Uniform semiclassical Gram matrices in a Lorentzian group-averaging quotient

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Abstract

We construct a prescribed family of finite-harmonic test states for four scalar principal-series representations of $\mathrm{PSL}(2,\mathbb{C})$, equipped with a single Haar-averaged inner product. For every target A in a fixed compact five-dimensional ball of positive definite trace-three matrices, the normalized quotient states obey $\|(O_{ij} - N^2 A_{ij})\psi_{N,A}\|_\eta \le CN^{3/2}$ for all six Lorentzian Gram insertions, with a common constant and threshold. We prove positivity by orbit factorization, establish a common operator domain, and derive the estimate using compact orbit lifts, a uniform cross-ratio chart and an H^{10} bound for the tail of the original angular cutoff. The numerical means are eventually positive definite, their relative fluctuations are $O(N^{-1/2})$, and their normalized shape map converges to the identity in C^1 at the same rate. It is consequently injective for large N and realizes every fixed strict interior target uniquely within the prescribed family. We further identify the mathematical quotient with the full-spin conformal partial-wave space, fix the literal sewn-coefficient Parseval normalization, and transfer the Gram actions on the original common domain. Reflection parity does not imply scalar internal spin: we construct an even semiclassical family, prove that the scalar sector is proper, and state the remaining conditions for scalar-projected residual control. No additional physical source identification is assumed.

Keywords: group averaging; Lorentzian Gram operators; semiclassical geometry; conformal harmonic analysis; operator domains.

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1 Introduction

A semiclassical geometric state must control several quantities that need not commute. In a noncompact group-averaging construction, this question is also tied to convergence, the null space of the averaged form, and the domain of the induced observables. A sharply localized function before averaging need not by itself solve any of those problems. The present paper constructs a prescribed family of actual finite-harmonic test states for which these analytic and geometric questions can be answered together.

Our setting is the diagonal action of $G = \mathrm{PSL}(2,\mathbb{C})$ on four copies of its boundary sphere, with scalar principal-series labels $\rho_i > 0$. We use one Haar average, an explicitly normalized external label measure, and the completion of the resulting positive quotient. The six observables are the Lorentzian Gram insertions $O_{ij} = K_i \cdot K_j - J_i \cdot J_j$ for $1 \le i < j \le 3$, with the diagonal convention $O_{ii} = K_i^2 - J_i^2 - I = \rho_i^2$. The target shapes range over the entire fixed five-dimensional compact ball

$$\mathcal{K} = \left\{ A_0 + H : H = H^T,\ \operatorname{tr} H = 0,\ \|H\|_F \le \frac{1}{24} \right\}, \qquad A_0 = \frac{4}{3} I - \frac{1}{3} e e^T, \quad e = (1,1,1)^T. \tag{1.1}$$

Every matrix in this set is positive definite and has trace three.

The main result constructs, from profiles fixed once for all N and A, normalized quotient vectors $\psi_{N,A}$ with finite angular cutoff N^3 and proves

$$\sup_{A \in \mathcal{K}} \left\| (O_{ij} - N^2 A_{ij}) \psi_{N,A} \right\|_\eta \le C N^{3/2}, \qquad 1 \le i \le j \le 3. \tag{1.2}$$

The threshold and constant are common to all six entries and all targets. The underlying construction localizes in a moving cross-ratio chart on scale $N^{-1/2}$, uses compact orbit support before averaging, and applies the original angular projector once. Its proof derives the horizontal operators on that compact lift, identifies their leading symbol at the prescribed classical center, and transfers the estimate to the finite harmonic state through an H^{10} tail bound.

The numerical mean $M_N = (\langle O_{ij} \rangle)$ is eventually positive definite, its trace scale is $S_N = \operatorname{tr} M_N/3 = N^2 + O(N)$, and the relative Hilbert-space fluctuation is $O(N^{-1/2})$. The shape map $B_N = M_N/S_N$ satisfies the stronger estimate

$$\|B_N - \mathrm{id}\|_{C^1(\mathcal{K})} \le C N^{-1/2}. \tag{1.3}$$

It is therefore eventually injective and bi-Lipschitz onto its image. Every fixed strict interior target has a unique preimage within this prescribed family for sufficiently large N. These statements use the same projected states as (1.2). The derivative argument distinguishes ambient differentiation at fixed spectral labels from the transported derivative in a moving chart; the distinction is necessary for the claimed rate.

We also identify the whole mathematical quotient with the full principal-series conformal partial-wave space. This is an exact comparison of the original averaged form and its operators, including all internal integer spins. We fix the normalization of the literal sewn coefficients, prove their Parseval identity, and transfer the six Gram actions on the original common domain. The resulting full-spin representation preserves (1.2) exactly. Projection to internal spin zero is an additional operation. We prove that this projection is proper even within the reflection-even sector and quantify its distance from the original unsymmetrized family. We give an explicit sufficient criterion for a projected semiclassical family, without claiming that the criterion has been verified.

1.1 Relation to previous work

Semiclassical tetrahedra and small relative geometric uncertainties are established subjects. Rovelli and Speziale constructed semiclassical tetrahedral states [1]; Livine and Speziale developed coherent intertwiners and analyzed their geometric semiclassical behavior [2]. Holomorphic

intertwiner coordinates and the cross-ratio description of tetrahedral shape appear in Freidel, Krasnov and Livine [3]. The broader area-normal description of quantum polyhedra is treated by Bianchi, Donà and Speziale [4]. Lorentzian coherent invariant tensors and their asymptotics are studied by Donà, Fanizza, Martin-Dussaud and Speziale [5]. Accordingly, neither geometric coherent states nor the inverse-square-root relative uncertainty scale is presented here as a new general principle. Our object is the specific uniform construction and its domain and representation control in the quotient defined below.

Group averaging and its relation to constraint quantization have an established framework [6]. Convergence for de Sitter free fields, including smooth multiparticle states, was studied in Ref. [16]. Positivity must be checked for the actual representation and test space, not inferred from the name of the procedure. The recent work of Marolf and Yu addresses positivity for de Sitter group averaging under explicit convergence conditions and also reports negative norms for some smaller-weight classes [7]. In particular, their tempered-representation result in Section 5.1 overlaps the four-scalar principal-series regime considered here, without by itself identifying our test domain and normalization. The present positivity proof is a direct orbit-factorization calculation for four scalar sphere representations; it is not a claim of a general new positivity theorem.

The representation-theoretic comparison uses the established $SL(2,\mathbb{C})$ Clebsch–Gordan and Racah analysis of Ismagilov [8] and Derkachov and Spiridonov [9], together with the principal-series conformal harmonic analysis of Karateev, Kravchuk and Simmons-Duffin [10]. The scalar Type-I Racah kernels of Krasnov and Louko [11] provide a useful normalization comparison for the spin-zero channel. The source completeness theorem is an explicit input, as presented by Karateev, Kravchuk and Simmons-Duffin, who refer to the classical treatment of Dobrev et al. [17]. Our additional work identifies its quotient norm, literal coefficients and operator domains with the original one-Haar construction. We do not infer full spin completeness from a scalar kernel alone.

Lorentzian Barrett–Crane models provide related geometric and representation-theoretic motivation [12], and integrability of relativistic spin networks is a separate established problem [13]. Group field theory realizations can impose simplicity, covariance, left invariance and additional source constraints in a specific order [14, 15]. Those choices do not follow solely from a unitary identification of the mathematical quotient used here. We therefore state the physical boundary of the result explicitly in Section 9.

1.2 Structure of the argument

Section 2 fixes the action, measures, positive quotient and common domain. Section 3 constructs the entire target geometry and the prescribed state family. The subsequent core proof establishes Theorem 3.2 by compact horizontal reduction and a quantitative projection-tail estimate. Section 6 derives the numerical geometry, covariance bounds, spectral-support controls and the four-leg numerical inertia. Section 7 proves (1.3). Section 8 gives the full-spin unitary comparison, operator transfer and scalar-projection analysis. All target-dependent estimates are uniform on the originally specified set (1.1).

2 The one-Haar model and its common domain

2.1 Representation and measure conventions

Let $Q = (S^2)^4$, and give each sphere its probability measure:

$$dm(q) = \prod_{i=1}^{4} \frac{d\Omega(q_i)}{4\pi}, \qquad \int_{S^2} d\Omega = 4\pi. \tag{2.1}$$

The group is $G = \mathrm{PSL}(2,\mathbb{C})$, acting diagonally by Möbius transformations. We use the stereographic coordinate $z = (n_x + in_y)/(1 - n_z)$; expressions at infinity are understood in the other sphere chart. For a determinant-one lift $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, set

$$\delta(g,z) = \frac{|az+b|^2 + |cz+d|^2}{1+|z|^2}. \tag{2.2}$$

This is independent of the two possible lifts of g. Directly from the Möbius derivative and the norm of the vector $(z,1)$,

$$d\Omega(gn) = \delta(g,n)^{-2} d\Omega(n), \qquad \delta(gh,n) = \delta(g,hn)\delta(h,n). \tag{2.3}$$

For $\rho = (\rho_1,\dots,\rho_4) \in (0,\infty)^4$, the scalar principal-series action on $L^2(Q,dm)$ is the inverse action

$$(T_\rho(g)u)(q) = \prod_{i=1}^{4} \delta(g^{-1},q_i)^{-1+i\rho_i} u(g^{-1}q_1,\dots,g^{-1}q_4). \tag{2.4}$$

Equation (2.3) proves both $T_\rho(g)T_\rho(h) = T_\rho(gh)$ and unitarity.

Write $K_c = \mathrm{PSU}(2)$, with Haar mass one, and $a_r = \mathrm{diag}(e^{r/2}, e^{-r/2})$. The Haar normalization throughout the paper is

$$\int_G f(g)\,dg = \frac{2}{\pi}\int_0^\infty \sinh^2 r\,dr \int_{K_c} dk_1 \int_{K_c} dk_2\, f(k_1 a_r k_2). \tag{2.5}$$

This formula is meant for nonnegative measurable or integrable f. The correspondingly normalized measure on $\mathrm{SL}(2,\mathbb{C})$ pushes forward to dg; no further factor of two occurs.

We record a coordinate form that fixes all subsequent coarea constants. If $(z_1,z_2,z_3) = (g0, g1, g\infty)$, then

$$dg = \frac{1}{4\pi^4}\frac{d^2z_1\,d^2z_2\,d^2z_3}{|z_1-z_2|^2|z_2-z_3|^2|z_3-z_1|^2}. \tag{2.6}$$

Here $d^2z = d(\mathrm{Re}\,z)d(\mathrm{Im}\,z)$. For completeness, the quotient by K_c is hyperbolic three-space of curvature -1, and (2.5) is $dg = (2\pi^2)^{-1} d\,\mathrm{vol}_{\mathbb{H}^3}\,dk$. In a local matrix lift $g = I + \left(\begin{smallmatrix} u & v \\ w & -u \end{smallmatrix}\right) + O(2)$, its PSL density is $\pi^{-4} d^2u\,d^2v\,d^2w$. Indeed, if $X = \frac{1}{2}\sum_j (\beta_j + i\vartheta_j)\sigma_j$, the real matrix Jacobian is $1/16$; normalized SU(2) Haar has local density $d^3\vartheta/(16\pi^2)$, and the two local sheets double this compact density on PSU(2). The local hyperbolic density is $d^3\beta$. Finally the first-order map to $(z_1,z_2,1/z_3) = (v, 1+2u+v-w, w) + O(2)$ has real Jacobian 4. The ratio of differentials in (2.6) is Möbius invariant, so this local calculation determines its global constant.

The external label measure is part of the model:

$$d\Pi(\rho) = \prod_{i=1}^{4} 4\rho_i^2\,d\rho_i. \tag{2.7}$$

No identification of this convention with an additional physical source measure is needed below. Our inner products are antilinear in the first argument. The one-Haar form is

$$\eta_\rho(u,v) = \int_G \langle u, T_\rho(g)v\rangle_{L^2(Q,dm)}\,dg, \qquad \eta(u,v) = \int d\Pi(\rho)\,\eta_\rho(u_\rho, v_\rho). \tag{2.8}$$

Definition 2.1 (Original smooth and finite-harmonic domains)**.** The space $\mathcal{D}_{\mathrm{sm},c}$ consists of fields $u(\rho,q)$ whose zero extensions to $\mathbb{R}^4 \times Q$ are smooth and have spectral support in a box $[0,R]^4$, for some finite R. In particular they are flat at each face $\rho_i = 0$; their support need not be separated from those faces. Define $\mathcal{D}_{R,L}$ by requiring, in addition, an expansion

$$u(\rho,q) = \sum_{\substack{0\le j_i\le L \\ -j_i\le m_i\le j_i}} f_{\mathbf{j},\mathbf{m}}(\rho)\prod_{i=1}^{4} Y_{j_i m_i}(q_i), \qquad \mathcal{D}_{\mathrm{fin}} = \bigcup_{R,L}\mathcal{D}_{R,L}, \tag{2.9}$$

with the same support and flatness conditions on every coefficient. There is no parity, fourth-leg, or diagonal-rotation restriction.

2.2 Positive orbit factorization

Let Q_* be the full-measure open subset of pairwise distinct ordered configurations, and put $Z = \mathbb{C} \setminus \{0, 1\}$. An ordered triple of distinct points determines exactly one projective transformation. Consequently

$$(g, w) \longmapsto g\, s(w), \qquad s(w) = (0, 1, \infty, w), \tag{2.10}$$

is a diffeomorphism from $G \times Z$ to Q_*. With quotient measure d^2w, the exact coarea density is

$$\begin{aligned} dm &= J(g, w)\, dg\, d^2w, \\ J(g, w) &= \frac{1}{(1 + |w|^2)^2} \prod_{i=1}^{4} \delta(g, s_i(w))^{-2}. \end{aligned} \tag{2.11}$$

For example, using (2.6) and $dm_{S^2} = d^2z/[\pi(1+|z|^2)^2]$, the triangular four-point Jacobian gives

$$J = 4|z_{12}z_{23}z_{31}|^2 |cw + d|^{-4} \prod_{i=1}^{4} (1 + |z_i|^2)^{-2}.$$

The first three factors reduce to $\frac{1}{4}\prod_{i=1}^{3} \delta(g, s_i)^{-2}$, and the fourth is $(1 + |w|^2)^{-2}\delta(g, w)^{-2}$, proving (2.11). With quotient area $d\Omega_w$ of mass 4π, the same product measure instead has $J = \frac{1}{4}\prod_i \delta^{-2}$.

Define

$$\begin{aligned} (W_\rho u)(g, w) &= \sqrt{J(g, w)}\, e^{i\theta_\rho(g,w)} u(gs(w)), \\ \theta_\rho(g, w) &= \sum_i \rho_i \log \delta(g, s_i(w)), \\ I_\rho u(w) &= \int_G (W_\rho u)(g, w)\, dg. \end{aligned} \tag{2.12}$$

The last integral is justified in the next proposition. The first map is unitary from auxiliary $L^2(Q, dm)$ onto $L^2(G \times Z, dg\, d^2w)$, and the cocycle gives

$$W_\rho T_\rho(h) = L_h W_\rho, \qquad (L_h F)(g, w) = F(h^{-1}g, w). \tag{2.13}$$

In particular the positive phase in W_ρ, and the negative phase in its inverse, are fixed by (2.4).

