hexagon:2611.01260v1

On deformations of McNulty-Weigert and other matrices

Banica and Codex

Abstract

We study deformations of McNulty-Weigert matrices constructed from bases in odd prime dimension p. For outer matrix F_2 and 4 distinct labels with square cross-ratio, we construct p^2+2 dephased real analytic families of dimension $p-1$ with distinct tangent spaces. Every first-order deformation extends through order 2. At the order-10 matrix H_5 the defect is 8 and the dephased local dimension is 4. We find 4 entries whose phases can be prescribed freely near H_5. Each choice has finitely many nearby labeled completions, and for a generic choice exactly 27 of them have a nontrivial permutation symmetry. For outer matrix F_2 a rectangle partition function gives the defect, and isolation is equivalent to the entrywise square being Hadamard. For outer matrix F_3 and 6 distinct labels, we prove isolation for every prime $5 \leq p \leq 31$. For coprime $N, M \geq 2$, every Diţă deformation of $F_N \otimes F_M$ has the Berezin spectrum of F_{NM}. Finally, a projective McNulty-Weigert matrix admits a master representative exactly when both label lists are constant and the outer matrix is equivalent to a Hadamard submatrix of a Fourier matrix of order prime to p.

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Introduction

A complex Hadamard matrix of order N is a matrix H whose entries belong to the unit circle $\mathbb{T}$ and whose rows are pairwise orthogonal. Throughout we use the scalar product

$$\langle x, y\rangle = \sum_k x_k \bar{y}_k,$$

linear in the left variable. Thus the orthogonality and norm conditions together read $HH^* = NI$. Multiplying rows and columns by phases gives elementary deformations of H. We fix the 1st row and column at their original values to remove these deformations. We call this choice dephasing. For the basic constructions and examples of real and complex Hadamard matrices, see [2]. The Fourier matrices provide a first class of examples for which the defect can be computed explicitly [3].

How many parameters can occur in a family through a given H? Differentiating row orthogonality gives a real space T_H of possible velocities. Its dimension $d(H)$ is the dephased defect of Tadej and Zyczkowski [42]. Every smooth family gives velocities in this space, but some velocities may be obstructed at higher orders. In particular the real local dimension $\dim_H$ is at most $d(H)$. Section 1 works this out first in order 4 and then at the Fourier matrices $F_N = (w^{jk})$ with $w = e^{2\pi i/N}$. There the analytic curves of Nicoara and White [35] give a basis of velocities. We ask which families of deformations retain a permutation symmetry. We also seek a formula for the defect from the entries which avoids solving the linearized equations.

McNulty and Weigert [31] build a Hadamard matrix out of smaller changes of basis. Section 2 gives their general construction. For our results we use the following bases in odd prime dimension p. The standard $p+1$ mutually unbiased bases of $\mathbb{C}^p$ are labeled by the points of $P^1(\mathbb{F}_p) = \mathbb{F}_p \cup \{\infty\}$. Write B_c for the unitary matrix whose columns form the basis with label c. For distinct labels c, d, every entry of $B_c^* B_d$ has modulus $1/\sqrt{p}$.

Definition A. *Let p be an odd prime and K a Hadamard matrix of order N. Choose lists $s = (s_i)$ and $t = (t_j)$ of N points each in $P^1(\mathbb{F}_p)$, with $s_i \neq t_j$ for every i, j. The McNulty-Weigert matrix is the block matrix*

$$H_K(s,t) = \left(\sqrt{p}\, K_{ij} B_{s_i}^* B_{t_j}\right)_{0 \le i,j < N},$$

a Hadamard matrix of order Np.

For example, with outer matrix $K = F_2$, put $C_{ij} = \sqrt{p}\, B_{s_i}^* B_{t_j}$. The construction then reads

$$H_{F_2}(s,t) = \begin{pmatrix} C_{00} & C_{01} \\ C_{10} & -C_{11} \end{pmatrix}.$$

Each block records the overlaps between a source basis and a target basis. We will also ask when a matrix of this kind comes from the construction of Avan, Fonseca, Frappat, Kulish, Ragoucy and Rollet [1]. We use the real exponents of [4].

Definition B. *A Hadamard matrix H is a master Hadamard matrix if its entries have the form*

$$H_{ij} = e^{2\pi i x_i y_j}$$

for real numbers x_i, y_j. We say that H has a master representative if it is equivalent to a master Hadamard matrix.

The Fourier matrix F_N gives an example with $x_i = i/N$ and $y_j = j$. Here and below, equivalence allows permutations of rows and columns and multiplication of individual rows and columns by phases. We begin with deformations when the outer matrix is F_2. For 4 distinct labels, the defect depends only on whether their cross-ratio

$$\lambda = \frac{(s_0 - t_0)(s_1 - t_1)}{(s_0 - t_1)(s_1 - t_0)}$$

is a square in $\mathbb{F}_p$, with the usual cancellation at ∞. If it is not a square the defect is zero. If it is a square the defect is $2(p-1)$, and a family of dimension $p-1$ was found in [10]. At $p = 3$ the only admissible cross-ratio is 2 and the defect is zero. At $p = 5$ the only admissible square is 4. Pairing the nonzero labels as sources $1, -1$ and targets $2, -2$ gives precisely this value. We denote the resulting matrix of order 10 by H_5 and display it in Section 2. There is also a reason to single it out in the Butson classification. Of the 51 classes in $B(10, 10)$ listed by Lampio, Östergård and Szöllősi [27], just 2 up to transposition have positive defect and admit no dephased affine line. One is H_5.

Theorem A. *Suppose the* 4 *labels of* $H_{F_2}(s,t)$ *are distinct and their cross-ratio is a square. There are* p^2+2 *dephased real analytic families through this matrix, each of dimension* $p-1$*, with pairwise distinct tangent spaces. Every first-order deformation extends through order* 2*.*

At H_5 the linearized equations allow 8 independent directions. How many parameters can we actually vary? Fix the 1st row and column, keep the labels, and consider the problem of completing a matrix when a few more entries are prescribed. Here a permutation symmetry is a nonidentity pair of row and column permutations that preserves the matrix up to row and column phases.

Theorem B. *The dephased local dimension at* H_5 *is* 4*. There are* 4 *entries whose phases can be varied independently near their original values. Each such choice has finitely many nearby Hadamard completions. For a generic choice exactly* 27 *of these completions have a permutation symmetry.*

Numbering from zero in the matrix of Section 2, take positions $(1,2)$, $(4,9)$, $(7,4)$ and $(8,3)$. Each of the 27 families of Theorem A gives one completion. Coincidences occur only on an exceptional set of phase choices of dimension at most 2. Up to equivalence these families have just 2 types, represented by a family fixed by translation of the inner indices and a family fixed by reversal.

There are also completions with no permutation symmetry. With the labels fixed, the first 27 families and at least 10 further branches give distinct components of the complexified space through H_5, each of real local dimension 4. On each of these further branches there are matrices arbitrarily close to H_5 whose only remaining symmetry involves complex conjugation. Whether every nearby matrix admits a symmetry when conjugation is allowed remains open.

The dimension statement also has an arithmetic side. In a dephased neighborhood of H_5, every other Hadamard matrix has an entry of infinite order. In order 14 we obtain local dimension 6 for every square cross-ratio. This gives a counterexample to the isolation conjecture for odd and even labels of [4]. For 4 distinct labels of square cross-ratio at an arbitrary prime p, the question suggested by these examples is whether the local dimension always equals $p-1$.

Can we find the defect without solving the linearized equations? For the matrices of Definition A with outer matrix F_2, it suffices to sum a weight over rectangles. On rows i,k and columns j,l, its flux is $H_{ij}\bar{H}_{kj}H_{kl}\bar{H}_{il}$. Squaring this phase gives the weight of the doubled rectangle. Its partition function is

$$R_2(H)=\sum_{ikjl}\left(H_{ij}\bar{H}_{kj}H_{kl}\bar{H}_{il}\right)^2,$$

where all ordered choices of indices are included, even when a row or a column is repeated.

Theorem C. *For every* $H=H_{F_2}(s,t)$ *of order* $n=2p$*, the doubled rectangle gives the defect by*

$$d(H)=(n-2)\left(\frac{R_2(H)}{n^3}-1\right).$$

The matrix is isolated precisely when its entrywise square is Hadamard.

At H_5 we have $R_2(H_5)=2000$, so Theorem C gives the defect 8 directly. To obtain the same number from the complex glow, multiply the rows and columns of H by independent uniform phases and let Ω_H be the sum of the resulting entries. At fixed order, Section 3 expresses $R_2(H)$ in terms of the moment $E\left(|\Omega_H|^8\right)$ by comparison with the Fourier matrix.

What does the flux retain of the 4 labels beyond the defect? At the normalized matrices $K_p(c)=H_{F_2}((\infty,0),(1,c))$, the distribution on one rectangle depends only on whether c is a square. Joining 3 rectangles in a cycle gives the third trace of the Berezin matrix. In Theorem 3.8 we compute it from c. For example, at $p=11$ the squares 3 and 5 give the same flux

distribution and defect 20, but their third traces are 319/4 and 22. Corollary 5.2 uses these values to distinguish the Hessians of the function $S \to Tr|S|$ on the phase torus.

For an outer Fourier matrix of odd prime order, [10] asks whether distinct labels always give isolation. The first case is F_3, with 3 sources and 3 targets.

Theorem D. *For each prime $5 \le p \le 31$, every McNulty-Weigert matrix $H_{F_3}(s,t)$ with 6 distinct labels has defect zero and is isolated.*

The labels here range over $P^1(\mathbb{F}_p)$. At $p = 5$ they fill the projective line. At larger primes we must also choose which 6 points to use.

A repeated label leads to a different calculation. For outer matrix F_2, either the source list or the target list is then constant. For example, move a repeated source label to ∞. The inner Fourier entry in block column j then acquires the phase $w_p^{t_j a^2}$ in row a, where $w_p = e^{2\pi i/p}$. This gives a deformation of a Fourier product. More generally, put $w_h = e^{2\pi i/h}$. The Diţă deformation of $F_N \otimes F_M$ that we use has entries

$$L_{ia,jb} = w_N^{ij} w_M^{ab} Q_{aj}, \qquad i,j \in \mathbb{Z}_N, \quad a,b \in \mathbb{Z}_M,$$

where the phases Q_{aj} are arbitrary.

Theorem E. *If $N, M \ge 2$ are coprime, every such Diţă deformation of $F_N \otimes F_M$ has the same Berezin spectrum as F_{NM}. In particular all the cyclic traces Z_r and the defect agree.*

The common defect is $D(N)D(M) - 2NM + 1$, where $D(h) = \sum_{k=0}^{h-1} \gcd(h,k)$. In particular, a repeated label in outer order 2 gives defect $2(p-1)$ and a dephased affine family of dimension $p-1$. Thus a matrix in this class is isolated exactly when all 4 labels are distinct and their cross-ratio is not a square.

For the question about master matrices in Definition B we keep track of which rows and columns form each rectangle. When both label lists are constant, the same change of basis occurs in every block of $H_K(s,t)$. The matrix is then equivalent to $K \otimes F_p$. This leaves a question about the outer matrix K itself.

Theorem F. *The matrix $H_K(s,t)$ of Definition A has a master representative exactly when both label lists are constant and K is equivalent to a Hadamard submatrix of a Fourier matrix of order prime to p.*

For $K = F_N$ this means that both lists are constant and $\gcd(N,p) = 1$. In that case the matrix is equivalent to F_{Np}. The same test on minors applies to the regular matrices classified in [17]. Here regularity means that each quotient of distinct rows splits into rotated copies of complete sets of r-th roots of unity, with r prime. Among regular matrices of order 7, only the equivalence class of F_7 has a master representative.

Can a square master Hadamard matrix be isolated? The conjecture in [4] says that in orders greater than one only the prime Fourier matrices are isolated. We prove this whenever the entries after dephasing are roots of unity of level m with at most 2 distinct prime divisors. Here m is the least common multiple of the entry orders. In Theorem 11.9 a complete cycle of columns gives actual deformations of every other matrix in this class.

The paper is organized as follows. Section 1 reviews the defect and its relation to isolation. Section 2 gives the McNulty-Weigert construction and works out examples using Gauss sums. Section 3 computes the rectangle flux sums and relates them to the defect and the complex glow. In Sections 4 and 5 we construct analytic families and show that every tangent direction extends through order 2 when the cross-ratio is a square. Section 6 determines the local dimension in orders 10 and 14. Section 7 classifies nearby matrices with a permutation symmetry and counts their completions from 4 entries. Section 8 constructs real deformations outside those families and bounds the number of components. Section 9 treats 6 distinct labels for F_3. Section 10 computes the Berezin spectrum of deformed Fourier products and applies the formula to repeated labels for F_2. Section 11 tests for master representatives and settles the isolation conjecture at levels with at most 2 distinct prime divisors.

1 The defect of a Hadamard matrix

Suppose we are given a complex Hadamard matrix H and want to vary one of its entries. Its modulus is already fixed, so we can only move it along the unit circle. That motion changes the scalar products of its row with every other row. Can we move the other entries so as to compensate? The defect of Tadej and Zyczkowski [42] answers the linear part of this question.

Multiplying a whole row or column by a phase always works. We remove these motions by making the first row and column equal to 1. The resulting matrix has entries $D(H)_{ij} = H_{ij}H_{00}/(H_{i0}H_{0j})$. Equivalently, when varying a given H we can keep its first row and column at their original values.

Definition 1.1. *Let H be a complex Hadamard matrix of order N. Its dephased linearized space T_H consists of the real arrays A with $A_{0j} = A_{i0} = 0$ and*

$$\sum_k H_{ik}\bar{H}_{jk}(A_{ik} - A_{jk}) = 0$$

for every pair of distinct rows i, j. Its dephased defect is $d(H) = \dim_{\mathbb{R}} T_H$.

The equation comes from differentiating row orthogonality. We will use it both to compute the defect and to bound the number of parameters in an actual family.

Proposition 1.2. *The defect is invariant under row and column phases and permutations. The local dimension of the dephased Hadamard matrices at H is at most $d(H)$. In particular $d(H) = 0$ implies that H is isolated.*

Proof. Write a nearby matrix as $(H_{ik}e^{iX_{ik}})$ with X real and its first row and column zero. Differentiate a row scalar product at $X = 0$ in the direction A. After division by i we obtain Definition 1.1. Taking real and imaginary parts for each $i < j$ gives a real matrix J_H with $(N-1)^2$ columns. In the equation for (i, j) the coefficient of A_{ik} is $H_{ik}\bar{H}_{jk}$ and that of A_{jk} is its negative. Thus

$$d(H) = (N-1)^2 - \dim \operatorname{Im} J_H.$$

Before dephasing, the elementary phase directions are $A_{ik} = a_i + b_k$. They have dimension $2N - 1$, since adding opposite constants to the lists leaves A unchanged. Fixing the first row and column selects one representative modulo these directions. Phases and permutations carry the full linear system and its phase directions isomorphically to those of the equivalent matrix. This proves invariance.

Choose orthogonality equations whose derivatives form a basis of the row space of J_H. The implicit function theorem gives a smooth manifold of dimension $d(H)$ cut out by those equations near H. Every nearby solution belongs to this manifold. The remaining equations can lower its dimension further. □

If a minor of J_H does not vanish at H, it cannot vanish sufficiently close to H, so the defect can jump up at a special point of a family. The smallest example where this happens has order 4. We first compute the defect at its real members.

Proposition 1.3. *Every real Hadamard matrix H of order N has dephased defect $(N-1)(N-2)/2$.*

Proof. Keep all the phase variables and put $B_{ik} = H_{ik}A_{ik}$. The equation for rows i, j says exactly that $(BH^t)_{ij} = (BH^t)_{ji}$. Entrywise multiplication by H and right multiplication by H^t are both invertible. Thus BH^t runs freely through the real symmetric matrices. Removing the $2N - 1$ phase directions leaves

$$d(H) = \frac{N(N+1)}{2} - (2N-1) = \frac{(N-1)(N-2)}{2}.$$ □

We can follow the jump explicitly by moving a single parameter around the unit circle.

Proposition 1.4. *For $q \in \mathbb{T}$, the dephased Hadamard matrix*

$$H(q) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & q & -1 & -q \\ 1 & -q & -1 & q \end{pmatrix}$$

has defect 1 for $q \neq \pm 1$ and defect 3 for $q = \pm 1$. Its local dimension is 1 at every q.

Proof. The terms in each scalar product cancel in pairs. Suppose first that q is not real, and number the rows from zero. For rows $2, 0$ the defect equation is

$$q(A_{21} - A_{23}) - A_{22} = 0.$$

Its imaginary part gives $A_{21} = A_{23}$, and hence $A_{22} = 0$. Likewise $A_{31} = A_{33}$ and $A_{32} = 0$. Comparing rows $1, 2$ gives $A_{11} = A_{13}$ and $A_{12} = 0$. Rows $0, 1$ then give $A_{11} = 0$, and rows $2, 3$ give $A_{21} = A_{31}$. The solutions are precisely

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & a & 0 & a \\ 0 & a & 0 & a \end{pmatrix} \qquad (a \in \mathbb{R}).$$

Moving q realizes this direction. At $q = \pm 1$ Proposition 1.3 gives defect 3. The order-4 classification recalled by Tadej and Zyczkowski [41] consists precisely of the matrices $H(q)$ up to equivalence. After dephasing there are only finitely many row and column permutations to consider, so the local dimension is still 1 at the real members. □

In this example the phases are linear in the parameter t when we put $q = e^{it}$. Such deformations are called affine. There is a direct test for them.

Proposition 1.5. *Let H be Hadamard and let A be real. The matrices $(H_{ik}e^{itA_{ik}})$ are Hadamard for all real t exactly when*

$$\sum_{k:\, A_{ik} - A_{jk} = b} H_{ik}\bar{H}_{jk} = 0$$

for each pair $i \neq j$ and each value b of the row difference. In particular, every real Hadamard matrix of order $N \geq 4$ belongs to a dephased affine family of dimension at least 1.

Proof. Grouping the scalar product of rows i, j by the value of their phase difference gives

$$\sum_{b} e^{itb} \left(\sum_{k:\, A_{ik} - A_{jk} = b} H_{ik}\bar{H}_{jk} \right).$$

The exponentials with distinct b are linearly independent, since successive derivatives at zero give a Vandermonde matrix. This proves the test. Taking only the first derivative gives the weaker linear equation of Definition 1.1.

For a real H, choose 2 columns. In half the rows their entries agree and in half they are opposite, by column orthogonality. Choose a phase $q \in \mathbb{T}$. In each row where the selected entries are opposite, multiply them both by q. Within either part the scalar products stay fixed. Between the parts the 2 selected terms already sum to zero, so their common phase has no effect. A cross-ratio using a row from each part and one column inside and one outside the selected pair acquires the factor q or $\bar{q}$. The motion therefore survives dephasing. This is a deformation of the kind studied by Szöllősi [38]. □

At a Fourier matrix the defect has an arithmetic interpretation. Transforming each row of A converts the linear equations into equalities along cycles.

Theorem 1.6. *For the Fourier matrix $F_N = (w^{ik})$ with $w = e^{2\pi i/N}$, we have*

$$d(F_N) = \sum_{r=0}^{N-1} \gcd(N, r) - 2N + 1,$$

where $\gcd(N, 0) = N$. More generally, for the character table F_G of a finite abelian group G the sum is $\sum_{g \in G} |G|/\operatorname{ord}(g)$ and $N = |G|$.

Proof. First keep all phases. Write $P_{ir} = \sum_k w^{rk} A_{ik}$ for the Fourier coefficients of row i. The equation for rows $i, i - r$ reads

$$P_{ir} = P_{i-r,r}.$$

For fixed r, the coefficient is constant on each of the $\gcd(N, r)$ cycles of addition by r in $\mathbb{Z}_N$. For $r = 0$ there are N free coefficients. To count the real parameters, we use the fact that the coefficients at r and $-r$ are conjugate. Distinct opposite frequencies together give $2\gcd(N, r)$ real parameters, and when $r = -r$ each coefficient is real. Subtracting $2N - 1$ phases gives the formula of Tadej and Zyczkowski [42]. For F_G, index rows by elements of G and columns by its characters. For a row difference g the same transform gives constancy on the cycles of translation by g. There are $|G|/\operatorname{ord}(g)$ of these, which gives the group formula. Tadej announced his Fourier product formulas in 2011 [39] and derived them using the Berezin transform [40]. An independent derivation appears in [3]. Nicoara and White [34] obtain the element-order formula for the commuting square of any finite group. □

Ma and Zhang [28] show that deleting any 2 rows of F_p, for $p \geq 5$ prime, gives a deformation. More generally, for $1 \leq m < p$ retained rows, each nonzero Fourier frequency gives differences along the paths left by cutting its translation cycle at the missing rows. Arbitrary edge differences on these paths can be prescribed. Pairing conjugate frequencies gives surjectivity onto the $m(m - 1)$ real orthogonality equations. The implicit function theorem and removal of the $m + p - 1$ row and column phases give dephased local dimension $(m - 1)(p - m - 1)$. At $m = p - 2$ this is $p - 3$, disproving the density conjecture of [4].

Even at order 9 the answer depends on the abelian group underlying the Fourier matrix. We will use F_6, F_{10} and F_{14} for comparison with the product bases of Section 2.

Proposition 1.7. *For a prime p the matrix F_p is isolated. For distinct primes p, r we have $d(F_{pr}) = 2(p - 1)(r - 1)$. At order 9 the cyclic Fourier matrix has defect 4, while $F_3 \otimes F_3$ has defect 16.*

Proof. For a prime p the gcd sum in Theorem 1.6 is $2p - 1$, so F_p is isolated. For distinct primes p, r the sum factors by the Chinese remainder theorem, giving $d(F_{pr}) = 2(p - 1)(r - 1)$. For instance the cycle counts at $N = 6$ are $6, 1, 2, 3, 2, 1$, so $d(F_6) = 15 - 11 = 4$. The same formula gives

N	4	5	6	7	8	9	10	14
$d(F_N)$	1	0	4	0	5	4	8	12

In $\mathbb{Z}_3^2$ the 8 elements other than zero have order 3, giving $d(F_3 \otimes F_3) = 9 + 8 \cdot 3 - 17 = 16$ by the group formula. □

Karabegov [24] studied the reconstruction of a unitary matrix from the moduli of its entries. In this formulation the normalized Hadamard matrices are precisely the unitaries whose squared moduli are all $1/N$. The defect has the following differential interpretation.

