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MAXIMAL SPATIAL FACTORS OF INDEPENDENT PRODUCTS AND UNIFORM WHIRLINESS

Abstract. We establish three results on whirly and uniformly whirly probability-preserving near-actions of Polish groups. First, for any finite or countable independent product of near-actions of a Polish group on standard probability spaces, the maximal spatial factor is the product of the maximal spatial factors (proved via the σ-algebra of bounded norm-continuous vectors); in particular, independent products of whirly near-actions are whirly, answering Problem 1 of Glasner and Weiss (2005). By contrast, a relatively independent square over a nonatomic spatial factor can acquire a strictly larger maximal spatial factor. Second, answering a question of Wu (2017), we construct a rigid whirly transformation whose cyclic closure is a Lévy group, yet which fails to be uniformly whirly even on pairs of sets of measure 1/2. Third, we construct a uniformly whirly near-action of a Polish group whose independent square is not uniformly whirly.

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1. Introduction

A classical theorem of Mackey [9] and Varadarajan [13] states that every continuous measure-preserving action of a locally compact Polish group on the measure algebra of a standard probability space (X, μ)—that is, every continuous homomorphism $G \to \mathrm{Aut}(X, \mu)$ into the automorphism group of (X, μ) equipped with its weak topology, called a *near-action*—admits a *spatial model*, meaning a jointly Borel pointwise action $G \times X \to X$ preserving μ that induces the same automorphisms modulo null sets. For non-locally-compact Polish groups, this correspondence fails in general. Glasner, Tsirelson, and Weiss [4] proved that every probability-preserving spatial action of a *Lévy group*—a Polish group approximated by an increasing dense union of compact subgroups exhibiting the concentration of measure phenomenon of Gromov and Milman [6] (Definition 2.5)—is trivial modulo null sets (its set of common fixed points has full measure). Some Lévy groups nevertheless admit nontrivial ergodic Gaussian near-actions.

To describe this failure of spatial realizability in dynamical terms, Glasner, Tsirelson, and Weiss [4] introduced *whirly* near-actions: probability-preserving near-actions of a Polish group G on (X, μ) in which arbitrarily small elements of G can move any positive-measure set to intersect any other positive-measure set, that is, for every identity neighborhood $U \subseteq G$ and all measurable sets $A, B \subseteq X$ of positive measure there exists $g \in U$ with $\mu(A \cap gB) > 0$. Writing $U_g f = f \circ g^{-1}$ for the Koopman operator of $g \in G$, Glasner and Weiss [5] showed that every probability-preserving near-action admits a unique *maximal spatial factor*, whose σ-algebra

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$\mathcal{D}_X = \sigma(\mathcal{C}_G(X))$ is generated by the space of bounded *norm-continuous vectors*

$$\mathcal{C}_G(X) = \{f \in L^\infty(X,\mu) : \lim_{g\to e} \|U_g f - f\|_\infty = 0\},$$

and characterized whirly near-actions as precisely those having no nontrivial spatial factor (equivalently, $\mathcal{C}_G(X)$ consists only of constant functions). Problem 1 in Section 6 of [5] asks whether the independent product of two whirly actions of a Polish group is whirly.

Our first main theorem answers this question affirmatively for finite and countable *independent products* $(\prod_{i\in I} X_i, \bigotimes_{i\in I} \mu_i)$ endowed with the coordinatewise *diagonal action* of G, without assuming that the factors are whirly, by determining the maximal spatial factor of any independent product.

Theorem 1.1. *Let G be a topological group, let I be a nonempty finite or countable index set, and for each $i \in I$ let $G \to \mathrm{Aut}(X_i,\mu_i)$ be a probability-preserving near-action on a standard probability space. Equip $X = \prod_{i\in I} X_i$ with the product probability measure $\mu = \bigotimes_{i\in I}\mu_i$ and the diagonal action of G. Then this defines a continuous probability-preserving near-action, and modulo μ-null sets*

$$\mathcal{D}_X = \bigotimes_{i\in I} \mathcal{D}_{X_i}.$$

In particular, when G is a Polish group, the maximal spatial factor of the independent product (X,μ) is the independent product of the maximal spatial factors of the (X_i,μ_i).

Specializing to the case where every factor has trivial maximal spatial factor yields an affirmative answer to Glasner and Weiss's Problem 1.

Corollary 1.2. *Let G be a Polish group and let I be a nonempty finite or countable index set. If $G \to \mathrm{Aut}(X_i,\mu_i)$ $(i \in I)$ are whirly probability-preserving near-actions on standard probability spaces, then the diagonal near-action on the independent product $(\prod_{i\in I} X_i, \bigotimes_{i\in I}\mu_i)$ is whirly.*

The proof of Theorem 1.1 uses *weighted partial integrals* (slices)

$$(S_h F)(x) = \int_Y F(x,y)h(y)\,d\nu(y) \qquad (h \in L^1(Y,\nu)).$$

Using L^1-continuity of the Koopman representation on Y, we show that slicing sends $\mathcal{C}_G(X\times Y)$ into $\mathcal{C}_G(X)$. Applying this to each coordinate of a finite or countable product forces every $F \in \mathcal{C}_G(X)$ to be fixed by every coordinatewise conditional expectation onto $\mathcal{D}_{X_i}$, and hence to be $\bigotimes_{i\in I}\mathcal{D}_{X_i}$-measurable.

It is natural to ask whether Theorem 1.1 extends to relatively independent products over a common spatial factor. We show in Proposition 3.2 that independence cannot be omitted: there is a continuous near-action of $G = L^0([0,1),\{-1,1\})$ on $X = [0,1)\times\{-1,1\}$ whose maximal spatial factor is the base $Z = [0,1)$, yet the relatively independent square $X \times_Z X$ acquires a nonconstant G-invariant function independent of Z and thus a strictly larger maximal spatial factor.

The second half of the article investigates a quantitative strengthening of whirliness introduced by Wu [14] in connection with whirly 3-interval exchange transformations. An automorphism $T \in \mathrm{Aut}(X,\mu)$ of a standard atomless probability space is *rigid* if there exist integers $n_j \to \infty$ with $T^{n_j} \to \mathrm{Id}$ in the weak topology. For a rigid T, whirliness of its *cyclic closure* $\Lambda(T) = \overline{\{T^n : n \in \mathbb{Z}\}} \le \mathrm{Aut}(X,\mu)$ means that for every weak identity neighborhood $U \subseteq \mathrm{Aut}(X,\mu)$ and all positive-measure sets $A, B \subseteq X$, there is an integer n with $T^n \in U$ and $\mu(T^n A \cap B) > 0$. Wu called T *uniformly whirly* if the supremum $\sup_{n\in\mathbb{Z}:\, T^n\in U} \mu(T^n A \cap B)$ stays bounded away from 0 uniformly over all measurable sets A, B of fixed positive measures $\alpha, \beta \in (0,1)$, and asked [14, after Definition 1.12] whether uniform whirliness is equivalent to whirliness.

This notion extends verbatim to any probability-preserving near-action of a Polish group G on a standard atomless probability space (X,μ) via the *uniform hitting capacity*

$$\kappa_U^X(\alpha,\beta) = \inf_{\substack{\mu(A)=\alpha\\ \mu(B)=\beta}} \sup_{g\in U} \mu(A\cap gB) \qquad (0<\alpha,\beta<1),$$

calling the near-action *uniformly whirly* if $\kappa_U^X(\alpha,\beta)>0$ for every identity neighborhood $U\subseteq G$ and all $\alpha,\beta\in(0,1)$. Our second main theorem answers Wu's question negatively, showing that having a Lévy cyclic closure does not imply uniform whirliness.

Theorem 1.3. *On a standard atomless probability space (X,μ), there exists a rigid whirly transformation $T\in \mathrm{Aut}(X,\mu)$ whose cyclic closure $\Lambda(T)$, equipped with the weak topology induced from $\mathrm{Aut}(X,\mu)$, is a Lévy group, and yet for some weak identity neighborhood $U\subseteq \mathrm{Aut}(X,\mu)$,*

$$\inf_{\mu(A)=\mu(B)=1/2} \sup_{n\in\mathbb{Z}:\, T^n\in U} \mu(T^nA\cap B) = 0.$$

In particular, T is not uniformly whirly.

The construction in Theorem 1.3 balances two competing scales inside a monothetic abelian Polish group G_w, obtained by completing the group of dyadic step functions $[0,1)\to\mathbb{T}$ under a weighted dyadic-energy metric D_w. When the weights w_m grow slowly enough ($2^{-n/2}\sum_{m<n} w_m\to 0$), Efron–Stein concentration on the dyadic tori $\mathbb{T}^{2^n}$ makes G_w a Lévy group, so every ergodic near-action of G_w is whirly by Glasner–Tsirelson–Weiss [4, 5]. When $w_m\to\infty$, elements in a D_w-ball stay uniformly close to their dyadic conditional expectations E_nf. We embed G_w as a closed subgroup of $\mathrm{Aut}(\Omega,\gamma)$ via a Gaussian realization that combines a faithful weakly mixing representation (from the Gaussian kernel $e^{-D_w^2}$) with multiplication on $L^2([0,1);\mathbb{C})$; in this realization, the median half-sets of the rank-2^n dyadic projections are uniformly almost invariant on every D_w-ball. Picking a topological generator t of G_w yields T.

Finally, in view of Corollary 1.2, it is natural to ask whether uniform whirliness itself is preserved by taking independent products. Our third main theorem shows that it is not, even for the independent square of a single near-action.

Theorem 1.4. *There exist a Polish group G and a uniformly whirly probability-preserving near-action of G on a standard atomless probability space (X,μ) such that the diagonal near-action of G on the independent square $(X\times X,\mu\times\mu)$ satisfies*

$$\kappa_V^{X\times X}\left(\tfrac12,\tfrac12\right)=0$$

for some open identity neighborhood $V\subseteq G$. In particular, the independent square fails to be uniformly whirly, although it is whirly by Corollary 1.2.

In the construction for Theorem 1.4, we let $G=(C\rtimes \mathrm{Aut}(Y,\nu))\times K$ act on $X=Y\times K$, where $Y=[0,1)$, $K=\{-1,1\}^{\mathbb{N}}$ is the Cantor group, and C is a weighted subgroup of $L^0(Y,K)$ whose coordinates become increasingly rigid in the topology of C. On a single copy of X, any identity neighborhood of G contains a subgroup leaving $S\times K$ invariant (for a suitably chosen small base set $S\subseteq Y$) and acting ergodically on $S\times K$—using base automorphisms and cocycles supported on S for the first M fiber coordinates together with global translations from K on the tail coordinates $j>M$—which yields a uniform lower bound on $\kappa_U^X(\alpha,\beta)$. On the independent square $X\times X$, however, the global fiber translations cancel in the relative coordinates z_nz_n', leaving those coordinates driven solely by the increasingly rigid cocycle c_n.

Organization. Section 2 sets up the definitions, background results, and the reduction from rigid transformations to their cyclic closures. Sections 3, 4, and 5 prove Theorems 1.1 (with Corollary 1.2 and Proposition 3.2), 1.3, and 1.4, respectively.

2. Preliminaries: near-actions, spatial factors, and whirliness

Throughout this article, (X, μ) denotes a *standard probability space*, that is, a standard Borel space equipped with a Borel probability measure μ; we omit the Borel σ-algebra from the notation when no confusion can arise. A probability measure μ is *atomless* (or *nonatomic*) if $\mu(\{x\}) = 0$ for every $x \in X$. All sets, functions, and σ-algebras on (X, μ) are considered modulo μ-null sets unless stated otherwise, and we write $L^p(X, \mu)$ $(1 \le p \le \infty)$ for the usual spaces of complex-valued functions (writing $L^p(X, \mu; \mathbb{R})$ when real-valued functions are required).

Automorphisms, near-actions, and spatial factors. Let $\mathrm{Aut}(X, \mu)$ be the group of all measure-preserving Borel automorphisms of (X, μ), where two automorphisms are identified if they agree μ-almost everywhere. Equipped with the *weak topology*—in which a net (or sequence) (T_ι) converges to T if and only if

$$\mu(T_\iota A \triangle TA) \longrightarrow 0 \qquad \text{for every measurable } A \subseteq X,$$

where $A \triangle B = (A \setminus B) \cup (B \setminus A)$—the group $\mathrm{Aut}(X, \mu)$ is a Polish group (a separable, completely metrizable topological group); see [8]. A basis of identity neighborhoods in $\mathrm{Aut}(X, \mu)$ is given by the sets

$$\mathcal{U}(\mathcal{P}, \eta) = \Big\{ T \in \mathrm{Aut}(X, \mu) : \sum_{B \in \mathcal{P}} \mu(B \triangle TB) < \eta \Big\},$$

as $\mathcal{P}$ ranges over finite measurable partitions of X and $\eta > 0$.

Each $T \in \mathrm{Aut}(X, \mu)$ induces its *Koopman operator* U_T on $L^p(X, \mu)$ $(1 \le p \le \infty)$ by $(U_T f)(x) = f(T^{-1}x)$. On $L^2(X, \mu)$ the assignment $T \mapsto U_T$ is a unitary representation, and weak convergence $T_\iota \to T$ in $\mathrm{Aut}(X, \mu)$ is equivalent to strong operator convergence $U_{T_\iota} \to U_T$ on $L^2(X, \mu)$ (and also to strong operator convergence on $L^1(X, \mu)$): indeed, for indicators $\|U_{T_\iota} \mathbf{1}_A - U_T \mathbf{1}_A\|_1 = \|U_{T_\iota} \mathbf{1}_A - U_T \mathbf{1}_A\|_2^2 = \mu(T_\iota A \triangle TA)$, and simple functions are dense in both L^1 and L^2 while the Koopman operators are isometries.

Definition 2.1 (Near-actions and spatial actions)**.** Let G be a topological group and (X, μ) a standard probability space.

