hexagon:2611.01262v1

THE ZILBER–PINK CONJECTURE FOR CURVES IN $Y(1)^3$

MANVIR JASWAL

Abstract. We prove the Zilber–Pink conjecture for curves in $Y(1)^3$. By known results it reduces to a conjecture of Daw, Orr and Papas, that an irreducible curve over $\overline{\mathbb{Q}}$ in $Y(1)^3$, not contained in a proper special subvariety, has only finitely many points whose coordinates are j-invariants of pairwise isogenous elliptic curves. We prove this conjecture by the G-function method, which bounds the height of every such point polynomially in its degree. The obstacle is the places of supersingular reduction, where the period equations involve a trace that changes with the place. Pairing each period equation with its Frobenius conjugate, as in OpenAI's preprint on curves in $\mathcal{A}_2$, eliminates the place-dependent trace.

Contents

1. Introduction 1
2. From the height bound to finiteness 5
3. The local equations 6
4. Nonvanishing of the systems 10
5. Proof of Theorem 1.3 13
Appendix A. Frame estimates 16
Declarations 18
References 18

1. Introduction

We identify $Y(1)$ with the affine line through the j-invariant, so that a point of $Y(1)^3 = \mathbb{A}^3$ is a triple of j-invariants of elliptic curves over $\mathbb{C}$. For $N \geq 1$ let $\Phi_N \in \mathbb{Z}[X, Y]$ be the modular polynomial of level N, so that $\Phi_N(j(E), j(E')) = 0$ if and only if there is an isogeny $E \to E'$ with cyclic kernel of order N. Special subvarieties are as in Habegger and Pila [9, §2.1]. Up to a permutation of the coordinates, a special subvariety is the image under j of a product of a CM point in some of the coordinates with, for each further block of coordinates, the image of $\mathbb{H}$ under $z \mapsto (g_k z)_k$ for elements $g_k \in \mathrm{GL}_2^+(\mathbb{Q})$. So a special curve in $Y(1)^3$ has one of three shapes. Either two coordinates are constant and CM, or one coordinate is constant and CM and the other two are joined by a modular polynomial, or the three coordinates are pairwise joined by modular polynomials. The curves of the third shape are the strongly special ones.

Let $C \subset Y(1)^3$ be an irreducible algebraic curve over $\mathbb{C}$ that is not contained in a proper special subvariety. The Zilber–Pink conjecture, in the form Habegger and Pila state for

Date: November 27, 2026.
2020 *Mathematics Subject Classification.* Primary 11G18; Secondary 14G35, 11G50, 11J91, 03C64.
Key words and phrases. Zilber–Pink conjecture, unlikely intersections, modular curves, G-functions, supersingular reduction.

curves in $Y(1)^n$ [9, §1], predicts that C meets the union of the special subvarieties of codimension at least two in only finitely many points. Pila proved this when C is not defined over $\overline{\mathbb{Q}}$ [16, Thm. 1.4]. For C defined over $\overline{\mathbb{Q}}$, Habegger and Pila proved finiteness for the points of C on special curves of the first two shapes [9, Thm. 2]. Daw, Orr and Papas observed that what remains is the following statement, which they conjectured [5, Conj. 1.2]. We prove it.

Theorem 1.1. *Let $C \subset Y(1)^3$ be an irreducible algebraic curve defined over $\overline{\mathbb{Q}}$ and not contained in a proper special subvariety. Then C contains only finitely many points (x_1, x_2, x_3) for which there are integers $M, N \geq 1$ with*

$$\Phi_M(x_1, x_2) = \Phi_N(x_2, x_3) = 0.$$

The condition says that x_1, x_2, x_3 are the j-invariants of pairwise isogenous elliptic curves. A point of C on a strongly special curve satisfies the condition, so with the two theorems just cited we obtain the following.

Corollary 1.2 (Zilber–Pink for curves in $Y(1)^3$)**.** *Let $C \subset Y(1)^3$ be an irreducible algebraic curve over $\mathbb{C}$ that is not contained in a proper special subvariety. Then C meets the union of the special subvarieties of $Y(1)^3$ of codimension at least two in only finitely many points.*

Corollary 1.2 was known over $\overline{\mathbb{Q}}$ for asymmetric curves, whose three coordinate functions do not all have the same degree [9, Thm. 1], for lines [4, Thm. 1.4], and under conditions on the closure $\overline{C}$ of C in $(\mathbb{P}^1)^3$. Daw and Orr treated the case where $\overline{C}$ contains (∞, ∞, ∞) [4, Thm. 1.3]. Papas treated the case where it contains, up to a permutation, a point $(\infty, \zeta_1, \zeta_2)$ or $(\infty, \infty, \zeta_1)$ with ζ_1, ζ_2 singular moduli [15, Thms. 1.4 and 1.5], and Daw, Orr and Papas the case where it contains a point with exactly one coordinate ∞ and the other two joined by a modular polynomial [5, Thm. 1.3]. Theorem 1.1 has no condition at the boundary.

All these proofs follow the strategy of Pila and Zannier [19], and so does ours. In our proof a bound for the height of the points, polynomial in their degree, gives large Galois orbits through an isogeny estimate, and, with an Ax-type statement for j [9, Prop. 3.1], these contradict the counting theorem of Pila and Wilkie [18] on a definable set. The height bound is where the difficulty lies. Following Daw, Orr and Papas, we call a point of $Y(1)^3$ *modular* if its coordinates are j-invariants of pairwise isogenous elliptic curves without CM. For a point $s = (x_1, x_2, x_3)$ with algebraic coordinates, $h(s) = \max_i h(x_i)$, where h is the absolute logarithmic Weil height.

Theorem 1.3. *Let K be a number field and $C \subset Y(1)^3$ an irreducible curve over $\overline{\mathbb{Q}}$, defined over K and not contained in a proper special subvariety, with a modular point $s_0 \in C(K)$. There are $c, \kappa > 0$, depending only on C, K and s_0, such that every modular point $s \in C$ satisfies*

$$h(s) \leq c\,[K(s) : K]^{\kappa}.$$

Write $s_0 = (b_1, b_2, b_3)$ and, for a modular point s, let $\Delta(s) = \max(M, N)$, where M and N are the degrees of the cyclic isogenies $E \to E'$ and $E' \to E''$ between elliptic curves E, E', E'' with j-invariants x_1, x_2, x_3; these curves have no CM, so the cyclic isogenies are unique up to sign. Daw, Orr and Papas proved $h(s) \leq c_3[\mathbb{Q}(s) : \mathbb{Q}]^{c_4}\Delta(s)^{c_5\delta}$ for the modular points s at which at most $\Delta(s)^{\delta}$ places v of $\mathbb{Q}(s_0, s)$ of supersingular reduction have $|x_i - b_i|_v < 1$ for $i = 1, 2, 3$ [5, Thm. 1.8], and finiteness of these points for some $\delta > 0$ [5, Thm. 1.6]. At such a place the period relations behind their bound have on the right a reduced trace that changes with the place, and this is why these places had to be counted; Daw and Papas

meet them in $\mathcal{A}_3$ and bound their number assuming a Lang–Trotter conjecture for pairs of elliptic curves [6, Thm. 1.1 and Conj. 5.1].

At a supersingular place we subtract from each relation its conjugate under a Frobenius lift there, as in OpenAI's preprint on curves in $\mathcal{A}_2$ [13]. The trace cancels, and the new equations depend on the place only through the restriction of the lift to a field of degree $O([K(s):K])$. So the places fall into $O([K(s):K])$ groups with one system of equations each. Weight each place by how close s is to s_0 there: some group carries at least the average weight of a group, a fraction $1/O([K(s):K])$ of the total, and the interpolation theorem of that preprint needs only that group.

Theorem 1.3 covers every modular point, with no factor in $\Delta(s)$. The hypothesis on s_0 is no restriction: if C has a modular point, we fix one as s_0 and enlarge K so that it is defined over K. Like the bound of Daw, Orr and Papas, Theorem 1.3 has no condition at the boundary of C, because the periods are expanded around the base point s_0, an interior point, as in [5, §1]. A coordinate may be constant on C, and then Theorem 1.3 needs only an isogeny estimate (Section 3.1).

1.1. **The supersingular places.** We recall the set-up of Daw, Orr and Papas. Fix $s_0=(b_1,b_2,b_3)$ as in Theorem 1.3, with all three coordinates nonconstant on C, and cyclic isogenies $f_{02}\colon E_{b_1}\to E_{b_2}$ and $f_{03}\colon E_{b_1}\to E_{b_3}$ of degrees N_{02} and N_{03}. After a fixed extension of K, f_{02} and f_{03} are defined over K. Near s_0 the curve carries elliptic families E_1,E_2,E_3 whose j-invariants are the coordinate functions. Horizontal transport $T_i(a)$ of first de Rham homology from the fiber of E_i at a nearby point to the fiber at s_0 is, in fixed algebraic frames, a matrix of G-functions in the value a of a local parameter on a chosen normalization branch at s_0 [5, Lemma 2.3]. For a point s we use these series at the places v where s is v-adically close to s_0, in the sense of Section 3.2; there $|a|_v<1$. For maps λ,λ' from the first homology of one elliptic curve to that of another put $\langle\lambda,\lambda'\rangle=\operatorname{tr}(\lambda^\vee\lambda')$, where $\lambda^\vee$ is the adjoint for the Weil pairings; for an isogeny it is the dual isogeny. Let s be a modular point, with cyclic isogenies $f_2\colon E_{1,s}\to E_{2,s}$ and $f_3\colon E_{1,s}\to E_{3,s}$ of degrees M_2 and M_3, let $L\supset K(s)$ be a field over which f_2 and f_3 are defined, of degree at most 4 over $K(s)$ since f_2 and f_3 are unique up to sign, and let A_k be the matrix of f_k in the frames at s. The period we use is

$$\langle f_{0k},Y_{1k}(a)A_k\rangle=\operatorname{tr}(f_{0k}^\vee\,T_k(a)\,A_k\,T_1(a)^{-1})\qquad(k=2,3),\tag{1.1}$$

the trace of $f_{0k}^\vee$ composed with f_k transported to the fibers at s_0.

At an archimedean place where s is close to s_0, de Rham transport is Betti transport, and (1.1) is an integer r_k with $|r_k|\le c\sqrt{N_{0k}M_k}$. Here and below $c,c',c_1,c_2,\dots$ denote positive constants that depend only on C, K, s_0 and choices made with them, and c and c' need not be the same at each occurrence. At a place v of good reduction where s is v-adically close to s_0, the curves $E_{i,s}$ and E_{b_i} have the same reduction $\overline{E}_i$, and by the crystalline comparison (1.1) is the reduced trace of $w_k=\bar f_{0k}^\vee\bar f_k\in\operatorname{End}(\overline{E}_1)$ [5, §3.2.1]. So at each of these places the period satisfies an equation whose right side is an integer that depends on the place. André's G-function method [1] turns such v-adic relations into a global one by multiplying them, and needs the product to have degree polynomial in $[K(s):K]$. Expanding around a point where the families degenerate, Daw and Orr avoid the product at the non-archimedean places with one relation valid at all of them [4, §1.B]. At the ordinary places Daw, Orr and Papas use the second isogeny. There w_2 and w_3 lie in an imaginary quadratic field, so they commute, and the commutator of the two transported endomorphisms vanishes, an equation with fixed coefficients [5, §3.3]. At a supersingular place, the endomorphism algebra of $\overline{E}_1$ over the algebraic closure of the residue field is the quaternion algebra $B_{p,\infty}$, and w_2 and

w_3 need not commute. The right side $\mathrm{trd}(w_2)$ is then an integer of absolute value at most $2\sqrt{N_{02}M_2}$ that depends on the place.