Lemma 1.8. *Let $U = H/\sqrt{N}$ and $f(V)_{ik} = |V_{ik}|^2$ for $V \in U_N$. Modulo row and column phases, $\ker df_U$ identifies with T_H by the correspondence $\dot{U}_{ik} = iA_{ik}U_{ik}$.*

Proof. Since $U_{ik} \neq 0$, the derivative $2\,Re(\dot{U}_{ik}\bar{U}_{ik})$ vanishes exactly when $\dot{U}_{ik} = iA_{ik}U_{ik}$ with A_{ik} real. The condition that $\dot{U}$ be tangent to U_N is $\dot{U}U^* + U\dot{U}^* = 0$. Substituting the entries gives the row equations of Definition 1.1. Dephasing removes the same $2N - 1$ directions as before. Tadej and Zyczkowski [42] use the map f to define the defect for more general unitaries. □

A direction in T_H need not extend to a curve. The next equation explains what it means for a direction to survive through order 2.

Proposition 1.9. *Let $A \in T_H$. The phases $tA_{ik} + t^2B_{ik}/2$ satisfy row orthogonality through order 2 exactly when the real dephased array B solves*

$$i\sum_k H_{ik}\bar{H}_{jk}(B_{ik} - B_{jk}) = \sum_k H_{ik}\bar{H}_{jk}(A_{ik} - A_{jk})^2$$

for all distinct rows i, j.

Proof. For rows i, j put $u_k = A_{ik} - A_{jk}$ and $v_k = B_{ik} - B_{jk}$. In their scalar product the exponential has the following expansion

$$e^{itu_k + it^2v_k/2} = 1 + itu_k + \tfrac{1}{2}t^2(iv_k - u_k^2) + O(t^3).$$

The constant and linear terms sum to zero. The quadratic term gives the assertion, with a single correction B required to work for all pairs. □

Higher orders can distinguish matrices with the same first-order behavior. At the order-6 matrix of Diţă, Bengtsson, Bruzda, Ericsson, Larsson, Tadej and Zyczkowski [6] extended all 4 parameters through order 3. At F_{10} the calculations of Barros e Sá and Bengtsson [5] first detected an obstruction at order 11. At F_{12} they found evidence for 2 families of dimension 13 meeting along a family of dimension 9.

At F_N the test of order 2 is always satisfied. We recall what is known about the directions which survive it.

Theorem 1.10. *For every N, derivatives of analytic curves through F_N span T_{F_N}, and every vector in T_{F_N} extends through order 2. If N is a prime power, F_N lies in an affine family of dimension $d(F_N)$. If N is neither a prime power nor twice a prime, some linearized directions are obstructed at higher orders.*

Proof. The spanning curves are constructed by Nicoara and White [35]. Their second-order theorem [36] gives the correction B above for every prescribed A, including sums of velocities of different curves. For prime powers, the affine Fourier family of Tadej and Zyczkowski [42] attains the defect. The higher-order obstructions in the stated orders are due to Hopkins and Nicoara [20]. □

McNulty and Weigert [33] ask whether an isolated Hadamard matrix can have positive defect. Write $B(N, m)$ for the Hadamard matrices of order N whose entries are m-th roots of unity. The example comes from the 2019 catalogue of Lampio, Östergård and Szöllősi, line 394 of the file for $B(13, 6)$. We record its entries by the exponents

$$H_{ij} = w^{E_{ij}}, \qquad w = e^{\pi i/3},$$

so the digits in the display below are read modulo 6.

Theorem 1.11 (Hall and Nicoara)**.** *The following $BH(13,6)$ matrix, written in logarithmic form as*

$$E = \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&2&2&2&3&3&4&4&4\\
0&0&0&1&3&4&4&5&1&4&2&2&3\\
0&0&2&4&4&1&3&4&3&0&1&5&2\\
0&0&3&3&2&0&2&5&5&2&4&2&5\\
0&2&3&0&3&5&0&3&1&2&5&5&3\\
0&2&5&3&5&4&2&2&5&0&0&2&3\\
0&2&5&4&1&0&5&3&3&4&3&1&1\\
0&3&1&3&5&3&1&5&0&3&3&5&1\\
0&3&4&0&2&4&2&0&3&0&2&4&0\\
0&4&1&4&3&3&5&1&3&2&0&1&5\\
0&4&3&1&5&2&4&3&1&0&4&2&0\\
0&4&3&2&1&1&5&1&5&4&1&4&3
\end{pmatrix},$$

has defect 1 *and is isolated among all complex Hadamard matrices of order* 13 *up to equivalence.*

Proof. For this matrix the dephased linearized equations have 144 real unknowns. Hall and Nicoara [18] compute their rank to be 143, so the solution space is a line. They also show that the only vector on this line satisfying their necessary condition of order 2 is the zero vector. A sequence of distinct dephased Hadamard matrices converging to H would give a unit vector on the line satisfying that condition. This proves isolation and gives a counterexample to Conjecture 1.8 of [4] in the square case. □

2 McNulty-Weigert matrices

A pair of orthonormal bases gives a unitary change-of-basis matrix. If all its entries have the same absolute value, multiplying by the square root of the dimension gives a Hadamard matrix. McNulty and Weigert [31] apply this observation to product bases. In $F_2 \otimes F_M$ the same basis in $\mathbb{C}^M$ is used twice. Their construction allows the choice of that basis to depend on the vector in the first factor.

Definition 2.1. *Let K be a Hadamard matrix of order N. Choose 2 lists of unitary basis matrices B_{s_i}, B_{t_j} of order M such that every entry of $B_{s_i}^* B_{t_j}$ has absolute value $1/\sqrt{M}$ for all i, j. The McNulty-Weigert matrix $H_K(s,t)$ is defined by the blocks*

$$H_K(s,t)_{ia,jb} = \sqrt{M}\, K_{ij} (B_{s_i}^* B_{t_j})_{ab},$$

with $0 \le i, j < N$ and $0 \le a, b < M$. A basis can occur more than once within either list.

Two bases satisfying this condition are called mutually unbiased. Their overlaps give every entry in the block the required modulus. In Definition A we take $M = p$ and choose from the standard $p + 1$ mutually unbiased bases. Orthogonality across different blocks follows from the tensor product.

Theorem 2.2. *The matrix $H_K(s,t)$ is Hadamard of order NM. Dividing it by $\sqrt{NM}$ gives the change-of-basis matrix between 2 orthonormal product bases.*

Proof. Take the standard basis (e_i) in $\mathbb{C}^N$ and the columns (κ_j) of $K/\sqrt{N}$. In $\mathbb{C}^N \otimes \mathbb{C}^M$ form the vectors

$$u_{ia} = e_i \otimes B_{s_i} e_a, \qquad v_{jb} = \kappa_j \otimes B_{t_j} e_b.$$

The u's form an orthonormal basis. Different i's are separated in the first factor, and for the same i the second factors form an orthonormal basis. The same argument applies to the v's. With the scalar product linear on the left we obtain

$$\sqrt{NM}\,\langle v_{jb}, u_{ia}\rangle = \sqrt{NM}\,\langle \kappa_j, e_i\rangle\langle B_{t_j}e_b, B_{s_i}e_a\rangle = \sqrt{M}\,K_{ij}(B_{s_i}^* B_{t_j})_{ab}.$$

The right-hand side has modulus 1 by hypothesis. The matrix on the left is $\sqrt{NM}$ times a unitary, proving both assertions. □

How large can a set of mutually unbiased bases be? In dimension d the upper bound is $d+1$, attained at prime powers, as discussed by Durt, Englert, Bengtsson and Zyczkowski [13]. In dimension 6, Zauner [48] conjectured that the maximum is 3. Jaming, Matolcsi, Móra, Szöllősi and Weiner [22] constructed a family of triples containing members of the Fourier family. OpenAI [37] announced a computer-assisted proof of the bound 3 under stated arithmetic and compiler assumptions, and Cárdenes Wuttig and Tindall [8] obtained it by classifying the triples.

For the rest of this section we take $M = p$, where p is an odd prime. We take the bases of Ivanović [21] and Wootters and Fields [46]. With $w = e^{2\pi i/p}$ they are

$$B_\infty = I, \qquad B_c = D^c F_p/\sqrt{p}, \qquad D = diag(w^{a^2})_{a\in\mathbb{F}_p}, \qquad c \in \mathbb{F}_p.$$

The finite-field construction is discussed more generally by Klappenecker and Rötteler [25]. The labels here form the projective line $P^1(\mathbb{F}_p) = \mathbb{F}_p \cup \{\infty\}$. Thus ∞ labels the standard basis and 0 labels the Fourier basis.

Proposition 2.3. *Any 2 distinct labels give mutually unbiased bases. For finite labels $s \neq t$, put $h = t - s$ and $G(h) = \sum_{x\in\mathbb{F}_p} w^{hx^2}$. The block $C_h = \sqrt{p}\,B_s^* B_t$ is the symmetric circulant matrix*

$$(C_h)_{ab} = \frac{G(h)}{\sqrt{p}}\, w^{-(b-a)^2/(4h)},$$

where the exponent is read in $\mathbb{F}_p$. We have $|G(h)| = \sqrt{p}$ and $G(h) = \chi(h)G(1)$, where χ is the quadratic character.

Proof. With source ∞ the block is $D^t F_p$ and its entries have modulus 1. With target ∞ take the adjoint. For finite labels the entry is $p^{-1/2}\sum_x w^{hx^2+(b-a)x}$. Completing the square and translating x gives the claimed formula. Its exponent is symmetric in a, b and depends only on their difference.

For the modulus put $u = x - y$, $v = x + y$. Since p is odd this change of variables is a bijection, and

$$|G(h)|^2 = \sum_{x,y} w^{h(x^2-y^2)} = \sum_u \sum_v w^{huv} = p.$$

Only $u = 0$ survives the sum over v. To determine the sign, recall that $\chi(0) = 0$. For $y \neq 0$, we have $\chi(y) = 1$ if y is a square and $\chi(y) = -1$ if y is not a square. There are $1 + \chi(y)$ square roots of y. Hence

$$G(h) = \sum_y (1 + \chi(y))w^{hy} = \chi(h)G(1)$$

by the substitution $y \to h^{-1}y$ and the vanishing of $\sum_y w^{hy}$. □

Quadratic Gauss sums also occur in the construction of Damski and Zyczkowski [11]. For primes $r < p$, a common permutation of coordinates takes r of their $r+1$ product bases in $\mathbb{C}^r \otimes \mathbb{C}^p$ to bases of maximally entangled vectors. The matrices of these r bases are all equivalent, after rescaling, to F_{rp}.

At $p = 3$ we need just 2 matrices to write down all the input bases. They are

$$F_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & w & w^2 \\ 1 & w^2 & w \end{pmatrix}, \qquad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & w & 0 \\ 0 & 0 & w \end{pmatrix}.$$

The 4 bases are $I, F_3/\sqrt{3}, DF_3/\sqrt{3}, D^2F_3/\sqrt{3}$. Since $G(1) = 1 + 2w = i\sqrt{3}$ and $G(-1) = -i\sqrt{3}$, Proposition 2.3 gives the changes from 0 to 1 and back, multiplied by $\sqrt{3}$, as

$$C_1 = i \begin{pmatrix} 1 & w^2 & w^2 \\ w^2 & 1 & w^2 \\ w^2 & w^2 & 1 \end{pmatrix}, \qquad C_{-1} = -i \begin{pmatrix} 1 & w & w \\ w & 1 & w \\ w & w & 1 \end{pmatrix}.$$

McNulty and Weigert [32] showed that, up to equivalence of the pair of product bases, the possibilities in order 6 reduce to the Fourier family and the matrix of Tao. We can see both here. First repeat the standard basis in the source.

Proposition 2.4. *For $p = 3$, $K = F_2$ and $s = (\infty, \infty)$, the target lists $(0,0)$ and $(0,1)$ give members of the dephased family*

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & w & w^2 & x & xw & xw^2 \\ 1 & w^2 & w & y & yw^2 & yw \\ 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & w & w^2 & -x & -xw & -xw^2 \\ 1 & w^2 & w & -y & -yw^2 & -yw \end{pmatrix}, \qquad x, y \in \mathbb{T},$$

at $(x, y) = (1, 1)$ and (w, w), respectively. Every member has defect 4.

Proof. Put $P = diag(1, x, y)$. The 4 blocks in row order are $F_3, PF_3, F_3, -PF_3$. Thus the matrix is Hadamard for every x, y. At $P = I$ it is $F_2 \otimes F_3$, equivalent to F_6. At $P = D$ the second block column uses the basis B_1. The coprime Fourier product formula of Theorem 10.1 gives

$$d = (2 + 1)(3 + 1 + 1) - 2 \cdot 6 + 1 = 4.$$

The 2 visible phases therefore account for half the defect. □

For the other choice all 4 labels are distinct. It is convenient to do the calculation once at a general odd prime before returning to order 6.

Proposition 2.5. *Let $K_p(c) = H_{F_2}((\infty, 0), (1, c))$ for $c \in \mathbb{F}_p - \{0, 1\}$. Put $V_h = (w^{h(b-a)^2})_{ab}$ and $k = -1/4$ in $\mathbb{F}_p$. Then*

$$K_p(c) \sim \begin{pmatrix} F_p & D^{c-1}F_p \\ V_k & -\chi(c)V_{k/c} \end{pmatrix}.$$

Its defect is $2(p-1)$ if c is a square and zero if c is not a square.

Proof. Multiply the upper block row by D^{-1} on the left, and the lower block row by $\sqrt{p}/G(1)$. Proposition 2.3 gives the displayed blocks. For the defect we use the formula in [10] for 4 distinct labels. It gives $p - 1$ times the number of fixed points of the projective involution interchanging the sources and interchanging the targets. Here the involution is $x \to c/x$, so

$$d(K_p(c)) = (p - 1)\#\{x \in \mathbb{F}_p : x^2 = c\}.$$

The equation has 2 roots if c is a square and no roots otherwise. In these coordinates the cross-ratio of the 4 labels is precisely c. □

At $p = 3$ there is just one choice, $c = 2 = -1$. It uses the entire projective line and recovers the matrix of Tao [43] by the construction of McNulty and Weigert [31].

Proposition 2.6. *With $w = e^{2\pi i/3}$, the matrix $K_3(2)$ is equivalent to*

$$T_6 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & w & w^2 & w^2 & w \\ 1 & w & 1 & w & w^2 & w^2 \\ 1 & w^2 & w & 1 & w & w^2 \\ 1 & w^2 & w^2 & w & 1 & w \\ 1 & w & w^2 & w^2 & w & 1 \end{pmatrix}.$$

We have $d(T_6) = 0$, so T_6 is isolated among complex Hadamard matrices of order 6.

Proof. In the dephased $K_3(2)$ take the rows in the order $0, 1, 3, 2, 4, 5$ and the columns in the order $0, 5, 3, 4, 2, 1$. This gives the displayed matrix. The entrywise quotient of any 2 distinct rows contains $1, w, w^2$ exactly twice each. For the defect, the squares modulo 3 are 0 and 1, so Proposition 2.5 gives zero at $c = 2$. Isolation follows from Proposition 1.2. □

For the full order-6 classification, Cárdenes Wuttig and Tindall [7] recover each equivalence class from a suitable dephased 3×3 corner. We next take an outer matrix with 3 rows. Here isolation can occur even when both lists contain a repetition.

Proposition 2.7. *Take $p = 3$, $K = F_3$, $s = (\infty, \infty, 1)$ and $t = (0, 0, 2)$. Write $F = F_3$, $w = e^{2\pi i/3}$ and $D = \mathrm{diag}(1, w, w)$. The resulting matrix S_9 of McNulty and Weigert [31] is equivalent to*

$$\begin{pmatrix} F & F & D^2F \\ F & wF & w^2D^2F \\ E & w^2E & -w\bar{E} \end{pmatrix}, \qquad E = \begin{pmatrix} 1 & w & w \\ w & 1 & w \\ w & w & 1 \end{pmatrix}.$$

Its defect is zero, so it is isolated among complex Hadamard matrices of order 9.

Proof. The first 2 source labels are ∞. Before multiplying by the entries of K, both block rows are F, F, D^2F. In the last block row the differences of labels are $-1, -1, 1$. Removing the common phase of C_{-1} and multiplying by the outer Fourier entries gives $E, w^2E, -w\bar{E}$.

For an exact defect calculation write each row quotient as $H_{ik}\bar{H}_{jk} = u_k + v_k w$, with $u_k, v_k \in \mathbb{Z}$. Since $1, w$ are linearly independent over $\mathbb{R}$, each pair $i < j$ gives 2 real equations. For each weight list put the weights in row i of the phase array, their negatives in row j, and delete the fixed coordinates. For the resulting integer matrices J the calculations are

matrix	size of J	$\det(J^tJ) \bmod 5$
S_9	72×64	3
T_6	30×25	3

Thus J has independent columns at S_9. The second line gives the same check directly from the entries of T_6 in Proposition 2.6. □

Starting at $p = 5$ we can obtain different defects by changing just the last target label. The blocks themselves make the comparison quite concrete.

Proposition 2.8. *In order 10 we have the representatives*

$$K_5(2) \sim \begin{pmatrix} F_5 & DF_5 \\ V_1 & V_3 \end{pmatrix}, \qquad K_5(4) \sim \begin{pmatrix} F_5 & D^3F_5 \\ V_1 & -V_4 \end{pmatrix},$$

with defects 0 and 8, respectively. In order 14 the corresponding examples $K_7(2)$ and $K_7(3)$ have defects 12 and 0.

Proof. In $\mathbb{F}_5$ we have $k = -1/4 = 1$ and $D = diag(1, w, w^4, w^4, w)$. The circulant V_h has first row $(1, w^h, w^{4h}, w^{4h}, w^h)$, with each subsequent row shifted one place to the right. Proposition 2.5 gives the blocks. The squares in $\mathbb{F}_5$ are $0, 1, 4$, so $x^2 = 2$ has no solutions and $x^2 = 4$ has the 2 solutions ± 2. In $\mathbb{F}_7$ the squares are $0, 1, 2, 4$, with $3^2 = (-3)^2 = 2$. The same count gives $2(7-1) = 12$ at $c = 2$ and zero at $c = 3$. McNulty and Weigert [31] already found examples of defects 8 and 12 in these orders. □

The product bases defining $K_5(4)$ belong to a triple of the form described by McNulty, Pammer and Weigert [30]. The first 2 bases use the label pairs $(\infty, 0)$ and $(1, 4)$. To obtain the 3rd, pair the qubit vectors $(1, \pm i)/\sqrt{2}$ with the unused bases B_2, B_3, respectively. Each qubit overlap has modulus $1/\sqrt{2}$ and each overlap in $\mathbb{C}^5$ has modulus $1/\sqrt{5}$. Yang, Liu, Li and Deng [47] showed that no vector is unbiased to all 3 bases of any such product triple in dimension 10.

For the local dimension problem it is convenient to use the sources $1, -1$ and the targets $2, -2$. At $p = 5$ this gives H_5 in a form with 4 circulant blocks. McNulty and Weigert [31] displayed an equivalent matrix and computed its defect to be 8. At $p = 7$ the same labels give, up to equivalence, a counterexample to the isolation conjecture for odd and even labels of [4].

Proposition 2.9. *Let $H_p = H_{F_2}((1,-1),(2,-2))$ for $p = 5, 7$. Removing a common phase gives*

$$H_5 \sim \begin{pmatrix} V_1 & -V_3 \\ -V_2 & -V_4 \end{pmatrix}, \qquad H_7 \sim \begin{pmatrix} V_5 & V_3 \\ -V_4 & V_2 \end{pmatrix},$$

where $V_h = (w^{h(b-a)^2})_{ab}$ is taken at the indicated prime. We have $d(H_5) = 8$ and $d(H_7) = 12$.

Proof. In order 10, choose $w = e^{2\pi i/5}$. After removing the common scalar the matrix is

$$H_5 = \left(\begin{array}{ccccc|ccccc} 1 & w & w^4 & w^4 & w & -1 & -w^3 & -w^2 & -w^2 & -w^3 \\ w & 1 & w & w^4 & w^4 & -w^3 & -1 & -w^3 & -w^2 & -w^2 \\ w^4 & w & 1 & w & w^4 & -w^2 & -w^3 & -1 & -w^3 & -w^2 \\ w^4 & w^4 & w & 1 & w & -w^2 & -w^2 & -w^3 & -1 & -w^3 \\ w & w^4 & w^4 & w & 1 & -w^3 & -w^2 & -w^2 & -w^3 & -1 \\ \hline -1 & -w^2 & -w^3 & -w^3 & -w^2 & -1 & -w^4 & -w & -w & -w^4 \\ -w^2 & -1 & -w^2 & -w^3 & -w^3 & -w^4 & -1 & -w^4 & -w & -w \\ -w^3 & -w^2 & -1 & -w^2 & -w^3 & -w & -w^4 & -1 & -w^4 & -w \\ -w^3 & -w^3 & -w^2 & -1 & -w^2 & -w & -w & -w^4 & -1 & -w^4 \\ -w^2 & -w^3 & -w^3 & -w^2 & -1 & -w^4 & -w & -w & -w^4 & -1 \end{array}\right).$$

Writing its upper blocks as A, B, their first rows are

$$(1, w, w^4, w^4, w), \qquad -(1, w^3, w^2, w^2, w^3).$$

The lower blocks are $B^*, -A^*$. The quadratic chirps give $AA^* = BB^* = 5I$, and the circulants commute. Thus the diagonal blocks of $H_5 H_5^*$ are $10I$ and the upper right block is $AB - BA = 0$. Allowing A, B to vary is the starting point of Section 4.