(i) A *probability-preserving near-action* (or simply *near-action*) of G on (X, μ) is a continuous group homomorphism $\alpha : G \to \mathrm{Aut}(X, \mu)$. When α is understood, we abbreviate $\alpha(g)A$ to gA and write $U_g f = f \circ \alpha(g)^{-1}$ for the associated Koopman operators; we call the near-action *ergodic* if every measurable set $A \subseteq X$ satisfying $\mu(gA \triangle A) = 0$ for all $g \in G$ has $\mu(A) \in \{0, 1\}$.

(ii) When G is a Polish group, a *probability-preserving spatial action* of G on (X, μ) is a jointly Borel map $G \times X \to X$, $(g, x) \mapsto g \cdot x$, satisfying the group action laws $e \cdot x = x$ and $(gh) \cdot x = g \cdot (h \cdot x)$ at every point $x \in X$, and preserving μ ($g_*\mu = \mu$ for every $g \in G$). Every spatial action induces a continuous near-action $G \to \mathrm{Aut}(X, \mu)$; see [4, 8]. A near-action is *spatial* (or *admits a spatial model*) if it is isomorphic, as an action on the measure algebra, to one arising from a spatial action on some standard probability space.

A *factor* of a near-action of G on (X, μ) corresponds to a G-invariant sub-σ-algebra $\mathcal{F}$ of the Borel σ-algebra of X (modulo null sets). To detect which factors admit spatial models, Glasner and Weiss [5] introduced the algebra of norm-continuous vectors.

Definition 2.2 (Norm-continuous vectors)**.** For a probability-preserving near-action of a topological group G on (X, μ), a bounded measurable function $f \in L^\infty(X, \mu)$ is *norm-continuous* if its orbit map $g \mapsto U_g f$ is continuous into $L^\infty(X, \mu)$ with the essential supremum norm $\|\cdot\|_\infty$

(equivalently, continuous at the identity $e \in G$, by invariance of $\|\cdot\|_\infty$). We denote the set of such functions and the σ-algebra they generate (modulo μ-null sets) by

$$\mathcal{C}_G(X) = \{f \in L^\infty(X,\mu) : \lim_{g\to e} \|U_g f - f\|_\infty = 0\}, \qquad \mathcal{D}_X = \sigma(\mathcal{C}_G(X)).$$

Explicitly, $\lim_{g\to e}\|U_g f - f\|_\infty = 0$ means that for every $\varepsilon > 0$ there exists an identity neighborhood $U \subseteq G$ such that $\|U_g f - f\|_\infty < \varepsilon$ for all $g \in U$. Note that $\mathcal{C}_G(X)$ is a G-invariant, unital, norm-closed $*$-subalgebra of $L^\infty(X,\mu)$, and a function lies in $\mathcal{C}_G(X)$ if and only if its real and imaginary parts do.

Whirliness and uniform whirliness. Intuitively, a near-action is *whirly* if arbitrarily small group elements already mix the space enough to push every positive-measure set into every other positive-measure set.

Definition 2.3 (Whirly near-action [4, 5])**.** A probability-preserving near-action of a Polish group G on a standard probability space (X,μ) is *whirly* if for every identity neighborhood $U \subseteq G$ and all measurable sets $A, B \subseteq X$ with $\mu(A) > 0$ and $\mu(B) > 0$, there exists $g \in U$ such that $\mu(A \cap gB) > 0$.

Norm-continuous vectors obstruct whirliness: if $f \in \mathcal{C}_G(X)$ is real-valued and nonconstant, then for some $c \in \mathbb{R}$ and $\varepsilon > 0$ both $A = \{f \le c - \varepsilon\}$ and $B = \{f \ge c + \varepsilon\}$ have positive measure; choosing an identity neighborhood $U \subseteq G$ on which $\|U_g f - f\|_\infty < \varepsilon$, we have $U_g f = f \circ g^{-1} \ge c + \varepsilon$ on gB and hence $f > U_g f - \varepsilon \ge c$ almost everywhere on gB, so $\mu(A \cap gB) = 0$ for all $g \in U$. Glasner and Weiss proved that norm-continuous vectors are the only obstruction to whirliness and generate the maximal spatial factor.

Theorem 2.4 (Glasner–Weiss [5, Theorems 1.7 and 3.10])**.** *Let G be a Polish group and $G \to \mathrm{Aut}(X,\mu)$ a probability-preserving near-action on a standard probability space.*

(i) The σ-algebra $\mathcal{D}_X = \sigma(\mathcal{C}_G(X))$ is the σ-algebra of the maximal spatial factor of the near-action: the factor corresponding to $\mathcal{D}_X$ admits a spatial model, and every factor admitting a spatial model is contained in $\mathcal{D}_X$ modulo μ-null sets.

(ii) The near-action is whirly if and only if $\mathcal{C}_G(X)$ consists only of constant functions, equivalently if and only if $\mathcal{D}_X$ is trivial modulo μ-null sets (that is, the near-action has no nontrivial spatial factor).

A standard source of whirly near-actions is concentration of measure.

Definition 2.5 (Lévy family and Lévy group [6])**.** A sequence $(K_n, d_n, h_n)_{n\ge 0}$ of compact metric spaces equipped with Borel probability measures h_n is a *Lévy family* if for every sequence of Borel sets $A_n \subseteq K_n$ with $\liminf_{n\to\infty} h_n(A_n) > 0$ and every $\varepsilon > 0$, the open ε-neighborhoods $B_{d_n}(A_n,\varepsilon) = \{x \in K_n : d_n(x, A_n) < \varepsilon\}$ satisfy

$$\lim_{n\to\infty} h_n\big(B_{d_n}(A_n,\varepsilon)\big) = 1.$$

A Polish group G is a *Lévy group* if it admits an increasing chain of compact subgroups $K_0 \le K_1 \le \cdots$ with dense union $\bigcup_{n=0}^\infty K_n$ such that $(K_n, d|_{K_n}, h_n)$ is a Lévy family, where d is a compatible right-invariant metric on G and h_n is the normalized Haar measure on K_n. (By uniform equivalence of compatible right-invariant metrics on G, this property does not depend on the choice of d.)

Theorem 2.6 (Glasner–Tsirelson–Weiss [4]; see [5, Theorem 3.11], [10, Theorem 1.2])**.** *Every ergodic probability-preserving near-action of a Lévy group on a standard probability space is whirly.*

We now turn to Wu's quantitative strengthening of whirliness, formulated first for general near-actions and then specialized to single transformations.

Definition 2.7 (Uniform hitting capacity and uniform whirliness)**.** Let a Polish group G have a probability-preserving near-action on a standard atomless probability space (X, μ). For an identity neighborhood $U \subseteq G$ and numbers $\alpha, \beta \in (0, 1)$, the *uniform hitting capacity* of U at masses (α, β) is

$$\kappa_U^X(\alpha, \beta) = \inf_{\substack{\mu(A)=\alpha \\ \mu(B)=\beta}} \sup_{g \in U} \mu(A \cap gB),$$

where the infimum runs over all measurable sets $A, B \subseteq X$ with $\mu(A) = \alpha$ and $\mu(B) = \beta$ (a nonempty family since μ is atomless). The near-action is *uniformly whirly* if $\kappa_U^X(\alpha, \beta) > 0$ for every identity neighborhood $U \subseteq G$ and all $\alpha, \beta \in (0, 1)$.

Every uniformly whirly near-action on a standard atomless probability space is whirly: any sets A, B of positive measure contain subsets $A' \subseteq A$, $B' \subseteq B$ of positive masses $\alpha, \beta \in (0, 1)$, and $\kappa_U^X(\alpha, \beta) > 0$ produces $g \in U$ with $\mu(A \cap gB) \geq \mu(A' \cap gB') > 0$. Notice also that because every identity neighborhood contains a symmetric identity neighborhood $U_0 = U_0^{-1}$ and $\mu(A \cap gB) = \mu(g^{-1}A \cap B)$, replacing $\mu(A \cap gB)$ with $\mu(gA \cap B)$ defines the exact same property.

In the setting of a single measure-preserving transformation $T \in \mathrm{Aut}(X, \mu)$ of a standard atomless probability space, the *cyclic closure* of T is the closed abelian subgroup

$$\Lambda(T) = \overline{\{T^n : n \in \mathbb{Z}\}} \leq \mathrm{Aut}(X, \mu),$$

endowed with the topology induced from the weak topology on $\mathrm{Aut}(X, \mu)$, and its *natural near-action* is the inclusion $\Lambda(T) \hookrightarrow \mathrm{Aut}(X, \mu)$. A topological group is *monothetic* if it contains a dense cyclic subgroup, and a generator of such a subgroup is a *topological generator*.

Definition 2.8 (Rigid, whirly, and uniformly whirly transformations [14])**.** Let (X, μ) be a standard atomless probability space and $T \in \mathrm{Aut}(X, \mu)$.

(i) T is *rigid* if there exists a sequence of integers $n_j \to \infty$ such that $T^{n_j} \to \mathrm{Id}$ in the weak topology of $\mathrm{Aut}(X, \mu)$.

(ii) A rigid transformation T is *whirly* if for every weak identity neighborhood $U \subseteq \mathrm{Aut}(X, \mu)$ and all measurable sets $A, B \subseteq X$ of positive measure, there exists $n \in \mathbb{Z}$ with $T^n \in U$ and $\mu(T^n A \cap B) > 0$.

(iii) A rigid transformation T is *uniformly whirly* if for every weak identity neighborhood $U \subseteq \mathrm{Aut}(X, \mu)$ and all $\alpha, \beta \in (0, 1)$,

$$\inf_{\substack{\mu(A)=\alpha \\ \mu(B)=\beta}} \sup_{n \in \mathbb{Z}:\, T^n \in U} \mu(T^n A \cap B) > 0.$$

The following lemma bridges Definition 2.8 and the group-action notions in Definitions 2.3 and 2.7; we will use it at the end of Section 4 to transfer properties of a monothetic Lévy group directly to a single rigid transformation.

Lemma 2.9 (Monothetic reduction for rigid transformations)**.** *Let (X, μ) be a standard atomless probability space and let $T \in \mathrm{Aut}(X, \mu)$ be rigid.*

(i) The positive powers $\{T^n : n \geq 1\}$ are dense in $\Lambda(T)$.

(ii) For every open identity neighborhood $U \subseteq \mathrm{Aut}(X, \mu)$ and all measurable sets $A, B \subseteq X$,

$$\sup_{g \in \Lambda(T) \cap U} \mu(gA \cap B) = \sup_{n \in \mathbb{Z}:\, T^n \in U} \mu(T^n A \cap B) = \sup_{n \geq 1:\, T^n \in U} \mu(T^n A \cap B).$$

(iii) Consequently, T is whirly (respectively, uniformly whirly) in the sense of Definition 2.8 if and only if the natural near-action of $\Lambda(T)$ on (X,μ) is whirly (respectively, uniformly whirly).

Proof. (i) Choose integers $n_j \to \infty$ with $T^{n_j} \to \mathrm{Id}$. For any $k \in \mathbb{Z}$, we have $n_j + k \geq 1$ for all sufficiently large j, and $T^{n_j+k} = T^{n_j}T^k \to T^k$. Thus every power T^k ($k \in \mathbb{Z}$) lies in $\overline{\{T^n : n \geq 1\}}$, so $\overline{\{T^n : n \geq 1\}} = \Lambda(T)$.

(ii) Since U is open and $\{T^n : n \geq 1\}$ is dense in $\Lambda(T)$, the set $\{T^n : n \geq 1\} \cap U$ is dense in $\Lambda(T) \cap U$. Because the map $g \mapsto \mu(gA \cap B)$ is continuous on $\mathrm{Aut}(X,\mu)$ (as $|\mu(gA \cap B) - \mu(hA \cap B)| \leq \mu(gA \triangle hA)$), its supremum over $\Lambda(T) \cap U$ equals its supremum over any dense subset, yielding both equalities.

(iii) Every identity neighborhood of $\Lambda(T)$ has the form $\Lambda(T) \cap U$ for an identity neighborhood U of $\mathrm{Aut}(X,\mu)$ and contains $\Lambda(T) \cap U_0$ for a symmetric open identity neighborhood $U_0 \subseteq U$. By monotonicity of the suprema in the neighborhood, it suffices to verify whirliness and uniform whirliness on symmetric open identity neighborhoods U_0, where $\sup_{g \in \Lambda(T) \cap U_0} \mu(A \cap gB) = \sup_{g \in \Lambda(T) \cap U_0} \mu(gA \cap B)$ by symmetry of U_0. Applying (ii) to U_0 yields both equivalences. □

3. Maximal spatial factors of independent products

In this section we prove Theorem 1.1, deduce Corollary 1.2, and demonstrate with Proposition 3.2 that independence cannot be relaxed to conditional independence over a nonatomic spatial factor.

The key mechanism in Theorem 1.1 is that partial integration against any integrable weight on one factor maps norm-continuous functions on a product into norm-continuous functions on the complementary factor. The proof relies only on measure preservation and strong L^1-continuity of the Koopman representation.

Lemma 3.1 (Partial integration preserves norm-continuity)**.** *Let G be a topological group with probability-preserving near-actions on standard probability spaces (X,μ) and (Y,ν). Write U_g, V_g, and W_g for the Koopman operators on X, Y, and the independent product $(X \times Y, \mu \otimes \nu)$ with the diagonal action (whose construction and continuity as a near-action are verified independently for all finite or countable products in Step 1 of the proof of Theorem 1.1). For $F \in L^\infty(X \times Y, \mu \otimes \nu)$ and $h \in L^1(Y,\nu)$, define the slice $S_hF \in L^\infty(X,\mu)$ by*

$$(S_hF)(x) = \int_Y F(x,y)h(y)\,d\nu(y).$$

Then $\|S_hF\|_\infty \leq \|F\|_\infty \|h\|_1$, the equivariance identity $U_g(S_hF) = S_{V_gh}(W_gF)$ holds for all $g \in G$, and

$$\|U_g(S_hF) - S_hF\|_\infty \leq \|W_gF - F\|_\infty \|h\|_1 + \|F\|_\infty \|V_gh - h\|_1. \tag{3.1}$$

In particular, if $F \in \mathcal{C}_G(X \times Y)$, then $S_hF \in \mathcal{C}_G(X)$ for every $h \in L^1(Y,\nu)$.