Proposition 2.2 (One-Haar positivity and the absolute integration bound)**.** *For bounded measurable u, v on Q, the group-and-configuration integral defining $\eta_\rho(u, v)$ is absolutely convergent, uniformly in real ρ, and*

$$\begin{aligned} \int_G \int_Q |u(q)|\, |T_\rho(g)v(q)|\, dm(q)\, dg \le C_{\mathrm{av}} \|u\|_\infty \|v\|_\infty, \qquad C_{\mathrm{av}} &= \frac{\pi^3}{15}, \\ \eta_\rho(u, v) &= \int_Z \overline{I_\rho u(w)}\, I_\rho v(w)\, d^2w, \\ \|I_\rho u\|_{L^2(Z)} &\le \sqrt{C_{\mathrm{av}}}\, \|u\|_\infty. \end{aligned} \tag{2.14}$$

The orbit integrals exist absolutely almost everywhere. These statements hold in any smooth orbit section when its exact coarea density is used.

Proof. For the Cartan parameter r, normalized integration over one sphere gives

$$\int_{S^2} \delta(g^{-1}, n)^{-1} \frac{d\Omega(n)}{4\pi} = \frac{1}{2} \int_{-1}^{1} \frac{dt}{\cosh r + t \sinh r} = \frac{r}{\sinh r}.$$

Real labels affect only the phase. The first estimate therefore follows from (2.5) and

$$\frac{2}{\pi}\int_0^\infty \frac{r^4}{\sinh^2 r}\,dr = \frac{2}{\pi}\,3\zeta(4) = \frac{\pi^3}{15}. \tag{2.15}$$

The evaluation is elementary: the nonnegative expansion $\sinh^{-2} r = 4\sum_{m\ge 1} m e^{-2mr}$ permits termwise integration.

To justify factorization in the correct order, first set all labels to zero and $u = v = 1$, which is allowed for this auxiliary estimate. Put $B(g,w) = \sqrt{J(g,w)}$. The group is unimodular because its semisimple Lie algebra has trace-free adjoint maps. Tonelli's theorem, followed by the measure-preserving substitution $k = h^{-1}g$, gives

$$\int_Z \Big(\int_G B(g,w)\,dg\Big)^2 d^2w = \int_Z\int_G\int_G B(g,w)B(h^{-1}g,w)\,dh\,dg\,d^2w = C_{\mathrm{av}}. \tag{2.16}$$

The final equality is precisely the preceding nonnegative group-and-configuration computation after the coarea change. Thus $\int B\,dg$ is finite almost everywhere and belongs to $L^2(Z)$. Since $|W_\rho u| \le \|u\|_\infty B$, both orbit integrals and their product now have an integrable absolute majorant. Complex-valued Fubini and (2.13) yield

$$\begin{aligned}\eta_\rho(u,v) &= \int_Z\int_G\int_G \overline{W_\rho u(g,w)}\, W_\rho v(h^{-1}g,w)\,dh\,dg\,d^2w\\ &= \langle I_\rho u, I_\rho v\rangle_{L^2(Z)}.\end{aligned}$$

The norm bound follows from (2.16). The same argument in another section uses only its density transformation and cocycle, so the last assertion follows as well. □

Integrating (2.14) against $d\Pi$ defines a positive semidefinite form on $\mathcal{D}_{\mathrm{sm,c}}$; a label box has measure $(4R^3/3)^4$. Write $\mathcal{H}_\eta$ for the completion of this test space modulo its null space. The null space is exactly $\ker I$, where $Iu(\rho,w) = I_\rho u_\rho(w)$. Moreover,

$$\mathcal{H}_\eta \simeq L^2\big((0,\infty)^4 \times Z, d\Pi(\rho)\,d^2w\big). \tag{2.17}$$

To see surjectivity after completion, take $f \in C_c^\infty((0,\infty)^4\times Z)$ and $a_G \in C_c^\infty(G)$ with $\int a_G\,dg = 1$. The inverse transform of $a_G(g)f(\rho,w)$, extended by zero, is a smooth compact-flat test: its configuration support is compactly contained in Q_*. Its orbit integral is f. Such f's are dense in the displayed L^2 space.

Remark 2.3. The norm in (2.17) is the form's own quotient norm. The proof does not assert that I is bounded on auxiliary $L^2(Q)$, or that the form is the quadratic form of a bounded operator there. In particular an auxiliary L^2 limiting argument cannot replace the absolute estimates in Proposition 2.2.

2.3 The six Gram operators on the original common domain

For $b \in \mathbb{R}^3$, let

$$R_b = (b\times n)\cdot\nabla_{S^2}, \qquad X_b = (b-(b\cdot n)n)\cdot\nabla_{S^2}.$$

Differentiating (2.4), with $T(\exp(tB_b)) = e^{-itK_b}$, gives

$$J_b = -iR_b, \qquad K_b = -iX_b + i(b\cdot n) + \rho(b\cdot n). \tag{2.18}$$

For the boost with forward field X_{e_z}, the action is $z \mapsto e^t z$, and $\delta(\exp(tB_z), n) = \cosh t + n_z \sinh t$, which also checks the sign.

The identities $\operatorname{div} R_b = 0$ and $\operatorname{div} X_b = -2b \cdot n$ show symmetry on $C^\infty(S^2)$. Writing sums over the three coordinate axes, one obtains

$$\begin{gathered}[J_a, J_b] = i\epsilon_{abc} J_c, \qquad [J_a, K_b] = i\epsilon_{abc} K_c, \qquad [K_a, K_b] = -i\epsilon_{abc} J_c, \\ J^2 = -\Delta_{S^2}, \qquad K^2 = -\Delta_{S^2} + 1 + \rho^2.\end{gathered} \tag{2.19}$$

For the last identities, use $\sum X_a^2 = \sum R_a^2 = \Delta_{S^2}$, $\sum n_a X_a = 0$, and $\sum X_a n_a = 2$. The six observables are

$$O_{ij} = K_i \cdot K_j - J_i \cdot J_j \quad (1 \le i < j \le 3), \qquad O_{ii} = K_i^2 - J_i^2 - \mathbf{1} = \rho_i^2 \mathbf{1}. \tag{2.20}$$

Set $O_{ji} = O_{ij}$ when a full 3×3 array is used. The diagonal subtraction in (2.20) is retained throughout.

Proposition 2.4 (Common domain, null descent, and fixed-vector graph density)**.** *Every fixed finite word in the generators preserves $\mathcal{D}_{\mathrm{sm},c}$ and $\mathcal{D}_{\mathrm{fin}}$; the finite harmonic cutoff may increase. Each O_{ij} commutes with the diagonal $T_\rho(G)$, is symmetric for η, and preserves its null space. It therefore defines a densely defined symmetric operator on $\mathcal{H}_\eta$. The completion of $\mathcal{D}_{\mathrm{fin}}/\ker\eta$ is the same space $\mathcal{H}_\eta$.*

More precisely, for a fixed $u \in \mathcal{D}_{\mathrm{sm},c}$, the tensor harmonic projections

$$P_L = \bigotimes_{i=1}^{4} \mathbf{1}_{\{j_i \le L\}} \tag{2.21}$$

satisfy $P_L u \to u$ in η and in the graph norm of every fixed finite ordered Gram word. No assertion of essential self-adjointness, or of symmetry of noncommuting products, is included.

Proof. The coefficients in (2.18) are polynomial in the labels; there are no label derivatives or inverse label powers. Thus support and flat boundary jets are preserved by every finite word. Rotations preserve harmonic degree. For a homogeneous harmonic polynomial H_j,

$$X_a(H_j|_{S^2}) = (\partial_a H_j)|_{S^2} - j n_a (H_j|_{S^2}),$$

and multiplication by n_a has only degrees $j-1, j+1$. Hence a boost changes degree by at most one. This proves the domain claims; it is not a statement that a finite boost preserves finite harmonic content.

Auxiliary symmetry of O_{ij} follows from (2.18), since different legs commute. The commutators in (2.19) show that the contracted Lorentz bilinear form $K_i \cdot K_j - J_i \cdot J_j$ commutes with all diagonal Lie generators. The group is connected, so it commutes with its diagonal action on smooth functions. For each fixed h, auxiliary integration by parts consequently gives

$$\langle O_{ij} u, T_\rho(h) v \rangle = \langle u, T_\rho(h) O_{ij} v \rangle.$$

Every inserted test is smooth and bounded, and all label support lies in a finite-measure box. Proposition 2.2 justifies the absolute group and label integrations, yielding $\eta(O_{ij}u, v) = \eta(u, O_{ij}v)$.

If n is null, domain preservation makes $O_{ij}^2 n$ admissible. Positivity and Cauchy–Schwarz for the semidefinite form give

$$\eta(O_{ij}n, O_{ij}n) = \eta(n, O_{ij}^2 n) = 0. \tag{2.22}$$

This is the required null descent; symmetry alone would not suffice without the square insertion. Composition then gives null descent for every ordered finite Gram word.

Finally $P_L u$ preserves the exact label support and every flat boundary jet. Smoothness on the compact label box times Q implies uniform convergence in each fixed angular C^m norm.

One way to see this is to apply sufficiently high powers of $1 - \sum_i \Delta_{S^2,i}$, whose norms are uniformly bounded on the box, and then use the spectral tail estimate and Sobolev embedding. Any fixed Gram word is a finite-order angular differential operator with uniformly bounded coefficients on this box. Thus its insertion into $u - P_L u$ tends uniformly to zero. The bound (2.14), integrated over the finite box, proves the claimed η and graph convergence. The quotient of $\mathcal{D}_{\mathrm{fin}}$ is consequently dense in the same completion. □

3 The full target set and the prescribed finite-harmonic family

3.1 Uniform geometry of the five-dimensional target set

Let $e = (1,1,1)^T$, and define

$$A_0 = \frac{4}{3} I_3 - \frac{1}{3} e e^T, \qquad \mathcal{K} = A_0 + \{H = H^T : \operatorname{tr} H = 0,\ \|H\|_F \le 1/24\}. \tag{3.1}$$

This is a compact convex set with nonempty relative interior in the five-dimensional trace-three affine space. Fix

$$\begin{aligned} C_0 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{pmatrix}, \quad Q_0 = C_0 A_0^{-1/2} \in \mathrm{SO}(3), \quad C(A) = Q_0 A^{1/2}, \\ c_i &= C(A) e_i \quad (i \le 3), \qquad c_4 = -C(A) e, \qquad a_i = |c_i|, \qquad q_i = c_i / a_i. \end{aligned} \tag{3.2}$$

Here $C_0^T C_0 = A_0$ and $\det C_0 = 4/(3\sqrt{3}) > 0$. These definitions give the exact identities

$$\sum_{i=1}^{4} a_i q_i = 0, \qquad a_i a_j q_i \cdot q_j = A_{ij}\ (i,j \le 3), \qquad \sum_{i=1}^{3} a_i^2 = 3. \tag{3.3}$$

Use the fixed base stereographic chart $z_i = (q_{i,x} + i q_{i,y})/(1 - q_{i,z})$, and define

$$\begin{aligned} \mu_A(z) &= \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}, \qquad \zeta_A = \mu_A(z_4), \\ s_A(x) &= (q_1, q_2, q_3, \mu_A^{-1}(\zeta_A + x)), \qquad |x| < r_0 = \frac{1}{64}, \\ \Phi_A(g, x) &= g\, s_A(x). \end{aligned} \tag{3.4}$$

The fourth component is interpreted as a sphere point, including when its affine coordinate is infinite. The symbol Φ_A for this coordinate map is distinct from the state $\Phi_{N,A}$ defined below.

Proposition 3.1 (Uniform chart, density, and finite mixed derivatives)**.** *On all of $\mathcal{K}$,*

$$\frac{7}{24} I_3 \preceq A \preceq \frac{11}{8} I_3, \qquad a_- := \sqrt{\frac{7}{8}} \le a_i \le a_+ := \sqrt{\frac{9}{8}}. \tag{3.5}$$

All four reference points are uniformly separated. The maps $\Phi_A : G \times D_{r_0} \to Q$ are diffeomorphisms onto their open images and depend smoothly on A.

Fix any compact set $\Gamma \subset G$. On a common compact group neighborhood of $\Gamma \cup \{e\}$, the moving coordinates, their inverses, and the coarea density

$$dm = H_A(g, x)\, dg\, d^2x \tag{3.6}$$

have all prescribed finite mixed parameter and coordinate derivatives uniformly bounded. The functions H_A are bounded above and away from zero there, and the same derivative conclusion holds for $H_A^{1/2}$, $H_A^{-1/2}$, and $\log \delta(g, s_{A,i}(x))$. The supports $\{\Phi_A(g,x) : g \in \Gamma,\ |x| \le r_0/2\}$ have a common positive collar inside the respective moving coordinate images.

Proof. The eigenvalues of A_0 are $1/3, 4/3, 4/3$, and $\|H\|_{\mathrm{op}} \le 1/24$. This proves the matrix bounds. For $i \le 3$, $23/24 \le a_i^2 = A_{ii} \le 25/24$, while $a_4^2 = e^T A e \in [7/8, 9/8]$. If $m = 7/24$, then

$$|q_i - q_j|^2 = |C(e_i/a_i - e_j/a_j)|^2 \ge m(a_i^{-2} + a_j^{-2}) \ge \frac{14}{27} \quad (i \ne j \le 3),$$
$$|q_i - q_4|^2 = |C(e_i/a_i + e/a_4)|^2 \ge \frac{6m}{a_+^2} = \frac{14}{9}.$$

Thus $d_0 = \sqrt{14/27}$ is a common separation. Also $\det[q_1\ q_2\ q_3] \ge m^{3/2}/a_+^3 > 0$.

We justify smooth uniform parameter dependence before using compactness. The derivative of $A^{1/2}$ in direction E solves $A^{1/2}X + XA^{1/2} = E$, with solution

$$X = \int_0^\infty e^{-tA^{1/2}} E e^{-tA^{1/2}}\, dt.$$

Its norm is at most $\|E\|/(2\sqrt{m})$. Each higher derivative solves the same Sylvester equation with a finite sum of lower derivatives on the right. Induction gives every prescribed finite derivative on a neighborhood of $\mathcal{K}$; the positive bounds for a_i then give the corresponding assertion for a_i, q_i.

