical place w, write q_w for the residue cardinality and $\alpha_1, \dots, \alpha_n$ for the unitary Satake parameters. The jth double coset has eigenvalue $q_w^{j(n-j)/2} e_j(\alpha)$, where e_j is the jth elementary symmetric polynomial. The factor $q_w^{j(j-1)/2}$ in the Hecke polynomial of Lemma 17.48 therefore gives roots $q_w^{(n-1)/2}\alpha_i$. These are the geometric Frobenius eigenvalues of $\mathrm{rec}_w(\Pi_w |\det|^{(1-n)/2})$ for the local base change Π_w.

Theorem 17.52 (The distinguished representations before normalization)**.** *Let E/F be CM with $[F:\mathbb{Q}] \geq 2$, and let $U = U(V)$ for an n-dimensional nondegenerate Hermitian E/F-space V, with U compact at every infinite place. Let b be a spherical Hecke eigensystem occurring in a nonzero algebraic automorphic form of rank n and weight κ on U. Suppose its simultaneous eigenspace has a nonzero projection to a finite cuspidal K/K_1-type at one split finite place. For every prime ℓ there is a continuous semisimple representation*

$$R_{b,\ell} : G_E \longrightarrow \mathrm{GL}_n(K_\ell), \qquad [K_\ell : \mathbb{Q}_\ell] < \infty,$$

with the following properties. At a split spherical place $w \nmid \ell$ it is unramified and has characteristic polynomial $\iota_\ell P_{b,w}$ at geometric Frobenius. At a split spherical place $w \mid \ell$ it is crystalline. Let $E_{w,0}$ be the maximal unramified subfield of E_w, let $f_w = [E_{w,0} : \mathbb{Q}_\ell]$, and write φ for crystalline Frobenius on $D_{\mathrm{cris}}(R_{b,\ell}|_{G_{E_w}})$. After extending the coefficient field to contain the images of $E_{w,0}$, the linear map φ^{f_w} on each component indexed by an embedding of $E_{w,0}$ has characteristic polynomial $\iota_\ell P_{b,w}$. It is de Rham at every place above ℓ. Its labelled Hodge–Tate multiset is

$$\{j - 1 - \lambda_j : 1 \leq j \leq n\} \quad \text{at } \Phi, \qquad \{\lambda_j + n - j : 1 \leq j \leq n\} \quad \text{at } \overline{\Phi},$$

where $\lambda_1 \geq \cdots \geq \lambda_n$ is the highest weight specified by κ at that infinite place, in the conventions of the present note.

Proof. Project the nonzero Hecke eigenform to the prescribed finite type. This preserves the Hecke eigensystem away from that place. At fixed level and weight the definite group has discrete spectrum. Choose an irreducible summand σ with nonzero projected component. At the chosen split place $U(F_v) = \mathrm{GL}_n(F_v)$, and Lemma 17.51 makes σ_v supercuspidal.

Labesse's base change applies to the inner form $U(V)$, since $[F:\mathbb{Q}] \geq 2$, its infinite component is discrete series, and compactness at infinity implies condition $(*)$ [Lab11, Corollary 5.3]. Its isobaric base change Π has conjugate-self-dual discrete blocks and agrees with σ_v at the split place. Multiple blocks give a subquotient of proper parabolic induction there. A single residual block does so as well by the Mœglin–Waldspurger classification [MW89, §I]. Supercuspidality excludes both cases. Thus Π is cuspidal, conjugate self-dual and cohomological.

Chenevier and Harris give the representation over a finite coefficient field and its de Rham property [CH13, Theorem 3.2.3(b) and Proposition 3.2.5]. At a spherical place $w \mid \ell$, part (c) gives

crystallinity and compares the smallest linear power φ^{f_w} with the normalized local parameter. The semilinear operator φ permutes the components indexed by the embeddings of $E_{w,0}$. Away from ℓ, the Weil–Deligne parameter is dominated by the normalized automorphic parameter. At a spherical place the latter has trivial inertia and zero monodromy, forcing both properties on the Galois side.

The Satake normalization above gives $P_{b,w}$ at each split spherical place. At infinity the scalar form is $L(\rho(x^{-1})f(g))$, so right translation acts on L by the dual of the coefficient representation ρ. Entrywise conjugation in ρ gives cohomological coefficient weights $(-\lambda_n, \dots, -\lambda_1)$ and $(\lambda_1, \dots, \lambda_n)$ at the two embeddings. Substitution in the formula $\{\mu_i + n - i\}$ of [CH13, (1.6)] gives the claimed multisets, with χ_ℓ of weight -1.

Lean, formal context. `AllRanks` (L782-L794), `FitTwistsExist` (L415), `Isotypic.attached_galRep`. Here the primitive split type permits cuspidal base change and Galois attachment. The formal theorem instead uses cohomology in odd rank, followed by theta lifts and cancellation of character summands. □

Application to the definite terminal main space. Every good label tuple in the definite terminal main space W_m has a primitive degree-r Green label at a chosen place above each of the two distinct auxiliary rational primes, as checked in the discussion of (10.2). One such place suffices for Theorem 17.52. A nonzero vector in the image of the sum of these tuple projectors has a nonzero component in at least one tuple, so the hypothesis applies to every main eigensystem used in `exists_isMainSummand_descentPair`. The field hypothesis also holds: the stipulated primitive 5040-th root gives $\zeta_5 \in E$ and consequently $\mathbb{Q}(\sqrt{5}) \subset F$. This constructs the raw R used in `exists_isMainSummand_descentPair`; twisting and the trace identity are handled by the preceding attachment lemmas. It does not by itself identify every prescribed ramified local type in `IsRepOfType`.

Repeated entries of λ cause no regularity obstruction. Indeed

$$(j - \lambda_{j+1}) - (j - 1 - \lambda_j) = 1 + \lambda_j - \lambda_{j+1} > 0.$$

The conjugate multiset is strictly decreasing in the analogous ordering. Thus the shifted Hodge–Tate weights are distinct without assuming distinct entries of the highest weight. The application is to the eigenspaces of W_m with the primitive type just verified. An arbitrary theta lift can have noncuspidal base change and requires a further argument.

Lemma 17.53 (Zero monodromy from domination or simple inertia)**.** *Let (r, N) and (r', N') be Frobenius-semisimple Weil–Deligne representations over an algebraically closed field of characteristic zero. If $(r, N) \prec (r', N')$ in the usual monodromy dominance order and $N' = 0$, then $N = 0$. Independently, if $r|_I$ is multiplicity-free as a representation of inertia, then $N = 0$. In particular, it suffices that some inertia element have pairwise distinct eigenvalues.*

Proof. Dominance bounds each largest Jordan block by the largest block of the same type on the right. If $N' = 0$, every such bound is 1, so $N = 0$. For the second assertion N commutes with inertia. Finite inertia is semisimple in characteristic zero, and Schur's lemma makes N scalar on each irreducible constituent of a multiplicity-free sum. Nilpotence forces every scalar to vanish. The same argument on the eigenlines of a single inertia element covers the simple-spectrum case.

Lean, formal context. `FitTwistsExist` (L415), `GaloisRep.Attached.local_global_of_eigencharacter`. The zero-monodromy conclusion here uses dominance of Weil–Deligne parameters. Lean obtains the required finite-type and annihilating-polynomial conditions directly and does not construct a Weil–Deligne parameter. □

Local parameters of cuspidal base change. Let π be a cohomological, conjugate-self-dual cuspidal representation of $\mathrm{GL}_n(\mathbb{A}_E)$, and take the semisimple attachment $r_\ell(\pi)$ with the unramified Frobenius normalization of Theorem 17.52. At every finite place w one has

$$\mathrm{WD}(r_\ell(\pi)|_{G_{E_w}})^{F\text{-}ss} \simeq \iota_\ell \,\mathrm{rec}_w(\pi_w |\det|^{(1-n)/2}). \tag{17.28}$$

For $n \geq 2$ and $w \nmid \ell$, apply [BLGGT14b, Theorem 2.1.1] to the polarized pair $(\pi, \delta_{E/F}^n)$, where $\delta_{E/F}$ is the quadratic character of F attached to E/F. For $w \mid \ell$, apply Caraiani's theorem to $\pi^\vee$, which is again cohomological, conjugate-self-dual and cuspidal [Car14, Theorem 1.1]. Her geometric normalization sends ρ_w to $\mathrm{rec}_w(\rho_w^\vee |\det|^{(1-n)/2})$. Thus the unramified polynomials agree with ours, and Lemma 17.46 identifies the semisimple Galois representations by Chebotarev density. In rank one the formula is local and global class field theory. Frobenius semisimplification retains the operator N, so the formula identifies monodromy as well as inertia. The integral length estimate in Corollary 17.88 also requires the lattice comparisons in Sections 15 and 16.

The primitive split type supplies the cuspidal base change for Theorem 17.52. For eigensystems on the larger spaces, the argument in Theorem 17.102 also allows noncuspidal base change. We first treat a lift whose Hecke polynomial has the full factorization through a main eigensystem.

17.8 The representation attached to a theta-lift eigensystem

For a system which is already known to satisfy the full theta Hecke factorization, its Galois representation can be constructed directly from the main row and the character string. This gives more than the trace comparison, without invoking an existence theorem for every ambient eigensystem of the larger unitary group.

Theorem 17.54 (Direct realization from the full Hecke factorization)**.** *Let $R : G_E \to \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell)$ be continuous and semisimple, with exact unramified Frobenius polynomial $P_{b,w}$ outside a finite set of split places. Write*

$$B_w(X) = X^r P_{b,w}(X^{-1}).$$

Let H_w be a monic degree-$r+t$ system, $t \geq 0$, whose reversed polynomial satisfies

$$X^{r+t} H_w(X^{-1}) = \prod_{0 \leq c < t} (1 - q_w^c \nu_w X)\, B_w(\beta_w X) \tag{17.29}$$

at those places. Let α, ψ, θ be continuous Galois characters unramified outside a finite set, with

$$\alpha(\phi_w)\nu_w = 1, \qquad \psi(\phi_w)\theta(\phi_w) = \beta_w/\nu_w,$$

where ϕ_w is geometric Frobenius and every scalar is read in $\overline{\mathbb{Q}}_\ell$. Then

$$S = \alpha^{-1} \otimes \left((\psi\theta \otimes R) \oplus \bigoplus_{0 \leq c < t} \chi_\ell^{-c} \right) \tag{17.30}$$

is continuous, semisimple and attached to H_w. It has a finite coefficient field. Any other continuous semisimple representation attached to that system is isomorphic to S after a finite coefficient enlargement.

Proof. At a good Frobenius, $\alpha^{-1} = \nu_w$ and $\alpha^{-1}\psi\theta = \beta_w$, while $\chi_\ell(\phi_w)^{-c} = q_w^c$. Thus

$$\begin{aligned} \det(1 - S(\phi_w)X) &= \det(1 - \beta_w R(\phi_w)X) \prod_{0 \leq c < t} (1 - \nu_w q_w^c X) \\ &= B_w(\beta_w X) \prod_{0 \leq c < t} (1 - \nu_w q_w^c X). \end{aligned}$$

Equation (17.29) identifies the full characteristic polynomial with H_w. Every operation in (17.30) preserves continuity and semisimplicity. Its image trace form is nondegenerate by Lemma 17.45. The exact polynomial identity outside finitely many places supplies the approximation condition of

attachment, with the identity map on G_E. Theorem 17.43 supplies the finite field. For another attached representation, the same Frobenius coefficient limits identify its traces with those of S at every Galois element. Lemma 17.46 proves the last assertion.

Lean, formal context. `FitTwistsExist` (L415). □

Corollary 17.55 (Local properties of a factored lift)**.** *Retain Theorem 17.54. At a fixed place, if R and all character factors in (17.30) are unramified, de Rham, crystalline, or potentially crystalline, respectively, then S has the corresponding property. For potential crystallinity one common finite local extension suffices. If their Hodge–Tate jumps at an embedding are J_R and $h_\alpha, h_\psi, h_\theta$, then*

$$J_S = \{u + h_\psi + h_\theta - h_\alpha : u \in J_R\} \sqcup \{c - h_\alpha : 0 \le c < t\}, \tag{17.31}$$

with multiplicities and the convention that χ_ℓ has jump -1. For each actual inertia operator, or each inertial period operator when defined, its spectrum is the corresponding scaled union of the spectra in (17.30).

Proof. The indicated categories are closed under finite direct sums, duals and tensor products. In the potentially crystalline case, take the compositum of the finitely many extensions making the factors crystalline. The period comparison then respects the direct-sum and tensor decompositions. Hodge–Tate jumps add under a character twist, and those of χ_ℓ^{-c} are c, giving (17.31). The same decomposition gives the spectrum assertion for both actual Galois operators and period operators.

Lean, formal context. `FitTwistsExist` (L415). □

Apply this construction to the raw main representation from Theorem 17.52, with the full factorization of an actual theta-lift eigensystem and its avatar normalizations. For that system the representation's existence is a deduction from those inputs; it need not be separately postulated at the larger rank. The factorization itself remains a premise for the specified automorphic form. The main local types are supplied by Theorem 17.76 and Corollary 17.77; Lemma 17.91 supplies the avatar local formulas. This direct-sum construction does not construct representations for arbitrary nonlift eigensystems in the ambient Hecke space; the separate construction is Theorem 17.102. Nor does $\det(1 - N(g)) = 0$ for all g, the weak-lift condition, supply the full factorization used here.

17.9 Integral Hecke operators from Frobenius attachment

We next remove the separate passage from Frobenius compatibility to integral Hecke operators at arbitrary Galois elements. The argument uses the actual finite lattice of forms, so it does not construct a matrix-valued representation over a Hecke algebra from its traces. The coefficient fields below are finite extensions of $\mathbb{Q}_\ell$, with their valuation topologies.

Lemma 17.56 (A finite evaluation lattice)**.** *Let O_0 be a finite coefficient valuation ring. Let Λ_a, $a \in A$, be finitely many finite free O_0-modules with actions of finite groups Γ_a. Then*

$$M = \bigoplus_{a \in A} \Lambda_a^{\Gamma_a}$$

is a finite free O_0-module. A finite sum of O_0-linear maps between the Λ_a, if it preserves the displayed invariant submodule, acts integrally on M. No division by any $|\Gamma_a|$ is needed.

Proof. The invariant submodule is the intersection of the kernels of $\gamma - 1$ in a finite free module. It is finitely generated because O_0 is noetherian, and free because it is torsion-free over a discrete valuation ring. The assertion about maps is their restriction to this submodule.

Lean, formal context. `AttachedRepresentations` (L230), `FitTwistsExist` (L415). □

For definite algebraic modular forms this is the evaluation model on the finite double-coset set: Γ_a is its rational stabilizer and Λ_a is the algebraic weight lattice in that frame. The counting argument preceding (10.1) gives this model for the definite terminal form. At a place away from ℓ, the unnormalised spherical Hecke operator is a finite sum of translations preserving these lattices. The coefficients of its standard Hecke polynomial multiply these sums by integral powers of the residue cardinality; allowing inverse powers away from ℓ also preserves integrality. Thus the following argument applies whenever these specified integral translation operators and the usual finite eigensystem decomposition have been supplied. The lattices here are the ones in that integral evaluation model; arbitrary independent choices of lattices would not imply preservation by translations. The argument adds no averaging denominator to the integral lattice.

Lemma 17.57 (The eigenvalue algebra is finite and closed)**.** *Let $M \neq 0$ be a finite free O_0-module, and let $H_0 \subseteq \mathrm{End}_{O_0}(M)$ be a commutative unital O_0-algebra whose generic fibre is reduced. There is a finite coefficient extension K/K_0, with valuation ring O, for which $H_0 \otimes_{O_0} K$ is split. Write X for its distinct eigensystems on M_K. Evaluation identifies $H = O \otimes_{O_0} H_0$ with a finite O-subalgebra*

$$\mathbb{T} \subseteq O^X.$$

The subalgebra $\mathbb{T}$ is compact, hence closed in K^X. Its elements act on M_O as the specified integral Hecke operators. They are determined by their eigenvalues on X.

Proof. The algebra H_0 is finite and torsion-free as a submodule of $\mathrm{End}_{O_0}(M)$. Its reduced generic fibre is a product of finite separable fields. Choose K splitting them all. Faithfulness makes every factor occur on M_K, so evaluation embeds H in K^X. Each eigenvalue is integral by Cayley–Hamilton on M_O, hence lies in O. Finally H is finite free over the compact ring O, and evaluation is continuous. Its image $\mathbb{T}$ is thus compact and closed in K^X.

Lean, formal context. `AttachedRepresentations` (L230), `FitTwistsExist` (L415). □

In the usual Hecke application, reducedness is also a linear algebra consequence of the adjoint relation. A commuting family of complex operators closed under Hermitian adjoints consists of normal operators and is simultaneously diagonalizable. Its algebra has no nonzero nilpotent. Reducedness then descends to the coefficient algebra by the faithful scalar extension. This is the same adjoint relation used in the local first-order argument; it is not a multiplicity-one assertion about automorphic representations.

Theorem 17.58 (Integral coefficients at every Galois element)**.** *Retain Lemma 17.57. Let G be a profinite group and let $R_x : G \to \mathrm{GL}_n(K)$, $x \in X$, be a finite family of continuous representations, with $n \geq 1$. Suppose the supplied Frobenius attachment has the following precise property. For each $g \in G$ there is a proper filter $\mathcal{F}_g$ on the same set of Frobenius data, and polynomials $H_w(T) \in \mathbb{T}[T]$, such that for every x and every coefficient index j,*

$$[T^j]H_w(T)(x) \longrightarrow [T^j]\det(T - R_x(g)) \quad \text{along } \mathcal{F}_g.$$

The polynomials need belong to $\mathbb{T}[T]$ only eventually along that filter. Then for every $g \in G$ and j,

$$C_j(g) := \left([T^j]\det(T - R_x(g))\right)_{x \in X} \in \mathbb{T}. \tag{17.32}$$

The same conclusion follows if exact attachment is given on a dense union of Frobenius conjugacy classes.

Proof. Since X is finite, coefficientwise convergence gives convergence in K^X. The integral Hecke coefficient tuples eventually lie in the closed set $\mathbb{T}$, which therefore contains their limit $C_j(g)$. In the case of exact attachment, C_j is a continuous class function, so $C_j^{-1}(\mathbb{T})$ is closed and contains the dense union of Frobenius classes. In the application the good split primes have density one among primes of E, since inert primes over F have squared norms. Chebotarev therefore gives the required density even after finitely many places have been omitted.

Lean. `AttachedRepresentations` (L230). Lean uses coefficient limits in a Frobenius filter. For exact attachment, continuity on the dense union of Frobenius classes suffices. □

Lemma 17.59 (A common normalizing character preserves integrality)**.** *Under Theorem 17.58, let $\eta : G \to K^\times$ be a continuous character, common to all systems. Then $\eta(g) \in O^\times$ for every g, and the characteristic-polynomial coefficient tuples of $N_x = \eta R_x$ belong to $\mathbb{T}$.*

Proof. The valuation of η maps the compact group G into a compact subgroup of $\mathbb{Z}$, hence is zero. The coefficient of T^j after twisting is $\eta(g)^{n-j}C_j(g)(x)$. The scalar belongs to $O^\times$ and $\mathbb{T}$ is an O-algebra.

Lean, formal context. `AttachedRepresentations` (L230), `FitTwistsExist` (L415). □

Corollary 17.60 (Integral determinant and trace operators)**.** *Let N_x be as in Lemma 17.59. For every $g \in G$, the functions*

$$x \longmapsto \operatorname{tr} N_x(g), \qquad \beta_g(x) = \det(1 - N_x(g))$$

belong to $\mathbb{T}$. Fix $\phi, \gamma \in G$ and a polynomial $p(Z) = \sum_i a_i Z^i \in O[Z]$. Then the functions

$$\begin{aligned} s_{p,g}(x) &= \operatorname{tr}\big(p(N_x(\phi))N_x(g)\big), \\ t_{p,g}(x) &= \operatorname{tr}\big(p(N_x(\phi))(N_x(\gamma) - I)N_x(g)\big) \end{aligned}$$

also belong to $\mathbb{T}$ and act integrally on the given lattice M_O.

Proof. The trace is the negative coefficient of T^{n-1}, and β_g is the sum of all characteristic-polynomial coefficients. For the other two assertions expand directly:

$$\begin{aligned} s_{p,g}(x) &= \sum_i a_i \operatorname{tr} N_x(\phi^i g), \\ t_{p,g}(x) &= \sum_i a_i \big(\operatorname{tr} N_x(\phi^i \gamma g) - \operatorname{tr} N_x(\phi^i g)\big). \end{aligned}$$

Each trace belongs to $\mathbb{T}$. Lemma 17.57 gives the action on M_O.

Lean, formal context. `AttachedRepresentations` (L230), `FitTwistsExist` (L415). □

Corollary 17.61 (Fixed congruence tests and the integral indicator)**.** *Under Theorem 17.58, there is a list of at most $\binom{|X|}{2}$ pairs (g_a, j_a), independent of depth, such that for every ideal $\mathfrak{a} \subseteq O$ and $x, y \in X$,*

$$\begin{aligned} C_j(g)(x) \equiv C_j(g)(y) \pmod{\mathfrak{a}} \quad &\text{for all } g, j \\ \iff \quad C_{j_a}(g_a)(x) \equiv C_{j_a}(g_a)(y) \pmod{\mathfrak{a}} \quad &\text{for every } a. \end{aligned}$$

For a fixed x_0, the indicator of its residual congruence class belongs to $\mathbb{T}$ and acts integrally on M_O. The determinant separator (25.7) belongs to that same algebra.

Proof. For each unordered pair $\{x, y\}$ on which some coefficient differs, choose a nonzero difference $C_j(g)(x) - C_j(g)(y)$ of least valuation. Such a value exists by discreteness. It generates the ideal of all coefficient differences for that pair. Thus congruence at its chosen test implies congruence at every test modulo any prescribed ideal of O. Collecting the chosen tests proves the equivalence and gives the bound $\binom{|X|}{2}$.

Apply the resultant construction of Lemma 25.2 to the chosen coefficients in $\mathbb{T}$. Its linear factors split over O, and differences between distinct residual values are units in O. The resulting projector is the indicator of the residual class of x_0 and preserves M_O. Corollary 17.60 puts each β_g, hence their product (25.7), in $\mathbb{T}$.

Lean. The coefficient and residual indicator appear in `charpoly_coeff_mem_heckeAlgebra` and `exists_indicator_isCongruent`. The pair count follows here from discreteness of the valuation. □

Theorem 17.58 and Corollary 17.60 supply the Hecke-algebra membership used in the tied traces and in the common Ribet polynomial, from the finite integral Hecke lattice and the rational Frobenius attachment. The finite congruence test gives no bound on the norms of the chosen Frobenius places and does not replace the separate quantitative prime-triangle construction. Neither an integral decomposition of a Galois lattice into eigensystems nor a Galois representation over $\mathbb{T}$ has been inferred. Matrix identities on actual integral families require the hypotheses of the separate specialization argument below.

17.10 The auxiliary quadratic relation on stable lattices

The auxiliary-prime relation concerns the actual Galois action away from the coefficient prime. We reconstruct its linear-algebraic and integral steps here. The arithmetic input is a local compatibility statement that includes zero monodromy. Merely specifying the semisimplified inertial spectrum does not supply that input.

Throughout this subsection, O is the valuation ring of a finite extension $K/\mathbb{Q}_\ell$. A lattice is a finite free O-module spanning its specified K-space. A finite local field F has residue characteristic different from ℓ, and I_F denotes its inertia group.

Lemma 17.62 (A finite inertial spectrum gives the quadratic)**.** *Let $R : G_F \to \mathrm{GL}_K(V)$ be continuous, let $a : G_F \to K^\times$ be a character, and suppose $R(I_F)$ is finite. Assume that, for every $g \in I_F$, all eigenvalues of $R(g)$ belong to $\{1, a(g)\}$. Then*

$$(R(g) - a(g)I)(R(g) - I) = 0 \qquad (g \in I_F). \tag{17.33}$$

If $a(g) = 1$, then $R(g) = I$. If $a(I_F) \subset O^\times$, the same identity holds on every R-stable lattice and every quotient of that lattice by a nonzero ideal of O.

Proof. Each $T = R(g)$ has finite order d. Its minimal polynomial divides $X^d - 1$, which has no repeated root in characteristic zero. Thus T is diagonalizable over $\overline{K}$. Its minimal polynomial divides the product of the distinct factors among $X - 1$ and $X - a(g)$. This proves the asserted quadratic and, when the two roots coincide, the stronger equality $T = I$.

For a stable lattice L, both factors in (17.33) lie in $\mathrm{End}_O(L)$. The map from this module to $\mathrm{End}_K(K \otimes_O L)$ is injective, so the identity already holds integrally. Reduction proves it on every quotient. This argument neither divides by the order of inertia nor by $a(g) - 1$.

Lean, formal context. `FitTwistsExist` (L415). □

When $a(g) \neq 1$, the two spectral projectors can require division by $a(g) - 1$. The quadratic polynomial itself has integral coefficients, so the relation above holds directly on the given stable lattice.

Lemma 17.63 (Transport to the conjugate auxiliary place)**.** *Let E be a CM field, choose complex conjugation c, and write $g^c = cgc^{-1}$. Let $R : G_E \to \mathrm{GL}_K(V)$ satisfy*

$$R^c \simeq R^\vee \otimes \chi_\ell^{1-s}$$

over $\overline{K}$. Let $a : G_E \to K^\times$ satisfy $a(g^c) = a(g)^{-1}$. At a place $w \nmid \ell$, suppose $R(I_w)$ is finite and $(R(g) - a(g)I)(R(g) - I) = 0$ for every $g \in I_w$. Then both assertions hold at the conjugate place $\bar{w}$.

Proof. The duality isomorphism is given by an invertible matrix B after a finite coefficient extension. For $g \in I_w$, the cyclotomic character is trivial, and applying duality to g^{-1} gives

$$R(g^{-c}) = BR(g)^{\mathsf{t}}B^{-1}.$$

As g ranges over I_w, the element g^{-c} ranges over $I_{\bar{w}}$. The displayed equality transfers finiteness of the inertial image. It also transfers every polynomial identity of $R(g)$. Since $a(g^{-c}) = a(g)$, it transfers exactly the stated quadratic. Its coefficients and matrices belong to K, so the identity descends from the temporary coefficient extension.

Lean, formal context. `FitTwistsExist` (L415). □

Lemma 17.64 (The two avatar restrictions at an auxiliary place)**.** *Use a fixed normalization of the local and global reciprocity maps. Let ϑ be a finite-order Hecke character of E with $\vartheta^c = \vartheta^{-1}$, and let a be the ℓ-adic avatar of ϑ^{-1}. Then*

$$a(g^{-c}) = a(g) \qquad (g \in G_E).$$

At $w \nmid \ell$, its restriction to inertia is the character of local units obtained from ϑ_w^{-1} by reciprocity and the chosen coefficient embedding. In particular it has finite image in $O^\times$ after adjoining its finitely many values.

Let ν be the ℓ-adic avatar of an algebraic Hecke character ψ. If ψ_w is trivial on $O_{E_w}^\times$, then $\nu|_{I_w} = 1$. For a normalizing character of the form

$$\psi(x) = \mu^{-1}(x/\bar{x})|x|^b,$$

this condition holds whenever both w and $\bar{w}$ avoid the finite conductor of μ.

Proof. Reciprocity is equivariant for the automorphism c. The equality $\vartheta^c = \vartheta^{-1}$ therefore gives $a(g^c) = a(g)^{-1}$ on the abelianized Galois group, hence on G_E. Replacing g by g^{-1} gives the first formula. Away from ℓ, the definition of the avatar on local units has no ℓ-adic algebraic correction. Local reciprocity identifies the image of local units with inertia in the abelianized group. This proves both inertia assertions. The image of a finite-order character consists of roots of unity, all of which are integral units.

For the final assertion, place a local unit in the w-component of an idele. Its quotient by its conjugate has unit components only at w and $\bar{w}$, and its idele norm is 1. Both local components of μ are trivial on units by the conductor assumption. Consequently ψ_w is trivial on units, as required. The unramifiedness conclusion is unchanged if the reciprocity convention is inverted consistently.

Lean, formal context. `FitTwistsExist` (L415). □

Theorem 17.65 (The auxiliary quadratic on every stable lattice)**.** *Let Q be a finite set of places of E, disjoint from their conjugates and from the places above ℓ. Let R, a and ν be as in Lemmas 17.63 and 17.64, and suppose ν is unramified at $Q \cup \bar{Q}$. For each $w \in Q$, assume the following local compatibility assertion:*

1. *the actual image $R(I_w)$ is finite;*

2. *for every $g \in I_w$, the eigenvalues of $R(g)$ are contained in $\{1, a(g)\}$.*

Put $U = \nu \otimes R$. After one finite coefficient enlargement, every specified U-stable lattice L satisfies

$$(U(g) - a(g)I)(U(g) - I) = 0 \quad \text{on } L \text{ and on } L/\mathfrak{a}L \tag{17.34}$$

for every $w \in Q \cup \bar{Q}$, every $g \in I_w$, and every nonzero ideal $\mathfrak{a}$ of O. If $a(g) = 1$, then $U(g) = I$. The conclusion is uniform in the congruence depth.

Proof. Lemma 17.62 gives the quadratic at Q. Lemma 17.63 gives it and finiteness of inertia at $\bar{Q}$. The character ν is trivial on each of these inertia groups, so $U(g) = R(g)$ there. Adjoin the finite set of values of a on these inertia groups. Now the matrices and polynomial coefficients preserve every chosen stable lattice. The injection of integral endomorphisms into rational endomorphisms proves the identity on that same lattice, and reduction proves all the quotient identities. The case $a(g) = 1$ follows from finite order, as in Lemma 17.62.

Lean, formal context. `FitTwistsExist` (L415). □

In the specified application, $a = \vartheta^{-1}$'s avatar, the normalizing character is ν, and the auxiliary-prime conditions exclude ℓ and the conductors occurring in Lemma 17.64. The local parahoric has two determinant characters, 1 and ϑ_w^{-1}. Their nontriviality as characters does not mean that their values are distinct on every inertial element. The proof above includes the elements on which they coincide and also the coefficient primes for which $a(g) - 1$ is not a unit.

Lemma 17.81 derives the indicated inertial spectrum and zero automorphic monodromy from this determinant-block type. Given an actual attached representation, compatibility after *Frobenius* semisimplification retains monodromy and gives both numbered assertions by Lemma 17.82. Suitable monodromy domination also suffices; a statement discarding monodromy does not. Theorem 17.105 supplies the global realization and the applicability of this local comparison to every required ambient system. The precise source step is local compatibility for the product label; the subsequent auxiliary quadratic argument uses its finite-type conclusion explicitly.

For the auxiliary quadratic relation the finite image of inertia comes from zero monodromy and the Weil–Deligne comparison. The semisimplified inertia spectrum alone would allow a nontrivial unipotent block for the eigenvalue 1. Theorem 17.65 gives the relation on each representation's stable lattice. Proposition 17.104 gives the associated Hecke actions on the separate lattice of forms.

17.11 The realized main rows and the remaining local comparisons

Corollary 17.66 (Filtration jumps and duality after normalization)**.** *For every eigensystem in the specified good subspace of the definite terminal form W_m, every coefficient prime ℓ, and every descent index j, Theorem 17.52 supplies a raw rank-r representation R. Put $t_* = 5r$, $a = (r - t_*)/2$, and let θ_j be the specified algebraic avatar in the main-row normalization. Use its local class-field formula: its jump at σ is $-a - \epsilon_\sigma w_j(v)$, and $\theta_j(g^c)\theta_j(g) = \chi_\ell(g)^{2a}$. Then $T_j = \theta_j \otimes R$ is a continuous semisimple representation over a finite coefficient field, its actual jumps are*

$$\left\{ \left\lfloor \frac{t_*}{2} \right\rfloor - \epsilon_\sigma(\lambda_q(v) + w_j(v)) : 0 \le q < r \right\}, \tag{17.35}$$

and

$$\operatorname{tr} T_j(g^c) = \chi_\ell(g)^{1-t_*} \operatorname{tr} T_j(g^{-1}). \tag{17.36}$$

Thus the jump multiset of Theorem 17.9 is realized by the normalized main row. All its low-jump and conjugate-pair counts apply to this row and every one of its stable lattices.

Proof. Write $\lambda_1^{\mathrm{alg}} \geq \cdots \geq \lambda_r^{\mathrm{alg}}$ for the algebraic highest weight at v. The Harish–Chandra parameter relation for the specified weight is

$$\lambda_q(v) = \lambda_{q+1}^{\mathrm{alg}} + \frac{r-1}{2} - q \qquad (0 \leq q < r).$$

Consequently the two multisets of Theorem 17.52 have the common description $\{(r-1)/2 - \epsilon_\sigma \lambda_q(v) : q < r\}$. Adding the avatar jump gives

$$\frac{r-1}{2} - a - \epsilon_\sigma(\lambda_q(v) + w_j(v)) = \frac{t_* - 1}{2} - \epsilon_\sigma(\lambda_q(v) + w_j(v)).$$

The integer t_* is odd, so this is (17.35). Raw conjugate trace duality is Theorem 17.49. The character product and Lemma 17.50 give (17.36). Tensoring by a character preserves semisimplicity, and Theorem 17.43 gives the finite field. The jump multiset does not depend on the stable lattice, by Lemma 17.17.

Lean, formal context. `FitTwistsExist` (L415). □

The avatar formula in this corollary is the rank-one local class-field formula identified after Lemma 17.50. The calculation checks its signs against the actual cohomological weight; it does not add a second Hodge–Tate comparison hypothesis for the rank-r row. The raw representation is de Rham everywhere above ℓ, and its crystallinity at split spherical places is part of Theorem 17.52. The following sections supply the off-level comparison at all finite places and the exact squarefree types at the old level.

Local types of the main rows. The attachment gives the jumps and duality of the main rows. To verify the remaining local hypotheses, we compare the integral models away from the level and the parahoric types at the level primes.

17.12 The CM extension is unramified at every finite place

The fields E and E^+ may ramify over $\mathbb{Q}$. The relative extension E/E^+ is nevertheless unramified at all finite places. The integral basis below includes the places above 2 and identifies the compact subgroups needed in the local comparison.

Lemma 17.67 (An integral CM basis of unit discriminant)**.** *Let E be a CM number field with maximal totally real subfield $F = E^+$. Suppose that E contains a primitive twelfth root of unity. There is $y \in \mathcal{O}_E$ such that*

$$E = F(y), \qquad \mathcal{O}_E = \mathcal{O}_F \oplus \mathcal{O}_F y, \qquad (y - \bar{y})^2 = -1.$$

In particular $\mathcal{O}_E$ is finite étale of rank two over $\mathcal{O}_F$, and E/F is unramified at every finite place. For the fields of Proposition 5.2 we have $12 \mid 5040 \mid q_1$.

Proof. Take $y = \zeta_{12}$. Since y^2 is a primitive sixth root of unity, $y^2 + y^{-2} = 1$, whence

$$(y - \bar{y})^2 = y^2 + y^{-2} - 2 = -1.$$

Thus $y \notin F$ and $E = F(y)$. The order $\mathcal{O}_F[y]$ has basis $1, y$ and discriminant -1. At any finite place the discriminant of this order is the relative discriminant of E/F times the square of the local index ideal. Both factors are integral, so both are units. This proves the integral-basis assertion and unramifiedness at every finite place, including those above 2.

Lean, formal context. Lean constructs an integral element with unit difference from its conjugate using a fourth and a seventh root of unity in `HasPrimitiveRoot5040.exists_isUnit_sub_star`. Here a primitive twelfth root gives the integral basis directly. □

Lemma 17.68 (The specified maximal compact is hyperspecial)**.** *Let v be a finite place of F, put $A = \mathcal{O}_{F_v}$, and let $B = \mathcal{O}_E \otimes_{\mathcal{O}_F} A$ for the extension in Lemma 17.67. Let V be an n-dimensional Hermitian E/F-space, $n \geq 1$, equipped at v with a frame x_v for which*

$$x_v^* J x_v = I_n.$$

Then the lattice $L_v = x_v B^n$ is self-dual for J, and its stabilizer in $U(V)(F_v)$ is a hyperspecial maximal compact. This includes residue characteristic 2.

Proof. The lattice B^n is self-dual for the standard Hermitian pairing, so the first assertion follows by the isometry x_v. Its stabilizer is the A-point group of

$$\mathcal{U}(R) = \{g \in \mathrm{GL}_n(B \otimes_A R) : \bar{g}^{\mathsf{t}} g = I_n\}$$

for A-algebras R. The algebra B/A is quadratic finite étale. After the faithfully flat étale extension $A \to B$, the algebra with involution splits as $B \times B$, with the involution exchanging factors. The displayed group then becomes $\mathrm{GL}_{n,B}$: one matrix is free and the other is its inverse transpose. Thus $\mathcal{U}$ is a smooth affine reductive A-group scheme, since these properties descend under a faithfully flat étale extension. The stabilizer is $\mathcal{U}(A)$, hence is hyperspecial by the definition of a hyperspecial subgroup. The splitting argument uses no invertibility of 2 in A.

Lean, formal context. `Defs` (L185-L228), `FitTwistsExist` (L415). □

The hypotheses of this lemma match the finite coordinate convention in the source. A pointed space has $x_f^* J x_f = 1$ at the finite adeles, and its finite adelic group is expressed in these coordinates as the standard unitary group. Its standard level is precisely the stabilizer of $\widehat{\mathcal{O}}_E^n$; the subgroup at a place is its factor supported at that place. See the standard integral level and the finite isometry in a pointed space. Thus this application concerns the specified self-dual lattice, not a choice of an unspecified maximal compact subgroup.

Theorem 17.69 (Spherical compatibility at all finite places of the main rows)**.** *Retain the good main eigensystems and the representations $R_{b,\ell}$ of Theorem 17.52. At every finite place v where the eigenform has the specified maximal level, the base change is spherical at every $w \mid v$ of E. Consequently $R_{b,\ell}$ is unramified at $w \nmid \ell$ and crystalline at $w \mid \ell$. Its Hodge–Tate jumps remain those in Theorem 17.52. No splitting of v in E/F is required for these conclusions.*

Proof. Lemmas 17.67 and 17.68 give an unramified unitary group and hyperspecial level at v. Projecting the given form onto a nonzero irreducible constituent preserves its invariance and its prescribed split cuspidal type. The latter forces cuspidal base change as in Theorem 17.52. Labesse's unramified local matching makes Π_w spherical at every $w \mid v$ [Lab11, Corollary 5.3]. Its parameter has trivial inertia and $N = 0$. Equation (17.28) therefore gives unramifiedness for $w \nmid \ell$. For $w \mid \ell$ the spherical clause of [CH13, Theorem 3.2.3(c)] gives crystallinity. The labelled jumps were computed in Theorem 17.52.

All choices of constituent giving the same eigensystem have the same Frobenius traces at good split primes. Their classes are dense as in Theorem 17.58, so this equality and Lemma 17.46 identify their semisimple attachments, so the conclusion holds for the chosen $R_{b,\ell}$.

Lean, formal context. `Defs` (L498-L518), `FitTwistsExist` (L415). The formal context supplies the levels and attachment. The proof here uses spherical base change and the crystalline clause of the published attachment theorem. □

Corollary 17.70 (Local conditions away from the level of the distinguished summands)**.** *Let $M = p_0 p_1$ be the product of the two old level primes. For a good terminal main eigensystem with*

the specified principal level at those primes and maximal level elsewhere, its raw representation R is unramified outside the places above ℓM, and crystalline at every place above ℓ not above M.

Let θ_j be its algebraic normalizing avatar. If its finite Hecke character is unramified outside M, then the normalized main row $T_j = \theta_j \otimes R$ has these same off-level properties. It has the actual jumps and conjugate trace duality of Corollary 17.66.

Proof. At a place not above M, principal-level conditions on the specified forms give maximal-level invariance. Apply Theorem 17.69. The avatar of an algebraic Hecke character unramified at a finite place is unramified there away from ℓ, and crystalline there above ℓ, by the rank-one local class-field compatibility. Tensoring preserves these properties. The jump and duality assertions are Corollary 17.66.

Lean, formal context. `RowsFromEigensystems` (L268-L281), `FitTwistsExist` (L415). □

For the main rows of the datum, the character condition in the corollary is the specified level condition on the normalizer: here θ_j is the avatar of $\mu_j^{-1}(x/\bar{x})|x|^a$, whose tame part is supported on the old chosen places and whose norm factor is trivial on local units. See the actual main avatar and its tame part and unramifiedness away from that tame support. The general algebraic avatar formula can also contain a splitting-character factor; that factor is unramified at finite places by the splitting-character calculation. The maximal-level assertion for the descended eigenform is also explicit in the chosen-level and unchosen-place invariance calculation.

There are therefore no nonsplit places ramified in E/E^+ to check for the actual family. The general caution after Theorem 17.52 concerns CM fields without the root-of-unity condition. The old level primes and the auxiliary ramified characters still require the separate finite-type comparisons considered next.

17.13 Green representations, mixed types and their local parameters

The local compatibility needed for a main row can be obtained from classical depth-zero local Langlands theory. The argument applies to every squarefree label at every chosen old level place. A primitive label is needed at one place to obtain the cuspidal global base change of Theorem 17.52; primitiveness is not required at the remaining places. We keep the normalization of the finite character, local reciprocity, and the coefficient-prime period operator explicit.

The finite character and its local parameter

Let F be a finite extension of $\mathbb{Q}_p$, with residue field $k = \mathbb{F}_q$, and put $K = \mathrm{GL}_n(\mathcal{O}_F)$. Choose a faithful character $T : \overline{k}^{\times} \longrightarrow \mathbb{C}^{\times}$. For $x \in k_d^{\times}$, where $k_d = \mathbb{F}_{q^d}$, write

$$T(x) = \exp\left(\frac{2\pi i b}{q^d - 1}\right), \qquad \theta_x(y) = T(y)^b \quad (y \in k_d^{\times}).$$

This is the source's `thetaChar`, not the map $y \mapsto T(xy)$; b is well defined modulo $q^d - 1$. Local reciprocity Art_F sends a uniformizer to geometric Frobenius. For the unramified extension F_d/F, write $\theta_x \circ \mathrm{Art}_{F_d}^{-1}$ on $I_F = I_{F_d}$ for the character obtained by reducing the unit part of the inverse Artin map to $k_d^{\times}$.

Lemma 17.71 (Identification of the finite Green character)**.** *Let $P \in k[X]$ be monic irreducible of degree d, with $P(0) \neq 0$, and let x be a root in k_d. The source's finite character $\tau_P =$ `greenTau`(T, P)*

is the cuspidal Deligne–Lusztig representation $(-1)^{d-1}R^{\theta_x}_{k_d^\times}$. *In particular its parameter is* θ_x, *with no inversion. For a monic squarefree*

$$c = \prod_{i=1}^{a} P_i, \qquad c(0) \neq 0, \qquad \sum_i d_i = n,$$

the finite Green representation τ_c *is*

$$\tau_c = \mathrm{Ind}_{\overline{P}}^{\mathrm{GL}_n(k)}(\tau_{P_1} \boxtimes \cdots \boxtimes \tau_{P_a}),$$

where the finite unipotent radical acts trivially. This induced representation is irreducible.

Proof. The pairing $(x, y) \mapsto \theta_x(y)$ on $k_d^\times$ is a perfect pairing: after choosing the generator sent by T to $\exp(2\pi i/(q^d-1))$, it is $(u, v) \mapsto \exp(2\pi iuv/(q^d-1))$. It commutes with q-power Frobenius. Hence θ_x has Frobenius orbit of length d.

We use the finite Green character theorem already used in the label construction: τ_P is an irreducible cuspidal character, and the Hall product over distinct irreducible factors is an irreducible character. To identify its convention, take a regular elliptic element $y \in k_d^\times$, acting on k_d as a k-vector space. Its invariant-subspace lattice consists of 0 and k_d, so the latter has Möbius value -1. In the definition $C_\theta = (\beta_\theta \mu) * 1$, the zero-space term contributes zero and the full-space term contributes $-\sum_{j=0}^{d-1} \theta_x(y^{q^j})$. Multiplication by $(-1)^d$, as in `greenTau`, therefore gives

$$\operatorname{tr} \tau_P(y) = (-1)^{d-1} \sum_{j=0}^{d-1} \theta_x(y^{q^j}).$$

The regular elliptic character characterization of the finite cuspidal representation identifies this with $(-1)^{d-1}R^{\theta_x}_{k_d^\times}$. The convention is that of [EGH13, proof of Proposition 2.4.1(i)]. Finally the Hall product used in `labelChar` is finite parabolic induction, which gives the displayed expression for τ_c. The squarefree Green character theorem gives its irreducibility. For $d = 1$ the formula reads $\tau_{X-x} = \theta_x$, so it also independently checks the absence of a contragredient.

Lean, formal context. `FitTwistsExist` (L415). □

Mixed types and their inertial parameters

For a character $\chi : k^\times \to \mathbb{C}^\times$, let x_χ be the unique element of $k^\times$ with $\theta_{x_\chi} = \chi$. It corresponds to `charName`. For $\mathrm{GL}_N(F)$ we use the cohomological normalization $\mathrm{rec}^{\mathrm{coh}}(\pi) = \mathrm{rec}_F(\pi \otimes |\det|^{(1-N)/2})$.

Let $c \in k[X]$ be monic and squarefree, with $c(0) \neq 0$ and degree $d \geq 0$; $c = 1$ is allowed. Let $\chi_1, \ldots, \chi_b : k^\times \to \mathbb{C}^\times$ correspond via T to $u_l = x_{\chi_l} \in k^\times$. Assume that the u_l are distinct and that $c(u_l) \neq 0$. Let $t_l > 0$ and $N = d + \sum_l t_l > 0$. Blocks of size zero may be deleted. With $\overline{P}$ the finite parabolic of block sizes $(d, t_1, \ldots, t_b)$, omitting a zero first block, put

$$\kappa(c; \chi, t) = \mathrm{Ind}_{\overline{P}}^{\mathrm{GL}_N(k)} \left(\tau_c \boxtimes \boxtimes_{l=1}^{b} (\chi_l \circ \det_{t_l}) \right). \tag{17.37}$$

The unipotent radical acts trivially in this formula. We also write $\kappa(c; \chi, t)$ for its inflation to $K = \mathrm{GL}_N(\mathcal{O}_F)$.

Lemma 17.72 (The compact projector for a mixed type)**.** *The representation* (17.37) *is irreducible. Write* $c = \prod_i P_i$, *with the* P_i *distinct irreducible of degrees* d_i. *Let* $J_0 \subset K$ *be the parahoric with quotient*

$$\prod_i \mathrm{GL}_{d_i}(k) \times \prod_l (k^\times)^{t_l}, \qquad \lambda_0 = \boxtimes_i \tau_{P_i} \boxtimes \boxtimes_l \chi_l^{\boxtimes t_l}.$$

The compact Hecke algebra is

$$\mathcal{H}(K, J_0, \lambda_0) \simeq \bigotimes_l \mathcal{H}(S_{t_l}, q).$$

Under the standard determinant-twist normalization on the equal-character factors, the product of their spherical idempotents has image $\kappa(c; \chi, t)$ *in* $\operatorname{Ind}_{J_0}^K \lambda_0$.

Proof. By Lemma 17.71 and induction in stages, the first factor in (17.37) can be refined into the individual cuspidal factors τ_{P_i}. Their finite cuspidal supports are distinct, and a factor of degree one agrees with χ_l exactly when $P_i = X - u_l$, which has been excluded.

Here is the finite Mackey calculation needed for both assertions. Double cosets of two block parabolics are described by matrices of nonnegative integers with the prescribed row and column sums. If a cuspidal block of degree d_i is split between two nonzero parts, taking invariants under the corresponding unipotent subgroup gives a proper Jacquet module of τ_{P_i}, hence zero. Thus a cuspidal block can intertwine only an entire block carrying the same cuspidal representation. Distinctness of the P_i fixes every such block. On the remaining one-dimensional slots, restriction to the diagonal torus allows exactly the permutations of equal characters. Consequently the nonzero compact intertwining double cosets for (J_0, λ_0) are indexed by $\prod_l S_{t_l}$, and each intertwining space has dimension one. Multiplication within a character block is the ordinary finite Iwahori multiplication after tensoring by $\chi_l \circ \det$; the other blocks are unchanged. This proves the displayed Hecke-algebra formula, including its parameter q rather than a parameter from a higher-degree block.

If the equal-character slots are first grouped into the factors $\mathrm{GL}_{t_l}(k)$ and given the determinant representation, the same calculation leaves only the identity double coset. The endomorphism algebra of the resulting finite induction is therefore one-dimensional. Characteristic-zero semisimplicity proves irreducibility of $\kappa(c; \chi, t)$.

Finally, the spherical projector on the lth character block is

$$e_l = \frac{\sum_{w \in S_{t_l}} T_w}{\sum_{w \in S_{t_l}} q^{\ell(w)}}.$$

Normalize the Iwahori subgroup of this block to have mass one. After the twist by $\chi_l \circ \det$, the simple generators satisfy $(T_a + 1)(T_a - q) = 0$. The double coset of w has volume $q^{\ell(w)}$, so e_l is the normalized determinant-character average on the full block. This also fixes the normalization under the covering Hecke maps: a support-preserving map sends T_a to cT_a, and the quadratic relation forces $c = 1$. The idempotent therefore selects $\chi_l \circ \det$ with its stated character. The product over l acts identically on the cuspidal factors, so induction in stages gives (17.37).

Lean, formal context. `LocalFormTypes` (L239-L249), `FitTwistsExist` (L415). □

Theorem 17.73 (A mixed type determines inertia and zero monodromy)**.** *Let* π *be an irreducible smooth admissible representation of* $\mathrm{GL}_N(F)$ *containing* $\kappa(c; \chi, t)$*. For a root* x_i *of each irreducible factor* P_i *of* c*, define*

$$\tau(c; \chi, t) = \left(\bigoplus_i \bigoplus_{j=0}^{d_i - 1} \theta_{x_i^{q^j}} \circ \operatorname{Art}_{F_{d_i}}^{-1} \right) \oplus \bigoplus_l (\chi_l \circ \operatorname{Art}_F^{-1})^{\oplus t_l},$$

as a representation of inertia. Then

$$\mathrm{rec}^{\mathrm{coh}}(\pi)|_{I_F} \simeq \tau(c; \chi, t), \qquad N_{\mathrm{rec}(\pi)} = 0.$$

Moreover $\kappa(c; \chi, t)$ *occurs in* $\pi|_K$ *with multiplicity one. The conclusion requires no genericity assumption on* π*.*

Proof. First consider a single cuspidal factor attached to an irreducible P_i of degree d_i. By Lemma 17.71 its finite type has parameter θ_{x_i}. Proposition 2.4.1(i) of [EGH13] identifies its inertial parameter, for geometric reciprocity, with the d_i conjugates of $\theta_{x_i} \circ \mathrm{Art}_{F_{d_i}}^{-1}$ and gives zero monodromy. The rectifying character in the induced Weil representation is unramified. Thus the character has the indicated direction on inertia.

For the full parahoric (J_0, λ_0) in Lemma 17.72, the level-zero cover theorem [Mor99, Lemma 3.6 and Theorem 4.8] identifies its Bernstein component. Its cuspidal support lies on the Levi

$$\prod_i \mathrm{GL}_{d_i}(F) \times \prod_l \mathrm{GL}_1(F)^{t_l}.$$

The first factors have the primitive inertial parameters just computed. The remaining factors are the t_l copies of $\chi_l \circ \mathrm{Art}_F^{-1}$. Two finite cuspidal characters give the same inertial orbit only when their label roots are Frobenius conjugate, by the perfect pairing in Lemma 17.71. Thus the only repeated inertial classes occur within the specified determinant blocks.

Dat's construction [Dat99, Theorem 4.1 and §4.2] takes the spherical idempotent on each repeated block and the identity on the single cuspidal factors. Its image in $\mathrm{Ind}_{J_0}^K \lambda_0$ is the compact type $\sigma(\tau(c;\chi,t))$ associated to zero monodromy [CEGGPS16, §3.6]. The compact calculation in Lemma 17.72 identifies this image with $\kappa(c;\chi,t)$. By [CEGGPS16, Theorem 3.7] an irreducible smooth representation contains this type precisely when it has inertial parameter $\tau(c;\chi,t)$ and zero monodromy, and then the multiplicity is one. The unramified twist in $\mathrm{rec}^{\mathrm{coh}}$ leaves both assertions unchanged.

Lean, formal context. The local type theorem treats the squarefree and determinant blocks together. The product label in Lean is given in `LocalFormTypes` (L239-L249) and `HasParahoricType.hasProductLabel`. □

Theorem 17.74 (Local parameter for every squarefree label)**.** *Let π be an irreducible smooth admissible complex representation of $\mathrm{GL}_n(F)$. Suppose $\mathrm{Hom}_K(\tau_c, \pi) \neq 0$ for the inflation of a squarefree label c as above. Choose a root x_i of each P_i. Then*

$$\mathrm{rec}_F(\pi)|_{I_F} \simeq \bigoplus_{i=1}^{a} \bigoplus_{j=0}^{d_i-1} \theta_{x_i^{q^j}} \circ \mathrm{Art}_{F_{d_i}}^{-1}, \qquad N = 0. \tag{17.38}$$

In particular wild inertia acts trivially, all the displayed inertia characters are distinct, and the inertia representation is finite. The statement is unchanged by an unramified twist of π, including the normalization $\pi \otimes |\det|^{(1-n)/2}$.

Proof. Take no determinant blocks in Theorem 17.73. Then $\kappa(c;\chi,t) = \tau_c$, and the inertial parameter is the displayed sum. Its characters are distinct because the factors of c have disjoint Frobenius orbits. Each is tame and of finite order.

Lean, formal context. `LocalFormTypes` (L239-L249). □

Exact comparison with the label polynomial

Lemma 17.75 (The positive tame exponent gives the label roots)**.** *Let c be a squarefree degree-n label, let f be a positive multiple of every irreducible-factor degree of c, and set $m = q^f - 1$. For $s \in I_F$, choose z with $z^m = \varpi$ and put*

$$t_f(s) = \overline{s(z)/z} \in k_f^\times, \qquad T(t_f(s)) = \exp(2\pi i a/m).$$

The value of a modulo m is the source's tame exponent. For the parameter in (17.38), its characteristic polynomial at s is

$$Q_{c,s}(X) = \prod_{\substack{x \in \overline{k} \\ c(x)=0}} (X - T(x)^a). \tag{17.39}$$

The product has degree n. The same formula after applying ι_ℓ holds over $\overline{\mathbb{Q}}_\ell$. In particular one may take $f = n!$.

Proof. Fix a root $x \in k_d^\times$ of a factor of c, put $h = m/(q^d - 1)$ and $z_d = z^h$, and observe

$$z_d^{q^d-1} = \varpi, \qquad t_d(s) = t_f(s)^h.$$

For geometric Artin reciprocity over F_d, the reduction of the unit corresponding to s is $t_d(s)$. The sign can be checked by the tame norm-residue formula: arithmetic reciprocity applied to a unit u sends $\varpi^{1/(q^d-1)}$ to a multiple with reduction $\bar{u}^{-1}$; inverting reciprocity changes this to $\bar{u}$. Equivalently this follows by skew symmetry of the norm-residue symbol and the arithmetic Frobenius action on a root of a unit. See, for example, Sharifi, *Algebraic Number Theory*, §9.3, Theorem 9.3.8, with the two reciprocity conventions distinguished as above.

Write $T(x) = \exp(2\pi ib/(q^d - 1))$. Direct calculation gives

$$\theta_x(t_d(s)) = T(t_f(s))^{hb} = \exp\left(\frac{2\pi iab}{q^d - 1}\right) = T(x)^a.$$

Apply this to all $x_i^{q^j}$. Their union is precisely the root set of c, yielding (17.39). Changing the root z does not change its inertia ratio, since prime-to-p roots of unity lie in the maximal unramified extension. Changing the residue embedding over k permutes the roots by Frobenius; the product is unchanged. Every degree d with $1 \le d \le n$ divides $n!$.

Lean, formal context. `FitTwistsExist` (L415). □

The finite projector introduces no second inversion. Its coefficient is

$$\frac{\dim \tau_c}{|\operatorname{GL}_n(k)|}\overline{\operatorname{tr} \tau_c(g)},$$

which is the ordinary central projector onto τ_c in a left representation of the finite group. Thus a nonzero projector for one label (`labelIdem` in Lean) or for a tuple selects the stated type. In rank one $\tau_{X-x_\chi} = \chi \circ \det$.

The actual main rows at every coefficient prime

Theorem 17.76 (All chosen old main types are realized)**.** *Let b be one of the good eigensystems in the definite terminal main space W_m treated in Theorem 17.52. Choose a nonzero full tuple projection with labels c_w at the chosen old level places, and choose the automorphic constituent from that projected space. Let $R_{b,\ell}$ be the resulting raw representation. At every such chosen split place w, write $F = E_w$, $q = |k_w|$ and $n = \deg c_w$. Then $\mathrm{WD}(R_{b,\ell}|_{G_F})$ has $N = 0$ and the tame inertia representation (17.38) for c_w.*

For $w \nmid \ell$, the actual restriction $R_{b,\ell}|_{I_F}$ is finite and tame, with characteristic polynomial $\iota_\ell Q_{c_w,s}$ for every $s \in I_F$. For $w \mid \ell$, the restriction is potentially crystalline, and the covariant potentially crystalline descent operator $\beta_{s,0}$ satisfies

$$\iota_\ell Q_{c_w,s}(\beta_{s,0}) = 0 \qquad (s \in I_F). \tag{17.40}$$

Consequently the source's `IsFiniteTypeAt` and `IsLabelledAt`, with `PolyKillsAt` and $f = n!$, hold at all these places.

Proof. The commuting finite projectors preserve the Hecke eigensystem. A nonzero component of their common image in an irreducible automorphic constituent retains every chosen type. There is a primitive label at a split place. The base change in Theorem 17.52 is therefore cuspidal, cohomological and conjugate self-dual. Denote it by Π. Labesse's split-place compatibility identifies Π_w with the relevant local constituent, including its finite K-type. Theorem 17.74 therefore applies at every chosen w, even when c_w is reducible. Its conclusion survives the unramified local normalization $|\det|^{(1-n)/2}$.

Equation (17.28) now gives $N = 0$ and the inertia representation (17.38) of this type at every chosen w.

Away from ℓ, the relation between a Galois representation and its Weil–Deligne parameter is $R(s) = r(s)\exp(t_\ell(s)N)$ on inertia; with $N = 0$ it is $r(s)$. This proves the actual finite tame inertia and (17.39), hence the annihilating relation by Cayley–Hamilton. At ℓ, potentially semistable with $N = 0$ means potentially crystalline. We use the covariant period space

$$D_{\mathrm{pcris}}(V) = \bigcup_{L/F \text{ finite}} (B_{\mathrm{cris}} \otimes_{\mathbb{Q}_\ell} V)^{G_L}.$$

On inertia its descent action is the diagonal action of s, not the action on a dual Hom-space. The source uses the same space with $B_{\max}$ and extends its action linearly over the period ring. In particular, for $d \in D_{\mathrm{pcris}}(V)$,

$$\beta_{s,0}(d) = (s \otimes R_{b,\ell}(s))d.$$

This is precisely the covariant Weil–Deligne inertia action. The usual B_{cris}–$B_{\max}$ crystalline comparison, already used in the period construction, identifies these finite-dimensional period modules and their descent actions. Thus (17.39) annihilates the period module and, since its periods span after extension of scalars, also its $B_{\max}$-linear extension. This proves (17.40). The construction is functorial in the local embedding, so the same identity holds for every local datum in `PolyKillsAt`. For an inertia element its Weil-pair exponent is zero, as required.

The conclusion also applies to any previously chosen continuous semisimple representation attached to this same eigensystem b. The next-to-leading Hecke coefficient and the attachment limits give equality of traces at every Galois element, by Lemma 17.47. Lemma 17.46 then identifies it with the full-tuple-selected representation. All the local properties above are invariant under this isomorphism.

Lean, formal context. `FitTwistsExist` (L415), `GaloisRep.Attached.local_global_of_eigencharacter`. Lean obtains the finite-type conditions from the Hecke eigencharacter. Here full local–global compatibility identifies the characteristic-zero inertial type on the Weil–Deligne parameter. □

At $w \mid \ell$ this theorem concerns finite inertia on the period module. The original ℓ-adic representation can have infinite inertia, already for a nonzero cyclotomic twist. The distinction is built into `PolyKillsAt`; replacing the period operator by the original matrix would be incorrect.

The conjugate old places are covered as well. The conjugate trace identity and Lemma 17.46 give the usual conjugate-dual isomorphism with its cyclotomic twist. That twist is unramified on actual inertia away from ℓ and has trivial Weil–Deligne inertia at ℓ. Consequently the polynomial for s at w also annihilates the operator for $cs^{-1}c^{-1}$ at $\overline{w}$, since duality and inversion cancel. The same argument applies after the main-row twist below, using its normalized duality.

Corollary 17.77 (The normalizing twist gives exactly the scaled label)**.** *Write* $\mu = O.\mathrm{u1Char}$ *and let* a *satisfy* $\mathrm{rank}(W) - O.\,\mathrm{rankDiff} = 2a$*. The main-row normalizing character is*

$$\theta = (O.\mathrm{hasWeight}^{-1}.\mathrm{toAlgebraic}(0, a)).\mathrm{avatar}, \qquad x \longmapsto \mu^{-1}(x/\overline{x})|x|^a$$

on the Hecke-character side, as in `IsRepOfType.isMainSummandLike_twistByChar`. *At a chosen old place its tame residue-field character is* $\xi_w = (O.\mathrm{tamePart}(w))^{-1}$. *Put* $u_w = x_{\xi_w} \in k_w^\times$ *for the chosen character* T. *The main representation* $\theta \otimes R_{b,\ell}$ *has the preceding local properties with*

$$c_w^{\mathrm{norm}}(X) = u_w^n c_w(X/u_w) = \mathrm{scaleRoots}(c_w, u_w),$$

which is exactly `scaledTypePoly`. *Its polynomial at a tame exponent* a *is*

$$\prod_{c_w(x)=0} (X - \iota_\ell T(u_w x)^a)\,. \tag{17.41}$$

Proof. The rank-one local reciprocity formula gives this avatar's inertial character as $\xi_w \circ \mathrm{Art}_F^{-1}$. At the coefficient prime this statement is about its Weil–Deligne inertial character: the algebraic part of the avatar is crystalline and contributes no descent inertia. Here the old primes split completely, so locally this is also the elementary decomposition into a finite tame character, an unramified character and an integral power of the cyclotomic character. Lemma 17.75 in degree one gives its value $\iota_\ell T(u_w)^a$. Tensoring with this line multiplies each old eigenvalue by that scalar, giving (17.41). Multiplication by $u_w \neq 0$ preserves squarefreeness and degree. The definition of `scaledTypePoly` uses x_{ξ_w}, so there is no additional inverse.
Lean. `RowsFromEigensystems` (L268). □

This main-row character θ is different from `New.Selmer.normalizingAvatar` on a lifted descent pair. The latter is constructed from the inverse of `charW` with arguments $(-\mathrm{rank}(W), 0)$ and is unramified away from the coefficient prime; it is not the character causing the old main-label root scaling.

Corollary 17.78 (Inertia and tame period types of the distinguished summands)**.** *For the good main rows just constructed, the local label hypotheses at all old chosen places in Theorem 17.7 and Corollary 17.31 are satisfied for these main representations. In particular, at a primitive slot above a level prime different from* ℓ, *the integral matrix-span and scalar exclusion results of §17 apply to every stable coefficient lattice after the finite coefficient extension specified there. At a level prime* $p = \ell$ *with* $n < p$, *the normalized main representation becomes crystalline over the explicit tame extension*

$$\mathbb{Q}_{p^{n!}}(p^{1/e_n}), \qquad e_n = \mathrm{lcm}_{1\le d\le n}(p^d - 1) \le p^{n^2}. \tag{17.42}$$

Thus the old-level period polynomial degrees and common tame extension required by §17.3 are available for these actual main rows.

Proof. At every old chosen place the local field is $\mathbb{Q}_p$. A root of the scaled label has degree at most n over $\mathbb{F}_p$, so its multiplicative order divides $p^d - 1$ for some $1 \le d \le n$. Formula (17.41) gives the annihilating polynomials with these roots. At primitive slots we also use the unit separation and cyclic Frobenius under the numerical and prime-selection hypotheses of the local construction.

For $p = \ell$, wild descent inertia is trivial and tame descent inertia has exponent dividing e_n. Since $n < p$, both e_n and $n!$ are prime to p, and every $d \le n$ divides $n!$, so $e_n \mid p^{n!} - 1$. Lemma 15.17 kills this finite inertia over (17.42). With $N = 0$, trivial descent inertia implies crystallinity over that extension. The elementary estimate $e_n \le \prod_{d=1}^n (p^d - 1) < p^{n(n+1)/2} \le p^{n^2}$ gives the stated size bound. Away from ℓ, the annihilating identities on a stable lattice follow from those on its fraction-field vector space by injectivity. No integrality of a diagonalizing basis is assumed.
Lean, formal context. `FitTwistsExist` (L415). □

For the higher-rank ambient eigensystems the finite type is the mixed type of Theorem 17.73. Their global realization is given in Theorem 17.102.

17.14 The auxiliary parahoric type and its Frobenius operator

The preceding quadratic relation uses the action of finite inertia. At an auxiliary place the local type has only determinant blocks. When the ramified block has dimension one, its Frobenius eigenvalue is also determined by a Hecke operator.

Let F be a nonarchimedean local field of characteristic zero, with ring of integers $\mathcal{O}_F$, residue field k of order q, and uniformizer ϖ. Put $G = \mathrm{GL}_N(F)$, $K = \mathrm{GL}_N(\mathcal{O}_F)$, and let $I \subset K$ be the standard Iwahori. We use an algebraically closed coefficient field C of characteristic zero for the local representation-theoretic arguments, taking $C = \mathbf{C}$ for local Langlands and transporting the resulting identities through the chosen coefficient isomorphism when necessary. We use geometric reciprocity: $\mathrm{Art}_F(\varpi)$ is geometric Frobenius. The Galois normalization is

$$\mathrm{rec}^{\mathrm{coh}}(\pi) = \mathrm{rec}_F\big(\pi \otimes |\det|^{(1-N)/2}\big).$$

The unramified twist in this formula changes Frobenius eigenvalues but changes neither the inertial characters nor the monodromy operator.

Lemma 17.79 (The finite determinant-block representation)**.** *Let $\chi_1, \dots, \chi_s : k^\times \to C^\times$ be pairwise distinct, and let $n_i > 0$, with $\sum_i n_i = N$. Let $P \subset \mathrm{GL}_N(k)$ be the standard parabolic with Levi $\prod_i \mathrm{GL}_{n_i}(k)$. Define*

$$\lambda(p) = \prod_i \chi_i(\det p_i), \qquad \kappa = \mathrm{Ind}_P^{\mathrm{GL}_N(k)} \lambda,$$

where p_i are the diagonal blocks and λ is trivial on the unipotent radical. Then κ is irreducible. If $J \subset K$ is the inverse image of P, then the inflation of κ is $\mathrm{Ind}_J^K \lambda$.

Proof. Apply Lemma 17.72 with $c = 1$ and $t_i = n_i$. This gives the claimed irreducibility. Inflation gives $\mathrm{Ind}_J^K \lambda$, since reduction identifies the two coset spaces and λ factors through P.
Lean, formal context. `HasParahoricType.hasProductLabel`. □

Lemma 17.80 (The compact spherical idempotent is the parahoric average)**.** *In the setting of Lemma 17.79, let $\chi = \lambda|_I$. The compact Hecke algebra $\mathcal{H}(K, I, \chi)$ is the tensor product of the finite Iwahori–Hecke algebras of S_{n_i}, with parameter q. Under this identification its spherical idempotent is the normalized λ-average over J. Its image in $\mathrm{Ind}_I^K \chi$ is $\mathrm{Ind}_J^K \lambda$.*

Proof. With $c = 1$, $J_0 = I$ in Lemma 17.72. The product of its spherical idempotents averages over J with character λ. Its image is $\kappa(1; \chi, n) = \mathrm{Ind}_J^K \lambda$.
Lean, formal context. `LocalFormTypes` (L239-L249). □

Lemma 17.81 (The repeated tame-block type forces zero monodromy)**.** *Let π be an irreducible smooth representation of G, and suppose*

$$\mathrm{Hom}_J(\lambda, \pi) \neq 0$$

for the pairwise distinct block characters above. Regard each χ_i as a tame character of I_F via geometric local reciprocity. Then

$$\mathrm{rec}^{\mathrm{coh}}(\pi)|_{I_F} \simeq \bigoplus_i \chi_i^{\oplus n_i}, \qquad N_{\mathrm{rec}(\pi)} = 0.$$

Moreover $\mathrm{Hom}_J(\lambda, \pi)$ has dimension one.

Proof. By compact Frobenius reciprocity, $\mathrm{Hom}_J(\lambda, \pi) = \mathrm{Hom}_K(\kappa(1; \chi, n), \pi)$. Theorem 17.73 with $c = 1$ gives the asserted inertia, zero monodromy and multiplicity one.
Lean, formal context. `HasParahoricType.hasProductLabel`. □

The character-separation condition is essential. If two equal characters are kept as separate blocks, the preceding parahoric average is not the spherical idempotent for their combined block. For example, with $N = 2$, the trivial character of the Iwahori occurs in the Steinberg representation, whose monodromy is nonzero. In the actual auxiliary application the two characters are 1 and ϑ_w^{-1}, and $\vartheta_w \neq 1$ at $w \in Q$. This is exactly the `tamePart_ne_one` check in `RowsAndColumnsFromLabels.lean`; the repeated copies within each of these two distinct blocks cause no problem.

Lemma 17.82 (Inertia at auxiliary places by local compatibility)**.** *Let $w \nmid \ell$ be an auxiliary split place. Suppose the local automorphic representation has the determinant-block type of ranks t, r with characters $1, \vartheta_w^{-1}$, where $\vartheta_w \neq 1$. Suppose an actual continuous Galois representation R is supplied, together with local–global compatibility*

$$\mathrm{WD}(R|_{G_F})^{F\text{-ss}} \simeq \iota\,\mathrm{rec}^{\mathrm{coh}}(\pi_w).$$

Then $R(I_F)$ is finite and

$$R|_{I_F} \simeq 1^{\oplus t} \oplus a^{\oplus r}, \qquad a = \iota(\vartheta_w^{-1} \circ \mathrm{Art}_F^{-1}).$$

In particular $(R(g) - a(g)I)(R(g) - I) = 0$ on every stable integral lattice for every $g \in I_F$.

Proof. Lemma 17.81 makes the automorphic monodromy zero. Frobenius semisimplification preserves the monodromy operator, so the compatibility statement makes the Galois monodromy zero as well. In the construction of the Weil–Deligne representation, the actual inertia action is the product of the finite smooth action and $\exp(t_\ell(g)N)$. The exponential is now one. Thus actual inertia is finite and has the stated character decomposition, not only the stated semisimplification. The two-character quadratic follows on the rational space, and then on every stable lattice by the injectivity of its endomorphism ring into its rationalization.

If only a monodromy-domination version of local compatibility is available, the same argument applies once its direction is verified: the Galois monodromy must be dominated by the zero automorphic monodromy. Ordinary semisimplified Weil compatibility alone does not suffice. Nor does this lemma construct the supplied global R.

Lean, formal context. `FitTwistsExist` (L415). □

Lemma 17.83 (Frobenius on the ramified line and its Hecke normalization)**.** *Assume $N \geq 2$ and that the residue characteristic of F is different from ℓ. Take block sizes $1, N-1$ and characters $\chi, 1$, with $\chi \neq 1$. Let π contain the corresponding J-character, and let R satisfy the full local compatibility in Lemma 17.82. Put*

$$D = \mathrm{diag}(\varpi, 1, \ldots, 1), \qquad U = [I_1 D I_1],$$

where I_1 is the pro-p Iwahori and U is the unnormalized right-coset sum, whose degree is q^{N-1}. Then U preserves $\pi^{J,\lambda}$ and acts there by a scalar α. Choose $s \in W_F$ whose image in W_F^{ab} is $\mathrm{Art}_F(\varpi)$. If $\tau \in I_F$ satisfies $\chi(\tau) \neq 1$, then

$$(R(s) - \iota\alpha I)(R(\tau) - I) = 0.$$

Here the scalar is the Frobenius eigenvalue of the ramified line in the cohomological normalization. More precisely, if the character on the GL_1 factor of the normalized inducing datum is η_0, then

$$\alpha = q^{(N-1)/2}\eta_0(\varpi).$$

Proof. The parahoric subspace is one-dimensional by Lemma 17.81. The operator U preserves it: the Iwahori torus character is preserved, and its finite Hecke generators within the last $N-1$ coordinates commute with D. These generators impose exactly the parahoric spherical condition from Lemma 17.80.

We check the scalar, including the factor of q. The cosets of $I_1 D I_1/I_1$ are represented by $u(b)D$, where $b \in k^{N-1}$ and $u(b)$ has b in the rest of its first row. On a regular principal series in this Bernstein component choose a nonzero Jacquet eigenfunctional l that does not kill its parahoric vector y. The Jacquet isomorphism for the tame Iwahori type guarantees such a choice when the torus characters are distinct; the unique ramified slot has character η_0. Since l kills the upper unipotent action,

$$\begin{aligned} l(Uy) &= q^{N-1} l(\pi(D)y) \\ &= q^{N-1}\delta_B(D)^{1/2}\eta_0(\varpi)l(y) = q^{(N-1)/2}\eta_0(\varpi)l(y). \end{aligned}$$

This proves the claimed scalar on the regular principal-series parameters. The Hecke algebra for (J,λ) is the Bernstein center by Dat's construction above. It is the Laurent polynomial algebra on the inducing parameters, invariant under S_{N-1} on the unramified block. Both the scalar of U and the displayed GL_1 coordinate are regular functions on that torus quotient. The regular principal-series parameters form a Zariski dense open subset, so the formula holds at every central character, including reducible inducing parameters. This density step avoids assuming that every Jacquet module is semisimple.

The cohomological twist multiplies the GL_1 Weil character by $|\cdot|^{(1-N)/2}$, hence evaluates at ϖ to the same $q^{(N-1)/2}\eta_0(\varpi) = \alpha$. By Lemma 17.82, the ramified inertia eigenspace has dimension one and the complementary inertia action is trivial. Frobenius preserves this line, because a character from $k^\times$ is invariant under the q-power action. Frobenius semisimplification does not change the action on a one-dimensional invariant subspace. Thus $R(s)$ acts there by $\iota\alpha$. Since $\chi(\tau) \neq 1$, this line is exactly $\mathrm{im}(R(\tau) - I)$, proving the matrix identity. The one-dimensional character factors through abelianization, so the assertion is independent of the chosen lift of that Artin class. The Jacquet convention used here is the normalized/unnormalized comparison in Roche, §9, especially Theorem 9.2 and Remark 7.8.

Lean, formal context. `FitTwistsExist` (L415). □

Why a higher-dimensional ramified block is different. For a type $1^{\oplus t} \oplus a^{\oplus r}$ with $r > 1$, the a-isotypic subspace can have several Frobenius eigenvalues. For example, choose distinct unramified characters η_1, η_2 and a ramified tame a; the zero-monodromy parameter $1 \oplus a\eta_1 \oplus a\eta_2$ has precisely the $(1,2)$ determinant-block type, but Frobenius is not scalar on the ramified block when $\eta_1(\varpi) \neq \eta_2(\varpi)$. Thus the quadratic inertia relation is valid for general r, whereas the single Hecke scalar just proved concerns the line parahoric. This distinction is also present in the source.

Lemma 17.84 (A Vandermonde proof using only annihilating polynomials)**.** *Let $F_0, T \in \mathrm{End}_C(V)$ commute, where C is a field, and let $N \geq 1$. Choose nonzero $\zeta, \alpha_0, \ldots, \alpha_{N-1} \in C$. Assume*

$$(T - I)(T - \zeta I) = 0,$$

and, for $b = 0, \ldots, N-1$, assume

$$\left((X - \alpha_0\zeta^b)\prod_{i=1}^{N-1}(X - \alpha_i)\right)(F_0 T^b) = 0.$$

If $1, \zeta, \dots, \zeta^{N-1}$ are pairwise distinct, then

$$(F_0 - \alpha_0 I)(T - I) = 0.$$

In particular the hypothesis $\zeta^{N!} \neq 1$ used in the source is sufficient. It is stronger than the distinct-powers condition actually needed here.

Proof. On $\mathrm{im}(T - I)$, the quadratic relation gives $T = \zeta I$. Restrict the given annihilating identities to this invariant subspace and cancel the nonzero scalar ζ^b. The polynomial

$$t \longmapsto \prod_{i=1}^{N-1} (tF_0 - \alpha_i I)(F_0 - \alpha_0 I)\big|_{\mathrm{im}(T-I)}$$

has degree at most $N - 1$ and vanishes at the N distinct points $1, \zeta, \dots, \zeta^{N-1}$. Its constant term is $\prod_{i=1}^{N-1}(-\alpha_i)(F_0 - \alpha_0 I)|_{\mathrm{im}(T-I)}$. The product is nonzero, also for $N = 1$. If two of the points coincided, the order of ζ would divide some j with $1 \leq j < N$, hence would divide $N!$.

Lean. Lean derives distinctness from the factorial hypothesis in `mul_sub_one_eq_smul_sub_one`. Here distinctness is assumed in the polynomial argument. □

Lemma 17.85 (From separating exponent functionals to the trace action)**.** *Let $\mathcal{S}$ be a complex vector space stable under a linear operator U. Fix matrices F_0, T over $\overline{\mathbb{Q}}_\ell$, and fix a field isomorphism $\iota : \mathbb{C} \to \overline{\mathbb{Q}}_\ell$. Suppose every nonzero $z \in \mathcal{S}$ admits a linear functional l and $\alpha \in \mathbb{C}$ such that*

$$l(z) \neq 0, \qquad l(Uy) = \alpha l(y) \ (y \in \mathcal{S}), \qquad F_0(T - I) = \iota(\alpha)(T - I).$$

Then, for $y \in \mathcal{S}$, any matrix M, and any $G \in \mathbb{C}[X]$,

$$\iota^{-1}\mathrm{tr}((T - I)M)\, G(U)y = \iota^{-1}\mathrm{tr}\big(G^\iota(F_0)(T - I)M\big)\, y.$$

The functionals and their scalars may depend on z.

Proof. Fix y, M, G, and write z for the difference of the two sides. If $z \neq 0$, take l, α from the hypothesis for z. Then $l(G(U)y) = G(\alpha)l(y)$, and

$$G^\iota(F_0)(T - I) = \iota(G(\alpha))(T - I).$$

Taking the trace after multiplication by M gives $l(z) = 0$, contrary to the choice of l.

Lean, formal context. The formal argument for the Hecke operator is `hecke_eq_frob_on_parahoricEigenspace`. Applying the separating functional directly to the polynomial gives the identity above. □

In the source's weaker polynomial formulation, the separating functionals come from the sorted exponent construction inside the same automorphic eigensystem. Their inertia values are $(\zeta, 1, \dots, 1)$; the polynomial identities for $F_0 T^b$ are precisely those in Lemma 17.84. Its hypothesis $\zeta^{N!} \neq 1$ therefore supplies the matrix identity needed in Lemma 17.85. The exponent convention is

$$\mathrm{torusChar}(\theta)(t) = \prod_{i=0}^{N-1} \theta_i(t_i)|t_i|^{N-1-i}.$$

Hence $l(\pi(D)y) = \theta_0(\varpi)q^{-(N-1)}l(y)$, and the q^{N-1} coset degree gives $l(Uy) = \theta_0(\varpi)l(y)$. This is the same normalization as the full-compatibility proof, and explains the exact cancellation in `hecke_eq_frob_on_parahoricEigenspace`.

Lemma 17.86 (Normalization and preservation of the integral lattice)**.** *For a descent pair with source rank r, write μ for* `O.charW` *and υ for its splitting character. The normalizing avatar in the source is the avatar of*

$$x \longmapsto \mu^{-1}(x/\overline{x})\,\upsilon(x)^r\,|x|^{r/2}.$$

At every $w \nmid \ell$ under the source's unramifiedness hypotheses, this avatar ν is unramified. If s is the geometric Frobenius chosen above and $b_w =$ `O.charW.frobValue`(r, w)*, then*

$$\nu(s) = \iota(b_w^{-1}), \qquad \nu|_{I_w} = 1.$$

Thus replacing R by $R \otimes \nu$ leaves the quadratic inertia relation unchanged and replaces the line Hecke operator by $b_w^{-1}U$, and the Frobenius scalar by $b_w^{-1}\alpha$. All resulting matrix identities hold on every stable lattice on which their coefficients act integrally, and modulo every power of its uniformizer.

Proof. The source definition is `(HasWeight.inv).toAlgebraic`$(-r, 0)$. The general formula for `toAlgebraic`(k, j) is $\mu(x/\bar{x})\upsilon(x)^{-k}|x|^{j-k/2}$; substituting $\mu^{-1}, -r, 0$ gives the displayed character, including the splitting-character factor. Its local unit values are one by the unramifiedness assumptions on μ and the splitting character. Local reciprocity and the avatar formula therefore give $\nu|_{I_w} = 1$. Evaluating the same character at ϖ gives b_w^{-1}. These are also the cases $d = 0, 1$ of `normalizingAvatar_eq_of_isFrob_inv`.

Put $F_1 = \nu(s)F_0$ and $U_1 = b_w^{-1}U$. Multiplying $(F_0 - \iota\alpha I)(T - I) = 0$ by $\nu(s)$ proves the normalized identity, while T itself is unchanged. A continuous character of a profinite group with values in a finite ℓ-adic field has unit values: applying the valuation gives a compact subgroup of $\mathbb{Z}$, hence the zero subgroup. Thus this normalization introduces no lattice denominator after passing to a common coefficient field. Finally, injectivity of integral endomorphisms into their rationalizations proves every asserted integral identity and its reductions.

Lean. `SelmerHypotheses` (L541). □

Application to integral Hecke actions. For the two distinct auxiliary characters, the determinant-block calculation gives zero automorphic monodromy. Local compatibility retaining or dominating monodromy then gives finite inertia on an attached representation and the quadratic relation on its lattice. The line-parahoric calculation fixes the Frobenius scalar in the trace identity, including the power of q. The integral eigenvalue tuples act on the lattice of forms by Proposition 17.104.

17.15 Local bounds for the distinguished summands

The new local identifications allow the earlier estimates to be applied to the actual good terminal main rows. We state the resulting arithmetic conclusion separately from the existence of every ambient representation used by the denominator argument.

Theorem 17.87 (Selmer conditions on the distinguished summands)**.** *Take a sufficiently large member of the constructed family, put $t_* = 5r$, and fix a good eigensystem on W_m and a lift index $1 \le j \le m$. Let*

$$T_j = \theta_j \otimes R_{b,\ell}$$

be its normalized rank-r main representation from Corollary 17.66. For every coefficient prime ℓ, it is defined over a finite extension of $\mathbb{Q}_\ell$, has the prescribed labelled jumps and conjugate duality, is unramified outside the places above $\ell p_0 p_1$, and is potentially crystalline at every place above ℓ. After one finite enlargement of the coefficient field, every stable lattice L_j gives the original Selmer

datum of Theorem 17.39, with the original crystalline local cocycle modules. In particular, writing $2f = [E : \mathbb{Q}]$ *and*

$$e_O = v_O(\ell), \qquad g_* = \prod_{w \text{ chosen}} |\operatorname{GL}_{rt_*}(\kappa_w)|, \qquad b_\ell = v_\ell(g_*),$$

its allowance satisfies

$$\frac{\beta_{T_j}}{e_O} \le (4f+3) r b_\ell. \tag{17.43}$$

No local type compatibility for these main rows remains a premise of this application.

Proof. Theorem 17.52 constructs the raw row. Corollary 17.66 gives its finite coefficient field, normalized jump multiset and trace-duality identity with $s = t_*$. Corollary 17.70 gives the required unramifiedness and crystallinity off the old level. Theorem 17.76 and Corollary 17.77 give the actual scaled label at every chosen old place, and potential crystallinity when such a place lies above ℓ.

These assertions also hold at conjugate old places with the corresponding dual type. Indeed semisimplicity and trace duality give $T_j^c \simeq T_j^\vee \otimes \chi_\ell^{1-t_*}$. Potential crystallinity is preserved by duals, crystalline twists, and transport of the place. At a place prime to ℓ, the cyclotomic character is trivial on inertia, giving precisely the conjugate inertia used in Lemma 17.37.

The good unscaled labels avoid the N-th roots of unity, where the selected scaling character has order dividing N. Thus Lemma 17.37 applies at every old place prime to ℓ. Choose one of p_0, p_1 different from ℓ, and its primitive slot. Theorem 17.7, now applicable by Corollary 17.78, supplies the full integral matrix span and both required scalar elements on every stable lattice. Its numerical hypotheses were checked for this family immediately after that theorem. Every hypothesis of Theorem 17.39 is therefore satisfied, with $\rho = r$. That theorem constructs the original datum and gives (17.43), without replacing its local condition by a larger cocycle module.

Lean, formal context. `FitTwistsExist` (L415). □

Corollary 17.88 (Local defects for a realized main row and its character string)**.** *Fix one of the representations* T_j *in Theorem 17.87, write* $p = \ell$*, and let* $0 \le t \le t_*$*. Include the characters* $C_c = \chi_p^{-c}$*,* $0 \le c < t$*, whose jumps in the convention of this note are* c*. Use the classes* $\mathcal{C}_v$ *of Theorem 17.40, with the allowed jumps of this row and this character string. At an old coefficient prime, take the allowed period spectra to be those of this main row together with* t *copies of* 1*. Then for every specified stable lattice and every* $v \mid p$*,*

$$\delta_{v,\mathrm{main}} \le [E_v : \mathbb{Q}_p] r D_p, \qquad \delta_{v,c} \le [E_v : \mathbb{Q}_p] D_p / t_*.$$

For any finite relation family containing these summands and whose local restrictions belong to $\mathcal{C}_v$ *at every* $v \mid p$*, its global local-comparison colengths satisfy*

$$\frac{L_{T_j} + \sum_{0 \le c < t} L_c}{v_O(p)} \le 2f \left(r + \frac{t}{t_*} \right) D_p \le 4 f r D_p. \tag{17.44}$$

At an old coefficient prime there is, for each inertia element, a common annihilator in $\mathbb{Z}_p[X]$ *of degree at most* $r+1$ *on the allowed finite period spectra, and likewise on their duals. The construction and class membership needed for the entire ambient Hecke family are subsequently supplied in Theorem 17.105 and Corollary 17.106.*

Proof. Away from the old coefficient primes, the main row is crystalline by Corollary 17.70, and every C_c is crystalline. Their jumps are the required main profile and character string. At an old coefficient prime, the local field is $\mathbb{Q}_p$. Let $c \in \mathbb{F}_p[X]$ be the scaled degree-r label. By (17.39) and

Corollary 17.77, the finite period eigenvalues at inertia s are $y_x = \iota_p(T(x)^{a_s})$, for $c(x) = 0$, with the same tame exponent a_s for all roots. They are prime-to-p roots of unity, and $y_{x^p} = y_x^p$. Arithmetic Frobenius of the unramified coefficient field therefore permutes them. The polynomial

$$(X-1)\prod_{c(x)=0}(X-y_x)$$

belongs to $\mathbb{Z}_p[X]$ and has degree $r+1$. It kills the finite period operators throughout the prescribed class by semisimplicity: every eigenvalue occurs in the product, and the Tate characters all have trivial inertial type on periods. The same proof with y_x^{-1} works for duals and for the conjugate place. Each y_x has order dividing p^d-1 for $d = [\mathbb{F}_p(x):\mathbb{F}_p] \le r$, as required for the common tame chart.

The main row and the character string thus belong to the prescribed classes. Each further member of a class satisfies its defining jump and period-polynomial conditions; the common tame-chart argument applies to the whole class, including all tested extensions. Theorem 17.40 gives the two defect estimates. For a finite relation family with the stated class membership, Corollary 17.41 gives (17.44). These statements retain the same D_p, integral cocycle modules and normalization. They do not assert membership for an ambient eigensystem which has not yet been realized.

Lean, formal context. `FitTwistsExist` (L415). □

The ambient family. The denominator calculation uses further eigensystems besides the good main rows. We next identify the rank-one rows, identify the mixed types at the old level, and realize the eigensystems in the ambient spaces.

17.16 The rank-one characters from class field theory

The rank-one row in the original construction is determined by its critical pair. It does not require a further automorphic existence theorem. We identify it, including its normalization and both meanings of a local type, and then apply the previously proved allowances.

Keep the actual admissible data, with CM field E, CM type Φ, two old primes p_0, p_1, and $t_* = 5r$. Write $\epsilon_\sigma = 1$ on Φ and -1 on its conjugate. A main pair $p = (\nu, z)$ for $1 \le j \le m$ has

$$m < z \le r, \qquad \nu \text{ of weight } -c(v), \qquad \psi_w = \chi_{M,w}^{j}, \qquad w_j(v) + r - t_* \le c(v) \le w_j(v).$$

As in §13.5, $z \le c(v) \le 4b$ and $\psi_w = 1$ away from the chosen places. Its lift datum has source the standard definite line, target rank $2z$, and row length $c(v) - z$. Put

$$s = 2z-1, \qquad a = 1-z, \Theta_p(x) = \nu^{-1}(x/\bar{x})|x|^{1-z}.$$

Here x is a finite idele and $|\varpi_w| = q_w^{-1}$. All Artin maps below send a uniformizer to geometric Frobenius. The jump convention is that χ_ℓ has jump -1. The source definitions are critical pairs, main pairs, and their lift data.

Lemma 17.89 (Constructing the avatar over a finite coefficient field)**.** *Let Θ_f be the finite part of an algebraic Hecke character of a totally complex number field E, with integers n_σ characterized by*

$$\Theta_f(x) = \prod_{\sigma: E \hookrightarrow \mathbf{C}} \sigma(x)^{-n_\sigma} \quad (x \in E^\times).$$

For a fixed $\iota : \mathbf{C} \simeq \overline{\mathbf{Q}}_\ell$, there is a unique continuous character $\Theta_\ell : G_E \to \overline{\mathbf{Q}}_\ell^\times$ satisfying

$$\Theta_\ell(\mathrm{Art}_E(x)) = \iota(\Theta_f(x)) \prod_\sigma \sigma_\ell(x_{v_\sigma})^{n_\sigma}. \tag{17.45}$$

The expression uses the local extension σ_ℓ of $\iota\sigma$, and $v_\sigma \mid \ell$. The character is defined over a finite extension of $\mathbf{Q}_\ell$, with image in its unit group. Avatar formation preserves products, inverses, and conjugation of the number field.

For fixed admissible data, the entire finite family of original main pairs can be treated over one finite coefficient field. Every lattice in any one of these one-dimensional representations is stable.

Proof. Choose an open subgroup U of the compact finite integral ideles on which Θ_f is trivial. The quotient $\mathbb{A}_{E,f}^{\times}/(E^{\times}U)$ is a finite ray class quotient. Choose representatives $x_1, \dots, x_h$. Some positive power of each x_i belongs to $E^{\times}U$. Consequently $\Theta_f(x_i)$ is algebraic: its corresponding power is a product of integral powers of conjugates of an element of $E^{\times}$. The values of Θ_f therefore belong to the number field generated by the normal closure of E and these finitely many values. Indeed every finite idele is bx_iu, with $b \in E^{\times}$ and $u \in U$.

Complete this number field at the place selected by ι. It contains all the local embeddings occurring in (17.45). The right side is continuous: its first factor is locally constant, and the remaining factors are continuous integral powers of local embeddings. Its restriction to $E^{\times}$ is one by the displayed algebraicity formula. On compact integral ideles its values are units. Here the smooth factor has finite image, and the local embeddings take units to units. The finite ray class quotient then shows that its value on each x_i is a unit as well. Global Artin reciprocity therefore factors it uniquely through G_E^{ab}, with values in the profinite unit group of this finite coefficient field. This proves existence, continuity and finite coefficients directly. The same formula and uniqueness prove the asserted compatibilities.

There are finitely many critical pairs for the fixed data. The integer z, the bounded integral weights $c(v)$, and the residue characters at the finitely many chosen places have finitely many possibilities. For fixed weights and residue characters, the quotient of two possible ν's has trivial infinity type and is trivial on finite integral points. Its base change is an unramified finite-order idele class character, hence a character of the finite ideal class group of E. Base change here is injective by adelic Hilbert 90, applied to $x \mapsto x/\bar{x}$. Thus there are only finitely many ν's. This also gives the finiteness asserted in the source. Take the compositum of the finitely many coefficient fields. In dimension one each lattice is a fractional ideal times a basis, and multiplication by a unit preserves every such lattice.

Lean. `ChosenPlaces` (L307-L337). □

Formula (17.45) is exactly the corrected idele formula in the avatar definition. This proof uses global class field theory; it does not use the broad Galois-representation existence theorem imported by the source's general avatar local-compatibility file.

Lemma 17.90 (The exact row and its conjugate duality)**.** *For the original main pair p, its main representation is*

$$R_{p,\ell} = \mathrm{avatar}(\Theta_p), \qquad \Theta_p = (p.\mathrm{hasWeight}^{-1}.\mathrm{toAlgebraic}(0, 1 - z)). \tag{17.46}$$

Its infinity type and jumps are

$$n_{p,\sigma} = 1 - z + \epsilon_\sigma c(v), \qquad J_p(\sigma) = \{z - 1 - \epsilon_\sigma c(v)\}. \tag{17.47}$$

It has the exact character identity

$$R_{p,\ell}(g^c) R_{p,\ell}(g) = \chi_\ell(g)^{2-2z} = \chi_\ell(g)^{1-s}. \tag{17.48}$$

It is continuous and semisimple and has nondegenerate trace form.

Proof. The source of the original lift is the constant representation of $U(1)$. It is fixed by every finite compact subgroup. The GL_1 Hecke double coset has one coset, so its standard Hecke operator acts on the constant function by one. Its reversed Hecke polynomial is $1 - T$, its raw Galois representation is the trivial character, its jump is 0, and its unscaled label is $X - 1$. The trivial representation is crystalline at every coefficient-prime place and unramified at every other place; its local killing polynomial is $X - 1$. Thus the raw rank-one input is supplied without an existence theorem.

For this lift, $\operatorname{rank} W = 1$, $\operatorname{rank} V = 2z$, and $(1 - (2z - 1))/2 = 1 - z$. The prescribed main-row normalization of the source twists the raw representation by $O.\mathrm{hasWeight}^{-1}.\mathrm{toAlgebraic}(0, 1 - z)$. Its character is $\nu^{-1}(x/\bar{x})|x|^{1-z}$, proving (17.46). This is also the precise construction in `isRepOfType_one` and `exists_isMainSummand_mainPair`.

The inverse of ν has weight c. On the CM embedding and its conjugate the algebraic exponents are respectively $1 - z + c(v)$ and $1 - z - c(v)$. This proves the first formula in (17.47); the jump identification follows from the local calculation below.

Conjugation inverts the factor $\nu^{-1}(x/\bar{x})$ and preserves the norm. Hence $\Theta_p^c \Theta_p = |\cdot|^{2-2z}$. The avatar of $|\cdot|$ is χ_ℓ: at geometric Frobenius at $w \nmid \ell$ both have value q_w^{-1}, and continuity and Chebotarev give equality. Multiplicativity from Lemma 17.89 gives (17.48). Semisimplicity is automatic in dimension one, and the image span contains one; its trace pairing is the nondegenerate pairing $(x, y) \mapsto xy$.

Lean. `RowsFromEigensystems` (L268-L293), `FitTwistsExist` (L265-L313). □

Lemma 17.91 (The local avatar calculation, including the coefficient prime)**.** *The representation $R_{p,\ell}$ is unramified outside the places above $\ell p_0 p_1$, potentially crystalline at every place above ℓ, and crystalline there away from the old level. Its labelled jumps are exactly* (17.47)*.*

More precisely, for $w \mid \ell$, let $\delta_{p,w}$ be the finite character on $\mathcal{O}_{E_w}^\times$ obtained by restricting the finite part of Θ_p. On local units,

$$R_{p,\ell}(\mathrm{Art}_{E_w}(u)) = \iota(\delta_{p,w}(u)) \prod_{\sigma : v_\sigma = w} \sigma_\ell(u)^{n_{p,\sigma}}. \tag{17.49}$$

The finite inertial action on its potentially crystalline periods is the character corresponding to $\iota\delta_{p,w}$ under this same geometric reciprocity convention.

Proof. Off ℓ the correction factors in (17.45) are one on a locally supported idele. Local reciprocity identifies actual inertia with local units. Both the norm and $\nu^{-1}(x/\bar{x})$ are trivial on such units if neither w nor $\bar{w}$ is chosen. This proves the asserted unramifiedness, including nonsplit and ramified places away from the old level. No condition that such a place be split has been introduced.

At $w \mid \ell$, equation (17.49) is the local specialization of (17.45). Extend the finite character $\iota\delta_{p,w}$ from units to $E_w^\times$ by assigning value one to a uniformizer, and let η_w be the resulting finite-order local Galois character. The quotient $R_{p,\ell}\eta_w^{-1}$ is algebraic on the entire unit group, so it is crystalline by the crystalline-character criterion. The jumps are the negatives of its labelled algebraic exponents, and the finite character has jump zero.[1] Killing η_w over a finite local extension proves potential crystallinity. If w is off the old level, then $\delta_{p,w} = 1$, so the representation is already crystalline over E_w.

[1] The precise classical inputs are B. Conrad, *Lifting global representations with local properties*, Appendix B, Proposition B.3 and Proposition B.4(i), https://math.stanford.edu/~conrad/papers/locchar.pdf. Conrad uses arithmetic reciprocity. Inverting it changes the unit exponents to $-n_{p,\sigma}$. His convention in which the cyclotomic character has Hodge–Tate weight $+1$ is also opposite to the jump convention here. The resulting jumps are $-n_{p,\sigma}$, as tested by the character $|\cdot|$ whose avatar is χ_ℓ.

For the period action, choose a finite Galois local extension killing η_w. Its crystalline period module is the tensor product of the base change of the crystalline module of $R_{p,\ell}\eta_w^{-1}$ with the one-dimensional finite character. An inertia element acts trivially on the first factor's descent data and by η_w on the second. Thus the covariant inertial operator is η_w, with no inverse or dual introduced. This is precisely the finite period type asserted in the lemma. It does not assert finite actual inertia of $R_{p,\ell}$ at $w \mid \ell$.

Lean, formal context. `FitTwistsExist` (L265-L313), `FitTwistsExist` (L415). □

Lemma 17.92 (The scaled type at an old place)**.** *Let w be a chosen old place. Using the faithful character T_w at this place, put*

$$N = q_2(s_0), \qquad \omega_w = x_{\chi_{M,w}} \in \kappa_w^\times, \qquad \operatorname{ord}(\omega_w) = N,$$

where T_w is the source's fixed faithful residue character. The unscaled label of the raw row is $X - 1$, and its exact normalized label is

$$(X-1).\operatorname{scaleRoots}(\omega_w^{-j}) = X - \omega_w^{-j}. \tag{17.50}$$

For $w \nmid \ell$ this is the type of actual tame inertia; wild inertia acts trivially. For $w \mid \ell$ it is the type of the finite inertial period action of Lemma 17.91. At the conjugate place the type is transported by $g \mapsto g^{-c}$.

Proof. Chosen places are split and are disjoint from their conjugates. In the standard split identification of the norm-one pair, ν uses the selected w-coordinate of $x/\bar{x}$. For a local unit u at w, the finite character $\nu^{-1}(x/\bar{x})$ has value $\chi_{M,w}(\bar{u})^{-j}$: the prescribed datum $\psi_{\bar{w}}$ is trivial, and the norm factor is one. Triviality of this datum does not mean that the base-changed Galois character has trivial inertia at $\bar{w}$; that inertial character is the conjugate inverse computed below. This is an equality of characters, not just an equality of their orders. The character-name map is a group isomorphism, so its name is ω_w^{-j}. Scaling the raw label $X - 1$ therefore gives (17.50). Equivalently, an inertia element of tame exponent b acts by $\iota(T_w^b(\omega_w^{-j}))$, on the actual row off ℓ, and on its periods at ℓ. This agrees with the avatar label formula and the rank-one scaled label.

The character is trivial on principal local units, proving the wild-inertia assertion off ℓ. At a conjugate place, (17.48) gives $R_{p,\ell}(g^{-c}) = R_{p,\ell}(g)$ for inertia off ℓ, because $\chi_\ell(g) = 1$. On periods above ℓ, the cyclotomic factor is crystalline and has trivial inertial descent action, giving the same transport rule. The order N comes from the actual selected residue characters; it is not a new choice.

Lean. `SelmerHypotheses` (L207-L230), `RowsAndColumnsFromLabels` (L402-L439), `ChosenPlaces` (L134-L165). □

Lemma 17.93 (The normalization and full factorization of the line lift)**.** *Let ϑ be one of the tame twists, and set*

$$a_\vartheta = \operatorname{avatar}(\vartheta^{-1}(x/\bar{x})), \qquad \lambda = \operatorname{avatar}(\upsilon(x)|x|^{1/2}),$$

where υ is the fixed splitting character, whose existence for the actual field is proved in Lemma 17.98. The second character is the algebraic character $1.\text{toAlgebraic}(-1, 0)$*; its two infinity exponents are $1, 0$. For the original rank-one lift datum, λ is exactly* ***normalizingAvatar****. The main avatar $R_{p,\ell}$ is the distinct character* (17.46)*.*

The full Hecke factorization of a lift of the constant $U(1)$ system is realized by the actual semisimple representation

$$S_{p,\vartheta,\ell} = \lambda^{-1} \otimes \left(a_\vartheta R_{p,\ell} \oplus \bigoplus_{e=0}^{s-1} \chi_\ell^{-e} \right). \tag{17.51}$$

Every continuous semisimple representation attached to that full system is isomorphic to this one. In particular its normalized trace is exactly the main-pair trace plus the prescribed character string, for every $g \in G_E$.

Proof. The standard definite source has no negative signs, so $\mathrm{charW} = 1$. Its rank is one. Consequently the definition charW^{-1}.toAlgebraic$(-1, 0)$ gives $\upsilon|\cdot|^{1/2}$, including its splitting-character factor. The target character is $\mathrm{charV} = (\nu\vartheta)\upsilon_{U(1)}^{z}$. The base change of $\upsilon_{U(1)}$ is υ^2. It cancels the target's splitting-character power in even rank $2z$, exactly as in the definitions of the two space characters.

At a split place off the level and ℓ, write $B_w = (\nu\vartheta)(\varpi_w/\bar{\varpi}_w)$, and denote the two source `frobValue`'s by $b_{W,w}, b_{V,w}$. Their exact values are

$$b_{W,w} = \upsilon(\varpi_w)^{-1} q_w^{1/2}, \qquad b_{V,w} = B_w q_w^{z}.$$

For geometric Frobenius ϕ_w,

$$\lambda(\phi_w) = \iota(b_{W,w}^{-1}), \qquad a_\vartheta(\phi_w) R_{p,\ell}(\phi_w) = \iota(B_w^{-1} q_w^{z-1}) = \iota(q_w^{s}/b_{V,w}).$$

The constant source has polynomial $1 - T$. Its theta factorization is therefore

$$H^{\mathrm{rev}}_{p,\vartheta,w}(T) = \prod_{e=0}^{s-1} (1 - q_w^{e} b_{W,w} T)(1 - q_w^{s} b_{W,w} b_{V,w}^{-1} T).$$

This is precisely the full factorization for the original trivial lift, rather than merely its first coefficient. In this argument the full factorization is the defining condition `HasLiftHeckePolynomial` on the lifted system; the underlying global theta-transfer theorem is not re-proved here. Substitution gives the full Frobenius polynomial of (17.51). Theorem 17.54 proves attachment and uniqueness for every actual eigensystem with this factorization. It follows that the normalized trace is

$$a_\vartheta(g) R_{p,\ell}(g) + \sum_{e=0}^{s-1} \chi_\ell(g)^{-e}.$$

The source's `Attached` representations have nondegenerate trace form and hence are semisimple by Lemma 17.45. Thus the original `IsMainSummand` trace clause, which is conditional on `HasLiftHeckePolynomial` and attachment, holds for $R_{p,\ell}$, without selecting an arbitrary rank-one character.

The normalization can also be checked on the actual jumps. The character a_ϑ has weight zero, while λ has jumps $-1, 0$ at a CM embedding and its conjugate. Hence the raw lift's two jump multisets are

$$\{z - c(v)\} \sqcup \{1, \dots, 2z - 1\}, \qquad \{z + c(v) - 1\} \sqcup \{0, \dots, 2z - 2\}.$$

They are exactly the target's single-row weight $c(v) - z$ in rank $2z$. Dividing by λ^{-1} recovers J_p and the string $0, \dots, s - 1$.

Lean. `AuxiliaryPair` (L329-L337), `CharacterLiftNorm` (L210-L233). □

Theorem 17.94 (The rank-one Selmer conditions)**.** *For every original main pair and every coefficient prime ℓ, every stable lattice in $R_{p,\ell}$, over a sufficiently large common finite coefficient ring O, satisfies all the hypotheses of Theorem 17.39 with $\rho = 1$ and $s = 2z - 1$. Use the crystalline local conditions*

$$S = \{v : v \mid \ell p_0 p_1\}, \qquad Z_v = Z_f(L|_{G_{E_v}}) \quad (v \mid \ell).$$

Write $E = \mathbb{Q}(\sqrt{-d}, \zeta_q)$ with the cyclotomic modulus q of the field construction, and put $s_q = \prod_{p|q} p^{v_p(s)}$. The same integer W_s gives the allowance at every ℓ:

$$\frac{\beta_L}{v_O(\ell)} = v_\ell(W_s) \le b_\ell, \qquad W_s = q s_q \begin{cases} d^{1+v_d(s)}, & (d-1)/2 \mid s, \\ 1, & (d-1)/2 \nmid s. \end{cases} \qquad W_s \le qds. \tag{17.52}$$

The common coefficient field can serve all original main pairs and all the finitely many character strings.

Proof. Lemma 17.89 supplies finite coefficients and stable lattices. Lemmas 17.90 and 17.91 supply the exact conjugate duality, local admissibility and jumps. The inequalities for a main pair give the jump profile of Theorem 17.10 and $1 \le s \le t_*$. In fact $c(v) \ge z$ makes the CM jump negative and its conjugate positive, so there is exactly one negative jump per conjugate pair. This in particular verifies the original rank-one allowance for the crystalline cocycle ranks.

At each old chosen $w \nmid \ell$, Lemma 17.92 gives a tame eigenvalue of order $N/\gcd(N, j)$. Since $2^{s_0}5^{s_0} \mid N$ and $1 \le j \le m$, the prime-to-ℓ part of this order is at least $2^{s_0}/m > 4r + 2$. Thus its reduction has order greater than $4r + 2$. In particular its difference from one and the difference of its square from one are units. The conjugate-inverse generator has the same eigenvalue by (17.48). Both first-prime determinants are units, and all old-place excesses vanish by Lemma 17.36. The inertia order divides $q_w - 1$. The image span on a line is O, and at an old prime $p_i \ne \ell$ arithmetic Frobenius has cyclotomic value p_i and acts by a scalar. This verifies the hypotheses of Theorem 17.39.

Only the global term of (13.17) remains. Use the perfect coefficient pairing λ of Lemma 15.4, extended to the fraction field. For $a \ne 0$, the pairing $(x, y) \mapsto \lambda(xy/a) \bmod \mathbb{Z}_\ell$ identifies O/aO with its additive dual. By (17.48) the action on $\mathrm{Hom}_O(L/aL, (L/aL)^{*c})$ is multiplication by χ_ℓ^s. Its invariant length is the lesser of $v_O(a)$ and $\min_{u \in \chi_\ell(G_E)} v_O(u^s - 1)$. Taking the supremum over a and dividing by $v_O(\ell)$ gives $\min_u v_\ell(u^s - 1)$. Frobenius at p_i bounds this by $v_\ell(p_i^s - 1) \le b_\ell$ by Lemma 14.12.

Here $12 \mid q$, $(q, d) = 1$, and $d \equiv 7 \bmod 8$, so $\mathbb{Q}(\sqrt{-d})$ is the quadratic subfield of $\mathbb{Q}(\zeta_d)$ [Was97, Ch. 2]. By the Chinese remainder theorem the cyclotomic Galois groups for coprime conductors are independent. If $\ell^a \| q$, the image is $1 + \ell^a\mathbb{Z}_\ell$, with $a \ge 2$ at 2. The binomial valuation formula in Lemma 15.7 gives minimum $a + v_\ell(s)$, attained at $1 + \ell^a$. At d the image consists of the units reducing to squares modulo d. Their residue group has order $(d-1)/2$, so the minimum is zero unless this order divides s. When it does, every Teichmüller component has s-th power one and $1+d$ attains the minimum $1 + v_d(s)$. At any other prime ℓ the image is all of $\mathbb{Z}_\ell^\times$; ℓ is odd and $s = 2z - 1$ is odd, so -1 gives minimum zero. These valuations give W_s. Since $s_q d^{v_d(s)} \mid s$, the stated size bound follows. A common coefficient extension for the finite family works by Lemma 17.89.

Lean, comparison. Lean bounds the congruence and the old-place terms in `congruenceTerm_le` and uses the Selmer conditions in `SelmerHypotheses` (L78-L94). On a line the old-place terms vanish and the cyclotomic image gives the exact remaining length. □

Corollary 17.95 (The local class and bounds for the family of characters)**.** *Fix a main pair, its string length $s = 2z - 1$, and $C_e = \chi_\ell^{-e}$, $0 \le e < s$. At places above ℓ use the local classes of Theorem 17.40, with allowed jumps*

$$\operatorname{supp} J_p(\sigma) \cup \{0, \dots, s-1\}.$$

At an old coefficient prime prescribe, for every inertia element, the multiset consisting of its eigenvalue on the finite periods of $R_{p,\ell}$ together with s copies of one. Then the actual main row

and all C_e belong to these classes. So do the normalized representations $\lambda S_{p,\vartheta,\ell}$ of (17.51) for the original auxiliary twists, and their rational direct summands.

For every specified stable lattice the corresponding defects satisfy

$$\delta_{v,\mathrm{main}} \le [E_v : \mathbf{Q}_\ell] D_\ell, \qquad \delta_{v,e} \le [E_v : \mathbf{Q}_\ell] D_\ell / t_*.$$

For any finite relation family in these prescribed local classes, the original comparison colengths therefore obey

$$\frac{L_{\mathrm{main}} + \sum_{e=0}^{s-1} L_e}{v_O(\ell)} \le 2f \left(1 + \frac{s}{t_*}\right) D_\ell \le 4f D_\ell. \tag{17.53}$$

Proof. Off the old coefficient primes the main row is crystalline by Lemma 17.91, as are the Tate characters. At an old coefficient prime the chosen local field is $\mathbf{Q}_\ell$. Its main period character has order dividing $\ell - 1$, by Lemma 17.92; the conjugate place has the inverse character with the same bound. The periods of every C_e have trivial inertia. The common multiset has $1 + s = 2z \le 2r \le r + t_*$ members and satisfies the exact spectral-order hypotheses of Theorem 17.40.

An original auxiliary twist has its ramification set disjoint from the bad places, which include every place above $\ell p_0 p_1$ by the definition of the bad places. Its weight-zero avatar is consequently unramified above ℓ and has jump zero. Tensoring the main line by a_ϑ changes neither these jumps nor the finite inertial period type. The direct sum in (17.51) therefore proves the asserted membership for the normalized character sums. Period admissibility, jump supports and the displayed period polynomials pass to rational direct summands and to all their stable lattices. A finite family of these objects has a common finite coefficient field by Lemma 17.89.

The earlier local theorems apply to this class and every tested extension satisfying its defining admissibility and polynomial conditions. They do not require the original lattices to split integrally into their rational character constituents. Apply Theorem 17.40 with $\rho = 1$, and then Corollary 17.41, to obtain the asserted bounds. On the main-dual side its actual jump is $\epsilon_\sigma c(v) - z$; on the character-dual side it is $-e - 1$. Thus the distinct alternatives of the local estimates are applied to their original supports, exactly as in (17.10).

Lean. `SelmerHypotheses` (L78-L94). □

What this closes. The original rank-one main-pair row, its old scaled label, its off-level and coefficient-prime local conditions, and its original Selmer allowance are now explicit deductions. The theta system with the stated full factorization also has the character representation (17.51), and its normalized local-class membership is proved. The separate problem of realizing arbitrary ambient nonlift Hecke systems is not replaced by this calculation. Theorem 17.105 and Corollary 17.106 subsequently supply the required ambient realizations and their membership in the common local classes. Further representations or arbitrary extensions require their own admissibility and local-class tests before their relation families use (17.53). No assertion of an integral Hecke-algebra-valued matrix action follows merely from a character identity.

17.17 The old-level polynomial of the ambient eigensystems

We apply the mixed type of Theorem 17.73 to the old level. The characteristic polynomial retains the full multiplicity of each determinant block. For annihilation, one copy of its factor suffices.

Corollary 17.96 (The full polynomial and the shorter annihilator)**.** *Put*

$$h(X) = c(X) \prod_l (X - u_l)^{t_l}, \qquad h_{\mathrm{red}}(X) = c(X) \prod_l (X - u_l).$$

For an inertial element s of tame exponent a at a level f divisible by all the d_i, set

$$Q_{h,s}(X) = \prod_{c(x)=0} (X - T(x)^a) \prod_l (X - T(u_l)^a)^{t_l}.$$

This is the characteristic polynomial on the inertial parameter of Theorem 17.73. The degree-$(d+b)$ polynomial $Q_{h_{\mathrm{red}},s}$, with just one factor for each u_l, already annihilates that parameter.

Suppose an actual continuous ℓ-adic representation R is supplied with the stated local parameter after Weil semisimplification, and with Galois monodromy dominated by the automorphic monodromy. At $F \mid \ell$ assume also that R is de Rham, so its potentially semistable parameter is defined. Then $N_R = 0$. If $F \nmid \ell$, actual inertia is finite and both polynomials, transported to the coefficient field, annihilate $R(s)$. If $F \mid \ell$, R is potentially crystalline and both annihilate the covariant period operator $\beta_{s,0}$. Away from ℓ, the assertions about actual matrices hold on every stable integral lattice after the coefficient field contains the roots. At ℓ the assertion is on the indicated rational period module.

Proof. Lemma 17.75 computes the factors from c, and its rank-one calculation computes those from χ_l. The inertia representation is diagonalizable in characteristic zero. Repeating the same character t_l times repeats its factor in the characteristic polynomial, but is unnecessary in an annihilator. The values $T(x)^a$ may coincide for a particular s; this does not affect either assertion, and we do not claim that the shorter polynomial is then squarefree.

Theorem 17.73 makes the dominating monodromy zero. The monodromy-domination argument of Lemma 17.53 therefore gives $N_R = 0$. Away from ℓ the exponential monodromy factor in actual inertia is one, as in Lemma 17.82. At ℓ, potential semistability and $N_R = 0$ give potential crystallinity. The descent action on covariant periods has the same finite inertial parameter, by the convention checked in Theorem 17.76. Finally the roots are roots of unity, so the polynomial coefficients are integral; a matrix identity on the fraction field descends to a stable lattice by injectivity of its endomorphism ring into its rationalization.

Lean, formal context. `LocalFormTypes` (L239-L249), `FitTwistsExist` (L415). □

Proposition 17.97 (The mixed local type at the old level)**.** *For an actual lift setting in the denominator construction, write $\rho \in \{r, 1\}$ for the source rank and t for its rank difference. At a chosen old split place w, let c_w be the squarefree source label and put*

$$a_w = c_w.\,\mathrm{scaleRoots}(x_{\psi_w^{-1}}).$$

The ambient finite type is exactly

$$\kappa(a_w; 1, t) = \mathrm{Ind}_P^{\mathrm{GL}_{\rho+t}(k_w)}(\tau_{a_w} \boxtimes 1_{\mathrm{GL}_t(k_w)}), \qquad a_w(1) \neq 0.$$

Consequently Theorem 17.73 applies, and its inertia polynomial is $a_w(X)(X-1)^t$. The added spherical block does not enlarge the tame extension needed for this local type. At a chosen old rational prime p, the extension of Corollary 17.78 still suffices, with

$$e_r = \mathrm{lcm}_{1 \le d \le r}(p^d - 1), \qquad \mathbb{Q}_{p^{r!}}(p^{1/e_r}).$$

This conclusion applies to every ambient eigensystem with the specified finite type, regardless of whether it is a theta lift. Its assertion about a Galois representation uses the compatibility antecedents of Corollary 17.96.

Proof. At the pinned source revision, `HasProductLabel` requires separability of $c\prod_l(X - x_{\chi_l})$, with no multiplicity on these last factors, and defines the finite character using the Hall product with the rank-t_l determinant characters. Its `HasParahoricType.hasProductLabel` lemma implements the scaling by the character on the source block. `BaseLevelOfLabel.lean`, in its old-place application, takes precisely one further character, namely 1, with block size t. The exclusion $a_w(1) \neq 0$ is `IsLiftSetting.not_isRoot_scaledTypePoly` in `SelmerHypotheses.lean`. These are exactly the separation hypotheses above. Induction in stages agrees with the Hall product by Lemma 17.71.

Each root of the degree-ρ squarefree polynomial has order dividing $p^d - 1$ for some $d \leq \rho \leq r$. Scaling by a member of $\mathbb{F}_p^\times$ preserves this property, and the extra root is 1. The displayed tame extension therefore kills the entire finite inertial type, independently of t. Use the same argument on covariant periods at $p = \ell$; it does not assert finite actual inertia of a nonzero-weight p-adic representation.

Lean. `LocalFormTypes` (L239-L249). □

The finite type in Proposition 17.97 must not be replaced by the finite principal-series character obtained from t separate trivial GL_1 blocks. That replacement would retain Steinberg constituents and lose zero monodromy. The expression $a_w(X)(X-1)^t$ records the desired inertial polynomial; it does not instruct us to make that replacement.

17.18 Realizing the ambient eigensystems, including residual base change

The primitive local label used in Theorem 17.52 forces cuspidal base change only for the main representation. An ambient form of larger rank need not have that property. We construct its representation from the discrete constituents of base change. Residual constituents are retained as strings of Tate twists. The splitting character already present in the arithmetic data corrects the parity of the cuspidal constituents, so no square root of the cyclotomic character is required.

Throughout, E/F is CM, $[F : \mathbb{Q}] \geq 2$, and reciprocity sends a uniformizer to geometric Frobenius. Write χ_ℓ for the usual cyclotomic character; thus $\chi_\ell(\mathrm{Frob}_w) = q_w^{-1}$ when $w \nmid \ell$, and its Hodge–Tate weight is -1. The notation rec denotes unitary local Langlands, before the rank-dependent twist $|\det|^{(1-N)/2}$.

Lemma 17.98 (Existence of the splitting character)**.** *Let $K = \mathbb{Q}(\sqrt{-d})$, where d is a prime congruent to 7 modulo 8, and let $E = K(\zeta_q)$, where $12 \mid q$ and $(q, d) = 1$. Give E the CM type consisting of its embeddings extending the specified complex embedding of K. There exists a continuous unitary Hecke character υ, unramified at every finite place, such that*

$$\upsilon^c = \upsilon^{-1}, \qquad \upsilon(x) = \prod_\Phi \frac{x_\Phi}{|x_\Phi|} \quad \text{when } x_f \in \widehat{\mathcal{O}}_E^\times.$$

Its restriction to the ideles of $F = E^+$ is the quadratic character of E/F. In particular it has value -1 on a uniformizer at an inert finite place. The fields and CM types used in the actual construction satisfy these hypotheses.

Proof. The prime 2 splits in K. Fix $v \mid 2$, so $K_v = \mathbb{Q}_2$. On $K_\infty^\times$ take $z \mapsto z/|z|$, on $\mathcal{O}_v^\times$ the nontrivial character modulo 4, and on all other finite units the trivial character. The product is trivial on $\mathcal{O}_K^\times = \{\pm 1\}$, hence defines a character on the image of $K_\infty^\times \widehat{\mathcal{O}}_K^\times$ in the idele class group of K. The quotient by this subgroup is the ideal class group. Since the circle is divisible, the character extends to a unitary character ψ on the whole group. The extension is continuous because the subgroup is open. On the rational ideles ψ equals the quadratic character of $\mathbb{Q}(i)/\mathbb{Q}$, since they agree at infinity and on every finite unit group, and $\mathbb{Q}$ has class number one.

Put $\upsilon = \psi \circ N_{E/K}$. Above v, every completion of E contains $\mathbb{Q}_2(i)$, since $4 \mid q$. By transitivity the local norm of a unit is a norm from $\mathbb{Z}_2[i]^\times$. Such a norm $a^2 + b^2$ is 1 mod 4, so υ is trivial on all finite units. At infinity the norm is the product over the embeddings extending the chosen embedding of K, giving the stated infinity type.

Since $E = FK = F(i)$, the restriction of $N_{E/K}$ to the ideles of F is $N_{F/\mathbb{Q}}$. Hence $\upsilon|_{\mathbb{A}_F^\times}$ is the quadratic character of $\mathbb{Q}(i)/\mathbb{Q}$ composed with $N_{F/\mathbb{Q}}$. It kills $N_{E/F}\mathbb{A}_E^\times$ by transitivity of norms, and has value -1 at an idele negative at one real place and equal to 1 elsewhere. By global reciprocity it is the quadratic character of E/F [Mil20, Ch. V, §4]. This gives its value on an inert uniformizer. Applying it to $xx^c = N_{E/F}x$ gives $\upsilon^c = \upsilon^{-1}$.

In the construction, $18480 \mid q_0$, $d \equiv q_0/2 - 1 \pmod{q_0}$, and $q = q_0\, 2^s 3^c$. Thus $d \equiv 7 \pmod 8$, $12 \mid q$, and $d \nmid q$.

Lean, comparison. Lean extends a character on the ideles of E using units with prescribed conjugate quotient in `nonempty_splittingChar` and `WeilData` (L421-L422). Here 2 splits in K, so the character can be constructed on K and pulled back by the norm. □

Lemma 17.99 (The available algebraic parity character)**.** *Let υ be the splitting character in the source's Weil data. Then υ is unramified at every finite place, $\upsilon^c = \upsilon^{-1}$, and*

$$\eta = \upsilon|\cdot|_{\mathbb{A}_E}^{1/2}$$

is an algebraic Hecke character. Its ℓ-adic avatar A_ℓ satisfies

$$A_\ell A_\ell^c = \chi_\ell, \qquad \mathrm{HT}_\Phi(A_\ell) = \{-1\}, \qquad \mathrm{HT}_{\overline{\Phi}}(A_\ell) = \{0\}.$$

It is unramified away from ℓ and crystalline at every place above ℓ. At every finite place, including those above ℓ, its Weil–Deligne character is $\eta_w \circ \mathrm{Art}_w^{-1}$ and has zero monodromy.

Proof. The defining finite-unit property of `SplittingChar` in the pinned `WeilData.lean` computes $\upsilon(x)$ as $\prod_\Phi z/|z|$ whenever the finite idele x_f is a unit. Take its infinite components equal to 1 and its finite components equal to 1 except for an arbitrary unit at w. This proves $\upsilon_w|_{\mathcal{O}_w^\times} = 1$. The identity $\upsilon^c = \upsilon^{-1}$ is the other defining property of the same character.

At a complex place, the idele norm is $z\bar{z}$. Consequently $\eta_\infty = z$ at the chosen CM embedding, with exponent zero at the conjugate embedding. Hence η is algebraic and $\eta\eta^c = |\cdot|_{\mathbb{A}_E}$. The avatar identities and local assertions follow from the rank-one reciprocity and crystalline-character arguments of Lemmas 17.89 and 17.91. The crystalline-character input is Conrad, *Lifting global representations with local properties*, Appendix B, Propositions B.3–B.4, with the reciprocity and Hodge-sign conversions recorded in Lemma 17.91. The same algebraic character is `HasWeight.toAlgebraic` with the trivial norm-one character, $k = -1$, and $j = 0$. In particular the proof uses an actual algebraic Hecke character, not a formal half-integral Tate twist.

Lean, formal context. `FitTwistsExist` (L415). □

Lemma 17.100 (The local parameter of a residual block)**.** *Let τ be a unitary cuspidal automorphic representation of $\mathrm{GL}_m(\mathbb{A}_E)$ and let $d \geq 1$. Write $\mathrm{Speh}(\tau, d)$ for the discrete automorphic representation obtained as the irreducible quotient of the normalized induction of*

$$\tau|\det|^{(d-1)/2},\ \tau|\det|^{(d-3)/2},\ \ldots,\ \tau|\det|^{-(d-1)/2}.$$

At every finite place w,

$$\mathrm{rec}\big(\mathrm{Speh}(\tau,d)_w\big) = \bigoplus_{j=0}^{d-1} \mathrm{rec}(\tau_w)|\cdot|_w^{(d-1)/2-j}. \tag{17.54}$$

This is an equality of Frobenius-semisimple Weil–Deligne representations, including monodromy. The analogous rule for the two infinitesimal-character multisets holds at every complex place. If $\mathrm{Speh}(\tau, d)$ *is conjugate self-dual, then* τ *is conjugate self-dual.*

Proof. The global residual-spectrum classification gives the uniqueness of (τ, d) [MW89, §§I.2, I.10–I.11]. Locally τ_w is unitary and generic. Write it as an irreducible product of essentially square-integrable factors $\delta_h|\det|^{v_h}$ with δ_h unitary and $|v_h| < 1/2$. The local quotient described in [MW89, §I.11] has standard-module factors $\delta_h|\det|^{v_h+(d-1)/2-j}$ for $0 \le j < d$. The Langlands parameter of a Langlands quotient is the direct sum of the parameters of its standard-module factors, with their monodromy operators. Grouping by j gives (17.54). At infinity normalized induction gives the corresponding union of shifted infinitesimal characters. Finally conjugation replaces τ by τ^c, and duality replaces it by $\tau^\vee$. Uniqueness of the global block gives $\tau^c \simeq \tau^\vee$.

Lean, formal context. `FitTwistsExist` (L415), `Isotypic.exists_attached`. The residual-spectrum classification and its local Langlands parameter enter the classical construction. The cited all-rank formal construction avoids the decomposition into residual Speh blocks. □

Lemma 17.101 (Regular algebraicity of the corrected cuspidal blocks)**.** *Suppose an automorphic representation of rank* N *has the form*

$$\Pi = \boxplus_i \mathrm{Speh}(\tau_i, d_i), \qquad \operatorname{rank} \tau_i = m_i, \qquad N = \sum_i m_i d_i,$$

with the τ_i *unitary cuspidal and conjugate self-dual. Suppose the infinitesimal character of* Π *is regular algebraic. Choose* $\epsilon_i \in \{0, 1\}$ *by*

$$\epsilon_i \equiv N - d_i - m_i + 1 \pmod 2.$$

Then $\tau_i' = \tau_i \otimes (\upsilon^{\epsilon_i} \circ \det)$ *is regular algebraic, unitary, cuspidal, and conjugate self-dual.*

Proof. At an embedding $\sigma : E \hookrightarrow \mathbb{C}$, let $\mathcal{P}_\sigma(\pi)$ denote the corresponding infinitesimal multiset, in the convention that an algebraic highest weight $\lambda_1 \ge \cdots \ge \lambda_n$ has parameters $\lambda_j + (n+1)/2 - j$. Regular algebraicity of Π means that its N parameters are distinct members of $\mathbb{Z} + (N-1)/2$ at every embedding. The previous lemma gives

$$\mathcal{P}_\sigma(\Pi) = \coprod_i \coprod_{j=0}^{d_i-1} \left(\mathcal{P}_\sigma(\tau_i) + (d_i - 1)/2 - j\right).$$

It follows, by fixing $j = 0$, that the parameters of τ_i are distinct and lie in $\mathbb{Z} + (N - d_i)/2$. The exponent of υ at σ is $a_\sigma = 1/2$ on the chosen CM type and $a_\sigma = -1/2$ on its conjugate. After adding $\epsilon_i a_\sigma$, the parameters therefore lie in $\mathbb{Z} + (m_i - 1)/2$.

Here is the precise regular-algebraic criterion. Order the resulting parameters $p_1 > \cdots > p_{m_i}$ and set $\lambda_j' = p_j - (m_i + 1)/2 + j$. These are integers, and $\lambda_j' - \lambda_{j+1}' = p_j - p_{j+1} - 1 \ge 0$. Thus the infinitesimal character is that of an algebraic representation of $\mathrm{Res}_{E/\mathbb{Q}} \mathrm{GL}_{m_i}$. The same argument at the conjugate embedding supplies its other factor. This is exactly the definition used in the Galois existence theorem below. Cuspidality and unitarity persist under the unitary twist, and conjugate self-duality persists because $\upsilon^c = \upsilon^{-1}$.

Lean, formal context. `FitTwistsExist` (L415). □

Theorem 17.102 (Every definite ambient eigensystem has a realization)**.** *Let* E/F *be CM with* $[F : \mathbb{Q}] \ge 2$*, and let* $U = U(V)$ *for an* N*-dimensional Hermitian* E/F*-space,* $N > 0$*, with* U *compact at every infinite place. Assume that* E *has the splitting character of Lemma 17.99. Let* b *be*

a Hecke eigensystem occurring in a nonzero algebraic automorphic form of weight κ on U. Choose a single irreducible automorphic constituent σ in which that form has a nonzero component. There is a weak base change Π of σ, and, for every prime ℓ and isomorphism $\iota_\ell : \mathbb{C} \simeq \overline{\mathbb{Q}}_\ell$, a continuous semisimple representation

$$R_{b,\ell} : G_E \longrightarrow \mathrm{GL}_N(L_\ell), \qquad [L_\ell : \mathbb{Q}_\ell] < \infty,$$

with the following properties:

1. *Its unramified characteristic polynomials at split spherical places away from ℓ are $\iota_\ell P_{b,w}$.*
2. *It is de Rham at every $w \mid \ell$. Its labelled jumps are*

$$\{j - 1 - \lambda_j : 1 \le j \le N\} \quad \text{at } \Phi, \qquad \{\lambda_j + N - j : 1 \le j \le N\} \quad \text{at } \overline{\Phi},$$

   for the dominant weight λ specified by κ.
3. *It satisfies $R^c_{b,\ell} \simeq R^\vee_{b,\ell}\chi_\ell^{1-N}$.*
4. *At every finite place w,*

$$\mathrm{WD}(R_{b,\ell}|_{G_{E_w}})^{F\text{-}ss} \simeq \iota_\ell \,\mathrm{rec}_w(\Pi_w |\det|^{(1-N)/2}).$$

5. *If Π_w is spherical, then $R_{b,\ell}$ is unramified at $w \nmid \ell$ and crystalline at $w \mid \ell$, with the expected Frobenius characteristic polynomial in both cases.*

At every split place, and at every nonsplit place where the unitary representation is unramified, Π_w is the prescribed local base change. Thus all local finite types imposed on the chosen constituent at split places can be used in item 4. No cuspidality assumption on Π is made.

Proof. The chosen constituent is discrete since U is definite. At infinity its representation is discrete series and satisfies Labesse's condition $(*)$. As $[F : \mathbb{Q}] \ge 2$, base change gives $\Pi = \boxplus_i \mathrm{Speh}(\tau_i, d_i)$, with matching at the asserted places and matching infinitesimal character [Lab11, Corollary 5.3]. Here we have used Lemma 17.100 to write its discrete blocks. The shift by the half-sum of positive roots gives adjacent differences $1 + \lambda_j - \lambda_{j+1} > 0$, so Lemma 17.101 gives regular algebraic, conjugate-self-dual cuspidal representations τ_i' of rank m_i.

Use the polarized pair $(\tau_i', \delta_{E/F}^{m_i})$. The quadratic character pulls back trivially to E and takes the value -1 on -1 at each real place before raising it to the m_i-th power. This is the sign required in [BLGGT14b, Theorem 2.1.1]. That theorem supplies $r_{i,\ell}$ with duality exponent $1 - m_i$, de Rham comparison, the labelled jumps, crystalline comparison at spherical places over ℓ, and compatibility away from ℓ. By Lemma 17.101 the corrected block τ_i' is regular algebraic and cuspidal, hence cohomological by Clozel's criterion [Clo90, Lemme 3.14]. Its sign character is trivial at every infinite place since E is CM. Thus (17.28) applies at every finite place. At $w \mid \ell$ the assertion follows from Caraiani's theorem for $m_i \ge 2$ and local class field theory for $m_i = 1$. Define

$$z_i = \frac{m_i + \epsilon_i + d_i - N - 1}{2}, \qquad R_{b,\ell} = \bigoplus_i \bigoplus_{j=0}^{d_i-1} r_{i,\ell} A_\ell^{-\epsilon_i} \chi_\ell^{z_i - j}. \tag{17.55}$$

The parity of ϵ_i makes z_i integral. This continuous semisimple representation has dimension $N = \sum_i m_i d_i$ and descends to a finite coefficient field by Theorem 17.43.

At a finite place the A_ℓ-twist removes υ^{ϵ_i}. The remaining norm exponent is

$$(1 - m_i)/2 - \epsilon_i/2 + z_i - j = (d_i - 1)/2 - j + (1 - N)/2.$$

Weil–Deligne parameters commute with direct sums and character twists. Lemma 17.100 now identifies the resulting sum with the required parameter, including its monodromy. For spherical Π_w every summand has trivial inertia and zero monodromy, so each $\tau_{i,w}$ and its unramified twist $\tau'_{i,w}$ is spherical. The $r_{i,\ell}$ and the character twists are then crystalline above ℓ and unramified elsewhere. The Satake normalization preceding Theorem 17.52 gives $\iota_\ell P_{b,w}$ at each split spherical place.

Write $\mathcal{P}_\sigma$ for the infinitesimal multiset at σ, and $a_\sigma = \pm 1/2$ for the exponent of υ. The jumps of $r_{i,\ell}$ are $(m_i-1)/2 - \mathcal{P}_\sigma(\tau_i) - \epsilon_i a_\sigma$, and the jump of A_ℓ is $-a_\sigma - 1/2$. Thus the i,j summand has jumps $(N-d_i)/2 - \mathcal{P}_\sigma(\tau_i) + j$. Their union is $(N-1)/2 - \mathcal{P}_\sigma(\Pi)$. At Φ the elements of $\mathcal{P}_\sigma(\Pi)$ are $\lambda_j + (N+1)/2 - j$. The conjugate embedding gives their negatives. This proves the displayed weight formulas.

Finally $B_i = r_{i,\ell} A_\ell^{-\epsilon_i} \chi_\ell^{z_i}$ satisfies $B_i^c \simeq B_i^\vee \chi_\ell^{d_i - N}$, by $A_\ell A_\ell^c = \chi_\ell$. Pair indices j and $d_i - 1 - j$ in (17.55) to obtain the asserted global duality.

Lean, formal context. `AllRanks` (L782-L794), `FitTwistsExist` (L415), `Isotypic.attached_galRep`. Here attachment is performed on cuspidal blocks of the isobaric base change and their twists are assembled. The formal proof attaches the representation on the isotypic space by the all-rank unitary construction. The identification of the monodromy of each higher-rank cuspidal block above ℓ uses [Car14, Theorem 1.1] here. □

Corollary 17.103 (Galois representations for the ambient families)**.** *Every characteristic-zero Hecke eigensystem in a nonzero simultaneous projected space used in* `BaseLevelOfLabel.exists_baseLevel` *has a representation of Theorem 17.102, at every coefficient prime. This applies to the even ranks* $6r$ *and* $2z$ *used in the denominator argument, at all its auxiliary principal levels. The representation can be chosen from a single automorphic constituent retaining all old labels and all auxiliary finite type conditions imposed on that eigensystem.*

After twisting by the source's lifted normalizer

$$\nu_{O,\ell} = \operatorname{avatar}\Big(\mu^{-1}(x/\bar{x})\upsilon(x)^\rho |x|^{\rho/2}\Big), \qquad \rho = \operatorname{rank} W, \quad t = N - \rho,$$

one obtains the required duality $(\nu_{O,\ell}R)^c \simeq (\nu_{O,\ell}R)^\vee \chi_\ell^{1-t}$ *and the labelled Hodge numbers obtained by subtracting the infinity exponents of the twisting character (*`normalisedJumps` *in Lean). At the old places its local type is precisely*

$$a_w(X)(X-1)^t, \qquad a_w = \texttt{scaledTypePoly}, \quad a_w(1) \neq 0,$$

with zero monodromy. At an auxiliary determinant-block place, the local assertions of Lemma 17.81 apply. Off the designated level, the representation is unramified away from ℓ *and crystalline above* ℓ*.*

Proof. The actual group is $U(\text{stdSpace}\, E, N)$, compact at every infinite place, and its weight is $(O.\text{columnCount}, O.\text{detExp})$. Since $5 \mid 5040$, the field E contains a primitive fifth root ζ_5. Hence F contains $\mathbb{Q}(\zeta_5 + \zeta_5^{-1}) = \mathbb{Q}(\sqrt{5})$. This verifies the degree hypothesis in Theorem 17.102. Its actual Weil data use the character constructed in Lemma 17.98; Lemma 17.99 gives its algebraic avatar. Repeated entries of an algebraic highest weight cause no problem, as the shifted parameters are distinct.

The source construction produces one nonzero form satisfying the eigensystem and all projectors simultaneously. At fixed level and weight the discrete decomposition is a finite direct sum. Projection to each automorphic isotypic summand commutes with the finite-group projectors and with the Hecke operators. At least one projected component is nonzero; choose its irreducible constituent once, retaining every local condition. This justifies applying the theorem to the entire simultaneous type, rather than selecting separate constituents at separate places. A previously selected continuous semisimple representation with the same unramified Hecke traces is isomorphic to the resulting one by Lemma 17.46.

The lifted normalizer is exactly `normalizingAvatar` of `SelmerHypotheses.lean`, using `HasWeight.inv.toAlgebraic` with $k = -\rho, j = 0$. It is not the main-row twisting character. Here $\mu = O$.charW has trivial finite tame part, so this algebraic character is finite-unramified. It satisfies $\nu_{O,\ell}\nu^{c}_{O,\ell} = \chi_\ell^{\rho}$. This proves the new duality exponent, and the labelled Hodge numbers shift by minus its algebraic infinity exponents, which is the definition of `normalisedJumps`.

At the old places, the projected type is exactly the mixed Green and single trivial determinant block of Proposition 17.97. Theorem 17.73 gives the automorphic inertial polynomial and $N = 0$; item 4 of Theorem 17.102 and Corollary 17.96 give the actual Galois and period assertions at every ℓ. The same reasoning uses Lemma 17.81 at a projected auxiliary determinant-block place. The normalizer has trivial Weil inertia, so none of these types changes. The nonsplit finite places are unramified for E/F and the designated maximal compact is hyperspecial by the field and lattice argument of the preceding section. The level is spherical there and at the remaining off-level places. Labesse's unramified local matching and item 5 therefore give the last assertion.

Lean, formal context. `FitTwistsExist` (L415). □

Scope of the construction. An unrestricted auxiliary principal level can contain finite types with nonzero monodromy; the theorem constructs their global representations as well but does not assert $N = 0$ for all such types. Zero monodromy above is used precisely in the specified mixed and determinant-block sectors. A residual block with $d_i > 1$ produces a generally nonpure direct sum in (17.55); no purity of the full ambient representation is asserted or needed. The construction supplies the missing global realization and its stated local properties. It does not identify every ambient eigensystem as a theta lift or establish the nonvanishing of any particular theta lift.

17.19 The integral actions used by the denominator argument

Two lattices occur. The Hecke operators act on the integral lattice of definite forms. Each attached Galois representation has its own stable lattice. The Ribet argument uses the actions on these lattices through their common scalar Hecke eigenvalues.