The differences $t_j - s_i$ are $1, -3, 3, -1$ in block order. Inserting the entries of F_2 gives

$$H_p = \begin{pmatrix} C_1 & C_{-3} \\ C_3 & -C_{-1} \end{pmatrix}.$$

Divide by $G(1)/\sqrt{p}$ and use Proposition 2.3. For instance, at $p = 7$ the first row of V_h is $(1, w^h, w^{4h}, w^{2h}, w^{2h}, w^{4h}, w^h)$, which determines all 49 entries of the block. The cross-ratio of the chosen lists is

$$\frac{(1-2)(-1+2)}{(1+2)(-1-2)} = \frac{1}{9}.$$

This is 4 modulo both 5 and 7. Since 4 is a square in both fields, the formula in [10] gives defect $2(p-1)$ in both cases. □

To recover the dephased form of H_5 from the order-10 matrix with defect 8 displayed by McNulty and Weigert [31], remove their unitary normalization and read the columns, numbered from 1, in the order

$$1, 4, 2, 5, 3, 6, 10, 9, 8, 7.$$

The rows are already in the required order. Egan, Flannery and Ó Catháin [14] report 7 classes in $B(10, 10)$ from the product basis construction in their electronic supplement. The classification of all Butson matrices at these parameters gives a further comparison.

Proposition 2.10. *Of the* 51 *equivalence classes in* $B(10, 10)$*,* 46 *admit a nonconstant dephased affine deformation. Of the other* 5*, one has defect zero and the remaining* 4 *have defect* 8*. These* 4 *form* 2 *transpose pairs, one represented by* H_5 *and* H_5^t*.*

Proof. Use the representatives in the electronic supplement of Lampio, Östergård and Szöllősi [27], counting from 1 in their file for $B(10, 10)$. In 46 representatives there are 2 rows or 2 columns whose entrywise quotients are a fixed phase times signs. For a column pair, multiply both entries in the rows carrying the minus sign by $q \in \mathbb{T}$. Between the 2 parts the contribution of these columns to a row scalar product is zero. Within a part it is unchanged. A rectangle using one row from each part and one of the selected columns detects the parameter after dephasing. Transposition gives the row case.

The exceptions have indices $45, 46, 48, 50, 51$. Expand the differentiated row equations for rational velocities in the basis $1, z, z^2, z^3$ with $z = e^{\pi i/5}$. In each case the resulting integer matrix has rank 81 modulo 101. Hence there is no rational velocity. Any affine line would give a rational velocity by the partition argument in Theorem 4.5. For the real defect use $t = 2\cos(\pi/5)$, with $t^2 = t + 1$. Take twice the real part and divide the imaginary part by $\sin(\pi/5)$. The equations have coefficients in $\mathbb{Q}(t)$. Replacing multiplication by each coefficient by its 2×2 rational matrix gives ranks $146, 146, 146, 146, 162$. Dividing by 2 gives the ranks on the 81 dephased variables. The last representative has only 5th roots of unity as entries. Dephasing after explicit permutations identifies H_5, H_5^t with entries $46, 48$, and transposition pairs 45 with 50. The program `catalogue.py` verifies these permutations and ranks. □

For comparison, the consecutive odd sources $(1, 3)$ and even targets $(2, 4)$ of [4] have cross-ratio $-1/3$. At $p = 5$ this is 3 and the defect is zero. The other possible even pair $(0, 2)$ gives the same value. Our H_5 therefore lies outside that conjecture. At $p = 7$ the same cross-ratio is 2, a square. In the convention of [10] the old basis with index k has label $x = k/2$. The projective transformation $x \to 4x/(x + 1)$ has square determinant and sends the 2 old pairs to $(-1, 1)$ and $(2, -2)$. Interchanging the block rows and absorbing a block sign gives H_7. Section 6 determines its local dimension.

3 Rectangle fluxes and the defect

Choose 2 rows and 2 columns of a Hadamard matrix. Multiply the entries around the resulting rectangle, taking the conjugate at every other corner. The resulting phase is unchanged when any row or column is multiplied by a scalar of modulus 1. In the spin model language of Jones [23], the indices are spins and the entries are weights. Summing the weights over the spins gives a partition function. Here we compute the flux distribution for the matrices of Definition A with outer matrix F_2. Its second moment will distinguish the isolated matrices from those which admit deformations.

Definition 3.1. *For a matrix* H *of order* n *with entries in* $\mathbb{T}$*, the flux of the rectangle with rows* i, k *and columns* j, l *is*

$$u_{ik;jl} = H_{ij}\bar{H}_{kj}H_{kl}\bar{H}_{il}.$$

The m*-th flux sum of* H *is* $R_m(H) = \sum_{ikjl} u_{ik;jl}^m$*, where* $m \geq 1$*. The indices run independently, including rectangles with a repeated row or column.*

Thus $R_m(H)/n^4$ is the m-th moment of the flux of a uniformly chosen ordered rectangle. All these moments determine its distribution on $\mathbb{T}$, since the moments with negative indices are their complex conjugates. In particular, R_2 is the partition function of the doubled rectangle. Row and column permutations preserve the sum, as do row and column phases.

Lemma 3.2. *Write $S_m = (H_{ij}^m)$. For any matrix H of order n with entries in $\mathbb{T}$, we have*

$$R_m(H) = Tr\big((S_m S_m^*)^2\big) = n^3 + \sum_{i\neq k}\Big|\sum_j H_{ij}^m \bar{H}_{kj}^m\Big|^2.$$

In particular $R_m(H) = n^3$ exactly when S_m is a Hadamard matrix.

Proof. For fixed i, k, the sum over j, l is the squared modulus of the scalar product of rows i, k of S_m. The n diagonal terms are each n^2. □

There are 2 ways to read the doubled rectangle. One uses the Jones transfer matrix and the other uses the energy of planar rotators [9]. Let a, b be independent vectors whose coordinates are independent uniform phases. We use the variable $\Omega_H = \langle Hb, a\rangle = \sum_{ij} \bar{a}_i H_{ij} b_j$ without dividing by n. Its law is the complex glow, and the energy of the rotators is proportional to its real part. The following count gives the part of the 8th moment that depends on H.

Proposition 3.3. *Let H, K be complex Hadamard matrices of the same order n. With the above normalization,*

$$E\left(|\Omega_H|^8\right) - E\left(|\Omega_K|^8\right) = 18\big(R_2(H) - R_2(K)\big).$$

Moreover $R_2(H) = n^3 Tr(T_2 F)$, where T_2 is the transfer matrix of H defined below and F interchanges its 2 indices.

Proof. Expand $\Omega_H^4 \bar{\Omega}_H^4$, marking the 4 factors of H by a plus sign and the 4 conjugate factors by a minus sign. The moment-cumulant formula for the row phases is a sum over partitions of these 8 positions. Only blocks with a constant row index contribute, by independence. Each block must contain equally many signs of each kind, by rotation invariance. The same rules apply independently to the column phases. We obtain one bipartite graph for each pair of partitions. Its vertices are the row and column blocks, and each position joins its 2 blocks. Its contribution is the sum over all labels of the product of the edge weights, multiplied by the cumulants of the blocks.

Parallel edges of opposite signs cancel because $|H_{ij}| = 1$. At a vertex of degree 2 the signs are opposite, and summing its label identifies the neighboring labels by $HH^* = H^*H = nI$. An isolated vertex contributes n. At each reduction the labels still to be summed form a smaller balanced graph, and the scalar factor introduced depends only on n. If edges remain after all reductions, each vertex incident to an edge has degree at least 4, and there must be at least 2 such vertices of each color. This uses all 8 edges. Each color has 2 vertices of degree 4, and each vertex sends both plus signs to one neighbor and both minus signs to the other. Thus the graph is the doubled rectangle.

There are $\binom{4}{2}^2/2 = 18$ row partitions into 2 blocks each containing 2 signs of each kind. For each, the column partition giving the doubled rectangle is forced, up to the order of its blocks. The cumulant of 2 copies each of a uniform phase and its conjugate is $1-2 = -1$, so the product of the 4 vertex cumulants is 1. All other graphs give the same value at H and K. This proves the moment identity.

For the transfer identity, put $\xi_{ik} = (H_{ij}\bar{H}_{kj})_j/\sqrt{n}$ and let P_{ik} be the projection onto ξ_{ik}. On pairs of indices put

$$(T_2)_{ab,cd} = \frac{1}{n} Tr(P_{ac} P_{bd}), \qquad F(e_i \otimes e_k) = e_k \otimes e_i.$$

Taking the trace after the flip gives $\sum_{ik} |\langle \xi_{ik}, \xi_{ki}\rangle|^2/n$. By the definition of the row quotients this is $R_2(H)/n^3$. □

We can compute every flux sum for the McNulty-Weigert matrices with outer matrix F_2. The field $\mathbb{F}_p$ enters twice. The quadratic character determines a sign in the matrix, and the parity of m determines whether this sign survives the entrywise power.

Theorem 3.4. *Let p be an odd prime, $n = 2p$, and let $H = H_{F_2}(s,t)$ with $s_i \neq t_j$ for all i, j. If the 4 labels are distinct and their cross-ratio is not a square, then*

$$\frac{R_m(H)}{n^3} = \begin{cases} 1 & p \nmid m, \\ n & p \mid m. \end{cases}$$

In every other case $R_m(H) = n^3 \gcd(m,n)$, as for F_n, for every integer $m \geq 1$.

Proof. Suppose first that the 4 labels are distinct. By the projective symmetry in [10] we can send the sources to $\infty, 0$ by an element of $PSL_2(\mathbb{F}_p)$, preserving equivalence. Indeed, after sending the first source to ∞, a translation sends the second to 0. Denote the targets by u, v. Both are different from zero and the cross-ratio is v/u. Write χ for the quadratic character, $k = -1/4$ in $\mathbb{F}_p$, and use the matrices D, V_h of Proposition 2.5. Its block calculation, with u, v in place of $1, c$, gives the representative

$$H = \begin{pmatrix} A & B \\ C & \epsilon D_0 \end{pmatrix}, \qquad A = F_p, \quad B = D^{v-u}F_p, \quad C = V_{k/u}, \quad D_0 = V_{k/v}, \quad \epsilon = -\chi(v/u).$$

Indeed, multiply the upper block row by D^{-u} on the left and the lower block row by $\sqrt{p}/G(u)$. All 4 unsigned blocks are Hadamard matrices of order p with entries among the p-th roots of unity. Orthogonality of the 2 block rows gives $AC^* + \epsilon B D_0^* = 0$.

If a source is repeated, we send it to ∞. The same block form then holds with $A = C = D^u F_p$, $B = D_0 = D^v F_p$ and $\epsilon = -1$, allowing $u = v$. If a target is repeated, take the adjoint, which preserves R_m by Lemma 3.2 and exchanges the lists. Thus the computation for the displayed block matrix covers all admissible choices of labels.

Suppose $p \nmid m$. On each unsigned block, raising entries to the m-th power is the automorphism σ_m sending w to w^m. It commutes with conjugation, so the diagonal blocks of $S_m S_m^*$ are nI_p. Its upper right block is

$$\sigma_m(AC^*) + \epsilon^m \sigma_m(BD_0^*) = (1 - \epsilon^{m+1})\sigma_m(AC^*).$$

If $\epsilon = 1$ or m is odd, this block vanishes and $R_m(H) = n^3$. If $\epsilon = -1$ and m is even, put $U = \sigma_m(AC^*)/p$. Being the product of 2 unitary matrices, U is unitary, and

$$S_m S_m^* = n \begin{pmatrix} I_p & U \\ U^* & I_p \end{pmatrix}.$$

Its square is $2n$ times itself, so $R_m(H) = 2n^3$.

Finally, if $p \mid m$, each unsigned block becomes the matrix of ones J_p. Hence S_m is the tensor product of J_p with the 2 by 2 sign matrix with rows $(1,1)$ and $(1, \epsilon^m)$. When $\epsilon^m = 1$ all entries of S_m are 1, giving $R_m(H) = n^4$. When $\epsilon^m = -1$, the sign matrix is F_2 and $R_m(H) = n^4/2$. This gives the formulas in both cases. For comparison, in $F_n^{\circ m}$ two rows indexed by i, k agree if $m(i-k) = 0$ modulo n and are orthogonal otherwise. Each row has $\gcd(m,n)$ choices of k giving equality, so its flux sums are $n^3 \gcd(m,n)$. □

The flux sums also give a short description of the distribution itself. Let ν_p be the uniform probability measure on the p-th roots of unity and δ_1 the unit mass at 1.

Proposition 3.5. *Choose i, k, j, l independently and uniformly for a matrix of Theorem 3.4. If the 4 projective labels are distinct and their cross-ratio is not a square, the law of $u_{ik;jl}$ is*

$$\frac{1}{n}\delta_1 + \left(1 - \frac{1}{n}\right)\nu_p.$$

In every other case it is the law of $e^{2\pi i xy/n}$, where x, y are independent and uniform in $\mathbb{Z}_n$.

Proof. The m-th moment of ν_p is 1 if $p \mid m$ and zero otherwise. The moments of the displayed measure are therefore 1 or $1/n$, as prescribed by Theorem 3.4 after division by n^4. For the second law, average first over y. The surviving x are exactly the $\gcd(m,n)$ solutions of $mx = 0$ in $\mathbb{Z}_n$, giving moment $\gcd(m,n)/n$. Both laws are determined by their moments. □

To recover the defect we need only the value at $m = 2$. For distinct labels the defect is known from the cross-ratio. When a label is repeated, the matrix is a deformed Fourier product, whose defect we compute in Section 10.

Theorem 3.6. *For the matrices H of Theorem 3.4 we have*

$$d(H) = (n-2)\left(\frac{R_2(H)}{n^3} - 1\right) = (n-2)\big(Tr(T_2F) - 1\big).$$

In this class H is isolated precisely when its entrywise square is Hadamard. At fixed order the 8th absolute moment of the complex glow determines the defect.

Proof. For distinct labels the defect is zero or $2(p-1)$ according as the cross-ratio is not a square or is a square, by [10]. For repeated labels Proposition 10.3 gives defect $2(p-1)$. These cases give $R_2(H)/n^3 = 1, 2, 2$, respectively, by Theorem 3.4. The isolation criterion follows from Proposition 10.3 and Lemma 3.2. Proposition 3.3 allows us to compare the 8th moment with that of F_n, for which $R_2(F_n) = 2n^3$. □

For example, at order 6 the doubled rectangle has value 216 at the Tao matrix and 432 at F_6. At order 10 it has value 1000 at $K_5(2)$ and 2000 at $K_5(4)$. Thus the corresponding 8th moments in order 10 differ by 18000. At H_5 the flux gives defect 8, while Theorem 6.3 gives local dimension 4.

Theorem 3.4 gives the same flux distribution for every $K_p(c)$ with c a square. We can ask how much of the cross-ratio survives if we join several rectangles before summing over their indices. For a Hadamard matrix H of order n, put

$$(\Lambda_H)_{ij,kl} = \frac{u_{ik;jl}}{n} = \frac{H_{ij}\bar{H}_{kj}H_{kl}\bar{H}_{il}}{n}, \qquad Z_r(H) = Tr(\Lambda_H^r).$$

This is the Berezin matrix of Karabegov [24], in the convention of [10]. Tadej [40] gives $d(H) = \dim\ker(\Lambda_H - I) - (2n-1)$. In the trace of the r-th power, consecutive rectangles share an index pair and the last rectangle closes the cycle. Each trace is invariant under Hadamard equivalence. The matrix Λ_H is unitary, since the inner product of its rows $(i,j),(a,b)$ factors into the orthogonality sums for rows i,a and columns j,b of H, times $H_{ij}\bar{H}_{ab}/n^2$. We have $Z_1(H) = n$ and $Z_2(H) = R_2(H)/n^2$, since Λ_H is symmetric. To compute Z_3 we use the decomposition into characters from [10], keeping the whole operator in each character space.

Lemma 3.7. *Let $H = H_K(s,t)$ be as in Definition A, with K of order N. Put $u_h = (1, 2h)$ for $h \in \mathbb{F}_p$ and $u_\infty = (0,1)$. For each i,j and $x \in \mathbb{F}_p^2$ write $x = au_{s_i} + bu_{t_j}$ and set*

$$h_{ij}(x) = ab\det(u_{s_i}, u_{t_j}), \qquad \Theta_x = diag\big(w^{h_{ij}(x)}\big), \qquad w = e^{2\pi i/p}.$$

Then Λ_H is similar to the direct sum of the p^2 matrices $\bar{\Theta}_x\Lambda_K$, each of order N^2.

Proof. Write u_α, v_β for the product bases of Section 2, so that $u_\alpha^* v_\beta = H_{\alpha\beta}/\sqrt{n}$ with $n = Np$. Let Q_α, P_β be the projections onto these vectors. For an arbitrary matrix Y form the arrays

$$A_{\alpha\beta} = Tr(YP_\beta Q_\alpha), \qquad B_{\alpha\beta} = Tr(YQ_\alpha P_\beta).$$

We have $A_{\alpha\beta} = \bar{H}_{\alpha\beta}u_\alpha^* Y v_\beta/\sqrt{n}$, so A determines Y. Expanding Y in the basis $u_\gamma v_\delta^*$ gives $B = \Lambda_H A$. The same identity for the outer matrix K relates the arrays $Tr(Y_0 G_j E_i)$ and

$Tr(Y_0 E_i G_j)$ for any matrix Y_0 of order N. Here E_i, G_j are the projections onto the standard basis and the columns of $K/\sqrt{N}$, respectively.

Let $Xe_k = e_{k+1}$, $Ze_k = w^k e_k$ and $D_{(a,b)} = w^{ab/2} X^a Z^b$. On the inner space the projections of the basis with label h span the same diagonal algebra as the matrices D_{au_h}, since D_{u_h} is diagonal in that basis with distinct eigenvalues. Change from the projections to these matrices in both arrays. For each i, j, the pair (a, b) can now be indexed by $x = au_{s_i} + bu_{t_j}$, since $s_i \neq t_j$. The multiplication rule

$$D_y D_z = w^{-\det(y,z)/2} D_{y+z}$$

gives factors $w^{h_{ij}(x)/2}$ in A and $w^{-h_{ij}(x)/2}$ in B, because the inner factors occur in opposite orders. The remaining entries are $Tr(Y_x G_j E_i)$ and $Tr(Y_x E_i G_j)$, where Y_x is determined by $Tr(Y_x T) = Tr(Y(T \otimes D_x))$ for every T. The D_x form a basis of the inner matrix algebra, so the Y_x vary independently. With $S_x = diag(w^{-h_{ij}(x)/2})$, the block taking the new A to the new B is $S_x \Lambda_K S_x$. Conjugating by S_x gives $S_x^2 \Lambda_K = \bar{\Theta}_x \Lambda_K$. □

We now use the labels $(\infty, 0)$ and $(1, c)$ of $K_p(c)$. Write χ for the quadratic character of $\mathbb{F}_p$, with $\chi(0) = 0$, and put $G_p = \sum_{a \in \mathbb{F}_p} w^{a^2}$. The classical Gauss sum evaluation gives $G_p = \sqrt{p}$ for $p = 1$ (4) and $G_p = i\sqrt{p}$ for $p = 3$ (4). Define

$$A(t) = 6\chi(1 + 8t) - 12\chi(5 + 4t),$$

$$B(t) = 3(1 + \chi(2))\, 1_{1+8t=0} - 6\chi(2)(1 + \chi(5))\, 1_{5+4t=0},$$

where each indicator is 1 if its equality holds in $\mathbb{F}_p$ and zero otherwise.

Theorem 3.8. *For $p \geq 5$ prime and $c \in \mathbb{F}_p - \{0, 1\}$ the third trace of $K_p(c)$ is*

$$Z_3(K_p(c)) = \frac{p}{8}\big(28 + A(c) + A(c^{-1}) + G_p B(c) + \bar{G}_p B(c^{-1})\big).$$

For $p = 3$ the only choice is $c = 2$, and $Z_3(K_3(2)) = -9$.

Proof. Write $x = (a, b)$. For the source ∞ and target t, Lemma 3.7 gives $-h_{ij}(x) = -2ta^2 + ab$. For source 0 and target t it gives $-ab + b^2/(2t)$. Denote the resulting 4 quadratic forms, in block order, by

$$f_1 = -2a^2 + ab, \quad f_2 = -2ca^2 + ab, \quad f_3 = -ab + b^2/2, \quad f_4 = -ab + b^2/(2c).$$

The block at x is $diag(w^{f_i})\Lambda_{F_2}$. The entries of $2\Lambda_{F_2}$ are signs, with a minus sign precisely when both binary indices change. The product of signs around a triple of states is $+1$ if a state is repeated and -1 if all 3 are distinct. In the latter case exactly one pair of states differs in both indices. Thus

$$8Z_3 = \sum_{a,b} \Big(\sum_i w^{3f_i} + 3 \sum_{i \neq j} w^{2f_i + f_j} - 6 \sum_{i<j<k} w^{f_i + f_j + f_k} \Big).$$

For $f = \sum_{i=1}^4 m_i f_i$ with $\sum_i m_i = 3$, its discriminant as a binary quadratic form is

$$\Delta = (m_1 + m_2 - m_3 - m_4)^2 + 4(m_1 + cm_2)(m_3 + m_4/c) = 9 + 4(c-1)m_2 m_3 + 4(c^{-1} - 1)m_1 m_4.$$

When both products $m_2 m_3$ and $m_1 m_4$ vanish, the 4 forms $3f_i$ and 8 forms $2f_i + f_j$ give discriminant 9 with total weight 28. The other forms come in the following pairs. The last column gives the coefficients of a^2 in the 2 forms, in the order shown.

forms	discriminant	weight of each	coefficients of a^2
$2f_2 + f_3,\ f_2 + 2f_3$	$1 + 8c$	3	$-4c,\ -2c$
$2f_1 + f_4,\ f_1 + 2f_4$	$1 + 8/c$	3	$-4,\ -2$
$f_1 + f_2 + f_3,\ f_2 + f_3 + f_4$	$5 + 4c$	-6	$-2(1 + c),\ -2c$
$f_1 + f_2 + f_4,\ f_1 + f_3 + f_4$	$5 + 4/c$	-6	$-2(1 + c),\ -2$

For a binary quadratic form with $\Delta \neq 0$, completing the square gives $\sum_{a,b} w^f = p\chi(\Delta)$. Indeed the 2 one-variable Gauss sums multiply using $G_p^2 = \chi(-1)p$ and $\bar{G}_p = \chi(-1)G_p$, by Proposition 2.3. If the coefficient of a^2 vanishes, one can instead sum first over a, using the mixed coefficient, which is different from zero. For $\Delta = 0$ and coefficient $d \neq 0$ of a^2, the form is d times a square, giving $p\chi(d)G_p$.