We make the device precise for a point s with $a \neq 0$. Let $\mathfrak{P}$ be such a place, over p, of a Galois closure of L over $\mathbb{Q}$, and choose a Frobenius lift σ in the decomposition group of $\mathfrak{P}$. It preserves $|\cdot|_{\mathfrak{P}}$, so it acts continuously on the completion and maps the convergent series (1.1) to the corresponding series for the conjugate data, and it fixes the rational number $\mathrm{trd}(w_k)$. Hence

$$\langle f_{0k}, Y_{1k}(a)A_k\rangle = \langle f_{0k}^{\sigma}, Y_{1k}^{\sigma}(aR)\,\sigma(A_k)\rangle \qquad (k=2,3), \qquad R=\sigma(a)/a, \tag{1.2}$$

where Y_{1k}^{σ} is the transport on the conjugate curve C^{σ} around $\sigma(s_0)$, and $|R|_{\mathfrak{P}} = 1$. We call (1.2) the *paired equations* at $\mathfrak{P}$. Daw, Orr and Papas multiply the local relations into one relation among values of fixed G-functions at the point, which André's Hasse principle bounds. The paired equations do not have this form: they evaluate the transports at a and at aR, where $R = \sigma(a)/a$ changes with the place, and their coordinates include R and the isogeny matrices. So we use the interpolation theorem of [13]. It asks that the system vanish identically on no branch over $a = 0$ of a subvariety through the point on which a varies, and it needs bounds on the sizes of all coordinates, which Appendix A gives. The isogeny degrees enter the bounds on sizes only through $\log M_2$ and $\log M_3$, and the supersingular trace values need not be counted.

The paired system, including the base-coordinate equations, can vanish identically on such a branch only when $(s, \sigma s)$ lies on a Hecke correspondence between C and C^{σ} that matches each coordinate with the same coordinate and contains a curve dominating both. We call such correspondences *exceptional*; there are finitely many. At their places the paired equations are transported back to C along the isogenies of the correspondence, and the resulting equations are not identities. For the transported equations, each of f_2 and f_3 comes with a sign $\varepsilon_k \in \{\pm 1\}$ that compares its transport along the correspondence at s with that of f_{0k} at s_0, and an identity forces $\varepsilon_2 = \varepsilon_3 = +1$. Let $\psi = \bar{\varphi}_1^{-1} F_1 \in \mathrm{End}(\overline{E}_1) \otimes \mathbb{Q}$, with $\bar{\varphi}_1$ the reduction of the first isogeny of the correspondence, of degree m_1 prime to p, and F_1 the relative Frobenius of $\overline{E}_1$. It conjugates w_k to $\varepsilon_k w_k$, so with both signs $+1$ it commutes with w_2 and w_3, which generate $B_{p,\infty}$ where they do not commute; hence $\psi \in \mathbb{Q}$. Then $\mathrm{nrd}\,\psi$ would be a rational square, but it is p/m_1 with $p \nmid m_1$.

The paired equations and the sign argument are in OpenAI's preprint [13], and the paired equations are also in [14]. The finiteness of the exceptional correspondences follows from a theorem of Pila and Tsimerman [17, Thm. 7.1]. What has to be done in $Y(1)^3$ is to write down the systems at each kind of place, to show that none of them vanishes identically on such a branch, and to bound the sizes of the coordinates in the frames of the elliptic families. Tate curves replace the auxiliary level structures of [13] at the places where E_{b_1} is potentially multiplicative.

The proof does not make the constants c and κ of Theorem 1.3 effective: Proposition 4.2 shows that the exceptional correspondences are finitely many but gives no bound for the degrees m_i of their isogenies, and the finite set S of primes of Section 3.2 and the data of the transported systems of Section 3.7 depend on them. The count in Theorem 1.1 uses [10, Cor. 7.2], a Pila–Wilkie-type bound whose constant is not explicit.

Section 2 deduces Theorem 1.1 from Theorem 1.3, and Sections 3, 4 and 5 and Appendix A prove Theorem 1.3.

2. From the height bound to finiteness

Theorem 1.3 and Gaudron and Rémond's estimate [8, Thm. 1.4] give $[\mathbb{Q}(s):\mathbb{Q}] \geq c'\Delta(s)^\delta$ for some $c', \delta > 0$, as in the proof of [4, Prop. 5.15], which [5, §5] follow for their Theorem 5.2. The count follows Habegger and Pila's proof of [9, Prop. 5.1], with this bound in place of [9, Lemma 4.2] and, as in the proof of [10, Thm. 10.1], [10, Cor. 7.2] in place of their block counting. The Ax-type input is [9, Prop. 3.1], which suffices for curves, in place of the weak complex Ax condition of [10].

Let $\mathcal{F} \subset \mathbb{H}$ be the fundamental domain of [9, §2.3], which meets every $\mathrm{SL}_2(\mathbb{Z})$-orbit in exactly one point; the restriction of j to $\mathcal{F}$ is definable in $\mathbb{R}_{\mathrm{an,exp}}$ [9, Thm. 4]. Write $J\colon \mathbb{H}^3 \to Y(1)^3$ for j in each coordinate and identify $\mathbb{C}$ with $\mathbb{R}^2$. For $k \geq 1$ and a set $X \subset \mathbb{R}^m \times \mathbb{R}^{m'}$, $X^\sim(k,T)$ is the set of its points whose first m coordinates are real numbers of degree at most k and k-height at most T, as in [10, §7]. The Ax-type proposition enters through the following lemma.

Lemma 2.1. *Let $C' \subset Y(1)^3$ be an irreducible curve whose coordinates are nonconstant and on which no $\Phi_N(x_i, x_k)$ with $i \neq k$ vanishes identically. Then no nonconstant continuous path in $J^{-1}(C')$ lies in an algebraic subset of $\mathbb{C}^3$ of dimension less than 3.*

Proof. Suppose that $\gamma\colon [0,1] \to J^{-1}(C')$ is such a path and that a polynomial $Q \neq 0$ vanishes on its image. Let Σ be the inverse image under J of the singular points of C' and of its points with a coordinate in $\{0, 1728\}$. It is countable, and $J^{-1}(C') \setminus \Sigma$ is a complex manifold of dimension one with countably many components. On each component Q vanishes identically or has countably many zeros. The image of γ is uncountable, so Q vanishes on a component, and a neighborhood in $J^{-1}(C')$ of a point z of that component lies in the hypersurface $\{Q = 0\}$ of $\mathbb{H}^3$. By [9, Prop. 3.1], C' lies in a geodesic subvariety of positive codimension. Such a subvariety fixes a coordinate or lies in some $\{\Phi_N(x_i, x_k) = 0\}$ [9, §2.1], and both are excluded. □

Proof of Theorem 1.1. If a coordinate of C is constant, C is asymmetric, and Theorem 1.1 for C is [9, Thm. 1]; so we assume that the coordinates of C are nonconstant. Since C lies in no proper special subvariety, no $\Phi_N(x_i, x_k)$ with $i \neq k$ vanishes on C. So the points in question lie in the finite sets $C \cap \{\Phi_M(x_1, x_2) = 0\}$ and are algebraic. An isogeny identifies endomorphism algebras, so at such a point either all coordinates are CM or none is, and the former points are finitely many [9, Lemma 4.1]. If C has no modular point we are done. Otherwise we fix one, s_0, and a number field K over which C and s_0 are defined.

Let $s = (x_1, x_2, x_3) \in C$ be modular, $D = [\mathbb{Q}(s):\mathbb{Q}]$, and E_x the curve of [5, §4.1] with j-invariant x. For curves without CM the homomorphisms $E_{x_1} \to E_{x_2}$ form a group $\mathbb{Z}g$ with g cyclic, and every multiple ng with $|n| > 1$ has noncyclic kernel; so M in Theorem 1.1 is $\deg g$, and likewise N. [8, Thm. 1.4] over $\mathbb{Q}(s)$ gives $M \leq 10^{13} D^2 \max(h_F(E_{x_1}), \log D, 1)^2$, where h_F is the stable Faltings height. Silverman's comparison, as stated in [9, (14)], gives $h_F(E_{x_1}) \leq c_1(1 + h(s))$, and Theorem 1.3 gives $h(s) \leq c[K(s):K]^\kappa \leq cD^\kappa$. With the same bound for N,

$$\Delta(s) \leq c_2 D^{2+2\max(1,\kappa)}, \qquad \text{so} \qquad [K(s):K] \geq \frac{D}{[K:\mathbb{Q}]} \geq c_3 \Delta(s)^\delta, \quad \delta = \frac{1}{2 + 2\max(1,\kappa)}.$$

For the count let $Z = \mathcal{F}^3 \cap J^{-1}(C)$ and

$$X = \{((A,B), z) \in \mathrm{GL}_2^+(\mathbb{R})^2 \times Z : \ z_2 = Az_1, \ z_3 = Bz_2\} \subset \mathbb{R}^8 \times \mathbb{R}^6,$$

where $\mathrm{GL}_2^+(\mathbb{R})$ acts by fractional linear transformations; X is definable in $\mathbb{R}_{\mathrm{an,exp}}$. The conjugates s' of s over K lie on C and are modular with the same M and N, and there are $[K(s):K]$ of them. Each has exactly one lift $z(s') \in \mathcal{F}^3$, and [9, Lemma 5.2], applied to the first two coordinates and to the last two with the same lift of the second, gives $A, B \in M_2(\mathbb{Z})$ with $\det A = M$, $\det B = N$, $z_2 = Az_1$, $z_3 = Bz_2$ and entries at most $c_4\Delta(s)^{10}$ in absolute value. The points $((A,B), z(s'))$ lie in $X^{\sim}(1,T)$, $T = c_4\Delta(s)^{10}$, and have at least $c_3\Delta(s)^{\delta}$ distinct projections to $\mathbb{R}^6$. Let c_X be the constant of [10, Cor. 7.2] for X, $k=1$ and $\varepsilon = \delta/20$. If $\Delta(s)$ is large enough, then $c_3\Delta(s)^{\delta} > c_X\, c_4^{\varepsilon}\Delta(s)^{\delta/2} = c_X T^{\varepsilon}$, and [10, Cor. 7.2] gives a continuous path $t \mapsto ((A(t),B(t)), z(t))$ in X whose matrix part is semialgebraic and whose part $z(t)$ is nonconstant.

The matrix part is a semialgebraic curve, so its Zariski closure $P \subset \mathbb{C}^8$ has dimension at most one. The path $z(t) = (z_1(t), A(t)z_1(t), B(t)A(t)z_1(t))$ lies in the closure of the image of $P \times \mathbb{C}$ under the rational map $(A,B,u) \mapsto (u, Au, BAu)$, which has dimension at most two, and this contradicts Lemma 2.1 for $C' = C$.

Hence Δ is bounded on the modular points of C. So M takes finitely many values, and each $C \cap \{\Phi_M(x_1,x_2) = 0\}$ is finite. □

3. The local equations

This section proves Theorem 1.3 when a coordinate of C is constant. In the other case it proves the equations that a modular point satisfies at the places where it is close to s_0 (Propositions 3.2 and 3.5), excludes the places where E_{b_1} is potentially multiplicative (Lemma 3.3), and sets up the four systems to which Theorem 5.1 is applied (Section 3.7). In Sections 3, 4 and 5, K, C and $s_0 = (b_1, b_2, b_3)$ are as in Theorem 1.3. We replace K by a finite extension whenever convenient; this changes $[K(s):K]$ by a bounded factor.

3.1. **A constant coordinate.**

Proof of Theorem 1.3 when a coordinate of C is constant. The constant value b is not CM, since C lies in no proper special subvariety, and the other coordinates are nonconstant, since two constant coordinates would be isogenous at s_0 and some Φ_m would vanish on C. Let s be a modular point and x one of its other coordinates. Write E_y for the curve of [5, §4.1] with j-invariant y. [8, Thm. 1.4] over $K(s)$ bounds the degree of the cyclic isogeny $E_b \to E_x$ by $c(1+\log[K(s):\mathbb{Q}])^2[K(s):\mathbb{Q}]^2$. Faltings's comparison, as stated in [9, p. 15], gives $h_F(E_x) \le h_F(E_b) + c(1+\log[K(s):\mathbb{Q}])$ for the stable Faltings height, and [9, (14)] turns this into $h(x) \le c(1+\log[K(s):\mathbb{Q}])$. So $h(s) \le c(1+\log[K(s):K])$. □

3.2. **Fixed data.** From now on the three coordinates are nonconstant on C. No $\Phi_N(x_i,x_k)$ with $i \ne k$ vanishes on C, and the same holds for every conjugate C^{σ}, $\sigma \in \mathrm{Hom}(K,\overline{\mathbb{Q}})$.