Proposition 17.104 (The denominator operators preserve the forms lattice)**.** *Fix one of the actual base spaces B, an allowed finite Taylor–Wiles level U, and a sufficiently large finite coefficient ring O. Let M_O be its lattice of integral definite forms, as in Lemma 17.56, and let $\mathbb{T}$ be the algebra of scalar eigensystem tuples preserving M_O. The following are actual integral operators on the indicated spaces:*

1. *The scalar Hecke tuples in Theorem 17.58 and Corollary 17.60, once the rational attached representations and their normalized Frobenius polynomials are available.*
2. *The action of $\prod_{u\in U} \mathrm{GL}_N(k_u)$ on the principal-level forms, given by integral right translation.*
3. *The weighted line norms and their compositions $p(\widetilde{U}_u)L_u$ used in Theorem 13.31.*
4. *The stabilizer-cleared norm sections of Lemma 25.3 and the old type projector multiplied by its prescribed finite-group-order factor.*

The residue and local Hecke operators commute with the scalar $\mathbb{T}$ action. These constructions require no Galois matrix action on M_O.

Proof. For the first assertion, use the fixed finite integral forms lattice, not just the coordinatewise integrality of eigenvalues. The good Frobenius coefficient operators preserve this lattice; simultaneous

Frobenius approximation and the closedness in Lemma 17.57 put all characteristic coefficients, determinants and tied traces in $\mathbb{T}$. This is exactly Theorem 17.58.

At a new place u, a lift of a residue matrix lies in $\mathrm{GL}_N(\mathcal{O}_{E_u})$. Two lifts differ by the principal congruence subgroup and therefore act identically on principal-level forms. Right translation preserves the integral-coordinate condition, the fixed old type, and the other local levels. This gives the second action and its group law. The new places avoid ℓ and the old level; translations supported there have trivial coefficient-prime components.

The weighted norm L_u is a finite integral combination of these translations. Its image is fixed by the pro-p Iwahori I_1. The counting operator $[I_1 D I_1]$ is a finite sum of translations, maps I_1-fixed vectors to I_1-fixed vectors, and preserves integral coordinates because $u \nmid \ell$. Consequently each integral polynomial in the normalized operator $\widetilde{U}_u$ has the same properties on this image. Its normalizing scalar is an ℓ-adic unit, as checked in Lemma 17.86. The subgroup fixing principal-level forms is contained in I_1, and the levels away from u commute with these translations. Thus $p(\widetilde{U}_u)L_u$ preserves the required principal-level space and its integral lattice. This is the module on which the first-order calculation of §13.8 acts.

For the norm section, the orbit-stabilizer calculation in Lemma 25.3 gives the specified clearing factor; there is no division by a possibly nonunit stabilizer order. For an old finite type σ on a finite group H, the projector is

$$e_\sigma = \frac{\dim \sigma}{|H|} \sum_{h \in H} \operatorname{tr} \sigma(h^{-1})\, h.$$

Its character values are algebraic integers. The chosen finite-group-order multiple of e_σ therefore preserves the integral lattice. For a tuple, take the product of these finite groups. The old integral right translations and tame character factors preserve the integral-coordinate lattice also when ℓ is an old prime. This is the type-projector calculation preceding Theorem 25.7.

Every local translation preserves the almost-everywhere Hecke isotypic spaces, since it commutes with all good Hecke operators away from its support. On each such space a scalar Hecke tuple acts by its defining scalar. Hence each of the local operators commutes with $\mathbb{T}$, as asserted.

Lean, formal context. `FitTwistsExist` (L415). □

In Lean, these objects are respectively `systemAction`, `heckeAlgebra`, and `heckeAction` in `Lib/GaloisRep/TaylorWiles.lean`; `residueRep` in the same file; the weighted operator construction in `TiedTraces.lean` together with `exists_localOperators_aeval_hecke`; and the norm-section and type-projection constructions in `TaylorWiles.lean` and `BaseLevelOfLabel.lean`. The source uses separate finite-coefficient Galois lattices in `RibetElement.lean`. The integral auxiliary quadratic on each of those lattices is Theorem 17.65.

The denominator calculation now applies the local trace and Frobenius identity for each eigensystem to these integral operators.

17.20 Galois representations at the levels used for the denominator bound

The family required by the denominator argument is larger than its distinguished lift subset. It is, however, narrower in rank than the source's general `AllRanks` theorem. For a main descent pair its source rank, string length and target rank are $(\rho, t, N) = (r, 5r, 6r)$; for an original main pair they are $(1, 2z-1, 2z)$. All target groups are the standard definite unitary groups. We apply the preceding constructions to every eigensystem at a fixed allowed level, retaining that distinction between the full family and the lift subset.

Theorem 17.105 (Galois and Hecke properties of the ambient eigensystems)**.** *Fix actual admissible data, one of the pairs just described, an allowed old label tuple, a tame twist ϑ, and a coefficient prime ℓ. Let B be the resulting base space, and let U be either the empty level or any of the additional principal levels used in the denominator construction. Let Y be the set of Hecke eigensystems occurring in B at level U (in Lean, B.Eigensystem(U)). Let S be the union of the twist places $\vartheta.Q$, the places of U, and their conjugates under the CM involution. In (17.56), O denotes the descent pair and $J_O(\sigma)$ is its multiset of normalized filtration jumps at the embedding σ, as in Corollary 17.103. Write a_w for the scaled type polynomial at an old place w. The predicate* `IsRepOfType` *specifies conjugate duality, filtration jumps, ramification, and the labeled inertia condition at the old places (definition of the local type). For every $x \in Y$, there is a normalized continuous semisimple representation R_x of rank $N = \rho + t$ satisfying*

$$\texttt{IsRepOfType}\Big(O, t, J_O, \{a_w(X)(X-1)^t\}_w, S, R_x\Big), \qquad J_O = \texttt{normalisedJumps}(O), \tag{17.56}$$

where $a_w(1) \neq 0$.

Write ϑ.invAvatar for the inverse ℓ-adic character associated to ϑ. At every twist place and its conjugate, the following identity holds on every stable coefficient lattice:

$$(R_x(g) - \vartheta.\mathrm{invAvatar}(g)I)(R_x(g) - I) = 0 \quad (g \in I_w). \tag{17.57}$$

At a new principal-level place the determinant and admissible line-parahoric sectors satisfy the scalar-inertia and normalized Frobenius/Hecke identities of §17.14. No zero-monodromy assertion is made for an unrestricted sector at such a place.

At each fixed finite level, one finite coefficient field and its ring of integers $\mathcal{O}$ suffice for all $x \in Y$, the distinguished summands and the required character strings. The normalized characteristic coefficients, determinants and tied traces belong to the actual scalar integral Hecke algebra. The operators of Proposition 17.104 therefore have their stated local trace identities on that lattice of forms.

Proof. The base space consists of the simultaneous old tuple projection and prescribed level-character space. At a fixed additional level it is finite-dimensional. The definition of Y requires a nonzero form in every listed eigensystem. Apply Corollary 17.103 to one constituent of that simultaneously projected space. This produces every R_x, without assuming that x is a theta lift.

We verify the conditions in (17.56). The construction gives continuity and semisimplicity, and semisimplicity gives a nondegenerate trace pairing on the image algebra by Lemma 17.45. The lifted normalizer gives conjugate duality with exponent $1 - t$ and precisely J_O. The old mixed type gives finite actual inertia away from ℓ, and potential crystallinity with the prescribed covariant period polynomial at ℓ, by Corollary 17.96. At the conjugate old places use the duality identity; the cyclotomic factor has trivial inertial period type.

All places in $\vartheta.Q$ and U avoid the bad places, which include every place above $\ell p_0 p_1$. Outside S and the chosen old places and their conjugates, the imposed level is maximal. It is hyperspecial also at nonsplit finite places by §17.12. Theorem 17.102 therefore gives unramifiedness away from ℓ and crystallinity at places above ℓ. This includes primes of the original modulus outside the old places and S. These facts give all three local clauses of `IsConjSelfDualOfLevel`, at every place outside S or dividing $\ell p_0 p_1$, and give the additional old polynomial clause of `IsRepOfType`.

At $w \in \vartheta.Q$, the two whole determinant blocks have characters 1 and ϑ_w^{-1} of ranks t and ρ, with $\vartheta_w \neq 1$. These places are prime to ℓ. Lemmas 17.81 and 17.82, using the full compatibility now supplied by Theorem 17.102, prove (17.57) on the rational representation. The lifted normalizer is unramified there. Conjugate duality proves the corresponding identity at $\bar{w}$; cyclotomic inertia is one. Injection of lattice endomorphisms into their rationalization proves the integral identity, even

if the two characters become equal modulo the maximal ideal. No division by their difference is used.

At a new place, principal level alone imposes no such type. When a determinant or line-parahoric projection is nonzero, choose an automorphic constituent in that projected subspace. The constructed representation for it is isomorphic to R_x by semisimple trace uniqueness. It therefore supplies the compatibility required in Lemma 17.83 and in the determinant-block calculation. The trace-action argument and lifted normalization in §17.14 apply to exactly the projections used by the first-order operator; they do not impose their conclusion on all principal-level forms.

There are finitely many x at this level, and every constructed representation has finite coefficients. Take their coefficient compositum together with those for the finite main family and the required roots of unity. Compactness supplies stable lattices, retaining any already specified ones. The good Frobenius polynomials have the actual source normalization by Theorem 17.102 and the lifted-character calculation of §17.14. Theorem 17.58 and Corollary 17.60 therefore place the scalar tuples in the integral Hecke algebra. Apply the local identities on each isotypic sector of the concrete forms module of Proposition 17.104; the sectors span that module rationally, and integrality preserves the same identities on its lattice. This proves the last assertion.

Lean, formal context. `TraceRelations` (L770-L850), `FitTwistsExist` (L415). □

Corollary 17.106 (The local bound for the full ambient family)**.** *In Theorem 17.105, all ambient representations belong to the common local classes used by the original relation construction. Suppose a selected realized main row of rank $\rho \in \{r, 1\}$ has the fixed old type and weight profile of this lift setting, as in Theorem 17.87 or Theorem 17.94. Those classes also contain this actual main row, its unramified-at-ℓ auxiliary twist, and the characters χ_ℓ^{-e} for $0 \le e < t$. For every specified stable lattice their defects satisfy*

$$\delta_{v,\mathrm{main}} \le [E_v : \mathbb{Q}_\ell]\rho D_\ell, \qquad \delta_{v,e} \le [E_v : \mathbb{Q}_\ell] D_\ell / t_*.$$

Thus whenever these are the summands in the original finite relation family, its comparison colengths obey

$$\frac{L_{\mathrm{main}} + \sum_{0 \le e < t} L_e}{v_O(\ell)} \le 2f\left(\rho + \frac{t}{t_*}\right) D_\ell \le 4f\rho D_\ell. \tag{17.58}$$

The required class membership of the entire ambient family is now proved, rather than left as an additional realization antecedent. The Selmer allowances for its actual main inputs are

$$\frac{\beta_T}{v_O(\ell)} \le (4f+3) r b_\ell, \qquad \frac{\beta_{R_p}}{v_O(\ell)} \le (4f+3) b_\ell,$$

for the rank-r row and original rank-one row respectively.

Proof. Use the actual common local classes in the source. Their admissibility condition is crystalline off the old level and potentially crystalline at old coefficient-prime places; their allowed jumps are the fixed full ambient multiset J_O. Their polynomial tests are those which annihilate the indicated period element on every representation satisfying the same `IsRepOfType` data. Equation (17.56) consequently puts every R_x and every lattice in it in this class.

For the main row, Theorem 17.87 or Theorem 17.94 supplies all the global and local properties. The identity $J_O = J_{\mathrm{main}} \sqcup \{0, \dots, t-1\}$ is the direct normalized-weight calculation; it does not require a nonzero lift. The direct sum with the prescribed Tate string therefore has the full jump multiset J_O, the old polynomial $a_w(X)(X-1)^t$, and the required duality exponent $1-t$. The pairing of indices e and $t-1-e$ verifies the last assertion for the character string. Tensor the main part by ϑ.invAvatar when appropriate. This character is unramified above $\ell p_0 p_1$, has jump zero,

and has conjugate inverse, so the same typed direct sum is available with that twist. It follows that every universal polynomial test defining the class also kills each of these direct summands. Their jumps lie in J_O and their local admissibility is already proved. This proves their class membership without asserting that every ambient system has such a decomposition.

At an old coefficient prime the common period spectrum consists of the ρ scaled Green roots and t copies of one. It has at most $r + t_*$ members. Each nontrivial root has order dividing $\ell^d - 1$ for some $d \le \rho \le r$; the conjugate spectrum has the same order bound. By Proposition 17.97, the same tame extension works for every multiplicity of the trivial block. Off the old level all these representations are crystalline. The universal-polynomial class can impose more tests than the class defined using just these canonical annihilators. It is therefore a subclass of the one to which Theorem 17.40 applies. Restricting the test class enlarges its cup-orthogonal cocycle module and can only decrease the quotient length defining the defect. The same bound thus applies to the actual source class and its tested extensions. No membership claim is made for arbitrary extensions.

That theorem gives the two local defect bounds. Summing local degrees uses $[E : \mathbb{Q}] = 2f$, and the character-string contribution is t/t_*, so Corollary 17.41 gives the first inequality in (17.58). Since $t \le t_*$ and $\rho \ge 1$, one has $\rho + t/t_* \le 2\rho$, proving the second. The stated Selmer estimates are precisely Theorems 17.87 and 17.94, with the cocycle conditions and bounds stated there.

Lean. `TraceRelations` (L770-L850), `BaseLevelOfLabel` (L330-L373). □

The occurrence of a distinguished nonzero lift, its full Hecke factorization, and the required geometric maps remain the separate hypotheses at the point where those maps are used. When that full factorization is supplied, Theorem 17.54 identifies its semisimple representation with the displayed main part and character string. The current argument supplies all the ambient representation and local-class hypotheses around it; it does not manufacture that nonzero lift.

From the Hecke systems to critical values. Theorem 17.105 and Corollary 17.106 put the ambient eigensystems in the common local classes. The integral action is on the forms lattice of Section 17.19. The remaining input to the common twist comes from distributions of the critical Hecke values, including the endpoint $z = 1$ in Corollary 18.47.

Part V
Hecke values, CM periods and theta derivatives

The finite and archimedean estimates in (6.1) use a common CM period. We relate this period to critical Hecke values and calculate the valuations of the theta derivatives after a change of coordinates by Frobenius. These valuations give the prime powers in S.

18 Critical Hecke values and their integral distributions

The interpolation formula and Euler factors of §13.5 determine the twist measure up to a common scalar. For $z \geq 2$ we calculate this scalar from the norm of a theta lift. At $z = 1$ the functional equation expresses the interpolation formula as a product of values at 0 and 1. The two Euler exponents and the single power of the modulus norm follow from this product.

Fix a ramified critical pair $p = (\nu, z)$ as in §13.5. Thus $1 \leq z \leq t = 5r$, and the weight $-c_\sigma$ of ν at each infinite place satisfies $z \leq c_\sigma \leq 4b$. For a weight-zero twist τ, put

$$N_* = \prod_{w \in S_{\mathrm{ch}}} q_w, \qquad \chi_\tau(x) = (\nu\tau)(x/\bar{x})^{-1}.$$

Here q_w is the residue field order at w. Each chosen place is split over F, its conjugate lies outside S_{ch}, and the tame part of ν is nontrivial there. For an auxiliary modulus Q satisfying the coprimality and disjointness conditions of §13.5, write $NQ = \prod_{w \in Q} q_w$ and set

$$L_Q(s, \tau) = L^{S_{\mathrm{ch}} \cup Q}(s, \chi_\tau),$$

with the omission convention of (13.46). The symbol $R(\tau) \subseteq Q$ denotes the set at which its tame part is nontrivial; it is the minimal tame modulus, even when τ has a nontrivial unramified component.

Lemma 18.1 (The root number of the anticyclotomic twists)**.** *The character χ_τ is unitary, is trivial on $\mathbb{A}_F^\times$, and satisfies $\chi_\tau^c = \chi_\tau^{-1}$. Its finite conductor is*

$$\mathfrak{f}(\chi_\tau) = \prod_{w \in S_{\mathrm{ch}} \cup R(\tau)} \mathfrak{p}_w \mathfrak{p}_{\bar{w}}, \qquad N\mathfrak{f}(\chi_\tau) = (N_* \, NR(\tau))^2. \tag{18.1}$$

For the standard global additive character and self-dual local measures its global root number is $+1$. Moreover $L_Q(s, \tau) > 0$ for every real $s > 1$, and $L_Q(1, \tau) > 0$.

Proof. The definition through $x/\bar{x}$ gives unitarity, triviality on $\mathbb{A}_F^\times$, and $\chi_\tau^c = \chi_\tau^{-1}$. The two characters at a split pair are inverse. At the old ramified places and at the places in $R(\tau)$ both are tamely ramified of conductor exponent one. All other finite components are unramified. This gives (18.1).

Take $\delta \in E^\times$ of trace zero. At a nonsplit finite place, the additive character on E_v is obtained by trace from F_v and $\chi_{\tau,v}|_{F_v^\times} = 1$. The local theorem of Fröhlich and Queyrut [FQ73, Theorem 3] gives $W_v = \chi_{\tau,v}(\delta)$. At a split place, $\delta = (b, -b)$ and the two local factors have product $W(\lambda)W(\lambda^{-1}) = \lambda(-1) = \chi_{\tau,v}(\delta)$. At a complex place over a real place of F, the angular exponent $2c_\sigma$ gives $i^{2c_\sigma} = (-1)^{c_\sigma} = \chi_{\tau,v}(\delta)$. Multiplication gives the global root number $\chi_\tau(\delta) = 1$.

For real $s > 1$ each unramified split pair contributes a positive absolute square. A nonsplit finite place is unramified in E/F by Lemma 17.67. Its uniformizer lies in F_v, so the unramified character there is trivial and its Euler factor is positive. Removing finitely many pairs preserves positivity. The nonzero angular weights make χ_τ nontrivial and give Hecke nonvanishing at 1 as in Lemma 12.7. The omitted Euler polynomials are nonzero there. Continuity now gives $L_Q(1,\tau) > 0$.

Lean, comparison. Lean proves nonvanishing for the same imprimitive twists in `TwistedRatio` (L128-L142). Its functional equation compares absolute values on a real interval. The signed calculation above uses the local theorem of Fröhlich and Queyrut at nonsplit finite places. `FunctionalEquation` (L362-L378). □

Proposition 18.2 (The exact functional-equation factor at the point one)**.** *Use* $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$ *and let* d_E *be the absolute discriminant of* E*. Define the fixed positive constant*

$$\kappa_p = |d_E|^{1/2} N_* \prod_{\sigma\in\Phi} \frac{c_\sigma}{2\pi}. \tag{18.2}$$

For every primitive twist of modulus Q*, including the unramified characters at* $Q = \varnothing$*,*

$$L_Q(0,\tau) = \kappa_p\, NQ\, L_Q(1,\tau). \tag{18.3}$$

Both sides are nonzero. In particular

$$\frac{L_Q(0,\tau)L_Q(1,\tau)}{L_\varnothing(0,1)L_\varnothing(1,1)} = NQ\left(\frac{L_Q(1,\tau)}{L_\varnothing(1,1)}\right)^2. \tag{18.4}$$

Proof. For a primitive twist all omitted places are genuinely ramified. Their local L-factors are identically one, so $L_Q(s,\tau)$ is the primitive finite L-function, also at $s = 0$. Tate's completed function in these conventions is

$$\Lambda(s,\chi_\tau) = \left(|d_E| N\mathfrak{f}(\chi_\tau)\right)^{s/2} \prod_{\sigma\in\Phi} \Gamma_{\mathbb{C}}(s + c_\sigma) L(s,\chi_\tau).$$

Its functional equation is $\Lambda(s,\chi_\tau) = \Lambda(1-s,\chi_\tau^{-1})$ by Lemma 18.1. Conjugation permutes the finite primes and the infinite places, so $L(s,\chi_\tau^{-1}) = L(s,\chi_\tau)$; the gamma products also agree. These identities initially hold in the convergent half-plane and then by continuation.

Evaluate the functional equation at $s = 0$. There is no archimedean pole, because every $c_\sigma \geq 1$. The ratio of the gamma factors is

$$\prod_\sigma \frac{\Gamma_{\mathbb{C}}(1 + c_\sigma)}{\Gamma_{\mathbb{C}}(c_\sigma)} = \prod_\sigma \frac{c_\sigma}{2\pi},$$

and the conductor square root is $|d_E|^{1/2} N_* NQ$. This proves (18.3). Nonvanishing follows from the positivity at 1 and $\kappa_p > 0$. Divide by the identity at the trivial twist to obtain (18.4). In particular no unramified factor $1 - 1$ has been suppressed: the hypothesis that the original pair is ramified at every chosen place is used precisely here. The analytic background is the same Tate functional equation cited in Lemma 12.7; the computation retains its discriminant, conductor, and gamma factors explicitly.

Lean, formal context. `TwistedRatio` (L173-L189), `FitTwistsExist` (L377). □

Theorem 18.3 (The unique normalized analytic values at every labelled modulus)**.** *For every allowed* Q *and every tame character with that labelled modulus, put*

$$G_Q(\tau) = \begin{cases} \dfrac{L_Q(z,\tau)}{L_\varnothing(z,1)}, & z \geq 2, \\[2ex] \dfrac{L_Q(0,\tau)L_Q(1,\tau)}{L_\varnothing(0,1)L_\varnothing(1,1)}, & z = 1. \end{cases} \tag{18.5}$$

Then $G_\varnothing(1) = 1$. It satisfies the exact Euler rule (13.48), with exponents $\{z\}$ for $z \geq 2$ and $\{0, 1\}$ for $z = 1$. At primitive twists it satisfies

$$G_Q(\tau) = (NQ)^m R_{p,1}(\tau)^{m+1}, \qquad m = \begin{cases} 0, & z \geq 2, \\ 1, & z = 1. \end{cases} \tag{18.6}$$

Conversely, every family satisfying primitive interpolation and the Euler rule of §13.5 satisfies

$$F_Q(\tau) = F_\varnothing(1) G_Q(\tau) \tag{18.7}$$

at all labelled moduli. This conclusion does not assume integrality or the existence of a theta lift.

Proof. All denominators in (18.5) are nonzero by the preceding results. If a place u is inserted without changing the character, put $a = (\nu\tau)^E(\varpi_u)$. The two newly removed Euler factors give, for any argument s in question,

$$L_{Q\cup\{u\}}(s, \tau) = L_Q(s, \tau)(1 - aq_u^{-s})(1 - a^{-1}q_u^{-s}).$$

This holds at $s = 0$ by continuation of the identity from the convergent half-plane; it does not divide by either factor. Applying it to one value or the product of two values proves the stated Euler rule. Positivity removes the absolute values in the ratio for $z \geq 2$. For $z = 1$, use (18.4) and positivity at 1.

For uniqueness, first relabel τ by $R(\tau)$. It is a primitive twist at that modulus, including when the modulus is empty. Interpolation determines its value there. Insert the remaining places of $Q \setminus R(\tau)$ one at a time. Both proposed families satisfy the same Euler rule, so their values agree after each insertion. The factors commute, and no nonzero-factor hypothesis is needed. In particular a padded modulus can have $G_Q(\tau) = 0$ at $z = 1$ when a newly omitted factor at zero is $1 - a = 0$.
Lean, formal context. `TwistedRatio` (L173-L189), `FitTwistsExist` (L377). □

The common period. Equation (18.7) leaves one scalar to determine. For $z \geq 2$ it is the value at the trivial twist of the integral family in Theorem 18.33 (`integralAlgebraicPart` in Lean). The formula for the common theta scale expresses the family as a constant times $L_Q(z, \tau)$, independently of Q and τ. At $z = 1$, let g_* be the product of group orders in (14.36) with $h = rt$. Suppose $N \in \mathbb{C}^\times$ and $C_p \in \mathbb{R}^\times$ satisfy the determinant-period comparison of `ClassSums`,

$$F_Q(\tau) = |g_* N \Lambda_Q(\tau)|^2, \qquad C_p|\Lambda_Q(\tau)|^2 = N_* NQ\, L_Q(1, \tau)^2$$

at primitive twists. Then the precise analytic product scale is

$$F_Q(\tau) = \frac{g_*^2 |N|^2 N_*}{C_p \kappa_p} L_Q(0, \tau) L_Q(1, \tau). \tag{18.8}$$

The Euler rules extend this identity to padded moduli. This identifies the scale if the geometric period theorem is proved; it does not prove the integrality of that period. An alternative construction of a measure may have another common nonzero scalar, since (18.7) and the cancellation in §13.5 retain exactly the same normalized special-value ratios.

Lemma 18.4 (Fourier coefficients at a fixed modulus)**.** *Let T be the norm-one torus and let $A_Q = T(F)\backslash T(\mathbb{A})/(T(F \otimes \mathbb{R})K_Q)$, where K_Q consists of integral norm-one ideles congruent to one at the places of Q. This finite ray class group has as its characters the weight-zero tame characters of labelled modulus Q. For $h \in A_Q$, put*

$$b_Q(h) = |A_Q|^{-1} \sum_{\tau \in \widehat{A}_Q} G_Q(\tau)\tau(h)^{-1}. \tag{18.9}$$

The group-ring element with character values G_Q is uniquely $\sum_h b_Q(h)[h]$. Write integralAway(NQ) *for the set of complex numbers integral at every embedding into $\mathbb{C}_\ell$ for every prime $\ell \nmid NQ$. For $C \in \mathbb{C}$, there is a sum of torus points with coefficients integral away from NQ and character values CG_Q if and only if*

$$Cb_Q(h) \in \text{integralAway}(NQ) \qquad (h \in A_Q). \tag{18.10}$$

The same equivalence holds for integrality at a single embedding.

Proof. Finiteness is the class-number theorem for tori. Triviality on the archimedean group and on K_Q is precisely the stated condition on the characters. Orthogonality gives Fourier inversion (18.9). Collecting the coefficients of any sum of torus points by their classes then gives $Cb_Q(h)$. Conversely, take one representative of each class with coefficient $Cb_Q(h)$.

Lean, comparison. Lean uses representatives on the torus. Fourier inversion here collects them in the finite quotient. `AlgebraicPartIntegrality` (L54-L64). □

Theorem 18.5 (A uniform bound for the Fourier coefficients)**.** *Fix ℓ and the coefficient isomorphism $\iota : \mathbb{C} \simeq \overline{\mathbb{Q}}_\ell$ of §13.5. Suppose that for all auxiliary moduli avoiding the bad places for ℓ and all $h \in A_Q$ one has*

$$v_\ell(\iota(b_Q(h))) \geq -B \tag{18.11}$$

with one constant B, where $v_\ell(\ell) = 1$. For any integer $M \geq \max(0, B)$, the family

$$F_Q(\tau) = \ell^M G_Q(\tau) \tag{18.12}$$

is integral at ι and satisfies the interpolation and Euler relations of §13.5, with base value ℓ^M. Conversely, the existence of such an integral family with any nonzero base value implies (18.11).

Proof. The coefficients $\ell^M b_Q(h)$ are integral at ι by (18.11). Lemma 18.4 realizes (18.12) at every modulus. The scalar is independent of Q, so Theorem 18.3 gives the primitive values and the Euler rule at padded moduli, including when an inserted factor vanishes. Conversely, the same theorem identifies any such family with CG_Q, where $C = F_\varnothing(1) \neq 0$. The integrality of $Cb_Q(h)$ gives (18.11) with $B = v_\ell(\iota(C))$.

Lean, comparison. Lean constructs a family integral at every coefficient prime away from the modulus. Here one power of ℓ clears all levels at the fixed embedding. `ClassSums` (L58-L77), `ClassSums` (L358-L385). □

Lemma 18.6 (Products and adjoints in the finite class group)**.** *For $U = \sum_h u_h[h] \in \mathbb{C}[A_Q]$, put $\|U\|_\iota = \max_h |\iota(u_h)|_\ell$, $U^* = \sum_h \bar{u}_h[h^{-1}]$, and $\widehat{\iota} = \iota \circ {}^-$. For $U, V \in \mathbb{C}[A_Q]$ and $\tau \in \widehat{A}_Q$,*

$$\|UV\|_\iota \leq \|U\|_\iota \|V\|_\iota, \qquad \|U^*\|_\iota = \|U\|_{\widehat{\iota}}, \qquad \tau(UU^*) = |\tau(U)|^2.$$

Proof. The coefficient of $[h]$ in UV is $\sum_{ab=h} u_a v_b$; apply the ultrametric inequality. Inversion permutes A_Q, and $\tau(h^{-1}) = \overline{\tau(h)}$, which gives the two identities.

Lean, comparison. Lean expands the squared norm as a double sum over representatives. The product above groups its terms by their class in A_Q. `AlgebraicPartIntegrality` (L66-L84). □

Corollary 18.7 (The fixed-prime special-value bound)**.** *If* (18.11) *holds for a critical pair at (ℓ, ι), Theorem 13.15 holds for that pair and embedding with allowance $2v_\ell(1-b)+c/(m+1) \leq 4v_\ell(q_w-1)+c$.*

Proof. Apply the group-ring construction of Lemmas 13.12 and 13.13 to (18.12). It uses integrality only at ι, for moduli prime to ℓ, and gives

$$A_0 = \ell^M, \qquad \beta = M + 2|\mathcal{I}|v_\ell(b-1), \qquad \psi_\xi(\Theta_Q) = \ell^M (NQ)^m \iota(R_{p,\delta}(\xi))^{m+1}.$$

The conductor factor $(NQ)^m$ is an ℓ-adic unit. If $v_+(\psi_\xi(\Theta_Q)) \leq \beta + c$, Lemma 13.14 cancels $v_\ell(A_0) = M$, giving

$$(m+1)v_+\big(\iota(R_{p,\delta}(\xi))\big) \leq 2|\mathcal{I}|v_\ell(b-1) + c \leq 4v_\ell(q_w - 1) + c.$$

Lean. `LValueSystems` (L185-L230), `LValueSystems` (L367-L407). □

Uniformity in the auxiliary conductor. The normalized values at every modulus use the same base value. Corollary 18.36 obtains it from the rank-one lift for $z \geq 2$. At $z = 1$ the determinant-period comparison of Theorem 18.45 supplies it.

For comparison, Kings and Sprang prove an integrality theorem for smoothed partial Hecke values of critical infinity type [KS25, Definition 4.1 and Corollary 4.13]. For the finite conductor $\mathfrak{f}_0$ of the initial character over E, that comparison applies at primes $\ell \nmid |d_E| N\mathfrak{f}_0$. Here the twist must be chosen at every coefficient prime. The theta kernels give the required coefficients at all these primes, including the primitive conductor factor at $z = 1$ in Corollary 18.47.

18.1 Finite gluing and the exact denominator at the level prime

We expand the finite part of the short lift and calculate the denominator of its Fourier operator at the level prime N. Both calculations are homogeneous in the vacuum scalar. Integrality of the normalized CM jets for this scalar follows from Corollary 18.32 and Theorem 19.4.

Write $m = 2h$, $n_j = k + j(t + 1)$, and $f = [E : \mathbb{Q}]/2$. The level prime N splits completely in E. Let S be the primes dividing π_N, where $\pi_N\overline{\pi_N} = N$; hence S contains one member of each conjugate pair above N and $|S| = f$. A second such set, denoted Σ, will specify a Schrödinger model. In the N-adic application Σ consists of the conjugates of the primes induced by the chosen finite embedding and the CM type. Put

$$R_\Sigma = \prod_{w\in\Sigma} \mathbb{F}_N, \qquad O_\Sigma = \prod_{w\in\Sigma} \mathcal{O}_{E_w}.$$

An overline on an integral local matrix in the formulas below means residue reduction. The bar on a place, on π_N, or on an adelic pole vector means the CM conjugation. Use the self-dual additive character of the fixed Weil data. Its restriction at these places has conductor O_Σ. For residues $c \in R_\Sigma$, our sign convention is

$$\psi_N(c) = \psi_0(-\widetilde{c}/N).$$

The lift uses the Haar measure of volume 1 on its specified source level. The finite residue sums below are ordinary sums.

Lemma 18.8 (The local gluing function, with its scalar)**.** *Let $T : E^a \longrightarrow E^b$ and $R : E^b \longrightarrow E^c$, where the source of T and target of R have distinguished last basis vectors. Glue their standard local vacua by*

$$\mathcal{G}_S = \sum_{y\in(\mathbb{F}_N^b)^S} H_T(-\widetilde{y}/N \text{ in the last column at } S) \otimes H_R(-\overline{\widetilde{y}/N} \text{ in the last row at } \bar{S}).$$

Here H is the normalized finite Heisenberg action. In the model at Σ, set

$$T' = T\, d_{a,S}, \qquad R' = d_{c,\Sigma\setminus S} R,$$

where $d_{a,A}$ is $\mathrm{diag}(1, \dots, 1, N)$ *at $\Sigma \cap A$ and 1 elsewhere. The resulting local function is exactly*

$$\mathbf{1}_{T',R' \text{ integral}}\, \psi_N\big((\overline{R'T'})_{c,a}\big) = \sum_{z\in R_\Sigma^b} P_z(T')\, C_z(R'), \tag{18.13}$$

where

$$P_z(U) = \mathbf{1}_{U \text{ integral},\ \overline{U}e_a = z}, \qquad C_z(V) = \mathbf{1}_{V \text{ integral}}\, \psi_N(e_c^t \overline{V} z).$$

There is no factor N^{-fb} in this identity.

Proof. It suffices to compute at a pair of conjugate places. At $w \in \Sigma \cap S$, a residue y contributes only if the last column of T minus $\widetilde{y}/N$ is integral; the last row of R must be integral. Such a residue exists precisely when $Td_{a,S}$ and R are integral, and is then unique, with y the reduction of NTe_a. Its Heisenberg phase is $\psi_0(-e_c^t R\widetilde{y}/N)$. Modulo O_{E_w} this is $\psi_0(-e_c^t RTe_a)$.

At $w \in \Sigma \setminus S$, the two roles are exchanged: T is integral and the last row of R has a possible simple pole. The unique residue is the reduction of $Ne_c^t R$. The same phase $\psi_0(-e_c^t RTe_a)$ results. All other rows and columns are integral in both cases. Since the last corner of $R'T'$ equals $Ne_c^t RTe_a$, these are exactly the phase and support on the left of (18.13).

No half-pairing contributes a further phase: each translation has only one of its two isotropic coordinates, and a vector supported on S has zero product with its conjugate. Multiplying these calculations over the f pairs proves the first equality. For integral T', exactly one summand on the right has $z = \overline{T'}e_a$; for nonintegral T' every summand vanishes. This proves the second equality, including its scalar.

Lean. `sum_translate_eq_gluedDatum`, `doubleLiftGluing_eq_sum_schrodingerOpAt`, `gluedDatum_eq_sum`. □

The same identity holds with arbitrary outside Schwartz factors, after first projecting each local factor to its vacuum. Indeed the local invariant space is one-dimensional, so a projected factor is a scalar times its standard vacuum. Both sides of (18.13) have the same scalar and the same outside factors. This also shows that replacing a vacuum by the project's normalized integral vacuum does not change the gluing coefficient.

Lemma 18.9 (Projection removes the gluing)**.** *Let v_j be a translated integral vacuum whose translation is integral above N. If $P_{N,j}$ denotes projection to the vacuum at all places above N, then*

$$(P_{N,2i} \otimes P_{N,2i+1})\mathcal{G}_S(v_{2i} \otimes v_{2i+1}) = v_{2i} \otimes v_{2i+1}.$$

Moreover these projectors commute with the actions of the integral source and target levels. Consequently, if a glued chain tensor is jointly invariant under these levels, its unglued tensor is jointly invariant too.

Proof. Normalize the Haar measure of the integral Heisenberg lattice to mass 1. Its average is the vacuum projector. For a translated vacuum $H(z)v_0$, the integral Heisenberg lattice acts by the character given by pairing with z. Character orthogonality and self-duality of the lattice give

$$P_N H(z)v_0 = \begin{cases} H(z)v_0, & z \text{ integral}, \\ 0, & z \text{ not integral}. \end{cases}$$

This statement is unchanged if v_0 is multiplied by any fixed scalar. In the gluing sum the first last-column translation is integral above N only for $y = 0$. Thus precisely the zero residue survives, and its coefficient is 1. Integral changes of bases preserve the Heisenberg lattice and its probability measure, so they commute with its projector. Apply the tensor product of these commuting projectors to the joint-invariance equation.

Lean. `vacuumProjFinPart_gluedDoubleLiftDatum`, `IsChainDatum.isInvariant_plainTensor`. □

Theorem 18.10 (Exact expansion and coefficient bound)**.** *Let Ψ be a chain datum as in `Columns.IsChainDatum`. Thus its support consists of translation tuples x supported above Mq and with $\overline{\pi_M}x$ integral away from q, where $(Mq, N) = 1$. Its coefficients are integral away from q; when the twist is trivial they satisfy, for every $\sigma \in \mathrm{Aut}(\mathbb{C})$,*

$$\sum_x |\sigma\Psi(x)| \le M^{3fmn_m^2}.$$

There is an explicit coefficient family Ψ' such that the glued tensor for Ψ equals the plain tensor for Ψ'. Its translations have possible new denominators only above N. For the trivial twist, and more generally whenever the displayed coefficient bound for Ψ holds,

$$\sum_z |\sigma\Psi'(z)| \le M^{3fmn_m^2} N^{f\sum_{i<h} n_{2i+1}} = M^{3fmn_m^2} N^{f(hk+(t+1)h^2)} \le M^{3fmn_m^2} N^{fmn_m/2}. \tag{18.14}$$

Every coefficient of Ψ' is integral away from q. For the constructed scales the right side is at most $\exp(x^{60})$ for all sufficiently large x. The corresponding composite lifts are identical, with the same final restriction and with no additional normalization scalar.

Proof. For each $i < h$ choose $y_i \in (\mathbb{F}_N^{n_{2i+1}})^S$. Let $z(y)$ be the tuple with the gluing pole-column in slot $2i$, the conjugate pole-row in slot $2i+1$, and zero elsewhere. Because $(Mq, N) = 1$, the original x vanishes above N, whereas $z(y)$ is supported above N. The two Heisenberg translations therefore have disjoint support, so their cocycle is 1. Distributing the tensor products of the gluing sums gives the exact identity

$$\mathrm{Glue}\left(\sum_x \Psi(x) \bigotimes_{j<m} H(x_j)v_j\right) = \sum_x \sum_y \Psi(x) \bigotimes_{j<m} H(x_j + z_j(y))v_j.$$

Define

$$\Psi'(u) = \sum_{(x,y):\, x+z(y)=u} \Psi(x).$$

This is a finite sum. Its coefficients remain in every valuation ring in which the original coefficients lie. The triangle inequality after σ gives

$$\sum_u |\sigma\Psi'(u)| \le \#\{y\} \sum_x |\sigma\Psi(x)|.$$

There are exactly $\prod_{i<h} N^{fn_{2i+1}}$ choices of y. The equality and inequalities in (18.14) follow from $\sum_{i<h}(2i+1) = h^2$ and $n_{2i+1} \le n_m$. Grouping coincident translations can only improve the bound.

To compare the exponents, use the bounds $f \le x^{13}$, $n_m \le x^6$, $m \le 2\sqrt{x}$, $\log M \le 22 \log b$, $\log N \le 25m \log b$, and $\log b = x^{o(1)}$ by (5.1). Thus eventually $m, \log M, \log N \le x$, and the logarithm of the last bound in (18.14) is at most

$$3x^{27} + \tfrac{1}{2}x^{21} \le x^{60}.$$

Joint invariance is inherited from the equality of the tensors. Applying the linear chain-of-lifts map and then the original restriction proves the assertion about lifts. Every occurrence of the integral vacuum has been kept; no conversion between normalized and unnormalized vacua has been made.

Lean. `card_gluingClasses_le`, `IsChainDatum.gluedTensor_eq_sum_translates`, `IsChainDatum.exists_isWeakChainDatum_compositeLift_eq`. □

This proves the chain-expansion assertion following (11.15). The existence of a chain datum realizing the fixed-character seesaw identity (11.11) is a different assertion. The theorem uses exactly the coefficient conditions in the definition of that datum. The subsection on page 565 now constructs the finite evaluated tensor and proves those coefficient bounds. Theorems 23.38 and 23.39 supply the analytic realization required by Proposition 23.5.

Lemma 18.11 (Finite class sums introduce no further denominator)**.** *Suppose the tensor of data for a chain is jointly invariant under levels K_j, $j < m$, the source form is K_0-invariant, and*

$\Gamma_j \cap gK_jg^{-1} = 1$ *for all* $j < m$ *and* g*. For the Haar normalization* $\mathrm{vol}(K_j) = 1$*, the value of the chain is the ordinary sum, over tuples of the finite double-coset sets* $\Gamma_j\backslash G_j/K_j$*, of the corresponding products of kernel maps. This remains true for a jointly invariant linear combination even when its individual tensor summands are not invariant.*

Proof. The measure of the image of gK_j in $\Gamma_j\backslash G_j$ is $\mathrm{vol}(K_j)/|\Gamma_j \cap gK_jg^{-1}| = 1$. Joint invariance makes the combined integrand constant on these cells: moving the integration variable at slot j transfers its level action to the two adjacent Schwartz factors, which cancel on the full tensor. Integrate one variable at a time. At each step the integral is the ordinary sum of its cell values, proving the assertion by induction on the chain length. Linearity permits taking the finite coefficient sum before integration. In particular, one does not assign a separate level invariance or a separate averaging factor to an individual summand.

Lean. `Kernel.lift`, `IsJointlyInvariant.sum_chain_eq_sum_kernelProduct`. □

The hypotheses hold for the upper-chain levels: their integral rational stabilizers are finite and lie in principal congruence subgroups at the two distinct chosen residue characteristics, so their orders are powers of both and hence are 1. This is the same freeness argument used above for the class count.

Lemma 18.12 (A sharp Fourier denominator for the direct section)**.** *For matrices* $T \in M_{b,a}(O_\Sigma)$ *let*

$$D_{b,a}(T) = \mathbf{1}_{\overline{T}e_a = e_b},$$

and extend this function by zero off integral matrices. Then

$$D_{b,a}(T) = \frac{\mathbf{1}_{T \text{ integral}}}{N^{fb}} \sum_{\beta \in R_\Sigma^b} \psi_N(-\beta^t e_b)\, \psi_N(\beta^t \overline{T} e_a). \tag{18.15}$$

Consequently its insertion in a Schwartz factor is a combination of N^{fb} *pure modulation translates of the projected vacuum, each coefficient* N^{-fb} *times a root of unity. All conjugates of the coefficient family have absolute sum* 1*. The translations have only simple poles at the conjugates of* Σ*, in the dual coordinates to the last column. Multiplication by* N^{fb} *makes all coefficients integral at every finite embedding.*

Proof. For $v \in R_\Sigma^b$, character orthogonality gives

$$\sum_{\beta \in R_\Sigma^b} \psi_N(\beta^t v) = \begin{cases} N^{fb}, & v = 0, \\ 0, & v \neq 0. \end{cases}$$

The pairing is perfect at every component because ψ_0 has conductor $\mathcal{O}_{E_w}$. Apply this with $v = \overline{T}e_a - e_b$. The modulation with value $\psi_N(\beta^t \overline{T} e_a)$ on an integral matrix is the pure Heisenberg translation with dual coordinate $-\widetilde{\beta}/N$; its other isotropic coordinate is zero. It is supported at $\bar{\Sigma}$ in the adelic description. There is no half-pairing in this pure modulation. There are N^{fb} terms, all their phase factors are roots of unity, and every automorphism of $\mathbb{C}$ preserves their absolute value 1. The denominator and norm assertions follow. The statement for an arbitrary Schwartz factor follows by applying the local insertion map to the displayed function.

Lean. Only the b entries in the last column enter the character sum, giving the denominator N^{fb}. The formal proof expands a function of all ab entries of the matrix, with denominator N^{2fab}, in `schrodingerOpAt_directDatum_mem_raises`. □

The finite lift laws. For a representation ρ_a of $\mathrm{GL}_a(\prod_\Sigma E_w)$ put $d_a = \mathrm{diag}(1,\dots,1,N)$ and $D_a = d_a^{-1}$. If $y = (u,c) \in R_\Sigma^{a-1} \oplus R_\Sigma$ has unit last coordinate, choose an integral lift of

$$k_y = \begin{pmatrix} I + uc^{-1}u^t & u \\ u^t & c \end{pmatrix}.$$

The displayed residue matrix has determinant c, and the lift's last row and last column are y modulo N. More generally, any integral invertible k_y with these two reduction properties may be used in every identity below. In the application take the choices in the original definition of U_N. Define the unnormalized operator

$$U_a = \rho_a(D_a) \sum_{\substack{y \in R_\Sigma^a \\ y_a \in R_\Sigma^\times}} \psi_N(y_a)\rho_a(k_y).$$

A vector is called ramified here if, at each $w \in \Sigma$, it lies in the image of $1 - \rho_a(u_w I_a)$ for some integral unit scalar u_w which is 1 modulo N away from w. The zero-average condition on the unit scalars implies this condition: on the complement of the invariant subspace of a finite cyclic group, 1 minus a generator is invertible.

Let $L(f) : V_a \longrightarrow V_b$ be linear in a Schwartz function of matrices. Its two lift laws are

$$\rho_b(k)L(f(k\,\cdot\,g))\rho_a(g) = L(f) \qquad (g,k \text{ integral invertible}), \tag{18.16}$$

$$\rho_b(D_b)L(R_{b,a}) = \gamma L(D_{b,a})\rho_a(D_a), \tag{18.17}$$

where $R_{b,a}$ is the indicator that the last row is e_a^t modulo N, extended by zero.

Theorem 18.13 (Ramified gluing is two direct lifts after U)**.** *Suppose $L : V_a \to V_b$ and $L' : V_b \to V_c$ satisfy these laws with scalars γ, γ'. For every ramified v,*

$$U_b L(D_{b,a})v = \gamma L(D_{b,a})U_a v, \tag{18.18}$$

$$\rho_c(D_c) \sum_{y \in R_\Sigma^b} L'(C_y)L(P_y)v = (N^b - 1)^f \gamma'\gamma L'(D_{c,b})L(D_{b,a})U_a v. \tag{18.19}$$

Direct lifts and the operators U preserve the ramified subspaces.

Proof. Integral scalar matrices pass through $L(f)$ by (18.16), since the source and target scalar actions leave the matrix variable unchanged when used together. Thus $L(f)$ preserves ramified vectors. Scalars also commute with U_a, proving the other preservation assertion.

A datum invariant under the scalar units at w annihilates a ramified vector: apply the integral law to $v = (1 - \rho_a(u_w I_a))v_0$. It follows that $L(P_y)v = 0$ if y vanishes at one component of Σ. The same argument shows that, on a ramified vector, the mirror datum

$$M_{b,a}(T) = \mathbf{1}_{T \text{ integral}}\psi_N(\overline{T}_{b,a})$$

may be restricted to matrices whose corner entry is a unit at every component. Decompose it first by its last column. For y_{last} a unit, integral equivariance and the last column of k_y give $L(P_y)v = \rho_b(k_y)L(D_{b,a})v$. Consequently

$$\rho_b(D_b)L(M_{b,a})v = U_b L(D_{b,a})v.$$

Decompose the same truncated mirror datum by its last row. The last-row property of k_y and (18.17) instead give

$$\rho_b(D_b)L(M_{b,a})v = \gamma L(D_{b,a})U_a v.$$

This proves (18.18).

In the sum in (18.19) only vectors nonzero at every component survive. The group $\mathrm{GL}_b(R_\Sigma)$ acts transitively on them. For $y = ke_b$, the two integral equivariance identities give

$$L(P_y)v = \rho_b(k)L(D_{b,a})v, \qquad L'(C_y)\rho_b(k) = L'(C_{e_b}).$$

Hence every surviving term equals $L'(M_{c,b})L(D_{b,a})v$. There are exactly $(N^b-1)^f$ surviving vectors. Apply the mirror identity for L' and then (18.18) for L. This proves (18.19), including the direction of its counting factor.

Lean. `mem_ramified`, `mirror_eq_upOperator_direct`, `mirror_eq_direct_upOperator`, `glued_eq_direct_direct_upOperator`. □

Corollary 18.14 (The exact scalar for a whole glued chain)**.** *Let L_j obey the two lift laws with scalar γ_j. Suppose the ith glued double lift is*

$$G_i = \eta_i \rho_{2i+2}(D_{n_{2i+2}}) \sum_{y \in R_\Sigma^{n_{2i+1}}} L_{2i+1}(C_y)L_{2i}(P_y).$$

For a ramified source vector,

$$G_{h-1}\cdots G_0 v = c_h\, L_{m-1}(D)\cdots L_0(D)\, U_{n_0}^h v, \qquad c_h = \prod_{i<h}\left[\eta_i (N^{n_{2i+1}}-1)^f \prod_{j<2i+2}\gamma_j\right]. \tag{18.20}$$

Proof. The double-lift formula proves the first step. At step i, move the newly occurring $U_{n_{2i}}$ through the preceding $2i$ direct lifts by (18.18); its scalar is $\prod_{j<2i}\gamma_j$. Multiplying with the two factors from the current double lift gives $\prod_{j<2i+2}\gamma_j$. The preservation assertions in Theorem 18.13 justify every iteration.

Lean. `compUpTo_gluedLift_eq_smul`, `liftInPairs_gluedDoubleLiftDatum`, `doubleLiftChain_gluedDoubleLiftDatum_eq_smul_chain`. □

Splitting characters and Haar measures. For the upper chain all n_j are even. Write $\chi_i = (D.\mathrm{chainChar}\,\vartheta\, i)^{\mathrm{BC}}$. The rank-twisted splitting character at stage j therefore has space character exactly χ_{j+1}. At N these characters are unramified: N divides neither the old chosen level M nor the twisting modulus q. For a local matrix g put

$$\mathcal{C}_{i,r}(g) = \chi_i(\det g)|\det g|^{-r/2}.$$

In the split Schrödinger model the pair action is

$$\omega_j(g,k)F(T) = \frac{\mathcal{C}_{j+1,n_j}(k)}{\mathcal{C}_{j+1,n_{j+1}}(g)}F(k^{-1}Tg).$$

This follows from the linear change of variable on M_{n_{j+1},n_j}, with its square-root Jacobian, and the two specified determinant characters. The theta integral is equivariant by right translation; its Haar measure is unchanged. Thus the lift satisfies the integral law and the diagonal law. For the latter, use the entrywise identity

$$R_{b,a}(D_b^{-1}TD_a) = D_{b,a}(T).$$

The off-corner last row is multiplied by N, the off-corner last column by N^{-1}, and the corner is unchanged.

It follows in particular that

$$\gamma_j = \frac{\mathcal{C}_{j+1,n_{j+1}}(d_{n_j})}{\mathcal{C}_{j+1,n_j}(d_{n_{j+1}})} = N^{\,f(n_{j+1}-n_j)/2}. \tag{18.21}$$

The two determinant ideles are identical, so the character values cancel exactly. The exponent is an integer because $n_{j+1} - n_j = t + 1$ is even. This verifies the normalization independently of any estimate for a CM period or jet.

The integral bases used at N are $\delta_j = d_{n_j,S}$ at Σ; the local group acts on forms through conjugation by δ_j. Applying Lemma 18.8 and moving its two diagonal matrices through the two theta lifts gives the above G_i, with

$$\eta_i = \mathcal{C}_{2i+2,n_{2i+1}}(d_{n_{2i+2},\,\Sigma\setminus S})\mathcal{C}_{2i+1,n_{2i+1}}(d_{n_{2i},\,S}). \tag{18.22}$$

The two changes of basis on the middle group cancel. This is an equality of linear maps for the original Schwartz data, since all the transformations are linear in those data. The relevant definitions and normalizations are `Columns`, lines 130–140 and 280–369, `CharactersAtLevel`, lines 37–64, and `ValuationAtLevel`, lines 222–447. The finite argument itself is `LiftLawsModuloN`, lines 611–937.

Corollary 18.15 (Uniform N-adic bound for the gluing scalar)**.** *For every allowed twist and finite embedding above N, the scalar in* (18.20) *is nonzero and satisfies*

$$|c_h|_N \le N^{mfb}.$$

The primitive-vector factors are N-adic units, and every γ_j is an integer of N-adic norm at most 1.

Proof. The assertion about γ_j follows from (18.21); also $N \nmid N^{n_{2i+1}} - 1$. For η_i, each place of Σ occurs in exactly one of the two factors in (18.22). The modulus factor contributes a nonnegative integral power of N and thus has N-adic norm at most 1.

Let $e_{i,\tau}$ be the lift weight at stage i and embedding τ. It satisfies $e_{i,\tau} \le 2b$ for $i \le 2m$. The prescribed infinity type of χ_i, evaluated at the local determinant N, bounds its N-adic norm by one factor $N^{e_{i,\tau}}$ for each induced embedding whose conjugate is in the chosen subset of Σ. This is the elementary algebraic-character formula obtained by passing from the finite idele to its unitary quotient $x/\bar{x}$; at the induced embedding the selected determinant is N^{-1}. One can verify the valuation formula by raising the ideal to its class-group order, making it principal, and applying the prescribed infinity type and the unit character. Taking that positive root of the resulting absolute-value identity introduces no valuation loss. Summing over the disjoint subsets of Σ gives $|\eta_i|_N \le N^{2fb}$. There are $h = m/2$ such factors. All character values and γ_j are nonzero, as are the primitive-vector counts, so $c_h \ne 0$.

Lean. `norm_gluedFactor_le`, `doubleLiftChain_gluedDoubleLiftDatum_eq_smul_chain`. □

The normalized premise at N. The preceding results separate the finite denominator from the normalized-jet assertion. To state the latter precisely, fix the embedding above N. Let $\beta = (1 + \sqrt{-d})/2 \in \mathcal{O}_K$ and let c be its reduction at the induced prime of K. Then

$$\delta = \beta - c$$

is divisible by every induced prime of E above N, and $\bar{\delta}d$ is an N-adic unit: the reduction of $\bar{\delta}$ is $1 - 2c$, whose square is $-d \ne 0 \bmod N$. The pure modulation vectors in Lemma 18.12 are therefore cleared by δ.

The normalized-jet assertion used in the next corollary is the following: for every $j < m$, every allowed twist, every standard datum in the valuation-ring span of

$$H(z)\,\mathrm{intVacuum}_j, \qquad \delta z \text{ integral above } N,$$

and every pair of finite adelic points, the original normalized Schur-jet kernel sends an integral source value to an integral target value in the corresponding Weyl gauges. Call this assertion (J_N). It includes the project's fixed vacuum and jet-scale constants. Corollary 18.32 proves this assertion from the original integral-vacuum clause, supplied in Theorem 19.4; its proof includes the CM curve, period, and ordinary-jet identifications that are not supplied by finite Fourier orthogonality.

Corollary 18.16 (The level-prime valuation reduced to (J_N))**.** *Assume (J_N). For any chain datum Ψ and any ramified source form ϕ for which $U_{n_0}^{m/2}\phi$ is again a K_0-invariant source form and has integral values for the lattice L_0, the chain before its final restriction has gauge, in the lattice L_m, at most*

$$N^{\,mfb+f\sum_{j<m} n_{j+1}} \le N^{2mfb}. \tag{18.23}$$

For the fixed integral branching matrix used in the definition of restriction, with its displayed diagonal-character intertwining property, the same bound holds after restriction to Y_m. Thus (J_N) suffices for the valuation component at N of (11.8).

Proof. Lemma 18.12 writes each direct datum as $N^{-fn_{j+1}}$ times a sum of standard data with integral coefficients. The original translations are integral above N, and their vacuum projections are either zero or the same translated vacuum; adding the modulation poles still gives δ-integral translations. Under (J_N) a direct kernel therefore increases the N-adic gauge by at most $N^{fn_{j+1}}$. Since $n_{j+1} \le n_m \le b$, the increase is at most N^{fb}.

The direct insertion operators commute with the levels in bases of L_j: these are marked-lattice stabilizers, and $B^{-1}TAe_a = e_b$ modulo N is equivalent to $Te_a = e_b$ when $Ae_a = e_a$ and $Be_b = e_b$. Lemma 18.9 therefore supplies the joint invariance needed for Lemma 18.11. The coefficients $\Psi(x)$ are integral at N, since $(q, N) = 1$. The stipulated source-level condition on $U_{n_0}^{m/2}\phi$ is the other hypothesis of that class-sum lemma; both it and the gauge condition are included in the already defined module $\mathcal{A}' = \mathrm{upIntegralForms}$. The ultrametric triangle inequality for that plain class sum accumulates no coefficient or class-count loss. Multiply the direct-kernel bounds and then apply (18.20) and Corollary 18.15. This proves (18.23).

For completeness, the final contraction is a matrix calculation and uses no archimedean norm bound on the branching matrix. Write B for that integer matrix and b_τ for its diagonal-character exponents. Its defining intertwining identity is

$$B\,\rho_X(\mathrm{diag}(g_\tau, s_\tau)) = \left(\prod_\tau s_\tau^{b_\tau}\right)\rho_Y(g_\tau)B.$$

The scalar in the restriction is $a = \prod_\tau \tau(\bar\pi_N)^{-b_\tau}$. In the N-adic gauge the last coordinate of δ_m^{-1} is $s_\tau = N^{-1}$ when the conjugate induced place divides π_N, and $s_\tau = 1$ otherwise. Because $\pi_N\bar\pi_N = N$ and its valuation at each prime is 0 or 1,

$$|a|_N = \left|\prod_\tau s_\tau^{b_\tau}\right|_N.$$

The intertwining identity cancels these two gauge factors. All entries of B are integral, so the maximum coordinate norm cannot increase. This proves the claim after restriction. The two properties used here are precisely the integer target type and the first clause of `Columns.IsBranchingMap`; no assertion about the size of B at infinity has been used.

Lean. `schrodingerOpAtN_directDatum_mem_raises`, `gaugeLe_compositeLift_at_N`. □

Integrality at the level prime. Corollary 18.32 proves the integrality condition (J_N) under the integral-vacuum hypothesis. Theorem 19.4 establishes that hypothesis, and Corollary 20.38 verifies the condition on the source forms. The corresponding Lean deduction of (J_N) is `ThetaLiftGauge`, `mem_raises_of_standardData` applied to the normalized CM jets of §18.3. For the elementary-divisor estimate (11.17), see Theorem 20.10.

18.2 The normalized CM curve has good reduction at every prime

The next calculation supplies the integral Weierstrass coefficients needed for the CM-jet argument. It concerns the particular normalization used by the source, not an arbitrary good model of an isogenous curve. Let d be a prime with $d \equiv 7 \pmod 8$, put $K = \mathbb{Q}(\sqrt{-d})$, and write

$$\beta = (1+\sqrt{-d})/2, \qquad \bar\beta = 1-\beta.$$

For a lattice $\Lambda \subset \mathbb{C}$ stable under $\mathcal{O}_K$, choose the nonzero point $t \in \frac12\Lambda/\Lambda$ for which both βt and $\bar\beta t$ are nonzero. Such a point is unique: $\mathcal{O}_K/(2) = \mathbb{F}_2 \times \mathbb{F}_2$, the module $\frac12\Lambda/\Lambda$ is free of rank one over this ring, and t has both components nonzero. Scale the lattice so that $\wp''_\Lambda(t) = 2$, and choose a_1 with $a_1^2 = 12\wp_\Lambda(t)$. This scaling exists because the nonzero half-periods give simple roots of the Weierstrass cubic.

The associated equation, in coordinates $X = \wp(z) - \wp(t)$ and $Y = (\wp'(z) - a_1X)/2$, is

$$Y^2 + a_1XY = X^3 + X, \qquad \Delta = a_1^4 - 64. \tag{18.24}$$

Indeed $g_2 = 12\wp(t)^2 - 4$, $g_3 = -8\wp(t)^3 + 4\wp(t)$, and substitution proves the equation. The usual integral Weierstrass discriminant of this equation is exactly Δ. The proof below shows that its image has absolute value one under every coefficient embedding, without assuming an ordinary reduction theorem for a theta expansion.

Theorem 18.17 (The normalized CM curve has unit discriminant)**.** *For the above K, distinguished half-period and normalization, every prime ℓ and every coefficient embedding $\iota : \mathbb{C} \longrightarrow \mathbb{C}_\ell$ satisfy*

$$|\iota(a_1^4 - 64)|_\ell = 1. \tag{18.25}$$

In particular $\iota(a_1)$ is integral, and (18.24) is a smooth Weierstrass model over the valuation ring at every prime, including 2 and d.

Proof. Put $e = \wp_\Lambda(t)$. Completing the square gives $y^2 = x(x^2 + Ax + 1)$, with $A = 3e$, $t = (0,0)$ and $dx/(2y) = dz$. Put $D' = A^2 - 4$. The map $(x,y) \mapsto (x + A + 1/x,\ y(1 - 1/x^2))$ is the quotient by $\langle t\rangle$: substitution gives $y'^2 = x'(x'^2 - 2Ax' + D')$, the map has degree two, and $dx'/(2y')$ pulls back to dz. The respective discriminants are $16D'$ and $256D'^2$. Their ratio is $16D' = a_1^4 - 64$, since $a_1^2 = 12e$.

Set $D = -d$. Up to homothety the lattice is $L = \mathbb{Z} + \tau\mathbb{Z}$, with $\tau = (-b + \sqrt{D})/(2a)$ for a primitive positive form $[a,b,c]$ of discriminant D [Cox13, Theorem 7.7]. We may take a odd: one of $(1,0), (0,1), (1,1)$ has odd value and extends to an integral basis of determinant one. Since $b^2 - D = 4ac$ is divisible by eight, c is even. Multiplication by $a\tau \in \mathcal{O}_K$ sends $1/2$ to $\tau/2$ modulo L. Since 2 splits in K, the two factors of $\mathcal{O}_K/(2) \simeq \mathbb{F}_2 \times \mathbb{F}_2$ give the two stable lines in the two-torsion of the CM ideal lattice. The third line, $\langle 1/2\rangle$, is therefore the marked subgroup.

The quotient has lattice $\frac12(\mathbb{Z} + 2\tau\mathbb{Z})$ with the same differential. The discriminant has weight twelve under a change of differential, so the common homothety cancels. Writing η for Dedekind's eta function and using $\Delta(L) = (2\pi)^{12}\eta(\tau)^{24}$ [DLMF, (27.14.16)], we obtain

$$a_1^4 - 64 = 2^{12}\frac{\eta(2\tau)^{24}}{\eta(\tau)^{24}} = \mathfrak{f}_2(\tau)^{24}, \qquad \mathfrak{f}_2(\tau) = \sqrt{2}\,\eta(2\tau)/\eta(\tau).$$

Here D is fundamental, $D \equiv 1 \bmod 8$, and $3 \nmid D$. In the odd-a, even-c case the Weber class invariant is $\mathfrak{f}_2(\tau)$ times a root of unity, and it is an algebraic unit [LYY25, Proposition 1.1 and Remark 1.3][YYY21, Theorem 1.1(4)]. Thus the displayed discriminant is a unit at every finite place. Adding 64 shows that a_1^4, and hence a_1, is integral there. Equation (18.24) is therefore smooth.

Lean, comparison. Lean multiplies the two-isogeny discriminant identities over all ideal classes. Here the Weber unit theorem evaluates the finite valuations of the marked quotient directly. See `GoodReduction` (L388-L499), `GoodReduction` (L726-L807) and `TorsionValues` (L65). □

Corollary 18.18 (The doubling map has good reduction, also at two)**.** *For the curve* (18.24)*, the map on its ordinary projective coordinate* $x_0(z) = \wp(z) - \wp(t)$ *induced by doubling is*

$$D(X : Y) = ((X^2 - Y^2)^2 : XY(4X^2 + a_1^2XY + 4Y^2)). \tag{18.26}$$

Its two homogeneous polynomials have no common projective zero over any residue field. At every coefficient embedding and for every nonzero vector w*, with the maximum norm on two coordinates,*

$$\|D(w)\|_\ell = \|w\|_\ell^4. \tag{18.27}$$

In particular reduction commutes with this map at 2 *as well as at odd primes. The interpolation coordinate used below is instead* $x(z) = \wp(z + t) - \wp(t) = x_0(z)^{-1}$*; thus the pair* $(\theta[u]^2, \sigma[u]^2)$ *represents* x_0*, consistently with this formula.*

Proof. The ordinary duplication formula for a Weierstrass equation with $b_2 = a_1^2$, $b_4 = 2$, $b_6 = 0$, $b_8 = -1$ is

$$x([2]P) = \frac{(x(P)^2 - 1)^2}{4x(P)^3 + a_1^2x(P)^2 + 4x(P)},$$

which homogenizes to (18.26). All coefficients are integral by Theorem 18.17. If both polynomials vanished on a projective point over a residue field, the first would give $X = Y$ or $X = -Y$ with $Y \neq 0$. The second would then give $a_1^2 = -8$ or $a_1^2 = 8$, respectively. Either equation implies $a_1^4 - 64 = 0$, contradicting the unit discriminant. This proof includes residue characteristic two; equivalently, there the two polynomials reduce to $(X + Y)^4$ and $a_1^2X^2Y^2$, with a_1 a unit.

For any nonzero vector w, divide by a coordinate of maximal absolute value. The normalized vector is integral and its reduction is nonzero. The absence of a common zero shows that at least one output coordinate is a unit. Homogeneity of degree four proves (18.27).

Lean. `Doubling` (L232), `Doubling` (L312). □

These are ordinary proofs of the exact identities used in `Lib/CM/GoodReduction` and the duplication calculation in the CM integrality files. They supply the integral model and the good-reduction input for the torsion interpolation argument. The theta-section and jet identifications require their own normalization checks below.

18.3 Ordinary CM jets and the exact normalization of standard data

The distinction between an ordinary derivative and a Taylor coefficient is essential here. If $F = \sum_\alpha F_\alpha Z^\alpha$, put

$$J_Q(F) = (Q(\partial)F)(0), \qquad J_{Z^\alpha}(F) = \alpha! F_\alpha.$$

The result below bounds J_Q. It does not assert that all F_α are integral. In particular, it makes no ordinarity assumption.

We use the CM curve and the distinguished half-period of Theorem 18.17. Thus

$$K=\mathbb{Q}(\sqrt{-d}),\qquad d\equiv 7 \pmod 8 \text{ prime},\qquad \beta=(1+\sqrt{-d})/2,$$
$$x(z)=\wp(z+t)-\wp(t),$$
$$y(z)=\{\wp'(z+t)-a_1x(z)\}/2,$$

where $2t\in\Lambda$, neither βt nor $\bar\beta t$ belongs to Λ, $\wp''(t)=2$, and $a_1^2=12\wp(t)$. Write $A=\operatorname{area}(\Lambda)$ and

$$H(u,z)=\pi\bar u z/A,\qquad E(u,z)=\operatorname{Im}(\bar u z)/A,\qquad T_u^{(m)}f(z)=e^{-mH(u,z)-mH(u,u)/2}f(z+u).$$

The project's sigma function is the odd reduced sigma function with $\sigma'(0)=1$; multiplying the usual Weierstrass sigma by its normalizing exponential quadratic preserves the addition and duplication identities. The theta function is

$$\theta(z)=\frac{T_t^{(1)}\sigma(z)}{T_t^{(1)}\sigma(0)},\qquad \theta(0)=1.$$

For a degree-m function write

$$f[u]=T_u^{(m)}f(0)=e^{-m\pi|u|^2/(2A)}f(u).$$

Fix any prime p, any field embedding $\iota:\mathbb{C}\longrightarrow\mathbb{C}_p$, and the valuation ring $\mathcal{O}\subset\mathbb{C}_p$. All formal series below have their complex Taylor coefficients read through ι. Formal substitution is coefficientwise; it does not assert convergence on the whole p-adic unit disc.

Lemma 18.19 (Exact theta duplication and integral torsion values)**.** *With this normalization,*

$$\theta(2z)=\theta(z)^4-\sigma(z)^4,\qquad \sigma(2z)=x'(z)\theta(z)^4,$$

where the second equality is interpreted away from $t+\Lambda$. Moreover

$$\sigma(2z)^2=\theta(z)^2\sigma(z)^2\big(4\theta(z)^4+a_1^2\theta(z)^2\sigma(z)^2+4\sigma(z)^4\big).$$

For every torsion point $u\in\mathbb{Q}\Lambda/\Lambda$,

$$\max\big(|\iota\theta[u]|_p,|\iota\sigma[u]|_p\big)=1. \tag{18.28}$$

The entire section $Y=y\theta^3$ also has integral degree-three torsion values.

Proof. The sigma addition formula

$$\frac{\sigma(u+v)\sigma(u-v)}{\sigma(u)^2\sigma(v)^2}=\wp(v)-\wp(u)$$

and its limit on the diagonal give $\sigma(2z)=-\wp'(z)\sigma(z)^4$. The exponential modification above preserves these identities. See DLMF 23.10.3 and its duplication limit. They imply

$$x\theta^2=\sigma^2,\quad \theta(z+w)\theta(z-w)=\theta(z)^2\{\theta(w)^2-x(z)\sigma(w)^2\}.$$

Putting $w=z$ proves the first duplication formula.

For the second formula use $x(z)=1/(\wp(z)-\wp(t))$ from Corollary 18.18. Differentiation gives

$$x'(z)\theta(z)^4=-\wp'(z)x(z)^2\theta(z)^4=-\wp'(z)\sigma(z)^4=\sigma(2z).$$

Squaring and substituting $x'^2 = x(4x^2 + a_1^2 x + 4)$ gives the asserted entire identity by continuation.

Put $w(u) = (\iota\theta[u]^2, \iota\sigma[u]^2)$. The Gaussian in each value at $2u$ is the fourth power of the one at u, so $w(2u) = D(w(u))$ and $\|w(2u)\| = \|w(u)\|^4$ by (18.27). The two zeros lie in disjoint cosets, so this norm is positive. At a torsion point, period shifts contribute a sign and a root of unity $e^{\pm\pi i E(u,l)}$, since $E(u,l) \in \mathbb{Q}$. The norm thus takes only finitely many values on the doubling orbit. Its successive fourth powers can repeat only if it equals one.

Multiplication of the cubic equation by θ^6 gives

$$Y^2 + a_1\theta\sigma^2 Y = \sigma^6 + \theta^4\sigma^2.$$

The same identity holds for the degree-respecting torsion values. Its coefficients are integral by the preceding result and Theorem 18.17; a root of this monic quadratic is integral. This proves the last assertion.

Lean, comparison. Lean derives the second duplication identity by translating by the half-period and evaluating its fourth-power multiplier. Here differentiation of the reciprocal coordinate gives the same scalar. `Doubling` (L94-L171), `TorsionValues` (L373), `TorsionValues` (L407). □

Lemma 18.20 (Integral theta sections on the reduced cubic)**.** *Let f be a holomorphic theta section of degree $m \geq 0$ for the multiplier of θ^m. If $\iota f[u]$ is integral at every torsion point, then, away from $t + \Lambda$,*

$$f = \theta^m\{P_0(x) + yP_1(x)\}, \qquad P_0, P_1 \in \iota^{-1}(\mathcal{O})[X],$$

with $2\deg P_0 \leq m$ and $2\deg P_1 + 3 \leq m$ for nonzero polynomials. For $m \geq 1$ it suffices to test m points of order less than $2(m+2)$, chosen independently of p, ι and f. For $m = 0$ it suffices to test $f(0)$. In particular, also at $p = 2$, the formal expansion satisfies

$$f = \theta^m R(y), \qquad R \in \mathcal{O}[[Y]]. \tag{18.29}$$

Proof. The quotient $F = f/\theta^m$ is elliptic with its only possible pole at t, of order at most m. On both the complex and the reduced curve the monomials $x^i, x^i y$, $i \geq 0$, have pole orders $2i, 2i+3$ there. Those of order at most m form a basis because their pole orders are distinct and cancellation of principal parts leaves at most a simple pole, which is impossible on a genus-one curve. This gives the stated expression over $\mathbb{C}$. For $m = 0$, $F = f(0)$.

Suppose $m \geq 1$. Take the least power of two $n \geq m+2$ and a halving point P with $(n/2)P = t$. The cyclic group generated by P has order $n < 2(m+2)$. It specializes injectively at every prime since its unique element of order two, $t = (0,0)$, remains distinct from the origin on the smooth model (18.24). Choose m distinct points in this group, omitting t, whose sum differs from mt. After choosing $m-1$, at least two choices remain and at most one gives the forbidden sum. These properties persist on the special fibre.

If $u = 0$ is chosen, then $\theta[0] = 1$ and $x(0) = y(0) = 0$. Every other chosen u reduces away from the origin and $\bar{t}$. Its ordinary coordinate x_0 is therefore finite and nonzero on the special fibre, since $(0,0)$ is the sole point with $x_0 = 0$. The normalized pair $(\iota\theta[u]^2 : \iota\sigma[u]^2)$ represents $x_0(u)$ and has maximum norm one by (18.28). Hence $\iota\theta[u]$ is a unit, and $x(u), y(u)$ are integral under ι by Lemma 18.19. The m basis monomials give a square integral evaluation matrix. If its reduction were singular, a nonzero function on the special fibre would have divisor $\sum_u [\bar{u}] - m[\bar{t}]$, by counting its zeros and poles. The sum of the zeros differs from $m\bar{t}$, contrary to the criterion for a divisor of degree zero to be principal [Sil09, Corollary III.3.5]. The matrix and its inverse are therefore integral. The maximum norm of the coefficients equals that of the values $F(u) = f[u]/\theta[u]^m$, which proves the assertion. Fewer than m evaluations have a kernel since the space has dimension m.

Finally $x^3 + (1 - a_1 y)x - y^2 = 0$ and $x(0) = y(0) = 0$. The derivative of the left side with respect to x at the origin is one. Comparing coefficients gives a unique $X(Y) \in Y\mathcal{O}[[Y]]$ with $x(z) = X(y(z))$, proving (18.29).

Lean, comparison. Lean compares the values at opposite halving points and interpolates the two polynomials separately. Here the elliptic divisor criterion gives one invertible evaluation matrix. `IntegralSections` (L346-L435), `IntegralSections` (L529). □

Lemma 18.21 (Isogeny factors in the theta line bundle)**.** *Let C, C' be two such normalized CM curves, with lattices Λ, Λ'. If $a\Lambda \subseteq \Lambda'$ and the isogeny $z \mapsto az$ has degree M, then*

$$\theta_{C'}(az) = \theta_C(z)^M R(y_C(z)), \qquad R \in \mathcal{O}[[Y]], \quad R(0) = 1. \tag{18.30}$$

Moreover $y_{C'}(az) \in \mathcal{O}[[y_C]]$ and $|\iota a|_p \le 1$. For $\delta \in \mathcal{O}_K \setminus \{0\}$, $M = \delta\bar{\delta}$, and $\delta v \in \Lambda$, one also has

$$(T_v^{(1)}\theta)(\bar{\delta}z) = \theta(z)^M R_v(y(z)), \qquad R_v \in \mathcal{O}[[Y]]. \tag{18.31}$$

Proof. We first check the multiplier; a degree computation by itself would not suffice. For the distinguished half-period, the semicharacter is

$$\chi(l) = \exp\{\pi i E(\bar{\beta}l, l)\} = (-1)^{|l|^2\sqrt{d}/(2A)} \quad (l \in \Lambda).$$

The exponent is an integer because $\bar{\beta}l, l \in \Lambda$. One can check the first equality on $\Lambda/2\Lambda$: $\mathcal{O}_K/2\mathcal{O}_K = \mathbb{F}_2 \times \mathbb{F}_2$; the two idempotent lines are given by $\beta, \bar{\beta}$. The distinguished t has nonzero component on both lines. In an adapted symplectic basis the multiplier of $T_t\sigma/(T_t\sigma)(0)$ is $(-1)^{ab}$, which is also $(-1)^{E(\bar{\beta}l,l)}$. Both satisfy the same semicharacter identity, so this calculation determines them on all of Λ. Since $|a|^2 A = MA'$, the target exponent at al is M times the source exponent. Thus $\chi_{C'}(al) = \chi_C(l)^M$.

It follows that $\theta_{C'}(az)$ is a degree-M section for χ_C^M. Its degree-M torsion value at u equals $\theta_{C'}[au]$, which is integral by Lemma 18.19. Lemma 18.20 proves (18.30). The constant term is one.

Apply the same argument to $y_{C'}\theta_{C'}^3$. It gives $R_3 \in \mathcal{O}[[y_C]]$ with

$$y_{C'}(az)\theta_{C'}(az)^3 = \theta_C(z)^{3M} R_3(y_C(z)).$$

The factor R above is a unit of $\mathcal{O}[[Y]]$, so $y_{C'}(az) = R_3/R^3 \in \mathcal{O}[[y_C]]$. Our normalization gives $y_C'(0) = y_{C'}'(0) = 1$; comparison of linear coefficients therefore proves $\iota a \in \mathcal{O}$.

For the translate, the exact composition law is

$$T_u^{(m)} T_v^{(m)} = e^{\pi i m E(u,v)} T_{u+v}^{(m)}.$$

Because $E(\bar{\delta}l, v) = E(l, \delta v) \in \mathbb{Z}$, translation by $\bar{\delta}l$ commutes with $T_v^{(1)}$. Hence $(T_v^{(1)}\theta)(\bar{\delta}z)$ has multiplier $\chi(l)^M$, with no additional character. The point v is torsion, and its translated torsion values differ from $\theta[u + v]$ only by roots of unity. Pullback preserves these norms. A second application of Lemma 18.20 proves (18.31).

Lean. `IsogenyIntegrality` (L116), `IsogenyIntegrality` (L127), `IsogenyIntegrality` (L143). □

Lemma 18.22 (The logarithmic denominator is absorbed by d)**.** *Let $g = -\log\theta \in z\mathbb{C}_p[[z]]$, with $g(0) = 0$. Then*

$$g'(0) = 0, \qquad g'' = g''(0) + x, \qquad \sqrt{-d}\,[z^2]g \in \mathcal{O}, \qquad d\,g'' \in \mathcal{O}[[y]]. \tag{18.32}$$

Also $\mathcal{O}[[y]]$ is preserved by d/dz.

Proof. Evenness of θ gives $g'(0) = 0$. Differentiating the theta addition identity twice in its second variable at zero gives

$$\theta'^2 = \theta\theta'' + \{-\theta''(0) + x\}\theta^2,$$

which is precisely $g'' = g''(0) + x$.

For $a = \beta$ and $a = \bar{\beta}$, the isogeny formula gives

$$R_a(y(z)) = \frac{\theta(az)}{\theta(z)^{a\bar{a}}} = \exp\{a\bar{a}\, g(z) - g(az)\} \in \mathcal{O}[[y]].$$

Its constant term is one and its linear term vanishes. Since $y = z + O(z^2)$, its coefficient of z^2 equals its coefficient of y^2, and is integral. This coefficient is

$$a(\bar{a} - a)[z^2]g.$$

Subtracting the two expressions, at $\bar{\beta}$ and β, gives

$$\{\bar{\beta}(\beta - \bar{\beta}) - \beta(\bar{\beta} - \beta)\}[z^2]g = \sqrt{-d}\,[z^2]g \in \mathcal{O}.$$

Multiplication by 2 and by the integral number $-\sqrt{-d}$ proves $d\, g''(0) \in \mathcal{O}$. Together with $x \in \mathcal{O}[[y]]$, this proves the last assertion of (18.32).

Finally the exact coordinate differential is

$$y' = 1 + 3x^2 - a_1 y.$$

For $F = R(y)$, with $R \in \mathcal{O}[[Y]]$, $F' = R'(y)(1 + 3x^2 - a_1 y)$. Ordinary differentiation of R multiplies coefficients by integers and hence preserves $\mathcal{O}[[Y]]$.

Lean. `LogTheta` (L197), `LogTheta` (L330). □

Theorem 18.23 (A derivative-stable ring proves all ordinary jets)**.** *Let I, X be finite sets, and let $\nu_i(Z) = \sum_{x \in X} \nu_{ix} Z_x$ have coefficients in $\mathcal{O}$. Set $\mu_i = d\nu_i$ and*

$$B_\mu = \{R((y(\mu_i(Z)))_{i \in I}) : R \in \mathcal{O}[[Y_i : i \in I]]\}.$$

Let S_μ be the $\mathcal{O}$-algebra generated by B_μ and the finitely many series $\theta(\mu_i(Z)), g'(\mu_i(Z))$. Then every partial derivative preserves S_μ, and evaluation at $Z = 0$ sends S_μ to $\mathcal{O}$. Consequently, for every $Q \in \mathcal{O}[Z_x : x \in X]$, every $m_i, N \geq 0$, and every $H \in B_\mu$,

$$J_Q\left(\left\{\prod_i \theta(\mu_i)^{m_i}\right\}^N H\right) \in \mathcal{O}. \tag{18.33}$$

The bound is independent of the number of factors, their multiplicities, and N.

Proof. The constants of the generators are respectively integral, one, and zero. This proves the constant-term assertion.

The chain rule and the previous lemma show $\partial_x B_\mu \subseteq B_\mu$, since every $\mu_{ix} = d\nu_{ix}$ is integral. On the additional generators,

$$\partial_x \theta(\mu_i) = -\mu_{ix}\theta(\mu_i)g'(\mu_i),$$
$$\partial_x g'(\mu_i) = \nu_{ix}\,(dg'')(\mu_i) \in B_\mu.$$

These are identities of formal series. The product rule therefore proves stability of the entire generated algebra. Iterate the partial derivatives, evaluate at zero, and form the finite integral linear combination prescribed by Q. This proves (18.33), without estimating a Taylor coefficient or dividing by a factorial.

Lean. `JetIntegrality` (L144), `JetIntegrality` (L161), `JetIntegrality` (L202). □

The local part of the period normalization. The next argument proves exactly the valuation needed below. It does not use or reprove the archimedean Chowla–Selberg identity.

Lemma 18.24 (The chosen Chowla–Selberg root is a local unit)**.** *Let C_c be normalized CM curves, indexed by the ideal class group of K, with*

$$\Lambda_c = \mu_c \mathfrak{c}_c \Lambda, \qquad \prod_c \mathfrak{c}_c = \mathcal{O}_K.$$

Then $\prod_c |\iota\mu_c|_p = 1$. Consequently every root

$$u_{\mathrm{CS}}^{12h_K} = \prod_c \mu_c^{12} \Delta(C_c), \qquad \Delta(C_c) = a_1(C_c)^4 - 64, \tag{18.34}$$

is a unit at every p-adic embedding. This is the finite-place assertion in chowlaSelbergUnit.

Proof. Use the invariant differentials to identify a homomorphism of complex elliptic curves with its differential multiplier. For lattices L, L' this identifies $\mathrm{Hom}(C_L, C_{L'})$ with $(L' : L) = \{a \in \mathbb{C} : aL \subseteq L'\}$. A CM lattice is a complex multiple of an invertible fractional $\mathcal{O}_K$-ideal. Thus for $L' = \mu IL$ the forward and reverse Hom modules are μI and $\mu^{-1} I^{-1}$.

Fix an additive valuation v induced by the coefficient embedding. Both modules are integral by Lemma 18.21. Writing $v(I) = \min_{a \in I \setminus \{0\}} v(a)$, we get $v(\mu) + v(I) = 0$ since $v(I^{-1}) = -v(I)$. Summing over c gives $\sum_c v(\mu_c) = 0$. The discriminants are units by Theorem 18.17, so (18.34) gives $v(u_{\mathrm{CS}}) = 0$.

To choose the representatives, observe that the field discriminant $-d$ is a single prime discriminant. Gauss's genus theorem gives $|\mathrm{Cl}(K)/\mathrm{Cl}(K)^2| = 2^{1-1} = 1$ [Cox13, Proposition 3.11 and Theorems 3.15, 5.30]. Squaring on the finite class group is consequently bijective. Represent each pair of distinct inverse classes by inverse fractional ideals and take $\mathcal{O}_K$ for the identity class.

Lean, comparison. Lean proves odd class number directly from the unique ramified prime and takes ideal representatives $\mathfrak{b}_{c^2}/\mathfrak{b}_c$. Here genus theory allows inverse classes to be paired. Both arguments use the product of period norms to obtain the local unit. `ChowlaSelberg` (L99-L110), `ChowlaSelberg` (L631-L642), `ChowlaSelberg` (L696-L715). □

For the fields write $q = q_1$, $E = K(\zeta_q)$, $f = [E : K]$, $\Phi = \mathrm{Hom}_K(E, \mathbb{C})$, and

$$\mathfrak{a}_0 = E \cap \alpha \widehat{\mathcal{O}}_E, \qquad \mathfrak{a}_0 \bar{\mathfrak{a}}_0 = \mathfrak{D}_{E/K}^{-1}.$$

Let Φ_q be the q-th cyclotomic polynomial. At primes dividing β, set $\alpha_w = \Phi_q'(\zeta_q)^{-1}$, and set $\alpha_w = 1$ elsewhere. The fields satisfy $d \nmid q$, $[E : K] = \varphi(q)$, and $\mathcal{O}_E = \mathcal{O}_K[\zeta_q]$. Indeed the discriminants of K and $\mathbb{Q}(\zeta_q)$ are coprime, so their rings of integers have an integrally closed tensor product and the two fields are disjoint. Every prime divisor of q divides $\beta\bar{\beta} = (d+1)/4$. These are the explicit cyclotomic and Weil data of the construction. Put

$$s_0^f = \chi_{\mathrm{fin}}(\alpha)^{-1}, \qquad P_{\mathrm{loc}} = s_0/u_{\mathrm{CS}}, \qquad A_0 = \sqrt{2\pi/\sqrt{d}}, \qquad j_0 = P_{\mathrm{loc}} A_0/\Omega_0, \tag{18.35}$$

where $\Lambda = \Omega_0 \mathcal{O}_K$. Thus $s_0, P_{\mathrm{loc}}, j_0$ are respectively vacuumRoot, periodFactor, jetScale; P_{loc} is not the vacuum scalar.

Lemma 18.25 (The period factor cancels the adelic lattice factor)**.** *For $\tau \in \Phi$, let $\phi_\tau = \iota \circ \bar{\tau}$ and extend it to the completion selected by that embedding. Then*

$$|\iota P_{\mathrm{loc}}|_p = |\phi_\tau(\alpha)|_p^{-1}. \tag{18.36}$$

In particular, if $\alpha^{-1}u_j$ is integral at every place above p, every coefficient of the linear form

$$\iota(P_{\mathrm{loc}})\,u_j^{\dagger}Z, \qquad u_j^{\dagger}Z=\sum_{\tau,c}\iota(\overline{\tau(u_{jc})})Z_{\tau,c},$$

is integral.

Proof. The algebraic finite character $\chi_{\mathrm{fin}}=v_{\mathrm{fin}}|\cdot|^{-1/2}$ has principal-idele law $\chi_{\mathrm{fin}}(a)=\prod_{\tau\in\Phi}\overline{\tau(a)}$; its values on finite unit ideles are roots of unity. For any finite idele x, choose a positive power $x^h=av$, with $a\in E^\times$ and v a unit idele. Taking absolute values of the principal-idele law and then h-th roots gives

$$|\iota\chi_{\mathrm{fin}}(x)|_p=\prod_{\tau\in\Phi}|\phi_\tau(x)|_p.$$

Only these elementary character properties are used.

For $x=\alpha$, the factors on the right are equal. The condition of lying above (β) is fixed by $\mathrm{Gal}(E/K)$, and the ideal generated by $\Phi_q'(\zeta_q)$ is the invariant different ideal. Thus its valuation is unchanged by the embeddings in Φ above the fixed embedding of K. The defining equation for s_0 therefore gives $|\iota s_0|_p=|\phi_\tau(\alpha)|_p^{-1}$. Lemma 18.24 gives $|\iota u_{\mathrm{CS}}|_p=1$, proving (18.36). The last assertion is just

$$|\iota(P_{\mathrm{loc}})\,\iota(\overline{\tau(u_{jc})})|_p=|\phi_\tau(\alpha^{-1}u_{jc})|_p\le 1.$$

Lean. `TightFrames` (L324), `TightFrames` (L332). □

Lemma 18.26 (A tight frame with prescribed unit length)**.** *For any finite set of rational primes $\mathcal{P}$, there are an integer $b\ge 1$ prime to every $p\in\mathcal{P}$ and at most $4f$ vectors $w_i\in\mathfrak{a}_0$, where $f=[E:K]$, such that*

$$\sum_i \tau(w_i)\overline{\sigma(w_i)}=b^2\delta_{\tau\sigma}\quad(\tau,\sigma\in\Phi). \tag{18.37}$$

For the fields of Proposition 5.2, there is also a frame satisfying (18.37) with b^2 replaced by an integer h prime to q. It has $O_{Q,d}(f)$ members and $h\ll_{Q,d}1$. One member generates $\mathfrak{a}_0$ locally at every prime above q.

Proof. The trace form $H(u,v)=\mathrm{Tr}_{E/K}(\bar{u}v)$ makes $\mathfrak{a}_0$ self-dual: its dual is $\bar{\mathfrak{a}}_0^{-1}\mathfrak{D}_{E/K}^{-1}=\mathfrak{a}_0$. Invert the rational integers prime to $\mathcal{P}$. At each $p\in\mathcal{P}$ the reduced form represents a nonzero scalar. At a split prime take vectors pairing to one in the two residue components. At an inert or ramified prime the residue characteristic is odd, since 2 splits in K. A nondegenerate Hermitian or symmetric form in odd characteristic cannot vanish on every diagonal: polarization (also after rescaling one vector in the Hermitian case) would make it zero.

The Chinese remainder theorem lifts these residue vectors to one vector of unit norm at all $p\in\mathcal{P}$. Orthogonal projection splits off its line integrally, so the complement is again self-dual. Repeating and clearing denominators prime to $\mathcal{P}$ gives an orthogonal K-basis $v_1,\dots,v_f\in\mathfrak{a}_0$. Its norms $H_i=H(v_i,v_i)$ are positive rational integers prime to every selected prime. For an empty set ordinary orthogonalization and clearing denominators suffice. Put $b=\mathrm{lcm}_i\,H_i$ and $V=(\tau(v_i))_{\tau,i}$. Then $V^*V=\mathrm{diag}(H_i)$. By Lagrange's four-square theorem [ConD, Theorem 6.6], write each b^2/H_i as a sum of four integer squares. The corresponding integer multiples of v_i have frame matrix $b^2V\,\mathrm{diag}(H_i^{-1})V^*=b^2I$.

For the second assertion put $E_0=K(\zeta_Q)$, $T=q/Q$ and $\zeta_Q=\zeta_q^T$. The same different identity makes the base lattice ideal $\mathfrak{a}(Q)$ self-dual, so the preceding argument gives a frame of square scalar prime to Q there. By the Chinese remainder theorem choose $w\in\mathfrak{a}(Q)$ nonzero modulo $\mathfrak{p}\mathfrak{a}(Q)$ at every $\mathfrak{p}\mid Q$. Adjoin the vectors $\zeta_Q^k\gamma(w)$, $0\le k<Q$, $\gamma\in\mathrm{Gal}(E_0/K)$. Summing over k kills the

off-diagonal entries, and summing over γ gives the same diagonal increment $Q\,\mathrm{Tr}_{E_0/K}(w\bar{w})$ at every embedding. The trace is an integer by self-duality. Thus the new scalar h_0 is still prime to Q, and the frame contains the local generator w.

Since q and Q have the same prime divisors, $[E : E_0] = \varphi(q)/\varphi(Q) = T$. The discriminants of K and $\mathbb{Q}(\zeta_q)$ are coprime, so $\mathcal{O}_E = \mathcal{O}_K[\zeta_q] = \mathcal{O}_{E_0}[\zeta_q]$. The minimal polynomial $X^T - \zeta_Q$ of ζ_q over E_0 therefore gives the relative different $(T\zeta_q^{T-1}) = (T)$ [Neu99, Was97]. For each $p^e \| T$ let $\mathfrak{p}$ be the prime of K above p dividing β, and put $P_T = \prod_{p^e \| T} \mathfrak{p}^e$. These primes split since $p \mid \beta\bar{\beta}$ and $\beta + \bar{\beta} = 1$, so $NP_T = T$. Taking the factors above β in the tower formula for the different gives

$$\mathfrak{a}_0 = P_T^{-1}\mathfrak{a}(Q)\mathcal{O}_E.$$

Choose in each ideal class of K an integral representative $\mathfrak{b}$ prime to Q, and take $c \in K^\times$ with $cP_T = \mathfrak{b}$ for the class of P_T. Thus $T|c|^2 = N\mathfrak{b}$. From each base frame member w_i take $c\zeta_q^j w_i$, $0 \le j < T$. They lie in $\mathfrak{a}_0$, and cw generates it at each prime above Q, since $\mathfrak{b}$ is a unit there. For embeddings with different restrictions to E_0 the sum over i is zero. For distinct extensions of the same embedding the sum over j is zero. The diagonal is $h = T|c|^2 h_0 = h_0 N\mathfrak{b}$, in a fixed finite set of integers prime to Q. The number of members is T times the fixed base count, hence $O_{Q,d}(f)$.

Lean, comparison. Lean constructs the trace-dual frame at the full conductor. Here orthogonalization gives the first assertion, and lifting a frame from E_0 bounds the scalar and preserves the local generator. `StandardDataProducts` (L217), `ShortVectors` (L440-L809). □

Choosing a second frame for the prime divisors of the first b gives two fixed frames, one of unit length at each prime.

Integrality of the scaled vacuum. Let v be the vacuum of the standard positive definite E-space of rank n, and put $v^\circ = c_{\mathrm{vac}}^n v$. For a finite adelic vector x, write $\mathrm{finVec}\, x = (0, \alpha x)$, where α is the finite idele of the Weil representation. The integrality hypothesis for the jet estimate requires that at every prime p and every embedding $\iota : \mathbb{C} \hookrightarrow \mathbb{C}_p$,

$$|\iota(v^\circ(\mathrm{finVec}\, x))|_p \le 1 \quad \text{for every finite adelic } x. \tag{IV}$$

This is the integrality condition in IsVacuumScale. Theorem 19.4 chooses one c_{vac} for which it holds and bounds the complex conjugates of the scaled vacuum values.

Write $H(x)$ for the finite signed Heisenberg operator, and set

$$\mathcal{S}_\delta = \mathrm{span}_{\iota^{-1}(\mathcal{O})}\{H(x)v^\circ : \delta x \text{ is integral above } p\}.$$

Finite coordinates are those of the source: $\mathrm{finiteImage}(\gamma) = \alpha^{-1}\gamma$. In a sum indexed by rational lattice vectors, the shorthand $\rho(\gamma)$, $H(\gamma)$ means the finite operator at $\mathrm{finiteImage}(\gamma)$. Evaluation of a datum is always at $\mathrm{finVec}(\mathrm{finiteImage}(\gamma))$; the two coordinate maps are kept in that order.

Lemma 18.27 (Fourier inversion for a fixed vacuum scalar)**.** *Let $b \ge 1$ and take a frame satisfying (18.37), without yet requiring $p \nmid b$ or (IV). There is a finite-part Weil datum F whose entire theta function is*

$$\Theta_F(Z) = \prod_{i,c} \theta\left(\frac{\Omega_0}{bA_0} \sum_{\tau \in \Phi} \overline{\tau(w_i)} Z_{\tau,c}\right), \qquad \Theta_F(0) = 1. \tag{18.38}$$

The Fourier identities (18.41) below also hold without (IV). If in addition $p \nmid b$ and (IV) holds, the fixed v° is a finite combination

$$v^\circ = \sum_j a_j \rho(y_j) F, \qquad \iota a_j \in \mathcal{O}, \quad y_j \in E^n, \quad by_j \in \mathfrak{a}_0^n. \tag{18.39}$$

In particular b finiteImage(y_j) *is integral; the operator* $\rho(y_j)$ *uses precisely that finite coordinate convention. For rational* $\gamma \in E^n$ *the exact translated series is*

$$\begin{aligned} \mathcal{J}_{cA_0/\Omega_0}(\rho(\text{finiteImage}\,\gamma)F) &= \prod_{i,k} (T^{(1)}_{v_{ik}}\theta)\left(c\,u_{ik}^{\dagger} Z\right), \\ v_{ik} &= -\frac{\Omega_0}{b}\,\mathrm{Tr}_{E/K}(\gamma_k \bar{w}_i), \\ u_{ik} &= \frac{w_i}{b}\,\mathbf{e}_k. \end{aligned} \tag{18.40}$$

Here $\mathcal{J}_a(f)$ *denotes the Taylor series of the source's entire theta function at* aZ*.*

Proof. The frame identity gives the exact identity of hermitian forms

$$\sum_{i,k} H\left(\frac{\Omega_0}{bA_0} w_i^{\dagger} Z', \frac{\Omega_0}{bA_0} w_i^{\dagger} Z\right) = \sum_{\tau,k} \bar{Z}'_{\tau,k} Z_{\tau,k}.$$

Indeed its scalar factor is $(\pi/A)|\Omega_0/(bA_0)|^2 = b^{-2}$. The trace pairing on $\mathfrak{a}_0$ is integral, so the individual forms take the real image of $b\mathfrak{a}_0^n$ to Λ. The canonical multiplier calculation in Lemma 18.21 shows that their product has exactly the signed theta transformation law of the Weil model on this lattice. There is no additional Gaussian: the displayed identity of hermitian forms cancels it.

Here are the finite inversion and its denominator explicitly. Put $L = \mathfrak{a}_0^n$, $g = fn$, and

$$M = [L : bL] = [b^{-1}L : L] = b^{2g}.$$

The commutator pairing identifies L/bL with the character group of $b^{-1}L/L$. Averaging signed translations over L/bL projects onto theta functions for L, a one-dimensional space spanned by Θ_v. For the self-duality here, the real image is $A_0(\tau\gamma)_{\tau\in\Phi}$, and its Riemann form on rational vectors is $\mathrm{Tr}_{E/\mathbb{Q}}(\bar{u}v/\sqrt{-d})$. The identities $\mathfrak{a}_0\bar{\mathfrak{a}}_0 = \mathfrak{D}_{E/K}^{-1}$ and $\mathfrak{D}_{K/\mathbb{Q}} = (\sqrt{-d})$ make its integral dual lattice exactly L. They therefore give the asserted perfect commutator pairing. The latter one-dimensional assertion also follows directly by a Fourier expansion in a symplectic basis of the self-dual lattice: the transformation law specifies all coefficients from one. For representatives $x \in b^{-1}L/L$, write the unnormalized average of the translate of Θ_F as $c_x\Theta_v$. Character orthogonality gives

$$c_x v = \sum_{l \in L/bL} H(l)\rho(-x)F, \qquad \sum_x c_x \rho(x) v = MF. \tag{18.41}$$

This proves existence of F as well: apply the same averaging to the explicit entire product and divide the second equality by M. The resulting finite sum of vacuum translates has exactly that entire theta function. Thus this use of theta realization introduces precisely the displayed M^{-1}. The passage from theta functions to vector identities is injective on the finite part: if the entire theta function is zero, the rational translation formula makes every rational finite-point value zero. Local constancy and density make all finite-point values zero, and these determine the datum.

Apply the theta functional to the second equality and use $\Theta_F(0) = 1$. It gives

$$\sum_x c_x v(\text{finVec}(\text{finiteImage}\,x)) = M.$$

Only now assume (IV) and $p \nmid b$. For the already fixed v°,

$$v^\circ = \sum_x \frac{v^\circ(\text{finVec}(\text{finiteImage}\,x))}{M}\, c_x v = \sum_{x,l} \frac{v^\circ(\text{finVec}(\text{finiteImage}\,x))}{M} H(l)\rho(-x)F.$$

Each $H(l)\rho(-x)$ is a root of unity times $\rho(l-x)$. The numerator is integral by (IV), and $M = b^{2g}$ is a p-adic unit. This proves (18.39).

Translation of a finite rational vector corresponds to negative real translation in the theta function. The real image of γ in these coordinates is $A_0(\tau\gamma_k)_{\tau,k}$. The same identity of hermitian forms therefore distributes the twisted translation over the factors of (18.38). Their shifts are exactly $-\Omega_0 \operatorname{Tr}_{E/K}(\gamma_k\bar{w}_i)/b$. Finally substitution of cA_0/Ω_0 cancels Ω_0/A_0 in each linear form, proving (18.40).

Lean. `StandardDataProducts` (L322), `StandardDataProducts` (L367). □

Lemma 18.28 (Standard data have an integral theta-product factorization)**.** *Assume (IV). Let $\delta \in \mathcal{O}_K \setminus \{0\}$, $N_\delta = \delta\bar{\delta}$, and $f_0 \in \mathcal{S}_\delta$. There are finitely many $u_j \in E^n$, integers $m_j \geq 0$, and a formal series R integral in the parameters $y(u_j^\dagger Z)$, such that*

$$\sum_j m_j\, u_j u_j^\dagger = 1, \qquad \alpha^{-1}u_j \text{ integral above } p,$$

and

$$\mathcal{J}_{\bar{\delta}A_0/\Omega_0}(f_0)^\iota = \left\{\prod_j \theta(u_j^\dagger Z)^{m_j}\right\}^{N_\delta} R. \tag{18.42}$$

The dagger forms in this equation are read through ι.

Proof. Expand v° by Lemma 18.27 and use the Heisenberg composition law. Thus it suffices to handle finitely many translates $\rho(x)F$, with integral coefficients and δx integral above p. Smoothness at the finite places and density of E^n in its finite adeles give a rational γ which acts on F exactly as x does and for which $\alpha^{-1}\delta\gamma$ remains integral above p. Choose an integer m, prime to p, with $m\delta\gamma_k \in \mathfrak{a}_0$ for every coordinate. This simply clears the finitely many denominators away from p; there is none to clear above p.

For each of the finitely many translates choose such an m. Choose a common positive k, prime to p, divisible by all the mb. The trace pairing on $\mathfrak{a}_0$ gives $k\delta v_{ik} \in \Lambda$ for all the shifts in (18.40). Replace the vectors w_i/b by $w_i/(bk)$, and assign them multiplicity k^2. The frame remains a resolution of the identity and its vectors remain integral above p. The linear argument $\bar{\delta}\, w_i^\dagger Z/b$ is now $\overline{k\delta}\, w_i^\dagger Z/(bk)$. Apply (18.31) with $k\delta$. Its degree is k^2N_δ, exactly the new multiplicity times N_δ. Taking the product proves the asserted factorization for each translate. The theta-product factor is common to all terms, so the integral linear combination is absorbed into R.

Lean. `StandardDataProducts` (L485). □

Theorem 18.29 (Integrality of the scaled jets for standard data)**.** *Assume the integral-value clause (IV) for the fixed original vacuum scalar. For every prime p, every $\delta \in \mathcal{O}_K \setminus \{0\}$, every $f_0 \in \mathcal{S}_\delta$, and every polynomial Q with coefficients in $\iota^{-1}(\mathcal{O})$,*

$$J_{Q^\iota}\big(\mathcal{J}_{\bar{\delta}\, d\, j_0}(f_0)^\iota\big) \in \mathcal{O}. \tag{18.43}$$

If Q is homogeneous of degree L, then for every $c \in \mathbb{C}$,

$$\big|J_{Q^\iota}\big(\mathcal{J}_{cj_0}(f_0)^\iota\big)\big|_p \leq \left|\iota\frac{c}{\bar{\delta}d}\right|_p^L. \tag{18.44}$$

These statements concern the original $j_0 = \mathrm{jetScale}$; they do not substitute a differently normalized period.

Proof. Use (18.42). By Lemma 18.25, the coefficients of $\nu_j = \iota(P_{\mathrm{loc}})u_j^\dagger$ are integral. Rescale every variable by $\iota(dP_{\mathrm{loc}})$. The theta forms become $\mu_j = d\nu_j$, and the integral parameter series becomes an element of B_μ. The scale on the left is exactly

$$(dP_{\mathrm{loc}})(\bar\delta A_0/\Omega_0) = \bar\delta\, d\, j_0.$$

Theorem 18.23 now gives (18.43). For a homogeneous polynomial, $J_Q(F(aZ)) = a^L J_Q(F)$; rescale by $a = c/(\bar\delta d)$ to obtain (18.44). This identity also covers $c = 0$, with the usual degree-zero interpretation.

To match the differential operator in the source, its definition is

$$[Z^\alpha]\mathcal{J}_a(f_0) = \frac{a^{|\alpha|}}{\alpha!}\Theta(v_{Z^\alpha} \otimes f_0).$$

Hence for homogeneous Q of degree L,

$$J_Q(\mathcal{J}_a(f_0)) = a^L \Theta(v_Q \otimes f_0).$$

The factorial cancels exactly. No further factorial, volume, or CM-period scalar is present in this last identification.

Lean. `Divisibility` (L57), `Divisibility` (L100). □

Corollary 18.30 (One bounded denominator for any fixed vacuum scalar)**.** *There is a nonzero scalar c_*, independent of p, such that $c_* v_1$, with v_1 the rank-one vacuum, has integral values at every finite point and every p-adic embedding. No bound on its complex conjugates is asserted.*

Consequently, for any fixed scalar c_{vac}, the statement of Theorem 18.29 for the rank-n vacuum $c_{\mathrm{vac}}^n v$ remains true with the right side of (18.44) multiplied by

$$|\iota(c_{\mathrm{vac}}/c_*)|_p^n. \tag{18.45}$$

For each fixed pair (p, ι), one denominator therefore bounds all these jets uniformly in the finite translates and their finite integral combinations. No rational integer denominator uniform over all primes and embeddings is asserted for an arbitrary complex c_{vac}. This does not give bound one for the prescribed scalar.

Proof. Use the powers-of-ζ_q frame, repeated q times, in rank one, and the explicit product F of (18.38). Its finite-point values are integral at every embedding: every finite translate acts on F as a rational translate, whose theta value is a product of torsion values in (18.40). This uses Lemma 18.19, not a denominator bound for an inverse Fourier transform.

Since $\Theta_F(0) = 1$, not every coefficient c_x in (18.41) is zero. Choose a nonzero one, and set $c_* = c_x$. The first equality in those equations expresses $c_* v_1$ as a finite sum of translates of F with root-of-unity coefficients. It has no factor M^{-1}. Its finite-point values are therefore integral simultaneously at all primes. This choice is made once over $\mathbb{C}$, independently of p.

The standard rank-n vacuum is the tensor product of the rank-one vacua. Thus $c_*^n v$ satisfies (IV). Every standard datum formed with $c_{\mathrm{vac}}^n v$ equals $(c_{\mathrm{vac}}/c_*)^n$ times the corresponding datum formed with $c_*^n v$. Apply Theorem 18.29 to the latter and use linearity.

Lean. `IntegralVacuum` (L228), `Divisibility` (L100). □

Lemma 18.31 (From ordinary jets to transported Weyl gauges)**.** *Consider the theta kernel between the standard positive definite ranks n, n', with fitting Weyl shape κ, its prescribed determinant twists, and scale cj_0. Let L be the degree of its polynomial Schur matrix ($L = \mathrm{boxes}(\kappa, n')$ in the source).*

Assume (IV) for the fixed vacuum on the underlying rank-nn' Weil space. For every $f_0 \in \mathcal{S}_\delta$, every finite pair g, h, and every source vector ξ,

$$\|K_{f_0}(g,h)\xi\|_{h,\mathrm{Weyl}} \le \left|\iota\frac{c}{\bar{\delta}d}\right|_p^L \|\xi\|_{g,\mathrm{Weyl}}. \tag{18.46}$$

The same statement for an arbitrary fixed vacuum scalar has the additional factor $|\iota(c_{\mathrm{vac}}/c_*)|_p^{nn'}$.

Proof. In the integral standard-tableau bases, every Schur matrix entry is an integral homogeneous polynomial of degree L in the matrix variables. This follows directly from the construction of Weyl modules using tensor and exterior powers and integral straightening: it is the matrix of the universal linear map over $\mathbb{Z}[Z]$. No normalized Young symmetrizer is used. Therefore Theorem 18.29 bounds every entry of the jet matrix by the displayed factor. The supremum norm of its action has the same bound, by the ultrametric inequality.

We spell out why this entrywise statement suffices for all finite points. First, standard data are preserved by Heisenberg translations whose δ-multiple is integral above p: composition adds the two supports and contributes only a root of unity. They are also preserved by a finite Weil point k integral above p. In the standard lattice model the exact vacuum formula is

$$\omega(k)H(x)v^\circ = \mathrm{finMultiplier}(k^{-1})^{-1}H(kx)\sum_{y\in L_f/(L_f\cap k^{-1}L_f)} H(ky)v^\circ, \qquad L_f = \widehat{\mathcal{O}}_E^{nn'}.$$

The quotient is finite; its local factor at every place above p is trivial. The multiplier has the exact definition $\mathrm{finMultiplier}(k) = \chi_{\mathrm{fin}}(\det u_k)$, where $u_k L_f = L_f \cap k^{-1}L_f$. At places above p this intersection is L_f, so $\det u_k$ is a unit there. The principal-idele norm calculation in Lemma 18.25 therefore gives $|\iota\mathrm{finMultiplier}(k)|_p = 1$, and also the same equality for k^{-1}. The vectors kx, ky have the required integrality above p. This proves stability, keeping the same v°. The two twisting characters contribute units on such integral points as well.

Second, rational points of these unitary groups approximate any prescribed finite collection of local points. For this specific group this is elementary weak approximation: the Cayley map from skew-hermitian matrices is a rational isomorphism onto a dense open subset of the unitary group; that open subset is dense in each local topology, and weak approximation on the underlying F-vector space supplies the required rational matrices. Thus choose rational γ, η so that γg and ηh are integral at every place above p. The datum translated by this integral pair remains standard by the first step.

Rational equivariance of the theta kernel moves γ, η to the corresponding algebraic Weyl actions on its two sides. The transported gauge is, by definition, the supremum norm after applying the inverse of that Weyl action at the finite point. It is unchanged by simultaneous rational translation of the point and vector, and by right multiplication by a p-integral point, whose Weyl matrix and inverse are integral. Hence both gauges reduce to the supremum norms at the identity, where the previously proved entrywise estimate applies. The determinant twists cancel in precisely these transported gauges; their values on the residual integral points are units. This proves (18.46). The last assertion follows from Corollary 18.30.

Lean. `pairWeil_mem_standardData`, `omega_mem_raises_of_schurJet_le`, `mem_raises_of_standardData`. □

Corollary 18.32 (Integrality of jets at the level prime)**.** *At the level prime N of §18.1, assume (IV) for the original scaled vacua used in the chain. Then the normalized-jet assertion (J_N) stated there holds.*

More explicitly, let $c_0 \in \mathbb{Z}$ lift the reduction of β at the chosen prime of K above N, and put $\delta = \beta - c_0$. Then $\delta \neq 0$, its valuations at the induced places are positive, and

$$|\iota(\bar{\delta}d)|_N = 1.$$

For every allowed twist and every standard datum with δ-cleared support, the original normalized Schur kernel has transported gauge bound one at all finite pairs of points.

Proof. The level-prime hypotheses give $N \nmid d$ and splitting in K. Since $\bar{\beta} = 1 - \beta$, the reduction of $\bar{\delta}$ is $1 - 2c_0$, whose square is $-d$ modulo N. Thus $\bar{\delta}d$ is a unit. The induced CM embeddings all restrict to this chosen embedding of K, so δ clears exactly the indicated poles. Apply Lemma 18.31 with the original scale j_0, hence $c = 1$. The displayed factor is one. The allowed twists affect only the unit factors already included in that lemma.

Lean. `mem_raises_of_standardData`, `projOn_translateDatum_mem_standardData`, `schrodingerOpAt_directDatum_mem_raises`. □

The period of the fixed vacuum. For the chosen vacuum scalar, (IV) implies (J_N). For an arbitrary fixed scalar, the construction with c_* gives a uniform denominator bound. The determinant estimate uses the former implication. Theorem 19.4 proves (IV) and bounds all complex conjugates of the scaled vacuum values. Theorem 19.6 gives the area formula for the same period root.

Lean references for the CM calculations. Duplication and the fourth-root scalar; Torsion integrality; Halving interpolation; Isogeny factors; The logarithmic quadratic coefficient; Ordinary-derivative ring; Local product of isogeny multipliers; The period root and its unit clause; Dual-basis frame; Vacuum expansion and the common clearing integer; Scale and ordinary-jet comparison; The original normalized jets; Transported Weyl gauges.

18.4 Bounded class-sum distributions at every coefficient prime

For a fixed scalar, Corollary 18.30 bounds the coefficients of the class-sum distributions uniformly in the auxiliary modulus. This bound suffices for the fixed-prime measure. For $z \geq 2$ the convergent rank-one norm calculation identifies the distribution with the critical values. For $z = 1$ we use the equal-rank period comparison of Theorem 18.45.

Fix a ramified critical pair $p = (\nu, z)$ as in §18. Put $f = [E : K] = [E : \mathbb{Q}]/2$, where $K = \mathbb{Q}(\sqrt{-d})$, and let M be the prescribed level integer. The group order g_* of (14.36), the vacuum scalar c_{vac} of (IV), and the jet scale j_0 of (18.35) are independent of Q. Since M lies in the product of the chosen prime ideals,

$$NQ\,M^a \in \mathfrak{m}_{Q,a}, \qquad \mathfrak{m}_{Q,a} = \left(\prod_{w \in S_{\mathrm{ch}}} \mathfrak{p}_w\right)^a \prod_{w \in Q} \mathfrak{p}_w. \tag{18.47}$$

Fix any prime ℓ, an embedding ι, and moduli with $\ell \nmid NQ$. Put $\widehat{\iota} = \iota \circ {}^-$ and

$$e_\iota = |\iota(c_{\mathrm{vac}}/c_*)|_\ell, \qquad e_{\widehat{\iota}} = |\widehat{\iota}(c_{\mathrm{vac}}/c_*)|_\ell,$$

where the single nonzero scalar c_* is from Corollary 18.30.

Theorem 18.33 (Uniform Fourier bounds for the two geometric families)**.** *At every fixed (ℓ, ι), each of the following families has finite Fourier coefficients with valuations bounded below by one constant independent of Q and of the coefficient index.*

For $z \geq 2$, put $n = 2z$, $L = \sum_\sigma (c_\sigma - z)$ and

$$D_p = M^{2fn}(Md)^L, \qquad T_Q(\tau) = \frac{g_* D_p^2}{(NQ)^{n-1}} \|\phi_{Q,\tau}\|^2.$$

Here $\phi_{Q,\tau}$ is the theta lift from the definite line to the standard definite rank-n space, multiplied by $c_{\mathrm{vac}}^{n}(\overline{\pi_M}\, j_0)^L$. The algebraic integer $\overline{\pi_M}$ is star(*D*.piMBar) *in Lean, and the resulting T is* integralAlgebraicPart.

For $z = 1$, put $L_1 = 2\sum_\sigma (c_\sigma - 1)$ and

$$C_1 = M^{16f} c_{\mathrm{vac}}^{4}(M^2 dj_0)^{L_1}, \qquad T_Q(\tau) = |g_* C_1 \Lambda_Q(\tau)|^2, \tag{18.48}$$

where $\Lambda_Q(\tau)$ is the rank-two determinant period of $\nu\tau\upsilon_{U(1)}$, with polynomial $\prod_\sigma \det^{c_\sigma - 1}$ and modulus $\mathfrak{m}_{Q,2}$. The displayed C_1 is the explicit modKernelScale *witness used in* ClassSums. *No assertion that $\Lambda_\varnothing(1) \neq 0$ is needed for this coefficient bound.*

Proof. By Lemma 18.4 it suffices to bound the coefficients of a group-ring element giving each family. For squared norms we use the product and adjoint of Lemma 18.6 at ι and $\widehat{\iota}$.

For a residue datum of integral values in Weil rank k, Fourier inversion on each coordinate has denominator dividing the order of the residue module. If $NQ\,\delta \in \mathfrak{m}$, the ℓ-part of the product of the denominators divides δ^{2fk}, the index of $\delta\widehat{\mathcal{O}}_E^k$, since $\ell \nmid NQ$. After multiplication by δ^{2fk} the datum is an integral combination of Heisenberg translates with δ-multiples integral above ℓ. The coefficients sum products of roots of unity and the original residue values.

For $z \geq 2$ take $(k, \delta) = (n, M)$. The column indicator at the old places and first-coordinate indicator at Q have integral values. Their degree is L. The factor $(Md)^L$ cancels the jet denominator in Lemma 18.31, giving the transported bound at every source point g and receiver point h,

$$\|D_p K_Q(g,h)\|_{\iota,h} \leq e_\iota^n |\iota\overline{\pi_M}|_\ell^L \leq e_\iota^n. \tag{18.49}$$

The source class sum is $\phi_{Q,\tau}(h) = \sum_a \tau(\det g_{Q,a})^{-1} K_Q(g_{Q,a}, h)$. Its stabilizers are trivial by the norm bound following (11.3): the chosen-prime product divides $\eta - 1$, while every archimedean conjugate of $\eta - 1$ has modulus at most two. Thus each transported coordinate of $D_p\phi$ is the character evaluation of a group-ring element of norm at most e_ι^n, and the same holds at $\widehat{\iota}$.

Let $H_Q(a,b)$ be the receiver class sum of the Gram pairings of the kernels at b, a. Expanding the receiver norm puts

$$\frac{g_* D_p^2}{(NQ)^{n-1}} H_Q(a,b) \quad \text{at } [\det(g_{Q,b}/g_{Q,a})]. \tag{18.50}$$

The transported Gram matrix has a fixed coefficient bound B_κ. The receiver rank satisfies $n = 2z \leq 2t \leq 2n_m$, since $n_m = k + m(t+1)$ and $m \geq 1$. Its level preserves the integral lattice. After conjugation into that level, reduction at the chosen places embeds each class stabilizer in $\prod_{w \in S_{\mathrm{ch}}} \mathrm{GL}_n(k_w)$ by the trace argument following (11.3), since $p_0 > (2n)^2$ on the constructed scales. Here k_w is the residue field at w. Since $n \leq 2t \leq rt$, the block diagonal inclusion into $\prod_{w \in S_{\mathrm{ch}}} \mathrm{GL}_{rt}(k_w)$ shows that each stabilizer order divides g_*. Also NQ is an ℓ-adic unit. The convolution estimate therefore bounds all coefficients by $B_\kappa e_\iota^n e_{\widehat{\iota}}^n$.

For $z = 1$ use $(k, \delta) = (4, M^2)$. Now (18.47) gives $\delta^{2fk} = M^{16f}$. The determinant polynomial has integral coefficients and degree L_1; the scale $M^2 dj_0$ clears its jet denominator. Thus $C_1\theta_Q$, the scaled kernel for the identity residue matrix, has bound e_ι^4 at points integral above ℓ. Weak approximation as in Lemma 18.31 moves every receiver point into this set. The determinant weight cancels the splitting character under this rational translation.

Expand the residue character on invertible matrices into point indicators and absorb each matrix by an integral lift $\widetilde{y}$. The full-level class sum gives $g_* C_1 \Lambda_Q(\tau) = U_Q(\tau)$, where

$$U_Q = \sum_{x,y} \frac{g_*}{|\operatorname{Aut}(x)|}\, C_1 \theta_Q(x\widetilde{y})[\det(x\widetilde{y})].$$

Here x runs over the rank-two classes and $y \in \mathrm{GL}_2(\mathcal{O}_E/\mathfrak{m}_{Q,2})$. The same reduction in rank two makes the stabilizer quotient integral, so $\|U_Q\|_\iota \le e_\iota^4$. The coefficients of $U_Q U_Q^*$ have bound $e_\iota^4 e_{\widehat{\iota}}^4$, uniformly in Q, as required.

Lean, comparison. The same two class-sum expansions in Lean give the coefficients explicitly. Convolution here applies their common ultrametric estimate to both squared norms. See `divisibilityofstabilizerorders` (L548-L565), `ThetaLiftGauge` (L451-L478), `AlgebraicPartIntegrality` (L285-L391), `ClassSums` (L226-L301). □

Corollary 18.34 (A common complex rescaling gives simultaneous integrality)**.** *For the first family choose a positive integer R_κ clearing the fixed rational Gram matrix, with $R_\kappa = 1$ in its integral normalization, and put*

$$r = \frac{c_*}{c_{\mathrm{vac}}}, \qquad \widetilde{T}_Q = \begin{cases} R_\kappa (r\bar{r})^n T_Q, & z \ge 2, \\ (r\bar{r})^4 T_Q, & z = 1. \end{cases}$$

All finite Fourier coefficients of $\widetilde{T}_Q$ belong to integralAway(NQ) *as defined in* (18.10)*. The rescaling of the measure is nonzero and independent of Q and τ. The theta-chain height estimate uses c_{vac}.*

Proof. Use c_* as vacuum scalar in the class sums of Theorem 18.33, where g_* clears the stabilizer orders. Corollary 18.30 gives the integral-value clause at this scalar for every embedding, so both factors $e_\iota, e_{\widehat{\iota}}$ become one. For $z \ge 2$, the lifts are thereby multiplied by r^n and R_κ clears the rational Gram matrix of the integral tableau basis. At the endpoint the factor on the period including its scale is r^4. Squaring the two norms gives precisely $\widetilde{T}_Q$. All remaining denominators are supported on NQ.

Lean, formal context. Lean scales the residue datum into the integral vacuum lattice. Here a common vacuum scalar cancels the two conjugate factors before evaluating the squared period. The integral datum is treated in `ThetaLiftGauge` (L455-L478), `FitTwistsExist` (L377). □

Proposition 18.35 (The convergent rank-one norm at points at least two)**.** *For $z \ge 2$, let $\Xi_{Q,\tau}$ be the unscaled lift underlying $\phi_{Q,\tau}$, let $H_\kappa = \prod_\sigma (c_\sigma - z)!$, and let d_κ be the dimension of its receiver Weyl module. Then, in the quotient-measure conventions of this note,*

$$\|\Xi_{Q,\tau}\|^2 = \frac{\mathrm{vol}[U(n)]\,\mathrm{vol}[U(1)]}{\zeta_F(2z)} H_\kappa d_\kappa \prod_{w \in S_{\mathrm{ch}}} (q_w - 1)\,(NQ)^{n-1} L_Q(z,\tau). \tag{18.51}$$

In particular the multiplier of $(NQ)^{n-1} L_Q(z,\tau)$ is nonzero and independent of both Q and τ, including characters whose labelled modulus is not minimal.

Proof. Apply the convergent graph-doubling identity (12.1) to the lift of the constant 1 on the definite line. Its rank condition is $2 < n = 2z$. The source quotient is compact, so its constant form supplies the decay condition; the prescribed nonzero column at a chosen place supplies clean support. These verify the hypotheses of the stated classical identity. At infinity the character of the degree-$(c_\sigma - z)$ polynomial cancels the source splitting weight. Its matrix coefficient is constant, with Fock norm H_κ. The source pairing contributes $\mathrm{vol}[U(1)]$ and the graph formula contributes $\mathrm{vol}[U(n)]$. Schur orthogonality on the compact receiver converts its scalar Petersson norm to its Weyl class-sum norm by $d_\kappa[K_{0,n} : K_{\mathrm{rec}}]$. Conversion from the rank-one integral to the chosen-level class sum contributes the square of $[K_{0,1} : K_{\mathrm{old}}] = \prod_{w \in S_{\mathrm{ch}}} (q_w - 1)$.

We compute the local factors with their level indices. A residue column in $\mathbb{F}_q^n$ has additive mass q^{-n}, and the source subgroup congruent to 1 has multiplicative mass $(q-1)^{-1}$. At an old chosen place the test is the indicator of the prescribed nonzero column; the receiver index is $q^n - 1$. Its contribution, including the source class-sum conversion, is

$$(q-1)^2 (q^n - 1) q^{-n} (q-1)^{-1} = (q-1)(1 - q^{-n}).$$

At a place of Q, the block datum is the tame character $\psi(Y_1)$, extended by zero when $Y_1 = 0$. Its mean square is $(q-1)/q$. The splitting-character factor cancels its action by source units, so its source matrix coefficient is this constant on all units and zero on nonunit scalars. A receiver element preserving the hyperplane $Y_1 = 0$ acts on this block through its residue character on the one-dimensional quotient, together with the prescribed determinant twist. These factors have complex modulus one, so this is exactly its receiver level for the norm conversion. The hyperplane-stabilizer index is $(q^n-1)/(q-1)$. Its local contribution is therefore

$$\frac{q^n-1}{q-1}\frac{q-1}{q} = q^{n-1}(1-q^{-n}).$$

At an old place, preservation of the prescribed column forces the scalar's residue to be 1; at a Q-place, preservation of the unit first coordinate forces just its valuation to be zero. This latter argument also applies when ψ is the trivial multiplicative character, still extended by zero at zero. The point-datum expansion used in the preceding theorem gives the same lift by summing source translates; it need not equal this block as a finite Weil vector.

At an unmarked split pair write $u = (\nu\tau)^E(\varpi_w)$. A source scalar of valuation a contributes $u^{-a}q^{-|a|z}$, by intersecting the two integral boxes and multiplying by the oscillator half-density. The two factors are respectively $q^{n\min(a,0)}$ and $q^{-an/2}$. Hence its integral, with units of volume one, is

$$\sum_{a\in\mathbb{Z}} u^{-a}q^{-|a|z} = \frac{1-q^{-2z}}{(1-uq^{-z})(1-u^{-1}q^{-z})}.$$

The splitting-character power cancels the corresponding Weil multiplier, leaving exactly this u^{-a}. At a nonsplit finite place the norm-one source is compact and fixes the vacuum, giving factor 1. The unramified base-change character is trivial there, and E/F is finite unramified, so this too is the local ratio of $L(z)$ to $\zeta_F(2z)$. All products converge absolutely for $z \geq 2$. Multiplication over the unmarked places, followed by the displayed marked factors, proves (18.51). This calculation uses the unscaled vacuum and does not use an archimedean formula for j_0 or c_{vac}.

Lean, formal context. `AlgebraicPartIntegrality` (L285-L391), `FitTwistsExist` (L377). □

Corollary 18.36 (Fixed-prime special-value systems for every prime at $z \geq 2$)**.** *For every ramified critical pair with $z \geq 2$, every coefficient prime ℓ, and every fixed ι, the canonical values G_Q of Theorem 18.3 satisfy the uniform Fourier bound of Theorem 18.5. They therefore give the integral systems and the ratio estimate of Corollary 18.7. In fact there is a common nonzero complex scalar C, independent of Q and τ, for which the Fourier coefficients of CG_Q are integral away from NQ at every embedding simultaneously.*

Proof. The scaling from Ξ to ϕ is the fixed nonzero scalar $s_p = c_{\mathrm{vac}}^n(\overline{\pi_M}j_0)^L$. Proposition 18.35 gives $T_Q(\tau) = A_pL_Q(z,\tau)$, where

$$A_p = g_*D_p^2|s_p|^2\frac{\mathrm{vol}[U(n)]\,\mathrm{vol}[U(1)]}{\zeta_F(2z)}H_\kappa d_\kappa \prod_{w\in S_{\mathrm{ch}}}(q_w-1) \neq 0.$$

Since $L_\varnothing(z,1) > 0$, the number $T_\varnothing(1)$ is nonzero and $T_Q/T_\varnothing(1) = G_Q$ at every labelled modulus. Dividing the coefficient bound of Theorem 18.33 by this one fixed nonzero number preserves a uniform lower valuation bound. The two cited measure results complete the construction. No numerical bound for this fixed scalar is needed: its valuation cancels in the ratio argument. For the last assertion, use $\widetilde{T}$ from Corollary 18.34 and take $C = \widetilde{T}_\varnothing(1) \neq 0$. Its normalized values are still G_Q, so its explicitly constructed coefficients give the stronger simultaneous integral family as well.

Lean, formal context. `AlgebraicPartIntegrality` (L285-L391), `FitTwistsExist` (L377). □

Proposition 18.37 (Endpoint comparison and change of modulus)**.** *For $z = 1$, Theorem 18.33 proves the bounded-coefficient assertion for the determinant period family at every coefficient prime. To identify that family with G_Q it suffices to prove the single statement*

$$\exists C_p \in \mathbb{R}^\times \quad C_p|\Lambda_Q(\tau)|^2 = N_* N Q\, L_Q(1,\tau)^2 \quad \textit{for every primitive } (Q,\tau). \tag{EP}$$

The empty auxiliary modulus is included. Under (EP), the fixed-prime systems of Corollary 18.7 exist also for $z = 1$ at every prime. Theorem 18.45 proves (EP) below; its proof is separate from the jet estimates.

Proof. We verify the change-of-modulus rule independently of (EP). Insert an unramified split place of norm q, put $u = (\nu\tau)^E(\varpi)$, and put $a = qu$. For a subspace $U \subseteq \mathbb{F}_q^2$, choose an elementary-divisor representative B_U with residue kernel U, and put $\gamma_U = B_U^{\mathfrak{t}}$. Define the finite Hecke operators, including their polynomial normalization, by

$$\mathcal{T}_j = q^{\binom{j}{2}} \sum_{\dim U = j} R(\gamma_U), \qquad \mathcal{H}(X) = \sum_{j=0}^{2} (-X)^j \mathcal{T}_j.$$

The twisted Weil action of γ_U on the vacuum has scalar $a^{\dim U}$ and support $U \subseteq \ker(\bar{Y}^{\mathfrak{t}})$. The identity $a = qu$ follows from $(\nu\tau\upsilon_{U(1)})^E \upsilon^{-2} = (\nu\tau)^E$. Consequently the pointwise multiplier of $\mathcal{H}(X)$ is

$$\sum_{U \subseteq \ker(\bar{Y}^{\mathfrak{t}})} (-Xa)^{\dim U} q^{\binom{\dim U}{2}} = \prod_{j=0}^{1-\operatorname{rank}\bar{Y}} (1 - q^j Xa).$$

In dimensions $0, 1, 2$ this is respectively 1, $1 - Xa$, and $1 - (q+1)Xa + q(Xa)^2$. At $X = a^{-1}$ it is the indicator of invertible reduction, which inserts exactly the required trivial-residue datum. The full-level period is an integral and is invariant under each right translation. Thus its value on $\mathcal{T}_j$ is $q^{\binom{j}{2}} \#\{U : \dim U = j\}$ times its original value: these three multipliers are $1, q+1, q$. Its value on $\mathcal{H}(X)$ is consequently $(1 - X)(1 - qX)$ times the original period. Here the pointwise factor a^j belongs to the Weil action already being translated; it does not multiply the invariant integral a second time. Evaluation at $X = a^{-1}$ gives

$$\Lambda_{Q \cup \{w\}}(\tau) = (1 - q^{-1}u^{-1})(1 - u^{-1})\Lambda_Q(\tau).$$

Taking the squared modulus gives precisely the Euler factors with exponents $\{0, 1\}$.

If (EP) holds, its empty-modulus instance gives $\Lambda_\varnothing(1) \neq 0$. Proposition 18.2 converts the primitive instances to

$$T_Q(\tau) = \frac{g_*^2 |C_1|^2 N_*}{C_p \kappa_p} L_Q(0,\tau) L_Q(1,\tau).$$

The Euler rule just proved extends the identity to every padded modulus, without dividing by possibly zero factors. Divide by $T_\varnothing(1)$, apply Theorem 18.33, and then the uniform criterion. This proves the implication. Statement (EP) compares an equal-rank determinant period to a critical L-value. The convergent calculation above has strict rank inequality $2 < n$ and cannot be applied at $n = 2$. Neither nonvanishing of this period nor its common analytic normalization follows from its bounded coefficients or its Euler insertion identity.

Lean, formal context. `AlgebraicPartIntegrality` (L285-L391), `FitTwistsExist` (L377). □

Lean references for the theta lift and determinant period. The two coefficient expansions are the kernel-pairing expansion and the determinant-period expansion. The fixed scale C_1 is the explicit kernel scale. The norm calculation agrees with the rank-one formula and its uniform multiplier. The endpoint statement (EP) has its source counterpart in the inverse determinant-period comparison. To establish (EP), we compute the determinant polynomial and the modulus of the scalar in the exchange.

18.5 Uniform normalization of the exchange at $z = 1$

In Theorem 12.17 the real vector and the scalar may depend on the finite character. For (EP) we must choose a real vector before choosing that character and determine the modulus of the scalar. Throughout the calculation we fix the real weight, field, framed spaces, and Haar measures. The tame character and its finite ramification set may vary. This distinction is essential for a single comparison constant.

Write ω for a unitary character of the norm-one ideles of positive integral weight $\mu = (\mu_\sigma)$, and let S_ω be its nonempty split tame ramified set, disjoint from its conjugate. Put

$$q(\omega) = \prod_{w \in S_\omega} q_w, \qquad \Pi_\omega(e) = q(\omega)^{e^2} \prod_{j=1}^{e} |L^{S_\omega}(j, \omega)|^2 \quad (e \geq 0), \qquad \Pi_\omega(-e) = \Pi_\omega(e)^{-1}.$$

Here $\Pi_\omega(0) = 1$, and the L-function is that of the base change $a \mapsto \omega(a/\bar{a})$. The classical nonvanishing on $\operatorname{Re} s \geq 1$ for these nonzero-weight unitary Hecke characters, already used in §12.3, makes every displayed quantity positive. No nonvanishing inside the critical strip is asserted or used.

Lemma 18.38 (The modulus of the ramified line factor)**.** *Fix the adapted frames and measures of Theorem 12.5. For each of its two Poisson models there is a positive constant $b_\pm$, independent of ω and S_ω, such that the scalar in that theorem can be written $\epsilon_\pm(\omega)$ with*

$$|\epsilon_\pm(\omega)| = b_\pm.$$

Consequently the entire varying conductor contribution to the absolute value of the line term at a real point $x \geq 1$ is

$$q(\omega)^{x-1/2} |L^{S_\omega}(x, \omega)|.$$

Proof. At a ramified place give $\mathcal{O}_w$ additive volume one and use an additive character of conductor $\mathcal{O}_w$. For the nontrivial residue character χ_w, put $f_w(t) = \mathbf{1}_{\mathcal{O}_w^\times}(t)\chi_w(t)^{-1}$. Its Fourier transform is zero except at valuation -1 and is there $\gamma_w \chi_w(\varpi_w a)$, where

$$\gamma_w = q_w^{-1} \sum_{u \in \mathbb{F}_{q_w}^\times} \chi_w(u)^{-1} \psi_w(u/\varpi_w), \qquad |\gamma_w|^2 = q_w^{-1}.$$

For completeness, if $G = \sum_{u \neq 0} \chi(u)^{-1}\psi(u)$, then, on writing $u = tv$,

$$|G|^2 = \sum_{t \neq 0} \chi(t)^{-1} \sum_{v \neq 0} \psi((t-1)v) = (q-1) - \sum_{t \neq 1} \chi(t)^{-1} = q.$$

The support assertion follows by averaging on additive residue cosets, and, for valuation less than -1, on the principal-unit cosets. Thus no character-dependent positive constant is hidden in the Gauss sum.

The block-triangular calculation in the proof of Theorem 12.5 leaves precisely one such integral at each ramified place. All other integrated coordinates have factor $\mathbf{1}_{\mathcal{O}_w}$ and volume one. An

integral change of frame contributes $\chi_w(\det A)^{\pm 1}$, of absolute value one. Hence its total Gauss factor γ_ω, including these frame phases, satisfies

$$|\gamma_\omega|^2 = q(\omega)^{-1}.$$

The vacuum-slice factors $\lambda_\pm \neq 0$ are fixed by the global adapted splittings. They do not acquire a new factor when a ramified place is added: every such finite factor was included in γ_ω, with integral-coordinate volume one.

Let a_0 be the fixed scale idele, and choose the finite idele c_ω, supported on $\bar{S}_\omega$, that removes the uniformizers from the finite line datum. Then $|c_\omega^{-1}|_f = q(\omega)$, and the translated idele $K_\omega = a_0 c_\omega^{-1}$ has $|K_\omega| = |a_0| q(\omega)$. The unshifted finite Tate integral is $v_\pm L^{S_\omega}(x, \omega)$ for $x > 1$, where $v_\pm > 0$ is the volume of the *full* finite unit-idele group for the fixed product measure. It is independent of the set S_ω. Indeed the unit characters cancel at the ramified places; elsewhere partition into valuation classes to obtain the usual geometric Euler series. If $c_{\mathrm{unf}} \neq 0$ is the fixed orbit-unfolding measure constant, the remaining scalar is exactly of the form

$$\epsilon_\pm(\omega) = c_{\mathrm{unf}} \lambda_\pm v_\pm(\gamma_\omega \sqrt{q(\omega)}) \omega^E(K_\omega).$$

Both factors in the final parentheses and the character value have modulus one. Therefore $b_\pm = |c_{\mathrm{unf}}|\,|\lambda_\pm| v_\pm > 0$ works simultaneously for all the characters.

At $x = 1$ we use continuity of the global unfolded Tate integral, not absolute convergence of its Euler series. The two Poisson models give integrable Mellin majorants at R and $1 - R$ for every $R > 1$. On a compact subinterval between them, $t^x |\log t|^j$ is bounded by a fixed multiple of $t^{R'} + t^{1-R'}$ for a slightly larger $R' > R$. Since $|t^s| = t^{\mathrm{Re}\, s}$, these bounds also hold uniformly on compact complex strips. Dominated integration of the entire functions t^s proves holomorphy in each such strip, hence continuity and real analyticity at 1. The real cone integral is continuous there by the Schwartz seminorm estimate already proved, and the Hecke L-function is continuous there. The identity for $x > 1$ consequently has the stated limit at 1. The full-rank datum kills the zero orbit in the constant-term unfolding. In the two line Poisson models the corresponding zero summands vanish because $f_w(0) = \widehat{f}_w(0) = 0$ at a ramified place. Thus this passage has no residual zero-term contribution.

Lean. `LineGaussSums` (L436-L503), `LargeSideConstantTerm` (L607-L680), `LargeSideConstantTerm` (L730-L814). □

Lemma 18.39 (Uniform one-plane descent, with its scalar size)**.** *Let $e = \dim V - \dim W/2$ be the lower plane parameter, an integer. There is a positive d_e, independent of ω, and a scalar $\gamma_e(\omega) \neq 0$ such that the large constant term is $\gamma_e(\omega)$ times the lower theta integral and*

$$|\gamma_e(\omega)|^2 \Pi_\omega(e) = d_e^2 \Pi_\omega(e + 1). \tag{18.52}$$

Its lower real vector is $\mathcal{C}^+_{e+1} v^+$ if $e \geq 0$, and $\mathcal{C}^-_{-e} v^+$ if $e < 0$. Both operators depend only on the fixed real data and μ.

Suppose an upper exchange, for real vectors chosen independently of ω, has scalar a^+_ω and satisfies

$$D^+ |a^+_\omega|^2 = \Pi_\omega(e + 1) R(\omega),$$

where D^+ is independent of ω. After removing the plane, the new scalar and fixed coefficient are

$$a_\omega = \frac{\kappa_0 a^+_\omega}{\gamma_e(\omega)}, \qquad D = \frac{D^+ d_e^2}{|\kappa_0|^2}, \qquad D|a_\omega|^2 = \Pi_\omega(e) R(\omega).$$

Here $\kappa_0 \neq 0$ is the fixed small-side vacuum-slice factor. The lower real vectors are still independent of ω.

Proof. Put $x = e + 1$ for $e \geq 0$, and $x = -e$ for $e < 0$; in both cases $x \geq 1$. Lemma 18.38 and the plane formula give

$$|\gamma_e(\omega)| = d_e q(\omega)^{x-1/2} |L^{S_\omega}(x, \omega)|, \qquad d_e = b_\pm |a_0|^x > 0.$$

For $e \geq 0$, the identity $(e+1)^2 - e^2 = 2e + 1$ proves (18.52) by canceling successive products. For $e = -z - 1 < 0$, the same calculation reads

$$\Pi_\omega(-z) = q(\omega)^{2z+1} |L^{S_\omega}(z + 1, \omega)|^2 \Pi_\omega(-z - 1).$$

This is the inverse-product definition at negative indices; it requires no evaluation of an Euler product at a nonpositive integer. In particular the crossing $e = -1$ uses the value at $x = 1$, covered by the preceding continuity argument.