Proof. By Fubini's theorem, S_hF is well-defined modulo μ-null sets and satisfies $|(S_hF)(x)| \leq \|F\|_\infty \int_Y |h(y)|\,d\nu(y)$ for μ-almost every $x \in X$, giving $\|S_hF\|_\infty \leq \|F\|_\infty \|h\|_1$.

Fix $g \in G$. Because (X,μ) and (Y,ν) are standard probability spaces, the measure-algebra automorphisms corresponding to g are represented by measure-preserving Borel automorphisms of X and Y. Using the measure-preserving substitution $y \mapsto g^{-1}y$ on (Y,ν), we obtain for μ-almost every $x \in X$

$$\begin{aligned}(U_g(S_hF))(x) &= \int_Y F(g^{-1}x,y)h(y)\,d\nu(y)\\ &= \int_Y F(g^{-1}x,g^{-1}y)h(g^{-1}y)\,d\nu(y) = (S_{V_gh}(W_gF))(x).\end{aligned}$$

Thus $U_g(S_hF) = S_{V_gh}(W_gF)$. Writing

$$U_g(S_hF) - S_hF = S_{V_gh}(W_gF - F) + S_{V_gh-h}F$$

and applying the L^∞-bound together with $\|V_gh\|_1 = \|h\|_1$ gives (3.1).

Finally, if $F \in \mathcal{C}_G(X \times Y)$, then $\|W_gF - F\|_\infty \to 0$ as $g \to e$, while $\|V_gh - h\|_1 \to 0$ as $g \to e$ by strong L^1-continuity of the Koopman representation V. Hence $\|U_g(S_hF) - S_hF\|_\infty \to 0$ as $g \to e$, so $S_hF \in \mathcal{C}_G(X)$. □

We now apply Lemma 3.1 to each coordinate of a finite or countable independent product.

Proof of Theorem 1.1. Write $U_g^{(i)}$ for the Koopman operator of $g \in G$ on (X_i, μ_i) and W_g for the diagonal Koopman operator on $(X, \mu) = \prod_{i\in I}(X_i, \mu_i)$. Because I is finite or countable, (X, μ) is a standard probability space.

Step 1: Construction and continuity of the diagonal near-action. For each $g \in G$ and $i \in I$, choose a Borel automorphism $T_g^{(i)}$ of (X_i, μ_i) representing the action of g on (X_i, μ_i), and let T_g be the product Borel automorphism $(x_i)_{i\in I} \mapsto (T_g^{(i)}x_i)_{i\in I}$ of (X, μ), which preserves $\mu = \bigotimes_{i\in I}\mu_i$. Because I is finite or countable, changing any representative $T_g^{(i)}$ on a μ_i-null set alters T_g only on a countable union of μ-null coordinate cylinders, so the class of T_g in $\mathrm{Aut}(X, \mu)$ is well-defined; for the same reason, for each fixed $g, h \in G$ we have $T_{gh} = T_gT_h$ modulo μ-null sets, defining a group homomorphism $G \to \mathrm{Aut}(X, \mu)$ with Koopman operators W_g. For any finite subset $J \subseteq I$ and bounded functions $f_i \in L^\infty(X_i, \mu_i)$ $(i \in J)$, consider the elementary tensor $\varphi = \bigotimes_{i\in J} f_i \in L^2(X, \mu)$ (understood with factor 1 in coordinates outside J). Writing $W_g\varphi - \varphi$ as a telescoping sum over $i \in J$ and using unitarity of $U_g^{(i)}$ yields

$$\|W_g\varphi - \varphi\|_2 \le \sum_{i\in J}\Big(\prod_{j\in J\setminus\{i\}}\|f_j\|_\infty\Big)\|U_g^{(i)}f_i - f_i\|_2 \longrightarrow 0 \qquad (g \to e).$$

Since finite linear combinations of such bounded elementary tensors are dense in $L^2(X, \mu)$ and the operators W_g are unitary, it follows that $g \mapsto W_g$ is strongly continuous on $L^2(X, \mu)$, so $G \to \mathrm{Aut}(X, \mu)$ is a continuous near-action.

Step 2: Coordinatewise containment. If $|I| = 1$, there is nothing to prove, so assume $|I| \ge 2$. For each $i \in I$, let $P_i = \mathbb{E}(\cdot \mid \mathcal{D}_{X_i})$ be the orthogonal projection of $L^2(X_i, \mu_i)$ onto $L^2(X_i, \mathcal{D}_{X_i}, \mu_i)$. Factor (X, μ) as $(X_i, \mu_i) \times (X_{\neq i}, \mu_{\neq i})$, where $X_{\neq i} = \prod_{j\in I\setminus\{i\}} X_j$ carries the product measure $\mu_{\neq i} = \bigotimes_{j\neq i}\mu_j$ and the diagonal near-action (continuous by Step 1). Let $\widetilde{P}_i$ be the orthogonal projection on $L^2(X, \mu)$ acting as P_i in coordinate i and as the identity in all other coordinates, that is, the conditional expectation onto $\mathcal{D}_{X_i} \otimes \bigotimes_{j\neq i}\mathcal{B}_{X_j}$ (where $\mathcal{B}_{X_j}$ is the Borel σ-algebra of X_j).

Let $F \in \mathcal{C}_G(X)$. By Lemma 3.1 applied to $X_i \times X_{\neq i}$, for every $h \in L^2(X_{\neq i}, \mu_{\neq i}) \subseteq L^1(X_{\neq i}, \mu_{\neq i})$ the slice S_hF belongs to $\mathcal{C}_G(X_i)$, and hence to $L^2(X_i, \mathcal{D}_{X_i}, \mu_i)$. Therefore, for every $u \in L^2(X_i, \mathcal{D}_{X_i}, \mu_i)^\perp$ and every $h \in L^2(X_{\neq i}, \mu_{\neq i})$, Fubini's theorem gives

$$\langle F,\, u \otimes \bar{h}\rangle_{L^2(X)} = \int_{X_i}\Big(\int_{X_{\neq i}} F(x_i, y)h(y)\, d\mu_{\neq i}(y)\Big)\overline{u(x_i)}\, d\mu_i(x_i) = \langle S_hF,\, u\rangle_{L^2(X_i)} = 0.$$

For any $f \in L^2(X_i, \mu_i)$ and $h \in L^2(X_{\neq i}, \mu_{\neq i})$, we have $(\mathrm{Id} - \widetilde{P}_i)(f \otimes \bar{h}) = ((\mathrm{Id} - P_i)f) \otimes \bar{h}$, where $(\mathrm{Id} - P_i)f$ lies in $L^2(X_i, \mathcal{D}_{X_i}, \mu_i)^\perp$. Because the span of the tensors $f \otimes \bar{h}$ is dense in $L^2(X, \mu)$ and $\mathrm{Id} - \widetilde{P}_i$ is a bounded orthogonal projection, the range $(\mathrm{Id} - \widetilde{P}_i)L^2(X, \mu)$ is the closed linear

span of all tensors $u \otimes \bar{h}$ with $u \in L^2(X_i, \mathcal{D}_{X_i}, \mu_i)^\perp$ and $h \in L^2(X_{\neq i}, \mu_{\neq i})$. Consequently $F \perp (\mathrm{Id} - \widetilde{P}_i) L^2(X, \mu)$, meaning that

$$\widetilde{P}_i F = F \qquad \text{for every } i \in I. \tag{3.2}$$

Step 3: Upper inclusion $\mathcal{D}_X \subseteq \bigotimes_{i\in I} \mathcal{D}_{X_i}$. Choose an increasing sequence of finite subsets $J_1 \subseteq J_2 \subseteq \cdots \subseteq I$ with $\bigcup_{n=1}^{\infty} J_n = I$ (if I is finite, take $J_n = I$ for all n), and set $Q_n = \prod_{i\in J_n} \widetilde{P}_i$. On any finite-coordinate tensor $\varphi = \bigotimes_{j\in A} \varphi_j$ (with $A \subseteq I$ finite and $\varphi_j \in L^2(X_j, \mu_j)$), the projections $\widetilde{P}_i$ act independently in each coordinate, so they commute and for every n with $J_n \supseteq A$ we have

$$Q_n \varphi = \bigotimes_{j\in A} (P_j \varphi_j).$$

Let $Q = \mathbb{E}(\cdot \mid \bigotimes_{i\in I} \mathcal{D}_{X_i})$ be the orthogonal projection of $L^2(X, \mu)$ onto $L^2(X, \bigotimes_{i\in I} \mathcal{D}_{X_i}, \mu)$. We claim that $Q\varphi = \bigotimes_{j\in A}(P_j\varphi_j)$ as well. Indeed, $\bigotimes_{j\in A}(P_j\varphi_j)$ is $\bigotimes_{i\in I} \mathcal{D}_{X_i}$-measurable, and for any finite-coordinate rectangle $R = \bigcap_{j\in B} \pi_j^{-1}(D_j)$ with $B \subseteq I$ finite and $D_j \in \mathcal{D}_{X_j}$ (where $\pi_j : X \to X_j$ is the coordinate projection), independence of the coordinates gives

$$\int_R \Big(\bigotimes_{j\in A} \varphi_j\Big)\, d\mu = \prod_{j\in A\cup B} \int_{X_j} \tilde{\varphi}_j \tilde{\mathbf{1}}_j \, d\mu_j = \prod_{j\in A\cup B} \int_{X_j} (P_j\tilde{\varphi}_j)\tilde{\mathbf{1}}_j \, d\mu_j = \int_R \Big(\bigotimes_{j\in A} P_j\varphi_j\Big)\, d\mu,$$

where $\tilde{\varphi}_j = \varphi_j$ for $j \in A$ and 1 otherwise, and $\tilde{\mathbf{1}}_j = \mathbf{1}_{D_j}$ for $j \in B$ and 1 otherwise (using $P_j 1 = 1$). Since such rectangles form a π-system generating $\bigotimes_{i\in I} \mathcal{D}_{X_i}$, the two integrals agree on every set in $\bigotimes_{i\in I} \mathcal{D}_{X_i}$ by Dynkin's π-λ theorem, confirming $Q\varphi = \bigotimes_{j\in A}(P_j\varphi_j) = Q_n\varphi$ for all n with $J_n \supseteq A$.

Because $\|Q_n\| \le 1$ and finite-coordinate tensors span a dense subspace of $L^2(X, \mu)$, it follows that $Q_n \to Q$ strongly on $L^2(X, \mu)$. For any $F \in \mathcal{C}_G(X)$, equation (3.2) gives $Q_n F = F$ for every n, and passing to the limit $n \to \infty$ yields $QF = F$. Thus every $F \in \mathcal{C}_G(X)$ is $\bigotimes_{i\in I} \mathcal{D}_{X_i}$-measurable modulo μ-null sets, so $\mathcal{D}_X \subseteq \bigotimes_{i\in I} \mathcal{D}_{X_i}$.

Step 4: Lower inclusion $\bigotimes_{i\in I} \mathcal{D}_{X_i} \subseteq \mathcal{D}_X$ and spatial conclusion. For any $i \in I$ and $f \in \mathcal{C}_G(X_i)$, the pullback $f \circ \pi_i \in L^\infty(X, \mu)$ satisfies

$$\|W_g(f \circ \pi_i) - f \circ \pi_i\|_\infty = \|U_g^{(i)} f - f\|_\infty \longrightarrow 0 \qquad (g \to e),$$

so $f \circ \pi_i \in \mathcal{C}_G(X)$. As these pullbacks generate $\bigotimes_{i\in I} \mathcal{D}_{X_i}$, we obtain $\bigotimes_{i\in I} \mathcal{D}_{X_i} \subseteq \mathcal{D}_X$, completing the proof that $\mathcal{D}_X = \bigotimes_{i\in I} \mathcal{D}_{X_i}$ modulo μ-null sets.

Finally, when G is a Polish group, Theorem 2.4(i) identifies $\mathcal{D}_{X_i}$ with the maximal spatial factor of (X_i, μ_i) and $\mathcal{D}_X$ with the maximal spatial factor of (X, μ), yielding the final assertion. □

Proof of Corollary 1.2. By Theorem 2.4(ii), each factor (X_i, μ_i) is whirly if and only if $\mathcal{D}_{X_i}$ is trivial modulo μ_i-null sets. By Theorem 1.1, $\mathcal{D}_X = \bigotimes_{i\in I} \mathcal{D}_{X_i}$ is therefore trivial modulo μ-null sets, so (X, μ) is whirly by Theorem 2.4(ii). □

We close this section by showing that independence in Theorem 1.1 cannot be replaced by conditional independence over a common spatial factor. Recall that if a near-action of G on (X, μ) factors onto a spatial factor (Z, λ) with disintegration $\mu = \int_Z \mu_z \, d\lambda(z)$ (so $g_*\mu_z = \mu_{gz}$ for λ-almost every $z \in Z$ and each $g \in G$), the *relatively independent square* $X \times_Z X$ is obtained by equipping the fibered square with the measure $\int_Z (\mu_z \otimes \mu_z)\, d\lambda(z)$ and the diagonal G-action.