Let $\nu\colon \widetilde{C} \to C$ be the normalization and $\tilde{s}_0 \in \widetilde{C}(K)$ a point over s_0. Fix an affine open $\widetilde{C}^0 \subset \widetilde{C}$ containing $\tilde{s}_0$ on which the functions $j_i = x_i \circ \nu$ avoid 0 and 1728, closed coordinates $u = (u_1,\dots,u_m)$ embedding $\widetilde{C}^0$ in $\mathbb{A}^m$, and $\theta \in K(\widetilde{C})$ regular on $\widetilde{C}^0$ with a simple zero at $\tilde{s}_0$. Let $E_i \to \widetilde{C}^0$ be the pullback by j_i of the family $Y^2+XY = X^3 - 36X/(j-1728) - 1/(j-1728)$ of [5, §4.1], with frame $\omega_i = dX/(2Y+X)$, $\xi_i = X\omega_i$ of $H^1_{\mathrm{dR}}(E_i/\widetilde{C}^0)$ and the dual frame of H^{dR}_1. In these frames a homomorphism between the first homologies of two fibers is a 2×2 matrix, and the pairing of Section 1.1 is $\langle \lambda, \lambda' \rangle = \mathrm{tr}(\mathrm{adj}(\lambda)\,\lambda')$, since the cup product of ω and ξ is the same for every curve of the family. Near $\tilde{s}_0$, $T_i(a) \in \mathrm{GL}_2(K[[a]])$ is the matrix of horizontal transport from the fiber at the point t with $\theta(t) = a$ to the fiber at $\tilde{s}_0$, so $T_i(0) = 1$, and $g_\nu(a) \in K[[a]]$ is the expansion of u_ν. As in Section 1.1, f_{02} and f_{03} are

generators of $\mathrm{Hom}(E_{b_1}, E_{b_2})$ and $\mathrm{Hom}(E_{b_1}, E_{b_3})$, of degrees N_{02} and N_{03}, and for a modular point s with a point $\tilde{s} \in \widetilde{C}^0$ over it, $a = \theta(\tilde{s})$, f_2 and f_3 are cyclic isogenies $E_{1,s} \to E_{2,s}$ and $E_{1,s} \to E_{3,s}$, of degrees M_2 and M_3, and A_k is the matrix of f_k. Thus (1.1) is $\langle f_{0k}, Y_{1k}(a)A_k\rangle$ with $Y_{1k}(a)A = T_k(a)AT_1(a)^{-1}$. For $\sigma \in \mathrm{Hom}(K, \overline{\mathbb{Q}})$ the same objects for C^σ and $\sigma(\tilde{s}_0)$ carry the superscript σ.

For a number field $F \supset K$ and a place v of F, $|\cdot|_v$ extends the usual absolute value of $\mathbb{Q}$, $n_v = [F_v : \mathbb{Q}_v]/[F : \mathbb{Q}]$, $\mathbb{C}_v$ is the completion of an algebraic closure of F_v, and $\beta_p = \log p/(p-1)$. Let S be a finite set of primes containing 2, 3, the primes below which $\widetilde{C}^0$, θ, the families and the frames do not extend smoothly over a neighborhood of $\tilde{s}_0$ or of its conjugates, the primes of bad reduction of the E_{b_i} and their conjugates, the primes at which some b_i or $b_i - 1728$ is not a unit, the primes dividing $N_{02}N_{03}$, and the primes at which any u_ν or its conjugate fails to extend regularly near the section $\tilde{s}_0$ or its conjugate; it is enlarged in Section 3.6. For v archimedean or above S we fix a closed disk $\Delta_v = \{|\theta|_v \le r_v\}$ about $\tilde{s}_0$ in $\widetilde{C}^0(\mathbb{C}_v)$ on which θ is injective, with $0 < r_v < 1$ and $r_v < \varrho/2$, where $\varrho > 0$ is fixed first below the radii of convergence of the germs $T_i^{\pm 1}$, g_ν and their conjugates at all these places, and subject to Lemmas 3.1 and 3.3 below; there are finitely many such disks up to the choice of F. For v above $p \notin S$, Δ_v is the residue disk of $\tilde{s}_0$, which θ maps isomorphically onto $\{|\theta|_v < 1\}$.

3.3. **Comparison.**

Lemma 3.1. *Let $F \supset K$ be a number field and v a finite place of F over p, with $p \notin S$, or with $p \in S$ and E_{b_1} potentially good at v; in the second case F contains a fixed finite extension of K and r_v is small enough. Let $t \in \Delta_v \cap \widetilde{C}^0(F)$, with $|\theta(t)|_v < e^{-\beta_p}$ if $p \notin S$. Then $E_{i,t}$ and $E_{i,\tilde{s}_0}$ have good reduction at v with the same special fiber $\overline{E}_i$, and there are isomorphisms $\Pi_t \colon H_1^{\mathrm{dR}}(E_{i,t}/F_v) \to H_i$, with H_i the dual of $H^1_{\mathrm{crys}}(\overline{E}_i/W(k_v)) \otimes F_v$, such that*

$$T_i(\theta(t)) = \Pi_{\tilde{s}_0}^{-1}\Pi_t.$$

For an isogeny g between such fibers, $\Pi \circ g_* = \bar{g}_* \circ \Pi$, where $\bar{g}$ is its reduction, and the Π_t respect the Weil pairings.

Proof. This is [5, Lemma 3.3 and §3.2.1] in our frames; we follow the argument of [13, Prop. 6.2]. If $p \notin S$, the families have a smooth model over the residue disk, and t and $\tilde{s}_0$ reduce to the same point; if $p \in S$, after a fixed extension E_{b_1} has good reduction, a fixed change of Weierstrass coordinates gives a model of good reduction for the fibers over a small disk Δ_v, and again the fibers over Δ_v have the same special fiber. The comparison isomorphisms of Berthelot and Ogus [2, Cor. 2.5] identify $H^1_{\mathrm{crys}}(\overline{E}_i/W(k_v)) \otimes F_v$ with H^1_{dR} of each lift, functorially for morphisms with given reductions [2, (2.4.5)]; dualizing gives Π_t and the second assertion, and the pairings are respected since cup products are. The map $t \mapsto \Pi_{\tilde{s}_0}^{-1}\Pi_t$ is horizontal, with value 1 at $\tilde{s}_0$ [12, Rem. 5.14.3]. By uniqueness of the Taylor solution it equals $T_i(\theta(t))$ wherever the series converges. If $p \notin S$ the Gauss–Manin matrix in the frames has v-integral coefficients on the residue disk, and the recursion for the coefficients $T_{i,n}$ of T_i gives $\|T_{i,n}\|_v \le |n!|_v^{-1} \le e^{n\beta_p}$, so the series converges for $|\theta(t)|_v < e^{-\beta_p}$; if $p \in S$ it converges on Δ_v by the choice of r_v. □

3.4. **The equations.**

Proposition 3.2. *Let s be a modular point, $\tilde{s} \in \widetilde{C}^0$ over it, $a = \theta(\tilde{s})$, $L \supset K(\tilde{s})$ a field of definition of f_2, f_3, $\widetilde{L}$ a Galois closure of L over $\mathbb{Q}$, and v a place of L with $\tilde{s} \in \Delta_v$. All equations below hold in $\mathbb{C}_v$.*

(a) *If v is archimedean, or v lies over $p \in S$ and E_{b_1} is potentially good at v, there are $r_2, r_3 \in \mathbb{Z}$ with $|r_k| \le c\sqrt{N_{0k}M_k}$ and*

$$\langle f_{0k}, Y_{1k}(a)A_k\rangle = r_k \qquad (k = 2, 3).$$

(b) *Let v lie over $p \notin S$ with $|a|_v < e^{-\beta_p}$, let $\overline{E}_1$ be the common reduction of $E_{1,s}$ and E_{b_1}, k_v the residue field and $w_k = \bar{f}_{0k}^{\vee}\bar{f}_k \in \mathrm{End}(\overline{E}_1 \otimes k_v)$. Then $\langle f_{0k}, Y_{1k}(a)A_k\rangle = \mathrm{trd}(w_k)$. If $w_2w_3 = w_3w_2$, then*

$$[P_2(a), P_3(a)] = 0, \qquad P_k(a) = N_{0k}^{-1} f_{0k}^{\vee}\, Y_{1k}(a)A_k.$$

(c) *Let $\mathfrak{P}$ be a place of $\widetilde{L}$ above v, with $v \mid p \notin S$ and $0 < |a|_{\mathfrak{P}} < e^{-\beta_p}$, and $\sigma \in D_{\mathfrak{P}}$. Then, with $R = \sigma(a)/a$,*

$$\langle f_{0k}, Y_{1k}(a)A_k\rangle = \langle f_{0k}^{\sigma}, Y_{1k}^{\sigma}(aR)\,\sigma(A_k)\rangle \qquad (k = 2, 3),$$

$|R|_{\mathfrak{P}} = 1$, and $\sigma(\tilde{s})$ lies in the disk $\Delta_{\mathfrak{P}}^{\sigma}$ about $\sigma(\tilde{s}_0)$.

Proof. (a) At an archimedean place, horizontal sections on the simply connected disk Δ_v are flat extensions of Betti classes [5, Lemma 3.3]. So $T_kA_kT_1^{-1}$ is the transport of $f_{k*}\colon H_1(E_{1,s}, \mathbb{Z}) \to H_1(E_{k,s}, \mathbb{Z})$ to an integral map $H_1(E_{b_1}, \mathbb{Z}) \to H_1(E_{b_k}, \mathbb{Z})$, and $r_k = \mathrm{tr}(f_{0k}^{\vee}T_kA_kT_1^{-1}) \in \mathbb{Z}$. An isogeny of degree n multiplies Hodge norms by $\sqrt{n}$, and transport over the compact disk Δ_v distorts them by a bounded factor, so $|r_k| \le 2\sqrt{N_{0k}}\, c\sqrt{M_k}$. At a place over S the argument of (b) applies on Δ_v by Lemma 3.1, and gives $r_k = \mathrm{trd}(w_k)$ with $|\mathrm{trd}(w_k)| \le 2\sqrt{N_{0k}M_k}$ by Cauchy–Schwarz for the positive definite form deg on $\mathrm{Hom}(\overline{E}_1, \overline{E}_k) \otimes \mathbb{R}$, whose polar form is $\langle\cdot,\cdot\rangle$.

(b) By Lemma 3.1, $T_kA_kT_1^{-1} = \Pi_{\tilde{s}_0}^{-1}(\Pi_s A_k \Pi_s^{-1})\Pi_{\tilde{s}_0} = \Pi_{\tilde{s}_0}^{-1}\bar{f}_{k*}\Pi_{\tilde{s}_0}$, and $f_{0k}^{\vee}$ realizes $\bar{f}_{0k}^{\vee}$. So $X = \mathrm{adj}(f_{0k})T_kA_kT_1^{-1}$ is the realization of w_k, which is defined over k_v because f_{0k} and f_k are defined over L. Realization is an injective homomorphism of algebras on the division algebra $\mathrm{End}(\overline{E}_1 \otimes k_v) \otimes \mathbb{Q}$, and it takes $w_k^{\vee}$ to $\mathrm{adj}(X)$, since the comparisons respect the pairings. As $w_k + w_k^{\vee} = \mathrm{trd}(w_k)$ and $X + \mathrm{adj}(X) = \mathrm{tr}(X)$ for 2×2 matrices, the left side $\mathrm{tr}(X)$ is $\mathrm{trd}(w_k)$. Likewise $P_k(a)$ realizes w_k/N_{0k}, which gives the commutator equation.

