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EXOTIC SMOOTH STRUCTURE DETECTED BY A FOUR-DIMENSIONAL STATE SUM

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Abstract. We show that a state sum on triangulations distinguishes two closed 4-manifolds that are homeomorphic but not diffeomorphic. The state sum is the bosonic shadow, at trivial fermionic background, of the construction of Tata, Kobayashi, Bulmash and Barkeshli for the even part of $\mathrm{SU}(2)_6$ with its time-reversal symmetry: a sum over labellings of the triangles and tetrahedra of a triangulation, weighted by products of F- and R-symbols and time-reversal phases. We prove that it is invariant under Pachner moves, deriving all Pachner identities from fifteen computationally verified isometry identities for a single 4-simplex and one closed contraction, and that it is independent of the branching and of the orientation frames. It is therefore an invariant of closed PL 4-manifolds, orientable or not. Its reduction to $\mathbb{F}_{113}$ is multiplicative under connected sums and takes the values 83 and 30 on two homeomorphic manifolds. Regina's documented constructions identify these with Kreck's manifolds $\mathbb{RP}^4\#K3$ and $\mathbb{RP}^4\#11(S^2\times S^2)$. To our knowledge this is the first separation of a closed exotic pair obtained from direct tensor contractions of a finite local Pachner-invariant state sum and a proved gluing formula.

1. Introduction

Kreck showed that $\mathbb{RP}^4\#K3$ and $\mathbb{RP}^4\#11(S^2\times S^2)$ are homeomorphic but not diffeomorphic [10]. We show that these two manifolds are distinguished by a *state sum*: a sum over labellings of the triangles and tetrahedra of a triangulation by a finite set, weighted by a product of local factors, one for each simplex, and unchanged under Pachner moves.

1.1. **The result.** Let X_R, X_H and X_K be the triangulations `rp4()`, `s2xs2()` and `k3()` of the class `Example4` in Regina 7.4.1 [13]; their isomorphism signatures are given in Section 10. Their f-vectors are

$$(3,8,12,10,4),\qquad (5,6,14,15,6),\qquad (24,27,108,135,54).$$

These are generalised triangulations: faces of the same pentachoron may be identified. Put

$$M_0 = X_R\#\underbrace{X_H\#\cdots\#X_H}_{11},\qquad M_1 = X_R\#X_K.$$

Theorem 1.1. *Let B_{113} be the invariant of closed PL 4-manifolds defined in Section 3, with values in $\mathbb{F}_{113}$.*

(a) *$B_{113}(X_R)=99$, $B_{113}(X_H)=2$ and $B_{113}(X_K)=99$.*

(b) *$B_{113}(M\#N)=B_{113}(M)\,B_{113}(N)$, so*

$$B_{113}(M_0)=99\cdot 2^{11}=30,\qquad B_{113}(M_1)=99\cdot 99=83.$$

(c) *M_0 and M_1 are homeomorphic.*

Consequently M_0 and M_1 are homeomorphic but not PL homeomorphic, and their smoothings are not diffeomorphic.

Indeed $2^{11}\equiv 14$, $99\cdot 14\equiv 30$ and $99^2\equiv 83 \pmod{113}$. The new input is the inequality $B_{113}(X_K)\neq 14 = B_{113}(X_H)^{11}$. At the prime 257 the same computation gives $B_{257}(X_K)=8$; both $99\in\mathbb{F}_{113}$ and $8\in\mathbb{F}_{257}$ are the residue of -2^{11}.

The manifold X_R is PL homeomorphic to $\mathbb{RP}^4$ (Proposition 10.1). According to the constructions documented in Regina 7.4.1, X_H and X_K are Burke's triangulations of the standard

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$S^2 \times S^2$ and K3 surface [1, 13]; with this identification, M_1 and M_0 are Kreck's manifolds. The homeomorphism statement in Theorem 1.1(c) uses the manifold and orientation checks together with the fundamental groups and intersection forms computed from the triangulations in Section 10. Section 8.1 describes triangulations of M_0 and M_1 with 262 and 90 pentachora.

Corollary 1.2 (Stable non-diffeomorphism)**.** *For every $k \geq 0$, the manifolds $M_0 \# k(S^2 \times S^2)$ and $M_1 \# k(S^2 \times S^2)$ are not PL homeomorphic, and their compatible smoothings are not diffeomorphic.*

Proof. Put $H = S^2 \times S^2$. Theorem 1.1 and Proposition 8.2 give

$$B_{113}(M_0 \# kH) = 30 \cdot 2^k, \qquad B_{113}(M_1 \# kH) = 83 \cdot 2^k.$$

Since 2 is a unit in $\mathbb{F}_{113}$, these values are distinct for every $k \geq 0$, excluding a PL homeomorphism and hence a diffeomorphism. □

Kreck already proved stable non-diffeomorphism for this pair [10, pp. 248, 259]; the corollary detects the same distinction by the state sum.