Proposition 3.2 (Failure for relatively independent products)**.** *Let $Z = [0,1)$ with Lebesgue measure m, let $K = \{-1,1\}$ with the uniform probability measure, and let $G = L^0(Z,K)$ be the group of measurable maps $Z \to K$ modulo null sets under pointwise multiplication, equipped with the complete invariant metric $d(g,h) = m(\{z \in Z : g(z) \neq h(z)\})$. Consider the continuous probability-preserving near-action of the Polish group G on $X = Z \times K$ (with product measure) given by*

$$g \cdot (z,t) = (z,\, g(z)t).$$

Then the maximal spatial factor of X is the base Z, that is, $\mathcal{D}_X = \sigma(z)$, whereas the relatively independent square $X \times_Z X = Z \times K^2$ with diagonal action $g \cdot (z,t_1,t_2) = (z,\, g(z)t_1,\, g(z)t_2)$ has

$$\mathcal{D}_{X\times_Z X} = \sigma(z, t_1t_2) \supsetneq \sigma(z).$$

Proof. It is standard that (G,d) is a Polish abelian group: if (g_k) is d-Cauchy, a subsequence (g_{k_j}) with $\sum_j d(g_{k_{j+1}}, g_{k_j}) < \infty$ stabilizes almost everywhere by the Borel–Cantelli lemma and gives the d-limit; finite step functions with rational endpoints form a countable dense subset. For any measurable set $A \subseteq X$, we have $gA \triangle hA \subseteq \{g \neq h\} \times K$, so $\mu(gA \triangle hA) \leq d(g,h)$; thus the action on X (and by the same estimate on $X \times_Z X$) is a continuous near-action.

Every bounded measurable function of z alone is G-invariant, hence lies in $\mathcal{C}_G(X)$, so $\sigma(z) \subseteq \mathcal{D}_X$. Conversely, let $F \in \mathcal{C}_G(X)$ and $\varepsilon > 0$. Choose $\delta > 0$ such that $d(g,1) < \delta$ implies $\|U_gF - F\|_\infty < \varepsilon$. Partition Z into finitely many measurable subsets $Z_1, \ldots, Z_r$ with $m(Z_j) < \delta$, and define $g_j \in G$ to equal -1 on Z_j and 1 off Z_j. Then $d(g_j,1) = m(Z_j) < \delta$, while on $Z_j \times K$ the automorphism g_j acts as the fiber flip $(z,t) \mapsto (z,-t)$. Hence $|F(z,-t) - F(z,t)| \leq \varepsilon$ for almost every $(z,t) \in Z_j \times K$. Taking the union over $j = 1, \ldots, r$ gives $\|F(z,-t) - F(z,t)\|_\infty \leq \varepsilon$, and letting $\varepsilon \to 0$ shows that $F(z,t) = F(z,-t)$ almost everywhere, so F is $\sigma(z)$-measurable. Thus $\mathcal{D}_X = \sigma(z)$.

On $X \times_Z X = Z \times K^2$, the function $(z,t_1,t_2) \mapsto t_1t_2$ is G-invariant because $g(z)t_1 \cdot g(z)t_2 = g(z)^2t_1t_2 = t_1t_2$. Hence $\sigma(z,t_1t_2) \subseteq \mathcal{D}_{X\times_Z X}$. Applying the exact same partition argument with the elements g_j shows that every $F \in \mathcal{C}_G(X \times_Z X)$ is almost everywhere invariant under the simultaneous flip $(z,t_1,t_2) \mapsto (z,-t_1,-t_2)$, so F is a function of (z,t_1t_2) alone. Thus $\mathcal{D}_{X\times_Z X} = \sigma(z,t_1t_2)$, which strictly contains $\sigma(z)$ because t_1t_2 is a uniform sign independent of z. By Theorem 2.4(i), these σ-algebras give the respective maximal spatial factors. □

4. A rigid whirly transformation that is not uniformly whirly

In this section we prove Theorem 1.3, answering Wu's question whether whirliness implies uniform whirliness. We first collect four auxiliary results:

(1) a coordinate variance and concentration inequality on product spaces (Lemma 4.1), used to prove that our dyadic-energy group G_w is a Lévy group;
(2) a general lemma showing that Lévy groups have no nontrivial continuous finite-dimensional unitary representations (Lemma 4.2), used to verify weak mixing of our Gaussian near-action;
(3) the Gaussian realization functor for orthogonal representations (Lemma 4.3), used to embed G_w as a closed subgroup of $\mathrm{Aut}(\Omega,\gamma)$;
(4) a quantitative Gaussian half-set estimate for finite-rank projections (Lemma 4.4), used to construct uniformly almost-invariant half-sets on balls of G_w.

Probabilistic and representation-theoretic tools. We begin with the classical Efron–Stein variance inequality [3, 12] together with a zero-mass tail bound, which will establish measure concentration on the approximating dyadic tori $\mathbb{T}^{2^n}$.

Lemma 4.1 (Coordinate variance and tail bound)**.** *Let $Y_1, \dots, Y_n$ be independent random variables and let $f = f(Y_1, \dots, Y_n)$ be a real square-integrable function.*

(i) Let $\mathrm{Var}_j(f)$ denote the variance of f with respect to Y_j with all other variables held fixed. Then

$$\mathrm{Var}(f) \le \sum_{j=1}^{n} \mathbb{E}[\mathrm{Var}_j(f)].$$

In particular, if changing Y_j alone alters the value of f by at most c_j, then $\mathrm{Var}(f) \le \frac{1}{4}\sum_{j=1}^{n} c_j^2$.

(ii) If in addition $f \ge 0$ and $\mathbb{P}(f = 0) \ge b > 0$, then for every $r > 0$,

$$\mathbb{P}(f \ge r) \le \frac{\mathrm{Var}(f)}{br^2}.$$

Proof. (i) We induct on n, the case $n = 1$ being an equality. For $n \ge 2$, let $g(Y_1, \dots, Y_{n-1}) = \mathbb{E}[f \mid Y_1, \dots, Y_{n-1}]$. By the law of total variance,

$$\mathrm{Var}(f) = \mathbb{E}[\mathrm{Var}_n(f)] + \mathrm{Var}(g).$$

By the induction hypothesis, $\mathrm{Var}(g) \le \sum_{j=1}^{n-1} \mathbb{E}[\mathrm{Var}_j(g)]$. For each $j < n$, fix all coordinates except Y_j and Y_n, and write $Z(Y_j, Y_n)$ for the resulting function of the two independent variables (Y_j, Y_n) and $m(Y_n) = \mathbb{E}_{Y_j}[Z(Y_j, Y_n)]$. By Jensen's inequality for $u \mapsto u^2$ and independence of Y_j and Y_n,

$$\mathrm{Var}_{Y_j}\big(\mathbb{E}_{Y_n}[Z]\big) = \mathbb{E}_{Y_j}\Big[\big(\mathbb{E}_{Y_n}[Z - m(Y_n)]\big)^2\Big] \le \mathbb{E}_{Y_j}\mathbb{E}_{Y_n}\big[(Z - m(Y_n))^2\big] = \mathbb{E}_{Y_n}\big[\mathrm{Var}_{Y_j}(Z)\big].$$

Taking expectations over the remaining coordinates gives $\mathbb{E}[\mathrm{Var}_j(g)] \le \mathbb{E}[\mathrm{Var}_j(f)]$, completing the induction. If changing Y_j alone varies f within an interval $[a, a + c_j]$ of length c_j, then $\mathrm{Var}_j(f) \le \mathbb{E}_j[(f - (a + c_j/2))^2] \le c_j^2/4$, yielding $\mathrm{Var}(f) \le \frac{1}{4}\sum_{j=1}^{n} c_j^2$.

(ii) Since $f \ge 0$ and $\mathbb{P}(f > 0) \le 1 - b$, the Cauchy–Schwarz inequality gives

$$(\mathbb{E}[f])^2 = \big(\mathbb{E}[f\mathbf{1}_{\{f>0\}}]\big)^2 \le \mathbb{P}(f > 0)\,\mathbb{E}[f^2] \le (1 - b)\,\mathbb{E}[f^2],$$

whence $\mathrm{Var}(f) = \mathbb{E}[f^2] - (\mathbb{E}[f])^2 \ge b\,\mathbb{E}[f^2]$. Markov's inequality now gives $\mathbb{P}(f \ge r) \le \mathbb{E}[f^2]/r^2 \le \mathrm{Var}(f)/(br^2)$. □

Next we record a general structural property of Lévy groups: they have no nontrivial continuous finite-dimensional unitary representations. This is an immediate consequence of Gromov and Milman's theorem [6] that every Lévy group is extremely amenable (a nontrivial continuous homomorphism $\pi : G \to \mathrm{U}(d)$ would give a fixed-point-free action on the compact group $\overline{\pi(G)}$), and also admits a direct proof from Definition 2.5.

Lemma 4.2 (No nontrivial continuous finite-dimensional unitary representations)**.** *Let G be a Lévy group. Every continuous homomorphism $\pi : G \to \mathrm{U}(d)$ ($d \ge 1$) is trivial. Consequently, if $\rho : G \to \mathrm{O}(E)$ is a strongly continuous orthogonal representation of G on a real Hilbert space E with no nonzero invariant vectors, then ρ has no nonzero finite-dimensional invariant subspace.*

Proof. Let d_G be a compatible right-invariant metric on G and $(K_n, d_G|_{K_n}, h_n)$ a Lévy family of increasing compact subgroups with dense union. Suppose for contradiction that $\pi : G \to \mathrm{U}(d)$ is continuous and nontrivial, and equip the compact group $L = \overline{\pi(G)} \le \mathrm{U}(d)$ with the compatible bi-invariant operator-norm metric $d_L(u, v) = \|u - v\|_{\mathrm{op}}$. By density of $\bigcup_n K_n$, there exist n_0 and $g_0 \in K_{n_0}$ with $c := d_L(\pi(g_0), I) > 0$. Because d_G and d_L are right-invariant and π is a continuous homomorphism, π is uniformly continuous: there is $\varepsilon > 0$ such that $d_G(x, y) < \varepsilon$ implies $d_L(\pi(x), \pi(y)) < c/3$.

Cover the compact space L by finitely many open balls $W_1, \dots, W_r$ of d_L-diameter at most $c/3$. For each $n \geq n_0$, choose j_n so that $A_n \coloneqq \pi^{-1}(W_{j_n}) \cap K_n$ has Haar measure $h_n(A_n) \geq 1/r$. Since $g_0 \in K_n$, the translate $g_0 A_n \subseteq K_n$ also has $h_n(g_0 A_n) = h_n(A_n) \geq 1/r$. For any $x \in A_n$ and $y \in g_0 A_n$, we have $\pi(y) \in \pi(g_0) W_{j_n}$, so $d_L(\pi(y), \pi(g_0)\pi(x)) \leq c/3$, and hence

$$d_L(\pi(y), \pi(x)) \geq d_L(\pi(g_0)\pi(x), \pi(x)) - \frac{c}{3} = c - \frac{c}{3} = \frac{2c}{3} > \frac{c}{3}.$$

Thus $d_G(y, x) \geq \varepsilon$, showing that $B_{d_G}(A_n, \varepsilon)$ is disjoint from $g_0 A_n$. Therefore $h_n(B_{d_G}(A_n, \varepsilon)) \leq 1 - 1/r$ for all $n \geq n_0$, contradicting that $(K_n, d_G|_{K_n}, h_n)$ is a Lévy family.

For the second assertion, if $V \subseteq E$ is a finite-dimensional $\rho(G)$-invariant subspace, then complexifying $\rho|_V$ yields a continuous homomorphism $G \to \mathrm{U}(V_{\mathbb{C}})$, which is trivial by the first part. Hence every vector in V is $\rho(G)$-invariant, so $V = \{0\}$. □

We now recall the standard Gaussian realization of orthogonal representations (see [2, Chapter 14], [8], [10, Section 6], and also [7], where related Gaussian embeddings and weak-mixing results are used to construct non-amenable whirly groups), summarizing the exact properties we need. Recall that a probability-preserving near-action of G on (X, μ) is *weakly mixing* if its Koopman representation on the zero-mean subspace $L^2_0(X, \mu) = \{f \in L^2(X, \mu) : \int_X f \, d\mu = 0\}$ has no nonzero finite-dimensional invariant subspace; in particular, it has no nonzero invariant vector in $L^2_0(X, \mu)$ and is therefore ergodic. For a Hilbert–Schmidt operator T between Hilbert spaces, we write $\|T\|_{\mathrm{HS}} = \big(\mathrm{Tr}(T^*T)\big)^{1/2}$ for its Hilbert–Schmidt norm.

Lemma 4.3 (Gaussian realization of orthogonal representations)**.** *Let E be a separable real Hilbert space, and endow its orthogonal group $\mathrm{O}(E)$ with the strong operator topology. There exist a standard probability space (Ω_E, γ_E), a real-linear isometry $W : E \to L^2(\Omega_E, \gamma_E; \mathbb{R})$ onto a closed subspace of centered jointly Gaussian random variables generating the Borel σ-algebra of Ω_E modulo null sets, and a topological group embedding*

$$\Gamma : \mathrm{O}(E) \longrightarrow \mathrm{Aut}(\Omega_E, \gamma_E)$$

with closed image, satisfying the intertwining relation

$$W(v) \circ \Gamma(u)^{-1} = W(uv) \qquad \text{for all } v \in E,\ u \in \mathrm{O}(E). \tag{4.1}$$

Moreover:

- *(i) If $E \neq \{0\}$, then γ_E is atomless.*
- *(ii) For an orthogonal direct sum $E = E_1 \oplus E_2$, the realizations can be chosen so that $(\Omega_E, \gamma_E) = (\Omega_{E_1} \times \Omega_{E_2}, \gamma_{E_1} \otimes \gamma_{E_2})$ and $\Gamma(u_1 \oplus u_2) = \Gamma(u_1) \times \Gamma(u_2)$ for all $u_i \in \mathrm{O}(E_i)$.*
- *(iii) If G is a topological group and $\rho : G \to \mathrm{O}(E)$ is a strongly continuous orthogonal representation with no nonzero finite-dimensional invariant subspace, then the Gaussian near-action $\Gamma \circ \rho : G \to \mathrm{Aut}(\Omega_E, \gamma_E)$ is weakly mixing, and in particular ergodic.*