(c) The automorphism σ of $\widetilde{L}$ preserves $|\cdot|_{\mathfrak{P}}$, so $|\sigma(a)|_{\mathfrak{P}} = |a|_{\mathfrak{P}}$, which gives $|R|_{\mathfrak{P}} = 1$; it commutes with reduction at $\mathfrak{P}$, so $\sigma(\tilde{s})$ and $\sigma(\tilde{s}_0)$ have the same reduction, as $\tilde{s}$ and $\tilde{s}_0$ do, which gives the last assertion; and σ extends continuously to $\widetilde{L}_{\mathfrak{P}}$. It maps the convergent series $\sum_n \langle f_{0k}, Y_{1k,n}A_k\rangle a^n$ to $\sum_n \langle f_{0k}^{\sigma}, Y_{1k,n}^{\sigma}\sigma(A_k)\rangle\sigma(a)^n$, which is the right side, and it fixes the left side, the integer $\mathrm{trd}(w_k)$ by (b). □

We call a place in (b) with $w_2w_3 = w_3w_2$ *commuting*. Every ordinary place is commuting, since then $\mathrm{End}(\overline{E}_1 \otimes \overline{k}_v) \otimes \mathbb{Q}$ is an imaginary quadratic field. At a non-commuting place $\overline{E}_1$ is supersingular, w_2 and w_3 generate $B = \mathrm{End}(\overline{E}_1 \otimes \overline{k}_v) \otimes \mathbb{Q} \cong B_{p,\infty}$, since a proper subalgebra of a quaternion division algebra is commutative, and the commutator equation need not hold. Part (b) is the computation of [5, Lemma 3.5 and (12)], where (12) is proved at the ordinary places, and part (c) and Proposition 3.5 below are equations (2) and (3) of [13, Prop. 6.4] in $Y(1)^3$. Only $\sigma \in D_{\mathfrak{P}}$ was used in the proof of (c); in the proof of Theorem 1.3, that σ induces the Frobenius of the residue field is used once, in Lemma 4.5.

3.5. **Potentially multiplicative places.** At the places where E_{b_1} is potentially multiplicative, Tate curves show that only finitely many modular points are close to s_0 (Lemma 3.3); [13, Prop. 4.5] does this in $\mathcal{A}_2$ with an auxiliary level structure.

Lemma 3.3. *Let w_0 be a finite place of K with $|b_1|_{w_0} > 1$. There is $r > 0$ such that, for every modular point s and every place w above w_0 of a finite extension of $K(\tilde{s})$ at which $|a|_w < r$ and $\tilde{s} \in \Delta_w$, $M_2 \le N_{02}$. Such points s lie in the finite set $C \cap \bigcup_{m \le N_{02}} \{\Phi_m(x_1, x_2) = 0\}$.*

Proof. Work over $\mathbb{C}_w$, with p the residue characteristic. An isogeny preserves potentially multiplicative reduction, so $|b_i|_w > 1$ for all i. Tate's curve $\mathbb{G}_m/q^{\mathbb{Z}}$, for $0 < |q| < 1$, has $j = q^{-1} + 744 + 196884q + \cdots$ [20, (18)]. Since $\mathbb{C}_w$ is algebraically closed, an elliptic curve over $\mathbb{C}_w$ with $|j| > 1$ is therefore isomorphic to it for $q = Q(1/j)$, where $Q \in \mathbb{Z}[[J]]$ inverts this series, and $|q| = |j|^{-1}$. On a disk $|a| < r_1$ on which $|j_i(\tilde{s}(a)) - b_i| < |b_i|$, the parameters $q_i(a) = Q(1/j_i(\tilde{s}(a)))$ are analytic with constant absolute values. The isogenies $\mathbb{G}_m/q_1^{\mathbb{Z}} \to \mathbb{G}_m/q_2^{\mathbb{Z}}$ are the maps induced by $t \mapsto t^n$ for the nonzero pairs $(n, k) \in \mathbb{Z}^2$ with $q_1^n = q_2^k$, and such a map has degree nk [20, Isogenies, Theorem]. Let (n_0, k_0) be the pair of $\pm f_{02}$ with $n_0 > 0$; then $k_0 > 0$ and $n_0 k_0 = N_{02}$. At a modular point s with $|a| < r_1$, some pair (n, k) with $n > 0$ has $q_1(a)^n = q_2(a)^k$, and comparing absolute values $nk_0 = kn_0$. So $\eta(a) = q_1(a)^{n_0} q_2(a)^{-k_0}$ satisfies $\eta(a)^n = 1$. Since $\eta(0) = 1$, there is $r \le r_1$ with $|\eta(a) - 1| < p^{-1/(p-1)}$ for $|a| < r$. A root of unity $\zeta \ne 1$ has $|\zeta - 1| \ge p^{-1/(p-1)}$, so $\eta(a) = 1$ and (n_0, k_0) gives an isogeny $E_{1,s} \to E_{2,s}$ of degree N_{02}. Since f_2 generates the homomorphisms, $M_2 \le N_{02}$, and $\Phi_{M_2}(x_1, x_2) = 0$. No $\Phi_m(x_1, x_2)$ vanishes on C, so the last set is finite. The disk is defined over $\mathbb{C}_w$, so r serves every finite extension. □

We take $r_v < r$ at the places over w_0, so that $\tilde{s} \notin \Delta_v$ there for every modular point s outside the finite set of Lemma 3.3.

3.6. **Exceptional correspondences.** For $g = (m_1, m_2, m_3)$ let $T_g \subset Y(1)^3 \times Y(1)^3$ be the locus $\Phi_{m_i}(y_i, y_i') = 0$, $i = 1, 2, 3$. We call a pair (σ, g), with σ an embedding of K, an *exceptional correspondence* if $T_g \cap (C \times C^\sigma)$ contains a curve whose projections to C and C^σ are both dominant. By Proposition 4.2 there are finitely many. We add to S the primes dividing $m_1 m_2 m_3$ for each of them and the primes of the following lemma for the pairs $(E_{b_i}, E_{\sigma(b_i)})$ and these m_i, for all $\sigma \in \mathrm{Hom}(K, \overline{\mathbb{Q}})$.

Lemma 3.4 (cf. [13, Lemma 6.3])**.** *Let E, E' be elliptic curves without CM over a number field F and $m \ge 1$. For all primes p outside a finite set and every place over p, every isogeny of degree m from the reduction of E to that of E' over the algebraic closure of the residue field is the reduction, for a fixed prolongation of the place to $\overline{F}$, of an isogeny $E_{\overline{F}} \to E'_{\overline{F}}$ of degree m.*

Proof. Exclude the primes dividing $6m$, those of bad reduction for E or E', those dividing $j(E')(j(E') - 1728)$, and those dividing $j(E/H) - j(E')$ for the finitely many subgroups $H \subset E[m]$ of order m with $j(E/H) \ne j(E')$. Let $\bar{\psi}$ have degree m. Its kernel is étale and lifts to a subgroup $H \subset E[m]$, and $\bar{\psi}$ factors as the reduction of $E \to E/H$ followed by an isomorphism $\bar{\iota}_1$ to the reduction of E'. So $j(E/H) \equiv j(E')$, hence $j(E/H) = j(E')$, and there is an isomorphism $\iota\colon E/H \to E'$ over $\overline{\mathbb{Q}}$. Since the reduction of E' has $j \notin \{0, 1728\}$ and $p \ge 5$, its automorphisms are ± 1, so $\bar{\iota}_1 = \pm\bar{\iota}$ and $\bar{\psi}$ is the reduction of $\pm\iota \circ (E \to E/H)$. □

Proposition 3.5. *In the situation of Proposition 3.2(c), suppose that $(s, \sigma(s)) \in T_g$ for an exceptional correspondence $(\sigma|_K, g)$, $g = (m_1, m_2, m_3)$, and let $\varphi_i\colon E_{i,s} \to E^{\sigma}_{i,\sigma(s)}$ be cyclic*

of degree m_i. There are isogenies $\varphi_{0i}\colon E_{b_i}\to E_{\sigma(b_i)}$ of degree m_i, from a finite set depending only on $(\sigma|_K, g)$, with the same reductions as φ_i at $\mathfrak{P}$, and

$$\langle f_{0k}, Y_{1k}(a)A_k\rangle = \frac{m_k}{m_1}\,\langle C_k, Y_{1k}(a)\widehat{A}_k\rangle \qquad (k=2,3), \tag{3.1}$$

where $C_k=\varphi_{0k}^{-1}f_{0k}^{\sigma}\varphi_{01}$ and $\widehat{A}_k$ is the matrix of $\hat{f}_k=\varphi_k^{-1}f_k^{\sigma}\varphi_1$. Moreover $C_k=\mu_{0k}f_{0k}$ and $\hat{f}_k=\mu_k f_k$ with $\mu_{0k},\mu_k\in\mathbb{Q}$ and $\mu_{0k}^2=\mu_k^2=m_1/m_k$.

Proof. At $\mathfrak{P}$ the curves $E_{i,\sigma(s)}^{\sigma}$ and $E_{\sigma(b_i)}$ have a common reduction, by Proposition 3.2(c), and $p\nmid m_i$. By Lemma 3.4 the reduction of φ_i is that of an isogeny φ_{0i} of degree m_i, one of the at most two in the group $\mathrm{Hom}(E_{b_i},E_{\sigma(b_i)})$ of rank one. Let Π and Π' be the comparisons of Lemma 3.1 for C and for C^{σ}. Then $T_i=\Pi_{\tilde{s}_0}^{-1}\Pi_s$ and $T_i^{\sigma}(aR)=\Pi'^{-1}_{\sigma(\tilde{s}_0)}\Pi'_{\sigma(s)}$, and functoriality with $\bar{\varphi}_i=\bar{\varphi}_{0i}$ gives

$$T_i^{\sigma}(aR)\,\varphi_i=\varphi_{0i}\,T_i(a).$$

Hence $Y_{1k}^{\sigma}(aR)\sigma(A_k)=I_0(Y_{1k}(a)\widehat{A}_k)$ and $f_{0k}^{\sigma}=I_0(C_k)$, with $I_0(x)=\varphi_{0k}x\varphi_{01}^{-1}$. Since $\varphi_{0k}^{\vee}\varphi_{0k}=m_k$ and $(\varphi_{01}^{-1})^{\vee}=\varphi_{01}/m_1$, we have $\langle I_0x,I_0y\rangle=(m_k/m_1)\langle x,y\rangle$, and (3.1) follows from Proposition 3.2(c). The groups $\mathrm{Hom}(E_{b_1},E_{b_k})\otimes\mathbb{Q}$ and $\mathrm{Hom}(E_{1,s},E_{k,s})\otimes\mathbb{Q}$ have dimension one, which gives μ_{0k} and μ_k, and comparing degrees gives their squares. □

3.7. **The four systems.** We use the following systems, each with a point $z=(a,U)$ of an affine space and the base equations $u_\nu-g_\nu(a)=0$, $1\le\nu\le m$. In each, a and the entries of u and A_2,A_3 are coordinates of z. By Proposition 5.4, (I) serves the archimedean places and those over S, (II) the commuting places, and (III) and (IV) the non-commuting ones.

(I) The *fixed* system with constants $(r_2,r_3)\in\mathbb{Z}^2$: $z=(a,u,A_2,A_3)$ and $\langle f_{0k},Y_{1k}(a)A_k\rangle-r_k=0$.

(II) The *commutator* system: $z=(a,u,A_2,A_3)$ and the four entries of $[P_2(a),P_3(a)]=0$.

(III) The *paired* system for $\rho\in\mathrm{Hom}(L,\overline{\mathbb{Q}})$, with $\sigma=\rho|_K$: $z=(a,u,A_2,A_3,R,\rho(u),\rho(A_2),\rho(A_3))$ with $R=\rho(a)/a$, the paired equations of Proposition 3.2(c) with $\rho(A_k)$ for $\sigma(A_k)$, and the second base equations $\rho(u)_\nu-g_\nu^{\sigma}(aR)=0$.