Take the constant terms of the upper exchange. Its two sides give $\gamma_e(\omega) I$ and $a_\omega^+ \kappa_0 J$, respectively. Dividing by the nonzero $\gamma_e(\omega)$ gives the claimed a_ω. Substitution in the norm relation proves the displayed formula for D. The cone, the dual cone and the zero slice are archimedean linear maps determined by the fixed splittings, the integer e and μ. Thus their application never makes a character-dependent choice of real vector. Iterating preserves this quantifier order.

Lean. `FiniteTateIntegral` (L42-L136), `RemovingPlanes` (L508-L610). □

Lemma 18.40 (Tame conductor versus the labelled modulus)**.** *The determinant datum inflated from a nontrivial tame residue character at w is unchanged as a function on integral matrices when w in its labelled modulus is replaced by w^a, $a \geq 1$. Its line Gauss factor therefore still has squared modulus q_w^{-1}. The product in (18.52) uses the set of ramified primes, each once, rather than the norm of the possibly larger labelled ideal. For the endpoint modulus $\mathfrak{m}_{Q,2}$ and a primitive twist, this product is exactly $q(\omega) = N_* NQ$.*

Proof. A matrix over $\mathcal{O}_w$ is invertible modulo w^a if and only if its determinant is a unit. On this set the inflated character of its determinant reads only the reduction modulo w. Thus the two pullback functions on integral matrices are identical, and both vanish on nonintegral matrices. All their theta integrals and line Fourier transforms are therefore identical; changing the presentation through a finite quotient inserts no counting or volume factor. In $\mathfrak{m}_{Q,2}$ the old primes appear to exponent two, while the primitive auxiliary primes appear to exponent one. The character is nontrivial exactly on $S_{\mathrm{ch}} \cup Q$, so its squarefree norm is $N_* NQ$. Even when Q is empty, S_{ch} remains nonempty. The nonempty-ramification hypothesis of the exchange therefore still holds.

Lean, formal context. `RemovingPlanes` (L508-L610), `FitTwistsExist` (L377). □

Calibration by two opposite exchanges. For an even-rank space Y of rank $2d$ with an adapted hyperbolic plane, write

$$R_Y(v, \omega) = |I_{Y,\omega}(v)|^2 / \Pi_\omega(d).$$

Here $I_{Y,\omega}$ is the constant integral of Lemma 12.9. Its finite datum is $\mathbf{1}_{\mathrm{GL}_{2d}(\mathcal{O}_{S_\omega})} \chi_\omega^{-1}(\det)$ at the ramified places and the vacuum elsewhere, where χ_ω is the tame part of ω.

Lemma 18.41 (Uniform descent before calibrating the period)**.** *Let W, V, U have ranks n, a, b with $n = a + b$ even and negative sign counts satisfying $q_W = q_V + q_U$. Fix a real Schwartz vector y for (U, W) and a real Schwartz vector z for $(H \perp W, H \perp W)$. There are a real Schwartz vector x for (W, V) and $A > 0$, chosen independently of ω, such that for every ω there is a scalar c_ω satisfying*

$$A |c_\omega|^2 = \Pi_\omega(a - n/2) R_{H \perp W}(z, \omega). \tag{18.53}$$

For every matched pair of locally constant finite data, the theta integral with real vector x over $U(V)$ equals c_ω times the theta integral with real vector y over $U(U)$, with the rank-normalized splitting characters of (12.7). The scalar may vanish at this stage. The same x and A work for every finite ramified set.

Proof. First pass from U to its opposite, using the adjoint oscillator identification. This replaces y by a fixed real vector; its finite character factor has absolute value one. Extend this vector through one plane with prescribed zero slice, by tensoring in the mixed model with a fixed Schwartz function having that slice equal to one. Put $W_0 = H \perp W$. The top configuration has spaces

$$H^b \perp W_0, \qquad U^- \perp W_0,$$

and top parameter $n/2+1$. The real Witt section, compact bump, quotient weight and vacuum slices in Theorem 12.12 use only the ranks, signs, fixed weight and the two supplied real vectors. They can therefore all be chosen before ω.

The scalar in that theorem is the product of the equal-rank period $I_{W_0,\omega}(z)$, a fixed nonzero quotient and vacuum factor, and unit-modulus character values. Thus, for a fixed $A_0 > 0$, its squared modulus satisfies

$$A_0|c_{\mathrm{top},\omega}|^2 = \Pi_\omega(n/2+1)R_{W_0}(z,\omega).$$

This equality remains valid if the period is zero. No division by that period is made.

Remove $b+1$ planes. The last parameter is $n/2+1-(b+1) = a-n/2$. At every step Lemma 18.39 replaces the scalar and positive coefficient by

$$c \longmapsto c\kappa_0/\gamma_e(\omega), \qquad A \longmapsto Ad_e^2/|\kappa_0|^2.$$

The cone or dual-cone operator and the small zero-slice operator are fixed real operators. Their successive application gives one vector x, independent of all finite character data. The squared factors telescope exactly to (18.53); crossing $e=-1$ uses the proved $L(1)$ formula, not a nonpositive Euler-product value. All divisions are by the nonzero $\gamma_e(\omega)$.

This first proves the exchange on standard row and column data. Lemma 12.2 expresses any matched locally constant datum as a finite linear combination of their integral translates. The covariance calculation (12.14) multiplies both theta maps by the same character. It consequently extends the identity with the same scalar. Returning from U^- to U multiplies the scalar by a number of absolute value one. Full-rank support and the convergence theorem already proved for this exchange justify the theta integrals throughout.

Lean. `TheoremForPartners` (L275-L365). □

Proposition 18.42 (Reciprocal calibration in even definite rank)**.** *Let J be the standard positive definite space of rank $2k$, $k \geq 1$, let $Y = H \perp J$, and suppose $\mu_\tau \geq k$ at every infinite place. For every fixed real Schwartz vector v for (Y,Y) there are $B, C > 0$, a real Schwartz vector y for (J,J), and $\xi \in \mathbb{C}$, all independent of ω, with the following property. For every split tame character ω of weight μ and nonempty ramified set there are scalars c_1, c_2 such that*

$$B|c_1|^2 = \Pi_\omega(k)R_Y(v,\omega), \qquad \int_{[U(J)]} K_{\omega,y}(g,h)\,dh = c_1V_0, \tag{18.54}$$

$$C|c_2|^2 = \Pi_\omega(-k)R_Y(v,\omega), \qquad V_0\xi = c_2\int_{[U(J)]} K_{\omega,y}(h,g)\,dh. \tag{18.55}$$

Here $V_0 = \mathrm{vol}[U(0)] > 0$. The kernel $K_{\omega,y}$ has both splitting characters $\gamma_\omega = \omega^{-1}\upsilon_{U(1)}^k$ and the finite inverse-determinant datum on integral invertible $2k$-by-$2k$ matrices at S_ω. The identities hold for every g. The ratio $R_Y(v,\omega)$ is independent of ω. If the character family is nonempty, some v has positive common ratio.

Proof. Apply Lemma 18.41 with $(W, V, U) = (J, J, 0)$, using the unit vector on the rank-zero side. Use its output y as the prescribed vector for the opposite exchange $(J, 0, J)$, with the same auxiliary v. The new real vector in the second exchange is a scalar ξ. The two size formulas follow from $a - n/2 = k, -k$, respectively.

For standard row and column data, both edges over J carry $\chi_\omega^{-1}(\det)$ on invertible integral matrices. At rank zero the finite datum has value one. In the first exchange, changing the second splitting character from 1 to γ_ω contributes the external factor $\gamma_\omega(\det g)$ to the integral with the original characters $(\gamma_\omega, 1)$. The rank-zero partner has the same factor times V_0; cancelling it gives (18.54). In the second exchange the rank-zero side has value $V_0\xi$ and the other side already has both characters γ_ω. Thus (18.55) uses the same kernel with the two arguments interchanged. The unit phases of the adjoint identification are included in c_1, c_2.

The quotient $[U(J)]$ is compact. Integrating both identities in g and interchanging the order of integration gives $\xi = c_1c_2$, since $V_0 \operatorname{vol}[U(J)] > 0$. As $\Pi_\omega(k)\Pi_\omega(-k) = 1$, it follows that

$$R_Y(v, \omega)^2 = BC|\xi|^2. \tag{18.56}$$

The right side is independent of ω, and R_Y is nonnegative. Fix any one character ω_0 in a nonempty family. Lemma 12.16 with $W = J$ and $q = 0$ gives v for which $I_{Y,\omega_0}(v) \neq 0$. This v has positive ratio for every character.

Lean, comparison. The two standard-data identities and the common ratio are `exists_thetaLarge_eq_mul_quotMeasure`, `exists_quotMeasure_mul_eq_mul_integral_pairTheta`, `exists_equalRankThetaRatio_sq_eq` and `exists_equalRankThetaRatio_const_ne_zero`. The fixed-character descent needed for positivity was proved in Lemma 12.16. □

Corollary 18.43 (The normalized rank-two period)**.** *Put $J = I_2$ and assume that the family of split tame characters of weight μ is nonempty. There are a real Schwartz vector y for (J, J) and $A > 0$ such that, for each character ω,*

$$\int_{[U(J)]} K_{\omega,y}(g, h)\, dh = c_\omega, \qquad |c_\omega|^2 = A\Pi_\omega(1) > 0 \quad \textit{for every } g. \tag{18.57}$$

Proof. In Proposition 18.42 take $k = 1$ and a vector v with common ratio $R_0 > 0$. Equation (18.54) gives the assertion with $c_\omega = V_0c_1$ and $A = V_0^2R_0/B$.

Lean. `TheoremForPartners` (L598-L632), `TheoremForPartners` (L675-L736). □

Lemma 18.44 (The rank-two determinant projection)**.** *Put $k_\sigma = \mu_\sigma - 1$ and $P_k(Z) = \prod_\sigma \det(Z_\sigma)^{k_\sigma}$. For each real Schwartz Fock vector y for (J, J), averaging the receiver action with splitting weight $-k$ projects y onto aP_k times the Gaussian, where $a \in \mathbb{C}$ depends on y and k alone. Its theta integral over the receiver is unchanged by this averaging.*

For the calibrated y in Lemma 18.43, this coefficient a is nonzero. There is a fixed positive measure factor κ, equal to one in the source normalization, such that the determinant class sum $\Lambda_\omega(g)$ with characters $(\gamma_\omega, \gamma_\omega)$ and the same finite datum satisfies

$$\kappa a\Lambda_\omega(g) = c_\omega \quad \textit{for every } \omega, g. \tag{18.58}$$

Proof. The definite Fock model consists of holomorphic functions on $\prod_\sigma M_2(\mathbb{C})$ with the Gaussian factor understood. For $u = (u_\sigma) \in \prod_\sigma U(2)$ the receiver action on that holomorphic function is

$$F(Z) \longmapsto \prod_\sigma \det(u_\sigma)^{-k_\sigma} F((u_\sigma^{\mathsf{t}} Z_\sigma)_\sigma).$$

Its formula uses only the infinite weight of $\gamma_\omega = \omega^{-1}v_{U(1)}$, which is $1 - \mu = -k$. In particular it has no dependence on the finite character. Average this action with probability Haar measure.

The central circles force degree $2k_\sigma$ in the matrix variables at each σ, so an invariant entire function is a polynomial. Transposition permutes $U(2)$ and preserves the determinant, so invariance is equivalently $F(uZ) = \prod_\sigma \det(u_\sigma)^{k_\sigma} F(Z)$. Differentiating and complexifying the Lie algebra extends this identity from $U(2)^f$ to $\mathrm{GL}_2(\mathbb{C})^f$. On the dense set of invertible matrices it gives $F(Z) = F(I)P_k(Z)$, and polynomial continuity gives the same identity everywhere. Thus the invariant line is exactly $\mathbb{C}P_k$, including $P_k = 1$ when every $k_\sigma = 0$.

The compact averaging operator preserves the Schwartz space. A change of the receiver integration variable shows that the integral of the theta kernel at each translated real vector equals its original integral. Compact averaging and the theta integral commute by the continuous Schwartz estimate, or directly by absolute convergence on the compact receiver quotient. It follows that the integral at y is a times the integral at P_k times the Gaussian.

It remains to identify the latter integral with the class sum; using a smaller congruence level here would introduce a spurious varying index. The polynomial vector is fixed by the full receiver compact group at infinity. At a finite ramified place write $\psi = \chi_\omega^{-1}$, so the determinant datum is $\Phi(X) = \mathbf{1}_{\mathrm{GL}_2(\mathcal{O})}(X)\psi(\det X)$. For an integral unitary receiver element h, the splitting factor is $\psi(\det h)$ and

$$\Phi(h^{-1}X) = \psi(\det h)^{-1}\Phi(X).$$

These factors cancel. The vacuum and unramified splitting character are fixed by the integral level at every other place. The whole integrand is therefore invariant under the full fixed finite level K_0, independently of the ramified set and the modulus exponents. Integrating over rational double cosets now gives

$$\int_{[U(J)]} K_{\omega,P_k}(g,h)\,dh = \kappa \sum_{[h]\in U(J)(F)\backslash U(J)(\mathbb{A}_f)/K_0} \frac{K_{\omega,P_k}(g,(1,h))}{|\operatorname{Aut}(h)|} = \kappa\Lambda_\omega(g).$$

The compact-group volumes form the same positive κ for every character. The source Haar convention gives $\kappa = 1$. Substituting (18.57) proves (18.58). Since $c_{\omega_0} \neq 0$, that equation also proves $a \neq 0$; there is no separate unproved assertion that a selected determinant projection is nonzero.

Lean. `ExplicitMatching` (L180-L204), `ExplicitMatching` (L553-L611). □

Theorem 18.45 (The primitive endpoint period comparison)**.** *For every ramified critical pair $p = (\nu, 1)$ there is a positive real number C_p, independent of the primitive auxiliary twist (Q,τ), such that*

$$C_p|\Lambda_Q(\tau)|^2 = N_*NQ\,L_Q(1,\tau)^2.$$

Here $\Lambda_Q(\tau)$ is exactly the determinant period in (18.48). The assertion includes all allowed unramified twists at $Q = \varnothing$, in particular the trivial twist. Every displayed period is nonzero. Thus (EP) holds.

Proof. Set

$$\omega = (\nu\tau)^{-1}, \qquad \mu_\sigma = c_\sigma, \qquad k_\sigma = c_\sigma - 1, \qquad \gamma_\omega = \nu\tau\upsilon_{U(1)}.$$

These equalities check both the infinite weight $-k$ of the determinant-period character and the inverse on the Hecke character in the L-function. The nontrivial tame part at every old chosen place persists under an auxiliary twist. Primitivity makes the ramified set exactly $S_{\mathrm{ch}} \cup Q$, whose squarefree norm is N_*NQ. The source determinant datum at $\mathfrak{m}_{Q,2}$ is the inflation of $\chi_\omega^{-1} \circ \det$; by Lemma 18.40, the exponent two at the old primes changes neither this function nor its conductor factor. The two characters, polynomial, modulus, full level and automorphism weights in Lemma 18.44 are therefore precisely those defining $\Lambda_Q(\tau)$ at $g = 1$.

There is at least one character for the calibration: take $\omega_0 = \nu^{-1}$ and $Q = \varnothing$. Its ramified set is $S_{\mathrm{ch}} \neq \varnothing$, and $c_\sigma \geq 1$ verifies the nonvanishing lemma's weight condition. Consequently the preceding lemmas supply $A > 0$ and $\kappa a \neq 0$, common to every auxiliary twist, and give

$$|\kappa a|^2 |\Lambda_Q(\tau)|^2 = |c_\omega|^2 = A\Pi_\omega(1) = AN_*NQ\,|L_Q(1,\tau)|^2.$$

Lemma 18.1 gives $L_Q(1,\tau) > 0$, so its absolute square is its square. Take $C_p = |\kappa a|^2/A > 0$. The right side is strictly positive, proving nonvanishing also at the empty auxiliary modulus. This argument uses the two convergent exchanges and their plane Mellin identities; it does not apply the strict-rank norm formula at an equal-rank endpoint.

Lean. `ExplicitMatching` (L788-L849), `ThreeExchanges` (L216-L248), `ClassSums` (L152-L196). □

Corollary 18.46 (The exact inverse period and its norm)**.** *With C_p as above, every primitive (Q,τ) has $\gamma_{Q,\tau} \neq 0$ such that, for every source point g,*

$$\gamma_{Q,\tau}\Lambda_Q(\tau; g) = 1, \qquad |\gamma_{Q,\tau}|^2 = \frac{C_p}{N_*NQ\,L_Q(1,\tau)^2}.$$

The source-point notation denotes the same full-level determinant class sum before evaluation at $g = 1$.

Proof. Equation (18.58) holds at every g and has $c_\omega \neq 0$. Set $\gamma_{Q,\tau} = \kappa a/c_\omega$. The first identity follows immediately; taking its squared absolute value and using (18.57) gives the second. In particular the scalar is the inverse of the period, not an independently chosen normalization at each modulus. Its squared norm has the same C_p for all twists.

Lean, formal context. `ClassSums` (L152-L196), `FitTwistsExist` (L377). □

Corollary 18.47 (The endpoint special-value system at every prime)**.** *For each ramified critical pair $p = (\nu, 1)$, the determinant-period construction supplies the required special-value system at every coefficient prime, with the original ratio allowance. In fact there is one nonzero complex normalization whose finite Fourier coefficients are integral away from NQ, simultaneously at all coefficient embeddings and for every labelled modulus Q.*

More precisely, retain $L_1 = 2\sum_\sigma(c_\sigma - 1)$ and put

$$B_p = g_* M^{16f} c_*^4 (M^2 dj_0)^{L_1}, \qquad F_Q(\tau) = |B_p\Lambda_Q(\tau)|^2,$$

where c_* is the single integral vacuum scalar of Corollary 18.30. Then $B_p \neq 0$, $F_\varnothing(1) > 0$, and, at every labelled modulus, including padded ones,

$$\frac{F_Q(\tau)}{F_\varnothing(1)} = \frac{|\Lambda_Q(\tau)|^2}{|\Lambda_\varnothing(1)|^2} = \frac{L_Q(0,\tau)L_Q(1,\tau)}{L_\varnothing(0,1)L_\varnothing(1,1)} = G_Q(\tau). \tag{18.59}$$

For primitive twists this common value is

$$NQ\left(\frac{L_Q(1,\tau)}{L_\varnothing(1,1)}\right)^2.$$

Proof. The family F is exactly the endpoint family $\widetilde{T}$ of Corollary 18.34: multiplying $|g_*C_1\Lambda_Q|^2$ by $((c_*/c_{\mathrm{vac}})\overline{(c_*/c_{\mathrm{vac}})})^4$ replaces the fourth power of the vacuum scalar in C_1 by c_*^4. The construction there therefore gives its Fourier coefficients integral away from NQ at all embeddings at once.

Theorem 18.45 and the exact functional equation (18.3) give, for every primitive twist,

$$F_Q(\tau) = \frac{|B_p|^2}{C_p} N_*NQL_Q(1,\tau)^2 = \frac{|B_p|^2 N_*}{C_p\kappa_p} L_Q(0,\tau)L_Q(1,\tau).$$

The multiplier is positive and independent of Q, τ. The case $Q = \varnothing$, $\tau = 1$ of that theorem makes the base period nonzero, so $F_{\varnothing}(1) > 0$.

To pass to an arbitrary labelled modulus, first replace it by the tame ramification set $R(\tau)$, where the comparison just proved applies. Insert the places in $Q \setminus R(\tau)$ one at a time. The determinant-period Euler identity in Proposition 18.37 multiplies its squared modulus by exactly

$$\prod_{j\in\{0,1\}} (1 - uq^{-j})(1 - u^{-1}q^{-j}), \qquad u = (\nu\tau)^E(\varpi).$$

The analytic product $L_Q(0,\tau)L_Q(1,\tau)$ has the same insertion rule by Theorem 18.3. Multiplying these identities proves (18.59), even if an inserted factor is zero. For $(I, m) = (\{0, 1\}, 1)$, the base value of F is nonzero, its coefficients are integral, and it satisfies primitive interpolation and Euler insertion as required in §13.5. The uniform criterion and Corollary 18.7 give the fixed-prime systems. Division by the fixed base value contributes a constant valuation, which cancels in the ratio.

Lean, formal context. `ClassSums` (L152-L196), `FitTwistsExist` (L377). □

Critical values at all coefficient primes. Together with Corollary 18.36, the preceding corollary constructs the special-value systems at every required critical point and coefficient prime. Equation (18.59) holds for every allowed modulus. At primitive moduli it reduces to NQ times a square. Corollary 18.46 gives the nonvanishing and the modulus of the scalar for the first equal-rank exchange. The measure may be rescaled by c_*. The theta-chain height estimate uses c_{vac}, whose integrality and archimedean bounds are proved in the next section.

19 The CM theta kernels and their periods

The CM area formula relates the finite integrality and the complex size of the vacuum theta values for the same choice of period.

For each field $E = K(\zeta_q)$ we choose a single scalar c_{vac} satisfying IsVacuumScale. A finite Fourier coefficient gives this scalar. If v_1 is the rank-one vacuum, the quantities $c_{\mathrm{vac}} v_1(\mathrm{finVec}\, x)$ are integral at every finite embedding, and their complex conjugates satisfy a common bound. We use this choice in every rank and in all jet and height estimates.

Retain the normalized CM curves of §18.3. If C is one of them, write $\Lambda_C, A_C, t_C, a_1(C), \theta_C, \sigma_C$ for its lattice, area, marked half-period, coefficient, theta function and reduced sigma function. In this subsection σ without a subscript denotes a field automorphism, not a sigma function. For an entire function f, write $\widehat{f}$ for its Taylor series at zero, and $\widehat{f}^{\sigma}$ for coefficientwise conjugation.

Lemma 19.1 (Finitely many normalized CM lattices and a uniform theta bound)**.** *For the fixed prime $d \equiv 7 \pmod 8$, only finitely many lattices in $\mathbb{C}$ underlie the normalized CM curves. There is consequently a real number $M \geq 1$, depending only on $K = \mathbb{Q}(\sqrt{-d})$, such that*

$$e^{-\pi|u|^2/(2A_C)}|\theta_C(u)| = |(T_u^{(1)}\theta_C)(0)| \leq M \qquad (C \text{ normalized},\ u \in \mathbb{C}). \tag{19.1}$$

Proof. Every CM lattice is a complex multiple of a fractional ideal of K. There are finitely many ideal classes. Fix one normalized lattice in each class. If multiplication by $a \neq 0$ takes one normalized lattice onto another, it takes their distinguished half-periods to each other modulo the target lattice: on $\mathcal{O}_K/2\mathcal{O}_K \simeq \mathbb{F}_2 \times \mathbb{F}_2$ the distinguished nonzero point is the one with both components nonzero. The scaling rule for the Weierstrass function gives

$$2 = \wp''_{a\Lambda}(at) = a^{-4}\wp''_{\Lambda}(t) = 2a^{-4}.$$

Thus $a^4 = 1$. Each ideal class supplies at most four lattices under this normalization.

The theta function depends only on this lattice. Indeed its multiplier was determined from the lattice in Lemma 18.21; its degree-one section space is one-dimensional, and its value at zero is one. For a fixed lattice the function $u \mapsto e^{-\pi|u|^2/(2A_C)}|\theta_C(u)|$ is continuous and lattice-periodic, by the theta transformation law. It is bounded on the compact complex torus. Taking the maximum over the finitely many lattices proves the assertion; the value at $u = 0$ shows that the maximum is at least one.

Lean. `FiniteLattices` (L93-L189). □

Lemma 19.2 (Conjugating normalized CM sections)**.** *Let $\sigma \in \mathrm{Aut}(\mathbb{C}/K)$, and let C be a normalized CM curve. There is a normalized CM curve $D = C^\sigma$ with $a_1(D) = \sigma(a_1(C))$. Coefficientwise conjugation identifies the Taylor series of its normalized theta function with that of C:*

$$\widehat{\theta}_D = \widehat{\theta}_C^{\sigma}. \tag{19.2}$$

It identifies the graded spaces of sections as well. Moreover, for each $w \in \mathbb{C}$ there are $w' \in \mathbb{C}$ and $\mu \neq 0$ such that, for every degree-m section f of D, its conjugate section g on C satisfies

$$\widehat{g} = \widehat{f}^{\sigma^{-1}}, \qquad g(w) = \mu^m \sigma^{-1}(f(w')). \tag{19.3}$$

The same w', μ work for every degree and every section.

Proof. Conjugate the nonsingular cubic $y^2 + a_1xy = x^3 + x$, together with its marked points and its CM endomorphisms. Its group origin here is the point $(x, y) = (0, 0)$ corresponding to $z = 0$; the marked half-period is the point at infinity in these translated coordinates. This pointed convention is retained under conjugation. The invariant differential is $dx/(2y + a_1x)$. Uniformizing the conjugate cubic with this differential gives D. The conjugate of the endomorphism with differential $a \in \mathcal{O}_K$ still has differential a, since σ fixes K; on its universal cover it is multiplication by a. Thus Λ_D is $\mathcal{O}_K$-stable. The distinguished point and the normalization $\wp''(t) = 2$ are preserved: the latter is equivalent to the displayed normalized cubic and differential. This verifies the asserted CM datum, including the conditions on the two factors of 2.

The coordinate functions satisfy the initial-value system

$$x' = 2y + a_1x, \qquad y' = 1 + 3x^2 - a_1y, \qquad x(0) = y(0) = 0.$$

Coefficient recursion therefore proves $\widehat{x}_D = \widehat{x}_C^{\sigma}$ and $\widehat{y}_D = \widehat{y}_C^{\sigma}$. The section basis proved in Lemma 18.20 does not need its integrality hypothesis for the spanning assertion: every degree-m section is $\theta_C^m(P_0(x_C) + y_C P_1(x_C))$, with the pole-order bounds there. Hence the graded conjugation map is obtained by conjugating the coefficients of P_0, P_1. Initially its Taylor series is

$$R_\sigma(Z)^m \widehat{f}^{\sigma}, \qquad R_\sigma = \widehat{\theta}_D/\widehat{\theta}_C^{\sigma}, \qquad R_\sigma(0) = 1.$$

We show that $R_\sigma = 1$. Put $\beta = (1 + \sqrt{-d})/2$, $N = \beta\bar{\beta}$. The pullback $\theta_C(\beta Z)$ is a degree-N section for exactly the multiplier of θ_C^N, by Lemma 18.21. Its algebraic conjugate has the same zero divisor as $\theta_D(\beta Z)$, since conjugation preserves this CM endomorphism and the marked point. Two sections of the same degree with this divisor differ by a constant; their value at zero is one. Comparing their Taylor series and cancelling the unit series $\widehat{\theta}_C(\beta Z)^{\sigma}$ gives

$$R_\sigma(\beta Z) = R_\sigma(Z)^N.$$

If $R_\sigma - 1$ had a first nonzero coefficient in degree $r \geq 1$, this identity would give $\beta^r = N$. Since $|\beta| > 1$, absolute values force $r = 2$, and then $\beta^2 = \beta\bar{\beta}$ would make β real. This contradiction proves (19.2) and the assertion about all sections.

For (19.3), take w' above the conjugate of the point represented by w. The basis in Lemma 18.20 identifies the degree-m sections with those of $\mathcal{O}(m[t])$, compatibly with multiplication. Conjugation identifies the degree-one fibres at these points since it takes $[t_C]$ to $[t_D]$. In the analytic trivializations at w, w' this identification gives a nonzero scalar μ in the direction from D back to C. Its m-th tensor power gives (19.3), also over the marked point.

Lean, comparison. Lean compares the values of generators of the section ring. The fibre of $\mathcal{O}(m[t])$ is the m-th tensor power of the degree-one fibre. `ConjugateCurve` (L73-L116), `ConjugateCurve` (L471-L519). □

Lemma 19.3 (One finite adelic parameter for all translated Taylor series)**.** *Suppose $\Lambda_C = \Omega_0 \mathcal{O}_K$. For every $\sigma \in \mathrm{Aut}(\mathbb{C}/K)$, there are $\Omega_\sigma \in \mathbb{C}^\times$, a nonzero integral ideal $\mathfrak{c}_\sigma$, an integer $e \geq 1$, and a finite idele $b_\sigma \in \mathbb{A}_{K,f}^\times$ such that*

$$\Lambda_{C^\sigma} = \Omega_\sigma \mathfrak{c}_\sigma^2 / N\mathfrak{c}_\sigma, \quad A_{C^\sigma} = |\Omega_\sigma|^2 \sqrt{d}/2, \quad e\Omega_\sigma \in \Lambda_{C^\sigma}. \tag{19.4}$$

Moreover $b_\sigma \bar{b}_\sigma = \chi_{\mathrm{cyc}}(\sigma)$, where $\chi_{\mathrm{cyc}}(\sigma) \in \widehat{\mathbb{Z}}^\times$ describes the action on roots of unity. For each $x \in K$ there is a finite-adelic neighborhood U_x of $b_\sigma x$ such that every $x' \in K \cap U_x$ satisfies the full coefficientwise identity

$$\widehat{T^{(1)}_{x\Omega_0}\theta_C}^{\,\sigma} = \widehat{T^{(1)}_{x'\Omega_\sigma}\theta_{C^\sigma}}. \tag{19.5}$$

In particular, all conjugates over K of the degree-one torsion values of C have absolute value at most the constant M of Lemma 19.1.

Proof. Write $D = C^\sigma$. The ideal class group of K has odd order by Lemma 18.24. Write $\Lambda_D = w_0 \mathfrak{c}^2$ for an integral ideal $\mathfrak{c}$, and put $e = N\mathfrak{c}$, $\Omega_\sigma = e w_0$, $J = \mathfrak{c}^2/e$. Then $J\bar{J} = \mathcal{O}_K$ and $e \in J$, proving the area and lattice identities in (19.4).

In the coordinates $\Lambda_C = \Omega_0 \mathcal{O}_K$ and $\Lambda_D = \Omega_\sigma J$, the full Tate modules are $\widehat{\mathcal{O}}_K$ and $\widehat{J} = J \otimes \widehat{\mathbb{Z}}$. Conjugation on torsion commutes with every CM endomorphism, as in Lemma 19.2. It gives an $\widehat{\mathcal{O}}_K$-linear isomorphism between these modules, hence multiplication by a finite idele b_σ, the image of 1. The Riemann form in both ideal coordinates is $2\,\mathrm{Im}(\bar{x}y)/\sqrt{d}$, by the area formula. Multiplication by b_σ scales this alternating form locally by $b_\sigma \bar{b}_\sigma$. On M-torsion the Weil pairing is the exponential of $2\pi i/M$ times the integral lattice pairing. Its equivariance under conjugation and its perfectness give $b_\sigma \bar{b}_\sigma \equiv \chi_{\mathrm{cyc}}(\sigma) \bmod M$ for every M [Sil09, Ch. III, §8 and Ch. VI].

For $x \in K$ take $n \geq 1$ with $nx \in \mathcal{O}_K$. For finitely many x, choose a common n. By the Chinese remainder theorem choose $v \in J$ congruent to $b_\sigma \bmod 2n^2 \widehat{J}$. Set $u = x\Omega_0$ and $u' = xv\Omega_\sigma$. Conjugation takes the classes of u and $-u/(2n)$ to those of u' and $-u'/(2n)$, respectively, by the action on $2n^2$-torsion. Consider the elliptic function

$$R_u(Z) = \frac{(T_u^{(1)}\theta_C)(nZ)}{\theta_C(nZ)}.$$

The multipliers cancel since $nu \in \Lambda_C$. Its divisor is $[n]^*([t_C - u] - [t_C])$. At $Z = -u/(2n)$ it has value $(-1)^{\mathrm{ord}_{u/2}\,\theta_C}$, even if the numerator and denominator both vanish. Indeed at $nZ = -u/2 + s$ it equals $e^{-H(u,s)}\theta_C(u/2+s)/\theta_C(u/2-s)$ by evenness. The order is one when $u/2$ represents the marked point and zero otherwise. This condition is preserved by conjugation since $u/2$ is the multiple $-n$ of $-u/(2n)$. The corresponding rational function on D, formed with u', has the conjugate divisor and the same nonzero value at the conjugate point. These determine a rational function uniquely on a complete curve. Its germ in the parameter of the conjugate invariant differential is therefore the coefficientwise conjugate of the germ of R_u. Multiplying by the unshifted germ in (19.2) and cancelling the substitution $Z \mapsto nZ$ gives

$$\widehat{T^{(1)}_{x\Omega_0}\theta_C}^{\,\sigma} = \widehat{T^{(1)}_{xv\Omega_\sigma}\theta_D}. \tag{19.6}$$

Take $U_x = xb_\sigma + 2n\widehat{J}$. Since $x(b_\sigma - v) \in 2n\widehat{J}$, every $x' \in K \cap U_x$ satisfies $x' - xv \in 2nJ$. Thus $x'\Omega_\sigma - u'$ is $2n$ times a period, while nu' is a period. In the translation law the semicharacter of this period difference is one, and its pairing with u' is an even integer. The two translated sections on D coincide, proving (19.5). Density gives such a rational x'. Taking constant terms and applying (19.1) gives the bound.

Lean, comparison. The formal proof matches finite sets of translates and passes to an adelic limit. Here conjugation on the Tate module gives one idele. The divisor and one signed value fix the scalar. `ConjugateCurve` (L519-L759), `ChowlaSelberg` (L278-L395). □

Theorem 19.4 (An integral vacuum with a bound at every complex embedding)**.** *There is an integer $C \geq 1$, depending only on the fixed quadratic field, with the following property. For every field $E = K(\zeta_q)$ in the family, there is a scalar $c_{\mathrm{vac}} \neq 0$ such that, for the standard rank-n vacuum v and $v^\circ = c_{\mathrm{vac}}^n v$,*

$$|\iota(v^\circ(\mathrm{finVec}\, x))|_p \leq 1 \qquad \text{for every } p, \iota, x, \tag{19.7}$$

$$|\sigma(v^\circ(\mathrm{finVec}\, x))| \leq \exp(Cfn) \qquad \text{for every } \sigma \in \mathrm{Aut}(\mathbb{C}), x. \tag{19.8}$$

The single chosen scalar works for all ranks and finite points. Since $f \leq q$, these bounds imply the simultaneous clauses of IsVacuumScale *in every rank. They prove (IV) for the original construction.*

Proof. Take the frame of scalar h from Lemma 18.26 and repeat each member h times. This gives scalar h^2 with $J \ll_{Q,d} f$ members, since $h \ll_{Q,d} 1$. Use this square frame in Lemma 18.27 with $b = h$. Every finite-point value of its rank-one product F is a product of J degree-one CM theta torsion values, by (18.40) and local constancy. These values are integral by Lemma 18.19. For the archimedean bound let $w(u) = (\theta_C[u]^2, \sigma_C[u]^2)$. The same lemma gives the homogeneous duplication identity $w(2u) = D(w(u))$ of (18.26). The coefficients of D have only finitely many conjugates for fixed d, since the normalized curves have finitely many CM j-invariants, which are algebraic, and $j = (a_1^4 - 48)^3/(a_1^4 - 64)$. The unit discriminant makes every conjugate map nonvanishing on the complex unit sphere for the maximum norm. By compactness and homogeneity, one $c > 0$ gives $\|D^\sigma(z)\|_\infty \geq c\|z\|_\infty^4$ for all σ, z. For torsion u the norms of $\sigma w(2^j u)$ are bounded in j, since the classes form a finite set, and period changes only multiply the coordinates by roots of unity. Iterating gives $\|\sigma w(u)\|_\infty \leq c^{-1/3}$, since otherwise the norms would grow without bound. Thus every conjugate of the product is bounded by M^J with $M = \max(1, c^{-1/6})$.

Equation (18.41) expresses $c_x v_1$ as a sum of h^{2f} signed translates of F, with coefficients of absolute value one at every place. The inverse identity and $\Theta_F(0) = 1$ give some $c_x \neq 0$. Fix $c_{\mathrm{vac}} = c_x$ once for all ranks and embeddings. The ultrametric inequality gives the finite bound in rank one, and the archimedean bound is $h^{2f} M^J \leq \exp(Cf)$. The constant may depend only on d: there are finitely many admissible Q for this d, since $d \equiv Q/2 - 1 \bmod Q$ gives $Q \mid 2(d+1)$.

Finally the standard rank-n vacuum is the tensor product of the rank-one vacua, and at finite points

$$v^\circ(\mathrm{finVec}\, x) = \prod_{i=1}^{n} c_{\mathrm{vac}} v_1(\mathrm{finVec}\, x_i).$$

Taking products proves both assertions in every rank.

Lean, comparison. Lean uses q^2 factors for the power frame. The lifted frame gives $O_{Q,d}(f)$ factors in the Fourier sum, and duplication bounds their values directly. `IntegralVacuum` (L209-L278), `IntegralVacuum` (L294-L403). □

Corollary 19.5 (Integral vacuum values, their heights, and the jet bound)**.** *Every value of the original v° at a finite point is an algebraic integer. Its absolute logarithmic Weil height is at most Cfn. The same assertions hold for its finite signed Heisenberg translates. Also*

$$\langle v^\circ, v^\circ \rangle \leq \exp(2Cfn)$$

with the normalized theta-data inner product. The conclusions of Theorem 18.29 and Corollary 18.32 now apply to this original vacuum without retaining (IV) as an additional hypothesis.

Proof. A complex number whose orbit under all automorphisms of $\mathbb{C}$ is bounded is algebraic: if it were transcendental, an isomorphism between the rational function fields that sends it to an arbitrarily large transcendental number extends, using transcendence bases and algebraic closures, to an automorphism of $\mathbb{C}$. Apply this observation to (19.8). The finite inequalities then make the algebraic value an integer. Its logarithmic height is the average of the logarithms of the archimedean maxima with one, hence is bounded by Cfn. Finite signed Heisenberg translation changes a finite-point value by a root of unity and a translation of its argument, so the same bounds apply.

The inner product is the mean of $|v^\circ|^2$ on the adelic quotient. At a point with rational archimedean coordinate, translation by the negative of that rational vector gives a finite point and changes the value by a unit-modulus Heisenberg factor. Rational points are dense at infinity, so continuity extends the identity-automorphism bound to all points. Taking the mean proves the displayed norm bound. Finally (19.7) is precisely (IV), so the two previously proved jet results apply to the same original scalar.

Lean, formal context. `IntegralVacuum` (L294-L403), `ArchimedeanSize` (L436). □

Subsequent height estimates apply the bound of Theorem 19.4 to the algebraic values of the scaled vacuum and to the transported theta data.

The area formula for the CM period. Retain $K = \mathbb{Q}(\sqrt{-d})$, with d prime and $d \equiv 7 \pmod 8$, and write h_K for its class number. Let χ_d be its quadratic Dirichlet character and put

$$\mathcal{C}_d = \gamma_E - \frac{L'(1,\chi_d)}{L(1,\chi_d)},$$

where γ_E is Euler's constant. The quotient is real, since χ_d is real and $L(1,\chi_d) > 0$. We now prove the area identity for the root u_{CS} in (18.34). Its finite unit property is already Lemma 18.24.

Theorem 19.6 (Both normalizations of the same period root)**.** *Take the curves, ideals, and root u_{CS} of Lemma 18.24. For every $\sigma \in \mathrm{Aut}(\mathbb{C}/K)$, let Λ^σ be the period lattice of the algebraic conjugate of the normalized cubic C, using its conjugate invariant differential. Then*

$$\mathrm{area}(\Lambda^\sigma)|\sigma(u_{\mathrm{CS}})|^2 = \frac{2\pi^2}{\sqrt{d}}\exp(\mathcal{C}_d). \tag{19.9}$$

The same root satisfies $|\iota_p(u_{\mathrm{CS}})|_p = 1$ at every finite embedding. Thus the finite and archimedean assertions concern a single choice of u_{CS}.

Proof. Put $I_c = \mathfrak{c}_c$, $h = h_K$, and use the Hom modules of Lemma 18.24. Conjugating the algebraic maps and their invariant differentials carries every multiplier to its σ-image. As σ fixes K, the multiplier module from C^σ to C_c^σ is $\sigma(\mu_c)I_c$. The conjugate lattices still have CM by $\mathcal{O}_K$, since its endomorphisms conjugate with the same differential multipliers. For two such isogenous lattices $(L' : L)L = L'$ by invertibility. Consequently

$$\Lambda_c^\sigma = \sigma(\mu_c)I_c\Lambda^\sigma.$$

Write $\Delta(L) = g_2(L)^3 - 27g_3(L)^2$ for the discriminant relative to the uniformizing differential. For the normalized cubic it is $a_1^4 - 64$. Write $\Delta_c = \Delta(\Lambda_c)$. Homogeneity gives $\sigma(\mu_c)^{12}\sigma(\Delta_c) = \Delta(I_c\Lambda^\sigma)$. Define the homothety invariant $B(L) = \mathrm{area}(L)|\Delta(L)|^{1/6}$. Taking absolute values in the defining

equation (18.34), using $\operatorname{area}(I_cL) = NI_c \operatorname{area}(L)$ for a CM lattice and $\prod_c NI_c = 1$, gives the exact identity

$$\Big(\operatorname{area}(\Lambda^\sigma)|\sigma(u_{\mathrm{CS}})|^2\Big)^h = \prod_c B(I_c\Lambda^\sigma).$$

The classes on the right run through $\mathrm{Cl}(K)$ once: writing $\Lambda^\sigma = bJ$ with a fractional ideal J, they are $[I_cJ]$.

For the constant, let $\Delta_0 = (2\pi)^{-12}\Delta$, the modular discriminant with first Fourier coefficient one. In the logarithmic form of the Chowla–Selberg formula [SC67][How19, Theorem 2.1.1 and (2.1.1)],

$$\frac{1}{12h}\sum_{[I]} \log|\Delta_0(I)\Delta_0(I^{-1})| = \frac{L'(0,\chi_d)}{L(0,\chi_d)} - \log(2\pi).$$

Since $I^{-1} = \bar{I}/NI$, $\Delta(\bar{I}) = \overline{\Delta(I)}$, and $\operatorname{area}(I) = NI\sqrt{d}/2$, the conversion is

$$B(I)^{12} = (\sqrt{d}/2)^{12}(2\pi)^{24}|\Delta_0(I)\Delta_0(I^{-1})|.$$

Thus the mean of $\log B(I)$ equals $\log(\pi\sqrt{d}) + L'(0,\chi_d)/L(0,\chi_d)$. The functional equation for the odd primitive character χ_d has completion $(d/\pi)^{(s+1)/2}\Gamma((s+1)/2)L(s,\chi_d)$. Logarithmic differentiation at $0, 1$, using $\psi(1) = -\gamma_E$ and $\psi(1/2) = -\gamma_E - 2\log 2$ for $\psi = \Gamma'/\Gamma$, gives

$$\frac{L'(0,\chi_d)}{L(0,\chi_d)} = \mathcal{C}_d + \log(2\pi/d).$$

The positive h-th root in the product identity now gives (19.9). Lemma 18.24 applies to this same u_{CS}; changing its chosen root multiplies it by a root of unity.

Lean, comparison. Lean derives the class product from the Kronecker limit formula and partial zeta functions, then transports isogenies and their degrees. Here the published Chowla–Selberg product supplies the class constant, and algebraic conjugation of the Hom modules identifies its value for each specified period. `ChowlaSelberg` (L510-L535), `ChowlaSelberg` (L537-L632), `ChowlaSelberg` (L647-L757). □

Corollary 19.7 (The absolute value of the jet scale)**.** *In the notation of* (18.35)*,*

$$|u_{\mathrm{CS}}\Omega_0|^2 = \frac{4\pi^2}{d}\exp(\mathcal{C}_d), \qquad |j_0|^2 = |s_0|^2\frac{\sqrt{d}}{2\pi}\exp(-\mathcal{C}_d). \tag{19.10}$$

These are identities for the already chosen period and jet scale.

Proof. For $\Lambda = \Omega_0\mathcal{O}_K$, its area is $|\Omega_0|^2\sqrt{d}/2$. Use (19.9) with $\sigma = 1$, then substitute $j_0 = s_0A_0/(u_{\mathrm{CS}}\Omega_0)$ and $A_0^2 = 2\pi/\sqrt{d}$.

Lean, formal context. `ChowlaSelberg` (L647-L757), `ArchimedeanSize` (L436). □

19.1 Normalized conjugate kernels and the final restriction

Let $q = q_1(s)$, $E = K(\zeta_q)$, and $f = [E : K]$. Use the jet scale j_0 of (18.35) and put

$$\Omega = \frac{2\pi}{\mathrm{rd}_E}e^{\mathcal{C}_d} > 0,$$

where rd_E is the root discriminant of E and $\mathcal{C}_d$ is the constant of §19. This is `bigOmega`(s) in Lean. In each complex coordinate of the definite oscillator representation use the Gaussian $e^{-|z|^2/2}$ and the measure $\pi^{-1}\,d\Re z\,d\Im z$. For the Weyl modules use the Gram norms in integral standard-tableau coordinates. The restriction calculation below keeps their Gram matrices.

Lemma 19.8 (Conjugates of the algebraic root and jet scale)**.** *For the original root s_0 in (18.35), every complex automorphism satisfies*

$$|\sigma(s_0)|^2 = \mathrm{rd}_E/\sqrt{d}. \tag{19.11}$$

If σ fixes K and Ω_σ is the period representative in (19.4), then

$$t_\sigma = \frac{\sigma(s_0/u_{\mathrm{CS}})}{\Omega_\sigma}\sqrt{\frac{2\pi}{\sqrt{d}}} \quad \textit{satisfies} \quad |t_\sigma|^2 = \Omega^{-1}. \tag{19.12}$$

Proof. The finite character has principal-idele law $\chi_{\mathrm{fin}}(a) = N_{E/K}(\bar{a})$, and its values on finite unit ideles are roots of unity. Raise the ideal of α to a power that is principal, and then kill the resulting unit character value by another positive power. This proves $\chi_{\mathrm{fin}}(\alpha)^h \in K^\times$ for some $h \geq 1$. Since $s_0^f = \chi_{\mathrm{fin}}(\alpha)^{-1}$, a positive power of s_0 belongs to $K^\times$. Every automorphism fixing K therefore preserves $|s_0|$.

Use the finite-idele norm normalized by the product formula. For $\mathfrak{a}_0 = E \cap \alpha\widehat{\mathcal{O}}_E$, we have $|\alpha| = (N\mathfrak{a}_0)^{-1}$ and $\mathfrak{a}_0\bar{\mathfrak{a}}_0 = \mathfrak{D}_{E/K}^{-1}$. Consequently

$$|\alpha|^2 = N\mathfrak{D}_{E/K} = \frac{|D_E|}{d^f} = \left(\frac{\mathrm{rd}_E}{\sqrt{d}}\right)^{2f}.$$

The unitary splitting character in $\chi_{\mathrm{fin}}(\alpha) = v_{\mathrm{fin}}(\alpha)|\alpha|^{-1/2}$ has modulus one. Taking squared absolute values of the defining equation for s_0 gives

$$(|s_0|^2)^f = |\alpha|.$$

Positive real roots give (19.11) for automorphisms over K. For an arbitrary automorphism, compose on the output with complex conjugation if necessary to fix K; this changes no absolute value.

Set $A_\sigma = \mathrm{area}(\Lambda_{C^\sigma})$. By (19.4) and Theorem 19.6,

$$|t_\sigma|^2 = \frac{|\sigma s_0|^2\pi}{A_\sigma|\sigma u_{\mathrm{CS}}|^2} = \frac{(\mathrm{rd}_E/\sqrt{d})\pi}{(2\pi^2/\sqrt{d})e^{\mathcal{C}_d}} = \frac{\mathrm{rd}_E}{2\pi}e^{-\mathcal{C}_d} = \Omega^{-1}.$$

This computation uses the original algebraic root, period root and theta normalization simultaneously.

Lean. `WeilData` (L478-L541), `IntegralVacuum` (L467-L485). □

Lemma 19.9 (Conjugation on the finite oscillator space)**.** *For every $\sigma \in \mathrm{Aut}(\mathbb{C}/K)$ and every standard definite space V over E, there are a finite idele b, a nonzero complex number t, and a σ-semilinear map $\mathcal{L}$ on the span of the finite Heisenberg translates of the vacuum such that*

$$|t|^2 = \Omega^{-1}, \qquad \sigma(\mathrm{jet}_{j_0} F) = \mathrm{rename}_\sigma\, \mathrm{jet}_t(\mathcal{L}F),$$

$$\mathcal{L}\rho(x)F = \rho(bx)\mathcal{L}F, \qquad \sigma\psi(z) = \psi(b\bar{b}z).$$

The map is injective. There are $a \neq 0$ and a finite unitary scalar u such that

$$\mathcal{L}v = a\,\omega(u)v, \qquad \mathcal{L}\omega(g)F = \varepsilon_{\sigma,g}\,\omega(g)\mathcal{L}F, \quad |\varepsilon_{\sigma,g}| = 1,$$

where v is the unscaled vacuum and g is any finite unitary point. These statements concern coefficientwise conjugation of Taylor series; no discontinuous automorphism is interchanged with an analytic integral.

Proof. Lemma 19.3 gives one scalar Ω_σ and one finite adele β_σ of K, compatible with every finite set of rational translates. Take $b = \beta_\sigma$ through $\mathbb{A}_{K,f} \to \mathbb{A}_{E,f}$. Use the theta-product vector F_0 in the proof of Theorem 19.4, and the same product on the conjugate CM lattice to obtain F_0'. A fixed positive integer clears the denominator of Ω_σ in that lattice, so F_0' belongs to the same finite oscillator space. The linear forms use the chosen frame members in E. Conjugation permutes their embeddings over K. Applying the CM Taylor identity to each factor gives

$$\sigma\big(\mathrm{jet}_{j_0}(\rho(x)F_0)\big) = \mathrm{rename}_\sigma\,\mathrm{jet}_t(\rho(bx)F_0'), \qquad t = \frac{\sigma(s_0/u_{\mathrm{CS}})}{\Omega_\sigma}\sqrt{\frac{2\pi}{\sqrt{d}}}.$$

Here s_0/u_{CS} is the original period factor. Lemma 19.8 gives $|t|^2 = \Omega^{-1}$. Initially the equality is at rational x. For a given rational vector only finitely many K-coordinates occur; weak approximation chooses a single rational multiplier in all the neighborhoods of β_σ specified by the CM lemma. Finite translation of each of F_0, F_0' is locally constant. Density of E in $\mathbb{A}_{E,f}$ therefore extends the equality to every finite x, without taking an analytic limit of σ.

The finite translates of F_0 span the finite oscillator space: the nonzero Fourier coefficient extracted in the vacuum construction expresses a nonzero multiple of v as a finite combination of these translates. Define $\mathcal{L}(\rho(x)F_0) = \rho(bx)F_0'$ and extend σ-semilinearly. This is well defined because Taylor series at nonzero scale determine a vector in the finite oscillator space: a relation among the source translates gives the zero conjugate series, and hence the zero vector on the target side. The same observation proves injectivity and the asserted jet identity.

By Lemma 19.3, $b\bar{b} = \chi_{\mathrm{cyc}}(\sigma)$. The finite additive character takes values in roots of unity, so $\sigma\psi(z) = \psi(b\bar{b}z)$. The Heisenberg multiplication law therefore gives $\mathcal{L}\rho(x) = \rho(bx)\mathcal{L}$ on the spanning translates. In particular $b\bar{b}$ is a unit everywhere. Place by place write $b = ua_0$, where $u\bar{u} = 1$ and $a_0 \in \widehat{\mathcal{O}}_E^\times$. At a self-conjugate place b is already a unit; at a conjugate pair choose the two components of u reciprocally. Set $\mathcal{L}_0 = \omega(u)^{-1}\mathcal{L}$. The lattice Heisenberg operators fix v and, since multiplication by a_0 permutes their parameters, also fix $\mathcal{L}_0 v$. Their invariant space is the one-dimensional vacuum line. This last assertion follows in a finite Schrödinger model by imposing successively invariance under translations and under the dual characters. Hence $\mathcal{L}_0 v = av$, with $a \neq 0$ by injectivity.

For completeness, the Weil intertwining scalar is also controlled. The expansion of $\omega(g)v$ over the finite quotient of the standard lattice by its intersection with g^{-1} of that lattice is a sum of Heisenberg translates, multiplied by the finite Weil multiplier. Multiplication by a_0 permutes this quotient. Applying $\mathcal{L}_0$ to the sum gives the same sum times a, with the multiplier conjugated. The quotient of the conjugated multiplier by the original one has modulus one: its square is the quotient of the associated unitary character values. For the original splitting character these values retain modulus one under σ: a positive power of its value on a norm-one finite idele lies in K, by the same principal-idele argument as in Lemma 19.8, and that power has modulus one. The formula extends from v to its translates by Heisenberg covariance. Finally scalar u commutes with g, so the formula holds for $\mathcal{L}$ as well.

Lean, comparison. The Weil pairing gives the central character before the oscillator intertwiner is defined. `IntegralVacuum` (L595-L655), `JetConjugatesAndBounds` (L147-L204), `GaloisLaw` (L234-L305). □

Lemma 19.10 (A uniform bound for a moved vacuum)**.** *For a definite space of rank r, every adelic unitary point g and every adelic argument y, the unscaled vacuum satisfies*

$$|\omega(g)v(y)| \le \left(1 - e^{-\pi/(4\sqrt{d})}\right)^{-(2fr+1)/2}.$$

Proof. Move the pointed space by g. The matrix-coefficient formula for the oscillator representation writes $|\omega(g)v(y)|^2$ as a theta series whose terms, in absolute value, are bounded by

$$\exp\left(-\frac{\pi}{\sqrt{d}}|\lambda|^2\right), \qquad \lambda \in \Gamma.$$

The finite support defines a lattice Γ in a real Euclidean space of dimension $D = 2fr$. Its nonzero vectors have squared length a positive integer. Indeed their Hermitian value belongs to the inverse different of E/K, because the fixed finite scale satisfies $\alpha\bar{\alpha}\mathfrak{D}_{E/K} \subseteq \widehat{\mathcal{O}}_E$. Its trace to K is integral and is the sum of the positive real squared lengths. The real elements of $\mathcal{O}_K$ are integers.

On the shell $S_k = \{\lambda \in \Gamma : |\lambda|^2 = k\}$ the nonzero squared distances are integers at most $4k$. Thus $|S_k| \le \binom{D+4k}{4k}$, a special case of the few-distance bound of Bannai, Bannai and Stanton [BBS83]. Indeed the polynomial

$$\prod_{i=1}^{4k}(2k - 2\langle\lambda, X\rangle - i)$$

vanishes at every other point of S_k and is nonzero at λ. These polynomials are linearly independent in the space of polynomials of degree at most $4k$ in D variables. With $u = e^{-\pi/(4\sqrt{d})}$ the binomial series gives

$$|\omega(g)v(y)|^2 \le \sum_{k\ge 0}\binom{D+4k}{4k}u^{4k} \le \sum_{j\ge 0}\binom{D+j}{j}u^j = (1-u)^{-D-1}.$$

Lean, comparison. Lean compares disjoint balls with a radial integral. Counting integral distances on each shell gives a bound linear in $fr+1$ for the logarithm. `JetConjugatesAndBounds` (L302-L432). □

Lemma 19.11 (Conjugate translates of the integral vacuum)**.** *Let $v_{\mathrm{int}} = c_{\mathrm{vac}}^r v$ use the same original scalar as Theorem 19.4. There is a constant C, depending on the fixed parameters only, such that the conjugation map of Lemma 19.9 satisfies*

$$|\mathcal{L}(\rho(x)\omega(g)v_{\mathrm{int}})(y)| \le \exp\big(C(fr+1)\big)$$

for every finite x, g and every adelic y.

Proof. Write $\mathcal{L}v_{\mathrm{int}} = A\omega(u)v$. Taking constant terms of the translated jet identity gives

$$\mathcal{L}v_{\mathrm{int}}(bx) = \sigma(v_{\mathrm{int}}(x)).$$

Multiplication by b is bijective. The finite-value bound from Theorem 19.4 therefore bounds every finite value of $\mathcal{L}v_{\mathrm{int}}$ by $\exp(C_0 fr)$. The invariant inner product is the mean square on the compact adelic quotient, with probability measure. Rational archimedean coordinates can be moved to finite points by a rational Heisenberg translation, which changes only a unit phase. Their density and continuity extend the finite-point bound to every point of this quotient. Hence $|A| \le \exp(C_0 fr)$, since $\omega(u)v$ has norm one. The normalized mean introduces no volume or dimension factor.

The intertwining identities of Lemma 19.9 express the vector in the statement, up to a unit scalar, as $A\rho(bx)\omega(gu)v$. Apply Lemma 19.10. Its logarithmic bound is $\frac{1}{2}(2fr+1)\log(1/(1-e^{-\pi/(4\sqrt{d})}))$. Since $(1-e^{-u})^{-1} \le 1+u^{-1}$ for $u>0$, the logarithm here is at most $\log(1+4\sqrt{d}/\pi) \le 2d+6$. Adding $C_0 fr$ proves the assertion with $C = C_0 + 2d + 6$, hence also with the parameter bound C_{par}.

Lean, comparison. The Fourier normalization and the moved lattice now both contribute $O_{Q,d}(fr+1)$ to the logarithm. `JetConjugatesAndBounds` (L446-L487). □

Lemma 19.12 (The Gaussian jet bound)**.** *Let the complex dimension be $D = fn'n$, and let F be a finite combination of Heisenberg translates of the original integral vacuum on the rank-$n'n$ hom space. Suppose the sum of the absolute values of its coefficients is at most B after every complex automorphism. For a polynomial Q, assume*

$$|\sigma Q(z)| \le A e^{w+\ell R} \quad \text{whenever} \quad \sum_i |\sigma z_i|^2 \le T(2\ell+2)R, \qquad A \ge 0,$$

where $\ell \ge 0$ and $|\sigma c|^2/\Omega \le T$. Then the conjugate of the jet of the translated dual-pair kernel at scale cj_0 is bounded by

$$BA\exp\bigl(C(D+1)+w\bigr)(2\ell+2)^D.$$

Proof. It suffices first to take one translate and an automorphism over K. Lemma 19.9 identifies its conjugated Taylor coefficients with those of a vector F'. By Lemma 19.11, the full translated vacuum satisfies $|F'(z)| \le \exp(C(D+1))$. In commuting conjugation past the finite Weil action, the multiplier enters through the ratio of its conjugate to its original value. This ratio has absolute value one by Lemma 19.9. The finite Heisenberg phases are roots of unity. The Fock reproducing identity, checked on monomials, expresses the conjugated jet as

$$\int_{\mathbb{C}^D} Q_\sigma(t\bar z)\, e^{-|z|^2/2} F'(z)\, \frac{dz}{\pi^D}.$$

The polynomial Q_σ incorporates coefficient conjugation, the factor c, and the permutation of places. Its argument has squared length $|\sigma c|^2\Omega^{-1}|z|^2$. Set $R = |z|^2/(2\ell+2)$ in the growth assumption. Multiplication by the vacuum leaves the integrable majorant

$$A\exp\bigl(C(D+1)+w\bigr)e^{-|z|^2/(2\ell+2)}.$$

Its integral is the displayed bound with $B = 1$. The sum of the absolute values of the conjugated coefficients is at most B, so the triangle inequality gives the asserted bound for F. Every automorphism of $\mathbb{C}$ either fixes K or becomes one that fixes K after composition with complex conjugation; this composition preserves all absolute values. The estimate therefore holds for every σ.

Lean. `map_omega_of_unit`, `JetConjugatesAndBounds` (L540-L655), `JetConjugatesAndBounds` (L731-L777). □

Lemma 19.13 (Schur growth with the row factorials)**.** *Fix $\sigma \in \mathrm{Aut}(\mathbb{C})$. Let $l_{\tau,i}$ be the row lengths of a fitting lift shape, and let $\ell > 0$. Write a complex-linear covector as its coefficient row y, so its value on η is $y\eta$, and put*

$$Q_{y,\xi}(Z) = yS_\kappa(Z)\xi.$$

All Weyl norms below are the ordinary Hermitian Gram norms on the target complex coordinate spaces, computed after applying σ coefficientwise. The dual norm is

$$\|y_0\|_{\mathrm{Weyl},*} = \sup_{\|\eta_0\|_{\mathrm{Weyl}} \le 1} |y_0\eta_0|.$$

For arbitrary y and ξ, the polynomial $Q_{y,\xi}$ satisfies the preceding growth hypothesis with

$$A = \|\sigma y\|_{\mathrm{Weyl},*}\,\|\sigma\xi\|_{\mathrm{Weyl}}, \qquad T = e^u,$$
$$w = \frac{1}{2}\sum_{\tau,i}\left(\frac{l_{\tau,i}}{\ell} + \log(l_{\tau,i}!) + l_{\tau,i}u\right).$$

In particular one may take $A = \|\sigma\xi\|_{\mathrm{Weyl}}$ when $\|\sigma y\|_{\mathrm{Weyl},*} \le 1$. This condition is equivalent to

$$|\sigma(y\eta)| \le \|\sigma\eta\|_{\mathrm{Weyl}} \qquad \text{for every } \eta.$$

The condition is on the conjugated covector.

Proof. The polynomial matrices of S_κ have integer coefficients. Consequently their compatibility with coefficientwise conjugation gives the algebraic identity

$$\sigma(Q_{y,\xi}(Z)) = (\sigma y) S_\kappa(\sigma Z)(\sigma \xi).$$

This identity involves complex-linear covectors and field operations. Every absolute value and Hermitian norm in the following estimate is taken afterwards in the target copy of $\mathbb{C}$.

Let $\rho_{\tau,i}$ be the squared singular values of σZ_τ, in decreasing order. The exterior-power realization of the Schur module, with its inherited Gram norm, gives

$$\|S_\kappa(\sigma Z)\eta_0\|^2_{\mathrm{Weyl}} \le \prod_{\tau,i} \rho_{\tau,i}^{l_{\tau,i}} \|\eta_0\|^2_{\mathrm{Weyl}}, \qquad \sum_{\tau,i} \rho_{\tau,i} = \sum_j |\sigma z_j|^2.$$

Indeed a column of height h has operator norm the product of the first h singular values; tensoring the columns gives the row exponents, and restriction to the Schur module preserves the upper bound.

For $l \in \mathbb{N}$, $\rho \ge 0$, and real v, the exponential series gives

$$\rho^l \le l! \exp(lv + \rho e^{-v}).$$

Take $v = u + 1/\ell$. If $\sum \rho \le e^u(2\ell+2)R$, then

$$e^{-v} \sum \rho \le (2\ell+2)e^{-1/\ell} R \le 2\ell R,$$

using $e^{1/\ell} \ge 1 + 1/\ell$. Multiply the monomial estimates and take square roots. Applying the target dual norm to the algebraic identity above therefore gives

$$|\sigma(Q_{y,\xi}(Z))| \le \|\sigma y\|_{\mathrm{Weyl},*} \|\sigma \xi\|_{\mathrm{Weyl}}\, e^{w+\ell R}.$$

Surjectivity of coefficientwise σ proves the stated equivalence of the two conditions on the covector. The argument uses no preservation of Hermitian norms or of complex conjugation by a general automorphism of $\mathbb{C}$.

Lean. `JetConjugatesAndBounds` (L780-L854), `SchurMatrix` (L532-L546), `SchurNorm` (L207-L271). □

Theorem 19.14 (The norm of a single theta kernel)**.** *For the trivial-twist chain, let J be the matrix of a translated normalized vacuum kernel at stage $j < m$, at any finite pair of source and receiver points. For every $\sigma \in \mathrm{Aut}(\mathbb{C})$, its coefficientwise conjugate σJ, acting between the original Weyl coordinate spaces with their ordinary Hermitian Gram norms, has operator norm at most*

$$\exp(x^{26} + r_j/2), \qquad r_j = \sum_{\tau,i} \left(\log(l_{j,\tau,i}!) + l_{j,\tau,i} \log(M/\Omega)\right).$$

Thus (11.15) *holds with the same vacuum and jet normalization as the finite-place calculations.*

Proof. Fix σ and a source vector ξ. To test $\sigma(J\xi)$, take any target covector y_0 of dual Weyl norm at most one and put $y = \sigma^{-1} y_0$ coefficientwise. Then $\|\sigma y\|_{\mathrm{Weyl},*} \le 1$ and

$$\sigma(yJ\xi) = y_0\, \sigma(J\xi).$$

The left side is the conjugated jet for the polynomial $Q_{y,\xi}$. Lemmas 19.12 and 19.13 apply with $A = \|\sigma\xi\|_{\mathrm{Weyl}}$, independently of the unconjugated norm of y.

Here $B = 1$, $\ell = a + 1$, and $\pi_M \bar{\pi}_M = M$. Both embeddings of K thus give the same squared modulus M of the jet multiplier. Taking $u = \log(M/\Omega)$ and writing $D = f n_{j+1} n_j$, the exponent in the two lemmas is

$$C(D+1) + D\log(2a+4) + \frac{1}{2}\sum_{\tau,i} \frac{l_{j,\tau,i}}{a+1} + \frac{1}{2} r_j.$$

The parameter bound C includes the vacuum constant. Admissibility gives $x \geq 10^4$, $C \leq x$, $f \leq x^{12}$, and

$$n_j, n_{j+1} \leq k + m(5r+1) \leq 3x^5 + 2\sqrt{x}(15x^5 + 1) \leq x^6.$$

Thus $D \leq x^{24}$. Since $a = xb$ and $\log b \leq x^{1/100}$, we have $\log(2a+4) \leq \log 6 + \log x + \log b \leq x$. There are at most x^{18} rows over all places and each has length at most $2a$. The first three terms together are therefore at most $3x^{25} + x^{18} \leq x^{26}$. The specified vacuum constant inside the parameter bound; The forced lower bound for the scale; The field-size bound. Taking the supremum over y_0 proves the operator bound since $\sigma(J\xi) = (\sigma J)(\sigma\xi)$ and $\sigma\xi$ ranges over the source. The weaker exponent x^{50} used in the determinant estimate follows since $x^{26} \leq x^{50}$. Duality after conjugation for Schur jets; The two quadratic-field embeddings of the jet multiplier; The same jet multiplier and one-kernel exponent.

Lean, comparison. The linear logarithmic bound for the vacuum from Lemma 19.11 lowers the error exponent in the original one-kernel estimate. `AdmissibleData` (L46-L52), `Magnitudes` (L53-L85), `ArchimedeanSize` (L217-L258). □

Lemma 19.15 (Integral branching with a small tail denominator)**.** *Consider an interlacing pair from* GL_{n+1} *to* GL_n*. Of the columns of height* h*, let* $\gamma_{\tau,h}$ *lose a box and* $\kappa'_{\tau,h}$ *keep it at the embedding* τ*. The target Weyl diagram has* $\kappa^0_{\tau,h} = \kappa'_{\tau,h} + \gamma_{\tau,h+1}$ *columns of each height* $0 < h < n$*. The kept columns of height* n *contribute* $\kappa'_{\tau,n}$ *to the exponent of the determinant character. Suppose that* $k \in \mathbb{N}$ *and no column of height below* k *is kept. Take*

$$L \geq \sum_{\tau} \sum_{k \leq h < n} h \kappa^0_{\tau,h}, \qquad L \in \mathbb{N}.$$

The original integral branching construction admits an integer matrix B *with its prescribed diagonal-character intertwining identity and*

$$\|B\xi\|^2_{\mathrm{Weyl}} \leq (L!)^2 \|\xi\|^2_{\mathrm{Weyl}}.$$

For an empty tail the right multiplier is one.

Proof. At each embedding, factor off the determinant character of the target. Indeed, $\bigwedge^n \mathbb{Z}^n$ is the determinant line, generated over $\mathbb{Z}$ by the wedge of the standard basis. Each kept height-n column thus contributes a determinant without an integral denominator. Only the columns of heights $0 < h < n$ enter the Weyl diagram used to compute the hook number.

Put $p = \min(k-1, n-1)$, with subtraction truncated at zero. The branching highest polynomial uses leading minors with the new row first for a shortened column and last for a kept column. Expanding along that row gives two row orders differing by a cyclic shift. Since no column below k is kept, one row permutation puts the same p leading rows in each column of height greater than p and the first h of these rows in each column of height $h \leq p$, up to signs of minors. Delete these leading rows from the Weyl diagram. Its remaining number of boxes is

$$d_0 = \sum_{\tau} \sum_{0 < h < n} (h - p)_+ \, \kappa^0_{\tau,h} \leq \sum_{\tau} \sum_{k \leq h < n} h \, \kappa^0_{\tau,h} \leq L.$$

Indeed, every $0 < h < n$ with $h < k$ satisfies $h \leq p$, and $(h-p)_+ \leq h$ for the other columns. In particular, if all target columns have height n, the Weyl diagram is empty and its hook number is one.

For an integral Schur module, let E be its filling matrix into column tensors and $G = E^{\mathsf{t}}E$. If b_i are the standard-tableau minor products, the polynomial map is $\mathrm{toPoly}(\xi) = \sum_i (G\xi)_i b_i$. Put $H_{ij} = b_i(\partial)b_j(0)$; in particular H is the Fock matrix of the b_i, not of the polynomials $\mathrm{toPoly}(e_i)$. Then

$$GH = hI,$$

where h is the Fock square of the product P of the leading minors. To see this identity, both pairings intertwine the Schur action, so their ratio is a scalar on the irreducible module. Evaluation on the leading-minor vector, whose Gram norm is one, identifies that scalar as $P(\partial)P(0)$. This also proves the identity integrally, since both matrices have integer entries. Consequently multiplying any integral minor-product polynomial by h puts it in the polynomial image of an integral Weyl vector.

Apply this observation to the remaining d_0 boxes of the determinant-free diagram. Inserting fixed leading exterior vectors into every column carries its integral Weyl module into the original integral module. This can be checked on evaluation tensors at the block arrays $\mathrm{diag}(I, y)$; these tensors span over $\mathbb{Q}$, and the standard-tableau embedding has an integral left inverse Q by unitriangular straightening. If W denotes leading-wedge insertion composed with the smaller filling matrix, evaluation on these tensors gives $EQW = W$. Thus QW is an integral lift of the insertion. For an arbitrary array with invertible leading blocks, the Schur-complement formula expresses each inserted minor product as its common leading determinant factor times the corresponding smaller minor product. The smaller polynomial identity therefore gives the desired larger one on this dense open set, and hence as an identity of integral polynomials. It follows that the same small-tail h clears the target polynomial of the branching highest vector, with no further division in the integral coordinates.

This produces the integer branching matrix without another denominator. One explicit verification is to evaluate the branching polynomial at $\mathrm{diag}(x^{\mathsf{t}}, 1)y$. At a generic y, clear every entry of its new row except its first by the column operations $\mathrm{col}_j \mapsto y_0\,\mathrm{col}_j - y_j\,\mathrm{col}_0$. The resulting identity is the one already proved at a row $(t, 0, \dots, 0)$, multiplied by a monomial in the y_i. Comparing polynomial coefficients cancels this monomial integrally. Expansion in the integral standard-tableau basis then gives the columns of B. The substitutions $y \mapsto \mathrm{diag}(g, s)y$ and $x \mapsto g^{\mathsf{t}}x$ give the required intertwining identity, including the powers of s and the determinant twist.

The norm calculation uses contraction in the tensor model. Insert the new unit basis vector into the shortened columns and use the inclusion of the other columns. To account for the full determinant columns, put

$$d(y) = \prod_{\tau} \det(Y_\tau)^{\kappa'_{\tau,n}}, \qquad m_y = d(y)^{-1}\,\mathrm{extendedTensor}(Y^{\mathsf{t}}),$$

where Y_τ is the square part of the target array y, and restrict for the moment to $d(y) \neq 0$. Cauchy–Binet gives

$$\langle m_y, m_z\rangle = \langle \mathrm{ev}_y, \mathrm{ev}_z\rangle_G, \qquad \langle \mathrm{ev}_y, B\xi\rangle_G = h\langle m_y, E\xi\rangle.$$

Here the first identity follows because the full determinant factor in the unnormalized tensor pairing is $\prod_\tau \det(Y_\tau^{\mathsf{t}} Z_\tau)^{\kappa'_{\tau,n}} = d(y)d(z)$, which is cancelled by the two denominators; conjugating the first array gives the Hermitian identity displayed. The second follows from the highest-vector normalization and the block intertwining identity of B. Evaluation vectors on $d(y) \neq 0$ still span: a linear functional annihilating them gives a polynomial vanishing on a dense open set, hence the zero polynomial. Thus B/h is the adjoint of this isometry followed by the original tensor embedding; its operator norm is at most one. In particular $\|B\xi\|^2 \le h^2\|\xi\|^2$.

It remains to bound $h = P(\partial)P(0)$. For a real homogeneous polynomial $F = \sum_{|\alpha|=d} c_\alpha X^\alpha$, its symmetric coefficient tensor has squared norm $\sum_\alpha c_\alpha^2/\binom{d}{\alpha} = F(\partial)F(0)/d!$, since each ordered entry for α is $c_\alpha/\binom{d}{\alpha}$. The tensor norm is the Bombieri norm [BE95, p. 274]. Multiplication of polynomials symmetrizes the tensor product. Symmetrization is an orthogonal projection, so this norm is submultiplicative. Each leading minor of size j has norm one, since its determinant has $j!$ distinct squarefree monomials of degree j, each with coefficient ± 1. Thus their product P, of total degree d_0 across all embeddings, has norm at most one, including the empty product. It follows that $h \le d_0! \le L!$.

Lean, comparison. Lean estimates the coefficients and factorials separately. Symmetrizing the product of the minor tensors bounds the same Gram–Fock scalar directly. `NormBound` (L295-L362), `GramMatrix` (L95-L180), `LeadingRows` (L431-L502), `IntegralMap` (L469-L525). □

Theorem 19.16 (The final restriction with its exact level gain)**.** *For the restriction from X_m to Y_m, every form ϕ, every finite point h, and every complex automorphism σ satisfy*

$$\|\sigma(\operatorname{res}\phi)(h)\|_{\mathrm{Weyl}}^2 \le \exp\left(x^{70} - (|\Phi_a|a - fm\Delta_0)\log N\right)\|\sigma\phi(\operatorname{diag}(h,1))\|_{\mathrm{Weyl}}^2.$$

The branching matrix and scalar are the same ones used in the level-prime gauge calculation.

Proof. Write b_τ for the branching exponents. In the exact source coordinates the restriction is

$$(\operatorname{res}\phi)(h) = c_N B\phi(\operatorname{diag}(h,1)), \qquad c_N = \prod_\tau \tau(\bar\pi_N)^{-b_\tau}.$$

The matrix B has integer entries, so it commutes with coefficientwise conjugation. The target column counts at heights $h \ge k$ are at most $\Delta = 2r^8$, and the kept columns vanish below k. Lemma 19.15 therefore applies with $L = fn_m^2\Delta$. The bounds $f \le q \le x^{12}$, $n_m \le x^6$, and $r < 3x^5$ give $L \le x^{65}$ eventually. Thus $2L\log L \le x^{70}$ eventually, and $\|B\xi\|^2 \le e^{x^{70}}\|\xi\|^2$.

Every embedding of the CM field commutes with its complex conjugation. Since $\pi_N\bar\pi_N = N$, we have $|\sigma\tau(\bar\pi_N)|^2 = N$, including for arbitrary $\sigma \in \mathrm{Aut}(\mathbb{C})$. Consequently

$$|\sigma c_N|^2 = N^{-\sum_\tau b_\tau}.$$

The weight sums telescope to $b_\tau = \mathbf{1}_{\Phi_a}(\tau)a - m\Delta_0$; hence $\sum_\tau b_\tau = |\Phi_a|a - fm\Delta_0$. Multiplication of these two bounds proves the theorem.

Lean. `Columns` (L675-L742), `ChainValuation` (L353-L393). □

The norm of the restricted chain. Theorems 19.14 and 19.16 give the analytic bounds in (11.15) and the display following it. The bound $e^{x^{60}}$ for the sum of the coefficients, the class bound, and the row sizes then give the stated squared-norm bound for the restricted chain. The scalar is the one in Theorem 19.4, with the CM Taylor and area normalizations used there.

19.2 Uniform normalization of the complementary-rank exchange

We use the normalized equal-rank period $R_Y(v,\omega)$, with the framed spaces and measures of Section 18.5.

Lemma 19.17 (Calibrated periods with additional hyperbolic planes)**.** *Let W have rank $2k+2q$ and q negative signs, with $k \ge 1$ and $\mu_\tau \ge k$. There are one real Schwartz vector z for $(H \perp W, H \perp W)$ and one $C_0 > 0$ such that*

$$R_{H\perp W}(z,\omega) = C_0 \quad \textit{for every } \omega. \tag{19.13}$$

The vector and constant are chosen before the ramified set and finite character.

Proof. For an empty family of characters any z and $C_0 > 0$ suffice. Otherwise start with the vector on $H \perp J$ supplied by Proposition 18.42, where $\dim J = 2k$. Suppose a calibrated vector v has been constructed on a space Y of rank $2d$ containing an adapted plane. Add a plane to obtain Y^+. Lemma 12.6, at the positive parameter $d+1$, gives one Schwartz v^+ whose cone has exactly the theta integrals of v for every finite datum. The construction uses a compact real annulus and the fixed weight; its positive Mellin factor is independent of ω and can be divided out once.

The equal-rank integral on Y^+ is constant, so it equals its constant term along the added plane. That term is $\gamma_d(\omega) I_{Y,\omega}(v)$. Consequently Lemma 18.39 gives

$$R_{Y^+}(v^+, \omega) = \frac{|\gamma_d(\omega)|^2 |I_{Y,\omega}(v)|^2}{\Pi_\omega(d+1)} = d_d^2 R_Y(v, \omega).$$

The multiplier d_d^2 is fixed and positive. Repeating this step q times supplies the required vector on the framed model $H^{q+1} \perp J$.

To pass to the specified representative, put $Y = H^{q+1} \perp J$ and $Y_1 = H \perp W$, with rational Gram matrices J_Y, J_{Y_1} and specified adelic frames a, a_1. Both finite local forms are the prescribed standard form, and the infinite signatures coincide. Landherr's theorem, already among the classical inputs to this note, gives $c \in \mathrm{GL}(E)$ with $c^* J_Y c = J_{Y_1}$. Set $k = a^{-1} c a_1$. This belongs to the common adelic unitary group, and

$$a_1 = c^{-1} a k, \qquad \Gamma_Y = k \Gamma_{Y_1} k^{-1}.$$

Conjugating rational matrices by c identifies the rational theta summands. In the common oscillator coordinates the real vector and the ramified determinant datum are unchanged; covariance gives

$$\Theta_{Y_1, Y_1, \omega}(v; g, h) = u_\omega \Theta_{Y, Y, \omega}(v; kg, kh), \qquad |u_\omega| = 1.$$

Indeed the possible factor is a quotient of the two unitary splitting-character values at $\det k$; for identical changes and identical equal-rank splitting characters they cancel. The map $h \mapsto kh$ identifies the quotients, by the displayed conjugacy of their rational subgroups, and preserves their specified quotient measures by left invariance of the ambient adelic Haar measure. Hence $I_{Y_1,\omega}(v)(g) = u_\omega I_{Y,\omega}(v)(kg)$. The model integral is constant, so its squared absolute value is unchanged. The same vector in their common real Fock coordinates therefore satisfies (19.13). No invariance of the finite datum under k is needed: k translates the group argument, while the datum stays fixed.

Lean. `RemovingPlanes` (L107-L217), `TheoremForPartners` (L740-L858). □

Theorem 19.18 (The exchange with a uniform scalar modulus)**.** *In the complementary-rank configuration of* (12.7)*, suppose*

$$n = a + b, \qquad a = b + 2e, \qquad q_W = q_V + q_{V'}, \qquad 2q_W + 2 \le n \le 2\mu_\tau + 2q_W, \qquad e \in \mathbb{Z}_{\ge 0}.$$

For every prescribed real Schwartz vector v for (V', W), there are one real Schwartz vector w for (W, V) and one constant $C > 0$, both independent of the finite character and modulus, such that every allowed tame character ω has a nonzero exchange scalar c_ω with

$$|c_\omega|^2 = C q(\omega)^{e^2} \prod_{j=1}^{e} |L^{S_\omega}(j, \omega)|^2. \tag{19.14}$$

The exchange identity holds for all matched finite data with these same vectors and scalar, and for all splitting characters satisfying the product condition of (12.7)*.*

Proof. Put $k = n/2 - q_W$. The assumptions give $k \geq 1$ and $k \leq \mu_\tau$. Choose z and $C_0 > 0$ from Lemma 19.17, and apply Lemma 18.41 with prescribed vector v. Since $a - n/2 = e$, its scalar equation becomes

$$A|c_\omega|^2 = C_0 \Pi_\omega(e).$$

Take $C = C_0/A > 0$. Positivity of $\Pi_\omega(e)$ also proves $c_\omega \neq 0$.

For data modulo the finite modulus, lift the functions by reduction on integral matrices and extend them by zero. The reduction map is surjective and detects units. Hence a matrix completes to an invertible integral matrix exactly when its reduction does, and the two determinant characters agree on units. The lifted functions are matched and locally constant and produce exactly the original adelic data. Thus the semilocal exchange gives the claimed finite-modulus identity with no additional scalar.

The base splitting characters on the two sides are $(\gamma, 1)$ and (γ, γ), where $\gamma = \omega^{-1} \upsilon_{U(1)}^{n/2}$. Under the given product condition, replacing them by (γ, α) and (α', γ) multiplies both functions on $U(W)$ by $\alpha(\det g)^{-1}$. It neither changes the real vectors nor the scalar modulus. This proves the full assertion on the original framed datum.

Lean. `TheoremForPartners` (L875-L974), `TheoremForPartners` (L975-L1105). □

Application to the theta lifts. In each square of the triangle, $e = t$ and the character is $\nu_{n,\xi}$ of (12.7). Its rank, sign, and weight satisfy the hypotheses checked there. By Theorem 19.18 the vector can be chosen independently of ξ, and the scalar c satisfies

$$|c|^2 = C q^{t^2} \prod_{j=1}^{t} |L^{\mathrm{m}}(j, \nu_{n,\xi})|^2.$$

For the fields of Proposition 5.2, the factors with $j \geq 2$ have product $1 + O_Q(q_1^{-1})$, uniformly in t and the character. Indeed summing the absolute Euler logarithms first over $j \geq 2$ costs at most $4 \log \zeta_E(2)$, including the squares: for each prime power norm $A \geq 2$, $\sum_{j \geq 2} A^{-j} \leq 2A^{-2}$.

The zeta value of the auxiliary field. Put $q = q_1 = 3 l_0 \, 2^s Q$ as in Proposition 5.2 and $F = \mathbb{Q}(\zeta_q)$. Since $[E : F] = 2$, the Euler factors give $\log \zeta_E(2) \leq 2 \log \zeta_F(2)$. By the cyclotomic decomposition law [Was97, Ch. 2], the contribution from $p \nmid q$ is

$$\varphi(q) \sum_{\substack{p \nmid q,\ a \geq 1 \\ p^a \equiv 1 \bmod q}} \frac{p^{-2a}}{a} \leq \varphi(q) \sum_{h \geq 1} (1 + hq)^{-2} \leq \zeta(2)/q.$$

For $p \mid q$, write $q' = q/p^{v_p(q)}$, $u = \mathrm{ord}_{q'}(p)$. There are $\varphi(q')/u$ primes of norm p^u, so their total contribution is $O(q' p^{-2u})$. If $p \neq 2, 3$, then $q' \gg_Q q$ and $p^u \geq q' + 1$. At 2 one has $u \geq 2 l_0$, by the order of 2 modulo $3 l_0$; at 3 one has $u \geq 2^{s-2}$, by the order of 3 modulo 2^s. Since $27 s < l_0 \leq 81 s$, in both cases $p^u \geq q^2$ eventually. Summing over the fixed primes dividing $6Q$ proves $\log \zeta_E(2) \ll_Q q_1^{-1}$.

Lean. The exact exchange scalar is in `TheoremForPartners` (L975-L1105). The estimate for its convergent Euler factors uses the cyclotomic conductor.

20 Frobenius, theta derivatives and all-prime divisibility

The elementary factorial estimate of Lemma 6.1 applies after a change of local coordinates. We compute the elementary divisors, count collisions in the Frobenius orbit, and pass the resulting valuation through the Hecke operator on the ramified source.

Frobenius collisions, marked primes, and the exceptional Hecke matrix. We use the fixed field notation $E = K(\zeta_{q_1})$, $K = \mathbb{Q}(\sqrt{-d})$ and $\Phi = \mathrm{Gal}(E/K) \simeq (\mathbb{Z}/q_1\mathbb{Z})^\times$. Here $q_0 \mid q_1$, $18480 \mid q_0$, $d \equiv q_0/2 - 1 \pmod{q_0}$, and $\Phi_a = \{\tau \in \Phi : \tau \equiv 1 \pmod{q_0}\}$. Write $f = |\Phi|$, $m = 2^h$ for the number of lifts, and $L = \log b$. The constructed conductor satisfies $2^{22h} \mid q_1$. An integer coprime to q_1 acts on Φ through its residue class; brackets indicate this action.

For a small nonsplit prime p let r_p be the least integer with $m \le p^{r_p}$. A collision outside $S \subset \Phi$ is a pair (τ, e) with $e > 0$, with e odd when p is inert, and

$$\tau, [-p^e]\tau \in \Phi_a \setminus S.$$

Put

$$\mathcal{D}_p(S) = \sum_{(\tau,e) \text{ a collision outside } S} \frac{a}{p^e}.$$

At $p = d$ all positive exponents are allowed in this definition. There is nevertheless no collision at that prime.

Lemma 20.1 (The collision loss away from exceptional primes)**.** *For an inert prime put $o_0 = \mathrm{ord}_{q_0}(-p)$. Every collision has odd exponent divisible by o_0. There are no collisions at $p = d$. If every collision outside S has exponent at least r, then*

$$\mathcal{D}_p(S) \le \frac{2fa}{p^r}.$$

Consequently, at a small nonsplit prime which is not exceptional,

$$m\mathcal{D}_p(\varnothing) \le \lambda_p, \qquad \lambda_p = \frac{16fa}{p} \begin{cases} m, & p \equiv \pm 1 \pmod{m}, \\ 1, & \textit{otherwise.} \end{cases}$$

Here exceptional means inert, $p \not\equiv \pm 1 \pmod{m}$, and o_0 odd with $o_0 \le r_p$.

Proof. For odd e, $[-p^e] = [-p]^e$. Since $\tau \equiv 1 \pmod{q_0}$, the partner lies in Φ_a exactly when $o_0 \mid e$. At $p = d$, the congruence for d gives $d^2 \equiv 1 \pmod{q_0}$. Its powers are 1 and $q_0/2 - 1$, neither of which is -1 modulo q_0. Thus $-d^e \not\equiv 1 \pmod{q_0}$.

At each of the f possible starts the sum is bounded by $a\sum_{e \ge r} p^{-e} \le 2ap^{-r}$, proving the tail estimate. If $p \equiv \pm 1 \pmod{m}$, use $r = 1$. In the remaining nonexceptional inert case, a collision forces o_0 to be odd. Nonexceptionality therefore gives $o_0 > r_p$, and every collision has $e \ge r_p + 1$. It follows that

$$m\mathcal{D}_p(\varnothing) \le \frac{2mfa}{p^{r_p+1}} \le \frac{2fa}{p} \le \lambda_p.$$

This is an arithmetic bound for the loss appearing in the nonsplit jet calculation; it makes no additional assertion about the normalized theta jets themselves.

Lean. `JetExponent` (L126-L183). □

Fix a coefficient embedding into a field over $\mathbb{Q}_p$, and write w_τ for the prime of E induced by τ. At an exceptional prime a full mark is a set Λ of primes induced by Φ_a, disjoint from its complex conjugate, which contains one prime from each conjugate pair that occurs.

Proposition 20.2 (Full marks and their Frobenius cycles)**.** *Let p be exceptional, put $o = \mathrm{ord}_{q_1}(-p)$, and $o_0 = \mathrm{ord}_{q_0}(-p)$. Then $m \mid o$, every prime above p differs from its conjugate, and full marks exist. For $S_\Lambda = \{\tau : w_\tau \in \Lambda\}$ one has*

$$\mathcal{D}_p(S_\Lambda) = 0.$$

For $w \in \Lambda$, the set

$$\{\tau \in \Phi_a : w_\tau = w \text{ or } w_\tau = \bar{w}\}$$

is one cycle under $[-p]^{o_0}$. Its length $\nu = o/o_0$ is even and divisible by m, and exactly $\nu/2$ of its members induce w. Put $P = p^{o_0}$. The residue cardinality at w is $Q = p^o = P^\nu$. After conjugating the embeddings inducing $\bar{w}$, the successive maps $\iota_s : \mathcal{O}_{E,w} \longrightarrow R$ into the coefficient valuation ring satisfy

$$\iota_{s+1}(z) \equiv \iota_s(z)^P \pmod{p}.$$

In particular the arithmetic Frobenius cycle needed for the exceptional edge operator is realized on the field.

Proof. The prime is odd and coprime to $q_1 d$. Since $p \not\equiv \pm 1 \pmod{2^h}$, both $v_2(p-1)$ and $v_2(p+1)$ are smaller than h. If o were odd, lifting the exponent in $(-p)^o \equiv 1 \pmod{2^{22h}}$ would give $22h \le v_2(p+1) < h$, a contradiction. For even o, the same formula gives

$$22h \le v_2(p-1) + v_2(p+1) + v_2(o) - 1.$$

Thus $v_2(o) \ge 20h + 3 \ge h$, and $m \mid o$.

The fields K and $\mathbb{Q}(\zeta_{q_1})$ are linearly disjoint in the fixed construction. Arithmetic Frobenius at p acts by conjugation on K and by p on roots of unity. Complex conjugation acts by conjugation on K and by -1 on roots of unity. It follows that even powers of $[-p]$ fix an induced prime and odd powers send it to its conjugate. If a prime equalled its conjugate, complex conjugation would be an odd power of arithmetic Frobenius. This would give $(-p)^j \equiv 1 \pmod{q_1}$ for an odd j, contrary to the even order o. The decomposition-group calculation also gives residue degree $\mathrm{lcm}(2, o) = o$.

Two embeddings induce the same prime or conjugate primes exactly when they differ by a power of $[-p]$. Starting in Φ_a, the powers which return to Φ_a are precisely the multiples of o_0. Their cycle therefore has length $\nu = o/o_0$. As o_0 is odd, $m \mid \nu$, and each successive step interchanges the two induced primes. The cycle contains each prime equally often. In particular, the set of induced primes is closed under conjugation; choosing one member of each pair gives Λ.

If a collision started outside S_Λ, its induced prime would be the unmarked member of its pair. Its odd-exponent partner would induce the marked member, and hence would lie in S_Λ. This is impossible.

Number the cycle by $\tau_s = [-p]^{o_0 s}\tau_0$, beginning at a member inducing w. Use τ_s at even s and $\tau_s \circ \mathrm{conj}$ at odd s to define ι_s. The step between these maps is the o_0-th power of arithmetic Frobenius after the conjugations cancel. Since $E_w/\mathbb{Q}_p$ is unramified, p is a uniformizer at w, and reduction of arithmetic Frobenius gives the displayed congruence on global integers. It extends to the localization $(\mathcal{O}_E)_w$ because the denominators map to units. For $z \in \mathcal{O}_{E,w}$ choose an element of $(\mathcal{O}_E)_w$ with the same residue modulo p in the completion. Its images under both integral embeddings agree modulo pR with the corresponding images of z, giving the required congruence in R/pR.

Lean. `MarkedPrimes` (L140-L312), `MarkOperator` (L213-L333). □

Proposition 20.3 (The scaled Hecke matrix on the source lattice)**.** *Let p be exceptional, $w \in \Lambda$, $r = r_p$, and*

$$A_w = a\#\{\tau \in \Phi_a : w_\tau = w\} = a\nu/2, \qquad Y_w = p^{A_w} T_{w,r}.$$

Then $Y_w\mathcal{M}_p \subseteq \mathcal{M}_p$ for the original integral source lattice. More explicitly, let X be its class set, let $W = \mathcal{O}_{E,w}$, $K_w = \mathrm{GL}_k(W)$, and put

$$D = \mathrm{diag}(pI_r, I_{k-r}), \qquad \mathcal{P} = K_w \cap DK_wD^{-1}.$$

Choose class representatives integral above p. For an edge $e = (x, \beta\mathcal{P})$ into x, write its rational part γ_e and its origin x_e so that

$$\gamma_e^{-1} = G_eD^{-1}\beta^{-1}, \qquad G_e \in K_w.$$

The map

$$e \longmapsto (x_e, G_e^{-\mathsf{t}}\mathcal{P})$$

is a bijection of $X \times K_w/\mathcal{P}$.

Let ρ_{cyc} be the tensor product of the degree-a symmetric-power representations of G in the even embeddings ι_s, and of $G^{-\mathsf{t}}$ in the odd embeddings. For a monomial α put

$$d(\alpha) = \sum_{s \text{ even}} \sum_{i \geq r} \alpha_{i,s} + \sum_{s \text{ odd}} \sum_{i < r} \alpha_{i,s}.$$

In the original monomial evaluation coordinates, the contribution from x_e to x is

$$U_e \otimes \left[\rho_{\mathrm{cyc}}(G_e)\,\mathrm{diag}_\alpha(p^{d(\alpha)})\rho_{\mathrm{cyc}}(\beta^{-1})\right]^{\mathsf{t}}, \qquad U_e \in \mathrm{GL}_H(R), \tag{20.1}$$

where H indexes the Weyl factors outside the cycle.

Proof. The source level is free by the argument preceding (11.5). Evaluation at class representatives therefore identifies $\mathcal{M}_p$ with $\prod_{x\in X} R^I$ in the transported original Weyl bases. At w, right cosets in K_wDK_w are βDK_w, indexed by $\beta\mathcal{P}$. Replacing the w-component of the class representative by βD, and its conjugate component by the inverse transpose, gives a point $\gamma_e g_{x_e} k_e$. Integrality of g_{x_e} and k_e at w gives $\gamma_e^{-1}\beta D = G_e \in K_w$.

Suppose two edges have the same origin and the same displayed origin coset. Then $G_e^{\mathsf{t}}G_{e'}^{-\mathsf{t}} \in \mathcal{P}$. The definition of $\mathcal{P}$ gives $DG_e^{-1}G_{e'}D^{-1} \in K_w$ by inverse transposition. Thus $\gamma_e\gamma_{e'}^{-1}$ is integral at w and $\bar{w}$; away from that pair, the edge decompositions show that it carries the representative of one terminus to the other modulo the original level. The termini coincide, and freeness forces $\gamma_e\gamma_{e'}^{-1} = 1$. The same displayed factorization now gives $\beta^{-1}\beta' \in \mathcal{P}$, so the edges agree. Injectivity between the finite sets of equal cardinality proves bijectivity. This is the two-sided coset coverage used in the finite-dimensional argument.

At a cycle embedding the initial highest weight consists of one row of length a. Its integral Schur basis is the monomial evaluation basis, whose matrix is the transpose of the symmetric-power matrix on polynomials. On an even letter, unitarity and the displayed inverse of γ_e show that multiplying the variable action by p replaces D^{-1} with pD^{-1}: the deep coordinates are $i \geq r$. On an odd letter, the variable action is the inverse transpose and the diagonal is D: the deep coordinates are $i < r$. Taking symmetric powers and then the tensor product multiplies the whole action by $p^{a\nu/2} = p^{A_w}$ and gives the bracket in (20.1). At every other prime above p, the rational part is integral invertible. All remaining nonzero-degree Weyl factors therefore give the matrix $U_e \in \mathrm{GL}_H(R)$. Summing the contributions is exactly the original Hecke double-coset sum. Every matrix in the sum is integral, which proves the asserted lattice preservation.

Lean. `HeckeEdges` (L290-L478), `HeckeEdges` (L512-L694), `HeckeEdges` (L769-L868). □

Elementary divisors of the edge matrix. For a finite free module M of rank d_X over a valuation ring $R \subset \mathbb{C}_p$, normalized by $v(p) = 1$, and an integral endomorphism T, define

$$\delta_j(T) = \min v(\text{maximal minors of } [T \mid p^j I]).$$

This is a determinant valuation, defined over nondiscrete valuation rings as well. A nonzero minimizing minor gives a full sublattice of $TM + p^j M$ whose determinant has this valuation.

Lemma 20.4 (An approximate inverse with few extra directions)**.** *Let $Q : R^{d_X} \to R^n$ have a left inverse R_0, let $A : R^n \to R^{d_X}$ and $B : R^{d_X} \to R^n$ be integral, and let $\eta, u \in R \setminus \{0\}$ with $u \in \eta\mathfrak{m}_R$. Suppose*

$$AB \equiv \eta I \pmod{u}, \qquad Q(R^{d_X}), B(R^{d_X}) \subseteq C \subseteq R^n,$$

where C is an R-submodule. Suppose a residue subspace $S \subseteq (R/\mathfrak{m}_R)^n$ satisfies

$$uh \in C \implies \bar{h} \in S, \qquad \dim S \le d_X + t.$$

Then there are at most t vectors $z_i \in R^{d_X}$ such that

$$\eta R^{d_X} \subseteq AQ(R^{d_X}) + \sum_i R z_i.$$

In particular, if $\eta = p^c$ with $c \ge 0$ an integer, then

$$\delta_j(AQ) \le d_X \min(j, c) + t(j - c)_+ \le c d_X + jt \qquad (j \ge 0).$$

Proof. Project B to the direct summand $\ker R_0$ by $B' = (1 - QR_0)B$. Smith reduction of this finite matrix over the valuation ring gives generators $t_i u_i$, with the nonzero u_i part of a basis of $\ker R_0$. Let $J = \{i : 0 \ne t_i \mid u\}$. For $i \in J$, $uu_i \in C$, because $t_i u_i \in C$. Also $uQ(R^{d_X}) \subset C$. The residue classes of the columns of Q together with the u_i, $i \in J$, are independent: apply the reduction of R_0 first, and then use the basis of its kernel. They all belong to S. Hence $|J| \le t$.

If $i \notin J$, then $u \mid t_i$ (including $t_i = 0$), since divisibility in a valuation ring is total. Thus

$$B(R^{d_X}) \subseteq Q(R^{d_X}) + \sum_{i \in J} R t_i u_i + uR^n.$$

Apply A and the congruence for AB. With $G = AQ(R^{d_X}) + \sum_{i \in J} RAt_i u_i$ this gives $\eta R^{d_X} \subseteq G + uR^{d_X}$. The image of ηR^{d_X} in R^{d_X}/G is finitely generated and equals a multiple of itself by an element of $\mathfrak{m}_R$. Nakayama's lemma makes it zero.

In Smith coordinates for AQ, project onto the coordinates whose diagonal entries lie in $p^c\mathfrak{m}_R$, including the zero entries. If there were more than t of them, a nonzero row over the fraction field would kill all the projected z_i. Divide the row by an entry of largest absolute value. Its image on the projected free module is now R, so the inclusion just proved would give $p^c R \subseteq p^c\mathfrak{m}_R$. Thus at most t Smith valuations exceed c. In Smith coordinates $\delta_j(AQ)$ is the sum of the Smith valuations truncated at j, since a minimizing minor may choose the diagonal entry or p^j in each row. This gives the bound.

Lean. `ElementaryDivisorTheorem` (L64-L226), `colengthDvd_mul_mul_of_smul_top_le`. The formal proof bounds the determinant by adjoining the extra columns. Counting the exceptional Smith valuations gives the truncated bound. □

The depth estimate by finite interpolation. The next calculation supplies the determinant losses at both ends of the edge factorization. It is independent of the construction of the approximate inverse.

Lemma 20.5 (A weighted interpolation bound for multiple zeros)**.** *Let F be a field, let $z^{(1)},\dots,z^{(q)}\in F^d$ be distinct, and choose positive integers $\kappa_1,\dots,\kappa_d$ with $q=\prod_i\kappa_i$. Give X^α the weight $\sum_i\alpha_i/\kappa_i$. Suppose, for every i, there is a polynomial G_i all of whose monomials have weight strictly less than one and such that*

$$(z_i^{(u)})^{\kappa_i}=G_i(z^{(u)})\quad(1\le u\le q).$$

If a nonzero polynomial vanishes to order at least t at every $z^{(u)}$, it has a monomial of weight at least t. Vanishing to order t means membership in the t-th power of the corresponding maximal ideal; no characteristic-zero interpretation of derivatives is needed.

Proof. Write $B_i=X_i^{\kappa_i}-G_i$. The products X^rB^a with $0\le r_i<\kappa_i$ and $a\in\mathbb{N}^d$ form a basis of $F[X]$. Indeed, their leading monomials for the given weight are $X^{r+\kappa a}$, with $(\kappa a)_i=\kappa_i a_i$. Each occurs once, and all remaining terms have strictly smaller weight. Induction on the weight, after clearing its fixed denominator, gives spanning and independence.

For $I=(B_1,\dots,B_d)$, the products with $|a|\ge t$ form a basis of I^t, by multiplying this basis by the generators of I^t. In particular $\dim_F F[X]/I=\prod_i\kappa_i=q$. Evaluation at the q points is onto F^q by the Chinese remainder theorem, so I is the product of their maximal ideals. Their powers are pairwise comaximal. Hence vanishing to order t at every point is equivalent to membership in I^t. A nonzero element of that ideal has a surviving leading monomial of weight at least t, as required.

Lean. `New.exists_le_weight_of_mem_vanishingIdeal_pow`, `Depth` (L612-L654). Lean counts truncated jets. The triangular basis computes all ideal powers at once. □

Theorem 20.6 (The total depth of a Frobenius-cycle representation)**.** *Let $\nu>0$ be even and let a Frobenius cycle over a residue field K of characteristic p have multiplier P and residue cardinality $Q=P^\nu$. Let V be the forms of degree a in each of ν letters, with k variables in each letter, and let $0<2r\le k$. The variable $X_{b,s}$ is deep if s is even and $b\ge r$, or if s is odd and $b<r$. Let V_t be the subspace of forms whose translates by $\mathrm{GL}_k(\mathcal{O}_{E,w})$ have only monomials of total deep degree at least t. Then, for every integer $T\ge1$,*

$$\sum_{t=1}^{T}\dim_K V_t\le\nu a\left(P^{-2r}+\frac{r^2}{kP}\right)\binom{a+k-1}{k-1}^{\nu}.\tag{20.2}$$

Proof. Give a deep variable with $b<2r$ the cost P^{-1} and a deep variable with $b\ge2r$ the cost P^{-2r}; give the shallow variables cost zero. We first prove that every nonzero $F\in V_t$ has a monomial of cost at least t.

Regard the shallow variables as algebraically independent scalars, and let K' be their rational-function field over K. This preserves a nonzero F as a polynomial in the deep variables. Consider matrices A supported in the upper right block with precisely the permitted positions

$$(b,b+r)\quad(0\le b<r),\qquad(b,j)\quad(0\le b<r,2r\le j<k).$$

Their entries range independently over the residue field $\mathbf{F}_Q$ of $\mathcal{O}_{E,w}$; lift each entry to $\mathcal{O}_{E,w}$ when applying the group action. Since $A^2=0$, the action of $(1+A)^{-1}$ fixes the shallow variables and translates the deep variables. Its translation point has coordinates

$$z_{j,s}(A)=\sum_{b<r}\iota_s(A_{bj})X_{b,s}\quad(s\text{ even},\ j\ge r),$$

$$z_{b,s}(A) = -\sum_{j\ge r} \iota_s(A_{bj}) X_{j,s} \quad (s \text{ odd},\ b < r),$$

with zero entries at the nonpermitted positions understood. The two signs are the ordinary and inverse-transpose actions on successive letters. By the definition of V_t, the specialized polynomial vanishes to order at least t at every one of these points.

Write $\kappa_{b,s} = P$ or P^{2r} according to the two kinds of deep variable. We claim that the points are distinct and that each $z_{b,s}(A)^{\kappa_{b,s}}$ is a polynomial of cost less than one in their coordinates, with coefficients in K' independent of A.

Fix a letter i and a high column $j \ge 2r$. For $u = 0, \dots, r-1$, choose $J_u \in \{2u, 2u+1\}$ so that $i - J_u$ is even. Frobenius gives

$$z_{j,i-J_u}(A)^{P^{J_u}} = \sum_{b<r} X_{b,i-J_u}^{P^{J_u}} \iota_i(A_{bj}).$$

The coefficient matrix $\left(X_{b,i-J_u}^{P^{J_u}}\right)_{u,b}$ is invertible over K'. Indeed the J_u are distinct; in its determinant expansion each term records, for each column b, both the letter and the exponent assigned to it. Distinct permutations give distinct monomials, even if a letter repeats modulo ν, so the determinant is nonzero. Since $J_u < 2r$, every read on the left has cost $P^{J_u - 2r} < 1$. Thus all $\iota_i(A_{bj})$, $j \ge 2r$, are linear combinations of coordinate powers of cost less than one.

The matched entry $\iota_i(A_{b,b+r})$ is then recovered from $z_{b+r,i}(A)$ if i is even. If i is odd, recover it from $z_{b,i}(A)$ after subtracting the high-column terms. The remaining shallow coefficient is a nonzero variable, hence invertible in K'. These additional reads have cost $P^{-1} < 1$. This recovers every parameter of A in every embedding; in particular the points are distinct.

Finally, raising a displayed coordinate formula to $\kappa_{b,s} = P^e$ replaces the embedded entries by those at $s + e$. Substituting the recovery formulas just proved expresses the result as a polynomial of cost less than one. There are $Q^{r(k-2r+1)}$ parameter choices. There are νr deep variables of order P and $\nu(k-2r)/2$ of order P^{2r}, so

$$\prod_{(b,s)\ \text{deep}} \kappa_{b,s} = P^{\nu r(k-2r+1)} = Q^{r(k-2r+1)}.$$

Lemma 20.5 proves the claimed costly monomial. Returning from K' to K preserves its cost, because only shallow variables entered the coefficient field.

It follows that the projection from V_t to the coefficients of monomials of cost at least t is injective. Sum these dimension bounds over $t \ge 1$. A monomial of cost c is counted at most $\lfloor c \rfloor \le c$ times. Over all monomials of degree a in each letter, symmetry makes the mean exponent of each variable equal to a/k. Their mean cost is consequently

$$\frac{a}{k}\left(\frac{\nu r}{P} + \frac{\nu(k-2r)}{2P^{2r}}\right) \le \nu a\left(P^{-2r} + \frac{r^2}{kP}\right),$$

where $r \ge 1$ was used in the last inequality. Multiplying by $\dim V = \binom{a+k-1}{k-1}^{\nu}$ proves (20.2).

Lean. `Depth` (L246-L665), `Depth` (L687-L796). □

Corollary 20.7 (The integral determinant losses at both ends)**.** *In the edge construction of Proposition 20.3, choose $\varpi \in R$ with $\varpi^2 = p$, and let $D_\varpi = \mathrm{diag}_\alpha(\varpi^{d(\alpha)})$. Form either stack of matrices $D_\varpi \rho_{\mathrm{cyc}}(g_e)$, with the untouched factors included, using $g_e = G_e^{\mathsf{t}}$ at origins or $g_e = \beta^{-1}$ at termini of the edges. Each stack has a factorization $Q_i D_i$, with Q_i admitting an integral left inverse and D_i square of full rank. Moreover, for one integer $e_0 \ge 0$,*

$$\det D_1 \det D_2 \mid p^{e_0}, \qquad e_0 \le c_{\mathrm{div}} D_X, \quad c_{\mathrm{div}} = \nu a\left(P^{-2r} + \frac{r^2}{kP}\right).$$

These factorizations use the original source lattices.

Proof. Finite Smith reduction over a valuation ring factors a full column-rank matrix as a primitive matrix with an integral left inverse times a square full-rank matrix. Apply this to each stack. Full column rank follows from the coset coverage in Proposition 20.3 and invertibility of $\rho_{\mathrm{cyc}}(g)$ over R before the depth scaling.

For $h \in \mathcal{P}$, the matrix $D_\varpi \rho_{\mathrm{cyc}}(h) D_\varpi^{-1}$ and its inverse are integral. At the variable level an entry which would lower depth has the factor $p = \varpi^2$ from the upper right block of h; the inverse-transpose action on odd letters has the corresponding opposite block. Taking symmetric powers preserves this integrality. Thus divisibility of the stack implies the same divisibility in every frame, by its parahoric-coset coverage.

Write the Smith valuations of one stack in the normalization $v_\varpi(\varpi) = 1$ as $\lambda_i \geq 0$. They are at most $M = \nu a$: after adjoining the identity frame, every input coefficient is multiplied by a power of ϖ of exponent at most M. For an integer $d \geq 0$, the reductions of input vectors whose stack values have valuation strictly greater than d form a space of dimension $\#\{i : \lambda_i > d\}$. Such a reduction belongs, in every class and every untouched coefficient, to V_{d+1}: every coefficient of depth at most d vanishes modulo the maximal ideal in every frame. The values λ_i need not be integers. What is used is the valid inequality

$$\sum_i \lambda_i \leq \sum_{d=0}^{M-1} \#\{i : \lambda_i > d\},$$

since the right side counts $\lceil \lambda_i \rceil$ for each $0 \leq \lambda_i \leq M$. Thus this argument does not assume that R is a discrete valuation ring.

Put $n_0 = \binom{a+k-1}{k-1}^\nu$ and let h_0 be the number of untouched coefficient indices. Both stacks have $D_X = |X| h_0 n_0$ input coordinates. Therefore

$$v_\varpi(\det D_i) = \sum_j \lambda_j \leq |X| h_0 \sum_{d=0}^{M-1} \dim V_{d+1} = e_0.$$

The number e_0 on the right is an integer and is the same for the two stacks. Since $v_p = v_\varpi/2$, their determinants have product valuation at most e_0, giving the divisibility by p^{e_0} over the valuation ring. Theorem 20.6 gives $e_0 \leq c_{\mathrm{div}} D_X$. No rounding upwards of $c_{\mathrm{div}} D_X$ is needed: the integer is obtained from the sum of residue-space dimensions.

Lean. `ElementaryDivisorTheorem` (L250-L476). □

Lemma 20.8 (The quantitative hypotheses for the edge construction)**.** *Put $\nu = 2\ell$, $Q = P^\nu$, and*

$$c_0 = k^2 \log_p Q, \qquad c_{\mathrm{prec}} = 5\nu c_0, \qquad c_{\mathrm{inv}} = 2\nu c_0.$$

Here $\log_p Q = o_0 \nu$ is an integer. Suppose $P \geq 5$, $r \geq 1$, $2r \leq k$, $0 < \theta \leq 1$, and

$$a^4 \leq Q, \qquad (\nu k \log Q/\theta)^5 P^{3r} \leq a.$$

Then the following precise inequalities hold:

$$1 < a < Q, \qquad \nu k(\ell+2)a + a + k < P^\ell,$$
$$24\nu k^2 c_{\mathrm{prec}} P^{2r} \leq \theta a, \qquad c_0 P^{2r} \leq a, \qquad c_{\mathrm{inv}} \leq 2c_{\mathrm{div}}.$$

The threshold $u = \varpi^{c_{\mathrm{prec}}-1}$, with $\varpi^2 = p$, satisfies $u \in p^{c_{\mathrm{inv}}} \mathfrak{m}_R$.

Proof. Write $L = \log Q$ and $v = \nu k L$. Since $Q \geq 5$ and $\theta \leq 1$, the degree assumption gives $a \geq (\nu k)^5$. Both ν, k are at least two. Consequently $Q \geq a^4 \geq 2^{40}$, so $L > 27$. We have $\ell + 2 \leq \nu k$ and $a \geq (\nu k)^2 + 2$. Since $k < a$,

$$\nu k(\ell + 2)a + a + k < ((\nu k)^2 + 2)a \leq a^2 \leq P^\ell.$$

Also $1 < a < a^4 \leq Q$.

The inequality $\log p > \frac{2}{3}$ gives

$$c_0 \leq \tfrac{3}{2}k^2 L, \qquad c_{\mathrm{prec}} \leq \tfrac{15}{2}kv.$$

The degree assumption, multiplied by θ^5, implies $v^5 P^{3r} \leq \theta a$. Since $v \geq 27\nu k$ and $\nu, k \geq 2$, one has $180\nu k^3 \leq v^4$. Hence

$$24\nu k^2 c_{\mathrm{prec}} P^{2r} \leq 180\nu k^3 v P^{2r} \leq v^5 P^{3r} \leq \theta a.$$

Similarly $c_0 \leq v^2$, and therefore $c_0 P^{2r} \leq v^5 P^{3r} \leq a$. This gives $\nu c_0 \leq \nu a P^{-2r} \leq c_{\mathrm{div}}$.

Finally $\nu c_0 \geq 2$, so $c_{\mathrm{prec}} - 1 = 5\nu c_0 - 1 \geq 4\nu c_0 + 1 = 2c_{\mathrm{inv}} + 1$. Thus $\varpi^{c_{\mathrm{prec}}-1}$ is divisible by $p^{c_{\mathrm{inv}}}\varpi$; since $\varpi \in \mathfrak{m}_R$, this is the claimed strict valuation inequality.

Lean. `ElementaryDivisorTheorem` (L382-L607). □

Corollary 20.9 (What suffices for the exact exceptional constant)**.** *For the cycle in Proposition 20.2, put*

$$c_{\mathrm{div}} = \nu a \left(P^{-2r_p} + \frac{r_p^2}{kP} \right), \qquad \theta = \frac{1}{m^2 f}.$$

Suppose the matrix in (20.1) has an integral pairing factorization

$$Y_w B_0 = D_2^{\mathsf{t}}(AQ)D_1$$

with D_1, D_2 of full rank,

$$\det D_1 \det D_2 \mid p^{e_0}, \qquad e_0 \in \mathbb{Z}_{\geq 0}, \qquad e_0 \leq c_{\mathrm{div}} D_X,$$

and with the hypotheses of Lemma 20.4 holding for $\eta = p^c$, $c \leq 2c_{\mathrm{div}}$, and a residue space of dimension at most $(1+\theta)D_X$. Then, for every integer $j \geq 0$,

$$\delta_j(Y_w) \leq \left(\frac{12A_w}{mp} + \frac{j}{m^2 f} \right) D_X.$$

More precisely, there is an integer $e \geq 0$ at most the displayed right side such that

$$\mathrm{ColengthDvd}(\mathcal{M}_p, Y_w\mathcal{M}_p + p^j\mathcal{M}_p, p^e).$$

The endpoint determinant bounds in this criterion are proved in Corollary 20.7. The integral average and residue condition are supplied by Theorem 20.28; their assembly on the source lattice is proved below.

Proof. For integral square matrices D, D' of full rank,

$$\delta_j(D'TD) \leq \delta_j(T) + v(\det D) + v(\det D').$$

For right multiplication, use Smith bases for D and multiply the selected columns of a minimizing minor by their diagonal entries; the cost is at most $v(\det D)$. Left multiplication multiplies a selected

determinant by $\det D'$ and sends its congruence columns into $p^j R^{D_X}$. The inclusion $Y_w B_0 M \subseteq Y_w M$ and the preceding lemma therefore give an integer colength exponent $e = e_0 + cD_X + j|J|$, where $|J| \le \theta D_X$ is the number of additional directions in that lemma. Hence $e \le (3c_{\mathrm{div}} + \theta j)D_X$, and the same bound holds for $\delta_j(Y_w)$.

The source scales give $k \ge mr_p^2$. Since $P \ge p$, $p^{r_p} \ge m$, and $r_p \ge 1$,

$$P^{-2r_p} \le \frac{1}{mp}, \qquad \frac{r_p^2}{kP} \le \frac{1}{mp}.$$

As $A_w = a\nu/2$, this gives $3c_{\mathrm{div}} \le 6\nu a/(mp) = 12A_w/(mp)$. The large-degree conditions imposed in the formal finite-field construction are explicitly

$$0 < 2r_p \le k, \quad 0 < \theta \le 1, \quad a^4 \le Q, \quad (\nu k \log Q/\theta)^5 P^{3r_p} \le a.$$

Lemma 20.8 verifies the precise numerical requirements of that construction.

Lean. `ElementaryDivisorTheorem` (L1040-L1214), `MarkOperator` (L50-L151). □

Theorem 20.10 (The exceptional-prime estimate on the source lattice)**.** *For every exceptional prime $p < x_1$, every marked $w \in \Lambda$, and every integer $j \ge 0$, the original scaled operator $Y_w = p^{A_w} T_{w,r_p}$ preserves $\mathcal{M}_p$ and admits an integer $e \ge 0$ such that*

$$e \le \left(\frac{12A_w}{mp} + \frac{j}{m^2 f}\right) D_X, \qquad \mathrm{ColengthDvd}(\mathcal{M}_p, Y_w\mathcal{M}_p + p^j\mathcal{M}_p, p^e).$$

This holds for all sufficiently large data in §5.2, uniformly in p, w, j and the allowed twists. More generally, for any fixed tuple the exact sufficient size conditions are

$$k \ge mr_p^2, \quad 2r_p \le k, \quad a^4 \le P^\nu, \quad (\nu k m^2 f \log(P^\nu))^5 P^{3r_p} \le a, \qquad P = p^{o_0}, \quad \nu = o/o_0.$$

Thus this proves the original (11.17), including its integer exponent and the coefficient 12*.*

Proof. Let $n_0 = \binom{a+k-1}{k-1}^\nu$, and write $h_0 = |H|$, so $D_X = |X|h_0 n_0$. Keep the original vertex and edge coordinates of Proposition 20.3. Use the integral diagonal weights b, b' of Lemma 20.21, with degree a and ν letters. On the cycle put $B_{\mathrm{cyc}} = \mathrm{diag}(b)$; it is integral, and $B_{\mathrm{cyc}} B'_{\mathrm{cyc}} = p^{\nu c_0} I$. Let B_0 be this same diagonal matrix at every vertex and every untouched coefficient. We do not invert B_0 as an integral change of basis.

Put $\Delta = \mathrm{diag}(\varpi^{d(\alpha)})$ with $\varpi^2 = p$, and form the two edge stacks

$$F_1(v)_e = (1_H \otimes \Delta\rho(G_e^{\mathsf{t}}))v_{x_e}, \qquad F_2(v)_e = (1_H \otimes \Delta\rho(\beta_e^{-1}))v_x.$$

Let $\mathcal{U}$ be block diagonal with block $U_e \otimes 1$, and let $\mathcal{B}, \mathcal{B}'$ be the edgewise copies of $1_H \otimes B_{\mathrm{cyc}}$ and $1_H \otimes B'_{\mathrm{cyc}}$. The exact adjoint identity $\rho(G)^{\mathsf{t}} B_{\mathrm{cyc}} = B_{\mathrm{cyc}}\rho(G^{\mathsf{t}})$ and the diagonal commutation with Δ give

$$Y_w B_0 = F_2^{\mathsf{t}} \mathcal{U} \mathcal{B} F_1. \tag{20.3}$$

Indeed its block on an edge is $U_e \otimes \rho(\beta_e^{-1})^{\mathsf{t}} \Delta^2 \rho(G_e)^{\mathsf{t}} B_{\mathrm{cyc}}$, which is precisely (20.1) multiplied on the right by B_0. This checks the transpose and the inverse at both ends.

By Corollary 20.7, write $F_i = Q_i D_i$ with integral left inverses $R_i Q_i = 1$ and one integer $e_0 \le c_{\mathrm{div}} D_X$ satisfying $\det D_1 \det D_2 \mid p^{e_0}$. Every value of every column of Q_i belongs to the halfway all-edge module $\mathcal{M}$ in (20.33). To see this explicitly, clear the finitely many denominators of its preimage in $D_i^{-1} R^{D_X}$ by a power of ϖ. The resulting integral vertex form is divisible by that power in every frame, by the two coset-covering bijectivities. At any chosen edge it is exactly the definition of $\mathcal{M}$.

Apply Theorem 20.28 with

$$\theta = (m^2 f)^{-1}, \quad c_{\mathrm{prec}} = 5\nu c_0, \quad c_{\mathrm{inv}} = 2\nu c_0, \quad u = \varpi^{c_{\mathrm{prec}}-1}.$$

Write $\mathcal{A}\sqsubseteq$ for the edgewise average, acting separately on each untouched coefficient. Define the two integral rectangular maps

$$A = Q_2^{\mathsf{t}}\mathcal{U}\mathcal{B}, \qquad B = \mathcal{U}^{-1}\mathcal{A}\sqsubseteq\mathcal{B}'R_2^{\mathsf{t}}.$$

The pairing congruence of that theorem, applied to the columns of Q_2, gives

$$AB \equiv p^{\nu c_0} Q_2^{\mathsf{t}}\mathcal{B}\mathcal{B}'R_2^{\mathsf{t}} = p^{2\nu c_0} I \pmod{\varpi^{c_{\mathrm{prec}}}}.$$

The matrices on the untouched factors commute with the cyclic matrices, which is why $\mathcal{U}$ cancels exactly here.

Let $\mathcal{C}$ and $\mathcal{S}$ be the direct sums of the local condition C and residue space S over the edges and the h_0 coefficients. Both $Q_1(R^{D_X})$ and $B(R^{D_X})$ lie in $\mathcal{C}$: the first by the all-edge property, the second by the image property of the average. Also $uh \in \mathcal{C}$ implies $\bar{h} \in \mathcal{S}$. The number of edges is $|X|[K_w : \mathcal{P}]$, so

$$\begin{aligned}\dim \mathcal{S} &= |X|h_0[K_w : \mathcal{P}] \dim S \\ &\le (1+\theta)|X|h_0 n_0 = (1+\theta)D_X.\end{aligned}$$

Lemma 20.8 gives $u \in p^{c_{\mathrm{inv}}}\mathfrak{m}_R$ and $c_{\mathrm{inv}} \le 2c_{\mathrm{div}}$. All hypotheses of Lemma 20.4 therefore hold for $Q = Q_1$.

Equation (20.3) is now $Y_w B_0 = D_2^{\mathsf{t}}(AQ_1)D_1$. Corollary 20.9 gives the required integer $e = e_0 + c_{\mathrm{inv}}D_X + j|J|$, where $|J| \le \theta D_X$. The integral inclusion $Y_w B_0 M \subseteq Y_w M$ keeps the bound on the original lattice. No ceiling of a real determinant estimate has been used.

We finally verify uniform availability of the displayed size conditions; this is the needed condition on the field size, not a hypothesis about zeros. The cycle construction gives $o_0 \le r_p$, $\nu \le f$, and $o_0\nu = o \ge m$. For the data of §5.2, $m \asymp \sqrt{x}$, $k \asymp x^5$, $f \le x^{13}$, $r_p \le 1 + \log_2 m = O(\log x)$, and

$$W = \log x_1, \qquad \log x \ll W \ll_{Q,d} (\log x)^2, \qquad \log b \asymp W^6.$$

The bounds are independent of $p < x_1$. Thus $k \ge m r_p^2$ and $2r_p \le k$ eventually. Also $\log(P^\nu) \ge m \log 13 > 4 \log a$ eventually, so $a^4 \le P^\nu$. For the other size condition use

$$\nu k m^2 f \log(P^\nu) \le r_p k m^2 f^3 \log x_1, \qquad P^{3r_p} \le x_1^{3r_p^2}.$$

The logarithm of their product in the stated fifth-power expression is $O(\log x + (\log x)^2 W) = O(W^3) = o(\log b)$. It is therefore at most $\log a$ for every sufficiently large x, uniformly over every exceptional prime and every marked cycle. All finite row and character choices are independent of these estimates. This proves the promised quantifiers.

Lean, formal context. `MarkOperator` (L50-L151), `LocalDivisibility` (L494). □

The scalar in the marked Hecke identity. We next prove the transfer identity itself, including the uniform power of p paid when it is iterated.

Lemma 20.11 (Balance at a marked pair)**.** *If w belongs to a marked set above an exceptional prime, no embedding inducing w or $\bar{w}$ is a b-place. The multisets of lift-character weights on those two sets of embeddings agree. Consequently the base-change value χ_w of every lift character at a uniformizer of w is a p-adic unit, and the determinant twists in a marked neighbor contribute valuation zero.*

Proof. A b-place has class -1 modulo q_0, whereas an a-place has class 1. Every embedding inducing w or $\bar{w}$ is on the $[-p]$-orbit of an a-place. If one of them were a b-place, then the cyclic group generated by $-p$ modulo q_0 would contain -1. Its order o_0 is odd, while -1 has order two because $q_0 > 2$. This is impossible.

The underlying weight is therefore a/x_1 at every such embedding. The lift weight adds the same $(j + \frac{1}{2})$ gap at all embeddings. Multiplication by $[-p]$ is a bijection between those inducing w and those inducing $\bar{w}$, proving equality of the two weight multisets. The p-adic avatar formula for an algebraic character evaluates the uniformizer valuation as the difference of the sums of its weights on these two sets. The difference is zero. The last diagonal weight of a Weyl representation is likewise a fixed affine function of the lift weight, so its sums agree. This proves the assertion about determinant twists as well.

Lean. `MarkedPrimes` (L343-L428), `ValuationAtNonSplitPrimes` (L414-L446). □

Theorem 20.12 (The marked Hecke polynomial passes through one lift)**.** *Let w be marked, $q = Nw = z^2$, and let a lift Θ_x go from rank n to rank $n + d_0$. Its datum is the original translated vacuum, with translation x occurring in a weak chain datum of level prime to p. Write*

$$\mathcal{T}_N(U) = \sum_{i=0}^{N} (-U)^i q^{\binom{i}{2}} T_{w,i}^{(N)}.$$

Then Θ_x preserves the w-spherical subspaces, and

$$\mathcal{T}_{n+d_0}(U)\Theta_x = R_w(U)\Theta_x \mathcal{T}_n(z^{d_0}U), \qquad R_w(U) = \prod_{j=0}^{d_0-1} (1 - q^j z^n \chi_w U). \tag{20.4}$$

Every coefficient of R_w is p-integral, its constant term is one, and it is independent of x.

Proof. We give the finite calculation, to specify both the normalization and the conjugate place appearing before the operator is moved across the lift. Let $F = E_w$, let W be its integer ring, and let its residue field be $\mathbb{F}_q$. For a subspace $U_0 \subset \mathbb{F}_q^N$, choose an integral matrix γ_{U_0} with elementary divisors p on U_0 and one on a complement. It has kernel U_0 after reduction and determinant valuation $\dim U_0$. For any matrix Y with N columns,

$$Y\gamma_{U_0}^{-1} \text{ is integral} \quad \Longleftrightarrow \quad Y \text{ is integral and } U_0 \subseteq \ker \bar{Y}.$$

This follows first in a basis adapted to U_0, and then by integral changes of basis.

If a vector space over $\mathbb{F}_q$ has dimension d, counting its subspaces gives

$$\sum_{U_0} (-X)^{\dim U_0} q^{\binom{\dim U_0}{2}} = \prod_{j=0}^{d-1} (1 - q^j X).$$

For completeness, if $g(d, i)$ counts the i-planes, intersecting with a fixed hyperplane gives $g(d, i) = g(d-1, i) + q^{d-i} g(d-1, i-1)$. After the displayed weighting, this recurrence multiplies the polynomial of dimension $d-1$ by $1 - q^{d-1}X$. Its initial value in dimension zero is one.

In the local Schrödinger realization the vacuum is the indicator of integral matrices. A target representative $\gamma_{U_0}^{\mathrm{t}}$ acts by

$$\alpha^{\dim U_0} \mathbf{1}_{Y^{\mathrm{t}}\gamma_{U_0}^{-1} \text{ integral}},$$

and a source representative $\gamma_{U_0}^{-1}$ acts by

$$\beta^{\dim U_0} \mathbf{1}_{Y\gamma_{U_0}^{-1} \text{ integral}}.$$

Here α is the value of the target splitting character at a uniformizer times $q^{n/2}$, and β is the corresponding source value times $q^{(n+d_0)/2}$. These scalars follow directly from the determinant character and the square root of the change in additive Haar measure in the Weil action. The chosen measures give the integral matrix lattice volume one. Thus neither a coset-count denominator nor an additional volume scalar is present.

For integral Y of residual rank s, the target Hecke polynomial on the vacuum is the multiplier $\prod_{j<n+d_0-s}(1-q^j\alpha U)$. The source polynomial at the conjugate place is the multiplier $\prod_{j<n-s}(1-q^j\beta V)$. Both are zero off the integral matrices. Splitting the first product after d_0 factors proves the identity on the vacuum with $V=q^{d_0}\alpha\beta^{-1}U$. It holds on every datum fixed by the local vacuum projection, since all other factors of such a datum are left untouched by this calculation.

The chain data have that projection property: $w,\bar{w}$ avoid the twist conductor and the chosen primes dividing MN, and their allowed translations are integral there. Both rank-twisted characters are unramified there for the same reason. Their rank twists cancel their respective splitting characters, so

$$\alpha=z^n\chi_w,\qquad \beta=z^{n+d_0}\chi_w,\qquad q^{d_0}\alpha/\beta=z^{d_0}.$$

The vacuum is fixed by both maximal compact subgroups. Equivariance of the theta kernel therefore preserves sphericity. Apply the equivariant bilinear theta pairing to the vacuum identity. A source Hecke sum at $\bar{w}$ on the datum becomes the Hecke sum at w on the input: invariance replaces each representative by its inverse, and these inverse cosets are precisely the cosets at w. The Haar normalization of the compact subgroup is one, so this passage introduces no scalar. We obtain (20.4).

Finally Lemma 20.11 gives $\chi_w\in R^\times$. Since q,z are integers, $R_w\in R[U]$ and $R_w(0)=1$. Its displayed formula depends on the stage and character but not on the supported translation x.

Lean. `HeckeTransfer` (L166-L439), `HeckeTransfer` (L488-L547), `ValuationAtNonSplitPrimes` (L372-L526). □

Corollary 20.13 (Exact integral transfer cost along the chain)**.** *Let $d_0=t+1$ and $r=r_p\le t$. For every $0\le i\le r$, there are p-integral coefficients, independent of x, such that*

$$p^{frt}\Theta_x T^{(n)}_{w,i}=\sum_{i'=0}^{i}c_{ii'}T^{(n+t+1)}_{w,i'}\Theta_x.$$

If $F^{(j)}_x$ is the first j lifts, then

$$p^{jfrt}F^{(j)}_x T^{(k)}_{w,r}=\sum_{i=0}^{r}c^{(j)}_i T^{(n_j)}_{w,i}F^{(j)}_x.$$

*For $j\le m$ the exponent is at most $rf(n_m+r)$ (*transferExponent *in Lean). The same identities hold for the jointly invariant finite sum defining the weak chain.*

Proof. Write $R_w(U)^{-1}=\sum_{h\ge0}b_hU^h$ formally. Its coefficients lie in R, because $R_w(0)=1$. Taking the coefficient of U^i after multiplying (20.4) by this inverse gives

$$z^{(t+1)i}q^{\binom{i}{2}}\Theta_x T_{w,i}=\sum_{i'=0}^{i}(-1)^{i+i'}b_{i-i'}q^{\binom{i'}{2}}T_{w,i'}\Theta_x.$$

Only $i'\le i$ occur and all coefficients on the right are integral and independent of x.

Proposition 20.2 gives $q=p^{2h}$ with $2h\le f$. The scalar on the left is $p^{hi(t+i)}$. Since $i\le r\le t$,

$$hi(t+i)\le 2hrt\le frt.$$

Multiply by the remaining integral power of p. Iteration multiplies the triangular coefficient matrices, so their entries remain integral and independent of x. It also preserves sphericity, and proves the assertion for j stages. Finally $jt \le n_m$ for $j \le m$, hence $jfrt \le rf(n_m + r)$. Independence of x permits summing the same identity against every coefficient of the weak chain datum.

Lean. `ValuationAtNonSplitPrimes` (L528-L734). □

Lemma 20.14 (Box charges at a marked neighbor)**.** *Let J_w be any set of at most r_p row indices at each $w \in \Lambda$. At embeddings inducing w, put $\varepsilon_\tau(b) = \mathbf{1}_{b \in J_w}$; at those inducing $\bar{w}$, put $\varepsilon_\tau(b) = -\mathbf{1}_{b \in J_w}$; elsewhere put zero. For a box in a column of height h, assign charge one precisely in either of the cases*

$$\tau \mid w, \quad h = 1, \quad b \notin J_w; \qquad \tau \mid \bar{w}, \quad b \in J_w.$$

Erasing a height-one column at S_Λ exempts a coupling only if at least one of its two boxes has charge one. Let $\mathrm{lowColumns}_j(\tau)$ *count the columns of heights $2, \dots, n_{j+1} - 1$ in the diagram at τ. On every column-strict tableau T, the sum of exponents and charges is at most*

$$\sum_{w \in \Lambda} A_w + r_p \sum_{\tau \in S_\Lambda} \mathrm{lowColumns}_j(\tau).$$

The determinant-character contribution is zero.

Proof. An odd Frobenius partner exchanges w and $\bar{w}$ and preserves the set J_w. At a removed height-one box over w, an index outside J_w is charged there. An index in J_w is charged at the other end over $\bar{w}$. This proves the first assertion, and its version with the ends reversed.

At an embedding over w, every height-one box has exponent plus charge one. Each higher column contributes at most $|J_w| \le r_p$, because its row indices are distinct. At an embedding over $\bar{w}$, negative exponents and positive charges cancel box by box. Elsewhere both are zero. Summing gives the displayed bound, since the number of height-one boxes over the marked primes is $\sum_w A_w$. Lemma 20.11 cancels the remaining determinant twist.

Lean. `ValuationAtNonSplitPrimes` (L77-L286). □

Divisibility at the exceptional and nonsplit primes. Theorem 20.6 counts the depths and Corollary 20.7 gives the two determinant factors. The residue-space bound of Theorem 20.20 and the integral average of Theorem 20.28 give Theorem 20.10, hence the exponent with coefficient 12 in (11.17). At nonsplit primes, Theorem 20.12 and Corollary 20.13 give the Hecke action on the theta chain. Lemma 20.14 bounds the contributions from the marked columns. The valuation estimate (20.61) of §20.3 and the collision estimate of Lemma 20.34 then give Theorem 20.36.

20.1 The weighted root lemma and the residue defect count

We determine the least weighted term along a Frobenius cycle and bound the dimension of the residue subspace defined by the averaging congruences. The dimension bound uses the volume of a cone, computed below. These are the statements `RootLemma` and `DefectCount` in Lean.

The Frobenius cycle and the exact defect. Let W be a commutative local ring with residue field of cardinality $Q = P^N$, where $P = p^e \ge 5$, p is prime, and $N > 0$. The letters $s \in \mathbb{Z}/N\mathbb{Z}$ are marked alternately even and odd; in particular N is even. Let R be a commutative ring, let $w \in R$ satisfy $w^2 = p$, and let $\iota_s : W \longrightarrow R$ be ring homomorphisms satisfying

$$p \mid \iota_{s+1}(z) - \iota_s(z)^P \qquad (z \in W).$$

On the polynomial ring in $x_{b,s}$, $0 \le b < k$, use the action $\rho(g)$ whose substitution matrix in letter s is $\iota_s(g)$ for even s, and $\iota_s(g^{-T})$ for odd s. Thus $x_{b,s}$ is sent to $\sum_c (\iota_s(g))_{cb} x_{c,s}$, with the indicated inverse transpose in an odd letter.

For a root (a, a'), $a \neq a'$, write

$$Y_s = x_{a',s}, \quad X_s = x_{a,s} \quad (s \text{ even}), \qquad Y_s = x_{a,s}, \quad X_s = x_{a',s} \quad (s \text{ odd}).$$

At a specified letter i, put $d_s = P^{[s-i]_N}$, where $[s-i]_N \in \{0, \dots, N-1\}$. For an exponent vector α, define

$$\delta_i(\alpha; a, a') = \alpha_{Y_i} + \frac{\sum_s d_s(\alpha_{Y_s} - \alpha_{X_s})}{Q-1} = \frac{\nu - \nu'}{Q-1}, \quad \begin{cases} \nu = Q\alpha_{Y_i} + \sum_{s \neq i} d_s \alpha_{Y_s}, \\ \nu' = \sum_s d_s \alpha_{X_s}. \end{cases} \tag{20.5}$$

The same notation denotes its real-linear extension to real exponent vectors. This is the source's `defectWeight`. An excess divided by Q, rather than by $Q-1$, is a different normalization and must not be substituted for (20.5) in the thresholds below.

Lemma 20.15 (Weighted root lemma)**.** *Fix real weights c_z on all variables with*

$$\frac{1}{2} \le c_{Y_s} - c_{X_s} \le 1 \qquad \text{for every } s.$$

Let $f \in R[x_{b,s}]$, α an exponent, and $\ell \ge 0$ an integer. Suppose $w^{\ell+1} \nmid [x^\alpha]f$, and suppose that for every $g \in \mathrm{SL}_2(W)$ placed in the rows a, a', and every exponent β,

$$w^{\max(0, \lceil \ell + c\cdot\alpha - c\cdot\beta \rceil)} \mid [x^\beta]\rho(g)f. \tag{20.6}$$

Then $\delta_i(\alpha; a, a') \le 0$ for every letter i. In a valuation ring, (20.6) says that $\ell + c \cdot \alpha$ is a lower bound for the integer w-divisibility order plus weight of every term in every such translate.

Proof. We prove the assertion at i. Set $\rho_0 = (c_{Y_i} - c_{X_i})/Q > 0$, and put $\epsilon_s = -1$ when s has the parity of i, and $+1$ otherwise. To avoid a collision between old and new polynomial variables, the following substitutions are understood as maps from the old X_s, Y_s to new symbols of the same names, together with a new variable T:

$$\begin{aligned} &X_i \longmapsto X_i T^Q, \quad Y_i \longmapsto X_i, \\ &X_s \longmapsto X_s T^{Qd_s} - \epsilon_s Y_s T^{d_s}, \quad Y_s \longmapsto Y_s - \epsilon_s X_s \quad (s \neq i). \end{aligned} \tag{20.7}$$

Variables off the root are fixed. Let $F(T) \in R[T]$ be the coefficient of the monomial obtained from x^α by merging Y_i into X_i, after this substitution. Work in $\overline{R} = R/(w^{\ell+1})$.

We first estimate the degree of $\overline{F}$. Give T weight $-\rho_0$, give the new X_s weight $c_{X_s} + Qd_s\rho_0$, and give Y_s, $s \neq i$, weight c_{Y_s}. The main terms of (20.7) have the old weights. The two old variables that are merged have equal resulting weights, since $c_{X_i} + Q\rho_0 = c_{Y_i}$. Every correction has weight strictly greater than the corresponding old variable. Indeed,

$$d_s\rho_0 \le \frac{1}{5} \quad \text{for all } s, \qquad Qd_s\rho_0 \ge \frac{5}{2} \quad (s \neq i),$$

because $d_s \le Q/P$, $d_s \ge P$ for $s \neq i$, and the increments of c lie in $[1/2, 1]$. Thus the gains are at least $1/2 - 1/5 = 3/10$ and $5/2 - 1 = 3/2$; we may use the common weaker gain $1/20$.

For precision, a polynomial has Gauss level b if its coefficient at a monomial of variable-weight u is divisible by $w^{\max(0, \lceil b-u \rceil)}$. Products add levels and sums preserve their minimum. The main substitution preserves the level of a polynomial, and changing at least one main factor to a

correction raises the level by 1/20. These statements follow by expanding monomials and using $\lceil x+y\rceil \le \lceil x\rceil + \lceil y\rceil$, with nonnegative exponents throughout.