Proof. Choose an orthonormal basis (e_k) of E and let $(\Omega_E, \gamma_E) = (\mathbb{R}^{\dim E}, \gamma_1^{\otimes \dim E})$, where γ_1 is the standard normal distribution $\mathcal{N}(0, 1)$ on $\mathbb{R}$ (if $E = \{0\}$, Ω_E is a singleton). Mapping e_k to the k-th coordinate functional extends uniquely to a real-linear isometry $W : E \to L^2(\Omega_E, \gamma_E; \mathbb{R})$ whose image consists of centered jointly Gaussian variables generating the measure algebra of Ω_E. For $u \in \mathrm{O}(E)$, choosing Borel representatives of $W(ue_k)$ defines a Borel map $\Psi_u : \Omega_E \to \Omega_E$, $\Psi_u(\omega) = (W(ue_k)(\omega))_k$. Because $(W(ue_k))$ has the same covariance matrix as $(W(e_k))$ and hence the same joint Gaussian law, Ψ_u preserves γ_E, so composition $f \mapsto f \circ \Psi_u$ is an L^2-isometry satisfying $W(e_k) \circ \Psi_u = W(ue_k)$ and therefore $W(v) \circ \Psi_u = W(uv)$ for all $v \in E$ by linearity and L^2-continuity. Consequently, for all $u_1, u_2 \in \mathrm{O}(E)$ and every k, we have $W(e_k) \circ (\Psi_{u_2} \circ \Psi_{u_1}) = W(u_2 e_k) \circ \Psi_{u_1} = W(u_1 u_2 e_k) = W(e_k) \circ \Psi_{u_1 u_2}$ almost everywhere, so

$\Psi_{u_2} \circ \Psi_{u_1} = \Psi_{u_1 u_2}$ modulo null sets (as the coordinate functionals separate points of Ω_E). In particular, $\Psi_u \circ \Psi_{u^{-1}} = \Psi_{u^{-1}} \circ \Psi_u = \mathrm{Id}$ modulo null sets. Setting $B_0 = \{\omega \in \Omega_E : \Psi_u(\Psi_{u^{-1}}(\omega)) = \omega = \Psi_{u^{-1}}(\Psi_u(\omega))\}$, $B_{n+1} = B_n \cap \Psi_u^{-1}(B_n) \cap \Psi_{u^{-1}}^{-1}(B_n)$, and $\Omega_0 = \bigcap_{n=0}^{\infty} B_n$ gives a conull Borel set Ω_0 invariant under Ψ_u and $\Psi_{u^{-1}}$ on which they are mutual Borel inverses; extending $\Psi_{u^{-1}}|_{\Omega_0}$ by the identity off Ω_0 defines $\Gamma(u) \in \mathrm{Aut}(\Omega_E, \gamma_E)$, the unique automorphism satisfying (4.1), and $u \mapsto \Gamma(u)$ is an injective group homomorphism.

By the Wiener chaos decomposition, $L^2(\Omega_E, \gamma_E)$ splits as the orthogonal direct sum $\bigoplus_{n=0}^{\infty} \mathcal{H}_n$, where $\mathcal{H}_0 = \mathbb{C}$ and $\mathcal{H}_n$ is unitarily equivalent to the n-th symmetric Hilbert tensor power of $E_{\mathbb{C}}$, on which the Koopman operator $U_{\Gamma(u)}$ acts as $u_{\mathbb{C}}^{\otimes n}$. Strong convergence $u_j \to u$ in $\mathrm{O}(E)$ therefore implies strong convergence $U_{\Gamma(u_j)} \to U_{\Gamma(u)}$ on every $\mathcal{H}_n$ and hence on $L^2(\Omega_E, \gamma_E)$; conversely, restricting $U_{\Gamma(u_j)} \to U_{\Gamma(u)}$ to $W(E)$ recovers $u_j \to u$ strongly on E. Thus Γ is a topological embedding. If $\Gamma(u_j) \to T$ in $\mathrm{Aut}(\Omega_E, \gamma_E)$, then $U_{\Gamma(u_j)}$ and $U_{\Gamma(u_j)}^{-1}$ converge strongly to U_T and $U_{T^{-1}}$; both limits preserve the closed real subspace $W(E)$, so $u \coloneqq W^{-1} U_T W \in \mathrm{O}(E)$ and $U_T = U_{\Gamma(u)}$ on $W(E)$, hence $T = \Gamma(u)$ since $W(E)$ generates the measure algebra. Thus $\Gamma(\mathrm{O}(E))$ is closed.

Assertion (i) is immediate since $W(v) \sim \mathcal{N}(0,1)$ for any unit vector $v \in E$, and (ii) follows by choosing an orthonormal basis adapted to $E_1 \oplus E_2$. For (iii), suppose ρ has no nonzero finite-dimensional invariant subspace; then its complexification $\rho_{\mathbb{C}}$ has none either (if $V \subseteq E_{\mathbb{C}}$ is finite-dimensional and $\rho_{\mathbb{C}}$-invariant, its real and imaginary parts span a finite-dimensional ρ-invariant subspace of E). We recall why the tensor product $\pi_1 \otimes \pi_2$ of two unitary representations on Hilbert spaces H_1, H_2 has no nonzero finite-dimensional invariant subspace whenever π_1 has none: identifying $H_1 \widehat{\otimes} H_2$ with the space of Hilbert–Schmidt operators from the conjugate space $\overline{H_2}$ to H_1 (under which $(\pi_1 \otimes \pi_2)(g)$ acts by $T \mapsto \pi_1(g) T \overline{\pi_2(g)}^{*}$, where $\overline{\pi_2}$ is the conjugate representation on $\overline{H_2}$), any finite-dimensional invariant subspace V with orthonormal basis $T_1, \dots, T_d$ defines a positive compact operator $A = \sum_{j=1}^{d} T_j T_j^*$ on H_1 that is independent of the chosen orthonormal basis of V, hence commutes with $\pi_1(G)$. If $V \neq \{0\}$, then $\mathrm{Tr}(A) = d > 0$, so by the spectral theorem for compact self-adjoint operators, A has a positive eigenvalue whose eigenspace is a nonzero finite-dimensional $\pi_1(G)$-invariant subspace of H_1, a contradiction. For each $n \geq 2$, the n-th chaos $\mathcal{H}_n$ is an invariant subspace of $E_{\mathbb{C}} \widehat{\otimes} E_{\mathbb{C}}^{\otimes(n-1)}$, so applying this fact with $\pi_1 = \rho_{\mathbb{C}}$ and $\pi_2 = \rho_{\mathbb{C}}^{\otimes(n-1)}$ shows that $\mathcal{H}_n$ has no nonzero finite-dimensional invariant subspace (and the case $n = 1$ is $\rho_{\mathbb{C}}$ itself). If $V \subseteq L_0^2(\Omega_E, \gamma_E) = \bigoplus_{n \geq 1} \mathcal{H}_n$ is a finite-dimensional invariant subspace, then for every $n \geq 1$ the orthogonal projection of V onto $\mathcal{H}_n$ is a finite-dimensional invariant subspace of $\mathcal{H}_n$ (since the chaos projections commute with the Koopman representation), hence trivial; thus $V = \{0\}$. Therefore $\Gamma \circ \rho$ is weakly mixing, and in particular ergodic. □

Our fourth ingredient shows that in the Gaussian realization of a complex Hilbert space $\mathcal{H}$, a finite-rank orthogonal projection P of rank $N \geq 2$ defines a natural half-set A whose displacement by a unitary $u \in \mathrm{U}(\mathcal{H})$ is controlled by the normalized Hilbert–Schmidt distance $\|uPu^* - P\|_{\mathrm{HS}}/\sqrt{N}$.

Lemma 4.4 (Gaussian half-set estimate for finite-rank projections)**.** *Let $\mathcal{H}$ be a separable complex Hilbert space, regarded as the real Hilbert space $\mathcal{H}_{\mathbb{R}}$ with inner product $\mathrm{Re}\langle \cdot, \cdot \rangle$, and let $W : \mathcal{H}_{\mathbb{R}} \to L^2(\Omega, \gamma; \mathbb{R})$ and $\Gamma : \mathrm{O}(\mathcal{H}_{\mathbb{R}}) \to \mathrm{Aut}(\Omega, \gamma)$ be its Gaussian realization from Lemma 4.3. For every orthogonal projection P on $\mathcal{H}$ of finite rank $N \geq 2$, there exists a measurable set*

$A \subseteq \Omega$ with $\gamma(A) = 1/2$ such that for every unitary operator $u \in \mathrm{U}(\mathcal{H})$,

$$\gamma(A \triangle \Gamma(u)A) \leq 3\left(\frac{\|uPu^* - P\|_{\mathrm{HS}}}{\sqrt{N}}\right)^{2/3}. \tag{4.2}$$

Proof. Define $Z : \mathcal{H} \to L^2(\Omega, \gamma; \mathbb{C})$ by

$$Z(v) = \frac{W(v) - iW(iv)}{\sqrt{2}}.$$

Then $Z(iv) = (W(iv) + iW(v))/\sqrt{2} = iZ(v)$, so Z is complex-linear, and the real and imaginary parts of $\{Z(v) : v \in \mathcal{H}\}$ are jointly centered Gaussian. Using $\mathbb{E}[W(v)W(w)] = \mathrm{Re}\langle v, w\rangle$, $\mathrm{Re}\langle iv, iw\rangle = \mathrm{Re}\langle v, w\rangle$, and $\mathrm{Re}\langle v, iw\rangle = -\mathrm{Re}\langle iv, w\rangle = \mathrm{Im}\langle v, w\rangle$ (with inner product linear in the first variable), a direct calculation yields

$$\mathbb{E}\big[Z(v)\overline{Z(w)}\big] = \langle v, w\rangle, \qquad \mathbb{E}\big[Z(v)Z(w)\big] = 0 \qquad (v, w \in \mathcal{H}).$$

In particular, for any orthonormal family (w_j) in $\mathcal{H}$, the variables $Z(w_j)$ are independent standard complex Gaussians (their real and imaginary parts are independent $\mathcal{N}(0, 1/2)$ variables), so $|Z(w_j)|^2$ are independent exponential random variables of mean 1 and variance 1.

For any finite-rank self-adjoint operator $B = \sum_j \lambda_j \langle \cdot, w_j\rangle w_j$ on $\mathcal{H}$ (where (w_j) is a finite orthonormal family of eigenvectors with nonzero real eigenvalues $\lambda_j \in \mathbb{R} \setminus \{0\}$, and the empty sum corresponds to $B = 0$), define the real quadratic form

$$Q_B = \sum_j \lambda_j |Z(w_j)|^2.$$

To see that Q_B is independent of the spectral decomposition and real-linear in B, let $V \subseteq \mathcal{H}$ be any finite-dimensional subspace containing $\mathrm{ran}(B)$, pick an orthonormal basis $(e_k)_{k=1}^d$ of V, put $z_k = Z(e_k)$, and write $[B]_{k\ell} = \langle Be_\ell, e_k\rangle$. Because each w_j lies in $\mathrm{ran}(B) \subseteq V$, expanding $w_j = \sum_k a_{kj} e_k$ gives $[B]_{k\ell} = \sum_j \lambda_j a_{kj}\overline{a_{\ell j}}$ and $Z(w_j) = \sum_k a_{kj} z_k$, whence

$$Q_B = \sum_j \lambda_j \Big|\sum_k a_{kj} z_k\Big|^2 = \sum_{k,\ell=1}^{d} [B]_{k\ell} z_k \overline{z_\ell}.$$

This shows that Q_B is independent of the spectral decomposition, and choosing V to contain $\mathrm{ran}(B_1) + \mathrm{ran}(B_2)$ shows that $B \mapsto Q_B$ is real-linear; in particular, $Q_{uPu^*} - Q_P = Q_{uPu^* - P}$. Moreover, from the spectral representation of B and independence of the $|Z(w_j)|^2$,

$$\mathbb{E}[Q_B] = \sum_j \lambda_j = \mathrm{Tr}(B), \qquad \mathrm{Var}(Q_B) = \sum_j \lambda_j^2 = \|B\|_{\mathrm{HS}}^2.$$

Now let P be an orthogonal projection of rank $N \geq 2$. Then Q_P is a sum of N independent mean-one exponential random variables, so it follows the $\mathrm{Gamma}(N, 1)$ distribution with density $p_N(x) = x^{N-1}e^{-x}/(N-1)!$ $(x > 0)$. Let m_N be the median of $\mathrm{Gamma}(N, 1)$ and set $A = \{Q_P > m_N\}$; because the distribution is continuous, $\gamma(A) = 1/2$. For any unitary $u \in \mathrm{U}(\mathcal{H})$, complex-linearity of u and (4.1) give $Z(v) \circ \Gamma(u)^{-1} = Z(uv)$. If $(w_j)_{j=1}^N$ is an orthonormal basis of $P\mathcal{H}$, then $(uw_j)_{j=1}^N$ is an orthonormal basis of $uPu^*\mathcal{H}$, so $Q_P \circ \Gamma(u)^{-1} = Q_{uPu^*}$, and hence

$$\mathbf{1}_{\Gamma(u)A} = \mathbf{1}_A \circ \Gamma(u)^{-1} = \mathbf{1}_{\{Q_{uPu^*} > m_N\}}.$$

Set $\Delta := \|uPu^* - P\|_{\mathrm{HS}}/\sqrt{N}$. If $\Delta = 0$, then $uPu^* = P$ and $\gamma(A \triangle \Gamma(u)A) = 0$. Assume $\Delta > 0$ and let $t > 0$. On the event $A \triangle \Gamma(u)A$, the threshold m_N lies between Q_P and Q_{uPu^*}, so either $|Q_P - m_N| \leq t$ or $|Q_{uPu^*} - Q_P| > t$. The density $p_N(x)$ attains its maximum at

$x = N-1$, where Stirling's inequality $(N-1)! \geq \sqrt{2\pi(N-1)}\,((N-1)/e)^{N-1}$ (for $N \geq 2$) gives

$$p_N(x) \leq p_N(N-1) = \frac{(N-1)^{N-1}e^{-(N-1)}}{(N-1)!} \leq \frac{1}{\sqrt{2\pi(N-1)}} \leq \frac{1}{\sqrt{\pi N}},$$

so $\gamma(|Q_P - m_N| \leq t) \leq 2t/\sqrt{\pi N}$. On the other hand, $Q_{uPu^*} - Q_P = Q_{uPu^*-P}$ has mean $\mathrm{Tr}(uPu^* - P) = 0$ and variance $\|uPu^* - P\|_{\mathrm{HS}}^2 = N\Delta^2$, so Chebyshev's inequality gives $\gamma(|Q_{uPu^*} - Q_P| > t) \leq N\Delta^2/t^2$. Combining both bounds and choosing $t = \sqrt{N}\,\Delta^{2/3}$ yields

$$\gamma(A \triangle \Gamma(u)A) \leq \frac{2t}{\sqrt{\pi N}} + \frac{N\Delta^2}{t^2} = \left(\frac{2}{\sqrt{\pi}} + 1\right)\Delta^{2/3} \leq 3\Delta^{2/3},$$

since $2/\sqrt{\pi} + 1 < 3$. □

The dyadic-energy Lévy group G_w. We now build a family of monothetic abelian Lévy groups G_w parameterized by a weight sequence $w = (w_m)_{m\geq 0}$. The group is obtained by completing the dyadic step functions $[0,1) \to \mathbb{T}$ under an invariant Hilbert-space metric that penalizes oscillation at dyadic scale 2^{-m} with weight w_m. Two competing growth conditions govern w_m: slow cumulative growth ($2^{-n/2}\sum_{m<n} w_m \to 0$) ensures concentration of measure (making G_w a Lévy group), while divergence ($w_m \to \infty$) forces elements in any metric ball to be uniformly close to their dyadic averages $E_n f$ (which will destroy uniform whirliness via Lemma 4.4). Both conditions hold simultaneously, for example, for $w_m = m+1$ or $w_m = 2^{\alpha m}$ $(0 < \alpha < 1/2)$.