(IV) The *transported* system for an exceptional $(\sigma|_K,g)$, with isogenies φ_{0i} and scalars μ_k as in Proposition 3.5: $z=(a,u,A_2,A_3,\widehat{A}_2,\widehat{A}_3)$ with $\widehat{A}_k=\mu_kA_k$, and (3.1).

The base equations hold at the places v with $\tilde{s}\in\Delta_v$, and in (III) the second ones where also $\rho(\tilde{s})\in\Delta_v^{\sigma}$, which is the case there by Proposition 3.2(c).

4. Nonvanishing of the systems

This section proves that the systems of Section 3.7, at the places where Section 5 uses them, satisfy hypothesis (iv) of Theorem 5.1 (Proposition 4.7), and that there are finitely many exceptional correspondences (Proposition 4.2). The sign argument is Lemma 4.5.

4.1. **Monodromy.** We use three facts of linear algebra over $\mathbb{C}$. (L1) For $0\ne Z\in M_2$, $\gamma\mapsto\mathrm{tr}(\gamma Z)$ is not constant on SL_2, since the $\gamma-1$ span M_2. (L2) For invertible B,X, the matrices $B\gamma X$, $\gamma\in\mathrm{SL}_2$, span M_2. (L3) The maps $Z\mapsto\gamma Z\gamma'^{-1}$, $(\gamma,\gamma')\in\mathrm{SL}_2^2$, span $\mathrm{End}(M_2)=M_2\otimes M_2^{\mathrm{op}}$.

Lemma 4.1. *Let W° be a smooth connected complex variety and $\tilde{s}\colon W^\circ\to\widetilde{C}^0$ a dominant morphism.*

(a) *The identity component of the Zariski closure of the monodromy of* $H_1(E_1)\oplus H_1(E_2)\oplus H_1(E_3)$ *pulled back to* W° *is* SL_2^3.

(b) *Let* $\sigma\in\mathrm{Hom}(K,\overline{\mathbb{Q}})$ *and let* $\tilde s'\colon W^\circ\to\widetilde C^{\sigma,0}$ *also be dominant. Let* $\mathfrak M$ *be the identity component of the Zariski closure of the monodromy of the six local systems* $H_1(E_i)$ *and* $H_1(E_i^\sigma)$ *pulled back by* $\tilde s$ *and* $\tilde s'$. *Then* $\operatorname{Lie}\mathfrak M$ *is a sum of simple ideals, each mapping isomorphically onto the* $\mathfrak{sl}_2$ *of one of the six factors or onto those of two factors* E_i, E_j^σ, *one from each side. For each such pair there is* $m\ge 1$ *with* $\Phi_m(j_i(\tilde s(w)),j_j^\sigma(\tilde s'(w)))=0$ *for all* $w\in W^\circ$.

Proof. (a) Over a finite étale cover of $\widetilde C^0$ the families E_i become isomorphic to the pullbacks of the Legendre family used by Habegger and Pila, since the j-invariants agree and avoid 0 and 1728. So for $\widetilde C^0$ in place of W° the assertion is [9, (10)], which applies because the coordinates of C are nonconstant and no two are related by a modular polynomial. A dominant morphism of irreducible normal varieties induces a homomorphism of fundamental groups with image of finite index, so passing to W° does not change the identity component.

(b) Let $\mathfrak g=\operatorname{Lie}\mathfrak M\subset\mathfrak{sl}_2^6$. By (a) on each side, $\mathfrak g$ maps onto each factor, so its radical maps to solvable ideals of $\mathfrak{sl}_2$, which are 0, and $\mathfrak g$ is semisimple. Each simple ideal maps isomorphically onto the factors it does not kill, and each factor receives exactly one simple ideal. By (a), no ideal reaches two factors of the same side. If an ideal maps onto the factors of E_i and E_j^σ, the isomorphism between them is defined over $\mathbb{Q}$ and inner, $\mathrm{Ad}(h)$ with $h\in\mathrm{Hom}(H_1E_i,H_1E_j^\sigma)\otimes\mathbb{Q}$, since $\mathfrak{sl}_2$ has no outer automorphisms and $\mathrm{PGL}_2(\mathbb{Q})$ lifts to $\mathrm{GL}_2(\mathbb{Q})$. Over a finite étale cover W'' of W° on which the monodromy lies in $\mathfrak M$, h is then a flat section, and the flat sections of $\mathrm{Hom}(H_1E_i,H_1E_j^\sigma)\otimes\mathbb{Q}$ form the line $\mathbb{Q}h$, the invariants of the graph of $\mathrm{Ad}(h)$ in $\mathrm{std}\otimes\mathrm{std}$. By the theorem of the fixed part [7, Cor. 4.1.2], applied to H^2 of $E_i\times_{W''}E_j^\sigma$, this line is a sub-Hodge structure; being of dimension one in $H^1\otimes H^1$, it is of type $(1,1)$, so h is of type $(0,0)$ in $\mathrm{Hom}(H_1,H_1)$ at every point. A Hodge class there is a rational multiple of an isogeny. The primitive integral multiple of h has elementary divisors $(1,m)$, so it is an isogeny with cyclic kernel, of constant degree m. □

4.2. **Finiteness of the exceptional correspondences.**

Proposition 4.2. *There are only finitely many exceptional correspondences.*

Proof. We deduce this from [17, Thm. 7.1]; [13, Prop. 3.4] gives a direct argument in $\mathcal{A}_2$. Fix σ; there are finitely many. The surface $C\times C^\sigma\subset Y(1)^6$ lies in no proper special subvariety. For $g=(m_1,m_2,m_3)$, T_g is special and irreducible, since each Φ_m is irreducible, and does not contain $C\times C^\sigma$. So a curve $Z\subset T_g\cap(C\times C^\sigma)$ dominating C and C^σ is a component of this intersection. It is atypical, as $1>2+3-6$, strongly atypical, as the coordinates of C and C^σ are nonconstant, and maximal, since a larger strongly atypical subvariety would be $C\times C^\sigma$ itself, which is not atypical. By that theorem for $C\times C^\sigma$ there are finitely many such Z. Each lies in T_g for one g only, since for $m\ne m'$ the irreducible polynomials Φ_m and $\Phi_{m'}$ have finitely many common zeros and y_i is nonconstant on Z. □

4.3. **Continuation.**

Lemma 4.3 (cf. [13, Lemma 7.1])**.** *Let one of the systems of Section 3.7 be given, with its point* z. *Let* $W\ni z$ *be an irreducible closed subvariety of the affine space of* z *on which* a *is nonconstant, and* D *a prime divisor of the normalization of* W *above a component of* $W\cap\{a=0\}$. *Suppose that every equation of the system vanishes identically on the branch of* W *at a general point of* D. *Then:*

(i) *the coordinates u define a nonconstant morphism $\tilde{s}\colon W \to \widetilde{C}$ with $\theta \circ \tilde{s} = a$, and for the paired system the coordinates $\rho(u)$ define a nonconstant $\tilde{s}'\colon W \to \widetilde{C}^{\sigma}$ with $\theta^{\sigma} \circ \tilde{s}' = aR$;*

(ii) *on a smooth connected dense open $W^{\circ} \subset W$ all equations continue analytically along paths, and continuation along a loop γ at $w \in W^{\circ}$ replaces each transport T_i by $T_i\eta_i(\gamma)$, where $\eta_i(\gamma)$ is parallel transport around the reversed loop γ^{-1} on the fiber at $\tilde{s}(w)$, and leaves the coordinates unchanged;*

(iii) *the matrices A_2, A_3 and, for the paired and transported systems, $\rho(A_k)$ and $\widehat{A}_k$, are invertible at a general point of W°.*

Proof. On the branch, $u = g(a)$, so the branch is mapped into $\widetilde{C}^0$ with $\theta(u) = a$. The branch is Zariski dense in W, so the equations of $\widetilde{C}^0$ in $\mathbb{A}^m$ and $\theta(u) - a$ vanish on W; this gives $\tilde{s}$, which is nonconstant because a is. The same argument gives $\tilde{s}'$ in the paired case. If $\tilde{s}'$ were constant, $aR = \theta^{\sigma}(\tilde{s}')$ would be a constant that tends to 0 on the branch, so aR and then R would vanish on W, whereas $R(z) = \rho(a)/a \neq 0$. Let $W^{\circ} \subset W$ be a smooth connected dense open on which $\tilde{s}$ and $\tilde{s}'$ land in $\widetilde{C}^0$ and $\widetilde{C}^{\sigma,0}$. On the branch, $T_i(a)$ is the transport from $\tilde{s}(w)$ to $\tilde{s}_0$, a horizontal section of the pulled-back Gauss–Manin connection; it continues along paths in W° as such, and around a loop it is multiplied on the right by the monodromy. The equations are polynomial in the coordinates and the transports, so they continue as identities. For (iii), the determinants are regular functions on W that do not vanish at z. □

4.4. **The signs.** For a transported system put $\varepsilon_k = (m_k/m_1)\mu_{0k}\mu_k \in \{\pm 1\}$, $k = 2, 3$.

Lemma 4.4 (cf. [13, Prop. 7.4])**.** *If (3.1) for one k and the base equations vanish identically on a branch as in Lemma 4.3, then $\varepsilon_k = +1$.*

Proof. Fix a general $w \in W^{\circ}$, put $X = A_k(w)$, $\widehat{X} = \widehat{A}_k(w)$, and let $Y(v) = T_k v T_1^{-1}$ with the continued transports at w. By Lemma 4.3 and Lemma 4.1(a),

$$\langle f_{0k}, Y(\gamma X \gamma'^{-1})\rangle = \frac{m_k}{m_1}\,\langle C_k, Y(\gamma \widehat{X} \gamma'^{-1})\rangle \qquad \text{for all } (\gamma', \gamma) \in \mathrm{SL}_2^2,$$

and by (L3) with $\gamma(\cdot)\gamma'^{-1}$ replaced by any linear map Φ of M_2. Put $\lambda_b(v) = \langle b, Yv\rangle$; $b \mapsto \lambda_b$ is injective, since the pairing is non-degenerate and Y is invertible. The identity says $\lambda_{f_{0k}} \otimes X = (m_k/m_1)\,\lambda_{C_k} \otimes \widehat{X}$ in $M_2^* \otimes M_2$. Both sides are nonzero, so $\lambda_{f_{0k}} = t\lambda_{C_k}$ and $tX = (m_k/m_1)\widehat{X}$ for a scalar t. The first relation says $f_{0k} = tC_k = t\mu_{0k}f_{0k}$, so $t = 1/\mu_{0k}$. The second says $\widehat{A}_k = (m_1 t/m_k)A_k$ at a general point of W, hence on W and at z, where $\widehat{A}_k = \mu_k A_k$. So $\mu_k = m_1/(m_k\mu_{0k})$ and $\varepsilon_k = 1$. □

At the point, (3.1) is an identity when $\varepsilon_k = +1$ and says $\mathrm{trd}(w_k) = 0$ when $\varepsilon_k = -1$.