The original base stereographic pole stays separated. Indeed, if $S = A^{1/2}$, $S_0 = A_0^{1/2}$, then $S(S - S_0) + (S - S_0)S_0 = A - A_0$, so

$$\|C(A) - C_0\|_{\mathrm{op}} \le \frac{1/24}{\sqrt{7/24} + \sqrt{1/3}} < \frac{1}{24}.$$

For a unit vector v, $|u/|u| - v| \le 2|u - v|$. The first three directions therefore move by less than $1/12$ and the fourth by less than $\sqrt{3}/12$. Consequently

$$|q_{i,z}| \le \frac{5\sqrt{3}}{12} < \frac{3}{4}, \qquad |z_i| \le \sqrt{7}, \qquad |z_i - z_j| \ge d_z := \sqrt{\frac{7}{54}}. \tag{3.7}$$

The last inequality follows from the spherical chord formula. Cross-ratio moduli are ratios of chord products, hence

$$\frac{7}{54} \le |\zeta_A| \le \frac{54}{7}, \qquad |\zeta_A - 1| \ge \frac{7}{54}.$$

For $w = \zeta_A + x$, $|x| \le r_0$, this implies

$$|w| < 8, \qquad |w|,\ |w - 1| \ge \epsilon_* := \frac{7}{54} - \frac{1}{64} = \frac{197}{1728}. \tag{3.8}$$

A projective inverse, valid also at an affine pole, is furnished by

$$U_A = \begin{pmatrix} z_3(z_2 - z_1) & -z_1(z_2 - z_3) \\ z_2 - z_1 & -(z_2 - z_3) \end{pmatrix}, \qquad |\det U_A| \ge d_z^3, \quad \|U_A\|_F \le M_U := 8\sqrt{7}. \tag{3.9}$$

Its least singular value is at least $s_U = d_z^3/M_U$. Writing

$$P(v) = \frac{(2\,\mathrm{Re}(v_1\overline{v}_2), 2\,\mathrm{Im}(v_1\overline{v}_2), |v_1|^2 - |v_2|^2)}{|v_1|^2 + |v_2|^2},$$

the fourth sphere point is $P(U_A(w, 1)^T)$, whose denominator is at least s_U^2. Every prescribed finite mixed A, x derivative is therefore bounded. In addition,

$$|P(U_A u) - P(U_A v)| = \frac{2|\det U_A|\,|\det(u, v)|}{\|U_A u\|\,\|U_A v\|}.$$

Using the three reference vectors $(0,1),(1,1),(1,0)$ and $\|(w,1)\| \le \sqrt{65}$, all moving-slice pair distances are at least

$$d_* = \min\left\{ d_0, \frac{2d_z^3\epsilon_*}{M_U^2\sqrt{130}} \right\} > 0. \tag{3.10}$$

The first three images under Φ_A determine g uniquely, and their cross ratio with the fourth determines x; thus the map is injective. Its differential is invertible: a Möbius infinitesimal field vanishing at three distinct points is zero, since its affine expression has degree at most two. The remaining two real slice directions move the fourth point independently. This proves local invertibility and hence the asserted diffeomorphism onto the image.

Choose relatively compact group neighborhoods G_1, G_2 with $\Gamma \cup \{e\} \subset G_1$ and $\overline{G}_1 \subset G_2$. There are constants $0 < m_\delta \le M_\delta < \infty$ bounding δ on $\overline{G}_2 \times S^2$. The chord transformation law

$$|gq - gq'| = \frac{|q - q'|}{\sqrt{\delta(g,q)\delta(g,q')}}$$

then gives separation at least d_*/M_δ in the compact orbit image. Equivalently, the vertical differential of the joint map $(A,g,x) \mapsto (A, \Phi_A(g,x))$ is invertible on this compact set. Its inverse matrix is uniformly bounded, and differentiating the inverse relation proves every prescribed finite bound on the inverse jets. A fixed finite group atlas and a fixed finite sphere atlas suffice; no single affine chart is asserted to contain the orbit image.

Here is a positive density formula, which also makes the density bounds explicit. Put $\Delta_A = (z_2 - z_1)(z_3 - z_1)(z_3 - z_2)$. At the identity, in Lie coordinates with infinitesimal Möbius field $\alpha + \beta z + \gamma z^2$, the Haar density is $(4\pi^4)^{-1} d^2\alpha\, d^2\beta\, d^2\gamma$. The first-three-point complex Jacobian is the Vandermonde Δ_A; the remaining complex block is $(\mu_A^{-1})'(w)$. Since $|\det U_A| = |\Delta_A|$,

$$\begin{aligned} H_A(e,x) &= \frac{4|\Delta_A|^4}{\prod_{i=1}^{3} (1+|z_i|^2)^2 \|U_A(\zeta_A + x, 1)^T\|^4}, \\ H_A(g,x) &= H_A(e,x) \prod_{i=1}^{4} \delta(g, s_{A,i}(x))^{-2}. \end{aligned} \tag{3.11}$$

The second identity is the area law and left Haar invariance. The formula remains smooth at an affine pole. For example, at the identity it is bounded below and above by the positive constants

$$c_- = \frac{4d_z^{12}}{8^6 M_U^4\, 65^2}, \qquad c_+ = \frac{4(2\sqrt{7})^{12} M_U^4}{d_z^{12}},$$

and throughout the compact product it lies between $c_- M_\delta^{-8}$ and $c_+ m_\delta^{-8}$. All denominators are separated from zero. The already established finite derivative bounds therefore prove the assertions for $H_A^{\pm 1/2}$ and $\log \delta$.

Finally the set

$$\{(A, \Phi_A(g,x)) : A \in \mathcal{K},\ g \in \Gamma,\ |x| \le r_0/2\}$$

is compactly contained in the joint moving-chart image obtained with A in a small open neighborhood of $\mathcal{K}$, $g \in G_1$, and $|x| < r_0$. Its distance from the complement is positive in a fixed product background metric. This proves the common collar, including smooth zero extension across the moving chart boundaries. □

3.2 The fixed profiles and exactly one harmonic cutoff

Fix, once and independently of A, N, profiles

$$\begin{gathered} a_G \in C_c^\infty(G), \quad \int_G a_G\, dg = 1, \qquad b \in C_c^\infty(D_1), \quad \|b\|_{L^2(\mathbb{R}^2)} = 1, \\ \beta \in C_c^\infty((-1/4, 1/4)), \quad \|\beta\|_{L^2(\mathbb{R})} = 1. \end{gathered} \tag{3.12}$$

No symmetry or centering condition on these profiles is imposed. Set $b_N(x) = \sqrt{N}\, b(\sqrt{N}x)$, and write $\rho_i = Na_i(A) + r_i$ on the packet. Define

$$F_{N,A}(\rho) = \prod_{i=1}^{4} \frac{\beta(\rho_i - Na_i(A))}{2\rho_i}, \qquad |r_i| < \frac{1}{4}. \tag{3.13}$$

The expression is extended by zero away from its support. For $N \geq 1$, $\rho_i \geq Na_- - 1/4 > 0$ there.

Use the actual combined coarea density of (3.6), and put

$$\begin{aligned} \theta_{A,\rho}(g,x) &= \sum_i \rho_i \log \delta(g, s_{A,i}(x)), \\ u^{\infty}_{N,A,\rho}(\Phi_A(g,x)) &= H_A(g,x)^{-1/2} e^{-i\theta_{A,\rho}(g,x)} a_G(g) b_N(x), \\ \Phi_{N,A} &= F_{N,A} u^{\infty}_{N,A}, \\ v_{N,A} &= F_{N,A} P_{N^3} u^{\infty}_{N,A}. \end{aligned} \tag{3.14}$$

The angular lift is extended by zero outside the moving chart. For $N \geq N_{\mathrm{supp}} := 2^{14}$, its shape support lies in $|x| \leq r_0/2$. Proposition 3.1 then proves joint smoothness of this zero extension. This uses the original profiles, without inserting a further cutoff.

For clarity, the use of H_A does not change the state under a different quotient density convention. If $dm = J_A(g,x)\, dg\, w_A(x) d^2x$, the same state satisfies $W_{A,\rho} u^{\infty} = a_G b_N / \sqrt{w_A}$. The identification $\widehat{W}_{A,\rho} = \sqrt{w_A} W_{A,\rho}$ is precisely

$$\widehat{W}_{A,\rho} u = \sqrt{H_A}\, e^{i\theta_{A,\rho}}\, u \circ \Phi_A, \qquad \widehat{I}_{A,\rho} u = \int_G \widehat{W}_{A,\rho} u\, dg. \tag{3.15}$$

It has flat quotient measure d^2x. In particular

$$\widehat{I}_{A,\rho} u^{\infty}_{N,A,\rho} = b_N, \qquad d\Pi(\rho) |F_{N,A}(\rho)|^2 = \prod_i |\beta(r_i)|^2\, d^4 r, \qquad \|\Phi_{N,A}\|_{\eta} = 1. \tag{3.16}$$

Only the product density H_A enters the uniform estimates.

The intermediate $\Phi_{N,A}$ belongs to $\mathcal{D}_{\mathrm{sm},c}$. The prescribed vector is exactly

$$v_{N,A} \in \mathcal{D}_{R_N, N^3}, \qquad R_N = Na_+ + \frac{1}{4}. \tag{3.17}$$

For each fixed N, these states depend smoothly on A, including in the η graph norm of any fixed finite Gram word: they extend smoothly to a neighborhood of $\mathcal{K}$, with a common finite label box, and Proposition 2.2 applies to their derivatives. Uniform derivative estimates as N varies require the quantitative arguments given below and in the later mean-map section.

Theorem 3.2 (Uniform approximation by the prescribed six-Gram family)**.** *With the measures, domains, target set, and fixed profiles defined in (2.1)–(3.17), there exist finite constants C, N_0, independent of $A \in \mathcal{K}$, the integer $N \geq N_0$, and the six index pairs, such that $[v_{N,A}] \neq 0$ in $\mathcal{H}_\eta$ and*

$$\psi_{N,A} = \frac{[v_{N,A}]}{\|v_{N,A}\|_{\eta}}, \qquad \max_{1 \leq i \leq j \leq 3} \|(O_{ij} - N^2 A_{ij}) \psi_{N,A}\|_{\eta} \leq C N^{3/2}. \tag{3.18}$$

Every ordered finite Gram word is defined on these vectors. The projector in the definition of $v_{N,A}$ is applied once, to the angular state, before any Gram insertion.

The theorem concerns the specified quotient and the unaltered family. The following proof supplies its geometric and analytic hypotheses; they are not additional assumptions. In particular, no commutation of P_{N^3} with a Gram, and no replacement of a Gram by POP, will be used.

4 Compact-orbit reduction and the exact principal center

4.1 An actual compact-orbit reduction

For fixed A, ρ, conjugate an original Gram by the moving transform: $\widehat{O}_{ij} = \widehat{W}_{A,\rho} O_{ij} \widehat{W}_{A,\rho}^{-1}$. The cocycle and (3.11) give $\widehat{W}_{A,\rho} T_\rho(h) = L_h \widehat{W}_{A,\rho}$. Thus $\widehat{O}_{ij}$ commutes with left translations in g.

Lemma 4.1 (Horizontal reduction with the original filtration)**.** *There are differential operators $D^{\mathrm{hor}}_{ij,A,\rho}$ on the shape disk such that, for smooth tests with compact orbit support,*

$$\widehat{I}_{A,\rho}(O_{ij} u) = D^{\mathrm{hor}}_{ij,A,\rho} \widehat{I}_{A,\rho} u. \tag{4.1}$$

Giving a derivative and a label factor each degree one, these operators have total degree at most two. After the exact substitution $\rho = Na(A) + r$, $|r_i| \le 1/4$, they take the form

$$D^{\mathrm{hor}}_{ij,A,Na+r} = N^2 V_{ij}(A,x) + N L^{(1)}_{ij}(A,r,x,\partial_x) + L^{(2)}_{ij}(A,r,x,\partial_x), \qquad \operatorname{ord} L^{(k)}_{ij} \le k. \tag{4.2}$$

Every prescribed finite mixed A, r, x derivative of their coefficients has a uniform bound on the closed shape disk and all of $\mathcal{K}$.

Proof. Each J in (2.18) has angular order one and label degree zero. Each K has angular order one plus multiplication of label degree at most one. Each Gram consequently has total filtered degree at most two. A label-independent coordinate change and conjugation by $H_A^{1/2}$ preserve that filtration. Conjugation by $e^{i\theta_{A,\rho}}$ does so as well, since a derivative of θ is linear in ρ.

Choose a fixed ordered basis $Y_1, \dots, Y_6$ of left-invariant group fields, whose flows are right multiplication. Using the ordered monomials $Y_1^{\beta_1} \cdots Y_6^{\beta_6}$, write

$$\widehat{O}_{ij} = \sum_{|p|+|\alpha|+|\beta| \le 2} c_{ij,p\alpha\beta}(A,x)\, \rho^p \partial_x^\alpha Y^\beta. \tag{4.3}$$

The coefficients are independent of g, because the operator commutes with all left translations. This follows first for its principal symbol, expressed in the invariant frame, and then inductively for the lower orders; commutators from ordering are absorbed into those lower orders. The coefficient bounds can therefore be checked at $g = e$, where Proposition 3.1 supplies all finite mixed derivatives.

Unimodularity implies $\int_G Y_j F \, dg = 0$ for $F \in C_c^\infty(G)$. Iterating this identity kills every nonempty ordered group word in (4.3). Shape derivatives pass through the compact orbit integral. Consequently

$$D^{\mathrm{hor}}_{ij,A,\rho} = \sum_{|p|+|\alpha| \le 2} c_{ij,p\alpha 0}(A,x) \rho^p \partial_x^\alpha,$$

which proves (4.1). In particular all derivatives of the compact group profile integrate to zero. Substituting $\rho = Na + r$, the filtration forces the N^2 term to have differential order zero and the N term to have order at most one. Smoothness of $a(A)$ gives the asserted uniform coefficient bounds in (4.2). □

4.2 Identification with the two-branch Ward expression

We give an explicit local identification, so that horizontal reduction does not require a separate Ward assumption. Put

$$h_i = \frac{1 - i\rho_i}{2}, \qquad D_i = 1 + |z_i|^2, \qquad \varphi(z) = \prod_i D_i^{-2h_i} u(z). \tag{4.4}$$

For one leg and $g^{-1} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, the planar action is

$$\varphi(z) \longmapsto |cz+d|^{-4h}\varphi\left(\frac{az+b}{cz+d}\right).$$

If $p(z)$ is a forward infinitesimal Möbius polynomial, its derivative is

$$d\pi(p) = -p(z)\partial_z - \overline{p(z)}\partial_{\bar z} - hp'(z) - h\overline{p'(z)}. \tag{4.5}$$

Both branches have weight h. Complex conjugation here applies to the geometric coefficients, not to h; for $p(z) = z$, the constant term is $-2h = -1 + i\rho$.

For $i \neq j$, direct contraction of the original generators in this gauge gives a holomorphic and an antiholomorphic copy of

$$\mathscr{O}_{ij}^{(z)} = (z_i - z_j)^2\partial_i\partial_j - 2h_j(z_i - z_j)\partial_i + 2h_i(z_i - z_j)\partial_j - 2h_ih_j. \tag{4.6}$$

One direct derivation uses $v(z) = ((1-z^2)/2,\ i(1+z^2)/2,\ z)$ and $L_a = v_a\partial_z + hv_a'$. The Gram contraction is $-2\sum_a(L_{ia}L_{ja} + \overline{L}_{ia}\overline{L}_{ja})$, where the barred operator keeps the same h_i. The identity $\sum_a v_a(z_i)v_a(z_j) = -\frac{1}{2}(z_i - z_j)^2$ and its derivatives give all four terms of (4.6). This also fixes its normalization.

Let w be the cross ratio in (3.4), and define

$$\begin{aligned} &(t_{12}, t_{13}, t_{14}, t_{23}, t_{24}, t_{34}) \\ &\quad = (-2h_1, 0, 0,\ h_1 - h_2 - h_3 + h_4,\ h_1 - h_2 + h_3 - h_4,\ -h_1 + h_2 - h_3 - h_4), \\ P_{\mathrm{flat}}(z) &= \prod_{i<j}|z_i - z_j|^{2t_{ij}}, \qquad \mathcal{P}(z) = \prod_i D_i^{2h_i}P_{\mathrm{flat}}(z). \end{aligned} \tag{4.7}$$

The powers of positive distances use their real logarithms. Each row sum is $\sum_{j\neq i} t_{ij} = -2h_i$. The two Möbius transformation identities for D_i and $z_i - z_j$ therefore imply

$$\mathcal{P}(gz) = \mathcal{P}(z)\prod_i \delta(g, z_i)^{1-i\rho_i}.$$

Hence $\mathcal{P}F(w, \bar w)$ is a local formal invariant section for (2.4).