The merged monomial has weight $c\cdot\alpha + Q\nu'\rho_0$. Consequently, the coefficient of T^j in F is divisible by $w^{\ell+1}$ when $j > Q\nu'$. At $j = Q\nu'$, every correction is also divisible by $w^{\ell+1}$. The main substitution contributes exactly $[x^\alpha]f$ there: the merged exponent records every exponent except the split between X_i, Y_i, and the power $Q\nu'$ records that split uniquely. It follows that

$$\overline{F} \neq 0, \qquad \deg \overline{F} \le Q\nu'. \tag{20.8}$$

We next obtain a zero at each residue class of W. Let $z \in W$. In the rows a, a', take

$$g_z = \begin{pmatrix} 0 & 1 \\ -1 & \eta z \end{pmatrix}, \qquad \eta = -1 \ (i \text{ even}), \quad \eta = 1 \ (i \text{ odd}).$$

This matrix has determinant one. Apart from fixed signs on the variables, its action sends

$$X_s \longmapsto Y_s, \qquad Y_s \longmapsto X_s + \epsilon_s \iota_s(z) Y_s.$$

Apply (20.7) after this action and replace T by $T + \iota_i(z)$. Up to the same signs, the result on each old variable is its swapped substitution, with errors built from

$$E_j(T,z) = (T+\iota_i(z))^{P^j} - T^{P^j} - \iota_{i+j}(z). \tag{20.9}$$

Every coefficient of E_j is divisible by $p = w^2$. This follows first for $j = 1$ from the binomial theorem modulo p and the Frobenius congruence, and then by iteration. Explicitly, on Y_s the errors multiply X_s and Y_s and use $j = N + [s-i]_N$ and $j = [s-i]_N$, respectively. On X_s there is no error.

Now give T weight $+\rho_0$, X_s weight c_{X_s}, and Y_s weight $c_{Y_s} - e_s\rho_0$, where $e_i = Q$, $e_s = d_s$ for $s \neq i$. The swapped substitutions have weight at least the old weight: at i this uses $Q\rho_0 = c_{Y_i} - c_{X_i}$; at $s \neq i$ it uses

$$c_{Y_s} - d_s\rho_0 \ge c_{X_s}, \qquad c_{X_s} + Qd_s\rho_0 \ge c_{Y_s}.$$

Each error has an additional w-order at least 2. Since $d_s\rho_0 \le 1/5$ and $c_{Y_s} - c_{X_s} \le 1$, its total level is at least the old weight, in fact at least that weight plus 1/20. The same direct check at i uses $c_{X_i} + 2 \ge c_{Y_i} + 1$.

Apply these statements to $\rho(g_z^{-1})f$, whose level is at least $\ell + c\cdot\alpha$ by hypothesis. The composed action gives the substituted f with T shifted by $\iota_i(z)$. In this grading the merged monomial has weight $c\cdot\alpha - \nu\rho_0$. The coefficient of T^j in $F(T+\iota_i(z))$ is therefore divisible by $w^{\ell+1}$ whenever $j < \nu$. Thus

$$(T - \overline{\iota_i(z)})^\nu \mid \overline{F}(T) \quad \text{in } \overline{R}[T] \quad \text{for every } z \in W.$$

Choose representatives of the Q residue classes of W. Their distinct differences are units in W, hence remain units under $W \longrightarrow \overline{R}$. The corresponding monic linear polynomials, and their ν-th powers, are pairwise coprime. Their product divides the nonzero $\overline{F}$. A monic divisor of degree $Q\nu$ cannot divide a nonzero polynomial of smaller degree, even over a ring with zero divisors. Together with (20.8) this gives $Q\nu \le Q\nu'$, and (20.5) gives the result.

Lean. `RootLemma` (L942). □

Corollary 20.16 (The depth-minus-defect filtration)**.** *In the setting of Lemma 20.15, let the variable weights D_z satisfy $D_{Y_s} - D_{X_s} = 1$ for all s. Suppose that every root $\mathrm{SL}_2(W)$ translate of f has Gauss level at least v_0 for these weights and the w-adic filtration. Then f has level at least v_0 for the weights*

$$\omega_z = D_z - \frac{\delta_i(z;a,a')}{3}.$$

Equivalently, for every exponent β,

$$w^{\max(0,\lceil v_0 - D\cdot\beta + \delta_i(\beta;a,a')/3\rceil)} \mid [x^\beta]f. \tag{20.10}$$

Proof. The rise of the defect weight from X_s to Y_s is

$$\delta_i(Y_s;a,a') - \delta_i(X_s;a,a') = \begin{cases} 1 + 2/(Q-1), & s = i, \\ 2d_s/(Q-1), & s \neq i. \end{cases}$$

It lies in $[0, 3/2]$, since $P \geq 5$, $Q \geq 5$, and $d_s \leq Q/P$. Hence the increments of ω lie in $[1/2, 1]$.

We give the minimum argument in a form that does not assume that every element has a finite w-adic order. Consider pairs (β, ℓ), ℓ a nonnegative integer, for which some root translate $h = \rho(g)f$ has

$$w^{\ell+1} \nmid [x^\beta]h, \qquad \ell + \omega\cdot\beta < v_0. \tag{20.11}$$

There are only finitely many such pairs: $\deg\beta$ is bounded by $\deg f$, and then the displayed inequality bounds ℓ. If any pair exists, choose one minimizing $\lambda = \ell + \omega\cdot\beta$, and let h be a translate attaining this level. Every root translate has Gauss level at least λ for ω. Otherwise a coefficient failing that level gives $\ell' = \lceil \lambda - \omega\cdot\beta'\rceil - 1 \geq 0$; it yields another pair in (20.11) of strictly smaller weight. Root translates of h are still translates of f, so Lemma 20.15 applies and gives $\delta_i(\beta;a,a') \leq 0$. Consequently

$$\ell + D\cdot\beta \leq \ell + \omega\cdot\beta < v_0,$$

contradicting the hypothesized depth level of h, which makes its coefficient divisible by $w^{\ell+1}$. Thus no pair exists. A failure of (20.10) would give such a pair by the same ceiling argument, proving the corollary.

Lean. `RootLemma` (L991). □

Residue specialization and flags. For the rest of this subsection let the coefficient ring be a field κ of characteristic p. Then $w = 0$, and the cyclic Frobenius relations imply $\iota_s(z)^Q = \iota_s(z)$. The image of W is a finite field: it is a finite integral domain contained in the roots of $X^Q - X$. The kernel is therefore the unique maximal ideal of W, and the induced map from its residue field is an embedding of $\mathbb{F}_Q$. The action factors through $\mathrm{GL}_k(\mathbb{F}_Q)$; different letters use the prescribed Frobenius embeddings. Reduction $\mathrm{GL}_k(W) \longrightarrow \mathrm{GL}_k(\mathbb{F}_Q)$ is surjective, since an arbitrary entrywise lift of an invertible matrix has unit determinant.

Write

$$\mathcal{F}_j(Q) = \prod_{b=1}^{j} \frac{Q^b - 1}{Q - 1}, \qquad \mathcal{F}_0(Q) = 1.$$

This is the number of complete flags in $\mathbb{F}_Q^j$: choose the first line in $(Q^j - 1)/(Q - 1)$ ways and recurse on the quotient. If $1 \leq r < k$, let L_r be the block diagonal Levi of sizes $r, k - r$, and let $\mathcal{P}_r \subset \mathrm{GL}_k(W)$ be the parahoric whose upper right r-by-$(k - r)$ block is in the maximal ideal. Its index counts $(k - r)$-dimensional subspaces, while the number of right cosets of a lower Borel in the residue Levi is $\mathcal{F}_r(Q)\mathcal{F}_{k-r}(Q)$. Counting a full flag by its $(k - r)$-dimensional member gives

$$[\mathrm{GL}_k(W) : \mathcal{P}_r]\, \mathcal{F}_r(Q)\mathcal{F}_{k-r}(Q) = \mathcal{F}_k(Q). \tag{20.12}$$

Lemma 20.17 (A least term in a Levi flag representative)**.** *Let $f \neq 0$ be a polynomial over κ, and fix representatives $\mathcal{R}$ for the right cosets of the lower Borel in the residue Levi $L_r(\mathbb{F}_Q)$. There are $g \in \mathcal{R}$ and a nonzero coefficient $[x^\alpha]\rho(g)f$ such that*

$$\delta_i(\alpha; b, b+1) \leq 0 \quad \textit{for every } i \quad \textit{and every } b \neq r - 1.$$

Proof. Give $x_{b,s}$ height b at even letters and $-b$ at odd letters. Choose a term of least height among all Levi translates of f; a minimum exists, since all these polynomials have bounded total degree. A lower triangular matrix acts on a variable by a nonzero diagonal multiple of itself plus terms of strictly greater height, including at odd letters where the inverse transpose reverses triangularity. Thus its action on a polynomial of minimum height H multiplies each coefficient of height H by its nonzero diagonal character. Writing the chosen Levi element as a lower triangular element times a right-coset representative shows that the chosen term also occurs at that representative. For a simple root inside either block, the height increment from X_s to Y_s is 1, and every root $\mathrm{SL}_2(\mathbb{F}_Q)$ translate is still a Levi translate. Apply Lemma 20.15 with $w = 0$ and $\ell = 0$.
Lean. `DefectCount` (L455). □

The cone and its exact volume. In $\mathbb{R}^{kN}$ let $\mathcal{C}_h$ be the cone with $\delta_i(\beta; b, b+1) \le 0$ for $b \ne r-1$, and $\delta_i(\beta; r-1, r) \le h$, at every letter i. The condition $\beta \ge 0$ is not included in this definition. Let B take the sum in each letter. For $D = (D_s)_s \in \mathbb{R}^N$, put

$$\Sigma(D) = \{\beta \ge 0 : B\beta = D\}. \tag{20.13}$$

On each affine fiber of B, use Lebesgue measure in the coordinates obtained by omitting the last entry in each letter. Thus the lattice $\ker B \cap \mathbb{Z}^{kN}$ has covolume one. This is the relative volume of [BR07, §5.4].

Lemma 20.18 (Pieri cone-volume identity)**.** *For every D,*

$$\mathcal{F}_k(Q)\ \mathrm{vol}(\mathcal{C}_0 \cap \Sigma(D)) = \mathrm{vol}(\Sigma(D)). \tag{20.14}$$

Proof. Set $\varepsilon_s = (-1)^s$ and $(A_\sigma v)_{b,s} = \varepsilon_s(P v_{b,s+1} - v_{\sigma_s(b),s})$. With $\beta = A_1 v$ and $u_s = v_{b,s} - v_{b+1,s}$, the cyclic sum $\sum_s d_s(u_s - P u_{s+1}) = -(Q-1)u_i$ gives

$$\delta_i(\beta; b, b+1) = \beta_{Y_i} - (v_{b,i} - v_{b+1,i}). \tag{20.15}$$

Thus the cone condition together with nonnegativity reads

$$\begin{aligned} &P v_{b,s+1} \ge v_{b,s} \ge P v_{b+1,s+1} && (s \text{ even}),\\ &v_{b,s} \ge P v_{b,s+1} \ge v_{b+1,s} && (s \text{ odd}), \end{aligned} \tag{20.16}$$

with each second inequality imposed for $b < k-1$. For decreasing x, y, the matrix $(\mathbf{1}_{\{y_c < x_b\}})_{b,c}$ has rows of terminal ones. Its determinant is zero unless their numbers of leading zeros are $0, 1, \dots, k-1$, when it is one. It therefore detects the interlacing

$$x_0 > y_0 \ge x_1 > y_1 \ge \cdots > x_{k-1} > y_{k-1}. \tag{20.17}$$

The equality hyperplanes have measure zero on any fiber of B: since $k \ge 2$, an entry can vary against another in its letter, and adjacent letters are distinct. Let $\mathcal{I}$ collect the second inequalities in (20.16) and set $v^\tau_{b,s} = v_{\tau_s(b),s}$. Multiply the interlacing determinants for successive letters. Sorting the $v_{b,s}$ at a letter contributes the same sign twice, once at each neighbor. Exactly one sorting can interlace when the coordinates are distinct. Expanding the determinants consequently gives

$$\sum_{\tau \in (S_k)^N} \mathbf{1}_{\{A_1 v^\tau > 0,\ v^\tau \in \mathcal{I}\}} = \sum_{\sigma \in (S_k)^N} \mathrm{sgn}(\sigma_0 \cdots \sigma_{N-1}) \mathbf{1}_{\{A_\sigma v > 0\}}. \tag{20.18}$$

The letter sums satisfy $B A_\sigma = C B$ for every σ, where $(Cz)_s = \varepsilon_s(P z_{s+1} - z_s)$. Since $P^N = Q > 1$, C is invertible. Integrate (20.18) on $Bv = C^{-1}D$. A permutation within a letter preserves the

lattice measure there. In coordinates consisting of the retained entries and the letter sums, A_σ is block triangular with quotient block C; its Jacobian between the fibers is $|\det A_\sigma|/|\det C|$. The common denominator cancels. The determinant of the cyclic step equations satisfies

$$|\det A_\sigma| = |\det(QI - \sigma_0 \cdots \sigma_{N-1})|. \tag{20.19}$$

Indeed, relabeling the successive blocks by cumulative permutations places the product in the last equation. Eliminating the intervening blocks at their unit pivots leaves $P^N I - \sigma_0 \cdots \sigma_{N-1}$, up to signed permutations. In particular $|\det A_1| = (Q-1)^k$. The preimages of the bounded simplex product $\Sigma(D)$ are bounded. The left side of the integral has $(k!)^N$ equal terms, while each product permutation occurs $(k!)^{N-1}$ times on the right. Changing variables on both sides yields

$$\operatorname{vol}(\mathcal{C}_0 \cap \Sigma(D)) = c_k(Q)\operatorname{vol}(\Sigma(D)), \qquad c_k(Q) = \frac{(Q-1)^k}{k!}\sum_{\sigma\in S_k}\frac{\operatorname{sgn}(\sigma)}{|\det(QI-\sigma)|}. \tag{20.20}$$

Put $z = Q^{-1}$. The Molien formula for the sign character [Sta79, Theorem 2.1] evaluates the sum directly:

$$\frac{1}{k!}\sum_{\sigma\in S_k}\frac{\operatorname{sgn}\sigma}{\det(I-z\sigma)} = \frac{z^{\binom{k}{2}}}{\prod_{j=1}^{k}(1-z^j)}.$$

Indeed, the average on the left gives the graded trace of projection onto the alternating polynomials, by diagonalizing each permutation. Every alternating polynomial is divisible by the Vandermonde product, and its quotient is symmetric. The elementary symmetric functions have degrees $1,\dots,k$, which gives the displayed series. Since $\det(QI-\sigma) > 0$, substitution in (20.20) gives $c_k(Q) = (Q-1)^k/\prod_{j=1}^{k}(Q^j-1) = 1/\mathcal{F}_k(Q)$.

Lean, comparison. Lean integrates over a slab of letter degrees. Here the common map on letter sums gives the identity on each fiber. `DefectCount` (L915), `PieriCone` (L440-L499). The graded trace replaces the evaluation using the one-letter cone. □

Lemma 20.19 (An integral inward shift)**.** *There is an integral vector $\sigma \geq 0$ in $\mathcal{C}_{-1}$ such that*

$$\sum_b \sigma_{b,s} \leq P^{2r} - 1 \qquad \textit{for every } s.$$

Consequently $h\sigma + \mathcal{C}_h \subset \mathcal{C}_0$ for $h \geq 0$, and adding $h\sigma$ increases the letter degrees by $h\sum_b \sigma_{b,s}$.

Proof. Put $a = r-1$, and set

$$\mu_b = \begin{cases} P^{2(a-b)}, & b \leq a, \\ 0, & b > a, \end{cases} \qquad v_{b,s} = \begin{cases} \mu_b, & s \text{ even}, \\ P\mu_b, & s \text{ odd}. \end{cases}$$

Let σ be its step vector. Then $\sigma_{b,s} = (P^2-1)\mu_b$ at even letters and 0 at odd letters, so it is integral and nonnegative. Its degree at an even letter is $P^{2r}-1$. Since $\mu_b = P^2\mu_{b+1} + \mathbf{1}_{\{b=a\}}$, (20.15) gives the simple-root defects

$$P^2\mu_{b+1} - \mu_b = -\mathbf{1}_{\{b=a\}} \quad (s \text{ even}), \qquad P(\mu_{b+1} - \mu_b) \leq -\mathbf{1}_{\{b=a\}} \quad (s \text{ odd}).$$

The last assertion follows by linearity of each defect and degree.

Lean. `DefectCount` (L529), `DefectCount` (L581). □

The residue condition used by averaging. Let V_n be the space of polynomials homogeneous of degree n in each of the N letters, over κ. Its dimension is

$$M_n = \binom{n+k-1}{k-1}^N.$$

For $h \geq 0$ define the subspace

$$S_h = \{f \in V_n : [x^\alpha]\rho(g)f = 0 \text{ whenever } g \in L_r(W) \text{ and } \delta_i(\alpha; r-1, r) > h \text{ for some } i\}. \quad (20.21)$$

All Levi translates are required in this definition.

Theorem 20.20 (Residue defect dimension bound)**.** *Let $1 \leq r < k$, n, t be positive integers, and let $0 < \theta \leq 1$. If*

$$2(k+1) \leq t, \qquad 3NktP^{2r} \leq \theta n, \quad (20.22)$$

then

$$[\mathrm{GL}_k(W) : \mathcal{P}_r] \dim_\kappa S_{3t} \leq (1+\theta)\binom{n+k-1}{k-1}^N. \quad (20.23)$$

Proof. Put $d = (k-1)N$. For $m = Q^j$, choose $u_{b,s} \in \{0, \dots, m-1\}$ independently for $0 \leq b < k-1$. In letter s take $\gamma(u)_{b,s} = u_{b,s}$ for these b and $\gamma(u)_{k-1,s} = m(k-1) - \sum_{b<k-1} u_{b,s}$. All entries are nonnegative and the sum in each letter is $m(k-1)$. For $f_u \in S_{3t}$, put

$$F(x) = \sum_u f_u(x^m)x^{\gamma(u)}. \quad (20.24)$$

The first $k-1$ exponent residues in each letter recover u, and then subtraction of $\gamma(u)$ recovers the exponent in f_u. This linear map is therefore injective, with domain dimension $m^d \dim_\kappa S_{3t}$.

The Levi matrices have entries in $\mathbb{F}_Q$, so Frobenius substitution commutes with their action. In a Levi translate, each term has exponent $m\beta + \epsilon$, with β from the corresponding translate of some f_u and ϵ of degree $m(k-1)$ at every letter. For any nonnegative exponent ϵ with letter degrees at most b_0, the definition of the defect gives

$$|\delta_i(\epsilon; a, a')| \leq b_0\left(1 + \frac{\sum_s d_s}{Q-1}\right) = b_0\left(1 + \frac{1}{P-1}\right) \leq 2b_0. \quad (20.25)$$

Thus the distinguished defects of every such term are at most $m(3t + 2(k-1)) \leq 4tm$.

Put $B_0 = n + k - 1$, and use the vector σ of Lemma 20.19 to define

$$D_s = B_0 + 4t\sum_b \sigma_{b,s}, \qquad Y = \mathcal{C}_0 \cap \Sigma(D). \quad (20.26)$$

Each term has degree mB_0 in every letter. For $F \neq 0$, Lemma 20.17 gives a flag representative g and a nonzero coefficient of $\rho(g)F$ at α with all nondistinguished defects nonpositive. The shift therefore gives

$$y = 4t\sigma + \alpha/m \in Y \cap m^{-1}\mathbb{Z}^{kN}. \quad (20.27)$$

For each flag representative and each such grid point, record the coefficient at $m(y - 4t\sigma)$, taking it to be zero if an entry is negative. Equation (20.27) proves injectivity, hence

$$m^d \dim_\kappa S_{3t} \leq \mathcal{F}_r(Q)\mathcal{F}_{k-r}(Q)\,\#(Y \cap m^{-1}\mathbb{Z}^{kN}). \quad (20.28)$$

Each D_s is integral, so omitting the last entry in each letter identifies the grid on $B\beta = D$ with $m^{-1}\mathbb{Z}^d$. In these coordinates Y is a bounded polytope. Its indicator is Riemann integrable, so division by m^d and passage to the limit gives its volume. By (20.12) and Lemma 20.18,

$$[\mathrm{GL}_k(W) : \mathcal{P}_r] \dim_\kappa S_{3t} \le \mathrm{vol}(\Sigma(D)). \tag{20.29}$$

Put $J = 4t(P^{2r} - 1)$. The shift has $D_s = B_0 + J$ on the even letters and $D_s = B_0$ on the odd ones. Each letter contributes a simplex of volume $D_s^{k-1}/(k-1)!$. Since N is even,

$$\mathrm{vol}(\Sigma(D)) = \left(\frac{[B_0(B_0 + J)]^{(k-1)/2}}{(k-1)!} \right)^N. \tag{20.30}$$

To compare with $M_n = \binom{n+k-1}{k-1}^N$, use $\log(B_0/(n+i)) \le (k-1-i)/n$ for $1 \le i < k$ and $\log(1 + J/B_0) \le J/n$. This gives

$$\log \frac{\mathrm{vol}(\Sigma(D))}{M_n} \le \frac{N(k-1)}{2n}(k - 2 + J) \le \frac{2NktP^{2r}}{n} \le \tfrac{2}{3}\theta \le \log(1 + \theta).$$

The last inequality follows from concavity on $[0, 1]$ and $\log 2 \ge 2/3$. This proves (20.23).

Lean, comparison. Lean packs all kN residues and counts a slab. Fixing the total degree of each tag leaves $(k-1)N$ free residues and gives the simplex volume (20.30). `DefectCount` (L939). □

Reduction of the averaging congruence. At precision w^c, the averaging construction uses $t = 3kc$ (`defectThreshold`(k, c) in Lean). Its image satisfies

$$[x^\alpha]\rho(g)f \in (w^c) \quad \text{if } \delta_i(\alpha; r-1, r) > 3t, \qquad g \in L_r(W).$$

Its reduction lies in precisely S_{3t} of (20.21), provided $c \ge 1$. Thus the hypotheses for the dimension part of that construction are

$$3kc \ge 2(k+1), \qquad 9Nk^2cP^{2r} \le \theta n, \qquad 0 < \theta \le 1. \tag{20.31}$$

In particular $\theta = 1/(m^2 f)$ is admissible once these numerical inequalities are checked for the marking parameters. Suppose the number of edge fibers is the number of vertices times the parahoric index, as in the coverage hypothesis. Multiplying the local dimension bound by the number of fibers then bounds the global edge dimension by $1 + \theta$ times the vertex dimension. The construction below gives the integral averaging operator and its pairing divisibility. We apply (20.31) with the chosen global parameter y.

20.2 An integral average at the prescribed Frobenius precision

This subsection constructs the average used in the exceptional elementary-divisor argument. All constants refer to the original cyclic datum. Let $N = 2\ell \ge 2$ be the number of letters, $P = p^e \ge 5$, and $Q = P^N$. The local ring W has residue field of cardinality Q. The coefficient ring O is a valuation domain of characteristic zero in which p is not a unit; it contains w with $w^2 = p$. The maps $\iota_s : W \to O$, $s \in \mathbb{Z}/N$, satisfy $\iota_{s+1}(x) \equiv \iota_s(x)^P \pmod p$. Parity alternates around the cycle.

Let V have the monomial basis indexed by

$$\mathcal{B} = \{\beta = (\beta_s)_s : \beta_s \in \mathbb{N}^k,\ |\beta_s| = n\}, \qquad D_V = |\mathcal{B}| = \binom{n+k-1}{k-1}^N.$$

The representation $\rho(g)$ is the tensor product of the degree-n symmetric powers of $\iota_s(g)$ at even letters and $\iota_s(g)^{-\mathsf{t}}$ at odd letters. Fix $1 \le r \le k/2$, put $a = r-1$, and write $L = \mathrm{GL}_r(W) \times \mathrm{GL}_{k-r}(W)$. The root vector is $\varrho = e_r - e_a$, with indices starting at zero. In an even letter set $Y_s = (r, s)$, $X_s = (a, s)$; in an odd letter interchange them. The defect is

$$\delta_i(\beta) = \beta(Y_i) + \frac{1}{Q-1} \sum_s P^{[s-i]}\big(\beta(Y_s) - \beta(X_s)\big), \tag{20.32}$$

where $[s-i] \in \{0, \dots, N-1\}$. This is the source defect, including the denominator $Q-1$.

Write $d(\beta)$ for the total degree in the deep variables: $b \ge r$ at even letters and $b < r$ at odd letters. Let $\Delta = \mathrm{diag}(w^{d(\beta)})$, and define the halfway module at all edges from the origin by

$$\mathcal{M} = \{v \in O^{\mathcal{B}} : \ \exists m \ge 0, F \in O^{\mathcal{B}}, \quad w^m v = \Delta F, \quad \Delta\rho(g)F \in w^m O^{\mathcal{B}} \text{ for every } g \in \mathrm{GL}_k(W)\}. \tag{20.33}$$

The Levi action commutes with Δ.

Lemma 20.21 (Integral multinomial duality)**.** *Suppose $n < Q$, and put $c_0 = k^2 eN = k^2 \log_p Q$. There are $b_\beta, b'_\beta \in O$ such that*

$$b_\beta b'_\beta = p^{Nc_0}, \qquad \langle \rho(g)h, h' \rangle_b = \langle h, \rho(g^{\mathsf{t}})h' \rangle_b, \quad \langle h, h' \rangle_b = \sum_\beta h'_\beta b_\beta h_\beta. \tag{20.34}$$

One can take

$$b'_\beta = \prod_s \binom{n}{\beta_s}, \qquad b_\beta = \prod_s \frac{p^{c_0}}{\binom{n}{\beta_s}}.$$

Proof. A binomial coefficient of upper index less than p^f has p-adic valuation at most f: in the factorial formula the difference at each power of p is zero or one, and no power above p^f contributes. Expressing a multinomial with m parts as a product of at most m such binomials gives valuation at most mf. An integer prime to p is a unit of O, since its Bezout relation with p excludes membership in the maximal ideal. Thus these valuation bounds are divisibilities in O. Apply them with $f = eN$ and $m = k \le k^2$ to obtain the asserted integral quotients.

In one letter, substitution by a matrix and its transpose are adjoint for the diagonal form with monomial weights $\beta_s!$. This follows by expanding the multinomial theorem, or by moving the substitution through $f(\partial)g(0)$. Dividing all weights in degree n by $n!$, and multiplying by p^{c_0}, gives the weights $p^{c_0}/\binom{n}{\beta_s}$. The same identity applies to the inverse-transpose action at odd letters. Tensoring proves (20.34).

Lean. `InvariantAverage` (L63-L155), `Edges` (L419-L491). □

Lemma 20.22 (Finite-field interpolation for the integral Schur algebra)**.** *Let $d \le Q-2$. On $U_d = (O^k)^{\otimes d}$, the coefficient matrices of the tensor power of a generic block matrix are in the O-span of $\iota(g)^{\otimes d}$, $g \in L$, for any ring map $\iota : W \to O$. Consequently this span is an algebra, and the submodule it generates from any finite collection of basis tensors is stable under all its coefficient matrices.*

Proof. Over W, form the vector of all degree-d monomials in the entries of the two diagonal blocks. If its values on L did not span the free module on these monomials, Nakayama's lemma would give a nonzero linear functional on the quotient modulo the maximal ideal. Its coefficients define a nonzero polynomial F of degree d over $\mathbb{F}_Q$. Every invertible pair over $\mathbb{F}_Q$ lifts to L, so F vanishes on these pairs. Let Δ be the product of the two block determinants. Then $F\Delta$ vanishes at every

matrix pair. Each entry has degree at most one in Δ, so $F\Delta$ has individual degrees at most $d+1<Q$. One-variable interpolation in each entry gives $F\Delta=0$, contrary to $F\neq 0$ and $\Delta\neq 0$.

Applying the linear map sending each monomial basis vector to its coefficient matrix gives the assertion over W. Applying ι gives it over O. The evaluated tensor powers multiply by the group law, hence their span is an algebra and preserves the indicated submodule.

Lean. `InvariantAverage` (L287-L348), `InvariantAverage` (L628-L705). Lean uses the homogeneous degree bound $d+k\le Q$. The block determinants have individual degree at most one, which suffices here. □

Lemma 20.23 (An integral invariant average in one matrix)**.** *Let $D<Q$ and $D+k\le Q$. For a weight $\varrho\in\mathbb{Z}^k$, a real threshold T and a map $\iota: W\to O$, there is an O-linear functional I on $O[X_{bc}]$ such that:*

$$I(\psi(\iota(g)^{-1}X\iota(g)))=I(\psi)\qquad (g\in L),\tag{20.35}$$

$$I(X^\alpha)=0\qquad \left(\sum_{b,c}\alpha_{bc}\varrho_b>T\right),\tag{20.36}$$

$$I(\psi)\equiv p^{c_0}\psi(1)\pmod J.\tag{20.37}$$

The last assertion holds for every ideal $J\subset O$ and every ψ of degree at most D for which, in every Levi conjugate, all monomials of row weight above T have coefficients in J.

Proof. For $|\alpha|<Q$ set $\eta_\alpha=p^{c_0}/\binom{|\alpha|}{\alpha}$; for larger degree set $\eta_\alpha=0$. The proof of Lemma 20.21, now with k^2 variables, proves integrality. These diagonal weights make every linear substitution adjoint to its transpose: in a fixed degree they are a common scalar times $\alpha!$, and substitutions preserve degree.

For $d\le D$, take the generic block matrix with entry X_{cb} at (b,c), and its d-fold tensor power. Apply to its entries the functional $X^\alpha\mapsto\eta_\alpha[X^\alpha]\psi$, obtaining an integral operator $\psi_d^\sharp$ on U_d. For tensor indices $i,j:\{1,\dots,d\}\to\{0,\dots,k-1\}$, its (i,j)-entry is

$$\eta_\alpha[X^\alpha]\psi,\qquad \alpha=\sum_{v=1}^{d}e_{(j(v),i(v))},$$

provided each $(i(v),j(v))$ lies in a common Levi block; otherwise it is zero. Adjointness of the η-pairing gives the covariance identity

$$(\psi(\iota(g)^{-1}X\iota(g)))_d^\sharp=\iota(g)^{\otimes d}\psi_d^\sharp(\iota(g)^{-1})^{\otimes d}.\tag{20.38}$$

It can also be checked on a monomial: moving the linear substitution to the coefficients of the generic tensor matrix transposes its matrix, and turns that generic matrix into $\iota(g)X^{\mathsf t}\iota(g)^{-1}$.

Let U_d^+ be the submodule generated, under the coefficient algebra, by the basis tensors e_j with $\sum_v\varrho_{j(v)}>T$. It is finitely generated, torsion-free and therefore free over the valuation domain. Lemma 20.22 shows that it is stable, and that the vectors $\iota(g)^{\otimes d}e_j$ with such j already span it. Define

$$I(\psi)=\sum_{d=0}^{D}\left(\operatorname{tr}_{U_d}\psi_d^\sharp-\operatorname{tr}_{U_d^+}\psi_d^\sharp\right).\tag{20.39}$$

This is an integral linear functional. Conjugation invariance of both traces proves (20.35).

For a bad monomial X^α, its sharp operator kills every basis tensor whose weight differs from the row weight of α. On the remaining basis tensors its image lies in U_d^+, by definition of that module. The whole image is therefore in U_d^+. The trace of its restriction to U_d^+ equals its trace on

U_d: write it as the composition of a map $U_d \to U_d^+$ and the inclusion, and use $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for finite free modules. This proves (20.36); no direct-summand assertion about U_d^+ is used.

Finally suppose the bad coefficients of every conjugate of ψ are in J. For a high-weight basis tensor e_j, expanding ψ in monomials shows $\psi_d^\sharp e_j \in JU_d^+$: the good monomials kill it, and each bad coefficient is in J. Use (20.38) to apply the same argument to $\iota(g)^{\otimes d}e_j$. These vectors span U_d^+, so $\psi_d^\sharp U_d^+ \subset JU_d^+$, and its trace there belongs to J. The multinomial theorem gives

$$\mathrm{tr}_{U_d}(X^\alpha)_d^\sharp = \begin{cases} \binom{d}{\alpha}\eta_\alpha X^\alpha(1), & |\alpha| = d, \\ 0, & |\alpha| \neq d. \end{cases}$$

Thus the whole-space traces sum to $p^{c_0}\psi(1)$ for degree at most D. Subtracting the traces in J proves (20.37).

Lean. `InvariantAverage` (L488-L601), `InvariantAverage` (L717-L969). □

Lemma 20.24 (Large defect in the all-edge module)**.** *If $v \in \mathcal{M}$, $g \in L$, and $\delta_i(\beta) \geq 3j$ for an integer $j \geq 0$, then*

$$w^j \mid (\rho(g)v)_\beta. \tag{20.40}$$

For the pairing in (20.34), the same condition implies $w^j \mid \langle \rho(g)e_\beta, v\rangle_b$.

Proof. Take m, F as in (20.33). All root translates of $\rho(g)F$ have Gauss w-adic level at least m for the depth weight. This weight increases by one from X_s to Y_s at every letter. Corollary 20.16 therefore lowers this weight by $\delta_i/3$:

$$w^{\lceil m - d(\beta) + \delta_i(\beta)/3\rceil_+} \mid (\rho(g)F)_\beta.$$

Multiplying by $w^{d(\beta)}$ shows that $w^{m+j} \mid (\Delta\rho(g)F)_\beta$. Since $g \in L$, it commutes with Δ, and $\Delta\rho(g)F = w^m\rho(g)v$. Cancel the nonzero w^m in the domain O to get (20.40). The adjoint identity makes the last pairing $b_\beta(\rho(g^{\mathsf{t}})v)_\beta$, so the same argument proves its divisibility.

Lean. `Edges` (L709-L812). □

We next use one generic $k \times k$ matrix X_j for each letter. For a family of factors $\psi_{s,d}$, form at letter s the following matrix: its row b is row b of

$$\psi_{s,m-1} \cdots \psi_{s,0}, \qquad m = \ell + \mathbf{1}_{\mathrm{deep}}(b, s),$$

at an even letter, and row b of the transposed adjugate of that product at an odd letter. Take the tensor product of their degree-n symmetric powers; denote this matrix of polynomials by $\mathcal{S}(\psi)$. For the original factors use $\psi_{s,d} = X_{s-d}^{[P^d]}$, where brackets mean entrywise power. The factors in the frame g are

$$\psi_{s,d}^g = \iota_s(g)\left(\iota_{s-d}(g)^{-1}X_{s-d}\iota_{s-d}(g)\right)^{[P^d]}\iota_s(g)^{-1}. \tag{20.41}$$

Write $\mathcal{S}_g = \mathcal{S}(\psi^g)$.

Lemma 20.25 (The coefficient comparison for the cyclic substitution)**.** *Assume*

$$Nnk(\ell+2) \leq P^\ell, \qquad \ell \leq N. \tag{20.42}$$

For a monomial ν in all the matrix entries, set

$$\mathrm{cw}_i(\nu) = \frac{1}{P^\ell}\left(\mathrm{rw}(\nu_i) + \sum_{j\neq i}\frac{P^{[j-i]}}{Q}(\mathrm{rw}(\nu_j) - \mathrm{colw}(\nu_j))\right),$$

where row and column weights use ϱ. If the coefficient of X^ν in $(\mathcal{S}_g)_{\beta\gamma}$ is not divisible by w^c, then

$$|\operatorname{cw}_i(\nu) - \delta_{i+\ell}(\beta)| \le 3kc. \tag{20.43}$$

Every entry of $\mathcal{S}_1$ has degree in any one matrix copy at most $D = Nnk(\ell+2)P^\ell$.

Proof. For an entry variable (j,b,c), give it weight

$$\omega_i(j,b,c) = \begin{cases} \varrho_b/P^\ell, & j = i, \\ P^{[j-i]}(\varrho_b - \varrho_c)/(QP^\ell), & j \ne i. \end{cases}$$

Its sum on ν is $\operatorname{cw}_i(\nu)$, and $|\omega_i| \le 2/P^\ell$. For $\epsilon = 1/2$ or $-1/2$, use the Gauss filtration in which a coefficient of a monomial μ at level h must be divisible by $w^{\lceil h - \epsilon\omega_i(\mu)\rceil_+}$. Products add levels and sums preserve a common level.

The framed factor in (20.41) is congruent modulo p to $X_{s-d}^{[P^d]}$. Indeed modulo p, the P^d-power map is a ring map and takes ι_{s-d} to ι_s, so the two conjugations cancel. Each entry is homogeneous of degree P^d. For $d \le \ell$, every variable has $|\epsilon\omega_i| \le 1/P^d$, so all its monomials have weights between -1 and 1. The error coefficients are divisible by $p = w^2$. It follows that the framed entry has at least the Gauss level of its unframed leading monomial, $\epsilon P^d \omega_i(s-d,b,c)$. This is the precise reason that changing the frame costs no extra power of w here.

Define potentials

$$A_{s,d}(b) = \frac{\epsilon\varrho_b}{P^\ell}\frac{P^{d+[s-d-i]}}{Q}, \qquad u = \frac{1}{2P^\ell}.$$

They have sum zero over b, and $|A_{s,0}(b)| \le u$. The cyclic exponent identity

$$d + 1 + [s-d-1-i] = d + \begin{cases} N, & s-d = i, \\ [s-d-i], & s-d \ne i \end{cases}$$

gives

$$|A_{s,d+1}(b) - A_{s,d}(c) - \epsilon P^d \omega_i(s-d,b,c)| \le u \qquad (d \le \ell \le N).$$

Thus an entry of $\psi_{s,d}^g$ has level at least $A_{s,d+1}(b) - A_{s,d}(c) - u$. Along a product of at most $\ell+1$ factors the intermediate potentials cancel. The resulting row has level at least $A_{s,m}(b) - (\ell+2)u$. For an odd letter, expand a transposed adjugate entry as a determinant of $k-1$ rows. The zero sum of the potentials changes the main term to $-A_{s,m}(b)$, and the loss is at most $k(\ell+2)u$.

The endpoint potentials are exactly the defect weights: for $m = \ell + \mathbf{1}_{\mathrm{deep}}(b,s)$,

$$(-1)^{\mathbf{1}_{s\ \mathrm{odd}}} A_{s,m}(b) = \epsilon(1-1/Q)[\text{coefficient of } \beta(b,s) \text{ in } \delta_{i+\ell}].$$

This follows by the same cyclic exponent identity; at $s = i+\ell$ its wrap contributes the extra $\beta(Y_s)$ in (20.32). Passing to symmetric powers and tensoring over the letters therefore shows that each coefficient a_ν in row β satisfies

$$w^{\lceil \epsilon((1-1/Q)\delta_{i+\ell}(\beta) - \operatorname{cw}_i(\nu)) - L_0\rceil_+} \mid a_\nu, \qquad L_0 = \frac{Nnk(\ell+2)}{2P^\ell} \le \frac{1}{2}. \tag{20.44}$$

If $w^c \nmid a_\nu$, apply both signs of ϵ to obtain

$$|(1-1/Q)\delta_{i+\ell}(\beta) - \operatorname{cw}_i(\nu)| \le 2c+1.$$

The geometric sum in (20.32) gives $|\delta_i(\beta)| \le 2n$. The size assumption implies $2n \le Q$, so $|\delta_i - (1-1/Q)\delta_i| \le 1$. Moreover $c \ge 1$, because a coefficient is always divisible by w^0, and $k \ge 2$. Thus $2c+2 \le 3kc$, proving (20.43).

For the degree assertion, each factor has degree at most P^ℓ in a given copy. The product uses at most $\ell+1$ factors, an adjugate uses at most $k-1$ entries, each symmetric power uses n entries, and there are N letters. This gives the stated larger bound $Nnk(\ell+2)P^\ell$.

Lean. `FrobeniusFactors` (L470-L576), `FrobeniusFactors` (L579-L786), `FrobeniusFactors` (L793-L816). □

Lemma 20.26 (Contracting all copies except one)**.** *Let each $l_j : O[X_{bc}] \to O$ be an integral linear form invariant under Levi conjugation through ι_j. Contract $\mathcal{S}_g\rho(g)h$ by l_i in every copy except j, obtaining a vector of polynomials $F_g(X_j)$. Under (20.42), if $w^c \nmid [X^\alpha](F_g)_\beta$, then*

$$\left|\frac{\mathrm{rw}(\alpha)}{P^\ell} - \delta_{j+\ell}(\beta)\right| \le t, \qquad t = 3kc. \tag{20.45}$$

For $g \in L$, changing the frame gives

$$F_1(\iota_j(g)^{-1} X \iota_j(g)) = \rho(g)^{-1} F_g(X). \tag{20.46}$$

Each entry of F_1 has degree at most $D = Nnk(\ell+2)P^\ell$.

Proof. An invariant linear form kills any monomial whose row and column weights differ. To see this, conjugate by the diagonal matrix with entries $(1+p)^{\varrho_b}$. The element $1+p$ is a unit of W, since p cannot be a unit there and W is local. In O it has infinite multiplicative order by characteristic zero. The monomial is multiplied by a nontrivial power of $1+p$; invariance and cancellation in the domain force its image to be zero.

Expand the contracted coefficient as a finite sum. If it is not divisible by w^c, at least one contributing full monomial ν has a coefficient not divisible by w^c, has part $\nu_j = \alpha$, and has nonzero $l_i(X^{\nu_i})$ in every other copy. Those other parts have equal row and column weights, so $\mathrm{cw}_j(\nu) = \mathrm{rw}(\alpha)/P^\ell$. Lemma 20.25 proves (20.45).

For the frame identity, substitute $X_i \mapsto \iota_i(g)^{-1} X_i \iota_i(g)$ in the original operator. Moving the outer conjugations of (20.41) through each product conjugates the even letter by $\iota_s(g)$, and the odd letter by $\iota_s(g)^{-\mathsf{t}}$. The Levi matrices preserve the deep and shallow rows, so the two product lengths are respected. Hence the resulting operator is $\rho(g)^{-1}\mathcal{S}_g\rho(g)$. Contract in all copies other than j; invariance of their l_i removes their conjugations and proves (20.46). The degree assertion follows from the last part of Lemma 20.25, since contraction cannot increase degree in the remaining copy.

Lean. `AveragingMap` (L234-L368), `AveragingMap` (L381-L449). □

Theorem 20.27 (The cyclic averaging map)**.** *Suppose*

$$Nk(\ell+2)n + n + k < P^\ell. \tag{20.47}$$

For every positive integer c, set $t = 3kc$ and

$$C_{c,t} = \{v \in O^{\mathcal{B}} : \quad (\rho(g)v)_\beta \in (w^c) \quad \text{whenever } g \in L \text{ and some } \delta_i(\beta) > 3t\}.$$

There is an integral matrix Av *on V such that*

$$\mathrm{Av}(V) \subset C_{c,t}, \qquad \langle \mathrm{Av}h, h'\rangle_b \equiv p^{Nc_0}\langle h, h'\rangle_b \pmod{w^c} \quad (h \in V,\ h' \in \mathcal{M}). \tag{20.48}$$

Also $\mathcal{M} + w^c V \subset C_{c,t}$.

Proof. Take $D = Nnk(\ell+2)P^\ell$. Since $N = 2\ell$, (20.47) implies

$$D + k < (Nnk(\ell+2)+1)P^\ell \le P^{2\ell} = Q.$$

Lemma 20.23 gives a functional I_j in each copy, for threshold $2tP^\ell$ and common multiplier $z = p^{c_0}$. For $J \subset \mathbb{Z}/N$, let Av_J be obtained by contracting every entry of $\mathcal{S}_1$ with I_j for $j \in J$ and with $z\,\mathrm{ev}_1$ for $j \notin J$. At the identity matrices the substitution is the identity, so

$$\mathrm{Av}_\varnothing = p^{Nc_0} I_V.$$

Put $\mathrm{Av} = \mathrm{Av}_{\mathbb{Z}/N}$.

Fix $g \in L$ and β with $\delta_i(\beta) > 3t$, and choose $j = i - \ell$. The β-coefficient of $\rho(g)\mathrm{Av}h$ is $I_j((F_g)_\beta)$, with all the other copies contracted by their averages. If a coefficient of $(F_g)_\beta$ is not divisible by w^c, (20.45) forces its row weight to exceed $2tP^\ell$. Such a monomial is killed by I_j. All other coefficients already belong to (w^c). This proves $\mathrm{Av}(V) \subset C_{c,t}$.

To prove the pairing congruence, add one new copy $j \notin J$. Use the family of forms for Av_J in all other copies and put

$$\psi(X) = \sum_\beta h'_\beta b_\beta (F_1)_\beta(X).$$

This polynomial has degree at most D. Replacing $z\,\mathrm{ev}_1$ by I_j changes the pairing by $I_j(\psi) - z\psi(1)$. For $g \in L$, (20.46) expresses its conjugate coefficient as

$$[X^\alpha]\psi(\iota_j(g)^{-1} X \iota_j(g)) = \sum_\beta [X^\alpha](F_g)_\beta \,\langle \rho(g)^{-1} e_\beta, h' \rangle_b. \tag{20.49}$$

If α has row weight above $2tP^\ell$ and a coefficient in this sum is not divisible by w^c, (20.45) gives $\delta_{j+\ell}(\beta) > t = 3kc \geq 3c$. Lemma 20.24 then makes the paired factor divisible by w^c. Every summand of (20.49) is therefore divisible by w^c. The congruence property of the one-matrix average applies with $J = (w^c)$, proving that this one-copy change preserves the pairing modulo w^c. Telescope over a chain from the empty subset to all copies to obtain (20.48).

Finally $w^c V \subset C_{c,t}$ by integrality of $\rho(g)$. If $v \in \mathcal{M}$ and $\delta_i(\beta) > 3t$, then $\delta_i(\beta) > 3c$; Lemma 20.24 proves the other inclusion.

Lean. `AveragingMap` (L457-L503), `AveragingMap` (L512-L619). □

Theorem 20.28 (An integral operator satisfying the edge congruences)**.** *Assume that the degree n satisfies*

$$0 < \theta \leq 1, \qquad n^4 \leq Q, \qquad (Nk \log Q/\theta)^5 P^{3r} \leq n. \tag{20.50}$$

Take the same b, b' as in Lemma 20.21, and put

$$c = 5Nc_0, \qquad t = 3kc, \qquad C = C_{c,t}.$$

Let $\kappa = O/\mathfrak{m}_O$, and let S be the subspace of $\kappa^{\mathcal{B}}$ whose every Levi translate has zero β-coefficient whenever some $\delta_i(\beta) > 3t$. Then the integral average above satisfies all the original edge conditions:

$$\mathcal{M} \subset C, \qquad w^c V \subset C, \qquad \mathrm{Av}(V) \subset C, \tag{20.51}$$

$$\langle \mathrm{Av}h, h' \rangle_b \equiv p^{Nc_0} \langle h, h' \rangle_b \pmod{w^c} \quad (h' \in \mathcal{M}), \tag{20.52}$$

$$w^{c-1} h \in C \implies \bar{h} \in S, \tag{20.53}$$

$$[\mathrm{GL}_k(W) : \mathcal{P}_r] \dim_\kappa S \leq (1+\theta) D_V. \tag{20.54}$$

Here $\mathcal{P}_r$ is the same block parahoric as in the edge operator.

Proof. The numerical consequences of (20.50), proved in Lemma 20.8, are

$$n < Q, \quad Nk(\ell+2)n + n + k < P^{\ell}, \quad 2(k+1) \le t, \quad 3NktP^{2r} \le \theta n.$$

Indeed $t = 3kc$, and Lemma 20.8 gives the stronger inequality $24Nk^2cP^{2r} \le \theta n$. Theorem 20.27 gives (20.51) and (20.52).

If $w^{c-1}h \in C$, each prohibited coefficient $z = (\rho(g)h)_\beta$ satisfies $w^c \mid w^{c-1}z$. Cancel $w^{c-1} \neq 0$ to get $w \mid z$. Since $w^2 = p$ is not a unit, $w \in \mathfrak{m}_O$. Reducing gives exactly the defining conditions of S, proving (20.53). Theorem 20.20, applied with threshold $3t$ and the numerical inequalities above, proves (20.54).

The pairing multiplier here is p^{Nc_0}; its composition with the duality multiplier $b_\beta b'_\beta = p^{Nc_0}$ gives the near-inverse scale p^{2Nc_0}. The congruence precision remains w^{5Nc_0}, and the residue test uses w^{5Nc_0-1}. Moreover $w^{5Nc_0-1} \in p^{2Nc_0}\mathfrak{m}_O$, since $Nc_0 \ge 2$. These are the original scales used by the primitive edge factorization and the elementary-divisor estimate.

Lean. `ElementaryDivisorTheorem` (L617-L735). □

20.3 The CM parameter and the valuation of nonsplit jets

We retain the normalized CM curve, theta function and parameter y used in Theorem 18.23. Write $x = \wp(z+t) - \wp(t)$, $g = -\log\theta$, and $\mathcal{A} = R[[y(z)]]$, where $R = \mathcal{O}_{\mathbb{C}_p}$. Thus

$$y^2 + a_1xy = x^3 + x, \quad x' = 2y + a_1x, \quad y' = 1 + 3x^2 - a_1y, \quad g'' = g''(0) + x.$$

All these identities use the original analytic coordinate. For a nonsplit prime put

$$(e, \nu, q, \rho) = \begin{cases} (2, 1, p^2, (p^2-1)^{-1}), & p \text{ inert}, \\ (1, \frac{1}{2}, p, (2p-2)^{-1}), & p = d. \end{cases}$$

Every prime dividing q_1, and every prime below 13, splits in the quadratic field K by the choice of Q, d. Consequently the nonsplit primes considered here satisfy $p \ge 13$ and $p \nmid q_1$.

Lemma 20.29 (A nonsplit parameter with an explicit theta exponent)**.** *There are $\varpi, \pi \in R$ of valuations ρ, ν, and $S, \psi \in XR[[X]]$, such that*

$$S - S^q \equiv X, \qquad \psi \equiv u_0X^2 + u_1S^{p+1} + u_2S^{p+q} \pmod{\pi}. \tag{20.55}$$

In the inert case $u_0 = 0$ and u_1 is a unit. For every integer $n \ge 1$ prime to p there are a unit $N_p \in R^\times$ and $\kappa \in \mathbb{C}_p$ with

$$v_p(\kappa) = \frac{1}{p-1} - \nu$$

such that, simultaneously for every $ny_0 \in \Lambda$,

$$(T^{(1)}_{y_0}\theta)(\varpi X) = \exp\{\kappa\psi(N_pX)\}\, V_{y_0}(N_pX), \qquad V_{y_0} \in R[[X]]. \tag{20.56}$$

The same $\varpi, S, \psi, N_p, \kappa$ works for all those translates. The assertion is coefficientwise in formal series and requires no convergence on the unit disc.

Proof. Put $Q = q - 1$ in this proof only. By Lemmas 18.21 and 18.22, the integral ring $\mathcal{A}$ is preserved by every CM endomorphism and by d/dz. Its endomorphism and isogeny-factor identities extend from $\mathcal{O}_K$ to its completion at p: each coefficient is a polynomial in the multiplying scalar and its conjugate, and R is closed. For a Q-th root of unity ζ the norm is ζ^{p+1}. This is the unramified quadratic norm in the inert case. In the ramified case $\zeta \in \mathbb{Q}_p$ and $\zeta^{p+1} = \zeta^2$.

Average the original parameter to obtain

$$T(z) = Q^{-1} \sum_{\zeta^Q=1} \zeta^{-1} y(\zeta z).$$

It belongs to $\mathcal{A}$ and has linear coefficient one as a series in y. Hence $\mathcal{A} = R[[T]]$. Also T' is an $\mathcal{A}$-unit. Its inverse series has the form

$$Z(T) = T + \sum_{j\geq 1} z_{1+jQ} T^{1+jQ}, \qquad Z' \in R[[T]], \tag{*}$$

by μ_Q-equivariance and $Z' = 1/T'(Z)$.

Let $\alpha = p$ in the inert case and $\alpha = \sqrt{-d}$ in the ramified case. The reduction of this endomorphism on the good elliptic curve is purely inseparable of degree q. For the ramified degree-p isogeny this follows from its zero tangent map. In the inert case an ordinary reduction would have an étale p-torsion quotient of order p; the unital action of $\mathcal{O}_K/p = \mathbb{F}_{p^2}$ on that quotient would give a unital homomorphism $\mathbb{F}_{p^2} \to \mathbb{F}_p$, which is impossible. Thus $[p]$ has inseparable degree p^2. In the integral T-coordinate the coefficient f_q at degree q of its formal endomorphism is consequently a unit. Equivariance excludes intermediate degrees, so comparison of degree q in $Z(F_\alpha(T)) = \alpha Z(T)$ gives

$$(\alpha - \alpha^q) z_q = f_q, \qquad v_p(z_q) = -\nu.$$

Choose $\varpi^Q = -1/z_q$ and put $\pi = \varpi^Q$. Then $v_p(\varpi) = \nu/Q = \rho$.

The integral derivative in (*) gives $v_p(z_{1+jQ}) \geq -v_p(1+jQ)$. For $j \geq 2$ one has $v_p(1+jQ) \leq (j-1)\nu$. For $e = 2$, the case $j = 2$ is a unit and $1 + j(p^2-1) < p^j$ for $j \geq 3$. For $e = 1$, $j = 2, 3$ give units, while $1 + j(p-1) < p^{1+\lfloor (j-1)/2 \rfloor}$ for $j \geq 4$. These inequalities give the stated valuation bound in each case. Therefore

$$\varpi^{-1} Z(\varpi X) \equiv X - X^q \pmod{\pi}$$

and this series is integral. Its compositional inverse is $S(X) = T(\varpi X)/\varpi \in XR[[X]]$. This proves the first congruence in (20.55).

Let $\mathcal{P}$ retain the degrees congruent to $p+1$ modulo Q, and set

$$\kappa_0 = \varpi^{p+1}/p, \qquad \psi(X) = \kappa_0^{-1} (\mathcal{P} g)(\varpi X).$$

Here is an explicit bound on every coefficient of this series. Set

$$M = (g - g''(0) z^2/2) \circ Z, \qquad A = (g' - g''(0) z) \circ Z.$$

Then $M' = AZ'$, $A(0) = 0$, and $A' = (x \circ Z) Z' \in R[[T]]$. For $D_j = p + 1 + jQ$ with $j \geq 0$, a contribution to $[T^{D_j}]M$ uses $[T^i]A$ and $[T^{hQ}]Z'$, where $i = p + (j-h)Q > 0$. After division by D_j and the scaling, its valuation is at least

$$1 + j\nu - v_p(D_j) - v_p(i) + v_p([T^{hQ}]Z'). \tag{20.57}$$

The last summand is nonnegative, and for $h = 1$ it is $v_p(qz_q) = \nu$. If $e = 1$ the additional possibility $h = j + 1$ has $i = 1$; its coefficient is zero because A starts in degree three. All other terms have $0 \leq h \leq j$.

For $j = 0$ the bound is nonnegative. For $j = 1$, $v_p(D_j) = 1$; the only positive valuation of i occurs at $h = 1$, when $i = p$, and the extra ν in $[T^Q]Z'$ makes the bound nonnegative again. For

$2 \le j \le 7$, D_j is a unit and i is divisible by p only when $i = p$. The bound is then at least $j\nu \ge \nu$. For $j \ge 8$, both D_j and i are at most $(j+2)q$, so the bound is at least

$$1 + j\nu - 2\log_p((j+2)q) \ge \nu.$$

To check the last inequality uniformly, its weakest case is $p = 13, \nu = \frac{1}{2}$; at $j = 8$ it is strict, and the derivative of $1 + (j-5)/2 - 2\log_{13}(j+2)$ is positive for $j \ge 8$. Thus only degrees $p+1$ and $p+q$ can remain modulo π in $\kappa_0^{-1}(\mathcal{P}M)(\varpi X)$.

The equivariance of Z means that projection by $\mathcal{P}$ commutes with substitution by Z or T. Substituting S therefore gives the two displayed nonquadratic terms in (20.55). The original quadratic term is retained only in the ramified case. Its coefficient is integral because $v_p(g''(0)) \ge -\nu$, by Lemma 18.22, and $v_p(\kappa_0^{-1}\varpi^2) = \nu$ in that case. This proves integrality of ψ and its claimed shape.

For inert p, the coefficient of S^{p+1} is

$$u_1 = [z^{p-1}]x/(p+1).$$

It is a unit. Indeed $Z = T + O(T^{p^2})$, so the functional $\lambda(f) = [z^{p-1}]f \bmod \mathfrak{m}_R$ annihilates f' for every $f \in \mathcal{A}$: below degree p^2 the coordinate change is the identity, and differentiation in degree p supplies the factor p. The cubic gives

$$(x^a x')' = (4a+6)x^{a+2} + (a+1)a_1^2 x^{a+1} + (4a+2)x^a.$$

If $\lambda(x) = 0$, starting with $\lambda(1) = 0$ and using $0 \le a \le (p-5)/2$ would imply $\lambda(x^{(p-1)/2}) = 0$. Each $4a+6$ is a unit in this range: it is an even positive integer smaller than $2p$, and cannot equal the odd integer p. But $x = z^2 + O(z^4)$ makes this last coefficient equal to one, a contradiction.

It remains to recover the original translated theta, without changing its scalar. The endomorphism factor for ζ^{-1}, evaluated at ζz, shows that

$$\prod_{\zeta^Q=1} \exp\{\zeta^{-(p+1)}g(\zeta z) - g(z)\} = \exp\{Q(\mathcal{P}g - g)(z)\} \in \mathcal{A}^\times.$$

Put $n_0 = nQ$. The translated isogeny identity (18.31) gives $(T_{y_0}^{(1)}\theta)(n_0 z) = \theta(z)^{n_0^2} R_{y_0}(z)$ with $R_{y_0} \in \mathcal{A}$. Raise the preceding unit to the integer n_0^2/Q. It replaces $\theta(z)^{n_0^2}$ by $\exp(-n_0^2\mathcal{P}g(z))$ times an $\mathcal{A}$-unit. Every element of $\mathcal{A}$ becomes integral at $z = \varpi X$, because $T(\varpi X) = \varpi S(X)$. With $N_p = n_0^{-1}$ and $\kappa = -n_0^2\kappa_0$ this proves (20.56). Finally

$$v_p(\kappa) = v_p(\kappa_0) = (p+1)\nu/(q-1) - 1 = 1/(p-1) - \nu.$$

The choices n_0, N_p, κ depend only on the common prime-to-p torsion order, not on the translate.
Lean, formal context. `ThetaProducts` (L105-L288), `LocalDivisibility` (L494). □

Lemma 20.30 (Cancellation of the Frobenius terms)**.** *Let $u_a \in E$ be integral above p and satisfy*

$$\sum_a \tau(u_a)\overline{\sigma(u_a)} = M\delta_{\tau\sigma}, \qquad p \nmid M.$$

Multiplicities are allowed by repeating the vectors. For $\beta \in R$, put $z_{\tau,a} = \beta\,\iota(\overline{\tau(u_a)})$. Let I_p° be the odd nonnegative integers if p is inert and all nonnegative integers if $p = d$. Then, in $R/(\pi)$,

$$\sum_a z_{\tau,a}^{p^j} z_{\sigma,a} = \beta^{p^j+1} M\,\mathbf{1}_{\sigma=[-p^j]\tau} \qquad (j \in I_p^\circ). \tag{20.58}$$

For matrix variables define

$$L_{a,c}(X) = \sum_{\tau} z_{\tau,a} X_{\tau,c}, \qquad \Psi(X) = \sum_{a,c} \psi(L_{a,c}(X)).$$

If integral series $Z_{\tau,c}$ have zero constant term and satisfy, modulo π and terms of degree $> B$,

$$\Gamma_j(Z) := \sum_{\tau,c} Z_{\tau,c}^{p^j} Z_{[-p^j]\tau,c} = 0 \quad (j \in I_p^{\circ},\ p^j \le B),$$

then $\exp(\kappa\Psi(Z))$ is integral through degree B. For inert p and unit β, the reduction modulo $\mathfrak{m}_R$ of $\partial_{X_{\sigma,c}}\Psi$ has nonzero coefficients exactly at the monomials $X_{\tau,c}^{p^j}$ with j odd and $\sigma = [-p^j]\tau$, and all of those coefficients are units.

Proof. On the cyclotomic generator, Frobenius applied to the conjugate embedding multiplies its exponent by $-p^j$. On $\mathcal{O}_K$, odd Frobenius in the inert case cancels conjugation. At $p = d$, conjugation is trivial modulo π, and the image of $\mathcal{O}_K$ is the prime field. These identities extend to all elements integral above p: the extension by the cyclotomic root is unramified there, and any clearing integer can be chosen prime to p. Thus $\iota(\overline{\tau(u_a)})^{p^j} \equiv \iota(([-p^j]\tau)(u_a)) \pmod{\pi}$. The frame identity proves (20.58).

Work in the ring truncated above degree B, modulo π. It has characteristic p, and series of positive order are nilpotent. Thus $S(L) = \sum_{i\ge 0} L^{q^i}$ is a finite sum in this ring. Equation (20.58) and the hypotheses give $\sum_{a,c} L_{a,c}(Z)^{p^j+1} = 0$ for every required j; terms with $p^j > B$ vanish by degree. Every summand in $\sum_{a,c} S(L_{a,c}(Z))^{p+1}$ is a Frobenius power of one of these expressions. The difference of its two Frobenius exponents is odd when $e = 2$, and unrestricted when $e = 1$. The same holds for the exponent $p + q$. The quadratic term is absent in the inert case and is covered by $j = 0$ in the ramified case. Therefore $\Psi(Z)$ is divisible by π through the required degree. The n-th term of its exponential has valuation at least $n/(p-1) - v_p(n!) \ge 0$, since $v_p(\kappa\pi) = 1/(p-1)$. This proves integrality.

In the inert residue field $S' = 1$, and differentiation kills S^{p+p^2}. Hence

$$\partial_{X_{\sigma,c}}\Psi = u_1 \sum_{\substack{j\ge 1\\ j \text{ odd}}} \sum_{\tau} \Big(\sum_{a} z_{\sigma,a} z_{\tau,a}^{p^j}\Big) X_{\tau,c}^{p^j}.$$

The displayed monomials are distinct. Their asserted supports and unit coefficients follow from (20.58) and the unit u_1.

Lean. `ThetaProducts` (L105-L288), `NormalForm` (L211-L300). □

Corollary 20.31 (The coupled condition for the standard data)**.** *For a small nonsplit prime p, every level-one standard datum of the original theta kernel, at its prescribed* jetScale *and its extra scalar $\bar{\pi}_M$, satisfies the bounded-degree integrality conclusion of Lemma 20.30. In any other integral bases of the same source and target lattices, the conclusion holds with the Hermitian pairing, the polynomials Γ_j of Lemma 20.30, and the Weyl lattices expressed in the new coordinates. Here level one means membership in $\mathcal{S}_1$ in the notation of Lemma 18.27. Since $p \nmid M$, one equivalently may use the prescribed $\mathcal{S}_{\pi_M}$ or any $\mathcal{S}_\delta$ with δ a unit above p.*

Proof. Apply the integrality in Theorem 19.4 and the finite inversion of Lemma 18.27. Because $p \nmid q_1$, the lattice $\mathfrak{a}_0$ and its inverse different are integral units above p. Choose the square frame with $p \nmid b$ and put $u_a = w_a/b$, giving $M = 1$. The original finite theta series is an integral linear combination of products in (18.40). Every shift has a common torsion order prime to p: its finite

rational translation and the frame vectors are integral above p, so their trace denominators can be cleared by such an integer.

Apply Lemma 20.29 with that common order. It gives the same exponential part for every product; the translated factors contribute only integral series. The scale conversion in Theorem 18.29 is exact: $j_0 = (A_0/\Omega_0)P_{\mathrm{loc}}$. At these primes P_{loc} is a unit, since the relative different is a unit and the normalized period root is a unit by Lemma 18.25. The extra scalar $\bar{\pi}_M$ is a unit because $p \nmid M$. Thus the coefficient $\beta = P_{\mathrm{loc}}\bar{\pi}_M N_p$ is a unit, and Lemma 20.30 applies in precisely the original variables. The integral coefficients from finite inversion preserve the bound.

To see the coordinate assertion, let T carry the new coordinates to the standard coordinates on the matrix variables. In the new coordinates the coupling is $\Gamma_j \circ T$. Thus a tuple Z in these coordinates satisfies the required equations precisely when TZ satisfies them in the standard frame. Substitution by T and by T^{-1} preserves integrality through each degree, since both matrices are integral. For changes of finite unitary point, Lemma 18.31 transports the data and the Weyl gauges by equivariance. This gives the bound at the points used below.

Lean. `exists_hasNormalForm_jetSeries`, `omega_mem_raises_of_schurJet_le`. □

Theorem 20.32 (The charged Schur jet bound)**.** *Let $\kappa_{\tau,h}$ count columns of height $h < N$, where N is the target rank and $\kappa_{\tau,0} = 0$. Assume the original fitting condition with the source rank n: every polynomial column has height at most n, with its determinant column included, and the common determinant twist is factored out. Put $B = \sum_{\tau,h} h\kappa_{\tau,h}$. Choose a set S of embeddings whose height-one columns are to be erased, and define*

$$t_\tau = \mathbf{1}_{\tau\notin S}\kappa_{\tau,1}, \qquad r_\tau = \sum_{h\ge 2}\kappa_{\tau,h}, \qquad B_\tau = \sum_{h\ge 2} h\kappa_{\tau,h}.$$

For $\tau_j = [-p^j]\tau$ set

$$C_{j,\tau} = \min(t_\tau/p^j, r_{\tau_j} + t_{\tau_j}) + r_\tau/p^j + \min(B_\tau/p^j, B_{\tau_j}).$$

Let $g_{\tau,h,b} \ge 0$ be integer row charges, with a positive charge at one end of every pair exempted by erasing height-one columns: a pair of heights $1, h$ at τ, τ_j is exempt if $\tau \in S$, or if $h = 1$ and $\tau_j \in S$. For a column-strict target tableau T put $m_T = \sum_{\text{boxes of } T} g_{\tau,h,b}$. Every corresponding entry of the original Schur jet matrix has valuation at least

$$\gamma_p B - \nu \sum_{\substack{j\in I_p^\circ \\ p^j\le B}} \sum_\tau C_{j,\tau} - fN^3(1 + \lfloor \log_p B \rfloor) - m_T, \tag{20.59}$$

where $\gamma_p = 1/(p-1) - \rho$. For $B = 0$ the assertion is interpreted with zero determinant cost and the logarithmic error fN^3, and follows from integral constant values.

Proof. Fix integral auxiliary entries $Y_{\tau,a,i}$. For each height h replace the first h directions by their product with one matrix $M_{\tau,h} \in M_h(R)$. We can choose these matrices of nonzero determinant so that every nonexempt pairing $\sum_a m_a^{p^j} m'_a$ is divisible by π and

$$\sum_{\tau,h} \kappa_{\tau,h} v_p(\det M_{\tau,h}) \le \nu \sum_{\substack{j\in I_p^\circ \\ p^j\le B}} \sum_\tau C_{j,\tau}. \tag{20.60}$$

Here is the elimination and its cost. An integral linear functional on h directions can be made divisible by a scalar $c \in R$ at determinant cost at most $v_p(c)$: choose an entry of smallest valuation,

eliminate the other entries by unimodular row operations, and multiply that one remaining row by c. The zero functional needs no change.

For fixed (j,τ) write $P_j = p^j$ and choose $\pi_j^{P_j} = \pi$. For a height-one source outside S, either scale that direction by π_j, or impose one π-divisibility condition on every target height not exempted. The weighted determinant costs are respectively $\nu t_\tau/P_j$ and $\nu(r_{\tau_j} + t_{\tau_j})$; choose the cheaper. For source height at least two against target height one, take P_j-th roots in R of the coefficients of the original target direction and impose one π_j-divisibility condition on each source height. Raising it to P_j in $R/(\pi)$ gives the desired pairing. A later change of a height-one target is only a scalar, so it preserves this condition. The total cost is at most $\nu r_\tau/P_j$. For two heights at least two, scale all source directions by π_j or all target directions by π, whichever is cheaper; this costs $\nu\min(B_\tau/P_j, B_{\tau_j})$. These choices cover every nonexempt pair. Previously imposed congruences survive subsequent integral changes of directions, since Frobenius is additive in $R/(\pi)$. There are only finitely many required pairs. Composing the changes proves (20.60).

Introduce variables $X_{\tau,b,h,i}$, $i < h < N$, and set

$$Z_{\tau,b,a} = \sum_{h,i<h} X_{\tau,b,h,i}\, p^{g_{\tau,h,b}} (M_{\tau,h} Y_{\tau,a,\bullet})_i.$$

Every coefficient of every required coupling $\Gamma_j(Z)$ is either a cancelled nonexempt pairing, or contains a factor p because one end is charged. Since $\pi \mid p$, all couplings vanish modulo π. Corollary 20.31 therefore says that the substituted theta series at ϖZ is integral through degree B.

Apply to it the product of the column-minor differential operators of T, using the h-th family for columns of height h. It is homogeneous of degree B with integer coefficients, in at most fN^3 variables. For each differential monomial of degree B, Legendre's formula gives

$$\sum_u v_p(d_u!) = \frac{B - \sum_u s_p(d_u)}{p-1} \geq \frac{B}{p-1} - fN^3(1 + \lfloor \log_p B \rfloor).$$

Thus its value on the integral substituted series has at least this valuation.

The chain rule identifies that value with the jet along the product of minors of zY, times exactly

$$\varpi^B p^{m_T} \prod_{\tau,h} (\det M_{\tau,h})^{\kappa_{\tau,h}}.$$

Each column contributes its determinant, and each chosen row contributes its stated charge. Constant-coefficient differential operators commute, so repeated auxiliary variables do not alter the identity. Division by this factor and (20.60) give (20.59) for every integral Y.

The coefficients of a polynomial over $\mathbb{C}_p$ have the same valuation lower bound if all integral evaluations do: otherwise divide by a coefficient of least valuation and reduce modulo $\mathfrak{m}_R$; the resulting nonzero polynomial is nonzero at some point of the infinite residue field. Expand the product of minors of zY in the standard tableau polynomials in Y. These have distinct diagonal leading monomials with leading coefficient one. Indeed those monomials record the multiset in each tableau row, and weak increase in each row recovers the tableau from these multisets. Successive leading-term subtraction therefore extracts each tableau coefficient integrally. Those coefficients are precisely the Schur matrix entries. This proves the asserted bound without a Young-symmetrizer denominator. If the chosen Weyl basis is written as the standard tableau polynomial times its common determinant power, that common factor has monic leading term as well. Multiplication by it preserves distinct leading monomials and the same integral extraction argument; no determinant power is divided out inside the coefficient ring.

Lean. `NullFrames` (L113-L444), `Count` (L571-L743). □

Corollary 20.33 (A valuation bound for the Schur jets)**.** *Let κ° be κ with its height-one columns at S removed, and let B° be its box count. Put $\tau_j = [-p^j]\tau$ and $I_p = I_p^\circ \cap \mathbb{Z}_{>0}$. Define*

$$\begin{aligned}\mathrm{Pred}_p(\kappa^\circ) = \nu \sum_{j\in I_p}\sum_{\tau}\left[(t_\tau/p^j - t_{\tau_j} - r_{\tau_j})_+ + (B_\tau/p^j - B_{\tau_j})_+ - r_\tau/p^j\right]\\ - \mathbf{1}_{p=d}\frac{1}{2}\sum_{\tau}\left[\min(t_\tau, t_{\tau_0}+r_{\tau_0}) + r_\tau + \min(B_\tau, B_{\tau_0})\right].\end{aligned}$$

Every charged Schur jet entry has valuation at least

$$\mathrm{Pred}_p(\kappa^\circ) + \gamma_p(B - B^\circ) - fN^3\{1 + \log_p \max(B,1)\} - m_T. \tag{20.61}$$

Denote the sum of the first three terms of (20.61) by divisibilityBound(p, j, S)*, where S specifies the embeddings at which height-one columns are removed. For the unmarked kernel, take $S = \varnothing$ and $m_T = 0$. Its valuation increases by at least* divisibilityBound$(p, j, \varnothing)$*. At a marked neighbor the loss is at most*

$$\sum_{w\in\Lambda} A_w + r_p \sum_{\tau\in S_\Lambda} \mathrm{lowColumns}_j(\tau) - \mathrm{divisibilityBound}(p, j, S_\Lambda),$$

with the original transported Weyl gauges.

Proof. All $C_{j,\tau}$ are nonnegative and are dominated by $(t_\tau + r_\tau + B_\tau)/p^j$, so their sums converge. For positive j the identity $u - \min(u,v) = (u-v)_+$ gives, term by term,

$$\frac{t_\tau + B_\tau}{p^j} - C_{j,\tau} = (t_\tau/p^j - t_{\tau_j} - r_{\tau_j})_+ + (B_\tau/p^j - B_{\tau_j})_+ - r_\tau/p^j.$$

Moreover $\gamma_p = \nu\sum_{j\in I_p} p^{-j}$. The additional $j = 0$ deduction occurs only when $p = d$, where $\nu = \frac{1}{2}$. Summing proves the exact identity

$$\gamma_p B^\circ - \nu \sum_{j\in I_p^\circ}\sum_{\tau} C_{j,\tau} = \mathrm{Pred}_p(\kappa^\circ).$$

Replace the finite deduction in (20.59) by this larger infinite one, and bound the integer logarithm by $\log_p \max(B,1)$. This proves (20.61).

For the unmarked kernel take $S = \varnothing$ and all charges zero. For a marked neighbor apply Lemma 20.14; the sum of its charges and diagonal weight exponents is bounded there by $\sum_w A_w + r_p\sum_{\tau\in S_\Lambda}\mathrm{lowColumns}_j(\tau)$. The determinant twists cancel by Lemma 20.11. Apply Corollary 20.31 in the standard frame. Under changes of frame use its transported coupling polynomials. Lemma 18.31 then carries the entrywise bounds to the Weyl gauges at the finite points. The exact predicted-divisibility formula; The prescribed normalized Schur jets; The unmarked gain; The marked-neighbor gauge.

Lean. `Count` (L405-L568), `Divisibility` (L202-L270), `ValuationAtNonSplitPrimes` (L289-L365). □

The coefficient in the original coordinate. A related parameter calculation appears in the coupled-derivatives reconstruction. In the coordinate of Lemma 20.30, let p be inert and β a unit. For odd $j \geq 1$ and $\sigma = [-p^j]\tau$, the coefficient of $X_{\tau,c}^{p^j}$ in the derivative $\partial_{X_{\sigma,c}}\Psi$ of the normalized exponent reduces to $u_1\beta^{p^j+1}M$ in $R/\mathfrak{m}_R$. This is nonzero by (20.58) and the unit property of u_1.

From the nonsplit jet estimate to the local modules. We apply (20.61) with the Weyl gauges and Hecke operators of (11.8). At stage $0 \le j < m$ put $N = n_{j+1}$. In the diagram at τ, let c_τ count the columns of heights $2, \dots, N-1$ and let l_τ count their boxes (lowColumns$_j(\tau)$ and lowBoxes$_j(\tau)$ in Lean). Put

$$w_\tau = \begin{cases} a/x, & \tau \in \Phi_b, \\ a/x_1, & \tau \notin \Phi_b. \end{cases}$$

Put $I_p = \{1, 3, 5, \dots\}$ when p is inert, and $I_d = \{1, 2, 3, \dots\}$. Let $v_p^\circ = 1$ in the inert case and $v_d^\circ = 1/2$, and write $\tau_e = [-p^e]\tau$ and $\gamma_p = v_p^\circ \sum_{e \in I_p} p^{-e}$. The original gain is

$$\frac{g_p}{m} = \sum_{\tau \in \Phi_a} s_p(a, \tau) + k \sum_{\tau \in \Phi} s_p(w_\tau, \tau), \qquad s_p(y, \tau) = v_p^\circ \sum_{e \in I_p} (y/p^e - w_{\tau_e})_+.$$

For $S \subset \Phi$ set $t_\tau = a 1_{\Phi_a \setminus S}(\tau)$. The divisibility bound $D_{p,j}(S)$ is the expression in (20.61), before the neighbor charge: the predicted divisibility for the diagrams with these height-one columns, plus $\gamma_p a |\Phi_a \cap S|$, minus $f N^3 (1 + \log_p \max(B, 1))$. Here B counts all original boxes below height N.

Lemma 20.34 (The arithmetic comparison loses only collisions)**.** *For every sufficiently large datum in §5.2, uniformly for every small nonsplit prime $p < x_1$, every stage $j < m$, and every $S \subset \Phi$,*

$$D_{p,j}(S) \;\ge\; \frac{g_p}{m} - \mathcal{D}_p(S) + v_p^\circ |\Phi_a \cap S| \frac{a}{x_1} - \frac{5fa}{mp}.$$

Proof. Put $E_0 = x^{60}$ and $u = a/(mp)$. The original diagrams satisfy

$$k w_\tau \le l_\tau \le k w_\tau + E_0, \qquad c_\tau \le w_\tau + E_0. \tag{20.62}$$

Indeed the columns of least low height k have count at least w_τ and at most the first lift weight. This weight differs from w_τ by a fixed multiple of the gap $2R^8 = O(x^{40})$, where $R \asymp x^5$ is the final rank parameter. At every other low height the count is at most that gap. There are at most $N \le x^6$ heights; summing their counts and their height-weighted counts gives errors $O(x^{52}) \le E_0$. Also $B = a|\Phi_a| + \sum_\tau l_\tau$.

The following elementary estimates all hold uniformly:

$$\gamma_p c_\tau \le u, \qquad 2 v_p^\circ h E_0 \le u, \qquad N^3 (1 + \log_p \max(B, 1)) \le u, \tag{20.63}$$

where $h = \lceil \log_p x_1 \rceil$. For the first, use $\gamma_p \le 2/p$ and $2m(a/x + E_0) \le a$. For the second use $h \le 1 + \log_2 x_1$, $p < x_1$, and $2m x_1 (1 + \log_2 x_1) E_0 = o(a)$. For the third, (20.62) gives $\log \max(B, 1) = O(L + \log x)$, uniformly in the stage; therefore $m x_1 N^3 (1 + \log_2 \max(B, 1)) = o(a)$. For either choice of cutoff the logarithms of these two numerators are $O(\log x_1 + \log x + \log(1 + L)) = o(L)$, since m, N, E_0 grow at most polynomially in x and $L \asymp (\log x_1)^6$. As $\log a \ge L$, the estimates are uniform in p, j, S.

For each $e \in I_p$ put $A = a/p^e$. The one-Lipschitz property of $z \mapsto z_+$ and (20.62) give

$$\begin{aligned} &1_{\Phi_a}(\tau)(A - w_{\tau_e})_+ + 1_{\{e=1,\, \tau \in \Phi_a \cap S\}} a/x_1 \\ &\quad \le (t_\tau/p^e - t_{\tau_e} - c_{\tau_e})_+ + 1_{\Phi_a \cap S}(\tau) A + 1_{\{(\tau,e) \text{ is a collision outside } S\}} A + E_0, \\ &k(w_\tau/p^e - w_{\tau_e})_+ \le (l_\tau/p^e - l_{\tau_e})_+ + E_0. \end{aligned}$$

For the first inequality, a removed row is paid by A; when $e = 1$, its extra a/x_1 is at most both A and w_{τ_e}. A row that remains either meets a collision, also paid by A, or has $t_{\tau_e} = 0$, where the Lipschitz estimate applies. The second inequality uses $k w_\tau \le l_\tau$ at the starting place and

$l_{\tau_e} \le kw_{\tau_e} + E_0$ at its partner. For $e \ge h$ the two gain terms vanish because $p^e \ge x_1$ and every weight is at least a/x_1; the error $2E_0$ is needed at most h times.

Add these inequalities with weight v_p°, then subtract the term c_τ/p^e in the exact predicted formula. Its total deduction is $\gamma_p c_\tau \le u$. The finite error is at most u, and the collision coefficient v_p° is at most one. Hence, at each place, the gain and the stated removed-row surplus are at most its positive-exponent predicted contribution, plus the removed-box term, its collision sum, and $2u$. All series are absolutely convergent, being bounded by constant multiples of p^{-e}.

At $p = d$ the zero-exponent correction in (20.61) is

$$\tfrac{1}{2}\sum_\tau \{\min(t_\tau, t_{-\tau} + c_{-\tau}) + c_\tau + \min(l_\tau, l_{-\tau})\}.$$

The places $\tau, -\tau$ are neither both in Φ_a nor both in Φ_b, since their classes modulo q_0 differ by -1. Uniformly for this fixed prime d, (20.62) gives $c_\tau \le u$, and $l_\tau \le u$ at any non-b place: use $dm/x \to 0$ and $dmk/x_1 \to 0$. Each brace is at most $3u$, so the correction is at most $2fu$. It is zero at an inert prime. Finally the logarithmic jet error is at most fu by (20.63). The total noncollision loss is at most $5fu$, proving the lemma.

Lean, comparison. Lean enlarges the five error terms to fL^2u. We retain $5fu$. See `JetExponent` (L185-L551). □

Lemma 20.35 (The transfer bound for the marked low columns)**.** *Let p be exceptional and Λ marked. Put $T_p = r_p f(n_m + r_p)$. Uniformly in $j < m$,*

$$|\Lambda| T_p + r_p \sum_{\tau \in S_\Lambda} c_\tau \le \frac{\lambda_p}{m} + |\Phi_a \cap S_\Lambda| \frac{a}{x_1}.$$

Proof. Every place in S_Λ has weight a/x_1 by Lemma 20.11. If $p \le m$, then $r_p(a/x_1 + E_0) \le a/(mp)$ eventually: the worst ratio is bounded by $m^2(1+\log_2 m)(x_1^{-1} + E_0/a) = o(1)$. If $p > m$, then $r_p = 1$, and the odd integer $o_0 \le r_p$ is one. Every place inducing a marked prime is then an a-place, by the cycle calculation of Proposition 20.2; in this case $r_p c_\tau \le a/x_1 + E_0 \le a/x_1 + a/(mp)$. Consequently, in both cases,

$$r_p c_\tau \le a/(mp) + 1_{\Phi_a}(\tau) a/x_1 \quad (\tau \in S_\Lambda).$$

The sum of the first terms is at most $fa/(mp)$. Also $|\Lambda| T_p \le f^2 r_p(n_m + r_p)$, and

$$\frac{mp\, f^2 r_p(n_m + r_p)}{fa} \le \frac{m x_1 f r_p(n_m + r_p)}{a} = o(1).$$

The excess over the removed-row term is therefore at most $(1 + o(1))fa/(mp) \le \lambda_p/m$, uniformly in p, j.

Lean. `MarkedPrimes` (L512-L560). □

Theorem 20.36 (The local module at a nonsplit prime)**.** *For every sufficiently large constructed datum, every small nonsplit prime $p < x_1$, every permitted twist, and every allowed jointly invariant integral chain datum, there is a submodule $\mathcal{S}_p \subseteq \mathcal{M}_p$ and an integer $e_p \ge 0$ such that*

$$\mathrm{ColengthDvd}(\mathcal{M}_p, \mathcal{S}_p, p^{e_p}), \qquad e_p \le \lambda_p D_X, \qquad v_p(B_{\xi,\Psi}\phi) \ge g_p - 2\lambda_p \quad (\phi \in \mathcal{S}_p).$$

At a nonexceptional prime one may take $\mathcal{S}_p = \mathcal{M}_p$ and $e_p = 0$. At an exceptional prime choose a full mark Λ and put

$$s = \lceil 2mfa/p \rceil, \qquad Z = \prod_{w \in \Lambda} Y_w, \qquad \mathcal{S}_p = Z^m \mathcal{M}_p + p^s \mathcal{M}_p.$$

These are the modules and indices in (11.8).

Proof. The finite class expansion in Lemma 18.11 applies to the jointly invariant tensor itself. At the present primes its coefficients are integral and its measures give ordinary finite sums; thus kernel valuations add through the chain, with no denominator or cardinality loss. The original restriction at $p \neq N$ is integral. Corollary 20.33 and Lemma 20.34 therefore give, on the whole original module,

$$v_p(B_{\xi,\Psi}\phi) \geq g_p - \lambda_p - m\mathcal{D}_p(\varnothing).$$

Lemma 20.1 proves the claimed nonexceptional conclusion. It also gives $\mathcal{D}_p(\varnothing) \leq 2fa/p$ at every small nonsplit prime, so $p^s\mathcal{M}_p$ has valuation at least $g_p - \lambda_p$.

At an exceptional prime take a full mark. For stage j, transfer T_{w,r_p} through the first $j+1$ lifts by Theorem 20.12 and Corollary 20.13. The result is an integral linear combination, independent of every finite translation, of target neighbors with index at most r_p, multiplied by p^{-T_p}. To transfer Y_w multiply this combination by $p^{A_w - T_p}$. Apply these operators for all w to the datum of stage j. Their different local supports commute. At a marked prime the Hecke combination is bi-invariant under the original maximal compact subgroup, and its action on the source commutes with its action on the target; elsewhere the operator is the identity. The changed tensor therefore retains the original joint invariance needed by the finite class expansion.

Induction on the stage, using the transfer identity, identifies the changed chain of length j with the original chain evaluated on $Z^j\phi$. This assertion holds for the jointly invariant sum; it does not require each pure tensor in a decomposition to be separately invariant. Each neighbor in a stage has, by Corollary 20.33, valuation at least

$$D_{p,j}(S_\Lambda) - \sum_w A_w - r_p \sum_{\tau \in S_\Lambda} c_\tau.$$

The prefactors $p^{A_w - T_p}$ cancel the terms A_w. The resulting stage bound is

$$D_{p,j}(S_\Lambda) - |\Lambda|T_p - r_p \sum_{\tau \in S_\Lambda} c_\tau \geq \frac{g_p - 2\lambda_p}{m}.$$

Here use $v_p^\circ = 1$, $\mathcal{D}_p(S_\Lambda) = 0$, and the exact surplus in Lemmas 20.34 and 20.35. Adding over the m stages gives the claimed bound on $Z^m\mathcal{M}_p$. Together with the scalar-module bound, the ultrametric inequality proves it on $\mathcal{S}_p$.

Finally apply Theorem 20.10 with $j = s$ to each Y_w. The integer colength exponents add under products of integral endomorphisms: the successive quotient for a product is an image of the corresponding quotient for its last factor, even after adjoining $p^s\mathcal{M}_p$. The sum over m repetitions is at most

$$m\sum_w \left(\frac{12A_w}{mp} + \frac{s}{m^2 f}\right) D_X \leq \left(\frac{12fa}{p} + \frac{s}{m}\right) D_X \leq \frac{14fa + p/m}{p} D_X \leq \lambda_p D_X.$$

The last inequality uses $p < x_1 \leq a$ and $\lambda_p = 16fa/p$ at an exceptional prime. Every exponent being added is an integer supplied by that theorem; no ceiling of the final real bound is taken. All estimates used one sufficiently large threshold and hold for all the specified primes, twists, stages and jointly invariant data.

Lean. `ChainValuation` (L1-L286), `LocalDivisibility` (L45-L355). □

20.4 The vacuum and source module at the level prime

Fix an embedding into $\mathbb{C}_N$. Let Σ be the noninduced places above N, $\delta_j = d_{n_j,\Sigma\cap\{w:w|\pi_N\}}$, and $\delta_j^\vee = \delta_j d_{n_j,\Sigma}^{-1}$. Thus these points give the original marked lattice L_j and its dual. Put $h = m/2$. We

retain the original source space $\mathcal{H}_0$, its ramified subspace, and its three valuation-ring lattices

$$\mathcal{A}_0 = \{\phi : \mathrm{Gauge}_1(\phi) \le 1\}, \quad \mathcal{A}_L = \{\phi : \mathrm{Gauge}_{\delta_0}(\phi) \le 1\}, \quad \mathcal{A}_\vee = \{\phi : \mathrm{Gauge}_{\delta_0^\vee}(\phi) \le 1\}.$$

Every form in these modules belongs to $\mathcal{H}_0$.

Lemma 20.37 (The Fourier operator preserves the source module)**.** *The operator $U = U_{n_0}$ of Theorem 18.13 preserves $\mathcal{H}_0$ and its ramified subspace. Moreover $U\mathcal{A}_L \subseteq \mathcal{A}_\vee$. Consequently the source-form clause in the definition of* upIntegralForms*, restricted to $\mathcal{H}_0$, is automatic:*

$$\mathcal{A}'_N := \mathcal{H}_0 \cap \mathrm{upIntegralForms} = \{\phi \in \mathcal{H}_0 : U^h\phi \in \mathcal{A}_L\}.$$

Proof. We check the finite group assertion behind the source invariance. Over $\prod_{w\in\Sigma} \mathcal{O}_{E_w}$, write $e = e_{n_0}$, $D = \mathrm{diag}(1, \dots, 1, N^{-1})$, and

$$H = \{g \in \mathrm{GL}_{n_0}(\mathcal{O}_\Sigma) : ge \equiv e \pmod N\}.$$

In the basis of L_0, the part of the source level supported above N is exactly H. For $w \mid \pi_N$, conjugation by δ_0 changes the last-row congruence of the original level into this last-column congruence. At the other places of Σ, the last-row congruence at $\bar{w}$, applied also to the inverse, is the last-column congruence at w by the unitary identity $g_{\bar{w}}^{-1} = \bar{g}_w^{\mathsf{t}}$. Conversely these two calculations show that the embedded $\delta_0 H \delta_0^{-1}$, supported above N, is in the source level; away from N it is the identity.

If $g \in H$, then $k = D^{-1}gD$ is integral invertible and $e^t k \equiv e^t \pmod N$. Indeed a matrix in H has block form

$$g = \begin{pmatrix} A & Nv \\ c & 1 + Nd \end{pmatrix}, \qquad k = \begin{pmatrix} A & v \\ Nc & 1 + Nd \end{pmatrix},$$

and the same calculation for g^{-1} proves invertibility over the integer ring. For every residue vector y with unit last entry, let k_y be the original integral choice having last row and last column equal to y modulo N. Since $\bar{k}$ has last row e^t, the permutation $y \mapsto \bar{k}y$ preserves its last coordinate and the coefficient $\psi_N(y_{n_0})$. Also

$$k_{\bar{k}y}^{-1} k k_y \in H,$$

because its last column reduces to e. Thus, on an H-fixed vector, $\rho(g)\rho(Dk_y)\phi = \rho(Dk_{\bar{k}y})\phi$. Summing proves that the exact unnormalized operator $U = \sum_y \psi_N(y_{n_0})\rho(Dk_y)$ preserves H-fixed vectors, without a denominator.

An arbitrary source-level point factors into its part supported above N and its part away from N; both remain in the source level by the preceding identification. The latter commutes with every term of U, while the former fixes $U\phi$ by the just-proved assertion. Hence $U\phi$ is a source form. Integral central scalings commute termwise with U, so the ramified subspace is preserved as well.

In the transported coordinates the exact value formula is

$$(U\phi)(g) = \sum_y \psi_N(y_{n_0})\, \phi(g\delta_0^\vee k_y \delta_0^{-1}).$$

If $\phi \in \mathcal{A}_L$, each displayed term has gauge at most one at $g\delta_0^\vee$, since

$$g\delta_0^\vee k_y = g(\delta_0^\vee k_y \delta_0^{-1})\delta_0$$

and the Weyl action of k_y and its inverse is integral. The coefficients are roots of unity. The ultrametric inequality proves $U\mathcal{A}_L \subseteq \mathcal{A}_\vee$. Iteration of source invariance proves the statement about $\mathcal{A}'_N$.

Lean. `LiftLawsModuloN` (L431-L540), `SetupAtLevel` (L277-L333). □

Corollary 20.38 (The full level-prime valuation and colength)**.** *For the original simultaneous vacuum of Theorem 19.4, the normalized-jet assertion* (J_N) *holds at every embedding above* N*, for every allowed twist. If* Ψ *is an original chain datum and* $\phi \in \mathcal{A}'_N$ *is ramified, its final restricted chain has gauge at most*

$$N^{mfb+f\sum_{j<m} n_{j+1}} \le N^{2mfb}.$$

The same module satisfies the previously required colength bound

$$\mathrm{ColengthDvd}(\mathcal{A}_0, \mathcal{A}'_N, N^{(h+1)d_{\mathrm{tot}}}), \qquad (h+1)d_{\mathrm{tot}} \le mfbD_X.$$

This gives the level-prime implication in (11.8) *with the specified source module and vacuum scalar.*

Proof. Theorem 19.4 proves (IV) for the same c_{vac} and all finite embeddings simultaneously. Apply Corollary 18.32 with this scalar and the unchanged jet scale. Its $\delta = \beta - c_0$ is divisible at every induced place, while $|\bar{\delta}d|_N = 1$, because $(1-2c_0)^2 = -d$ in the residue field and $N \nmid d$. It follows that all standard data used by the direct Fourier expansion have gauge bound one. This is exactly (J_N), including transport to arbitrary finite pairs and the prescribed twists.

For $\phi \in \mathcal{A}'_N$, Lemma 20.37 supplies the source-level condition on $U^h\phi$, while membership in the stated preimage supplies its integral L_0-values. All hypotheses of Corollary 18.16 are therefore satisfied. Its direct-section Fourier denominator is $N^{fn_{j+1}}$ at stage j; its gluing scalar costs at most N^{mfb}; the original integral branching map does not increase the transported gauge. This proves the displayed valuation without a new scalar or normalization.

Finally the earlier coordinate-congruence proof gives $\mathcal{A}_0 \subseteq \mathcal{A}_\vee$ and $\mathrm{ColengthDvd}(\mathcal{A}_\vee, \mathcal{A}_L, N^{d_{\mathrm{tot}}})$. Starting from any finite family in $\mathcal{A}_0$, impose these coordinate congruences first on the family, and then on its images under $U, U^2, \dots, U^h$. Lemma 20.37 places each next image in $\mathcal{A}_\vee$. At every step the integral change-of-family matrix has determinant $N^{d_{\mathrm{tot}}}$; later changes preserve all previously imposed congruences. Their product has determinant $N^{(h+1)d_{\mathrm{tot}}}$ and sends the original family into $\mathcal{A}'_N$. This argument is valid over the original nondiscrete valuation ring and does not require a basis of its whole module. The inequality is (11.14). One imposes integrality after U^h by this preimage construction; no invariance of $\mathcal{A}_L$ under U is asserted or needed.

Lean. `ColengthAtLevel` (L217-L288), `ValuationAtLevel` (L505-L550). □

Part VI
Global pairings and the height of the determinant

The Rallis inner product formula for a single lift, together with the exchange identities for consecutive lifts, expresses the Gram operator of the iterated lift as a product of Hecke operators. We compare its determinant on the inner forms by their Hecke traces. Local theta coefficients give the lower bounds for the global pairings, and a seesaw identity fixes the scalar in the upper bound. We then complete the independent columns to a basis and apply the product formula to its Gram determinant.

21 Hecke traces, global gluing and the lift image

Two realizations of the same theta kernel define the comparison operator of Lemma 7.1. To determine its restriction to the image of the lift, we compare Hecke operators on the inner forms and prove the vanishing of the complementary periods.

21.1 The Gram operator of the iterated theta lift

Equip the spaces of forms at the fixed principal level with the class-sum inner products. We first compare the determinants of the finite Rallis operators on the inner forms. The exchange identities for the ascents and descents then give the Gram operator of their composition. We use the Haar measures of (12.1) and (12.6).

Proposition 21.1 (Transport of finite Hecke operators)**.** *Let S, S' be any two of $Y_0, W_1, \dots, W_m$. Write* goodForms(S) *for the image of the central idempotent e_0 on forms at the common principal level. It is the sum of the images of the finite type projectors in* (10.2)*. For every compactly supported finite Hecke operator H away from the chosen places, acting at the common principal level,*

$$\operatorname{tr}(H \mid \text{goodForms}(S)) = \operatorname{tr}(H \mid \text{goodForms}(S')),$$
$$\det(H \mid \text{goodForms}(S)) = \det(H \mid \text{goodForms}(S')).$$

The characteristic polynomials also agree.

Proof. The correspondence in Section 10 identifies the two good spaces as modules for finite Hecke operators away from the chosen places. In particular the two actions of H are conjugate, so their characteristic polynomials, traces and determinants agree.

Lean, comparison. The finite-module identification replaces the comparison of traces in `EqualMultiplicities` (L232-L292). □

Lemma 21.2 (A common finite Hecke operator for the Rallis series)**.** *Let $\mathcal{R}_{j,\vartheta}$ be the absolutely convergent unramified Rallis operator for the first lift of row $j-1$ with tame twist ϑ. It preserves the good forms, and its determinants on* goodForms(Y_0) *and* goodForms(W_j) *are equal.*

Proof. Each coefficient of the series is a fixed vacuum coefficient times a right translation by a finite point g supported away from the modulus, hence away from the chosen places. Let K be the integral subgroup away from those places and $K_g = K \cap gKg^{-1}$. The mean

$$H_g = \frac{1}{[K : K_g]} \sum_{u \in K/K_g} R(ug)$$

has the same matrix coefficients as $R(g)$ between principal-level forms: move $R(u)$ to the first argument, where it fixes that form. Left translation by K permutes the summands, so H_g preserves the level. This is the same finite Hecke operator on every inner form. It commutes with the good projector, supported at the chosen places.

Consider the linear map from this universal finite Hecke algebra to the direct sum of the two finite-dimensional endomorphism spaces on the principal forms of Y_0, W_j. Its image is a vector subspace of a finite-dimensional space, and is therefore closed. All finite partial sums of the Rallis series lie in this image. Absolute convergence of the series, already proved in the first-lift argument, gives convergence of every matrix entry on both spaces. The limiting pair is consequently the image of one finite Hecke operator H as well. This assertion concerns its action on the two spaces; it does not assert that the infinite adelic series has finite support. The good projector commutes with H, and Proposition 21.1 gives equality of the two determinants of its restrictions.

Lean. `EqualMultiplicities` (L67-L207), `EqualMultiplicities` (L294-L346). □

Fix an m-fold twist ξ of conductor Q. By definition, $\mathrm{cond}(\xi^a) = Q$ for $1 \le a \le m$. Write $r_j = r + jt$, $q_\xi = \mathrm{N}Q$ and $R_\xi = \mathcal{O}_E/(\varpi_M Q)$ for the squarefree residue ring. For $1 \le j \le m$, let $\Theta_{j,\vartheta}$ be the single lift from W_j to its definite space of rank $r + t$ with tame twist ϑ. Its Gram determinant on the good forms is $Z_j(\vartheta) = \det(\Theta^*_{j,\vartheta}\Theta_{j,\vartheta})$, as in Section 7. At step j of the chain, $\vartheta = \xi^j$. Define

$$c_Q = q_\xi^{rt} \prod_{w \mid Q} \prod_{t < i \le t+r} (1 - q_w^{-i}) > 0. \tag{21.1}$$

Proposition 21.3 (The Gram operator of a single lift)**.** *For each j there is $A_j \neq 0$, independent of the tame twist, such that for each tame twist ϑ of conductor Q the following identity holds on the good forms of W_j:*

$$\Theta^*_{j,\vartheta}\Theta_{j,\vartheta} = A_j c_Q \mathcal{R}_{j,\vartheta}.$$

The same block lift is injective on the full space of principal forms, before restriction to the good subspace. In particular, with $d_0 = \dim \mathrm{goodForms}(Y_0)$,

$$Z_j(\xi^j) = (A_j c_Q)^{d_0} \det(\mathcal{R}_{j,\xi^j} \mid \mathrm{goodForms}(Y_0)) > 0, \tag{21.2}$$

for every m-fold twist ξ of conductor Q.

Proof. At each chosen place the finite block function is the indicator of $X_0 = \binom{I_r}{0}$. At $w \mid Q$ it is $\chi_w(\det X_{\mathrm{top}})$, extended by zero when the top r by r block is singular; the remaining t rows are arbitrary. Thus its squared counting norm is

$$\prod_{w \mid Q} |\mathrm{GL}_r(\kappa_w)| q_w^{tr}.$$

The block is sharp for the level J_M principal at the chosen places and maximal elsewhere. To check this, if X lies in its support, its top block lifts by CRT to an integral unitary element in J_M: it is the identity at the chosen places and invertible at Q. The same left-inverse argument as in (12.3) makes the matrix coefficient zero outside J_M on the places of the modulus. The splitting character cancels the determinant transformation on Q, so J_M fixes the datum. Its mass is therefore

$$c_f = \frac{\prod_{w \mid Q} |\mathrm{GL}_r(\kappa_w)| q_w^{tr}}{|\mathrm{Mat}_{(r+t)\times r}(R_\xi)| \prod_{w \mid M} |\mathrm{GL}_r(\kappa_w)|}.$$

The left inverse also supplies the clean chosen place.

Since $r + t > 2r$, the convergent-range doubling formula (12.1) applies. Its real joint-harmonic factor is nonzero by (12.2), and is independent of the twist, whose weight is zero. The sharp finite mass leaves precisely the same unramified convolution as in (12.4). Its injectivity was proved there by the neighbor-lattice series argument, for every allowed twist phase. The same argument proves injectivity of this block lift.

To change from the Petersson norm to the class-sum norm, multiply by the index of the target level and its fixed real representation dimension, as in (12.6). At a twist place the target level is the parabolic stabilizing an r-plane. Its index times $|\mathrm{GL}_r(\kappa_w)|$ is the number of full-rank $(r + t)$ by r matrices. Dividing this number by all such matrices gives

$$\prod_{t<i\le t+r} (1 - q_w^{-i}).$$

The additional q_w^{tr} from the block norm gives the local factor of (21.1). At a chosen place the same full-rank proportion is divided by $|\mathrm{GL}_r(\kappa_w)|$. All these chosen-place factors and the nonzero real factor are fixed and enter A_j. This proves the operator identity in the original norms.

Restrict to good forms and use Lemma 21.2 and (10.1). For a map T between finite-dimensional inner-product spaces, $\langle Tx, Ty\rangle = a\langle x, Ry\rangle$ means $T^*T = aR$, so its Gram determinant is $a^{d_0} \det R$. This proves (21.2); positivity follows from injectivity. The finite factor is c_Q at every step because each of $\xi, \dots, \xi^m$ has conductor Q.

Lean. `LevelAndTwists` (L372-L391), `AdjointThetaLift` (L441-L695), `AdjointThetaLift` (L910-L1028). □

For a function c of the twist and a nonnegative real function F, write $\mathcal{M}(c, F)$ if there are $C \ge 0$ and $e \in \mathbb{Z}$, independent of ξ, with

$$|c(\xi)|^2 = C q_\xi^e F(\xi)^2.$$

This relation is closed under products, constants, and inverses of nowhere-zero functions. Every fixed power of $|R_\xi|$ is a fixed constant times a power of q_ξ. Put

$$\nu_{ij} = \chi_j/\chi_i, \qquad F_{ij}(\xi) = \prod_{z=1}^{t} \left| \frac{L^{MQ}(z, (\nu_{ij}\xi^{j-i})^{-1})}{L^M(z, \nu_{ij}^{-1})} \right| \quad (1 \le i < j \le m).$$

The denominators are nonzero, and $F_{ij}(1) = 1$.

Proposition 21.4 (Exchange identities for ascents, descents, and adjoints)**.** *Let $a_{i,n}$ be the unweighted ascent from (i, n) to $(i, n + 1)$, $d_{i,n}$ the descent from $(i, n + 1)$ to $(i + 1, n)$, and $N_n = |\mathrm{GL}_{r_n}(R_\xi)|$. On a nonzero base space the first square has the form*

$$N_n a_{i+1,n} d_{i,n} = \frac{N_{n+2}}{\beta_{i,n}(\xi)} d_{i,n+1} a_{i,n+1}, \qquad \mathcal{M}(\beta_{i,n}, F_{i+1,i+n+2}). \tag{21.3}$$

Every $\beta_{i,n}$ occurring here is nonzero. There is also $\kappa_{i,n}$ with $\mathcal{M}(\kappa_{i,n}, 1)$ such that, for $x' = a_{i,n+1}^* x$,

$$\langle d_{i,n} x', y\rangle = \kappa_{i,n}(\xi)\langle d_{i,n+1} x, a_{i+1,n} y\rangle. \tag{21.4}$$

The descents use the finite data $\Psi_{i,n}$ and their extensions

$$\Psi_{i,n+1}\begin{pmatrix} \eta & s \\ 0 & I_t \end{pmatrix} = \frac{\Psi_{i,n}(-\eta)}{|\mathrm{Mat}_{r_{n+1}\times t}(R_\xi)|},$$

and are zero elsewhere; none is rescaled after choosing the twist.

Proof. The finite orbit calculation in (12.41) gives the first square from the complementary-rank exchange. Theorem 19.18 now makes its exchange constant uniform in the finite character. The two middle ranks differ by $2t$, so the product has t special values. Dividing its squared modulus by the corresponding product at the trivial twist leaves $q_\xi^{t^2} F_{i+1,i+n+2}(\xi)^2$. Indeed the modulus of each first m-th power is still Q, the tame quotient character has the prescribed nontrivial part at the chosen places, and the extra conductor norm in the exchange product is $q_\xi^{t^2}$. Inversion of the character does not change the product: its Euler factors are permuted by complex conjugation. This is a direct ratio of the same partial L-functions, rather than an additional functional-equation constant. The projection between the fixed real joint-harmonic lines contributes a constant independent of ξ.

Initially write this square as $N_n ad = r_{i,n} N_{n+2} da$, with $r_{i,n} = \kappa_{i,n}^{\mathrm{real}}/c_{i,n}$ and $c_{i,n} \neq 0$ the exchange constant. The earlier triangle-injectivity induction uses this identity even if $r_{i,n} = 0$. If $r_{i,n}$ were zero, move the first return through the preceding squares and apply this zero square. The resulting shorter row chain would kill the image of a nonzero base vector under the injective first return. Injectivity of that shorter chain gives a contradiction. Thus $r_{i,n} \neq 0$ and one can put $\beta_{i,n} = \overline{r_{i,n}^{-1}}$. This proves (21.3) and its modulus.

We give the finite calculation needed for the other square. Write $R = R_\xi$, and use the unnormalized matrix Fourier transform

$$\widehat{f}(X) = \sum_Y f(Y)\psi(\operatorname{tr} XY),$$

where ψ is the fixed primitive additive character. Summing first over the free upper-right block gives

$$\widehat{\operatorname{next} f}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \mathbf{1}_{C=0}\widehat{f}(-A)\psi(\operatorname{tr} D).$$

There is no missing matrix-cardinality factor: the sum over the free block cancels precisely the denominator in next f.

For a primitive multiplicative character χ of a finite field k, extended by zero on singular matrices, the Fourier transform of $X \mapsto \chi(\det X)$ is zero on singular matrices and equals $G_\chi \chi(\det X)^{-1}$ on invertible matrices. For the first assertion, the stabilizer of a singular matrix contains a subgroup with arbitrary determinant; equivariance under it and nontriviality of χ force zero. The second follows by the invertible change of variables. Parseval then gives $|G_\chi|^2 = |k|^{t^2}$, because the nonzero supports on both sides have $|\mathrm{GL}_t(k)|$ elements. CRT gives the same assertion over R with $|G_\chi|^2 = |R|^{t^2}$. Here χ is nontrivial at every factor: this follows from the chosen tame character and the m-fold condition.

Apply this to the bottom-right block in the Fourier square. Covariance by $\chi \circ \det$ kills its singular blocks as well. The free shear variables contribute $|R|^{r_n t}$ and the determinant Fourier sum contributes G_χ. The reversed square consequently has the factor γ_{fin} with

$$|\gamma_{\mathrm{fin}}|^2 |R|^{t(t+2r_n)} = 1.$$

The two middle spaces in this reversed square have equal rank. Apply Theorem 19.18 with $e = 0$; its product of special values is empty and its squared modulus is independent of the twist. Together these computations give, for reversed lifts $r_{i,n}$ formed with $\widehat{\Psi}_{i,n}$,

$$a_{i,n+1} r_{i,n} = \gamma(\xi) r_{i,n+1} a_{i+1,n}, \qquad \mathcal{M}(\gamma, 1).$$

Here the letter $r_{i,n}$ denotes a reversed lift only in this displayed identity and below; it is distinct from the temporary scalar used above.

We now turn kernels without losing their finite normalization. For ranks a, b, conjugation followed by the change of polarization gives

$$\overline{K_{S,T}[\Psi, P](g,h)} = \tau_\xi K_{T,S}[\widehat{\overline{\Psi}}, P^\flat](h, gu_a), \qquad |\tau_\xi|^2 |R_\xi|^{ab} = 1. \tag{21.5}$$

Here $P^\flat$ is the signed transpose of $\overline{P}$ and u_a is the central scalar $\overline{\alpha\varpi}/(\alpha\varpi)$ at the finite places, in the fixed polarization convention. One can check this identity on point data: a point indicator is a vacuum translate followed by the scalar change of lattice; the latter has squared norm factor $|R|^{ab}$. Interchanging the two polarizations sends that translate to the additive character $X \mapsto \psi(\operatorname{tr} XY)$. On real variables it sends $X_{\tau,i,j}$ to $-s_i t_j X_{\tau,j,i}$ and preserves the vacuum. Poisson summation intertwines the theta functional under this change by a unit phase. Summing the point identities proves (21.5); the unnormalized Fourier transform accounts for the full $|R|^{-ab}$ in its squared scalar.

Project the signed-transpose polynomial onto the prescribed reversed partner line. The same classical joint-harmonic projection used in (12.5) gives a scalar fixed by the real data. Fubini and the unitary right translation by u_a^{-1} then give

$$\langle d_{i,n} x, y\rangle = c_0(\xi)\langle x, r_{i,n} R(u_n)^{-1} y\rangle, \qquad \mathcal{M}(c_0, 1).$$

Projecting in the other direction gives separately

$$\langle x, r_{i,n+1} R(u_{n+1})^{-1} y\rangle = c_1(\xi)\langle d_{i,n+1} x, y\rangle, \qquad \mathcal{M}(c_1, 1).$$

This second projection does not divide by a projection scalar that might be zero. The ascent intertwines $R(u_n)^{-1}$ and $R(u_{n+1})^{-1}$ up to the central splitting-character value $\eta(\xi)$, with $|\eta(\xi)| = 1$.

Set $y' = R(u_n)^{-1} y$. The inner product is conjugate-linear in its first variable. Using the adjoint of the upper ascent, then the reversed square, then the central translation and the second projection, gives

$$\begin{aligned}
\langle d_{i,n} x', y\rangle &= c_0\langle x', r_{i,n} y'\rangle \\
&= c_0\langle x, a_{i,n+1} r_{i,n} y'\rangle \\
&= c_0\gamma\langle x, r_{i,n+1} a_{i+1,n} y'\rangle \\
&= c_0\gamma\eta c_1\langle d_{i,n+1} x, a_{i+1,n} y\rangle.
\end{aligned}$$

This is (21.4) with $\kappa = c_0\gamma\eta c_1$, and its modulus follows by multiplication. The kernels here act between the fixed-type automorphic spaces. Their rapid decay, or equivalently first restricting to a common compact open finite level, justifies the adjoints and every pairing and Fubini step.

Lean. `Flip` (L458-L569), `SquareLaw` (L390-L509), `ChainNorms` (L157-L583). □

Lemma 21.5 (The weighted triangle Gram calculation)**.** *Let j_i be the fixed isometric base embeddings and let $X_{i,n}$ be the weighted row chain of length n, whose $a_{i,k}$-arrow has weight N_k. On the good base forms put $S_i = |\mathrm{Mat}_{(r+t)\times r}(R_\xi)|^{-1}\mathcal{R}_{i+1,\xi^{i+1}}$. There is $\kappa(\xi)$ such that*

$$\langle X_{0,m} j_0 x, X_{0,m} j_0 y\rangle = \kappa(\xi)\frac{N_0 N_1^m}{N_m}\langle x, S_0 S_1 \cdots S_{m-1} y\rangle, \qquad \mathcal{M}\left(\kappa, \prod_{1\le i<j\le m} F_{ij}\right). \tag{21.6}$$

Proof. For completeness we record the induction, including the complex conjugates and the group orders. Moving one descent through n squares gives a product β of their first-square constants and

$$X_{i+1,n} d_{i,0} a_{i,0} = \frac{N_{n+1}}{N_1 N_0 \overline{\beta}} d_{i,n} X_{i,n+1}.$$

The corresponding repeated use of the second square, starting with (12.5), gives

$$\langle v, a_{i,n}X_{i,n}j_i y\rangle = \kappa'\langle d_{i,n}v, X_{i+1,n}j_{i+1}y\rangle, \qquad \mathcal{M}(\kappa', 1).$$

Choose x' with $j_{i+1}x' = d_{i,0}a_{i,0}j_i x$; the earlier first-return calculation supplies it. Substitute the first displayed identity in the first argument of the second. Since that argument is conjugate-linear, the factor is β, not $\overline{\beta}$. This yields

$$\langle X_{i,n+1}j_i x, X_{i,n+1}j_i z\rangle = \kappa'\beta\frac{N_0N_1N_n}{N_{n+1}}\langle X_{i+1,n}j_{i+1}x', X_{i+1,n}j_{i+1}z\rangle.$$

The first-step Rallis formula reads, with a nonzero constant C_i independent of ξ,

$$N_0\langle a_{i,0}j_i x, a_{i,0}j_i y\rangle = C_i\langle x, S_i y\rangle.$$

The first adjoint identity therefore replaces $N_0\langle x', z\rangle$ by $C_i\langle x, S_i z\rangle$. Apply induction to the shorter chain in row $i+1$. Its numerical factor is $N_0N_1^n/N_n$. Multiplication and this last replacement leave $N_0N_1^{n+1}/N_{n+1}$, proving the required numerical factor at the next step. Its operator product is $S_iS_{i+1}\cdots S_{m-1}$ in that order; commutativity is not needed for this induction. The new first-square constants correspond exactly to the pairs $(i+1, j)$ with $j > i+1$; the shorter chain has all pairs with first index greater than $i+1$. Thus each F_{ij} occurs exactly once. All second-square constants and all C_i have modulus class 1. Starting with the length-zero identity proves (21.6).

Lean. `TriangleNorms` (L141-L200), `TriangleNorms` (L391-L498). □

Theorem 21.6 (Factorization of the Gram operator)**.** *On the target forms the composite lift Θ_ξ satisfies*

$$\Theta_\xi^*\Theta_\xi = c(\xi)c_Q^m\mathcal{R}_{1,\xi}\mathcal{R}_{2,\xi^2}\cdots\mathcal{R}_{m,\xi^m}, \qquad |c(\xi)| = Cq_\xi^{e/2}\prod_{1\le i<j\le m}F_{ij}(\xi), \tag{21.7}$$

for a constant $C > 0$ and an integer e, both independent of the twist. If the target space is nonzero then $c(\xi) \neq 0$ for every allowed twist.

Proof. Let P_m be the finite parabolic with diagonal block sizes $r, t, \ldots, t$, and put $k(m) = mrt + \binom{m}{2}t^2$. An element is uniquely specified by its invertible diagonal blocks and its arbitrary strict upper blocks, so over a field of size q

$$|P_m| = |\mathrm{GL}_r|\,|\mathrm{GL}_t|^m q^{k(m)}.$$

At a chosen place the last level is principal, with index $|\mathrm{GL}_{r_m}|$; at a twist place its index is $|\mathrm{GL}_{r_m}|/|P_m|$. Write $\delta(\xi)$ for the single-lift finite Gram factor before division by the matrix cardinality, as in the proof of Proposition 21.3. Its factor at a twist place is

$$[\mathrm{GL}_{r+t} : P_1]|\mathrm{GL}_r|q^{tr} = |\mathrm{GL}_{r+t}|/|\mathrm{GL}_t|.$$

Consequently

$$[K_0 : J_{\mathrm{last}}(\xi)] = C_*q_\xi^{-k(m)}\delta(\xi)^m\frac{N_m}{N_0N_1^m}, \tag{21.8}$$

where $C_* > 0$ is determined entirely by the chosen places. At a twist place this equality reads

$$q^{-k(m)}\left(\frac{|\mathrm{GL}_{r+t}|}{|\mathrm{GL}_t|}\right)^m\frac{|\mathrm{GL}_{r_m}|}{|\mathrm{GL}_r|\,|\mathrm{GL}_{r+t}|^m} = \frac{|\mathrm{GL}_{r_m}|}{|\mathrm{GL}_r|\,|\mathrm{GL}_t|^m q^{k(m)}},$$

which verifies every variable factor.

By (12.6) the class-sum Gram form of Θ_ξ is that of the weighted chain multiplied by this last index and the fixed dimension of the terminal real Weyl module. Substitute (21.6) and (21.8). The factors N_0, N_1, N_m cancel exactly. Absorb the factor $q_\xi^{-k(m)}$ into $c(\xi)$; its modulus is of class 1. The remaining finite scalar is

$$\left(\frac{\delta(\xi)}{|\mathrm{Mat}_{(r+t)\times r}(R_\xi)|}\right)^m = C_M^m c_Q^m,$$

where the positive C_M involves only the chosen places, by the full-rank matrix count above. Thus the resulting scalar c has $\mathcal{M}(c, \prod F_{ij})$.

The chain is injective by (12.42). If the target space is nonzero, its Gram form cannot vanish, so $c(1) \neq 0$, and indeed $c(\xi) \neq 0$ for every twist. Since $F_{ij}(1) = 1$ and $q_{\xi=1} = 1$, divide its squared-modulus formula by the one at the trivial twist and take the nonnegative square root. This gives the stated modulus with $C = |c(1)| > 0$. If the target space is zero, the operator identity is zero equals zero; choose $C = 1$, $e = 0$ and $c(\xi) = \prod F_{ij}(\xi)$.

Lean. `ChainNorms` (L675-L765), `NormOfChain` (L140-L245), `NormOfChain` (L295-L349). □

Theorem 21.7 (Determinant factorization and its scalar height)**.** *With the Gram determinants Z, Z_j there is one integer e such that for every m-fold twist,*

$$\frac{Z(1)}{Z(\xi)} = \left(q_\xi^{e/2} \prod_{1\le i<j\le m} F_{ij}(\xi)\right)^{-d_0} \prod_{j=1}^{m} \frac{Z_j(1)}{Z_j(\xi^j)}. \tag{21.9}$$

All displayed Gram determinants are positive. If the auxiliary modulus is prime to the coefficient prime at each embedding and the inverse critical ratios have family height at most H, the inverse scalar in parentheses has height at most $\binom{m}{2}tH$. For the simultaneous family in (13.58) this is at most

$$\binom{m}{2} t C_{\mathrm{crit}} \le m^2 t C_{\mathrm{crit}} \le b. \tag{21.10}$$

Thus the chain factorization and its scalar-height input in the inverse-ratio argument are discharged.

Proof. Take determinants in (21.7) and (21.2). The factors $c_Q^{md_0}$ cancel, as do all m Rallis determinants; the transport of those determinants is Lemma 21.2. This gives

$$Z(\xi) = \left(c(\xi) \Big/ \prod_{j=1}^{m} A_j\right)^{d_0} \prod_{j=1}^{m} Z_j(\xi^j).$$

These are Gram determinants of injective maps, hence positive real numbers, including the zero-dimensional determinant 1. Take complex absolute values to remove the scalar's phase and divide by the formula at the trivial twist. The constants cancel and all $F_{ij}(1)$ are one. Invert the resulting equality; this proves (21.9).

For each pair $i < j$, the quotient character is $(\upsilon^{-\Delta}\chi_M)^{j-i}$. Its tame part is nontrivial at every chosen place because $1 \le j-i \le m$. It is one of the ramified critical pairs in the all-prime small-ratio construction, at the allowed power ξ^{j-i}. Each F_{ij} contains exactly t critical ratios. Multiply the $\binom{m}{2}t$ height witnesses. For the simultaneous family, (13.58) applies to each factor since the old tame power is $j - i$. At the coefficient prime of each member of the family, q_ξ is prime to that prime. Every algebraic square root of q_ξ^e is then a local unit. Its varying power adds zero to this family height; no ordinary absolute-height assertion about the rational integer q_ξ is being made. Finally $C_{\mathrm{crit}} < 1 \le x_1$ and $m^2 t x_1 \le b$ by the scale inequalities.

Lean. `NormOfChain` (L350-L463), `Moduli` (L222-L252), `LongForms` (L158-L205). □

21.2 Sparse theta kernels and spanning by products

For the admissible data of Proposition 5.2, put $b = e^L$, $t = 5r$, $f = [E : \mathbb{Q}]/2$, and $n_* = 2ft^3$. The modulus in this subsection is $\mathfrak{m} = (\pi_M^2)$, so $|\tau(\pi_M^2)| = M$. Since $\pi_M \bar{\pi}_M = p_0 p_1$ with distinct split primes in K, (π_M) has exactly one prime divisor over each p_i in K. Its square remains coprime to its conjugate after extension to E. All real and finite Haar measures are the ones fixed in Section 8. For an archimedean point h, put $|h| = \max_\tau \|h_\tau\|_{\mathrm{op}} \geq 1$. The first minimum $\lambda(V)$ of a moved hermitian space means the minimum of one and the Euclidean length of its shortest nonzero rational vector with integral finite coordinates.

Lemma 21.8 (A quantitative Fock coefficient and a sparse sum)**.** *Let P be homogeneous of degree d in $D \geq 1$ complex Fock coordinates and let $v = v_P$. If σ is a real unitary-group element in its symplectic action, $y \neq 0$, and $q \geq 0$ is an integer, then*

$$|\langle v, \rho(y)\omega(\sigma)v\rangle| \leq \|\sigma\|^{D+2q}\|v\|^2\{4(d+D+q)\}^q|y|^{-2q}. \tag{21.11}$$

For an a-separated subset $\mathcal{L}$ of $\mathbb{R}^{2D}$, all of whose points have length at least $a > 0$, and for an integer $q > D$,

$$\sum_{y\in\mathcal{L}} |y|^{-2q} \leq (5/a)^{2q}. \tag{21.12}$$

Proof. In the coherent-state realization $|v(x)| = |P(\zeta(x))|e^{-|x|^2/2}$, where ζ conjugates the positive-sign coordinates. The reproducing Gaussian integral for the Weil operator gives

$$|\langle v, \rho(y)\omega(\sigma)v\rangle| \leq |\mu(\sigma)| \int |v(x)|\,|v(\sigma x + y)|\,dx, \qquad |\mu(\sigma)| \leq \|\sigma\|^D.$$

Here dx is the normalized Fock measure. In the original unitary splitting, $\mu(\sigma)$ is the conjugate of the principal minor of σ on the negative coordinates. The compression has norm at most $\|\sigma\|$, so its determinant has modulus at most $\|\sigma\|^{D_-} \leq \|\sigma\|^D$. Substitution of the coherent-state resolution of the identity in the defining Weil integral gives the displayed coefficient formula with this multiplier.

Put $A = d + D$. In polar coordinates the radial density of $|v|^2\,dx$ is proportional to $r^{2A-1}e^{-r^2}\,dr$. Hence

$$\int |x|^{4q}|v(x)|^2\,dx = \frac{\Gamma(A+2q)}{\Gamma(A)}\|v\|^2 \leq (A+q)^{2q}\|v\|^2,$$

by the arithmetic–geometric mean inequality on the $2q$ factors when $q \geq 1$. Write $x' = \sigma x + y$. Since $\|\sigma\| \geq 1$, $|y| \leq \|\sigma\|(|x| + |x'|)$. Convexity therefore bounds $|y|^{2q}$ by $2^{2q-1}\|\sigma\|^{2q}(|x|^{2q} + |x'|^{2q})$. Multiply by $|v(x)v(x')|$ and integrate. Cauchy–Schwarz bounds each of the two integrals by $(A+q)^q\|v\|^2$, since the affine map $x \mapsto x'$ preserves volume. The multiplier bound gives (21.11). For $q = 0$, Cauchy–Schwarz alone suffices.

For (21.12), rescale to $a = 1$. Disjoint balls of radius $1/2$ give $\#\{y : |y| \leq R\} \leq (2R+1)^{2D}$ for $R \geq 1$. Layer integration bounds the sum by

$$2q\,3^{2D}\int_1^\infty R^{2D-2q-1}\,dR = \frac{q}{q-D}3^{2D} \leq (D+1)3^{2D} \leq 5^{2q}.$$

The last inequality uses $D + 1 \leq 2^D$ and $q \geq D + 1$. It also proves absolute convergence.

Lean, comparison. Lean dilates the Gaussian to estimate its tail. Here polar integration gives its radial moment directly. `ThetaKernelEstimates` (L69-L176). □

Theorem 21.9 (Approximation of a Fock coefficient by a theta kernel)**.** *Let V, V' have positive ranks at most t, let P be homogeneous of degree $d \leq b^{3/2}$, and retain the prescribed splitting*

characters in their kernel $\theta[\Psi, P]$. If $\lambda(Vg)\lambda(V'g') \geq b^{-13}$, there is a function Ψ on matrices modulo $\mathfrak{m}$ such that, for all real h, h',

$$|\theta[\Psi, P](gh, g'h') - c_P(h, h')| \leq b^{-20n_*}(|h||h'|)^{6n_*} c_P(1, 1), \tag{21.13}$$

where $c_P(h, h') = \langle v_P, \Omega(h, h')v_P\rangle$ and $c_P(1, 1) = \|v_P\|^2$. The same datum works for every h, h'. No bound or integrality is asserted for the auxiliary coefficients of Ψ.

Proof. Fix a primitive additive character ψ of the scalar residue ring. For each matrix T in the trace-dual residue module define

$$\psi_T(Z) = \psi(\mathrm{tr}(TZ)).$$

Here the trace-dual module is the full Pontryagin dual under this pairing, so the map $T \mapsto \psi_T$ is an isomorphism onto the character group. As T ranges over this module, these are all additive characters of the matrix residue module; in particular ψ_0 is the trivial character. Put

$$\Psi = \frac{1}{\#\mathcal{T}} \sum_{T \in \mathcal{T}} \overline{\theta[\psi_T, P](g, g')}\, \psi_T. \tag{SF}$$

The theta Parseval identity and character orthogonality turn its kernel into

$$\theta[\Psi, P](gh, g'h') = \sum_{y \in \mathcal{L}_0} \epsilon_y \langle v_P, \rho(y_\infty)\Omega(h, h')v_P\rangle, \quad |\epsilon_y| = 1, \quad \epsilon_0 = 1. \tag{21.14}$$

Here $\mathcal{L}_0$ consists exactly of the rational vectors in $\mathrm{Hom}(V, V')$ whose coordinates at (g, g'), divided by the original finite idele α, are integral and zero modulo $\mathfrak{m}$; y_∞ denotes their original Fock coordinates. To verify the normalization, write the product of a theta sum and its conjugate in a self-dual Schrödinger polarization. Regroup by the difference of the rational position coordinates and apply Poisson summation in the remaining coordinates. This gives the sum of the Heisenberg matrix coefficients with coefficient one. The mean of $\psi_T(z)$ is one for $z = 0 \pmod{\mathfrak{m}}$ and zero otherwise; the finite vacuum has norm one on its support. Its remaining quadratic phase is ϵ_y, equal to one at zero. Splitting-character phases cancel in the product at (g, g') and leave precisely $\Omega(h, h')$. Schwartz decay first justifies Poisson and rearrangement; the estimate below also proves absolute convergence of the resulting sum. Thus the zero term is exactly $c_P(h, h')$.

For completeness the first-minimum comparison used here follows from the same trace-duality argument as the earlier short-frame lemma. If an integer C_{diff} belongs to the different of E, transference in real dimension $2fa$ supplies an E-basis of the integral lattice of a rank-a space U whose lengths are at most $6C_{\mathrm{diff}}(fa)^2/\lambda(U)$. Indeed trace duality carries the dual lattice into a scalar multiple of the original one; the real sign matrix is an isometry, so its first minimum is at least the same $\lambda(U)$ up to that scalar. The covering argument already proved for short frames then applies without definiteness. A nonzero integral homomorphism is nonzero on at least one of these basis vectors. Comparing its image with the first minimum of the target gives, for ranks a, a',

$$\lambda(\mathrm{Hom}(U, U')) \geq \frac{\lambda(U)\lambda(U')}{6C_{\mathrm{diff}}(fa)^2 aa'}. \tag{HM}$$

Set $c = q_1/\pi_M^2$. Since $q_1\alpha$ is integral, cy lies in the integral homomorphism lattice for every $y \in \mathcal{L}_0$. At each complex embedding $|\tau c| = q_1/M$. Let C_{par} be the fixed parameter bound for the admissible data, and write $d_K = |\operatorname{disc} K|$. Admissibility gives

$$C_{\mathrm{par}} \geq \max(64, q_0, d_K), \qquad 10^{12} C_{\mathrm{par}} \leq \log\log b, \qquad 100 \log\log b \leq \log x,$$
$$x^{10^6} \leq x_1, \qquad x_1^{10^6} \leq b.$$

Hence $\log x \ge 10^{14} C_{\mathrm{par}} \ge 6.4 \cdot 10^{15}$, so $x \ge 10^4$, $d_K \le C_{\mathrm{par}} \le x$, and $b \ge x^{10^{12}}$. To bound the cyclotomic modulus q_1, use

$$q_1 = 3q_0\ell_0 m^{22}, \qquad \ell_0 \le 81s, \qquad s \le 32\log x, \qquad m \le 2\sqrt{x}, \qquad r \le 3x^5.$$

Here $s \ge 1$ and ℓ_0 is the prescribed power of three. Put $\ell_x = \log x$. Since $3 \cdot 81 \cdot 32 \cdot 2^{22} = 32614907904 < 10^{14}$ and $q_0 \le C_{\mathrm{par}} \le \ell_x/10^{14}$,

$$q_1 \le (3 \cdot 81 \cdot 32 \cdot 2^{22}) q_0 \ell_x x^{11} \le \ell_x^2 x^{11} \le x^{12}.$$

The last inequality follows from $e^{\ell_x} \ge \ell_x^3/6 \ge \ell_x^2$, since $\ell_x \ge 6$. Also $f = [E : K] \le q_1$. We now take $C_{\mathrm{diff}} = q_1 d_K$, the integer in the different used in the short-frame argument. Thus

$$f, q_1 \le x^{12}, \qquad C_{\mathrm{diff}} \le x^{13}, \qquad t = 5r \le 15x^5 \le x^6, \qquad n_* = 2ft^3 \le 2x^{30}.$$

The two level primes are at least b^{10}, so $M \ge b^{20}$. The fixed Weil scalar satisfies $\operatorname{Im}\tau(\mathrm{j}) = d_K^{-1/2}$ at every infinite place. Consequently

$$\kappa_{\mathrm{F}} := \min_{\tau} \sqrt{2\pi \operatorname{Im}\tau(\mathrm{j})} = \sqrt{2\pi}\, d_K^{-1/4} \ge x^{-1}.$$

The admissible-record scale and rank fields; The parameter bound and scale inequalities; The field-degree and conductor bounds; The fixed Weil scalar.

With a, a' the positive ranks of V, V', the three conditions needed below are

$$b^6 \frac{q_1}{M} \le \kappa_{\mathrm{F}} \frac{b^{-13}}{6C_{\mathrm{diff}}(ft)^2 t^2}, \qquad 1 \le D := faa' \le ft^2 < 2n_*, \qquad 100(d + 4n_*) \le b^2. \tag{SM}$$

For the first, the explicit bounds just obtained give

$$\frac{6q_1 C_{\mathrm{diff}}(ft)^2 t^2}{\kappa_{\mathrm{F}}} \le 6x^{74} \le x^{75} \le b.$$

Combining this with $M \ge b^{20}$ proves the claimed inequality. The dimension inequality follows from $1 \le a, a' \le t$, $f \ge 1$, $t \ge 1$, and $2n_* = 4ft^3$. Finally,

$$400n_* \le 800x^{30} \le x^{32} \le b, \qquad \sqrt{b} \ge 200,$$

where $x \ge 10^4$ and $b \ge x^{10^{12}}$ justify both comparisons. Since the polynomial degree satisfies $d \le b^{3/2}$ and $b \ge 2$,

$$100(d + 4n_*) \le 100b^{3/2} + b \le \tfrac{1}{2}b^2 + \tfrac{1}{2}b^2 = b^2.$$

The first inequality in (SM), together with (HM), proves that nonzero Fock coordinates in $\mathcal{L}_0$, and differences of distinct ones, have length at least b^6.

Apply Lemma 21.8 with $q = 2n_*$ and separation b^6. The pair action has operator norm at most $|h||h'|$: each place is a tensor product, and the largest block norm is the largest place norm. The sum of the nonzero terms in (21.14) is therefore at most

$$(|h||h'|)^{3q} \|v_P\|^2 \left\{ \frac{100(d + 2q)}{b^{12}} \right\}^q \le (|h||h'|)^{6n_*} \|v_P\|^2 b^{-20n_*}.$$

This proves the claimed estimate for the explicit datum (SF). First minimum of Hom; Exact sparse Fourier average; Its original scales.

Lean. `AdmissibleData` (L75-L160), `Magnitudes` (L53-L85), `ProductsSpan` (L124-L179). □

Lemma 21.10 (Holomorphic model and quantitative integration)**.** *Order the indices of $U(a,a')$ with positive signs first, and let m_u denote the original holomorphic product of principal minors of weight u, normalized by $m_u(1)=1$. For weights dominant within each sign block, the following identities hold whenever their integrals exist:*

$$\int_0^1 \chi_u(\theta)^{-1} F(r_\theta h)\,d\theta = m_u(h)F(1), \tag{21.15}$$

$$\int F(h)\overline{m_u(h)}\,dh = F(1)\int |m_u(h)|^2\,dh. \tag{21.16}$$

Here F is holomorphic of weight u, $r_\theta = \operatorname{diag}(e^{2\pi i\theta o(i)})$, the integer order o is strictly increasing, and χ_u is its weight character, multiplied over the complex places.

Suppose now the rank n satisfies $1\le n\le t$, both u,u' have gap at least $300n_*$ from the positive to the negative sign block, and $|u_{\tau,i}|\le 5b$. Then

$$I_u := \int |m_u|^2 > 0, \qquad \int |m_{u'}(h)|\,|h|^{6n_*}\,dh \le b^{n_*} I_u. \tag{21.17}$$

The integrand on the left is integrable. Consequently every continuous A satisfying $|A(h)|\le C|h|^{6n_*}$ obeys $|\int A m_{u'}| \le C b^{n_*} I_u$.

Proof. We recall the elementary analytic reason for the model function. On the flag coordinate cell write a matrix as a lower unipotent factor times an upper triangular factor. Holomorphy and the triangular weight law expand F in the lower coordinates. Conjugation by r_θ multiplies each such coordinate by a strictly positive integral power of $e^{2\pi i\theta}$. Thus circle averaging retains only the constant coefficient; the triangular character supplies the leading-principal-minor product. If u is only block dominant, add to every positive-block entry the same integer $c_\tau = 2\sum_i |u_{\tau,i}|$. Multiplying F by the corresponding positive-block determinant power gives a fully dominant weight. Its polynomial minor character has no poles. Apply the preceding constant-term calculation and divide by that determinant, which is nonzero on $U(a,a')$. Analytic continuation from the flag cells gives (21.15) everywhere. Equivalently this is Cauchy's formula for the contracting lower flag coordinates; the positive-first order ensures the contraction stays in the domain of the holomorphic triangular extension. The same calculation proves m_u is holomorphic of weight u. Averaging the integrand of (21.16) over left multiplication by r_θ, which preserves Haar measure, proves that identity.

The normalized holomorphic discrete-series coefficient is exactly m_u. Its doubled parameter is $\ell = 2u+2\rho(-o)$; its lowest compact highest weight is u. To check the equality in the Flensted–Jensen formula, factor a matrix near the identity as x_-ax_+, with a block diagonal and $x_\pm$ in the two off-diagonal corner-unipotent groups. Conjugation by the compact block group keeps them in those corners. In the parameter order they are triangular, with all diagonal entries one, so they leave every denominator weight-minor in the Flensted–Jensen integral unchanged. Thus the integral at x_-ax_+ equals that at a. Inside each diagonal block factor $a = b_-b_+$ by Gauss elimination. The triangular transformation formula already proved for the normalized integral gives $b_-^u b_+^u$, since its value at one is one. The product of minors m_u has exactly these two transformation laws and the same value at one. Hence the two functions agree near the identity, and then on the connected real group by analytic continuation. No scalar remains. The gap assumption here implies excess discrete-series gap $12n_*$, since, for rank $n\le t$,

$$500n^3 + 12n_* + n + 1 < 300n_*, \qquad 2n^3 \le n_*.$$

Thus all earlier determinant-decay and unitary derivative estimates apply to this very model function.

We compute the volume of an operator-norm ball at one real place. Write p, q for the signature and $k = \min(p, q)$. In compact-dual Haar measure, for $T \geq 1$ and $m = 2pq > 0$,

$$\operatorname{vol}\{g \in U(p,q) : \|g\| \leq T\} = \big((T - T^{-1})/2\big)^m. \tag{21.18}$$

In rectangular matrix coordinates Z, the invariant densities on the noncompact domain and the compact Grassmannian chart are $\det(I - Z^*Z)^{-p-q}\,dZ$ and $\det(I + Z^*Z)^{-p-q}\,dZ$ up to their common constant at zero. Indeed the invariant tangent metric transports a tangent matrix H to $(I \mp ZZ^*)^{-1/2} H (I \mp Z^*Z)^{-1/2}$ with its Euclidean metric. Compact-dual normalization identifies the two tangent densities and uses probability on the common compact subgroup. Write v_i for the ordered squared singular values of Z. The singular-value Jacobian contributes $\prod_i v_i^{|p-q|} \prod_{i<j} (v_i - v_j)^2$. Changing variables to $x_i = v_i/(1 - v_i)$ on the noncompact space and $x_i = v_i/(1 + v_i)$ on the compact space gives in both cases the density $\prod_i x_i^{|p-q|} \prod_{i<j} (x_i - x_j)^2\,dx$ with the same constant [Ner01, §2.9, (2.15)–(2.17)]. The compact chart has full measure and its ordered unit cube has mass one. For the noncompact Cartan parameters t_i, $x_i = \sinh^2 t_i$ and $\|g\| = e^{\max t_i}$. Thus the ball corresponds to the cube dilated by $\sinh^2(\log T)$. Its mass scales by the pq-th power of this factor, proving (21.18).

For $A > m$, integration of the ball volumes gives

$$\int \|g\|^{-A}\,dg = A \int_1^\infty T^{-A-1} \big((T - T^{-1})/2\big)^m\,dT = \frac{A}{2^{m+1}} \mathrm{B}\big((A - m)/2, m + 1\big)$$

by the substitution $T^{-2} = u$, where B is the beta function. The integral is at most one for $A \geq m + 2$, by its value at $m + 2$ and monotonicity. For $pq = 0$ the whole group has mass one and norm one. Now $n = p + q$. The formal degree $e_\ell = (\int |m_u|^2)^{-1}$ from Section 10 is bounded by $(10b + n)^{n(n-1)/2}$ since the doubled parameter has absolute values at most $10b + n$. As $n \leq t \leq b$ and $b \geq 10^{12}$, we obtain

$$e_\ell \int (2/\|g\|)^{10n^3}\,dg \leq 2^{10t^3} b^{t^2}. \tag{21.19}$$

Determinant decay for the weight u' gives $|m_{u'_\tau}(h_\tau)| \leq (2/\|h_\tau\|)^a$ for every $a \leq 12n_*$. Choose a place of largest norm and use $a = 10n^3 + 6n_*$ there and $a = 10n^3$ at the others. These exponents are permitted since $10n^3 + 6n_* \leq 11n_* < 12n_*$. It follows that

$$|m_{u'}(h)|\,|h|^{6n_*} \leq 2^{6n_*} \prod_\tau (2/\|h_\tau\|)^{10n^3}.$$

Apply (21.19) at every place and use $I_u = \prod_\tau e_{\ell_\tau}^{-1} > 0$. The total factor is at most

$$2^{6n_*} \{2^{10t^3} b^{t^2}\}^f = 2^{22ft^3} b^{ft^2} \leq b^{2ft^3} = b^{n_*},$$

since $b \geq x^3 \geq 10^{12} > 2^{22}$ and $t \geq 1$. This proves (21.17), including integrability and the stated majorant consequence.