1.2. **Context.** Reutter proved that semisimple *oriented* 4-dimensional topological field theories cannot distinguish closed oriented smooth 4-manifolds that are homeomorphic by an orientation-preserving homeomorphism [15]. This applies to the Crane–Yetter–Kauffman state sums of ribbon fusion categories, whose value in the modular case is a function of the signature and the Euler characteristic [4, 5]. The state sum used here is unoriented: it uses the time-reversal symmetry of the category, and the manifolds in Theorem 1.1 are nonorientable.

For closed connected 4-manifolds with finite fundamental group, Reutter and Schommer-Pries showed that indistinguishability by semisimple field theories is equivalent to stable diffeomorphism. In particular, they give unoriented theories that distinguish homeomorphic smooth manifolds [16]. These are finite path integrals defined in homotopy-theoretic terms.

Tata, Kobayashi, Bulmash and Barkeshli constructed state sums for fermionic symmetry-protected phases [18]. They argued that their Pin^+ state sum carries the information of the η-invariant of the Pin^+ Dirac operator, which detects exotic structures [17], and described it as "perhaps the first example of a combinatorial state sum for a TQFT" with this property. That argument depends on the bordism invariance of the state sum, for which they did not have a proof; in [18, Section VIII.C] they also raise the possibility that the bosonic shadow detects exotic structure. We take their local tensor for the super-modular category $\mathrm{SO}(3)_3$, the even part of $\mathrm{SU}(2)_6$, with time-reversal symmetry and fermionic background $f = 0$, prove the invariance of the resulting state sum from the local data, and evaluate it.

Debray, Ye and Yu develop the related anomaly-indicator construction, retaining bosonization and invertibility as conjectures, and identify the evaluation on K3 as an important problem [6, Section 7].

Ren and Willis gave an analysis-free proof that exotic compact orientable 4-manifolds exist, using Khovanov skein lasagna modules of knot traces [14]; their manifolds have boundary.

To our knowledge, Theorem 1.1 is the first separation of a closed exotic pair obtained from direct tensor contractions of a finite local Pachner-invariant state sum and a proved gluing formula. The contractions evaluate the three summands; the connected-sum identity gives the values on the pair.

1.3. **Structure of the proof.**

(i) *Invariance* (Sections 3–6). A positivity argument derives every ordered Pachner identity from the isometry identities of a single 4-simplex and one closed contraction. An untwisting identity extends these to the orientation background needed on nonorientable manifolds, and a gauge identity removes the dependence on the frames. Stellar subdivisions connect generalised triangulations with simplicial complexes, and different branchings with each other.

(ii) *Triangulations without branching* (Section 7). The triangulation X_K admits no branching. We give the basis changes for re-ordering the vertices of a tetrahedron; with them the oriented state sum can be evaluated with arbitrary local vertex orders.

(iii) *Gluing and values* (Sections 8–9). A cap and a cylinder identity give the product formula (b), and tensor-network contractions give the three values in (a).

(iv) *Topology* (Section 10). The vertex links, fundamental groups and intersection forms of the three triangulations, together with Freedman's theorem and Kreck's orientation-reversal argument, give (c).

1.4. **Computations.** Several steps are identities between tensors with finitely many entries, or properties of the three triangulations, and were verified by computer. These are (4.2), (4.3) and (2.1)–(2.3), verified in exact arithmetic in $\mathcal{R}$; the reordering identities of Section 7 and the cap and cylinder identities of Section 8, verified over the stated finite fields; the three values of Section 9, with $B(X_H) = 2$ computed in $\mathcal{R}$; and the facts about the triangulations used in Section 10. These are finite premises of the proof. Tensor identities are compared in every coordinate. The direct comparisons of Remark 4.6 check particular Pachner identities separately from the positivity argument; they share the local tensor data. The second-prime computations establish the corresponding statements over $\mathbb{F}_{257}$; the distinction in Theorem 1.1 uses $\mathbb{F}_{113}$. Appendix A lists the computations. The accompanying `anc/` directory contains the programs, their required input data, and the retained result records listed in its README.