Define D_{ij}^{Ward} by the local differential identity

$$O_{ij}(\mathcal{P}F) = \mathcal{P}D_{ij}^{\mathrm{Ward}}F. \tag{4.8}$$

This definition is explicit through (4.6): conjugate its holomorphic and antiholomorphic copies by P_{flat}, then apply them to $F(w, \bar w)$. For example, if

$$\ell_i = \partial_i \log P_{\mathrm{flat}} = \sum_{k\neq i}\frac{t_{ik}}{z_i - z_k}, \qquad w_i = \partial_i w, \qquad w_{ij} = \partial_i\partial_j w, \qquad z_{ij} = z_i - z_j,$$

the holomorphic coefficients of $d_{ij}\partial_w^2 + b_{ij}\partial_w + c_{ij}$ are exactly

$$\begin{aligned} d_{ij} &= z_{ij}^2 w_i w_j, \\ b_{ij} &= z_{ij}^2(w_{ij} + \ell_i w_j + \ell_j w_i) - 2h_j z_{ij} w_i + 2h_i z_{ij} w_j, \\ c_{ij} &= z_{ij}^2(\ell_i\ell_j + t_{ij}/z_{ij}^2) - 2h_j z_{ij}\ell_i + 2h_i z_{ij}\ell_j - 2h_ih_j. \end{aligned} \tag{4.9}$$

The other branch replaces the geometric variables by their conjugates and again retains the same weights. Invariance makes the coefficients functions of $w, \bar w$; the formula can be evaluated in any local projective frame.

On a finite sphere chart where $\mathcal{P}$ is defined, set

$$S_{A,\rho}(x) = H_A(e,x)^{1/2}\mathcal{P}(s_A(x)). \tag{4.10}$$

It is smooth and nonvanishing locally. The density and phase transformation laws give the pointwise identity

$$\widehat{W}_{A,\rho}(\mathcal{P}F)(g,x) = S_{A,\rho}(x)F(\zeta_A + x, \overline{\zeta_A + x}),$$

independent of g. Apply the local conjugated differential expression (4.3) to this group-constant function. The nonempty group words vanish pointwise; comparing with (4.8) yields

$$D^{\mathrm{hor}}_{ij,A,\rho} = S_{A,\rho} D^{\mathrm{Ward}}_{ij} S^{-1}_{A,\rho}, \tag{4.11}$$

with the translation $w = \zeta_A + x$ understood. This is an identity of local differential expressions. The formal section was not integrated or admitted to a Hilbert space; the actual integration identity on compact-orbit tests was already proved in Lemma 4.1. Projective sphere charts handle an affine pole, and the compact differential expression extends there. For estimates we use H_A, not separate bounds on a choice of $S_{A,\rho}$ or on a split quotient density.

4.3 Weighted closure gives the exact center

Lemma 4.2 (Principal-center identity)**.** *The leading coefficient in* (4.2) *satisfies*

$$V_{ij}(A,0) = A_{ij}, \qquad |V_{ij}(A,x) - A_{ij}| \le C_{\mathrm{cen}}|x| \quad (A \in \mathcal{K},\ |x| \le r_0) \tag{4.12}$$

for all six index pairs, with one constant C_{cen}*.*

Proof. Let $q_A = s_A(0)$. On the moving physical coordinate image define

$$\Theta_A(\Phi_A(g,x)) = \sum_{i=1}^{4} a_i(A) \log \delta(g, s_{A,i}(x)).$$

At $g = e$ this function is zero for every x, so its two shape derivatives vanish. Its rotational derivatives vanish because $\delta(k,n) = 1$ for rotations. For a forward boost in direction b,

$$\frac{d}{dt}\Theta_A(\exp(tB_b)q_A)\Big|_{t=0} = b \cdot \sum_i a_i q_i = 0$$

by (3.3). The six orbit directions and two shape directions form a basis of $T_{q_A}Q$, by Proposition 3.1. Thus

$$d_q\Theta_A(q_A) = 0. \tag{4.13}$$

The parameter A is held fixed in this physical derivative.

The actual inverse transform has phase $e^{-iN\Theta_A}$, and

$$e^{iN\Theta_A}(-iX)e^{-iN\Theta_A} = -iX - N(X\Theta_A).$$

By (2.18), the order-N multiplier in a transformed boost on leg i is $a_i q_i - X_i\Theta_A$, and that in a transformed rotation is $-R_i\Theta_A$. At q_A, equation (4.13) makes these $a_i q_i$ and zero, respectively. Consequently the derivative-free order-N^2 term of the original off-diagonal Gram is $a_i a_j q_i \cdot q_j = A_{ij}$. Derivatives of an order-N phase coefficient contribute only order N. Coordinate reordering and density conjugation cannot add order-N^2 terms from the other filtered pieces. The group-independent, derivative-free coefficient retained by horizontal reduction is therefore exactly this value. For the diagonal, $O_{ii} = \rho_i^2$ gives $V_{ii} = a_i^2 = A_{ii}$ directly.

The uniform first x-derivative bounds from Lemma 4.1 now give the second assertion by the mean-value estimate on the disk. □

Lemma 4.3 (Residual of the genuine smooth intermediate)**.** *There is $C_{\mathrm{sm}} < \infty$, independent of A, N and the six pairs, such that for $N \geq 2^{14}$,*

$$\max_{i \leq j \leq 3} \|(O_{ij} - N^2 A_{ij})\Phi_{N,A}\|_\eta \leq C_{\mathrm{sm}} N^{3/2}. \tag{4.14}$$

Proof. In two shape dimensions the fixed bump has the exact scaling laws

$$\|b_N\|_2 = 1, \qquad \|\partial^\alpha b_N\|_2 = N^{|\alpha|/2}\|\partial^\alpha b\|_2, \qquad \||x| b_N\|_2 = N^{-1/2}\||x| b\|_2. \tag{4.15}$$

The $N^2(V_{ij} - A_{ij})$ contribution in (4.2) is therefore $O(N^{3/2})$. The term $N L_{ij}^{(1)} b_N$ is $O(N^{3/2} + N)$, and $L_{ij}^{(2)} b_N$ is $O(N + \sqrt{N} + 1)$. All constants are uniform in A, r. Use (4.1), (3.16), and the positive factorization to obtain the fixed-label residual bound. Integration against $d\Pi |F_{N,A}|^2$, whose total mass is one, gives (4.14). For a diagonal one may also use directly $\rho_i^2 - N^2 A_{ii} = 2N a_i r_i + r_i^2 = O(N)$. □

5 The single harmonic cutoff and completion of the proof

Give Q its product round metric. The Sobolev norm $H^s(Q)$ is defined by the spectral power of $1 - \sum_{i=1}^4 \Delta_{S^2,i}$; its integer norms are equivalent to the usual finite sums of covariant-derivative L^2 norms.

Lemma 5.1 (Tenth-order bound for the actual moving lift)**.** *For the profiles and lift in (3.14), there is a constant C_{10} such that*

$$\sup_{\substack{A \in \mathcal{K} \\ \rho \in \operatorname{supp} F_{N,A}}} \|u^\infty_{N,A,\rho}\|_{H^{10}(Q)} \leq C_{10} N^{10} \qquad (N \geq 2^{14}). \tag{5.1}$$

Proof. On the common compact collar, $\rho = N a(A) + r$, $|r_i| \leq 1/4$, implies $|\rho| \leq CN$. Every fixed positive derivative of the phase is therefore $O(N)$. Apply an angular differential word of order $k \leq 10$ to the actual formula (3.14). The chain and product rules give a finite sum of terms consisting of a uniformly bounded smooth coefficient, a factor N^p, a fixed derivative of a_G, and $\partial_x^\alpha b_N$, with $p + |\alpha| \leq k$. The phase has modulus one, and the remaining density, chart, and inverse-chart derivatives are bounded by Proposition 3.1. Integration in the compact group coordinate and (4.15) bound each term in $L^2(Q)$ by

$$C N^{p + |\alpha|/2} \leq C N^k.$$

Equivalently, the amplitude $N^{1/2}$ is supported in shape volume $O(N^{-1})$, so no extra half power remains in the L^2 norm. The fixed smooth supports and the common collar justify the same calculation after zero extension. Summing the finitely many derivative norms proves (5.1). □

Lemma 5.2 (One-cutoff state and Gram tails)**.** *For any fixed $0 < \varepsilon < 1$ and $k = 0, 1, 2$,*

$$\sup_{\substack{A \in \mathcal{K} \\ \rho \in \operatorname{supp} F_{N,A}}} \|(1 - P_{N^3}) u^\infty_{N,A,\rho}\|_{C^k(Q)} \leq C_{k,\varepsilon} N^{3k - 8 + \varepsilon}. \tag{5.2}$$

In particular, on fixing $\varepsilon = 1/2$, there are common constants $E, G, T < \infty$ such that, for $e_{N,A} = v_{N,A} - \Phi_{N,A}$,

$$\begin{aligned} \|e_{N,A}\|_\eta &\leq E N^{-15/2}, \\ \|O_{ij} e_{N,A}\|_\eta &\leq G N^{-3/2}, \\ \|(O_{ij} - N^2 A_{ij}) e_{N,A}\|_\eta &\leq T N^{-3/2}. \end{aligned} \tag{5.3}$$

For general fixed ε, the last two rates are $O_\varepsilon(N^{-2+\varepsilon})$.

Proof. The real dimension of Q is eight. Use the strict Sobolev embedding $H^s(Q) \hookrightarrow C^k(Q)$ with $s = k + 4 + \varepsilon/3$. On the complement of P_{N^3}, at least one harmonic degree is $N^3 + 1$ or larger, so the total Laplace eigenvalue is at least $(N^3 + 1)(N^3 + 2)$. The spectral norm estimate and Lemma 5.1 give

$$\begin{aligned}\|(1 - P_{N^3})u^\infty\|_{C^k} &\le C[1 + (N^3 + 1)(N^3 + 2)]^{(s-10)/2}\|u^\infty\|_{H^{10}} \\ &\le CN^{3s-20} = CN^{3k-8+\varepsilon}.\end{aligned}$$

The strict inequality $s > k + 4$ is essential; no endpoint $\varepsilon = 0$ assertion is being made.

The original generator filtration gives, on the packet,

$$\|O_{ij}h\|_{C^0(Q)} \le C_O\big(N^2\|h\|_{C^0} + N\|h\|_{C^1} + \|h\|_{C^2}\big). \tag{5.4}$$

Indeed its second angular order has label degree zero, its first angular order has label degree at most one, and its zeroth order has label degree at most two. There are only six operators and $|\rho| \le CN$, so a common constant suffices.

Let $h = (P_{N^3} - 1)u^\infty$, so $e = F_{N,A}h$. The Grams have no label derivatives and therefore commute with multiplication by $F_{N,A}$. The absolute η bound and the unit mass of $d\Pi|F|^2$ convert each uniform C^0 estimate to the corresponding η estimate. With $\varepsilon = 1/2$, the three angular tails have powers $N^{-15/2}, N^{-9/2}, N^{-3/2}$. In (5.4) their contributions are $N^{-11/2}, N^{-7/2}, N^{-3/2}$, respectively. This proves the first two bounds in (5.3). Since $\sup_{\mathcal{K}} |A_{ij}| < \infty$, subtracting $N^2 A_{ij} e$ gives the third. The same computation with general ε gives the stated rates.

This estimates $O_{ij}(1 - P_{N^3})u^\infty$ directly. No commutator with P_{N^3} has been discarded, and horizontal reduction has only been applied to the genuine compact-orbit intermediate, whose support is controlled. □

Proof of Theorem 3.2. By (3.16) and (5.3),

$$\|v_{N,A}\|_\eta \ge 1 - EN^{-15/2}.$$

Take one integer

$$N_0 \ge \max\left\{2^{14}, \left\lceil (2E)^{2/15} \right\rceil\right\}. \tag{5.5}$$

Then the norm is at least $1/2$, simultaneously for all $A \in \mathcal{K}$ and all integers $N \ge N_0$. In particular the quotient class is nonzero. The smooth residual and centered graph tail give

$$\|(O_{ij} - N^2 A_{ij})v_{N,A}\|_\eta \le C_{\mathrm{sm}} N^{3/2} + TN^{-3/2}. \tag{5.6}$$

After division by the lower norm bound,

$$\|(O_{ij} - N^2 A_{ij})\psi_{N,A}\|_\eta \le \frac{C_{\mathrm{sm}} N^{3/2} + TN^{-3/2}}{1 - EN^{-15/2}} \le 2(C_{\mathrm{sm}} + T)N^{3/2}.$$

Taking $C = 2(C_{\mathrm{sm}} + T)$ proves the uniform conclusion for all six pairs. The finite-domain assertion follows from (3.17) and Proposition 2.4. □

Remark 5.3 (Scope of the uniform theorem). The constants depend on the once-fixed profiles and the displayed model conventions. Their finiteness and a common onset are proved; a numerically sharp onset is not required. The theorem is an approximate simultaneous-eigenvector statement in a Hilbert quotient, not a claim that the six operators commute or possess a joint spectral measure. It requires neither essential self-adjointness of their minimal realizations nor an additional scalar-channel constraint. The full-spin native comparison below is a separate unitary identification of this same quotient and its common test action.

6 Numerical geometry and fluctuation bounds

Throughout this section the state is the actual normalized vector $\psi_{N,A}$ of Theorem 3.2, with its single angular cutoff. All statements are uniform for $A \in \mathcal{K}$. Matrix norms without a subscript are operator norms; $\|\cdot\|_F$ is the Frobenius norm. The operators act on the common quotient domain established above. In particular, the following expectations do not presume joint diagonalizability of the six Gram operators.

6.1 Positive mean matrices and relative fluctuations

Set $O_{ji} = O_{ij}$ and define the real symmetric numerical matrices

$$M_{N,ij}(A) = \eta(\psi_{N,A}, O_{ij}\psi_{N,A}), \qquad S_N(A) = \frac{\operatorname{tr} M_N(A)}{3}, \qquad B_N(A) = \frac{M_N(A)}{S_N(A)}. \tag{6.1}$$

The word "numerical" distinguishes M_N and its matrix functions from operator-valued matrices. We use the convention that η is antilinear in its first argument. Symmetry of O_{ij} makes every entry in (6.1) real.

Corollary 6.1 (Mean geometry and relative fluctuations)**.** *There is a common N_1 such that, for all integers $N \geq N_1$ and every $A \in \mathcal{K}$,*

$$\|M_N - N^2 A\| \leq 3CN^{3/2}, \qquad \lambda_{\min}(M_N) \geq \frac{7}{48}N^2, \tag{6.2}$$

$$S_N = N^2 + O(N), \qquad B_N = A + O(N^{-1/2}). \tag{6.3}$$

Let U be the matrix of quotient-Hilbert-space vectors $U_{ij} = (O_{ij} - M_{N,ij})\psi_{N,A}$, and put

$$\sigma_N^2 = \sum_{i,j=1}^{3} \|U_{ij}\|_\eta^2, \qquad \delta_N^2 = \sum_{i,j=1}^{3} \|(M_N^{-1/2} U M_N^{-1/2})_{ij}\|_\eta^2. \tag{6.4}$$

Then $\sigma_N = O(N^{3/2})$ and $\delta_N = O(N^{-1/2})$.