Lemma 4.5 (cf. [13, Lemma 7.3])**.** *Let $\mathfrak{P}$ be a non-commuting place over $p \notin S$ as in Proposition 3.5, with $\sigma \in D_{\mathfrak{P}}$ inducing the p-th power map on the residue field. Let $F_i\colon \overline{E}_i \to \overline{E}_i^{(p)}$ be the relative Frobenius, where $\overline{E}_i^{(p)}$ is the reduction of $E^{\sigma}_{i,\sigma(s)}$, and $\psi_i = \bar{\varphi}_i^{-1}F_i \in \mathrm{End}(\overline{E}_i \otimes \overline{k}) \otimes \mathbb{Q}$. Then $\mathrm{nrd}\,\psi_i = p/m_i$ and $\psi_1 w_k \psi_1^{-1} = \varepsilon_k w_k$ for $k = 2, 3$. In particular $\varepsilon_2 = \varepsilon_3 = +1$ is impossible.*

Proof. Since σ induces the Frobenius, the reduction of f_k^{σ} is $\bar{f}_k^{(p)} = F_k \bar{f}_k F_1^{-1}$, by naturality of the relative Frobenius, and likewise for f_{0k}^{σ}. With $\bar{\varphi}_{0i} = \bar{\varphi}_i$, the reductions of $\hat{f}_k$ and C_k are $\psi_k \bar{f}_k \psi_1^{-1} = \mu_k \bar{f}_k$ and $\psi_k \bar{f}_{0k} \psi_1^{-1} = \mu_{0k}\bar{f}_{0k}$. Dualizing the second, with $\psi^{\vee} = \mathrm{nrd}(\psi)\psi^{-1}$, gives $\psi_1 \bar{f}_{0k}^{\vee} \psi_k^{-1} = \mu_{0k}(m_k/m_1)\bar{f}_{0k}^{\vee}$, and multiplying, $\psi_1 w_k \psi_1^{-1} = \varepsilon_k w_k$. If both signs are $+1$,

ψ_1 commutes with w_2 and w_3, which generate B at a non-commuting place, so $\psi_1 \in \mathbb{Q}$ and $\operatorname{nrd}\psi_1 = \psi_1^2 = p/m_1$ is a square in $\mathbb{Q}$, which is impossible as $p \nmid m_1$. □

Remark 4.6. For $\sigma = \mathrm{id}$ and $g = (1,1,1)$, T_g is the diagonal, which contains the diagonal of $C \times C$, so $(\mathrm{id}, (1,1,1))$ is always exceptional. Now let $\sigma \in D_{\mathfrak{P}}$ be as in Proposition 3.2(c) and suppose that it induces the p-power Frobenius on the residue field. The paired equations can be identities: if $\sigma|_L = \mathrm{id}$, then $R = 1$ and both sides coincide. Such places are commuting. Indeed the residue field of L at $\mathfrak{P}$ is then $\mathbb{F}_p$, so $\overline{E}_1$ and w_2, w_3 are defined over $\mathbb{F}_p$ and commute with the Frobenius endomorphism π of $\overline{E}_1$. If $\overline{E}_1$ is supersingular, $\pi^2 = -p$ as $p \geq 5$, so $\pi \notin \mathbb{Q}$; if w_2, w_3 did not commute, π would be central in B, hence in $\mathbb{Q}$.

4.5. **Nonvanishing.**

Proposition 4.7. *Let s be a modular point and one of the systems of Section 3.7 be given, with its point z: a fixed system, the commutator system, a paired system with $(s, \rho(s))$ on no T_g with $(\rho|_K, g)$ exceptional, or a transported system with the data of Proposition 3.5 at a non-commuting place and a lift of the p-power Frobenius as in Lemma 4.5. Then for every W and D as in Lemma 4.3, some equation of the system is not identically zero on the branch of W at a general point of D.*

Proof. Suppose all equations vanish identically there. If $\tilde{s}$ is as in Lemma 4.3, it is dominant, and by Lemma 4.1(a) the continued identities hold with (η_1, η_2, η_3) ranging over SL_2^3. Fix a general $w \in W^\circ$, let $X_k = A_k(w)$, invertible, and let T_i be the continued transports at w.

Fixed system. The first equation becomes $\operatorname{tr}(f_{02}^\vee T_2 \eta_2 X_2 \eta_1^{-1} T_1^{-1}) = r_2$. With $\eta_1 = 1$ the left side is $\operatorname{tr}(\eta_2 Z)$ with $Z = X_2 T_1^{-1} f_{02}^\vee T_2 \neq 0$, which is not constant by (L1).

Commutator system. Conjugating by $T_1\eta_1$ and taking $\eta_1 = 1$, the equations become $[B_2\eta_2 X_2, B_3\eta_3 X_3] = 0$ for all η_2, η_3, with $B_k = N_{0k}^{-1} T_1^{-1} f_{0k}^\vee T_k$ invertible. By (L2) the $B_2\eta_2 X_2$ span M_2, so every $B_3\eta_3 X_3$ is scalar; but these span M_2 too.

Paired system. By Lemma 4.3, $\tilde{s}'$ is nonconstant, hence dominant; let $\mathfrak{M}$ be as in Lemma 4.1(b). The first paired equation continues to $\operatorname{tr}(f_{02}^\vee T_2 \eta_2 X_2 \eta_1^{-1} T_1^{-1}) = \operatorname{tr}((f_{02}^\sigma)^\vee T_2' \eta_2' X_2' \eta_1'^{-1} T_1'^{-1})$ on $\mathfrak{M}$, where primes refer to C^σ and $X_2' = \rho(A_2)(w)$. If the simple ideal at the factor E_2 reaches neither E_1^σ nor E_2^σ, the corresponding subgroup of $\mathfrak{M}$ acts through SL_2 on H_1E_2 and trivially on H_1E_1, $H_1E_1^\sigma$, $H_1E_2^\sigma$, and by (L1) it changes the left side and not the right. So E_2, and likewise E_1, is paired with E_1^σ or E_2^σ, and symmetrically. The second paired equation gives the same for $\{E_1, E_3\}$ and $\{E_1^\sigma, E_3^\sigma\}$. As E_1 has one partner, it is E_1^σ, and then E_2 and E_3 are paired with E_2^σ and E_3^σ. By Lemma 4.1(b) there is $g = (m_1, m_2, m_3)$ with $\Phi_{m_i}(j_i(\tilde{s}), j_i^\sigma(\tilde{s}')) = 0$ on W°. The closure of the image of W in $C \times C^\sigma$ is irreducible, contained in $T_g \cap (C \times C^\sigma)$, which is a proper closed subset of the surface, and has nonconstant projections. So it is a curve dominating both factors, (σ, g) is exceptional, and the image $(s, \rho(s))$ of z lies in T_g, contrary to the hypothesis.

Transported system. By Lemma 4.4, $\varepsilon_2 = \varepsilon_3 = +1$, which Lemma 4.5 excludes. □

5. Proof of Theorem 1.3

5.1. **The interpolation theorem.**

For a tuple $x = (x_1, \dots, x_r)$ over a number field F let $h_{\mathrm{aff}}(x) = \sum_v n_v \log \max(1, |x_1|_v, \dots, |x_r|_v)$, and for a polynomial let h_{aff} be that of its coefficients. A weighted collection of places of F may split a place into places of a larger field, with weights n_v that add up to the weight of the place they split. We use the following theorem of OpenAI's preprint, in our notation.

Theorem 5.1 ([13, Thm. 2.1]). *Fix integers $N, \chi_1, b \geq 1$, a real number $C_0 \geq 2$, and a finite list of regular algebraic germs and of horizontal matrix germs, in rational frames, of connections regular at their centers, on fixed algebraic curves over number fields, with their duals and inverses. There is C_1, depending only on these data, with the following property. Let $d, h \geq 2$, let $z = (a, U) \in \mathbb{A}^N(\overline{\mathbb{Q}})$ with $a \neq 0$, let V be a weighted collection of places of a number field F containing the coordinates of z, and let*

$$E_\alpha(a, U) = \sum_{j \geq 0} a^j e_{\alpha,j}(U) \qquad (1 \leq \alpha \leq \chi \leq \chi_1)$$

be such that each equation $E_\alpha(a, U) = 0$ holds in $\mathbb{C}_v$ for every $v \in V$, and:

(i) *$h_{\mathrm{aff}}(z) \leq C_0 d^{C_0} h$, $|a|_v < 1$ for $v \in V$, $\sum_{v \in V} n_v \log |a|_v^{-1} \geq h/(C_0 d^{C_0})$ and $\sum_{v \in V} n_v \log^+ \|U\|_v \leq C_0 d^{C_0} (1 + \log h)^{C_0}$, where $\log^+ \|U\|_v = \log \max(\{1\} \cup \{|U_i|_v : 1 \leq i < N\})$;*

(ii) *the $e_{\alpha,j}$ are polynomials over F with $\deg e_{\alpha,j} \leq C_0(1 + j)$ and $h_{\mathrm{aff}}(e_{\alpha,j}) \leq C_0(1 + j + d)^{C_0}(1 + \log h)^{C_0}$, and there are $b_v \geq 0$ with $\sum_{v \in V} n_v b_v \leq C_0 d^{C_0}(1 + \log h)^{C_0}$ and*

$$\Big|\sum_{j \geq \ell} a^j e_{\alpha,j}(U)\Big|_v \leq |a|_v^\ell \exp((\ell + 1) b_v) \qquad (\ell \geq 0,\ v \in V);$$

(iii) *as analytic functions of (a, U), the E_α are polynomial combinations of degree at most b of the coordinates and the fixed germs evaluated at a, and possibly at aR for a coordinate R of U, with constant coefficients that may depend on z;*

(iv) *for every irreducible closed $W \subset \mathbb{A}^N_{\overline{\mathbb{Q}}}$ containing z on which a is nonconstant, and every prime divisor D of the normalization of W above a component of $W \cap \{a = 0\}$, some E_α is not identically zero on the branch of W at a general point of D.*

Then $h \leq C_1 d^{C_1}$.

5.2. **Heights.** Let $\overline{C}$ be the smooth projective curve containing $\widetilde{C}$ and $h_{[\tilde{s}_0]}$ a Weil height for the divisor $[\tilde{s}_0]$. In what follows we leave out the finitely many modular points at a singular point of C and those whose points over them lie outside $\widetilde{C}^0$ or have $a = 0$. For a remaining modular point s let $h = h(s) + c_0$, where c_0 is a suitable constant, and $d = \max(2, [K(s) : K])$, and let $L = K(s)(f_2, f_3)$, so that $[L : \mathbb{Q}] \leq 4[K : \mathbb{Q}]d$ and, by Gaudron and Rémond's estimate with Silverman's comparison as in Section 2, $M_2, M_3 \leq c(dh)^c$.

Lemma 5.2. (a) *For each rational function y on $\overline{C}$ there is c_y such that $h(y(\tilde{s})) \leq c_y h$ whenever y is regular at $\tilde{s}$.* (b) *$h_{[\tilde{s}_0]}(\tilde{s}) \geq h/c - c$.*

Proof. For an effective divisor D' of degree n on $\overline{C}$, $(n + 1)[\tilde{s}_0] - D'$ has positive degree, hence is ample, so $h_{D'} \leq (n + 1) h_{[\tilde{s}_0]} + c$. With $D' = j_i^*(\infty)$ this gives $h(x_i) \leq c\, h_{[\tilde{s}_0]}(\tilde{s}) + c$, which is (b). Conversely $2j_1^*(\infty) - [\tilde{s}_0]$ has positive degree, so $h_{[\tilde{s}_0]} \leq 2h(x_1) + c$, and $h(y(\tilde{s})) = h_{y^*(\infty)}(\tilde{s}) + O(1) \leq (\deg y + 1) h_{[\tilde{s}_0]}(\tilde{s}) + c$. □

For Lemma 5.3 we follow the proof of [13, Lemma 6.1]; its error term $c(1 + \log dh)^2$ becomes $c(1 + \log dh)$ here, by Mertens's theorem in (a). Let $P(a)$ be the set of primes $p \notin S$ below a place v of L with $|a|_v < 1$.

Lemma 5.3. (a) $\sum_{p \in P(a)} \beta_p \leq c(1 + \log dh)$. (b) *Let V_0 be the set of places v of L with $\tilde{s} \in \Delta_v$ that are archimedean, or lie over $p \in S$ with E_{b_1} potentially good at v, or lie over*

$p \notin S$ with $|a|_v < e^{-2\beta_p}$. Outside a finite set of s,

$$\sum_{v\in V_0} n_v \log|a|_v^{-1} \ge h/c - c(1+\log dh).$$

Proof. (a) If $|a|_v < 1$ at $v \mid p$, then $n_v \log|a|_v^{-1} \ge n_v \log p/e_v \ge \log p/[L:\mathbb{Q}]$. So $\sum_{p\in P(a)} \log p \le X := \max\{2, [L:\mathbb{Q}]h(a)\}$, and $X \le cdh$ by Lemma 5.2(a). The primes of $P(a)$ above X are at most $X/\log X$ in number, each with $\beta_p \le 2\log X/X$, and $\sum_{p\le X} \beta_p \le 2\log X + c$ by Mertens's theorem.