2. The local data

2.1. **Fusion rules and dimensions.** The label set is $\mathcal{L} = \{0, 2, 4, 6\}$ (doubled spins), with fusion multiplicities

$$N_{ab}^{c} = 1 \iff |a-b| \le c \le \min(a+b, 12-a-b), \qquad N_{ab}^{c} = 0 \text{ otherwise.}$$

All labels are self-dual, and 6 is a transparent fermion. Put $u = \sqrt{1+\sqrt{2}} > 0$. The quantum dimensions and the global dimension are

$$(d_0, d_2, d_4, d_6) = (1, u^2, u^2, 1), \qquad D^2 = \sum_a d_a^2 = 8 + 4\sqrt{2} = 4(1+u^2), \quad D > 0.$$

The coefficients below lie in $\mathcal{R} = \mathbb{Z}[\tfrac{1}{2}, u, i]/(u^4 - 2u^2 - 1,\ i^2 + 1)$, with D adjoined if an odd power of D is needed. Formal complex conjugation fixes u and D and sends $i \mapsto -i$.

2.2. F **and** R**.** Let $r = \sqrt{2+\sqrt{2}}$ and $([0], \dots, [7]) = (0, 1, r, u^2, \sqrt{2}\,r, u^2, r, 1)$, the quantum integers at $q = e^{\pi i/8}$. Write $[0]! = 1$ and $[n]! = \prod_{j=1}^{n}[j]$, and take positive real square roots in Δ and the dimension factors. For an admissible triple put

$$\Delta(a,b,c)^2 = \frac{[\frac{a+b-c}{2}]!\,[\frac{a-b+c}{2}]!\,[\frac{-a+b+c}{2}]!}{[\frac{a+b+c}{2}+1]!}.$$

Write $F(a,b,c;q;e,h) = (F_q^{abc})_{eh}$. It vanishes unless (a,b,e), (e,c,q), (b,c,h) and (a,h,q) are admissible, in which case

$$F(a,b,c;q;e,h) = (-1)^{\frac{a+b+c+q}{2}}\sqrt{[e+1][h+1]}\,\Delta(a,b,e)\Delta(c,q,e)\Delta(b,c,h)\Delta(a,q,h)$$
$$\cdot \sum_z \frac{(-1)^z[z+1]!}{\prod_{j=1}^{4}[z-A_j]!\prod_{j=1}^{3}[C_j-z]!}.$$

with $A = (\frac{a+b+e}{2}, \frac{c+q+e}{2}, \frac{b+c+h}{2}, \frac{a+q+h}{2})$, $C = (\frac{a+b+c+q}{2}, \frac{a+c+e+h}{2}, \frac{b+q+e+h}{2})$, and z running from $\max A$ to $\min(C_1, C_2, C_3, 6)$. With $\zeta_8 = (1+i)/\sqrt{2}$,

$$R(a,b;c) = (-1)^{\frac{a+b-c}{2}}\,\zeta_8^{(c(c+2)-a(a+2)-b(b+2))/8}$$

on admissible triples, and $R = 0$ otherwise. There are 224 nonzero F and 24 nonzero R. The tables satisfy the pentagon and hexagon equations, unitarity and ribbon balance.

2.3. **Time reversal.** The symmetry $\mathbf{T}$ acts by $T(0)=0$, $T(2)=4$, $T(4)=2$, $T(6)=6$. The phase $U(a,b;c)=U_{\mathbf{T}}(a,b;c)$ vanishes on inadmissible triples. It equals i on $(2,4,6),(4,6,2),(6,2,4),(2,2,2),(4,4,4)$. It equals $-i$ on $(2,6,4),(6,4,2),(4,2,6)$ and on all permutations of $(2,2,4)$ and $(2,4,4)$, and it equals 1 on all other admissible triples. Finally, $e_a:=\eta_a(\mathbf{T},\mathbf{T})$ equals -1 for $a=6$ and 1 otherwise. Write $\widetilde{U}(a,b;c):=U(Ta,Tb;Tc)$. The data satisfy the following identities, the first whenever its four fusion triples are admissible and the other two on admissible triples.

$$F(Ta,Tb,Tc;Tq;Te,Th)=\overline{F(a,b,c;q;e,h)}\,\frac{\widetilde{U}(b,c;h)\,\widetilde{U}(a,h;q)}{\widetilde{U}(a,b;e)\,\widetilde{U}(e,c;q)}, \tag{2.1}$$

$$\frac{U(Tb,Ta;Tc)}{U(Ta,Tb;Tc)}\,R(Ta,Tb;Tc)=\overline{R(a,b;c)}, \tag{2.2}$$

$$U(a,b;c)\,\overline{U(Ta,Tb;Tc)}=e_ae_b/e_c. \tag{2.3}$$

2.4. **Specialisations.** The defining relations have solutions in the two finite fields we use:

p	u	i	$\sqrt{2}=u^2-1$	D	D^2
113	17	15	62	16	30
257	33	16	60	48	248

Every denominator occurring in the tables is a unit at these primes. The table of a coefficient and that of its formal conjugate are reduced separately; we never assume an automorphism of $\mathbb{F}_p$ that sends i to $-i$.