Suppose $p \geq 5$. The forms of discriminant 9 contribute $28p$. The entries in the table with discriminant different from zero contribute $p(A(c) + A(c^{-1}))$. If $1 + 8c = 0$, the 2 coefficients in the first row become $1/2, 1/4$, whose characters sum to $1 + \chi(2)$. If $5 + 4c = 0$, the third row gives $1/2, 5/2$, with character sum $\chi(2)(1 + \chi(5))$. This gives $pG_pB(c)$. At the zeros of the second and fourth discriminants, their character sums are $\chi(-1)(1+\chi(2))$ and $\chi(-2)(1+\chi(5))$, respectively. Their contribution is $p\bar{G}_pB(c^{-1})$. Adding all contributions and dividing by 8 gives the formula, including when several discriminants vanish.

Finally take $p = 3, c = 2$. The 4 forms $3f_i$ vanish identically and contribute 36. The 8 forms of discriminant 9 with distinct i, j are differences $f_j - f_i$ of rank 1. They cancel in opposite pairs since $\chi(-1) = -1$. In the table the first 2 discriminants are 2 modulo 3 and the last 2 are 1. Thus $8Z_3 = 36 - 36 - 72 = -72$. □

Away from the zeros of $1 + 8t$ and $5 + 4t$ for $t = c, c^{-1}$, the answer uses just 4 quadratic characters. At $p = 11$, the cross-ratios 3 and 5 are both squares. Their single-rectangle flux distributions agree by Theorem 3.4, and both matrices have defect 20. Theorem 3.8 gives

$$Z_3(K_{11}(3)) = 319/4, \qquad Z_3(K_{11}(5)) = 22.$$

The correction can also distinguish inverse cross-ratios. For $c = 7$ in $\mathbb{F}_{11}$ we have $5 + 4c = 0$ and $c^{-1} = 8$. The respective traces are $(11 + 33i\sqrt{11})/2$ and $(11 - 33i\sqrt{11})/2$.

The conjugate values for 7 and 8 have an explanation which applies to cycles of any length.

Corollary 3.9. *For every odd prime p and $c \in \mathbb{F}_p - \{0, 1\}$, the matrix $K_p(c^{-1})$ is equivalent to the entrywise conjugate of $K_p(c)$. In particular we have*

$$Z_r(K_p(c^{-1})) = \overline{Z_r(K_p(c))}$$

for every integer $r \geq 1$.

Proof. Complex conjugation fixes the standard basis. For finite t it sends $B_t = D^t F_p/\sqrt{p}$ to B_{-t} up to the column permutation $a \to -a$. Thus conjugating $K_p(c)$ gives source labels $(\infty, 0)$ and target labels $(-1, -c)$. Apply the projective transformation $t \to -1/t$, represented by a matrix of determinant 1. By the projective symmetry of [10] this preserves Hadamard equivalence. The sources are now $(0, \infty)$ and the targets are $(1, c^{-1})$. Interchanging the 2 block rows restores the source order. On the outer matrix F_2 this interchange merely changes the sign of its second column, which we absorb in the second block column. This proves the equivalence. Conjugating the entries conjugates every rectangle flux, and hence every cyclic trace. □

4 Analytic families and their tangent spaces

The order-10 examples of Section 2 show what happens to the defect when we change a single basis label. We now take the case of a positive defect and ask which of its directions give actual families. We work with the product bases u_{ia}, v_{jb} and write $w = e^{2\pi i/p}$.

For the families constructed here we take $K = F_2$ and assume the 4 labels are distinct. Their cross-ratio is

$$\lambda = \frac{(s_0 - t_0)(s_1 - t_1)}{(s_0 - t_1)(s_1 - t_0)},$$

with the usual cancellation when a label is ∞. The projective involution interchanging s_0, s_1 and t_0, t_1 has 2 fixed points precisely when λ is a square. We assume that the cross-ratio is

a square throughout this section. The fixed points correspond to 2 lines L, M in $\mathbb{F}_p^2$. We will associate to each line an analytic family through H and identify its tangent space.

Write $G = \mathbb{F}_p^2$. The operators $D_{(a,b)} = w^{ab/2} X^a Z^b$ on $\mathbb{C}^p$, with $Xe_k = e_{k+1}$ and $Ze_k = w^k e_k$, give an action of G by conjugation. The determinant pairing identifies G with its character group. In this identification, the characters trivial on a line L are indexed by the same line L. The signs in the identification can be chosen either way.

Theorem 4.1. *Through H there are 2 real analytic dephased families of dimension $p-1$. Their tangent spaces T_L, T_M have zero intersection and their sum is the full dephased linearized space of dimension $2(p-1)$.*

Proof. Conjugation by $I \otimes D_z$ permutes the rank-one projections onto the vectors u_{ia} and likewise those onto the vectors v_{jb}. We therefore obtain an action of G on the linearized equations at H.

We recall the character calculation in the proof of the cross-ratio theorem in [10]. In the Weyl expansion of a linearized variation, the trivial character contributes 3 phase directions. On the 4 input lines each point other than the origin contributes one phase direction. Away from the origin on either fixed line, the remaining contribution has complex dimension 1. Indeed, off the 4 input lines the kernel equation is a diagonal system with a rank-one correction. For the system to be consistent, the 4 cotangents must sum to zero. This happens precisely when the involution fixes the direction. The phase directions listed account for $3 + 4(p-1) = 4p-1$ dimensions. Thus, after removing phases, the complexified tangent space has precisely the characters

$$(L - \{0\}) \ \cup \ (M - \{0\}),$$

once each. Opposite characters are complex conjugates of each other. In particular the real invariant tangent space for each of the subgroups L, M has dimension $p-1$.

Choose L and use an $SL_2(\mathbb{F}_p)$ change of coordinates taking it to infinity. Such changes transport the product bases and Weyl symmetries by the same Clifford unitary, up to monomial changes of bases [10]. A translation then makes the 4 labels $s, -s, t, -t$. In these coordinates the analytic construction in [10] uses matrices

$$\begin{pmatrix} A & B \\ B^* & -A^* \end{pmatrix}, \qquad AA^* + BB^* = 2pI,$$

with A, B circulant. Varying the phases of the 1st row of B, with its 1st entry fixed, gives $p-1$ independent dephased parameters. At the initial quadratic chirp the derivative of the autocorrelations of the 1st row of A is invertible after fixing its 1st entry, which is the implicit-function argument producing A.

Every block throughout this family is invariant under simultaneous cyclic shifts of its row and column indices. These shifts are the action of Z on all 4 finite bases, since $ZB_c e_a = B_c e_{a+1}$. The whole family is therefore fixed by L modulo phases. Its tangent space has the full dimension of the L-invariant part. Repeating the construction with M gives the other part, whose character support is disjoint from the first. □

The 2 tangent spaces just found are complementary. Can a single analytic family have tangent directions from both? In the coordinates $s, -s, t, -t$, let $Pe_a = e_{-a}$ and put $R = diag(P, P)$. Reversing all inner row and column indices acts on matrices by $U \to RUR$. We write r for the resulting involution modulo row and column phases. At H each chirp block has entries depending only on $(b-a)^2$, so r fixes H.

Theorem 4.2. *Let H have 4 distinct labels and square cross-ratio as in Theorem 4.1. There are p^2 further real analytic dephased families through H, each of dimension $p-1$. Their tangent spaces are pairwise distinct, and each is the direct sum of a $(p-1)/2$-dimensional subspace of T_L and a $(p-1)/2$-dimensional subspace of T_M.*

Proof. We first compute the tangent space fixed by r. We have $PD_zP = D_{-z}$, and P sends the column indexed by a of each finite basis to the column indexed by $-a$. Thus r exchanges the character spaces indexed by x and $-x$. Each such space on $L - \{0\}$ or $M - \{0\}$ has complex dimension 1 by Theorem 4.1. For each pair the involution fixes a line whose projection onto either character space is the whole space. Returning to the real spaces gives

$$T_H^r = T_L^r \oplus T_M^r, \qquad \dim T_L^r = \dim T_M^r = (p-1)/2.$$

We will realize the whole of T_H^r as the tangent space of one family.

At the original matrix, both circulant chirps A, B are symmetric. Multiply the lower block row by -1 and divide by $\sqrt{2p}$. Our base point is now the unitary

$$U_0 = \frac{1}{\sqrt{2p}} \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix}.$$

Put $J = \left(\begin{smallmatrix} 0 & I_p \\ -I_p & 0 \end{smallmatrix} \right)$. We vary U in the group

$$K = \{U \in U_{2p} : UR = RU,\ UJ = J\bar{U}\}.$$

The second equation means that U commutes with the quaternionic structure $v \to J\bar{v}$. The $+1$ and -1 eigenspaces of R have quaternionic dimensions $m = (p+1)/2$ and $m-1$, respectively. Hence K is the product of the quaternionic unitary groups $Sp(m)$ and $Sp(m-1)$. A skew-Hermitian quaternionic matrix of order h has $3h + 4\binom{h}{2} = h(2h+1)$ real parameters. This gives

$$\dim K = m(2m+1) + (m-1)(2m-1) = p^2 + p + 1.$$

On K consider the real analytic map $F(U)_{ij} = |U_{ij}|^2 - 1/(2p)$. Let s interchange the upper and lower halves of the index set. The squared moduli are invariant under simultaneous interchange by s in the rows and columns, as well as simultaneous reversal by r. The 4 elements $1, s, r, sr$ fix $(2p)^2, 0, 4, 0$ entries, respectively. Averaging these numbers gives p^2+1 entry orbits. There are m row orbits and m column orbits. Unitarity makes all the row and column sums of F zero. Among invariant arrays these are $2m-1$ independent conditions. Indeed the sums can be independently prescribed subject to their totals agreeing, and averaging over the group preserves invariant prescriptions. Thus F takes values in a space E of dimension

$$\dim E = p^2 + 1 - (2m-1) = p^2 - p + 1.$$

We check that the derivative at U_0 maps onto E. Each vector in its kernel is a first-order Hadamard variation. After dephasing it belongs to T_H^r. To count the kernel of dephasing on these variations, write a phase variation as $i(a_i + b_j)(U_0)_{ij}$. Reversal makes both lists even in the inner indices. Interchange of the block halves with conjugation requires

$$a_{si} + a_i = c, \qquad b_{sj} + b_j = -c.$$

Adding opposite constants to the lists sets $c = 0$. The upper half of each list is now an arbitrary even function on $\mathbb{F}_p$, and the lower half is its negative. All these $2m = p+1$ variations are realized by diagonal row and column phases preserving K. Consequently

$$\dim \ker dF_{U_0} \le (p+1) + \dim T_H^r = 2p = \dim K - \dim E.$$

The opposite inequality holds for every linear map from $T_{U_0}K$ to E. We conclude that dF_{U_0} is onto, and that the dephasing of its kernel is all of T_H^r. By the real analytic implicit function theorem the flat unitaries in K form a smooth manifold of dimension $2p$ near U_0. Take an analytic slice in this manifold complementary to the $p+1$ phase directions. Multiplication by

$\sqrt{2p}$, restoration of the block row sign and dephasing embed this slice locally as the required family of dimension $p-1$.

It remains to count the families. Write g_z for the action of $z \in G = \mathbb{F}_p^2$ on the dephased germ. Indeed $I \otimes D_z$ permutes each product basis up to phases. The resulting row and column operations fix H, and their actions modulo phases compose by addition in G. The translate of the family just constructed has tangent space

$$W_z = g_z T_H^r = T_H^{r_z}, \qquad r_z = g_z r g_z^{-1} = g_{2z} r.$$

The projection of W_z onto each character space on either line is the whole character space, since this holds for T_H^r and translations act by scalars on characters. If $W_z = W_{z'}$, then $r_z r_{z'} = g_{2(z-z')}$ fixes this space pointwise. All the characters on both lines must therefore take value 1 at $2(z-z')$. The 2 lines span the character group, so $z = z'$ because p is odd. This gives p^2 different tangent spaces. Since g_z preserves T_L and T_M, each translate has the asserted dimensions in those spaces. □

At $p = 5$ the group $Sp(3) \times Sp(2)$ has dimension 31. The 21 independent modulus equations leave 10 parameters, of which 6 are row and column phases. Its dephased family therefore has dimension 4. The 25 translates are related by symmetries of H, and are distinct as germs with the row and column labels fixed.

We can also see where these families meet. Denote the original families by F_L, F_M, and the translate with tangent space W_z by F_z. All intersections below are taken as germs at H.

Proposition 4.3. *Each F_z meets each of F_L, F_M in a smooth family of dimension $(p-1)/2$. For $z \neq z'$, the intersection of F_z and $F_{z'}$ is also smooth of dimension $(p-1)/2$ when $z - z'$ lies on L or M. For all other differences the 2 families meet only at H.*

Proof. The germ fixed by L is precisely F_L. Indeed, the action is linear on logarithmic phases after dephasing, and the derivative of the Hadamard equations restricted to the fixed space has kernel T_L. Independent equations therefore cut out a smooth germ of dimension $p-1$ containing the fixed solutions. The embedded family F_L already has this dimension and consists of fixed solutions, so it fills that germ. The same reasoning applies to M, and to r_z with kernel W_z. Thus F_z is the whole germ fixed by r_z.

The involution r_z normalizes L and hence preserves F_L. A finite-order analytic action fixing a point on a smooth manifold can be linearized there by averaging local coordinates with their transforms under the derivative action. Its fixed locus is smooth with the fixed tangent space. Applying this to r_z on F_L gives dimension $(p-1)/2$, and likewise for F_M.

A point fixed by r_z lies in $F_{z'}$ exactly when it is also fixed by $g_{2(z-z')}$. If $z - z' \in L - \{0\}$, this translation generates L. The intersection is then $F_z \cap F_L$, and the same argument works on M. In every other direction the translation fixes only the zero tangent vector, since each character indexed by $(L \cup M) - \{0\}$ takes a value different from 1 on that translation. Its fixed Hadamard germ is therefore a point. □

Corollary 4.4. *Near H, the union of the $p^2 + 2$ families is exactly the set of points fixed by some nonidentity element of the group generated by G and r. The same description holds in the complexified dephased germ.*

Proof. The group elements are the translations g_z and the reflections $g_z r$. Since p is odd, $g_z r = r_{z/2}$ fixes precisely $F_{z/2}$. A nonzero translation on L or M generates that line, so its fixed germ is F_L or F_M. Every other nonzero translation fixes only H, by Proposition 4.3. These fixed-germ arguments also apply in complex logarithmic coordinates. The derivative restricted to the fixed space has the complexified fixed tangent space as kernel, and the complexification of the indicated real family fills the resulting smooth germ. □

The families through a matrix with square cross-ratio need not have any affine directions. To see this, it suffices to look at tangent vectors with rational entries.

Theorem 4.5. *Let $H = H_{F_2}(s,t)$ have 4 distinct labels in $P^1(\mathbb{F}_p)$. Its dephased tangent space contains no nonzero rational array. In particular no nonconstant dephased affine deformation passes through H.*

Proof. For a cross-ratio that is not a square, the whole tangent space is zero. In the square case Theorem 4.1 gives the direct sum $T_L \oplus T_M$. Averaging the translation action over L projects onto T_L, and averaging over M projects onto T_M. These operations preserve rational arrays, since on phase variations the translations are permutations followed by dephasing. It suffices to treat a rational tangent vector fixed by one of the 2 lines.

As in Theorem 4.1, send this line to infinity and write the labels as $s, -s$ and $t, -t$. The changes of basis preserve phase variations up to permutations and dephasing. Taking the average before dephasing gives a representative invariant under simultaneous shifts of the inner row and column indices. Its 4 blocks are circulant. Within one block row write α_j, β_j for the 2 rational lists of phase velocities. Up to constant phases the underlying rows in the 2 blocks are translates of w^{aj^2}, w^{bj^2} with $a, b \neq 0$, $a \neq \pm b$. Indeed the 2 coefficients are $-1/(4(t-h))$ and $-1/(4(-t-h))$ for $h = s$ or $-s$, and h, t are nonzero.

For $k \neq 0$ put $D_k\alpha(x) = \alpha_{x+k/2} - \alpha_{x-k/2}$, and likewise for β. Comparing 2 rows at displacement k and centering the inner index gives

$$\sum_x w^{2akx} D_k\alpha(x) + \sum_x w^{2bkx} D_k\beta(x) = 0.$$

A rational relation among $1, w, \ldots, w^{p-1}$ has all coefficients equal. Here their sum is zero by telescoping, so every coefficient is zero. With $c = a/b$ this gives $D_k\alpha(x) = -D_k\beta(cx)$ for every x, k.

Take the Fourier transform in x. At frequency $n \neq 0$, the multipliers of the 2 differences are $w^{nk/2} - w^{-nk/2}$ and $w^{nk/(2c)} - w^{-nk/(2c)}$. If one of the paired Fourier coefficients at $n, n/c$ were nonzero, both would be nonzero. Dividing the equality for $k = 2$ by that for $k = 1$ would give

$$w^{n/2} + w^{-n/2} = w^{n/(2c)} + w^{-n/(2c)}.$$

The 2 pairs are then the roots of the same quadratic of constant term 1. This forces $c = \pm 1$, which is excluded. Thus both velocity lists are constant, in each of the 2 block rows. Write the resulting block constants as d_{ij}. Orthogonality between the block rows implies $d_{00} - d_{10} = d_{01} - d_{11}$, since the contribution from either block column before multiplication by the outer signs is a nonzero entry of $pB_s^* B_{-s}$. The constants are row and column phases, so their dephasing is zero.

Finally suppose a real velocity gives an affine line. For each row pair, partition the columns by equal row differences as in Proposition 1.5. The equalities within the parts are rational linear equations on the velocity. Every solution of these equalities gives an affine line, since each part already has zero scalar product. A nonzero dephased real solution would therefore give a nonzero rational one. □

This gives a neighborhood of H_5 in which every other dephased Hadamard matrix has an entry of infinite order. We use the representative in $B(10, 10)$ obtained from the chirps in Proposition 6.1, with the common Gauss factor removed.

Corollary 4.6. *Fix the first row and column at this representative of H_5. If a Butson matrix of any exponent has entries $(H_5)_{ij} e^{iA_{ij}}$ with $|A_{ij}| < \pi/420$ for every i, j, then $A = 0$.*

Proof. In each scalar product of distinct rows of the proposed Butson matrix, partition the 10 terms into minimal vanishing subsums. Mann [29] proved that the quotient of any 2 terms of a minimal relation among k roots of unity with rational coefficients has order dividing the

product of the primes at most k. Thus every quotient within a part is a 210th root of unity, since $2 \cdot 3 \cdot 5 \cdot 7 = 210$. At H_5 the corresponding quotient is a 10th root and hence a 210th root as well. Its phase changes by a sum of 4 signed entries of A, of absolute value strictly less than $2\pi/210$. The quotients therefore agree, and the row differences of A are constant on each part. Each part also sums to zero at H_5, since its terms differ from those at the proposed matrix by one common phase. Proposition 1.5 gives an affine line through H_5, so Theorem 4.5 gives $A = 0$. □

The same proof applies to any Butson representative in Theorem 4.5. For order n and exponent m, replace 210 by the least common multiple of m and the product of the primes at most n. In particular H_5 is the only Butson matrix in a sufficiently small dephased neighborhood of H_5.

5 The equations of order 2

Let S be a matrix of order n with entries on the unit circle. We consider the function $M(S) = Tr|S|$ from the dual model in [15], where $|S| = (S^*S)^{1/2}$. The sum of the squared singular values is n^2, so Cauchy–Schwarz gives $M(S) \le n\sqrt{n}$. Equality holds precisely at the Hadamard matrices. What do the second derivatives of M at such a maximum tell us about the matrix?

Theorem 5.1. *Let H be a complex Hadamard matrix of order n. In the full real phase coordinates $S_{ij} = H_{ij}e^{iA_{ij}}$, with squared norm $\sum_{ij} A_{ij}^2$, the Hessian of M at H is*

$$\operatorname{Hess}_H M = \frac{\operatorname{Re} \Lambda_H - I}{2\sqrt{n}},$$

where Re *denotes the entrywise real part. Its kernel has dimension $2n - 1 + d(H)$.*

Proof. Put $E = SS^* - nI$. Its diagonal entries vanish. Near H we can expand the square root of $nI + E$. Since $Tr(E) = 0$ and $Tr\sqrt{SS^*} = M(S)$, we obtain

$$n\sqrt{n} - M(S) = \frac{Tr(E^2)}{8n^{3/2}} + O(\|E\|_F^3), \qquad \|E\|_F^2 = Tr(E^2) = R_1(S) - n^3.$$

The last identity follows from Lemma 3.2. To compute the Hessian, take $S_{ij}(t) = H_{ij}e^{itA_{ij}}$ and put $V = (H_{ij}A_{ij})H^*$. Then $E'(0) = i(V - V^*)$ and

$$Tr(VV^*) = n\sum_{ij} A_{ij}^2, \qquad Tr(V^2) = \sum_{ikjl} H_{ij}\bar{H}_{kj}H_{kl}\bar{H}_{il}A_{ij}A_{kl} = nA^t\Lambda_H A.$$

Here A in the last expression is the real vector indexed by the pairs (i, j). It follows that $\|E'(0)\|_F^2 = 2nA^t(I - \operatorname{Re}\Lambda_H)A$. The second derivative of $M(S(t))$ is $-\|E'(0)\|_F^2/(4n^{3/2})$, giving the asserted matrix. Its kernel consists precisely of the solutions of $E'(0) = 0$, the linearized row equations before dephasing. The row and column phases contribute $2n - 1$ dimensions and the dephased solutions contribute $d(H)$. □

Thus the defect counts the zero eigenvalues after the row and column phases are removed, as observed in [15]. The cyclic traces from Section 3 also give information about the other eigenvalues.