(b) Choose local Weil functions λ_v for $[\tilde{s}_0]$ from the section $\tilde{s}_0$ on the integral model at every v over $p \notin S$, so that $\sum_v n_v\lambda_v(\tilde{s}) = h_{[\tilde{s}_0]}(\tilde{s}) + O(1) \ge h/c - c$. For v over $p \notin S$ the model is smooth near $\tilde{s}_0$ with θ a local equation of the section $\tilde{s}_0$, so $\lambda_v(\tilde{s}) = \log|a|_v^{-1}$ on the residue disk and $\lambda_v(\tilde{s}) = 0$ off it. For v archimedean or over S, $\lambda_v = \log|\theta|_v^{-1} + O(1)$ on Δ_v and $\lambda_v \le c$ off it, uniformly in the field. At the places over a place w_0 as in Lemma 3.3, $\tilde{s} \notin \Delta_v$ outside a finite set. The places over $p \notin S$ with $e^{-2\beta_p} \le |a|_v < 1$ contribute at most $2\beta_p$ for each $p \in P(a)$, which (a) bounds. □

5.3. **The groups.** For each place $\mathfrak{P}$ of a Galois closure $\widetilde{L}$ of L over $\mathbb{Q}$, above $p \notin S$, fix $\sigma_{\mathfrak{P}} \in D_{\mathfrak{P}}$ inducing the p-th power map on the residue field. We split each place of V_0 over $p \notin S$ that is not commuting into the places of $\widetilde{L}$ above it.

Proposition 5.4 (cf. [13, Prop. 6.5])**.** *The places of V_0, split in this way, fall into at most cd groups. To each group belongs one of the systems of Section 3.7, with its point z, such that every equation of the system holds in $\mathbb{C}_v$ at every place v of the group, and such that the hypotheses of Proposition 4.7 hold:*

(I) *at the places that are archimedean or over S, the fixed system with the constants of Proposition 3.2(a), one group for each pair (r_2, r_3);*

(II) *at the commuting places, the commutator system;*

(III) *at a non-commuting $\mathfrak{P}$ with $(s, \sigma_{\mathfrak{P}}(s))$ on no T_g with $(\sigma_{\mathfrak{P}}|_K, g)$ exceptional, the paired system with $\rho = \sigma_{\mathfrak{P}}|_L$;*

(IV) *at the other non-commuting $\mathfrak{P}$, the transported system of Proposition 3.5.*

Proof. The equations hold by Propositions 3.2 and 3.5, since $|a|_v < e^{-2\beta_p}$ at the places over $p \notin S$. The point of a paired system depends on $\mathfrak{P}$ only through ρ, and so does the condition in (III), since $\sigma_{\mathfrak{P}}(s) = \rho(s)$. The data $(\sigma_{\mathfrak{P}}|_K, g, (\varphi_{0i}), (\mu_k))$ of the systems in (IV) lie in a finite set, by Proposition 4.2 and since the φ_{0i} and μ_k do. The places of L over ∞ and over each prime of S are at most $[L:\mathbb{Q}]$ in number, and ρ ranges over the $[L:\mathbb{Q}]$ embeddings of L, so there are at most $(2+|S|)[L:\mathbb{Q}] + c \le c'd$ groups. □

5.4. **Sizes.**

Lemma 5.5 (cf. [13, Prop. 6.5])**.** *Let $z = (a, U)$ be the point of a group of Proposition 5.4 and V its places. In these estimates put $r_2 = r_3 = 0$ outside system* (I)*. Then:*

(a) $h_{\mathrm{aff}}(z) \le c(h + \log d)$;

(b) $\log^+\|U\|_v = 0$ *for v over $p \notin S$,* $\log^+\|U\|_v \le \frac{1}{2}\log\max(M_2, M_3) + c$ *for v archimedean, and* $\log^+\|U\|_v \le c$ *for v over S; so* $\sum_{v\in V} n_v \log^+\|U\|_v \le c(1+\log dh)$;

(c) $\deg e_{\alpha,j} \le 2(1+j)$ *and* $h_{\mathrm{aff}}(e_{\alpha,j}) \le c(1+j)^2 + \log\max(1, |r_2|, |r_3|) \le c(1+j+d)^2(1+\log h)$;

(d) *the tail bound of Theorem 5.1(ii) holds with* $b_v = \beta_p$ *for* v *over* $p \notin S$ *and* $b_v = c + 2\log^+ \|U\|_v + \log\max(1, |r_2|_v, |r_3|_v)$ *at the other places, and* $\sum_{v\in V} n_v b_v \le c(1+\log dh)$.

Proof. (a) By Lemma A.3, $h_{\mathrm{aff}}(A_k) \le \frac{1}{2}\log M_k + c(1+h)$, which is at most $c(h+\log d)$. By Lemma 5.2(a), $h(a), h(u_\nu) \le ch$. Conjugation preserves heights, $h(R) \le 2h(a)$, and $\widehat{A}_k = \mu_k A_k$ with μ_k in a finite set.

(b) At v over $p \notin S$ with $\tilde{s} \in \Delta_v$, $j_i(\tilde{s}) \equiv b_i$ with b_i and $b_i - 1728$ units, so the model of $E_{i,s}$ in Section 3.2 has integral coefficients and unit discriminant, and $\|A_k\|_v \le 1$ by Lemma A.1(a). Since $\sigma_{\mathfrak{P}}$ preserves $|\cdot|_{\mathfrak{P}}$, $\|\rho(A_k)\|_{\mathfrak{P}} = \|A_k\|_{\mathfrak{P}}$; also $|R|_{\mathfrak{P}} = 1$ in (III), while $|\mu_k|_v = 1$ in (IV) as $p \nmid m_1 m_k$, u is integral on the residue disk, and $|a|_v < 1$. At an archimedean v, $j_i(\tilde{s})$ stays in a compact subset of $\mathbb{C} \smallsetminus \{0, 1728\}$, so Lemma A.2 gives the bound for A_k, and a, u are bounded on Δ_v. At v over S, a fixed change of coordinates takes the model to one of good reduction on Δ_v, and Lemma A.1(a) gives $\|A_k\|_v \le c$. The archimedean weights add up to 1 and those over S to at most $|S|$; and $\max(M_2, M_3) \le c(dh)^c$.

(c) Each fixed germ $Z(a) = \sum_n Z_n a^n$ has $\|Z_n\|_v \le |n!|_v^{-1}$ at the places over $p \notin S$, by the proof of Lemma 3.1, and $\|Z_n\|_v \le c\varrho^{-n}$ at the finitely many others, with ϱ as in Section 3.2. So $h_{\mathrm{aff}}(Z_n) \le \log n! + c(1+n)$. The coefficient of a^j in an equation is a sum of at most $c(1+j)^5$ products of at most six such coefficients with indices adding up to j, of the fixed constants, of powers of R where it occurs, and of the coordinates, which occur with degree at most 2, and in the paired equations at most $j+1$. The constants r_k are integers with $|r_k| \le c\sqrt{N_{0k}M_k} \le c(dh)^c$.

(d) At v over $p \notin S$, all coordinates and fixed constants have absolute value at most 1, and in system (III) the factor R satisfies $|R|_v = 1$, so $|e_{\alpha,j}(U)|_v \le e^{j\beta_p}$, and since $|a|_v e^{\beta_p} < 1$,

$$\Big|\sum_{j\ge\ell} a^j e_{\alpha,j}(U)\Big|_v \le \sup_{j\ge\ell}(|a|_v e^{\beta_p})^j \le |a|_v^\ell e^{\ell\beta_p}.$$

At the other places $|a|_v \le r_v$, with $r_v < 1$ and $r_v < \varrho/2$ by Section 3.2, and the geometric series gives the bound with the stated b_v. The sum over v is at most $\sum_{p\in P(a)} \beta_p + c + 2\sum_v n_v \log^+ \|U\|_v + \log\max(1, |r_2|, |r_3|)$, which (b), (c) and Lemma 5.3(a) bound. □

5.5. **Proof of Theorem 1.3.** We also leave out the finite sets of Lemmas 3.3 and 5.3; the heights of the points left out are bounded. Let s be another modular point. By Lemma 5.3(b), either $h \le cd^c$, or the places of V_0 carry $\sum_{V_0} n_v \log|a|_v^{-1} \ge h/c$, and then, by Proposition 5.4, one group V carries at least $h/(cd)$. We apply Theorem 5.1 to this group, with $F = L$ for systems (I) and (II) and $F = \widetilde{L}$ otherwise, its system and its point. The fixed germs are $T_i^{\pm 1}$, g_ν and their conjugates by $\mathrm{Hom}(K, \overline{\mathbb{Q}})$, the dimension N, the number of equations and the degree $b \le 6$ of the combinations are bounded, and the paired systems evaluate germs at aR, where R is a coordinate of z. Hypotheses (i) and (ii) hold by Lemma 5.5 for a suitable fixed C_0, (iii) by construction, and (iv) by Proposition 4.7. So $h \le C_1 d^{C_1}$, and Theorem 1.3 follows. □

Appendix A. Frame estimates

This appendix bounds the matrices of isogenies in the frames of Section 3.2, which gives the size estimates of Lemma 5.5.

For a Weierstrass model $Y^2 + a_1XY + a_3Y = X^3 + a_2X^2 + a_4X + a_6$ of an elliptic curve put $\omega = dX/(2Y + a_1X + a_3)$ and $\xi = X\omega$. For an isogeny $f\colon E \to E'$ we write $[f^*]$ for the matrix of f^* on H^1_{dR} in such frames; the matrix of f_* on H_1^{dR} in the dual frames is its transpose. A change of coordinates $X = u^2X' + r$, $Y = u^3Y' + su^2X' + t$ gives $\omega = u^{-1}\omega'$

and $\xi = u\xi' + (r/u)\omega'$, a change of frame of determinant 1 whose entries have absolute values at most e^ℓ, $\ell = \log\max(1, |u|, |u|^{-1}, |r/u|)$.

Lemma A.1. *Let F be a finite extension of $\mathbb{Q}_p$ and $f\colon E_1 \to E_2$ an isogeny of elliptic curves over F.*

(a) *If E_1, E_2 have good reduction, then $[f^*]$ in the frames of models with integral coefficients and unit discriminant has entries in $\mathcal{O}_F$.*

(b) *If E_i is Tate's curve $\mathbb{G}_m/q_i^{\mathbb{Z}}$ with the model of [20, (13)], and f lifts to $t \mapsto t^{\pm n}$ with $q_1^n = q_2^k$, then*

$$f^*\omega_2 = \pm n\,\omega_1, \qquad f^*\xi_2 = \pm\big((n\,s(q_2) - k\,s(q_1))\,\omega_1 + k\,\xi_1\big), \qquad s(q) = -2\sum_{m\ge1}\frac{q^m}{(1-q^m)^2},$$

and these entries lie in $\mathcal{O}_F$.

Proof. (a) Let $\mathcal{E}$ be an elliptic curve over $\mathcal{O}_F$ with zero section O. Then $H^1_{\mathrm{dR}}(\mathcal{E}/\mathcal{O}_F) \cong H^0(\mathcal{E}, \Omega^1_{\mathcal{E}}(2O))$ [11, (A1.2.3), p. 164]. With the Weierstrass parameter $T = -X/Y$ at O, $X = T^{-2} + \cdots$ and $\omega = (1 + \cdots)dT$ with integral coefficients, so ξ has leading term $T^{-2}dT$. For $\eta \in H^0(\mathcal{E}, \Omega^1(2O))$ with T^{-2}-coefficient μ, $\eta - \mu\xi$ has at most a simple pole, hence none by the residue theorem, and lies in $H^0(\mathcal{E}, \Omega^1) = \mathcal{O}_F\omega$. So (ω, ξ) is a basis over $\mathcal{O}_F$, also for $p = 2, 3$. The models are Néron models of their generic fibers, so f extends to them [3, §1.2, Prop. 8], and f^* preserves the lattices $H^1_{\mathrm{dR}}(\mathcal{E}_i/\mathcal{O}_F)$.