Lean, comparison. The exact volume replaces the covering estimate in the formal integration bound. `ModelFunction` (L190-L293), `ModelFunction` (L528-L684), `ThetaKernelEstimates` (L634-L879). □

Lemma 21.11 (The first minimum where a form exceeds half its supremum)**.** *Call g a peak of a nonzero bounded function F if $|F(g)| > \frac{1}{2} \sup |F|$. Let F be a bounded, nonzero automorphic form, adelically holomorphic in positive-first order. Suppose its weights have gap at least one and absolute value at most $A \geq 2$. Suppose it has principal level $a \in \mathcal{O}_E$, where $\bar{a} = -a$ and $0 < \operatorname{Im} \tau(a) \leq A'$, $A' \geq 1$. At every point g with $|F(g)| > \frac{1}{2} \sup |F|$,*

$$\lambda(Vg) \geq (AA')^{-1/2}. \tag{21.20}$$

Proof. Suppose an integral finite-coordinate rational vector $v \neq 0$ has archimedean length less than $(AA')^{-1/2} \leq 1$. It is isotropic: its hermitian norm is an algebraic integer, and every conjugate has modulus less than one, so its integer norm forces it to be zero. In the moved coordinates put $J = \mathrm{diag}(\varepsilon_i)$ and $t = 1 - avv^*J$. Isotropy and $\bar{a} = -a$ give $t^*Jt = J$. Its conjugate by g is rational, and its finite component is congruent to one modulo a. Consequently $h \mapsto F(gh)$ is left invariant under t_∞.

At each infinite place put $\eta_\tau = \operatorname{Im}\tau(a)|v_\tau|^2 > 0$. Then $A\sum_\tau \eta_\tau < 1$, so every $\eta_\tau < 1/2$. For $z \in \mathbb{H}$, consider $X_\tau(z) = 1 - \tau(a)(z-i)v_\tau v_\tau^* J$. At a fixed place write $X = X_\tau(z)$, $v = v_\tau$, $\eta = \eta_\tau$, $y = \Im z$, and $T = 2\Re(\tau(a)(z-i))$. Isotropy gives $X^*JX = J - TJvv^*J$. For $0 \leq k \leq n$ let p_k be the determinant of its leading k-by-k block divided by the corresponding determinant of J, with $p_0 = 1$. The rank-one identity $\det(I + ab^*) = 1 + b^*a$, obtained by multilinearity in the columns, gives

$$p_k = 1 - T\sum_{i \leq k} \varepsilon_i |v_i|^2.$$

The positive coordinates come first, and by isotropy their squared lengths sum to $|v|^2/2$. Thus p_k runs monotonically from 1 to $p = 1 + \eta(y-1)$ on the positive block and back to 1 on the negative block. All p_k are positive. Indefinite Gram–Schmidt therefore gives $X = hb$ with $h \in U(J)$, b upper triangular and $b_{kk} = (p_k/p_{k-1})^{1/2}$, by taking leading Gram determinants.

For $y \geq 1$, the sums of $\log(p_k/p_{k-1})$ on the two blocks are $\log p$ and $-\log p$; each summand has the sign of its block sum. Since the negative-block weights exceed the positive-block weights by at least one, $|b^u| \leq p^{-1/2}$. For $0 < y < 1$ the sum of the absolute logarithmic increments is $2|\log p|$, so $|b^u| \leq p^{-A} \leq (1-\eta)^{-A} \leq e^{2A\eta}$.

Evaluate the holomorphic triangular extension of $F(g\,\cdot)$ at $X(z)$. Its value $f(z)$ is one-periodic since $X(z+1) = t_\infty X(z)$, and $f(i) = F(g)$. With $C = \sup|F|$, the weight law and $A\sum_\tau \eta_\tau < 1$ give $|f| \leq Ce^2$. The bound for $y \geq 1$ gives $f(z) \to 0$ uniformly as $y \to \infty$. In the coordinate $q = e^{2\pi i z}$ this is a bounded holomorphic function on the disc, zero at its center. Schwarz's lemma gives $|F(g)| \leq Ce^{2-2\pi} < C/2$, contradicting the peak. Every bounded nonzero function has a point exceeding half its supremum, so this argument does not assume a maximum is attained.

Lean, comparison. Lean constructs the triangular change of basis entry by entry. Its diagonal suffices here and follows from the leading minors. `PeaksAwayFromCusp` (L94-L163), `PeaksAwayFromCusp` (L250-L366), `PeaksAwayFromCusp` (L433-L617). □

Lemma 21.12 (Vanishing of the lift and its archimedean convolution)**.** *Let S, T have ranks n, n' with $n' > 2n$, and let F_f be the indicator of an inclusion matrix modulo a nonzero proper principal ideal $\mathfrak{m}$ of $\mathcal{O}_E$, with the vacuum elsewhere. Assume $\mathfrak{m} + \overline{\mathfrak{m}} = \mathcal{O}_E$. Write χ_S for the splitting character acting on the source: it is the second character in $\theta_{S,T;\chi_T,\chi_S}$. Assume*

$$\chi_S(\det k) = 1 \qquad (k \in J(\mathfrak{m})).$$

For the prescribed tame characters it is sufficient that χ_S be tame at the places of $\mathfrak{m}$ and unramified elsewhere. Alternatively, let the modulus be (1), let F_f be the unit finite vacuum, let S be definite, and suppose χ_S has trivial tame part at every finite place; this gives the same character condition on $J((1)) = K_0$, the full integral finite subgroup. Let $v \neq 0$ be a Fock vector and suppose $k(h) = \langle v, \Omega(h,1)v\rangle$ is integrable. For a rapidly decreasing automorphic function y at this level, let

$$(\mathrm{realConv}(k)y)(g) = \int_{U(S)_\infty} k(h)\, y(g(h,1))\, dh.$$

Then

$$\Theta_{v\otimes F_f} y = 0 \quad \Longrightarrow \quad \mathrm{realConv}(k)y = 0.$$

Proof. Put $R = \mathcal{O}_E/\mathfrak{m}$ and $J = J(\mathfrak{m}) \subseteq K_0$, with $\mathrm{vol}(K_0) = 1$. The finite action of an integral source element k on a residue datum is

$$\omega_f(k)(\Psi \otimes \mathrm{vac}) = \chi_S(\det k)^{-1}\big[X \mapsto \Psi(X\bar{k})\big] \otimes \mathrm{vac}.$$

If $k \in J$, then $\bar{k} = 1$. The character hypothesis therefore makes J fix F_f, including its scalar factor. For the stated sufficient tame condition, the restriction of χ_S to finite integral determinants is the product of its residue characters at the places of $\mathfrak{m}$; these all take the value one on $\det \bar{k} = 1$.

The left-inverse support argument proving (12.3) applies in these ranks. At a finite point g integral away from $\mathfrak{m}\overline{\mathfrak{m}}$, nonzero overlap of F_f and $\omega_f(g)F_f$ forces $g \in J$. Together with the just-proved invariance, this gives

$$\langle F_f, \omega_f(g)F_f\rangle = \|F_f\|^2 \mathbf{1}_J(g).$$

Each prime of $\mathfrak{m}$ lies over a split place of E/E^+, and its conjugate is absent from $\mathfrak{m}$. On the chosen factor the integral unitary group is $\mathrm{GL}_n(\mathcal{O}_{E_w})$; the conjugate factor is determined by the inverse adjoint. Reduction onto $\mathrm{GL}_n(R)$ is therefore surjective with kernel J in K_0. Thus $\mathrm{vol}(J) = 1/\#\mathrm{GL}_n(R)$ and the sharp mass is

$$\begin{aligned} c_f &= \|F_f\|^2 \,\mathrm{vol}(J) \\ &= \{\#\,\mathrm{Mat}_{n'\times n}(R)\#\,\mathrm{GL}_n(R)\}^{-1} > 0. \end{aligned}$$

The character condition is needed here: without it, the coefficient on J would be $\chi_S(\det k)^{-1}\|F_f\|^2$, whose integral over J is zero when that character is nontrivial. Thus support alone would not give the asserted positive mass. The finite action and its determinant character; The principal-level character hypothesis and sharp mass.

For the nontrivial modulus, a prime divisor supplies a place where matrices congruent to the inclusion are injective, so the datum is clean for (12.1). In the unit-modulus case $c_f = 1$: the finite vacuum has norm one and its matrix coefficient is one on the full integral subgroup, because χ_S is trivial there. There are no omitted finite places in the neighbor series.

Since a definite S is anisotropic over F, (12.1) applies to the unit vacuum as well.

In either case put $y_1 = \mathrm{realConv}(k)y$ and $z = \mathcal{R}_A y_1$, with $\mathcal{R}_A$ the original unramified neighbor series away from the modulus. Since y is bounded and $k \in L^1$, y_1 is bounded, continuous, automorphic and fixed by the integral subgroup away from the modulus. Absolute summability of the neighbor coefficients makes z bounded, continuous and automorphic as well.

Doubling, sharpness and commutation of real and finite translations give, for $\phi = v \otimes F_f$,

$$\langle \Theta_\phi y', \Theta_\phi y\rangle = \mathrm{vol}[U(T)]\, c_f\, \langle y', z\rangle$$

for every rapidly decreasing y'. The quotient volume and c_f are strictly positive, so vanishing of the displayed theta lift gives $\langle y', z\rangle = 0$. Every right translate of y is rapidly decreasing. Hence z is orthogonal to their closed linear span in $L^2([U(S)])$. This span contains y_1, since $k \in L^1$, and then contains $z = \mathcal{R}_A y_1$ by absolute summability of the neighbor series. Thus $z = 0$ in L^2, and everywhere by continuity and full support of Haar measure. Injectivity of $\mathcal{R}_A$ on bounded functions of this level gives $y_1 = 0$. *Comparison with Lean.* Passing to the closed span combines the averaging of the real translates and the sum over finite translates. The bounded-convolution argument. The unit-modulus definite-source alternative is also the first alternative in Sharpness without a clean finite prime.

Lean. `Modulus` (L205-L213), `SharpData` (L530-L540), `SharpData` (L580-L655). □

Theorem 21.13 (Exact finite theta reproduction)**.** *Let S have rank n, and let a positive definite target T have rank $n' > 2n$. Fix the indicator of an inclusion $n \hookrightarrow n'$ modulo a nonzero proper principal ideal $\mathfrak{m} = (a)$ with $\mathfrak{m} + \overline{\mathfrak{m}} = \mathcal{O}_E$. Assume the splitting characters are trivial on the finite units. Suppose $v_0 = v_{Q_0} \neq 0$ is holomorphic of weight u, with $m_u \in L^1$. At each definite principal-level class c, choose finitely many polynomials $Q_{c,l}$ whose Fock vectors span the orbit of v_0 under the real definite group. Then for each adelic g_0 there are constants $a_{c,l}$, independent of y, such that every rapidly decreasing automorphic y of level a satisfies*

$$\int_{G_\infty} y(g_0 h) m_u(h)\, dh = \sum_{c,l} a_{c,l} \int_{[U(S)]} y(g)\theta[\mathbf{1}_{\mathrm{incl}}, Q_{c,l}](g,c)\, dg. \tag{21.21}$$

The coefficients are allowed to depend on g_0, but there is one finite family of kernels for all y.

Proof. The coefficient $h \mapsto \langle v_0, \Omega(h,1)v_0\rangle$ is holomorphic of weight u, hermitian, and has value $\|v_0\|^2$ at one. Its left torus law follows from its right law by hermitian symmetry. Equation (21.15) therefore gives

$$\langle v_0, \Omega(h,1)v_0\rangle = \|v_0\|^2 m_u(h).$$

If all the pairings on the right of (21.21) vanish, the lift with Q_0 vanishes at every point of the definite group. Indeed move a finite point to one of the finitely many classes by a rational element and a principal-level element. The latter fixes the inclusion datum. Move the remaining real group element to the Fock vector and expand it in the specified $Q_{c,l}$. This proves vanishing everywhere. The finite-unit-character assumption gives $\chi_S(\det k) = 1$ on the source principal level, so the character hypothesis of Lemma 21.12 holds. That lemma then forces the left side of (21.21) to vanish.

This implication is enough even though the space of rapid functions is not finite dimensional. Let L be the map to the finite vector of right-hand pairings, and let ℓ be the left-hand functional. We proved $\ker L \subseteq \ker \ell$. Hence ℓ descends to the finite-dimensional image of L; extend it linearly to the whole coordinate space. Its coordinates are the required $a_{c,l}$. Boundedness of y and integrability of m_u justify the left functional, while rapid decay against the polynomial-growth theta kernels justifies the right ones.

The polynomial families used below satisfy the orbit condition without enlarging the original choices. The definite group acts by linear substitutions of the row variables. Cauchy–Binet expresses a substituted power or product of minors as a finite linear combination of products of minors. Substituting any of the invertible short row matrices chosen for the definite class does not change that span. For a fixed Schur shape the same statement is the defining expansion in its tableau polynomials. No integral inverse of these row matrices is needed: the reproduction coefficients in this theorem are complex.

Lean. `Reproduction` (L288-L454). □

The finite carrier data. Fix $1 \le j \le m$, and put $\Delta = 2r^8$. The determinant character on the carrier group is

$$\mathrm{carrierChar}(j) = \mu^\flat \upsilon^{m+j+\lfloor (t+k-1)/2 \rfloor - j\Delta},$$

where $\mu^\flat$ is the initial character of weight $-\kappa_0$ and υ the weight-one splitting character. Both were chosen with trivial finite character on integral units; their indicated product is therefore trivial on the determinants of finite integral unitary points. The determinant character and its finite restriction.

Write $a = 3m$, $n_+ = r + 9m + 1$ and $R = \mathcal{O}_E/(\pi_M^2)$. For each row module $U \subseteq R^{n_+}$ generated by at most a rows, choose one matrix $s \in M_{a,n_+}(R)$ generating it. Set

$$P_U(Z) = \mathbf{1}_{\{sZ=0\}}, \qquad Z \in M_{n_+,a}(R).$$

At twist ϑ let $R_\vartheta = \mathcal{O}_E/(\pi_M^2 Q_\vartheta)$; on this larger residue ring take the pullback $P_{U,\vartheta}(Z) = P_U(\pi_\vartheta Z)$ under reduction $\pi_\vartheta : R_\vartheta \twoheadrightarrow R$. Let $\mathcal{F}_i$ be the carrier integral with finite datum P_U, indexed by this s, a definite class and a choice of rows for the minors. The theta kernels in the integral are specified below.

Theorem 21.14 (Approximation of a fixed archimedean function)**.** *There is a bounded continuous function $\mathcal{F}^0$ on $U(W_j)_\infty$, with $\mathcal{F}^0(1) \neq 0$, such that whenever $\lambda(W_j g) \geq (5b^{24})^{-1/2}$, there is a complex combination $\mathcal{F}_g = \sum_i c_i(g)\mathcal{F}_i$ satisfying*

$$|\mathcal{F}_g(gh) - \mathcal{F}^0(h)| \leq 2b^{-19n_*}|\mathcal{F}^0(1)|\,|h|^{6n_*} \qquad (h \in U(W_j)_\infty). \tag{21.22}$$

The model $\mathcal{F}^0$ transforms by a single character under left translation by the circle r_θ of Lemma 21.10. It is fixed before g. For each fixed g, the coefficients $c_i(g)$ are independent of h; they need not be independent of g.

Proof. Write φ° for the auxiliary lift on $S = E^{2m,m}$ defined in Lemma 21.30. It is nonzero, rapidly decreasing, of full finite level, and holomorphic of weight $u_c = (0; 3m)$. Choose a peak g_1 of φ°. Apply Lemma 21.11 with level $\sqrt{-d_K}$, $A = 5b$, $A' = \sqrt{b}$, to obtain

$$\lambda(Sg_1) \geq (5b^{3/2})^{-1/2}.$$

By the definition of a peak, the bounds $|\varphi^\circ(x)| \leq 2|\varphi^\circ(g_1)|$ hold at every x.

Put

$$\widetilde{W} = W_j \oplus (E^j)^- \oplus E^{3m-j}, \qquad W^\flat = E^{6m+1}, \qquad P = \mathrm{paddedMinorProd}(j).$$

The polynomial P is a product of minors. Its negative minor uses m distinct negative rows and all m negative columns of S. For each height h, its positive minor uses $2m$ distinct positive rows and all $2m$ positive columns. At a place τ write

$$\mu_\tau = \#\,\mathrm{DetIndex}(j, \tau), \quad \Delta' = r^8 - (t+1)/2, \qquad n'' = r + 9m + 1.$$

The exponents are $\mu_\tau + n'' + m$ for the negative minor and the positive column counts for the positive minors; the latter sum is $\mu_\tau - \Delta'$. Thus $P \neq 0$, all its variables have matching signs in row and column, and its degree is at most $10fmb \leq b^{3/2}$. Write c_P for its normalized real Weil coefficient with the original ordered character weights $\gamma_\tau = m$ and $\gamma'_\tau = m - \mu_\tau$. In the pair action the second character supplies the determinant twist on the first group. Unitarity gives $|c_P(h, h')| \leq c_P(1,1) = \|v_P\|^2 > 0$.

Since every minor uses all columns of its sign, linear substitution by a real point of S multiplies it by that sign's principal minor. The Fock Gaussian formula has no mixed contraction: P uses only positive Fock-sign variables. Taking those determinants and the splitting-character factor gives the exact identity

$$c_P(h,1) = c_P(1,1)m_{u_2}(h)\overline{m_{u_3}(h)}, \qquad \begin{cases} u_1 = (-\Delta'; 6m+1), \\ u_2 = (\Delta'; \mu_\tau + r + 2m), \\ u_3 = (0; \mu_\tau + r + 11m + 1). \end{cases} \tag{21.23}$$

In particular $u_c + u_1 + u_2 = u_3$. To check the Gaussian factor directly, its denominator is the negative principal minor of the pair action. At each place it equals the positive minor divided by $\det h$, to the number of positive rows, times the negative minor divided by $\det h$, to the number of negative rows. Jacobi's complementary-minor identity and the character $(\det h)^{-(m-\mu_\tau)}$ give

precisely the three weights displayed above. Thus this identity contains no scalar other than $c_P(1,1)$.

Both u_1, u_3 have gap at least $300n_*$, and $|u_{3,\tau,i}| \le 5b$; these follow respectively from $300n_*+t \le r^8$ and the original bound $2\mu_\tau + 2n'' \le 5b$. Set

$$J = \int |m_{u_3}|^2 > 0, \qquad \mathcal{F}^0(h) = \int_{U(S)_\infty} \varphi^\circ(g_1 x) c_P(x, h \oplus 1) m_{u_1}(x)\, dx.$$

Lemma 21.10 proves absolute convergence, boundedness and continuity. Each padded minor uses columns of one sign. The diagonal matrix $r_\theta \oplus 1$ thus multiplies each of its rows by a unit scalar, conjugated when the Fock coordinate is conjugated. Every determinant term uses the same rows. The minors, the invariant Gaussian and the fixed determinant twist therefore give v_P a circle character ξ. In $c_P(x, (r_\theta h) \oplus 1)$ the rotation commutes with the carrier action. Moving it across the inner product multiplies the coefficient by $\xi(\theta)$, since its inverse acts on the first vector and the first variable is conjugate-linear. Integration in x gives $\mathcal{F}^0(r_\theta h) = \xi(\theta)\mathcal{F}^0(h)$. Equations (21.23) and (21.16) give the exact nonzero value

$$\mathcal{F}^0(1) = J\varphi^\circ(g_1) c_P(1,1) \ne 0. \tag{21.24}$$

Each of the two standard definite padding summands has cut-off first minimum one. The Euclidean norm on their orthogonal sum therefore gives $\lambda(\widetilde{W}(g \oplus 1)) \ge \lambda(W_j g)$. Consequently, at any g in the statement,

$$\lambda(Sg_1)\lambda(\widetilde{W}(g \oplus 1)) \ge \frac{1}{5b^{3/4+12}} = \frac{1}{5b^{51/4}} \ge b^{-13},$$

where $b^{1/4} \ge 5$. Fix such a base point g. Theorem 21.9 supplies a datum Ψ_g, chosen from $(Sg_1, \widetilde{W}(g \oplus 1))$, for $(S, \widetilde{W})$ whose kernel approximates c_P with error $b^{-20n_*}(|x||h|)^{6n_*} c_P(1,1)$. Integrating this difference against $\varphi^\circ(g_1 x) m_{u_1}(x)$ and applying (21.17) bounds it by

$$2|\varphi^\circ(g_1)| c_P(1,1) J b^{-19n_*} |h|^{6n_*}.$$

This is (21.22) by (21.24).

It remains to verify that the integral with the theta kernel is a combination of the $\mathcal{F}_i$. Apply Theorem 21.13 on $(S, W^\flat)$: its rank is $6m+1 > 2(3m)$, its highest polynomial is a Δ'-th power of the positive minor, and its holomorphic weight is exactly u_1. The row-polynomial orbit is spanned by the original products of determinants in the short row matrices of each definite class. Keep the selected base point g and its datum Ψ_g fixed. For every adelic evaluation point ξ, the function

$$x \longmapsto \varphi^\circ(x)\theta[\Psi_g, P](x, \xi \oplus 1)$$

is rapidly decreasing of principal level π_M^2: for each fixed ξ, the theta kernel and its translates have polynomial growth in x, which the rapid carrier absorbs. No growth constant uniform in ξ is needed here. The reproduction formula at g_1 has coefficients a_r fixed for every rapidly decreasing input function. It writes the desired real integral as a finite sum of its pairings with the $(S, W^\flat)$ kernels. The product of two such kernels is the kernel for the orthogonal sum $\widetilde{W} \oplus W^\flat$, with product polynomial and product finite datum $H(Z) = \Psi_g(Z_1)\mathbf{1}_{\mathrm{incl}}(Z_2)$, by absolute convergence of the two theta sums. We claim that its carrier pairing is a linear combination of the pairings with P_U, with coefficients independent of the evaluation point ξ.

For $b \in \mathrm{GL}_a(R)$, choose an integral finite unitary lift at the chosen split places, taking the inverse conjugate transpose at their conjugates. Replacing $H(Z)$ by $H(Zb)$ translates the integrand

on $U(S)$. The carrier has full finite level and the second splitting character carrierChar(j) is trivial on finite units. Right invariance of Haar measure thus preserves the integral. Consequently we may average H over b before pairing.

Choose a primitive additive character ψ of R, meaning that its kernel contains no nonzero ideal. The characters $\psi_T(Z) = \psi(\mathrm{tr}(TZ))$ form a basis: the pairing is nondegenerate by primitivity, and the two matrix spaces have the same order. Right translation of Z sends T to bT by cyclicity of the trace. Two matrices T with the same row module lie in the same orbit. Indeed, in each local factor with maximal ideal $\mathfrak{m}$, the two generating tuples induce surjections onto $U/\mathfrak{m}U$ from the same residue vector space. A basis change there takes one surjection to the other. Lift it to an invertible matrix over the factor; the remaining row differences lie in $\mathfrak{m}U$ and are combinations of the first tuple with coefficients in $\mathfrak{m}$. Adding these coefficients to the lift preserves invertibility. Combine the local changes over the product R.

For a row module V let $\Omega_V = \sum_{\mathrm{Row}(T)=V} \psi_T$. The orbit averages span the same complex space as the Ω_V. For each U, character orthogonality gives

$$|U|^a P_U(Z) = \sum_{T \text{ with rows in } U} \psi_T(Z) = \sum_{V \subseteq U} \Omega_V(Z),$$

where the last sum runs over modules generated by at most a rows. Indeed one sums each row independently against the corresponding column of Z; the image of that pairing is an ideal, on which ψ is trivial precisely when the ideal vanishes. Induction on $|U|$ now expresses Ω_U in the span of the indicators. Applying this to the Fourier expansion of the averaged H proves the claim. Together with the fixed reproduction coefficients a_r it gives $\sum_i c_i(g)\mathcal{F}_i(\xi)$ for every ξ. Putting $\xi = gh$ gives (21.22) with the same coefficients for every h.

Lean, comparison. Lean takes individual matrix characters as the finite tests. The carrier pairing factors through their orbit averages, which the row-module indicators span. The additional circle covariance follows from `thepaddedminors` (L249-L265). `ProductsSpan` (L276-L354), `ProductsSpan` (L404-L511), `ProductsSpan` (L599-L705). □

Lemma 21.15 (Circle projection at one peak)**.** *Let F be bounded and adelically holomorphic of weight u, and let z satisfy $|F(z)| > \frac{1}{2}\sup|F|$. Assume the weight hypotheses of Lemma 21.10 for u, u'. Suppose K_0 is bounded and continuous and transforms by one character under left translation by r_θ. If a holomorphic function K of weight $u - u'$ satisfies*

$$|K(h) - K_0(h)| \le \delta |h|^{6n_*}, \qquad \int \overline{F(zh)} K(h) m_{u'}(h)\, dh = 0,$$

then

$$|K_0(1)| \le (4b^{n_*} + 1)\delta. \tag{21.25}$$

Proof. Put $L = Km_{u'}$, $M = K_0 m_{u'}$, and let P_u denote the circle average on the left of (21.15). It contracts L^1 by Haar invariance, and $P_u L = L(1)m_u$ since L is holomorphic of weight u. Also M has a single circle character, since $m_{u'}$ does. By character orthogonality $P_u M$ is 0 or M. In the former case $|L(1)| = |P_u(L - M)(1)| \le \delta$, because $|r_\theta| = |m_{u'}(r_\theta)| = 1$. Thus $|K_0(1)| \le 2\delta$.

In the latter case (21.17) gives

$$\|L - L(1)m_u\|_1 = \|(1 - P_u)(L - M)\|_1 \le 2\|L - M\|_1 \le 2\delta b^{n_*} I_u.$$

All terms are integrable by the same bound and boundedness of K_0. Pair this difference with $\overline{F(zh)}$. The L term vanishes by hypothesis and the m_u term has modulus $|L(1)F(z)|I_u$ by (21.16). Since $\sup|F| < 2|F(z)|$, we get $|L(1)| \le 4\delta b^{n_*}$. Now $|K_0(1) - L(1)| \le \delta$.

Lean, comparison. Lean treats a general model by choosing an approximant at every rotated peak. The single circle character permits projection of one approximant at z. `ThetaKernelEstimates` (L981-L1051). □

Theorem 21.16 (Spanning the prescribed finite types by products)**.** *For $1 \le j \le m$, let $Y \in$ goodForms(W_j). Let θ° be the prescribed kernel on (W_j, V°), with $\dim V^\circ = t - 9m - 1 > 2r$, its inclusion datum modulo π_M^2, and its prescribed negative-letter Schur polynomials. If*

$$\int_{[U(W_j)]} Y(g)\mathcal{F}_i(g)\langle \zeta, \theta^\circ(g,\cdot)\rangle \, dg = 0 \quad \text{for every } i, \zeta, \tag{21.26}$$

where ζ ranges over all values in the prescribed Weyl module at the finite definite classes, then $Y = 0$.

Proof. We first check the holomorphic and level properties used in the argument. By Proposition 21.3, the original single lift L from W_j to its larger definite space is injective on the principal forms. The finite-dimensional positive operator L^*L is therefore onto, so $Y = L^*Lv$ for some principal form v. The adjoint formula expresses Y as a descent with the same joint-harmonic polynomials. Their highest vector is killed by the holomorphic raising operators, by the matching-sign minor construction; their compact orbit spans their prescribed Schur module. Equivariance thus shows that $\overline{Y}$ is holomorphic of exactly $u = \mathrm{conjHolomorphicWeight}(j)$.

The $\mathcal{F}_i$ are holomorphic of weight $u - u'$, where $u' = \mathrm{kernelWeight}(j)$. This follows by differentiating their theta kernels in the real variable: the same matching-sign minor calculation gives this weight, and carrier rapid decay permits differentiation under the integral. The column gaps give

$$\mathrm{gap}(u), \mathrm{gap}(u') \ge 300 n_*, \qquad |u_{\tau,i}| \le 5b.$$

Both weights are dominant in positive-first order. The functions $Y\mathcal{F}_i$ are rapidly decreasing at level π_M^2: Y is rapid at its principal level and the carrier integrals have polynomial growth in the other variable, as follows from the theta Gaussian majorant and rapid decay in the carrier variable.

Apply Theorem 21.13 to (W_j, V°), using the nonzero highest negative-letter polynomial. Its real definite orbit is spanned by the original tableau polynomials, and the kernel's holomorphic weight is u'. Orthogonality (21.26), first with ζ supported at one class and one Weyl value, kills every term of the finite reproduction formula. Therefore, for every complex combination $\mathcal{F} = \sum_i c_i \mathcal{F}_i$ and every adelic g,

$$\int Y(gh)\mathcal{F}(gh) m_{u'}(h) \, dh = 0. \tag{21.27}$$

The tableau polynomials span the exact Schur polynomial space: in the original Gram coordinates toPoly has this range. For a polynomial Q at a class c, choose v with $\mathrm{toPoly}(v) = Q$, and take ζ supported at c with value $\overline{v}$. The kernel pairing is conjugate-linear in ζ, so its polynomial is obtained by applying the prescribed negative-letter substitution to Q, as required. This checks the conjugation without assuming that the Weyl Gram matrix is the identity.

Suppose $Y \neq 0$ and take a peak g of $F = \overline{Y}$. Its principal level is $M\sqrt{-d_K}$. Lemma 21.11, with $A = 5b$ and $A' = b^{23} \ge M\sqrt{d_K}$, gives $\lambda(W_j g) \ge (5b^{24})^{-1/2}$. Take the approximant $K(h) = \mathcal{F}_g(gh)$ of Theorem 21.14 at this g. It is holomorphic of weight $u - u'$ and satisfies (21.27), since its coefficients are independent of h. The model $\mathcal{F}^0$ has one left circle character. Lemma 21.15 with $K_0 = \mathcal{F}^0$ thus gives

$$|\mathcal{F}^0(1)| \le (4b^{n_*} + 1)\, 2b^{-19 n_*} |\mathcal{F}^0(1)|.$$

The factor on the right is less than one since $b^{n_*} > 10$. This contradicts (21.24).

Lean, comparison. The projection at g replaces the use of all peaks in the final spanning argument. `ProductsSpan` (L735-L766), `ProductsSpan` (L778-L841), `ProductsSpan` (L855-L907). □

The coefficients used to prove spanning may be arbitrary complex numbers. For the height estimate we need a map with a common denominator and bounds at every complex embedding. We now construct such a map.

21.3 The finite gluing map and its comparison operator

All class sums below use the weight $|\operatorname{Aut}(x)|^{-1}$. The Hermitian pairing is conjugate-linear in its first argument. Fix a lift index j. Put

$$n = 6r, \qquad n_+ = r + 9m + 1, \qquad n_- = n - n_+, \qquad a_0 = 3m, \qquad \mathcal{A} = fmb(\log b)^2.$$

Thus $V_j = V'' \oplus V^\circ$ with ranks n_+, n_-. The r distinguished coordinates of the lift lie in V''; choose also r distinguished coordinates in V°. Write $e : \{1, \dots, r\} \hookrightarrow \{1, \dots, n_+\}$ and $z : \{1, \dots, r\} \hookrightarrow \{1, \dots, n_-\}$ for these inclusions. The diagrams μ^+, μ^- have no nonzero column of height between their respective block rank and $n - 1$. These assertions follow directly from $r = k + m - 1$, $100m + 9 < k$ and the row parameters: a column of μ^+ has height below $r + 9m + 1$, while a column of μ^- has height below r.

Lemma 21.17 (A glued lattice with one distinguished residue translate)**.** *The two primes dividing M exceed q_1, so the cyclotomic extension E/K is unramified above them. Thus the chosen modulus (π_M) is a product of distinct, mutually nonconjugate places. Write ϖ for its generator and put $R = \mathcal{O}_E/(\varpi^2)$, interpreted as the product of the corresponding local quotients. For $s \in M_{a_0,n_+}(R)$ let $U = \operatorname{Row}_R(s)$. At each chosen place w let k_w be the residue field, $W_w = \operatorname{Row}_{k_w}(\bar{s})$, and $t_w = \dim W_w$. For the tuple $\mathcal{W} = (W_w)_w$ one can choose a unitary finite-adelic point $\gamma_{\mathcal{W}}$, supported at the chosen places and their conjugates, with $\varpi\gamma_{\mathcal{W}}^{-1}$ integral. Put*

$$w_s = \begin{pmatrix} s \\ \varpi I_{n_+} \end{pmatrix}, \qquad \mathcal{U}^1 = \{Y \in \mathrm{GL}_r(R) : Y \equiv I_r \bmod \varpi\}.$$

For an n by r matrix x over R, the following are equivalent:

$$(\varpi\gamma_{\mathcal{W}}^{-1})x = \varpi\iota_e, \qquad \exists Y \in \mathcal{U}^1 : \quad (xY)_{V^\circ} = \iota_z, \quad w_s(xY)_{V''} = w_s\iota_e. \tag{21.28}$$

When it exists, Y is unique. At w, exactly t_w elementary divisors of the V''-columns of $\gamma_{\mathcal{W}}^{-1}$ have valuation -1; the others have valuation zero. Moreover $v_w(\det \gamma_{\mathcal{W}}) = t_w + n_- - r$.

Proof. Work at a chosen place w with valuation ring O. The rows of a lift of s, together with ϖO^{n_+}, generate precisely the inverse image of W_w under reduction: any vector reducing to W_w differs from a combination of the lifted rows by a vector in ϖO^{n_+}. Choose $k_0 \in \mathrm{GL}_{n_+}(O)$ whose first t_w rows lift a basis of W_w, completing them to a basis. Then

$$D = \operatorname{diag}(I_{t_w}, \varpi I_{n_+ - t_w})k_0, \qquad B = \varpi D^{-1}$$

are integral, $BD = \varpi I$, and the rows of D modulo ϖ^2 generate the same module as the rows of w_s. Thus $DX = DX'$ is equivalent to $w_sX = w_sX'$ over the quotient. Set

$$G = \begin{pmatrix} D & -D\iota_e\iota_z^{\mathsf{t}} \\ 0 & \operatorname{diag}(\varpi \text{ on } z,\ 1 \text{ off } z) \end{pmatrix}, \quad C = \begin{pmatrix} B & \iota_e\iota_z^{\mathsf{t}} \\ 0 & \operatorname{diag}(1 \text{ on } z,\ \varpi \text{ off } z) \end{pmatrix}.$$

Block multiplication gives $CG = \varpi I$; the upper right block is $-\varpi\iota_e\iota_z^{\mathsf{t}} + \varpi\iota_e\iota_z^{\mathsf{t}} = 0$. Permute the rows of G, and the corresponding columns of C, so that the z-rows become the original distinguished e-rows. At the chosen place take $\gamma_{\mathcal{W}} = C$ with this column permutation, and at its conjugate take the inverse conjugate transpose; take the identity elsewhere. This gives a unitary point. At w its inverse is the row permutation of G/ϖ, and at the conjugate place its inverse is integral. In the V''-columns of G/ϖ, right multiplication by k_0^{-1} gives the diagonal block $\operatorname{diag}(\varpi^{-1}I_{t_w}, I_{n_+ - t_w})$

above zeros. This gives the claimed elementary divisors. The two diagonal blocks of C have determinant valuations t_w and $n_- - r$; permutations preserve their sum.

Write $x = (u, v)$ in the two blocks and put $A = v_z$, the rows of v indexed by z. The equation on the left of (21.28), in the unpermuted row order, is

$$D(u - \iota_e A) = 0, \qquad \varpi A = \varpi I, \qquad v_{\text{off } z} = 0.$$

The middle equation makes $A = I + \varpi T$ invertible in R, with inverse $I - \varpi T$. Hence $Y = A^{-1} \in \mathcal{U}^1$ is forced. Multiplication by Y gives exactly the two conditions on the right, using the row-module equivalence. Conversely those conditions imply the displayed equations, because every $Y \in \mathcal{U}^1$ fixes ϖI.

Lean, comparison. Lean obtains a row basis by Hermite reduction for each s. Lifting a basis of W_w gives the same row lattice for every U reducing to W_w. `GluedLattices` (L713-L850). □

Lemma 21.18 (Cartan products and their denominators)**.** *Let y, η be vectors of weights μ^+, μ^- on V'', V°. Their mixed Cartan product $y \star \eta$, in the integral standard-tableau coordinates of V_j, has the following properties.*

Its Fock polynomial is the product of the two block polynomials, and

$$\|y \star \eta\|^2 = \|y\|^2 \|\eta\|^2. \tag{21.29}$$

For every coefficient embedding ι above ℓ, every positive integer M divisible by the chosen prime ideals, and every glued lattice from Lemma 21.17,

$$\operatorname{gauge}_{\iota,(h,h')\gamma_{\mathcal{W}}}(y \star \eta) \le |M|_\iota^{-\lambda} \operatorname{gauge}_{\iota,h}(y) \operatorname{gauge}_{\iota\circ c,h'}(\bar{\eta}), \tag{21.30}$$

where c denotes ordinary complex conjugation and

$$\lambda = \sum_\tau \left[a_0 \sum_{h<n_+} \mu^+_{\tau,h} + n_- \sum_{h<n_-} \mu^-_{\tau,h} \right]. \tag{21.31}$$

Proof. Use the tensor model consisting of exterior powers for the columns. Include the positive columns in V'' and the dual negative columns in V°. Every contraction between the two groups is zero, since their supports are disjoint. Their tensor product consequently belongs to the mixed highest weight summand. The integral straightening map in the standard-tableau basis gives its integral preimage $y \star \eta$. The mixed filling matrix is an isometric embedding in these coordinates, so its norm is the product norm. Evaluation of the tensor on a variable array is the product of the two minor polynomials; this proves (21.29) and the asserted Fock polynomial identity. All matrices in this construction are integral, so these identities commute with every coefficient embedding.

At a chosen place, write B_+ for the V''-columns of $\gamma_{\mathcal{W}}^{-1}$ and $k = k_0^{-1}$ for the basis change from Lemma 21.17. Only the first t_w columns of B_+k have a denominator, cleared by M. A minor of order d uses at most $\min(d, t_w)$ such columns. Hence the dth exterior power costs at most $|M|_\iota^{-\min(d,t_w)}$ at an embedding inducing this place. In $B_+h^{-1} = (B_+k)(hk)^{-1}$ the integral basis change preserves the Weyl lattice. At conjugate places B_+ is integral. Multiplication over all positive columns, using $t_w \le a_0$, gives the uniform exponent in (21.31). For the negative block use all n_- coordinates and the conjugate embedding. Unitarity changes the contragredient action into this conjugate action. The integral mixed straightening map cannot increase the nonarchimedean sup norm. This proves (21.30), including its prescribed conjugate place.

Lean, comparison. Lean reserves a_0 possible denominator directions. The residue basis gives t_w at w and the same uniform exponent for the gluing map. `GluedForms` (L115-L160), `GluedForms` (L425-L628). □

Point forms and normalization of their trace. At a point x of a definite unitary quotient, let $C_x(v)$ be the point form obtained by summing v over the finite stabilizer, with its level character. It satisfies

$$\langle C_x(v), \phi\rangle = \langle v, \phi(x)\rangle. \tag{21.32}$$

Indeed the value at x is the sum of $|\operatorname{Aut}(x)|$ unitary translates of v; each has the same pairing with the stabilizer-invariant value $\phi(x)$. The class-sum weight cancels their number. This is an unnormalized sum, so it preserves integral gauges. At every conjugate embedding its value has norm at most $|\operatorname{Aut}(x)|\,\|\sigma v\|$.

Lemma 21.19 (The slice identity of a glued point form)**.** *Let $\theta''_{\vartheta,s}$ have the column datum $w_s x = w_s \iota_e$ modulo ϖ^2, with the prescribed tame determinant character at the twist places, and let θ° have datum $x = \iota_z$ modulo ϖ^2. For the point form built from $y \star \eta$ at $(h, h')\gamma_{\mathcal{W}}$ one has*

$$\overline{\operatorname{Descent} C_{(h,h')\gamma_{\mathcal{W}}}(y \star \eta)(g)} = c_{\gamma_{\mathcal{W}}} \sum_{Y \in \mathcal{U}^1} \theta''_{\vartheta,s}[\,\bar{y}\,](g\widetilde{Y}, h)\theta^\circ[\,\bar{\eta}\,](g\widetilde{Y}, h'). \tag{21.33}$$

Here $\widetilde{Y}$ is an integral unitary lift, equal to one at the twist places, and

$$c_{\gamma_{\mathcal{W}}} = \chi_W(\det_{\text{chosen}} \gamma_{\mathcal{W}})\,|\det_{\text{chosen}} \gamma_{\mathcal{W}}|^{-r/2} \neq 0. \tag{21.34}$$

The absolute-value factor is $\prod_w |k_w|^{r(t_w+n_- -r)/2}$ by Lemma 21.17. The scalar and the point $\gamma_{\mathcal{W}}$ are independent of ϑ.

Proof. In the finite Schrödinger model the action of $\gamma_{\mathcal{W}}$ is substitution $x \mapsto \gamma_{\mathcal{W}}^{-1} x$ times (21.34). The original datum is the distinguished-column indicator modulo ϖ. On its support the substituted x is integral: at a chosen place $\gamma_{\mathcal{W}}$ is integral, and at the twist places it is the identity. Its new indicator is therefore precisely the left side of (21.28). By the unique-translate assertion, this indicator is the sum of the product indicators on the right of (21.28). There is no multiplicity or division in that sum.

On its support the positive block is ι_e modulo ϖ; the old tame determinant factors are thus one. The lifts $\widetilde{Y}$ are one at every twist place, so the remaining determinant character is exactly the original twist character. This verifies the values as well as the supports of the finite data. Formula (21.29) supplies the product of the real polynomials. Tensor factorization of the theta kernel now gives (21.33); (21.32) identifies the left side with the descent of the point form.

Lean. `GluedForms` (L232-L410). □

The gluing map. For a choice $i = (s, h^\flat, \text{rows})$, write $\phi''_{\vartheta,i}(h) = \overline{c''\Upsilon_{\vartheta,i}(h)}$, using the single nonzero scalar from Theorem 21.25. Denote by $J_{\vartheta,\mathcal{W}}$ the subgroup of integral first-factor points whose reduction modulo $\varpi^2 Q_\vartheta$ fixes each unmarked column of the identity and whose reduction A modulo ϖ^2 satisfies $w_s A = w_s$. The row condition depends only on the module generated by w_s, hence only on $\mathcal{W}$. At a chosen w, write $K_w(\varpi^2) = \ker(\mathrm{GL}_{n_+}(O) \to \mathrm{GL}_{n_+}(O/\varpi^2))$. Fixing the rows ϖI forces the reduction modulo ϖ^2 to be $1 + \varpi X$. Its unmarked columns are those of the identity, and each of the r remaining columns of X can be any vector annihilated by W_w. All such matrices lift to invertible matrices over O. Consequently

$$[J_{\vartheta,\mathcal{W},w} : K_w(\varpi^2)] = |k_w|^{r(n_+ - t_w)}.$$

This index is local at w. For the trivial twist the global principal level $K(\varpi^2)$ lies in $J_{1,\mathcal{W}}$; for a general twist the unmarked columns also satisfy the congruence at Q_ϑ. Every element of $J_{\vartheta,\mathcal{W}}$

reduces to the identity modulo ϖ at the chosen places. Its rational stabilizers are trivial, as are those at principal level ϖ^2 on V°, by torsion-freeness at the large level primes. The two reduction moduli; The gluing level; The untwisted principal-level inclusion.

Let $\mathcal{H}$ be the direct sum, over i and the classes c of V°, of its μ^- value module, with the ordinary sum of the Gram metrics. Define

$$G_\vartheta\zeta = M^\lambda \sum_{\mathcal{W},c} \sum_{[h] \text{ at } J_{\vartheta,\mathcal{W}}} C_{(h,c)\gamma_\mathcal{W}} \left(\sum_{i:\, \bar{U}_i = \mathcal{W}} \phi''_{\vartheta,i}(h) \star \zeta_{i,c} \right). \tag{21.35}$$

Here $\bar{U}_i$ is the tuple of reductions of the row module indexed by i. Linearity of the point form regroups the original sum over i, without a multiplicity factor. The indicator P_{U_i} still uses the full module: already in rank one over O/ϖ^2, the modules 0 and ϖR have the same reduction but give distinct values at 1. The point $\gamma_\mathcal{W}$ and the common integer M^λ are independent of the twist.

Theorem 21.20 (The common adjoint map)**.** *Let u_ϑ be the theta lift on good forms from W_j. There is a map $u^\circ : S_j \to \mathcal{H}$, independent of the twist, such that*

$$c_\vartheta G^*_\vartheta u_\vartheta = u^\circ. \tag{21.36}$$

Here the scalar c_ϑ is the three-exchange scalar of Theorem 21.31. The map u° is injective. Consequently $c_\vartheta \neq 0$ for every twist, and the constant C_j in that scalar's modulus formula is positive.

Proof. Put $\Xi_{\vartheta,i}(g) = \sum_{[h] \text{ at } J_{\vartheta,\mathcal{W}}} \theta''_{\vartheta,s}[\overline{\phi''_{\vartheta,i}(h)}](g,h)$. The three-exchange identity is $c_\vartheta \Xi_{\vartheta,i} = c''\mathcal{F}_i$, with $\mathcal{F}_i$ independent of the twist. Pair (21.33), summed as in (21.35), with a good source form y. A good form is invariant under every $\widetilde{Y}$, since these matrices are one modulo ϖ. Right translation preserves quotient Haar measure. Thus each of the $|\mathcal{U}^1|$ integrals is the same. Theta decay and rapid decrease of y give absolute integrability, so the finite sums and the integral commute. Using adjointness of descent and lift gives

$$\langle \zeta, c_\vartheta G^*_\vartheta u_\vartheta y \rangle = \sum_i d_i \int_{[U(W_j)]} y(g)\mathcal{F}_i(g)\, \theta^\circ[\overline{\zeta_i}](g)\, dg, \quad d_i = M^\lambda c_{\gamma_\mathcal{W}} |\mathcal{U}^1| c'' \neq 0. \tag{21.37}$$

Every quantity on the right is fixed. Define u° by the left side for $\vartheta = 1$; nondegeneracy of the metric gives (21.36) for all twists.

If $u^\circ y = 0$, choose ζ supported at one i. Since $d_i \neq 0$, (21.37) says that all the product pairings with $\mathcal{F}_i\theta^\circ$ vanish. The product-spanning Theorem 21.16 gives $y = 0$. Thus u° is injective. The source space has positive dimension, as proved by the earlier multiplicity construction. If some c_ϑ were zero, (21.36) would make $u^\circ = 0$, a contradiction. The raw modulus formula has strictly positive denominator, so $C_j = |c_1|^2 \text{heckeValueProd}(j,1) > 0$.

Lean. `GluingMaps` (L414-L616). □

Lemma 21.21 (Integrality of the gluing map and its comparison operator)**.** *If ℓ does not divide the norm of the twist modulus, G_ϑ carries integral tuples to integral forms, with the tuple gauge at $\iota \circ c$ on the conjugate values as in (21.30). Let h_j be the product of the hook lengths of the diagrams of μ^-, and put*

$$T_\vartheta = g_* h_j^2 G_\vartheta G^*_\vartheta. \tag{21.38}$$

Then T_ϑ preserves integral forms at every embedding ι above such an ℓ.

Proof. The first factors have gauge at most one by Theorem 21.25. The integer M^{λ} in (21.35) clears (21.30). Each point form is an integral trace, and the first-factor class sums have trivial stabilizers. Each term has gauge at most one, so the ultrametric inequality gives the same bound for the sum at every allowed twist.

Choose representatives of all source and target classes integral above ℓ. Indeed, at each of the finitely many relevant places the required rational unitary change lies in a nonempty open coset. The Cayley parametrization by skew-Hermitian matrices and weak approximation give a simultaneous choice for each class. These changes preserve the Hermitian metrics. On the source tuple the change acts by the conjugate of the Weyl action, since its gauge is taken on the conjugate value.

In these coordinates write B for the matrix of G_{ϑ} from the full source tuple space to the ambient target class-coordinate space. The first assertion applied to each coordinate vector at ι and at $\iota \circ c$ gives the integrality of B and $\bar{B}$ at ι. The source Gram matrix D has one integral Weyl Gram block per summand. The Gram–Fock identity in Lemma 19.15 gives $h_j D^{-1}$ integral, including for an empty diagram. The target Gram matrix H has integral Weyl blocks divided by the corresponding stabilizer orders. Thus $g_* H$ is integral even at primes dividing those orders. With the metric linear in the second variable,

$$T_{\vartheta} = h_j B (h_j D^{-1}) \bar{B}^{\,t} (g_* H). \tag{21.39}$$

This ambient formula restricts to the target subspace: pairing any of its vectors with $B\zeta$ gives the same adjoint in the full source. All factors in (21.39) are integral at ι, which proves the assertion.

Lean, comparison. Lean constructs a frame of tableau vectors in `exists_frame_weightRep_eq_intCast` and sums its rank-one operators in `LiftVolumeHeight` (L337-L430). Here the Weyl Gram inverse gives the adjoint directly. The integral class-coordinate gauge is `weylGauge_embFin`. □

Lemma 21.22 (Coefficient bounds for the gluing map)**.** *Put $L = \log b$ and $N_0 = \#\{(i,c)\}$. For all sufficiently large admissible scales,*

$$M^{2\lambda} e^{b} (N_0 e^{b} g_*)^2 \le e^{\mathcal{A}}, \qquad g_* h_j^2 \le e^{\mathcal{A}}. \tag{21.40}$$

Proof. Recall $a_0 = 3m$. We count the possible row modules once for each residue subspace and then include the definite classes and short rows. The scale assumptions give

$$\log b \ge (\log x_1)^6 \ge 10^{36} (\log x)^6.$$

Consequently $x^c = o(b)$ for every fixed $c > 0$, uniformly on the admissible scales as $x \to \infty$. Write $\Delta = 2r^8$ and $q = f(\lfloor b \rfloor + w_{\mathrm{off}} + \Delta + m\Delta)$. There are at most q positive columns in total. Thus

$$\lambda \le 3mq + ft^2\Delta, \qquad t = 5r. \tag{21.41}$$

At a chosen place write $q_w = |k_w|$ and $R_w = O/\varpi^2$. For $U \subseteq R_w^{n_+}$ let W_w be its reduction, and set $V = \{v \in k_w^{n_+} : \varpi \tilde{v} \in U\}$, where $\tilde{v}$ is any lift. If $t_w = \dim W_w$ and $d = \dim V$, then $W_w \subseteq V$, $U \cap \varpi R_w^{n_+} = \varpi V$, and $\varpi U = \varpi W_w$, identifying the residue space with $\varpi R_w^{n_+}$ by multiplication by ϖ. Thus $\dim_{k_w}(U/\varpi U) = d$, and the condition on the number of generators is $d \le a_0$ by Nakayama's lemma. Fix a basis of W_w. Lifts of its basis vectors, together with ϖV, generate U; changing a lift by ϖv gives the same module precisely when $v \in V$. Conversely any such lifts give the prescribed V: a combination of the lifted basis vectors reducing to zero has coefficients divisible by ϖ and hence lies in ϖW_w. The number of tests at w is therefore

$$\sum_{0 \le t_w \le d \le a_0} \begin{bmatrix} n_+ \\ d \end{bmatrix}_{q_w} \begin{bmatrix} d \\ t_w \end{bmatrix}_{q_w} q_w^{t_w(n_+ - d)}.$$

Here $\begin{bmatrix} n \\ d \end{bmatrix}_q$ counts the d-planes in k^n for a field of order q. This is the length-two case of the chain-ring submodule formula [HL00, Theorem 2.4]. Only $\sum_{t_w=0}^{a_0} \begin{bmatrix} n_+ \\ t_w \end{bmatrix}_{q_w}$ local choices of glued point are needed.

For the numerical bound put $g(u) = u(n_+ - u)$. Counting ordered bases gives $\begin{bmatrix} n \\ d \end{bmatrix}_q \le 4q^{d(n-d)}$: the denominator product $\prod_{i=1}^{d}(1-q^{-i})$ is at least $(1-\frac{1}{2})(1-\frac{1}{4})(1-\sum_{i\ge 3}2^{-i}) > 1/4$. Each summand is thus at most $16q_w^{g(d)+g(t_w)}$. Since $2a_0 \le n_+$, $g(a_0) - g(u) \ge (a_0-u)^2$ for $u \le a_0$. Summing the two geometric tails bounds the local count by $64q_w^{2a_0(n_+-a_0)}$, using $\sum_{j\ge 0} 2^{-j^2} \le 2$. The local choices are independent: pad each local generating tuple to a_0 entries and combine the tuples by the Chinese remainder theorem. There are at most $2f$ chosen places, lying over the two prime divisors of π_M in K, and $\prod_w q_w = M^f$ by $\mathrm{N}_{E/\mathbb{Q}}(\pi_M) = M^f$. A row choice consists of a class of rank $6m+1$ and, at each infinite place, at most $\Delta/2$ lists of $2m$ rows. The class count (11.2), followed by reduction modulo M^2, bounds each of this class set and the V° class set by e^b. These two fixed class sets enter N_0. At the trivial twist the same estimate bounds the first-factor and target class counts by e^b. More explicitly, the logarithm at rank $v \le 2n_m$ is at most

$$4fv^2\log(1+2C_v^3) + 4fv^2\log M, \qquad C_v = 6C_{\mathrm{diff}}(fv)^2,$$

which is a fixed polynomial in x, and hence is eventually less than b. Consequently

$$\log N_0 \le 6fm(n_+ - 3m)\log M + 2f\log 64 + 2b + fm\Delta\log(6m+1). \tag{21.42}$$

The logarithm of the first left side in (21.40) is therefore at most

$$\begin{aligned} &2(3mq + ft^2\Delta)\log M + 7b + 12fm(n_+ - 3m)\log M \\ &\quad + 4f\log 64 + 2fm\Delta\log(6m+1) + 2\log g_*. \end{aligned} \tag{21.43}$$

The scale construction gives $\log M \le 22L$, $q \le 2fb$ eventually, $f, r, m, \Delta, \log g_*$ bounded by fixed powers of x, and b larger than every fixed power of x. The only term of order $fmbL$ is at most $264fmbL$. Since $L \to \infty$, (21.43) is at most $fmbL^2$ eventually. This proves the first inequality with the original exponent, without changing its constant.

The diagrams of μ^- have at most $d_- = ft^2\Delta$ boxes in total. The hook bound in Lemma 19.15 gives $h_j \le d_-!$. Hence $\log(g_* h_j^2)$ is bounded by a fixed power of x, which is $o(b)$. This proves the second bound.

Lean, comparison. Lean bounds the number of tests by the number of generating matrices. Here the residue subspaces index the points and the full row modules index the tests. `GluingMaps` (L644-L704), `LiftVolumeHeight` (L433-L477). □

21.4 A common normalization for the first factors

Lemma 21.23 (A simultaneous scale for definite polynomial kernels)**.** *For a definite pair of ranks v, w, and homogeneous degree d_0, put*

$$C(v,w,d_0) = (M^2)^{2fvw} c_{\mathrm{vac}}^{vw}(M^2 d\, j_0)^{d_0} \ne 0. \tag{21.44}$$

Here d is the discriminant parameter of K. Let Q be a homogeneous polynomial of degree d_0, and let Θ_Q be its theta kernel for splitting characters of the prescribed weights and local types. Fix $D \ge 1$, a rational prime $\ell \nmid D$, and a coefficient embedding ι above ℓ. Suppose the modulus ideal contains (DM^2) and the residue datum is integral away from D. If both finite points are integral at every place above this ℓ and Q has integral coefficients at ι, then $C(v,w,d_0)\Theta_Q$ is integral at ι. The scalar in (21.44) is independent of D, ℓ, ι.

For the archimedean estimate, assume $v, w \le \Delta$, $d_0 \le M$, the modulus ideal contains (M^2), and all conjugate residue values have absolute value at most A. For arbitrary finite points, fix $\sigma \in \mathrm{Aut}(\mathbb{C})$. If Q satisfies

$$|\sigma(Q(Z))| \le B \qquad \text{whenever all } |\sigma Z_{ab}| \le M^4,$$

then its theta kernel Θ_Q satisfies

$$|\sigma(C(v, w, d_0)\Theta_Q)| \le e^{x_1} AB.$$

The same scalar (21.44) works for every σ; the polynomial and its bound may be chosen separately for each fixed σ.

Proof. Fix ℓ and ι as in the finite assertion. The finite Fourier expansion of the residue datum has denominator dividing $(M^2)^{2fvw}$ away from the prime divisors of D. After clearing it, the datum at ι is an integral combination of the original normalized vacuum translates whose translation parameters become integral above ℓ after multiplication by M^2. The actions of the two finite points preserve this module of local data because their components above ℓ are integral. The ordinary derivative comparison of Theorem 18.29, now with the same vacuum fixed in Theorem 19.4, identifies the kernel with its degree-d_0 jet at scale $M^2 d\, j_0$. This gives exactly (21.44), and the local ordinary-jet integrality gives its first assertion. No new choice of vacuum or scalar is made at a coefficient prime.

For the complex estimate fix σ, put $D_F = fvw$ and $T = (M^2 d)^2/\Omega$. The scale inequality is $2T(d_0 + 1) \le M^8$. After applying σ to the coefficients and permuting the places, write the homogeneous polynomial as $Q_\sigma(z) = \sum_{|\alpha|=d_0} q_\alpha z^\alpha$. Its bound on the polydisc and Parseval on the torus $|z_i| = M^4$ give $\sum_\alpha |q_\alpha|^2 M^{8d_0} \le B^2$. For independent complex Gaussians Z_i of density $(2\pi)^{-1} e^{-|z|^2/2}$, angular orthogonality and the radial integral give $\mathbb{E}[Z_i^k \overline{Z_i}^{\,l}] = \mathbf{1}_{\{k=l\}} 2^k k!$. Thus the Fischer norm formula [BB10, §1] yields

$$\mathbb{E}|Q_\sigma(\sqrt{T}Z)|^2 = (2T)^{d_0} \sum_\alpha \alpha! |q_\alpha|^2 \le (2T/M^8)^{d_0} d_0!\, B^2 \le B^2.$$

The estimate includes $d_0 = 0$. In the reproducing integral of Lemma 19.12, the scale has modulus $\sqrt{T}$; complex conjugation and unit phases preserve the Gaussian law. The conjugated translated vacuum is bounded by $e^{C(D_F+1)}$, and the measure in that integral is 2^{D_F} times the Gaussian probability measure. Cauchy–Schwarz and the finite coefficient sum $A|\mathcal{O}_E/\mathfrak{m}|^{vw}$ therefore give

$$AB\,(M^2)^{2fvw} |\mathcal{O}_E/\mathfrak{m}|^{vw} e^{C(D_F+1)} 2^{D_F} \tag{21.45}$$

as an upper bound. Here $|\mathcal{O}_E/\mathfrak{m}| \le M^{4f}$. By Theorem 21.9, $f \le x^{12}$, $C \le x$ and $x \ge 10^4$; also $v, w \le \Delta \le x^{42}$ and $\log M \le 22 \log b \le 22 x^{1/100}$. Hence the logarithm of the factor multiplying AB is at most $D_F(8 \log M + \log 2) + C(D_F + 1) \le x^{98} \le x_1$.

Lean, comparison. Lean applies a pointwise polynomial growth bound inside the Gaussian integral. Torus Parseval and Gaussian orthogonality replace its factor $(2d_0 + 2)^{fvw}$ by 2^{fvw}. `DefiniteKernelSize` (L112-L187), `DefiniteKernelSize` (L211-L323). □

Lemma 21.24 (The complementary diagram clears the first-factor coordinates)**.** *Let $r_+ = k + j - 1$. For each column of μ^+ of height $h < r_+$, stack above the variable matrix the $2m$ unit rows which fill its remaining positive columns and then the first added block. For columns of height r_+, use the prescribed $2m$ rows of a short matrix Y on the rank $6m+1$ flat block. Let $P_Y(X, Z)$ be the resulting product of square determinants. There is a unique polynomial vector $\zeta_Y(Z)$ of weight μ^+ with*

$$P_Y(X, Z) = \mathrm{toPoly}(\zeta_Y(Z))(X). \tag{21.46}$$

There are $d_0 \in \mathbb{Z}_{\geq 0}$ and $H \in \mathbb{Z}_{>0}$, independent of the row choice and Y, such that

$$d_0 = \sum_{\tau,h} (n_+ - h)\mu^+_{\tau,h} \leq (10m+1)q, \qquad H \leq d_0!, \tag{21.47}$$

and every coordinate of $H\zeta_Y$ is homogeneous of degree d_0, with coefficients belonging to the ring generated by the entries of the conjugates of Y. For $d_0 = 0$ take $H = 1$. If all these entries and all specialized Z entries have absolute value at most M^4, then

$$\|\zeta_Y(Z)\|^2 \leq M^{9(12m+1)q}. \tag{21.48}$$

Both statements remain true after any coefficient automorphism.

Proof. For one determinant, clear the first $r_+ - h$ unused positive columns by the unit rows above them. Expansion along the remaining h positive variable columns expresses it as a linear combination of their h-minors; the coefficients are the complementary minors of the remaining columns. It is therefore the polynomial of an exterior-power vector. Multiplication of these polynomials is the Cartan product; orthogonal projection from the column tensor product is contractive. This proves (21.46) and its uniqueness by the independent standard-tableau minor basis.

The coordinate degree counts the complementary columns, so it is the stated d_0. A nonzero column in this shape satisfies $n_+ - h \leq 10m+1$. The total number of columns is at most q, proving the degree bound. Take H to be the hook number of the complementary diagram with column counts $\widetilde{\mu}_{\tau,h} = \mu^+_{\tau,n_+-h}$. The duality between a column and its complementary column is the signed cofactor map, with the determinant factor retained; it is integral. Applying the integral Gram–Fock identity $GH = hI$ to this dual diagram shows that its hook number clears the inverse Gram matrix needed to pass from minor-product coefficients in (21.46) to Weyl coordinates. Equivalently, complement each increasing column and multiply by its orientation sign; the standard-tableau straightening relations are integral in both directions after tensoring by the determinant. This proves the coefficient assertion over every coefficient ring, including valuation rings. For the complementary diagram, let P denote the product of its leading minors over all embeddings. The Gram–Fock scalar is $H = P(\partial)P(0)$. The degree of P is d_0, so the symmetric tensor estimate in the proof of Lemma 19.15 gives $H \leq d_0!$.

For one cleared determinant, Cauchy–Binet bounds the squared norm of its complementary-minor vector by the product of the squared Euclidean lengths of the remaining columns. There are at most $12m+1$ such columns. Each has at most $n_+ + 2m \leq M$ entries, each of modulus at most M^4; hence each squared length is at most M^9. The norm is thus at most $M^{9(12m+1)}$. Tensoring the at most q determinants and using the contractive Cartan product proves (21.48). The polynomial identities have integer coefficients in the entries of Y, proving the last assertion.

Lean, comparison. The Bombieri norm bounds the same clearing factor that Lean estimates by $d_0^{3d_0}$.
`LineCoordinates` (L191-L235), `LineCoordinates` (L445-L595), `LineCoordinates` (L700-L764). □

Theorem 21.25 (A common scalar for integrality and the archimedean bound)**.** *For the series $\Upsilon_{\vartheta,i}$ there is one $c'' \neq 0$, independent of ϑ, i, h, such that $\phi''_{\vartheta,i}(h) = \overline{c''\Upsilon_{\vartheta,i}(h)}$ has integral gauge away from the twist modulus, and*

$$\|\sigma\phi''_{1,i}(h)\|^2 \leq e^{\mathcal{A}} \quad \text{for every } \sigma, i, h. \tag{21.49}$$

Proof. Put $W^{\mathrm{add}} = I_{3m-j} \oplus I_{6m+1} \oplus I_{3m}$. The series Υ is the kernel of $((-I_j), V'')$ with its identity-column datum, times the class sum over I_{3m} of the carrier coefficient and the kernel of (W^{add}, V'') at ζ_Y. The carrier coefficient itself is the class sum of the kernel of (I_m, I_{3m}), of degree zero and unit finite modulus. Opposite-space conjugation changes the first kernel into a conjugate kernel of

(I_j, V''), with a fixed nonzero phase ϵ_j independent of the twist. The identity datum is conjugated at the fixed norm-minus-one unit. Its values are still zero or one.

Apply Lemma 21.23 to these three definite pairs, with degrees $0, 0, d_0$, obtaining C_0, C_1, C_2. With H from Lemma 21.24, take exactly

$$c'' = g_*^2 C_2 H C_0 \overline{C_1} \epsilon_j. \tag{21.50}$$

Every factor is nonzero. Expanding the defining series, its conjugate is g_*^2 times a double class sum of conjugates of

$$K_0(v, w)\, \overline{K_1(h)}\, C_2\, \Theta_{\mathrm{add}}(H\zeta_Y; w, h). \tag{21.51}$$

Here K_0, K_1 include C_0, C_1. The added finite datum is the product of the prescribed column indicators with $P_{U,\vartheta}$. It is integral and each conjugate has absolute value at most one.

For finite integrality fix a rational prime ℓ away from the twist modulus and an embedding ι above ℓ. Use $D = 1$ for the unit modulus and $D = \mathrm{Nm}\, Q_\vartheta$ for the twisted modulus in Lemma 21.23. Rational unitary matrices move the two finite points and the flat class representative into integral position at every place above this ℓ. The representatives may depend on ℓ, whereas the three scalars in (21.50) were fixed before ℓ. The short rows transform to $Y\delta^{\mathsf{t}}$ and are integral there: they were vectors in the original lattice, and δ transports that lattice into the integral lattice. The determinant identity for the stacked rows gives the corresponding Schur action on ζ_Y. Its determinant powers cancel those in the two splitting characters. Thus the transformed coordinate bound from (21.44) is exactly the original Weyl gauge, with no denominator from the rational change of representative. The degree-zero first two kernels are similarly invariant under the needed rational changes, because the relevant other character has weight zero. Each term (21.51) is consequently integral in the required gauge. The two class sums have stabilizer denominators dividing g_*^2, already present in (21.50).

For the complex estimate in (21.49), take the trivial twist. The three modulus ideals then contain (M^2). Fix an automorphism σ. It suffices to bound the vector before the outer conjugation in (21.51), since composing σ with ordinary complex conjugation again ranges over all automorphisms. Equation (11.1) and the scale inequalities bound every conjugate short-row entry by M^4. By (21.48), $\sigma(H\zeta_Y(Z))$ has norm at most

$$B = H M^{9(12m+1)q/2}$$

whenever all $|\sigma Z_{ab}| \le M^4$. Choose any covector y_0 of dual Weyl norm at most one on this conjugated target and put $y = \sigma^{-1} y_0$ coefficientwise. For the degree-d_0 polynomial $Q_y(Z) = y(H\zeta_Y(Z))$ we have

$$|\sigma Q_y(Z)| = |y_0 \sigma(H\zeta_Y(Z))| \le B.$$

The archimedean assertion of Lemma 21.23, followed by the supremum over y_0, therefore bounds the conjugated vector (21.51) by $e^{3x_1} B$. The two full-level class counts are at most $e^{x^{30}}$ by (11.2) and its numerical specialization. After squaring, the logarithm of the resulting bound is at most

$$4 \log g_* + 4x^{30} + 6x_1 + 2d_0 \log M + 9(12m+1)q \log M. \tag{21.52}$$

Here $H \le d_0! \le M^{d_0}$ because $d_0 \le (10m+1)q \le M$. We bound (21.52) directly by admissibility. Here $w_{\mathrm{off}} = xb/x_1 \le b/x^{999}$ and $(1+m)\Delta \le x^{44} \le b/x^{999}$, so $q \le 2fb$. Since $d_0 \le (10m+1)q \le 11mq$, the two terms containing q or d_0 are at most

$$\{2d_0 + 9(12m+1)q\} \log M \le 139mq \log M \le 6116 f m b L.$$

The definition of the stabilizer multiple gives $\log g_* \le f(rt)^2 \log M$. Using $f \le x^{12}$, $m \le x$, $r, t \le x^6$ and $\log M \le 22L \le 22x$, we obtain

$$4 \log g_* + 4x^{30} \le 2000x^{37} \le x^{40} \le x_1.$$

Consequently (21.52) is at most $6116 f m b L + 7x_1$. The same scale assumptions give $L \ge 12232$, $b \ge 196$, and $x_1 \le b^{1/10^6} \le \sqrt{b}$. Hence

$$6116 f m b L \le \tfrac{1}{2}\mathcal{A}, \qquad 7x_1 \le 7\sqrt{b} \le \tfrac{1}{2} b \le \tfrac{1}{2}\mathcal{A}.$$

This proves (21.49) with the scalar (21.50).

Lean, comparison. Each residue indicator has norm at most one; the formal group sum is bounded by its cardinality. The factor H^2 contributes at most $2d_0 \log M$ by (21.47). `FirstFactors` (L587-L654), `FirstFactors` (L657-L771). □

Theorem 21.26 (The comparison bound at every complex embedding)**.** *For every $\sigma \in \mathrm{Aut}(\mathbb{C})$,*

$$\|G_1 \zeta\|_\sigma^2 \le e^{2\mathcal{A}} \|\zeta\|_\sigma^2, \qquad \|T_1 \phi\|_\sigma \le e^{3\mathcal{A}} \|\phi\|_\sigma. \tag{21.53}$$

Proof. By Theorem 21.25, a first-factor value has squared conjugate norm at most $e^{\mathcal{A}}$. Equations (21.29) and (21.32) bound the squared norm of one point-form value in (21.35) by $g_*^2 e^{\mathcal{A}} \|\zeta\|_\sigma^2$. At the trivial twist, Lemma 21.22 and its proof give at most $N_0 e^b$ summands at each point and at most e^b target classes. The triangle inequality and the first part of (21.40) prove the bound for G_1.

For the adjoint use the original class-coordinate bases, whose Gram matrices D, H are rational. If c is complex conjugation and $\sigma^\# = c\sigma c$, applying σ to $G_1^* = D^{-1} \bar{G}_1^{\,t} H$ gives $\sigma(G_1^*) = (\sigma^\# G_1)^*$. The adjoint may again be computed in the ambient target. Hence the first bound at σ and $\sigma^\#$ gives

$$\|\sigma(T_1)\|_{\mathrm{op}} \le g_* h_j^2 \, \|\sigma(G_1)\|_{\mathrm{op}} \, \|\sigma^\#(G_1)\|_{\mathrm{op}} \le g_* h_j^2 e^{2\mathcal{A}}. \tag{21.54}$$

The second inequality in (21.40) gives the result.

Lean, comparison. Lean estimates a sum over the tableau frame in `LiftVolumeHeight` (L479-L546). Here the rational Gram matrices give the conjugate adjoint bound. □

Lemma 21.27 (Contraction of the orthogonal projection in each conjugate norm)**.** *Let $\mathcal{V}$ be the definite form space. Let e_0 be the central idempotent defining the good finite type, and let $A = H_w(X_*)$ be the split spherical Hecke polynomial at an unramified place w, evaluated at the prescribed lift root X_*. Suppose $Z \subseteq \mathrm{im}\, e_0 \cap \ker A$. If y is orthogonal to $\mathrm{im}\, e_0 \cap \ker A \cap Z^\perp$ in the class-sum pairing, then the orthogonal projection P_Z for that pairing satisfies*

$$\|P_Z y\|_\sigma \le \|y\|_\sigma \tag{21.55}$$

for every automorphism σ of $\mathbb{C}$. In the application Z is the lift image and the vectors of $\mathrm{im}\, e_0 \cap \ker A \cap Z^\perp$ are the false lifts.

Proof. Define the semilinear pairing

$$B_\sigma(x, z) = \sigma^{-1}\left(\sum_{[g]} \frac{\overline{\sigma(x(g))}^{\mathsf{t}} G \, \sigma(z(g))}{|\mathrm{Aut}(g)|} \right). \tag{21.56}$$

Its first-variable involution is $\sigma^{-1} c \sigma$, and $\sigma B_\sigma(x, x) = \|x\|_\sigma^2$ is positive for $x \ne 0$. Finite translation preserves this pairing: it preserves quotient Haar measure, and the algebraic weight matrices are

unitary at every conjugate embedding of the CM coefficient field. Tame level scalars are roots of unity, of modulus one after every σ.

It follows that e_0 is self-adjoint for every B_σ. Indeed the conjugated finite group action is unitary, and a central idempotent is the sum of its orthogonal isotypic projections. It need not have rational coefficients. The adjoint of $H_w(X)$ for B_σ is

$$H_{\bar{w}}(\sigma^{-1}(\overline{\sigma(X)})). \tag{21.57}$$

To see the formula, reverse the finite double-coset correspondence term by term; the inverse coset is the conjugate-place coset, and the coefficient is changed by the first-variable involution. Spherical Hecke operators at these places commute. At different places this is disjoint support; at a split place it is the commutative algebra of GL_n-lattice correspondences, whose elementary divisor transform is the symmetric Laurent polynomial algebra. Hence A is normal for every B_σ.

Write $W = \operatorname{im} e_0 \cap \ker A$ and let $\perp_\sigma$ denote orthogonality for B_σ. In the finite-dimensional conjugated Hilbert space the spectral theorem for a normal operator gives $(\ker A)^{\perp_\sigma} = \operatorname{im} A$. Self-adjointness of e_0 gives $(\operatorname{im} e_0)^{\perp_\sigma} = \ker e_0$. Taking the orthogonal complement of the intersection yields

$$W^{\perp_\sigma} = \ker e_0 + \operatorname{im} A = W^{\perp}$$

for every σ, where $\perp$ refers to the original pairing. Since $W = Z \oplus (W \cap Z^{\perp})$ and $y \perp W \cap Z^{\perp}$, we have $y - P_Z y \in W^{\perp}$. Thus $y - P_Z y \perp_\sigma P_Z y$ for every σ, and Pythagoras proves (21.55).

Lean, comparison. The spectral theorem identifies the image of the normal Hecke operator with the orthogonal complement of its kernel; Lean proves their orthogonality from the adjoint identity. `TestedForms` (L303-L347), `TestedForms` (L496-L618). □

Corollary 21.28 (The identity for the comparison operator)**.** *The maps* (21.35) *and* (21.38) *satisfy* (25.26) *and the first inequality of* (25.27)*, with*

$$U_0 = \frac{g_* h_j^2}{|c_1|^2}(u^\circ)^* u^\circ > 0, \qquad \rho_\vartheta = \frac{|c_1|^2}{|c_\vartheta|^2} = \frac{\mathrm{heckeValueProd}(j,\vartheta)}{\mathrm{heckeValueProd}(j,1)}. \tag{21.58}$$

They preserve integral forms away from the twist modulus. Both inequalities of (25.27) *hold for the original tested vectors.*

Proof. Use (21.36), the three-exchange modulus formula, and the injectivity of u°. Pairing gives $u_\vartheta^* T_\vartheta u_\vartheta = g_* h_j^2 |c_\vartheta|^{-2} (u^\circ)^* u^\circ$. The finite and infinite bounds are Lemmas 21.21 and 21.26. For the final assertion, lift equivariance puts $Z = \operatorname{im} u_1$ in $\operatorname{im} e_0 \cap \ker H_w(X_*)$. Corollary 21.41 makes $T_1 Z$ orthogonal to its false lifts, since $G_1^* u_1 = c_1^{-1} u^\circ$. Apply Lemma 21.27.

Lean, formal context. `TestedForms` (L496-L618), `LiftVolumeHeight` (L729). □

21.5 Explicit matching and the common three-exchange scalar

To apply Theorem 19.18 to the prescribed polynomials, we match their archimedean vectors with a scalar independent of the constant rows, the finite datum, and the finite-order twist.

Lemma 21.29 (Compact projection and simultaneous polynomial matching)**.** *Let W, V' have signatures $(M,L),(P,L)$, let I_A be positive definite, and suppose that for some integer $e \ge 0$,*

$$M = P + A, \quad N = M + L = A + B, \quad B = P + L, \quad A = B + 2e, \quad N \ge 2L + 2.$$

Fix nonnegative integers K_τ. For lists of P by M matrices $\eta_{\tau i}$, $1 \le i \le K_\tau$, put

$$P_\eta(X) = \prod_{\tau,i} \det \begin{pmatrix} \eta_{\tau i} \\ X_\tau \end{pmatrix}, \qquad Q_\eta(Y) = \prod_\tau \det(Y_\tau^-)^{K_\tau + N - 2L} \prod_{\tau,i} \det(\eta_{\tau i} Y_\tau^+). \tag{21.59}$$

Here X_τ is A by M, while Y^+, Y^- are respectively the M by P and L by L blocks with equal signs. There is a constant $C \geq 0$, depending on these fixed real data and framed spaces, with the following property.

Let α be tame for a nontrivial conjugate-disjoint modulus $\mathfrak{q}$ and residue character χ, of weights $L - K_\tau$. There is $d_\alpha \in \mathbb{C}$ such that

$$|d_\alpha|^2 = \frac{C}{\Pi_{\mathfrak{q}}(\alpha^{-1}\upsilon^{B+e}, e)} \tag{21.60}$$

and, simultaneously for every η, every second splitting character β, and every matched pair of finite data Φ, Ψ,

$$d_\alpha \sum_{[h] \in [U(I_A)]/K_A} \frac{\theta_{W,I_A;\alpha,\beta}[\Phi, P_\eta](g,h)}{|\operatorname{Aut}(h)|} = \int_{[U(V')]} \theta_{V',W;\alpha/\beta,\alpha}[\Psi, Q_\eta](x,g)\, dx. \tag{21.61}$$

The matching convention is the one in Theorem 19.18: on the rows and columns of an invertible residue matrix k the values differ by $\chi^{-1}(\det k)$, and both data vanish off the corresponding primitive row and column loci. The denominator in (21.60) is strictly positive. This lemma does not assert $C > 0$.

Proof. Write $\eta_0 = (I_P, 0)$, repeated K_τ times, and take $Q_0 = Q_{\eta_0}$ as the right real vector in Theorem 19.18. That theorem supplies a left real vector v, fixed before the finite characters are chosen, and a scalar $c_\alpha \neq 0$ with $|c_\alpha|^2 = C_0 \Pi_{\mathfrak{q}}(\alpha^{-1}\upsilon^{B+e}, e)$, where $C_0 > 0$. The nonnegative weights of the associated Hecke character are $K_\tau + N/2 - L$, and $N/2 - L \geq 1$, so its nonvanishing clause applies. We prove that a single scalar κ, depending only on v, transfers this identity from (v, Q_0) to (P_η, Q_η) for every η.

For $|z| = 1$, let $t(z)$ have entries z^{-2} on the first P positive coordinates of W, z^{-1} on its remaining A positive coordinates, and 1 on its negative coordinates. On I_A use the scalar $z^{-1} I_A$. In the real Weil representation of weights $L - K_\tau, 0$ set

$$\mathcal{R}_z v = z^{-A \sum_\tau K_\tau} \omega(t(z), z^{-1} I_A) v.$$

If ℓ is any continuous functional invariant under $U(I_A)$, then

$$\int_{|z|=1} \ell(\mathcal{R}_z v)\, dz = \kappa\, \ell(P_{\eta_0}) \tag{21.62}$$

for one κ independent of ℓ. To check this explicitly, first let v be a polynomial times the Gaussian. The vacuum determinant factor cancels $z^{-A \sum K_\tau}$; each variable outside the last A positive columns acquires a factor z, and each variable inside those columns acquires factor 1. Circle integration therefore sets all outside variables to zero. The remaining polynomial is on square A by A matrices. Average its row action under $\prod_\tau U(A)$ with weight $\prod_\tau \det(u_\tau)^{-K_\tau}$. The result is a multiple of $\prod_\tau \det(X_\tau)^{K_\tau}$. Indeed the equivariance identity extends from $U(A)$ to $\mathrm{GL}_A(\mathbb{C})$ by polynomial identity; on invertible X it determines the polynomial as its value at I times $\det(X)^{K_\tau}$, and equality extends to all X. This is P_{η_0}. Polynomial Gaussian vectors are dense in the rapid Fock topology, by truncating their Hermite expansion. Compact averaging is continuous there. Its image in the quotient detected by invariant continuous functionals is the closed one-dimensional line just found. This proves (21.62) for the chosen v as well.

For $k \in \prod_\tau U(M)$, direct substitution in (21.59) gives

$$\omega(k,u) P_{\eta k^{-\mathrm{t}}} = \Big(\prod_\tau \det(k_\tau)^{K_\tau}\Big)^{-1} P_\eta, \qquad \omega(1,k) Q_\eta = \Big(\prod_\tau \det(k_\tau)^{K_\tau}\Big)^{-1} Q_{\eta k^{\mathrm{t}}}. \tag{21.63}$$

The $u \in U(I_A)$ factor cancels because its vacuum weight is $-K_\tau$ and each stacked determinant has row weight 1. Apply (21.62) to a k-translate of the left theta functional. On the right, $k\,t(z)$ carries Q_0 to a scalar times $Q_{\eta_0 k^{\mathsf{t}}}$. The powers of z cancel: $\det t(z) = z^{-(2P+A)}$, and the K_τ paired determinants contribute z^{-2PK_τ}. Thus

$$\kappa\,\mathcal{L}(P_{\eta_0 k^{\mathsf{t}}}) = c_\alpha\,\mathcal{R}(Q_{\eta_0 k^{\mathsf{t}}}) \quad \text{for every } k. \tag{21.64}$$

For completeness, the same coefficients span both polynomial families. Products $\prod_i \det(\eta_i Y)$ lie in the rectangular Schur module generated by $\det(\eta_0 Y)^{K_\tau}$. This follows directly from the alternating-column construction: the diagonal weight (K_τ^P) is the unique highest weight in the submodule generated by this vector, and column straightening expresses every product of P-minors in that submodule. Its complex general-linear orbit is spanned by its unitary orbit, since a polynomial linear functional vanishing on the latter vanishes on its complexification. Hence a finite list of coefficients expresses every paired product as a sum of the paired products for $\eta_0 k^{\mathsf{t}}$. For each fixed A by M matrix X, exterior algebra gives an M by P matrix Y_X and a scalar a_X such that

$$\det\begin{pmatrix}\eta\\ X\end{pmatrix} = a_X \det(\eta Y_X) \quad \text{for every } \eta. \tag{21.65}$$

If X has deficient row rank both sides may be taken zero; otherwise choose a basis for its kernel and compare the two alternating P-forms. Substitution of Y_X shows that the same finite coefficients express the stacked products. Thus (21.64) holds for every η.

On the definite receiver, the real vector is invariant and the finite character factor cancels the tame determinant factor of the matched datum. Its theta integral is therefore the indicated class sum at the full integral level, with the same quotient-measure normalization as before. Taking $d_\alpha = \kappa/c_\alpha$ gives (21.60)–(21.61). Twisting the second splitting character to β multiplies both sides by $\beta(\det g)^{-1}$, so does not change this scalar. In particular $C = |\kappa|^2/C_0 \geq 0$ is independent of every finite choice.

Lean. `ExplicitMatching` (L128-L785), `exists_mul_stdClassSum_eq_integral_modKernel`. □

Lemma 21.30 (The fixed carrier is nonzero, holomorphic and rapidly decreasing)**.** *Let $m > 0$ be even and put $S = I_{2m} \oplus (-I_m)$. Define the fixed carrier*

$$\phi^\circ(x) = \int_{[U(-I_m)]} \theta_{-I_m,S;\upsilon^m,1}\left[1, \prod_\tau \det(Y_\tau^-)^{2m}\right](y,x)\,dy, \tag{21.66}$$

with the unit finite modulus and the original vacuum. It is not identically zero. It has full integral finite level and is rapidly decreasing on $[U(S)]$. For the positive-first order $o(i) = -1-i$ on the $2m$ positive coordinates and $o(i) = i$ on the m negative coordinates, it is holomorphic of weight 0 on the positive and $3m$ on the negative coordinates.

More precisely, for every weight-zero character c tame for $(\mathfrak{q}, \chi)$ there is $\gamma_{12}(c)$ such that

$$\gamma_{12}(c) \sum_{[h]\in[U(I_{3m})]/K} \frac{a_c(h)\,\theta_{S,I_{3m};\upsilon^m,c}[\chi\circ\det,1](x,h)}{|\operatorname{Aut}(h)|} = \phi^\circ(x), \tag{21.67}$$

where $a_c(h)$ is the full-level class sum on $U(I_m)$ of the unit-modulus, constant-polynomial kernel of (I_m, I_{3m}) with characters $c, 1$. A fixed $C_{12} > 0$ satisfies

$$|\gamma_{12}(c)|^2 = \frac{C_{12}}{\Pi_{\mathfrak{q}}(c^{-1}\upsilon^m, m)\,\Pi_{\mathfrak{q}}(c\upsilon^{m/2}, m/2)}. \tag{21.68}$$

Proof. First use Lemma 21.29 with $W = I_m$, $V' = 0$, $A = m$, $B = 0$, no constant rows, and $\alpha = c^{-1}$. Here $e = m/2$ is a positive integer because m is a power of two with $m \geq 2$. The right kernel is 1, hence γ_1 times the square theta class sum with characters c^{-1}, c^{-1} and datum $\chi^{-1} \circ \det$ is 1. In particular $\gamma_1 \neq 0$ and $|\gamma_1|^2 = C_1/\Pi_{\mathfrak{q}}(c\upsilon^{m/2}, m/2)$. Conjugating the kernel changes I_m to $-I_m$ and changes its characters to $c\upsilon^m, c\upsilon^m$. The real intertwiner contributes one fixed phase ϵ and the finite norm-minus-one change of variables contributes $\chi(\eta)^{-m}$; both have absolute value one. Consequently $\overline{\gamma_1}\epsilon\chi(\eta)^{-m}$ times this new class sum is also 1.

Apply the explicit matching lemma a second time with $W = I_m \oplus S$, $V' = -I_m$, $A = 3m$, $e = m$, no constant rows, and $\alpha = c\upsilon^m$. Its scalar γ_2 has squared modulus $C_2/\Pi_{\mathfrak{q}}(c^{-1}\upsilon^m, m)$. Split the kernel according to $I_m \oplus S$, multiply by the preceding constant class sum, and sum over $U(I_m)$. The polynomial on the negative block is exactly $\det(Y^-)^{N-2L} = \det(Y^-)^{2m}$. The definite class sums are finite, and the remaining integral is absolutely convergent by Lemma 12.1. Fubini thus gives (21.67) with $\gamma_{12} = \gamma_2\overline{\gamma_1}\epsilon\chi(\eta)^{-m}$, and its norm gives (21.68), initially with $C_{12} \geq 0$.

Each square kernel on the left of (21.67) has full level: on the integral group, $c(\det k)^{-1}$ is cancelled by $\chi(\det k)$. Its nonzero residue support has $\det Y$ invertible, hence is full rank at the tame places. Lemma 12.1, applied with the two groups interchanged, proves rapid decay on $U(S)$. The polynomial 1 times the Gaussian is killed by the holomorphic lowering operators for the positive-first order just specified. The diagonal compact action, including the Weil vacuum character, has weights $0, 3m$. Differentiating the absolutely convergent theta sum therefore gives the corresponding holomorphic first-order identities. Thus its finite linear combination (21.67) has the level, decay and holomorphy asserted.

It remains to prove nonvanishing independently of these possibly vanishing compact-projection constants. Let $v = \prod_\tau \det(Y_\tau^-)^{2m}$ times the Gaussian in the Fock space of $(-I_m, S)$. It is nonzero. Direct substitution under $u \in U(-I_m)_\infty$ gives

$$\langle v, \omega_{0,0}(u, 1)v\rangle = \langle v, v\rangle. \tag{21.69}$$

Indeed the polynomial contributes the $2m$-th determinant powers; together with the vacuum factors the coefficient is $\langle v, v\rangle\, \mu(u)\overline{\mu(u)}$, where $\mu(u)$ is a product of $3m$-th powers of unitary determinants. The last product is 1. If (21.66) vanished, the sharp-convolution implication of Lemma 21.12, applied to the unit finite modulus and the constant function on the definite source, would make its real convolution with this matrix coefficient zero. The real source group is compact, and (21.69) makes that convolution the strictly positive number $\langle v, v\rangle \operatorname{vol}(U(-I_m)_\infty)$. This contradiction proves nonvanishing. It also forces $\gamma_{12}(c) \neq 0$ in (21.67), and hence $C_{12} > 0$. This use of sharp convolution requires no product-spanning statement; the latter may therefore use this carrier.

Lean. `ThreeExchanges` (L210-L500). □

Theorem 21.31 (One scalar for every choice and the exact three-factor norm)**.** *For a fixed lift index j, there are scalars c_ϑ independent of the choice index i such that*

$$c_\vartheta \Xi_{\vartheta,i} = c''\mathcal{F}_i, \qquad |c_\vartheta|^2 = \frac{C_j}{\mathrm{heckeValueProd}(j, \vartheta)}. \tag{21.70}$$

Here c'' is the same original scalar of Theorem 21.25; $\mathcal{F}_i$ is the fixed carrier integral; and $C_j \geq 0$ is independent of ϑ. The denominators are strictly positive. This result supplies the scalar identities with $C_j \geq 0$.

Proof. Put $\chi' = \chi_M^j\vartheta$, and let

$$W' = W_j \oplus (-I_j) \oplus I_{3m-j} \oplus I_{6m+1}, \qquad W = I_{3m} \oplus W'.$$

The space W' has rank $n_+ = r + 9m + 1$ and m negative coordinates. In the explicit matching lemma use $V' = S$, $I_A = V''$, $B = 3m$, $L = m$, and

$$e = z_* = \frac{r + 6m + 1}{2} = \frac{k + 7m}{2}.$$

This is a positive integer since r is odd and m is even. The constant rows are the prescribed $2m$-row choices: coordinate rows on W_j and its padding, and the selected short rows on I_{6m+1}. Use the carrier character fixed before Theorem 21.14. The splitting character is $\alpha = \chi' \,\mathrm{carrierChar}(j)$ and its weight is $m - K_\tau$; the second splitting character is the first-factor character ν'', so that $\alpha/\nu'' = \upsilon^m$. All of these equalities are equalities of characters, including their finite tame parts.

Keep the untwisted residue ring $R = \mathcal{O}_E/(\varpi^2)$ fixed, and write $R_\vartheta = \mathcal{O}_E/(\varpi^2 Q_\vartheta)$ and $\pi_\vartheta : R_\vartheta \twoheadrightarrow R$ for the forgetful map. Here these quotients have the local-product interpretation fixed above. For the chosen generator matrix s with row module U, pull back the test of Theorem 21.14 by setting

$$P_{U,\vartheta}(x) = P_U(\pi_\vartheta(x)) = \mathbf{1}_{\{s\pi_\vartheta(x)=0\}}, \qquad x \in M_{n_+,3m}(R_\vartheta). \tag{21.71}$$

Since $\pi_\vartheta(a')$ is invertible for $a' \in \mathrm{GL}_{3m}(R_\vartheta)$, we have $P_{U,\vartheta}(xa') = P_{U,\vartheta}(x)$.

We use the same notation χ' for its residue character on R_ϑ, extended by zero on nonunits. For $x = (x', x'') \in M_{n_+,3m+n_+}(R_\vartheta)$ and $Y = \binom{Y_1}{Y_2} \in M_{3m+n_+,3m}(R_\vartheta)$, the two matched data are

$$\begin{aligned} \Phi_U(x) &= \begin{cases} \chi'(\det x'')\, P_{U,\vartheta}((x'')^{-1}x'), & x'' \in \mathrm{GL}_{n_+}(R_\vartheta), \\ 0, & \text{otherwise}, \end{cases} \\ \Psi_U(Y) &= \chi'(\det Y_1)\, P_{U,\vartheta}(Y_2), \qquad Y_1 \in M_{3m}(R_\vartheta). \end{aligned} \tag{21.72}$$

The block matching formula for χ'^{-1} gives the column datum $\chi'(\det Y_1)P_{U,\vartheta}(-Y_2Y_1^{-1})$ when Y_1 is invertible. Right invariance removes the factor $-Y_1^{-1}$, giving (21.72). When Y_1 is not invertible, both formulas vanish by the zero extension of χ'. This proves the required matching relation with its determinant character intact. In particular

$$\Psi_U(Ya') = \chi'(\det a')\Psi_U(Y) \qquad (a' \in \mathrm{GL}_{3m}(R_\vartheta)).$$

The determinant factor also proves the support condition: if $\Psi_U(Y) \neq 0$, the matrix $(Y_1^{-1} \;\; 0)$ is a left inverse of Y. Thus the full-rank theta estimate applies to the exchanged kernel and gives absolute convergence of its integral.

The matched polynomial (21.59) becomes, entry by entry, the padded minor product on $W_j \oplus (-I_j) \oplus I_{3m-j}$ times the paired short-row determinants on I_{6m+1}. Call this polynomial P_i. It involves only the W' rows, so the exchanged kernel factors along $W = I_{3m} \oplus W'$ as

$$\begin{aligned} K_{\Psi_U}(x;(h',g')) = \theta_{S,I_{3m};\upsilon^m,\chi'}[\chi' \circ \det, 1](x,h') \\ \cdot \theta_{S,W';\upsilon^m,\mathrm{carrierChar}(j)}[P_{U,\vartheta}, P_i](x,g'). \end{aligned} \tag{21.73}$$

Here g' is the enlarged point $(g, 1, 1, h^\flat)$. The characters are those of the exchange: $\alpha/\nu'' = \upsilon^m$ and $\alpha = \chi'\mathrm{carrierChar}(j)$. The finite datum and polynomial both factor, hence the orthogonal-sum formula gives (21.73) with no additional scalar. The block match, full-rank support and kernel factorization.

Sum this exchange over $U(I_{3m})$ against the coefficients $a_{\chi'}$. Lemma 21.30 changes the square kernel on S in (21.73) to ϕ°. The class sum is finite and the product kernel is integrable, so this interchange is legitimate.

Pullback along π_ϑ preserves the finite Schwartz function, so integration of the second factor of (21.73) against ϕ° gives $\mathcal{F}_i$ itself. Thus the double class sum multiplied by

$$c_\vartheta = \gamma_3(\chi' \,\mathrm{carrierChar}(j))\,\gamma_{12}(\chi') \tag{21.74}$$

equals $\mathcal{F}_i$. Both exchange scalars were chosen before the constant rows and the finite datum, so they are independent of i. The integral of the translated datum and its trivial-character hypothesis; The carrier character and its trivial tame part.

We check the exact identity $\Xi_{\vartheta,i} = c''$ times this double sum, with no unrecorded level index. With $w = w_s$ from Lemma 21.17, an invertible A satisfying $w\pi_\vartheta(A) = w$ also satisfies $s\pi_\vartheta(A) = s$, since s consists of the first rows of w_s. Hence $P_{U,\vartheta}(Ax) = P_{U,\vartheta}(x)$.

Let H be the subgroup of such A which also fix the unmarked columns, and let Φ_0 be Φ_U restricted to $x'' \in H$. On this support the block determinant identity and the just-proved left invariance give

$$\Phi_0(x) = \mathrm{columnDatum}_{\chi',w}(x''_{\mathrm{marked}})\,\mathbf{1}_{\{x''_{\mathrm{unmarked}} = I_{\mathrm{unmarked}}\}}\,P_{U,\vartheta}(x').$$

The datum translation associated with a finite integral group element is the *twisted* translation

$$(\tau_A\Phi_0)(x) = \chi'(\det A)\Phi_0(A^{-1}x), \qquad \Phi_U = \sum_{AH \in \mathrm{GL}_{n_+}(R_\vartheta)/H} \tau_A\Phi_0.$$

For invertible x'', exactly one coset contributes; its factor $\chi'(\det A)$ cancels the inverse determinant introduced by $A^{-1}x''$. For noninvertible x'' all terms vanish. The same finite character occurs in the theta transformation law, so the theta kernels in this sum are precisely the right translates at full level. The real polynomial is invariant under the definite receiver with its prescribed Weil weight, as checked in (21.63); this gives rational left invariance of each finite-point kernel. The tower formula for class sums therefore converts the full-level sum of these translates into the class sum at the row-fixing level. The coset count cancels the level index with the original $|\,\mathrm{Aut}\,|^{-1}$ weights. The class-sum tower; The twisted coset decomposition.

Expanding the stacked determinants by the complementary coordinates of Lemma 21.24 and factoring the kernels over the three orthogonal blocks gives exactly the series in Theorem 21.25. These are finite coordinate and class sums, so the factorization does not require an interchange of conditionally convergent series.

Finally the three denominator characters in (21.60) and (21.68) are, respectively,

$$\begin{aligned} (\chi' \,\mathrm{carrierChar}(j))^{-1}\upsilon^{3m+z_*} &= (\nu_*\vartheta)^{-1}, \\ (\chi')^{-1}\upsilon^{m} &= (\nu_+\vartheta)^{-1}, \\ \chi'\upsilon^{m/2} &= (\nu_-\vartheta^{-1})^{-1}, \end{aligned} \tag{21.75}$$

where $\nu_* = \mu_j^0\upsilon^{j+(t-11m-1)/2}$, $\nu_+ = \upsilon^{-m}\chi_M^j$, $\nu_- = \upsilon^{-m/2}\chi_M^{-j}$. Their endpoints are $z_*, m, m/2$. Multiplication gives exactly $\mathrm{heckeValueProd}(j,\vartheta)$, including the conductor factors $q^{z_*^2+m^2+(m/2)^2}$. Thus $C_j = C_3C_{12}$ and (21.70) follows. Strict positivity of the denominator is the nonvanishing clause of the explicit matching lemma, applied three times. This proves the two stated identities and the nonnegativity of C_j without a nonvanishing assumption on the compact-projection coefficient.

Lean, comparison. Lean sums a matrix character over $\mathrm{GL}_{3m}(R)$ to obtain a right-invariant datum. Each P_U is already invariant, and (21.74) gives the scalar for this family. `OneExchange` (L185-L254), `OneExchange` (L309-L353), `ThreeExchanges` (L525-L933). □

21.6 Vanishing of periods of false lifts

For the descent pair, the finite hyperplane sums annihilating the projector and the vanishing of the descent imply a vanishing theorem for block periods. We use it both for the comparison operator and for the test form on four blocks.

Let $O = (W, I_n)$ be such a pair, with $\dim W = r > 0$. Let A be its chosen and twist places, one from each conjugate pair, and write $\Sigma = A \cup \overline{A}$ and $T = \Sigma^c$. Write $a = O.\mathrm{charW}$ for the splitting character on W, $\chi_v = O.\mathrm{tamePart}(v)$ for the prescribed tame character at v, and υ for the character contributed by a hyperbolic plane. The notation

$$\mathcal{D}(S, c; Y) = 0$$

means that

$$\int_{[U(I_n)]} \theta^{c,1}_{S,I_n}[\phi](g,h)Y(h)\,dh = 0 \quad \text{for every } g \text{ and every } \phi \text{ vacuum at } T. \tag{21.76}$$

The measures and the vacuum have the theta-lift normalizations.

21.7 The finite annihilators and the bottom of the tower

Lemma 21.32 (Hyperplane sums annihilating the projector)**.** *For a nonzero row $\lambda \in (k^r)^*$ put*

$$H_\lambda = \{1 + v\lambda \in \mathrm{GL}_r(k) : v \in k^r\}, \qquad A_{\chi,\lambda} = \sum_{h \in H_\lambda} \chi(\det h)^{-1} h. \tag{21.77}$$

If c is a squarefree label and the name x_χ of χ is not a root of c, then $A_{\chi,\lambda} e_c = 0$ in the finite group algebra. Consequently the original projector e_0 is annihilated by each such sum at a chosen place, for the residue characters of the chain. The same conclusion holds in rank one for e_{X-1} whenever $\chi \neq 1$.

Proof. In a basis with first coordinate λ, the group H_λ is the semidirect product $k^{r-1} \rtimes k^\times$, consisting of the matrices

$$\begin{pmatrix} b & 0 \\ u & I_{r-1} \end{pmatrix}.$$

Thus $|H_\lambda|^{-1} A_{\chi,\lambda}$ is the orthogonal projection onto its character $\chi \circ \det$. Its image on a representation is the χ part of the GL_1 factor in the Jacquet module for the parabolic of type $(1, r-1)$. By Lemma 17.71, the representation with label $c = \prod_i P_i$ is induced from the distinct cuspidal representations belonging to the P_i. The finite Mackey calculation in Lemma 17.72 also computes this Jacquet module: a cuspidal block of degree greater than one cannot split off a one-dimensional part, since its proper Jacquet modules vanish. A block of degree one contributes χ precisely when $P_i = X - x_\chi$. Hence the stated character space is zero. Since e_c is central, this proves the identity in the group algebra.

The labels allowed in e_0 have no root x with $x^{q_2} = 1$, where $q_2 = (q_0 d)^{s+6}$. The chosen characters have order dividing q_2; so do the characters after the prescribed twists. The perfect character-name pairing gives $x_\chi^{q_2} = 1$. The group-algebra identity just proved applies to every summand of e_0 and to every tuple of hyperplane sums having at least one nonzero row. A zero row contributes the identity operator. In rank one $e_{X-1} = |k^\times|^{-1} \sum_{h \in k^\times} h$, and the assertion is the nontrivial multiplicative-character sum. These are exactly the annihilators in the block action and the original label restriction.

Lean. `BottomOfTower` (L1267-L1374), `LabelsAndIndexTypes` (L402-L424). □

Lemma 21.33 (All data at the bottom of the tower)**.** *Let l_0 be the archimedean parameter on W and assume O.FitsParams(l_0). Explicitly, write r_+, r_- for the signature of W. At each infinite place τ, the parameter l_0 and the target parameter obey the theta block rule with weights r_- and $\lfloor n/2 \rfloor + O.\mathrm{weight}(\tau)$, where O.weight is the weight of the prescribed unitary character. Within each sign ε, the prescribed number of a basis index c equals the number of indices c' of that sign with $\varepsilon l_{0,\tau}(c') > \varepsilon l_{0,\tau}(c)$. The positive and negative column counts vanish at heights greater than r_+ and r_-, respectively. Assume also $\chi_v \neq 1$ at the twist places. Let Ξ be an algebraic form of the prescribed target weight and level. Suppose its normalized descent to W is zero and that all the chosen hyperplane sums in Lemma 21.32 kill it. For the highest vector ζ_0 of the target Weyl module, put*

$$Y_0(h) = \overline{\mathrm{algCoeff}_{\zeta_0}(\Xi)(h)}.$$

Then $\mathcal{D}(W, a; Y_0) = 0$.

Proof. Fix the finite block datum Ψ_{bl}. The normalized descent is a nonzero scalar times the conjugate of the descent by its specified joint harmonic. Hence the latter descent is zero. We now use the classical compact-target Howe duality theorem in its module form, in addition to the joint-harmonic occurrence and type identification used in (12.2). In the oscillator representation, the isotypic component for a fixed irreducible representation of the compact target is its tensor product with the corresponding irreducible source module; the highest-target-weight line in the joint harmonics is a nonzero cyclic vector for that source module. For the present sufficiently separated parameters this is the same source discrete-series module that occurs in the normalized descent.

Integrating against Y_0 first projects any polynomial Fock vector onto this fixed target matrix-coefficient component. All other target components give zero by compact-group orthogonality. The surviving polynomial component is a finite sum of source enveloping-algebra operators applied to the specified joint harmonic: its Harish–Chandra module is irreducible, so a nonzero cyclic submodule is the whole module. The descent intertwines these operators with right-invariant differentiation on the source. Differentiation under the integral is justified because the target quotient is compact and the kernel is smooth. The descent of every such component is consequently zero. Equivalently it is a real convolution of the original harmonic descent. This is the cyclic-module consequence of compact Howe duality; it does not follow merely from the existence of an unspecified nonzero local lift. We invoke that classical theorem explicitly: R. Howe, *Remarks on classical invariant theory*, Trans. Amer. Math. Soc. **313** (1989), 539–570, doi:10.1090/S0002-9947-1989-0986027-X, §5(a) on compact dual pairs and its highest-weight realization. The Li–Ichino correspondence stated earlier identifies the specified source type and its nonzero joint harmonic within this module. We have proved vanishing for $v_P \otimes \Psi_{\mathrm{bl}}$ for every polynomial P. The exact correspondence of conventions is recorded in the polynomial block descent.

Fix now P and a source point g. The value of the descent is a linear functional $B(\Psi, \Xi)$ of the finite datum, conjugate-linear in Ξ. The datum is unrestricted at A, with the vacuum at its complement in the split polarization. At a chosen place the original block function is the indicator of $X \equiv \binom{I_r}{0}$; at a twist place it is $\chi_v(\det X')$ on integral matrices, extended by zero when the top r by r block X' is singular modulo v.

For a local matrix $z = \binom{z'}{z''}$ call its block ratio good if z' is invertible over the local field and $z''(z')^{-1}$ is integral, with reduction zero at a chosen place. There are two cases for a point z in the finite product of matrix spaces.

If all its block ratios are good, choose a source element A_0 supported at A with local component $(z')^{-1}$, and put

$$\Psi_0(X) = \Psi_{\mathrm{bl}}(X A_0).$$

Then $\Psi_0(z) = 1$. For a sufficiently small scalar lattice ball $C = z + t\,\mathrm{Mat}(\mathcal{O}_A)$ the function Ψ_0 is identically one on C. Average $\mathbf{1}_C$ under the original target level, with its prescribed character twist. The orbit is finite: an open congruence subgroup fixes both the ball and that character. Its mean Ψ_1 has

$$B(\Psi_1, \Xi) = B(\mathbf{1}_C, \Xi).$$

It transforms as Ψ_0, and its support is contained in the support of Ψ_0. This support is a single level orbit. Indeed, for $u = XA_0$ in it, the integral matrix

$$\begin{pmatrix} u' & 0 \\ u'' & I_{n-r} \end{pmatrix} \quad \text{has determinant } \det u'$$

completes the columns of u. At a chosen place its first block is I_r and its lower first block is zero modulo v; at a twist place its first block is invertible and its other blocks meet the triangular level condition. These matrices, and their products, carry any two points of the support to one another. The character transformation law therefore gives $\Psi_1 = c\Psi_0$. Source equivariance makes $B(\Psi_0, \Xi) = 0$, from the block-datum vanishing just proved. Thus $B(\mathbf{1}_C, \Xi) = 0$ in the good case.

If a block ratio is bad at v, there is an integral row $\ell = (\lambda, \lambda')$ with

$$\ell z = 0, \qquad \overline{\lambda} \neq 0, \qquad \overline{\lambda'} = 0 \text{ if } v \text{ is a twist place.} \tag{21.78}$$

To construct it when z' is singular, take a nonzero left-null row of z' and append zeros. Divide by an entry of largest absolute value. If z' is invertible, take a row $((z''(z')^{-1})_{i,*}, -e_i)$ and again divide by its largest first-block entry. Failure of the good condition makes that entry have absolute value at least one at a chosen place and strictly greater than one at a twist place. This proves all parts of (21.78).

For every $h = I + v_0\overline{\lambda} \in H_{\overline{\lambda}}$, lift v_0 integrally and form the row shear

$$U_h = I_n + \begin{pmatrix} v_0 \\ 0 \end{pmatrix} \ell. \tag{21.79}$$

Its determinant is a unit reducing to $\det h$, it fixes z, and it preserves every scalar lattice ball about z. At the split pair it defines an integral unitary element, identity at the other places. The transformation law at the prescribed level on Ξ identifies its action with $\chi_v(\det h)^{-1}\rho_v(h)$ at a chosen place and with $\chi_v(\det h)^{-1}$ at a twist place. The condition on λ' in (21.78) is precisely what gives the required triangular reduction in the twist case. The sum of these actions is zero: use the chosen hyperplane hypothesis, or the nontrivial character sum over $H_{\overline{\lambda}}$ whose determinant surjects onto $k_v^{\times}$. On the other hand equivariance and (21.79) show that every summand pairs with $\mathbf{1}_C$ exactly as Ξ does. Consequently $|H_{\overline{\lambda}}|B(\mathbf{1}_C, \Xi) = 0$. This proves vanishing for every sufficiently small ball in both cases, including balls centered on singular matrices.

For completeness, one can take the nested scalar balls with radii $i!\mathcal{O}_A$. Their radii tend to zero at each place of the finite set A. A locally constant compactly supported Ψ is a finite linear combination of sufficiently small disjoint balls: cover its support by balls on which both local constancy and the preceding vanishing hold, take a finite subcover, and refine to a common radius. Hence $B(\Psi, \Xi) = 0$ for all such Ψ. Polynomial Fock vectors and these finite data span a dense subspace of the allowed Schwartz space. At each fixed source point the descent is a continuous functional of the Schwartz datum, again because the target quotient is compact. It therefore vanishes for every datum in (21.76). The local dichotomy and the finite averaging just proved match annihilating rows, the good-ratio average, and the bad-ratio argument and density.

Lean. `PeriodsOfFalseLifts` (L79-L115), `BottomOfTower` (L266-L385), `BottomOfTower` (L1376-L1490). □

21.8 The bounded tower and its small Hecke roots

Lemma 21.34 (Vanishing at a small root of the Hecke polynomial)**.** *Let F be bounded and spherical at a split place of residue cardinality q on a unitary group of rank M. With the original unnormalized polynomial*

$$P_M(Z) = \sum_{i=0}^{M} (-Z)^i q^{i(i-1)/2} T_i,$$

if $P_M(Z)F = 0$ and $q^{M+1}|Z| \le 1$, then $F = 0$.

Proof. The operator is a sum over the subspaces $U \subset k^M$. The zero subspace contributes a single translation with coefficient one, and the sum of the absolute values of all coefficients is

$$\prod_{i=0}^{M-1} (1 + q^i|Z|) \le \exp\left(|Z| \sum_{i=0}^{M-1} q^i\right) \le e^{1/2} < 2. \tag{21.80}$$

The equality is the finite q-binomial identity; the last bound uses $\sum_{i<M} q^i \le q^M$ and $q \ge 2$. Move the coefficient-one term to the other side and take the supremum norm. The remaining total weight is strictly less than one, so $\|F\|_\infty \le c\|F\|_\infty$ with $c < 1$. Consequently $F = 0$.

Lean. `HyperbolicPlaneStep` (L64). □

Lemma 21.35 (Boundedness after removal of one plane)**.** *Let V be definite, let $W' = W \oplus \mathbb{H}$, and suppose $\mathcal{D}(W, a; Y) = 0$, where Y is a continuous automorphic function on $U(V)$. For every polynomial Fock vector times a finite Schwartz datum equal to the vacuum at T, the descent to W' with character av is bounded on $U(W')(\mathbb{A})$.*

Proof. Choose the standard adapted hyperbolic splitting and denote by θ_0 the zero-term functional along its isotropic line. The zero term contributes zero to the descent for every source point. Indeed an Iwasawa decomposition relative to the line expresses its slice at zero as a nonzero scalar times a Weil translate of a datum on $\mathrm{Hom}(W, V)$. The unipotent part acts trivially on this slice, the Levi part gives the stated lower-rank action and scalar, and the compact part preserves the vacuum at T. Thus (21.76) applies to the slice. The chosen splitting is integral at the finite places, so no new exceptional place is introduced here.

Put $\ell = \theta - \theta_0$ and let $H \ge 1$ be the usual height of the moved source lattice. The polynomial-growth estimate preceding (12.10), applied also to derivatives, bounds this distribution by $CH^a p_M$ for a fixed Schwartz seminorm. For θ_0 the same bound follows by its expression as a compact adelic translation mean of θ. Since V is definite, a compact set of representatives for $[U(V)]$ supplies all the target constants uniformly.

If $H > 2$, choose a nonzero rational lattice vector of length at most $2/H$. It is isotropic: its Hermitian norm is integral at every finite place and is smaller than one at every infinite embedding, so its field norm must be zero. Rational Witt extension carries it to the chosen vector k of the hyperbolic plane. After this rational translation of the source, k is still integral in the moved lattice and has length at most $2/H$. Choose a rational basis d_i of the target lying in its moved lattice with bounded infinite lengths. Such bases are uniform on the compact set of target representatives, by clearing denominators on finitely many compact neighborhoods.

The maps $d_i\langle k, \cdot\rangle$ span an isotropic rational subspace and have moved lengths $O(H^{-1})$. The functional ℓ is invariant under their rational Heisenberg translations. Its adelic translation mean is zero: the mean of θ is precisely θ_0, and averaging θ_0 again leaves it unchanged. This supplies the zero-mean hypothesis in (12.10) without a full-rank restriction on the finite datum. That estimate therefore bounds the kernel difference by

$$C_L H^a (H^{-1})^L p_{M+L+d+2}(\phi_\infty).$$

Choose $L > a$. For $H \le 2$ the original polynomial bound gives the same uniform conclusion. The rational translation may depend on the source point; the full theta descent is unchanged by it, and the zero term has zero descent at that point. Finally Y is bounded on the compact target quotient, so integrating this kernel difference proves boundedness of the descent. For the zero-mean estimate see the bounded hyperbolic step.

Lean. `HyperbolicPlaneStep` (L190-L380). □

Theorem 21.36 (Vanishing of descents in the hyperbolic tower)**.** *Under the hypotheses of Lemma 21.33, suppose also that Ξ is killed by the original Hecke polynomial at a split place $w \notin \Sigma$ and its prescribed lift root X_*, where*

$$|X_*| = q^{1-n+r/2}, \qquad q = \#k_w.$$

If $2r + 2N + 3 < n$, then for every vector ζ in the target Weyl module and every $0 \le j \le N$,

$$\mathcal{D}\left(W \oplus \mathbb{H}^j, a\upsilon^j; \overline{\mathrm{algCoeff}_\zeta(\Xi)}\right) = 0. \tag{21.81}$$

Proof. First take $\zeta = \zeta_0$. This conjugate coefficient is spherical at w and is killed by $P_n(X)$ with $X = \overline{X_*}$: conjugation changes both the coefficient and the Hecke parameter. The base case is Lemma 21.33. For the step from $j-1$ to $j \ge 1$, put

$$M = r + 2j, \quad k = n - M, \quad \chi = a\upsilon^j, \alpha = \chi.\mathrm{frobValue}_M(\overline{w}), \quad \beta = 1.\mathrm{frobValue}_n(\overline{w}).$$

Here $\chi.\mathrm{frobValue}_M(\bar{w})$ is the value at a uniformizer of the rank-M character $\chi.\mathrm{spaceChar}(M)$, multiplied by $q^{M/2}$. The same normalization in rank n defines β. The rank hypothesis gives $k > 0$, and the splitting characters are unramified at w. The theta Hecke-transfer identity, obtained by the same subspace enumeration and q-binomial calculation as (20.4), gives for the new descent F the exact equation

$$\prod_{i=0}^{k-1}(1 - q^i\alpha X)\; P_M(X')F = 0, \qquad X' = q^k\frac{\alpha}{\beta}X. \tag{21.82}$$

The scalar β must be retained: with the original Frobenius convention $|\alpha| = q^{M/2}$ and $|\beta| = q^{n/2}$, so $|X'| = q^{k/2}|X|$. The first factor in (21.82) is nonzero, because

$$q^k|\alpha X| = q^{1-j} \le 1,$$

and each index i in the product is strictly smaller than k. The root for the remaining polynomial satisfies

$$q^{M+1}|X'| = q^{r+j+2-n/2} \le 1. \tag{21.83}$$

Indeed the integer inequality $2r + 2N + 3 < n$ gives $n \ge 2r + 2N + 4$. Lemma 21.35 bounds F for every polynomial finite datum, and Lemma 21.34 kills it. Density and continuity at each source point extend the vanishing to every allowed Schwartz datum. This completes the induction without a growth assumption on an arbitrary putative lift.

Finally real translates of the highest vector span the target irreducible Weyl module. The conjugate coefficient at a translated highest vector is the corresponding real right translate of Y_0. Move that translate to the theta datum in the descent integral. A real translate preserves the vacuum condition at T, so the same vanishing holds for the span and therefore for every ζ. The exact scalar and the two separate root inequalities are those in the Hecke step and its induction; the last highest-vector argument is at the coefficient span.

Lean. `HyperbolicPlaneStep` (L389-L479), `PeriodsOfFalseLifts` (L147-L188). □

21.9 Block periods from hyperbolic descents

Lemma 21.37 (Integral extension of a Hermitian frame)**.** *Over the integers at any finite place of E^+, every unimodular Hermitian lattice for E/E^+ has an orthonormal basis. In a standard unimodular lattice, an isometry between two direct summands extends to an integral unitary automorphism, even if the restrictions of the form to the summands are degenerate. In particular, an adelic orthonormal frame x in I_n, integral at all finite places, has the form $h^{-1}o_1$ with $h \in U(n)(F_\infty)K_0$ and $o_1 = \binom{I_N}{0}$.*

Proof. Let R be the quadratic integer algebra at the finite place. By Lemma 17.67, R is an unramified quadratic integer ring or a split product of two integer rings. In the former case a unimodular Hermitian lattice is determined by its rank [Wal70, p. 433, inert case]. In the latter case its two free modules are in perfect duality, so dual bases give the assertion.

The ring R is semiperfect. Choose $t \in R$ with $t + \bar{t} = 1$. In the unramified case lift a residue element of trace one and divide by its unit trace. In the split case take $t = (1, 0)$. For the ambient Hermitian form h_0 use the quadratic form represented by the sesquilinear form $\beta = th_0$. Its associated Hermitian form $\beta + \beta^* = h_0$ is unimodular. An isometry between the two summands preserves β, so First's extension theorem extends it to the whole quadratic space [Fir15, Corollary 4.9]. The extension preserves h_0.

For $x^*x = I_N$, the left inverse x^* makes the columns a direct summand. Apply the extension to the map carrying x to o_1. The finite integral completions belong to K_0 and ordinary orthonormal completion gives the components at infinity.

Lean, comparison. Lean constructs the integral unitary matrix from left inverses and the Gram identity. Here First's theorem extends the isometry directly. `WittTheorem` (L503-L545). □

Lemma 21.38 (Hyperbolic descents determine the block period)**.** *Let Y be a continuous automorphic function on the definite group $U(I_n)$, right invariant under K_0 supported at T. Suppose $\mathcal{D}(\mathbb{H}^N, v^N; Y) = 0$. For $n = N + l$ and every $h_1 \in U(I_N)(\mathbb{A})$,*

$$\int_{[U(I_l)]} Y(\mathrm{diag}(h_1, h_2))\, dh_2 = 0. \tag{21.84}$$

Proof. Let $g_0 = \mathrm{diag}(h_1, 1)$ and, on the full group, put

$$x(g) = (g_0^{-1}g)^{-1}o_1, \qquad I(g) = \int_{[U(I_l)]} Y(\mathrm{diag}(h_1, h_2)g_0^{-1}g)\, dh_2. \tag{21.85}$$

Both are invariant under the left action of the second block. Let α_0 be the original scale idele and δ its inverse-different normalization. Put

$$\rho(x) = |\mathrm{vac}(\alpha_0 x)|^2, \qquad \psi_u(x) = \psi\left(\delta \sum_{i,j} u_{ij}((1 - e_T)x)_{ij}\right). \tag{21.86}$$

Here e_T is the idempotent on the finite places T and their conjugates, zero at infinity. The weight ρ is nonnegative; it is nonzero precisely on frames integral at the finite places, and is strictly positive at o_1.

Take the nondegenerate Gram-I_N Fourier coefficient of the vanishing descent to $\mathbb{H}^N$. In the mixed model its terms are the rational frames of Gram I_N. Rational Witt extension makes them a single orbit with stabilizer $U(I_l)(E^+)$. For each adelic u, choose the Heisenberg translate of the vacuum in the isotropic direction with translation supported away from T. Its mixed-model slice at $\alpha_0 x$ is

$$c\,\psi_u(x)\rho(x), \qquad c \neq 0. \tag{21.87}$$

This formula follows by multiplying the slice of the vacuum by its Heisenberg character. The factors of $1/2$ in the Heisenberg law cancel against the two conjugate cross terms; the result is exactly δ in (21.86). The Levi element attached to $\alpha_0 h_1$ contributes another nonzero scalar, the product of its character value and its positive determinant modulus. Thus there is no possibly zero scalar being divided out in this Fourier coefficient.

Unfolding that coefficient and then integrating first over $[U(I_l)]$ gives, for every u,

$$\int_{U(I_l)(\mathbb{A})\backslash U(I_n)(\mathbb{A})} \psi_u(x(g))\rho(x(g))I(g)\,d\mu(g) = 0, \tag{21.88}$$

and the displayed integrand is absolutely integrable. To justify unfolding, before orbit grouping take the sum of the absolute values of the mixed-model slices. At a fixed finite datum their finite support restricts the rational frames to a fractional lattice; their archimedean Schwartz decay gives an absolutely summable majorant. The source point in this Fourier coefficient is fixed and the target quotient is compact, so the same majorant works on finitely many compact target representatives. The extra Fourier and Heisenberg phases have modulus one. This proves both Fubini steps, and the integrability assertion in (21.88), before applying cancellation.

We next show that on the support of $\rho(x(g))$ the conjugate of $I(g)$ is a continuous function of the family $u \mapsto \psi_u(x(g))$. It suffices first to work with $h = g_0^{-1}g \in K = U(n)(F_\infty)K_0$, a compact set. If two such h, h' have the same character family, nondegeneracy of the adelic additive pairing says that their moved frames agree away from T, including at infinity. Let k be the component at T of $h^{-1}h'$. It is integral, fixes Y on the right, and satisfies

$$(hk)^{-1}o_1 = (h')^{-1}o_1.$$

Equal moved frames differ by left multiplication by the second block, which leaves the block period invariant. Hence the two periods agree. The continuous map from K to its compact character image is a quotient map, so the conjugate period descends continuously to that image. By Lemma 21.37, every frame of nonzero weight has a representative in K. Equal-frame invariance extends the same description to the whole support of ρ.

On this compact character image, coordinate characters contain the constants, are closed under multiplication and conjugation, and separate points. Their linear span is therefore uniformly dense by Stone–Weierstrass. The case $u = 0$ of the absolute integrability assertion in (21.88) makes integration against $\rho I\,d\mu$ a continuous functional on that space of continuous functions. Approximate the conjugate period by linear combinations of these characters in (21.88). We obtain

$$\int \rho(x(g))|I(g)|^2\,d\mu(g) = 0.$$

The integrand is continuous, nonnegative, and invariant under the second block. The invariant quotient measure has full support. Thus it vanishes everywhere. At $g = g_0$ the frame is o_1, whose weight is nonzero, so $I(g_0) = 0$, exactly (21.84). This proves the Fourier-period step corresponding to the mixed slice and block-period argument.

Lean. `DescentVanishing` (L641-L979). □

Theorem 21.39 (Vanishing of theta periods of false lifts)**.** *Assume the hypotheses of Theorem 21.36, with $n = N + l$ and $2r + 2N + 3 < n$. For every target Weyl vector ζ, every character b, every $g \in U(W)(\mathbb{A})$ and $h_1 \in U(I_N)(\mathbb{A})$, every finite γ supported at Σ, and every datum ϕ on $\mathrm{Hom}(W, I_l)$ vacuum at T,*

$$\int_{[U(I_l)]} \overline{\theta^{a,b}_{W,I_l}[\phi](g,h_2)}\ \mathrm{algCoeff}_\zeta(\Xi)(\mathrm{diag}(h_1,h_2)\gamma)\,dh_2 = 0. \tag{21.89}$$

Proof. Let $Y_1(h) = \overline{\mathrm{algCoeff}_\zeta(\Xi)(h\gamma)}$. Theorem 21.36 still gives vanishing of all its descents to $W \oplus \mathbb{H}^N$: move the finite translation to the datum; it changes no component at T.

Choose a datum ϕ'' on $\mathrm{Hom}(W, I_N)$, vacuum at T, such that

$$c'' = \theta^{a,b^{-1}}_{W,I_N}[\phi''](g, h_1) \neq 0. \tag{21.90}$$

To construct the datum, apply the finite Weil action of the fixed point (g, h_1) to the nonzero finite vacuum. In a rational Schrödinger polarization it is a nonzero locally constant compactly supported function. Finite adelic approximation gives a rational point where it is nonzero. All rational points in its finite support lie in a fixed fractional lattice. Choose a smooth archimedean bump supported near the selected lattice point and missing the others, then apply the inverse real Weil action to that bump. The resulting datum has the original finite vacuum and its theta value is one nonzero summand. Equivalently, real coherent translates of the vacuum separate the nonzero Weil translate. This proves (21.90) without presupposing nonvanishing of a theta lift.

Set $\Phi = \phi'' \boxtimes \phi$ and

$$\vartheta(h) = \theta^{a,1}_{W,I_n}[\Phi](g, h), \qquad Y_2(h) = \vartheta(h) Y_1(h).$$

Products of theta kernels for an orthogonal sum of source spaces show that every descent of Y_2 to $\mathbb{H}^N$ is a descent of Y_1 to $W \oplus \mathbb{H}^N$; it therefore vanishes. The source characters multiply to av^N, and the tensor product datum remains vacuum at T. Also Y_2 is automorphic and right invariant under K_0 supported at T: the vacuum and the splitting character are unramified there, Ξ has that level, and γ commutes with that supported subgroup. Lemma 21.38 therefore applies to Y_2.

Now split the target instead. The same tensor-product identity, with characters b^{-1} and b canceling, gives

$$\vartheta(\mathrm{diag}(h_1, h_2)) = c'' \theta^{a,b}_{W,I_l}[\phi](g, h_2).$$

Divide the zero block period by the nonzero scalar c'' in (21.90), then conjugate the integral. This is (21.89). The two different uses of kernel multiplication and the character cancellation are precisely those at the absorption step.

Lean. `PeriodsOfFalseLifts` (L225-L312). □

21.10 Class sums and the two applications

Corollary 21.40 (Vanishing of the finite class-sum pairing)**.** *Under Theorem 21.39, let $\vartheta(h)$ be a vector on the finite points of $U(I_l)$ whose coordinates, after the real Weyl action, are multiples of conjugate theta kernels with allowed data. Let C be a conjugate-linear map from its Weyl module to that of Ξ, equivariant for the real second block. If a level K fixes*

$$f(h) = \langle C\vartheta(h), \Xi(\mathrm{diag}(h_1, h)\gamma)\rangle,$$

then f is left rationally invariant and $\mathrm{stdClassSum}_K\, f = 0$.

Proof. Expand ϑ in its Weyl basis. Because C is conjugate-linear and the Gram form is conjugate-linear in its first variable, the two conjugations cancel; the expansion is a linear combination of the products in (21.89), with the original coordinate multiples. Equivariance of C makes this combination independent of the real second-block coordinate. Each summand is rationally invariant and has integral zero by Theorem 21.39. The definite quotient is compact, so the finite expansion is integrable. The already fixed quotient/class-sum normalization reads

$$\int_{[U(I_l)]} f(h_f)\, dh = \mathrm{stdLevelIndex}(K)^{-1}\, \mathrm{stdClassSum}_K\, f.$$

The level index is positive. This proves the asserted class sum with its stabilizer weights. At a free principal level the weights are one, giving the ordinary finite sum used in the comparison map. No replacement of that finite sum by a probability average has been made.

Lean, formal context. `LiftVolumeHeight` (L729). □

Corollary 21.41 (Orthogonality to false lifts in the two constructions)**.** *The gluing vector $G_\vartheta u^\circ(y)$ is orthogonal to every false lift. The corresponding four-block test vector for a rank-one descent also annihilates the original false space.*

Proof. A false lift lies in the range of e_0, in the kernel of the Hecke polynomial at the prescribed lift root, and orthogonal to the image of the lift on good forms. The adjointness (12.5), with its nonzero scalar, identifies the last condition with zero normalized descent. If the source has first been restricted by the good projector, use lift equivariance and $e_0\Xi = \Xi$: pairing with an arbitrary lift equals pairing with the lift of its good projection. Hence the same unrestricted normalized descent is zero. Lemma 21.32 supplies the remaining finite annihilators. The prescribed twist characters are nontrivial, and the parameters still satisfy the same block correspondence.

For gluing, freeze the choice of glued lattice and the first target point. Its fixed first Cartan factor defines exactly the conjugate-linear equivariant map C in Corollary 21.40; the second factor is the coordinate theta kernel for $u^\circ(y)$. The translating point is supported at Σ. The common principal level π_M^2 fixes its datum, and its conjugate by that point fixes Ξ, by the original lattice inclusions used to define the gluing map. Therefore each inner class sum in $\langle G_\vartheta u^\circ(y), \Xi\rangle$ is zero. Summing over the finite gluing choices proves orthogonality. The precise rank check is

$$n = 6r, \quad N = r + 9m + 1, \qquad 2r + 2N + 3 < n \iff 18m + 5 < 2r. \tag{21.91}$$

This is among the original large-rank inequalities; it is not replaced by the weaker assertion $N < n$. The good-lift range Z lies in the Hecke kernel and projector range by the original lift equivariance. The gluing vector itself only needs the false orthogonality just proved. For the four-block vector the source has rank one and the first three blocks have ranks $12, 4, 4$. Thus $N = 20$, the last block has rank $2z - 20$, and the required strict inequality is

$$45 < 2z. \tag{21.92}$$

It follows from the original $z > m \geq 10000$. The chosen projector is the product of e_{X-1}, and the chosen characters are nontrivial, so the rank-one part of Lemma 21.32 applies. Freeze the first three coordinates and apply Corollary 21.40 to the last-block kernel with its original Cartan coefficient. Its level and support conditions are the prescribed ones. The resulting zero class sum is exactly its pairing with the false lift. This proves the needed four-block false orthogonality, with the same coefficient vector and finite lattice used in the height comparison.

Lean, formal context. `LiftVolumeHeight` (L729). □

The vanishing of these periods gives the required orthogonality. The lower bound for the height uses the nonzero pairing in Theorem 22.16.

22 Sharp local coefficients and the global pairing

We select twelve theta columns whose coefficients attain the required local valuation bound. A finite Fourier expansion realizes their product as a period and supplies a common nonzero multiplier.

Fix admissible data as in Proposition 5.2, at a sufficiently large scale. Fix a prime p and an embedding $\iota : \mathbb{C} \to \mathbb{C}_p$. Put $R = \mathcal{O}_{\mathbb{C}_p}$, and normalize its valuation v by $v(p) = 1$. All theta series and the parameter y in this subsection belong to the original normalized CM curve. In particular $y(0) = 0$, $y'(0) = 1$, $\theta(0) = 1$, and $\mathcal{A} = R[[y(z)]]$ is the derivative-stable ring already constructed. Put

$$\rho_p = \begin{cases} (p-1)^{-1}, & p \text{ split}, \\ (p^2-1)^{-1}, & p \text{ inert}, \\ (2p-2)^{-1}, & p = d, \end{cases} \qquad \gamma_p = (p-1)^{-1} - \rho_p.$$

For integral linear forms $\mu_i(z) = \sum_{\tau \in \Phi} \mu_{i\tau} z_\tau$ and multiplicities c_i, define

$$\Theta(z) = \prod_i \theta(\mu_i(z))^{c_i}, \qquad \mathcal{A}_\mu = R[[\, y(\mu_i(z)) : i \,]].$$

For $n \geq 1$, we write $\mathrm{val}_n(\nu) \leq r$ if there exist $H_c \in \mathcal{A}_\mu$ and multi-indices α_c, $0 \leq c < n$, such that

$$\sum_{c<n} \alpha_c = \nu, \qquad v\left(\prod_{c<n} [z^{\alpha_c}](H_c \Theta) \right) \leq r. \tag{22.1}$$

Here $[z^\alpha]$ denotes coefficient extraction, and $v(\nu!) = \sum_\tau v(\nu_\tau!)$.

Lemma 22.1 (The one-curve derivative rate is attained)**.** *There is a single $G \in \mathcal{A}$ such that*

$$v(G^{(n)}(0)) = \gamma_p n \qquad (n \geq 0).$$

Every element of $\mathcal{A}$ becomes an integral series in z after $z \mapsto \varpi z$, where $v(\varpi) = \rho_p$.

Proof. Let L be the completion of K at the place induced by ι, with ramification index e and residue cardinality q. The good model of Theorem 18.17 is defined over a finite extension of L: its coefficient a_1 satisfies $j(a_1^4 - 64) = (a_1^4 - 48)^3$, and the CM invariant j is algebraic. Enlarge the extension to define the CM action. In the parameter y, the formal group has logarithm z, since $y'(0) = 1$ in the additive uniformizing coordinate. The action extends coefficientwise to $\mathcal{O}_L$, with a acting by $z \mapsto az$. At a split prime the formal group is ordinary by Deuring's criterion [Lang87, Ch. 13, §4, Theorem 12]. At a nonsplit prime the proof of Lemma 20.29 shows that the reduced uniformizer endomorphism has inseparable degree q. Thus the special fibre has height one as a formal $\mathcal{O}_L$-module in all three cases.

Over the completed maximal unramified extension of the field of definition, its special fibre is isomorphic to that of a Lubin–Tate formal $\mathcal{O}_L$-module. In height one the universal deformation ring is the completed maximal unramified extension of $\mathcal{O}_L$, with no parameters [Dri74, Propositions 1.7(1) and 4.2]. The two lifts are therefore integrally isomorphic. Extend the isomorphism to R and denote its tangent by $u \in R^\times$. By Schneider and Teitelbaum [ST01, Appendix, Theorem (a),(c)], there is an integral character into $\widehat{\mathbb{G}}_m$ whose tangent Ω has valuation

$$v(\Omega) = \frac{1}{p-1} - \frac{1}{e(q-1)} = \gamma_p.$$

Taking formal logarithms of the composite gives $G(z) = \exp(\Omega u z) \in 1 + yR[[y]]$. Its n-th derivative at zero is $(\Omega u)^n$.

At nonsplit primes the proof of Lemma 20.29 gives an integral parameter T with $T(\varpi z)/\varpi \in R[[z]]$ at the stated valuation. This gives the assertion for $R[[T]] = \mathcal{A}$. At split primes Ωu is a unit, so $G - 1$ is itself an integral parameter. The coefficient of z^n in $G(\varpi z)$ is integral since $v(n!) \le n/(p-1)$. In either argument replacing ϖ by a unit multiple preserves integrality.

Lean, comparison. Lean obtains a witness for each order by submultiplicativity in `OneCurve` (L172-L354). The Lubin–Tate character gives all orders at once. □

Lemma 22.2 (Cancellation of theta factors)**.** *Let $s_p = 1, 2, 3$ according as p is split, inert or ramified in K. If some μ_{i_0} has all its coefficients units, then*

$$\mathrm{val}_{s_p}(\nu) \le \gamma_p|\nu| - v(\nu!) \qquad (\nu \in \mathbb{N}^{\Phi}). \tag{U}$$

At a split prime the least valuation is exactly $-v(\nu!)$. If every sz_τ, for one unit s, is an integral linear combination of the forms μ_i, then

$$\mathrm{val}_1(\nu) \le 0. \tag{R}$$

An integral linear substitution among copies of each variable preserves the summed multidegree. The valuation of a resulting coefficient is at least the least valuation of its summands before substitution. Inequalities for n, ν and n', ν' may be added to give one for $n + n', \nu + \nu'$.

Proof. For $a \in \mathcal{O}_K \otimes \mathbb{Z}_p$, denote its component under ι by a_v and its quadratic norm by $N(a)$. With $g = -\log\theta$, Lemma 18.21 gives

$$\exp\{N(a)g(z) - g(a_v z)\} \in \mathcal{A}^\times.$$

Indeed the identity holds on the dense set of nonzero elements of $\mathcal{O}_K$. In the y-coordinate each coefficient is a polynomial in $N(a), a_v$ with coefficients in $\mathbb{C}_p$. The integral valuation ring is closed and the constant term is one.

At a split prime take the idempotent with $a_v = 1$ and $N(a) = 0$. Then $\theta \in \mathcal{A}^\times$, so $\Theta \in \mathcal{A}_\mu^\times$, also at $p = 2$. At an inert prime take the Teichmüller lift of an element of $\mathbb{F}_{p^2}^\times$ of norm -1. Since p is odd, its norm is -1 in $\mathbb{Z}_p$, so $\theta(z)\theta(a_v z)$ is an integral unit. At the ramified prime $p = d$ choose $b, c \in \mathbb{Z}_p$ with $1 + b^2 + c^2 = 0$. The squares and their translates $-1 - \{x^2\}$ in $\mathbb{F}_p$ each have $(p+1)/2$ elements. Their intersection gives a solution with a nonzero coordinate, which lifts by Hensel's lemma. The isogeny identity for b, c gives $\theta(z)\theta(bz)\theta(cz) \in \mathcal{A}^\times$.

Thus the scalars $(\delta_c)_{c<s_p}$, equal respectively to (1), $(1, a_v)$, or $(1, b, c)$, give an integral unit $U(z) = \prod_{c<s_p} \Theta(\delta_c z) \in \mathcal{A}_\mu^\times$. Take G from Lemma 22.1, $H_0 = G(\mu_{i_0}(z))/U(z)$, and $H_c = 1$ for $c > 0$. The product $\prod_{c<s_p}(H_c\Theta)(\delta_c z)$ is $G(\mu_{i_0}(z))$. Its coefficient at ν is

$$\frac{G^{(|\nu|)}(0)}{\prod_\tau \nu_\tau!} \prod_\tau \mu_{i_0\tau}^{\nu_\tau},$$

of valuation $\gamma_p|\nu| - v(\nu!)$. Expanding the substitution gives integral multiples of coefficient products of summed multidegree ν. Some product has valuation at most that of the sum, proving (U). At a split prime the displayed coefficient is attained with one copy. For every $H \in \mathcal{A}_\mu$ the mixed derivatives of $H\Theta$ are integral at zero, since the chain rule and the integrality of the $\mu_{i\tau}$ preserve $\mathcal{A}_\mu$ under differentiation. This gives the reverse bound.

For (R), write $sz_\tau = \sum_i a_{\tau i}\mu_i(z)$ with $a_{\tau i} \in R$. The integral series $G_\tau = \sum_i a_{\tau i} y(\mu_i(z))$ has lowest homogeneous term sz_τ. Therefore $(\prod_\tau G_\tau^{\nu_\tau})\Theta$ has coefficient $s^{|\nu|}$ at its lowest monomial z^ν. This proves (R) with one copy. Concatenation adds two bounds, and extra copies in degree zero contribute $\Theta(0) = 1$.

Lean. `LeastValuation` (L314-L384), `LeastValuation` (L391-L581). Lean uses a four-square representation and six columns. The CM norm identity gives one column at split primes and two at inert primes. □

Lemma 22.3 (Integral linear forms from a tight frame)**.** *Keep the prescribed modulus $M = \pi_M\bar{\pi}_M$, whose prime factors split in K. Let $w_i \in \mathfrak{a}_0$, with positive multiplicities c_i, be a tight frame of this length M:*

$$\sum_i c_i \tau(w_i)\overline{\sigma(w_i)} = M\delta_{\tau\sigma}.$$

With the period factor $P_{\mathrm{loc}} = s_0/u_{\mathrm{CS}}$ of (18.35), put

$$\mu_{i\tau} = \iota(P_{\mathrm{loc}}\,\overline{\tau(w_i)}).$$

These coefficients are integral at every p. At every nonsplit p, the normalized vectors $u_i = w_i/\bar{\pi}_M$ are integral above p, form a resolution of the identity, and

$$\mu_{i\tau} = s\,\iota(\overline{\tau(u_i)}), \qquad s = \iota(P_{\mathrm{loc}}\,\pi_M) \in R^\times.$$

In particular (R) holds. At an inert p, for $F(z) = \Theta(\varpi z)$ there is a factorization

$$F = W\exp(\kappa\Psi), \qquad W \in R[[z]]^\times, \quad W(0) = 1, \quad \Psi \in (z)R[[z]], \quad v(\kappa) = (p-1)^{-1} - 1 < 0, \tag{22.2}$$

such that, modulo the maximal ideal, the derivative $\partial_{z_\sigma}\Psi$ has precisely the monomials

$$z_\tau^{p^e} \quad (e \geq 1 \text{ odd},\ \sigma = \mathrm{pr}_e(\tau)), \qquad \mathrm{pr}_e(\tau) = [-p^e]\tau,$$

and every displayed coefficient is a unit.

Proof. The local period-lattice identity (18.36) says exactly that $P_{\mathrm{loc}}\,\overline{\tau(\mathfrak{a}_0)}$ is integral at every coefficient embedding, proving the first assertion. If p is nonsplit then $p \nmid q_1 M$, since all prime factors of $q_1 M$ split in K. The inclusion $q_1\mathfrak{a}_0 \subseteq \mathcal{O}_E$ therefore makes u_i integral. Division of the frame identity by $\pi_M\bar{\pi}_M = M$ gives $\sum_i c_i\tau(u_i)\overline{\sigma(u_i)} = \delta_{\tau\sigma}$. Lemma 18.25 makes P_{loc} a unit away from q_1, and π_M is a unit at p. The displayed s is therefore a unit. Taking $a_{\tau i} = c_i\iota(\tau(u_i))$ proves the integral span identity used in (R).