Corollary 5.2. *The matrices $K_{11}(3)$ and $K_{11}(5)$ of order 22 have the same rectangle flux distribution and defect 20. The Hessians of M at these matrices have different spectra.*

Proof. Both cross-ratios are squares in $\mathbb{F}_{11}$, so the first assertions follow from Theorems 3.4 and 3.6. Write $C = \operatorname{Re} \Lambda_H$. The Berezin matrix is symmetric and unitary, so $C = (\Lambda_H + \Lambda_H^{-1})/2$. Cubing and taking traces gives

$$4Tr(C^3) - 3Tr(C) = \operatorname{Re}\, Z_3(H), \qquad Tr(C) = n.$$

By Theorem 3.8, the right side of the first identity is $319/4$ at $K_{11}(3)$ and 22 at $K_{11}(5)$. Thus the spectra of C differ, and Theorem 5.1 gives the result. □

More generally the real parts of the cyclic traces determine the whole real phase Hessian spectrum at fixed order n. Indeed, if T_r is the Chebyshev polynomial defined by $T_r(\cos t) = \cos(rt)$, then

$$Tr\big(T_r(C)\big) = \operatorname{Re} Z_r(H), \qquad C = I + 2\sqrt{n}\, \operatorname{Hess}_H M.$$

The eigenvalues of C are the real parts of the unitary eigenvalues of Λ_H. Since T_r has degree r and nonzero leading coefficient, the traces for $1 \le r \le n^2$ successively give the first n^2 power sums of C. Newton's identities then give its characteristic polynomial.

We now follow a phase direction in the kernel of the Hessian. For an analytic trial deformation $X(t)$ through H, the row equations hold through order r precisely when $R_1(X(t)) - n^3 = O(t^{2r+2})$. Indeed this quantity is the sum of the squared moduli of the scalar products of distinct rows. The expansion in the proof of Theorem 5.1 gives the same criterion with $n\sqrt{n} - M(X(t))$ in its place. At order 2 we therefore ask whether we can choose a quadratic phase correction for which the energy loss is $O(t^6)$.

Theorem 4.1 provides analytic curves in each of the 2 tangent spaces T_L, T_M. We now take a sum of directions from these spaces and ask what the quadratic orthogonality equations require of their sum.

Theorem 5.3. *Let H be a matrix of Theorem 4.1 whose cross-ratio is a square. Each dephased linearized direction at H admits a quadratic correction satisfying the Hadamard equations through order 2.*

Proof. Put $n = 2p$ and write u_α, v_β for the product basis vectors of Section 2. The normalized matrix $H/\sqrt{n}$ has entries $\langle v_\beta, u_\alpha \rangle$. For a Hermitian matrix Y near zero we vary the 2nd basis by e^{iY} and measure the change in the squared moduli by putting

$$U(Y)_{\alpha\beta} = \langle e^{iY} v_\beta, u_\alpha \rangle, \qquad \Phi(Y)_{\alpha\beta} = |U(Y)_{\alpha\beta}|^2 - 1/n.$$

Since $U(Y)$ is unitary, $\Phi(Y)$ belongs to the real space E of arrays with all row and column sums zero. Solving $\Phi(Y) = 0$ gives the Hadamard matrices near H.

Let D_u, D_v be the real spaces of Hermitian matrices diagonal in the respective bases. They intersect in $\mathbb{R}I$. Indeed, if a matrix diagonal in both has eigenvalues a_α on u_α and b_β on v_β, then

$$(a_\alpha - b_\beta)\langle v_\beta, u_\alpha \rangle = 0$$

for all α, β, and every overlap has positive modulus. We take $S = (D_u + D_v)^\perp$ in the Hermitian matrices, with scalar product $Tr(YZ)$. Left and right diagonal phases together with $Y \in S$ give local coordinates: the derivative of $(D, Y, D') \to e^{iD} e^{iY} e^{iD'}$ at zero is $i(D + Y + D')$, an isomorphism after fixing one scalar. Thus S gives coordinates modulo phases, and

$$J = D\Phi(0)|_S : S \to E, \qquad \ker J \simeq T_H.$$

We seek the 2nd derivative B of a path with prescribed derivative $A \in \ker J$. Taylor expansion gives

$$\Phi(tA + t^2 B/2) = \frac{t^2}{2}\big(JB + D^2\Phi(0)(A, A)\big) + O(t^3).$$

It is enough to show that the symmetric bilinear map

$$Q(A,C) = D^2\Phi(0)(A,C) + J(S) \quad \text{in } E/J(S)$$

vanishes on $\ker J$. We use the action of $G = \mathbb{F}_p^2$ on both sides. Conjugation by $I \otimes D_z$ permutes the vectors of each basis up to phases. Taking absolute values removes these phases, so Φ, its derivative J and the induced map Q respect the action of G.

To identify the characters in $E/J(S)$, first restore the phase variables in S and the row and column sums in E. The complexified spaces on both sides then carry 4 copies of the regular representation $\mathbb{C}[G]$. On the matrix side we have the Weyl decomposition

$$M_{2p}(\mathbb{C}) = M_2(\mathbb{C}) \otimes M_p(\mathbb{C}), \qquad M_p(\mathbb{C}) = \bigoplus_{x \in G} \mathbb{C}D_x,$$

where conjugation has a different character on each $\mathbb{C}D_x$. On the array side each of the 4 blocks is a single free G-orbit. Indeed, a translation fixing an entry of block (i,j) belongs to both s_i and t_j, hence is zero. The block has $p^2 = |G|$ entries.

The spaces to be removed are also isomorphic as G-representations. On the matrix side a pair of lists (a,b) gives the sum of diagonal matrices in the u and v bases. On the array side the same lists give $(a_\alpha + b_\beta)_{\alpha\beta}$, the orthogonal complement of E. Both maps have kernel $a_\alpha = c$, $b_\beta = -c$. Thus S and E have the same character multiplicities. Subtracting the dimension of the image of J in each character gives

$$\dim(E/J(S))_x = \dim(\ker J)_x \qquad (x \in G),$$

where a subscript denotes the corresponding space after complexification. By Theorem 4.1 and its character calculation, these dimensions are 1 on $(L - \{0\}) \cup (M - \{0\})$ and zero elsewhere.

We can now check Q on the 2 summands of $\ker J = T_L \oplus T_M$. Each $A \in T_L$ is the derivative of a curve in the corresponding analytic family. Differentiating $\Phi = 0$ twice along this curve gives $Q(A,A) = 0$. Polarization gives $Q(T_L, T_L) = 0$, and the same argument applies to T_M.

For the mixed term, take A_x of character $x \in L - \{0\}$ and C_y of character $y \in M - \{0\}$. Equivariance gives

$$Q(A_x, C_y) \in (E/J(S))_{x+y} = 0,$$

since L and M are distinct lines and $x + y$ lies on neither. Bilinearity now gives $Q = 0$ on the whole kernel.

Returning to the real spaces, for every $A \in \ker J$ we can therefore solve $JB = -D^2\Phi(0)(A,A)$ with $B \in S$. The unitary path $U(tA + t^2B/2)$ has all entries of modulus $1/\sqrt{n} + O(t^3)$. Restoring each modulus to $1/\sqrt{n}$ changes the matrix by $O(t^3)$. Multiplying by $\sqrt{n}$ and dephasing gives the required phase deformation through order 2. □

We can give the coefficient of order 4 in the rectangle energy a geometric meaning at any Hadamard matrix. Let H have order n, take $A \in T_H$, and vary the real array B with its first row and column fixed at zero. For $i \neq k$ write

$$L(B)_{ik} = i \sum_j H_{ij}\bar{H}_{kj}(B_{ij} - B_{kj}), \qquad C(A)_{ik} = \sum_j H_{ij}\bar{H}_{kj}(A_{ij} - A_{kj})^2.$$

We give these arrays the squared norm $\|Y\|^2 = \sum_{i \neq k} |Y_{ik}|^2$ and take the image of L over the reals.

Proposition 5.4. *For $X(t)_{ij} = H_{ij} \exp(itA_{ij} + it^2B_{ij}/2)$, the least possible coefficient of t^4 in $R_1(X(t)) - n^3$, as B varies, is*

$$\frac{1}{4} \operatorname{dist}\big(C(A), \operatorname{Im} L\big)^2.$$

The least coefficient of t^4 in $n\sqrt{n} - M(X(t))$ is the displayed value divided by $8n^{3/2}$. Both minima vanish exactly when A extends through order 2.

Proof. For $i \neq k$ the constant and linear terms in the row scalar product vanish. By Proposition 1.9 its quadratic term is $t^2(L(B)_{ik} - C(A)_{ik})/2$. Lemma 3.2 sums the squared moduli of these terms. Minimizing over B gives the squared distance to a real linear subspace, and the minimum is attained. Here $X(t)X(t)^* - nI = O(t^2)$, so the remainder in the energy expansion of Theorem 5.1 is $O(t^6)$. This gives the second minimum. □

At the Hall and Nicoara matrix of Theorem 1.11, this minimum is strictly positive in either unit direction on its tangent line [18]. Indeed their necessary condition uses only the terms through order 2 in unitarity and the prescribed entry moduli. At the matrices of Theorem 5.3 it is zero in every direction. In particular the distinction between defect 8 and local dimension 4 at H_5 first becomes visible beyond this quadratic correction.

6 Local dimension in orders 10 and 14

The families of Section 4 give lower bounds on the local dimension. Section 5 shows why the equations through order 2 give no improvement on the defect bound. We now use higher-order terms at the matrices $H_p = H_{F_2}((1,-1),(2,-2))$ of Section 2, for $p = 5, 7$. Their cross-ratio is $1/9$ and their defect is $d_p = 2(p-1)$. We first obtain a strict upper bound in both orders, then determine the dimension in both cases.

Proposition 6.1. *The dephased local dimensions satisfy*

$$4 \leq \dim_{H_5} \leq 7, \qquad 6 \leq \dim_{H_7} \leq 11.$$

In particular the local dimension is strictly smaller than the defect at both matrices.

Proof. The lower bounds follow from Theorem 4.1. We give the exact finite-field certificates for the upper bounds. Write the perturbed entries as $H_{ak}\exp(z_{ak})$, with the 1st row and column of z zero. Real phases correspond to purely imaginary z. The holomorphic equations for the complexification are

$$E_{ab}(z) = \sum_k \frac{H_{ak}}{H_{bk}} \exp(z_{ak} - z_{bk}) = 0 \qquad (a \neq b).$$

Both orders of every pair are included. At imaginary z these equations come in conjugate pairs. Their complex Jacobian J therefore has the rank of the real phase system, namely $r = (2p-1)^2 - d_p$ by [10].

Choose an $r \times r$ minor of J whose determinant does not vanish. The corresponding r equations give a holomorphic graph solving for the pivot coordinates in terms of the other d_p coordinates. If every other equation vanished on this graph it would give a full-dimensional complex solution germ. It suffices to find a Taylor coefficient of a residual that differs from zero. This also proves the strict real bound. Indeed the free coordinates of real solutions lie in $i\mathbb{R}^{d_p}$. If a holomorphic germ vanishes on that whole real slice, it vanishes identically. Thus either the real or the imaginary part of our residual restricts to a real analytic function that does not vanish identically. Its zero set has real dimension at most $d_p - 1$.

Choose an auxiliary prime $p' = 1\ (p)$ and reduce $w = e^{2\pi i/p}$ to a primitive p-th root in $\mathbb{F}_{p'}$. Up to a common Gauss scalar the 4 blocks of H_p are $C_1, C_{-3}, C_3, -C_{-1}$ in row order, where

$$(C_h)_{ab} = \left(\frac{h}{p}\right) w^{-(b-a)^2/(4h)}.$$

The table specifies the image of w for each reduction. We select the pivot minor after reduction. Its determinant is a unit in the localization of $\mathbb{Z}[w]$ at the corresponding prime ideal, so the same minor defines the graph in characteristic zero.

Index matrix entries from zero. Order the unfixed coordinates row by row and the ordered pairs (a, b) lexicographically. Scan columns for pivots, choosing the 1st available row after previous eliminations. Set the free coordinates in increasing order equal to $t, 2t, \ldots, d_p t$. At degree k write D_k for the array of differences $z_{aj,k} - z_{bj,k}$ of the degree-k log coefficients, where the subscript j indexes the summand in E_{ab}. The exponential coefficients satisfy $P_0 = 1$ and

$$P_k = D_k + \frac{1}{k} \sum_{l=1}^{k-1} l D_l P_{k-l}$$

entrywise. Solving the pivot equations for D_k means solving for the pivot entries of z with the same invertible minor at every degree. Summing P_k with weights $\frac{H_{aj}}{H_{bj}}$ then checks all the other equations. The table gives the first residual that differs from zero in each field.

p	p'	image of w	r	degree	pair	residue
5	101	95	73	6	$(7, 4)$	62
7	211	58	157	5	$(11, 6)$	6

The accompanying program `mixed_jets.py` verifies the ranks, inverts the selected minors, and checks these residues. Through the indicated degree, all divisions in the recursion are by units at the chosen prime. These residues show that the corresponding coefficients do not vanish in characteristic zero. Earlier coefficients which vanish in the finite field need not vanish in characteristic zero. The calculation gives the strict dimension bounds without assigning an intrinsic first obstruction degree. □

The reduction in Proposition 6.1 detects a term of degree 6 at H_5. We can now locate the first obstruction in characteristic zero.

Proposition 6.2. *At H_5 every real dephased tangent direction extends through order 5. Some directions fail to extend through order 6.*

Proof. Use the 73 pivot rows and columns of Proposition 6.1 at 101, $w = 95$. This time compute the homogeneous residuals on the entire graph of the 8 free logarithms over $\mathbb{Q}(w)$, where $w = e^{2\pi i/5}$. The following finite interpolation checks their vanishing. Set the first free coordinate to 1 and let the remaining 7 be integers at least zero with sum at most 5. There are 792 such points. For each, run the exponential recursion of Proposition 6.1 through degree 5. In the basis $1, w, w^2, w^3$ all 4 coefficients of every residual are zero at every point. The program `order10_jets.py` uses the pivot inverse with common denominator 1000 and checks the identities with integers. Values on this simplex determine a polynomial of degree at most 5, by successive finite differences. Homogeneity restores the first coordinate. In degree 6, the free direction $(1, 2, \ldots, 8)$ gives a nonzero residual, as can also be seen from Proposition 6.1.

For a real phase velocity, take the free logarithms purely imaginary. Through degree 5 the graph solves every ordered row equation. Applying $z \to -\bar{z}$ interchanges the ordered equations and preserves these free logarithms. Uniqueness in the pivot equations makes the resulting jet real in phase coordinates. Finally the leading sextic on a curve depends only on its velocity, since all lower residuals vanish identically. Multiplying $(1, 2, \ldots, 8)$ by i preserves its nonvanishing and gives a real phase direction obstructed in degree 6. □

One residual on a line rules out dimension 8 at H_5. To reach dimension 4, we cut the 8 free coordinates by 4 linear equations and prove that the resulting germ is a point. We use multiplicative coordinates in this calculation so that all power series can be reduced modulo 101 in every degree.

Theorem 6.3. *Let $H = H_{F_2}(s, t)$ be an order-10 matrix with 4 distinct labels in $P^1(\mathbb{F}_5)$. Its dephased real local dimension is 4 when the cross-ratio is a square and zero otherwise. In the square case the labeled dephased germ has at least 27 distinct smooth complex components of dimension 4, each obtained by complexifying a real analytic family through H.*

Proof. We start with H_5 and its 81 dephased variables. Replace each entry H_{aj} by $H_{aj}(1+X_{aj})$, where $X_{0j} = X_{a0} = 0$. In these coordinates the complex equations near the origin are

$$\sum_j \frac{H_{aj}}{H_{bj}} \cdot \frac{1+X_{aj}}{1+X_{bj}} = 0 \qquad (a \neq b).$$

Their Jacobian at zero is the same matrix J as above. Use $p' = 101$, $w = 95$ modulo p', and the same pivot rule as in Proposition 6.1. The 73 pivot equations determine 73 variables as power series in the other 8. In row order on the 9×9 unfixed entries, starting at zero, the free indices are

$$70, 71, 75, 76, 77, 78, 79, 80.$$

Write x for these 8 coordinates in the indicated order. We take the slice

$$x = \begin{pmatrix} I_4 \\ C \end{pmatrix} u, \qquad C = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{pmatrix}, \qquad u = (u_1, u_2, u_3, u_4)^t.$$

This imposes exactly 4 equations on the free coordinates.

Here is the calculation on the slice over $\mathbb{F}_{101}$. Write $X_{aj,\alpha}$ for the coefficient of u^α in X_{aj} and $V_{abj,\alpha}$ for its coefficient in $(1+X_{aj})/(1+X_{bj})$. Multiplying back by $1+X_{bj}$ gives $V_{abj,0} = 1$ and

$$V_{abj,\alpha} = X_{aj,\alpha} - X_{bj,\alpha} - \sum_{\substack{\beta+\gamma=\alpha \\ |\beta|,|\gamma|>0}} X_{bj,\beta} V_{abj,\gamma} \qquad (|\alpha| > 0).$$

At each monomial the sum is known from smaller degrees. The free entries of X are given by the slice, so we obtain the pivot entries by multiplying the right-hand side of the pivot system by the inverse of the fixed 73×73 minor. Summing $V_{abj,\alpha}$ over j with weights $\frac{H_{aj}}{H_{bj}}$ gives the residual coefficient for row pair (a, b).

All residual coefficients through degree 5 vanish over $\mathbb{F}_{101}$. In degree 6 the residual forms span a space of dimension 8. Let I_6 be the homogeneous ideal generated by these sextics in $\mathbb{F}_{101}[u_1, u_2, u_3, u_4]$. Multiplying the sextics by all monomials of degree $e-6$ and eliminating in the degree-e monomial basis gives

degree e	6	7	8	9	10	11
number of monomials	84	120	165	220	286	364
dimension of $(I_6)_e$	8	32	80	160	272	364

The program `mixed_jets.py` verifies the table by the recursion above. A second reduction at $p' = 151$, $w = 8$ gives the same table. In particular I_6 contains every degree-11 monomial. In the local ring of the sliced residual equations over $\mathbb{F}_{101}$ this gives $\mathfrak{m}^{11} = \mathfrak{m}^{12}$, where $\mathfrak{m}$ is generated by the u_i. Indeed one can replace each sextic by its full residual combination, introducing only terms of higher degree. Nakayama's lemma now gives $\mathfrak{m}^{11} = 0$.

We explain why this bounds the dimension in characteristic zero. Work over $\mathbb{Z}[w]$ localized at $(101, w-95)$, with $w = e^{2\pi i/5}$. The pivot determinant is a unit, and the recursion uses no other divisions. Thus the sliced residual series are defined over this local ring in every degree. For any k, truncate at total degree $k-1$. The truncated residual ideal is the span of all monomial multiples of the residuals in this finite space. Its coefficient matrix has at least the same rank over $\mathbb{C}$ as after reduction. Consequently the dimension of the truncated quotient over $\mathbb{C}$ is bounded, for every k, by the number of monomials of degree at most 10 in 4 variables. The successive dimensions therefore stabilize. Nakayama's lemma gives a power of the maximal ideal equal to zero over $\mathbb{C}$ as well. This proves that the complex slice has dimension zero.

By Krull's height theorem, imposing 4 equations can lower the local dimension by at most 4. The complex Hadamard equations before slicing thus have local dimension at most 4. In logarithmic coordinates the real solutions lie in $i\mathbb{R}^{81}$. The complexification of a smooth real analytic stratum of the real solution set is a germ contained in the complex solution set. Its complex dimension equals the real dimension of the stratum. The real local dimension is therefore also at most 4. Theorem 4.1 supplies the matching lower bound.

For 4 distinct labels the cross-ratio belongs to $\mathbb{F}_5 - \{0, 1\}$, whose only square is 4. This is the cross-ratio of H_5. Ordered configurations with the same cross-ratio differ by an element of $PGL_2(\mathbb{F}_5)$. By the projective symmetry in [10], their matrices differ by Hadamard equivalence and possibly a cyclotomic Galois conjugation. The multiplicative equations above have algebraic coefficients. Conjugation preserves the dimensions of all their truncated local rings, hence also their complex local dimension. The real lower bound still follows at each matrix from Theorem 4.1. If the cross-ratio is not a square, the defect is zero by [10].

Finally, Theorems 4.1 and 4.2 give $2 + 5^2 = 27$ real analytic families through each matrix whose cross-ratio is a square. Each complexifies to a smooth germ of dimension 4. Indeed its 4 independent real phase derivatives remain independent over $\mathbb{C}$ in logarithmic coordinates. A smooth germ of dimension 4 contained in a complex analytic germ of dimension 4 is an irreducible component, since a proper analytic subset of an irreducible germ has smaller dimension. The 27 real tangent spaces are pairwise distinct and remain so after complexification. This distinguishes the 27 components. □

At order 14 the same method encounters a difference. The first forms on a slice of the desired dimension have common projective zeros. We must test whether any of these directions extends to a curve.