Proof. For each entry, Cauchy–Schwarz and Theorem 3.2 give $|M_{N,ij} - N^2 A_{ij}| \leq CN^{3/2}$. A 3×3 matrix with entries bounded by c has operator norm at most $3c$. The lower eigenvalue bound follows from $A \geq (7/24)I$ and, for sufficiently large common N, $3CN^{-1/2} \leq 7/48$.

For a symmetric operator and a unit vector, centering at its expectation minimizes the squared residual over real constants. Consequently $\|U_{ij}\|_\eta \leq CN^{3/2}$, which proves the absolute fluctuation bound. View U as a vector in the Hilbert tensor product of the numerical matrix space with $\mathcal{H}_\eta$. Left and right multiplication by the fixed numerical matrix $M_N^{-1/2}$ have product norm at most $\|M_N^{-1/2}\|^2 = 1/\lambda_{\min}(M_N)$. Thus

$$\delta_N \leq \frac{\sigma_N}{\lambda_{\min}(M_N)} = O(N^{-1/2}). \tag{6.5}$$

This operation uses a numerical matrix and makes no functional-calculus claim about noncommuting Gram operators. The stronger trace estimate in (6.3) follows from Proposition 6.4 below. Dividing the mean estimate by that positive trace scale proves the last assertion of (6.3). □

Proposition 6.2 (Classical cofactor reconstruction)**.** *For a positive definite 3×3 matrix M, define*

$$f(M) = \sqrt{\det M}\, M^{-1}. \tag{6.6}$$

If E is an invertible matrix of three edge vectors and $C = \operatorname{cof} E$ is the corresponding matrix of oriented area vectors, then $M = C^T C$ satisfies $f(M) = E^T E$. Moreover, $f(tM) = t^{1/2} f(M)$ for $t > 0$, and the states above obey

$$\|f(B_N(A)) - f(A)\| = O(N^{-1/2}) \tag{6.7}$$

uniformly on $\mathcal{K}$.

Proof. The cofactor identity $C = (\det E)E^{-T}$ gives $C^TC = (\det E)^2(E^TE)^{-1}$ and $\det(C^TC) = (\det E)^4$. Substitution proves the reconstruction identity, including either orientation of E. The homogeneity follows directly from the determinant and inverse. By Corollary 6.1, B_N and A lie eventually in one compact subset of the positive definite cone. The derivative of f is bounded on a slightly larger compact convex subset of that cone. The mean value estimate and (6.3) imply (6.7). □

The normalization $\operatorname{tr} B_N = 3$ isolates shape. Turning the scale S_N into a physical length requires an additional physical normalization; no such normalization is part of this mathematical construction. Likewise, $f(M_N)$ is a classical readout of an expectation matrix, not the expectation of a newly defined quantum metric operator. At the central target A_0, the construction simply specializes the same family $\psi_{N,A}$; it does not identify that specialization with any separately prescribed packet.

Proposition 6.3 (Deterministic stability of the descriptor)**.** *Let $M > 0$ and let $E = E^T$ satisfy $\|M^{-1/2}EM^{-1/2}\| \le r < 1$. Then $M + E > 0$, and every eigenvalue of*

$$f(M)^{-1/2} f(M+E) f(M)^{-1/2}$$

lies in the interval

$$\left[\frac{1-r}{\sqrt{1+r}}, \frac{1+r}{\sqrt{1-r}}\right]. \tag{6.8}$$

Proof. Set $R = M^{-1/2}EM^{-1/2}$. Direct substitution in (6.6) gives

$$f(M)^{-1/2} f(M+E) f(M)^{-1/2} = \sqrt{\det(I+R)}\,(I+R)^{-1}.$$

If $\lambda_1, \lambda_2, \lambda_3 \in [1-r, 1+r]$ are the eigenvalues of $I+R$, the corresponding eigenvalues of this matrix are $\sqrt{\lambda_j\lambda_k/\lambda_i}$, where $\{i,j,k\} = \{1,2,3\}$. Their extremal bounds are exactly (6.8). □

This deterministic perturbation statement is distinct from the Hilbert-valued fluctuation bound (6.5). The latter does not supply a simultaneous classical distribution for all six operators.

6.2 Spectral covariance and a protected direction

Index the six independent entries by $\alpha \in \{11, 22, 33, 12, 13, 23\}$ and write

$$\Gamma_{\alpha\beta} = \operatorname{Re}\eta\big((O_\alpha - M_\alpha)\psi, (O_\beta - M_\beta)\psi\big). \tag{6.9}$$

This real covariance matrix is positive semidefinite, even though the operators need not commute.

Proposition 6.4 (Diagonal and total covariance)**.** *The diagonal multipliers $O_{ii} = \rho_i^2$ imply*

$$(N - \tfrac{1}{4})^2 \le S_N \le (N + \tfrac{1}{4})^2, \tag{6.10}$$

$$\sum_{i=1}^{3} \Gamma_{ii,ii} \le \frac{3}{4}N^2, \tag{6.11}$$

$$\operatorname{tr}\Gamma \le \frac{3}{4}N^2 + 3C^2N^3, \qquad \sigma_N^2 \le \frac{3}{4}N^2 + 6C^2N^3. \tag{6.12}$$

Proof. The normalized state defines the probability measure

$$d\mu_{N,A}(\rho) = \frac{\|I_\rho v_{N,A}(\rho)\|_2^2}{\|v_{N,A}\|_\eta^2}\, d\Pi(\rho). \tag{6.13}$$

Its support lies in $|\rho_i - Na_i(A)| \le 1/4$. Thus the numerical diagonal means and variances are those of the scalar random variables ρ_i^2 under this one measure. Since $\sum_{i=1}^3 a_i^2 = 3$ and $\sum_{i=1}^3 a_i \le 3$, pointwise bounds on $\frac{1}{3}\sum_{i=1}^3 \rho_i^2$ give (6.10). The interval occupied by ρ_i^2 has width Na_i, so its variance is at most $N^2a_i^2/4$. This proves (6.11). The off-diagonal centered residuals have squared norm at most C^2N^3. Counting them once in $\operatorname{tr}\Gamma$ and twice in σ_N^2 gives (6.12). □

We next record a quotient identity. It is an identity of induced Gram operators on the common test domain; it does not assert that individual rotation or boost generators act on the quotient.

Lemma 6.5 (Total Casimir identity)**.** *On the original common quotient domain,*

$$O_{12} + O_{13} + O_{23} = \frac{\rho_4^2 - \rho_1^2 - \rho_2^2 - \rho_3^2 - 2}{2}. \tag{6.14}$$

Proof. Haar invariance of the absolutely convergent orbit map gives $I_\rho T_\rho(h)u = I_\rho u$. Differentiation on the smooth test space therefore gives $I_\rho X_{\mathrm{tot}} w = 0$ for each total Lie algebra generator and each test vector w occurring below. In any fixed component, $[K^a_{\mathrm{tot}}, K^a_4] = [J^a_{\mathrm{tot}}, J^a_4] = 0$. Expanding $(K_{\mathrm{tot}} - K_4)^2 - (J_{\mathrm{tot}} - J_4)^2$ and moving total components to the left consequently shows that the total Casimir of legs $1, 2, 3$ equals that of leg 4 after applying I_ρ. The single-leg convention is $K_i^2 - J_i^2 = \rho_i^2 + 1$. Expanding the three-leg expression gives $\sum_{i=1}^3(\rho_i^2+1) + 2\sum_{i<j\le 3} O_{ij} = \rho_4^2 + 1$, which is (6.14). Each differentiated vector remains in the original common domain, so every step is justified before passage to the quotient. □

Proposition 6.6 (A protected off-diagonal covariance direction)**.** *Let Γ_{off} be the 3×3 covariance of (O_{12}, O_{13}, O_{23}), and let $s = (1,1,1)^T/\sqrt{3}$. Then*

$$\mathrm{Var}_\psi(O_{12} + O_{13} + O_{23}) \le \frac{N^2}{16}\Big(\sum_{i=1}^4 a_i\Big)^2, \tag{6.15}$$

$$\left\| \frac{\Gamma_{\mathrm{off}}}{N^3} s \right\| \le \frac{C(3 + \sqrt{9/8})}{4\sqrt{N}}. \tag{6.16}$$

In particular every limit of the bounded matrices $N^{-3}\Gamma_{\mathrm{off}}$ annihilates s.

Proof. The multiplier on the right of (6.14) ranges over an interval of width at most $(N/2)\sum_{i=1}^4 a_i$. The variance bound for a bounded real variable proves (6.15). Also $s^T\Gamma_{\mathrm{off}} s = \mathrm{Var}(O_{12} + O_{13} + O_{23})/3$ and $\|\Gamma_{\mathrm{off}}\| \le \mathrm{tr}\,\Gamma_{\mathrm{off}} \le 3C^2N^3$. For a positive semidefinite matrix G, $\|Gs\|^2 \le \|G\| s^T G s$. Apply this inequality to $G = N^{-3}\Gamma_{\mathrm{off}}$, then use $\sum_{i=1}^4 a_i \le 3 + \sqrt{9/8}$. □

This conclusion is asymptotic. It does not impose rank two on the finite-N off-diagonal covariance matrix.

6.3 A small-label obstruction and the fourth leg

Proposition 6.7 (Fixed spectral support cannot produce scale divergence)**.** *Fix a label box $0 \le \rho_i \le R_i$. Every finite Gram word preserves this spectral support. Any normalized vector obtained within that support has*

$$S \le \frac{R_1^2 + R_2^2 + R_3^2}{3}. \tag{6.17}$$

Thus a construction restricted to a fixed label box cannot have $S \to \infty$.

Proof. The Gram operators contain angular derivatives and multiplication by spectral labels, with no derivatives in ρ. They therefore preserve the stated support, as do finite compositions. The exact identity $O_{ii} = \rho_i^2$ and the probability measure defined as in (6.13) give (6.17). □

Proposition 6.8 (General small-label control)**.** *For any normalized vector in the original common test domain whose spectral support obeys $\rho_4^2 \le R_4^2$, its numerical three-leg mean matrix satisfies*

$$\lambda_{\min}(M) \le \frac{R_4^2 - 2}{3}. \tag{6.18}$$

Thus $R_4^2 < 2$ forces a negative direction, while $R_4^2 = 2$ excludes positive definiteness.

Proof. Let $e = (1,1,1)^T$. The diagonal identity and (6.14) give $e^T M e = \langle \rho_4^2 \rangle - 2$. The Rayleigh quotient at $e/\sqrt{3}$ proves the bound. □

The label restriction is essential. This proposition does not exclude positive mean matrices for unrestricted external labels, and it is fully compatible with Theorem 3.2.

Proposition 6.9 (The ten Gram insertions and numerical inertia)**.** *On the same unrestricted four-leg test space define, for* $1 \le a, b \le 4$,

$$Q_{aa} = K_a^2 - J_a^2 - I = \rho_a^2, \qquad Q_{ab} = K_a \cdot K_b - J_a \cdot J_b \quad (a \neq b).$$

These ten symmetric insertions preserve the common domain and descend to the quotient. On that domain,

$$\sum_{b=1}^{4} Q_{ab} = -I. \tag{6.19}$$

If the three-leg numerical mean M *of a normalized state is positive definite, the full numerical mean is*

$$G_4 = \begin{pmatrix} M & -Me - e \\ -e^T M - e^T & e^T M e + 2 \end{pmatrix}. \tag{6.20}$$

It has inertia $(3,1,0)$ *and the exact eigenpair* $G_4 \mathbf{1}_4 = -\mathbf{1}_4$*. For the family of the main theorem, all ten residuals are* $O(N^{3/2})$ *about the rank-three classical matrix* $\widetilde{A} = [I \mid -e]^T A [I \mid -e]$.

Proof. The finite-word, symmetry and null-space arguments for the six operators in Proposition 2.4 apply verbatim after relabeling any pair of legs. No restriction on the fourth leg's angular harmonics was imposed. This proves the domain assertion before the additional insertions are used. For a fixed component, commute K_a past K_{tot}, and similarly for J_a. The commutators with the same component vanish. The differentiated Haar identity used in Lemma 6.5 then yields $I_\rho \sum_b Q_{ab} u = -I_\rho u$, proving (6.19). The first three row identities determine the off-diagonal block of (6.20); the fourth determines its lower-right entry.

The Schur complement of M in G_4 equals

$$e^T M e + 2 - (Me + e)^T M^{-1} (Me + e) = -4 - e^T M^{-1} e < 0.$$

Congruence reduction therefore gives three positive and one negative eigenvalue. The row-sum identity supplies the exact negative eigenpair. Finally, in the quotient, $Q_{i4} = -I - \sum_{j=1}^{3} O_{ij}$ and $Q_{44} = 2I + \sum_{i,j=1}^{3} O_{ij}$. The six residual bounds imply bounds $1 + 3CN^{3/2}$ and $2 + 9CN^{3/2}$ for these additional entries. Both are $O(N^{3/2})$. □

The inertia statement concerns the numerical expectation matrix. It makes no assertion about a joint spectrum of the ten operators. The order-one negative eigenvalue is also compatible with the rank-three nonnegative classical limit $N^{-2} G_4 \to \widetilde{A}$.

7 First derivatives of the same mean map

The uniform norm estimate alone does not imply injectivity of the numerical readout. We now prove a first-derivative estimate for exactly the family constructed above, without replacing its profiles or its angular cutoff. All derivatives in A are in the trace-zero symmetric tangent space to $\mathcal{K}$, with its Frobenius norm. The construction and its estimates extend to a neighborhood of this compact set, so boundary derivatives are unambiguous.

Theorem 7.1 (Uniform first-derivative control)**.** *For the numerical mean and normalization in (6.1),*

$$\|N^{-2}M_N - \mathrm{id}\|_{C^1(\mathcal{K})} \le CN^{-1/2}, \tag{7.1}$$

$$\|N^{-2}S_N - 1\|_{C^1(\mathcal{K})} \le CN^{-1}, \tag{7.2}$$

$$\|B_N - \mathrm{id}\|_{C^1(\mathcal{K})} \le CN^{-1/2}. \tag{7.3}$$

The constants depend on the fixed profiles and common chart, but not on A or N.

7.1 Fixed-label differentiation and transported differentiation

Write $\Xi_A(g,x) = gs_A(x)$ for the common product chart, $h_A(g,x) = J_A(g,x)w_A(x)$ for its combined density, and $\mathcal{B}_N(g,x) = a_G(g)b_N(x)$, where $b_N(x) = \sqrt{N}\,b(\sqrt{N}x)$. Abbreviate $u = u^\infty$ for the angular lift in formula (3.14), so that

$$u_{N,A,\rho}(\Xi_A(g,x)) = h_A(g,x)^{-1/2}e^{-i\theta_{A,\rho}(g,x)}\mathcal{B}_N(g,x). \tag{7.4}$$

There are two distinct derivatives. The derivative D_A^ρ fixes the physical configuration q and the spectral label ρ. Once $\rho = Na(A) + r$ has been substituted, differentiation at fixed q, r is

$$D_A^r = D_A^\rho + N\sum_{i=1}^{4}(D_A a_i)\partial_{\rho_i}. \tag{7.5}$$

We keep these operations separate throughout.