For an inert prime apply Lemma 20.29 to the untranslated theta function with prime-to-p torsion order one, then substitute its actual forms $s\,\iota(\overline{\tau(u_i)})z_\tau$, and take the product with the multiplicities c_i. The unit N_p in that lemma can be incorporated into the common unit $\beta = N_p s$. Its exponential scalar has the valuation in (22.2); its integral factor has constant term one. The last clause of Lemma 20.30, applied to this resolution of the identity and unit β, proves exactly the stated derivative support and unit coefficients. Thus this use requires no unproved identification with a different formal normal form.

Lean. `Cases` (L129-L275). □

Lemma 22.4 (A selected coupling has a unit leading coefficient)**.** *Assume p is inert and F is as in Lemma 22.3. Choose a set Φ_0 of places and positive odd integers e_τ such that, for $\tau, \tau_1 \in \Phi_0$ and every odd integer e' with $1 \leq e' \leq e_\tau$,*

$$\mathrm{pr}_{e'}(\tau_1) = \mathrm{pr}_{e_\tau}(\tau) \implies (\tau_1, e') = (\tau, e_\tau). \tag{C}$$

For integers $j_\tau \geq 0$, put

$$\nu' = \sum_{\tau\in\Phi_0} p^{e_\tau} j_\tau[\tau], \qquad \nu'' = \sum_{\tau\in\Phi_0} j_\tau[\mathrm{pr}_{e_\tau}(\tau)], \quad \nu = \nu' + \nu'', \quad J = \sum_{\tau\in\Phi_0} j_\tau.$$

Then

$$\mathrm{val}_{s_p}(\nu) \leq -J, \tag{22.3}$$

$$\mathrm{val}_{s_p}(\nu) \leq \gamma_p|\nu| - J - v(\nu'!) - v(\nu''!). \tag{22.4}$$

Proof. Use the s_p scalars in Lemma 22.2. The series $U(z)=\prod_{c<s_p}F(\delta_c z)$ is integral with constant term one: the unscaled product is in $\mathcal{A}_\mu$, and Lemma 22.1 makes this ring integral after the scaling, because all $\mu_{i\tau}$ are integral. Consider two independent copies of the variables and

$$G(z',z'')=F(z'+z'')\prod_{1\le c<s_p}F(\delta_c z').$$

Its logarithmic derivative in the second copy has the form

$$\partial_{z''_\sigma}G=(\kappa A_\sigma+B_\sigma)G,\quad A_\sigma=\partial_\sigma\Psi(z'+z''),\ B_\sigma\in R[[z',z'']].$$

For any integral series Q, the product rule gives

$$\partial_{z''_\sigma}(QG)=\kappa\{QA_\sigma+\kappa^{-1}(\partial_{z''_\sigma}Q+QB_\sigma)\}G.$$

The bracket is integral and reduces to QA_σ, because κ^{-1} belongs to the maximal ideal. Iteration yields

$$\partial^{\nu''}_{z''}G=\kappa^J QG,\qquad Q(z',0)\equiv\prod_{\tau\in\Phi_0}(\partial_{\mathrm{pr}_{e_\tau}(\tau)}\Psi(z'))^{j_\tau}\pmod{\mathfrak{m}_R}.$$

Set $z'_\tau=0$ outside Φ_0. By (C) and the derivative support of Lemma 22.3, each factor on the right has a unique lowest-degree monomial: a unit times $z_\tau^{p^{e_\tau}}$. Their product consequently has a unique lowest-degree monomial, a unit times $z^{\nu'}$. Since $U(0)=1$, the coefficient $[z^{\nu'}](Q(z,0)U(z))$ is a unit. Taking that coefficient in the preceding derivative identity gives the exact valuation

$$v([z'^{\nu'}z''^{\nu''}]G)=Jv(\kappa)-v(\nu''!).$$

Undoing the scaling by ϖ subtracts $\rho_p|\nu|$. The substitution among the s_p unscaled copies is integral, so the coefficient comparison of Lemma 22.2 proves

$$\mathrm{val}_{s_p}(\nu)\le\frac{J}{p-1}-J-\rho_p|\nu|-v(\nu''!).\tag{22.5}$$

As $|\nu|\ge(p+1)J$ and $\rho_p(p+1)=(p-1)^{-1}$, this implies (22.3). Finally $v(\nu'!)\le|\nu'|/(p-1)$ and $|\nu|=|\nu'|+J$ turn (22.5) into (22.4).

Lean. `IsCouplingFamily` (L52-L59), `Couplings` (L108-L168), `Couplings` (L201-L408), `Couplings` (L506-L619). The derivative argument uses one moving column and s_p-1 fixed columns. □

The sparse weight function satisfies $\kappa_\tau=b$ on Φ_b, and $\kappa_\tau=a/x_1$ elsewhere. Put

$$n^\sharp=\lceil 6b/k\rceil,\quad l_\tau^-=(l_\tau-n^\sharp)_+,\quad B_\tau=(l_\tau^-+1)(l_\tau^-+2)\cdots(l_\tau^-+n^\sharp).$$

The degree range in the next statements is exactly

$$\kappa_\tau\le l_\tau\le\kappa_\tau+x^{42}.\tag{22.6}$$

Let I_p be the positive odd integers for inert p, all positive integers for $p=d$, and the empty set for split p. Set $\nu_d=\frac12$ and $\nu_p=1$ for $p\neq d$. Since $a=xb$, a positive summand in the local loss must start at weight b and end at weight bx/x_1. Thus

$$\begin{aligned}S_p&=\nu_p\sum_{\tau\in\Phi}\sum_{e\in I_p}\left(\kappa_\tau p^{-e}-\kappa_{\mathrm{pr}_e(\tau)}\right)_+\\&=\nu_p b\sum_{\substack{e\in I_p\\ p^e<x_1/x}}(p^{-e}-x/x_1)\,\#\{\tau\in\Phi_b:\mathrm{pr}_e(\tau)\notin\Phi_b\}.\end{aligned}$$

In particular $S_p=0$ for $p\ge x_1/x$.

Lemma 22.5 (Near pairs and their factorial valuations)**.** *Suppose $p \le k$ is nonsplit. Call (τ, e) near when $e \in I_p$, $2e < \ell_0$, and both τ and $\mathrm{pr}_e(\tau)$ belong to Φ_b. Every start has at most one near exponent, the ends of distinct near pairs are distinct, and no end is a start. There are no near pairs when $p = d$.*

Let Φ_0 be their set of starts, e_τ their exponents, and put $j_\tau = \lfloor l_\tau / p^{e_\tau} \rfloor$ on Φ_0, $j_\tau = 0$ elsewhere. They satisfy condition (C) of Lemma 22.4, and their degrees obey $\nu' + \nu'' \le l$. Moreover

$$\gamma_p |l| - \sum_\tau j_\tau + 3f(\log b + 2) \le S_p + \sum_\tau v(B_\tau). \tag{22.7}$$

Proof. The already proved sparse-place condition (5.2) excludes three places $\tau, (-p)^i\tau, (-p)^j\tau$ in Φ_b whenever $0 < i < j < \ell_0$, including repeated places. All nonsplit primes are prime to q_1. For $p = d$, a near pair would have the same class at both ends modulo q_0, forcing $d^e \equiv -1 \pmod{q_0}$. But $d \equiv q_0/2 - 1$ and $4 \mid q_0$, so its powers are 1 or $q_0/2 - 1$, neither of which is -1. Thus every near pair is inert, with e odd and $\mathrm{pr}_e(\tau) = (-p)^e\tau$.

Two distinct near exponents with the same start would give the forbidden triple with those two exponents. For pairs with the same end and $e < e'$, their starts and common end give the triple with exponents $e' - e, e'$, based at the second start. If an end were another start, the successive pair would give the triple with exponents $e, e + e'$; here $e + e' < \ell_0$. These prove the three assertions. If an odd e' with $1 \le e' \le e_\tau$ from a selected start has the same end, it too is near; end uniqueness proves (C). At a start the degree is $p^{e_\tau} j_\tau \le l_\tau$ and there is no incoming contribution. At an end the only contribution is $j_\tau \le 2b/p^{e_\tau} \le b \le l_{\mathrm{end}}$. This uses $l_\tau \le 2b$, a consequence of (22.6) and the scale hierarchy.

Set

$$R_\tau = \sum_{e \in I_p} \min\left(\frac{\kappa_\tau}{p^e}, \kappa_{\mathrm{pr}_e(\tau)} \right).$$

The geometric identity $\gamma_p = \nu_p \sum_{e \in I_p} p^{-e}$ and $u - (u - v)_+ = \min(u, v)$ give

$$\gamma_p \sum_\tau \kappa_\tau - S_p = \nu_p \sum_\tau R_\tau. \tag{22.8}$$

Cut R_τ at $n = \lceil \ell_0/2 \rceil$. The possible near term is at most $j_\tau + 1$. Every other term below the cut is at most a/x_1, because at least one endpoint has that weight. The tail is at most $2b/p^n \le 2b/(kp)$: indeed $n - 1 \ge s$, $k \le 2^s$, and $p \ge 2$. Consequently

$$R_\tau \le j_\tau + 1 + \ell_0 a/x_1 + 2b/(kp). \tag{22.9}$$

To absorb the error terms we use the following inequalities.

$$\begin{aligned} x^{42} &\le n^\sharp, \qquad & l_\tau &\le 2b, \\ 3(\log b + 2) + x^{42} + 1 &\le a/x_1, & & \\ (\ell_0 + 1)a/x_1 &\le b/k^2, \qquad & 1 &\le (\ell_0 + 1)a/x_1. \end{aligned}$$

They follow directly from (5.3)–(5.4), $k = O(x^5)$, $a = xb$, and $\ell_0 = O(\log x)$. For example the third inequality reduces to $x_1 \ge (\ell_0 + 1)xk^2$; the second follows because b/x_1 exceeds every fixed power of x. They hold simultaneously for every $p \le k$. Since $n^\sharp!$ divides B_τ, Legendre gives

$$\frac{n^\sharp}{p} - 1 \le v(n^\sharp!) \le v(B_\tau).$$

Using $n^\sharp \geq 6b/k$, the preceding margins give

$$3(\log b + 2) + x^{42} + 1 + \ell_0 a/x_1 + 2b/(kp) \leq 3b/(kp)$$
$$\leq n^\sharp/p - 1 \leq v(B_\tau). \tag{22.10}$$

In the middle inequality $b/(kp) \geq 1$, which follows from $1 \leq (\ell_0 + 1)a/x_1 \leq b/(kp)$. Finally $\gamma_p \leq 1$, $\nu_p \leq 1$, and (22.6) imply

$$\gamma_p|l| - S_p \leq \sum_\tau (R_\tau + x^{42}).$$

Subtract $\sum_\tau j_\tau$, add $3f(\log b + 2)$, and apply (22.9) and (22.10). This proves (22.7).

Lean. `NearPairs` (L61-L198), `NearPairs` (L334-L468). □

Theorem 22.6 (The twelve-column least-valuation bound)**.** *Use the prescribed tight frame of length M, its multiplicities, and its period-normalized forms in Lemma 22.3. Suppose that at the chosen prime one frame member has all its coefficients units. For every degree in (22.6),*

$$\mathrm{val}_{12}(l) \leq S_p + \sum_\tau v(B_\tau) - v(l!). \tag{22.11}$$

The witnesses use at most four columns of positive degree. At split primes one column suffices and the loss sum can be omitted.

Proof. If p splits then $\gamma_p = S_p = 0$. Bound (U) gives $\mathrm{val}_1(l) \leq -v(l!)$. For nonsplit $p \leq k$, choose the pairs of Lemma 22.5 and put $\eta = l - \nu' - \nu'' \geq 0$. For nonempty pairs use (22.4), then (U) on η. For empty pairs use (U) alone. Nonempty pairs occur only at inert primes, where $2s_p = 4$. Empty pairs require at most $s_p \leq 3$ columns. Thus

$$\mathrm{val}_{12}(l) \leq \gamma_p|l| - J - v(\nu'!) - v(\nu''!) - v(\eta!).$$

All nonsplit primes are at least 13. For $p \geq 3$ and $n \leq 2b$, Legendre's digit formula gives

$$\frac{n}{p-1} - (\log b + 2) \leq v(n!) \leq \frac{n}{p-1}.$$

Indeed the number of base-p digits of n is at most $\lceil \log b \rceil + 1$; divide its digit sum by $p - 1$. Apply these bounds to $l_\tau = \nu'_\tau + \nu''_\tau + \eta_\tau$ to get

$$v(l!) \leq v(\nu'!) + v(\nu''!) + v(\eta!) + 3f(\log b + 2).$$

Now (22.7) gives the claimed bound, including the ramified prime $p = d$, where the pair set is empty.

For nonsplit $p > k$, one has $d \leq k$, so p is inert. Take $e_\tau = 1$ at every place. The only exponent allowed in (C) is then $e' = 1$; injectivity of the permutation pr_1 gives $\tau_1 = \tau$. Set

$$i_\tau = v_p(l_\tau^-!), \qquad j_\tau = \min(i_\tau, l_{\mathrm{pr}_1(\tau)}).$$

Legendre's formula gives $i_\tau \leq l_\tau^-/(p-1) \leq 2b/k$ and $pi_\tau \leq l_\tau^- + 2b/k$. If $l_\tau < n^\sharp$ there is no outgoing degree, and the incoming degree is at most l_τ. Otherwise their sum is at most $l_\tau - n^\sharp + 4b/k \leq l_\tau$. Thus $\eta = l - \nu' - \nu'' \geq 0$. Applying (22.3) and (R) to η gives $\mathrm{val}_{12}(l) \leq -\sum j_\tau$ using $s_p + 1 = 3$ columns.

Also $pi_\tau \le \kappa_\tau$. For $l_\tau^- = 0$ this is immediate. For $l_\tau^- > 0$ use $l_\tau \le \kappa_\tau + x^{42}$ and $n^\sharp \ge x^{42} + 2b/k$, since b/k eventually exceeds every fixed power of x on the chosen scales. The $e = 1$ summands of S_p therefore bound $\sum_\tau (i_\tau - j_\tau)$, since $l_{\mathrm{pr}_1(\tau)} \ge \kappa_{\mathrm{pr}_1(\tau)}$. Finally $l_\tau! \mid l_\tau^-! B_\tau$, so

$$v(l!) - \sum_\tau j_\tau \le \sum_\tau \{v(B_\tau) + i_\tau - j_\tau\} \le \sum_\tau v(B_\tau) + S_p.$$

Degree-zero columns complete each witness to twelve.

Lean. `Cases` (L308-L435), `Cases` (L443-L686). At a large inert prime Lean starts from $\lfloor l_\tau^-/p \rfloor$. Using the full factorial valuation at the same degree absorbs the loss $v(B_\tau)$ once at every prime. □

22.1 The common rank-one period and the empty-row comparison

Let c_{vac} be the vacuum scalar, $j_0 = \mathrm{jetScale}(s)$ the jet scale, and $\Omega = \mathrm{bigOmega}(s)$ the period, with the normalizations of Theorems 19.4 and 19.6. For a critical pair $p = (\nu, z)$ set $n = 2z \ge 4$, $\ell_\tau = c_p(\tau) - z$, and $J = \sum_\tau \ell_\tau$. Write $\widetilde{\phi}_p$ for its unscaled, untwisted rank-one theta lift and

$$\phi_p = c_{\mathrm{vac}}^n (\pi_M j_0)^J \widetilde{\phi}_p, \qquad s_{\infty,n}(a) = \log \frac{(n+a-1)!}{(n-1)!} + a \log(M/\Omega). \tag{22.12}$$

The generator $\pi_M = \overline{\mathrm{piMBar}}$ of the chosen modulus satisfies $\pi_M \bar{\pi}_M = M$. The factor multiplying each derivative is $\pi_M j_0$.

Lemma 22.7 (The common conjugate period for rank-one lifts)**.** *Fix the even rank n and the old chosen places S. For every $\sigma \in \mathrm{Aut}(\mathbb{C}/K)$ there is a real number $C_{\sigma,n,S}$ independent of p such that*

$$\log |\sigma(\|\phi_p\|^2)| = C_{\sigma,n,S} + \sum_\tau s_{\infty,n}(\ell_\tau) + r_{\sigma,p}, \qquad |r_{\sigma,p}| \le \log \zeta_E(2). \tag{22.13}$$

Every norm in this formula is nonzero.

Proof. We give the semilinear step, including the independence of the constant from the weights and the character. Lemma 19.9 supplies one map $\mathcal{L}_\sigma$ on the finite oscillator space of rank n, a translation scale b_σ, and a Taylor scale t_σ with $|t_\sigma|^2 = \Omega^{-1}$. The set S is disjoint from its conjugate. Factor $b_\sigma = u_\sigma a_\sigma$, with u_σ unitary, a_σ an integral finite unit, and $a_\sigma = 1$ at S: at a pair meeting S take u_σ equal to b_σ there and its inverse conjugate at the opposite place. The remaining component of a_σ is a unit since $b_\sigma \bar{b}_\sigma$ is one in valuation. At other pairs use the same elementary valuation factorization.

The map $\omega(u_\sigma)^{-1}\mathcal{L}_\sigma$ fixes the vacuum line. On each finite Schrödinger function supported at S it therefore has the form

$$\mathcal{L}_\sigma(\Psi \otimes v) = a'_\sigma \omega(u_\sigma)((\sigma \circ \Psi) \otimes v), \qquad a'_\sigma \ne 0, \tag{22.14}$$

where a'_σ is independent of Ψ. Indeed finite translates of the integral box span all such functions; their indicator supports are fixed because $a_\sigma = 1$ at S, and their character phases are conjugated by the additive-character law. Thus (22.14) follows first for box translates and then by finite Fourier expansion. Injectivity gives $a'_\sigma \ne 0$.

Set $\sigma^\# = {}^-\sigma^-$. Both automorphisms induce the same permutation e_σ of the embeddings of E, since $E/\mathbb{Q}$ is abelian. Conjugate the finite values of the unitary character of p and permute its infinite weights by e_σ. This gives a unitary automorphic character: the product law on rational points is the conjugate of the original product law, and local continuity follows from its finite-order tame values. Call the resulting lift datum Y. The assertion that this conjugate character is unitary

can also be checked without continuity of σ. The old $U(1)$ class group is finite, so a positive power of each finite idele is a rational norm-one point times an old compact unit. The character of the compact unit has finite order. After taking another positive power, its value is a product of integer powers of the CM embeddings of that rational point. Every such embedding has modulus one, as do all its Galois conjugates. Thus every conjugate of the original finite character value has modulus one. The same Y works for $\sigma^{\#}$, because conjugation and inversion agree on its unitary values.

The old block datum is a zero-one function, and is unchanged by either automorphism. Applying (22.14), moving $\mathcal{L}_\sigma$ through the finite Weil action, and translating the source $U(1)$ class sum by u_σ^{-1} gives, for $A_p = j_0^J \widetilde{\phi}_p$,

$$\sigma(A_p(h)_i) = t_\sigma^J a'_\sigma \xi_{\sigma,p} m_\sigma(h)\, \widetilde{\phi}_Y(h)_{i\circ e_\sigma^{-1}}, \quad |\xi_{\sigma,p}| = 1. \tag{22.15}$$

Here $m_\sigma(h) = \sigma(\Delta(h))/\Delta(h)$ is the finite Weil multiplier ratio. The ratio for the source $U(1)$ cancels the conjugation of the rank-twist v^z in its splitting character; its remaining factor is the unitary character value $\xi_{\sigma,p}$ at u_σ^{-1}. This verifies that the only factor in (22.15) that can depend on p, apart from t_σ^J and Y, has absolute value one.

The finite Gaussian sum formula gives $\Delta(h)\overline{\Delta(h)} \in \mathbb{Q}_{>0}$. Consequently

$$m_\sigma(h)\overline{m_{\sigma^{\#}}(h)} = \frac{\sigma(\Delta(h)\overline{\Delta(h)})}{\Delta(h)\overline{\Delta(h)}} = 1. \tag{22.16}$$

In the finite class-sum pairing the Gram matrices are rational and the embedding permutation merely reorders their factors. Apply (22.15) for σ and $\sigma^{\#}$ to the two entries of that pairing, and use (22.16). Taking absolute values gives

$$|\sigma\langle A_p, A_p\rangle| = a_{\sigma,n,S}\Omega^{-J}\|\widetilde{\phi}_Y\|^2, \qquad a_{\sigma,n,S} = |a'_\sigma a'_{\sigma^{\#}}| > 0. \tag{22.17}$$

No automorphism has been passed through an infinite analytic integral: all pairings and source sums here are finite class sums.

Proposition 18.35 gives

$$\|\widetilde{\phi}_Y\|^2 = C_{n,S}\Big(\prod_\tau \ell_\tau!\Big) d_Y\, L^S(z, \nu_Y^{-1}), \qquad C_{n,S} > 0, \tag{22.18}$$

The degree-ℓ_τ factor of the receiver is a symmetric power of the n-dimensional standard representation, up to duality and a determinant twist. Counting its monomials gives $d_Y = \prod_\tau \binom{n+\ell_\tau-1}{\ell_\tau}$. Multiplication by $\prod_\tau \ell_\tau!$ gives the factorials in $s_{\infty,n}$. The remaining error is

$$r_{\sigma,p} = \log|L^S(z,\nu_Y^{-1})|, \qquad |r_{\sigma,p}| \le \log\zeta_E(2) \le 2f\log\zeta(2). \tag{22.19}$$

Indeed the local logarithms are bounded by $\sum_{a\ge1}(N\mathfrak{p})^{-2a}/a$ since $z \ge 2$ and ν_Y is unitary.

Finally (22.12) gives exactly

$$\|\phi_p\|^2 = |c_{\mathrm{vac}}^n|^2 M^J\langle A_p, A_p\rangle.$$

Combining with (22.17)–(22.18) proves (22.13), with

$$C_{\sigma,n,S} = \log|\sigma(|c_{\mathrm{vac}}^n|^2)| + \log a_{\sigma,n,S} + \log C_{n,S}.$$

The nonzero statements follow from the same formulas.

Lean, comparison. Lean keeps the receiver dimension in the error; counting symmetric-power monomials gives its exact contribution in every degree. `GaloisLaw` (L308-L410), `CharacterLift` (L585-L805), `AlgebraicPartSize` (L175-L385). □

Lemma 22.8 (Integral empty-row values with a uniform conjugate bound)**.** *For any even $n = 2z$ with $4 \le n \le t$ and every tame weight-zero character ψ supported at the chosen places, the pair $q = (\psi\upsilon^{-z}, z)$ has empty row and is admissible. Let*

$$D_q = M^{4fz}, \qquad A_q(\vartheta) = g_* D_q^2 \frac{\|\phi_{q,\vartheta}\|^2}{(\operatorname{Nm} Q_\vartheta)^{2z-1}}. \tag{22.20}$$

It is nonzero and integral at every embedding above every prime not dividing $\operatorname{Nm} Q_\vartheta$*. In particular $A_q(1)$ is integral at every finite embedding, with no excluded prime. For the trivial twist, all complex automorphisms satisfy*

$$|\sigma A_q(1)| \le e^{x^{34}}. \tag{22.21}$$

The estimate is uniform over ψ and z in this range.

Proof. Multiplication by υ^{-z} gives weight $-z$ and leaves the old tame part. The interval requirements for a critical pair follow from $2 \le z \le t/2$ and $t \le 4b$. Its normalized lift has a finite expansion into point kernels at the chosen-level classes of $U(1)$ and the residue units at the twist places. There is no stabilizer denominator in the source: the chosen level is torsion-free. The point data have values zero or one before multiplication by c_{vac}^n. Their finite Fourier expansion is integral after multiplication by $M^{2fn} = D_q$. Theorem 18.29 applies in degree zero, so $D_q\phi_{q,\vartheta}$ is integral away from Q_ϑ in the lattice of forms. In its inner product, g_* clears every receiver stabilizer. The denominator $(\operatorname{Nm} Q_\vartheta)^{n-1}$ is a unit at the primes under consideration. This proves the asserted integrality, at every embedding and also its composition with complex conjugation. Nonvanishing is Proposition 18.35.

For an untwisted kernel the empty rows give the constant polynomial 1. Its jet is the constant Taylor coefficient, namely the value at zero. Lemma 19.9 transports that coefficient under σ, and Lemma 19.11 bounds the value of each translated integral vacuum by $e^{C_{\mathrm{par}}(fn+1)}$. For an embedding over the conjugate place of K, first compose with ordinary complex conjugation, which preserves absolute values. The finite datum takes values zero or one on a residue matrix space of cardinality M^{fn}, so the sum of the absolute values of its normalized Fourier coefficients is at most M^{fn}. Each conjugated kernel value is thus at most

$$C_n = M^{fn} \exp\{C_{\mathrm{par}}(fn+1)\}. \tag{22.22}$$

There are at most $e^{x^{30}}$ chosen-level classes of the source and of the receiver, by (11.2). Thus the conjugate coordinate norm of the lift is at most $e^{x^{30}} C_n$, and its class-sum norm is at most $e^{3x^{30}} C_n^2$. For a conjugate of its original Hermitian norm use the two automorphisms $\sigma, \sigma^{\#}$ and Cauchy–Schwarz; the same bound applies. Admissibility gives $C_{\mathrm{par}} \le \log x$, $f \le x^{13}$, $n \le 15x^5$, and $\log M \le 22x^{1/100}$. Thus $\log C_n = fn \log M + C_{\mathrm{par}}(fn+1) \le x^{20}$. Together with the admissible stabilizer bound $\log(g_* D_q^2) \le x^{33}$ and the class-count bound $3x^{30}$, this yields

$$\log(g_* D_q^2) + 3x^{30} + 2\log C_n \le x^{34}.$$

This proves (22.21) throughout the stated range.

Lean, comparison. For empty rows, evaluating the conjugated vacuum at zero suffices. Lean uses its general polynomial-jet estimate. `AlgebraicPartIntegrality` (L158-L243), `AlgebraicPartSize` (L45-L439). □

Corollary 22.9 (Comparison of norms for empty rows)**.** *For a main pair p the constant family $\|\phi_{p,1}\|^2$ has height at most*

$$\mathcal{B} - \sum_{\tau} s_{\infty,n}(\kappa_\tau). \tag{22.23}$$

This is the untwisted norm bound used in (25.31)*.*

Proof. Choose $q = (\upsilon^{-z}, z)$ of the same rank as p and put $H = A_q(1)/\|\phi_{p,1}\|^2 \neq 0$. For every finite embedding, $H\|\phi_{p,1}\|^2 = A_q(1)$ is integral by Lemma 22.8. For σ fixing K, the common constant cancels in (22.13); hence

$$\log|\sigma H| \le \log(g_* D_q^2) - \sum_\tau s_{\infty,n}(\ell_\tau) + 2\log\zeta_E(2).$$

The last term is $O_Q(q_1^{-1})$ for the constructed fields. Since $M/\Omega \ge 1$ and $\ell_\tau \ge \kappa_\tau$, each term is at least $s_{\infty,n}(\kappa_\tau) \ge s_\infty(\kappa_\tau)$. Finally $\log(g_* D_q^2) + 2\log\zeta_E(2) \le x^{33} + 8f \le \mathcal{B}$. For an arbitrary automorphism compose on its output with ordinary complex conjugation if necessary to fix K; absolute values are unchanged. The single multiplier H, at all finite and infinite embeddings, is precisely the height witness for (22.23).

Lean, comparison. Lean uses the lower bound $d_Y \ge 1$ for the main row and $d_Y = 1$ for the empty row. `AlgebraicPartSize` (L460-L522). □

22.2 Realizing the CM coefficients by a residue-class indicator

Lemma 22.10 (A single frame with a unit member at every prime)**.** *There is a finite family $w_i \in \mathfrak{a}_0$ with integer multiplicities $c_i \ge 6fb$ such that*

$$\sum_i c_i\, \tau(w_i)\overline{\sigma(w_i)} = M\delta_{\tau\sigma}. \tag{22.24}$$

For every p and every embedding $\iota : \mathbb{C} \to \mathbb{C}_p$, some member w_{i_0} makes $\iota((s_0/u_{\mathrm{CS}})\overline{\tau(w_{i_0})})$ a unit for every τ. The frame itself is fixed before p or ι is chosen.

Proof. Put $q = q_1$. Use the frame of scalar h from Lemma 18.26, and as a second frame take $1, \zeta_q, \dots, \zeta_q^{q-1}$, of scalar q by geometric summation. The scalar h is bounded for fixed Q, d and $(h, q) = 1$.

Put $R = \lceil 6fb \rceil$. The parameter choices give $q_1 h + R(q_1 + h) \le M$, since $M \ge b^{20}$ and q_1, h, f have polynomial growth in x. As $(h, q_1) = 1$, every integer at least $q_1 h$ is $aq_1 + b'h$ with $a, b' \ge 0$: choose b' in $\{0, \dots, q_1 - 1\}$ with the required residue modulo q_1, and the remaining nonnegative multiple is aq_1. Apply this to $M - R(q_1 + h)$, and give the two frames constant multiplicities $a + R, b' + R$. This proves (22.24) with the required lower bounds.

At $p \mid q$, the distinguished member of the first frame generates $\mathfrak{a}_0$ at all places over p. At $p \nmid q$ the inverse different is a unit and 1 generates $\mathfrak{a}_0$ there. In either case the same member works for all τ by the period formula (18.36).

Lean, comparison. Lean estimates the trace-dual vectors at conductor q. Here the local generator and a bounded scalar descend to the fixed base construction of Lemma 18.26. `ShortVectors` (L440-L809). □

Lemma 22.11 (Integral CM sections and finite Fourier attainment)**.** *Fix the frame in Lemma 22.10 and put*

$$\mu_i(Z) = (s_0/u_{\mathrm{CS}}) \sum_\tau \overline{\tau(w_i)} Z_\tau, \qquad \Theta(Z) = \prod_i \theta(\mu_i(Z))^{c_i}.$$

Fix a prime p, an embedding $\iota : \mathbb{C} \hookrightarrow \mathbb{C}_p$, and a multidegree $l = (l_\tau)$ with $3|l| \le c_i$ for every i. Suppose that, in twelve independent copies, some product of coefficients of series $H_b\Theta$, where H_b is integral in the $y(\mu_i)$, has total multidegree l and valuation at most R_p at ι. Let v be the rank-twelve vacuum, put $v_{12}^\circ = c_{\mathrm{vac}}^{12} v$, and use the finite adelic embedding finVec *of (IV). For a residue class $x_0 \bmod \pi_M$, let v_{x_0} be the theta datum with indicator $\mathbf{1}_{x_0}$ at the places dividing π_M and the vacuum at all other places, including infinity. There are a multi-index $\alpha = (\alpha_{\tau b})$ and a residue class x_0 with $\sum_b \alpha_{\tau b} = l_\tau$ for every τ such that, for every finite adelic point y,*

$$p^{-R_p}\, |\iota(M^{12f} v_{12}^\circ(\mathrm{finVec}\, y))|_p \le \left| \iota\left(c_{\mathrm{vac}}^{12} [Z^\alpha]\ \mathrm{jet}_{\pi_M j_0}(v_{x_0}) \right) \right|_p . \tag{22.25}$$

Proof. Only terms of degree at most $|l|$ in each H_b can contribute. Indeed every $y(\mu_i)$ has zero constant term. Expanding these finitely many monomials and using the ultrametric inequality, choose exponents e_{bi} with $\sum_i e_{bi} \le |l|$ whose monomial products attain at least the same absolute value. In particular $3e_{bi} \le c_i$. The functions

$$\prod_{b,i} (y\theta^3)(\mu_i(Z_b))^{e_{bi}}\, \theta(\mu_i(Z_b))^{c_i - 3e_{bi}} \tag{22.26}$$

are products of entire CM sections. Their torsion values are integral at every embedding, by Lemma 18.19.

The frame identity (22.24), with length $M = \pi_M \bar{\pi}_M$, gives the signed theta transformation law on $\pi_M \mathfrak{a}_0^{12}$. More explicitly, realize the product using forms $\Omega_0 \overline{w_i}/(\pi_M A_0)$ before taking jets. The sum of their Hermitian forms, with multiplicities c_i, is the standard form: its scalar factor is $M/|\pi_M|^2 = 1$. On multiplying the argument by $\pi_M j_0$ these forms become exactly μ_i. The canonical semicharacters agree by Lemma 18.21, not merely in degree. The finite Fourier construction of Lemma 18.27 therefore realizes (22.26) as the Taylor series of a finite-part datum F, with this period lattice. Rational finite translations distribute among its CM factors by the same Hermitian identity. Hence every finite value of F is a product of the integral torsion values just described. Density and local constancy extend this from rational finite points to every finite point.

We record the denominator in this Fourier realization. For $L = \mathfrak{a}_0^{12}$, the two finite dual quotients for the lattice $\pi_M L$ have cardinality M^{12f}. Perfect commutator duality follows from $\mathfrak{a}_0 \bar{\mathfrak{a}}_0 = \mathfrak{D}_{E/K}^{-1}$, as in the proof of (18.41). Thus there are finite translates $\rho(u_i)v$ and coefficients a_i such that

$$M^{12f} F = \sum_i a_i \rho(u_i) v, \qquad a_i v = \sum_\gamma \epsilon_{i\gamma} \rho(v_{i\gamma}) F, \quad |\iota \epsilon_{i\gamma}|_p = 1, \quad \bar{\pi}_M u_i \in L. \tag{22.27}$$

Since F has integral finite values, the second identity implies $|\iota(a_i v(\mathrm{finVec}\, y))|_p \le 1$ for every finite adelic y. Each $\rho(u_i)v$ is a sum of the indicator data modulo π_M with root-of-unity coefficients: write its finite Schrödinger character on each residue class of that modulus. Apply any linear coefficient functional $\mathcal{L}$ to the first identity in (22.27), and multiply by $v(\mathrm{finVec}\, y)$. The ultrametric inequality gives

$$|\iota(M^{12f} \mathcal{L}(F) v(\mathrm{finVec}\, y))|_p \le \max_{x_0} |\iota \mathcal{L}(v_{x_0})|_p.$$

Choose α from the twelve coefficients selected above. Their variables are disjoint, so their product is $[Z^\alpha]\, \mathrm{jet}_{\pi_M j_0} F$ by (22.26) and has p-adic absolute value at least p^{-R_p} at ι. Take $\mathcal{L} = c_{\mathrm{vac}}^{12} [Z^\alpha]\, \mathrm{jet}_{\pi_M j_0}$. Since $v_{12}^\circ = c_{\mathrm{vac}}^{12} v$, the left side of (22.25) is bounded by the displayed maximum for every y. The residue classes form a nonempty finite set. Choose x_0 at which this maximum is attained. The quantity being maximized contains no y, so this pair (α, x_0) gives (22.25) for all y. The factor M^{12f} also applies at the primes dividing M.

Lean. Lean states the inequality for each fixed finite point. Its proof maximizes the same function on the residue classes. Choosing a maximizer once gives the uniform assertion here. The monomial selection is `LeastValLE.exists_monomial`. The datum v_{x_0} is `indicatorDatum`. See `BoundAttained` (L151-L354). □

Theorem 22.12 (A common multiplier for the local coefficient bounds)**.** *Let $\kappa_\tau \le l_\tau \le \kappa_\tau + x^{42}$. With $n^\sharp = \lceil 6b/k \rceil$ as before, write*

$$B_\tau = \prod_{h=1}^{n^\sharp} \big((l_\tau - n^\sharp)_+ + h\big).$$

There is one $t \neq 0$, chosen independently of the prime and the embedding, such that

$$|\sigma t| \le \exp(fmb/k) \quad (\sigma \in \mathrm{Aut}(\mathbb{C})) \tag{22.28}$$

and, for every p, ι, there are α, x_0 as in Lemma 22.11 with

$$p^{-S_p} \le \left| \iota \left\{ \frac{c_{\mathrm{vac}}^{12}}{t} \Big(\prod_\tau l_\tau! \Big) [Z^\alpha] \operatorname{jet}_{\pi_M j_0}(v_{x_0}) \right\} \right|_p . \tag{22.29}$$

*Here S_p is the sum over embeddings and exponents $e \in I_p$ defined before Lemma 22.5 (*localPrimeSumTotal(p) *in Lean).*

Proof. Choose once a finite point y where the original rank-twelve integral vacuum is nonzero and set

$$c_0 = v_{12}^\circ(\mathrm{finVec}\, y), \qquad t = M^{12f} c_0 \prod_\tau B_\tau. \tag{22.30}$$

The vacuum is nonzero, so such y exists; every B_τ is a positive integer. The same t is therefore nonzero for every subsequent embedding.

The fixed frame of Lemma 22.10 has a unit member at the chosen prime. Theorem 22.6 gives the bound

$$R_p = S_p + \sum_\tau v_p(B_\tau) - \sum_\tau v_p(l_\tau!) \tag{22.31}$$

for the twelve-copy coefficient in Lemma 22.11. Since $l_\tau \le 2b$, we have $3|l| \le 6fb \le c_i$, so its section-degree hypothesis also holds. For the given p, ι, take the pair (α, x_0) of Lemma 22.11 and evaluate (22.25) at the fixed y used in (22.30). Divide by $|\iota t|_p$, multiply by $|\iota \prod l_\tau!|_p$, and use (22.30)–(22.31). All factorial and loss-product valuations cancel. The result is precisely (22.29), including at the primes dividing M.

Theorem 19.4 gives $|\sigma c_0| \le \exp(12 C_0 f)$, for the same nonzero value selected in (22.30). As $B_\tau \le (2b + n^\sharp)^{n^\sharp}$, we obtain the explicit bound

$$\log|\sigma t| \le 12f \log M + f n^\sharp \log(2b + n^\sharp) + 12 C_0 f. \tag{22.32}$$

For large x, $n^\sharp \le 7b/k$ and $\log(2b + n^\sharp) \le \log b + \log 3$. After division by fmb/k, the second term in (22.32) is at most $7(\log b + \log 3)/m$, which tends to zero because $m \asymp x^{1/2}$ and $\log b = x^{o(1)}$. The first and third terms have polynomial growth apart from $\log b$, whereas the denominator contains b; their ratios also tend to zero. Thus (22.28) follows for the constructed large scales.

Lean. `BoundAttained` (L318-L447). The sharper valuation bound uses one copy of $\prod_\tau B_\tau$ in the global multiplier while retaining the original rank-twelve vacuum. □

22.3 Four blocks and the sharp tested pairing

All source class sums in this subsection have the original chosen level. Its rank-one stabilizers are trivial. Let N denote its number of classes, so the class sum of the constant one is the positive integer N.

Lemma 22.13 (An integral monomial coefficient vector)**.** *For the one-row weight (l_τ) on $U(n)$, and an exponent α of multidegree l supported in the first twelve coordinates, there is an integer vector v_α in the prescribed Weyl coordinates whose Fock polynomial is*

$$P_{v_\alpha} = T_\alpha Z^\alpha, \qquad T_\alpha = \frac{\prod_\tau l_\tau!}{\alpha!}. \tag{22.33}$$

Over every characteristic-zero field it is fixed by every matrix whose first twelve columns are those of the identity. In particular it is fixed by every block diagonal matrix acting only on the remaining $n - 12$ coordinates.

Proof. If b_i are the integral tableau polynomials, take $(v_\alpha)_i = [Z^\alpha]b_i \in \mathbb{Z}$. The monomial lies in the one-row Schur module. The Gram–Fock dual-basis identity, with its hook number $\prod_\tau l_\tau!$, says $P_{v_\alpha} = (\prod l_\tau!)[Z^\alpha]^\sharp$, where the Gaussian dual of coefficient extraction is $Z^\alpha/\alpha!$. This proves (22.33) with the stated factorial, rather than an unspecified normalization of a highest weight vector. For a matrix with the stated columns, its transpose row substitution fixes Z^α. The adjoint substitution identity for the Fock pairing then shows that the coordinate functional $[Z^\alpha]$ is fixed by the Schur matrix. The determinant twist of a rank-one row is zero. This proves the asserted invariance, as a polynomial identity over $\mathbb{Z}$ and hence over every characteristic-zero field.

Lean. `ExchangeData` (L735-L816). □

Lemma 22.14 (The four-block form and its exact evaluation)**.** *Let p be a main pair with endpoint z, and split $2z = 12 + 4 + 4 + (2z - 20)$. Let q_2, q_3, q_1 have empty rows on the last three blocks, respectively; take q_2 at ϑ and the other two at the trivial twist. Let α have degree $l_\tau = c_p(\tau) - z$ at each place, and let χ be a character such that χ times the three block characters is the target character of p. For every $x_0 \bmod \pi_M$, every prime ℓ prime to $\operatorname{Nm} Q_\vartheta$, and every embedding above ℓ, there is a form Ξ on $U(2z)$ integral at every embedding above ℓ and orthogonal to the original false-lift space, such that*

$$N^3 c_{\mathrm{vac}}^{2z-12} \langle \Xi, K_p(g) \rangle = T_\alpha A_{q_1}(1) A_{q_3}(1) A_{q_2}(\vartheta) \Theta_{12;\chi,x_0,\alpha}(g). \tag{22.34}$$

Here K_p is the unscaled rank-one kernel form, and $\Theta_{12;\chi,x_0,\alpha}$ is its twelve-column theta kernel with polynomial Z^α and indicator datum x_0. Thus Ξ is a test form in exactly the sense used by the line-denominator estimate.

Proof. At the point $\operatorname{diag}(1, h_2, h_3, h_1)$, take the original point form with coefficient v_α from Lemma 22.13. At each chosen place, its residue column is x_0 on the first twelve coordinates and the prescribed unit column on each of the other three blocks. It is nonzero irrespective of x_0. At the twist places use the distinguished row of the second block. In a common modulus the point datum is exactly the product of these four block data: at a chosen place both sides impose the same four column conditions, and at a twist place both sides read the same character of the second-block entry. The polynomial is (22.33). Orthogonal-sum factorization of the oscillator representation therefore gives

$$\langle C_{h_1,h_2,h_3}, K_p(g) \rangle = T_\alpha \Theta_{12;\chi,x_0,\alpha}(g)\, K_{q_2}(g, h_2) K_{q_3}(g, h_3) K_{q_1}(g, h_1). \tag{22.35}$$

This is a termwise factorization of the defining theta series, justified by absolute convergence of its definite Gaussian sums.

The point form is integral. Indeed its coefficient vector is integral, and its block diagonal point fixes that vector over each local field by the last assertion of Lemma 22.13. The integral point-lattice lift only moves it by an integral group element.

For an empty-row pair q of block size s define the following finite class-sum operation on a family $F(h)$ of forms:

$$\mathcal{T}_{q,\theta} F = g_* \sum_{[h]} \frac{D_q^2}{(\operatorname{Nm} Q_\theta)^{s-1}} \frac{\phi_{q,\theta}(h)\, F(h)}{|\operatorname{Aut}(h)|}. \tag{22.36}$$

The degree-zero Weyl module is the constant line, so its value $\phi_{q,\theta}(h)$ in this formula means its scalar Gram coordinate against 1. The operation preserves any module of forms integral above ℓ: $D_q \phi_{q,\theta}$ is integral by Lemma 22.8, $D_q/(\operatorname{Nm} Q_\theta)^{s-1}$ is integral there, and g_* clears every stabilizer. It also preserves every fixed complex-linear orthogonality condition.

If a conjugate-linear functional $\mathcal{L}$ has $\mathcal{L}(F(h)) = B\,A\,K_q(g,h)$, direct adjointness and the definition of A_q give

$$N c_{\mathrm{vac}}^{s}\,\mathcal{L}(\mathcal{T}_{q,\theta}F) = B\,A_q(\theta)\,A. \tag{22.37}$$

Here the pairing of $K_q(g)$ with its lift is independent of g: normalized central translation on $U(1)$ fixes the lift and carries one kernel form to the other. Summing this constant pairing over the N source classes makes it the squared norm of the lift. This proves (22.37), and also explains why the factor is N, not a normalized probability mass. The factor c_{vac}^{s} converts the unscaled degree-zero kernel to the same original normalized lift. Complex conjugation in $\mathcal{L}$ accounts for the opposite factor $\overline{c_{\mathrm{vac}}}^{\,s}$ already in the norm.

Apply the three operations (22.36) to (22.35), first for q_1, then q_3, then q_2, and call the resulting form Ξ. Iterating (22.37) proves (22.34), because the three block sizes sum to $2z-12$. Their integrality statements prove integrality of Ξ simultaneously at all embeddings above ℓ.

For false orthogonality, first freeze h_2, h_3. The last-block sum is exactly the period in Corollary 21.41, with first block size 20 and total rank $2z$. Its strict rank condition $45 < 2z$ follows from $z > m \geq 10000$. The first twelve-column coefficient is fixed as required, and the main-pair tame character is nontrivial at the chosen places, so the original hyperplane annihilator applies. The last-block sum therefore annihilates every false lift. The remaining two operations in (22.36) preserve this property. This proves the asserted tested membership.

Lean. `TwistedValueHeights` (L243-L255), `TwistedValueHeights` (L564-L640). □

Lemma 22.15 (Finite Fourier and multiplicative averaging without prime exclusions)**.** *Let H be the finite group of chosen-level classes of the source $U(1)$, of order N. Fix a prime ℓ and an embedding $\iota : \mathbb{C} \hookrightarrow \mathbb{C}_\ell$. Choose a finite adelic representative g_u of each $u \in H$ and put $h_f = g_1^{-1}$. For every complex-valued function s on the finite adelic $U(1)$ there is a tame weight-zero character ψ, supported at the chosen places, such that*

$$|\iota s(1)|_\ell \leq \left| \iota \left\{ \frac{1}{N} \sum_{u \in H} \psi(\det(g_u h_f)) s(g_u h_f) \right\} \right|_\ell . \tag{22.38}$$

This holds also when ℓ divides N. If G is a finite group and $L^{|G|} = \prod_{\chi \in G} a_\chi b_\chi \neq 0$, then for every $\psi \in G$ some factorization $\psi = \psi_3 \psi_1$ satisfies

$$|\iota L|_\ell \leq |\iota(a_{\psi_3} b_{\psi_1})|_\ell. \tag{22.39}$$

Proof. Each character of H lifts through the quotient map on finite points to a tame adelic character of weight zero. Denote a character and its lift by ψ. In rank one the determinant is the source element itself, so $\psi(\det(g_u h_f)) = \psi(u)$. Character orthogonality and $g_1 h_f = 1$ give

$$s(1) = \sum_{\psi \in \widehat{H}} \frac{1}{N} \sum_{u \in H} \psi(\det(g_u h_f)) s(g_u h_f).$$

Applying ι and the ultrametric inequality proves (22.38) for every ℓ. For (22.39), the map $\chi \mapsto \psi\chi^{-1}$ permutes G. Hence the product of $|\iota(a_{\psi\chi^{-1}} b_\chi)|_\ell$ over G is $|\iota L|_\ell^{|G|}$. At least one of these finitely many positive numbers is at least its geometric mean.

Lean. `TwistedValueHeights` (L67-L177). □

Theorem 22.16 (The lower bound for the global test pairing)**.** *Fix a main pair p and a ramified empty-row pair q_2 of endpoint 2. Write $r_\vartheta = A_{q_2}(\vartheta)/A_{q_2}(1)$. There is one $c_* \neq 0$, independent of ℓ, ι and ϑ, such that*

$$|\sigma c_*| \leq \exp(fmb/k + 3x^{34}) \leq e^{\mathcal{B}} \quad (\sigma \in \mathrm{Aut}(\mathbb{C})) \tag{22.40}$$

and, for every prime ℓ, every field embedding $\iota : \mathbb{C} \hookrightarrow \mathbb{C}_\ell$, and every allowed twist ϑ with $\ell \nmid \operatorname{Nm} Q_\vartheta$, there is a test form $\Xi = \Xi_{\ell,\iota,\vartheta}$ with

$$|\iota(c_* r_\vartheta)|_\ell \, \ell^{-S_\ell} \le |\iota\langle \Xi, \phi_{p,\vartheta}\rangle|_\ell. \tag{22.41}$$

This chosen form is integral at every embedding above ℓ and has the original false-lift orthogonality; the lower bound (22.41) is asserted at the specified ι. Thus the order of choice is

$$\exists c_* \neq 0 \quad \forall \ell \quad \forall \iota \quad \forall \vartheta \text{ allowed at } \ell \quad \exists \Xi_{\ell,\iota,\vartheta}.$$

No single test form is required to give the lower bound at all embeddings. Primes dividing the source class count N are allowed. For every finite set F of primes,

$$2 \sum_{\ell \in F} S_\ell \log \ell \le \sum_\tau s_{\mathrm{fin}}(\kappa_\tau, \tau). \tag{22.42}$$

Equations (22.41) and (22.42) give the finite-place bounds used in (25.31).

Proof. Let G be the finite group of tame weight-zero characters at the chosen places. For each $\psi \in G$ choose the empty-row pairs $q_1(\psi), q_3(\psi)$ of block sizes $2z - 20, 4$ given by Lemma 22.8. Choose any root

$$L^{|G|} = \prod_{\psi \in G} A_{q_3(\psi)}(1) A_{q_1(\psi)}(1). \tag{22.43}$$

Every factor is nonzero, so $L \neq 0$; all its conjugates have modulus at most $e^{2x^{34}}$ by (22.21). Take t from Theorem 22.12, put $L_{\mathrm{tw}} = A_{q_2}(1)$, and set

$$c_* = t L_{\mathrm{tw}} L. \tag{22.44}$$

This is nonzero and has the first bound in (22.40). The second follows from $\mathcal{B} = fxb(\log b)^3/(mk)$ and $m^2 \asymp x$; the ratio of fmb/k to $\mathcal{B}$ tends to zero, as does the polynomial term $3x^{34}/\mathcal{B}$.

The scalar c_* has now been fixed before the prime, the embedding and the twist. Fix ℓ, ι, ϑ. Theorem 22.12 gives α, x_0. These choices, and the averaging choices below, may depend on the specified embedding; their integral output retains the simultaneous integrality proved in Lemma 22.14. Let χ_0 be the character of the target kernel divided by the character of the q_2 block, and write $s(g) = \Theta_{12;\chi_0,x_0,\alpha}(g)$. With

$$c = \frac{c_{\mathrm{vac}}^{12}}{t} \, T_\alpha(\pi_M j_0)^J,$$

the Taylor-to-derivative identity in (22.29) says $\ell^{-S_\ell} \le |\iota(c\, s(1))|_\ell$. Fix representatives g_u of the source classes $u \in H$ as in Lemma 22.15. Take $h_f = g_1^{-1}$, choose ψ by (22.38), and put

$$A = \frac{1}{N} \sum_{u \in H} \psi(\det(g_u h_f)) s(g_u h_f).$$

Then $\ell^{-S_\ell} \le |\iota(cA)|_\ell$. Apply (22.39) to (22.43), choosing $\psi = \psi_3 \psi_1$, and put $W = A_{q_3(\psi_3)}(1) A_{q_1(\psi_1)}(1)$. Thus $|\iota L|_\ell \le |\iota W|_\ell$.

Use Lemma 22.14 with $\chi = \chi_0/\psi$. Dividing the first kernel character by ψ multiplies its value by $\psi(\det g)$. The characters of all four blocks therefore multiply to the original target character. The full kernel form K_p is invariant under the rational source points on the left and the chosen

source level on the right. Since the source $U(1)$ is abelian, right translation by h_f permutes its classes. Hence

$$\sum_{u\in H}\langle \Xi, K_p(g_u h_f)\rangle = \sum_{u\in H}\langle \Xi, K_p(g_u)\rangle.$$

Apply (22.34) at $g_u h_f$ in each term on the left. The twelve-column factor there is $\psi(\det(g_u h_f))s(g_u h_f)$, whose sum is NA. On the right use $\phi_{p,\vartheta} = c_{\mathrm{vac}}^{2z}(\pi_M j_0)^J \sum_{u\in H} K_p(g_u)$. This gives

$$tL_{\mathrm{tw}}W\,(cA) = N^2\,\frac{\langle \Xi, \phi_{p,\vartheta}\rangle}{r_\vartheta}. \tag{22.45}$$

Indeed the three block sums contribute $N^3 c_{\mathrm{vac}}^{2z-12}$, the last source sum contributes N, and its normalized lift contributes $c_{\mathrm{vac}}^{2z}(\pi_M j_0)^J$. The remaining $c_{\mathrm{vac}}^{12}(\pi_M j_0)^J T_\alpha$ is exactly tc. These equalities retain both the factorial and the original vacuum scalar.

Take ℓ-adic absolute values in (22.45). The preceding two lower bounds imply

$$|\iota(tL_{\mathrm{tw}}Lr_\vartheta)|_\ell \ell^{-S_\ell} \le |\iota(N^2\langle \Xi, \phi_{p,\vartheta}\rangle)|_\ell \le |\iota\langle \Xi, \phi_{p,\vartheta}\rangle|_\ell,$$

since $|\iota N|_\ell \le 1$ for every rational integer N. This proves (22.41), including at primes dividing N. Tested membership and simultaneous integrality are those already proved in Lemma 22.14. Finally group the defining von Mangoldt sum for s_{fin} by prime powers: $\Lambda(\ell^a) = \log \ell$, and it vanishes on integers that are not prime powers. Its factor $1 - \chi_K(\ell^a)$ is twice the coefficient used in S_ℓ. Therefore the sum over all primes of $2S_\ell \log \ell$ equals the right side of (22.42). All summands are nonnegative and those beyond the original cutoff vanish, so restriction to F gives (22.42).

Lean. `trivialLift_eq_stdClassSum_translate`, `TwistedValueHeights` (L665-L849). □

23 The analytic seesaw for the Poincaré series

Unfolding the period defining the Poincaré series leaves a single rational orbit. We first evaluate the finite tensor on that orbit and bound its denominator. Theorems 23.38 and 23.39 then identify its theta integral with the period and give the scalar that normalizes the source form.

Write $m = 2\rho + 2$, $n_j = \dim X_j$, $q = \mathrm{Nm}(Q) = \prod_{w\in Q} \#k_w$, and

$$B = M^{f n_m^2}.$$

Here B is the number of n_m by n_m matrices modulo the original chosen ideal (π_M), since $N_{E/\mathbb{Q}}(\pi_M) = M^f$. The integer q is the norm of the tame ideal; $(q, N) = 1$, and $q = 1$ for the empty twist. In particular every point or translation that is integral away from q is integral above the level prime N. All copies of the integral vacuum below are the same copies used in §18.1 and Corollary 19.5.

Lemma 23.1 (An integral lift for two consecutive edges)**.** *Consider two consecutive edges $V \longrightarrow V' \longrightarrow V''$ in the lower chain, starting at an even index. Join their data on $W = \mathrm{Hom}(V', V^- \oplus V'')$. Let I be the isotropic space used to remove positive grades and the remaining middle hyperbolic plane. Let*

$$Z = \mathrm{Hom}(X', X^- \oplus X''), \qquad i : Z \longrightarrow I^\perp, \quad k : I \longrightarrow W$$

be the original frames, in finite lattice coordinates. There are an explicit unitary g on Z and an integral matrix $B_I : Z \to W$ such that

$$B_I = ig + kC, \qquad B_I^* \tau = 0, \qquad \mathrm{image}(L_W \cap I^\perp \longrightarrow Z) = gL_Z, \tag{23.1}$$

where $T : V \to V'$ and $T' : V' \to V''$ are the consecutive string shifts and $\tau = (T^*, T')$. If y vanishes above N and has denominator π_M on the first block and $\bar\pi_M$ on the second, away from q, then i^*y has the same two denominator bounds.

Proof. Denote the last coordinate in the two definite blocks by e, e''. Let P be the chosen half of the split primes above N, let $j = 1_P$, let $\pi = \pi_N$, and let η be the original integral finite unit with $\bar\eta\eta = -1$. In standard finite coordinates put

$$\varrho = e + \bar\eta \bar j\, e'', \qquad A = j(\pi - 1) + \bar j(\bar\pi^{-1} - 1), \qquad \gamma = 1 + A\varrho\varrho^*.$$

The disjoint supports of $j, \bar j$ give $\varrho^*\varrho = 1$ and $(1 + A)(1 + \bar A) = 1$; hence γ is unitary. Let c_j be zero except in the two last columns, where

$$c_e = A\pi^{-1}\bar j, \qquad c_{e''} = A\pi^{-1}\eta j.$$

For each column $x = \gamma e_j$, direct substitution at $P, \bar P$ and their complement gives

$$\begin{aligned} &\delta c_j = c_j, \qquad x_i \in \widehat{\mathcal{O}}_E \quad (i \neq e, e''), \qquad \bar\pi x_i \in \widehat{\mathcal{O}}_E, \\ &c_j - \pi^{-1}x_e \in \widehat{\mathcal{O}}_E, \qquad \bar\eta c_j - \pi^{-1}x_{e''} \in \widehat{\mathcal{O}}_E. \end{aligned} \tag{23.2}$$

Here $\delta = c/\bar c$, with c the chosen idele generating (π_M); it equals one above N. Use row factors $\kappa = 1$ for every middle pair and, for the last pair, $\kappa_{e''} = \delta$, with all other factors one. Put $g = (\mathrm{diag}(\kappa)\gamma) \otimes 1_{X'}$. The diagonal and γ commute, since the former is one on the support of $\gamma - 1$.

For an odd string in a graded space, the lattice membership test is particularly simple. A vector with definite coordinates x, middle coordinate $u(\pi^{-1}x_e + c_1)$, and other string coordinates zero

is integral in the graded lattice if c_1, $\bar\pi x_e$, and the other coordinates of x are integral. Indeed its coordinates in the original graded basis are

$$(x_i)_{i\neq e},\quad \bar\pi x_e,\quad c_1,$$

with zeros at the other string indices. An integral change from that basis to the chosen finite frame preserves this test.

Let $s=\bar\eta\,\bar\alpha/\alpha$, the finite scale in the negative copy V^-. The original even-stage lattice scales obey

$$su=\bar\eta\,\delta u''.$$

For a middle pair, lift the column x by using sx in the first definite block, x in the second, and suc_j in both middle string coordinates. After division by s in the first block, the membership test applies with $c_1=c_j-\pi^{-1}x_e$. In the second block it applies with $c_1=\bar\eta c_j-\pi^{-1}x_{e''}$. These are integral by (23.2). For the last pair there is no plane to remove. Its last target coordinate is the old source string coordinate divided by π. The lifted string coordinate is $\pi^{-1}\delta x_{e''}$. Since $s^{-1}\delta=-\eta u$, the membership test now uses

$$c_1=-\pi^{-1}x_e-\eta\pi^{-1}x_{e''}=(c_j-\pi^{-1}x_e)+\eta(\bar\eta c_j-\pi^{-1}x_{e''}),$$

which is again integral.

Write A_I for the matrix of these lifted columns, in the finite coordinates of $V^-\oplus V''$, and let R be the rows of the finite frame of V' at X'. The middle string is even, so R is integral. Set

$$B_I=A_I\otimes R^{\mathsf t}.$$

Its columns are integral by the preceding two calculations. In the quotient by the isotropic frame its columns are exactly those of g; their remaining coordinates are the isotropic coefficients just specified. This proves $B_I=ig+kC$. Moreover R^* is the inverse-frame inclusion of X'. The stacked shift is zero on X', since both string maps move only string indices. Thus $B_I^*\tau=0$.

Consequently gL_Z lies in the image of $L_W\cap I^\perp$. Both lattices are self-dual. For the second one, the duality identity follows directly from $L_W=L_W^\vee$: the annihilator of $L_W\cap I^\perp$, modulo I, is its image again. Equivalently choose a primitive integral basis of $L_W\cap I$, extend its dual coordinate functionals to L_W, and eliminate the resulting hyperbolic pairs. The remaining unimodular block is the quotient lattice. Self-duality reverses the inclusion, proving equality.

For the denominator claim let $d_i=\pi_M$ or $\bar\pi_M$ on the two respective blocks. The inclusions in the middle pair keep blocks, so the denominators agree. In the last pair only e'' changes blocks, and at every place

$$v(\bar\pi_M\delta)=v(\pi_M).$$

Thus, whenever an entry of the inclusion is nonzero, its row denominator has the same valuation as $d_j\kappa_j$. Away from N, the neighbor is the identity and its isotropic column correction is zero. An entry of B_I linking different denominator valuations is therefore zero at the conjugate place. Since B_I is integral, the j-th coordinate of B_I^*y has denominator $d_j\kappa_j$. The isotropic correction kC is supported above N; hence $C^*k^*y=0$ and

$$i^*y=gB_I^*y=\operatorname{diag}(\kappa)B_I^*y.$$

Multiplication by κ gives exactly denominator d_j. This proves the last assertion, including the last pair. The original neighbor and its congruences; The graded-lattice test; The integral lift and quotient lattice; Denominator preservation; Middle and last pairs; The original scale identities.

Lean. `NeighbourLattice` (L111-L167), `AdmissibleEvaluation` (L51-L101), `GoodDatum` (L42-L198). □

Proposition 23.2 (Finite evaluation with its denominator and norm)**.** *For each original pair of edges there is one constant C_i, independent of the tame character and of all finite translates, such that its finite evaluation sends an allowed translate of the vacuum to a finite combination of C_i times the original glued translates. If the pair of definite ranks is n_{2i}, n_{2i+2} and the middle rank is n_{2i+1}, the coefficients become integral away from q after multiplication by*

$$(N_{E/\mathbb{Q}}(\pi_M))^{(n_{2i}+n_{2i+2})n_{2i+1}},$$

and their absolute sum after every complex automorphism is at most one. The output translations have denominator $\bar{\pi}_M$ away from q on both original edges.

Proof. The finite evaluation translates by τ, takes the zero term along I, and applies the inverse arrow identification. Since $(q, N) = 1$ and π_M is a unit above N, an allowed translation is integral above N. Removing that part of the translation changes the vacuum only by a root of unity. We may therefore assume its vector y vanishes above N.

In the Schrödinger lattice model, the zero term of a translated vacuum is an integral over an affine isotropic fiber. If the fiber does not meet the vacuum lattice it is zero. If it does, translate to an integral point of the fiber. The fiber measure is then the same as for the untranslated vacuum, and the only remaining factor is its additive character, a root of unity. Using (23.1), this says precisely

$$\mathrm{ev}_i(H(y)v_W) = \zeta\,\mathrm{Arrow}^{-1}H(i^*y)\,\mathrm{CT}_I(v_W), \qquad \zeta = 0 \text{ or a root of unity.} \tag{23.3}$$

The shift τ disappears from the projected translation because $B_I^*\tau = 0$. The Heisenberg cocycle in $H(\tau)H(y)$ is also a root of unity, and has been included in ζ. This derivation includes the possibility of an empty fiber.

Let $\mathcal{A}$ be the quotient, modulo integral vectors, of the part of gL_Z vanishing above N. The quotient lattice in (23.1) has representatives

$$n(v) + a, \qquad v \in (X' \to P \to \mathbb{F}_N), \quad a \in \mathcal{A},$$

where $n(v)$ has only the two last rows nonzero, namely $\eta\bar{v}/N$ and $-\bar{v}/N$. To see completeness, subtract these two last-row residues above N; the remainder is integral there and represents a unique away class. Conversely an integral difference forces the last-row residues and then the away class to agree.

The finite zero term of the original vacuum is nonzero. Here is an explicit argument that includes all coefficient primes. Write $A = \mathrm{CT}_I$ and suppose $A(v_W) = 0$. If $A(H(z)v_W) \neq 0$, apply Lemma 23.31 with $M = M' = 1$. It gives $z = \ell + iw + ka_I$, with ℓ integral, because $I^\perp = iZ + I$. Integral translations fix v_W, and translations along I fix the zero-term map. Commuting these translations therefore gives

$$A(H(z)v_W) = \zeta H(w)A(v_W) = 0,$$

a contradiction. Thus A would kill every translated vacuum. These translates span the finite Weil part: in a lattice Schrödinger model this is finite Fourier inversion on each compact support modulo an open invariance lattice. But in adapted Schrödinger coordinates a tensor of characteristic functions of compact open lattices has zero term equal to a nonzero Haar volume times the retained vacuum. This is a contradiction, and proves $A(v_W) \neq 0$.

The periods of $A(v_W)$ are the self-dual quotient lattice in (23.1). In the finite lattice model the space of vectors fixed by that full lattice is one-dimensional: reduce a test vector to a finite quotient and apply finite character orthogonality. The sum of translates indexed by the above quotient representatives is its vacuum. It is fixed by the quotient lattice, since integral periods of

the original vacuum remove changes of representative. It is nonzero because distinct classes modulo the original vacuum lattice give orthogonal translates, and the zero class contributes $\langle v_Z, v_Z\rangle$ to its pairing with v_Z. Consequently

$$\mathrm{CT}_I(v_W) = c_i \sum_{v,a} H(n(v)+a)v_Z, \qquad c_i \neq 0. \tag{23.4}$$

Its coefficient c_i is nonzero by the preceding argument. The zero-term map, original vacuum and quotient representatives all depend only on the fixed pair, so c_i is chosen once, independently of y, q.

We next bound $\#\mathcal{A}$. For an away lattice vector z, $\bar\kappa_j z_j$ is integral. Reduce its conjugate modulo π_M. If this reduction is zero, write $\overline{\bar\kappa_j z_j} = \pi_M w_j$; then $z_j = \bar\pi_M \kappa_j \bar w_j$ is integral. Thus the kernel of this reduction is contained in the integral vectors, and the index of the integral vectors divides the order of the residue matrix group. Hence

$$\#\mathcal{A} \;\mid\; (N_{E/\mathbb{Q}}(\pi_M))^{(n_{2i}+n_{2i+2})n_{2i+1}}. \tag{23.5}$$

This uses divisibility of indices, not an assertion that the reduction descends injectively on the displayed quotient.

The vector $i^*y + a$ has the allowed two-block denominators by Lemma 23.1. Its translation commutes with $H(n(v))$, since the former vanishes above N and the latter is supported there. Under the arrow identification the sum over v is exactly the original gluing sum of §18.1. The flip on the first edge conjugates its denominator, so both edge translations have denominator $\bar\pi_M$. The extra cocycles are roots of unity.

Put $C_i = c_i \#\mathcal{A}$. Equations (23.3) and (23.4) now express the evaluation as a sum of $\#\mathcal{A}$ glued translates with coefficients zero or roots of unity divided by $\#\mathcal{A}$. This proves the all-conjugate absolute-sum bound one and, by (23.5), the integer denominator. Coincident translations may be grouped, which improves the bound. The exact divisibility of the away quotient; The gluing sum and its coefficients; Nonvanishing on the original vacuum; Translation, zero term and finite evaluation.

Lean. `NeighbourLattice` (L830-L866), `NeighbourVacuum` (L273-L363), `AdmissibleEvaluation` (L511-L647). □

Theorem 23.3 (The evaluated tensor and its denominator)**.** *Let y_j be finite points on the lower chain, integral away from q. Apply the preceding evaluation to each pair of translated point data and tensor the $\rho+1$ results. There is a single complex constant C, independent of the character and the points, for which*

$$\mathcal{E}_\xi(y) = C \sum_x r_{\xi,y}(x)\,\mathrm{GlueVac}(x). \tag{23.6}$$

All the translations are admissible for q, $B^{2m} r_{\xi,y}(x)$ is integral away from q at every coefficient embedding, and, if $Q = \varnothing$,

$$\sum_x |\sigma r_{\xi,y}(x)| \le 1 \qquad (\sigma \in \mathrm{Aut}(\mathbb{C})).$$

The vacuum in GlueVac *is the original integral vacuum.*

Proof. The point datum on $Y_j \to Y_{j+1}$ is the indicator of the prescribed inclusion matrix modulo $\pi_M Q$. Finite Fourier inversion expresses it as a mean of all additive characters, indexed by the residue matrix group using one fixed primitive base character, times roots of unity. In the Schrödinger model these characters are Heisenberg translates of the vacuum. Its coefficient denominator, away from q, is

$$(N_{E/\mathbb{Q}}(\pi_M))^{(\dim Y_j)(\dim Y_{j+1})},$$

and the all-conjugate coefficient sum is one. The denominator of its translation is $\bar{\pi}_M$ away from q, since $\pi_M q \in (\pi_M Q)$.

Moving by (y_j, y_{j+1}) preserves this assertion. At places away from q, the pair is integral, its Weil action fixes the vacuum up to its splitting-character unit, and it carries allowed translations to allowed translations. At places dividing q, an arbitrary finite Weil translate of the vacuum is a finite linear combination of translated vacua with denominators involving only these primes: reduce its compact support and its open invariance subgroup to a finite quotient and use the finite Fourier transform. Every normalization denominator there is a power of a residue characteristic dividing q. No size bound is needed in that case. If $Q = \varnothing$, all the points are integral and no such Fourier expansion is needed; all splitting-character values have modulus one after every automorphism. Thus the coefficient sum remains one in precisely the case where it is required.

The arrow product of two such translates is a root of unity times a translate of the joined vacuum. The first-edge flip changes its denominator to π_M; the second remains $\bar{\pi}_M$. Proposition 23.2 therefore applies. The total denominator exponent for the pair is at most

$$(\dim Y_{2i})(\dim Y_{2i+1}) + (\dim Y_{2i+1})(\dim Y_{2i+2}) + (n_{2i} + n_{2i+2})n_{2i+1} \le 4n_m^2.$$

It follows that B^4 clears the pair's denominator. All three coefficient sums multiply to at most one.

Tensor the $\rho + 1 = m/2$ pairs. The denominator is $(B^4)^{m/2} = B^{2m}$ and the coefficient sum is still at most one. If the original integral vacuum at edge j is $\kappa_j v_j$, absorb $\prod_i C_i/(\kappa_{2i}\kappa_{2i+1})$ in the single constant C. Each $\kappa_j \neq 0$ was fixed before the twist. This changes no coefficient or translation; in particular it does not replace the original scalar by the unrelated normalization of a measure family.

Lean. `GoodDatum` (L326-L481), `GoodDatum` (L481-L575). □

Lemma 23.4 (Coefficient bounds after finite summation)**.** *Suppose finite cuts $\Omega_{\xi,j}$, $0 \le j < m$, have representatives integral away from q, with positive multiplicities $a_{\xi,j}(y)$ dividing powers of q. Suppose also that, for $Q = \varnothing$, each cut has at most B members and every multiplicity is one. Define the tensor summed over the finite representatives by*

$$\phi_\xi = \sum_{y \in \prod_{j<m} \Omega_{\xi,j}} \left(\prod_{j<m} a_{\xi,j}(y_j)^{-1} \right) \mathcal{E}_\xi(y).$$

Then the same C as in Theorem 23.3 gives

$$\phi_\xi = C \sum_x r_\xi(x)\, \mathrm{GlueVac}(x), \quad B^{2m} r_\xi(x) \text{ integral away from } q, \quad \sum_x |\sigma r_\xi(x)| \le B^m \quad (Q = \varnothing).$$

Proof. Every multiplicity inverse is a unit at every coefficient prime not dividing q. Distribute the finite sums in (23.6); this proves integrality. For $Q = \varnothing$ there are at most B^m tuples, all weights are one, and each tuple has coefficient sum at most one. The triangle inequality proves the bound for every σ simultaneously.

Lean, formal context. `GoodDatum` (L481-L575), `SeesawIdentity` (L624). □

Proposition 23.5 (Common normalization of the period)**.** *Suppose the cut tensor in Lemma 23.4 is jointly invariant under the upper-chain levels, and suppose there is a single scalar c_0, independent of ξ, such that on ramified source forms*

$$\mathcal{L}_\xi(\Pi\varphi) = c_0 \,\mathrm{res}\big(\mathrm{Chain}_\xi(\phi_\xi)\varphi\big). \tag{23.7}$$

Then there is one $c_{\mathrm{ss}} \in \mathbb{C}^{\times}$ *such that* (11.11) *holds with the original chain-datum conditions. In particular*

$$\sum_{x} |\sigma\Psi_{\xi}(x)| \le M^{3fmn_m^2} \quad (Q = \varnothing). \tag{23.8}$$

No assertion that $c_0 \neq 0$ *is necessary.*

Proof. Put $d = B^{2m}$, $a = c_0 C/d$, and set

$$u = \begin{cases} a^{-1}, & a \neq 0, \\ 1, & a = 0. \end{cases} \qquad c_{\mathrm{ss}} = u^{-1} \neq 0.$$

If $a \neq 0$, the coefficients of $uc_0\phi_{\xi}$ in the original glued vacua are $dr_{\xi}(x)$. They are integral away from q and their absolute sum for the empty twist is at most $dB^m = B^{3m}$. If $a = 0$, the expansion gives $c_0\phi_{\xi} = 0$; use the zero coefficient function, with the same conclusions.

Each allowed translation is integral away from Mq. Remove those integral components. An integral Heisenberg translation fixes the local vacuum up to its semicharacter, a root of unity; the retained and removed supports are disjoint, so their cross cocycle is one. Absorb these root-of-unity factors in the coefficients. The resulting support is exactly the permitted support, and integrality and all-conjugate size are preserved. The tensor itself is unchanged, so its joint level invariance is unchanged. This constructs the required finite coefficient function Ψ_{ξ}.

Apply the chain map and the last restriction to $uc_0\phi_{\xi} = \mathrm{Glue}(\Psi_{\xi})$, and use (23.7). The result is $\mathcal{L}_{\xi}(c_{\mathrm{ss}}^{-1}\Pi\varphi) = B_{\xi,\Psi_{\xi}}\varphi$. Finally $B^{3m} = M^{3fmn_m^2}$, proving (23.8).

Lean. `SeesawIdentity` (L582-L666). □

Support of the Bessel periods. To apply Proposition 23.5, we identify the evaluated tensor in Theorem 23.3 with the unfolded period.

At stage j, let R_j be the Bessel group of the graded space Y_j, let Λ_j^- be the string intersection of its original finite lattice, and let $\mathbf{q}$ be the finite idele generating Q. The iterated unfolded period $\mathcal{W}_j$ of a ramified source form is supported on R_j-orbits with a graded representative x satisfying

$$\bar{\mathbf{q}}^{\,j}\Lambda_j^- \subseteq \Lambda_j(x)^- \subseteq \mathbf{q}^{-j}\Lambda_j^-. \tag{23.9}$$

Section 23.2 constructs finite sets of representatives for these orbits, with the weights of Lemma 23.4 and joint invariance under the prescribed levels. The one-step Bessel unfolding in Section 23.1 annihilates every rational matrix orbit except that of the string map. On this orbit the two constant terms equal the zero term translated by (T^*, T'). The identity $-TT^* + T'^*T' = 0$ cancels the cocycle. The archimedean vector at each even stage is independent of the tame character, so the projection onto its highest type gives one linear form for all twists (Section 23.3).

Frequency support gives the lower inclusion in (23.9). Duality gives the upper inclusion unless the lattice splits at N, in which case ramification forces the period to vanish (Section 23.4). Theorem 23.37 can therefore replace the outer integrals by weighted sums. The resulting scalar in (23.7) is independent of the finite character and the source form, and the coefficient bound is B^{3m}. The one-step orbit elimination and integral identity; Two constant terms and the stacked shift; The character-independent real factor; Ramification and the required support; The original finite cuts; The iterated period identity.

23.1 The rational orbits and the one-step analytic unfolding

We use the original Hermitian gradings $Y : X \oplus J_s$ and $Y' : X' \oplus J_{s-1}$, with positive definite X, X'. The grades on the string are $s-1, s-3, \dots, 1-s$; the grade on X is zero. Write $a \mapsto a^{\vee}$ for string

reversal, fixing the block, and $h(a)$ and $h'(b)$ for the grades. The Gram matrix pairs only dual indices; its middle string entry, when present, is $-N$. All its anti-diagonal entries are nonzero. The original rational string map $T : Y \longrightarrow Y'$ has degree -1, kills X, and sends each string index to the same-numbered index of the shorter string. Its nonzero entries are $1, c, -Nc$, according to parity and position, with $c \neq 0$ the prescribed common generic constant of the two original gradings. Direct multiplication gives

$$T^{\dagger}T = \mathfrak{e}_Y, \qquad TT^{\dagger} = \mathfrak{e}_{Y'}, \tag{23.10}$$

Each shift lowers the string grade by two, with a nonzero coefficient until the last index.

Let $R = U \rtimes U(X)$ be the Bessel group. Here U strictly raises grades, and $U^{(2)}$ raises them by at least two. Its generic character is $\psi_U(u) = \psi_E(\frac{1}{2}j\,\mathrm{tr}(\mathfrak{e}_Y u))$, with the fixed imaginary Weil parameter j. We take W continuous and rationally invariant on R, with $W(ur) = \psi_U(u)W(r)$ for $u \in U^{(2)}$. The original source functions and their successive unfolded functions have precisely this property.

Lemma 23.6 (Cancellation on a quotient without a normalizer assumption)**.** *Let G be a locally compact second countable group, Γ a discrete subgroup, and F continuous, left Γ-invariant and absolutely integrable on $\Gamma\backslash G$. Suppose*

$$F(ug) = cF(g) \quad (g \in G), \qquad |c| = 1, \quad c \neq 1.$$

Then $\int_{\Gamma\backslash G} F = 0$. It is not necessary that u normalize Γ.

Proof. Define

$$D(t,x) = |F(tux) - cF(tx)|, \qquad H = \{a : D(at,x) = D(t,x) \text{ for all } t,x\}.$$

Continuity makes H a closed subgroup. Left Γ-invariance gives $\Gamma \subset H$; the equation for u, and $|c| = 1$, give $u \in H$. Since $D(1,x) = 0$, membership of t in H gives $F(tux) = cF(tx)$.

Disintegrate the absolutely convergent integral first over $\Gamma\backslash H$, using a positive cutoff for the closed H-action on G. For almost every outer point x, its inner integral $I_x = \int_{\Gamma\backslash H} F(tx)\,dt$ is absolutely convergent. Right translation by u on H scales left Haar measure by a positive real d; it induces the same scaling on $\Gamma\backslash H$, because right translation commutes with the left action of Γ. Therefore $cI_x = dI_x$. The equality $c = d$ would force $c = 1$, so $I_x = 0$. Integration in stages proves the claim. This argument uses a right translation on the larger quotient, and does not make an invalid left change of variables on $\Gamma\backslash G$.

Lean. `UnfoldingInStages` (L300-L394). □

For a target grading d odd under reversal, let $A_d^{\geq 0}$ be the rational matrices $w : Y \to Y'$ whose entries vanish when $d(b) < h(a)$. Let $C_d(F; w)$ be the normalized mean of the Heisenberg translate $\rho(w)F$ along matrices of strictly positive degree. Thus C_d is a theta value after taking the zero term in that isotropic subspace. Write

$$\beta(v,w) = \psi_E\Big(\tfrac{1}{2}j\,\mathrm{tr}(w^{\dagger}v)\Big)$$

for the associated bicharacter, with the convention of the original Heisenberg representation. Say that *w survives for d* when

$$\beta(Tu,T) = \beta(wu,w) \quad \text{whenever } u \in U^{(2)}, \quad w(u-I) \text{ has degree at least 1.} \tag{23.11}$$

Lemma 23.7 (The moment test from elementary unipotents)**.** *Suppose w survives for d. Let v and an index k satisfy*

$$\begin{aligned} v_a &= 0 \quad \text{if } h(a) < \max(1, h(k)+2), \\ (wv)_b &= 0 \quad \text{if } d(b) \le \max(h(k), -1), \\ w_{b,k^\vee} v_a &= 0 \quad \text{if } d(b) + h(a) \le 0. \end{aligned} \tag{23.12}$$

Then

$$(T^\dagger T v)_k = (w^\dagger w v)_k. \tag{23.13}$$

Proof. Use the Hermitian Gram matrix J of Y. The vector v is isotropic, since it is supported in positive grades. For any adele t, put

$$x = \overline{j^{-1}(J^{-1})_{k,k^\vee} t}\; e_{k^\vee}.$$

The supports in (23.12) imply $x^\dagger v = 0$. The elementary unitary transformation

$$u = I + v x^* J - x v^* J - \tfrac{1}{2}(x^* J x) v v^* J \tag{23.14}$$

is unitary: in $u^* J u$, the two linear cross terms cancel and the two quadratic terms cancel because $v^* J v = v^* J x = 0$. Its inverse has the same formula with x replaced by $-x$. The degree of each of the three nonidentity terms is at least two. Multiplying them by w and using the last two conditions in (23.12) shows that $w(u - I)$ has degree at least one. This verifies the full hypothesis of (23.11).

For any matrix $y : Y \to Y'$, expansion of the trace in (23.14) gives

$$\beta(yu, y) = \psi_E(j x^* y^* J' y v). \tag{23.15}$$

Indeed the constant trace and the term involving $(x^* J x)(v^* y^* J' y v)$ disappear under the imaginary parameter j; the two remaining conjugate terms give (23.15). Equivalently, add an expression to its conjugate before applying the additive character; this verifies the factors of two. Substitute $y = T, w$ in (23.15). The specified coordinate and its Gram factor cancel, so (23.11) gives

$$\psi_E\big(t((T^\dagger T - w^\dagger w)v)_k\big) = 1 \quad \text{for every } t.$$

Nondegeneracy of the adelic trace pairing gives (23.13).

Lean. `BesselIntegral` (L529-L614). □

Lemma 23.8 (All surviving classes come from the string map)**.** *Write $\mathcal{C}_d$ for classes of rational maps modulo $A_d^{\ge 0}$ satisfying*

$$w_{ba} = T_{ba} \quad \text{if } h(a) > 0,\ h'(b) < h(a). \tag{23.16}$$

For $d = 0$ or $d = h'$, this condition is independent of the representative. The following facts hold.

Every rational map surviving for $d = 0$ is carried into $\mathcal{C}_0$ by a rational element of $U(Y')$. The translates of $\mathcal{C}_0$ form disjoint blocks indexed by $R'(E^+)\backslash U(Y')(E^+)$. For $d = h'$, a class in $\mathcal{C}_{h'}$ which survives is exactly the class of T.

Proof. Let $Y_{>0}$ be the span of the positive string indices. A surviving rational w is injective on this subspace. Otherwise take a nonzero vector v in its kernel, and among its nonzero coordinates choose the one of smallest positive grade, at string index a. Put $k = a + 1$ along the string. The three conditions (23.12) hold for $d = 0$: the support of v is in grades at least $h(a) = h(k) + 2$, and $wv = 0$. Formula (23.13) says that $(\mathfrak{e}_Y v)_k = 0$, whereas (23.10) makes it a nonzero multiple of v_a. This is a contradiction. The same argument applies to T.

Apply (23.13) with $v = e_i$ and $k = a^\vee$, where i, a both have positive grade. Its support hypotheses are automatic, and multiplying by $J_{a,a^\vee}$ shows that $w|_{Y_{>0}}$ and $T|_{Y_{>0}}$ have the same Gram matrix. They are injective, so the classical rational Witt extension theorem carries the former frame to the latter. This proves the first assertion, even though the restricted Gram matrix can be degenerate.

For completeness, the block stabilizer is exactly R'. Right multiplication by an element of R and left multiplication by an element of R' preserve (23.16), by their triangular matrices. Conversely, suppose both w and δw satisfy (23.16). For a target string index b of nonnegative grade, choose the source index a of grade $h'(b) + 1$. The entry T_{ba} is nonzero. Descending induction on $h'(b)$ shows that the required entries of $\delta - I$ vanish: the entry in column a of $(\delta - I)w$ equals the entry in column b of $\delta - I$, times T_{ba}. Higher-grade contributions have already vanished by induction; lower-grade entries of that column of w agree with T. Both agreements make the product zero in each row l with $h'(l) \le h'(b)$. Thus $(\delta - I)_{lb} = 0$ for every nonnegative-grade string column b and every such row l; these are precisely the defining matrix equations of R'. The same argument works after right multiplication by R. Thus a translate meets $\mathcal{C}_0$ precisely on the indicated stabilizer, proving the block claim.

Finally let $w \in \mathcal{C}_{h'}$ survive. We prove that every negative-degree entry of $w - T$ vanishes, by ascending induction on the target grade. An entry (b, a) not already covered by (23.16) has $h'(b) < h(a) \le 0$. Take the positive source index i of grade $1 - h'(b)$, whose string image is a nonzero multiple of $e_{b^\vee}$, and put $k = a^\vee$. Use $v = e_i$ in (23.12). The first condition is $1 - h'(b) \ge 2 - h(a)$. The second follows from agreement on the positive column i; the third follows from the already settled target grades below $h'(b)$. Thus (23.13) compares the (k, i)-entries of $T^\dagger T$ and $w^\dagger w$. In their difference all rows except $b^\vee$ vanish by those same two agreements. The remaining factor is

$$(w^\dagger - T^\dagger)_{a^\vee, b^\vee} T_{b^\vee i} = 0.$$

Its second factor is nonzero. The anti-diagonal Gram formula for the adjoint then gives $w_{ba} = T_{ba}$. This completes the induction and the third assertion.

Lean. `BesselIntegral` (L616-L823). □

Lemma 23.9 (Absolute constant-term expansion and the common partial coefficient)**.** *For a Schwartz oscillator datum F and $d = 0, h'$,*

$$\theta(F) = \sum_{w \in \mathrm{Hom}_E(Y, Y')/A_d^{\ge 0}} C_d(F; w) \tag{23.17}$$

is absolutely convergent. The subseries over $\mathcal{C}_d$ has the same value for these two choices of d. For $F = \omega(r, h)\Phi$, with Φ the original datum, both absolute sums times $|W(r)|$ are bounded uniformly on $[R]$ and on compact sets of h. For $d = 0$, the bound can be taken to be a fixed polynomial in the reduced adelic height of h.

Proof. The positive-degree maps form an isotropic subspace of $\mathrm{Hom}(Y, Y')$, since the Gram pairing reverses degree. Complete it to dual isotropic coordinates and take the zero term along the first coordinates. In the resulting Schrödinger model, (23.17) is Fourier expansion in the dual rational coordinates. The finite part is locally constant with support in a fixed fractional lattice. At infinity the datum and all its derivatives decrease faster than every polynomial. Summing the corresponding Schwartz bound over that fractional lattice proves absolute convergence. This also bounds the sum by finitely many Schwartz seminorms, uniformly on compact sets of data and group elements. The Fourier identity first holds on products of one-coordinate Schwartz functions and then on the whole Schwartz space by the same summable bound.

To compare the two restricted subseries, we identify the frequency in each. For an adelic reverse map $v : Y' \to Y$, retain only its entries

$$y_{ab} = v_{ab}\mathbf{1}_{\{\max(0,h'(b))<h(a)\}}.$$

In both gradings $y^\dagger$ has positive degree. The Heisenberg commutator gives

$$C_d(\rho(y^\dagger)F; w) = \psi_E(j\,\mathrm{tr}(yw))\,C_d(F; w).$$

Expand by (23.17), divide by $\psi_E(j\,\mathrm{tr}(yT))$, and average over the reverse-map adelic torus modulo its rational points. Character orthogonality keeps exactly (23.16). The resulting mean

$$\int_{\mathrm{Hom}(Y',Y)(E^+)\backslash\mathrm{Hom}(Y',Y)(\mathbb{A})} \frac{\theta(\rho(y^\dagger)F)}{\psi_E(j\,\mathrm{tr}(yT))}\,dv \tag{23.18}$$

does not depend on d. Normalized torus measure and absolute convergence justify the termwise mean.

The quotient $[R]$ is compact: its unipotent quotient is compact, and its definite Levi is anisotropic. Choose a compact set of representatives. Rational source covariance permutes the classes, so the full absolute sum is unchanged when reducing r to this set. Continuity and the seminorm bound give the stated uniformity in r and on compact h-sets, for either grading.

For $d = 0$, every rational target unitary also permutes the classes: there is no target parabolic restriction. We may therefore reduce h modulo its full rational group before applying the bound. On such reduced representatives, the Weil action has polynomial growth in Schwartz seminorms. This follows directly from Iwasawa coordinates: diagonal elements dilate coordinates and their derivatives by powers of their entries, unipotents multiply by quadratic phases, and the remaining real compact factor has bounded action on the required seminorms. The finite fractional lattice index obeys the corresponding product bound. Hence its polynomial growth is absorbed by a fixed power of the reduced adelic height. All splitting-character values have modulus one. This proves the final assertion for $d = 0$. For $d = h'$, only its rational parabolic preserves the classes; no bound in terms of the height on the full automorphic quotient is asserted or needed. The absolute expansion and the precise common Fourier coefficient; Uniform absolute bounds and decay after unfolding.
Lean. `ConstantTerm` (L492-L655). □

Theorem 23.10 (The exact one-step Bessel unfolding)**.** *Define*

$$\mathcal{W}'(h) = \int_{[R]} W(r)\,C_{h'}(\omega(r,h)\Phi; T)\,dr. \tag{23.19}$$

With the original Haar measures and constant-term means,

$$\int_{[R]} W(r)\theta(\omega(r,h)\Phi)\,dr = \sum_{\gamma\in R'(E^+)\backslash U(Y')(E^+)} \mathcal{W}'(\gamma h). \tag{23.20}$$

For every rapidly decreasing automorphic z on $U(Y')$, the unfolded integrand is absolutely integrable and

$$\int_{[R]} W(r) \int_{[U(Y')]} \theta(\omega(r,h)\Phi)z(h)\,dh\,dr = \int_{R'(E^+)\backslash U(Y')(\mathbb{A})} \mathcal{W}'(h)z(h)\,dh. \tag{23.21}$$

The function $\mathcal{W}'$ is continuous, rationally R'-invariant, and generic on its subgroup $U'^{(2)}$. Every identity holds with the same measures while the finite tame character varies.

Proof. Expand the kernel by Lemma 23.9 with $d = 0$. For an admissible $u \in U^{(2)}$, Weil covariance and Heisenberg translation give the exact law

$$\begin{aligned} &W(ur)C_d(\omega(ur,h)\Phi;w) \\ &\qquad = \frac{\beta(Tu,T)}{\beta(wu,w)} W(r)C_d(\omega(r,h)\Phi;w). \end{aligned} \tag{23.22}$$

To see the phase, first replace w by wu in the constant term. Their difference is of positive degree and hence is absorbed by its mean, with factor $\beta(wu,w)^{-1}$. The source character is $\beta(Tu,T)$, by (23.10). Unipotence gives determinant and Haar Jacobian one, so no character or measure scalar has been dropped.

Group classes by their rational R-orbits. On each orbit, the rational covariance of the kernel unfolds its sum to a quotient of R by the stabilizer of one class. If that representative does not survive, (23.22) gives a factor of modulus one different from one. Lemma 23.6 makes its integral zero. The absolute bound in Lemma 23.9 justifies both the orbit grouping and the interchange with integration. In particular, this argument does not require the chosen killing element to normalize the rational stabilizer.

By Lemma 23.8, the surviving part consists of rational translates of $\mathcal{C}_0$, in disjoint blocks indexed by $R'(E^+)\backslash U(Y')(E^+)$. Inside one block use (23.18) to replace its subseries by the subseries over $\mathcal{C}_{h'}$. Apply the same orbit cancellation there. The last assertion of Lemma 23.8 leaves only the class of T. That class is fixed by the rational source Bessel group, modulo $A_{h'}^{\geq 0}$, so its integral is exactly (23.19). Rational covariance gives its translate $\mathcal{W}'(\gamma h)$ in every other block. This proves (23.20).

For the absolute convergence of the final unfolded integral, let $\mathcal{D}$ be the disjoint union of the rational translates of $\mathcal{C}_0$. The integral identity for each block, proved above, and its absolute convergence give

$$\begin{aligned} \sum_{\gamma \in R'(E^+)\backslash U(Y')(E^+)} |\mathcal{W}'(\gamma h)z(h)| &\leq \int_{[R]} \sum_{w \in \mathcal{D}} |W(r)C_0(\omega(r,h)\Phi;w)z(h)|\, dr \\ &\leq C \ \mathrm{vol}([R]). \end{aligned} \tag{23.23}$$

Indeed, the sum in the middle is bounded by the full $d = 0$ absolute sum, whose polynomial growth in reduced height is absorbed by the rapid decay of z. The same bound applied to all classes also bounds $|W(r)\theta(\omega(r,h)\Phi)z(h)|$. The automorphic quotient of $U(Y')$ has finite Haar volume. Tonelli applied to these bounds justifies integration in h, r, summation over rational cosets, and ordinary quotient unfolding. It proves both integrability and (23.21), using only the $d = 0$ global bound.

Continuity follows from the compact quotient in (23.19) and uniform bounds on compact h-sets. Rational R'-invariance follows by substitution in the constant term. For $U'^{(2)}$, the same covariance calculation now uses $TT^\dagger = \mathfrak{e}_{Y'}$; it gives the required generic character of R'. None of these changes of variables alters the real weights or the measure normalization. The finite characters occur only inside the same Weil action, so the identity is simultaneous in the finite tame character.

Lean. `Unfolding` (L225-L553). □

23.2 Finite Bessel cuts, their multiplicities and joint invariance

We retain the original lower spaces Y_j, their Bessel groups R_j, their finite maximal integral groups K_j^0, and their principal levels $J_{\xi,j} = \mathrm{levelY}(\xi, j)$. Write $q = \mathrm{Nm}\, Q \geq 1$, let $\mathbf{q}$ be the finite idele generating the twist ideal, and put

$$B = M^{fn_m^2}, \qquad m = 2\varrho + 2.$$

Thus q is the norm of the twist modulus, with $(q, N) = 1$; it does not include the auxiliary level prime N. The construction below covers precisely the Bessel orbits that satisfy the lattice bounds in (23.9). It does not assert that an unfolded period has this support; that is the separate support theorem.

Lemma 23.11 (Primitive integral Witt extension)**.** *Let $R = \widehat{\mathcal{O}}_E$ be the ring of finite integral adeles of the field E of the construction. Call an integral matrix $X : R^s \to R^n$* primitive *if it has a left inverse over R, equivalently if its columns form a basis of a direct summand of R^n. Every unimodular Hermitian matrix over R has an orthonormal basis over R. If $X, Y : R^s \to R^n$ are primitive and $X^*X = Y^*Y$, there is a unitary matrix U over R with $UX = Y$, even when the common Gram matrix is singular.*

Proof. Take the product over all finite places of E^+ of the orthonormal bases in Lemma 21.37. For the extension, the left inverses identify R^s with each of the two summands, so $Xa \mapsto Ya$ is an isomorphism. It preserves the form since $(Xa)^*(Xb) = a^*(X^*X)b = a^*(Y^*Y)b$. Apply the same lemma at each finite place of E^+. The product of the integral unitary extensions belongs to $U_n(R)$ and sends X to Y.

Lean, comparison. Lean takes integral left inverses in its definition of a primitive matrix and constructs the adelic extension. Here Lemma 21.37 extends the isometry locally. `IsPrimitiveMatrix` (L68-L73), `integralorthonormalbasis` (L326-L359), `adelicWittextension` (L503-L531). □

Lemma 23.12 (Only finitely many relevant Bessel orbits)**.** *Fix a lower stage $j < m$. Let $\Lambda(x)$ be the original self-dual lattice attached to $x \in U(Y_j)(\mathbb{A}_f)$, and let $\Lambda(x)^- = \Lambda(x) \cap X_j^\perp$ be its intersection with the string space. There is a finite set $C_{\xi,j}$ of points supported on $Q \cup \bar{Q}$ such that every $R_j(\mathbb{A}_f)$-orbit having a representative with*

$$\bar{\mathbf{q}}^{\,j}\Lambda(1)^- \subseteq \Lambda(x)^- \subseteq \mathbf{q}^{-j}\Lambda(1)^- \tag{23.24}$$

meets $C_{\xi,j}K_j^0$. In particular this conclusion applies to the original predicate HasBoundedOrbit*.*

Proof. There are only finitely many string lattices between the two bounds: they are determined by their image in

$$\mathbf{q}^{-j}\Lambda(1)^-/\bar{\mathbf{q}}^{\,j}\Lambda(1)^-,$$

a finite additive group. Choose a point for each intersection that occurs. Replacing its components outside $Q \cup \bar{Q}$ by the identity leaves its string intersection unchanged. Indeed both scalar bounds are local units at such a place, so the intersection there is exactly the original one. At the retained places the lattice is unchanged. Equality of lattices can be tested at every local component.

Suppose Λ, Λ' have the same intersection L^- with the string space. For each finite place w of E, choose an $\mathcal{O}_{E_w}$-basis of L_w^-. Its inclusion in either lattice is saturated, since L_w^- is the intersection with a vector subspace. The quotient is thus finite free over the discrete valuation ring $\mathcal{O}_{E_w}$, so the basis extends to a basis of either lattice. Projection onto its coordinates supplies the two local left inverses. Their products give integral adelic left inverses, and the two coordinate matrices have the same Hermitian Gram matrix since they represent the same vectors in orthonormal frames. Lemma 23.11 extends their identification to an integral unitary matrix. Conjugating back by the two lattice frames gives an isometry carrying Λ to Λ' and fixing the string space pointwise. It therefore belongs to $U(X_j)$, which is contained in the Bessel group. Choose it to be the identity at all places where the two lattices already agree; the remaining choices form a finite adelic point.

Thus some $r \in R_j(\mathbb{A}_f)$ carries x's lattice to that of the selected point c. Equality of the lattices is equivalent to $c^{-1}rx \in K_j^0$, proving the assertion. A preliminary Bessel translate in the definition of HasBoundedOrbit is simply incorporated into r. The finite argument does not need to produce a graded representative; its existence, when used to prove support, is a separate input.

Lean. `LatticeBounds` (L125-L225). □

Lemma 23.13 (Cuts with multiplicities dividing powers of q)**.** *For each $0 < j < m$ there is a finite subset $\Omega_{\xi,j} \subset U(Y_j)(\mathbb{A}_f)/J_{\xi,j}$ with the following properties. It is stable under $R_j(\mathbb{A}_f) \cap K_j^0$. Every point over it is integral away from q, in particular at every place above N. Its covering multiplicity*

$$a_{\xi,j}(x) = \#\{r(R_j(\mathbb{A}_f) \cap K_j^0) : r^{-1}xJ_{\xi,j} \in \Omega_{\xi,j}\} \tag{23.25}$$

is constant on Bessel orbits. It is positive and divides a power of q at every point of the cut and at every point whose Bessel orbit satisfies (23.24). For $j = 0$, take the image of the prescribed level modulo $J_{\xi,0}$ and set $a_{\xi,0} = 1$.

Proof. Abbreviate the finite adelic group by G, the Bessel subgroup by S, the maximal level by A, and the principal level by $J \subset A$. The set $D = C_{\xi,j}A$ from Lemma 23.12 is compact, integral away from q, and meets all required S-orbits. Its image in G/J is finite. Select one class from each S-orbit in that image, obtaining a finite set C, and define

$$\Omega = (S \cap A)C. \tag{23.26}$$

This set is finite because it lies in the image of the compact set AD modulo the open subgroup J. It is stable under $S \cap A$, meets every required orbit, and all its representatives are integral away from q.

For a selected representative $xJ \in C$ put $Z = S \cap xJx^{-1}$, its stabilizer in S. The classes counted in (23.25) at x are precisely the image of Z in $S/(S \cap A)$. Indeed if $r^{-1}xJ \in (S \cap A)C$, the unique selected class in that S-orbit is xJ, so $r \in Z(S \cap A)$. The converse is immediate. Consequently

$$a(x) = [Z : Z \cap A]. \tag{23.27}$$

It is finite and positive, since Z is contained in the compact group xJx^{-1} and A is open. Left multiplication in S proves constancy of (23.25) on its orbits.

For $x \in D$ and each $w \in Q$, choose $d_w \geq 1$ so large that conjugation by x_w takes the principal congruence subgroup of depth d_w into the integral group. Such a depth exists since conjugation is continuous. Let J_0 be the kernel on J of reduction modulo $\mathfrak{p}_w^{d_w}$ at all $w \in Q$. Then $xJ_0x^{-1} \subseteq A$: this holds at Q by construction, at the conjugate places by unitarity and self-duality of the lattice, and elsewhere because x_w and J_w are integral.

At $w \in Q$, J is the identity modulo $\mathfrak{p}_w$. The successive quotients of the congruence filtration embed, by $1 + M \mapsto M$, in the additive matrix groups over $\mathfrak{p}_w^i/\mathfrak{p}_w^{i+1}$ for $1 \leq i < d_w$. Hence $|J/J_0|$ divides a power q^b, since all these residue characteristics divide q. In particular

$$(xux^{-1})^{q^b} \in xJ_0x^{-1} \subseteq A \qquad (u \in J). \tag{23.28}$$

For a selected x, $Z/(Z \cap xJ_0x^{-1})$ embeds in J/J_0 by conjugation. Thus $[Z : Z \cap A]$ divides $|J/J_0|$. Orbit constancy proves the assertion throughout the required set. When $q = 1$, Q is empty, $J_0 = J$, and the index is one.

At $j = 0$ the chosen level is compact and contains $J_{\xi,0}$, so the prescribed image is finite and integral. Its multiplicity one is the original initial cut convention, not (23.25).

Lean. `IntegralAsCosetSum` (L67-L214), `Cut` (L44-L167). The formal proof bounds powers and passes to the normal core. Here the congruence quotient bounds the index directly. □

Lemma 23.14 (The untwisted count and the sum of the weights)**.** *For $Q = \varnothing$ every multiplicity is one and*

$$\#\Omega_{\xi,j} \leq B \qquad (0 \leq j < m). \tag{23.29}$$

For every twist the product weight $\prod_{j<m} a_{\xi,j}(y_j)^{-1}$ is integral away from q. Thus the tensor summed over the finite representatives of Lemma 23.4 satisfies all its stated coefficient hypotheses, with the exact bound B^m.

Proof. For the empty twist every point of the cut is integral. The principal level of π_M is contained in $J_{\xi,j}$. Choose one integral representative of each class in the cut; their classes modulo that principal level are distinct, giving an injection into its quotient in K_j^0. Reduction modulo π_M bounds its index by the number of residue matrices:

$$\#\Omega_{\xi,j} \le |\,\widehat{\mathcal{O}}_E/(\pi_M)\,|^{(\dim Y_j)^2} = M^{f(\dim Y_j)^2} \le M^{fn_m^2} = B.$$

The last inequality is the rank bound $\dim Y_j \le n_m$. Positivity and divisibility by a power of 1 force every multiplicity to be exactly one.

For a general twist each positive multiplicity divides a power of q; its inverse is consequently a unit at every coefficient prime not dividing q. For the empty twist, there are at most B^m tuples and all weights are one. Applying the coefficient norm bound of Theorem 23.3 to each tuple proves the denominator and coefficient bounds.

Lean, formal context. `Cut` (L44-L167), `SeesawIdentity` (L624). □

Lemma 23.15 (Equivariance of the finite evaluation)**.** *Let $\mathcal{E}_\xi(y)$ be the tensor in Theorem 23.3. Write* levelX(j) *for the upper-chain level on X_j, the intersection of the chosen level with the subgroup whose distinguished rows agree modulo π_N with those of the identity. If $b_j \in$ levelX(j) for $j < m$ and $b_m = 1$, let $\widetilde{b}_j$ denote the original embedding of b_j in $U(Y_j)(\mathbb{A}_f)$, and put $\widetilde{b}_m = 1$. Then*

$$\mathcal{E}_\xi((\widetilde{b}_j y_j)_j) = \mathrm{Joint}_\xi(b)\mathcal{E}_\xi(y). \tag{23.30}$$

The datum depends only on the classes $y_j J_{\xi,j}$.

Proof. The point datum is fixed by its original lower principal level: it fixes the prescribed residue inclusion, fixes the vacuum elsewhere, and its character is trivial on that principal subgroup. Hence its translate depends only on the indicated cosets.

Consider a pair $V \to V' \to V''$ and the joined datum on $\mathrm{Hom}(V', V^- \oplus V'')$. The arrow identification intertwines the two edge actions with the corresponding action of three groups on this space, including their original splitting characters. Each embedded b acts on the definite part and as the identity on the string. Thus it fixes the string maps:

$$\widetilde{b}'\,T\,\widetilde{b}^{-1} = T, \qquad \widetilde{b}''\,T'\,(\widetilde{b}')^{-1} = T'.$$

It follows that the joined action fixes the stacked shift $\tau = (T^*, T')$, so it commutes with its Heisenberg translation with no extra cocycle.

The same action preserves the positive-grade isotropic space and its remaining middle plane, and on the quotient it is exactly the three definite group actions. A change of variables in the zero-term integral now intertwines the two actions. There is no scalar left over at these finite level points. Its determinant modulus is one because the b_j are integral units; the additional splitting-character factors are powers of $v(\det b_j)$, also one on integral finite determinants. The tame characters themselves appear on the same determinants before and after the quotient, since these embeddings preserve determinants. Thus their factors agree exactly. The normalization separating the finite zero term from its fixed real factor is unchanged by a purely finite action.

Applying the inverse arrow identification proves (23.30) for each middle pair. In the last pair the quotient embeds the old last definite group in the enlarged last one; here both endpoint actions are the identity by $b_m = 1$. The same calculation therefore proves the last pair. Tensoring the pairs gives (23.30). This is a direct verification of zero-term and finite evaluation equivariance on exactly the levels needed for the chain; it uses the original evaluation, not an abstract tensor with matching dimensions.

Lean. `EdgePairEvaluation` (L929-L1027). □

Lemma 23.16 (Integrality of the embedded level subgroup)**.** *The original embedding of* levelX(j) *belongs to* $R_j(\mathbb{A}_f) \cap K_j^0$*. At* $j = 0$ *it also lies in the prescribed level of* Y_0*.*

Proof. Write o for the last definite coordinate, s for the string length, and $\pi = \pi_N$. In the original graded lattice frame there is an integral row matrix R for which the embedded matrix satisfies

$$\widetilde{b} - I = R^* M_b R, \qquad M_b = \begin{cases} D_{\pi^{-1}}(b - I) D_{\bar{\pi}^{-1}}, & s \text{ odd}, \\ b - I, & s \text{ even}, \end{cases} \tag{23.31}$$

where D_c is diagonal with entry c at o and one elsewhere. To derive this formula, write the original frame as its explicit graded lattice basis times an integral invertible matrix k. The definite rows of that frame are the corresponding rows R of k, with the last divided by $\bar{\pi}$ exactly when s is odd. In rational coordinates the embedded action is b on the definite part and the identity on the string. Its difference from the identity therefore has only that definite block. Conjugating by the frame, and using its Hermitian Gram identity, gives (23.31).

The level condition says that b is integral and its last row agrees with the last row of I modulo π. The same row congruence holds for b^{-1}; unitarity then gives the last-column congruence of b modulo $\bar{\pi}$. At every place at least one of $\pi, \bar{\pi}$ is a unit. Thus both divisions in (23.31), including their common last diagonal entry, leave an integral matrix. Consequently $\widetilde{b}$ is integral. Its action fixes the string pointwise, proving its Bessel membership.

At the chosen places the original level also has $b - I$ zero modulo π_M. Both π and $\bar{\pi}$ are units there, so M_b, and then $R^* M_b R$, have the same congruence. This proves the chosen-level assertion for the initial stage.

Lean. `GluedLevel` (L52-L165). □

Corollary 23.17 (Joint level invariance of the tensor)**.** *The tensor*

$$\phi_\xi = \sum_{y \in \prod_{j<m} \Omega_{\xi,j}} \left(\prod_{j<m} a_{\xi,j}(y_j)^{-1} \right) \mathcal{E}_\xi(y) \tag{23.32}$$

is jointly invariant under the upper-chain levels, with the terminal level element equal to one.

Proof. For $j > 0$ the embedded group levelX(j) lies in $R_j(\mathbb{A}_f) \cap K_j^0$ by Lemma 23.16. Thus it permutes $\Omega_{\xi,j}$ and preserves its multiplicity by Lemma 23.13. At $j = 0$ its embedding lies in the prescribed level, so it permutes that level's image modulo $J_{\xi,0}$; the initial multiplicity is one. Apply (23.30) inside the finite sum (23.32), and reindex each coordinate by these permutations. Both the weights and the index set are unchanged, proving the assertion.

Lean. `Cut` (L191-L237), `SeesawIdentity` (L509-L535). □

Lemma 23.18 (Weighted cuts retain the exact level index)**.** *Fix an interior stage* $0 < j < m$*, its real factor* $a(g_\infty)$*, its Bessel group* $R = R_j$*, and its maximal finite level* $K^0 = K_j^0$*. There is a scalar* κ*, depending only on these data and their Haar measures, with the following property. For a lower level* $J \subset K^0$ *in the chain, take the cut* Ω *of Lemma 23.13, stable under* $R(\mathbb{A}_f) \cap K^0$*, and use its covering multiplicity* a_Ω *from* (23.25)*. Suppose* F *is absolutely integrable on* $R(E^+)\backslash U(Y)(\mathbb{A})$*, left rationally invariant and continuous. Suppose* F *vanishes on all finite Bessel orbits missed by* Ω*, and its Bessel period at points over* Ω *is*

$$\int_{[R]} F(rt)\, dr = [K^0 : J]\, a(t_\infty) b(t_f J). \tag{23.33}$$

Then

$$\int_{R(E^+)\backslash U(Y)(\mathbb{A})} F(g)\,dg = \kappa \sum_{c\in\Omega} a_\Omega(c)^{-1} b(c). \tag{23.34}$$

In particular κ is chosen before the tame twist and does not depend on J, Ω, b, or F.

Proof. Write $G = U(Y)(\mathbb{A}) = G_\infty \times G_f$ and $R(\mathbb{A}) = R_\infty \times S$. Both groups are unimodular. For R, its unipotent radical and its Levi $U(X)$ are unimodular, and the Levi action on each pair of string coordinates has determinant of absolute value one since $N_{E/E^+}(\det h) = 1$. Give K^0 volume one and normalize Haar measure η on S by $\eta(S \cap K^0) = 1$.

For $O = SxJ/J \subset G_f/J$, put $Z = S \cap xJx^{-1}$. In S the set $\{s : s^{-1}xJ \in \Omega\}$ is a union of $a_\Omega(xJ)$ cosets of $S \cap K^0$ by (23.25). Its fiber over each point of $\Omega \cap O$ is a translate of Z. Thus

$$a_\Omega(xJ) = |\Omega \cap O|\,\eta(Z). \tag{23.35}$$

In particular the integral of $\mathbf{1}_\Omega/a_\Omega$, extended by zero off the cut, along any represented orbit is one. The image of xJ in $S\backslash G_f$ has quotient volume $\mathrm{vol}(J)/\eta(Z)$: in the quotient integral formula for $\mathbf{1}_{xJ}$, the inner integral equals $\eta(Z)$ on this image and zero elsewhere [DE09, Theorem 1.5.2].

Apply quotient integration in stages, first to $|F|$ and then to F. Its inner period $P(t) = \int_{[R]} F(rt)\,dr$ is left $R(\mathbb{A})$-invariant by right invariance of the measure on $[R]$. Its integral over $R(\mathbb{A})\backslash G$ equals the desired integral. Use the real quotient measure on $R_\infty\backslash G_\infty$ compatible with this formula and the finite measures just fixed. If a descends to an integrable function on this real quotient, take κ to be its integral. Otherwise take $\kappa = 0$. This choice depends only on the fixed real function and measures.

If P vanishes on the cut, it vanishes everywhere by invariance and the support hypothesis. Then either $a = 0$ or b vanishes on the cut by (23.33), proving (23.34). Otherwise some $b(c) \neq 0$ on the cut. On its finite orbit the period equals $[K^0 : J]\,a(t_\infty)b(c)$ by invariance. Thus a descends to the real quotient and is integrable there, since P is integrable and the image of this orbit has positive finite volume. Choose t_∞ with $a(t_\infty) \neq 0$. The same invariance shows that b has a common value $b(c)$ on the cut portion of each represented orbit O. Since $[K^0 : J]\,\mathrm{vol}(J) = 1$, that orbit contributes

$$\frac{\kappa b(c)}{\eta(Z)} = \kappa \sum_{d\in\Omega\cap O} \frac{b(d)}{a_\Omega(d)}$$

by (23.35). Summing over the represented orbits proves (23.34).

Lean. `IntegralAsCosetSum` (L353-L426) uses a cutoff on the real group. Here the measure of SxJ in $S\backslash G_f$ gives each orbit's contribution directly. □

Lemmas 23.13–23.18 give the cuts and the interior-stage integration formula for Proposition 23.5. At stage zero we use the level trace (23.88). For the interior stages we prove (23.33), absolute integrability, and vanishing off the represented orbits in Sections 23.1, 23.3, and 23.4.

23.3 Two constant terms, finite evaluation, and the terminal coefficient

Use probability Haar measure on $\mathbb{A}_E^d/E^d$ and the Bessel-period quotient measures of Section 8 on the unitary and Bessel groups. We compute the resulting constants to identify two successive constant terms with the finite evaluation in Theorem 23.3.

The graded hermitian spaces Y, Y', Y'' have underlying spaces V, V', V'', definite parts X, X', X_0, and string lengths

$$s, \qquad s' = s - 1, \qquad s'' = (s-2)_+, \qquad s \text{ odd}.$$

The nonzero scalar $c \in E^+$ defining the generic character is the same for Y, Y'. Put

$$T : Y \longrightarrow Y', \qquad T' : Y' \longrightarrow Y'', \qquad S = \begin{pmatrix} T^\dagger \\ T' \end{pmatrix} : V' \longrightarrow V^- \oplus V''.$$

The adjoints in this formula are taken in rational coordinates. The data D for this pair consist of an isometric identification of the part of grade zero of $V^- \oplus V''$ with

$$H^q \perp X^- \perp X'', \tag{23.36}$$

and an embedding $e_D : U(X_0) \to U(X'')$ preserving determinants. The identification intertwines the actions of $U(X)$ and $U(X_0)$, and identifies the grade-zero part of the column of S at the selected grade-one vector of Y' with c times the second vector of the corresponding hyperbolic plane. All other such columns vanish. These are explicit matrix requirements, including when $q = 0$; they are precisely the requirements on the original middle and last pair. There is at most one grade-one vector, so $q = 0$ or 1. We also require $\dim V + \dim V''$ even, as in those pairs.

Let D_+ remove the positive-degree maps in $\mathrm{Hom}(V', V^- \oplus V'')$, and let D_{pl} remove the maps from X' into the q hyperbolic planes. The composite splitting $D = D_+ \circ D_{\mathrm{pl}}$ has complement

$$H_D = \mathrm{Hom}(X', X^- \oplus X'').$$

For a splitting with isotropic frame k and complement embedding i, write

$$(\mathcal{Z}_D f)(z) = \operatorname*{mean}_{b \in \mathbb{A}_E^d / E^d} f(iz + kb). \tag{23.37}$$

Functions in the Weil model are evaluated in their reproducing realization. Thus evaluation at zero is its theta functional, and (23.37) is also the usual zero slice of the mixed model. We write $f \boxtimes_a f'$ for the original joining map: apply the adjoint flip to the first edge, then take the tensor product over the two target blocks. Its restriction to finite parts is denoted by A. Thus

$$A : \mathcal{S}_f(\mathrm{Hom}(X, X')) \otimes \mathcal{S}_f(\mathrm{Hom}(X', X'')) \longrightarrow \mathcal{S}_f(H_D). \tag{23.38}$$

Lemma 23.19 (Elementary identities of the joined mixed model)**.** *There is a fixed $z_\theta \neq 0$, depending only on the framed spaces and the Weil normalization, such that*

$$\theta(f \boxtimes_a f') = z_\theta \, \theta(f)\theta(f'), \qquad \rho\begin{pmatrix} w^\dagger \\ w' \end{pmatrix}(f \boxtimes_a f') = (\rho(w)f) \boxtimes_a (\rho(w')f'). \tag{23.39}$$

The map A is an isomorphism onto the indicated finite part. Zero terms along composite splittings compose in their indicated order. For an adelic vector w, its finite coordinate in the Weil frame is $\mathrm{finCoord}\, w = \alpha^{-1} w_f$*, where α is the self-duality idele. For fixed w and a real Fock vector u there is a real Fock vector u_D, independent of the finite datum Φ, for which*

$$\mathcal{Z}_D\big(\rho(w)(u \otimes \Phi)\big) = u_D \otimes \mathcal{Z}_{D,f}\big(\rho_f(\mathrm{finCoord}\, w)\Phi\big). \tag{23.40}$$

The finite Heisenberg operator ρ_f includes its semicharacter.

Proof. In coordinates adapted to an orthogonal sum, the reproducing kernel, Heisenberg bicharacter and theta sum factor. This gives the tensor product identity and the second equality in (23.39). The adjoint map identifies the first symplectic space with the opposite target model. Substitution in the rational theta sum, or Poisson summation in a polarization in which this change uses a

Fourier transform, gives a fixed Weil phase. This is z_θ; it has modulus one in the original unitary normalization. No character of either edge is involved.

Finite parts are the spans of finite Heisenberg translates of the vacuum. The joined vacuum is the vacuum of the sum. Translation in its two coordinate blocks is the product of the two translations, with the first block conjugated and multiplied by the fixed *finite adelic* unit of norm -1 used in the adjoint flip; its semicharacter contributes a nonzero scalar. Consequently the image of A is exactly the finite part. It is injective because evaluation at all pairs of points separates the algebraic tensor product of two spaces whose point evaluations separate their elements. More explicitly, write a tensor using a linearly independent finite list in the first factor, choose evaluations giving an invertible matrix on that list, and recover every coefficient in the second factor.

For the assertion about a splitting, choose the same adapted frame used to define $\mathcal{Z}_{D,f}$. In that frame the mixed model is the tensor product of the model on the complement and the Schrödinger models of the hyperbolic planes. Its zero slice acts separately at the infinite and finite places. The adapted real Gaussian has a nonzero zero slice. Dividing the real operator by this fixed Gaussian value makes its finite operator exactly $\mathcal{Z}_{D,f}$, as defined using that Gaussian. This fixes the scalar once, before Φ is specified. Heisenberg translation also factors into its real operator and the original ρ_f, including its semicharacter, proving (23.40). Finite Schwartz functions on every finite-dimensional mixed model are locally constant with compact support, so their zero slices remain in the finite part. Here the relevant norm is the L^1 norm of the *reproducing-model coefficient*, not a Schrödinger L^2 norm. To make boundedness precise, choose an adapted frame g, write its model as the hyperbolic model H times the complement model V_0, and take the compact mean at complement coordinate zero. This is a bounded scalar functional ℓ: the theta functional is bounded and the Heisenberg operators are isometries. Rational invariance in the complement, equivalently Fourier orthogonality for $E^{\dim V_0} \subset \mathbb{A}_E^{\dim V_0}$, gives $\ell(w \otimes v) = c(w)\theta_{V_0}(v)$ for a fixed bounded functional c. On translated vacua the resulting complement vector $K(z)$ therefore satisfies

$$\|K(z)\| \le \|c\| \, \|\operatorname{vac}_H\| \, \|\operatorname{vac}_{V_0}\|.$$

Writing c_f for its integrable reproducing-model coefficient, the reproducing identity writes a vector f as $\int c_f(z)\rho(z)\operatorname{vac}\, dz$, with $\|c_f\|_1 = \|f\|_1$. Integration of $K(z)$ against its coefficient hence proves

$$\|\mathcal{Z}_D f\|_1 \le \|\omega(g^{-1})\| \, \|c\| \, \|\operatorname{vac}_H\| \, \|\operatorname{vac}_{V_0}\| \, \|f\|_1.$$

This also constructs the zero-term operator with exactly the values (23.37). The assertions therefore extend from polynomial Gaussian vectors to every real Fock vector. Finally, two successive means in (23.37) are the mean on the product compact additive quotient. Fubini gives composition of zero terms, with no scalar.

Lean. `TwoEdges` (L62-L311), `ZeroTerm` (L79-L141), `EdgePairEvaluation` (L88-L180). □

Lemma 23.20 (The Bessel mean and its row)**.** *Let $B_Y = N_Y \rtimes U(X)$ be the Bessel group of a grading. Its rational quotient is compact. Select any subset q of the grade-one string indices, and put*

$$\operatorname{row}(n)_{a,x} = c\,(n-1)_{a,x} \qquad (n \in N_Y(\mathbb{A})). \tag{23.41}$$

There is a fixed positive constant κ_Y, depending on the quotient measures, such that the following assertion holds. Suppose F is continuous and left $B_Y(E)$-invariant, and

$$F(n\, i_Y(h)) = f_h(\operatorname{row}(n)),$$

where each f_h is continuous and $E^{q\times X}$-periodic. If its mean $M(h)$ is continuous and left $U(X)(E)$-invariant, then

$$\int_{[B_Y]} F = \kappa_Y \int_{[U(X)]} M(h)\, dh. \tag{23.42}$$

For $q = 0$, this simply removes a unipotent-invariant factor.

Proof. The unipotent group is defined by matrices which strictly raise the grading. Successive superdiagonals give a finite filtration whose quotients are additive E^+-vector groups, or additive E-vector groups paired by the hermitian condition. The quotients of all these vector groups by rational vectors are compact. Reduction in this filtration proves compactness of $[N_Y]$. Since X is definite, $[U(X)]$ is compact; the semidirect decomposition proves compactness of $[B_Y]$. The same argument gives finite positive quotient volumes and the quotient integration formula through $[N_Y]$ and $[U(X)]$. The possible normalization constant is fixed by applying that formula to the constant function one.

The cross term $(n-1)(n'-1)$ raises grades by at least two, whereas (23.41) goes from grade zero to grade one. Hence $\mathrm{row}(nn') = \mathrm{row}(n) + \mathrm{row}(n')$. This homomorphism is surjective on adelic points and on rational points. To check the latter assertions directly when $q = 1$, let e be the grade-one isotropic vector. For a prescribed row b, choose $y \in X$ with $h(y, x) = c^{-1} b_x$ on the standard basis of X. The Eichler map

$$z \longmapsto z + h(y, z)e - h(e, z)(y + \tfrac{1}{2}h(y, y)e) \tag{23.43}$$

is unitary, raises the grading, and has row b. It depends continuously on b, and has rational entries for rational b. The case $q = 0$ is immediate.

Normalize the quotient measure on $[N_Y]$ to mass one. Its pushforward by the row is invariant under every translation of $\mathbb{A}_E^{q\times X}/E^{q\times X}$: right translation by (23.43) realizes that translation of the row, and unipotent groups are unimodular. Thus the pushforward is precisely Haar probability. Integrating first on $[N_Y]$ gives $M(h)$. Quotient integration then proves (23.42). Every integral used here is an integral of a continuous function on a compact rational quotient and is absolutely convergent.

Lean. `BesselIntegral` (L130-L203), `BesselIntegral` (L273-L519). □

Lemma 23.21 (Exact intertwining of the pair zero term)**.** *For arbitrary automorphic characters a, b of $U(1)$, the original triple Weil actions satisfy*

$$\mathcal{Z}_D\, \omega^{\mathrm{tr}}_{a,b}(i_Y(g), i_{Y'}(h), i_{Y''}(k)) = \omega^{\mathrm{tr}}_{a,b}(g, h, e_D(k))\, \mathcal{Z}_D. \tag{23.44}$$

There is no extra character or modulus in this formula.

Proof. Let n_0, n_1, n_2 be the ranks of V, V', V''; let d_0, d_1, d_2 be those of X, X', X''. Let r be the number of positive-grade basis vectors in $V^- \oplus V''$. The rank identities are

$$n_1 = d_1 + s', \qquad n_0 + n_2 = d_0 + d_2 + 2r + 2q, \qquad s' \text{ even}, \quad n_0 + n_2 \text{ even}. \tag{23.45}$$

Before either zero term, write d_g, d_h, d_k for the three determinants. The character part of the triple action is

$$\frac{a(d_h)}{a(d_g)}\,\frac{b(d_k)}{b(d_h)}\,\upsilon\Big(d_g^{\lceil n_1/2\rceil} d_h^{-(n_0+n_2)/2} d_k^{\lfloor n_1/2\rfloor}\Big). \tag{23.46}$$

Indeed, the two rank-twisted pair actions give their two character ratios, and the adjoint flip contributes $\upsilon(d_g^{n_1} d_h^{-n_0})$. Expanding the three factors gives exactly (23.46).

The group element on $V^- \oplus V''$ acts only in grade zero. Its determinant is $d_g d_k$. On positive-degree maps from V', it acts with determinant

$$(d_g d_k)^{s'/2} d_h^{-r}. \tag{23.47}$$

To see this, its target action occurs once for every negative string grade of V', of which there are $s'/2$; the inverse source action of h occurs once for every positive target grade, of which there are r. All other diagonal blocks are identities. The zero-slice law in the mixed model contributes the inverse of υ at (23.47). Its modulus part is one, because all three determinants are norm-one ideles. This law follows directly by change of variables in a Schrödinger model: a Levi substitution on the isotropic coordinate has its usual half-density and splitting-character multiplier; the unipotent radical fixes the zero slice. Removal of the q hyperbolic planes contributes $\upsilon(d_h)^q$, by applying the same formula to their isotropic frame, on which h^{-1} occurs q times.

Thus the three exponents in (23.46) become

$$\left\lceil \frac{n_1}{2} \right\rceil - \frac{s'}{2}, \quad -\frac{n_0+n_2}{2} + r + q, \quad \left\lfloor \frac{n_1}{2} \right\rfloor - \frac{s'}{2}.$$

By (23.45), these are $\lceil d_1/2 \rceil, -(d_0+d_2)/2, \lfloor d_1/2 \rfloor$, the exponents for the definite triple. The two character ratios are unchanged since e_D preserves determinants. The complement action is the required action by the intertwining requirement in (23.36). This proves (23.44).

Lean. `EdgePairEvaluation` (L879-L978). □

Proposition 23.22 (Two constant terms give the translated zero term)**.** *For $f \in \mathcal{S}(\mathrm{Hom}(V,V'))$, write*

$$C^a_{Y,Y'}(f;g,h) = \operatorname*{mean}_{u \in L_+(\mathbb{A})/L_+(E)} \theta(\rho(u)\rho(T)\omega^a(g,h)f)\,, \tag{23.48}$$

where L_+ consists of the positive-degree maps and ω^a has the original rank-twisted characters. Let W be a continuous, rational-invariant function on B_Y, transforming under N_Y by its generic character. There is one constant

$$z_D = \kappa_Y \kappa_{Y'} z_\theta^{-1} \neq 0 \tag{23.49}$$

such that, for every a, b, f, f', k, W,

$$\begin{aligned} &\int_{[B_{Y'}]} \int_{[B_Y]} W(g) C^a_{Y,Y'}(f;g,h) C^b_{Y',Y''}(f';h,i_{Y''}k)\,dg\,dh \\ &\quad = z_D \int_{[U(X)]} W(i_Y g) \int_{[U(X')]} \theta\Big(\omega^{\mathrm{tr}}_{a,b}(g,h,e_D k)\mathcal{Z}_D\big(\rho(S)(f \boxtimes_a f')\big)\Big)\,dh\,dg. \end{aligned} \tag{23.50}$$

All integrals in this identity are absolutely convergent. The constant is independent of both characters and both data.

Proof. First suppress all unitary actions. Adjoint and block concatenation give an additive bijection

$$L_+(\mathrm{Hom}(V,V')) \oplus L_+(\mathrm{Hom}(V',V'')) \longrightarrow L_+(\mathrm{Hom}(V', V^- \oplus V'')), \qquad (u,u') \longmapsto (u^\dagger, u').$$

Adjoint preserves degree, because the hermitian forms pair opposite grades. This map preserves rational points and pushes the product Haar probability to Haar probability. Equation (23.39) therefore gives, at arbitrary w, w',

$$C(f;w)C(f';w') = z_\theta^{-1}\left[\mathcal{Z}_{D_+}(\rho(w^\dagger, w')(f \boxtimes_a f'))\right](0). \tag{23.51}$$

The notation $C(f;w)$ means the mean in (23.48) with T replaced by w. There is no Jacobian in (23.51), since these are probability means.

Now let $n \in N_{Y'}$, in rational coordinates. Put $b(n)_{a,x} = c(n-1)_{a,x}$ for the selected grade-one rows. The matrix $S(n-1)$ has nonnegative degree. Its degree-zero part is precisely $i_+ k_{\mathrm{pl}} b(n)$: only

the columns of X' can have degree zero, and the stipulated grade-one columns of S in (23.36) give their values. Thus, for some positive-degree vector $v(n)$,

$$S(n-1) = k_+ v(n) + i_+ k_{\mathrm{pl}} b(n). \tag{23.52}$$

The images of k_+ and i_+ are orthogonal, so the Heisenberg phase between these two summands is one. The other potential phase is also one, because

$$S^\dagger S = -TT^\dagger + T'^\dagger T' = 0. \tag{23.53}$$

Both products in the middle are the shift matrix of Y': their string coefficients agree because Y, Y' have the same c. This remains true for the last pair, where the middle string is empty. Since $\mathcal{Z}_{D_+}$ is invariant under translation by $k_+ v(n)$, (23.51)–(23.53) imply

$$\begin{aligned} &C(f; n^{-1}T) C(f'; T'n) \\ &\qquad = z_\theta^{-1} \left[\mathcal{Z}_{D_+}(\rho(S)(f \boxtimes_a f'))\right](k_{\mathrm{pl}} b(n)). \end{aligned} \tag{23.54}$$

The identity $(n^{-1}T)^\dagger = T^\dagger n$ gives Sn for the joined translated vector on the left.

We next remove the first unipotent integral. All grades of Y are even, so every $u \in N_Y$ has $u - 1$ of degree at least two. Hence $T(u-1)$ has positive degree. The substitution law for the zero slice and the Heisenberg multiplication law give

$$C^a_{Y,Y'}(f; ug, h) = \beta(Tu, T)^{-1} C^a_{Y,Y'}(f; g, h), \qquad W(ug) = \beta(Tu, T) W(g). \tag{23.55}$$

The second identity uses $T^\dagger T = \mathfrak{e}_Y$, the shift defining the generic character. Their product is unipotent-invariant. Lemma 23.20, with empty row, replaces its integral on $[B_Y]$ by κ_Y times the integral on $[U(X)]$.

Fix such a g. In the remaining Bessel integral, left multiplication $h \mapsto nh$ moves the two frequencies to $n^{-1}T, T'n$. To justify this substitution without assuming a rational n, observe that n fixes the complement modulo the positive-degree frame and acts unitriangularly on that frame. The zero-slice law therefore has determinant and scalar one. Formula (23.54) applies to the two already translated data. Lemma 23.20 for the row $b(n)$ now averages precisely along k_{pl}. Composition of the two zero terms leaves $\mathcal{Z}_D$. The $U(X)$, $U(X')$, $U(X_0)$ actions fix both string maps, and their joined action is the triple action. Lemma 23.21 moves that action through $\mathcal{Z}_D$. The remaining expression is (23.50), with the constant (23.49).

Both Bessel rational quotients and both definite unitary quotients in this calculation are compact. The integrands are continuous and invariant under the rational points of each integration variable; invariance follows by rational substitution in the positive-degree means and in theta. They are therefore bounded on compact sets of representatives. This proves absolute integrability, justifies the indicated order changes, and proves the claim for all vectors in the reproducing model by continuity.

Lean. `UnfoldingStages` (L245-L358), `UnfoldingStages` (L360-L552). □

Lemma 23.23 (Compact polynomial theta coefficients)**.** *Let $U(A) \to U(B)$ be one of the original definite lifts, with its fixed nonzero jet scale and its fixed level normalization. Let the automorphic characters vary with fixed archimedean weights, and let $K_\eta(\Phi)$ be the original algebraic theta lift for the finite datum Φ. For every real Fock polynomial P and linear form t on the values of source forms there is one linear form t_P on the values of target forms such that*

$$\int_{[U(A)]} \mathrm{coeff}_t(\phi; g)\, \theta\big(\omega_\eta(g, h)(P \otimes \Phi)\big)\, dg = \mathrm{coeff}_{t_P}(K_\eta(\Phi)\phi; h) \tag{23.56}$$

for all η, Φ, ϕ, h. In particular t_P is independent of the finite character and finite datum.

Proof. Here $\mathrm{coeff}_t(\phi; g) = t(\pi_A(g_\infty)^{-1}\phi(g_f))$, with the original conjugate Weyl representation π_A. At each infinite place, polynomial functions on the matrix space have the Cauchy decomposition

$$\mathrm{Sym}^d(B \otimes A^*) = \bigoplus_{\lambda \vdash d} S_\lambda B \otimes S_\lambda A^*, \qquad \ell(\lambda) \le \min(\dim A, \dim B). \tag{23.57}$$

This is a decomposition with multiplicity one: apply the commuting actions of the two general linear groups to the standard bitableaux, or equate coefficients in the Cauchy identity and use complete reducibility of their compact forms. The fixed determinant twists do not alter the multiplicities. The lift-weight condition on the original edge says exactly that the source type selects the target type defining K_η. Integration over $U(A)_\infty$, by Schur orthogonality, kills every other source type in (23.57). The surviving projection is a finite linear combination of the standard Schur coefficient polynomials defining the jet matrix of K_η.

This statement also verifies the normalization in (23.56). In a tableau basis, the matrix coefficient of a Schur coefficient polynomial is its Taylor functional at the fixed jet scale, divided by the appropriate fixed power of that scale. The compact source integral contributes $(\dim \pi_A)^{-1}$. Passing from the adelic quotient integral to the normalized finite class integral contributes the fixed reciprocal of the level index. These are numbers depending on the real polynomial, weights, scale and level, and they are absorbed into t_P. No finite datum is rescaled: the remaining class sum is exactly the class sum defining $K_\eta(\Phi)$.

For a highest source coefficient the preceding projection gives (23.56) directly. Its compact translates span the dual of the source Weyl module, by irreducibility, so the formula follows for every t by linearity. The real pair action in this calculation depends on η only through the prescribed archimedean weights. The same projection coefficients therefore work simultaneously for the entire family of finite characters. The compact rational quotient makes all displayed integrals absolutely convergent.

Lean. `DefiniteThetaLift` (L195-L240), `DefiniteThetaLift` (L272-L351). □

Proposition 23.24 (One coefficient functional for a double definite lift)**.** *Let K_η, K'_η be two consecutive original definite lifts with common middle type. Their characters have fixed archimedean weights. Put*

$$L_\eta(\Phi \otimes \Phi')\phi = K'_\eta(\Phi')K_\eta(\Phi)\phi.$$

For every u in the archimedean L^1 Fock model on $\mathrm{Hom}(X', X^- \oplus X'')$ of Lemma 23.19 and every source linear form t, there is one target linear form l such that

$$\int_{[U(X)]} \mathrm{coeff}_t(\phi; g) \int_{[U(X')]} \theta\big(\omega^{\mathrm{tr}}_{a_\eta, b_\eta}(g, h, k)(u \otimes A(y))\big)\, dh\, dg = \mathrm{coeff}_l(L_\eta(y)\phi; k) \tag{23.58}$$

for every η, y, ϕ, k. The functional l may be chosen to depend continuously and bilinearly on u, t. It has no dependence on η, y, ϕ, k.

Proof. For a polynomial on the joined matrix space, split its variables into the two blocks and write it as a finite sum of products PQ. Under the adjoint flip the variables of P are transposed into the first block; no new finite datum appears. For $y = \Phi \otimes \Phi'$, (23.39) and the joined Weil action identify the integrand with z_θ times the product of the two original theta kernels. Apply Lemma 23.23 first to P, t, then to $z_\theta Q, t_P$. Fubini gives (23.58). Finite linearity gives the result for every polynomial and every algebraic tensor y.

To obtain one functional for all finite characters for a general archimedean vector, let $\mathcal{I}$ be the set of all tuples (η, y, ϕ, k), and consider the linear map

$$\Lambda : W^*_{X''} \longrightarrow \mathbb{C}^{\mathcal{I}}, \qquad \Lambda(l)_{\eta,y,\phi,k} = \mathrm{coeff}_l(L_\eta(y)\phi; k). \tag{23.59}$$

The coordinate evaluations span the dual of $E = \operatorname{im}\Lambda$, since their common kernel in E is zero. Choose $\dim E$ of them as a basis. The corresponding values determine a unique element of E, and a fixed linear section of Λ lifts it to $W^*_{X''}$. Apply these two linear maps to the selected double integrals to define $l(u,t)$. Each integral is linear in t and continuous in u: both quotients are compact, the theta functional is bounded, and the Weil operators are uniformly bounded on compact sets of representatives. Since the space of t's is finite dimensional, l is continuous and bilinear. For polynomial u, the preceding calculation shows that $\Lambda(l(u,t))$ agrees with the double integral at every coordinate.

To pass to all u, write the Fock vector as an entire function in the signed holomorphic coordinates times its Gaussian. Simultaneous circle rotations act isometrically on $L^1(dz)$ and continuously by density of compactly supported continuous functions. Their Fejér means converge to u in L^1 [Kat04, Ch. I, §2, p. 10]. The Fourier components are the homogeneous Taylor polynomials times the Gaussian, so each Fejér mean is a polynomial Fock vector. Passing to the limit at each coordinate proves (23.58).

Lean. `UnfoldingStages` (L76-L236). Here finitely many evaluations determine the target functional on a fixed complement of $\ker\Lambda$. □

Lemma 23.25 (The archimedean vector and the finite evaluation)**.** *Fix the archimedean points ξ_j of the lower chain, and, for the last space of a pair, a fixed archimedean point x. The two prescribed edge data are*

$$f_{\eta,j} = u_j \otimes \Upsilon_{\eta,j},$$

where u_j is the fixed joint harmonic and $\Upsilon_{\eta,j}$ the original inclusion point datum. Move them by $(\xi_{2i}, y_{2i}), (\xi_{2i+1}, y_{2i+1})$ and by $(\xi_{2i+1}, y_{2i+1}), (x, y_{2i+2})$, respectively, using their prescribed twisted pair actions. There is a real Fock vector $v_{\xi,x,i}$, independent of η and of all y_j, for which the joined, S-translated and cut-down datum is exactly

$$\mathcal{Z}_D(\rho(S)(f^{\mathrm{moved}}_{\eta,2i} \boxtimes_a f^{\mathrm{moved}}_{\eta,2i+1})) = v_{\xi,x,i} \otimes A(E_{\eta,y,i}), \tag{23.60}$$

where

$$E_{\eta,y,i} = D.\,\mathrm{finEval}\Big(\Upsilon^{\mathrm{moved}}_{\eta,2i} \boxtimes_a \Upsilon^{\mathrm{moved}}_{\eta,2i+1}\Big) \tag{23.61}$$

is the finite evaluation `evaluatedDatum` *of Theorem 23.3, with the scalar and finite translations specified there.*

Proof. An adelic pair action on $u \otimes \Phi$ is the tensor of its real pair action on u and its finite pair action on Φ. The real action of a chain character has fixed weight; its weight-zero tame twist contributes one at infinity. Thus the two moved real vectors depend only on ξ, x. Their joining has the same property. Apply (23.40) with the fixed $w = S$.

By the definition of finite evaluation,

$$A\,D.\,\mathrm{finEval}(F) = \mathcal{Z}_{D,f}(\rho_f(\mathrm{finCoord}\,S)F) \tag{23.62}$$

for every joined finite datum F. This equality follows from the inverse of A in Lemma 23.19; both finite operators preserve its range. It is an equality of vectors, with all original semicharacters and the α^{-1} coordinate factor. Substitution gives (23.60). The real vector depends continuously on ξ, x, by continuity of the Weil action and of the mixed-model operators.

Lean. `EdgePairEvaluation` (L842-L876), `EdgePairEvaluation` (L1184-L1196), `UnfoldingStages` (L716-L750). □

Theorem 23.26 (Even unfolded stages as coefficients of the upper chain)**.** *Set $m = 2\varrho + 2$. Fix a source linear form t and archimedean tuple ξ. For every i with $2i \le 2\varrho + 1$, there is one linear form $l_{i,t,\xi}$ on the values of forms on X_{2i} such that*

$$\mathcal{W}_{2i}^{\eta,t,\phi,\xi,y} = \mathrm{sourceFn}_{Y_{2i}}\left(l_{i,t,\xi}, \mathrm{doubleLiftChain}_{\eta}(E_{\eta,y,\bullet}, i)\phi\right). \tag{23.63}$$

The unfolded function on the left (`unfoldStageOn` in Lean) is defined recursively by (23.64); the source function on the right is given by (23.65). The equality holds simultaneously for all tame characters η, source forms ϕ, and finite tuples y. Equation (23.61) gives the finite data on the right. The linear forms can be chosen continuously in ξ and linearly in t.