Theorem 6.4. *Let $H = H_{F_2}(s, t)$ have 4 distinct labels in $P^1(\mathbb{F}_7)$. If the cross-ratio is a square, its dephased real and complex local dimensions are both 6, and the labeled complex germ has at least 51 smooth irreducible components of dimension 6. If the cross-ratio is not a square, H is isolated.*

Proof. We first work at H_7. Reduce at $p' = 211$, with $w = 58$, and use the multiplicative coordinates and pivot rule of Theorem 6.3. Of the 169 unfixed entries, the 12 free indices are

$$153, 154, 155, 160, 161, 162, 163, 164, 165, 166, 167, 168.$$

Write the free coordinates as $x = \binom{u}{Cu}$ with $u = (u_0, \ldots, u_5)^t$ and

$$C = \begin{pmatrix} 18 & 13 & 11 & 28 & 11 & 8 \\ 27 & 1 & 13 & 9 & 0 & 22 \\ 30 & 24 & 9 & 16 & 26 & 9 \\ 13 & 28 & 6 & 28 & 20 & 13 \\ 24 & 12 & 11 & 19 & 28 & 22 \\ 20 & 19 & 26 & 13 & 23 & 13 \end{pmatrix}.$$

This cuts the graph by 6 linear equations. Denote its residual series by $R_{ab}(u)$, one for each ordered row pair. The coefficient recursion of Theorem 6.3 gives zero through degree 4 in this reduction. The homogeneous parts of degree 5 span a space of dimension 12.

Let $S = \mathbb{F}_{211}[u_0, \ldots, u_5]$ and let I be the ideal generated by these quintics. We give the finite calculations identifying their common projective zeros. The program `order14.py` starts with this matrix and slice. It uses `order14_linear.py` for the linear algebra of the projective directions. Multiplication of the quintics by the degree-6 monomials gives a matrix with 5544 rows and 4368 columns. Its kernel is the dual of $(S/I)_{11}$. In each subsequent degree a functional

is determined by its 6 restrictions to the multiples of u_j. Each restriction belongs to the preceding dual space, and their values must agree on common monomials. These linear equations give

e	11	12	13	14
$\dim(S/I)_e$	90	38	20	20.

Applying the pivot rule to a dual basis selects 20 monomials forming a basis in the degree-13 quotient. Their products by u_0 form a basis in degree 14. Thus at a projective zero with $u_0 = 0$ every degree-14 monomial would vanish, which is impossible.

Set $u_0 = 1$. In the chosen bases multiplication by u_0 identifies the 2 quotient spaces. Write M_j for multiplication by u_j under this identification, acting on evaluation vectors, and put $A = \sum_{j=0}^{5} jM_j$. The characteristic polynomial of A has 9 distinct linear factors, one quadratic and 3 distinct irreducible cubics. The program expresses each M_j as a polynomial $g_j(A)$ of degree at most 19 by linear elimination. At any projective zero the vector of its degree-13 basis values is nonzero, and M_j acts on this vector by its coordinate u_j. Hence every direction occurs among

$$(1, g_1(\alpha), \ldots, g_5(\alpha)), \qquad \det(\alpha I - A) = 0.$$

Substitution verifies that all 20 are zeros. They are distinct since $\alpha = \sum_j j g_j(\alpha)$.

For each of these points write $u = t(1, v)$ and divide the residuals by t^5. At $t = 0$ their derivatives in the 5 coordinates of v have rank 5 over the field of the point. Thus 5 independent residuals determine a unique power series $v(t)$ through the point. We test the others by substituting $u = t(1, v(t))$ in the original ratio recursion. At step h, the coefficient of t^{5+h} is a known vector plus the same derivative matrix times the new coefficient of v. The table gives the first inconsistent equation order, counting powers before division by t^5. Each irreducible factor is tested over its own field.

field degree of the point	failure in degree 6	failure in degree 9
1	2	7
2	0	2
3	0	9

These counts are over the algebraic closure of $\mathbb{F}_{211}$. To see why they suffice, suppose the sliced germ contained a curve. Normalizing a formal branch gives a power series parametrization. Its leading direction is a zero of the quintics, so u_0 has minimal order and is nonzero along the branch. Taking $t = u_0$, the ratios $v_j = u_j/u_0$ are regular there. The formal implicit function theorem forces them to be the series just determined, composed with u_0 in the branch parameter. A remaining residual has a nonzero leading term, which stays nonzero under this substitution. This rules out the branch, including when u_0 has order greater than 1.

The sliced local ring in this reduction therefore has finite length. The truncation argument of Theorem 6.3, now over $\mathbb{Z}[w]$ localized at $(211, w-58)$, bounds every characteristic-zero truncation by this length. The complex slice has dimension zero. Since it is cut by 6 equations, the complex local dimension at H_7 is at most 6. Theorem 4.1 supplies a real family of dimension 6, proving equality for both dimensions.

The cross-ratio of H_7 is $1/9 = 4$ in $\mathbb{F}_7$. The other admissible square is $2 = 4^{-1}$. Corollary 3.9 and the projective symmetry used in Theorem 6.3 transfer the complex dimension to every ordered configuration of either cross-ratio. The real lower bound holds at each one. The $7^2 + 2$ families of Theorems 4.1 and 4.2 now complexify to distinct smooth irreducible components by the dimension argument of Theorem 6.3. If the cross-ratio is not a square, the defect is zero by [10], which gives isolation. □

There is also a way to specify a nearby matrix by a few of its entries, with finitely many choices left over. Here the row and column labels matter. Use the displayed circulant representative in Section 2, with its first row and first column held fixed.

Proposition 6.5. *For sufficiently small deformations of H_5, prescribing the entries in positions*

$$(1,2), \quad (4,9), \quad (7,4), \quad (8,3)$$

leaves at most 449 *labeled dephased completions, with indices starting at zero. They are distinguished by the last* 4 *entries of the last row. The other entries are single-valued analytic functions of these* 8 *entries on the pivot graph. The bound holds in the complexified germ as well.*

Proof. On the graph of the 73 pivot equations used in Theorem 6.3, the 4 selected entries have differential of rank 4 at H_5. More precisely, on the first 4 of the 8 free variables their differential is invertible modulo 101. The last 4 free variables are precisely the entries in positions $(9,6), (9,7), (9,8), (9,9)$, relative to their base values. Thus the 4 prescribed entries and these 4 last-row entries give local analytic coordinates on the pivot graph. To recover a completion one solves the pivot equations there and tests the remaining 17 ordered row equations in the 4 last-row variables. Each root near the base point gives exactly one complex completion. The real Hadamard matrices are those with unit-modulus entries. On the complex pivot graph the fibre over the original 4 entries is smooth of dimension 4. We intersect this fibre with the remaining orthogonality equations. Since the residuals vanish through degree 5 on the whole graph, their degree-6 parts on the fibre are obtained by restricting the leading sextics to its tangent plane. Thus it suffices to use the kernel of the 4 selected rows of the tangent matrix as the linear substitution.

The program `mixed_jets.py` carries out this substitution. As in Theorem 6.3, modulo 101 the quotient by the restricted sextics has dimensions $76, 88, 85, 60, 14, 0$ in degrees 6 through 11. The matrices defining the substitution have unit pivot minors, so the ranks of the homogeneous products can only increase in characteristic zero. The central fibre of the full complex germ therefore has length at most

$$\binom{9}{4} + 76 + 88 + 85 + 60 + 14 = 449.$$

The binomial term counts degrees 0 through 5. An isolated central fibre makes the projection finite as a map of analytic germs, by Weierstrass finiteness. By Nakayama's lemma the local algebra is generated by at most 449 elements over the convergent series ring of the 4 entries. The generators still suffice on small representatives of the finite map, so every nearby fibre has length at most 449. Every real phase completion is one of these points. To identify equivalent completions, including ones over different prescribed entries, one can use the finite group in the next section. □

7 Permutation symmetries and components

The dephased space has dimension 4 at H_5. We now keep track of which families pass through the same point. We first determine all ordinary symmetries of H_5. This accounts for every nearby matrix which retains such a symmetry, since there are only finitely many permutations of the rows and columns.

Theorem 7.1. *Modulo scalar phases acting trivially, the group of row and column symmetries on the dephased germ at H_5 is $\mathbb{F}_5^2 \rtimes Q_8$, where Q_8 is the quaternion group of order* 8*. A sufficiently close dephased Hadamard matrix has a nonidentity permutation symmetry exactly when it lies in the union of the* 27 *smooth families of Theorem A. Up to equivalence there are* 2 *types of such families.*

Proof. Index an inner row or column by (h, j), where h is its basis label and $j \in \mathbb{F}_5$. The translations $g_{(a,b)}$ add $b - 2ha$ to j. Consider also the permutations

$$P(h,j) = (-1/h, j/h), \qquad Q(h,j) = (-h, 2j).$$

Inserting them in the chirp formula changes only row and column phases. They normalize the translations, satisfy $P^2 = Q^2 = r$ with $r(h, j) = (h, -j)$, and $PQ = Q^{-1}P$. They generate a quaternion group which intersects the translations only in the identity. This gives 200 permutation pairs.

Here is an exhaustive check of the upper bound. Write the exponents of H_5 modulo 10 and dephase at $(0, 0)$ by subtracting the first row and column and adding back the corner. Try all 100 choices for the image of the corner and dephase the candidate matrix there. The image of row 1 must have the prescribed multiset of entries. Within its equal entries try every column bijection, fixing the chosen corner column. The remaining rows are distinct and their images are then determined. The program `local_symmetries.py` finds exactly 200 pairs and verifies that the displayed generators give them all. In a sufficiently small neighborhood only permutations fixing the base point can fix a matrix or identify 2 matrices in that neighborhood. This follows by continuity and finiteness after dephasing.

Every nonidentity translation is covered by Corollary 4.4. An element over r is a translated reversal. Every element over a member of order 4 of Q_8 has square over r and hence fixes only points already fixed by a translated reversal. Conversely each of the 27 families is fixed by a nonidentity permutation. The 25 reversal families are conjugate by translations. The map P exchanges the translation lines, indexed by $0, \infty$, and hence the other 2 families. These types are distinct since an order-5 translation cannot fix an entire reversal family, by Proposition 4.3. □

We can now count the symmetric completions when the 4 phases in Proposition 6.5 are prescribed. Write K for a dephased matrix near H_5 and put

$$\pi(K) = (K_{1,2}, K_{4,9}, K_{7,4}, K_{8,3}).$$

We use the representative of H_5 displayed in Section 2. The indices start at zero.

Theorem 7.2. *On each of the* 27 *smooth families with a permutation symmetry, the* 4 *entries in* π *give real analytic coordinates near* H_5*. Every choice of these entries sufficiently near* $\pi(H_5)$ *in* $\mathbb{T}^4$ *has exactly* 27 *distinct nearby labeled completions with a nonidentity permutation symmetry, except on a set of real dimension at most* 2*.*

Proof. It suffices first to check that $d\pi$ is invertible on each of the 27 tangent spaces. We use logarithms of the entries relative to their values at H_5, so that $d\pi$ reads off the same 4 coordinates. On a real phase space this amounts to using the arguments of the entries as local coordinates on $\mathbb{T}^4$.

Here is the finite calculation. Let V be the matrix whose columns form a basis of the kernel of the dephased Jacobian in Proposition 6.1, reduced at $w = 95$ modulo 101. Regard each column as a 10 by 10 array by restoring its zero first row and column. For an array X write

$$D(X)_{ij} = X_{ij} - X_{i0} - X_{0j} + X_{00}.$$

For a permutation pair g, apply g to the indices of each column of V. Compute a basis W_g of the kernel of $D(gV) - V$. The 4 rows of VW_g selected by π give a square matrix A_g. For both kernels take the nonpivot coordinates in increasing order as the free coordinates, using the pivot convention of Proposition 6.1.

Take the translations $g_{(1,0)}, g_{(0,1)}$ and the 25 reversals given on a block of label h by $j \to -j - b + 2ha$, with $a, b \in \mathbb{F}_5$. For the translations the determinants of A_g are $96, 4$ in that order. For the reversals their product is 92, all modulo 101. These computations are given in `entry_charts.py`. In particular the only array satisfying the tangent equations, the fixed equations and $d\pi = 0$ is zero after reduction. The combined equations thus have full column rank in characteristic zero as well. Each fixed tangent space has real dimension 4 by Theorems 4.1 and 4.2. The inverse function theorem gives one completion on each family for every choice of the 4 phases in a common neighborhood.

By Theorem 7.1 this list exhausts the nearby completions with a permutation symmetry. A completion is counted twice only if it lies in a pairwise intersection of the families. Each intersection has dimension at most 2 by Proposition 4.3 and $T_L \cap T_M = 0$. Its image under the coordinate chart on either family has the same dimension. Taking the union over all pairs gives the stated exceptional set. □

Do the smooth families of Section 4 exhaust the components in orders 10 and 14? To answer this we return to the quaternionic structure used to construct F_0. In that construction we imposed both quaternionic symmetry and reversal. We now impose just the quaternionic symmetry and look at how its components meet.

Use the symmetric labels $s, -s$ and $t, -t$ and the normalized matrix U_0 from the proof of Theorem 4.2. Such coordinates exist when the cross-ratio is a square, by Theorem 4.1. Recall the matrices

$$J = \begin{pmatrix} 0 & I_p \\ -I_p & 0 \end{pmatrix}, \qquad R = diag(P, P), \qquad Pe_a = e_{-a}.$$

The matrices J and R commute. Both J and JR are real signed permutation matrices with square $-I$, and their underlying permutations, denoted by σ, have no fixed indices. For either choice $D = J, JR$ we have $U_0 D = D\bar{U}_0$. Denote the induced involution on dephased matrices by τ for J and by τr for JR. In complex logarithmic coordinates $U_{ij} = (U_0)_{ij} \exp X_{ij}$, their actions are holomorphic. In each case they send X_{ij} to $-X_{\sigma i, \sigma j}$ modulo row and column terms.

Lemma 7.3. *Suppose the complexified dephased germ has dimension $p - 1$. For each of the involutions $\tau, \tau r$, its fixed locus is the support of a local complete intersection of dimension $p-1$. In particular its irreducible components are connected in codimension 1.*

Proof. Fix D and work in the complex symplectic group defined by $U^t D U = D$. This is smooth of dimension $p(2p + 1)$. Write $V = (U^{-1})^t$, so $V = DUD^{-1}$ on the group. The complexified flatness equations for a normalized matrix are

$$m_{ij}(U) = U_{ij} V_{ij} - \frac{1}{2p} = 0.$$

Indeed, at a solution they prescribe $V_{ij} = 1/(2pU_{ij})$, and $UV^t = I$ gives exactly the ordered row equations. The array m is invariant under simultaneous permutation by σ of its row and column indices, since $V = DUD^{-1}$ interchanges the entries in each pair up to the same sign. Its row and column sums vanish by $UV^t = V^t U = I$. There are $2p^2$ entry orbits, with p row orbits and p column orbits. The sums impose $2p-1$ independent linear conditions on invariant arrays. Indeed row and column sums with equal totals can be prescribed on all arrays, and averaging gives the invariant prescriptions. Thus exactly

$$e = 2p^2 - 2p + 1$$

coordinates suffice for the target of the equations.

We remove the diagonal row and column scalings inside the group. They have $2p$ independent parameters, with factors on paired indices reciprocal. To see that these are all the lost directions, take a scaling variation $\delta U_{ij} = (a_i + b_j)(U_0)_{ij}$. Differentiating $V = DUD^{-1}$ gives

$$a_i + a_{\sigma i} + b_j + b_{\sigma j} = 0$$

for every i, j. Adding opposite constants to a and b makes both paired sums zero. A smooth local slice transverse to the scaling orbits therefore embeds by dephasing, by the rank theorem. Its dimension is $p(2p + 1) - 2p$.

The flat part of the slice gives precisely the fixed germ. In fact a fixed logarithmic class has the representative obtained by averaging X_{ij} with $-X_{\sigma i, \sigma j}$. For this representative $X_{\sigma i, \sigma j} = -X_{ij}$.

Flatness and the identity at U_0 then give $V = DUD^{-1}$ entry by entry. Conversely a flat symplectic matrix has this logarithmic symmetry. The slice consequently defines the fixed locus by e equations in a smooth space of dimension $e + p - 1$. Every component has dimension at least $p - 1$, and the dimension of the full germ supplies the opposite bound. The equations have height e and form a regular sequence in the regular local ring. Hartshorne's connectedness theorem [19] now applies. In the graph whose vertices are the irreducible components, joining 2 vertices when their intersection has dimension $p - 2$, every vertex can be reached from every other. □

For $p \geq 5$, the families of Section 4 meet in dimension at most $(p-1)/2 < p-2$. The fixed locus just found contains several of them. This forces other components, and the translations distinguish some of these from one another.

Theorem 7.4. *Let $p \geq 5$ and suppose the 4 labels have square cross-ratio. If the complexified dephased germ has dimension $p - 1$, it has at least $p^2 + 2p + 2$ distinct irreducible components of that dimension. In particular there are at least 37 in order 10 and 65 in order 14 under these hypotheses.*

Proof. Write $g_{(a,b)}$ for a translation. On a basis with finite label h it shifts the inner index by $b - 2ha$, up to phases, by the action of $X^a Z^b$ on B_h. The permutation in J interchanges the block halves, replacing h by $-h$. Reversal r negates the inner index. Thus on the dephased logarithmic classes we have

$$\tau g_{(a,b)} \tau^{-1} = g_{(-a,b)}, \qquad (\tau r) g_{(a,b)} (\tau r)^{-1} = g_{(a,-b)}.$$

In particular each involution commutes with a line of p translations and has p distinct conjugates by the translation group.

The family F_0 lies in both fixed loci by its construction from matrices satisfying $UR = RU$ and $UJ = J\bar{U}$. Each fixed locus therefore contains the p families $F_z = g_z F_0$ from its commuting line. They have distinct tangent spaces and dimension $p - 1$, so their complexifications are components of that fixed locus. Any 2 distinct members of the original $p^2 + 2$ families meet in dimension at most $(p-1)/2$, by Proposition 4.3 and $T_L \cap T_M = 0$. The same bound holds over $\mathbb{C}$ by the fixed-germ argument there. Hence in the graph of Lemma 7.3 there is no edge between any 2 original components. Since the graph is connected and contains at least 2 such vertices, each of them has a neighbor outside the original list. Every such neighbor has dimension $p - 1$, so it is also a component of the full germ.

Apply this argument to all the translates of the 2 fixed loci. The $2p$ involutions have the forms

$$g_{(a,0)}\tau, \qquad g_{(0,b)}\tau r, \qquad a, b \in \mathbb{F}_p.$$

The product of distinct involutions from the same list is a nonidentity translation. A product across the lists is a translated reflection $g_z r$. By Corollary 4.4 every point fixed by any such product lies in the original families. Thus a component outside them can be fixed pointwise by at most one of the $2p$ involutions. This gives at least $2p$ additional components. Theorems 6.3 and 6.4 verify the dimension hypothesis in the stated orders. □

In order 10 each of the 25 reflection components F_z consequently meets at least 2 further components in dimension 3, one from each kind of quaternionic involution.

8 Real branches in order 10

The preceding section gives at least 10 complex components through H_5 beyond the 27 permutation families. To obtain new Hadamard matrices we must find real phases on these components. We first construct a real curve leaving all 27 families. Along this curve the defect drops from 8 to 4, which gives a real family of the full local dimension.

Theorem 8.1. *At H_5 at least 37 distinct irreducible components of the labeled complexified dephased germ have real local dimension 4. For each of the 10 quaternionic involutions of Theorem 7.4, one such component lies in its fixed locus outside the original list. On each of these additional components there are regular real points of defect 4 arbitrarily close to H_5.*

Proof. Use $D = J$ in Lemma 7.3 and write $U_{ij} = (U_0)_{ij}e^{iP_{ij}}$ in real phases. Write i' for the index obtained by interchanging the 2 block halves. We impose $P_{i'j'} = -P_{ij}$ and use the top 5 rows as coordinates. Paired row and column phases allow us to set

$$P_{0j} = 0 \ (0 \le j < 5), \qquad P_{i0} = 0 \ (1 \le i < 5), \qquad P_{05} = 0.$$

Indeed the conditions at $(0,0)$ and $(0,5)$ first fix the paired scalings of row 0 and column 0. The other 8 conditions fix the remaining scalings. This leaves 40 phase coordinates, ordered by row and then by column, starting from zero.

Put $E = UU^* - I$. For every $0 \le a < b < 5$ take the pairs (a,b), and then, in a second list, the pairs $(a,b+5)$, in lexicographic order within each list. For each pair (i,j) use the real and imaginary parts of $10E_{ij}$, in that order. These 40 real functions determine E. Its diagonal is zero and it is Hermitian and invariant under quaternionic conjugation. In particular its upper left block is Hermitian and its upper right block is skew-symmetric.

In fact 36 equations suffice near U_0. The columns have norm 1, so invertibility gives

$$(U^*E(I+E)^{-1}U)_{jj} = 0.$$

To see this, the left side is the diagonal of $U^*U - I$. Set the first 34 functions and functions $34, 36$ equal to zero. Regard the remaining 4 coordinates, numbered $35, 37, 38, 39$, as independent entries of E in the last display, with U as parameter. At $(U_0, 0)$ the first 4 diagonal equations have an invertible derivative in these entries. It is the matrix with entries $(U_0^*E_hU_0)_{jj}$ for $0 \le j < 4$, where E_h is the Hermitian quaternionic coordinate matrix of the omitted entry. The exact check is included in `real_branches.py`. The zero matrix solves these equations for every U. By the implicit function theorem it is the only sufficiently small solution for the remaining entries.

The first 34 equations have an invertible minor on the first 34 phases. We solve for these and denote the remaining 6 phases, in order, by u. Write $R_1(u), R_2(u)$ for the resulting equations numbered $34, 36$. We take a real slice in these free phases. With $s = 2\cos(\pi/10)$ and $f = s^2 - 2 = (1+\sqrt{5})/2$, put

$$a = (2, 2, -f, 0, f+1, 0)^t, \quad b = (0, f, 0, 1, f+1, 0)^t, \quad c = (0, 1, 2, 0, -1, 3)^t.$$

Set $u = t(xa+yb+c)$. On this slice both residuals are divisible by t^6. Let F, G be the coefficients of t^6 in R_1, R_2. On the line $y = -x$ they coincide and give

$$F(x,-x) = G(x,-x) = x^2\left(-1284 + \frac{1812}{5}s^2 + \left(-\frac{1496}{5} + \frac{2564}{25}s^2\right)x\right).$$

The root

$$x_0 = \frac{181695s^2 - 690810}{88409}$$

is nonzero. At $(x_0, -x_0)$ the Jacobian determinant of (F, G) in (x, y) is nonzero. For example it reduces to 6 under $s \to 9$ modulo 101.