Lemma 7.2 (Differentiated angular and ambient bounds)**.** *For every tangent vector H, uniformly on the spectral packet,*

$$\|u_{N,A,\rho}\|_{H^{10}(Q)} \le CN^{10}, \qquad \|D_A^\rho u_{N,A,\rho}[H]\|_{H^{10}(Q)} \le CN^{11}\|H\|_F, \tag{7.6}$$

$$\|D_A^\rho \Phi_{N,A}[H]\|_\eta \le CN\|H\|_F, \qquad \|D_A^\rho(O_{ij}\Phi_{N,A})[H]\|_\eta \le CN^3\|H\|_F. \tag{7.7}$$

Also $\|O_{ij}\Phi_{N,A}\|_\eta \le CN^2$.

Proof. All required mixed A, g, x jets of the chart, its inverse and its density are bounded on the common compact collar of Proposition 3.1. The phase is linear in ρ, hence an angular derivative of its exponential costs at most $O(N)$, while a derivative of the shape bump costs $O(\sqrt{N})$. The amplitude is $O(\sqrt{N})$ and its angular support has volume $O(N^{-1})$. Thus a kth angular derivative has supremum $O(N^{k+1/2})$ and L^2 norm $O(N^k)$ for $k \le 10$. A fixed-label parameter derivative adds at most one factor N, including motion of the inverse chart. This proves (7.6). Since $\partial_{\rho_i}u = -i(\partial_{\rho_i}\theta)u$ and the multiplier has bounded angular jets, its H^{10} norm is $O(N^{10})$; hence the same $O(N^{11})$ parameter bound holds after the transport in (7.5).

The supremum majorant is not sharp enough for the ambient eta bounds, so we derive the transport formula. Define the smooth vector field

$$Z_H = (d_{(g,x)}\Xi_A)^{-1}D_A\Xi_A[H].$$

At fixed physical q, the product coordinates have derivative $-Z_H$. Let $\widehat{W}_A$ denote multiplication by $h_A^{1/2}e^{i\theta_{A,\rho}}$ after pullback by Ξ_A, the unitary product-coordinate factor used in the orbit reduction. Direct differentiation of (7.4) gives

$$\widehat{W}_A D_A^\rho u[H] = -Z_H\mathcal{B}_N - \left\{\tfrac{1}{2}(D_A[H] - Z_H)\log h_A + i(D_A[H] - Z_H)\theta_{A,\rho}\right\}\mathcal{B}_N. \tag{7.8}$$

Here the phase derivative fixes ρ. Every coefficient of Z_H is bounded by $C\|H\|_F$, and the phase term is $O(N)\|H\|_F$.

Integrate (7.8) over the fixed compact group support. The resulting shape function $f_H = \widehat{I}_A D_A^\rho u[H]$ is a sum of bounded-coefficient first derivatives of b_N and multipliers of the form $p_0(A,x) + \sum_i \rho_i p_i(A,x)$ times b_N. All the needed coefficient jets are uniform and linear in H. Consequently

$$\|\partial_x^k f_H\|_2 \le C_k N^{1+k/2} \|H\|_F, \qquad 0 \le k \le 2. \tag{7.9}$$

The horizontal Gram operator from Lemma 4.1 has coefficients of order $N^2, N, 1$ at differential orders $0, 1, 2$, respectively. Applying it to (7.9) proves $\|D_A^\rho u[H]\|_{\eta,\rho} \le CN\|H\|_F$ and $\|O_{ij} D_A^\rho u[H]\|_{\eta,\rho} \le CN^3 \|H\|_F$. Applying it to $\widehat{I}_A u = b_N$ gives $\|O_{ij} u\|_{\eta,\rho} \le CN^2$.

It remains to differentiate the external packet. Let $F^{[i]}$ be $F_{N,A}$ with its ith numerator β replaced by β', keeping the same denominators $2\rho_i$. At fixed ρ,

$$D_A^\rho F[H] = -N \sum_i D_A a_i[H] F^{[i]}, \qquad \|F^{[i]}\|_{L^2(d\Pi)} = \|\beta'\|_2. \tag{7.10}$$

There is no division by β at its zeros. The bounds above, the compact profile supports and (7.10) give (7.7). At fixed ρ the operator O_{ij} itself is independent of A, so no operator derivative is omitted. □

For each fixed N, all parameter-dependent spectral supports lie in one compact subset of $(0,\infty)^4$. The smooth zero extensions and the Haar majorant in Proposition 2.2 justify differentiation of the eta pairing, giving

$$D_A \eta(f,g)[H] = \eta(D_A^\rho f[H], g) + \eta(f, D_A^\rho g[H]). \tag{7.11}$$

The differentiated projected vectors are still finite harmonic vectors with smooth compact spectral coefficients. The smooth intermediate derivatives have compact orbit support. Thus all pairings and finite Gram insertions used here lie in the previously established domains.

7.2 The smooth numerical mean in moving coefficients

Let M_N^{sm} be the mean computed with the unit vector $\Phi_{N,A}$. Changing variables to $\rho = Na(A) + r$ gives the exact fixed density

$$|F_{N,A}|^2 d\Pi = \omega(r) d^4 r, \qquad \omega(r) = \prod_i |\beta(r_i)|^2.$$

Consequently

$$M_{N,ij}^{\mathrm{sm}}(A) = \int \omega(r) \, \langle b_N, \mathscr{D}_{ij}(A, Na(A) + r) b_N \rangle_2 \, d^4 r, \tag{7.12}$$

where the horizontal operator has the form

$$\mathscr{D}_{ij}(A, Na + r) = N^2 V_{ij}(A,x) + N L_{1,ij}(A,r) + L_{2,ij}(A,r). \tag{7.13}$$

Here L_1 has order at most one and L_2 has order at most two. Their coefficients and first parameter jets are uniformly bounded. The center identity of Lemma 4.2 and its derivative read

$$V_{ij}(A,0) = A_{ij}, \qquad D_A V_{ij}(A,0)[H] = H_{ij}.$$

The mixed jets therefore give

$$|V_{ij}(A,x) - A_{ij}| \le C|x|, \qquad |D_A V_{ij}(A,x)[H] - H_{ij}| \le C|x|\|H\|_F. \tag{7.14}$$

At fixed (r,x), differentiating (7.12) uses $\partial_A \mathscr{D}|_{\rho,x}[H] + N \sum_i D_A a_i[H] \, \partial_{\rho_i} \mathscr{D}$, where the first term fixes (ρ, x) in the transported coefficients. For example, ρ_i^2 gives $2N\rho_i D_A a_i[H]$ with leading term $N^2 H_{ii}$, which requires spectral transport.

Since the support has $|x| \le N^{-1/2}$ and $\|\partial_x^k b_N\|_2 = O(N^{k/2})$, equations (7.13)–(7.14) yield

$$\left\| (\mathscr{D}_{ij}(A, Na + r) - N^2 A_{ij}) b_N \right\|_{C^1(\mathcal{K}; L^2(\omega\, dr\, d^2x))} \le C N^{3/2}. \tag{7.15}$$

Pairing with the fixed unit function b_N proves

$$\|N^{-2} M_N^{\mathrm{sm}} - \mathrm{id}\|_{C^1} \le C N^{-1/2}. \tag{7.16}$$

For the trace, put $\mu_k = \int r^k |\beta(r)|^2 dr$. The exact diagonal formula gives, without assuming the profile is even or centered,

$$S_N^{\mathrm{sm}} = N^2 + \frac{2N\mu_1}{3} \sum_{i=1}^{3} a_i + \mu_2, \qquad \|N^{-2} S_N^{\mathrm{sm}} - 1\|_{C^1} \le C N^{-1}. \tag{7.17}$$

7.3 Differentiation of the original projection tail

Set $t = (P_{N^3} - I)u$ and $e = v_{N,A} - \Phi_{N,A} = F_{N,A} t$. For $0 < \epsilon < 1$, the strict Sobolev embedding on the eight-dimensional compact manifold Q, with index $k + 4 + \epsilon/3$, and the complement eigenvalue bound for the tensor cutoff imply

$$\|t\|_{C^k} \le C_\epsilon N^{3k-8+\epsilon}, \qquad \|D_A^\rho t[H]\|_{C^k} \le C_\epsilon N^{3k-7+\epsilon} \|H\|_F, \quad 0 \le k \le 2. \tag{7.18}$$

We used (7.6) and the fact that P_{N^3} is independent of A. The triangular differential bound

$$\|O_{ij} h\|_\infty \le C(N^2 \|h\|_\infty + N \|h\|_{C^1} + \|h\|_{C^2})$$

is applied directly; no commutation of O_{ij} with the cutoff is used. Combining (7.18), (7.10) and the supremum-to-eta majorant gives

$$\begin{aligned} \|e\|_\eta &\le C_\epsilon N^{-8+\epsilon}, & \|O_{ij} e\|_\eta &\le C_\epsilon N^{-2+\epsilon}, \\ \|D_A^\rho e[H]\|_\eta &\le C_\epsilon N^{-7+\epsilon} \|H\|_F, & \|O_{ij} D_A^\rho e[H]\|_\eta &\le C_\epsilon N^{-1+\epsilon} \|H\|_F. \end{aligned} \tag{7.19}$$

For instance $D_A e = (D_A F)t + F(D_A t)$; both terms cost exactly one additional power of N. These are strict, nonendpoint estimates. We may fix $\epsilon = 1/2$ in all applications.

7.4 Transfer by eta symmetry and proof of the theorem

Fix one Gram entry and abbreviate $m = \eta(\Phi, O\Phi)$, $Z = \eta(v, v)$, $\Delta m = \eta(v, Ov) - m$. Symmetry on the common domain gives the exact identity

$$\Delta m = 2 \operatorname{Re} \eta(e, O\Phi) + \eta(e, Oe). \tag{7.20}$$

This places the operator on the smooth vector in the cross term. Using Lemma 7.2, (7.19) and the product rule (7.11), we obtain

$$|\Delta m| \le C_\epsilon N^{-6+\epsilon}, \qquad \|D_A \Delta m\| \le C_\epsilon N^{-5+\epsilon}, \tag{7.21}$$

$$|Z - 1| \le C_\epsilon N^{-8+\epsilon}, \qquad \|D_A Z\| \le C_\epsilon N^{-7+\epsilon}. \tag{7.22}$$

For clarity, the two largest derivative terms in the first line have bounds $N^{-7+\epsilon} N^2$ and $N^{-8+\epsilon} N^3$; the tail–tail terms are smaller. For the second line use $Z - 1 = 2 \operatorname{Re} \eta(e, \Phi) + \eta(e, e)$ and the exact identity $\|\Phi\|_\eta = 1$.

For common sufficiently large N, Z is bounded away from zero. Since $m, D_A m = O(N^2)$ by (7.16), the formula

$$M_{N,ij} - m = \frac{\Delta m - m(Z - 1)}{Z}$$

proves

$$\|M_N - M_N^{\mathrm{sm}}\|_{C^0} \le C_\epsilon N^{-6+\epsilon}, \qquad \|M_N - M_N^{\mathrm{sm}}\|_{C^1} \le C_\epsilon N^{-5+\epsilon}. \tag{7.23}$$

Equations (7.16), (7.17) and (7.23) imply (7.1) and (7.2). Write $\widetilde{M} = N^{-2}M_N$ and $\widetilde{S} = N^{-2}S_N$. Since $\widetilde{S} \ge 1/2$ eventually,

$$B_N - A = \frac{(\widetilde{M} - A) - A(\widetilde{S} - 1)}{\widetilde{S}}.$$

The product and quotient rules prove (7.3), completing the proof of Theorem 7.1.

Remark 7.3 (Which residual is differentiated). The C^1 estimate (7.15) is in the moving coefficient space with fixed r and fixed formula $b_N(x)$. That eta-isometric identification depends on A. Differentiating the full ambient vector $(O - N^2A)\Phi$ at fixed physical configuration adds the transport and phase terms in (7.8), which can cost another factor N. We neither need nor assert an ambient $O(N^{3/2})$ bound for that residual derivative. The proof uses the moving formula only for the smooth numerical mean, and the separately derived ambient bounds only for projection transfer.

Corollary 7.4 (Injectivity and strict interior preimages)**.** *For all sufficiently large common N, B_N is bi-Lipschitz onto its image. If A_* is any fixed strict interior point of $\mathcal{K}$, then for all sufficiently large N there is exactly one $A \in \mathcal{K}$ such that $B_N(A) = A_*$.*

Proof. Put $E_N = B_N - \mathrm{id}$ and $d_N = \sup_A \|D_A E_N\|$. Theorem 7.1 gives $d_N \le CN^{-1/2} < 1$ eventually. A segment in the convex set $\mathcal{K}$ and the fundamental theorem of calculus give

$$(1 - d_N)\|A - A'\|_F \le \|B_N(A) - B_N(A')\|_F \le (1 + d_N)\|A - A'\|_F.$$

No assumption that $B_N(\mathcal{K}) \subseteq \mathcal{K}$ is required. Let the distance of A_* to the boundary of this ball in its trace-three affine space be $d > 0$. Once $\sup_A \|E_N(A)\|_F < d$, the map $T(A) = A_* - E_N(A)$ takes $\mathcal{K}$ into itself and is a contraction. Its unique fixed point is the asserted preimage. The threshold is existential and depends on the unchanged profiles and chart constants; no numerical value of the minimum admissible N is certified. □

8 Principal-series comparison and scalar channels

The quotient in Proposition 2.2 has a concrete principal-series realization. We identify its normalization, establish the integral formulas on the original test domain, and transport all six Gram operators. The scalar internal channel is then a closed subspace of this realization. Its projection is an additional operation on the states of Theorem 3.2.

8.1 The full shape space and a fixed Ward gauge

Throughout this section, inner products are antilinear in their first argument. Put

$$Z = \mathbb{C} \setminus \{0, 1\}, \qquad \delta_i = 1 - i\rho_i, \qquad h_i = \widetilde{h}_i = \delta_i/2, \qquad D(z) = 1 + |z|^2. \tag{8.1}$$

In particular the two chiral weights are equal, not complex conjugates. The compact-to-plane unitary determined by the inverse action in Section 2 is

$$(U_4 u)(z_1, \dots, z_4) = \pi^{-2} \prod_{i=1}^{4} D(z_i)^{-\delta_i} u(n(z_1), \dots, n(z_4)), \qquad \frac{d\Omega}{4\pi} = \frac{d^2 z}{\pi D(z)^2}. \tag{8.2}$$

Use the ordered cross ratio

$$w = \frac{(z_4 - z_1)(z_2 - z_3)}{(z_4 - z_3)(z_2 - z_1)}.$$

A covariant four-point structure is written $P_0(z_i)F(w,\bar w)$, where

$$\begin{aligned} P_0(z_i) &= \prod_{i<j}|z_i-z_j|^{2t_{ij}},\\ (t_{12},t_{13},t_{14},t_{23},t_{24},t_{34}) &= (-2h_1,0,0,h_1-h_2-h_3+h_4,\\ &\qquad h_1-h_2+h_3-h_4,-h_1+h_2-h_3-h_4). \end{aligned} \tag{8.3}$$

The relation with the coarea shape coordinate of Proposition 2.2 is the fixed multiplier

$$S_\rho(w)=2^{\delta_2}(1+|w|^2)^{\delta_4-1}|1-w|^{\delta_1-\delta_2+\delta_3-\delta_4},\qquad |S_\rho(w)|=2. \tag{8.4}$$

Powers of positive real quantities in these formulas use the real logarithm.