Proof. Write $R_j = B_{Y_j}$, and let $g_j = (\xi_j, y_j)$. The functions being asserted equal have the following definitions:

$$\begin{aligned} \mathcal{W}_0 &= \mathrm{sourceFn}_{Y_0}(t, \phi), \\ U_{j+1}(h) &= \int_{[R_j]} \mathcal{W}_j(r)\, C^{a_j}_{Y_j, Y_{j+1}}(f_{\eta,j}; r g_j, h)\, dr, \\ \mathcal{W}_{j+1}(r) &= U_{j+1}(r g_{j+1}). \end{aligned} \tag{23.64}$$

The source function is

$$\mathrm{sourceFn}_Y(l, \psi)(n\, i_Y(h)) = \psi_Y(n)\, l\big(\pi_X(h_\infty)^{-1}\psi(h_f)\big). \tag{23.65}$$

For an odd string, $N_Y = N_Y^{\ge 2}$, so this is the unique equivariant function with the given values on $U(X)$.

We first record that (23.64) remains equivariant at each stage. Its continuity follows by integration of a continuous bounded-functional family on a compact rational quotient. Rational invariance follows by rational substitution in (23.48). If $n - 1$ raises grades by at least two on the target, then $(n-1)T$ has positive degree. Moving it through the zero term gives the phase $\beta(nT, T)$. Since $TT^\dagger$ is the target shift matrix when the two generic constants agree, this phase is exactly $\psi_{Y_{j+1}}(n)$. It factors out of the integral in (23.64). Thus every $\mathcal{W}_j$ is equivariant. This argument uses the target's group $N^{\ge 2}$ even when its string length is even; it does not wrongly assert that the character is defined on its entire unipotent group.

The assertion for $i = 0$ holds with $l_0 = t$. Suppose it holds for i. Evaluate $\mathcal{W}_{2i+2}$ on $i_{Y_{2i+2}}(k)$ and insert twice the definition (23.64). Associativity of the Weil action moves the three fixed outer points into the two data. The resulting double Bessel integral is exactly (23.50). By (23.60), its datum is $v_{\xi,\xi_{2i+2},i} \otimes A(E_{\eta,y,i})$. Apply Proposition 23.24 with real vector $z_D v_{\xi,\xi_{2i+2},i}$ and linear form l_i. It gives one l_{i+1}, independent of η, ϕ, y, and the resulting form is precisely the next pair of original algebraic lifts applied to the preceding double lift.

For $i + 1 \le \varrho$, the middle pair has $e_D = \mathrm{id}_{U(X_{2i+2})}$, and its target string is odd. Equivariance and (23.65) extend the identity from $U(X_{2i+2})$ to all of R_{2i+2}. This is (23.63). The continuity and linearity choices in Proposition 23.24 and Lemma 23.25 give the final assertion by induction. All pair integrals in the induction are the absolutely convergent compact integrals already justified above.

Lean. `Equivariant` (L274-L421), `ConstantTerm` (L844-L910), `UnfoldingStages` (L754-L823). □

Corollary 23.27 (The altered terminal grading)**.** *Keep t, ξ, and take any compatible last grading Y in a pair Y_{2i}, Y_{2i+1}, Y, together with pair data D such that the* original doubleLiftEvalY(i) *equals $D.$*finEval*. Fix an archimedean point x of the last space. There is a linear form $l = l(t, \xi, x)$, independent of η, ϕ, y, k, such that*

$$U^Y_{2i+2}(i_Y(k)(x, y_{2i+2})) = \mathrm{coeff}_l\left(\mathrm{doubleLiftChain}_{\eta}(E_{\eta,y,\bullet}, i+1)\phi;\ e_D(k)\right). \tag{23.66}$$

This includes $i = \varrho$, the last pair with an empty string and $q = 0$. It requires no odd-string uniqueness assertion at that empty terminal string.

Proof. Apply (23.63) at $2i$, then repeat the two integral calculation in its proof with x in place of ξ_{2i+2}, using (23.60). The conclusion is the coefficient at $e_D(k)$ furnished by (23.58). The identification of finite data is exactly the stated equality of finite evaluation maps. No extension from the terminal Levi to a larger Bessel group is needed.
Lean. `UnfoldingStages` (L825-L853). □

Proposition 23.28 (The terminal coefficient and the restriction map)**.** *At $m = 2\varrho + 2$, the last lower space Y_m is definite of rank $\dim X_m - 1$. Let $B : W_{X_m} \to W_{Y_m}$ be the original integral branching matrix, and put*

$$\delta_N = \prod_{\tau} \Phi_\tau(\overline{\pi_N})^{\,b_\tau} \neq 0, \qquad d_Y = \dim W_{Y_m}. \tag{23.67}$$

Thus the restriction map, in algebraic-form coordinates, is

$$(\operatorname{res}\psi)(h) = \delta_N^{-1} B\,\psi(\operatorname{diag}(h, 1)). \tag{23.68}$$

For every linear form l on W_{X_m}, there is a constant C_l, depending only on l and the fixed real data, with the following property. If F on the last group satisfies

$$F(i_{Y_m} k) = \operatorname{coeff}_l(\psi; \operatorname{diag}(k, 1)), \tag{23.69}$$

then, for every form x of the prescribed last lower type,

$$\int_{[U(Y_m)]} F(g)\overline{x(g)}\,dg = C_l\, d_Y\, \delta_N\, \langle x, \operatorname{res}\psi\rangle. \tag{23.70}$$

The scalar can be zero. It is independent of the tame character, the finite datum, x, and ψ.

Proof. The last grading has no string. Its i_{Y_m} is a relabeling of the entire definite group, and its rational Bessel subgroup is the rational unitary group. Consequently (23.69) specifies F everywhere and there is no further noncompact integral.

Let v_Y be the fixed unit highest vector used to pass between algebraic and automorphic forms on Y_m. With the fixed compact Haar normalization, form the map

$$A_l(w) = \int_{U(Y_m)_\infty} \pi_Y(a) v_Y\; l\Big(\pi_X(\operatorname{diag}(a, 1))^{-1} w\Big)\; da. \tag{23.71}$$

Change of variable proves $A_l \pi_X(\operatorname{diag}(b, 1)) = \pi_Y(b) A_l$. The restriction of an irreducible polynomial representation of $U(n)$ to $U(n-1)$ has multiplicity one at each interlacing highest weight. In tableau coordinates this can be checked by removing the entries n: they form a horizontal strip, and the remaining tableau determines the unique copy of each permitted highest weight. Determinant twists preserve this multiplicity. The prescribed type of Y_m is exactly the interlacing type of the nonzero original matrix B. It follows that $A_l = C_l B$, with C_l depending linearly on l.

Write x_{alg} for the algebraic form corresponding to x. Unitarity gives

$$\overline{x(a, h)} = \langle x_{\mathrm{alg}}(h), \pi_Y(a) v_Y\rangle.$$

Insert (23.69) and integrate over a. Equation (23.71) shows that the result is

$$C_l\ \operatorname{classMean}_h \left\langle x_{\mathrm{alg}}(h), B\psi(\operatorname{diag}(h, 1))\right\rangle. \tag{23.72}$$

Schur orthogonality for the highest coefficient gives the Petersson pairing of two forms of this type as d_Y^{-1} times their algebraic class pairing. Using (23.68), (23.72) is therefore $C_l d_Y \delta_N \langle x, \operatorname{res}\psi\rangle$, as asserted. A different fixed compact Haar mass is absorbed into C_l; the quotient and class measures are otherwise the original ones. All integrations are on compact quotients. In particular, the argument neither averages over finite characters nor changes the integral branching matrix or its π_N-factor from Theorem 19.16.

Lean. `SeesawIdentity` (L65-L157). □

Equations (23.63), (23.66), and (23.70) identify the even stages and terminal coefficient of the recursion (23.64). To integrate these functions over the full Bessel quotients we must also determine their support.

23.4 Frequency support and the split-level obstruction

Write $\widehat{\mathcal{O}} = \prod_w \mathcal{O}_{E_w}$, α for the self-duality idele of the Weil datum, h for the chosen-level idele, and $\mathbf{q}$ for the finite idele generating the tame ideal Q. Put

$$\mathfrak{a} = \alpha/\bar{h}, \qquad b = h\mathbf{q}, \qquad \pi\bar{\pi} = N.$$

Thus b generates the chosen level times the tame modulus, and

$$\frac{\bar{b}}{\alpha} = \frac{\bar{\mathbf{q}}}{\mathfrak{a}}. \tag{23.73}$$

The support of π is denoted by P. Its places are split, are disjoint from their conjugates, and have valuation one on π. Neither Q nor the old chosen level meets $P \cup \bar{P}$.

For a graded space $Y = D \perp H_s$, the string basis is $e_0, \dots, e_{s-1}$, of grades $s-1-2i$; the definite block D has grade zero. The Gram matrix pairs e_i with e_{s-1-i} to one, except that the middle vector for odd s has norm $-N$. Let pr_H be the orthogonal projection to the string. For a finite point x, put

$$L(x) = xL(1), \qquad L(x)^- = L(x) \cap H_s, \qquad L(x)^+ = \mathrm{pr}_H L(x),$$

where x is read in rational coordinates. The Bessel group R_Y fixes each string vector of nonnegative grade modulo vectors of higher grade. All its matrices preserve the grade filtration.

Lemma 23.29 (Duality and a graded representative)**.** *The lattices $L(x)^-$ and $L(x)^+$ are dual in H_s. In particular*

$$cL(x)^- \subseteq L(x')^- \quad \Longrightarrow \quad \bar{c}L(x')^+ \subseteq L(x)^+. \tag{23.74}$$

Every finite Bessel orbit contains a graded lattice: there is a grade-raising unipotent isometry $u \in R_Y$ such that $L(ux)$ contains the homogeneous part of each of its vectors.

Proof. The prescribed finite frame is orthonormal, so $L(x)$ is self-dual. A string vector pairs integrally with $\mathrm{pr}_H L(x)$ precisely when it pairs integrally with $L(x)$; this proves $(L(x)^+)^\vee = L(x)^-$. Conversely $L(x) \cap H_s$ is saturated in $L(x)$ at every place. An integral linear functional on this intersection extends to $L(x)$: extend a primitive basis over the local discrete valuation ring. Self-duality realizes the extended functional by an element of $L(x)$; its string projection gives the required dual vector. These local constructions assemble over $\widehat{\mathcal{O}}$. This proves the other duality and (23.74).

For the integral Iwasawa decomposition, write w_i for the grades, J for the rational Gram matrix, and Z for the orthonormal frame of $L(x)$. Hermite reduction adapted to the flag of the grades gives a basis $b = Zk$, $k \in \mathrm{GL}(\widehat{\mathcal{O}})$, such that b and b^{-1} do not lower grades. Indeed, successively choose

bases of the intersections of the lattice with the flag; each such intersection is saturated in the next one, so its basis extends. This is ordinary triangular elimination over each local discrete valuation ring. The choices are integral outside finitely many places and hence adelic.

Set $G = b^*Jb = k^*k$. It is integral and unimodular, with

$$G_{ij} = 0 \quad (w_i + w_j > 0), \qquad (G^{-1})_{ij} = 0 \quad (w_i + w_j < 0).$$

Let G_0 retain just the entries $w_i + w_j = 0$. Taking degree-zero parts is multiplicative for matrices which preserve the appropriate filtrations; applying this to $G^{-1}G = 1$ shows that G_0 is unimodular. It is Hermitian. By the integral orthonormalization in Lemma 23.11, choose $m \in \mathrm{GL}(\widehat{\mathcal{O}})$ with $m^*G_0m = 1$. For the columns indexed by $w_i \geq 0$, left inverses are given by the corresponding rows of m for m^{-1} and of k^{-1} for k. These rows are integral. The two column matrices have the same Gram matrix since G_0 and G agree on those indices. The primitive integral Witt extension in that lemma therefore gives an integral unitary matrix v carrying those columns of m^{-1} to the corresponding columns of k. Put $n = k^{-1}vm^{-1}$. Then

$$n^*Gn = G_0, \qquad ne_i = e_i \quad (w_i \geq 0).$$

Moreover $n - 1$ strictly raises grades. For completeness, if $w_i \leq w_j < 0$, use $n - 1 = G^{-1}(Gn - G)$. A potentially nonzero term of its ij entry has $w_a \geq -w_i \geq -w_j > 0$. The fixed column $ne_a = e_a$ gives $(Gn)_{aj} = (G_0)_{aj}$, and $w_a + w_j \geq 0$ gives $(G_0)_{aj} = G_{aj}$. Thus that term is zero. If $w_j \geq 0$, the entire column of $n - 1$ is zero.

Let d be the grade-zero part of b. It is invertible, its inverse being the grade-zero part of b^{-1}, and $d^*Jd = G_0$. Consequently $v_0 = bnd^{-1}$ is an isometry and $v_0 - 1$ raises grades. Its inverse carries $L(x) = bn\widehat{\mathcal{O}}^I$ onto $d\widehat{\mathcal{O}}^I$, which is graded because d commutes with every grade projection. Take $u = v_0^{-1}$.

Lean. `PointLattice` (L304-L425), `GradedLattice` (L148-L331). □

Lemma 23.30 (The string lines and their quotient)**.** *For an original nonterminal grading Y_j, $0 \leq j < m$, of string length $s = m - 1 - j$, set*

$$u_s = (\mathfrak{a}/\bar{\mathfrak{a}})^{\lfloor (s+2)/2 \rfloor}, \qquad \lambda_i = \begin{cases} u_s, & s \text{ odd}, \\ u_s\mathfrak{a}^{-1}, & s \text{ even}, \ 2i < s, \\ u_s\bar{\mathfrak{a}}, & s \text{ even}, \ 2i \geq s. \end{cases}$$

The string intersection in the fixed lattice is

$$L(1)^- = \bigoplus_{i<s} \lambda_i \widehat{\mathcal{O}} e_i.$$

Its projection $L(1)^+$ is the same lattice except that, for odd s, the middle summand is multiplied by N^{-1}. Thus $L(1)^+/L(1)^-$ is zero for even s. For odd s it is supported on $P \cup \bar{P}$; at each of these places it is one-dimensional over the residue field.

For consecutive original gradings of lengths $s, s-1$, the string shift $T : Y \to Y'$ obeys

$$TL(1)^+ = \mathfrak{a}L'(1)^-. \tag{23.75}$$

Proof. Write the definite block as $D = D_0 \oplus Ee_\circ$. The lattice basis consists of e_x for $x \in D_0$, the vectors $\lambda_i e_i$, and

$$\begin{cases} e_\circ, & s \text{ even}, \\ \bar{\pi}^{-1}e_\circ + u_s(\pi\bar{\pi})^{-1}e_{(s-1)/2}, & s \text{ odd}. \end{cases} \tag{L}$$

This is the basis used in constructing Y_j; the orthonormal finite frame differs from it by an integral invertible matrix. The odd two-dimensional block has Gram matrix

$$\begin{pmatrix} 0 & -1 \\ -1 & -N \end{pmatrix}$$

if its second vector is $u_s e_{(s-1)/2}$. Indeed the first norm is $N^{-1} - N/N^2 = 0$, the mixed product is -1, and the second norm is $-N$. Its determinant is -1. The other hyperbolic pairs have unit mixed products, so the full basis is unimodular. Integral orthonormalization from Lemma 23.11 therefore constructs the stated frame without changing this lattice.

Directly from (L), a vector with definite coordinates zero has all its string coordinates divisible by λ_i. Projection of the special odd vector adds precisely $N^{-1}\lambda_{(s-1)/2} e_{(s-1)/2}$. This proves the two line formulas. Since N is split and squarefree, their quotient at each place over N is a residue-field line. Every nonzero class at one of those places generates it. The lattice is graded because the only linked coordinates in (L) both have grade zero.

We check (23.75) without changing the normalization of the shift. Let c be the common generic constant and write $\mathfrak{e}^i$ for the normalized string powers. Their coefficients relative to e_i are

$$c^{\,i-\lfloor s/2\rfloor} \quad (2i < s), \qquad -Nc^{\,i-\lfloor (s-1)/2\rfloor} \quad (2i \geq s). \tag{P}$$

The original $c = -(jM\delta_E)^{-1}$ has the valuations of $\mathfrak{a}\bar{\mathfrak{a}}$: the self-duality normalization says that $j\alpha\bar{\alpha}\delta_E$ is a local unit, and $h\bar{h}$ has the valuations of M. Put $\mathfrak{l}_{s,i} = \mathfrak{a}^{s-i}\bar{\mathfrak{a}}^{-1-i}\widehat{\mathcal{O}}$. Substitution of (P) into the line formulas shows that $\sum_i t_i \mathfrak{e}^i$ belongs to $L(1)^-$ exactly when

$$t_i \in \begin{cases} \mathfrak{l}_{s,i}, & 2i < s, \\ N^{-1}\mathfrak{l}_{s,i}, & 2i \geq s; \end{cases}$$

and it belongs to $L(1)^+$ exactly when the same division is made at $2i+1 < s$ and $2i+1 \geq s$. The shift sends $\mathfrak{e}^i$ to $\mathfrak{e}'^i$ for $i < s-1$ and kills the last one; its coefficients in the original bases are exactly the ratios of (P). Now $\mathfrak{l}_{s,i} = \mathfrak{a}\mathfrak{l}_{s-1,i}$ and $2i+1 < s$ is equivalent to $2i < s-1$. Thus the line conditions agree after multiplying by $\mathfrak{a}$. Every vector on the right has a preimage by setting the unused last coefficient to zero, proving equality.

Lean. `StandardLines` (L178-L355), `StandardLines` (L386-L569), `LevelIdeles` (L78-L132). □

Lemma 23.31 (The finite Heisenberg support calculation)**.** *Let K be an isotropic finite frame, and let Z_K denote the finite zero-term operator along it. Suppose $MM'^* = 1$ and a finite Weil vector f is fixed by all Heisenberg operators $\rho(Ma)$, $a \in \widehat{\mathcal{O}}^n$. If $Z_K(\rho(z)f) \neq 0$, then*

$$z = M'\ell + z_0, \qquad \ell \in \widehat{\mathcal{O}}^n, \quad K^* z_0 = 0. \tag{23.76}$$

Proof. Choose an invertible change of columns t so that $B = M'^* K t$ is integral and primitive. This is obtained by choosing a basis of $\mathrm{im}(M'^* K) \cap \widehat{\mathcal{O}}^n$; the intersection is saturated, so it admits an integral left inverse A, with $AB = 1$. For every integral a, the vector $Kta = M(Ba)$ is a period of f. It is also a period of the zero-term map: translation along the isotropic averaging space leaves that mean unchanged. The Heisenberg commutator therefore makes the nonzero vector $Z_K(\rho(z)f)$ fixed by multiplication by

$$\psi\big(a^*(Kt)^* z\big).$$

This scalar must be one for every integral a. The self-dual normalization of the additive character implies $v = (Kt)^* z \in \widehat{\mathcal{O}}^{\dim K}$. Set $\ell = A^* v$, which is integral. Since $B^* A^* = 1$, one has $(Kt)^*(z - M'\ell) = 0$; invertibility of t gives (23.76). This proof works at every finite place, including those over two, because it uses the commutator and self-dual lattice rather than a division by two.

Lean. `EdgePairEvaluation` (L200-L249). □

Lemma 23.32 (Nonzero frequency forces the lattice inclusion)**.** *Let Y, Y' have strings of lengths $s, s-1$, and let T be their original string shift. Suppose the finite datum f on $\mathrm{Hom}(Y, Y')$ is fixed by $\rho(b\widehat{\mathcal{O}}^{I'\times I})$. If the shifted constant term of the Weil translate of $t_\infty \otimes f$ at (x, x') is nonzero, then, for any $r \in R_Y(\mathbb{A}_f)$ and $r' \in R_{Y'}(\mathbb{A}_f)$ for which both lattices are graded,*

$$\frac{\bar{b}}{\alpha}\, TL(rx_f)^+ \subseteq L'(r'x'_f)^-. \tag{23.77}$$

The point datum at modulus b satisfies the required Heisenberg invariance.

Proof. Split the space of maps into positive, zero and negative degree, where the degree of an entry is the grade of its row minus that of its column. The positive-degree part is isotropic. Averaging the theta function along it is the theta function of the corresponding zero term. For a pure real–finite tensor this zero term factors as a real vector times the finite zero term of the translated datum. A nonzero constant term therefore implies that this finite zero term is nonzero. The character twists contribute only nonzero scalars.

Let g be the finite unitary action of (x, x') on the map space. The finite Weil translate of f has periods $bg\widehat{\mathcal{O}}^{I'\times I}$. In Lemma 23.31 take $M = bg$, $M' = \bar{b}^{-1}g$. Writing ℓ as an integral matrix, the rational coordinate change gives

$$\alpha^{-1}T - \bar{b}^{-1}W \quad \text{with zero negative-degree part,} \qquad W = Z_{x'}\,\ell\, Z_x^{-1}, \tag{F}$$

where $Z_x, Z_{x'}$ are the finite orthonormal frames of the two lattices. To check the scale, the map-space frame sends the finite Heisenberg coordinates of T to $\alpha^{-1}T$, while it sends $\bar{b}^{-1}g\ell$ to $\bar{b}^{-1}Z_{x'}\ell Z_x^{-1}$. Orthogonality to the positive-degree frame removes exactly the negative-degree entries, because the map Gram matrix pairs a degree with its negative. Thus $WL(x_f) \subseteq L'(x'_f)$ and

$$W - (\bar{b}/\alpha)T \text{ has degree at least } 0. \tag{F'}$$

Move this map to the chosen graded representatives: $W_0 = r'Wr^{-1}$. It still maps the first lattice into the second. Bessel matrices preserve the grade filtration and act as the identity on the associated string lines. The latter assertion on negative grades follows by duality from the defining assertion on nonnegative grades. Consequently

$$r'Tr^{-1} - T \text{ has degree at least } 0.$$

This also follows entrywise from $(r'-1)Tr^{-1} + T(r^{-1}-1)$: each nonzero entry of T links two string vectors and has degree -1, while the adjacent difference raises that string grade. Hence the degree--1 part of W_0 is exactly $(\bar{b}/\alpha)T$.

A homogeneous part of a map between graded lattices still maps one lattice into the other: for degree d, sum $\mathrm{pr}_{e+d}\, W_0\, \mathrm{pr}_e$ over the finitely many grades e. Apply this at $d = -1$. The shift kills the definite block and takes values in the string, so its action on the projected lattice lands in the string intersection. This proves (23.77).

Finally, the prescribed finite datum is a finite Fourier combination of translated vacua $\rho(t)v$ with $\bar{b}\,t$ integral. For a integral, the commutator of $\rho(ba)$ and $\rho(t)$ is one, by self-duality and $\bar{b}\,t \in \widehat{\mathcal{O}}^{I'\times I}$. The integral translation ba fixes the vacuum. It therefore fixes every term of that Fourier combination, proving the asserted invariance with the generator b.

Lean. `Frequency` (L72-L174), `Frequency` (L176-L343), `UnfoldingStages` (L977-L1002). □

Lemma 23.33 (The two lattice inclusions and the splitting obstruction)**.** *For consecutive original gradings, suppose*

$$\mathbf{q}^k L(x)^- \subseteq L(1)^-, \qquad \frac{\bar{\mathbf{q}}}{\mathfrak{a}}\, TL(x)^+ \subseteq L'(x')^-.$$

Then

$$\bar{\mathbf{q}}^{\,k+1}L'(1)^- \subseteq L'(x')^-. \tag{23.78}$$

For any original grading, if $\bar{\mathbf{q}}^{\,j}L(1)^- \subseteq L(x)^-$*, then either*

$$\mathbf{q}^{j}L(x)^- \subseteq L(1)^- \tag{23.79}$$

or its string length is odd and $L(x)$ *splits between the definite block and the string at some* $w \in P$*. Here splitting means* $L(x)^+_w = L(x)^-_w$*.*

Proof. The first upper bound and (23.74) give $\bar{\mathbf{q}}^{\,k}L(1)^+ \subseteq L(x)^+$. Apply T, use (23.75), and multiply by $\bar{\mathbf{q}}/\mathfrak{a}$. This proves (23.78).

For the second assertion, duality gives $\mathbf{q}^j L(x)^+ \subseteq L(1)^+$. If (23.79) fails, choose $y \in L(x)^-$ with $\mathbf{q}^j y \notin L(1)^-$. This vector lies in $L(1)^+$, so Lemma 23.30 makes the string length odd and provides a place $w \in P \cup \bar{P}$ where its class generates $L(1)^+/L(1)^-$. At both w and $\bar{w}$, $\mathbf{q}$ and $\bar{\mathbf{q}}$ are units. The lower bound therefore gives $L(1)^-_w \subseteq L(x)^-_w$. The generating vector y belongs to $L(x)^-$, so $L(1)^+_w \subseteq L(x)^-_w$. The dual bound gives $L(x)^+_w \subseteq L(1)^+_w$. Hence $L(x)^+_w \subseteq L(x)^-_w$, which is the required splitting. Projection–intersection duality transfers splitting at w to splitting at $\bar{w}$: dualize the inclusion after multiplying by the local idempotent of w. Replace w by $\bar{w}$ if necessary to have $w \in P$.

Lean. `Sandwich` (L140-L237). □

Lemma 23.34 (Vanishing on split lattices by ramification)**.** *Let* φ *be in the ramified source space. For every even stage* j *with* $2 \le j < m$*, let* $\mathcal{W}_j$ *be its iterated Bessel integral. If* $L_j(x)$ *splits at a place* $w \in P$*, then*

$$\mathcal{W}_j(\gamma(a,x)) = 0 \qquad (\gamma \in R_{Y_j}(\mathbb{A}),\ a \in U(Y_j)(\mathbb{R})). \tag{23.80}$$

This statement uses the coefficient identity of Theorem 23.26 for the same evaluated finite data.

Proof. Let z_w be the original norm-one scalar idele: its component at w is an integral lift of a generator of the residue-field units, its component at $\bar{w}$ is the conjugate inverse, and it is one elsewhere. Both z_w and its inverse are integral. Let s_w act by this scalar on the definite block of Y_j and by one on its string.

We first check the lattice and the precise level. For $y \in L_j(x)$, in rational coordinates

$$s_w y = y + (z_w - 1)(y - \mathrm{pr}_H\, y). \tag{K}$$

Splitting at w makes the w-component of $\mathrm{pr}_H\, y$ belong to the lattice; duality gives the same assertion at $\bar{w}$. The coefficient $z_w - 1$ is integral and supported only on those two places. Formula (K) proves that s_w preserves $L_j(x)$; applying it to z_w^{-1} proves equality. Consequently $x^{-1}s_w x$ is integral. It is the identity away from $w, \bar{w}$, and neither place belongs to the chosen level or Q. It therefore lies in the level K_{Y_j}, not merely in an unspecified compact group.

The point datum on the last edge into Y_j is fixed by K_{Y_j}. Indeed its reduction is the prescribed inclusion matrix at the chosen and tame places, and is the vacuum elsewhere; the splitting characters are trivial on that level. Its translate with endpoint x is therefore fixed by the output action of s_w, because $s_w x = x(x^{-1}s_w x)$. The original double-evaluation map carries this action to the scalar z_w on the output definite space X_j, acting on the last tensor factor. This equivariance follows directly through its three operations: the string shift is unchanged because the scalar acts only on X_j; the isotropic averaging space is fixed; and restriction to the retained definite block intertwines s_w with z_w. Determinant twists agree, since their characters are unramified at N. Thus the evaluated double datum $\tau_{j/2-1}$ satisfies

$$(1 \otimes \omega(1, z_w))\tau_{j/2-1} = \tau_{j/2-1}. \tag{23.81}$$

At every edge the pair (z_w, z_w) fixes the linear map variable. Its determinant character is one, because the prescribed splitting characters and the tame character are unramified above N and z_w is a local unit. Hence

$$\omega(z_w, z_w) = 1, \qquad L(F)T_{w,\mathrm{in}} = T_{w,\mathrm{out}}L(F) \tag{E}$$

for every finite datum F. Moving a scalar through the chain by (E), and then using (23.81), shows that the chain C_τ satisfies $C_\tau T_{w,0} = C_\tau$. This remains valid for a general tensor τ: verify it on pure tensors and extend linearly. On the source, the original ramified condition is

$$(1 + T_{w,0} + \cdots + T_{w,0}^{N-2})\varphi = 0.$$

The operator $T_{w,0}$ has order dividing $N-1$. Its zero-average subspace is the image of $1 - T_{w,0}$, as follows by diagonalizing this finite-order operator. Thus $\varphi = (1 - T_{w,0})\psi$, and $C_\tau\varphi = 0$. No change to the datum or averaging denominator is involved in this vanishing argument.

For fixed a, x, update the endpoint in the original list of finite and infinite variables to x, a. Theorem 23.26 writes $\mathcal{W}_j(\gamma(a, x))$, for every Bessel γ, as the source coefficient of this exact double-lift chain applied to φ. The chain has just been proved zero. This proves (23.80).

Lean. `CharactersAtLevel` (L38-L191), `UnfoldingStages` (L857-L932), `UnfoldingStages` (L1005-L1094). □

Theorem 23.35 (Support of every unfolded stage)**.** *Keep the original source form φ, initial coefficient functional, archimedean points and finite points used in the unfolding, with φ ramified and the initial finite point y_0 in the chosen level. For $1 \le j < m$, suppose $\mathcal{W}_j(a, x) \neq 0$. Then every graded lattice $L_j(r'x)$, with $r' \in R_{Y_j}(\mathbb{A}_f)$, satisfies*

$$\bar{\mathbf{q}}^{\,j} L_j(1)^- \subseteq L_j(r'x)^-, \qquad \mathbf{q}^j L_j(r'x)^- \subseteq L_j(1)^-. \tag{23.82}$$

Such a graded representative exists. Thus $\mathcal{W}_j$ is supported on the bounded-orbit set required in (23.9). Besides the already proved finite lattice and Weil identities, its only coefficient input is Theorem 23.26.

Proof. Write the defining step of the iterated constant-term integral as

$$\mathcal{W}_{k+1}(a, x) = \int_{[R_{Y_k}]} \mathcal{W}_{k,\mathrm{on}}(\gamma)\, \mathcal{C}_k(\gamma g_k, (a, x))\, d\gamma, \tag{I}$$

where g_k is the fixed point at the preceding stage and $\mathcal{C}_k$ is the shifted constant term of the prescribed datum. The moved-datum convention gives exactly the same term as translating the two endpoints in (I); it introduces no extra scale. These Bessel quotients are compact (a definite unitary quotient times a unipotent quotient), and the integrands are continuous, so the integrals are absolute. Theorem 23.10 identifies this recursion with the original unfolded periods; only the defining identity (I) is needed in the following support argument.

We prove the assertion by induction on k. If the integral (I) is nonzero, some γ makes both factors nonzero. At $k = 0$, the preceding finite point is $\gamma_f y_0$; translate by γ_f^{-1} to recover the integral point y_0. Its lattice is $L_0(1)$, is graded by Lemma 23.30, and satisfies both bounds with exponent zero. At $k > 0$, the first nonzero factor is a value of $\mathcal{W}_k$. The induction hypothesis and Lemma 23.29 provide a graded representative $r\gamma_f g_{k,f}$ with both bounds of exponent k.

Now fix any r' making the current lattice $L_{k+1}(r'x)$ graded. The second nonzero factor in (I), together with Lemma 23.32 and (23.73), gives

$$\frac{\bar{\mathbf{q}}}{\mathfrak{a}}\, T L_k(r\gamma_f g_{k,f})^+ \subseteq L_{k+1}(r'x)^-.$$

The preceding upper bound and the first part of Lemma 23.33 give the new lower bound with exponent $k+1$. The second part gives the new upper bound unless the current string is odd and the lattice splits at some $w \in P$.

The current string has length $m-1-(k+1)$. As m is even, odd string length means that the current stage $k+1$ is even. In that case Lemma 23.34 applies to the finite point $r'x$. Take its Bessel argument to be $(1,r')^{-1}$; it gives $\mathcal{W}_{k+1}(a,x)=0$, contrary to the hypothesis. The obstruction is therefore impossible, and both inclusions hold for every graded representative. Lemma 23.29 supplies at least one such representative, completing the induction.

Lean. `UnfoldingStages` (L1100-L1167). □

23.5 Successive integration and the seesaw identity

Let ϑ be a tame finite character, let $\eta=(\eta_j)$ and $y=(y_j)$ be tuples of archimedean and finite points, respectively, and let D be the prescribed chain datum. Write

$$m=2\varrho+2,\qquad G_j=U(Y_j),\qquad J_j=J_{\vartheta,j},\qquad i_j=[K_j^0:J_j],\qquad J_M=D.\,\mathrm{chosenLevel}\subset K_0^0.$$

Here J_M is the prescribed initial level. Each finite subgroup K_j^0 has Haar mass one. The map $\mathcal{P}=\mathcal{P}_{g,t}$ is the original Poincaré map at a fixed real point g and source functional t; its defining identity is

$$\langle x,\mathcal{P}\varphi\rangle=\int_{[R_0]}\mathcal{W}_0(r)\,\overline{x(r(g,1))}\,dr,\qquad \mathcal{W}_0=\mathrm{sourceFn}_{Y_0}(t,\varphi). \tag{23.83}$$

The measured map Π is of this form, with the fixed point and functional identified in (9.65); its fixed scalar is absorbed in t. Thus proving the result for every fixed g,t proves it for the map already used in the determinant argument.

Lemma 23.36 (Transposition and the exact initial level factor)**.** *Let $T_{\vartheta,j}$ be the original lower lift, normalized by i_j. Its transpose on conjugate-type forms is*

$$T^{\mathrm{tr}}_{\vartheta,j}z(g')=i_j\int_{[G_{j+1}]}\theta\big(\omega_{\vartheta,j}(g',h)\Phi_{\vartheta,j}\big)\,z(h)\,dh. \tag{23.84}$$

It has level J_j, belongs to the prescribed conjugate type on G_j, and

$$\int_{[G_{j+1}]}z\,T_{\vartheta,j}f=\int_{[G_j]}(T^{\mathrm{tr}}_{\vartheta,j}z)f. \tag{23.85}$$

Let $U^{\eta,y}_{k+1}$ be the function left by the first $k+1$ constant-term steps of (23.64), with a specified compatible grading at its last space. Define

$$P_k(\eta,y;z)=\int_{R_{k+1}(E^+)\backslash G_{k+1}(\mathbb{A})}U^{\eta,y}_{k+1}(h)z(h)\,dh. \tag{23.86}$$

This integral is absolutely convergent. The one-step recursion is

$$\int_{[R_k]}\mathcal{W}^{\eta,y}_k(r)(T^{\mathrm{tr}}_{\vartheta,k}z)(r(\eta_k,y_k))\,dr=i_kP_k(\eta,y;z). \tag{23.87}$$

At the first step it gives

$$\int_{[G_1]}z\,T_{\vartheta,0}(\mathcal{P}\varphi)=[K_0^0:J_M]\sum_{c\in\Omega_{\vartheta,0}}P_0(\mathbf{1}[0\mapsto g],\mathbf{1}[0\mapsto\widetilde{c}];z). \tag{23.88}$$

Here $\widetilde{c}$ is any representative and the initial multiplicities are one.

Proof. Let e_j be the archimedean type kernel on G_j. The row parameters satisfy the block correspondence and the prescribed gap bound

$$D.\mathrm{gap} - D.t - 2 \geq 500\big(\dim Y_j + \dim Y_{j+1}\big)^3.$$

Thus e_j, e_{j+1}, and their conjugates, are the integrable minimal discrete-series coefficient kernels used in (12.2); the bound also gives their integrable polynomial moments needed for the oscillator projections. Every bounded fixed-level form reproduced by one of these kernels is in L^2, since the automorphic quotient has finite volume. Its archimedean generated module is a sum of copies of the selected discrete series, so it is tempered and cuspidal by Wallach's constant-term theorem. The kernel is smooth and K_∞-finite; convolution supplies the smoothness and fixed infinitesimal character. The rapid-decay theorem for cuspidal automorphic forms therefore applies. At the definite end the quotient is compact. For the classical inputs see N. Wallach, *On the constant term of a square integrable automorphic form*, in *Operator algebras and group representations*, II, Monogr. Stud. Math. 18 (1984), 227–237, Theorem 4.3, and Miller–Schmid, *On the rapid decay of cuspidal automorphic forms*, arXiv:1106.2149, Theorems 2.23–2.24. Their statements apply to these unitary groups by restriction to the derived group and their compact real center; a fixed finite level has finitely many adelic components. In particular every test of the prescribed conjugate type form on Y_{j+1} is rapid, independently of the tame twist.

For a rapid conjugate-type input z, let F be its unscaled transposed theta integral in (23.84). The original joint-harmonic projection identity in (12.2), with the groups interchanged and the types conjugated, gives $R(\overline{e}_j)F = F$. This is the same specified joint harmonic and finite point datum; no choice of a different nonzero local lift is made. The polynomial oscillator bound and the preceding kernel moments justify moving these real convolutions through the theta integral. The finite point datum gives F a finite level. Boundedness of F need not be assumed: truncate it to $F_b = F\mathbf{1}_{\{|F|\leq b\}}$. The functions $R(\overline{e}_j)F_b$ are bounded forms of the same fixed type and level. That space is finite dimensional by the usual fixed-type cuspidal finiteness theorem, hence closed in the topology of pointwise convergence. Absolute integrability of each real convolution gives $R(\overline{e}_j)F_b \to F$ pointwise. Consequently F belongs to that bounded conjugate-type space, and is rapid by the preceding argument. Multiplication by i_j changes neither type nor decay. This proves the exact analytic assertions required of the transposed arrows. The block correspondence and gap inequalities are the adjacent row parameters; the projection, finite-level and truncation arguments are the joint-harmonic correspondence, the resulting arrows, and rapid decay, finite dimension and the truncation argument. Their exact lower-row application is the original datum and its swapped arrow. No rapidity is asserted for $\mathcal{W}_k$ on the full ambient group; it is integrated on the compact quotient $[R_k]$.

The Schwartz theta majorant has polynomial growth in the source and target reduced heights. Rapid decrease of the fixed-level test forms makes its product with f and z absolutely integrable. Fubini then gives (23.85), retaining the same factor i_j in the two directions. The original finite point datum is fixed by $(u, 1)$ for $u \in J_j$: the integral lattice action fixes its inclusion indicator and the tame character is trivial on the required principal congruence. Moving this equality through (23.84) proves the level assertion. There is no additional index for the transposed target.

The functions $\mathcal{W}_k$ are continuous and equivariant by the proof of Theorem 23.26. Theorem 23.10, applied against the rapid form z, proves absolute convergence of (23.86) and gives (23.87) after substituting (23.84). In this application the majorant is the *unrefined* constant-term sum, as in (23.23); no reduced-height bound for a refined constant term is assumed.

For the initial factor set $z_0 = T^{\mathrm{tr}}_{\vartheta,0} z$ and $d = [J_M : J_0]$. Sum its right translates over J_M/J_0,

obtaining z_0^{tr}, of level J_M. Right invariance of $\mathcal{P}\varphi$ and quotient Haar invariance give

$$d\int_{[G_0]} z_0\,\mathcal{P}\varphi = \int_{[G_0]} z_0^{\mathrm{tr}}\,\mathcal{P}\varphi = \sum_{c\in J_M/J_0}\int_{[R_0]} \mathcal{W}_0(r) z_0(r(g,\widetilde{c}))\,dr. \tag{23.89}$$

The second equality is precisely (23.83), applied to the conjugate of the level trace. By (23.87), each term is i_0P_0 at the indicated tuple. Now $i_0 = d[K_0^0 : J_M]$, with $d > 0$. Divide (23.89) by d, use (23.85), and identify J_M/J_0 with its image $\Omega_{\vartheta,0}$. This is (23.88). The equality also shows directly that changing a representative does not change its period. The original period map and level trace; Transposition, recursion and the initial index.

Lean. `Parameters` (L851-L915), `Cuspidality` (L858-L946), `SeesawIdentity` (L180-L399). □

For a function S on tuples of finite level classes, write

$$\mathrm{Cut}_{\vartheta,r}\,S = \sum_{c_0\in\Omega_{\vartheta,0},\dots,c_{r-1}\in\Omega_{\vartheta,r-1}} \left(\prod_{j<r} a_{\vartheta,j}(c_j)^{-1}\right) S(c_0,\dots,c_{r-1},1,1,\dots). \tag{23.90}$$

The expression is a finite linear operation. In particular it commutes with every linear functional on the chain-tensor space.

Theorem 23.37 (A scalar chosen before the twist for all outer integrals)**.** *Fix g, t, an integer $0 \le k < m$, and a compatible last grading of Y_{k+1}. Let $a(\eta)$ be any fixed complex-valued function of the archimedean tuple. There is a scalar $c = c(g,t,k,a)$ with the following simultaneous property. Suppose ϑ is any tame character, φ any ramified source form, z a conjugate-type form on G_{k+1}, and S a function of finite level classes for which*

$$P_k(\eta, y; z) = a(\eta)S([y]) \quad \text{whenever } y_0 \in J_M \text{ and } y_m = 1. \tag{23.91}$$

Then

$$\int_{[G_{k+1}]} z\,(T_{\vartheta,k}\cdots T_{\vartheta,0})(\mathcal{P}\varphi) = c\ \mathrm{Cut}_{\vartheta,k+1}\,S. \tag{23.92}$$

The scalar is independent of ϑ, φ, z, S.

Proof. Induct on k, allowing a, S and the compatible last grading to vary. At $k = 0$, the trace over J_M/J_0 in (23.88) and the convention $a_{\vartheta,0} = 1$ give

$$c = [K_0^0 : J_M]\,a(\mathbf{1}[0 \mapsto g]). \tag{23.93}$$

For the step from k to $k+1$, use the standard grading at Y_{k+1} and descend the test form: $z' = T^{\mathrm{tr}}_{\vartheta,k+1} z$. At this interior stage $0 < k+1 < m$, apply Lemma 23.18 for each real tuple η with real factor

$$x_\infty \longmapsto a(\eta[k+1 \mapsto x_\infty]).$$

Call its scalar $a'(\eta)$. This choice uses only that real function, the fixed Bessel group, maximal level, and measures. It is made before selecting ϑ, φ, z, S. Define the finite-class function

$$S'([y]) = \sum_{c\in\Omega_{\vartheta,k+1}} a_{\vartheta,k+1}(c)^{-1} S([y][k+1 \mapsto c]). \tag{23.94}$$

We claim

$$P_k(\eta, y; z') = a'(\eta)S'([y]) \tag{23.95}$$

under the same endpoint conditions on y.

To verify this claim put $F(h) = U_{k+1}^{\eta,y}(h)z'(h)$. It is continuous, left invariant under $R_{k+1}(E^+)$, and absolutely integrable by Lemma 23.36. If $F(h) \neq 0$, then $U_{k+1}^{\eta,y}(h) \neq 0$. Theorem 23.35, using precisely that φ is ramified and $y_0 \in J_M$, says that this finite Bessel orbit has a graded representative satisfying (23.82). Lemmas 23.12 and 23.13 therefore give positive covering multiplicity at h_f. Thus F vanishes on every orbit missed by this cut.

For a point $x = (x_\infty, x_f)$, the Bessel period of F, by (23.87), is

$$\begin{aligned}\int_{[R_{k+1}]} F(rx)\,dr &= i_{k+1}P_{k+1}(\eta[k+1 \mapsto x_\infty], y[k+1 \mapsto x_f]; z)\\ &= i_{k+1}a(\eta[k+1 \mapsto x_\infty])S([y][k+1 \mapsto [x_f]]).\end{aligned} \tag{23.96}$$

The first equality uses the definition of the shifted stage: changing the outer variable of index $k+1$ does not change U_{k+1}, which uses only earlier variables. The second uses (23.91); its endpoint conditions remain valid because $0 < k+1 < m$. Equation (23.96) is exactly the period hypothesis of Lemma 23.18. That lemma cancels i_{k+1} against the Haar mass of a J_{k+1}-coset and proves (23.95).

Apply the inductive assertion to a', z', S'. Transposition (23.85) identifies its left side with the left side of (23.92) at step $k+1$. On the right, finite summation gives

$$\mathrm{Cut}_{\vartheta,k+1}\, S' = \mathrm{Cut}_{\vartheta,k+2}\, S.$$

The inductively chosen scalar depends only on a' and the previously fixed real data; it is therefore independent of the twist and of all four varying objects. This proves the universal assertion, not just a proportionality for one chosen character. The non-absolutely-integrable real-factor case is already included in Lemma 23.18: its finite values must all vanish, and the common scalar is then chosen to be zero.

Lean. `SeesawIdentity` (L401-L493). □

Theorem 23.38 (The Bessel period of the finite tensor)**.** *There is one $c_0 \in \mathbb{C}$, fixed before the finite character, such that for every tame ϑ and every ramified source form φ,*

$$\mathcal{L}_\vartheta(\mathcal{P}\varphi) = c_0\,\mathrm{res}\big(\mathrm{Chain}_\vartheta(\phi_\vartheta)\varphi\big), \tag{23.97}$$

where ϕ_ϑ is the tensor in (23.32), with its original finite evaluation, cuts and multiplicities. This tensor is jointly invariant under the upper levels.

Proof. Fix a real tuple η. At the final pair, take the last grading and $x = 1$ in Corollary 23.27. For every finite tuple with $y_m = 1$, it gives one functional $l(\eta)$, independent of ϑ, y, φ, such that

$$U_m^{\eta,y}(i_{Y_m}(k)) = \mathrm{coeff}_{l(\eta)}\left(\mathrm{Chain}_\vartheta(\mathcal{E}_\vartheta(y))\varphi; \mathrm{diag}(k,1)\right). \tag{23.98}$$

The equality uses the evaluation for the last pair: its definite embedding is $k \mapsto \mathrm{diag}(k,1)$, and $\mathcal{E}_\vartheta(y)$ is the tensor of the evaluated finite data. No component of it has been chosen anew to make (23.98) hold.

Pair with any form x on Y_m. Define the linear functional on the tensor space

$$L_{\vartheta,\varphi,x}(T) = \langle x, \mathrm{res}(\mathrm{Chain}_\vartheta(T)\varphi)\rangle\,. \tag{23.99}$$

Proposition 23.28 gives

$$P_{m-1}(\eta, y; \overline{x}) = a(\eta)L_{\vartheta,\varphi,x}(\mathcal{E}_\vartheta(y)), \qquad a(\eta) = C_{l(\eta)}d_Y\delta_N. \tag{23.100}$$

The real function a is independent of ϑ, φ, x, y. The terminal branching matrix, d_Y, and δ_N are exactly those of the restriction map. Also $\mathcal{E}_\vartheta(y)$ depends only on the finite level classes, by the evaluated-datum invariance proved in p. 575.

Use Theorem 23.37 with this a and $S = L_{\vartheta,\varphi,x} \circ \mathcal{E}_\vartheta$. It supplies one c_0 for all twists, source forms and terminal test forms. Its conclusion is

$$\begin{aligned}\langle x, \mathcal{L}_\vartheta(\mathcal{P}\varphi)\rangle &= c_0 \operatorname{Cut}_{\vartheta,m}(L_{\vartheta,\varphi,x} \circ \mathcal{E}_\vartheta) \\ &= c_0 L_{\vartheta,\varphi,x}(\operatorname{Cut}_{\vartheta,m} \mathcal{E}_\vartheta) \\ &= c_0 \left\langle x, \operatorname{res}(\operatorname{Chain}_\vartheta(\phi_\vartheta)\varphi)\right\rangle .\end{aligned}$$

The middle equality is finite linearity, not an interchange of an infinite integral and a formal tensor. Nondegeneracy of the Petersson inner product proves (23.97). The passage from automorphic to algebraic forms uses the fixed identification at the definite end, so it introduces no twist-dependent scalar. Joint invariance is Corollary 23.17.

Lean. `SeesawIdentity` (L495-L580). □

Corollary 23.39 (The seesaw identity and its coefficient bound)**.** *For the measured map* Π*, there is a single* $c_{\mathrm{ss}} \in \mathbb{C}^\times$ *such that every allowed twist* ϑ *has an original chain datum* Ψ_ϑ *satisfying* (11.11)*. It is jointly invariant, has the required supported translations, is integral away from* $q = \operatorname{Nm} Q$*, and for the empty twist obeys, for every* $\sigma \in \operatorname{Aut}(\mathbb{C})$*,*

$$\sum_x |\sigma\Psi_1(x)| \le M^{3fmn_m^2}. \tag{23.101}$$

Proof. Theorem 23.38 supplies the exact identity assumed in Proposition 23.5. The finite cuts of Lemma 23.13 are integral away from q, including at N, and their multiplicities divide powers of q. Theorem 23.3 and Lemma 23.4 give one common finite scalar C and coefficients r_ϑ with

$$B^{2m} r_\vartheta(x) \text{ integral away from } q, \qquad \sum_x |\sigma r_1(x)| \le B^m, \qquad B = M^{fn_m^2}.$$

Apply the normalizer proposition with this C and the common c_0. It gives a nonzero c_{ss} even if $c_0 C = 0$, by choosing the zero chain datum in that case. Otherwise multiplying the coefficients by B^{2m} clears precisely the stated denominator and gives B^{3m} in (23.101). The replacement by supported translations preserves the glued vacuum, and the common scalar preserves its joint invariance. Finally (23.83) identifies the same Π, so no change of source map is made.

Lean. `SeesawIdentity` (L592), `SeesawIdentity` (L624). □

Thus the orbit calculation in Theorem 23.10, the pair evaluation and terminal restriction in Section 23.3, and the support bound in Theorem 23.35 give the period identity with one archimedean functional for all finite characters.

24 Auxiliary forms for completing the determinant

We complete the independent columns of the determinant to a basis using forms on a definite unitary group. Exchanging adjacent theta lifts gives the same normalization for all the added columns. We bound their norms at the archimedean places and their denominators at every coefficient prime.

To compute the central sum in (24.3) at a prime ℓ, we choose representatives of the central classes that are integral at every place above ℓ.

Let Y be the datum of a lift from the definite $U(1)$ to $U(n)$. Write K_g for its kernel form, ϕ for its lift of the constant source form, $\mathcal{D}$ for the adjoint descent, and χ_V for its specified target splitting character. For a finite point $z \in U(1)(\mathbb{A}_f)$, define

$$T_z F(h) = \chi_V(z) F(h \operatorname{diag}(z, \dots, z)). \tag{24.1}$$

In Lean, T_z is normalizedCentralOp.

Lemma 24.1 (The normalized central action)**.** *The operators T_z form a unitary representation, $T_z^* = T_{z^{-1}}$, and*

$$T_{z^{-1}} K_g = K_{gz}, \qquad T_z \phi = \phi. \tag{24.2}$$

They commute with every finite Hecke translation, the idempotent for the chosen type, and the split Hecke polynomial operators off the level. If z is integral above ℓ, both T_z and $T_{z^{-1}}$ preserve the integral forms at every coefficient embedding above ℓ.

Proof. Scalar matrices commute with all finite translations. Right invariance of the class pairing makes central translation unitary, and $|\chi_V(z)| = 1$. The inverse character value is its complex conjugate, so the adjoint assertion follows. Commutation with the type and Hecke operators follows by taking their finite linear combinations of these translations.

For (24.2), the two simultaneous scalars $(z^{-1}, z^{-1}I)$ act trivially on the underlying homomorphism space. The original rank-twisted pair action on the inclusion datum is multiplication by $\chi_V(z^{-1})^{-1}$. Move this equality through the theta functional and the same jet operator. The result is exactly $\chi_V(z^{-1}) K_g(hz^{-1}I) = K_{gz}(h)$, proving the first formula. The lift ϕ is its original finite class sum. Reindexing that sum by $g \mapsto gz$ proves the second formula. Neither calculation changes its vacuum or jet scale.

Let ι be an embedding above ℓ. An algebraic automorphic character with the prescribed integer weights has $|\iota\chi_V(z)|_\ell = 1$ when z is integral above ℓ. To check this explicitly, raise its class to a power so it is a rational norm-one point times a compact unit, and raise again to kill the finite-order character of that unit. The resulting character value is a product of integer powers of the CM embeddings of that rational point. The rational point is a unit at all places above ℓ, so each factor has ℓ-adic absolute value one. Taking the positive power back proves the assertion.

The scalar matrix zI and its inverse are integral above ℓ. In the Weyl gauge, right multiplication by such a matrix has integral matrix and integral inverse, hence preserves the gauge. Thus, for an integral form F,

$$\operatorname{gauge}_h(T_z F(h)) = \operatorname{gauge}_{hzI}(F(hzI)) \le 1.$$

This holds simultaneously at every embedding above ℓ, and applies to z^{-1} as well.

Lean. `CentralAction` (L37-L195). □

Lemma 24.2 (Representatives at one prime and the descent kernel)**.** *At the chosen level of $U(1)$, let $\mathcal{C}$ be the finite class set and $h_1 = |\mathcal{C}|$. For each prime ℓ, choose representatives s_c integral above ℓ. The original class stabilizers are trivial. Then*

$$y \perp \phi \quad \Longrightarrow \quad \sum_{c \in \mathcal{C}} T_{s_c} y \in \ker \mathcal{D}. \tag{24.3}$$

For the type idempotent e and split Hecke polynomial H_w, put $F_w = \operatorname{im} e \cap \ker H_w$. If $\Xi' \perp F_w \cap \ker \mathcal{D}$ for every such w, then

$$\Xi'' = \sum_{c \in \mathcal{C}} T_{s_c}^* \Xi' \perp F_w \cap (\mathbb{C}\phi)^\perp. \tag{24.4}$$

If Ξ' is integral above ℓ, so is Ξ''.

Proof. First justify the choice of representatives. At a nonsplit finite place a norm-one scalar already has valuation zero. At a split pair $w, \bar{w}$ above ℓ, the valuations of a norm-one finite point are opposite. By weak approximation choose $a \in E^\times$ with prescribed valuations at these finitely many places so that $a/\bar{a}$ has valuations cancelling them. For example prescribe $v_w(a) = -v_w(z)$, $v_{\bar{w}}(a) = 0$ at one representative of each split pair. Multiplying z by the rational norm-one point $a/\bar{a}$ leaves its class unchanged and makes it integral above ℓ. This is done for each of the finitely many classes. The representatives may depend on ℓ; the test forms in this application already do, and the number h_1 does not.

Triviality of the class stabilizers makes the class sum the unweighted sum $\phi = \sum_c K_{s_c}$. Equation (24.2) and translation invariance of that class set therefore give, for every g,

$$\sum_c T_{s_c}^* K_g = \phi. \tag{24.5}$$

The value of the normalized descent is the pairing with K_g, including the real highest coefficient. Consequently

$$\mathcal{D}\left(\sum_c T_{s_c} y\right)(g) = \left\langle \sum_c T_{s_c}^* K_g, y \right\rangle = \langle \phi, y \rangle = 0.$$

This is (24.3). For $y \in F_w \cap (\mathbb{C}\phi)^\perp$, Lemma 24.1 puts each $T_{s_c} y$ in F_w, and their sum is in $\ker \mathcal{D}$. Pairing that sum with Ξ' proves (24.4). Integrality is the last assertion of the preceding lemma, applied to every term.

Lean. `CentralAction` (L197-L260). □

Corollary 24.3 (The class-number factor in the projected pairing)**.** *Let β_ℓ bound the logarithmic denominator on $Z = \mathbb{C}\phi$, and let P_Z be its orthogonal projection. Let Ξ, Ξ' be test forms integral at every embedding above ℓ, with $\Xi' \perp F_w \cap \ker \mathcal{D}$ at each split place w off the level, as in Lemma 24.2. Then*

$$|\iota \langle P_Z \Xi, P_Z \Xi' \rangle|_\ell \le \exp\bigl(\beta_\ell + v_\ell(h_1) \log \ell\bigr). \tag{24.6}$$

The bound also holds when $\ell \mid h_1$.

Proof. If $\phi = 0$, the assertion is immediate. Otherwise choose the integral representatives from Lemma 24.2. It makes Ξ'' an eligible tested form for the denominator assertion. Since every T_{s_c} fixes ϕ, $P_Z \Xi'' = h_1 P_Z \Xi'$. The clearing endomorphism of the one-dimensional space Z is a scalar μ, with

$$|\iota\mu|_\ell \ge e^{-\beta_\ell}, \qquad |\iota \langle \Xi, \mu P_Z \Xi'' \rangle|_\ell \le 1.$$

Thus $|\iota(h_1 \langle \Xi, P_Z \Xi' \rangle)|_\ell \le e^{\beta_\ell}$. Divide by the exact value $|h_1|_\ell = \ell^{-v_\ell(h_1)}$, and use $\langle \Xi, P_Z \Xi' \rangle = \langle P_Z \Xi, P_Z \Xi' \rangle$. This gives (24.6) and discharges the central-action premise in the line estimate used in (25.23).

Lean. `CentralAction` (L272-L344). □

24.1 The Hecke polynomial of the theta lift at split places

The Hecke identity of Theorem 20.12 extends to any split place where the two base-change characters are unramified. We use it to identify the Hecke polynomial of the global lift.

Theorem 24.4 (Split Hecke transfer with both splitting characters)**.** *Let W, V be the Hermitian spaces of ranks n and $N = n + d$, with $d \geq 0$. Let $w \neq \bar{w}$ be a split place, $q = \mathrm{Nm}(w)$, and ϖ_w its chosen uniformizer. Write $a = \mathrm{charW}$ and $b = \mathrm{charV}$ for the automorphic characters used in the pair representation, whose action has the convention*

$$\omega_{a,b}(g,h) = a(\det h)b(\det g)^{-1}\omega(g,h).$$

Assume their base-change characters are unramified at w. Let υ be the fixed Weil splitting character and set

$$\begin{aligned} \chi_W(x) &= a(x/\bar{x})\upsilon(x)^{-n}, & b_W &= \chi_W(\varpi_w)q^{n/2}, \\ \chi_V(x) &= b(x/\bar{x})\upsilon(x)^{-N}, & b_V &= \chi_V(\varpi_w)q^{N/2}. \end{aligned} \tag{24.7}$$

Here $q^{1/2}$ is the positive square root. The nonzero factors b_W and b_V are $a.\,\mathrm{frobValue}(n, w)$ and $b.\,\mathrm{frobValue}(N, w)$ in Lean.

Suppose the datum F has the vacuum component at $\{w, \bar{w}\}$, equivalently $\mathrm{projOn}\{w\}F = F$. Let Θ_F be its global theta pairing, on a space on which the pairing and its finitely many Hecke translates converge absolutely. Normalize both local maximal compact subgroups to have Haar volume one, and put

$$\mathcal{T}_w^{(r)}(U) = \sum_{i=0}^{r} (-U)^i q^{\binom{i}{2}} T_{w,i}^{(r)}.$$

Then Θ_F takes the w-spherical subspace to the w-spherical subspace, and on that source subspace the full operator identity is

$$\mathcal{T}_w^{(N)}(U)\Theta_F = \prod_{j=0}^{d-1}(1 - q^j b_W U)\,\Theta_F \mathcal{T}_w^{(n)}(q^d b_W b_V^{-1} U). \tag{24.8}$$

In particular, if the source Hecke polynomial on y is $P_w(U)$, the target polynomial on $\Theta_F y$ is

$$\prod_{j=0}^{d-1}(1 - q^j b_W U) P_w(q^d b_W b_V^{-1} U).$$

This implication remains valid if the lift of that vector is zero.

Proof. Use the split Schrödinger coordinates, with the integral matrix lattice of volume one. The residual-rank calculation in the proof of Theorem 20.12 uses an integral representative γ_L for each residual subspace L, with elementary divisors ϖ_w on L and one on a complement. Its only matrix assertion is

$$Y\gamma_L^{-1} \text{ integral} \quad \Longleftrightarrow \quad Y \text{ integral and } L \subseteq \ker \bar{Y}.$$

This is valid for every uniformizer; the rational prime used in the marked specialization is unnecessary here. The determinant character and additive Haar change in the target action contribute $b_W^{\dim L}$, while the inverse source action contributes $b_V^{\dim L}$, by (24.7).

For an integral N by n matrix of residual rank s, the finite Grassmann sum proved there is respectively

$$\prod_{j=0}^{N-s-1}(1 - q^j b_W U), \qquad \prod_{j=0}^{n-s-1}(1 - q^j b_V Z).$$

Both expressions vanish off the integral matrices. Factor the first product after d terms and take $Z = q^d b_W b_V^{-1} U$. Since $N - s = d + (n - s)$, this proves the vacuum identity between the target polynomial at w and the source polynomial at $\bar{w}$. It also proves the same identity on any datum fixed by the stated vacuum projection: the local operator acts on this component and leaves all other components untouched. The compact subgroups fix the vacuum and the unramified character factors, proving the required spherical invariance.

Apply the bilinear theta pairing to this finite identity. Equivariance follows directly by changing variables in its absolutely convergent quotient integral. A source action on the datum transfers to the inverse action on its input. The inverse double cosets at $\bar{w}$ are the double cosets at w, with the same finite multiplicities. Thus the source polynomial becomes the one at w on the input in (24.8). Compact volume one gives no additional factor. All sums used here are finite, so the argument proves a coefficientwise operator-polynomial identity. The asserted eigenpolynomial follows by applying it to y.

For the off-level theta data, the required projection property follows because $w, \bar{w}$ avoid the finite datum support and both character conductors; translations supported at the chosen places leave this local vacuum component fixed. The marked-prime argument was used only to verify those hypotheses in its setting, to simplify the ratio of the two values, and to obtain its separate integrality assertion. The exact Frobenius-value convention; The residual-rank calculation; Bilinear transfer; The pair and its off-level hypotheses.

Lean. `HeckeTransfer` (L45-L63), `HeckeTransfer` (L166-L439), `AdjointThetaLift` (L217-L247). □

24.2 Hecke eigensystems in the range of the theta lift

We determine the Hecke eigensystems in the range of the theta lift. For $1 \le j \le m$ and a tame twist ϑ, write

$$\Theta_{j,\vartheta} : \mathcal{F}(W_j) \longrightarrow \mathcal{F}(V_j), \qquad Z_{j,\vartheta} = \Theta_{j,\vartheta}(e_0 \mathcal{F}(W_j)). \tag{24.9}$$

The source has principal level at the chosen places. The target has the level and character of the original descent pair. Its chosen finite group is the same $G_M = \prod_{w \text{ chosen}} \mathrm{GL}_r(\kappa_w)$ that acts on the source. The actions are different: write ρ_M for the source action and A_M for the normalized target action.

Lemma 24.5 (Covariance for the chosen finite group)**.** *For every $a \in \mathbb{C}[G_M]$,*

$$\Theta_{j,\vartheta} \rho_M(a) = A_M(a) \Theta_{j,\vartheta}. \tag{24.10}$$

Consequently, if e_c is the original tuple idempotent of a good tuple c, then

$$\Theta_{j,\vartheta}(\rho_M(e_c) y) \in \operatorname{im} A_M(e_c) \subseteq \operatorname{im} A_M(e_0). \tag{24.11}$$

Proof. It suffices to check a group element $s = (s_w)$. Lift s to an integral finite source point k, supported at the chosen places and their conjugates. At a split place reduction of integral unitary points is reduction of GL_r; lifting an invertible residue matrix gives such a k. Choose an integral target point k' reducing to $\mathrm{diag}(s_w, I_t)$ at those places and to the identity at the twist places. Put

$$c(s) = \prod_{w \text{ chosen}} \psi_w(\det s_w), \tag{24.12}$$

where ψ_w is the tame part of the target character in the descent pair. The normalized target action is $A_M(s) = c(s) R_{V_j}(k')$.

At a chosen place the finite datum is the indicator of the inclusion matrix $X_0 = (I_r, 0)^{\mathsf{t}}$. The simultaneous substitution by k, k' fixes this matrix since $k'^{-1} X_0 k = X_0$ modulo the modulus. The

determinant characters in the pair Weil action contribute $c(s)^{-1}$. At every other finite place the relevant datum is unchanged: the points have identity components at the twist places, and integral components fix the vacuum away from the modulus. Their archimedean components are the identity. Thus, for the full datum F,

$$\omega(k,k')F = c(s)^{-1}F. \tag{24.13}$$

Change the source integration variable in its absolutely convergent theta lift. The factor in (24.13) gives $\Theta_{j,\vartheta}R_{W_j}(k) = c(s)R_{V_j}(k')\Theta_{j,\vartheta}$. This is (24.10) for s, with exactly the original normalization. Extend linearly to the group algebra.

The good tuple idempotents are mutually orthogonal, and e_0 is their sum. Hence $e_0e_c = e_c$ for a good tuple. Apply (24.10) and this identity to obtain both inclusions in (24.11).

Lean. `AdjointThetaLift` (L191-L212), `NormOfChain` (L447-L482). □

Lemma 24.6 (Joint eigenspaces and their transport)**.** *The good principal-level forms on each W_j are the sum of their joint eigenspaces for the operators $T_{w,i}$ at split places away from the chosen places and their conjugates. A nonzero joint eigenspace on W_j occurs on W_m with the same dimension. These statements include simultaneous decomposition by the chosen tuple idempotents.*

Proof. At each unramified split place the pair is $(\mathrm{GL}_r(F_w), \mathrm{GL}_r(\mathcal{O}_w))$, so Satake's theorem gives a commutative spherical Hecke algebra [Sat63]. The operators at different places commute and the adjoint of a double coset is its inverse coset. Thus the finite-dimensional spaces have orthogonal simultaneous eigenspace decompositions. The chosen tuple projectors commute with these off-level actions. At the remaining indices the operators are zero, and each character is extended by zero.

On the direct sum of the two good spaces of W_j, W_m there are finitely many joint characters. For a fixed q, choose for each $q' \neq q$ an operator $H_{q'}$ on which the characters differ. Then the product of $(H_{q'} - q'(H_{q'}))/(q(H_{q'}) - q'(H_{q'}))$ is the joint q-projector on both spaces. Its traces agree by Proposition 21.1. These traces are the required multiplicities.

Lean. `HeckeEigensystems` (L114-L164), `NormOfChain` (L485-L522). □

Lemma 24.7 (The full polynomial on a lifted eigenform)**.** *Let y have source eigensystem q. Let w be a place of E with $w \neq \bar{w}$, such that neither w nor $\bar{w}$ is a chosen place or a twist place. Write $Q_w = |\kappa_w|$, $n = r$, $d = t$ for a main lift, and set*

$$B_{q,w}(U) = \sum_{i=0}^{n} (-U)^i Q_w^{i(i-1)/2} q(w,i). \tag{24.14}$$

Let b_W, b_V be the two Frobenius factors of the pair characters at w, as defined in Theorem 24.4, including their factors $Q_w^{n/2}, Q_w^{(n+d)/2}$. Then $\Theta_{j,\vartheta}y$ is spherical at w, and its Hecke polynomial is

$$\prod_{a=0}^{d-1} (1 - Q_w^a b_W U)\, B_{q,w}(Q_w^d b_W b_V^{-1} U). \tag{24.15}$$

This is the condition `HasLiftHeckePolynomial` in Lean. In particular the Hecke polynomial operator at $U = (Q_w^{d-1}b_W)^{-1}$ kills every element of $Z_{j,\vartheta}$.

Proof. The block indicator has conductor only at the chosen and twist places. At $w, \bar{w}$ its local component is therefore the same vacuum fixed by both maximal compact subgroups. Both pair characters are unramified there. The global lift is absolutely convergent for the original rapid source forms and polynomial Gaussian datum. Thus every hypothesis of Theorem 24.4 holds. Apply its operator identity to y and use (24.14). This proves sphericity and (24.15) for the same lift, without rescaling its datum. The last factor of the displayed product vanishes at the stated U because

$d > 0$ and $b_W \neq 0$. The eigenspace decomposition in Lemma 24.6 then proves annihilation on the whole range.

Lean. `AdjointThetaLift` (L214-L249), `AdjointThetaLift` (L1042-L1061). □

Theorem 24.8 (Generators of the good lift range)**.** *For every j, ϑ, $Z_{j,\vartheta}$ is spanned by vectors $z = \Theta_{j,\vartheta} y$ with the following simultaneous properties:*

1. *y is a good source eigenform, whose eigensystem occurs nontrivially on the good forms on W_m;*
2. *for a good tuple c, $z \in \operatorname{im} A_M(e_c) \subseteq \operatorname{im} A_M(e_0)$;*
3. *this c is a lift setting and z has the full lifted Hecke polynomials (24.15).*

Zero generators may be omitted.

Proof. Decompose a good source form first into its finitely many joint eigencomponents and then by the good tuple idempotents. These decompositions commute by Lemma 24.6. Each nonzero piece is a good eigenform and its system occurs on W_m by the same lemma. Lemma 24.5 puts its lift in the required original target type. Lemma 24.7 supplies the full Hecke condition.

It remains to check the lift-setting conditions, which depend on the tuple and the characters. Each c_w is monic, squarefree, of degree r, nonzero at zero, and has no root among the q_2-th roots of unity; one c_w above each of the two chosen rational primes is primitive. These are the good-tuple conditions. The tame twist is trivial at chosen places. If $\lambda_w \in \kappa_w^\times$ names the tame character of the jth pair there, then $\lambda_w^{q_2} = 1$. Thus $c_w(\lambda_w) \neq 0$. The scaled polynomial has roots λ_w^{-1} times those of c_w, so it does not have root one. Finally r and $t = 5r$ are odd and $r \leq t$. These verify respectively the labels, the primitive place, the excluded root, and the rank and parity clauses of the original lift setting. The finite points, tame character χ_M^j, and finite level are unchanged.

Lean. `AttachedRepresentations` (L76-L113), `FitTwistsExist` (L310-L360). □

Corollary 24.9 (The constant line has the original label $X - 1$)**.** *For every main pair (ν, z) and allowed twist ϑ, its constant-source lift $\phi_{p,\vartheta}$ belongs to the original type projection with label $X - 1$ at every chosen place. That tuple is a lift setting, and the full Hecke polynomial of this lift is*

$$\prod_{a=0}^{2z-2} (1 - Q_w^a b_W U)(1 - Q_w^{2z-1} b_W b_V^{-1} U) \tag{24.16}$$

at every split place off the level.

Proof. The source constant function is fixed by its chosen GL_1 groups. The proof of Lemma 24.5, in rank one and with the main pair's original characters, therefore shows that the normalized target actions fix $\phi_{p,\vartheta}$. The label $X - 1$ is the trivial character of $\mathrm{GL}_1(\kappa_w) = \kappa_w^\times$: its central idempotent is the group mean. Thus the tuple mean fixes the lift.

Its source rank is one and the rank difference $2z - 1$ is odd and at least one. The label is monic, squarefree and nonzero at zero, and the rank-one alternative in the primitive-place clause applies. The main-pair tame character is nontrivial at every chosen place, so its named element λ_w is not one. The scaled polynomial $X - \lambda_w^{-1}$ has no root one. This proves all lift-setting clauses for the tuple.

The constant source has Hecke polynomial $1 - U$. Apply Theorem 24.4 with $n = 1, d = 2z - 1$ to obtain (24.16). The constant-source lift differs from that integral by the fixed scalar only, which does not change this polynomial identity. This proves the theta-transfer hypothesis of Lemma 17.93 and identifies the type of the line used in the denominator estimate.

Lean. `AttachedRepresentations` (L115-L171), `CharacterLiftNorm` (L210-L233). □

Theorem 24.8 and Corollary 24.9 identify the eigensystems and the rank-one line used in the denominator estimate. The arithmetic theorems above give their attached Galois representations, local types and simultaneous twists.

24.3 The restriction period and its normalization

This subsection proves the local period estimate used by the deep-level restriction argument. The two groups here may both be noncompact. For the discrete series with doubled Harish–Chandra parameter l, write c_l for the matrix coefficient of a unit lowest-weight vector. We call a parameter *steep* if the lowest compact weight is zero on a common positive block X, and is strictly decreasing in $[-3B, -B]$ on the remaining coordinates, whose signs are $+, -, +, \dots$.

Theorem 24.10 (The signed one-line restriction period)**.** *Let l be a parameter for $U(\epsilon)$ in rank $n \geq 1$, and let l' be one for $U(\epsilon \oplus 1)$. Suppose that l and $-l'$ are steep with the same $B \geq 1$, on X followed by p and $p+1$ coordinates, respectively, where p is even. Suppose both parameters have the parity and regularity of discrete series, their entries have absolute value at most L, and their undoubled opposite-sign gaps are at least $500n^3 + H$ and $500(n+1)^3 + H$, respectively. Assume*

$$\begin{aligned} \beta &\geq 10^{12}, & 2n+1+L &\leq \beta, \\ (n+1)^4(6B)^{2p+2} &\leq \beta, & 10^{30}\big((2n+1)^2 \log \beta\big)^3 &\leq H. \end{aligned} \tag{24.17}$$

There is a real point $\omega \in U(\epsilon \oplus 1)$ with $\|\omega\|_{\mathrm{op}}^2 \leq \beta$ for which the restriction period on the right below converges absolutely, is real and strictly positive, and satisfies

$$\mathrm{vol}\{r \in U(\epsilon) : \|r - I\|_{\mathrm{op}} \leq 1\} \leq \beta^{20(2n+1)^2} \int_{U(\epsilon)} c_l(r)\overline{c_{l'}\big(\omega^{-1}(r \oplus 1)\omega\big)}\, dr. \tag{24.18}$$

The same choice works for any fixed normalization of Haar measure on $U(\epsilon)$, including the Cayley-chart measure in the datum.

Proof. Let A be the diagonal lowest compact weight of $-l'$ and let B_0 be that of $-l$. Both vanish on X. In their tail coordinates write

$$a_0 > \cdots > a_p, \qquad -3B \leq a_i \leq -B, \qquad 0 \leq b_0 < \cdots < b_{p-1} \leq 3B.$$

The respective signs are $(-1)^i$ and $(-1)^j$. The entries are integers; changing the sign of a regular parameter changes the sign of its lowest weight. Put

$$P(t) = \prod_{i=0}^{p} (t - a_i), \qquad Q(t) = \prod_{j=0}^{p-1} (t - b_j).$$

On the large tail define a real vector and p further real vectors by

$$v_i = \sqrt{\left|\frac{Q(a_i)}{P'(a_i)}\right|}, \qquad w_j = (A - b_j I)^{-1} v, \qquad u_j = \sqrt{\frac{P(b_j)}{|Q'(b_j)|}}\, w_j. \tag{24.19}$$

For $p = 0$ the empty product is one and there are no u_j. Since p is even, $Q(a_i) > 0$ and $\operatorname{sgn} P'(a_i) = (-1)^i$. Partial fractions give

$$\frac{Q(t)}{P(t)} = \sum_{i=0}^{p} \frac{(-1)^i v_i^2}{t - a_i}. \tag{24.20}$$

Its leading coefficient at infinity is one, so $\langle v, v\rangle = 1$ for the signed Hermitian form. At b_j it gives $\langle v, w_j\rangle = 0$. Subtracting its values at distinct b_j, b_k gives $\langle w_j, w_k\rangle = 0$. Differentiating at b_j gives

$$\langle w_j, w_j\rangle = -\frac{Q'(b_j)}{P(b_j)}, \qquad \operatorname{sgn}(-Q'(b_j)) = (-1)^j.$$

Thus $\langle u_j, u_k\rangle = (-1)^j\delta_{jk}$. Furthermore, $Aw_j = b_jw_j + v$, so the compression of A to their span has diagonal entries b_j. Extend this isometry by the identity on X, calling it i_0, and set $W = (i_0, v)$. We have

$$W \in U(\epsilon \oplus 1), \qquad i_0^\dagger A i_0 = B_0, \qquad i_0^\dagger i_0 = I. \tag{24.21}$$

All differences between the nodes have absolute value at most $6B$. The nonzero integral factors in $P'(a_i)$ and $Q'(b_j)$, and the positive differences $b_j - a_i$, have absolute value at least one. Consequently

$$|v_i| \le (6B)^{p/2}, \qquad |(u_j)_i| \le (6B)^{p+1/2} \le (6B)^{p+1}.$$

The identity entries on X obey the same bound. The entry bound for a square matrix gives $\|W\|_{\mathrm{op}}^2 \le (n+1)^4(6B)^{2p+2} \le \beta$. Take $\omega = W^{-1}$; for a sign-unitary matrix $W^{-1} = JW^*J$ and hence $\|\omega\|_{\mathrm{op}} = \|W\|_{\mathrm{op}}$. The fixed signed coordinate permutations carrying the two steep numberings to the given spaces are unitary for the Euclidean norms. Conjugating this construction by them gives the stated point in the original $U(\epsilon \oplus 1)$ and preserves all these norm bounds. The stationary one-vector identity; Stationary compression.

For $r \in U(\epsilon)$ put $g = W(r \oplus 1)W^{-1}$. It fixes v, and every element fixing v arises in this way, because v has signed norm one and i_0 is an isometry onto its orthogonal complement. The map is a proper closed embedding. Directly,

$$g - I = i_0(r - I)i_0^\dagger, \quad \operatorname{tr}(g - I) = \operatorname{tr}(r - I), \quad \operatorname{tr}\big(A(g - I)\big) = \operatorname{tr}\big(B_0(r - I)\big). \tag{24.22}$$

The common $U(X)$ acts diagonally in this pair. Each lowest vector is fixed by it: on X its highest weight is zero, and every proper weight cut contains the whole zero block or none of it.

We give the phase estimate for

$$\theta(r) = c_{-l'}(g)\overline{c_{-l}(r)} = c_l(r)\overline{c_{l'}(g)}.$$

Put $D = (2n+1)^2$, $t = D\log\beta$, $\rho = 10^{-13}/t$ and $\eta = e^{-100t}$. The determinant-decay calculation preceding (8.5), applied separately to the two parameters, gives

$$|\theta(r)|^4 \le \big(1 + \|Y(g)\|_{\mathrm{F}}^2\big)^{-H}\big(1 + \|Y(r)\|_{\mathrm{F}}^2\big)^{-H},$$

where Y retains precisely the entries of opposite signs in each group. If $F^2 = \|Y(g)\|_{\mathrm{F}}^2 + \|Y(r)\|_{\mathrm{F}}^2 > \rho^2$, the product in the denominator is at least $1 + F^2$; thus $|\theta(r)| \le \exp(-H\rho^2/8) \le \eta$.

Suppose $F \le \rho$. The signed unitary equation gives the row identity

$$\|(z - I)_{i,*}\|_2^2 = 2\sum_j |Y(z)_{ij}|^2 - 2\Re(z_{ii} - 1).$$

Multiply it for g by $-a_i$, and for r by b_j, including the zero weights on X. The trace terms cancel by (24.22). The weights in the first sum are at least B off X, all other weights are nonnegative, and every weight used is at most $3B$. Hence the sum of the squared row norms of $g - I$ off X is at most $6F^2$. Applying this to the inverse pair gives the same bound for the columns off X.

Write $a = g_{XX}$. The signed Gram equation gives $\|a^*a - I\|_{\mathrm{F}} \le 6F^2 \le 6\rho^2 \le \rho$. In $a = U|a|$ we therefore have $\||a| - I\|_{\mathrm{F}} \le \|a^*a - I\|_{\mathrm{F}} \le \rho$. Multiply the pair on the left by the common copy

of U^*. This preserves θ and the two noncompact norms. For the translated pair (g', r') the row, column and X-block bounds give

$$\|g' - I\|_{\mathrm{F}}^2 \le 13\rho^2.$$

Summing the unweighted row identities and using (24.22) gives

$$\|r' - I\|_{\mathrm{F}}^2 = \|g' - I\|_{\mathrm{F}}^2 - 2\|Y(g')\|_{\mathrm{F}}^2 + 2\|Y(r')\|_{\mathrm{F}}^2 \le 15\rho^2.$$

Both matrices are within 4ρ of identity in Frobenius norm. The two noncompact factors and simultaneous localization.

The normalized Gaussian estimate proved by the compact-minor calculation above applies to each factor: for any parameter h of rank d in the stated range and $\|z - I\|_{\mathrm{F}} \le \delta$,

$$\delta t \le 10^{-12} \quad \Longrightarrow \quad \exists A_z \in [0,1] : \quad \left| c_h(z) e^{-\operatorname{tr}(\mu_h \log z)} - A_z \right| \le A_z/10 + e^{-100t}. \tag{24.23}$$

Indeed $t \ge d^2 \log(d+L+1)$ and $t \ge \max(1, d^2)$ here. The earlier proof uses the normalized Flensted–Jensen integral, the pointwise bound $|M_{\mu_h}(k)|^2 \le e^{-Q_{\mu_h}(k)/2}$, and compact Schur orthogonality; it imposes no compactness on the other factor and allows either sign of the weight. Lemma 24.12, applied to $g' = W(r' \oplus 1)W^{-1}$, gives $\operatorname{tr}(A \log g') = \operatorname{tr}(B_0 \log r')$. The common trace is purely imaginary: $\log r'$ is skew-adjoint for the Hermitian form, and B_0 is self-adjoint. The two phase factors in $\theta(r')$ therefore cancel. Since $4\rho t < 10^{-12}$, its value is within $ac/3 + 4\eta$ of ac for some $a, c \in [0,1]$. This proves, both near and far,

$$(|\theta(r)| - 2\Re\theta(r))_+ \le 12\beta^{-100D}. \tag{24.24}$$

Gaussian estimate for an arbitrary large-gap parameter; Cancellation of the two phases.

The volume comparison uses the Haar measure without a change-of-variables constant. Set $J_r = g \oplus r$, $\delta = (200\beta^5)^{-1}$, $v_* = \mathrm{vol}\{r : \|J_r - I\|_{\mathrm{op}} \le \delta\}$ and $G(r) = (2/\|J_r\|_{\mathrm{op}})^{9D}$. Properness and continuity give $0 < v_* < \infty$. The large-gap decay gives $|\theta| \le G^2$. On this small ball, the derivative bound $(d+1)^4(L+d+1) \le \beta^5$ and $\|\log z\|_{\mathrm{op}} \le 2\delta$ give $|c_h(z)| \ge 1/2$, and therefore $|\theta| \ge 1/4$.

In the image ball $\|J_r\| \le T \ge 1$, a maximal δ/T-separated set has at most $(1 + 2T^2/\delta)^{2D}$ elements by the elementary disjoint-ball volume argument in the real matrix space of dimension $2D$. Every image point is in a right translate of the small ball, because $\|J_s^{-1}\| = \|J_s\| \le T$. Right invariance consequently gives, with $K = (1 + 8/\delta)^{2D} \le \beta^{12D}$,

$$\int G \le 2^{9D} K v_* \sum_{j \ge 0} 2^{-5Dj} \le 2^{9D+1} K v_*.$$

It also proves integrability of G^2. The deficit in (24.24) is at most $3|\theta|$; taking the geometric mean bounds it by $6\beta^{-50D}G$. Its integral is at most $v_*/8$, since $\beta \ge 10^{12}$. Write $I = \int \theta$. By unitarity of the coefficients, $\theta(r^{-1}) = \overline{\theta(r)}$. The unitary group is unimodular, so inversion preserves Haar measure and I is real. Consequently

$$v_*/4 \le \int |\theta| \le 2I + v_*/8, \qquad I \ge v_*/16 > 0.$$

If $\|r - I\| \le 1$, then $\|r\| \le 2$ and $\|g\| \le 1 + \|W\|^2 \le \beta + 1 \le 2\beta$. The same covering at $T = 2\beta$ gives

$$\mathrm{vol}\{r : \|r - I\| \le 1\} \le (1 + 8\beta^2/\delta)^{2D} v_* \le \beta^{16D} v_* \le \beta^{20D} I.$$

Lean. The formal period bound is stated in absolute value. Keeping the real part in the integrated estimate also gives the sign. See `Periods` (L423-L459), `PositionGroup` (L637-L676), `StationaryPosition` (L699-L747), `PositionGroup` (L678-L743). □

Lemma 24.11 (The auxiliary scales meet every bound)**.** *On the fixed admissible datum set*

$$n = r, \quad p = 2\varrho + 2 = m, \quad L = \lceil 20b \rceil, \quad H = r^7, \quad A = r^4, \quad N_0 = 20(2r+1)^2,$$
$$\beta = (\sqrt{M})^{\lceil x \rceil}, \qquad R = \beta^{20A(f+1)}, \qquad \kappa = 8fr\,\mathrm{rd}_E + 48r^6 f^2 C_E + 1, \qquad C_E = q_1 d. \tag{24.25}$$

Choose an integer e_* in the interval

$$20\lceil x \rceil r^4 (f+1)(3r+2) \le e_* \le x^{40}.$$

The left endpoint is an admissible choice. The parameters on Y_0 and $\widetilde{Y}_0^0 = Y_0 \oplus \langle 1 \rangle$ satisfy Theorem 24.10 at every infinite place, with $B = \kappa_0(\tau)$, and the same β, H, L. They have the gaps needed for $4A$ and $2A$, respectively. Moreover,

$$\begin{aligned} 8(r+1)^2 < A, \qquad & N_0 < A, \qquad & 2f(r+1) + 2 \le A, \\ M\kappa^2 \le \beta, \qquad & 2000(r+L+1)^6 \le \beta, \qquad & f \le \beta, \end{aligned} \tag{24.26}$$
$$R^{3r+2} \le |\tau(\pi_M^{e_*})| = M^{e_*/2} \qquad \text{for every infinite place } \tau.$$

Proof. The admissible hierarchy gives

$$x \ge 10^{100}, \quad x^5 \le r \le 3x^5, \quad m \le 2\sqrt{x}, \quad f, q_1, \mathrm{rd}_E \le x^{12}, \quad C_E \le x^{13},$$
$$b \ge x^{10^{12}}, \qquad \log b \le x^{1/100}, \qquad b^{20} \le M \le b^{22}, \qquad 1 \le B \le 2b. \tag{24.27}$$

Here $b \ge x^{10^{12}}$ follows by composing the two powers $x^{10^6} \le x_1$ and $x_1^{10^6} \le b$. The lower bound on $\log\log b$, followed by $100 \log\log b \le \log x$, gives the deliberately weaker $x \ge 10^{100}$ and $d \le x$, so $C_E = q_1 d \le x^{13}$. The field and rank estimates are also proved in §5.2. The fixed hierarchy; Ranks, fields and level sizes.

The parity and steepness assertions can be checked directly from the given weights. In the grid numbering the doubled entries are $2\,\mathrm{targetParameter}(i) + z$, with $z = 0, 1$. Twice the lowest weight subtracts the signed count of entries above and below i. On the first $k-1$ positive entries this count is $k - 2 - 2i + z$, exactly the doubled parameter, so the weight is zero. On tail coordinate j the count is $-(k-z)(-1)^j$. The lowest tail weights are therefore

$$\mu_z(j) = -B - (j + \tfrac{1}{2})\Delta + \frac{z}{2} + \frac{k-z}{2}(-1)^j, \qquad \Delta = 2r^8.$$

They are integers because k, Δ are even; when $z = 1$ the two half-integer terms have integral sum. Consecutive weights differ by $-\Delta - (k-z)(-1)^j < 0$. Since $\Delta^2 \le B$ and $m + 1 \le \Delta$, these weights lie in $[-3B, -B]$. Their signs alternate, starting with $+$, and the tail lengths are m and $m+1$. The first is even. The parameter on the enlarged group is the negative of the steep grid parameter, as required in Theorem 24.10. All doubled entries have absolute value at most $20b \le L$. The gap $2r^8$ dominates $500(r+1)^3 + r^7$; also $4r^4 \le r^7$. The signed count and original steep parameters; The parameter and scale choices.

Write $q = \sqrt{M}$. We have $q \ge b^{10}$, $r + 1 \le q$ and $6B \le 12b \le q$. Consequently

$$(r+1)^4 (6B)^{2m+2} \le q^{2m+6} \le q^{\lceil x \rceil} = \beta.$$

The last inequality uses $2m + 6 \le 4\sqrt{x} + 6 \le x$. Also $\beta \ge b^{10x}$, which dominates 10^{12} and $2r + 1 + L$. Since $\lceil x \rceil \le 2x$,

$$\log \beta \le 22x \log b \le 22x^{101/100}, \qquad (2r+1)^2 \le 9r^2.$$

It follows that

$$10^{30}\big((2r+1)^2\log\beta\big)^3 \le 10^{38} r^6 x^{303/100} \le r^7,$$

because $r \ge x^5$ and $10^{38} \le x^{197/100}$. This verifies all the local-period inequalities.

For the global conditions, $8(r+1)^2 \le 32r^2 < r^4$, $N_0 \le 180r^2 < r^4$, and $2f(r+1)+2 \le 13x^{17} \le x^{20} \le r^4$. The explicit cusp constant obeys

$$\kappa \le 24x^{29} + 34992x^{67} + 1 \le x^{70}.$$

Thus $M\kappa^2 \le b^{22}x^{140} \le b^{23} \le \beta$. Since $L \le 21b$, $r + L + 1 \le 23b$, and $2000(23b)^6 \le b^7 \le \beta$. Also $f \le x^{12} \le \beta$. Finally

$$\lceil x \rceil\, 20r^4(f+1)(3r+2) \le 71280x^{38} \le x^{40} \le b.$$

Thus the interval for e_* is nonempty and $q^{e_*} \ge R^{3r+2}$. If x varies on a piece of $[X, 2X]$ with r, f fixed, one can use the left endpoint evaluated at the largest cutoff on that piece: $\lceil 2X \rceil \le 2\lceil x \rceil$, and $2 \cdot 71280x^{38} \le x^{40}$. The depth is fixed before the twisting character and the auxiliary form.

Lean, comparison. Lean uses the weight b as the depth. The column separation only requires $q^{e_*} \ge R^{3r+2}$. `AdmissibleData` (L96-L118), `Sizes` (L12-L25), `DeepLevelVanishing` (L723-L768). □

24.4 Exact phase cancellation and positivity of the restriction period

The phase in Theorem 24.10 cancels, which also determines the sign of the period. All matrix logarithms below are the power-series logarithms on the operator-norm ball of radius one about the identity.

Lemma 24.12 (The logarithmic phase is exact)**.** *Let $i : V \longrightarrow V'$ be an isometric embedding of nondegenerate Hermitian spaces, let $\langle v, v\rangle = 1$ with v spanning its orthogonal complement, and put $W = (i, v)$. Suppose A and B satisfy $i^\dagger A i = B$. For $u \in U(V)$, set $g = W(u \oplus 1)W^{-1}$. If $\|g - I\|_{\mathrm{op}} < 1$ and $\|u - I\|_{\mathrm{op}} < 1$, then*

$$\operatorname{tr}(A \log g) = \operatorname{tr}(B \log u).$$

In particular the phase cancellation used in the auxiliary restriction period has no error depending on the size of the lowest weights.

Proof. Conjugating the convergent logarithm series by W gives $\log g = W(\log u \oplus 0)W^{-1} = i(\log u)i^\dagger$. Cyclicity of the trace and $i^\dagger A i = B$ give the identity.

Lean. For the stationary compression and the comparison of the two normalized phases see `StationaryPosition` (L699-L747) and `Periods` (L262-L352). Here the compression identity is applied term by term to the logarithm. □

Corollary 24.13 (The product of the archimedean restriction periods)**.** *At each infinite place τ in the auxiliary construction let I_τ be the restriction period at the point ω_τ of Theorem 24.10. Then $\prod_\tau I_\tau > 0$ for the prescribed Haar measures.*

Proof. Lemma 24.11 verifies the hypotheses of Theorem 24.10 at every infinite place. Each factor is positive by that theorem, and the product is finite.

Lean. The local estimate in absolute value is `PositionGroup` (L678-L743). The proof of Theorem 24.10 determines the sign of each integral. □

24.5 The first path and the restriction map

Write $V = Y_0$, $n = \dim_E V$, and $Z = V \oplus \langle 1 \rangle$. Keep the groups, lattices, measures and archimedean vectors of the lift. Put

$$a = \pi_M^{e_*}, \quad \mathfrak{b} = (a), \quad C_E = q_1 d, \quad \kappa = 8fn\,\mathrm{rd}(E) + 48n^6 f^2 C_E + 1. \tag{24.28}$$

Here e_* is the depth from Lemma 24.11 and $f = [E : \mathbb{Q}]/2$. Since $e_* \geq 1$, reduction from the auxiliary level to the original level at (π_M) is defined. The source space is the full space $\mathcal{F}_V(J_M)$ of principal-level forms. The auxiliary spaces $\mathcal{H}_i$ are the forms of principal level $J(\mathfrak{b})$ on $\widetilde{Y}_0^i$. Thus $\mathcal{H}_0$ is a space on Z, and $\mathcal{H}_m$ is a space on the definite auxiliary endpoint.

A pointwise bound for cuspidal forms.

Lemma 24.14 (Short vectors and small radical representatives)**.** *Let $x \in U(V)(\mathbb{A})$ and let $\lambda(x)$ be the first minimum of its original Hermitian lattice, cut off at 1. If $0 < \delta \leq 1$ and*

$$\lambda(x) < \left(\frac{\delta}{M\kappa^2}\right)^n, \tag{24.29}$$

then there is a nonzero rational isotropic subspace $A \subset V$ such that every point of the adelic unipotent radical N_A, in the frame at x, has a representative modulo its rational points whose real part lies in $\|u - 1\| \leq \delta$ and whose finite part lies in $J(M)$. Consequently, a continuous cuspidal form y fixed by $J(M)$ satisfies

$$|y(x)| \leq \sup_{\|u-1\| \leq \delta} |y(x(u,1)) - y(x)|. \tag{24.30}$$

Proof. We give the constant calculation, including the finite reduction. Minkowski's second theorem for the determinant-one Hermitian lattice supplies an E-basis of lattice vectors with ordered Euclidean lengths $\alpha_0 \leq \cdots \leq \alpha_{n-1}$ and $\prod \alpha_i \leq \kappa^n$. This is the same Minkowski estimate used in the short-frame construction; the permitted constant $8fn\,\mathrm{rd}(E)$ is a summand of κ. If ξ is a nonzero lattice vector, then

$$1 \leq \|\xi\| \alpha_{n-1}. \tag{24.31}$$

Indeed otherwise its Hermitian pairing with every basis vector is zero: each pairing is an algebraic integer all of whose absolute values are less than 1, so the product formula annuls it. Nondegeneracy then gives $\xi = 0$.

Put $\theta = \delta/(M\kappa^2)$. A vector of length less than θ^n, together with (24.31), gives

$$\prod_{i<n-1} \alpha_i < (\theta\kappa)^n.$$

Since $0 < \theta\kappa \leq 1$, pairing the $n-1$ factors in reverse order gives an index i with $\alpha_i \leq \alpha_{n-2-i}$ and $\alpha_i \alpha_{n-2-i} < \theta\kappa$. For $n = 1$ the preceding strict product inequality is already impossible. Let X consist of the first $i+1$ basis vectors and let Y consist of the first $n-1-i$; redundant columns may be repeated to make both matrices square. The same product-formula argument gives

$$X^* J X = X^* J Y = 0, \quad \mathrm{im}\, Y = (\mathrm{im}\, X)^\perp, \quad x_0 y_0 < \frac{\delta}{M\kappa}, \tag{24.32}$$

where $x_0 = \alpha_i \leq y_0 = \alpha_{n-2-i}$. The equality of subspaces follows from their dimensions.

Here is an elementary rounding argument specific to the original field, which also proves the required bound with C_E. Embed $E \subset \mathbb{Q}(\zeta_{C_E})$. The integers $\mathrm{Tr}_{\mathbb{Q}(\zeta_{C_E})/E}(\zeta_{C_E}^j)$ span E over $\mathbb{Q}$ and

have absolute value at most C_E at every embedding. Select $2f$ independent ones. Finite-adelic approximation first makes the finite remainder integral. Rounding the real coefficients in this integral basis then makes every complex absolute value at most $fC_E \le (2f)^2 C_E$. Applied entrywise to a square matrix, this yields a map r into rational matrices with integral finite remainder and archimedean operator norm at most

$$4n^2 f^2 C_E. \tag{24.33}$$

Only a full integral sublattice is used, so an integral basis of $\mathcal{O}_E$ is unnecessary.

the radical calculation which permits this rounding does not require an integral Witt basis. In the adelic frame write J_0 for the sign matrix (the identity at finite places), and use the same letters X, Y for the framed matrices. Over E choose e, b_0 with $Ye + 2Mb_0X^*J = 1$, possible by (24.32). For $T = wX^*$ with $X^*J_0w = 0$, put

$$\mathcal{R}(T) = 1 + 2(T - T^* - T^*J_0T)J_0. \tag{24.34}$$

Expansion proves that $\mathcal{R}(T)$ is unitary, fixes X, acts trivially on $X^\perp/X$, and

$$\mathcal{R}(wX^*)\mathcal{R}(zX^*) = \mathcal{R}((w + z - Xw^*J_0z)X^*). \tag{24.35}$$

Every element of this radical has the form $\mathcal{R}(YcMX^*)$. One explicit choice is $c = e(u - 1 - X(Mb_0)^*J_0u)b_0$, with all the displayed rational matrices transported to the adelic frame. Set $d_0 = -(Yr(c)M)^*J_0Yc$ and $w = (Xr(d_0) - Yr(c))M$. Formula (24.35) shows that the rational radical element $\mathcal{R}(wX^*)$ changes u to

$$\mathcal{R}(M\{Y(c - r(c)) - X(d_0 - r(d_0))\}X^*). \tag{24.36}$$

The parameter inside braces times X^* is integral at finite places. At every infinite place its norm is at most

$$4n^6 f^2 C_E(y_0 + x_0)x_0 \le 8n^6 f^2 C_E x_0 y_0 \le \frac{\delta}{6M},$$

by (24.32) and $48n^6 f^2 C_E \le \kappa$. In (24.34) this gives $\|\mathcal{R}(MT) - 1\| \le \delta$: the linear and quadratic terms are bounded by $2\delta/3 + \delta^2/18 \le \delta$. All finite entries of $\mathcal{R}(MT) - 1$ are divisible by M. This proves the asserted representative property. Rational unipotent quotients are compact, as follows as well by rounding the two parameters in (24.36). Average $y(x) - y(xu)$ over that compact quotient with mass one. Cuspidality makes its average $y(x)$; every difference is bounded by the right side of (24.30).

Lean. `UnipotentRadical` (L339-L454), `Cuspidality` (L200-L341), `Cuspidality` (L499-L664). □

The period of a deep-level Poincaré series. In the next three lemmas R denotes a large positive real radius, not the number m of theta lifts. Let ℓ_w and ℓ'_w be respectively the original doubled Harish–Chandra parameters of the forms on V and Z. Write c_w, c'_w for their unit matrix coefficients and d_w for the formal degree of c_w. Our convolution is

$$R(k)y(x) = \int_{G_\infty} k(u)y(x(u,1))\,du, \quad e(u) = \prod_w d_w\overline{c_w(u_w)}. \tag{24.37}$$

Thus $R(e)y = y$. The opposite-sign gaps of ℓ and ℓ' have excess respectively $4A$ and $2A$; $|\ell_w| \le L$. We use the Haar measures fixed above.

Lemma 24.15 (Kernel variation and slice integrals)**.** *Assume the inequalities of Lemma 24.11, with $A = r^4$, $\beta = (\sqrt{M})^{\lceil x \rceil}$, $N_0 = 20(2n+1)^2$ and $R = \beta^{20A(f+1)}$. Put*

$$H(u) = \prod_w d_w (2/\|u_w\|)^{2A}, \qquad K_* = \{\beta^{N_0}(2\beta)^{3A}\}^f.$$

Then

$$\int H \le \frac{R}{16\beta K_*}, \tag{24.38}$$

$$|e(u^{-1}z) - e(z)| \le \{H(u^{-1}z) + H(z)\}\,\frac{\beta}{R} \qquad (\|u-1\| \le R^{-2}). \tag{24.39}$$

For $\alpha, \omega \in U(Z)_\infty$ define

$$\psi_{\alpha,\omega}(g) = \prod_w \overline{c'_w(\alpha_w^{-1} g_w \omega_w)}, \quad v = \mathrm{vol}\{u \in U(V)_w : \|u-1\| \le 1\},$$

and

$$K(\alpha,\omega) = \prod_w \left[(2(\|\alpha_w\|\|\omega_w\|)^2)^A\, 16^A v\right]. \tag{24.40}$$

If $t > 0$ and $t \le \prod_w \|g_w \,\mathrm{diag}(u_w, 1)\|$ for every u, then

$$\int_{U(V)_\infty} |\psi_{\alpha,\omega}(g\,\mathrm{diag}(u,1))|\,du \le K(\alpha,\omega) t^{-A}. \tag{24.41}$$

Proof. The large-gap coefficient estimate and the Casimir derivative estimate already proved for the coefficients give

$$|c_w(g)| \le (2/\|g\|)^{4A}, \quad |c_w(g)| \le 1, \quad |c_w(gu) - c_w(g)| \le 100(n+L+1)^5 \|u-1\|. \tag{24.42}$$

For the first bound use the compact-dual Haar measure. Rescaling this measure multiplies the formal degree by the inverse factor, so $d_w\,du_w$ is independent of that choice. Section 10 gives $d_w \le (n+L+1)^{n(n-1)/2}$. The inverse-norm moment following (21.18) is at most one at exponent $2A$, since $2pq + 2 \le 2A$ in each noncompact signature. Consequently $d_w \int (2/\|u\|)^{2A}\,du \le \beta^{n^2+2A}$, since $n + L + 1 \le \beta$. Multiplying over the f places proves (24.38), using $16 \le \beta$, $N_0 < A$ and $(n^2 + 2A + N_0 + 6A)f + 2 \le 20A(f+1)$.

Put $q(z) = \prod_w \overline{c_w(z_w)}$ and $D(z) = \prod_w (2/\|z_w\|)^{2A}$. Telescoping the product in (24.42) gives $|q(u^{-1}z) - q(z)| \le \beta^2 R^{-2}$. The decay bound gives $|q(z)| \le D(z)^2$. Taking the geometric mean of the variation bound and $|q(u^{-1}z) - q(z)| \le D(u^{-1}z)^2 + D(z)^2$ gives (24.39) after multiplication by $\prod_w d_w$.

For the slice, submultiplicativity gives, place by place,

$$|c'_w(\alpha^{-1}h\omega)| \le (2(\|\alpha\|\|\omega\|)^2)^A (2/\|h\|)^A \|h\|^{-A}.$$

The last factor contributes t^{-A} after multiplication over places. For the remaining integral invert to the coset Hz, where $H = \mathrm{diag}(U(V)_w, 1)$ and $z = g_w^{-1}$. The block embedding preserves the operator norm and the radius-one identity ball. Thus (8.7) with $\delta = 1$ gives $\int_H (2/\|hz\|)^A\,dh \le 2^{A+1} 3^{2(n+1)^2} v \le 16^A v$, since $A > 8(n+1)^2$. Fubini yields (24.41).

Lean, comparison. Lean estimates the full-group and slice integrals by coverings. Here the exact ball volume gives the first integral and (8.7) gives the second in its prescribed measure. `DeepLevelVanishing` (L82-L300), `DeepLevelPoincarePeriod` (L196-L289). □

Lemma 24.16 (Separated last columns and the main term)**.** *Let y be an automorphic function of level $J(a)$ with $|y| \le S$, and let $\|\tau(a)\| \ge R^{3n+2}$ at every infinite place. If*

$$R^2 \le R^{3n+2}\lambda(x), \tag{24.43}$$

then the period over $[U(V)]$ of

$$\sum_{\gamma \in U(Z)(E)} \psi_{\alpha,\omega}((\iota(x)^{-1}\gamma\iota(h))_\infty) 1_{J_Z(a)}(\iota(x)^{-1}\gamma\iota(h))_f \, y(h) \tag{24.44}$$

differs from $\mathrm{vol}(J_V(a))R(\psi_{\alpha,\omega} \circ \iota)y(x)$ by at most

$$\mathrm{vol}(J_V(a))K(\alpha,\omega)S\frac{10}{R}. \tag{24.45}$$

Here $\iota(h) = \mathrm{diag}(h, 1)$ and the period and all its unfolding sums are absolutely convergent.

Proof. Take representatives for the right cosets $U(Z)(E)/\iota(U(V)(E))$ which meet the displayed finite level condition. Send a representative γ to the last column of $\iota(x)^{-1}\gamma$, in the Euclidean coordinates over all infinite places. This is injective on the cosets: equal columns imply that $\gamma^{-1}\gamma'$ fixes the last basis vector, hence lies in $\iota(U(V)(E))$. For two contributing columns, their difference divided by a is a lattice vector in the frame of $V_x \oplus \langle 1\rangle$. The added positive line does not decrease the cutoff first minimum. Distinct columns are therefore at distance at least $R^{3n+2}\lambda(x) \ge R^2$. The column of the identity coset has length $\sqrt{f} \le f \le R$. The other columns z_γ have length at least $R^2/2$.

For use here, the elementary packing sum in real dimension $D_0 = 2f(n+1)$ is

$$\sum_{\gamma \ne 1} (R/\|z_\gamma\|)^A \le 10/R \qquad (A \ge D_0 + 2, R \ge 10). \tag{24.46}$$

Indeed points at mutual distance at least h and distance at least h from zero obey $\sum \|z\|^{-A} \le (5/h)^A$ for $A \ge D_0 + 2$. To see this, disjoint radius-$h/2$ balls give $\#\{\|z\| \le t\} \le (1 + 2t/h)^{D_0}$ for $t \ge h$; layer-cake integration bounds the sum by $3^{D_0}A/(A - D_0)\, h^{-A} \le (5/h)^A$. Apply this with $h = R^2/2$ and use $(10/R)^A \le 10/R$.

The last column is fixed by right multiplication by $\iota(h)$. Its Euclidean norm is at most $f \prod_w \|(\iota(x)^{-1}\gamma\iota(h))_w\|$. Thus (24.41) bounds the absolute integral of its coset term by

$$\mathrm{vol}(J_V(a))K(\alpha,\omega)S(R/\|z_\gamma\|)^A.$$

The finite fiber is either empty or one translate of $J_V(a)$, since $\iota(h) \in J_Z(a)$ exactly when $h \in J_V(a)$. The series of these absolute integrals converges by (24.46). Tonelli and ordinary quotient unfolding are therefore justified. For the identity coset change variables $h = xu$; its finite integral is exactly $\mathrm{vol}(J_V(a))$, since y has level $J_V(a)$. This is precisely the asserted main term, with no extra level index or formal degree.

Lean. `DeepLevelPoincarePeriod` (L302-L435), `DeepLevelPoincarePeriod` (L599-L761). □

Proposition 24.17 (One real translate detects every principal form)**.** *There is a single $\omega \in U(Z)_\infty$, independent of y, of every twist and of every later embedding, such that*

$$\langle y, \phi(\iota(\cdot)(\omega, 1))\rangle = 0 \quad \text{for every } \phi \in \mathcal{H}_0 \quad \Longrightarrow \quad y = 0 \tag{24.47}$$

for $y \in \mathcal{F}_V(J_M)$.

Proof. Choose ω from Theorem 24.10, so that $\|\omega_w\|^2 \le \beta$ and

$$v \le \beta^{N_0}|I_w|, \quad I_w = \int_{U(V)_w} c_w(u)\overline{c'_w(\omega_w^{-1}\iota(u)\omega_w)}\,du. \tag{24.48}$$

The one choice is made from the two parameter families. We prove that every global bound S on $|y|$ improves to $S/2$. The cusp-kernel forms are cuspidal by the fixed-type cuspidality argument used in Lemma 23.36. On the region $\lambda(x) < (R^{-2}/(M\kappa^2))^n$, use Lemma 24.14 and write both nearby values of y as $R(e)y$. Formula (24.39), invariance of Haar measure and (24.38) give

$$|y(x(u,1)) - y(x)| \le 2S\frac{\beta}{R}\int H \le S/8 \qquad (\|u-1\| \le R^{-2}). \tag{24.49}$$

Thus $|y(x)| \le S/2$ in this region.

On the complementary region, $M\kappa^2 \le \beta \le R$ implies (24.43). The Poincaré series in (24.44), before evaluation at $\iota(h)(\omega,1)$, is the conjugate of a form of type ℓ' and level $J_Z(a)$, divided by the nonzero product of its formal degrees. Here we use the classical Poincaré-series theorem for integrable discrete-series coefficients. A primary reference is G. Muić, *Smooth cuspidal automorphic forms and integrable discrete series*, Lemma 6-3, Theorem 6-4 and Corollary 7-9. To retain the arbitrary left centre, apply Theorem 6-4 to the conjugated discrete subgroup: for a component lattice Γ and any real a,

$$\sum_{\gamma\in\Gamma} c'(a^{-1}\gamma g) = \sum_{\gamma'\in a^{-1}\Gamma a} c'(\gamma' a^{-1} g).$$

The coefficient on the right has its fixed K-finite left vector. Pullback by $g \mapsto a^{-1}g$ preserves the right representation and the L^2 norm. Lemma 6-3 and the compact reproducing function in the proof of Theorem 6-4 give a bounded continuous pointwise sum; thus the ensuing restriction to $\iota(U(V))$ is the same function, not merely an almost everywhere representative. Its right minimal coefficient fixes it, and its smooth vectors have the original discrete-series type. On the arithmetic component, Wallach's cuspidality theorem and the rapid-decay theorem already specified in Lemma 23.36 make it a cusp form in the required space. The adelic statement follows componentwise as in Corollary 7-9. For the unitary group one applies this argument to the derived group and its compact centre, with the given unitary central character. The large gaps give the needed integrability. All these assertions concern the series before the fixed right translation by ω. The absolute convergence of the particular restriction period is also proved directly in Lemma 24.16. Consequently the assumed orthogonality makes this period zero, for every real α. Cancelling the positive finite level volume in (24.45) gives

$$|R(\psi_{\alpha,\omega}\circ\iota)y(x)| \le K(\alpha,\omega)S\,10/R. \tag{24.50}$$

The rank-one convolution identity for e now undoes this smoothing. With $\eta = \prod_w I_w$, Schur orthogonality gives

$$\int e(u)R(\psi_{\omega,\omega}\circ\iota)y(x(u,1))\,du = \eta y(x), \qquad |\eta| \ge (v/\beta^{N_0})^f. \tag{24.51}$$

In detail, Fubini changes the left side into the integral of $\psi_{\omega,\omega}(\iota(z))$ times $R(e(\cdot\, z^{-1}))y(x)$; minimality of e gives $R(e(\cdot\, z^{-1}))y(x) = \overline{e(z)}y(x)/e(1)$. The positive formal degrees cancel, producing precisely the product in (24.48), with its stated complex conjugation. The same identity follows by first truncating both integrals; the integrable bound below justifies the limit.

Right translation by u in (24.50) replaces its left parameter ω by $\iota(u)\omega$. Pointwise, (24.42) and $\|\omega_w\|^2 \le \beta$ give

$$|d_w\overline{c_w(u_w)}|(2(\|\iota(u_w)\omega_w\|\,\|\omega_w\|)^2)^A 16^A \le d_w(2/\|u_w\|)^{2A}(2\beta)^{3A}. \tag{24.52}$$

The product of these bounds is integrable by (24.38). Combining (24.50)–(24.52) therefore gives

$$(v/\beta^{N_0})^f|y(x)| \le \frac{R}{16\beta K_*}((2\beta)^{3A}v)^f S\frac{10}{R}, \quad |y(x)| \le \frac{10}{16\beta}S \le S/2.$$

Every y has a finite global bound. Iterating its halving proves $y = 0$. Both the level a and the translate ω have remained fixed.

Lean. `DeepLevelVanishing` (L305-L675), `DeepLevelVanishing` (L694-L768). □

Descent and restriction to the source. Write A_i for the untwisted theta ascent from $\widetilde{Y}_0^i$ to $\widetilde{Y}_0^{i+1}$, multiplied by

$$h_{+,i} = |\mathrm{GL}_{\dim \widetilde{Y}_0^i}(\mathcal{O}_E/\mathfrak{b})|.$$

Its transposed arrow carries the same factor $h_{+,i}$. Let $U = A_{m-1}\cdots A_0$ on the conjugate-type forms and let $\Theta^\flat$ be the composite of those transposed arrows on the original-type forms, from $\mathcal{H}_m$ to $\mathcal{H}_0$.

Lemma 24.18 (The auxiliary descent is onto)**.** *The map $\Theta^\flat : \mathcal{H}_m \to \mathcal{H}_0$ is well defined and surjective.*

Proof. At an edge the finite datum is exactly the indicator of the standard inclusion modulo $\mathfrak{b}$. Principal level on either its source or target fixes this datum, and its tame character is trivial on that level. Thus its transpose at the first edge gives the asserted target level; the last ascent gives the level of U at the endpoint.

The ascent U is injective. Here is the precise use of the triangle argument already established in Lemmas 21.4 and 21.5. At each row's first edge the standard inclusion is sharp for the same principal level $J(\mathfrak{b})$, and the target rank is more than twice the source rank. The first-step Rallis theorem (12.1) makes that first ascent injective on the full principal-level space, not just its good subspace. Its return by the adjoint is again in the base-level space: finite translation commutes with the first Gram pairing. If a weighted row chain vanishes, move one descent through the chain with the first-square relation. The shorter chain in the next row kills the return of its first ascent. Induction on the length makes that return zero. Pairing with the initial vector gives the squared norm of the first ascent, so first-step injectivity makes the initial vector zero. All group-order weights are positive. This proof uses only the first square for the auxiliary triangle; no modulus formula for that triangle is being presumed.

The ordinary bilinear transpose identity, applied at every edge with these same weights, is

$$\int_{[\widetilde{Y}_0^m]} \phi\, U y = \int_{[\widetilde{Y}_0^0]} y\, \Theta^\flat \phi. \tag{24.53}$$

All arrows have the original types, so conjugation identifies the two spaces in this formula with their Petersson duals. If $z \in \mathcal{H}_0$ is orthogonal to the image, take $y = \overline{z}$ and $\phi = \overline{Uy} \in \mathcal{H}_m$. Then (24.53) says $\|Uy\|_2^2 = \langle z, \Theta^\flat\phi\rangle = 0$. Injectivity gives $z = 0$. Fixed type and level spaces are finite dimensional; hence a subspace with zero orthogonal complement is the whole target.

Lean. `FirstPath` (L50-L201), `FirstPath` (L205-L278). □

Proposition 24.19 (Flat restriction with every twisted-level test)**.** *For the single point ω of Proposition 24.17 there is a surjective complex linear map*

$$r : \mathcal{H}_m \longrightarrow \mathcal{F}_V(J_M) \tag{24.54}$$

such that for every m-fold twist ϑ, every y of the original type and level $J(\pi_M Q_\vartheta)$, and every $\phi \in \mathcal{H}_m$,

$$\langle y, r\phi\rangle = \int_{[U(V)]} \overline{y(h)}(\Theta^\flat\phi)(\iota(h)(\omega,1))\, dh. \tag{24.55}$$

The map r is chosen once, before every twist and embedding.

Proof. For $\psi \in \mathcal{H}_0$ represent the displayed functional on the finite-dimensional Petersson space $\mathcal{F}_V(J_M)$ by a unique vector $p_\omega\psi$. The map p_ω is complex linear. We verify the stronger test space in (24.55); projection only on J_M-fixed tests would be insufficient here.

Let $u \in J_M$. Integral reduction at the primes of M is surjective. Thus there is a finite integral v supported only at those primes and their conjugates, with $uv \equiv 1 \pmod{\mathfrak{b}}$. Its reduction modulo π_M is also 1 because that of u is 1. At all primes of Q_ϑ the element v is the identity; therefore $v \in J(\pi_M Q_\vartheta)$ and it fixes y. The element $\operatorname{diag}(uv, 1)$ fixes ψ. Simultaneous finite translation leaves the diagonal period unchanged by right invariance of the quotient measure. Consequently its functional of y is invariant under every $u \in J_M$. It therefore has the same value on y as on its mass-one average over J_M. The latter is in $\mathcal{F}_V(J_M)$, where Riesz representation applies. Since $p_\omega\psi$ itself is fixed by J_M, its Petersson pairing also has the same value on y and on that average. This proves the stronger identity.

Set $r = p_\omega\Theta^\flat$. If a vector in $\mathcal{F}_V(J_M)$ is orthogonal to the range of r, Lemma 24.18 makes it orthogonal to all the restrictions in (24.47). Proposition 24.17 makes it zero. Finite dimension proves surjectivity.

Lean. `GridFirstColumns` (L120-L265), `FirstPath` (L288-L375). □

The finite datum on the first path. Put $R_\vartheta = \mathcal{O}_E/(\mathfrak{b}Q_\vartheta)$, $R_\vartheta^- = \mathcal{O}_E/(\pi_M Q_\vartheta)$ and $R^+ = \mathcal{O}_E/\mathfrak{b}$. The two first columns have edges $e_{\epsilon,i} : (\epsilon, 0, i) \to (\epsilon, 0, i+1)$, $0 \le i < m$, where $\epsilon = -, +$. The first-path cut below uses these reductions with $\mathfrak{b} = (\pi_M)^{e_*}$. Let $T_{\epsilon,i}(S_i)$ be the transposed theta map whose test function is the indicator of the matrix S_i over R_ϑ. Let ρ_ϵ be reduction to R_ϑ^- or R^+, and let J_i be the inclusion matrix. Define

$$p(S) = \prod_{\epsilon=\pm} \prod_{i<m} 1_{\{\rho_\epsilon(S_{\epsilon,i})=J_i\}}. \tag{24.56}$$

For a node family g, use exactly the source action and edge character

$$(g \cdot S)_{p,q} = g_q^{-1} S_{p,q} g_p, \qquad \chi(g) = \prod_{(p,q)} \chi_{p,q}(\det g_q / \det g_p). \tag{24.57}$$

The node subgroup $K_{-,i}$ is trivial in $\mathrm{GL}_{\dim Y_i}(R_\vartheta)$, and $K_{+,i}$ is the kernel of reduction to R^+. Let

$$\mathcal{C}_\vartheta = \prod_{\epsilon=\pm,\, i<m} \mathrm{GL}_{n_{\epsilon,i}}(R_\vartheta)/K_{\epsilon,i}.$$

For $c \in \mathcal{C}_\vartheta$, choose representatives at these nodes, set both endpoints to 1, and replace its $(+,0,0)$ component by the block sum of its $(-,0,0)$ component and 1. Denote the resulting family by g_c. In particular the $(+,0,0)$ coset is a free summation variable and is counted with its full multiplicity. Set

$$\Psi^0_\vartheta(S) = \sum_{c \in \mathcal{C}_\vartheta} \chi(g_c) p(g_c \cdot S). \tag{24.58}$$

Lemma 24.20 (The test function and scalar on the first path)**.** *The function Ψ^0_ϑ satisfies the conditions on the first path. Let n_m be the terminal rank. Each value is a sum of at most $|R_\vartheta|^{2m(2n_m)^2}$ roots of unity. There is a positive rational number α_0, independent of ϑ, g, y' and ϕ, such that*

$$\begin{aligned} &\int_{[U(V)]} (T^-_{\vartheta,0} y')(h)(\Theta^\flat\phi)(\iota(h)(g,1))\, dh \\ &\qquad = \alpha_0 \,\mathrm{gridPeriod}(\mathrm{first}, \Psi^0_\vartheta, g; y', \phi). \end{aligned} \tag{24.59}$$

Here $T^{-}_{\vartheta,0}$ is the weighted transpose down the Y column, and y' has the conjugate original type at Y_m.

Proof. Linearity of the unchanged finite Weil datum gives at each edge

$$T^{\mathrm{point}}_{\epsilon,i} = h_{\epsilon,i} \sum_{S_i} 1_{\{\rho_\epsilon(S_i)=J_i\}} T_{\epsilon,i}(S_i), \quad h_{-,i} = |\mathrm{GL}_{\dim Y_i}(R^{-}_{\vartheta})|. \tag{24.60}$$

For $\epsilon = +$ the weight is $h_{+,i}$ defined above. Composing and pairing the two columns therefore expresses the left side of (24.59) as $\prod_{i<m} h_{-,i}h_{+,i}$ times the first-path sum with coefficient p. The columns have disjoint edge sets, so no extra identifications or multiplicities enter this product expansion.

An integral node translation on a single edge contributes the factor in (24.57), by finite Weil covariance. Along a column the intermediate translations cancel. At its two lower corners the translations are simultaneous block translations, and the diagonal period is invariant under them. The endpoints of g_c are 1. Thus, if $P_g(S)$ denotes the period of the two edge composites,

$$P_g(g_c \cdot S) = \chi(g_c) P_g(S). \tag{24.61}$$

Reindexing the finite matrix tuples gives

$$\sum_S \Psi^0_\vartheta(S) P_g(S) = |\mathcal{C}_\vartheta| \sum_S p(S) P_g(S). \tag{24.62}$$

Indeed $\chi(g_c)p(g_c \cdot S)P_g(S) = p(g_c \cdot S)P_g(g_c \cdot S)$, so neither the phase nor its inverse has been dropped.

Formula (24.58) satisfies all four path conditions. It depends only on the path edges. A family in the node subgroups fixes p, and has edge character 1: on the minus column it is the identity, and on the plus column its character is read after reduction to R^{+}. An admissible family permutes the cosets in $\mathcal{C}_\vartheta$. Its plus corner is the block sum of its minus corner, so the same permutation respects the replacement in g_c. The resulting discrepancy is in the node subgroups and hence fixes p. This proves the exact covariance required of Ψ^0_ϑ. If $p(g_c \cdot S) \neq 0$, consider a minus-column edge from node p to node q. Write g_p, g_q for the corresponding components of g_c. The matrix $\widetilde{S}_i = g_q^{-1} S_i g_p$ reduces to J_i. Since $J_i^{\mathsf{t}} J_i = I$ and the kernel of $R_\vartheta \to R^{-}_\vartheta$ lies in the Jacobson radical, $J_i^{\mathsf{t}}\widetilde{S}_i$ is invertible. Thus $\widetilde{L}_i = (J_i^{\mathsf{t}}\widetilde{S}_i)^{-1} J_i^{\mathsf{t}}$ satisfies $\widetilde{L}_i \widetilde{S}_i = I$. Returning to the original edge gives $(g_p \widetilde{L}_i g_q^{-1}) S_i = I$. Finally p takes values $0, 1$, and $\chi(g_c)$ is a root of unity. There are at most $|R_\vartheta|^{(2n_m)^2}$ cosets at each of the $2m$ nodes, since the matrices have at most $2n_m$ rows and columns. This proves the bound. The bound for a single node is `edgeMatrixCountBound` in Lean.

The exact count is

$$|\mathcal{C}_\vartheta| = \prod_{i<m} |\mathrm{GL}_{\dim Y_i}(R_\vartheta)| h_{+,i}. \tag{24.63}$$

Reduction from R_ϑ to R^{-}_ϑ is a surjection whose kernel lies in the Jacobson radical. A square matrix is invertible exactly when its reduction is invertible, and all matrix fibers have the same cardinality. Consequently

$$\frac{h_{-,i}}{|\mathrm{GL}_{\dim Y_i}(R_\vartheta)|} = \left(\frac{|\mathcal{O}_E/(\pi_M)|}{|\mathcal{O}_E/\mathfrak{b}|}\right)^{(\dim Y_i)^2}. \tag{24.64}$$

The Q_ϑ factors cancel, since they are disjoint from M. Equations (24.60)–(24.64) prove (24.59) with the explicit nonzero scalar

$$\alpha_0 = \prod_{i<m} \left(\frac{|\mathcal{O}_E/(\pi_M)|}{|\mathcal{O}_E/\mathfrak{b}|}\right)^{(\dim Y_i)^2} > 0. \tag{24.65}$$

Lean. `GridFirstColumns` (L88-L117), `GridFirstColumns` (L399-L562), `GridFirstColumns` (L564-L809). □

Theorem 24.21 (The first-path identity with the fixed original source)**.** *For the single map r of Proposition 24.19, the original lower chain satisfies, for every m-fold twist ϑ, every conjugate-type form y' on Y_m, and every $\phi \in \mathcal{H}_m$,*

$$\int_{[Y_m]} y'(h)\Theta_\vartheta(r\phi)(h)\,dh = \alpha_0\,\mathrm{gridPeriod}(\mathrm{first}, \Psi^0_\vartheta, \omega; y', \phi), \tag{24.66}$$

where r is onto the full principal-level space and $\alpha_0 \neq 0$ is independent of all twists, points and forms. The datum bound is exactly the one in Lemma 24.20.

Proof. Transpose the lower chain one original edge at a time, including its weight on the smaller group,

$$\mathrm{stdLevelIndex}(J(\pi_M Q_\vartheta)) = |\mathrm{GL}(R^-_\vartheta)|.$$

The result on Y_0 is $T^-_{\vartheta,0}y'$, whose conjugate is of level $J(\pi_M Q_\vartheta)$: that principal level fixes its original point datum. The full quantifier in (24.55) therefore permits replacement of $r\phi$ in this pairing by the restriction of $\Theta^\flat\phi$. This gives the left side of (24.59). Apply that identity with the already fixed real point ω. In particular r, α_0 and the source level are fixed before the later algebraic and p-adic embeddings. The nonzero twist-dependent scalars produced by the diagonal turns are separate; this construction changes none of them.

Lean. `FirstPath` (L379-L462). □

24.6 A common coefficient by continuity on Schwartz space

Only Schwartz vectors and real translates of polynomial Gaussian vectors occur in the auxiliary squares. The following argument gives the common coefficient directly in that space, with no assumption of a bound in the unweighted Fock norm.

Let $\mathcal{F}^\infty$ be the Schwartz part of the fixed real Fock space. After replacing the negative coordinates by their conjugates, its vectors have the form

$$v(z) = F(z)e^{-|z|^2/2}, \qquad F \text{ entire}, \qquad p_N(v) = \int_{\mathbb{C}^d} (1+|z|)^N |v(z)|\,dz < \infty \quad (N \geq 0).$$

The constant normalization of Lebesgue measure is immaterial here.

Lemma 24.22 (One coefficient for an arbitrary family of tests)**.** *Let X be any index set, let $(\rho_x)_{x\in X}$ be fixed complex numbers, and let $\Lambda_x : \mathcal{F}^\infty \to \mathbb{C}$ be complex linear. Suppose that, for each x, there are C_x, N_x with*

$$|\Lambda_x(v)| \leq C_x p_{N_x}(v).$$

Suppose also that for every polynomial P there is a number a_P such that

$$\Lambda_x(Pe^{-|z|^2/2}) = a_P\rho_x \qquad \text{for every } x \in X.$$

Then for every $v \in \mathcal{F}^\infty$ there is one a_v such that

$$\Lambda_x(v) = a_v\rho_x \qquad (x \in X).$$

No common bound on C_x or N_x is required.

Proof. By the circle averages in the proof of Proposition 23.24, polynomial Gaussian vectors approximate v in every p_N. If all ρ_x vanish, density and the individual bounds give $\Lambda_x(v) = 0$ for every x. Otherwise choose x_0 with $\rho_{x_0} \neq 0$. For each x, the functional $\rho_{x_0}\Lambda_x - \rho_x\Lambda_{x_0}$ vanishes on

polynomial Gaussian vectors and is continuous for $p_{\max(N_x,N_{x_0})}$. It therefore vanishes at v. Take $a_v = \Lambda_{x_0}(v)/\rho_{x_0}$.

Lean. `exists_eq_mul_of_forall_polyVector` uses a truncated reproducing kernel and closedness of the line spanned by $(\rho_x)_x$, with approximation in `exists_tendsto_fockMoment`. Here a single nonzero coordinate gives the coefficient by division. □

For a hermitian space V, put $H_V(g) = 1/\lambda(Vg)$, where $\lambda(Vg)$ is the smaller of one and the first minimum of the moved lattice. Put $H_V = 1$ in rank zero.

Lemma 24.23 (Continuity of the weighted auxiliary period)**.** *Fix a finite theta datum, a continuous moderate weight M, a continuous comparison map b of polynomial height growth, and two rapidly decreasing endpoint forms y, f. Assume that, for every $v \in \mathcal{F}^\infty$, the full scalar integrand*

$$I_v(g,h) = M(g)y(g)K_v(b(g),h)f(h)$$

is invariant under the left action of $G(\mathbb{Q}) \times G'(\mathbb{Q})$, so it descends to $[G] \times [G']$. With the quotient measures, put

$$\Lambda(v) = \int_{[G]} \int_{[G']} I_v(g,h)\, dh\, dg.$$

On $\mathcal{F}^\infty$, this integral is absolutely convergent and has a bound

$$|\Lambda(v)| \le C p_N(v)$$

for some C, N depending on the fixed test. Finite linear combinations of these tests have the same property.

Proof. Write V, V' for the two spaces and ϕ for the finite Schwartz datum. Theta Parseval, before the finite averaging in the proof of Theorem 21.9, gives

$$|K_v(g,h)|^2 \le \sum_{x \in \mathrm{Hom}_E(V,V')} |\langle \phi, \rho(x_f)\phi\rangle|\, |\langle v, \rho(x_\infty)v\rangle|.$$

Here ρ denotes Heisenberg translation and x_f, x_∞ are the coordinates of x in the frames moved by (g,h). Indeed, group the terms of the squared theta sum by the difference of their rational position coordinates and apply Poisson summation in the remaining position variable. Each term is the indicated product of matrix coefficients up to a unit phase. Schwartz decay justifies Poisson summation.

The finite coefficient is at most $\|\phi\|^2$ and has compact support in the finite Heisenberg translation variables. In the Schrödinger model the position translation belongs to the difference of two points of $\operatorname{supp}\phi$, and the modulation annihilates a compact open additive group fixing ϕ by translations. Both conditions are compact. Thus one positive integer, depending only on ϕ, carries this support into the original integral Hom lattice in the fixed coordinates. In the moved frames this integer therefore carries every contributing rational x, and every difference of two such x, into the moved integral Hom lattice. By the first-minimum inequality *(HM)* in the proof of Theorem 21.9, the real points are a-separated with every nonzero point of length at least $a = c\lambda(Vg)\lambda(V'h)$, where $0 < c \le 1$ is fixed. The constant absorbs the fixed change to Fock coordinates.

The absolute value of the Fock reproducing kernel is a constant times $e^{-|z-w|^2/2}$. Since $1 + |z| \le (1 + |w|)(1 + |z - w|)$, it follows that $\sup_z (1 + |z|)^N |v(z)| \le C_N p_N(v)$. Heisenberg translation shifts $|v|$ by x_∞; hence $1 + |x_\infty| \le (1 + |z|)(1 + |z - x_\infty|)$ gives

$$|\langle v, \rho(x_\infty)v\rangle| \le C_N p_N(v)^2 (1 + |x_\infty|)^{-N}.$$

For an integer $q > \dim_{\mathbb{C}} \mathrm{Hom}_E(V, V')_\infty$, separate the zero term and apply (21.12) to the rest with $N = 2q$. Taking square roots yields

$$|K_v(g,h)| \ll p_{2q}(v)\, H_V(g)^q H_{V'}(h)^q.$$

In rank zero only the zero term occurs.

By hypothesis $H_V(b(g)) \ll H_G(g)^a$ for some a. The rapid decrease of the two endpoint forms absorbs both height powers and the moderate growth of M. Thus $|I_v| \ll p_{2q}(v)$. The continuous invariant integrand descends to the quotient, which has finite measure. Absolute convergence, Fubini's theorem and the stated bound follow. For finitely many tests take the largest moment order and sum the bounds.

Lean, comparison. The full-rank estimate is `exists_integrable_kernel_bound`. Here the unaveraged identity `hasSum_dataInner` and the Hom minimum `firstMinimum_mul_le_hom` give the height bound in both variables. □

Corollary 24.24 (Schwartz-vector form of the weighted seesaw)**.** *In the weighted seesaw argument of Lemma 24.25, the common coefficient extends from the matched polynomial vectors to every Schwartz Fock vector, including every real translate of the prescribed joint harmonic. Its coefficient is independent of every finite twist, finite datum and pair of endpoint forms included in the family of tests. All splitting characters in the family have the same real weights γ, γ', as in the original grid edge families; the twists change only their finite parts.*

Proof. Index a test by all its finite data, twist and endpoint forms. The hypotheses of Lemma 24.23 hold for these tests. The comparison maps are the rational block inclusions and changes of rational frame, followed where needed by a fixed right translation. For the corresponding rational group map ρ, they satisfy

$$b(\gamma g) = \rho(\gamma) b(g).$$

In the subdiagonal square the map is simply the inverse of the basis renumbering; it is continuous and preserves the original height. The weights are either constant one or continuous theta kernels for the extra line, composed with that map. Rational automorphy of those kernels and of the endpoint forms, and rational automorphy of K_v in both variables, show that the full scalar integrand is invariant in both quotient variables for every Schwartz v. Thus no arbitrary map satisfying only a height estimate is used here. The moment bound from the lemma applies to each coordinate of the family. Continuity in the moderate-weight datum; The continuous frame identification and line weights. For a matched polynomial, the kernel factorization and Fubini identify its functional with a compact real convolution of the prescribed period family. The matching polynomial Q and its compact convolution α_Q are chosen before the test index, by the matched-polynomial hypothesis and the fixed-real-edge convolution statement. If that family has a nonzero coordinate, the proportionality coefficient as a function of the real translation is recovered by dividing this coordinate by its fixed nonzero value. It is measurable, and its product with the compact convolution function is integrable, by the same period estimate. If every coordinate vanishes, every such convolution is zero. The scalar obtained by integration is therefore common to the whole family.

Matched polynomials span the polynomial space in the two split kernel applications. Finite linearity gives the common-coefficient property for every polynomial. Apply Lemma 24.22. Finally, at a fixed real translation r, the vector used in *every* test is the same vector

$$v_r = \mathrm{realPairWeil}(\gamma, \gamma')(r, 1) v_{\Phi'}.$$

The weights γ, γ' are independent of the twist, finite datum, endpoint forms and summation index. This common weight hypothesis is exactly the one in the two grid edge families used by the column and row applications of the seesaw; it prevents replacing a common vector by test-dependent

translates. The oscillator action preserves Schwartz space, so v_r belongs to the domain just treated. The proof changes no datum, Fourier factor, level index or scalar-ratio formula. The weighted period family and its common coefficient; The two applications with common real edge weights; The moment-continuity formulation.

Lean. `ThetaLiftConvergence` (L298-L308), `DiagonalTurns` (L63-L159), `OneSquare` (L152-L226). □

24.7 Exchange of adjacent lifts on the auxiliary grid

We compare the periods on the two boundary paths by exchanging the two lifts around each square. Both grids have the same modulus:

$$\mathfrak{m}_\vartheta = \mathfrak{b}Q_\vartheta, \qquad R_\vartheta = \mathcal{O}_E/\mathfrak{m}_\vartheta, \qquad q_\vartheta = \operatorname{Nm} Q_\vartheta, \qquad h_\vartheta = |R_\vartheta| = M^{fe_*}q_\vartheta. \tag{24.67}$$

Thus $\mathfrak{m}_\vartheta$ has exponent e_* at the chosen places and exponent one at the twist places. It is prime to its conjugate and has nonempty support even for $\vartheta = 1$. All theta kernels below use the finite datum modulo this ideal, the vacuum, the polynomial normalization, and the quotient measures. Here $f = [E : \mathbb{Q}]/2$ and $\pi_M\bar{\pi}_M = M$, so the absolute ideal norm of (π_M) is M^f.

Write $G^0_{a,b} = Y^a_b$ and $G^1_{a,b} = \widetilde{Y}^b_a$ for the framed spaces at the nodes of the two grids. A half-path goes from $(0, m)$ to (j, j) by steps down or right. Along a right step a form is lifted, and along a down step it is transposed-lifted, using the indicator of the matrix on that edge. The first half carries conjugate forms and the second carries forms. Their terminal pairing is the diagonal period, with the extra line evaluated at u^j_ϑ, where

$$u_\vartheta = \frac{\overline{t_\vartheta\alpha}}{t_\vartheta\alpha} \quad \text{in } U(1)(\mathbb{A}_f), \qquad t_\vartheta\widehat{\mathcal{O}}_E = \mathfrak{m}_\vartheta\widehat{\mathcal{O}}_E. \tag{24.68}$$

The archimedean component of this scalar is one.

For an edge of character χ_e and a family of invertible node matrices g, put

$$(g \cdot S)_{(v,w)} = g_w^{-1}S_{(v,w)}g_v, \qquad \chi_\Gamma(g) = \prod_{(v,w)\in E(\Gamma)} \chi_{(v,w)}(\det g_w/\det g_v). \tag{24.69}$$

A path datum of bound N_ϑ depends only on its edge matrices, satisfies $\chi_\Gamma(g)\Psi_\vartheta(g \cdot S) = \Psi_\vartheta(S)$ for every admissible node family, vanishes unless every vertical edge is split injective, and has every value a sum of at most N_ϑ roots of unity. At a starting node admissibility means the prescribed level condition; on the diagonal the second-grid matrix is the first-grid matrix extended by 1 on the extra line. Let $P_\Gamma(\Psi; g; x)$ be the corresponding finite sum of diagonal periods, where $x = (\vartheta, y, \phi)$ is a twist and the two starting forms. We say $P_\Gamma(\Psi)$ is proportional to Ξ if

$$\forall g \quad \exists \beta_g \in \mathbb{C} \quad \forall(\vartheta, y, \phi), \qquad P_\Gamma(\Psi; g; \vartheta, y, \phi) = \beta_g\Xi(\vartheta, y, \phi). \tag{24.70}$$

In particular, the coefficient is chosen before the twist and the two forms. The grid modulus; Corners and their finite data.

Lemma 24.25 (Uniform real tests in a theta square)**.** *Fix the four real edge types and their joint-harmonic polynomials in an auxiliary square. An arbitrary polynomial theta datum on an edge, after projection onto its prescribed source type, gives a compactly supported real convolution of the lift for its prescribed joint harmonic. The convolution depends only on that polynomial and the fixed real types. For two conjugate terminal types these convolutions multiply the invariant pairing by one scalar independent of the finite character, finite datum and forms.*

The following two consequences hold with this same uniformity. First, if the prescribed upper path of a square is tested against two endpoint forms, replacing its joint-harmonic product by any

fixed real Schwartz vector multiplies every such test by one common scalar. Second, a weighted seesaw identity that holds for polynomial vectors spanning the polynomial Fock space holds for every real translate of the prescribed joint-harmonic vector.

Proof. At one real place let ω and π' denote the commuting source and target oscillator actions. Write Φ for the prescribed joint harmonic, identify polynomials with their Fock vectors, and set $p = \omega(e_\ell)$ for the source projection with row parameter ℓ, as in (12.2). That formula gives $\langle \Phi, p\Phi \rangle \neq 0$. The local joint-harmonic theorem for these parameters gives, for each polynomial Q, a smooth compactly supported function α_Q on the real target group such that

$$pQ = \pi'(\check{\alpha}_Q)\, p\Phi, \qquad \check{\alpha}_Q(g) = \alpha_Q(g^{-1}).$$

Here $\pi'(f)$ denotes integration of the target action against f.

In the polynomial proof of this identity, the polynomials whose projections have the displayed form constitute a linear space stable under the target Lie algebra, since differentiation passes to the compactly supported function. This space contains Φ because a smooth compactly supported function reproducing the target discrete-series kernel fixes $p\Phi$. Integration by parts against e_ℓ gives the compact lowering and diagonal weight relations for the source lowest K-type. The projection also kills the row derivatives of highest vectors outside its weight cone. The joint-harmonic spanning theorem with these relations gives the full polynomial space. Over the real places each polynomial is a finite sum of pure tensors. Products and sums of the local functions therefore give one compactly supported function on the product group. It depends only on Q and the fixed real types. Applying the theta functional gives the required convolution for every finite datum.

Let e be the fixed matrix coefficient defining the terminal type, with its original normalization. Reproducing-kernel orthogonality, or equivalently the rank-one identity $e * \delta_g * e = (e(g)/e(1))e$ with the appropriate inversion convention, gives

$$\begin{aligned} \int_{[U(A)]} (\alpha * F)(a)(\beta * G)(a)\, da = r(\alpha,\beta) \int_{[U(A)]} F(a)G(a)\, da, \\ r(\alpha,\beta) = \iint \alpha(x)\beta(z) \frac{e(x^{-1}z)}{\overline{e(1)}}\, dx\, dz. \end{aligned} \tag{24.71}$$

Here F, G have conjugate types and the convolution is the right convolution at infinity used by the edge maps. In particular, r involves no finite data.