Here are details of the certificate. We use $s^4 = 5s^2 - 5$ and do the phase calculation over $\mathbb{Q}(s)$. The real and imaginary parts of $w = e^{2\pi i/5}$ are $(f-1)/2$ and $s/2$. For each row pair and column, expand the exponential of its logarithmic difference. If L_k and V_k are the coefficients in the logarithm and the exponential, respectively, then $V_0 = 1$ and

$$V_n = L_n + \frac{1}{n}\sum_{k=1}^{n-1} kL_kV_{n-k}.$$

At each order solve for the pivot phases using the same 34 by 34 minor. Summing with weights $10(U_0)_{aj}(\bar{U}_0)_{bj}$ gives the residuals. The coefficient of t^n has total degree at most n in x, y. Interpolation at the 28 integer pairs $x, y \geq 0$, $x + y \leq 6$, verifies the vanishing through order 5, the displayed restriction in degree 6 and the derivative determinant. Thus the divided residuals $t^{-6}R_j(t(xa + yb + c))$ are real analytic near $(0, x_0, -x_0)$. Their derivative in x, y is invertible there. The real analytic implicit function theorem gives $x(t), y(t)$ and an actual curve of flat matrices.

Write P_1 for its phase velocity. It is computed by the pivot equations from $u'(0) = x_0(a - b) + c$. To test equality modulo row and column phases, subtract row 0 and column 0 and add back the $(0, 0)$ entry. For each of the 49 nonidentity translations and translated reversals of Section 4, the difference between its transform of P_1 and P_1 survives this operation. For a translation with parameters (d, e), the inner index on a block with label h changes by $e - 2hd$, as in Theorem 7.4. The program checks all these linear comparisons in $\mathbb{Q}(s)$. Hence the velocity belongs to none of the 27 fixed tangent spaces, and the curve leaves all the original families.

It remains to verify the defect on the curve. This is a calculation in the full dephased space. Use the 73 pivot equations and 8 free complex logarithms of Proposition 6.1. Denote the residual map on that graph by R and its homogeneous part of degree k by $R^{(k)}$. Extend the reduction $s \to 9$ modulo 101 by $i \to 91$, so $w \to 95$. The velocity in the 8 free logarithms reduces to

$$v = (98, 69, 4, 75, 65, 37, 66, 10)^t.$$

In this field $R^{(1)}, \ldots, R^{(5)}$ vanish identically. One can check this by putting the first free coordinate equal to 1 and evaluating the remaining 7 coordinates on the integer simplex of total degree at most 5, using the preceding recursion. At v, the derivative of $R^{(6)}$ has rank 4. To compute each column, evaluate the sextics at $v + he_j$ for $0 \leq h \leq 6$ and differentiate the interpolating polynomial at $h = 0$.

For the passage to the real curve, all the denominators in these recursions are units at the chosen prime. This holds also for the 34 by 34 minor and the derivative determining $x(t), y(t)$, as verified in the certificate. Thus the coefficients of $x(t), y(t)$ and of the pivot phases through order 5 reduce at this prime. Indeed they use only the exponential expansion through order 10 and the same 2 invertible derivatives. Along the reduced curve the first term of dR is $t^5 dR^{(6)}(v)$. For instance the rows numbered $67, 74, 69, 75$ and columns $0, 1, 2, 3$ give the derivative minor

$$\begin{pmatrix} 35 & 22 & 19 & 83 \\ 96 & 2 & 75 & 11 \\ 60 & 81 & 81 & 99 \\ 32 & 31 & 57 & 42 \end{pmatrix}$$

modulo 101. The corresponding minor of dR has a nonzero coefficient of t^{20} after reduction and hence is not identically zero in characteristic zero. For all sufficiently small real $t \neq 0$, the full system therefore has rank at least $73 + 4 = 77$ on the 81 dephased phases.

The complex curve lies in a component of the quaternionic fixed locus, all of whose components have dimension 4 by Lemma 7.3. At these real points the full tangent dimension is at most $81 - 77 = 4$. Equality follows, and the full complex germ is smooth there. Its real form in phase coordinates is therefore smooth of real dimension 4. This proves the assertion for one component fixed by J.

Translations give components of the same kind for the 5 involutions conjugate to τ. A further symmetry of H_5 interchanges the 2 lists of involutions. On an inner index j belonging to a source or target block with label h, take the permutation

$$(h, j) \longrightarrow (-1/h, j/h)$$

in $\mathbb{F}_5$. It preserves both label lists and sends the simultaneous block interchange to block interchange followed by reversal. The chirp formula of Proposition 6.1 verifies that the permuted base

matrix differs from U_0 by row and column phases. Equivalently, the quotient of the 2 matrices has all its dephased entries equal to 1, which the program checks directly. The permutation conjugates the translations to translations and commutes with reversal. It therefore preserves the original list of 27 components and takes the component just obtained to an additional one fixed by τr. Translating again gives the other 5. Their distinctness follows from Theorem 7.4. □

The conclusions at H_5 apply to every order-10 configuration with 4 distinct labels and square cross-ratio. Indeed the projective symmetry in Theorem 6.3 leaves only cyclotomic conjugates to consider. In this case they give equivalences or adjoints. The automorphism $w \to w^2$ multiplies each finite basis label by 2 and permutes the columns within each basis, up to a common Gauss factor in the chirp formula. It takes $(1,-1)$ to $(2,-2)$ and $(2,-2)$ to $(-1,1)$. Interchanging the 2 block columns restores $(1,-1)$ and changes only a block row sign in F_2. The new matrix is therefore equivalent to $H_{F_2}((2,-2),(1,-1)) = H_5^*$. Since 2 generates $\mathbb{F}_5^\times$ and taking the adjoint preserves real deformations, the real dimension conclusions transfer as well.

The additional components were found one at a time using symmetries. We can also put an upper bound on their total number by intersecting the full germ with 3 linear equations.

Proposition 8.2. *The number of irreducible components of complex dimension* 4 *through* H_5 *is between* 37 *and* 331*. Each of the* 10 *quaternionic fixed loci has local multiplicity* 36 *at* H_5 *for the equations of Lemma 7.3.*

Proof. Use the free multiplicative coordinates of Theorem 6.3. In their 8-dimensional space take the 5-dimensional linear section

$$u = \begin{pmatrix} v \\ Cv + tb \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{pmatrix}, \qquad b = (2,7,17,31)^t.$$

Here v has 4 coordinates and t is one more variable. Work first over $\mathbb{F}_{101}$. Let I be the ideal generated by the leading sextics on this section. At $t = 0$ the quotient by these sextics is zero in degree 11, by Theorem 6.3. Thus in the graded quotient by I, or in any further quotient, multiplication by t maps each degree onto the next from degree 10 onwards.

Choose 8 residuals whose leading sextics are independent. Multiplying them by the 210 monomials of degree 6 gives 1680 rows. On the 1820 monomials of degree 12 they have rank 1469. Hence there are 211 relations among the rows. Apply each relation to the same multiples of the full residuals. Their degree-12 terms cancel, and their degree-13 terms come from the septics. The program `order10_jets.py` verifies the following dimensions by the ratio recursion of Theorem 6.3 and elimination over $\mathbb{F}_{101}$.

	degree 12	degree 13
dimension modulo the sextic multiples	351	351
additional independent leading terms used	0	20

The associated graded quotient by the full residuals therefore has dimension at most 331 in degree 13 and every subsequent degree. Its truncation at degree k has length at most $331k + O(1)$. The characteristic-zero truncation has at most the same length by the coefficient-matrix comparison of Theorem 6.3, which works in every degree in multiplicative coordinates. The section has dimension at most 1, and cutting the full 4-dimensional germ by 3 equations gives dimension at least 1. Its multiplicity, the linear coefficient of the truncation length, is at most 331. Cutting by a linear equation with a dimension drop of 1 can only increase multiplicity. Indeed multiplication by the equation gives, for each k, a surjection from the quotient modulo $\mathfrak{m}^{k-1}$ onto its image modulo $\mathfrak{m}^k$. The difference of the two successive lengths is therefore a lower bound for the section length modulo $\mathfrak{m}^k$. Comparing leading coefficients proves the assertion. Applying it to the 3

cuts bounds the multiplicity of the full germ by 331. Every component of maximal dimension contributes at least 1 by the additivity formula for multiplicities. The lower bound is Theorem 8.1.

For the fixed locus of J, the minimal presentation in Theorem 8.1 has 2 equations in 6 variables. Both have order at least 6. To see this directly from the full graph, choose 8 minimal generators of its local analytic ring which transform linearly under the involution. One obtains them by averaging an arbitrary lift of a tangent coordinate basis with its transform, using the linear action on the basis. The change of generators has invertible derivative. Thus the ideal of relations still has order at least 6. The fixed scheme is obtained by setting the coordinates on which the involution acts by -1 equal to zero. Restricting the relations preserves this order, as does changing to the 6 free phases of Theorem 8.1.

On the plane spanned in those phases by $(1,2,3,4,5,6)^t$ and $(2,3,5,7,11,13)^t$, the 2 sextics have binary resultant 14 modulo 101 in the convention of `component_slice.py`. They have no common factor on the plane or in the 6 variables. The leading forms therefore form a regular sequence and generate the associated graded ideal. Indeed any cancellation between their leading multiples is a multiple of the relation $(G,-F)$ for the 2 coprime forms F,G, and subtracting the corresponding relation on the full equations raises the order. The Hilbert series is $(1-z^6)^2/(1-z)^6$, giving multiplicity 36. The symmetries at the end of Theorem 8.1 transport this computation to all 10 fixed loci. □

We can also see several real deformations at once in a single fixed locus. The next slice makes the ambiguity in counting components explicit, since distinct curves on the slice can belong to the same component of the full germ.

Proposition 8.3. *In the* 6 *free phases for the fixed locus of* J *in Theorem 8.1, use its vectors* a,b *and set* $c'=(0,2,3,0,-4,5)^t$. *The* 3*-dimensional linear section spanned by* a,b,c' *contains* 5 *smooth real analytic curve germs through* H_5 *with distinct tangent lines, all outside the original* 27 *tangent spaces.*

Proof. Put $u=t(xa+yb+c')$. As in Theorem 8.1, divide the 2 residuals by t^6 and evaluate at $t=0$. The resulting polynomials in x,y contain only terms of degrees 2 and 3, by exact interpolation at the same 28 points. Set $y=xz$ and divide by x^2 for $x\neq 0$. The equations now read

$$P(z)+xQ(z)=0, \qquad S(z)+xT(z)=0.$$

Elimination gives $E=PT-SQ$ over $\mathbb{Q}(\sqrt{5})$. The program `component_slice.py` verifies $\deg E=5$, $\gcd(E,PQ)=1$, and opposite signs at the endpoints of each of

$$(-13,-12), \quad (-5,-4), \quad (-7/2,-3), \quad (-3,-2), \quad (-3/2,-1/2).$$

Thus the 5 roots are real and simple. At each one $x=-P(z)/Q(z)$ is defined and nonzero. Solving the first equation for x reduces the second to $-E/Q$, so the derivative in x,z is invertible. The implicit function theorem gives a real analytic curve with this initial slope. The c' coefficient of its velocity is 1, so the distinct slopes give distinct tangent lines.

To exclude the original families, lift a,b,c' to tangent arrays by the first 34 equations. For each of the 49 nonidentity symmetries in Corollary 4.4, apply its difference from the identity to these arrays and dephase. In exact arithmetic, either the third difference lies outside the span of the first 2, or it vanishes and the first 2 are independent. Thus no $xa+yb+c'$ with $x\neq 0$ is fixed. This test concerns the tangent lines themselves and precedes any choice of higher coefficients. □

Theorem 7.1 now shows that on each of the 10 additional real components there are points with trivial ordinary stabilizer arbitrarily near H_5. Their surviving symmetry involves conjugation. How much of the germ is covered when we also allow this operation?

Write D for dephasing in logarithmic coordinates, so that $D(X)_{ij} = X_{ij} - X_{i0} - X_{0j} + X_{00}$. An ordinary permutation pair acts by $D((X_{r(i)c(j)}))$. For a symmetry involving conjugation the sign is reversed on real phases. The same enumeration as in Theorem 7.1, with the exponents negated, gives another 200 pairs. If the ordinary stabilizer of a matrix is trivial, at most one such symmetry can fix it, and that symmetry must square to the identity. Conjugating the signed pairs in `local_symmetries.py` by the 200 ordinary pairs gives exactly 3 classes of involutions, of size 10 each. The numbers of fixed row and column indices are respectively $(0,0), (6,0), (0,6)$. The first class consists of the quaternionic involutions above. Thus every point with a conjugating symmetry and trivial ordinary stabilizer can be moved into one of just 3 fixed loci.

Proposition 8.4. *There is a real dephased phase velocity at H_5 with trivial stabilizer even on allowing complex conjugation, for which the Hadamard equations admit real corrections through order* 14*.*

Proof. Use the free logarithms of Proposition 6.1 in their stated order. Let V be the full linearized array with free coordinates $1, \dots, 8$. With $w = e^{2\pi i/5}$ take the velocity

$$v = \frac{1}{5} \sum_{k=0}^{4} (w^k + w^{-k}) \big(g_{(k,0)} V + g_{(0,k)} V \big)$$

where the translations act on dephased arrays. This selects the frequencies $1, -1$ on each translation line. Its free coordinates reduce to $(65, 43, 39, 2, 67, 2, 27, 27)$ at $w = 95$ modulo 101. Of the 400 signed permutation actions above only the identity fixes the reduced array, and hence only the identity can fix v.

Let R be the residual on the graph of the 73 pivot equations in logarithms and let R_6 be its leading sextic. In the free coordinates start with $u(t) = tv$. At order $k > 6$ a correction $t^{k-5}b$ changes the coefficient of degree k by $dR_6(v)b$ and leaves all preceding coefficients fixed. Rows $67, 69, 74, 75$ and columns $0, 2, 3, 4$ give a minor invertible modulo 101, in the indexing of Proposition 6.1. Set the other coordinates of b to zero. Solve these 4 equations at each step in $\mathbb{Q}(w)$ and substitute in *all* the ordered row equations. The program `symmetry_jets.py` checks that every residual coefficient through degree 14 is zero. It also checks for each full logarithmic coefficient X_k that $\bar{X}_k = (-1)^{k+1} X_k$. Replacing t by it therefore gives real phase coefficients to the asserted order. □

For these prescribed free coordinates there is a single convergent candidate to test. Substitute $u(t) = t(v + y)$, with y supported on the 4 correction columns, and divide the selected residuals by t^6. Their derivative in y at $(0,0)$ is the invertible minor just used. The implicit function theorem gives a unique analytic $y(t)$ with $y(0) = 0$. It gives an actual deformation precisely if the other residuals vanish identically along it. The finite calculation proves their vanishing to the indicated order only. Whether any actual deformation through H_5 escapes all the conjugating and ordinary fixed loci remains open, as does the full decomposition up to symmetry. At H_7 the connectedness argument gives 14 additional complex components. Do their real loci also attain the full dimension 6?

9 Configurations of 6 distinct points

We now take $K = F_3$. For distinct input labels, [10] asks whether the resulting matrices are always isolated when the outer Fourier order is an odd prime. The experiments there used random configurations in outer orders 3, 5 and 7, with $p \le 17$. We can settle the following finite range exhaustively.

Theorem 9.1. *For each prime $5 \le p \le 31$, every McNulty-Weigert matrix with outer matrix F_3 and 6 distinct projective labels has defect zero. In particular all these matrices of order $3p$ are isolated.*

Proof. By projective invariance of the defect [10], we may send the ordered triple of source points to $(\infty, 0, 1)$. The 3 target labels then run through all ordered triples of distinct elements of $\mathbb{F}_p - \{0, 1\}$. We keep the block order throughout.

Write $u_c = (1, 2c)$ for finite c and $u_\infty = (0, 1)$. We need roots of 2 different orders here. Put $w_3 = e^{2\pi i/3}$ and $w_p = e^{2\pi i/p}$. For $x \in \mathbb{F}_p^2$ set

$$c_{ij}(x) = \frac{\det(x, u_{t_j}) \det(u_{s_i}, x)}{\det(u_{s_i}, u_{t_j})}, \qquad \Lambda_{ij,kl} = \frac{w_3^{(i-k)(j-l)}}{3}, \qquad \Theta_x = diag(w_p^{c_{ij}(x)}).$$

Taking the eigenspace for eigenvalue 1 in Lemma 3.7 gives the sector formula of [10], namely

$$d(H) = \sum_{x \in \mathbb{F}_p^2} \dim \ker(\Lambda - \Theta_x) - (6p - 1).$$

The contribution at zero is 5, since F_3 has defect zero. On the 6 selected lines each point other than the origin contributes at least one, from the phases of the corresponding basis. These contributions account for $5 + 6(p - 1) = 6p - 1$. Thus we need to certify dimension 8 for the image of $\Lambda - \Theta_x$ for $x \neq 0$ on the 6 selected lines, and dimension 9 for x off those lines. One representative of each direction suffices. Indeed, multiplying x by $b \neq 0$ multiplies all c_{ij} by b^2, and the cyclotomic Galois automorphism sending w_p to $w_p^{b^2}$ can be chosen to fix w_3.

For each auxiliary prime p' in the table choose the least integers greater than one of orders 3 and p modulo p' as the images of w_3, w_p. If a minor does not vanish after reduction, it cannot vanish over $\mathbb{Q}(w_3, w_p)$. The program `distinct_points.py` uses Gaussian elimination on the 9-dimensional sectors. When a reduction falls short, the program checks the remaining sectors at the next prime. Each sector reaches its required dimension at one of the listed primes.

p	ordered configurations	directions across configurations	primes p' used
5	6	36	1531
7	60	480	2143
11	504	6048	3301
13	990	13860	4057
17	2730	49140	5101, 5407
19	4080	81600	5701, 6043
23	7980	191520	7039, 7177
29	17550	526500	9049, 9397
31	21924	701568	9859, 10789

The middle columns are $(p-2)(p-3)(p-4)$ and $(p+1)(p-2)(p-3)(p-4)$, respectively. This exhausts the normalized ordered configurations and all their projective directions. The sector formula now gives defect zero in each case. □

There is also a test in characteristic zero for the directions off the 6 selected lines. It records a restriction on the exponents of any sector that could contribute to the defect.

Proposition 9.2. *Fix $p \geq 5$ and a direction x off the selected lines. If $\ker(\Lambda - \Theta_x) \neq 0$, some nonempty subset of the 9 exponents $c_{ij}(x)$ has sum zero in $\mathbb{F}_p$.*

Proof. Represent the exponents by integers in $\{0, \ldots, p-1\}$ and form

$$P(T) = \det\left(\Lambda - diag(T^{c_{ij}(x)})\right).$$

Reduce modulo $T^p - 1$ to degree less than p, writing the remainder as $\sum_{j=0}^{p-1} b_j T^j$. Its coefficients belong to $\mathbb{Q}(w_3)$. The minimal polynomial of w_p over this field is $1 + T + \cdots + T^{p-1}$, since $[\mathbb{Q}(w_{3p}) : \mathbb{Q}(w_3)] = p - 1$. Thus $P(w_p) = 0$ forces all b_j to coincide. But $P(1) = \det(\Lambda - I) = 0$,

so their common value must be zero. In the determinant expansion, choosing no diagonal powers contributes $\det \Lambda \neq 0$ to b_0. Every other term contributing to b_0 selects a nonempty subset of exponents with sum zero modulo p. At least one such subset is therefore needed. □

For example, take sources $0, 1, 2$ and targets $4, 5, 6$. At $x = u_3$ the exponents are the reductions of the positive rational numbers $2(t_j - 3)(3 - s_i)/(t_j - s_i)$. Multiplication by 30 gives the integers

$$45, 72, 90, 40, 60, 72, 30, 40, 45.$$

Their sum is 494. Suppose $p > 494$. At u_3 every nonempty subset has sum different from zero modulo p. Each subset therefore gives a nonzero homogeneous quadratic in x with at most 2 projective zeros. The 9 individual exponents vanish only on the selected lines. For this configuration and every prime $p > 494$, at most $2(2^9 - 1 - 9) = 1004$ directions off those lines can contribute to the defect.

For $N = 3$, can one replace the finite list of certificates in Theorem 9.1 by a proof valid for every prime $p \geq 5$? For $K = F_N$ with N any odd prime and all $2N$ projective labels distinct, the isolation question of [10] remains open in general. The sector formula there reduces the stronger claim $d(H) = 0$ to eigenvalue multiplicities for matrices of order N^2.

10 Repeated labels and Fourier products

For outer matrix F_2, a repeated label means that all sources coincide or all targets coincide. The defect in this case completes the calculation in Theorem 3.6. We first identify the Fourier factor for an arbitrary outer matrix K. Suppose the source list is constant. A projective transformation in $PSL_2(\mathbb{F}_p)$ can send its common value to ∞. Such a transformation acts on all the bases by the same unitary, up to phases and permutations within each basis, and hence preserves the equivalence class of $H_K(s, t)$ by [10]. All target labels are now finite. With $w = e^{2\pi i/p}$ the entries become

$$H_{ia,jb} = K_{ij} w^{ab} Q_{aj}, \qquad Q_{aj} = w^{t_j a^2}.$$

In fact we can give each Q_{aj} any value in $\mathbb{T}$. To check orthogonality of rows $(i, a), (i', a')$, sum over b first. The sum vanishes if $a \neq a'$. If $a = a'$, the phases cancel and the sum over j gives the orthogonality of the rows of K. This is the deformation of a tensor product studied by Diţă [12].