Proposition 8.1 (The completed shape image)**.** *The map I_ρ induces the unitary identification*

$$\mathcal{H}_\eta \simeq \int_{(0,\infty)^4}^{\oplus} L^2(Z,d^2w)\,d\Pi(\rho),\qquad d\Pi(\rho)=\prod_{i=1}^4 4\rho_i^2\,d\rho_i. \tag{8.5}$$

Both signs of $\operatorname{Im} w$ *occur. In the conformal quotient normalization of [10], the map*

$$f\longmapsto 4P_0S_\rho^{-1}f \tag{8.6}$$

is an onto isometry from a shape fiber to the Hilbert space of covariant four-point structures with external dimensions δ_i.

Proof. The image assertion is (2.17); its compact-orbit lifts and Proposition 2.4 give the same completion from either original test domain. For the normalization, fix a finite triple $p=(p_1,p_2,p_3)$ and set $\Delta_p=(p_1-p_2)(p_2-p_3)(p_3-p_1)$. Direct substitution of (8.3)–(8.4) in the original coarea formula gives, for $F_f=P_0S_\rho^{-1}f$,

$$\|f\|_2^2=4\int_{\mathbb{C}}|\Delta_p|^2|F_f(p_1,p_2,p_3,z)|^2\,d^2z. \tag{8.7}$$

The quotient pairing in [10, (2.11), (2.13), Appendix A] has factor $2^{-2}/\operatorname{vol}SO(1)=1/4$ in the same section. Its squared norm is therefore $1/16$ of (8.7). Covariance determines a structure almost everywhere from its section value, which also proves surjectivity of (8.6). □

8.2 Sewn kernels on the original test domain

Set $\alpha_i=\bar\alpha_i=i\rho_i$, and label the internal principal series by

$$a=\nu+is,\qquad \bar a=-\nu+is,\qquad \nu\in\mathbb{Z},\quad s>0,\quad \kappa=\nu^2+s^2. \tag{8.8}$$

Here the bar denotes the second exponent, not complex conjugation. For exponent pairs, write $[z]^\alpha=z^\alpha\bar z^{\bar\alpha}$; the integer difference $\alpha-\bar\alpha$ makes this single-valued. Specializing the ordered kernel of [9, (17), (21)], with its unit scalar prefactor, gives

$$\begin{aligned} W(\alpha_i,\alpha_j,a;z_i,z_j,y) &= [z_j-z_i]^{-(1+\alpha_i+\alpha_j+a)/2}[y-z_i]^{-(1+\alpha_i-\alpha_j-a)/2}\\ &\quad\times[z_j-y]^{-(1-\alpha_i+\alpha_j-a)/2},\\ \overline{W(\alpha_i,\alpha_j,a)} &= W(-\alpha_i,-\alpha_j,-a). \end{aligned} \tag{8.9}$$

These ordered powers retain the spin phases. Define

$$K_{\rho,a}(z_i)=\int_{\mathbb{C}}W(-\alpha_1,-\alpha_2,-a;z_1,z_2,y)W(-\alpha_3,-\alpha_4,a;z_3,z_4,y)\,d^2y, \tag{8.10}$$

$$K^c_{\rho,a}(q)=\prod_i D(z_i)^{\delta_i}K_{\rho,a}(z_i),\qquad \Xi_{\rho,a}(w)=S_\rho(w)\frac{K_{\rho,a}(z_i)}{P_0(z_i)}. \tag{8.11}$$

The last quotient depends only on $w,\bar w$. The first internal leg has dimension $1+is$ and spin ν; the second is its dual.

Lemma 8.2 (Absolute sewing)**.** *With* $dm_4 = \prod_i d\Omega_i/(4\pi)$,

$$\|K^c_{\rho,a}\|_{L^1(dm_4)} \le \pi^5. \tag{8.12}$$

For every bounded original angular test, the five-point sewn integral is absolutely convergent and defines

$$c_u(\rho;\nu,s) = \int_{\mathbb{C}^4} \overline{K_{\rho,a}(z_i)}(U_4u)(z_i)\, d^8z = \pi^2 \int_{(S^2)^4} \overline{K^c_{\rho,a}(q)}u(q)\, dm_4(q), \tag{8.13}$$

$$|c_u(\rho;\nu,s)| \le \pi^7\|u_\rho\|_\infty, \qquad c_u(\rho;\nu,s) = \pi^2\langle \Xi_{\rho,a}, I_\rho u\rangle_{L^2(Z)}. \tag{8.14}$$

The last integral is absolutely convergent on this test domain.

Proof. On the unitary line the modulus of (8.9) is $(|z_i - z_j||z_i - y||z_j - y|)^{-1}$. For the full chordal distance $d_{ij} = |n_i - n_j|$, stereographic compactification therefore gives

$$|K^c_{\rho,a}(q)| \le 16 \int_{S^2} \frac{d\Omega_y}{d_{12}d_{1y}d_{2y}d_{34}d_{3y}d_{4y}}. \tag{8.15}$$

The triangle integral

$$J(y) = \int_{(S^2)^2} \frac{d\Omega_1 d\Omega_2}{d_{12}d_{1y}d_{2y}}$$

equals $2\pi^4$. To justify both its finiteness and this value, use $C_t(n,m) = (1 - 2t\, n\cdot m + t^2)^{-1/2}$, $0 < t < 1$. Its degree-ℓ spherical-harmonic eigenvalue is $4\pi t^\ell/(2\ell+1)$; hence the regularized triangle is

$$J_t(y) = 16\pi^2 \sum_{\ell\ge 0} \frac{t^{3\ell}}{(2\ell+1)^2} \longrightarrow 2\pi^4.$$

Fatou's lemma first gives $J(y) < \infty$. For $t \ge 1/2$, $C_t(n,m) \le \sqrt{2}/|n-m|$, so dominated convergence then proves equality. Tonelli's theorem applied to (8.15) yields

$$\frac{16}{(4\pi)^4} \int_{S^2} J(y)^2\, d\Omega_y = \pi^5.$$

This estimate holds with the absolute value taken before the y integration; it proves the required five-point Fubini statement as well. Equation (8.2) gives the factor π^2 in (8.13). Covariance and the fixed gauge give $W_\rho K^c_{\rho,a}(g,w) = \Xi_{\rho,a}(w)$. The absolute coarea change of variables then proves (8.14). □

For later use, let $T_{ij,a}f = \int_{\mathbb{C}^2} W(\alpha_i,\alpha_j,a)f$ and

$$(U_{ij}f)(z_i,z_j) = \pi^{-1}D(z_i)^{-\delta_i}D(z_j)^{-\delta_j}f(n(z_i),n(z_j)).$$

The same triangle estimate gives, for a bounded compact-picture pair test,

$$|(T_{ij,a}U_{ij}f)(y)| \le \frac{\pi^3}{1+|y|^2}\|f\|_\infty. \tag{8.16}$$

Consequently its literal pair transform is an $L^2(\mathbb{C}, d^2y)$ vector. If $u = \sum_r f_r(12)g_r(34)$, as for every finite-harmonic test, then absolute sewing reads

$$c_u(a) = \sum_r \int_{\mathbb{C}} (T_{12,a}U_{12}f_r)(y)(T_{34,-a}U_{34}g_r)(y)\, d^2y. \tag{8.17}$$

This is the bilinear pairing of contragredient representations. The formula requires no restriction of an arbitrary auxiliary L^2 field to a diagonal in two independent spectral variables.

8.3 Exact Plancherel normalization

We use two established harmonic-analysis inputs. The pair transform and its shadow-compatible completeness are [9, (17)–(29)]. For four-point structures we use the principal-series expansion and orthogonality in [10, (2.36)–(2.40)]. The latter explicitly excludes the possible additional discrete terms when all external representations are scalar principal series.

Theorem 8.3 (The normalized native transform)**.** *The coefficients* (8.13) *descend to eta classes and extend to an onto unitary*

$$\begin{aligned} \mathcal{A}_{12} &: \mathcal{H}_\eta \longrightarrow \mathscr{H}_{12}, \\ \mathscr{H}_{12} &= L^2\left((0,\infty)^4 \times \mathbb{Z} \times (0,\infty), d\Pi(\rho)\,\frac{\nu^2+s^2}{16\pi^8}\,ds\right), \\ (\mathcal{A}_{12}[u])(\rho;\nu,s) &= c_u(\rho;\nu,s). \end{aligned} \tag{8.18}$$

In particular, on the original test domain,

$$\eta(u,v) = \int d\Pi(\rho) \sum_{\nu\in\mathbb{Z}} \int_0^\infty \frac{\nu^2+s^2}{16\pi^8}\overline{c_u(\rho;\nu,s)}c_v(\rho;\nu,s)\,ds. \tag{8.19}$$

Proof. The $d=2$ measure of [10, (3.54)], for a signed one-dimensional $SO(2)$ character, is $\mu(1+is,\nu)=\kappa/(8\pi^3)$, with spectral differential $ds/(2\pi)$. Each of our raw three-point structures has quotient pairing $1/4$. Their partial-wave orthogonality therefore gives

$$\langle K_{\rho,a}, K_{\rho,b}\rangle_{\text{quotient}} = \frac{\pi^4}{\kappa}\delta_{\nu\nu'}\delta(s-s').$$

By Proposition 8.1, the corresponding shape norm has an additional factor sixteen. Orthogonality and the cited completeness thus give the onto shape transform

$$\begin{aligned} (\mathcal{T}_\rho f)(\nu,s) &= \langle \Xi_{\rho,a}, f\rangle, \\ \|f\|_2^2 &= \sum_\nu \int_0^\infty \frac{\nu^2+s^2}{16\pi^4}|(\mathcal{T}_\rho f)(\nu,s)|^2\,ds. \end{aligned} \tag{8.20}$$

These are generalized-kernel identities, and inversion holds in L^2.

Initially the coefficient pairing is immediate for compact smooth shape functions. For an original test u, choose smooth compact coarea cutoffs $0 \le \chi_R(g,w) \le 1$ tending to one and set

$$u_R(q) = \chi_R(g(q),w(q))u(q)$$

on the free chart, extended by zero. Then u_R is an original smooth test and $I_\rho u_R$ is compact and smooth in w. The absolute bound in Proposition 2.2 gives $A_u(\rho,w) = \int_G |W_\rho u(g,w)|\,dg \in L^2(d\Pi\,d^2w)$. Thus dominated convergence proves $Iu_R \to Iu$ in L^2. On the other hand, (8.12) and $|u_R| \le |u|$ give $c_{u_R} \to c_u$ pointwise in the native labels. The Hilbert transforms converge by (8.20); an almost-everywhere convergent subsequence identifies both limits. Thus $c_u = \pi^2\mathcal{T}_\rho I_\rho u$ on the actual domain. This supplies the additional π^4 in the squared-coefficient denominator of (8.19). The kernels are measurable in the external labels, as follows from the absolutely convergent power integrals and a dense collection of compact shape tests. The fiberwise unitaries consequently integrate with the unchanged measure $d\Pi$. Surjectivity follows from (8.5) and fiberwise completeness. □

Remark 8.4 (Shadow and scalar normalization)*.* Shadow equivalence sends (ν,s) to $(-\nu,-s)$. It does not remove one sign of ν at fixed $s>0$. For completeness, an explicit phase check uses

$$\mathfrak{a}(\alpha) = \frac{\Gamma(1-\bar\alpha)}{\Gamma(\alpha)}, \qquad A_{ij}(a) = \pi\,\frac{\mathfrak{a}((1+\alpha_i-\alpha_j-a)/2)\mathfrak{a}(1+a)}{\mathfrak{a}((1+\alpha_i-\alpha_j+a)/2)}.$$

The shadow identity of [9, (23)–(24)] implies, in the ordered convention above,

$$K_{\rho,-a} = \gamma_\rho(a) K_{\rho,a}, \quad c_u(-a) = \gamma_\rho(a)^{-1} c_u(a), \quad \gamma_\rho(a) = \frac{\kappa}{\pi^2} A_{12}(a) A_{34}(-a), \quad |\gamma_\rho(a)| = 1. \tag{8.21}$$

One obtains this by moving the two shadow operators through the bilinear pairing (8.17); their kernels are transpose-symmetric because 2ν is even. Thus a full cover $s \in \mathbb{R}$ has half the measure in (8.19) and the constraint (8.21). The auxiliary pair Plancherel density on $s > 0$ is instead $\kappa/(4\pi^4)$.

For $\nu = 0$, the four-point kernel of [11, (4.14), (6.1)], with $\Lambda = 1 + is$, is

$$D_{\mathrm{KL},\Lambda} = C(\Lambda, \delta_1, \delta_2) C(2 - \Lambda, \delta_3, \delta_4) K_{\rho,is}, \tag{8.22}$$

where

$$C(x,y,z) = \frac{\pi}{2} \frac{\Gamma((x+y+z-2)/2)\Gamma((x+y-z)/2)\Gamma((x+z-y)/2)\Gamma((y+z-x)/2)}{\Gamma(x)\Gamma(y)\Gamma(z)}.$$

Using this scalar kernel in place of $K_{\rho,is}$ changes the coefficient by the conjugate of the displayed factor and changes its spectral weight accordingly.

8.4 Gram insertions and all three pairings

The native transform must also identify the unbounded operators on the declared core. Let $C_i^{\mathrm{hol}} = h_i(h_i - 1)$ and denote by C_{ij}^{hol} the Casimir for the sum of the two chiral representations. The differential realization of the original Gram has

$$O_{ij}^{\mathrm{hol}} = -(C_{ij}^{\mathrm{hol}} - C_i^{\mathrm{hol}} - C_j^{\mathrm{hol}}), \qquad O_{ij} = O_{ij}^{\mathrm{hol}} + O_{ij}^{\mathrm{anti}}. \tag{8.23}$$

This fixes the sign and the additive constant. Write $\mathcal{Q}_{12}$ for $i(O_{12}^{\mathrm{hol}} - O_{12}^{\mathrm{anti}})$, a symmetric quadratic polynomial in the same derived generators. In the shape gauge it is $iS_\rho(D_{12} - \widetilde{D}_{12})S_\rho^{-1}$. Here $\widetilde{D}_{12}$ replaces w, ∂_w by $\bar{w}, \partial_{\bar{w}}$, keeping the same h_i.