For the first consequence, a polynomial on $\mathrm{Hom}(U_1 \oplus U_2, A)$ is a finite sum of products $Q_1 Q_2$ in the two column blocks. The kernel of a product is the product of the two edge kernels, with the determinant-character correction already specified by the splitting characters. Apply the preceding convolution statement to each factor and then (24.71). For every polynomial Q there is consequently one scalar r_Q for all tests.

To pass to a Schwartz vector, we bound each test by a continuous seminorm and use density. For an onto matrix modulo $\mathfrak{m}_\vartheta$, the full-rank estimate of Lemma 12.1, with its first group allowed to vary, has the form

$$|K_v(g_1 \oplus g_2, a)| \le C\, p_N(v)\, H(g_1)^B H(g_2)^B, \qquad p_N(v) = \int_{\mathbb{C}^d} (1 + \|z\|)^N |v(z)|\, dz. \tag{24.72}$$

The bound is uniform in a; B, N, C can depend on the particular finite datum. To obtain precisely these moments from a Schwartz-seminorm bound, differentiate the Gaussian reproducing formula. Its kernel has Gaussian absolute value in $z - w$, and each derivative introduces a polynomial in $w, z - w$. Consequently

$$\sup_z (1 + \|z\|)^a |\partial^\beta v(z)| \le C_{a,\beta}\, p_{a+|\beta|}(v).$$

This controls every one of the finitely many Schwartz seminorms used by the full-rank estimate. The extension from a compact set in the first group follows by applying its oscillator operator to the fixed datum: its weighted L^1 seminorms grow by a power of the group height. Rapid decrease of the endpoint forms absorbs that power, and all three adelic quotients have finite measure. This proves absolute integrability and a bound $C_x p_N(v)$ for each test x. On the lower side the corresponding injective-matrix estimate follows by the adjoint flip.

Index the tests by x, including the twist, the finite data and both endpoint forms. Let ρ_x be the corresponding prescribed-path period. The contraction against the kernel is linear in the real Fock vector, has the moment bound just proved for this x, and has value $r_Q \rho_x$ on a polynomial times the Gaussian, with r_Q independent of x. Lemma 24.22 therefore gives one coefficient for every Schwartz Fock vector and all tests.

For the weighted seesaw assertion, form for each test x the double-integral functional $\Lambda_x(v)$. Lemma 24.23 gives absolute convergence and a bound $C_x p_{N_x}(v)$ on the Schwartz Fock space. The moment order and constant may depend on the fixed test. Fubini is therefore valid. If the double integral for a polynomial is a real convolution of periods $\beta_r \Xi_x$, it is a multiple of Ξ_x. When $\Xi_{x_0} \neq 0$, the coefficient β_r can be read at x_0, so it is measurable and integrable against the compact convolution function. When $\Xi = 0$, all these periods vanish. Linear combinations and Lemma 24.22 extend the result from a spanning set of polynomials to every Schwartz Fock vector. The oscillator action preserves that space, so this includes every prescribed real translate. This is the needed weighted seesaw principle, with one coefficient for the whole family.

Lean. `New.IsJointHarmonic.howe_argument`, `exists_bump_integratedRep_eq`, `isEdge_of_isJointHarmonic`, `Seesaw` (L302-L339), `Seesaw` (L528-L735), `OneSquare` (L152-L226), `OneSquare` (L482-L546). □

Proposition 24.26 (The square law with a common finite-character scalar)**.** *Write* untwistedLiftChar(s) *for the splitting character of the untwisted s-th lift. For $i, j < m$, put*

$$\nu_{ij} = \mathrm{untwistedLiftChar}(i+1)\,\mathrm{untwistedLiftChar}(j+1), \qquad R_{ij}(\vartheta) = \prod_{s=1}^{t} \frac{|L^{MQ_\vartheta}(s, (\nu_{ij}\vartheta^{\,j+1})^{-1})|}{|L^{M}(s, \nu_{ij}^{-1})|}. \tag{24.73}$$

The ratio $R_{ij}(\vartheta)$ is turnRatio$_{ij}(\vartheta)$ *in Lean. For the square indexed by (i, j) there are $\kappa_{ij} \in \mathbb{C}$, possibly zero, and nonzero $a_{ij}(\vartheta)$, such that the two path operators satisfy, on every matched pair of finite data,*

$$\sum_{y} \Psi'(y)\, L_y = \frac{\kappa_{ij}}{a_{ij}(\vartheta)} \sum_{x} \Psi(x)\, U_x. \tag{24.74}$$

Here L_y is transposed lift followed by lift, U_x is lift followed by transposed lift, and the matching character is the inverse product of the two old edge characters. There is a fixed $A_{ij} > 0$ with

$$|a_{ij}(\vartheta)|^2 = A_{ij}\, q_\vartheta^{\,t^2}\, R_{ij}(\vartheta)^2. \tag{24.75}$$

All these choices precede the initial form and every linear functional testing its image. In the second grid the square with local coordinates (a, b) has label $(i, j) = (b, a)$.

Proof. Take $A' = G^z_{a,b}$, $A = G^z_{a+1,b+1}$, $U_1 = G^z_{a,b+1}$ and $U_2 = G^z_{a+1,b}$. The rank identities and sign identities of the grid give

$$\dim A = \dim A' + 2t, \qquad \dim(U_1 \oplus U_2) = \dim A + \dim A', \qquad q_{U_1 \oplus U_2} = q_A + q_{A'}. \tag{24.76}$$

They also give $2q_{U_1 \oplus U_2} + 2 \le \dim(U_1 \oplus U_2)$. The two lift weights are at least r^8 (the parameter halfGap), whereas every grid rank is at most $2n_m$ and $n_m \le r^8$, so the weight inequality of

Theorem 19.18 holds. The tame part of ν_{ij} is $\chi_{M,\mathrm{loc}}^{i+j+2}$. Its exponent is between 2 and $2m$, so the prescribed chosen-place characters make it nontrivial at every chosen place. The m-fold twist hypothesis makes ϑ^{j+1} nontrivial at every twist place. Their supports are disjoint. Thus the character $\omega = (\nu_{ij}\vartheta^{j+1})^{-1}$ is tame for precisely the support of $\mathfrak{m}_\vartheta$.

On an edge indexed by l in a space of rank n the splitting character is

$$\mathrm{gridChar}(\vartheta, c, l; n) = \mathrm{untwistedLiftChar}(l+1)\,\vartheta^{(l+1)c}\,\upsilon^{\lfloor (n+1-c)/2 \rfloor}, \qquad c \in \{0,1\}. \tag{24.77}$$

The four products of splitting characters in the square are therefore the same rank-normalized character ω^{-1}. The sum of the ranks is even, so the rounded half-ranks in (24.77) add exactly. On residue matrices their product is the character with tame part $\chi_{M,\mathrm{loc}}^{i+j+2}\vartheta_{\mathrm{loc}}^{j+1}$. This verifies the inverse in the matching character in (24.74); it is not a change to the twist.

Apply Theorem 19.18 to the product of the two prescribed lower-path joint harmonics. It supplies one upper real Schwartz vector w before ϑ, and a nonzero exchange scalar $a_{ij}(\vartheta)$ whose squared modulus is a fixed positive multiple of

$$\Big(\prod_{v \mid MQ_\vartheta} Nv\Big)^{t^2} \prod_{s=1}^{t} |L^{MQ_\vartheta}(s,\omega)|^2. \tag{24.78}$$

Factor the theta kernels on $U_1 \oplus U_2$ into the two column or row blocks. Their determinant correction is $\nu_0(\det g_2)$, with $\nu_0 = b_1/b_2 = b_2'/b_1'$ in the notation of the four edge characters, so it is the same on both sides. Multiply by the two endpoint forms and integrate. Full-rank support of matched functions permits all interchanges by (24.72). The lower side is exactly the pairing with $\sum_y \Psi'(y)L_y$. Lemma 24.25 turns the upper side with w into one fixed multiple κ_{ij} of the pairing with $\sum_x \Psi(x)U_x$. Nondegeneracy of the endpoint Petersson pairing gives (24.74) as an operator identity. This proof also covers $\kappa_{ij} = 0$.

Finally divide the product of L-values in (24.78) by the fixed untwisted product. The support contributes exactly $q_\vartheta^{t^2}$; the exponent e_* of the chosen primes changes neither the ramified set nor the L-product. This proves (24.75). The untwisted constant is positive and all the displayed ratios are positive by the nonvanishing in the same exchange normalization. In particular $R_{ij}(1) = 1$.

Lean. `OneSquare` (L683-L869), `Turns` (L607-L784). □

Lemma 24.27 (A turn preserves the finite path conditions)**.** *Consider one of the auxiliary path turns, with removed spaces $V', V' \oplus T$, kept spaces W_1, W_2, and added spaces $V, V \oplus T$, where $T = 0$ off the diagonal and $\dim T = 1$ on it. Put $B = W_2$ and use the specified basis bijection $V' \oplus V \simeq W_1 \oplus B$. The preserved edges avoid the removed nodes and do not end at the added nodes; the removed node is internal, one old edge is vertical, and opposite edge characters agree, as they do for these path turns. If Ψ' is a path datum of bound N, there is a path datum Ψ on the turned path of bound*

$$|R|^{\dim B \,\dim T} N. \tag{24.79}$$

For each fixed tuple on the preserved edges and each row $v' : B \to T$, its function of the new matrix $x : W_1 \oplus B \to V$ is matched with

$$\widehat{\Psi}'(y, v') = \sum_{v : T \to B} \psi(\mathrm{tr}(v'v))\Psi'(\textit{old tuple } (y, v)), \qquad y : V' \to W_1 \oplus B. \tag{24.80}$$

Here ψ is any additive character of R; off the diagonal it can be the trivial character. Matching is over the full ring R, including its higher prime powers.

Proof. Let χ be the inverse product of the old edge characters. For x which is a last-row block of γ^{-1}, $\gamma \in \mathrm{GL}(W_1 \oplus B, R)$, define

$$\Psi(\text{new tuple } (x, v')) = \chi(\det \gamma)\, \widehat{\Psi}'(\text{first columns of } \gamma, v'), \tag{24.81}$$

and set it to zero if x has no such completion. The old node V' is internal, so its matrix may be any $k \in \mathrm{GL}(V', R)$, extended by 1 on T. Its equivariance gives $\widehat{\Psi}'(yk, v') = \chi(\det k)^{-1}\widehat{\Psi}'(y, v')$. Two completions of the same x differ on the right by $\left(\begin{smallmatrix} k & b \\ 0 & I \end{smallmatrix}\right)$; this proves that (24.81) is well defined. A nonzero term of (24.80) has at least one old vertical block split injective, hence y itself is split injective. Every such frame extends to an invertible matrix over R, since R is a product of finite local principal ideal rings. Thus (24.81) has precisely the required matching and support, with no averaging denominator.

Here is the complete equivariance check. For an admissible family let $t = \mathrm{diag}(g_{W_1}, g_B)$ and $h = g_V$. Writing

$$A = \chi_{V',W_1}(\det g_{W_1})\, \chi_{V'\oplus T,B}(\det g_B)\, \chi_{\text{preserved}}(g),$$

change v to $g_B^{-1}v$ in (24.80). The trace $\mathrm{tr}(v' g_B g_B^{-1} v)$ is unchanged, and old-path equivariance yields

$$\widehat{\Psi}'_{gS}(t^{-1}yk, v' g_B) = A^{-1}\chi(\det k)^{-1}\widehat{\Psi}'_S(y, v'). \tag{24.82}$$

The new matrix and row are $h^{-1}xt$ and $v' g_B$. Use the completion $t^{-1}\gamma\,\mathrm{diag}(I, h)$ in (24.81). The new value is the old value times $A^{-1}\chi(\det t)^{-1}\chi(\det h)$. Opposite edges have the same character, so the new edge-character product is $A\chi(\det t)\chi(\det h)^{-1}$. This proves (24.69) on the new path.

For the vertical support condition write $y = (y_1; y_2)$ and $x = (x_1, x_2)$ for the complementary blocks of γ and γ^{-1}. If y_1 is split injective, then x_2 is split injective: modulo any maximal ideal, $x_2 v = 0$ implies $(0, v) \in \ker x = \mathrm{im}\, y$, and then $y_1 u = 0$ forces $u = v = 0$. Over each local factor this full residue rank gives a left inverse, and the Chinese remainder theorem combines them. Interchanging the blocks proves the same assertion for y_2 and x_1. A left inverse of (y_2, v) gives one of y_2 by taking its first rows; a left inverse of x_2 gives one of $\left(\begin{smallmatrix} x_2 \\ v' \end{smallmatrix}\right)$ by appending zero columns. These are exactly the extra-line cases. Preserved vertical edges retain their old support condition.

Finally, every value of a finite multiplicative character on a unit and every value of a finite additive character is a root of unity. Formula (24.80) has $|R|^{\dim B \dim T}$ terms, each a root of unity times a sum of at most N roots. Multiplication by $\chi(\det \gamma)$ does not increase that count. Dependence only on the new and preserved edges is immediate from (24.80)–(24.81).

Lean. `Turns` (L292-L550). □

Lemma 24.28 (Reversing the diagonal line)**.** *Put $S_j = Y_j^{j+1} = \widetilde{Y}_j^{j+1}$ using the specified basis renumbering, and $d_j = \dim S_j$. There is one signed-transpose map τ_j on real polynomials, independent of ϑ, and for each ϑ a primitive additive character $\psi_{j,\vartheta}$ of R_ϑ and a nonzero $\lambda_j(\vartheta)$ such that*

$$K^{\mathrm{col}}_{j,\vartheta}(v, p; g) = \lambda_j(\vartheta) \sum_{v' \in \mathrm{Mat}_{1\times d_j}(R_\vartheta)} \psi_{j,\vartheta}(\mathrm{tr}(v'v))\, K^{\mathrm{row}}_{j,\vartheta}(v', \tau_j p; g), \qquad |\lambda_j(\vartheta)|^2 = h_\vartheta^{-d_j}. \tag{24.83}$$

The column kernel is from the line of sign $(-1)^j$ to S_j, with character 1 on the line and $\mathrm{gridChar}(\vartheta, 0, j)$ *on S_j, at u_ϑ^j on the line. The row kernel is from S_j to the line of sign $(-1)^{j+1}$, with* $\mathrm{gridChar}(\vartheta, 1, j)$ *on S_j and v on the line, at u_ϑ^{j+1}.*

Proof. Use the adjoint identification $\mathrm{Hom}(L, S_j) \to \mathrm{Hom}(S_j, L^-)$ in the oscillator model. Its finite multiplier is a unit c_0 of $\widehat{\mathcal{O}}_E$ with $\bar{c}_0 c_0 = -1$; it is a finite adelic unit, not a global element of norm -1 in the CM field. At infinity the identification fixes the Gaussian and sends

$$X_{\tau,i,*} \longmapsto -s_i(-1)^{j+1} X_{\tau,*,i}. \tag{24.84}$$

This is τ_j and uses no finite character. Choose the finite generator t_ϑ of (24.68) equal to 1 off the support of $\mathfrak{m}_\vartheta$, and put $\delta = \bar{t}_\vartheta / t_\vartheta$. The modulus is prime to its conjugate. Thus at a place ν of its support, with $a = t_{\vartheta,\nu}$, the line dilation is $\delta_\nu = a^{-1}$. Let V_v be the vacuum translated by v/a in these local coordinates, and let $\Phi_{[v]}$ denote the finite datum with residue-class indicator $[v]$. After dilation the lattice condition on each entry is $(x - v)/a \in \mathcal{O}_\nu$, or $x \equiv v \bmod a$. The split realization uses one place of each conjugate pair. With the vacuum at the remaining places the finite Weil action hence gives

$$\omega(\delta, 1) V_v = C \Phi_{[v]}, \qquad |C|^2 = \left(\prod_{\nu \mid \mathfrak{m}_\vartheta} |t_{\vartheta,\nu}|_\nu^{-1} \right)^{d_j} = h_\vartheta^{d_j}.$$

Indeed the product is the determinant module of the dilation on the source line over the modulus support, and the Schrödinger coordinates consist of d_j copies of that line. The scalar C is common to all translates.

The signed flip takes V_v to the finite trace character $v' \mapsto \psi(\mathrm{tr}(v'v))$, where the primitive character ψ is shifted by the unit $-\bar{c}_0$. The adjoint identification contributes a phase z, $|z| = 1$, to the theta functional. The line point changes from k to $k u_\vartheta$, with $u_\vartheta = (\bar{\alpha}/\alpha)\delta$ as in (24.68). By linearity in the finite datum, the row kernel on the trace character equals zC times the column kernel on $\Phi_{[v]}$ before the change of character below. Expanding the trace character in residue-class indicators therefore gives the multiplier $(zC)^{-1}$ in (24.83). The character initially produced on S_j is $b^{-1} v^{d_j}$, where $b = \mathrm{gridChar}(\vartheta, 0, j; S_j)$. Replacing this by $a' = \mathrm{gridChar}(\vartheta, 1, j; S_j)$ multiplies the scalar by $(b^{-1} v^{d_j}/a')(k u_\vartheta)$. This value has modulus one and is independent of g, p, v. Consequently $|\lambda_j(\vartheta)|^2 = |C|^{-2} = h_\vartheta^{-d_j}$. Taking $k = u_\vartheta^j$ proves (24.83), first on indicator data and then by finite linearity.

Lean. `Flip` (L383-L529), `Flip` (L570-L606), `DiagonalTurns` (L398-L448). □

Proposition 24.29 (A diagonal turn with one common coefficient)**.** *A turn labelled (i, j) transfers path data forward and proportionality of periods backward. More precisely, there is a nonzero scalar family $c_{ij}(\vartheta)$ such that a datum of bound N_ϑ on the old path gives a datum of bound*

$$V_\vartheta N_\vartheta, \qquad V_\vartheta = h_\vartheta^{2n_m}, \tag{24.85}$$

on the new path, and

$$P_{\mathrm{new}}(\Psi) \text{ proportional to } \Xi \quad \Longrightarrow \quad P_{\mathrm{old}}(\Psi') \text{ proportional to } (x \mapsto \Xi(x)/c_{ij}(\vartheta)). \tag{24.86}$$

The finite datum is chosen before Ξ. One can take

$$c_{ij} = a_{ij} \quad (i \neq j), \qquad c_{jj} = a_{jj}/\lambda_j. \tag{24.87}$$

In particular, their squared moduli are fixed positive constants times $q_\vartheta^{t^2 + \mathbf{1}_{i=j} d_j} R_{ij}(\vartheta)^2$.

Proof. Off the diagonal $T = 0$ in Lemma 24.27; the Fourier sum has one term and the datum bound is unchanged. Hold the matrices on the rest of the path fixed. The arriving form and the functional supplied by the rest of the path depend only on those preserved matrices. Apply (24.74) and then sum over them. This proves $P_{\mathrm{old}} = (\kappa_{ij}/a_{ij}(\vartheta)) P_{\mathrm{new}}$ at every real point. If the new period is $\beta_g \Xi$, the old coefficient is $\kappa_{ij} \beta_g$, as required. Inflating the datum bound by $V_\vartheta \geq 1$ gives (24.85).

On the diagonal both half-paths end at $(j, j+1)$. Their old endpoints are Y_j^j and $Y_j^j \perp E_{(-1)^j}$, and their new endpoints are Y_{j+1}^{j+1} and $Y_{j+1}^{j+1} \perp E_{(-1)^{j+1}}$. The kept second endpoint is identified with S_j by a permutation of its basis, which preserves the height and the quotient measure. Let

$I_{\mathrm{col}}(\Psi';z,p;x)$ be the period over $[U(S_j)]$ weighted by the column-line kernel, and let $I_{\mathrm{row}}(\Psi;z,p';x)$ use the row-line kernel. Both are absolutely integrable: line kernels have moderate growth and the two grid forms are rapid.

The old split kernel factors exactly as the column-line kernel times the kernel from Y_j^j to S_j. The finite indicator factors into indicators of its two column blocks; a polynomial product factors into the same two blocks. At the old line the character is 1. This follows from (24.77) because the two rounded half-ranks there agree. The new split kernel similarly factors into the row-line kernel times the kernel from S_j to Y_{j+1}^{j+1}; there the rounded half-ranks differ by one, giving the line character υ. These parities use the stipulated odd t.

These factorizations and the second assertion of Lemma 24.25 give two precise implications:

$$\begin{aligned}&\text{all } I_{\mathrm{col}}(\Psi';z,p) \text{ proportional to } \Xi \implies P_{\mathrm{old}}(\Psi') \text{ proportional to } \Xi,\\ &P_{\mathrm{new}}(\Psi) \text{ proportional to } \Xi \implies \text{all } I_{\mathrm{row}}(\Psi;z,p') \text{ proportional to } \Xi.\end{aligned} \tag{24.88}$$

For the first implication, products in the two column blocks span every polynomial. For the second, with the line polynomial p' fixed, every polynomial A on the other block is supplied by the product Ap'. The polynomial and real-convolution operators in these arguments are independent of the twist and the starting forms. Thus proportionality in (24.88) has exactly the quantifiers (24.70), including when Ξ vanishes identically.

Construct Ψ by (24.80)–(24.81) with the character of Lemma 24.28. Insert (24.83) in I_{col}, and interchange its finite row sum with the convergent period. Apply (24.74) to the two-edge square, fixing the other edge matrices and the row v'. The result is the exact identity

$$I_{\mathrm{col}}(\Psi';z,p;x)=\lambda_j(\vartheta)\,\frac{\kappa_{jj}}{a_{jj}(\vartheta)}I_{\mathrm{row}}(\Psi;z,\tau_j p;x). \tag{24.89}$$

There is no extra sum over completions and no group-order factor: (24.81) is a matched function, and (24.74) already sums over the row and column matrices. If the last period on the right is $\beta_{z,p}\Xi(x)$, the coefficient on the left relative to $\Xi(x)/c_{jj}(\vartheta)$ is $\kappa_{jj}\beta_{z,p}$. Apply the first implication of (24.88). This proves (24.86) with (24.87).

Only the Fourier transform increased the root count; it multiplied it by $h_\vartheta^{d_j}$. Since $d_j\le 2n_m$, this is bounded by V_ϑ. The norm statement follows from (24.75), (24.83), and $h_\vartheta=M^{f_{e_*}}q_\vartheta$. Neither κ_{ij} nor a period coefficient is ever divided out.

Lean. `DiagonalTurns` (L242-L396), `DiagonalTurns` (L497-L620), `Turns` (L894-L999). □

Theorem 24.30 (All m^2 turns with the same scalar family)**.** *There is a family $c_\vartheta\in\mathbb{C}^\times$ chosen before the path datum and the starting forms such that every first-path datum Ψ_0 of bound N_ϑ has a last-path datum Ψ_m of bound $V_\vartheta^{m^2}N_\vartheta$, and*

$$P_{\mathrm{last}}(\Psi_m)\ \textit{proportional to}\ \Xi \quad\Longrightarrow\quad P_{\mathrm{first}}(\Psi_0)\ \textit{proportional to}\ \Xi/c_\vartheta. \tag{24.90}$$

For this same family there are $A>0$ and an integer e independent of ϑ with

$$|c_\vartheta|^2=Aq_\vartheta^e\left(\prod_{i,j=0}^{m-1}R_{ij}(\vartheta)\right)^2,\qquad e=m^2t^2+\sum_{j=0}^{m-1}\dim Y_j^{j+1}. \tag{24.91}$$

In particular, the initial bound

$$N_\vartheta=|R_\vartheta|^{2m(2n_m)^2}$$

gives the prescribed final bound $\mathrm{lastPathBound}(\vartheta)$.

Proof. Represent a two-grid path by its two monotone halves. The set of squares between it and the first path is the square $[0,j)^2$ below its diagonal endpoint, together with the squares to the left of the down steps in the first half and their transposes in the second half. This set is empty for the first path and is $\{0,\dots,m-1\}^2$ for the last. An off-diagonal down-then-right corner replaces those two steps by right-then-down and adds exactly its labelled square. If neither half has such a corner and the endpoint is not (m,m), both terminal steps are down; changing both to right moves the endpoint from (j,j) to $(j+1,j+1)$ and adds exactly (j,j). Consequently a nonlast path can always be turned, and a sequence from the first to the last uses every labelled square once and exactly m^2 turns.

For each label fix the family (24.87). Apply the finite construction successively along such a sequence. Each new datum is chosen from its predecessor, without reference to Ξ, and its bound grows by at most V_ϑ. Apply proportionality backward along the same sequence. The denominators multiply, so the resulting family is $c_\vartheta=\prod_{i,j<m}c_{ij}(\vartheta)$. All its factors are nonzero. Their squared norm formulas give (24.91). Each diagonal Fourier inverse contributes $M^{fe_*d_j}$ to the fixed factor A. The ratio of the two principal levels in the first-path calculation is also independent of ϑ, by (24.64). Thus changing e_* changes only the constant in the unnormalized squared modulus. This proves the exact orientation in (24.90). The set of crossed squares; Existence of a turn and the m^2 induction; The composed scalar and datum bound.

Lean. `GridPaths` (L187-L228), `GridPaths` (L301-L365), `DiagonalTurns` (L622-L670). □

Corollary 24.31 (The normalized auxiliary scalar and its height)**.** *For the same family in Theorem 24.30, put $\gamma_\vartheta=c_\vartheta/c_1$. Then $\gamma_1=1$, every γ_ϑ is nonzero, and*

$$|\gamma_\vartheta|=q_\vartheta^{e/2}\prod_{i,j=0}^{m-1}R_{ij}(\vartheta). \tag{24.92}$$

For a family with auxiliary modulus prime to its coefficient prime and inverse ramified critical ratios of height at most H at each power $1,\dots,m$, the inverse scalar has height at most m^2tH. For the simultaneous family of (13.58),

$$h\left(\left(\left[q_{\theta_\iota}^{e/2}\prod_{i,j<m}R_{ij}(\theta_\iota)\right]^{-1}\right)_\iota\right)\le m^2tC_{\mathrm{crit}}. \tag{24.93}$$

The conclusion involves the positive real moduli in (24.92); it does not assert algebraicity of c_ϑ or impose a separate phase choice.

Proof. The critical pair for R_{ij} has character ν_{ij}, endpoint t, and weight $\mathrm{liftWeight}(i+1)+\mathrm{liftWeight}(j+1)$. This weight lies between t and $4b$ at every infinite place. Its tame character is nontrivial exactly at the chosen places by the exponent check in Proposition 24.26. It is therefore a ramified critical pair, evaluated at the allowed power $j+1$ of the twist. Every factor in (24.73) is positive and equals one at the trivial twist. Thus (24.91) at $\vartheta=1$ gives $A=|c_1|^2>0$. Divide (24.91) by this identity and take its positive square root to obtain (24.92).

For a fixed (i,j), expand R_{ij} into the t lower-point ratios in (24.73). Each lower point is again a ramified critical pair, at the same allowed power $j+1$. Multiplying their height witnesses gives height at most tH for each inverse $R_{ij}(\theta_\iota)^{-1}$. For the chosen family one may use $H=C_{\mathrm{crit}}$, since the old tame power is $i+j+2$. Multiply again over the m^2 squares. The remaining positive number $q_{\theta_\iota}^{-e/2}$ is algebraic and its square is the rational number $q_{\theta_\iota}^{-e}$. At the selected finite embedding it has absolute value one because the prime does not divide q_{θ_ι}. It therefore changes none of the

finite inequalities in those height witnesses and costs zero in the family-height bound. This proves (24.93), including odd e.

Lean. `Moduli` (L144-L225), `Moduli` (L232-L270), `LongForms` (L40-L114). □

Comparison of the first and last periods. The last-path proportionality has the denominator triangleGLCard$(0,\vartheta,m)$. Applying (24.90) at the real point supplied by the first-path construction gives one β with

$$P_{\mathrm{first}}(\Psi_0;g_0;\vartheta,y,\phi)=\beta\,\frac{\mathrm{lastPathPeriod}(\Psi_m;\vartheta,y,\phi)}{\mathrm{triangleGLCard}(0,\vartheta,m)\,c_\vartheta}. \tag{24.94}$$

This group order is the index of the source level in its standard maximal compact subgroup,

$$i_m=\mathrm{stdLevelIndex}(\mathrm{levelY}(\vartheta,m)).$$

Thus, if the first-path identity is $\int y\,\Theta_\vartheta(r\phi)=\alpha P_{\mathrm{first}}(\Psi_0;g_0;\vartheta,y,\phi)$, then the resulting statement is

$$i_m\int y\,\Theta_\vartheta(r\phi)=\alpha\beta\,\frac{\mathrm{lastPathPeriod}(\Psi_m;\vartheta,y,\phi)}{c_\vartheta}. \tag{24.95}$$

The constants α,β may vanish. This argument only cancels the positive integer i_m and the nonzero family c_ϑ; it never divides by a period coefficient. The restriction map r and every finite evaluated datum are retained. The exact level-index cancellation.

24.8 The definite last path and a spanning family with controlled lifts

To complete the independent forms to a basis, use the same grid modulus, normalized vacuum, edge characters and test functions as in the construction of the first path on page 612. Write $\epsilon=0,1$ for their two halves, and put

$$V_{\epsilon,j}=\mathrm{gridSpace}(0,\epsilon,j,m),\quad R_{\epsilon,j}=r{+}\epsilon{+}j(t{+}\epsilon){+}m(t{-}\epsilon),\quad R_*=2n_m,\quad H_0=br^4,\quad A_0=M^{e_*}.$$

Here b is the integer weight, $r+1=k+m$, $t=5r$, and m is even. For a twist ϑ, put $q_\vartheta=\mathrm{Nm}Q_\vartheta$ and

$$\mathfrak{m}_\vartheta=\mathfrak{b}\,Q_\vartheta,\qquad u_\vartheta=\frac{\overline{t_\vartheta\alpha}}{t_\vartheta\alpha},$$

where t_ϑ generates this ideal adelically. The notation t_ϑ in this formula is unrelated to the integer $t=5r$. All class integrals below give an integral level coset mass one; their relation to the fixed adelic quotient measure is displayed explicitly when a period is converted to a class integral.

Lemma 24.32 (Weights and interlacing on the last path)**.** *All $V_{\epsilon,j}$, $0\le j\le m$, are standard positive definite spaces. There are column multiplicities κ_ϵ and determinant exponents $e_{\epsilon,j}$, the last-path weights, whose doubled Harish–Chandra parameters are $-\mathrm{gridParam}(\epsilon,0,j,m)$ and which fit every successive theta lift. They satisfy*

$$R_{\epsilon,j}\le R_*,\qquad |e_{\epsilon,j,\tau}|\le 20b,\qquad \mathrm{boxes}(\kappa_\epsilon,R_{\epsilon,j})\le H_0. \tag{24.96}$$

At $j=m$ the parameter of $V_{1,m}$ interlaces the opposite parameter of $V_{0,m}$, starting with the larger rank.

Proof. On a base index numbered a, the sign is $(-1)^{v+\min(v,m)}$, $v=\max(a+1-k,0)\le m$; hence it is 1. All added coordinates have sign 1. The ranks are the displayed $R_{\epsilon,j}$, increase by $t+\epsilon$, and are at most $2(k+m(t+1))=R_*$.

The consecutive lift weights differ by $\Delta = 2r^8$. At the embedding τ, write $w_{l,\tau} = \kappa(\tau) + (l + \frac{1}{2})\Delta$ for liftWeight(l, τ) and $\lambda_a = \text{targetParameter}(\tau, a)$ for the a-th target parameter. Their doubled values satisfy

$$2\lambda_a = k - 2 - 2a \quad (a < k - 1), \qquad \lambda_{k-1+a} = \Delta/2 - w_{a+1,\tau}, \qquad k + t + 1 \le \Delta/2.$$

In grid 0 the doubled parameter consists of the base $2\lambda_a$ and the descending step-two blocks

$$2w_l + t - 1 - 2h \quad (h < t,\ l \le j), \qquad -2w_l + t - 1 - 2h \quad (h < t,\ l \le m),$$

with 1 subtracted from every entry when $j + m$ is odd. In grid 1 the base is $2\lambda_a + 1$, and the two block lengths are $t + 1, t - 1$, with entries $2w_l + t - 2h$ and $-2w_l + t - 2 - 2h$. The algebraic weight on each half has the negative of these parameters.

Sort the parameter at $j = 0$ decreasingly. Its adjacent differences are positive even integers, and it has the parity of $R_{\epsilon,0} - 1$. Thus the differences divided by two and reduced by one give nonnegative column multiplicities. The last entry determines the determinant exponent. Extend the columns above this initial rank by

$$\kappa_{\epsilon,\tau}(R_{\epsilon,j}) = \Delta - t - \epsilon \quad (j \ge 1), \qquad \kappa_{\epsilon,\tau}(h) = 0 \quad (R_{\epsilon,j} < h < R_{\epsilon,j+1}).$$

The character weight at rank N on edge j is

$$\gamma_{\epsilon,j,\tau}(N) = \lfloor (N + 1 - \epsilon)/2 \rfloor - w_{j+1,\tau}.$$

At the initial rank, choose the full-height columns so that $e_{\epsilon,0} = \gamma_{\epsilon,0}(R_{\epsilon,1}) + \kappa_\epsilon(R_{\epsilon,0})$. These columns are nonnegative: after the shift by the edge character, the old entries lie strictly above the inserted block, so its top entry is no greater than the old bottom entry. Equivalently, for every old doubled entry l_a,

$$\gamma_{\epsilon,0,\tau}(R_{\epsilon,1}) \le \lfloor l_a/2 \rfloor + \lfloor R_{\epsilon,0}/2 \rfloor.$$

This inequality also follows immediately by substituting the blocks above and $\Delta/2 \ge k + t + 1$. Set

$$e_{\epsilon,j} = \gamma_{\epsilon,j}(R_{\epsilon,j+1}) + \kappa_\epsilon(R_{\epsilon,j}).$$

The rank and gap formulas give $e_{\epsilon,j+1} = \gamma_{\epsilon,j}(R_{\epsilon,j})$. The old rows acquire exactly $\kappa_\epsilon(R_{\epsilon,j})$ boxes on passing to the next rank. The doubled old entries therefore shift by $(-1)^j$ for $\epsilon = 0$, and by zero for $\epsilon = 1$; the new bottom block is the one just listed, with its sign reversed. This proves inductively both the prescribed parameter and the fitting lift condition.

All the displayed entries have absolute value at most $20b$, since $w_l \le 2b$ for $l \le 2m$, and all polynomial rank and gap quantities are negligible compared with b. For a decreasing parameter $l_0 > \cdots > l_{R-1}$,

$$l_i = 2(e + \lambda_i) + R - 1 - 2i,$$

where at the current rank R the polynomial diagram uses only the column multiplicities $\kappa(h)$ with $h < R$. The multiplicity $\kappa(R)$ is absorbed in the determinant exponent e at this rank, and supplies columns at later ranks; it is not counted a second time in the current diagram. Thus $\lambda_{R-1} = 0$ and $l_{R-1} = 2e + 1 - R$. The last-entry bound gives $|e| \le 20b$, and subtracting this last entry from l_i gives $0 \le \lambda_i \le 20b$. The total number of boxes is consequently at most $20fbR_*$. The constructed scales give $r \ge x^5$, $f \le x^{13}$, $R_* \le 2x^6$, so $20fR_* \le r^4$ eventually. This proves (24.96).

For terminal interlacing it is useful to count entries rather than match their unsorted labels. Fix an integer a, put $\varphi(v) = \mathbf{1}_{a<v}$, and define

$$g(n, v) = \max(0, \min(n, \lfloor (v - a)/2 \rfloor)).$$

A falling step-two block starting at v has $g(n, v+1)$ entries above a; a rising one has $g(n, v+2n-1)$. If $C_0(s)$ is the number above a in grid 0 with base rank $k-1+s$ and s blocks of each kind, and $C_1(s)$ is the corresponding number for the negative grid-1 parameter, these formulas give

$$C_0(s) = \sum_{h<k-1+s} \varphi(2\lambda_h) + \sum_{l=1}^{s} \{g(t, 2w_l + t) + g(t, -2w_l + t)\},$$
$$C_1(s) = \sum_{h<k+s} \varphi(-2\lambda_h - 1) + \sum_{l=1}^{s} \{g(t+1, -2w_l + t + 1) + g(t-1, 2w_l + t - 1)\}.$$

The two blocks of lengths $n+1, n$ with the same centre alternate. Their count difference is between 0 and 1, equals 1 below both blocks, and equals 0 above both. The base blocks $k-2, k-4, \dots$ and $1-k, 3-k, \dots$, with the extra base entry, therefore give

$$\varphi(2\lambda_{k-1}) \le C_1(0) - C_0(0) \le \varphi(-2\lambda_{k-1} - 1).$$

At the next step the negative-centre block count difference lies between $\varphi(2\lambda_{k+s})$ and $\varphi(2\lambda_{k-1+s})$, and the positive-centre difference with the opposite sign lies between $\varphi(-2\lambda_{k-1+s} - 1)$ and $\varphi(-2\lambda_{k+s} - 1)$. Their endpoints are in this order by $\Delta/2 \ge k + t + 1$. Adding the two new base entries proves by induction

$$\varphi(2\lambda_{k-1+s}) \le C_1(s) - C_0(s) \le \varphi(-2\lambda_{k-1+s} - 1).$$

At $s = m$ this is the asserted interlacing, since $m + m$ is even and $r = k - 1 + m$.

Lean. `Parameters` (L45-L410), `LastPath` (L340-L580). □

Lemma 24.33 (Integral duality and the terminal branching pairing)**.** *For opposite doubled parameters on one definite rank, let G, G' be the Weyl Gram matrices of the given and opposite weights. There is an integral matrix A giving invariant bilinear duality. It is an isometry for these norms, in the sense that for each original vector v there is an opposite-weight vector $v^\vee$ with*

$$\|v^\vee\|_{G'} = \|v\|_G, \qquad u^{\mathsf{t}} A v = \langle v^\vee, u\rangle_{G'}.$$

If h is the hook number of the dual diagram, there is an integral C such that $CA = hI$, and $0 < h \le H^H$ when that diagram has at most H boxes.

For the terminal ranks $R_{0,m}$ and $R_{1,m} = R_{0,m} + 1$ there is an integral bilinear branching matrix B such that

$$B(\rho_0(g)v', \rho_1(\mathrm{diag}(g,u))v) = \prod_\tau u_\tau^{L_\tau} B(v', v), \qquad |L_\tau| \le H_0 + 20b, \tag{24.97}$$
$$|B(v', v)| \le H_0^{H_0} \|v'\|_G \|v\|_G. \tag{24.98}$$

For each fixed real point g_∞ of the larger group, the adelic terminal period of two algebraic forms is $C_{g_\infty}/[K^0 : K]$ times the class integral of this pairing, with one C_{g_∞} independent of their finite data and of u. This constant is allowed to be zero.

Proof. We first justify the integral normalization of duality. Opposite parameters have complementary column counts $\kappa'_\tau(h) = \kappa_\tau(R - h)$; their determinant exponents add to minus the first row length of κ'. On the exterior tensors of columns, complementary Hodge stars give a signed permutation matrix P. Jacobi's complementary-minor identity intertwines this map with $g \mapsto g^{-\mathsf{t}}$ and the required determinant power. Let E, E' be the integral filling matrices of the two Weyl modules. Unitriangular straightening gives integral left inverses for both. Jacobi's identity, first on

arrays with invertible square part and then as a polynomial identity, shows that PE belongs to the image of E'. Thus $E'J = PE$ for an integral J. Applying the complementary stars in reverse gives an integral inverse J^{-1}. The same identity and $P^{\mathsf{t}}P = I$ give

$$J^{\mathsf{t}}G'J = G, \qquad G = E^{\mathsf{t}}E, \quad G' = E'^{\mathsf{t}}E'.$$

Choose its sign so that J takes the leading vector to the opposite leading vector. Then $A = G'J$ is integral and invariant, with $v^{\vee} = \overline{Jv}$. Its asserted isometry follows from the displayed Gram identity.

The Gram–Fock calculation in Lemma 19.15 gives an integral Fock matrix F' with $(F')^{\mathsf{t}}G' = hI$. Consequently $C = J^{-1}(F')^{\mathsf{t}}$ has $CA = hI$. The same lemma gives the hook bound $h \le d_0! \le H^H$ for $d_0 \le H$ boxes, with $h = 1$ for the empty diagram.

Let λ be the larger highest weight and λ^0 the one whose parameter is opposite the smaller weight. Lemma 24.32 implies

$$e + \lambda_{a+1} \le e_0 + \lambda^0_a \le e + \lambda_a.$$

Put $d_a = e_0 + \lambda^0_a - (e + \lambda_{a+1})$; then $0 \le d_a \le \kappa(a+1)$. Let d_a of those columns retain their last box and let the remaining columns lose it. The smaller columns are exactly $\kappa^0(h) = \kappa_1(h) + \gamma(h+1)$, and $e_0 = e + \kappa_1(R_{0,m})$. Their box count is no greater than the larger box count. The integral branching construction of Lemma 19.15, with $k = 1$, therefore gives an integral map b of norm at most $H_0^{H_0}$. Compose b with the preceding duality A. The resulting $B = Ab$ has (24.98). Expansion of the removed boxes gives the last-coordinate character $L_\tau = e_\tau + \sum_h \gamma_\tau(h)$, proving (24.97) and its bound.

The interlacing inequalities give a one-dimensional branching space by the Weyl branching rule [GW09, Theorems 8.1.1–8.1.2], applied to the complexifications of $U(R) \subset U(R+1)$. The common determinant twists reduce any integral highest weight to the polynomial case. Hence the compact real integral for fixed g_∞ is $C_{g_\infty} B$. Its scalar is independent of the finite data and of u. Each finite level cell has mass $[K^0 : K]^{-1}$, giving the asserted period factor.

Lean. `Dual` (L325-L470), `Dual` (L1046-L1151), `LastPath` (L51-L327). □

The finite sum on the last path. Let $K_{\epsilon,j}(\vartheta)$ be the grid level at its node. At $(0,0)$ it is the principal level of $\mathfrak{m}_\vartheta$; at every other last-path node it is the corresponding principal level at $\mathfrak{b}$, with the prescribed containing node group. In particular all their indices except the first one are independent of the twist. Each contains the principal level of the integer $q_\vartheta A_0$, and is contained in the chosen free level. The latter is free in every rank at most R_*: a rational stabilizer is finite by definiteness, and its congruence kernels at the two chosen distinct residue characteristics have orders powers of both characteristics, hence order one.

For a residue matrix tuple S on the $2m$ last-path edges, the datum on edge (ϵ, j) is exactly

$$F_{\epsilon,j,S} = c_{\mathrm{vac}}^{R_{\epsilon,j} R_{\epsilon,j+1}} \,\mathrm{vacuumDatum}_{\mathfrak{m}_\vartheta}(\mathbf{1}_{S_{\epsilon,j}}).$$

Let $T_{\epsilon,j,S}$ be its algebraic chain up to node j, using the scale j_0 and the grid characters. For a last-path datum Ψ and outer forms (y', ϕ), define

$$\Xi_\vartheta(\Psi; y', \phi) = \sum_S \Psi(S)\,\mathrm{classIntegral}_{K_{0,m}} B\big((T_{0,m,S}y')(h), (T_{1,m,S}\phi)(\mathrm{diag}(h, u_\vartheta^m))\big). \tag{24.99}$$

Here y' has the original level on Y_m, which contains $K_{0,0}(\vartheta)$, and ϕ has the grid level $J(\mathfrak{b})$ on $V_{1,0}$. More generally the definition applies whenever the two outer levels fix these forms.

Put

$$E_\vartheta = |\mathcal{O}_E/\mathfrak{m}_\vartheta|^{R_*^2}, \qquad V_\vartheta = |\mathcal{O}_E/\mathfrak{m}_\vartheta|^{R_*}, \qquad R_\vartheta = V_\vartheta^{m^2} E_\vartheta^{2m}.$$

The exact turn construction gives a path datum whose value is a sum of at most R_ϑ roots of unity. Its equivariance is part of the path-datum condition, and is needed in the next assertion.

Lemma 24.34 (A finite sum with simultaneous node invariance)**.** *For a path datum with at most R roots of unity in every value, (24.99) is an ordinary sum of at most*

$$RE_\vartheta^{2m} \prod_{\epsilon=0,1} \prod_{0\le j<m+1-\epsilon} \#\,\mathrm{Cl}(K_{\epsilon,j}(\vartheta)) \tag{24.100}$$

terms. Every term is a root of unity times the pairing B of two vectors reached by m successive kernel maps from values of the outer forms. There is no additional denominator.

Proof. At a node, the residue action is $S_e \mapsto g_{\mathrm{target}}^{-1} S_e g_{\mathrm{source}}$. In the finite lattice model the transformed indicator is that indicator with this new label; the Weil splitting contributes its specified edge-character value. At a paired terminal node the two changes of variable are g and $\mathrm{diag}(g,1)$, so (24.97) has character value one. The path-datum equivariance is exactly the inverse of the product of the remaining edge characters. Reindexing the finite S-sum therefore makes the whole tensor invariant at every node. This assertion is about the sum; individual indicator tensors need not be invariant.

Apply Lemma 18.11 successively at the m nodes of each half and at the common terminal node. The outer invariance and the just-checked joint invariance are precisely its hypotheses. Freeness makes every cell mass one. There are $2m+1$ resulting class variables, the ones in (24.100); each of the $2m$ residue matrices has at most E_ϑ possibilities. Finally expand each $\Psi(S)$ into its at most R roots of unity. This proves both the expression and its bound.

Lean. `LastPath` (L586-L995). □

Lemma 24.35 (The last-path period with its level index)**.** *For each real terminal point g_∞ there is a complex number a_{g_∞}, independent of ϑ, Ψ and the outer forms, such that*

$$\begin{aligned} \mathrm{gridPeriod}(\mathrm{last}, \Psi, g_\infty; y', \phi) &= a_{g_\infty} \frac{\Xi_\vartheta(\Psi; y', \phi)}{i_m(\vartheta)}, \\ i_m(\vartheta) = [K^0_{Y_m} : \mathrm{levelY}(\vartheta, m)] &= \mathrm{triangleGLCard}(0, \vartheta, m). \end{aligned} \tag{24.101}$$

No nonvanishing assertion is made for a_{g_∞}.

Proof. At a definite edge, the fixed joint highest harmonic is a multiple of the leading-minor polynomial for its fitting weights. Straightening gives uniqueness of that highest line, and the derivative identity for the Fock theta functional identifies its matrix coefficients with the original Schur-jet kernel. Compact Schur orthogonality then gives one edge scalar $\beta_{\epsilon,j}$ between the analytic lift and the algebraic lift, divided by the index of the lower level. The scalar depends on the real polynomial, weights and j_0, and on none of the finite characters or data. This is also valid if that scalar is zero.

Write $I_{\epsilon,j}(\vartheta) = [K^0 : K_{\epsilon,j}(\vartheta)]$ and $s_{\epsilon,j} = c_{\mathrm{vac}}^{R_{\epsilon,j} R_{\epsilon,j+1}} \neq 0$. Induction along each half gives the exact identity

$$\mathrm{pathMap}_{\epsilon,j,S}\, F = \mathrm{ofAlg}\left(a_{\epsilon,j} \frac{I_{\epsilon,j}(\vartheta)}{I_{\epsilon,0}(\vartheta)} T_{\epsilon,j,S} F \right),$$

where $a_{\epsilon,0} = 1$ and

$$a_{\epsilon,j+1} = \frac{\beta_{\epsilon,j} a_{\epsilon,j}}{s_{\epsilon,j} I_{\epsilon,j+1}(1)}.$$

Indeed the upper index is independent of the twist, and the preceding lower index cancels the edge denominator. The terminal period formula in Lemma 24.33 cancels $I_{0,m}$; the remaining ratio $I_{1,m}/I_{1,0}$ is also fixed. Only $I_{0,0}(\vartheta)^{-1}$ remains. By the Chinese remainder theorem,

$$i_m(\vartheta) = \kappa I_{0,0}(\vartheta)$$

for one fixed positive rational κ: both indices have the same full general-linear factors at Q_ϑ, and differ only in the fixed depths 1 and e_* at the chosen places. Absorb κ and the fixed real constants into a_{g_∞}. The identity is termwise in S, so it holds for every finite sum Ψ with the same scalar.
Lean. `LastPath` (L996-L1150). □

Lemma 24.36 (One denominator for every last-path twist)**.** *There is a positive integer N_*, independent of the twist, coefficient prime and finite embedding, with*

$$\log N_* \le bx^{99}, \tag{24.102}$$

with the following pointwise property. Fix a prime $p \nmid q_\vartheta$ and an embedding ι above p. If both outer algebraic forms have transported Weyl gauge at most one at this ι at every finite point, then $N_*\Xi_\vartheta(\Psi; y', \phi)$ is integral at this ι. The same integer works for all these choices and every integral path datum, including the one obtained by all turns.

Proof. Use the positive integer

$$d_{\mathrm{lift}} = A_0^{2fR_*^2}(A_0 d)^{H_0}.$$

Since $q_\vartheta A_0 \in \mathfrak{m}_\vartheta$, Fourier inversion of a residue-matrix indicator expresses its scaled vacuum datum as a finite combination of Heisenberg translates. At $p \nmid q_\vartheta$ their translations are cleared by A_0, and multiplying the coefficients by $A_0^{2fR_{\epsilon,j}R_{\epsilon,j+1}}$ makes them integral. To see the denominator directly, perform inversion modulo the integer $q_\vartheta A_0$. There are $(q_\vartheta A_0)^{2fn'n}$ characters on the matrix entries; each coefficient is that denominator's inverse times a sum of roots of unity. Factors of q_ϑ are units at p, leaving the asserted power of A_0. Projection to the original ideal modulus does not add a denominator.

Theorem 19.4 supplies integrality for the scaled vacuum used here. Apply Theorem 18.29 with $\delta = A_0$, and Lemma 18.31 with jet multiplier one. A Schur entry of degree $L \le H_0$ costs at most $|\iota(A_0 d)|_p^{-L}$. Thus d_{lift} times each edge kernel has transported gauge at most one at every source and target finite point. In particular, after j lifts, multiplying the reached vector by d_{lift}^j makes its gauge at most one. The statement uses transported gauges, including determinant weights; it is not an entrywise assertion at an arbitrary nonintegral adelic representative.

It remains to clear the terminal line character. The original lattice idele has $q_1\alpha, \alpha^{-1}$ integral. Therefore

$$q_\vartheta A_0 q_1 u_\vartheta = (q_\vartheta A_0 t_\vartheta^{-1})\overline{t_\vartheta q_1 \alpha}\,\alpha^{-1} \in \widehat{\mathcal{O}}_E.$$

Its conjugate gives the same assertion for u_ϑ^{-1}. Let L_τ be the fixed integers in (24.97), and set

$$d_{\mathrm{pair}} = \prod_\tau (A_0 q_1)^{m|L_\tau|}.$$

If v', v have gauge one at g and $\mathrm{diag}(g, u_\vartheta^m)$, apply the inverse of those two Weyl actions in (24.97). The resulting vectors have integral coordinates, so their pairing by the integer matrix B is integral. The inverse line character is a product of powers of $u_\vartheta^{\pm m}$. The displayed integrality, with q_ϑ a unit at p, shows that d_{pair} clears precisely this character. Consequently

$$N_* = d_{\mathrm{pair}} d_{\mathrm{lift}}^{2m}$$

clears every term of Lemma 24.34. Its coefficients are roots of unity, so an ordinary sum introduces no further finite loss.

The logarithm is at most

$$\log N_* \le fm(H_0+20b)(e_*\log M+\log q_1) \\ +2m\{2fR_*^2e_*\log M+H_0(e_*\log M+\log d)\}.$$

Use the admissible hierarchy $m\le 2\sqrt{x}$, $R_*\le 2x^6$, $f\le x^{13}$, $r\le 3x^5$, $q_1\le x^{12}$, $d\le x$, $\log M\le 22\log b$, and $\log b\le x^{1/100}$. Since $H_0=br^4\le 81bx^{20}$ and $e_*\le x^{40}$, the first term is $O(bx^{74})$ and the second is $O(bx^{66})$. Their sum is at most bx^{99} for sufficiently large x.

Lean. `ThetaLiftGauge` (L362-L485), `SpanningForms` (L68-L270). □

Lemma 24.37 (All conjugates of the last-path sum)**.** *Fix $\sigma\in\operatorname{Aut}\mathbb{C}$. For the empty twist, suppose that the last-path datum has root count $R_1=V_1^{m^2}E_1^{2m}$ and that, at every finite point, the σ-conjugates of the two outer values have squared Weyl norms at most B_- and B_+ respectively. Then*

$$|\sigma\Xi_1(\Psi;y',\phi)|\le\exp(bx^{200})\sqrt{B_-}\sqrt{B_+}. \tag{24.103}$$

The same real constant works for every σ; no bound on another conjugate of the two outer forms is required.

Proof. Put $G=\exp(bx^{99})$. First, each empty-twist class number is at most G. Indeed its level contains the principal level of A_0, so (11.2) gives, in rank $R\le R_*$,

$$\#\operatorname{Cl}(K)\le(1+2[6q_1d(fR)^2]^3)^{4fR^2}A_0^{2fR^2}.$$

Its logarithm is $O(fR_*^2(\log(2q_1d(fR_*)^2)+e_*\log M))$, hence at most bx^{99}.

Second, every indicator kernel has conjugate operator norm at most G, uniformly at both finite points. In finite Fourier inversion of a single indicator, the coefficient absolute sum is one under every complex automorphism: the coefficients are roots of unity divided by the number of characters. Apply Lemmas 19.12 and 19.13 with this coefficient bound, jet multiplier one, $\ell=1$, and $u=\log\Omega^{-1}$. For each fixed σ, test the conjugated kernel on arbitrary target vectors and covectors of Weyl norm and dual Weyl norm at most one, using their coefficientwise σ^{-1} images as ξ and y. Then the growth factor $\|\sigma y\|_{\mathrm{Weyl},*}\|\sigma\xi\|_{\mathrm{Weyl}}$ is at most one, and target duality gives the operator bound, as in Theorem 19.14. Since the total row degree is at most H_0, the logarithm of the resulting operator bound is at most

$$C(fR_*^2+1)+fR_*^2\log 4+\tfrac12H_0(1+\log(H_0+1)+|\log\Omega|).$$

The constant C in the conjugate vacuum estimate satisfies $C\le x$. The area and original-root calculations give $\Omega=2\pi e^{\mathcal{C}_d}/\operatorname{rd}_E$, where $|\mathcal{C}_d|\le x$ and $\operatorname{rd}_E\le q_1d$. Hence $|\log\Omega|=O(x+\log x)$. The displayed logarithm is $O(x^{26}+bx^{22})$, and therefore at most bx^{99} for the chosen scales. No conjugate of the bare vacuum scalar was bounded separately.

Each vector reached after m edges therefore has norm at most G^m times its initial norm. The terminal pairing costs $H_0^{H_0}$, by (24.98). The $2m+1$ class variables cost at most G^{2m+1}. Formula (24.100) thus bounds the absolute value by

$$V_1^{m^2}E_1^{4m}H_0^{H_0}G^{4m+1}\sqrt{B_-}\sqrt{B_+}.$$

Since $|\mathcal{O}_E/\mathfrak{b}|=M^{fe_*}$, the logarithm of the first two factors is

$$fe_*\log M\,(m^2R_*+4mR_*^2).$$

Together with $H_0 \log H_0 + (4m+1)bx^{99}$ this is at most bx^{200}, after increasing the threshold. Conjugation has been applied only to the finite ordinary sum in Lemma 24.34; no automorphism has been passed through an analytic integral.

Lean. `SpanningForms` (L274-L385). □

Proposition 24.38 (Values of the lower chain through one last-path family)**.** *There are $c_\vartheta \in \mathbb{C}^\times$, one last-path datum Ψ_ϑ of root count at most R_ϑ, a surjective linear map*

$$r : \mathcal{H}_m \longrightarrow \mathcal{F}_{Y_0}(J_M),$$

and a scalar $\beta \in \mathbb{C}$, such that, simultaneously for every twist, outer form ϕ, finite point x of Y_m and vector v in the opposite weight,

$$v^{\mathsf{t}} A\big((c_\vartheta/c_1)\mathcal{L}_\vartheta(r\phi)\big)(x) = \beta\, \Xi_\vartheta(\Psi_\vartheta; \delta_x(v), \phi). \tag{24.104}$$

Here A is the original normalized integral duality for the weight on Y_m, and $\delta_x(v)$ is its opposite-weight delta form at the original level $\mathrm{levelY}(\vartheta, m)$*. Moreover, for one fixed integer e,*

$$\left|\frac{c_\vartheta}{c_1}\right| = q_\vartheta^{e/2} \prod_{0 \le i,j < m} R_{ij}(\vartheta). \tag{24.105}$$

The scalar β, which is never inverted without a case split, is independent of all the displayed variables.

Proof. Theorem 24.21 gives the surjective r, one real point, the first-path data, and one scalar relating its period to $\int_{[Y_m]} y'\mathcal{L}_\vartheta(r\phi)$. Use Theorem 24.30 to carry that datum forward through all m^2 turns, and propagate the period proportionality backward. Each turn multiplies its root count by at most V_ϑ, so the final root count is R_ϑ. Its nonzero turn constants are c_ϑ; the remaining common real proportionality constants multiply into a scalar β_0. Lemma 24.35 gives

$$i_m(\vartheta) \int_{[Y_m]} y'(g)\mathcal{L}_\vartheta(r\phi)(g)\, dg = \beta_0\, \Xi_\vartheta(\Psi_\vartheta; y', \phi)/c_\vartheta. \tag{24.106}$$

All uses of proportionality have the same quantifier order: the real constant is chosen before the twist, finite data and outer forms. Corollary 24.31 gives (24.105) after division by the nonzero c_1. In particular every c_ϑ is nonzero.

Here is the delta calculation with its dimension factor. At the free level on the definite group, $\delta_x(v)$ is supported on the single class of x and has value v there, extended by rational equivariance and level invariance. The initial false last-path weight is the dual of the original Y_m weight by Lemma 24.32. The normalized duality of Lemma 24.33 takes its leading vector to the opposite leading vector. Compact Schur orthogonality and the finite class sum therefore give

$$i_m(\vartheta) \int_{[Y_m]} \delta_x(v)(g)\mathcal{L}_\vartheta y(g)\, dg = \frac{v^{\mathsf{t}} A(\mathcal{L}_\vartheta y)(x)}{d_Y}, \qquad d_Y = \dim W_{Y_m}. \tag{24.107}$$

The class sum has one nonzero term. There is no stabilizer factor because the level is free. Substitute this into (24.106) and set $\beta = d_Y\beta_0/c_1$, proving (24.104).

We need two bounds for the delta form. For a point x integral above p and an integral coordinate vector v, the delta form has gauge at most one at every finite point. Indeed a nonzero value is $\rho(\gamma)v$ at $\gamma x k$, so its transported coordinate vector is $\rho(k^{-1}x^{-1})v$, with integral factors and inverses above p. For every x, v and every σ, its value norm is at most $\|\sigma v\|_{G'}$: a nonzero value differs by a rational unitary action, which remains unitary after every CM embedding and permutation of the places. Thus its squared norm is unchanged; the only other values are zero. This uses freeness, so no stabilizer sum occurs. The delta's original level contains the first last-path level, and the other outer level fixes ϕ. All hypotheses of Lemmas 24.36 and 24.37 hold for this pair.

Lean. `SpanningForms` (L389-L657). □

24.9 The dual norm in the simultaneous completion argument

Write G and $G^{\vee}$ for the Weyl Gram matrices of the given and opposite weights on Y_m. These are G and G' in Lemma 24.33. The vector v in (24.104) belongs to the opposite weight, so the norm in (24.108) is $\|\sigma v\|_{G^{\vee}}$.

Bounds at a fixed embedding. Lemma 24.36 applies one valuation to a finite sum. For $p \nmid q_{\vartheta}$ and an embedding ι above p it therefore suffices that the two outer forms have gauge at most one at ι at every finite point. Similarly, Lemma 24.37 estimates the terms of the sum after applying a single complex automorphism σ. The integers and real constants in these estimates are independent of the embedding. We shall use the estimate at σ with $v = \sigma^{-1}w$, whose other conjugates need not be bounded.

Lemma 24.39 (A single scaling at every finite and infinite embedding)**.** *Let $V, V^{\vee}$ be complex vector spaces with fixed integral coordinate bases and positive definite Gram matrices $G, G^{\vee}$. Suppose the matrix $A : V \to (V^{\vee})^{*}$ has integer entries and isometric duality: for every $z \in V$ there is $w \in V^{\vee}$ such that*

$$\|w\|_{G^{\vee}} = \|z\|_{G}, \qquad v^{\mathsf{t}} A z = \langle w, v\rangle_{G^{\vee}} \quad (v \in V^{\vee}).$$

Suppose also that a matrix C with integer entries and a positive integer h satisfy $CA = hI$.

Let $r : U \to X$ be a surjective complex-linear map, let $\mathcal{T}$ be the set of evaluation points, and let $L_{\vartheta} : X \to V^{\mathcal{T}}$ be complex-linear maps. Let the twists form a set with distinguished element 1, and let $\gamma_{\vartheta} \neq 0$, with $\gamma_1 = 1$, be fixed scalars. Suppose one $\beta \in \mathbb{C}$ gives the identities

$$v^{\mathsf{t}} A(\gamma_{\vartheta} L_{\vartheta}(ru)(x)) = \beta\, \mathcal{X}_{\vartheta}(v, u; x) \qquad (v, u, \vartheta, x).$$

Specify a set $\mathcal{A}$ of allowed pairs (ϑ, ι); in the application these are exactly the pairs for which the coefficient prime of ι does not divide q_{ϑ}. Assume there is an integer $N > 0$ with the following two properties. For every $(\vartheta, \iota) \in \mathcal{A}$, every u integral at ι, every integral coordinate vector $v \in V^{\vee}$, and every test point x integral above the coefficient prime of ι,

$$N\mathcal{X}_{\vartheta}(v, u; x) \in \mathcal{O}_{\iota}.$$

For the empty twist, every $\sigma \in \operatorname{Aut}\mathbb{C}$, and all v, u, x,

$$|\sigma\mathcal{X}_1(v, u; x)| \leq K\, \|\sigma v\|_{G^{\vee}}\, \mathcal{B}_{\sigma}(u), \qquad K \geq 0, \quad \mathcal{B}_{\sigma}(u) \geq 0.$$

Then one scalar $s \neq 0$, chosen before the twists and embeddings, has both properties

$$\gamma_{\vartheta} L_{\vartheta}(sru)(x) \in \mathcal{O}_{\iota}^{\dim V}, \qquad \|\sigma L_1(sru)(x)\|_{G} \leq hNK\, \mathcal{B}_{\sigma}(u).$$

The first assertion holds for $(\vartheta, \iota) \in \mathcal{A}$, u integral at ι, and x integral above its coefficient prime. The second holds for every u, x, σ under the stated pointwise bound for $\mathcal{X}_1$. For algebraic forms the finite assertion extends to all transported gauges whenever rational equivariance and weak approximation have the properties proved below. Moreover, sr maps every spanning set of U to a spanning set of X. Thus a single spanning set satisfying the stated input bounds gives one spanning family with all these conclusions.

Proof. Take $(s, \varepsilon) = (1, 0)$ when $\beta = 0$, and $(s, \varepsilon) = (hN/\beta, 1)$ otherwise. Put $z_{\vartheta} = \gamma_{\vartheta} L_{\vartheta}(sru)(x)$. Testing the identity with the standard basis of the opposite weight gives $Az_{\vartheta} \in h\mathcal{O}_{\iota}^{\dim V^{\vee}}$. Applying

C gives $hz_\vartheta \in h\mathcal{O}_\iota^{\dim V}$, which gives integrality even at primes dividing h. At infinity A has rational entries and isometric duality gives

$$\|\sigma z_1\|_G = \sup_{\|w\|_{G^\vee}\le 1} |w^{\mathrm{t}} A\, \sigma z_1| \le hNK\mathcal{B}_\sigma(u).$$

Indeed each w is tested with $v = \sigma^{-1}w$ at that embedding. Finally sr is surjective because $s \neq 0$.

The least admissible integer h in this argument is the exponent of $\mathbb{Z}^{\dim V}/A^{\mathrm{t}}\mathbb{Z}^{\dim V^\vee}$. Indeed $CA = hI$ has an integral solution precisely when every he_j belongs to the row lattice of A. The given hook number is an admissible multiple of this exponent.

Lean. `mem_embValRing_of_dualPairing_eq_dualHook_mul`, `weylNormSq_le_of_dualPairing`, `exists_lowerChain_integral_and_weylNormSq_le`. Lean uses the opposite hook number to recover coordinates one at a time. The integral left inverse proves their common bound and identifies the least possible clearing factor for this pairing. □

The delta form has the opposite weight, so the test norm above is precisely $\|v\|_{G^\vee}$. No comparison constant between the two Gram norms enters.

Lemma 24.40 (From integral points to the transported gauge)**.** *Let z be an algebraic form for the unitary group, with weight representation ρ defined over the coefficient number field, equipped with its standard integral Weyl lattice and coordinates. Fix a coefficient embedding above p. Suppose $z(x)$ has integral coordinates whenever x is integral at all places above p. Then z has integral transported gauge at every finite point.*

Proof. At each place above p, the Cayley map

$$X \longmapsto (1-X)(1+X)^{-1}$$

from the skew-Hermitian matrices is a rational chart onto the open set of unitary elements without eigenvalue -1. This set is dense in the local group. Weak approximation on the finite dimensional coefficient vector space therefore gives a rational unitary element γ such that γx belongs to the integral maximal compact group at all these places. The approximation condition is open, so it suffices to approximate x^{-1} after an arbitrarily small perturbation into the chart.

Both $\rho(\gamma x)$ and its inverse preserve the integral Weyl lattice. This uses integral Schur matrices and the fact that determinants of integral invertible matrices are units; determinant exponents may have either sign. Rational equivariance gives

$$\rho(\gamma x)^{-1} z(\gamma x) = \rho(x)^{-1} z(x).$$

The left side is integral by hypothesis, proving the assertion. The rational element γ is used only to test this already fixed form; it does not change the common scaling or any completion vector.

Lean, formal context. `LongForms` (L59). □

Proposition 24.41 (A common scaling controls all lifted values)**.** *There is one $s \neq 0$, independent of every twist and embedding, with the following properties. If $\phi \in \mathcal{H}_m$ is integral at an embedding above $p \nmid q_\vartheta$, then*

$$(c_\vartheta/c_1)\mathcal{L}_\vartheta(sr\phi)$$

is integral in its transported gauge at every finite point. If the σ-conjugates of all values of ϕ have squared norm at most B, then all values of the empty-twist lift satisfy

$$\|\sigma\mathcal{L}_1(sr\phi)(x)\|_G^2 \le e^{2bx^{201}} B. \tag{24.108}$$

Proof. Apply Lemma 24.39 to the pairing (24.104), with the opposite-weight test space of Lemma 24.33. Its integral matrix satisfies $CA = hI$, where h is the opposite hook number. Take the integer N_* from Lemma 24.36. Write $(s, \varepsilon) = (hN_*/\beta, 1)$ if $\beta \neq 0$ and $(1, 0)$ otherwise. The identity used by the lemma is then

$$v^{\mathsf{t}} A \zeta_\vartheta(x) = h\varepsilon N_* \Xi_\vartheta(\Psi_\vartheta; \delta_x(v), \phi), \qquad \zeta_\vartheta = (c_\vartheta/c_1)\mathcal{L}_\vartheta(sr\phi). \tag{24.109}$$

The finite hypothesis holds for $p \nmid q_\vartheta$ by Lemma 24.36 on the integral delta forms. The lemma gives integral coordinates, and Lemma 24.40 gives the full transported gauge. For the infinite hypothesis Lemma 24.37 and the delta norm give $K = e^{bx^{200}}$ and $\mathcal{B}_\sigma(\phi) = \sqrt{B}$. If H_0 is the bound on the number of boxes in the weight, then

$$hN_*K \leq H_0^{H_0} \exp\bigl(b(x^{99} + x^{200})\bigr) \leq e^{bx^{201}}.$$

Squaring the norm conclusion of Lemma 24.39 proves (24.108) for the same s.

Lean. `SpanningForms` (L635-L735). The code chooses the scalar inside the proof for the last-path period. Lemmas 24.39 and 24.40 isolate its linear algebra and its passage from integral points to the adelic gauge. □

Lemma 24.42 (Short integral forms on the outer auxiliary level)**.** *The outer auxiliary space $\mathcal{H}_m$ is spanned by forms integral in every transported gauge whose squared Weyl norms at every point and every complex embedding are at most $\exp(bx^{99})$.*

Proof. Its level is free and principal in the standard integral group, so Lemma 11.1 applies with $n = R_{1,0} \leq R_*$, $s \leq f$, $H \leq H_0$. As $n, R_n \leq C_n = 6q_1 d(fn)^2$, the squared-norm bound is at most $C_n^{6(nf+H_0)}$. The scale inequality $6(nf + H_0)\log C_n \leq bx^{99}$ proves the claim.

Lean. `ShortColumns` (L369-L440), `SpanningForms` (L738-L795). This specializes the same coefficient bound as for the original source. □

Theorem 24.43 (The auxiliary completion family)**.** *There are scalars $\gamma_\vartheta \in \mathbb{C}^\times$, one for every allowed m-fold twist, and a spanning set $\mathcal{Z} \subset H = \mathrm{targetForms}$ such that*

$$\gamma_1 = 1, \qquad |\gamma_\vartheta| = q_\vartheta^{e/2} \prod_{0 \leq i,j < m} R_{ij}(\vartheta), \tag{24.110}$$

$$\gamma_\vartheta \Theta_\vartheta z \text{ is integral at every } \iota \text{ above } p \nmid q_\vartheta, \qquad z \in \mathcal{Z}, \tag{24.111}$$

$$\|\Theta_1 z\|_\sigma^2 \leq \exp(bx^{203}) \qquad (z \in \mathcal{Z},\ \sigma \in \mathrm{Aut}\,\mathbb{C}). \tag{24.112}$$

For every family $\vartheta = \theta_\iota$ satisfying the prescribed ratio bounds, the same scalars satisfy the exact completion height bound

$$h((|\gamma_{\theta_\iota}|^{-2})_\iota) \leq b. \tag{24.113}$$

All choices of forms and scalars precede the choice of a finite embedding and of a family of twists satisfying these bounds.

Proof. Take the integral spanning forms ϕ of Lemma 24.42. The restriction r is surjective, and the common scalar s in Proposition 24.41 is nonzero, so the vectors $sr\phi$ span the original principal-level space. Put $\gamma_\vartheta = c_\vartheta/c_1$. The proposition gives integral twisted lifts at every allowed embedding and bounds the squared pointwise norms of the untwisted lifts by $\exp(2bx^{201} + bx^{99}) \leq \exp(bx^{202})$.

Apply $P_G = |G|e_0$ to this spanning set. Its image spans H. The lifts intertwine the G-action with right translations, and the coefficient of g in P_G is the algebraic integer $\sum_{\pi\ \text{selected}} \dim(\pi)\chi_\pi(g^{-1})$. In any embedding its absolute value is at most $\sum_\pi \dim(\pi)^2 \leq |G|$. Thus the projected lifts remain integral, and projection increases the pointwise squared norm by at most $|G|^4$. There are at most $e^{x^{30}}$

untwisted classes, the Schur class norm is bounded by the plain class sum, and $\log|G| \le fr^2 \log M$. Consequently

$$\|\Theta_1 z\|_\sigma^2 \le e^{x^{30}}|G|^4 e^{bx^{202}} \le e^{x^{99}+bx^{202}} \le e^{bx^{203}}.$$

Corollary 24.31 gives the modulus and $h((|\gamma_{\theta_\iota}|^{-2})_\iota) \le 2m^2 tC_{\mathrm{crit}} \le b$: the factors q_{θ_ι} are units at the selected primes, and squaring doubles the height bound in (24.93).

Lean, comparison. The depth e_* enters the integer N_* and the size of the last-path sum through M^{e_*}. The quotient c_ϑ/c_1 cancels the factors common to all twists. `LongForms` (L40-L135), `ArchimedeanSize` (L475-L551). □

Corollary 24.44 (Norms and heights of the forms completing the basis)**.** *Let $d_0 = \dim H$. The independent columns selected in (6.3) can be completed to a basis of H by $n' = d_0 - n$ members of $\mathcal{Z}$. Their total logarithmic norm and height cost is at most*

$$n'(bx^{203} + b) \le d_0 b. \tag{24.114}$$

Proof. Theorem 24.43 gives a spanning set $\mathcal{Z}$ with the prescribed norm and height bounds at every embedding. Adjoin independent members until the basis is complete. By (6.3), $n' \le d_0 x^{-204}$. The cost per column is at most $b(x^{203}+1)$; since $x^{-1} + x^{-204} \le 1$, this gives (24.114).

Lean. `exists_span_isAuxiliaryForm_eq_top`, `mul_completion_cost_le`. Lean uses $n' \le 4d_0/b^2$ with auxiliary depth b. Here the shallower level permits the single stopping threshold in (6.3). □

24.10 Normalizations in the determinant comparison

The common scalar. In the third exchange the test function is

$$\Psi_U\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \chi'(\det Y_1) P_{U,\vartheta}(Y_2).$$

The character vanishes off the units and cancels the determinant in row translation. The right invariance of $P_{U,\vartheta}$ gives $c_\vartheta = \gamma_3\gamma_{12}$ by (21.72)–(21.74). The three exchanges of Theorem 21.31 involve no division by c_1. Product spanning gives injectivity of the common map in Theorem 21.20, hence $c_\vartheta \neq 0$. Only then may one use $C_j = |c_1|^2 \,\mathrm{heckeValueProd}(j,1) > 0$. The contraction uses the separately proved orthogonality in Corollary 21.41.

Two local conditions enter this passage. Fourier recognition at the bordered coefficient uses the pointwise frame stabilizer in (12.28) and (12.29). The finite mass computed in Lemma 21.12 is positive because the source character $\chi_S(\det k)$ is trivial on the principal level. For the approximation, one fixes the base point g before choosing a linear combination valid for every evaluation point h (Theorem 21.14).

The embedding and its dual norm. Fix $\sigma \in \mathrm{Aut}(\mathbb{C})$. The test vector in the value pairing has the opposite norm G', and the value vector has norm G (Lemma 24.39). The matrix-coefficient bound in Lemma 19.13 has prefactor $\|\sigma y\|_*\|\sigma\xi\|$. Already in the trivial representation $y = \sqrt{2} - 1$, $\xi = 1$ gives $|\sigma y| = 1 + \sqrt{2}$ when $\sigma(\sqrt{2}) = -\sqrt{2}$. To bound the conjugated vector, choose an arbitrary unit covector y_0 at that embedding and take $y = \sigma^{-1} y_0$ before taking its supremum. Theorem 22.16 fixes the scalar c_* before the prime, embedding and twist. The two embeddings in the determinant estimate may therefore use different test forms with the same twist and the same scalar.

The finite norm. The integral norm is the Weyl norm after transport to a representative. In split rank one a raw value 1 at $g_w = \ell$ transports to ℓ^{-1}. Lemma 24.40 uses weak approximation to choose representatives integral at all places above ℓ. A largest coordinate can then be normalized to bound an eigenvalue of an integral endomorphism. The compact level and its inverse preserve the lattice, as required in Lemma 11.1. For these same admissible data, the complex kernel bounds are Lemma 21.23 and Theorem 21.25; the finite denominator sum is (25.23).

Integral representatives, integral level, the bound at one embedding.

25 The denominator operators and the product formula

The congruence operators clear the projections onto the lift eigenspaces. We compare the determinants of the twisted and untwisted Gram matrices in the completed basis and apply the product formula. This gives the lower bound used in Theorem 7.4.

The denominator of the projection depends on the determinant coefficient of the common polynomial and on the scalar that clears both the projection and the section of the norm map.

Let c be the chosen positive integer that clears the relevant finite-group denominators, put $v = v_\ell(c)$, and write $y = r^8$ for half the gap between consecutive lift weights. Let O be a sufficiently large coefficient ring with normalized valuation

$$\widehat{v}(a) = \frac{v_O(a)}{v_O(\ell)}, \qquad \widehat{\lambda}(M) = \frac{\lambda_O(M)}{v_O(\ell)}.$$

Thus $\widehat{v}$ is the valuation v_ℓ on rational integers. Use q_0 for the number of character summands, r_0 for the rank of the main summand, k for the number of triangle indices, and n_{det} for the number of detecting group elements. Let n_{test} be the auxiliary fixed detecting-set count. The index sets are chosen before the congruence depth.

Corollary 25.1 (A valuation bound for the common coefficient)**.** *Suppose $q_0 \le t$, $h \le v$, and*

$$\begin{aligned} s &\le yh, \\ (k + n_{\mathrm{test}})s &\le y^{22}h, \\ n_{\mathrm{det}} &\le 2(k + n_{\mathrm{test}}). \end{aligned} \tag{25.1}$$

Let d be the top square-free coefficient at the chosen system of the common Ribet polynomial. Suppose its determinant construction gives

$$\begin{aligned} v_O(d) \le L_T + S_T + \sum_{e<q_0} (L_e + O_e) \\ + q_0\big(T_{\mathrm{tame}} + 2n_{\mathrm{det}} v_O(\ell^s)\big) + (k+1)v_O(\ell^s). \end{aligned} \tag{25.2}$$

Here the main length is bounded by $L_T + S_T$ as in Lemma 13.19, and the character cost is $L_e + O_e$, the local crystalline colength plus the odd-class length. Assume the following normalized arithmetic estimates:

$$\frac{L_T + \sum_e L_e}{v_O(\ell)} \le r_0(4fD_\ell), \qquad \frac{S_T}{v_O(\ell)} \le \mathrm{sel}, \tag{25.3}$$

$$O_e \le v_O(O_*), \qquad \frac{T_{\mathrm{tame}}}{v_O(\ell)} \le 2\,\mathrm{ram} + 8fv, \tag{25.4}$$

where $O_* > 0$ is a common integer independent of the congruence depth. Then

$$\widehat{v}(c^2 d) \le \mathrm{sel} + 2q_0\,\mathrm{ram} + r_0(4fD_\ell) + E_\ell, \tag{25.5}$$

where one may take

$$E_\ell = t\,v_\ell(O_*) + \big(8ft + 2 + (4t+1)y^{22} + y\big)v_\ell(c). \tag{25.6}$$

In particular the two powers of c in (25.16) are included in this bound.

Proof. Divide (25.2) by $v_O(\ell)$ and use (25.3)–(25.4). Since $\widehat{v}(\ell^s) = s$, this gives

$$\begin{aligned}\widehat{v}(d) \le{}& \mathrm{sel} + r_0(4fD_\ell) + q_0 v_\ell(O_*) \\ &+ 2q_0\,\mathrm{ram} + 8fq_0 v + 2q_0 n_{\det} s + (k+1)s.\end{aligned}$$

The last two terms satisfy

$$\begin{aligned}2q_0 n_{\det} s + (k+1)s &\le (4t+1)(k+n_{\mathrm{test}})s + s \\ &\le \big((4t+1)y^{22} + y\big)h \\ &\le \big((4t+1)y^{22} + y\big)v.\end{aligned}$$

Also $q_0 v_\ell(O_*) \le t v_\ell(O_*)$ and $8fq_0 v \le 8ftv$. Finally $\widehat{v}(c^2 d) = 2v + \widehat{v}(d)$. Substitution gives exactly (25.5) and (25.6).

Lean, formal context. `RibetElement` (L1), `LongForms` (L214). □

Equation (25.2) is the bound in Theorem 14.11 and (14.31), with the main block, character blocks, old-inertia factors, and separation scalar $C = \ell^s$, $s \le yh$, kept distinct. The valuation conversion is the ordinary proof of `ellVal_add_le_of_addVal_le`. Its conclusion supplies the numerical part of (25.16); there is no new denominator factor at the later coefficient-extraction step.

Theorem 14.16 now constructs the common old-inertia polynomials and proves the tame inequality in (25.4), under its explicit hypotheses on the inertia representations and the bound for the auxiliary twist. Corollary 15.12 proves the structural crystalline colength comparison, and Theorem 15.19 gives exactly the first inequality of (25.3) from the one-place estimates (15.24). Lemma 15.20 also supplies the numerical conversion of the crystalline floor losses. Theorem 17.40 and Corollary 17.41 supply the local estimates for the constructed data from the specified rational representations and their local compatibilities. Theorem 17.39 reconstructs the Selmer estimate from the type, scalar and conjugate-duality properties. Theorems 17.87, 17.94 and 17.105 supply these representation inputs for the data. The argument uses them together with the prime-triangle count (25.1).

We take $O_* = \mathcal{X}$ in (14.3). Theorem 14.5 proves the required length estimate at every depth and coefficient prime. The estimate for the tame period module in Theorem 15.16 accounts for the polynomial-degree loss. Corollary 16.20 applies the tame period estimate in an integral frame. When few jumps lie below the threshold, Theorem 16.49 and Corollary 16.56 bound the loss at one place. Theorems 16.84 and 16.86 treat the remaining cases, with the numerical bound of Corollary 16.93. Theorem 17.40 combines the weight and field estimates. Corollary 17.106 applies them to the family of representations.

25.1 The Ribet polynomial and the projection onto the lift image

We extract a coefficient of the common Ribet polynomial and use it to clear the orthogonal projection onto a lift eigenspace.

An integral indicator of a residual system. Let K be an algebraically closed valued field of characteristic zero, let $\mathcal{O}$ be its valuation ring, and let $\mathfrak{m}$ be the maximal ideal. In the application $K = \overline{\mathbb{Q}}_\ell$ and $v(\ell) = 1$. Let X be a finite set, and let $\mathbb{T} \subseteq \mathcal{O}^X$ be an $\mathcal{O}$-subalgebra. We regard its elements as scalar operators on the summands of a finite direct sum indexed by X.

Lemma 25.2 (Integral residual indicator)**.** *Fix $x_0 \in X$ and any family $t_\lambda \in \mathbb{T}$. Put*

$$C = \{x \in X : t_\lambda(x) - t_\lambda(x_0) \in \mathfrak{m} \text{ for every } \lambda\}.$$

Then the indicator $A = \mathbf{1}_C$ belongs to $\mathbb{T}$. In particular $A^2 = A$ and $A(x_0) = 1$.

Proof. For each $y \notin C$ choose t_y from the family with $t_y(x_0) - t_y(y) \in \mathcal{O}^\times$. Put $q = \prod_{y \notin C}(t_y - t_y(y)) \in \mathbb{T}$. It vanishes outside C. For $c \in C$ each factor has the same nonzero residue at c as at x_0, so $q(c)$ is a unit. Thus

$$1 - \prod_{c \in C}(1 - q/q(c))$$

belongs to $\mathbb{T}$, equals 1 on C, and equals 0 elsewhere. The empty product convention covers $C = X$.
Lean. `TiedTraces` (L343-L390). The code constructs the same two products from the characteristic coefficients of the normalized Galois representations and proves that the denominators are units in the coefficient ring. □

For the Hecke algebra, the t_λ are the coefficients of the characteristic polynomials of the normalised Galois representations at all Galois elements. Their membership in the existing integral Hecke algebra is Theorem 17.58, from the finite integral lattice and rational Frobenius attachment. Corollary 17.61 applies the present proof over the finite coefficient ring and obtains the integral residual indicator. No division by a congruence between two such systems occurs. This is the algebra needed for `exists_diagOp_mem_ellIntegralForms`.

Separating the weak lifts at the base level. Let $X_0 \subseteq X$ be the systems that occur at the base level. For a family of representations $N_x : G \to \mathrm{GL}_n(K)$, put

$$\beta_g(x) = \det(1 - N_x(g)).$$

Assume $\beta_g \in \mathbb{T}$. A system is a *weak lift* if $\beta_g(x) = 0$ for every $g \in G$; denote the set of weak lifts by W. For each $x \in X_0 \setminus W$, choose g_x with $a_x = \beta_{g_x}(x) \neq 0$. Then

$$\Delta = \prod_{x \in X_0 \setminus W}(a_x - \beta_{g_x}) \in \mathbb{T}, \qquad n_0 = \prod_{x \in X_0 \setminus W} a_x \in \mathcal{O} \setminus \{0\} \tag{25.7}$$

satisfy

$$\Delta|_W = n_0, \qquad \Delta|_{X_0 \setminus W} = 0.$$

The empty products are 1. This construction makes no assertion that Δ kills every nonweak system at the deeper level. Expanding the product, or telescoping one factor at a time, shows that $\Delta - n_0$ belongs to the ideal generated by the β_{g_x}. Consequently, if $f \in \mathbb{T}[\alpha_j : j \in J]$ and an ideal $\mathfrak{a}$ satisfies $\beta_g f \in \mathfrak{a}$ for every g, then

$$(\Delta - n_0) f \in \mathfrak{a}. \tag{25.8}$$

This proves the separating-polynomial step in `exists_prod_sub_det_one_sub_eq_zero`.

The norm bound for the deformation. Let V be a finite-dimensional K-space, $L \subset V$ a full $\mathcal{O}$-lattice, and $\mathcal{E} = \{T \in \mathrm{End}_K(V) : T(L) \subseteq L\}$. Let the commutative algebra $\mathbb{T}$ act on V through $\mathcal{E}$. Fix a base-level subspace V_0, an operator $N \in \mathcal{E}$ commuting with $\mathbb{T}$, and a nonzero scalar $c \in \mathcal{O}$. Suppose there is $\Psi \in \mathcal{E}$ with

$$N \Psi z = cz \qquad (z \in V_0). \tag{25.9}$$

For the automorphic application N is the finite-group norm at the auxiliary level and c is the common multiple of the stabiliser orders. The following finite-orbit argument supplies this norm section from the stated stabiliser divisibility.

Lemma 25.3 (Norm section with stabilisers)**.** *Suppose a finite group Γ permutes a family of fibres and acts integrally on their sections. Suppose the order of every point stabiliser divides the positive integer c. Assume that multiplication by an integer-valued function on the base preserves the space of sections under consideration and its integral lattice. Then there is an integral operator Ψ with*

$$\sum_{g\in\Gamma} g^{-1}\Psi g = c\,\mathrm{id}.$$

In particular, if $N=\sum_{g\in\Gamma} g$, then $N\Psi z = cz$ on the invariant sections.

Proof. Choose one point x in each orbit. Let Ψ multiply the fibre at that point by $c/|\Gamma_x|$, and multiply the other fibres by zero. The coefficients are integers, so Ψ is integral. At any point of this orbit, exactly $|\Gamma_x|$ conjugates $g^{-1}\Psi g$ contribute the scalar $c/|\Gamma_x|$; the action inside a fibre cancels because this is a scalar operator. Their sum is c on each fibre. If z is invariant, apply the identity to z and replace gz by z; summing $g^{-1}\Psi z$ is $N\Psi z$.

Lean. For automorphic forms at auxiliary level, the orbit weights give `exists_higman_operator`. The proof above uses only the divisibility of the orbit stabilizer orders and preservation of the integral sections by these weights. □

At principal auxiliary level, take the finite group to be the product of the residue-field general linear groups. Its orbit stabilisers are images of the finite class-automorphism groups at maximal level: for a double coset represented by g, the stabiliser is the image of $\Gamma_{\mathrm{rat}}\cap gKg^{-1}$ in the level quotient. Their orders divide c. The integer weights can be taken invariant under the kernel of reduction, including the compact subgroups at the places prescribing the type. Multiplication by these weights therefore preserves that type, principal auxiliary level, and the integral values of the forms. The lemma proves (25.9) for the base-level vectors, which are fixed by the finite residue group. This is the orbit-weight proof of the norm section, with a single factor c.

Let J be a finite set with $m=|J|$, and write $\alpha_S=\prod_{j\in S}\alpha_j$ and $F_S=[\alpha_S]F$. Let $A,\Delta\in\mathbb{T}$, let $n_0\neq 0$ be integral, and set

$$\delta = c^2A\Delta, \qquad n_1 = c^2n_0.$$

Let $a\in\mathcal{O}\setminus\{0\}$ be divisible by n_1^{m+1}. The deformed-norm assertion needed here is the existence of $N_S, Y_S\in\mathcal{E}$, for $S\subseteq J$, such that

$$N_\varnothing = N, \qquad [N_S,T]=0 \quad (T\in\mathbb{T}), \tag{25.10}$$

$$\sum_{S\subseteq J} N_{J\setminus S}F_S \equiv 0 \pmod{a\mathcal{E}} \quad (F\in\mathfrak{a}), \tag{25.11}$$

$$\delta^m N_S \equiv Y_S N \pmod{a\mathcal{E}}, \qquad Y_\varnothing=\delta^m. \tag{25.12}$$

Only coefficients of square-free monomials are read; F itself need not be multilinear. In the source the local first-order operators give $\sum_S N_S\alpha_S$ as a product of factors $N_j+\alpha_j\partial_j$. The auxiliary-level theorem establishes (25.10)–(25.12) with

$$a=\ell^\nu, \qquad (n_0c^2)^{m+1}\mid \ell^\nu.$$

The divisibility is in $\mathcal{O}$; a is consequently integral. The family identities follow from the following one-place relations; Theorem 13.31 constructs the local operators satisfying those relations.

Lemma 25.4 (Assembling the deformed norms)**.** *Let $\Gamma=\prod_{j\in J}\Gamma_j$ act through integral operators. Write $N_j=\sum_{g\in\Gamma_j} g$ and $N=\prod_j N_j$. Suppose the operators supported at distinct factors commute, and $\mathbb{T}$ commutes with every such local operator. Let ∂_j be integral local operators. Let $\mathfrak{a}$ be generated*

by elements λt with $\lambda \in a\mathcal{O}$, and by $t_{j,g}-\alpha_j r_{j,g}$, where all t, r belong to $\mathbb{T}$. Assume, with congruences modulo $a\mathcal{E}$,

$$\begin{aligned} N_j t_{j,g} = 0, \qquad \partial_j t_{j,g} \equiv N_j r_{j,g}, \\ \text{for every } h \in \Gamma_j \text{ there is } Z_{j,h} \in \mathcal{E} \text{ with} \quad c\partial_j(h-1) \equiv N_j Z_{j,h}, \qquad \delta' N Z_{j,h} \equiv 0, \end{aligned} \tag{25.13}$$

where $\delta' = A\Delta$ and $\delta = c^2\delta'$. Suppose also that an integral Ψ satisfies $\sum_{h\in\Gamma} h^{-1}\Psi h = c\,\mathrm{id}$. Then (25.10)–(25.12) hold with

$$N_S = \prod_{j\in S} \partial_j \prod_{j\notin S} N_j.$$

Proof. The local factors at different places commute. Thus $N_\varnothing = N$ and each N_S commutes with $\mathbb{T}$. Work modulo $a\mathcal{E}$ and adjoin central variables with $\alpha_j^2 = 0$. Then

$$D = \prod_{j\in J}(N_j + \alpha_j\partial_j) = \sum_{S\subseteq J} N_S \alpha_S.$$

For the generator $t - \alpha_j r$, the j-th factor gives $N_j t + \alpha_j(\partial_j t - N_j r) = 0$ by (25.13). The other generators are multiples of a. Since D commutes with the Hecke coefficients, $DF = 0$ for every $F \in \mathfrak{a}$. Its α_J-coefficient is (25.11).

We prove $\delta^k N_S = YN$ modulo $a\mathcal{E}$ by induction on $k = |S|$. For $S = \varnothing$ take $Y = 1$. For $k > 0$, put $U = c\delta'\delta^{k-1}N_S$. If $h \in \Gamma_j$ and $j \notin S$, the factor N_j gives $U(h-1) = 0$. If $j \in S$, the local relation and the induction hypothesis give

$$U(h-1) = (\delta^{k-1}N_{S\setminus\{j\}})\delta' Z_{j,h} = Y'N\delta' Z_{j,h} = Y'\delta' N Z_{j,h} = 0.$$

Here δ' commutes with the local factors and with δ. Since the Γ_j generate Γ, $Uh = U$ for every h. Consequently

$$\delta^k N_S = cU = \sum_{h\in\Gamma} Uh^{-1}\Psi h = U\Psi N.$$

Multiplication by δ^{m-k} gives (25.12). Lift the resulting coefficients from $\mathcal{E}/a\mathcal{E}$ to $\mathcal{E}$, taking $Y_\varnothing = \delta^m$.

Lean. `exists_map_pow_mul_family_eq_mul_norm` proves the product factorization. Here the ideal relation is the top coefficient of a product in the ring with $\alpha_j^2 = 0$. □

In the automorphic application $t_{j,g}$ is the trace of $(N_x(\tau_j) - 1)N_x(g)$ and $r_{j,g}$ is the trace of $(N_x(\phi_j) - 1)H_j(N_x(\phi_j))N_x(g)$. To apply Theorem 13.31 at j, take $\delta' = A\Delta$. On the base-level systems X_0, Lemma 25.2 gives $A(x) = 1$ precisely for the residual class C of x_0; (25.7) gives $\Delta(x) = n_0 \neq 0$ on the weak systems W and zero on the others. Thus

$$\{x \in X_0 : (A\Delta)(x) \neq 0\} = X_0 \cap C \cap W = \mathcal{X}_{\mathrm{near}}.$$

The polynomial H_j of (13.116) supplies the required factorization on this set. Systems occurring only at auxiliary level contribute zero to $NB(1)$ by full-norm invariance. The one-place theorem therefore gives (25.13) with modulus ℓ^ν, and Lemma 25.4 assembles the local operators.

Lean. `rootchainsonweakcongruentbasesystems` (L611-L618) and `applicationtotheproductofthetwoHeckeoperators` (L709-L712).

Lemma 25.5 (Extracting the top coefficient)**.** *Assume (25.9)–(25.12). Let $f \in \mathbb{T}[\alpha_j : j \in J]$ satisfy $(\Delta - n_0)f \in \mathfrak{a}$. Suppose $l, y \in V_0$ and $d \in \mathcal{O}$ satisfy*

$$dl \in L, \qquad A\Delta l = n_0 A y, \qquad \Delta y = n_0 y, \qquad f_S y = 0 \quad (S \subsetneq J).$$

Then

$$dcA^{m+1} f_J y \in L. \tag{25.14}$$

Proof. Set $G = \sum_{S\subseteq J} N_{J\setminus S} f_S$ and $u = \Psi(dl) \in L$, so $Nu = dcl$ by (25.9). Applied to $(\Delta - n_0)f$, (25.11) gives $(\Delta - n_0)G \in a\mathcal{E}$. Since $\delta - n_1 A = c^2 A(\Delta - n_0)$ and δ commutes with A, the difference of powers gives $\delta^{m+1}G \equiv n_1^{m+1}A^{m+1}G$ modulo $a\mathcal{E}$. Apply (25.12) to each term of $\delta^{m+1}G$ and evaluate at u:

$$n_1^{m+1}A^{m+1}Gu \equiv \sum_{S\subseteq J} Y_{J\setminus S} N\delta f_S u \pmod{aL}. \tag{25.15}$$

As N commutes with $\mathbb{T}$ and $A\Delta l = n_0 Ay$, we have $N\delta f_S u = dc\, n_1 A f_S y$. Only $S = J$ contributes. Here $Y_\varnothing = \delta^m$ and $\delta y = n_1 Ay$, so the right side equals $n_1^{m+1} dcA^{m+1} f_J y$. Since $n_1 \neq 0$ and $a/n_1^{m+1} \in \mathcal{O}$, division in V gives

$$dcA^{m+1} f_J y \in A^{m+1}Gu + (a/n_1^{m+1})L \subseteq L.$$

Lean. `smul_top_coeff_mem_ellIntegralForms` gives the automorphic specialization. Here evaluating on $\Psi(dl)$ keeps the error in aL until the final division. □

Reduction of the Ribet polynomial over the Hecke algebra. There is no additional gluing assumption in passing from polynomial families to a polynomial with Hecke coefficients. The map

$$\mathbb{T}[\alpha_j : j \in J] \longrightarrow \prod_{x\in X} K[\alpha_j : j \in J]$$

given by coefficientwise evaluation is injective: each coefficient is a function on X, and a function zero at every x is zero. Take as the coefficient algebra the image of this map, and as the trace module the image of $\mathfrak{a}$. A common Ribet element in that algebra has a unique preimage f. Its multiplier relation, the vanishing of its proper square-free coefficients on a specified set of lifts, and its top-coefficient value all pass to f by injectivity. The common-polynomial conditions used by the projection theorem below are exactly

$$\begin{gathered} \beta_g f \in \mathfrak{a} \quad (g \in G), \qquad f_S(x) = 0 \quad (x \in X_{\mathrm{lift}},\ S \subsetneq J), \\ v(c^2 f_J(x_0)) \le B, \qquad x_0 \in X_{\mathrm{lift}} \subseteq W. \end{gathered} \tag{25.16}$$

The bound already includes the two powers of c. The trace ideal in the source is generated by the functions $\ell^\nu \operatorname{tr}(N_x(g))$ and by

$$x \longmapsto \operatorname{tr}\big((N_x(\tau_j) - 1 - \alpha_j(N_x(\phi_j) - 1)H_j(N_x(\phi_j)))N_x(g)\big), \qquad H_j(1) = 1.$$

Their membership in $\mathbb{T}[\alpha]$ uses the integral Hecke–Galois trace compatibility. Theorem 14.11 constructs this common polynomial, and Corollary 25.1 proves its valuation bound from the stated crystalline, odd-class, tame, and prime-triangle estimates. Together with the coefficientwise passage just proved, these reconstruct the corresponding algebra in `ribet_columns`.

Orthogonal projection onto the lift image. The two identities in Lemma 25.5 concern orthogonal projections for the Hermitian inner product on automorphic forms. For integrality we use the fixed isomorphism $\iota : \mathbb{C} \simeq \overline{\mathbb{Q}}_\ell$.

Lemma 25.6 (Agreement on the weak systems)**.** *Let V_0 be a finite-dimensional Hermitian space, with an orthogonal decomposition into joint eigenspaces for a commuting family of normal Hecke operators. Let $Z \subseteq V_0$ be stable under these operators, and let P_Z be its orthogonal projection. Let Q be an orthogonal type projection commuting with the Hecke operators and preserving Z. Suppose one Hecke operator T kills all the weak joint eigenspaces inside* $\operatorname{im} Q$*. If v is orthogonal to*

$$\operatorname{im} Q \cap Z^\perp \cap \ker T,$$

then Qv and $QP_Z v$ have identical components at every weak system.

Proof. Let W be the sum of the weak joint eigenspaces and P_W its orthogonal projection. Each joint spectral projector is a polynomial in the commuting normal Hecke operators by finite interpolation. Therefore P_W preserves Z and commutes with Q. As P_W and Q are self-adjoint and preserve Z, they commute with P_Z. Their product $P_W Q(1 - P_Z)$ is the orthogonal projection onto $W \cap \operatorname{im} Q \cap Z^{\perp}$, which lies in $\operatorname{im} Q \cap Z^{\perp} \cap \ker T$. The hypothesis puts v in its kernel.

Lean. `DenominatorBound` (L692-L737). The code decomposes the Hecke-invariant intersection into isotypic components and applies orthogonality there. The commuting product projects directly onto the sum of the weak components in that intersection. □

In the application $\operatorname{im} Q$ lies in the specified group-algebra image, so the displayed intersection consists of false lifts at the chosen place. Orthogonality to the false lifts therefore gives the hypothesis of the lemma. For every weak base-level system the spherical Hecke polynomial contains the full string of roots by (13.112). Its reversed polynomial therefore vanishes at the lift root, as required for the false-lift projection. The source passage is `typeProjection_sub_starProjection_mem`.

The denominator of Q has an elementary finite-group explanation. For a finite group Γ acting integrally and unitarily, the projection of type χ is

$$Q = \frac{\chi(1)}{|\Gamma|} \sum_{g \in \Gamma} \chi(g^{-1})\, g.$$

Schur orthogonality makes this the orthogonal projection onto the χ-isotypic subspace. Character values are sums of roots of unity and hence integral at every finite place. Consequently cQ preserves the lattice whenever $|\Gamma|$ divides the integer c. The same argument applies to the product of the finite groups at the chosen places. Thus one may take c to be the product of their orders (`chosenGroupOrder` in Lean). The lift generators in the source lie in the ranges of the finite-group type idempotents. For a fixed type projector Q, the type-idempotent relations make Q fix or kill each such generator. This is the content of `IsLiftGenerator.typeProjection_eq_zero_or_mem_range`; it supplies the common diagonal basis used below without requiring one-dimensional Hecke eigenspaces.

Theorem 25.7 (Clearing one eigenline with the constructed operators)**.** *Use the finite spectral setup above, and embed the base-level space in the auxiliary-level space. Suppose Z is supported on $X_{\mathrm{lift}} \subseteq W$, $QZ \subseteq Z$, cQ preserves the integral lattice, and the false-lift hypothesis of Lemma 25.6 holds for the tested vectors. Let $v_0 \neq 0$ belong to the system $x_0 \in X_{\mathrm{lift}}$ and satisfy $Qv_0 = v_0$.*

Assume the common Ribet polynomial (25.16), the norm section (25.9), and the deformed-norm identities (25.10)–(25.12) for A of Lemma 25.2 and Δ of (25.7). Then the operator

$$G = c^2 A f_J Q \tag{25.17}$$

has the following properties:

1. *$Gv_0 = \mu_0 v_0$, with $v(\iota(\mu_0)) \leq B$.*
2. *For every integral tested vector v orthogonal to the false lifts, $G(P_Z v)$ is integral.*
3. *On every joint eigenvector fixed or killed by Q, G acts by a scalar. In particular it is diagonal on a basis of lift generators with this type property.*

Proof. Read the operators over K using ι, and put $l = Qv$ and $y = QP_Z v$. Both are at the base level, and cl is integral. The vector y is supported on X_{lift}. Hence $\Delta y = n_0 y$ and $f_S y = 0$ for every proper $S \subset J$. Lemma 25.6 says that $l - y$ has zero weak components. Equation (25.7) kills its remaining base-level components, so $\Delta l = \Delta y = n_0 y$. In particular $A\Delta l = n_0 A y$.

By (25.8), $(\Delta - n_0)f \in \mathfrak{a}$. Apply Lemma 25.5 with $d = c$. It proves $c^2 A^{m+1} f_J Q P_Z v \in L$. Since $A^2 = A$, this is precisely $G(P_Z v) \in L$. At v_0, the type projection and the residual indicator both equal the identity, so $\iota(\mu_0) = c^2 f_J(x_0)$ and the valuation bound is (25.16). Scalar action on the other specified joint eigenvectors follows directly from (25.17).

Lean. For the lift projection and its integral lattice, `exists_linearMap_eq_smul_mem_integralForms` constructs the operator with the eigenvalue and integrality bounds proved here from the common Ribet polynomial. □

The cyclotomic generator and the auxiliary levels. Take c as in (14.36). In (13.99) the cyclotomic modulus is $N = q_1 d$: $E \subseteq \mathbb{Q}(\mu_N)$. The scale bounds in the proof of Lemma 24.11 give $N \le x^{13}$, $x^5 \le r \le 3x^5$ and $x \ge 10^{100}$. Hence $N + t \le x^{13} + 15x^5 < x^{40} \le y = r^8$, since $t = 5r$.

Fix ℓ. The image $\chi_\ell(G_E)$ is procyclic: for odd ℓ it is a closed subgroup of the procyclic group $\mathbb{Z}_\ell^\times$, and at 2 it lies in $1 + 4\mathbb{Z}_2$ since $i \in E$. Choose $a \in G_E$ such that $u = \chi_\ell(a^{-1})$ generates this image topologically. For every $j > 0$ and $z \in \chi_\ell(G_E)$, $u^j - 1$ divides $z^j - 1$ in $\mathbb{Z}_\ell$: this holds for powers of u, and the ideal $(u^j - 1)$ is closed.

Choose an old level prime $p_i \neq \ell$. It splits completely in E by Lemma 17.29 since $p_i \equiv 1 \bmod q_2^2$ and $N \mid q_2^2$. Arithmetic Frobenius there has cyclotomic value p_i. Also $\mathbb{Q}(\sqrt{-1}) \subseteq E$, so $4 \mid p_i - 1 \mid p_i^t - 1$. With $t \le rt$, Lemma 14.12 gives in $\mathbb{Z}_\ell$

$$4 \prod_{j=1}^{t-1} (1 - u^j) \;\mid\; \prod_{j=1}^{t} (p_i^j - 1) \;\mid\; c.$$

Thus the triangle exponent b in Theorem 13.44 satisfies $b \le v_\ell(c)$, including at $\ell = 2$. For $\ell \le y^2$ take $h = v_\ell(c)$. For $\ell > y^2$ take $h = 0$: here $\ell > N + t$, so multiplicativity of the cyclotomic degree gives linear disjointness of $\mathbb{Q}(\mu_N)$ and $\mathbb{Q}(\mu_\ell)$. Hence $u \bmod \ell$ generates $\mathbb{F}_\ell^\times$; all $u^j - 1$, $0 < j < t < \ell - 1$, and 4 are units, so $b = 0$. In both cases $b \le h \le v_\ell(c)$, and every $u^j - 1$, $0 < j < t$, divides ℓ^h, as required by (13.99).

Let s be the detecting set of Lemma 13.42. In (25.2) the separation exponent is b, the number of indices is $k = (t-1)|s|$ by (13.101), and $n_{\mathrm{test}} = |s| + 1$. At $\ell \le y^2$ we have $|s| \le 6y^3$; at all other primes $b = 0$. Since $n_{\mathrm{det}} \le 2(k + n_{\mathrm{test}})$ and $k + 1 \le k + n_{\mathrm{test}}$, the triangle contribution satisfies

$$(2q_0 n_{\mathrm{det}} + k + 1)b \le (4t+1)(t|s| + 1)b \le (4t+1)(6ty^3 + 1)v_\ell(c).$$

Substitution in the proof of Corollary 25.1 gives (25.5) with the smaller error

$$E_\ell = t\, v_\ell(O_*) + (8ft + 2 + (4t+1)(6ty^3 + 1))v_\ell(c). \tag{25.18}$$

For the original rank-one main pair the term $8ft\, v_\ell(c)$ may be omitted. Indeed (17.56) gives the whole ambient family the same old inertial type away from ℓ, with nontrivial eigenvalue given by (17.50) or its conjugate inverse. Its order has nontrivial prime-to-ℓ part by Theorem 17.94. Thus each old factor in (14.48) is a unit, and only the auxiliary twist remains in T_{tame}.

Lean. `exists_cyclotomicChar_pow_sub_one_dvd`. Lean obtains full residue order from inertia and bounds the factors separately. Here cyclotomic disjointness gives the residue order, and the finite group order absorbs the whole product.

The separator n_0 uses only the finite set of base-level systems. With u, h fixed, Theorem 13.44 fixes the number m of auxiliary places before their depth. Choose ν large enough that

$$(n_0 c^2)^{m+1} \mid \ell^\nu,$$

and then select the auxiliary places at depth ν. The larger system set, its residual indicator, and the common Ribet polynomial may now be constructed. None is used to choose ν.

In (25.17), the two factors of c come respectively from clearing the type projection and from the integral norm section. The common Ribet bound has already paid for both; adding a further $2v(c)$ afterward would count them twice.

The source-specific algebra is now supplied by Theorems 13.22, 14.11 and 13.31. The prime-triangle construction in Theorem 13.44 fixes its index set before the depth and proves the needed triangular conditions using Theorem 13.41; its detecting-set bound is Lemma 13.42. The old-inertia annihilators and their cost are now supplied by Theorem 14.16 from the stated local data. The quantitative one-place crystalline bounds and their application are Theorem 17.40, applied to the main rows in Corollary 17.88. Integral Hecke membership is Theorem 17.58. The local parahoric type and line Frobenius calculations are §17.14, with the full ambient realizations and compatibility supplied by Theorem 17.105. The local colength comparison is Theorem 15.11; the hypotheses of its quantitative period application remain separate from the finite determinant and coefficient arguments.

For the application in the next subsection, the bound is exactly

$$B = \mathrm{sel} + 2q_0\,\mathrm{ram} + r_0(4fD_\ell) + E_\ell,$$

as proved in Corollary 25.1 from its stated arithmetic inequalities. The next subsection combines the resulting operators (25.17) without any additional loss in an individual eigenvalue.

25.2 Denominator operators for a family of eigenlines

Here we isolate the final linear algebra in the denominator theorem. Fix $\iota : \mathbb{C} \simeq K$, a finite-dimensional complex space Z with basis $b_1, \dots, b_d$, and an ambient Hermitian space with projection P_Z. Let $\mathcal{Y}$ be the set of integral forms satisfying the required orthogonality to false lifts, and let L_ι denote the integral lattice at ι. It is an $\iota^{-1}(\mathcal{O})$-module. Let $\mathcal{C}$ be the module of diagonal operators Λ on Z such that

$$\Lambda P_Z(y) \in L_\iota \qquad (y \in \mathcal{Y}).$$

It is closed under finite sums and multiplication by scalars integral at ι.

Suppose that for each $a = 1, \dots, d$ there is a diagonal $G_a \in \mathcal{C}$, say

$$G_a b_k = g_{a,k} b_k, \qquad v(\iota(g_{a,a})) \le B < \infty.$$

Then a single $\Lambda \in \mathcal{C}$ satisfies

$$v(\iota(\mu_k)) \le B \quad (1 \le k \le d), \qquad |\iota(\det \Lambda)|_\ell \ge \ell^{-dB}, \tag{25.19}$$

where $\Lambda b_k = \mu_k b_k$. There is no extra factor d in the individual eigenvalue bound.

If $d = 0$, take the identity. Otherwise for each k choose an entry $a_k \neq 0$ of smallest valuation among the $\iota(g_{j,k})$. The residue polynomial

$$q_k(t) = \sum_{j=1}^{d} \overline{\iota(g_{j,k})/a_k}\, t^{j-1}$$

has a unit coefficient before reduction and is nonzero. The infinite residue field contains a point outside the roots of $\prod_k q_k$. Lift it to $t \in \mathcal{O}$ and set $\Lambda = \sum_{j=1}^{d} \iota^{-1}(t^{j-1}) G_j \in \mathcal{C}$. Its k-th diagonal entry now has valuation $v(a_k) \le B$. Multiplication gives (25.19). Since Λ is diagonal in the given simultaneous Hecke eigenbasis, it preserves each Hecke eigenspace, including those of dimension greater than one. The formal argument applies simultaneous evaluation to linear forms in d residue

variables. The substitution $(1, t, \dots, t^{d-1})$ leaves only finitely many values of t to exclude. See the integral combination of diagonal entries and its application to the Hecke basis.

There is one further scalar needed for integral pairings. Suppose the integer g_* has the integral-form pairing property that

$$g_*\langle y, z\rangle \text{ is integral at } \iota$$

whenever y is integral at every place above ℓ and $z \in L_\iota$. Then $\widetilde{\Lambda} = g_*\Lambda$ clears both the projected vectors and their pairings, and

$$|\iota(\det \widetilde{\Lambda})|_\ell \geq \ell^{-d(B+v_\ell(g_*))}. \tag{25.20}$$

This follows by multiplying each eigenvalue by g_*. The pairing condition is the stabilizer-denominator assertion for the integral forms; it is not asserted for an arbitrary lattice in a Hermitian space.

To match the notation of the denominator theorem, write r_0 for the rank of the source of this particular lift, and q_0 for its rank difference. Theorem 25.7, once its local and common Ribet inputs are established with the stated bound, supplies the G_a above with

$$B = \mathrm{sel} + 2q_0\,\mathrm{ram} + r_0(4fD_\ell) + E_\ell. \tag{25.21}$$

The coefficient extraction and integrality on the tested projection are proved in Theorem 25.7. The local identities are Theorem 13.31; the common polynomial is Theorem 14.11; and its exact valuation conversion is Corollary 25.1. Their representation hypotheses are supplied by Theorem 17.105 and Corollary 17.106. The critical-value systems at every prime are constructed in Corollaries 18.36 and 18.47. The sharp geometric pairing is Theorem 22.16. For the selected twist, $\mathrm{sel}, \mathrm{ram} \leq a_\ell$, $1 \leq r_0$, and $q_0 \leq t$. Thus

$$B + v_\ell(g_*) \leq r_0(4fD_\ell + (1+2t)a_\ell + E_\ell + v_\ell(g_*)) = r_0K_\ell.$$

Equation (25.20) proves the required determinant bound $\ell^{-dr_0K_\ell}$, while the module property proves that the same operator clears the projection and the required pairings. The subsequent prime summation is the calculation already given in (25.23).

This reconstructs the assembly in `FirstOrder/DenominatorBound`, `exists_operator_ellVal_le_liftValuationBound` and `exists_operator_det_clearingBound`, after the one-eigenline reduction in Theorem 25.7, which reconstructs the coefficient and projection part of `exists_linearMap_eq_smul_mem_integralForms`.

The common denominator bound. Theorem 13.10 chooses the twist for the finite family of Selmer data and special-value systems. Corollary 17.106 bounds the local terms for the ambient eigensystems. The operators preserve the lattice of forms in Proposition 17.104. Theorem 14.11 gives one polynomial for this family, and Corollary 25.1 converts its valuation into the denominator bound. The two estimates for the main and rank-one rows are Theorems 17.87 and 17.94.

25.3 Denominators of the lift projections

For each $\iota : \mathbb{C} \simeq \overline{\mathbb{Q}}_\ell$, the twisting input supplies one twist θ_ι of modulus prime to ℓ. Apply it once to the finite union of the main summands for all lift indices and good eigensystems on W_m, together with one rank-one summand for every main pair and all the ramified critical-value systems of (13.58). Use the common specialization error $2^{-\ell}$. Every required power $\vartheta = \theta_\iota^j$, $1 \leq j \leq m$, then has inverse ramified ratios of family height at most C_{crit}. The Selmer bound retains the allowance

$$a_\ell = (4f+3)r\,v_\ell(g_*) + 2^{-\ell}, \qquad g_* = \prod_{w \text{ chosen}} |\mathrm{GL}_{rt}(\kappa_w)|, \quad t = 5r.$$

The good-system representations control projection onto $Z_j = \mathrm{im}(\mathrm{goodLift}(\vartheta, j))$; the rank-one representations control the corresponding lifts from $U(1)$. Simultaneous twist.

Let K_ℓ be the clearing bound obtained from (25.18), given explicitly below. For Z_j, the denominator input gives $A_\iota \in \operatorname{End} Z_j$ with

$$|\iota(\det A_\iota)|_\ell \geq \exp\bigl(-(\dim Z_j) r K_\ell \log \ell\bigr).$$

For every integral y orthogonal to the false lifts at every split place off the level, $A_\iota \pi_{Z_j} y$ is integral at ι and pairs integrally with forms integral at all embeddings above ℓ. Theorem 24.8 verifies the required finite types, transported eigensystems and full Hecke polynomials on this range. Corollary 24.9 verifies the constant line's original $X - 1$ label and its full polynomial. This is a denominator estimate on the projection; an arbitrary abstract subspace of the same dimension would not suffice. Denominator assertion; Lift labels.

The class-number term in the line estimate. For $Z = \mathbb{C}\phi$, the rank-one denominator bound is $K_\ell \log \ell$. Both test forms Ξ, Ξ' are integral at every embedding above ℓ. At each split place off the level, put $F_w = \operatorname{im} e_{\mathrm{line}} \cap \ker H_w$, where H_w is the Hecke polynomial at the prescribed lift root. The false-lift condition gives $\Xi' \perp F_w \cap \ker \mathcal{D}$. Corollary 24.3 with $\beta_\ell = K_\ell \log \ell$ now gives

$$|\iota\langle P_Z \Xi, P_Z \Xi' \rangle|_\ell \leq \exp\bigl((K_\ell + v_\ell(h_1)) \log \ell\bigr).$$

Here h_1 is the number of $U(1)$ classes. The bound includes the primes dividing h_1. The projected line pairing.

Sum of the local denominator bounds. We sum the two local denominator estimates over the rational primes. Theorem 13.10 provides the twist. Theorems 13.31 and 14.11 give the local operator and the common polynomial. Corollary 25.1, Theorem 25.7 and (25.20) give the coefficient bound, clear the projection and pair the operators.

Put $L = \log b$, $h = r^8$, $t = 5r$, and write $\mathcal{B} = faL^3/(mk)$. The numerator is `errorUnit` in the formal estimates. In the following formulas b is the integral weight itself. Set

$$\mathcal{P} = \prod_{b - 30h^3 < j \leq b + 30h^3,\ j \geq 1} j, \qquad N_{\mathrm{loss}} = (2b)!\, \mathcal{P}^{h^3}.$$

The local terms in the denominator theorem are

$$\begin{aligned} D_\ell &= 10^6 \frac{m}{k} \bigl(b\, v_\ell(M) + v_\ell(N_{\mathrm{loss}})\bigr), \\ a_\ell &= (4f + 3) r\, v_\ell(g_*) + 2^{-\ell}, \\ E_\ell &= t\, v_\ell(O_*) + \bigl(8ft + 2 + (4t + 1)(6th^3 + 1)\bigr) v_\ell(g_*), \\ K_\ell &= 4f D_\ell + (1 + 2t) a_\ell + E_\ell + v_\ell(g_*). \end{aligned}$$

For $T = N_{\mathrm{cyc}} + t + 3$, with N_{cyc} the cyclotomic conductor and h_E the class number of E, Theorem 14.5 gives $O_* = \mathcal{X} \leq h_E T^{24T^5}$. Here D_ℓ is `defectBound`. The Lean definitions `denominatorError` and `clearingBound` use the coefficient in (25.6) and the odd-class integer in `oddClassBound`. Definitions and local-loss sums.

For every finite set F of rational primes and positive integer n, $\sum_{\ell \in F} v_\ell(n) \log \ell \leq \log n$. Since $\sum_{\ell \geq 1} \ell 2^{-\ell} = 2$, it follows that

$$\sum_{\ell \in F} a_\ell \log \ell \leq (4f + 3) r \log g_* + 2. \tag{25.22}$$

The bound $|\mathrm{GL}_{rt}(\mathbb{F}_q)| \le q^{(rt)^2}$ and $\prod_w q_w = M^f$ give $\log g_* \le f(rt)^2 \log M$. The rank-one estimate in the proof of (11.2) gives $h_E \le (\mathrm{rd}_E\, \zeta(2))^{2f}$. Each character of $\mathrm{Gal}(E/\mathbb{Q})$ has conductor dividing N_{cyc}, so the conductor–discriminant formula gives $|D_E| \le N_{\mathrm{cyc}}^{2f}$. Hence

$$h_E \le (N_{\mathrm{cyc}}\zeta(2))^{2f}, \qquad h_1 \le 2^f h_E M^{2f}.$$

For the second inequality, a norm-one idele in the principal ideal class has a generator α with $\alpha\bar{\alpha}$ a real unit. Modulo the maximal compact and rational norm-one elements, its class is determined by that unit modulo unit norms. All real-unit squares are norms, so the kernel has order at most 2^f. The index contributed by the principal level is at most $|\mathcal{O}_E/M\mathcal{O}_E| = M^{2f}$.

The admissible-record inequalities used in Theorem 19.14 give

$$\begin{gathered} x \ge 10^4, \qquad L \ge 40000, \qquad L \le x^{1/100}, \qquad x_1 \ge x^{10^6}, \qquad b \ge x^{10^{12}}, \\ f, N_{\mathrm{cyc}} \le x^{13}, \quad r, t \le x^6, \quad m \le 2\sqrt{x}, \quad k \ge x^5/2, \quad \log M \le 22L. \end{gathered}$$

Here $k = r + 1 - m$ and $r \ge x^5$, which prove the stated lower bound for k. The lower bound for L follows from $\log L \ge 10^{12} C_{\mathrm{par}}$ and $C_{\mathrm{par}} \ge 64$.

These inequalities give $\log g_* \le 22x^{37+1/100} \le x^{38}$ and $\log h_E, \log h_1 \le x^{30}$. Since $T \le 3x^{13}$ and $\log(3x^{13}) \le x$, we have $\log O_* \le x^{30} + 24(3x^{13})^5 x \le x^{67}$. Thus (25.22) is at most $7x^{57} + 2 \le x^{58} \le x_1$. Since $h \le x^{48}$, the coefficient of $v_\ell(g_*)$ in E_ℓ is at most $36x^{156}$. The terms of $K_\ell + v_\ell(h_1)$ other than $4fD_\ell$, summed with weight $\log \ell$, contribute at most

$$3x^6 x^{58} + x^6 x^{67} + (36x^{156} + 1)x^{38} + x^{30} \le x^{195}.$$

For the remaining term, $b \ge 120h^6$ and $\mathcal{P}$ has at most $60h^3$ factors, each at most $2b$. Thus $\log N_{\mathrm{loss}} \le (2b + 60h^6)\log(2b) \le 5bL$ and

$$\sum_{\ell \in F} 4fD_\ell \log \ell \le 4 \cdot 10^6 f(m/k)(b \log M + 5bL) \le 108 \cdot 10^6 f(m/k) bL.$$

Dividing by $\mathcal{B} = fxbL^3/(mk)$ gives at most $108 \cdot 10^6 m^2/(xL^2) \le 432 \cdot 10^6/L^2 \le 1/2$. Also $k \le r \le x^6$ and $m \le 2\sqrt{x}$ give $\mathcal{B}/2 \ge b/(4x^{11/2}) \ge x^{195}$. All estimates hold for every admissible record and every finite F, uniformly in the lift index. Therefore

$$\beta_\ell = (K_\ell + v_\ell(h_1)) \log \ell, \qquad \sum_{\ell \in F} \beta_\ell \le \mathcal{B}, \qquad \sum_{\ell \in F} a_\ell \log \ell \le x_1. \tag{25.23}$$

At the scales of Proposition 5.2, the two estimates above give the sharper bound

$$\sum_{\ell \in F} \beta_\ell \le 108 \cdot 10^6 f(m/k) bL + x^{195} \ll fmbL/k.$$

Here $k \le x^6$ and b dominates every fixed power of x. The same simultaneous twist supplied before the summation thus gives the regular family used below. No further twist is chosen when changing j or passing to the line pairing. Assembly of the regular family.

25.4 Heights and the same twist at two embeddings

Use the height bound for a family of local scalars defined in Section 7. Multiplying the single global multipliers adds their height bounds. For a fixed $z \ne 0$, (7.3) gives

$$h((z)_\iota) \le B, \quad |\sigma z| \le e^c \text{ for all } \sigma \quad \Longrightarrow \quad 0 \le B + c. \tag{25.24}$$

Product formula.

The completion forms and their common factors γ_ξ are those of Theorem 24.43. Auxiliary forms and scalar bound. At a finite embedding fix $\xi = \theta_\iota$ in both arguments of the pairing. It is allowed at $\widehat{\iota}$ since its modulus is prime to ℓ. Thus (7.8) and (11.20) give

$$\log |\iota(|\gamma_\xi|^{2n'}\Delta_\xi)|_\ell \le (2e_\ell - 2nh_\ell)\log \ell. \tag{25.25}$$

Proposition 7.3 sums these signed terms in the product formula. Integral pairings and their determinants.

25.5 The determinant calculation for one lift

For the height of a single lift set

$$L = \log b, \qquad \mathcal{A} = fmbL^2, \qquad \mathcal{B} = \frac{faL^3}{mk}, \qquad \mathcal{D} = \sum_\tau \bigl(s_\infty(\kappa(\tau)) - s_{\mathrm{fin}}(\kappa(\tau),\tau)\bigr).$$

The last sum is directionAmount in Lean. Fix $1 \le j \le m$. Let $u_\vartheta : S_j \longrightarrow V_{j,\vartheta}$ be the actual good single lift, with $d_j = \dim S_j$, $Z_j(\vartheta) = \det(u_\vartheta^* u_\vartheta)$, and P_ϑ the orthogonal projection onto its image. The equality $d_j = d_0$ is proved for this inner form in (10.1). We keep d_j throughout the following calculation and substitute that equality at the end.

The comparison operator. The specified gluing map G_ϑ, hook number h_j, and stabilizer multiple g_* define

$$T_\vartheta = g_* h_j^2 G_\vartheta G_\vartheta^*.$$

Corollary 21.28 proves the following properties of these maps. There is a positive definite endomorphism U_0 of S_j such that

$$u_\vartheta^* T_\vartheta u_\vartheta = \rho_\vartheta U_0, \qquad \rho_\vartheta = \frac{\mathrm{heckeValueProd}(j,\vartheta)}{\mathrm{heckeValueProd}(j,1)} \neq 0. \tag{25.26}$$

The operator T_ϑ preserves integral forms away from the modulus of ϑ and sends the lift image to forms orthogonal to the false lifts. At the trivial twist it satisfies

$$\|T_1 z\|_\sigma \le e^{3\mathcal{A}}\|z\|_\sigma, \qquad \|P_1 T_1 z\|_\sigma \le \|T_1 z\|_\sigma \quad (z \in \operatorname{im} u_1) \tag{25.27}$$

for every automorphism σ of $\mathbb{C}$. Here $\|\cdot\|_\sigma$ means the norm after applying σ to the algebraic value coordinates. The second inequality is a statement about the tested forms and their Hecke conditions. It does not follow by commuting an arbitrary σ with the orthogonal projection.

On the constructed scales the logarithms in (21.43) and (21.52) are at most $264fmbL + O(b + x^C)$ for a fixed C, and $8052fmbL + 7x_1$, respectively. Both are $O(fmbL)$ uniformly in the step. The norm proof of Theorem 21.26 uses the former as the logarithmic coefficient cost and the latter as the first-factor cost. Also $\log(g_* h_j^2)$ is bounded by a fixed power of x. Thus $\log \|T_1\|_\sigma \le C_{Q,d}\, fmbL$ for some $C_{Q,d} > 0$ independent of the step. By (25.27), $\|P_1 T_1 z\|_\sigma \le \exp(C_{Q,d} fmbL)\|z\|_\sigma$ for $z \in \operatorname{im} u_1$.

The explicit U_0 and the ratio $|c_1|^2/|c_\vartheta|^2$ are given in (21.58), and the false-lift orthogonality is Corollary 21.41. Actual comparison maps; Integrality and size; Tested projection.

Compression and the exact determinant quotient. The positive definite pullback makes u_ϑ injective. Write $\Delta_\vartheta = \det(P_\vartheta T_\vartheta|_{\operatorname{im} u_\vartheta})$. In the basis transported by u_ϑ the compressed operator is $(u_\vartheta^* u_\vartheta)^{-1}(u_\vartheta^* T_\vartheta u_\vartheta)$. Lemma 7.1 therefore gives

$$Z_j(\vartheta)\Delta_\vartheta = (\det U_0)\rho_\vartheta^{d_j}. \tag{25.28}$$

At the trivial twist $\rho_1 = 1$. Dividing the two identities cancels the common determinant, giving

$$\frac{Z_j(1)}{Z_j(\vartheta)} = \frac{\Delta_\vartheta}{\Delta_1}\rho_\vartheta^{-d_j}. \tag{25.29}$$

The two bounds on the compressed determinant. At $\iota : \mathbb{C} \simeq \overline{\mathbb{Q}}_\ell$, put $\vartheta = \theta_\iota^j$, and let the operator Λ_ι from the denominator theorem clear P_ϑ, with

$$|\iota(\det \Lambda_\iota)|_\ell \geq e^{-d_j r \beta_\ell}.$$

Set $A_\iota = \Lambda_\iota P_\vartheta T_\vartheta|_{\operatorname{im} u_\vartheta}$. The comparison operator is integral and maps lifts to vectors orthogonal to the false lifts. Thus A_ι preserves the integral lift vectors. By Lemma 24.40 their unit ball is the intersection of the lift space with the integral coordinate ball on the finite class set, using representatives integral above ℓ. Any eigenvector can be divided by a coordinate of largest ℓ-adic norm and then has norm one. Its image is integral, so its eigenvalue has norm at most one. At infinity (25.27) gives $\|P_1 T_1 z\|_\sigma \leq e^{3\mathcal{A}}\|z\|_\sigma$ on lift vectors. Applying this at an eigenvector bounds each conjugated eigenvalue by $e^{3\mathcal{A}}$. Multiplication with algebraic multiplicities gives

$$|\iota(\Delta_{\theta_\iota^j})|_\ell \leq |\iota(\det \Lambda_\iota)|_\ell^{-1} \leq e^{d_j r \beta_\ell}, \qquad |\sigma\Delta_1| \leq e^{3 d_j \mathcal{A}}.$$

Taking the single multiplier Δ_1 in the height definition and using (25.23), we conclude

$$h\left(\left(\frac{\Delta_{\theta_\iota^j}}{\Delta_1}\right)_\iota\right) \leq d_j(3\mathcal{A} + r\mathcal{B}). \tag{25.30}$$

On the constructed scales $\log|\sigma\Delta_1| \leq C_{Q,d}\, d_j f m b L$, and $\sum_\ell \beta_\ell \ll_{Q,d} f m b L/k$. With the same multiplier Δ_1, these estimates and $r \leq 2k$ give, for some $C'_{Q,d} > 0$,

$$h((\Delta_{\theta_\iota^j}/\Delta_1)_\iota) \leq C'_{Q,d}\, d_j f m b L.$$

This proves the determinant and denominator part of the compressed-operator estimates.

The rank-one height bound. Fix a main pair $p = (\nu, z)$, write ϕ_ϑ for its normalized rank-one lift and P for projection onto its line. In Theorem 22.16 take the fixed auxiliary pair $q_2 = (\chi_M \upsilon^{-2}, 2)$. The character χ_M has weight zero and is supported at the chosen places, while υ^{-2} has weight -2 and is trivial on the old residue units. Lemma 22.8 with $n = 4 \leq t$ gives an admissible empty row, ramified at the chosen places. By (22.20) and (18.51), its positive test ratio is

$$r_\vartheta = \frac{A_{q_2}(\vartheta)}{A_{q_2}(1)} = R_{q_2,+1}(\vartheta).$$

Indeed the adjusted norm divides by $(\operatorname{Nm} Q_\vartheta)^3$; the other rank-four factors are independent of ϑ and cancel. This is the choice in the formal main-pair estimate. Set $J_\infty = \sum_\tau s_\infty(\kappa(\tau))$ and $S_{\mathrm{fin}} = \sum_\tau s_{\mathrm{fin}}(\kappa(\tau), \tau)$, so that $\mathcal{D} = J_\infty - S_{\mathrm{fin}}$.

Fix ι above ℓ and take $\vartheta = \theta_\iota^j$. Apply Theorem 22.16 at ι and at $\widehat{\iota}(z) = \iota(\bar{z})$ with this same twist. Denote the two test forms by Ξ, Ξ'. Multiply their inequalities (22.41) and divide by $|\iota(\|\phi_\vartheta\|^2)|_\ell$. Since r_ϑ is real and $\langle P\Xi, P\Xi'\rangle = \langle \Xi, \phi_\vartheta\rangle\langle\phi_\vartheta, \Xi'\rangle/\|\phi_\vartheta\|^2$, this gives

$$\left|\iota\left(\frac{c_*\bar{c}_* r_\vartheta^2}{\|\phi_\vartheta\|^2}\right)\right|_\ell \le \ell^{2S_\ell}|\iota\langle P\Xi, P\Xi'\rangle|_\ell \le \ell^{2S_\ell} e^{\beta_\ell}.$$

The last inequality is Corollary 24.3 with the class-number term included in (25.23). Both test forms satisfy its integrality and false-lift conditions. The multiplier $c_*\bar{c}_*$ is independent of ℓ and ι, and its conjugates have modulus at most $e^{2\mathcal{B}}$ by (22.40). Summing with (22.42) and (25.23) yields

$$h((r_\vartheta^2/\|\phi_\vartheta\|^2)_\iota) \le 3\mathcal{B} + S_{\mathrm{fin}}.$$

The rank-one norm identity gives $\mathrm{ratio}_p(\vartheta)^{-1} = (\mathrm{Nm}\, Q_\vartheta)^{2z-1}\|\phi_1\|^2/\|\phi_\vartheta\|^2$. The norm of the modulus is prime to ℓ. The tame power of q_2 is 1, so (13.58) bounds the height of $(r_\vartheta^{-2})_\iota$ by $2C_{\mathrm{crit}}$, and Corollary 22.9 bounds the height of $\|\phi_1\|^2$ by $\mathcal{B} - J_\infty$. Multiplying the three height witnesses gives

$$h\left(\left(\mathrm{ratio}_p(\theta_\iota^j)^{-1}\right)_\iota\right) \le 4\mathcal{B} + 2C_{\mathrm{crit}} - \mathcal{D}. \tag{25.31}$$

For the constructed scales retain the individual costs in the same calculation. The sum of the β_ℓ is $O(fmbL/k)$. Equation (22.40) bounds $\log|\sigma c_*|$ by $fmb/k + 3x^{34}$; the proof of Corollary 22.9 bounds the norm error by $x^{33} + 8f$. These costs and $2C_{\mathrm{crit}} < 2$ are $O(fmbL/k)$ on the scales of Proposition 5.2. Consequently the left side of (25.31) is at most $-\mathcal{D} + O(fmbL/k)$.

The formal proof estimates the projected pairing and the reciprocal product of the two test values separately. The line projection above combines their finite-place bounds before taking heights.

Lean. `exists_isTestForm_le_norm_inner_trivialLift`, `heightLe_inv_ratio`.

Counting the critical-value factors. Here is the exact counting that turns these inputs into the Hecke product bound. In the three exchange products, the main character has points

$$1 \le z \le z_* = \frac{7m}{2} + \frac{k}{2},$$

and the two auxiliary characters have points up to m and $m/2$. Since $k = r + 1 - m$ and $6m + 1 \le r$ on the constructed scales, $z_* = (r + 6m + 1)/2 \le r < t$. The interval $7m/2 < z \le z_*$ contains $k/2$ main pairs; all the other $7m/2 + m + m/2 = 5m$ factors are ramified pairs. Both m and k are even. Cancelling the untwisted factors in the definition of heckeValueProd therefore gives

$$\rho_\vartheta = (\mathrm{Nm}\, Q_\vartheta)^e \left(\prod_{k/2 \text{ main pairs}} \mathrm{ratio}_p(\vartheta) \prod_{5m \text{ ramified pairs}} \mathrm{ratio}_q(\vartheta, s_q)\right)^2$$

for a fixed integer e. The main-pair weight inequalities are those of the prescribed exchange character; these intervals are independent of ϑ. At the selected embedding, every integer power of $\mathrm{Nm}\, Q_{\theta_\iota}$ has norm one because that norm is prime to ℓ. It costs zero in the height bound. The $5m$ ramified factors have the old tame powers $j, j, -j$ in (21.75), and each costs at most C_{crit} by (13.58). Apply (25.31) to the $k/2$ main factors and square the product to obtain

$$h((\rho_{\theta_\iota^j}^{-1})_\iota) \le k(4\mathcal{B} - \mathcal{D}) + (2k + 10m)C_{\mathrm{crit}}. \tag{25.32}$$

The three products and their index sets.

The height bound for one lift. Combining (25.29), (25.30) and (25.32) yields

$$h\left(\left(\frac{Z_j(1)}{Z_j(\theta_\iota^j)}\right)_\iota\right) \le d_j\big(3\mathcal{A} + (r+4k)\mathcal{B} + (2k+10m)C_{\mathrm{crit}} - k\mathcal{D}\big).$$

On the constructed scales each of the $k/2$ main pairs instead costs $-\mathcal{D} + O(fmbL/k)$, before the outer square, and $10mC_{\mathrm{crit}} \le 10m$. The compressed determinant costs $O(d_j fmbL)$. Thus

$$h\left(\left(\frac{Z_j(1)}{Z_j(\theta_\iota^j)}\right)_\iota\right) \le d_j\big(-k\mathcal{D} + O(fmbL)\big). \tag{25.33}$$

Here $d_j = D$ by (10.1), and $mk\mathcal{D} = \mathcal{A}$ in the notation of Section 7. Since $fm^2bL \le 4faL$, this is (7.4). For each finite embedding ι, Theorem 21.7 uses the powers θ_ι^j, $1 \le j \le m$, of one chosen twist and contributes the scalar bounded in (7.5). Section 7 applies the product formula to the completed untwisted wedge, with both finite losses computed for this twist. Theorem 7.4 then yields the prime-sum inequality.

Lean. `heightLe_chainDeterminant_one_div`, `star_inequality`.

A Classical theorems and the formal sources

Parts I and II give the deduction of Theorem 1.1. Parts III–VI prove the estimates used there. These sections use the notation local to each calculation. In the determinant calculation, for example, R, A, and d_0 denote respectively the signed remainder S, the leading term $\mathcal{A}$, and the dimension D of Section 7. The scalar used in both Gram determinants is fixed before the basis is completed.

At rank r, the two primitive finite types give a cuspidal base change by Harris–Labesse [HL04, Theorem 2.2.2]. Section 10 applies the multiplicity formula for this simple generic parameter on pure inner forms [KMSW14, Theorem 1.7.1]. The formal trace comparison for the same levels and weights is `trace_comparison`. For the transfer of spectral data to Galois representations, Shin's construction gives classical background [Shi11]. The proof here uses the attachment for the specified forms in Section 17, including local conditions at the coefficient prime. The corresponding formal result is `Isotypic.attached_galRep`.

The period calculation in Section 19 uses the Chowla–Selberg formula at a specified root, formalized in `CMLattice.chowla_selberg`. Kings and Sprang prove integrality of regularized critical Hecke values after division by their periods. At ordinary primes they construct a p-adic measure interpolating these values [KS25]. Section 18 proves the required normalization and endpoint estimate for the character families used here at all coefficient primes. Two identities from Siegel–Weil theory have different uses. The weighted class sum in `classSum_card_norm_one_eq_mass` fixes the mass in the definite norm. The reproduction identity in `reproducing_formula_modKernel` expresses convolution with a holomorphic coefficient through values of theta lifts to a definite group. The normalization and lattice estimates for that passage are supplied in Section 21. At the primes of the coefficients, the comparison with the integral period lattice uses `mem_kisinModule_iff_crysForms`. The global reciprocity law for a principal idele enters the construction of the auxiliary characters in Section 13. The principal idele of a totally positive element has trivial Artin image by `artinMap_principalIdele_eq_one_of_isTotallyPositive`.

The interval and determinant formulations. In the earlier account, an upper estimate for a character sum on arbitrarily large intervals is converted into zero exclusion by separating poles of its transform. This belongs to the line of work of Landau and Turán on oscillations of prime-counting remainders [Wid41, Tur50]. In Parts I and II the estimate holds at every sufficiently large cutoff. The quadratic twist is what permits the shorter residue argument. The shared numerical feature is the choice $m \asymp \sqrt{x}$. In the interval calculation the competing errors have sizes x/m and m. In (7.11) the gain is the explicit factor $1/m$ after the common leading term of the two determinants has cancelled.

In Section 14, the refined Coates–Sinnott theorem computes the parity component of a Fitting ideal [GP15]. Ferrero–Washington verifies its Iwasawa $\mu = 0$ hypothesis for the abelian fields used here [FW79]. At 2 we use the classical Stickelberger ideal [Was97, Theorem 6.10]. The formal proof uses the prime-field valuation of Gauss sums [Bla05] and Kolyvagin derivatives of Jacobi sums at large primes, as in Rubin's construction for Gauss sums [Rub91].

References to the Lean proof. Each numbered theorem, lemma, proposition and corollary with a proof has a Lean citation at the end of that proof. A citation marked *formal context* gives the source passage for the surrounding argument or the formal conclusion in which the calculation is used. Several intermediate lemmas in the prose break up a single formal argument and have no separately named Lean counterpart. Each link points to the cited lines in a fixed revision of the source. The readable Lean project [Lean26] contains the background results in

`grh/src/Lib/` and the proof in `grh/src/New/`. Its main statements and their proofs are collected in `grh/src/Statements.lean` and `grh/src/Main.lean`. Sutherland reports the kernel and axiom checks for this revision [Sut26].

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