Let $\mathcal{D}_m$ denote the partition of $[0,1)$ into the 2^m half-open dyadic intervals $[j2^{-m}, (j+1)2^{-m})$ $(0 \leq j < 2^m)$, let E_m be the conditional expectation operator onto $\sigma(\mathcal{D}_m)$ on $L^2([0,1);\mathbb{C})$, and let K_n be the group of $\sigma(\mathcal{D}_n)$-measurable functions $[0,1) \to \mathbb{T}$ under pointwise multiplication. Then $K_n \cong \mathbb{T}^{2^n}$, $K_0 \leq K_1 \leq \cdots$, and we set $K_* = \bigcup_{n=0}^{\infty} K_n$.

Proposition 4.5 (The dyadic-energy Lévy group)**.** *Let $w = (w_m)_{m\geq 0}$ be a sequence of positive real numbers satisfying*

$$\lim_{n\to\infty} 2^{-n/2} \sum_{m=0}^{n-1} w_m = 0. \tag{4.3}$$

On K_*, define

$$D_w(f,g)^2 = \|f-g\|_2^2 + \sum_{m=0}^{\infty} w_m \|f\bar{g} - E_m(f\bar{g})\|_2^2. \tag{4.4}$$

Then D_w is an invariant metric on K_*, and its completion G_w is a nondiscrete, abelian, monothetic Lévy group with the following properties:

(i) G_w admits a continuous injective homomorphism into $L^0([0,1),\mathbb{T})$ (circle-valued measurable functions modulo null sets, equipped with the topology of convergence in measure), and (4.4) holds for all $f, g \in G_w$; in particular,

$$\|f - E_n f\|_2^2 \leq \frac{D_w(f,1)^2}{w_n} \qquad \text{for all } f \in G_w,\ n \geq 0.$$

(ii) There exist a separable real Hilbert space $\mathcal{K}$ and a map $b : G_w \to \mathcal{K}$ such that $D_w(f,g) = \|b(f) - b(g)\|_{\mathcal{K}}$ for all $f, g \in G_w$.

(iii) The topological generators of G_w form a dense G_δ subset of G_w.

Proof. Step 1: Hilbert embedding, metric structure, and (i)–(ii). For each $m \geq 0$, let λ_m be the Borel probability measure on $[0,1)^2$ supported on $\bigcup_{I\in\mathcal{D}_m}(I \times I)$ with density 2^m relative to planar Lebesgue measure (equivalently, λ_m chooses an interval $I \in \mathcal{D}_m$ uniformly and then picks (x,y) uniformly and independently in $I \times I$). Both marginals of λ_m are Lebesgue measure

on $[0,1)$. For any $f \in L^0([0,1),\mathbb{T})$, define $\Phi_f(x,y) = f(x)\overline{f(y)}$. Then for any $f,g \in L^0([0,1),\mathbb{T})$, writing $h = f\bar{g}$ (so $|h| = 1$), we have

$$\int_{[0,1)^2} \Phi_h \, d\lambda_m = \sum_{I \in \mathcal{D}_m} 2^{-m} \left| \frac{1}{|I|} \int_I h(x)\,dx \right|^2 = \|E_m h\|_2^2,$$

and therefore, since E_m is an orthogonal projection and $\|h\|_2 = 1$,

$$\|\Phi_f - \Phi_g\|_{L^2(\lambda_m)}^2 = \int_{[0,1)^2} (2 - 2\operatorname{Re}\Phi_h)\, d\lambda_m = 2(1 - \|E_m h\|_2^2) = 2\|h - E_m h\|_2^2.$$

Let $\mathcal{K} = L^2([0,1);\mathbb{C}) \oplus \bigoplus_{m=0}^{\infty} L^2([0,1)^2, \lambda_m; \mathbb{C})$, regarded as a separable real Hilbert space with inner product $\operatorname{Re}\langle\cdot,\cdot\rangle$. For $f \in K_*$, all but finitely many terms $f - E_m f$ vanish, and by the preceding identity $\|\Phi_f - 1\|_{L^2(\lambda_m)}^2 = 2\|f - E_m f\|_2^2$, so

$$b(f) \coloneqq \Big(f - 1,\ \big(\sqrt{w_m/2}\,(\Phi_f - 1)\big)_{m \ge 0} \Big)$$

belongs to $\mathcal{K}$, and $\|b(f) - b(g)\|_{\mathcal{K}}^2 = D_w(f,g)^2$. Hence D_w is a metric on K_*, since the first coordinate separates functions modulo null sets. Because $|f| = |g| = 1$ implies $\|f - g\|_2 = \|f\bar{g} - 1\|_2$, (4.4) shows that D_w is invariant under multiplication and inversion (complex conjugation), whence $D_w(fg, f'g') \le D_w(f,f') + D_w(g,g')$. Thus multiplication and inversion extend continuously to the completion G_w of (K_*, D_w), making G_w a complete abelian metric group.

On each K_n, all terms in (4.4) with $m \ge n$ vanish; the remaining finite sum is continuous for the compact torus topology on $K_n \cong \mathbb{T}^{2^n}$ and dominates the Hausdorff L^2-metric, so D_w induces the compact torus topology on K_n. In particular, each K_n is separable in D_w, so $K_* = \bigcup_{n=0}^{\infty} K_n$ is separable and G_w is a Polish abelian group; moreover, $G_w \supseteq K_0 \cong \mathbb{T}$ is nondiscrete.

If (f_j) is a D_w-Cauchy sequence in K_*, then $(b(f_j))$ converges in $\mathcal{K}$ to some vector $v = (v_*, (v_m)_{m\ge0})$. From the first coordinate, $f_j \to f \coloneqq 1 + v_*$ in $L^2([0,1);\mathbb{C})$, so $|f| = 1$ almost everywhere (passing to an almost-everywhere convergent subsequence). For each fixed $m \ge 0$, the bound $|\Phi_{f_j}(x,y) - \Phi_f(x,y)| \le |f_j(x) - f(x)| + |f_j(y) - f(y)|$ and the marginal property of λ_m yield $\|\Phi_{f_j} - \Phi_f\|_{L^2(\lambda_m)} \le 2\|f_j - f\|_2 \to 0$. Thus $v_m = \sqrt{w_m/2}\,(\Phi_f - 1)$ for all m, so $v = b(f) \in \mathcal{K}$. It follows that G_w injects continuously into $L^0([0,1),\mathbb{T})$ (with group operations agreeing with pointwise multiplication and conjugation), that $D_w(f,g) = \|b(f) - b(g)\|_{\mathcal{K}}$ and (4.4) hold for all $f,g \in G_w$, and that dropping all terms except $m = n$ in $D_w(f,1)^2$ gives $\|f - E_n f\|_2^2 \le D_w(f,1)^2/w_n$. This proves (i) and (ii).

Step 2: G_w is a Lévy group. Put $N = 2^n$ and $\Sigma_n = 1 + \sum_{m=0}^{n-1} w_m$. For $f,g \in K_n$ and $h = f\bar{g}$, we have $E_m h = h$ for all $m \ge n$, while for $m < n$ the orthogonal projection E_m fixes constants, so $\|h - E_m h\|_2 \le \|h - 1\|_2 = \|f - g\|_2$. Hence

$$D_w(f,g)^2 \le \Sigma_n \|f - g\|_2^2 = \frac{\Sigma_n}{N} \sum_{j=1}^{N} |f_j - g_j|^2, \tag{4.5}$$

where $f_1, \dots, f_N \in \mathbb{T}$ are the values of f on the N intervals of $\mathcal{D}_n$. In particular, if $f,g \in K_n$ differ in at most k coordinates, then $D_w(f,g)^2 \le 4\Sigma_n k/N$.

Let h_n be the normalized Haar measure on $K_n \cong \mathbb{T}^N$ (the product of N uniform measures on $\mathbb{T}$). Let $A_n \subseteq K_n$ be Borel sets with $\liminf_{n\to\infty} h_n(A_n) > 0$, so $h_n(A_n) \ge a > 0$ for all sufficiently large n, and fix $\varepsilon > 0$. For each such n, by inner regularity of h_n choose a compact subset $C_n \subseteq A_n$ with $h_n(C_n) \ge a/2$, and let $L_n(x)$ be the Hamming distance from $x \in \mathbb{T}^N$ to C_n—the least number of coordinates of x that must be changed to land in C_n. The function L_n

takes values in $\{0,1,\dots,N\}$, and for each integer $0 \le k \le N$ the sublevel set $\{L_n \le k\}$ is a finite union of preimages $\pi_S^{-1}(\pi_S(C_n))$ under coordinate projections $\pi_S : \mathbb{T}^N \to \mathbb{T}^S$ ($|S| = N-k$), hence is closed (as C_n is compact); thus L_n is Borel measurable. Changing a single coordinate changes L_n by at most 1, and $L_n = 0$ on C_n. Lemma 4.1 therefore gives $\mathrm{Var}(L_n) \le N/4$ and

$$h_n(L_n \ge k) \le \frac{N}{2ak^2} \qquad \text{for every } k \ge 1.$$

Put $k_n = \lfloor \varepsilon^2 N/(5\Sigma_n) \rfloor$. By (4.3), $\Sigma_n/\sqrt{N} \to 0$, so $k_n/\sqrt{N} \to \infty$; in particular $k_n \ge 1$ for large n. Whenever $L_n(x) \le k_n$, there exists $y \in C_n \subseteq A_n$ differing from x in at most k_n coordinates, so by (4.5) we have $D_w(x,y)^2 \le 4\Sigma_n k_n/N \le \frac{4}{5}\varepsilon^2 < \varepsilon^2$, meaning $x \in B_{D_w}(A_n,\varepsilon)$. Therefore

$$1 - h_n\big(B_{D_w}(A_n,\varepsilon)\big) \le h_n(L_n > k_n) \le \frac{N}{2ak_n^2} \longrightarrow 0 \qquad (n \to \infty).$$

Thus $(K_n, D_w|_{K_n}, h_n)$ is a Lévy family. Since D_w is a compatible invariant metric and $\bigcup_n K_n = K_*$ is dense in G_w, G_w is a Lévy group.

Step 3: Topological generators form a dense G_δ. Choose a countable dense sequence $(a_j)_{j\ge1}$ in K_*. The set of topological generators of G_w is

$$\bigcap_{j,\ell\ge1} \bigcup_{m\in\mathbb{Z}} \{t \in G_w : D_w(t^m, a_j) < 1/\ell\}.$$

Each union is open in G_w. To verify density, let $V \subseteq G_w$ be nonempty and open, and fix $j,\ell \ge 1$. Because K_* is dense and the subgroups K_n increase, there is n such that $V \cap K_n \ne \emptyset$ and $a_j \in K_n$. Since D_w induces the compact torus topology on $K_n \cong \mathbb{T}^{2^n}$, Kronecker's theorem implies that the elements of K_n whose angular coordinates together with 2π are $\mathbb{Q}$-linearly independent are dense in K_n and generate dense cyclic subgroups of K_n. Picking such an element $t \in V \cap K_n$, some integer power t^m satisfies $D_w(t^m, a_j) < 1/\ell$. By the Baire category theorem, the intersection is a dense G_δ subset of G_w, so G_w is monothetic. □

Proof of Theorem 1.3. We now combine Proposition 4.5 with Lemmas 4.2, 4.3, and 4.4 to construct the desired transformation T.

Proof of Theorem 1.3. Fix any weight sequence $w = (w_m)_{m\ge0}$ of positive real numbers satisfying (4.3) and $\lim_{n\to\infty} w_n = \infty$, for instance $w_m = m+1$, and let G_w be the monothetic abelian Lévy group of Proposition 4.5.