Theorem 8.5 (Actual-core intertwining and recoupling)**.** *For $u \in \mathcal{D}_{\mathrm{fin}}$,*

$$\mathcal{A}_{12}(O_{12}u) = m_{12}\mathcal{A}_{12}u, \qquad m_{12}(\rho;\nu,s) = \frac{s^2 - \nu^2 - \rho_1^2 - \rho_2^2 - 1}{2}, \tag{8.24}$$

$$\mathcal{A}_{12}(\mathcal{Q}_{12}u) = \nu s\, \mathcal{A}_{12}u. \tag{8.25}$$

In particular the right sides belong to $\mathscr{H}_{12}$. The analogous unitary transforms $\mathcal{A}_{13}$ and $\mathcal{A}_{23}$, obtained by the indicated leg permutations, give recoupling unitaries

$$\mathcal{R}_{ij} = \mathcal{A}_{ij}\mathcal{A}_{12}^{-1}, \qquad ij = 13, 23.$$

In $\mathscr{H}_{12}$, define

$$\mathsf{O}_{ii} = M_{\rho_i^2}, \qquad \mathsf{O}_{12} = M_{m_{12}}, \qquad \mathsf{O}_{ij} = \mathcal{R}_{ij}^{-1} M_{m_{ij}} \mathcal{R}_{ij} \quad (ij = 13, 23). \tag{8.26}$$

With maximal multiplication domains these are self-adjoint extensions of the transported original symmetric Grams. They share the invariant dense subdomain $\mathcal{A}_{12}(\mathcal{D}_{\mathrm{fin}}/\ker\eta)$, which is preserved by all six actions and every fixed finite word in them.

Proof. At fixed external labels a finite-harmonic vector is a finite sum of pair tensors as in (8.17). Each pair factor is a smooth vector of the auxiliary unitary group representation. The unitary Clebsch–Gordan intertwiner therefore intertwines its derived enveloping-algebra action. Its literal integral equals that unitary transform: cut off the plane input, use auxiliary L^2

convergence, and use (8.16) for pointwise convergence; a subsequence identifies the limits. The same argument applies after a Casimir insertion.

The internal chiral eigenvalues are $(a^2-1)/4$ and $(\bar{a}^2-1)/4$. Substitution in (8.23) gives (8.24) and (8.25) before the absolutely convergent pair contraction. The inserted source vector is again in the original finite core. Applying Theorem 8.3 to it proves that each displayed multiplier is in spectral L^2, so this argument establishes actual multiplication-domain membership as well as a differential identity. Fixed-vector harmonic approximation from Proposition 2.4, and closedness of multiplication, extend the identities to the smooth core.

Relabeling the legs proves the other channel identities. Unitary conjugation of real multiplication is self-adjoint on its maximal domain; the core identities prove the extension statement. Invariance and the finite-word assertion follow from the corresponding original-core properties in Proposition 2.4. □

No essential self-adjointness of the minimal original operators is needed here. Nor does (8.26) imply that adjacent pair Grams commute. The construction selects explicit native extensions and proves their agreement on the original core.

Corollary 8.6 (Full-channel transport of the semiclassical family)**.** *Let $\zeta_{N,A} = \mathcal{A}_{12}\psi_{N,A}$, with the states of Theorem 3.2. Then*

$$\|\zeta_{N,A}\|_{\mathscr{H}_{12}} = 1, \qquad \|(\mathsf{O}_{ij} - N^2 A_{ij})\zeta_{N,A}\|_{\mathscr{H}_{12}} \le CN^{3/2} \tag{8.27}$$

uniformly for $A \in \mathcal{K}$ and $1 \le i \le j \le 3$.

Proof. The original once-projected vectors belong to $\mathcal{D}_{\mathrm{fin}}$. Apply the exact unitary and core identities to the residual in Theorem 3.2. □

8.5 Reflection, the scalar channel, and an even family

Let R reflect every sphere variable in the plane whose stereographic action is $z \mapsto \bar{z}$; it acts linearly on function values. The change $g \mapsto \bar{g}$ preserves Haar measure and intertwines the reflected inverse action. The spherical measure is also preserved. Consequently R is an eta-unitary involution, commutes with all six Grams, and commutes with every product spherical-harmonic cutoff. Write $P_+ = (1+R)/2$.

Proposition 8.7 (An actual even finite-harmonic family)**.** *For all sufficiently large N,*

$$\psi^+_{N,A} = \frac{P_+\psi_{N,A}}{\|P_+\psi_{N,A}\|_\eta} \tag{8.28}$$

is well-defined uniformly on $\mathcal{K}$, belongs to the original finite-harmonic domain, and satisfies all six estimates of Theorem 3.2, with a uniform change of constant. Moreover

$$\sup_{A\in\mathcal{K}} \left| \|P_+\psi_{N,A}\|_\eta^2 - \frac{1}{2} \right| \longrightarrow 0. \tag{8.29}$$

Proof. The central normals of Proposition 3.1 obey $\sum_i a_i(A)q_i(A) = 0$ with $a_i(A) > 0$, and $q_1(A), q_2(A), q_3(A)$ are linearly independent. Four such normals cannot lie on a circle. Indeed a circle lies in a plane $n \cdot q = b$, and positive weighted closure forces $b = 0$; this would contradict the independence of the first three normals. Thus their central cross ratio is nonreal. Continuity and compactness give a uniform positive lower bound on its absolute imaginary part. The shrinking shape support of the smooth packet $\Phi_{N,A}$ and its reflected support are therefore disjoint, uniformly for large N. By Proposition 2.2, $\eta(\Phi_{N,A}, R\Phi_{N,A}) = 0$.

The uniform tail estimate in Lemma 5.2 gives $\|v_{N,A} - \Phi_{N,A}\|_\eta = o(1)$, while $\|\Phi_{N,A}\|_\eta = 1$. Expanding this cross inner product and using the eta isometry of R shows $\langle \psi_{N,A}, R\psi_{N,A}\rangle_\eta = o(1)$

uniformly. Since P_+ is an orthogonal projection, this proves (8.29). Reflection preserves the finite-harmonic domain and commutes with the single cutoff already used to define $v_{N,A}$. Finally

$$(O_{ij} - N^2 A_{ij})\psi^+_{N,A} = \frac{P_+(O_{ij} - N^2 A_{ij})\psi_{N,A}}{\|P_+\psi_{N,A}\|_\eta},$$

which proves the residual estimates. □

Proposition 8.8 (The scalar subspace and its limits)**.** *The operator*

$$\Pi_0 = \mathcal{A}_{12}^{-1}\mathbf{1}_{\{\nu=0\}}\mathcal{A}_{12} \tag{8.30}$$

is a bounded orthogonal projection in $\mathcal{H}_\eta$. It is the kernel projection of the particular self-adjoint extension $\mathsf{Q}_{12} = \mathcal{A}_{12}^{-1}M_{\nu s}\mathcal{A}_{12}$. Its range is a proper closed subspace of the reflection-even eta space. For the original, unsymmetrized family,

$$\liminf_{N\to\infty}\inf_{A\in\mathcal{K}} \operatorname{dist}_\eta(\psi_{N,A}, \operatorname{ran}\Pi_0) \geq \frac{1}{\sqrt{2}}. \tag{8.31}$$

Proof. The multiplier construction proves boundedness and orthogonality. For $\nu \neq 0$, the zero set of νs has spectral measure zero, which proves the kernel statement. Reflection interchanges the two chiral exponents in (8.9), sending ν to $-\nu$ at fixed s. Scalar kernels are even; hence $\Pi_0 P_+ = P_+\Pi_0 = \Pi_0$.

Properness can also be seen directly on smooth tests. In the Ward gauge the holomorphic pair operator is

$$D_{12} = w(1-w)\partial_w^2 + [1 + h_1 - h_2 - h_3 + h_4 - 2(1 + h_1 - h_2)w]\partial_w + 2h_1(h_2 - 1). \tag{8.32}$$

Choose a reflection-even compact smooth cutoff χ in Z, equal to one on a nonreal open patch and its reflection, and put $F = \chi(w + \bar{w})$. On that patch,

$$(D_{12} - \widetilde{D}_{12})F = -[\,2 - i(\rho_1 - \rho_2)\,](w - \bar{w}) \neq 0. \tag{8.33}$$

Then $S_\rho F$, with compact smooth external-label support, has a compact-orbit lift as in Proposition 8.1; the lift may be symmetrized. It is in the smooth original domain, and Theorem 8.5 puts it in $\operatorname{Dom}\mathsf{Q}_{12}$ with nonzero image. It is therefore an even eta vector outside $\operatorname{ran}\Pi_0$.

Finally $\|\Pi_0\psi_{N,A}\|_\eta \leq \|P_+\psi_{N,A}\|_\eta$. The Pythagorean identity for Π_0 and (8.29) imply (8.31). □

The distance bound concerns $\psi_{N,A}$, not the even family $\psi^+_{N,A}$. The example (8.33) proves a proper-range statement; it does not estimate the distance of that even family from the scalar subspace.

8.6 A sufficient condition for scalar-projected families

Projection onto a fixed channel is a further construction. The following criterion isolates exactly the additional estimates it would require, without adding them as assumptions to the preceding theorems. All operators in this paragraph are the native extensions of Theorem 8.5, pulled back to $\mathcal{H}_\eta$. Let j range over the six entries $11, 22, 33, 12, 13, 23$.

Proposition 8.9 (Normalized projection and finite-core return)**.** *Suppose that the scalar mass satisfies*

$$p_{N,A} = \|\Pi_0\psi^+_{N,A}\|_\eta > 0$$

and that $\Pi_0\psi^+_{N,A}$ belongs to all six Gram domains. A sufficient uniform estimate for the normalized projections $\varphi_{N,A} = \Pi_0\psi^+_{N,A}/p_{N,A}$ is

$$\sup_{A\in\mathcal{K}} \frac{\|\Pi_0(O_j - N^2A_j)\psi^+_{N,A}\|_\eta + \|[O_j,\Pi_0]\psi^+_{N,A}\|_\eta}{p_{N,A}} \le CN^{3/2} \qquad \text{for every } j. \tag{8.34}$$

No uniform positive lower bound on $p_{N,A}$ is required. If, in addition, for each N the map

$$A \longmapsto (\varphi_{N,A}, O_1\varphi_{N,A}, \dots, O_6\varphi_{N,A})$$

is continuous and takes values in the closure in $\mathcal{H}_\eta^7$ of

$$\{(u, O_1u, \dots, O_6u) : u \in \mathcal{D}_{\mathrm{fin}}/\ker\eta\},$$

then there is a normalized finite-harmonic family $\widehat{\varphi}_{N,A}$ satisfying the same uniform $O(N^{3/2})$ six-Gram residual estimate. For each N, one common finite harmonic cutoff and one common compact external-label set suffice for all $A \in \mathcal{K}$.

Proof. On the stated domains,

$$(O_j - N^2A_j)\varphi_{N,A} = \frac{\Pi_0(O_j - N^2A_j)\psi^+_{N,A} + [O_j,\Pi_0]\psi^+_{N,A}}{p_{N,A}}. \tag{8.35}$$

This proves the first assertion. For the second, the continuous image of compact $\mathcal{K}$ is compact in the joint graph norm. Approximate a finite cover of this image by vectors from $\mathcal{D}_{\mathrm{fin}}/\ker\eta$, and use a subordinate partition of unity on $\mathcal{K}$. The resulting finite combination $u_{N,A}$ has joint graph error at most $\delta_N = N^{-2}$, uniformly in A. The finitely many approximants have a common finite harmonic bound and compact label support. Since $\mathcal{K}$ is bounded,

$$\|(O_j - N^2A_j)(u_{N,A} - \varphi_{N,A})\|_\eta \le (1 + N^2 \sup_{\mathcal{K}} |A_j|)\delta_N = O(1).$$

Moreover $\|u_{N,A}\|_\eta \ge 1 - \delta_N$. Normalizing $u_{N,A}$ proves the assertion. □

The diagonal Grams and O_{12} commute with Π_0; the adjacent channels 13 and 23 require the estimates in (8.34). Neither positive scalar mass with these normalized estimates nor the joint graph-closure condition has been established here for $\psi^+_{N,A}$. A small mixed-Casimir residual alone cannot supply them: the multiplier νs has no gap on the complement of $\nu = 0$. The finite-core conclusion of Proposition 8.9 is existential. It imposes no N^3 cutoff rate, preserves no specified packet profile, and need not preserve exact scalar-channel membership. These distinctions do not affect the original family theorem, its even variant, or their full-channel unitary transport.

9 Scope and further questions

The main theorem is a statement in the explicitly defined one-Haar quotient $\mathcal{H}_\eta$. Its states are actual finite harmonic test vectors before passage to that quotient. Their external spectral support moves to scale N, while its profile is fixed. The angular cutoff is applied once to construct the states; the observables are the original uncompressed Gram operators. This distinction is what allows the projection error and the operator domain to be controlled separately.

The full-spin transform of Theorem 8.3 is an isometry of this mathematical quotient. It preserves the norm of every residual in the main theorem and identifies all six Gram actions on the specified common domain. The maximal multiplication operators and their unitary channel transforms give self-adjoint realizations. Equality of their maximal domains with the graph

closures of the original minimal operators, and essential self-adjointness on that original domain, are stronger assertions not established here.

Scalar internal spin and reflection parity are also different conditions. Proposition 8.7 constructs an actual even family with the same semiclassical order, whereas the spin-zero projection is a proper subspace of the even quotient. For the unsymmetrized family, the distance bound (8.31) rules out an interpretation as an asymptotically scalar vector. For the even family, the remaining question is the normalized projected residual: both the surviving mass and the adjacent-channel residual relative to that mass matter. Proposition 8.9 states sufficient conditions, including the required joint graph-domain condition. The present paper does not replace those conditions by a parity argument.

The distinction from a physical source space is equally concrete. For example, the Lorentzian Barrett–Crane group field theory discussed in Ref. [14] introduces simplicity and right covariance in Section 2.1, and a further adjoint or left invariance in Sections 3.3–3.4. Its reduced spin representation retains an internal (ρ_{12}, ν_{12}) sum. Such additional constraints and reductions require their own intertwining, domain and measure analysis. The mathematical results above do not automatically identify our vectors with an admissible source sector of that model, nor do they derive a gravitational dynamics or an absolute physical metric scale.

There are useful quantitative questions within the mathematical model itself. The proofs give an existential common threshold N_0, with its dependence on fixed chart and profile constants exposed through the estimates, but do not certify a numerical minimum. The $N^{-1/2}$ rate is sufficient for the uniform geometry and injectivity results; optimality is not asserted. The fixed-support obstruction in Proposition 6.7 explains why the moving spectral support is necessary for the scale divergence within this type of construction. The small-label obstruction in Proposition 6.8 and the exact negative eigenvalue of the four-leg mean matrix provide additional consistency checks without extending those controls into unrestricted no-go theorems.

10 Conclusion

We have constructed a uniform finite-harmonic semiclassical family for six Lorentzian Gram insertions on a positive one-Haar quotient, over a fixed compact five-dimensional set of shapes. The proof includes the actual orbit geometry, common domain, phase-center identity, smooth residual estimate and the tail of the original angular projector. The same family yields positive numerical mean matrices, vanishing relative fluctuations, a stable classical cofactor readout, and an eventually injective C^1 shape map. Its full-spin conformal partial-wave description is obtained with an explicit Parseval normalization and an operator-domain comparison. The scalar-projection and physical-source questions have precise remaining conditions and are not premises of these results.

Declaration of AI assistance

OpenAI ChatGPT was used extensively in developing and checking mathematical arguments, drafting and editing the manuscript, locating and checking bibliographic sources, and preparing computational checks. The human author directed the research and approved the scope and limitations of the mathematical claims in the earlier manuscript. This editorial revision retains that scientific scope. The checks undertaken in preparing the manuscript include internal AI-assisted analytic reviews and finite symbolic and exact-arithmetic checks. They do not constitute independent human peer review or full machine-formal verification of the proofs. The author is responsible for the content and the decision to submit.

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