3. The state sum

3.1. **Triangulations.** A *generalised triangulation* is a finite set of 4-simplices with affine identifications of facets in pairs. A *branching* is a total order of the vertices of each 4-simplex, compatible with all facet identifications. We assume throughout that open faces are embedded, that every face has trivial stabiliser (no face is identified with itself by a nontrivial symmetry), and that the link of every face is a PL sphere, so that the realisation is a closed PL 4-manifold. Faces are the cells of the realisation; different faces may have the same vertices. Each triangle ijk (with $i<j<k$) carries a label $a_{ijk}\in\mathcal{L}$, and each tetrahedron $ijkl$ carries a channel $b_{ijkl}\in\mathcal{L}$.

3.2. **Orientation background.** Choose a local orientation, called a *frame*, at each vertex of K. Define $g_{ij}\in\mathbb{Z}/2$ by transporting the frame along the edge ij and comparing it with the frame at the terminal vertex. Thus g is flat ($g_{ij}+g_{jk}=g_{ik}$ on each triangle) and represents w_1. An edge whose two ends are the same vertex can have $g_{ij}=1$, since the comparison includes the transport along the edge. Labels of triangle ijk and tetrahedron $ijkl$ are expressed in the frames of their last vertices k and l. Each pentachoron σ carries a sign $\epsilon_\sigma=\pm1$, which compares its local order with the frame at its last vertex. Writing $p(\sigma,\tau)$ for the position of the vertex of σ omitted by τ, the signs are compatible with g when, for the two pentachora σ,σ' meeting along a tetrahedron τ,

$$\epsilon_\sigma(-1)^{p(\sigma,\tau)+g_{l_\tau m_\sigma}}=-\epsilon_{\sigma'}(-1)^{p(\sigma',\tau)+g_{l_\tau m_{\sigma'}}}. \tag{3.1}$$

Here m_σ is the last vertex of σ and l_τ that of τ. On an oriented manifold we may take $g=0$, and then ϵ_σ is the usual comparison of the order with the orientation.

3.3. **The local tensor.** The tetrahedron channel is constrained by $N^{b_{ijkl}}_{a_{ijl},a_{jkl}} = N^{b_{ijkl}}_{a_{ikl},T^{g_{kl}}a_{ijk}} = 1$. For a positively signed ordered pentachoron 01234,

(3.2)

$$\begin{aligned} Z^{+}_{g}(01234) = \Phi \sum_{x,y} d_x \; & F(a_{024},a_{234},a'_{012};x;b_{0234},y)\, R(a'_{012},a_{234};y) \\ & \cdot \overline{F(a_{024},a'_{012},a_{234};x;b_{0124},y)} \\ & \cdot F(a_{014},a_{124},a_{234};x;b_{0124},b_{1234})\, \overline{F(a_{014},a_{134},T^{g_{34}}a_{123};x;b_{0134},b_{1234})} \\ & \cdot F(a_{034},T^{g_{34}}a_{013},T^{g_{34}}a_{123};x;b_{0134},T^{g_{34}}b_{0123}) \\ & \cdot \overline{F(a_{034},T^{g_{34}}a_{023},a'_{012};x;b_{0234},T^{g_{34}}b_{0123})}, \end{aligned}$$

where $a'_{012} = T^{g_{24}}a_{012}$ and

$$\Phi = e^{g_{23}g_{34}}_{a_{012}} \left[\frac{U_{g_{34}}(a_{013},a_{123};b_{0123})}{U_{g_{34}}(a_{023},T^{g_{23}}a_{012};b_{0123})}\right]^{*g_{34}}, \qquad U_0 = 1,\ U_1 = U,$$

and where $z^{*0} = z$ and $z^{*1} = \bar{z}$. For a negatively signed pentachoron one takes $Z^{-}_{g} = \overline{Z^{+}_{g}}$. At $g = 0$ we write $W := Z^{+}_{0}$; this is the tensor of [18, arXiv v2, Eq. (97)] with $f = 0$.

Remark 3.1. In the second F-symbol of (3.2) the argument $a'_{012} = T^{g_{24}}a_{012}$ is forced by the neighbouring R-symbol and by the channel b_{0124}, which involve a'_{012}. The formula printed in [18, arXiv v2, Eq. (97)] has a_{012} at this place; the two agree for $g = 0$. Equation (96) there gives the diagrammatic normalisation.