Step 1: A faithful orthogonal representation with no invariant vectors. By Proposition 4.5(ii), $D_w(f,g) = \|b(f) - b(g)\|_{\mathcal{K}}$ for a map $b : G_w \to \mathcal{K}$ into a real Hilbert space. Following Schoenberg [11], consider the kernel

$$k(f,g) = e^{-D_w(f,g)^2} = e^{-\|b(f)\|_{\mathcal{K}}^2}\, e^{-\|b(g)\|_{\mathcal{K}}^2} \sum_{r=0}^{\infty} \frac{2^r}{r!} \langle b(f), b(g)\rangle_{\mathcal{K}}^r \qquad (f,g \in G_w).$$

For every $r \ge 0$, $\langle b(f), b(g)\rangle_{\mathcal{K}}^r = \langle b(f)^{\otimes r}, b(g)^{\otimes r}\rangle_{\mathcal{K}^{\otimes r}}$ is positive definite, so k is a *positive definite kernel* on G_w (meaning $\sum_{j,\ell=1}^m c_j c_\ell k(f_j,f_\ell) \ge 0$ for all finite tuples $f_1,\dots,f_m \in G_w$ and $c_1,\dots,c_m \in \mathbb{R}$). Let F be the real Hilbert space obtained by completing the linear span of formal symbols $\{v_g : g \in G_w\}$ with inner product $\langle v_f, v_g\rangle = k(f,g)$, modulo null vectors. By invariance of D_w, the formula $\rho(a)v_g = v_{ag}$ defines an orthogonal representation $\rho : G_w \to \mathrm{O}(F)$. For every $h \in G_w$,

$$\|\rho(g)v_h - v_h\|^2 = 2\big(1 - e^{-D_w(g,1)^2}\big),$$

so ρ is strongly continuous on the dense span of $\{v_g\}$ and hence on F; moreover F is separable because $g \mapsto v_g$ is continuous and G_w is separable.

Let $F_0 \subseteq F$ be the orthogonal complement of the subspace of $\rho(G_w)$-invariant vectors, let $\rho_0 : G_w \to \mathrm{O}(F_0)$ be the restriction of ρ, and let v_0 be the orthogonal projection of v_1 onto F_0. Because $\rho(g)$ fixes $v_1 - v_0$, we have $\rho_0(g)v_0 - v_0 = \rho(g)v_1 - v_1$, and therefore

$$\|\rho_0(g)v_0 - v_0\|^2 = 2(1 - e^{-D_w(g,1)^2}) \qquad \text{for all } g \in G_w. \tag{4.6}$$

By construction, ρ_0 has no nonzero invariant vector.

Step 2: The multiplication representation and Gaussian near-action. Let $\mathcal{H} = L^2([0,1);\mathbb{C})$, and for $f \in G_w \subseteq L^0([0,1),\mathbb{T})$ let $M_f \in \mathrm{U}(\mathcal{H})$ be the unitary multiplication operator $(M_f\xi)(x) = f(x)\xi(x)$. As $D_w(f,1) \to 0$, we have $\|f - 1\|_2 \to 0$, so $M_f\xi \to \xi$ in $\mathcal{H}$ for every bounded $\xi \in \mathcal{H}$ and hence for all $\xi \in \mathcal{H}$; thus $M : G_w \to \mathrm{U}(\mathcal{H})$ is strongly continuous. Since the constant function $-1 \in K_0 \leq G_w$ satisfies $M_{-1} = -I$, the realified orthogonal representation $M_{\mathbb{R}} : G_w \to \mathrm{O}(\mathcal{H}_{\mathbb{R}})$ has no nonzero invariant vector.

Consider the orthogonal direct sum $\pi = \rho_0 \oplus M_{\mathbb{R}}$ on $E = F_0 \oplus \mathcal{H}_{\mathbb{R}}$, and let

$$\alpha = \Gamma \circ \pi : G_w \longrightarrow \mathrm{Aut}(\Omega,\gamma)$$

be its Gaussian realization given by Lemma 4.3. Since π has no nonzero invariant vector and G_w is a Lévy group, Lemma 4.2 shows that π has no nonzero finite-dimensional invariant subspace. By Lemma 4.3, (Ω,γ) is a standard atomless probability space, α is weakly mixing (hence ergodic), and α is the independent product of the Gaussian factors of ρ_0 and $M_{\mathbb{R}}$. Since G_w is a Lévy group, Theorem 2.6 implies that the near-action α is whirly.

Moreover, $\alpha : G_w \to \mathrm{Aut}(\Omega,\gamma)$ is a topological group embedding with closed image. Indeed, by (4.1) and (4.6),

$$\|U_{\alpha(g)}W(v_0) - W(v_0)\|_2^2 = \|\rho_0(g)v_0 - v_0\|^2 = 2(1 - e^{-D_w(g,1)^2}) \qquad (g \in G_w).$$

Thus α is injective, and $\alpha(g_j) \to \mathrm{Id}$ in $\mathrm{Aut}(\Omega,\gamma)$ forces $D_w(g_j,1) \to 0$; combined with continuity of α, this shows that α is a homeomorphism onto its image. If $(\alpha(g_j))$ converges in $\mathrm{Aut}(\Omega,\gamma)$, then $\alpha(g_j^{-1}g_k) \to \mathrm{Id}$ as $j,k \to \infty$, so $D_w(g_j,g_k) \to 0$; by completeness of G_w, (g_j) converges to some $g \in G_w$, and $\alpha(g_j) \to \alpha(g)$. Hence $\alpha(G_w)$ is closed in $\mathrm{Aut}(\Omega,\gamma)$.

Step 3: Uniformly almost-invariant half-sets on D_w-balls. For each $n \geq 1$, put $N = 2^n \geq 2$ and let P_n be the rank-N orthogonal projection of $\mathcal{H} = L^2([0,1);\mathbb{C})$ onto the $\sigma(\mathcal{D}_n)$-measurable step functions. In the orthonormal basis $e_I = 2^{n/2}\mathbf{1}_I$ $(I \in \mathcal{D}_n)$ of $P_n\mathcal{H}$, the operator $P_nM_fP_n$ is diagonal with entries $\langle M_fe_I, e_I\rangle = |I|^{-1}\int_I f(x)\,dx$, so $\|P_nM_fP_n\|_{\mathrm{HS}}^2 = N\|E_nf\|_2^2$. Consequently, for every $f \in G_w$ with $D_w(f,1) < 1$, Proposition 4.5(i) gives

$$\begin{aligned}\|M_fP_nM_f^* - P_n\|_{\mathrm{HS}}^2 &= \mathrm{Tr}\big((M_fP_nM_f^* - P_n)^2\big) = 2N - 2\,\mathrm{Tr}(M_fP_nM_f^*P_n)\\ &= 2N - 2\|P_nM_fP_n\|_{\mathrm{HS}}^2 = 2N(1 - \|E_nf\|_2^2) = 2N\|f - E_nf\|_2^2 \leq \frac{2N}{w_n}.\end{aligned}$$

Applying Lemma 4.4 to the projection P_n on the Gaussian factor $(\Omega_{\mathcal{H}_{\mathbb{R}}}, \gamma_{\mathcal{H}_{\mathbb{R}}})$ of $M_{\mathbb{R}}$ yields a measurable set $\widetilde{A}_n \subseteq \Omega_{\mathcal{H}_{\mathbb{R}}}$ with $\gamma_{\mathcal{H}_{\mathbb{R}}}(\widetilde{A}_n) = 1/2$. Pulling it back to $A_n := \Omega_{F_0} \times \widetilde{A}_n \subseteq \Omega = \Omega_{F_0} \times \Omega_{\mathcal{H}_{\mathbb{R}}}$ gives $\gamma(A_n) = 1/2$ and, since $\alpha(f) = \Gamma(\rho_0(f)) \times \Gamma(M_f)$,

$$\begin{aligned}\sup_{D_w(f,1)<1} \gamma(A_n \triangle \alpha(f)A_n) &= \sup_{D_w(f,1)<1} \gamma_{\mathcal{H}_{\mathbb{R}}}(\widetilde{A}_n \triangle \Gamma(M_f)\widetilde{A}_n)\\ &\leq 3\left(\frac{\sqrt{2}}{\sqrt{w_n}}\right)^{2/3} = 3\cdot 2^{1/3}\,w_n^{-1/3} \longrightarrow 0 \qquad (n \to \infty).\end{aligned} \tag{4.7}$$

Step 4: Passing to a topological generator. By Proposition 4.5(iii), G_w has a topological generator $t \in G_w$. Set $(X,\mu) = (\Omega,\gamma)$ and $T = \alpha(t) \in \mathrm{Aut}(X,\mu)$. Since α is a topological

embedding with closed image, the cyclic closure of T is $\Lambda(T) = \overline{\{\alpha(t^m) : m \in \mathbb{Z}\}} = \alpha(G_w)$, and $\alpha : G_w \to \Lambda(T)$ is a topological group isomorphism. Thus $\Lambda(T)$ is a Lévy group, and its natural near-action is isomorphic to α, hence whirly.

To see that T is rigid, observe that if 1 were isolated in $\{t^m : m \in \mathbb{Z}\}$, then $\{t^m : m \in \mathbb{Z}\}$ would be a discrete—hence closed—subgroup of G_w, forcing $G_w = \{t^m : m \in \mathbb{Z}\}$ to be countable, which contradicts $K_0 \cong \mathbb{T} \subseteq G_w$. Hence there is a sequence of pairwise distinct nonzero integers m_j with $t^{m_j} \to 1$; passing to a subsequence with $|m_j| \to \infty$ and replacing m_j by $|m_j|$ (using $D_w(t^{-m}, 1) = D_w(t^m, 1)$) yields a sequence $n_j \to \infty$ with $t^{n_j} \to 1$, so $T^{n_j} \to \mathrm{Id}$ in $\mathrm{Aut}(X, \mu)$. Thus T is rigid, and by Lemma 2.9(iii) it is whirly.

Finally, because α is a topological embedding, there exists a weak identity neighborhood $U \subseteq \mathrm{Aut}(X, \mu)$ such that $\alpha(G_w) \cap U \subseteq \alpha(\{f \in G_w : D_w(f, 1) < 1\})$. For each $n \geq 1$, take A_n from Step 3 and $B_n = X \setminus A_n$, so $\mu(A_n) = \mu(B_n) = 1/2$. Whenever $T^m = \alpha(f) \in U$, we have $D_w(f, 1) < 1$ and $T^m A_n \cap B_n = T^m A_n \setminus A_n \subseteq T^m A_n \triangle A_n$, so (4.7) yields

$$\sup_{m \in \mathbb{Z} : T^m \in U} \mu(T^m A_n \cap B_n) \leq \sup_{D_w(f,1)<1} \mu(A_n \triangle \alpha(f) A_n) \longrightarrow 0 \qquad (n \to \infty).$$

Hence $\inf_{\mu(A)=\mu(B)=1/2} \sup_{m \in \mathbb{Z} : T^m \in U} \mu(T^m A \cap B) = 0$, and T is not uniformly whirly. □

5. Loss of uniform whirliness under independent squares

In contrast to Corollary 1.2, uniform whirliness does not pass to independent products, even to the independent square of a single near-action. We first record two auxiliary lemmas: one for selecting a small base subset carrying a proportional share of given fiber densities, and one extracting a quantitative hitting bound from ergodicity of any group action.

Lemma 5.1 (Simultaneous subset selection)**.** *Let (Y, ν) be a standard atomless probability space, let $f_1, \ldots, f_r : Y \to [0, 1]$ be measurable functions with positive integrals $\alpha_j = \int_Y f_j \, d\nu > 0$, and let $t, \delta \in (0, 1)$. Then there exists a measurable set $S \subseteq Y$ such that*

$$\nu(S) \leq (1 + \delta)t \qquad \text{and} \qquad \int_S f_j \, d\nu \geq (1 - \delta)t\alpha_j \quad \text{for all } 1 \leq j \leq r.$$

Proof. Because (Y, ν) is atomless, for each integer $k \geq 1$ we may partition Y into k measurable sets $Y_1, \ldots, Y_k$ of measure $1/k$. Let $\xi_1, \ldots, \xi_k$ be independent Bernoulli random variables with $\mathbb{P}(\xi_i = 1) = t$, and form the random measurable set $S = \bigcup_{i : \xi_i = 1} Y_i$. For any measurable function $f : Y \to [0, 1]$ with integral $\alpha = \int_Y f \, d\nu > 0$ (including $f \equiv 1$ with $\alpha = 1$), the random variable $Z_f = \int_S f \, d\nu = \sum_{i=1}^k \xi_i \int_{Y_i} f \, d\nu$ has expectation $\mathbb{E}[Z_f] = t\alpha$ and variance

$$\mathrm{Var}(Z_f) = t(1 - t) \sum_{i=1}^k \Big(\int_{Y_i} f \, d\nu \Big)^2 \leq \frac{1}{4} \sum_{i=1}^k \frac{1}{k^2} = \frac{1}{4k}.$$

By Chebyshev's inequality, $\mathbb{P}(|Z_f - t\alpha| > \delta t\alpha) \leq (4k\delta^2 t^2 \alpha^2)^{-1} \to 0$ as $k \to \infty$. Applying a union bound over the $r + 1$ functions $1, f_1, \ldots, f_r$, we find that for sufficiently large k the desired inequalities hold simultaneously with positive probability. □

Remark 5.2. By Lyapunov's convexity theorem for atomless vector measures, one can in fact achieve exact equality $\nu(S) = t$ and $\int_S f_j \, d\nu = t\alpha_j$; the probabilistic proof above suffices for our purposes.

The next lemma is a standard consequence of the Alaoglu–Birkhoff mean ergodic principle [1], valid for an arbitrary ergodic group action with no amenability or topological assumptions.

Lemma 5.3 (Ergodic hitting bound)**.** *Let a group L act by measure-preserving automorphisms on a probability space (Z, ρ), and suppose the action is ergodic. Then for all measurable sets $A, B \subseteq Z$,*

$$\sup_{g \in L} \rho(A \cap gB) \geq \rho(A)\rho(B).$$

Proof. Let $M \subseteq L^2(Z, \rho)$ be the closed convex hull of $\{U_g \mathbf{1}_B : g \in L\}$, where $U_g f = f \circ g^{-1}$. Because M is a nonempty closed convex subset of the Hilbert space $L^2(Z, \rho)$, it contains a unique element f_0 of minimal norm. Each U_g is unitary and maps M onto itself, so $U_g f_0 \in M$ has the same norm as f_0, forcing $U_g f_0 = f_0$ for all $g \in L$. By ergodicity, f_0 is constant almost everywhere. Since U_g, convex combinations, and L^2-limits preserve the integral, every element of M has integral $\rho(B)$, and hence $f_0 = \rho(B)$ almost everywhere.