(b) In Tate's parametrization $\omega = dt/t$ [20, (9)], and $X = \mathcal{X}_q(t) + s(q)$ with $\mathcal{X}_q(t) = \sum_{m\in\mathbb{Z}} q^m t/(1-q^m t)^2$ [20, (1)]. The case of $t \mapsto t^{-n}$ follows from that of t^n by composing with -1, which acts by -1 on H^1_{dR}. Clearly $(t^n)^*(dt/t) = n\,dt/t$. Put

$$G_q(t) = \sum_{m\ge0}\frac{q^m t}{1-q^m t} + \sum_{m<0}\frac{1}{1-q^m t};$$

then $dG_q = \mathcal{X}_q(t)\,dt/t$ and $G_q(qt) = G_q(t) + 1$. So $(t^n)^*(\mathcal{X}_{q_2}dt/t) = d(G_{q_2}(t^n))$, and since $(q_1 t)^n = q_2^k t^n$, the function $G_{q_2}(t^n) - kG_{q_1}(t)$ is invariant under $t \mapsto q_1 t$. Such q_1-periodic rigid meromorphic functions on $\mathbb{G}^{\mathrm{an}}_{m,F}$ are rational functions on E_1 [20], so $(t^n)^*(\mathcal{X}_{q_2}dt/t) \equiv k\,\mathcal{X}_{q_1}dt/t$ in $H^1_{\mathrm{dR}}(E_1)$, and adding $n\,s(q_2)\,dt/t$ gives the formula. As $|s(q)| \le |q| < 1$ and $n, k \in \mathbb{Z}$, the entries are integral. □

Lemma A.2. *Give H^1_{dR} of a complex elliptic curve the Hodge norm, for which $H^{1,0} \perp H^{0,1}$ and $\|\eta\|^2 = |\int \eta\wedge\bar\eta|$ on each part, and H^{dR}_1 the dual norm. An isogeny of degree n multiplies these norms by $\sqrt{n}$. For the curve of Section 3.2 with invariant j, the frame (ω, ξ) and its dual frame have norms at most $\exp F_\infty(j)$, where*

$$F_\infty(j) \le c + \tfrac14\log^+|1/j| + \tfrac14\log^+|1/(j-1728)| + \tfrac12\log(1+\log^+|j|).$$

So each entry of $[f^*]$ for an isogeny $f\colon E_{j_1} \to E_{j_2}$ of degree n has absolute value at most $\sqrt{n}\exp(F_\infty(j_1) + F_\infty(j_2))$.

Proof. If f is $z \mapsto \gamma z$ from $\mathbb{C}/\Lambda_1$ to $\mathbb{C}/\Lambda_2$ with $\gamma\Lambda_1 \subset \Lambda_2$ of index n, then $|\gamma|^2\mathrm{covol}(\Lambda_1) = n\,\mathrm{covol}(\Lambda_2)$, and f^* preserves the Hodge decomposition; this gives the first claim. Let τ be in the standard fundamental domain with $j(\tau) = j$ and $q = e^{2\pi i\tau}$. On $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ the model of [11, (A1.2.3), p. 160] has $c_4 = E_4(\tau)$, $c_6 = -E_6(\tau)$ and $\omega = 2\pi i\,dz$, with $X = (2\pi i)^{-2}\wp(z) - 1/12$. Our model has $c_4 = j/(j-1728)$, $c_6 = -j/(j-1728)$ and the same $b_2 = 1$, so the change between them has $u^2 = E_4/E_6$ and $r = (u^2-1)/12$. From $E_4^3 = j\Delta$, $E_6^2 = (j-1728)\Delta$, the boundedness of E_4, E_6 on the fundamental domain and

$|\Delta(\tau)| \geq c/(1+|j|)$ we get $\max(|u|, |u|^{-1}, |r/u|) \leq c(1+|1/j|+|1/(j-1728)|)^{1/4}$. Here $\|dz\|^2 = \|d\bar{z}\|^2 = 2\operatorname{Im}\tau$, and by Legendre's relation $[\wp\, dz] = (-\eta_1 + \pi/\operatorname{Im}\tau)\, dz - (\pi/\operatorname{Im}\tau)\, d\bar{z}$, where the quasi-period $\eta_1 = (\pi^2/3)E_2(\tau)$ is bounded. So $2\pi i\, dz$ and $X \cdot 2\pi i\, dz$ have norms at most $c(\operatorname{Im}\tau)^{1/2}$, and so does the dual frame, since the cup product of the two is a fixed nonzero constant. Finally $|j| \geq e^{2\pi \operatorname{Im}\tau} - c$, so $\operatorname{Im}\tau \leq c(1+\log^+|j|)$. □

Lemma A.3. *Let s be a modular point with a point $\tilde{s} \in \widetilde{C}^0$ over it. Then $h_{\mathrm{aff}}(A_k) \leq \frac{1}{2}\log M_k + c(1+h)$ for $k = 2, 3$.*

Proof. Let $F \supset L$ be a finite extension over which $E_{1,s}$ and $E_{k,s}$ have good or split multiplicative reduction at every finite place; the sums below do not change. At a finite place v let $\ell_v(E)$ be the ℓ of the change from the model of Section 3.2 to a model with integral coefficients and unit discriminant, if E has good reduction, or to Tate's model, if E has split multiplicative reduction. By Lemma A.1, $\log^+ \|A_k\|_v \leq \ell_v(E_{1,s}) + \ell_v(E_{k,s})$. In the first case $u^{12} = \Delta/\Delta'$ with $\Delta = j^2/(j-1728)^3$ and $|\Delta'|_v = 1$, and $r = (u^2 b_2' - 1)/12$ with $|b_2'|_v \leq 1$, so $\ell_v \leq \log|12|_v^{-1} + \frac{1}{12}(2|\log|j|_v| + 3|\log|j-1728|_v|)$. In the second case $|c_4|_v = |j/(j-1728)|_v = 1 = |E_4(q)|_v$, so $|u|_v = 1$, $r = (u^2-1)/12$ and $\ell_v \leq \log|12|_v^{-1}$. Since $\sum_v n_v |\log|y|_v| = 2h(y)$ and $h(j-1728) \leq h(j) + \log 3456$, the finite places contribute at most $c(1+h(x_1)+h(x_k))$. At the archimedean places, Lemma A.2 and $\sum_{v|\infty} n_v \log^+|1/y|_v \leq h(y)$ give at most $\frac{1}{2}\log M_k + c(1+h)$, using $\sum_{v|\infty} n_v \log(1+\log^+|j|_v) \leq \log(1+h(j))$ by concavity. □

Declarations

Use of AI. This paper and its results were produced by an AI research system that the author designed, built and directs, running on Anthropic's Claude Opus 5.5 in Claude Code and OpenAI's GPT-6.2 Astra in Codex. The author chooses the problems and prioritizes the work. Separate AI agents checked each result. The system also searched the literature and checked the citations against reliable sources. The author read the paper and takes full responsibility for all of the paper's contents.

References

1. Yves André, *G-functions and geometry*, Aspects Math., vol. E13, Friedr. Vieweg & Sohn, Braunschweig, 1989, doi:10.1007/978-3-663-14108-2.
2. Pierre Berthelot and Arthur Ogus, *F-isocrystals and de Rham cohomology. I*, Invent. Math. **72** (1983), no. 2, 159–199, doi:10.1007/BF01389319.
3. Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud, *Néron models*, Ergeb. Math. Grenzgeb. (3), vol. 21, Springer, Berlin, 1990.
4. Christopher Daw and Martin Orr, *Zilber–Pink in a product of modular curves assuming multiplicative degeneration*, Duke Math. J. **174** (2025), no. 13, 2877–2926, doi:10.1215/00127094-2025-0011, arXiv:2208.06338v4.
5. Christopher Daw, Martin Orr, and Georgios Papas, *Some new cases of Zilber–Pink in $Y(1)^3$*, 2026, Preprint, arXiv:2510.09603v2.
6. Christopher Daw and Georgios Papas, *Lang–Trotter phenomena and unlikely intersections*, 2026, Preprint, arXiv:2605.00759v2.
7. Pierre Deligne, *Théorie de Hodge. II*, Publ. Math. Inst. Hautes Études Sci. **40** (1971), 5–57, doi:10.1007/BF02684692.
8. Éric Gaudron and Gaël Rémond, *Théorème des périodes et degrés minimaux d'isogénies*, Comment. Math. Helv. **89** (2014), no. 2, 343–403, doi:10.4171/CMH/322, arXiv:1105.1230v1.
9. Philipp Habegger and Jonathan Pila, *Some unlikely intersections beyond André–Oort*, Compos. Math. **148** (2012), no. 1, 1–27, doi:10.1112/S0010437X11005604.

10. ———, *O-minimality and certain atypical intersections*, Ann. Sci. Éc. Norm. Supér. (4) **49** (2016), no. 4, 813–858, doi:10.24033/asens.2296, arXiv:1409.0771v1.
11. Nicholas M. Katz, *p-adic properties of modular schemes and modular forms*, Modular functions of one variable, III (Antwerp, 1972), Lecture Notes in Math., vol. 350, Springer, Berlin, 1973, doi:10.1007/978-3-540-37802-0_3, pp. 69–190.
12. Arthur Ogus, *F-isocrystals and de Rham cohomology. II. Convergent isocrystals*, Duke Math. J. **51** (1984), no. 4, 765–850, doi:10.1215/S0012-7094-84-05136-6.
13. OpenAI, *Elliptic squares and Zilber–Pink for curves in* $\mathcal{A}_2$, 2026, OpenAI Math Release preprint, dated September 24, 2026, in `openai/math` on GitHub, commit `3344da59` (November 25, 2026), https://github.com/openai/math/tree/3344da59991ed94f9ba711217b29c1d760badfa6/preprints/Elliptic-Squares-and-Zilber-Pink-for-Curves-in-A2-September-24-2026.
14. ———, *Quaternionic division points on curves in the Siegel threefold*, 2026, OpenAI Math Release preprint, dated September 24, 2026, in `openai/math` on GitHub, commit `3344da59` (November 25, 2026), https://github.com/openai/math/tree/3344da59991ed94f9ba711217b29c1d760badfa6/preprints/Quaternionic-Division-Points-on-Curves-in-the-Siegel-Threefold-September-24-2026.
15. Georgios Papas, *Zilber–Pink in* $Y(1)^n$*: Beyond multiplicative degeneration*, 2026, Preprint, arXiv:2402.09487v3.
16. Jonathan Pila, *On a modular Fermat equation*, Comment. Math. Helv. **92** (2017), no. 1, 85–103, doi:10.4171/CMH/407, arXiv:1508.05050v1.
17. Jonathan Pila and Jacob Tsimerman, *Ax–Schanuel for the j-function*, Duke Math. J. **165** (2016), no. 13, 2587–2605, doi:10.1215/00127094-3620005, arXiv:1412.8255v2.
18. Jonathan Pila and Alex J. Wilkie, *The rational points of a definable set*, Duke Math. J. **133** (2006), no. 3, 591–616, doi:10.1215/S0012-7094-06-13336-7.
19. Jonathan Pila and Umberto Zannier, *Rational points in periodic analytic sets and the Manin–Mumford conjecture*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. **19** (2008), no. 2, 149–162, doi:10.4171/RLM/514, arXiv:0802.4016v1.
20. John Tate, *A review of non-Archimedean elliptic functions*, Elliptic curves, modular forms, & Fermat's last theorem (Hong Kong, 1993), Ser. Number Theory, I, Int. Press, Cambridge, MA, 1995, pp. 162–184.

Toronto, Canada
Email address: `manvirjaswal@icloud.com`
URL: `https://manvirjaswal.github.io`