The dephased parameters can be read off from the matrix itself. The cross-ratio for rows $(0, a), (0, 0)$ and columns $(j, 0), (0, 0)$ is

$$\frac{Q_{aj} Q_{00}}{Q_{a0} Q_{0j}}.$$

Thus we can vary $(N - 1)(p - 1)$ independent phases through the given matrix, fixing Q_{a0} and Q_{0j}. A constant target list gives the same conclusion by taking the adjoint, since $H_K(s, t)^* = H_{K^*}(t, s)$.

Tadej [40] computes the Berezin spectrum for Fourier matrices and their ordinary Kronecker products. How does the defect vary with the phases of a deformed tensor product? This question was raised in [3], and the linear equations are discussed in [2]. For cyclic Fourier factors of coprime orders, we can follow every eigenvalue as the phases vary. Write $w_h = e^{2\pi i/h}$ and $D(h) = \sum_{k=0}^{h-1} \gcd(h, k)$, including $\gcd(h, 0) = h$.

Theorem 10.1. *Let $L_{ia,jb} = w_N^{ij} w_M^{ab} Q_{aj}$, with $i, j \in \mathbb{Z}_N$, $a, b \in \mathbb{Z}_M$ and arbitrary $Q_{aj} \in \mathbb{T}$. When N and M are coprime, Λ_L has the same spectrum, with multiplicities, as $\Lambda_{F_{NM}}$. In particular $d(L) = D(N)D(M) - 2NM + 1$.*

Proof. Fix $k \in \mathbb{Z}_M$ and $l \in \mathbb{Z}_N$. Consider arrays of the form

$$A_{ia,jb} = w_N^{il} w_M^{bk} Y_{aj}.$$

These spaces give a direct sum over k, l by Fourier expansion in i, b. In the rectangle defining Λ_L, let i', a' and j', b' be the indices summed over. The factors involving i', b' have exponents $i'(l - j + j')$ modulo N and $b'(k - a + a')$ modulo M. Summing forces $j' = j - l$, $a' = a - k$ and cancels the factor $1/(NM)$. The resulting operator on Y is

$$(T_{kl}Y)_{aj} = U_{aj}(k, l)Y_{a-k,j-l}, \qquad U_{aj}(k, l) = \frac{Q_{aj}Q_{a-k,j-l}}{Q_{a-k,j}Q_{a,j-l}}.$$

Let O run over the orbits of translation by (k, l) in $\mathbb{Z}_M \times \mathbb{Z}_N$. On such an orbit the operator is a cyclic shift with the indicated weights. Its characteristic polynomial is $z^{|O|} - \prod_{(a,j)\in O} U_{aj}(k, l)$. Thus for arbitrary orders we have

$$\det(zI - \Lambda_L) = \prod_{k,l} \prod_{O} \left(z^{|O|} - \prod_{(a,j)\in O} U_{aj}(k, l) \right).$$

When M, N are coprime, the orders of k, l are coprime as well. Each orbit is then the rectangle $(a + \langle k \rangle) \times (j + \langle l \rangle)$. Shifting either coordinate permutes the rectangle, so each product of weights equals 1. This is exactly the polynomial at $Q_{aj} = 1$, where $L = F_N \otimes F_M$ is equivalent to F_{NM}. Each rectangle contributes one eigenvalue 1. There are $\gcd(M, k) \gcd(N, l)$ rectangles, giving $D(M)D(N)$ such eigenvalues in all. Subtracting the $2NM - 1$ phases gives the defect. □

In particular, every cyclic partition function Z_r is constant along these coprime deformations. By Theorem 5.1 the full real phase Hessian also has constant spectrum. For orders with a common factor the orbit formula explains how the defect can drop as the phases vary.

Corollary 10.2. *Let m_k be the order of k in $\mathbb{Z}_M$ and n_l the order of l in $\mathbb{Z}_N$. Outside a finite union of proper closed subsets of the parameter torus, the dephased defect is*

$$d(L) = \sum_{\gcd(m_k, n_l)=1} \frac{MN}{m_k n_l} - 2MN + 1.$$

In particular for $N = M = p$ prime, both the generic defect and the generic real local dimension are $(p - 1)^2$.

Proof. An orbit contributes one eigenvalue 1 precisely when its product of weights equals 1. For coprime orders the orbit is a rectangle, with product 1 identically. Otherwise $O - (k, 0)$ and $O - (0, l)$ are disjoint from O. Indeed $(k, 0)$ lies in $\langle (k, l) \rangle$ exactly when the congruences $h = 1$ (m_k), $h = 0$ (n_l) have a common solution. In the product of weights the numerator thus uses the entries on O twice, while the denominator uses entries outside O. This is a nonconstant monomial in the phases. The equation setting it equal to 1 cuts out a proper closed subset. Removing these finitely many subsets leaves exactly the orbits counted in the formula.

For equal prime orders the surviving pairs are those with $k = 0$ or $l = 0$. The multiplicity of 1 is $p^2 + 2p(p - 1)$, giving defect $(p - 1)^2$. The same number of independent dephased phases varies freely in the Fourier product, so the dimension bound is attained. □

Proposition 10.3. *Let p be an odd prime and let $H = H_{F_2}(s, t)$ have order $2p$. If $s_0 = s_1$ or $t_0 = t_1$, then $d(H) = 2(p - 1)$ and H belongs to a dephased affine family of dimension $p - 1$. For all admissible lists, H is isolated exactly when its 4 labels are distinct and their cross-ratio is not a square.*

Proof. For a constant source list take $N = 2$, $M = p$ in Theorem 10.1. We have $D(2) = 3$, $D(p) = 2p - 1$, and hence

$$d(H) = 3(2p - 1) - 4p + 1 = 2(p - 1).$$

The independent phases above give the family. Taking the adjoint settles a constant target list. These are all admissible repetitions, since $s_i \neq t_j$ for every i, j. For 4 distinct labels, if the cross-ratio is not a square then the defect is zero by [10]. If it is a square, Theorem 4.1 gives the required families. □

11 Master Hadamard matrices

The moments R_m in Section 3 sum over all rectangles. Here we keep the row and column indices of each rectangle. For a master Hadamard matrix in the sense of Definition B, the flux on rows i, k and columns j, l is $e^{2\pi i(x_i - x_k)(y_j - y_l)}$. Its exponent is a product of a row difference and a column difference. This gives a test for a master representative and bears on the isolation question in [4]. We use the real exponents of [4] in the construction of Avan, Fonseca, Frappat, Kulish, Ragoucy and Rollet [1]. Dephasing at (i_0, j_0) replaces x_i, y_j by $x_i - x_{i_0}, y_j - y_{j_0}$. The same observation applies to rectangular matrices.

Lemma 11.1. *Choose any list of ordered pairs of rows and any list of ordered pairs of columns. Form the array of their cross-ratios*

$$u_{ik;jl} = \frac{H_{ij} H_{kl}}{H_{il} H_{kj}}.$$

If all these numbers are m-th roots of unity and the matrix has a master representative, then every 2×2 minor of their exponent matrix vanishes modulo m.

Proof. For a master representative the array of exponents has the real lift

$$m(x_i - x_k)(y_j - y_l).$$

Its entries are integers and the array is an outer product over the reals. Reduce its vanishing minors modulo m. Row and column phases cancel from each ratio, and permutations simply relabel the pairs. □

For a whole dephased Butson matrix one can recover a cyclic Fourier submatrix.

Proposition 11.2. *A dephased square Hadamard matrix with entries in the N-th roots has a master representative if and only if it is a submatrix of F_N up to row and column permutations. In particular its order is at most N.*

Proof. A master representative gives the dephased integer lift $N(x_i - x_0)(y_j - y_0)$ of rank at most one. An integer matrix of rank one is an outer product of integer vectors. Indeed, choose a column containing an entry different from zero and divide by the gcd of its entries to obtain a primitive vector. Every other column is a rational multiple of that vector, and Bezout's identity makes the multiple integral. Reducing the 2 integer vectors modulo N gives the row and column indices of F_N. Orthogonality excludes repetitions in either list. The zero lift in the square Hadamard case has order one. Conversely every such Fourier submatrix has the required exponential form. □

This excludes the isolated matrix in $B(13, 6)$ of Hall and Nicoara [18] and the defect-zero matrices in $B(19, 6)$ from [44] as possible master representatives. Proposition 11.2 also gives a way to formulate the isolation question in the Butson case. Which square Hadamard submatrices of F_N are isolated among all Hadamard matrices of their order?

The tensor product with a Fourier matrix imposes an additional arithmetic condition on its other factor.

Lemma 11.3. *Let K be a square Hadamard matrix and $m \geq 2$. The matrix $K \otimes F_m$ has a master representative if and only if K is equivalent to a Hadamard submatrix of some F_l with* $\gcd(l, m) = 1$.

Proof. Dephase K so that its 1st row and column consist of ones, and put $w = e^{2\pi i/m}$. For any entry $q = K_{ij}$ with $i, j \neq 0$, take rows $(0,0), (i,0), (0,1)$ and columns $(0,0), (j,0), (0,1)$ of the tensor product. They give the corner

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & q & 1 \\ 1 & 1 & w \end{pmatrix}.$$

If the tensor product has a master representative, dephasing this corner gives a real lift of its arguments in turns, of rank at most 1. In the lower right square the lift has the form

$$\begin{pmatrix} u & b \\ c & h + 1/m \end{pmatrix}, \qquad e^{2\pi i u} = q, \qquad b, c, h \in \mathbb{Z}.$$

Its determinant gives $u = mbc/(mh+1)$. Thus the order of q divides $|mh+1|$, which is prime to m. Taking the least common multiple of the entry orders shows that K has entries among the l-th roots for some l prime to m. Restriction to the rows and columns with inner index zero gives a master representative up to equivalence for K itself. Proposition 11.2 puts K inside F_l.

Conversely, if K is equivalent to such a submatrix, its tensor product with F_m is equivalent to a submatrix of $F_l \otimes F_m$. By the Chinese remainder theorem this is a submatrix of F_{lm} up to permutations, so it has a master representative. □

Theorem 11.4. *Let p be an odd prime. The matrix $H_K(s,t)$ has a master representative exactly when both label lists are constant and K is equivalent to a Hadamard submatrix of some F_l with $\gcd(l,p) = 1$. For $K = F_N$ the criterion is that both lists be constant and $\gcd(N,p) = 1$. In that case $H_{F_N}(s,t)$ is equivalent to F_{Np}.*

Proof. Suppose $t_j \neq t_{j'}$. Choose a single block row i and send s_i to ∞ by a projective change in $PSL_2(\mathbb{F}_p)$ as in Section 10. In the new coordinates every target label is finite. The entries of this block row are $K_{ij} w^{t_j a^2 + ab}$ with $w = e^{2\pi i/p}$. Choose inner rows $a = 0, 1, -1$ and columns $(j,0), (j,1), (j',0)$. Dephasing these 3 rows and columns gives

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & w & w^{\delta} \\ 1 & w^{-1} & w^{\delta} \end{pmatrix}, \qquad \delta = t_{j'} - t_j.$$

The exponent minor is $2\delta \neq 0$ in $\mathbb{F}_p$, contrary to Lemma 11.1. If the source list varies, take the adjoint, which exchanges the lists and preserves the existence of a master representative.

If both lists are constant, send the common source to ∞ and denote the common target by c. Every block is a scalar multiple of $diag(w^{ca^2}) F_p$, so $H_K(s,t)$ is equivalent to $K \otimes F_p$. Lemma 11.3 proves the first assertion. For $N \geq 2$, the matrix F_N has a cross-ratio of order N, whereas all cross-ratios of a submatrix of F_l are l-th roots. Thus $N \mid l$ is necessary, and we may take $l = N$ whenever $\gcd(N,p) = 1$. The final equivalence follows from the Chinese remainder theorem, and for $N = 1$ it holds directly. □

In particular, for $N > 1$ every McNulty-Weigert matrix in this construction with a master representative belongs to a dephased family of dimension at least $(N-1)(p-1)$ by Section 10. Thus this class contains no isolated master Hadamard matrices for $N > 1$.

We next apply the test to regular square matrices of order 7 and regular partial matrices of width 11.

Proposition 11.5. *Up to equivalence, the only regular master Hadamard matrix of order 7 is F_7. Also no member of the 4-parameter 5-row family $H(t) = [F_5 \mid diag(1, t_1, t_2, t_3, t_4) V]$ of [45] has a master representative.*

Proof. The classification [17] gives F_7, the Petrescu family, and its conjugate up to equivalence. Complex conjugation preserves the master property. In the dephased representative R_7^q from that classification, rows $1, 3, 4$ and columns $1, 2, 3$ (here numbered from one) give the following submatrix, independent of q:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & w & w^4 \\ 1 & w^2 & w^3 \end{pmatrix}, \qquad w = e^{2\pi i/6}.$$

The exponent minor of its lower right corner is $3 - 8 = -5$ modulo 6. In the 5-row example, take the 1st 3 rows and 1st 3 columns of the block V, canceling the row phases by dephasing. With $w = i$ the corresponding exponent minor is $2 \cdot 2 - 3 \cdot 3 = -5$ modulo 4. Both exclusions follow from Lemma 11.1, and F_7 is a master Hadamard matrix. □

For regular matrices of order 8, the classification in [17] gives explicit families. Which members pass the minor test of Lemma 11.1, and which of those admit a master representative? In order 4 the test leads to a complete answer for the standard family. In the notation of [4], the matrix $H(q)$ of Proposition 1.4 is $F_{2,2}^q$, with $q \in \mathbb{T}$ and rows $(1, 1, 1, 1)$, $(1, -1, 1, -1)$, $(1, q, -1, -q)$ and $(1, -q, -1, q)$.

Proposition 11.6. *The matrix $F_{2,2}^q$ has a master representative exactly when q is a root of unity whose order is divisible by 4.*

Proof. For the middle 2 rows and columns the dephased entries form the matrix

$$\begin{pmatrix} -1 & 1 \\ q & -1 \end{pmatrix}.$$

In a master representation choose lifts of their arguments in turns, giving a real matrix of rank one. Its diagonal entries are half-integers and its upper right entry is an integer. That integer cannot be zero, since neither diagonal entry is zero. The lower left entry is the quotient of that product by the integer. Thus it is rational and q has finite order. Put $m = \operatorname{lcm}(2, \operatorname{ord}(q))$. The exponent minor modulo m is $(m/2)^2$. Lemma 11.1 implies $4 \mid m$, hence $4 \mid \operatorname{ord}(q)$.

Conversely, take $N = \operatorname{ord}(q)$ divisible by 4, and use q as the primitive N-th root of unity defining F_N. This choice permutes the rows of the standard Fourier matrix. Choose row indices $0, N/2, 1, 1 + N/2$ and column indices $0, 1, N/2, 1 + N/2$. Evaluating q^{ab} gives the displayed 4 rows, because $(N/2)^2$ is a multiple of N. □

The conjecture on isolated master Hadamard matrices in [4] asks whether the Fourier matrices F_p, for p prime, exhaust the isolated square master matrices.

For a Butson matrix the level m is the least common multiple of the entry orders after dephasing the first row and column to ones. It is invariant under equivalence, since it is also the least common multiple of the orders of the rectangle fluxes. Proposition 11.2 puts a master matrix of level m in the form

$$H_{ab} = w^{ab}, \qquad a \in A, \quad b \in B, \quad w = e^{2\pi i/m},$$

where $A, B \subset \mathbb{Z}_m$. Translating either index set changes only row or column phases. A complete cycle of columns gives the following deformation.

Lemma 11.7. *Suppose that B contains a coset C of the subgroup of prime order p in $\mathbb{Z}_m$. If $B = C$, then H is equivalent to F_p. If B strictly contains C, then H belongs to a dephased affine family of dimension at least $p - 1$.*

Proof. Write $C = b_0 + (m/p)\mathbb{Z}_m$ and $w_p = e^{2\pi i/p}$. On these columns row a reads

$$w^{ab_0}\big(1, w_p^a, \dots, w_p^{(p-1)a}\big).$$

Orthogonality of the columns with indices $b_0, b_0 + m/p$ gives $\sum_{a\in A} w_p^a = 0$. Since $1 + X + \cdots + X^{p-1}$ is the minimal polynomial of w_p, the p residue classes occur equally often among the rows. If $B = C$, squareness gives one row per class and the displayed rows give F_p.

Otherwise choose the first column outside C and translate the first row index to zero. For phases $t_0 = 1, t_1, \dots, t_{p-1} \in \mathbb{T}$, multiply row a on the columns in C by $t_{a \bmod p}$. For a, a' in the same residue class the phases cancel in their scalar product. For different residues the contribution of C was already zero, since

$$\sum_{b\in C} w^{(a-a')b} = w^{(a-a')b_0} \sum_{j=0}^{p-1} w_p^{(a-a')j} = 0.$$

The first row and column are fixed. Each free phase occurs on its own nonempty set of entries, so the $p - 1$ parameters are independent. □

At prime power level we can say more about the size of the family. Fan, Fan and Shi [16] describe the index sets of Hadamard submatrices of F_{p^k} by their base-p digits. We give the short counting argument for both index sets, since the order of the digits tells us which phases can vary together.

Proposition 11.8. *Let p be prime. Every square Hadamard submatrix of F_{p^k} of order greater than one has order p^r for some $1 \le r \le k$. Up to equivalence, it belongs to a dephased affine family through F_{p^r} of dimension*

$$d(F_{p^r}) = 1 + p^{r-1}\big(r(p-1) - p\big),$$

the defect of the Fourier matrix of its own order.

Proof. For the row set A, let I be the set of positions where 2 row indices first differ when their digits are read from low to high, starting at position zero. If $|I| = r$, there are at most p^r rows. At a position outside I the preceding digits determine the next digit. For each $u \in I$ row orthogonality gives a zero of the column polynomial $P_B(X) = \sum_{b\in B} X^b$ at a primitive p^{k-u}-th root, using representatives $0 \le b < p^k$. The distinct cyclotomic polynomials $\Phi_{p^{k-u}}$ therefore divide P_B jointly. Each takes the value p at 1. Hence p^r divides $|B| = |A| \le p^r$. Equality forces all p choices at every position in I after every allowed prefix.

The same argument gives r branching positions for B. To identify them put $v = k - 1 - u$ for $u \in I$. Reduce P_B modulo $X^{p^{v+1}} - 1$. The remainder is a multiple of $1 + X^{p^v} + \cdots + X^{(p-1)p^v}$ with quotient of degree less than p^v. Hence above every occupied residue modulo p^v the p next digits occur equally often. These r positions v account for all the branching in B.

Write $I = \{u_1 < \cdots < u_r\}$, and put $v_i = k - 1 - u_i$. Label the rows by their free digits x_i at u_i and the columns by y_i at v_i. With $w_p = e^{2\pi i/p}$, consider the matrices

$$K_{xy} = w_p^{\sum_i x_i y_i} \prod_{i=1}^{r-1} Q_i(x_1, \dots, x_i\,;\, y_{i+1}, \dots, y_r),$$

where the values of every Q_i are arbitrary phases. If x, x' first differ at i, the factors with index less than i cancel in their scalar product. The remaining factors are independent of y_i, and summing $w_p^{(x_i - x_i')y_i}$ over that digit gives zero.

To recover H, translate both index sets to contain zero. Group the digits of a from u_i up to the next free digit, and group those of b in the same way starting at each v_j. This gives

$$a = \sum_i p^{u_i} \alpha_i(x_1, \dots, x_i), \qquad b = \sum_j p^{v_j} \beta_j(y_j, \dots, y_r),$$

where $\alpha_i = x_i \ (p)$ and $\beta_j = y_j \ (p)$. In ab/p^k the terms with $i > j$ are integers. The diagonal terms give the displayed power of w_p. Each remaining term has $i < j$, so its phase can be included in Q_i. The same family contains F_{p^r} by taking all digits free in modulus p^r.

To count phases after dephasing, normalize Q_i to be 1 when $x_i = 0$ or when $(y_{i+1}, \dots, y_r) = 0$. For the first condition, move the value at $x_i = 0$ into Q_{i-1}, working in decreasing order of i. At $i = 1$ the removed factor is a column phase. The second condition costs only row phases. The remaining parameters are independent. Indeed, after dividing by the later factors, compare the phase products at x_i and at 0 with all other row digits fixed. This recovers Q_i in decreasing order of i. The dimension is therefore

$$\sum_{i=1}^{r-1}(p-1)p^{i-1}(p^{r-i}-1) = 1 + p^{r-1}\big(r(p-1)-p\big).$$

This agrees with the Fourier defect formula of Tadej and Zyczkowski [42]. □

With 2 prime divisors we can still force a complete cycle. This gives the following case of the conjecture in [4], with isolation taken among all dephased complex Hadamard matrices.

Theorem 11.9. *Let H be a square master Hadamard matrix of order greater than one. Suppose its dephased entries are roots of unity whose orders divide an integer with at most 2 distinct prime divisors. Then H is isolated if and only if it is equivalent to F_p for some prime p. Every other such matrix admits a nonconstant dephased affine deformation.*

Proof. Use the least level m and write $H = (w^{ab})_{a\in A, b\in B}$ as above, with $0 \in A, B$. For every prime $p \mid m$, the set A occupies at least 2 residue classes modulo p. Otherwise all a would be divisible by p and every entry would be an (m/p)-th root.

We claim that some difference $a - a'$ is prime to m. For a prime power m this follows at once. For distinct prime divisors p, l, take a_1, a_2 differing modulo p. If they agree modulo l, take a_3 in a different class modulo l. It must differ modulo p from at least one of a_1, a_2. This gives the required pair in both cases.

Its orthogonality equation is

$$\sum_{b\in B} w^{(a-a')b} = 0.$$

Lam and Leung [26] proved that when m has at most 2 distinct prime divisors, every such sum splits into rotated complete cycles of prime length. Multiplication by $a - a'$ permutes $\mathbb{Z}_m$ and carries cosets of each subgroup to cosets of the same subgroup. Hence B contains a complete prime cycle C. Lemma 11.7 gives a dephased affine deformation if C is proper, and a prime Fourier matrix if $C = B$. Finally $d(F_p) = 0$ gives isolation. □

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