Set $s = \sup_{g \in L} \rho(A \cap gB) = \sup_{g \in L} \langle U_g \mathbf{1}_B, \mathbf{1}_A \rangle$. The half-space $\{f \in L^2(Z, \rho) : \mathrm{Re}\langle f, \mathbf{1}_A \rangle \leq s\}$ is closed and convex, and contains $\{U_g \mathbf{1}_B : g \in L\}$, so it contains M. Evaluating at $f_0 = \rho(B)$ yields $\rho(A)\rho(B) = \langle f_0, \mathbf{1}_A \rangle \leq s$. □

We now construct the near-action of Theorem 1.4, using the Cantor group $K = \{-1, 1\}^{\mathbb{N}}$ as the fiber.

Proof of Theorem 1.4. Step 1: The Polish group G. Let (Y, ν) be $[0, 1)$ with Lebesgue measure, let $\mathcal{A} = \mathrm{Aut}(Y, \nu)$ with the weak topology, and let $K = \{-1, 1\}^{\mathbb{N}}$ with the product topology, coordinatewise multiplication, and normalized Haar probability measure m (the product of uniform measures on $\{-1, 1\}$). Fix any sequence of positive weights $w_n \to \infty$ (for instance, $w_n = n$). Define

$$C = \Big\{ c = (c_n)_{n \geq 1} \in L^0(Y, \{-1, 1\})^{\mathbb{N}} : \sum_{n=1}^{\infty} w_n \, \nu(\{c_n = -1\}) < \infty \Big\},$$

with coordinatewise multiplication and distance

$$D(c, d) = \sum_{n=1}^{\infty} w_n \, \nu(\{c_n \neq d_n\}).$$

For $c, d, h \in C$, because $\{c_n d_n = -1\} \subseteq \{c_n = -1\} \cup \{d_n = -1\}$ and $\{c_n \neq h_n\} \subseteq \{c_n \neq d_n\} \cup \{d_n \neq h_n\}$, C is an abelian group (every element is its own inverse) and D is a translation-invariant metric on C, making the group operations continuous. To verify completeness, let $(c^{(j)})$ be D-Cauchy. Each coordinate $(c_n^{(j)})_j$ is Cauchy in $L^0(Y, \{-1, 1\})$ under the complete metric $(a, b) \mapsto \nu(\{a \neq b\})$ and thus converges to some $c_n \in L^0(Y, \{-1, 1\})$. For any fixed j and M,

$$\sum_{n=1}^{M} w_n \, \nu(\{c_n^{(j)} \neq c_n\}) = \lim_{k \to \infty} \sum_{n=1}^{M} w_n \, \nu(\{c_n^{(j)} \neq c_n^{(k)}\}) \leq \limsup_{k \to \infty} D(c^{(j)}, c^{(k)}).$$

Letting $M \to \infty$ gives $\sum_{n=1}^{\infty} w_n \, \nu(\{c_n^{(j)} \neq c_n\}) \leq \limsup_{k \to \infty} D(c^{(j)}, c^{(k)}) < \infty$. Since $\{c_n = -1\} \subseteq \{c_n^{(j)} = -1\} \cup \{c_n^{(j)} \neq c_n\}$, it follows that $\sum_{n=1}^{\infty} w_n \, \nu(\{c_n = -1\}) \leq D(c^{(j)}, 1) + \limsup_{k \to \infty} D(c^{(j)}, c^{(k)}) < \infty$, so $c \in C$, and hence $D(c^{(j)}, c) \leq \limsup_{k \to \infty} D(c^{(j)}, c^{(k)}) \to 0$ as $j \to \infty$. Sequences $c \in C$ with $c_n = 1$ for all sufficiently large n and each $\{c_n = -1\}$ a finite union of rational intervals form a countable dense subset of C, so (C, D) is a Polish group.

The group $\mathcal{A} = \mathrm{Aut}(Y, \nu)$ acts on C by automorphisms $\beta_T(c) = c \circ T^{-1}$ (applied coordinatewise). Each β_T is a D-isometry since T preserves ν. For fixed $c \in C$ and $\varepsilon > 0$, choose N so

that $\sum_{n>N} w_n \nu(\{c_n = -1\}) < \varepsilon/4$; then for any $T, T' \in \mathcal{A}$, writing $B_n = \{c_n = -1\}$,

$$D(\beta_T(c), \beta_{T'}(c)) \le \sum_{n=1}^{N} w_n \, \nu(TB_n \triangle T'B_n) + \frac{\varepsilon}{2},$$

which is less than ε whenever T' lies in a sufficiently small weak neighborhood of T. Because β_T is isometric, $(c, T) \mapsto \beta_T(c)$ is jointly continuous. Therefore the semidirect product $C \rtimes_\beta \mathcal{A}$, with multiplication $(c, T)(d, S) = (c\,(d \circ T^{-1}), TS)$, and the direct product

$$G = (C \rtimes_\beta \mathcal{A}) \times K$$

are Polish groups.

Step 2: The near-action on $X = Y \times K$. Equip $X = Y \times K$ with the product probability measure $\mu = \nu \times m$, a standard atomless probability space. Define the action of $g = (c, T, k) \in G$ on $(y, z) \in Y \times K$ by

$$g \cdot (y, z) = \big(Ty,\ k\, c(Ty)\, z\big),$$

where multiplication in the second coordinate is coordinatewise in K (after choosing representatives of T and the countably many c_n). This preserves μ because T preserves ν and fiberwise translations preserve m. A direct check verifies the group law modulo null sets:

$$(c, T, k) \cdot \big((d, S, k') \cdot (y, z)\big) = \big(TSy,\ kk'\, c(TSy)\, d(Sy)\, z\big) = \big(c\,(d \circ T^{-1}),\, TS,\, kk'\big) \cdot (y, z).$$

To check continuity of $G \to \mathrm{Aut}(X, \mu)$, let $B \subseteq Y$ be measurable and let $Z \subseteq K$ be a cylinder depending only on coordinates $1, \dots, N$. For any $g = (c, T, k) \in G$ lying in the open identity neighborhood $\{k_1 = \cdots = k_N = 1\}$, the set $g(B \times Z)$ agrees with $(TB) \times Z$ outside $\{y \in Y : \exists\, j \le N,\ c_j(y) = -1\} \times K$, and hence

$$\mu\big(g(B \times Z) \triangle (B \times Z)\big) \le \nu(TB \triangle B) + \sum_{j=1}^{N} \nu(\{c_j = -1\}) \le \nu(TB \triangle B) + \Big(\max_{1 \le j \le N} w_j^{-1}\Big) D(c, 1),$$

which tends to 0 as $g \to e$. Since finite disjoint unions of such cylinder sets approximate every measurable subset of X in μ-measure, $G \to \mathrm{Aut}(X, \mu)$ is continuous.

Step 3: One copy of X is uniformly whirly. Let $U \subseteq G$ be an identity neighborhood. By definition of the product topology on G, U contains a box

$$\{c \in C : D(c, 1) < \varepsilon\} \times \mathcal{U}(\mathcal{P}, \eta) \times K_{>M},$$

where $\varepsilon, \eta > 0$, $\mathcal{P}$ is a finite measurable partition of Y, $M \ge 0$ is an integer, and $K_{>M} = \{k \in K : k_1 = \cdots = k_M = 1\}$. Put $\Sigma_M = \sum_{j=1}^{M} w_j$ (with $\Sigma_0 = 0$) and choose $t \in (0, 1/2)$ depending only on U such that

$$2\Sigma_M t < \varepsilon \qquad \text{and} \qquad 4t < \eta.$$

Now let $E, F \subseteq X$ be measurable sets with $\mu(E) = \alpha \in (0, 1)$ and $\mu(F) = \beta \in (0, 1)$. Define the fiber-mass functions $a, b : Y \to [0, 1]$ by $a(y) = m(\{z \in K : (y, z) \in E\})$ and $b(y) = m(\{z \in K : (y, z) \in F\})$, so $\int_Y a\, d\nu = \alpha$ and $\int_Y b\, d\nu = \beta$. By Lemma 5.1 (with $\delta = 1/2$), there exists a measurable set $S \subseteq Y$ with

$$0 < p := \nu(S) \le 2t, \qquad \mu(E \cap (S \times K)) = \int_S a\, d\nu \ge \frac{t\alpha}{2}, \qquad \mu(F \cap (S \times K)) = \int_S b\, d\nu \ge \frac{t\beta}{2}.$$

Consider the subgroup $L_S \le G$ consisting of all $g = (c, T, k) \in G$ such that:

- $T \in \mathcal{A}$ is supported on S ($Ty = y$ for almost every $y \notin S$);
- $c_j = 1$ almost everywhere off S for $1 \le j \le M$, and $c_j = 1$ almost everywhere for $j > M$;
- $k \in K_{>M}$.

Because such T preserve S, L_S is indeed a subgroup of G. Every $(c,T,k) \in L_S$ lies in U:

$$D(c,1) = \sum_{j=1}^{M} w_j\, \nu(\{c_j = -1\}) \le \Sigma_M p \le 2\Sigma_M t < \varepsilon,$$

and for each $B \in \mathcal{P}$ we have $B \triangle TB = (B \cap S) \triangle T(B \cap S) \subseteq S$, so

$$\sum_{B\in\mathcal{P}} \nu(B \triangle TB) \le \sum_{B\in\mathcal{P}} \big(\nu(B\cap S) + \nu(T(B\cap S))\big) = 2\nu(S) = 2p \le 4t < \eta.$$

Thus $L_S \subseteq U$.

The set $S \times K$ is L_S-invariant, and we claim that L_S acts ergodically on $(S \times K, p^{-1}\mu)$. For any $r \in K$, define $c \in C$ by setting $c_j = r_j$ on S and 1 off S for $1 \le j \le M$, and $c_j = 1$ for $j > M$; define $k \in K_{>M}$ by $k_j = r_j$ for $j > M$. Then $(c, \mathrm{Id}, k) \in L_S$ acts on $S \times K$ as the constant fiber translation $(y,z) \mapsto (y, rz)$. Consequently, for any L_S-invariant function $f \in L^2(S \times K, p^{-1}\mu)$ and every $r \in K$, we have $f(y,rz) = f(y,z)$ for $(p^{-1}\mu)$-almost every $(y,z) \in S \times K$; applying Fubini's theorem to $(r,y,z) \mapsto f(y,rz)$ on $K \times (S \times K)$ shows that $f(y,z) = \int_K f(y,rz)\, dm(r) = \int_K f(y,u)\, dm(u)$ almost everywhere, so f depends only on y. Since L_S also contains all automorphisms of Y supported on S—which act ergodically on the atomless space $(S, p^{-1}\nu)$—f is constant almost everywhere on $S \times K$.

Applying Lemma 5.3 to L_S on $(S \times K, p^{-1}\mu)$ with the sets $E \cap (S \times K)$ and $F \cap (S \times K)$ and multiplying through by p yields

$$\begin{aligned}\sup_{g\in U} \mu(E \cap gF) &\ge \sup_{g\in L_S} \mu\big(E \cap gF \cap (S \times K)\big) \\ &\ge \frac{\mu(E \cap (S \times K))\, \mu(F \cap (S \times K))}{p} \ge \frac{(t\alpha/2)(t\beta/2)}{2t} = \frac{t\alpha\beta}{8}.\end{aligned}$$

Because t depends only on U, we obtain $\kappa_U^X(\alpha,\beta) \ge t\alpha\beta/8 > 0$. Thus the near-action on (X,μ) is uniformly whirly.

Step 4: The independent square $X \times X$ fails to be uniformly whirly. Let $V = \{(c,T,k) \in G : D(c,1) < 1\}$, an open identity neighborhood of G. Write points of $X \times X$ as $((y,z),(y',z'))$. For each $n \ge 1$, define the relative sign $r_n((y,z),(y',z')) = z_n z_n' \in \{-1,1\}$ and put $E_n = \{r_n = 1\}$ and $F_n = \{r_n = -1\} = (X \times X) \setminus E_n$. Under $\mu \times \mu$, z_n and z_n' are independent uniform signs, so $(\mu\times\mu)(E_n) = (\mu\times\mu)(F_n) = 1/2$.

For any $g = (c,T,k) \in G$ acting diagonally on $X \times X$, the global rotation factor $k_n \in \{-1,1\}$ satisfies $k_n^2 = 1$ and therefore cancels:

$$r_n(g(y,z), g(y',z')) = (k_n c_n(Ty) z_n)(k_n c_n(Ty') z_n') = c_n(Ty)\, c_n(Ty')\, r_n((y,z),(y',z')).$$

Hence a point $((y,z),(y',z'))$ lies in $g^{-1}E_n \triangle E_n$ if and only if $c_n(Ty)c_n(Ty') = -1$, which forces at least one of $c_n(Ty)$ or $c_n(Ty')$ to equal -1. By the union bound and ν-invariance of T,

$$(\mu\times\mu)(E_n \triangle gE_n) = (\mu\times\mu)(g^{-1}E_n \triangle E_n) \le 2\nu(\{c_n \circ T = -1\}) = 2\nu(\{c_n = -1\}).$$

Since $(\mu\times\mu)(E_n) = (\mu\times\mu)(gE_n) = 1/2$ and $F_n = E_n^c$, we have $(\mu\times\mu)(E_n \cap gF_n) = (\mu\times\mu)(E_n \setminus gE_n) = \frac{1}{2}(\mu\times\mu)(E_n \triangle gE_n)$. For every $g = (c,T,k) \in V$, it follows that

$$(\mu\times\mu)(E_n \cap gF_n) \le \nu(\{c_n = -1\}) \le \frac{D(c,1)}{w_n} < \frac{1}{w_n}.$$

Taking the supremum over $g \in V$ and letting $n \to \infty$ (using $w_n \to \infty$) yields $\kappa_V^{X\times X}(1/2,1/2) = 0$. Thus the diagonal near-action on $X \times X$ is not uniformly whirly, even though it is whirly by Corollary 1.2. □

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