3.4. **The closed sum.** For a branched generalised triangulation K with compatible (g,ϵ), put

$$B(K) = D^{\,2(N_0-N_1)-\chi(K)} \sum_{a,b} \frac{\prod_{t\in K_2} d_{a_t}}{\prod_{\tau\in K_3} d_{b_\tau}} \prod_{\sigma\in K_4} Z^{\epsilon_\sigma}_{g,\sigma}, \tag{3.3}$$

where N_i is the number of i-simplices. The coefficients lie in $\mathcal{R}[D]$, and B_p denotes the image under the specialisation of Section 2.

Theorem 3.2. *B, and hence each B_p, depends only on the PL homeomorphism type of the closed 4-manifold $|K|$. It is independent of the triangulation, the branching, the frames and the representative g of w_1.*

The proof occupies Sections 4–6.

4. Pachner identities from unitarity

In this section $g = 0$, and we work in the real field $\mathbb{K} = \mathbb{Q}(u,D) \subset \mathbb{R}$ with $\mathbb{K}(i) \subset \mathbb{C}$. Here every d_a, every $\sqrt{d_a}$ and D are positive, and formal conjugation is complex conjugation.

4.1. **Boundary vectors.** Let X be a triangulated oriented 4-dimensional piece with boundary. Let $\mathcal{H}(\partial X)$ be the finite-dimensional $\mathbb{K}(i)$-vector space with orthonormal basis the fusion-admissible labellings λ of the boundary triangles and tetrahedra. Define

$$\Psi_X(\lambda) = D^{E(X)}\sqrt{\mu_{\partial X}(\lambda)} \sum_{\text{interior labels}} \frac{\prod_{\text{int. } t} d_{a_t}}{\prod_{\text{int. } \tau} d_{b_\tau}} \prod_{\sigma\subset X} W^{\epsilon_\sigma}_{\sigma}, \tag{4.1}$$

where $\mu_{\partial X}$ is the product of the boundary triangle dimensions divided by the product of the boundary tetrahedron dimensions, and

$$E(X) = 2\big(N_0(X) - N_1(X)\big) - \big(N_0(\partial X) - N_1(\partial X)\big) - \chi(X).$$

If two pieces are glued along their whole common boundary, the square-root weights combine to interior weights and the exponents add up to the closed exponent. Hence the closed sum is the bilinear pairing of the two boundary vectors. Reversing all orientations complex-conjugates Ψ_X.

4.2. **Attachment operators.** Let S be a nonempty proper set of facets of the ordered 4-simplex, P_S their union (a 3-ball), $\bar{S}$ the complement, and $\Gamma = \partial P_S = \partial P_{\bar{S}}$. For a labelling e of the triangles of Γ, let $\mathcal{H}_S(e)$ have as orthonormal basis the admissible labellings x of the interior triangles and the tetrahedra of P_S. Let $\mu_S(x)$ be the product of the interior triangle dimensions of P_S divided by the product of its tetrahedron dimensions. Put $\kappa_1 = \kappa_4 = 3$, $\kappa_2 = \kappa_3 = 1$, and for $r = |S|$

$$A^{\epsilon}_S(y, x; e) = D^{-\kappa_r} \sqrt{\mu_S(x)\, \mu_{\bar{S}}(y)}\; W^{\epsilon}(x, y, e) \colon\ \mathcal{H}_S(e) \to \mathcal{H}_{\bar{S}}(e).$$

Attaching a pentachoron to a piece X along a copy of P_S in ∂X acts on boundary vectors by $\Psi_X \mapsto (A^{\epsilon}_S \otimes \mathrm{id})\Psi_X$. Indeed, the face counts of the attachment give $\Delta E = -\kappa_r$, and the labels of ∂X outside P_S are fixed parameters.

Lemma 4.1 (finite isometry identities)**.** *For each of the five singletons and the ten two-element sets S, for both signs ϵ, and for every interface labelling e,*

$$(A^{\epsilon}_S)^{\dagger} A^{\epsilon}_S = \mathrm{id}_{\mathcal{H}_S(e)}. \tag{4.2}$$

Equivalently, gluing two copies of the ordered pentachoron with opposite signs along $\bar{S}$, and summing all labels except those of P_S and Γ, gives $D^{2\kappa_r}\mu_S(x)^{-1}\delta_{x,x'}$ on admissible states and 0 elsewhere.

Proof. The fifteen identities for $\epsilon = +1$ were computed in exact arithmetic in $\mathcal{R}$, comparing all coordinates; formal conjugation gives $\epsilon = -1$. The result has 160 nonzero entries among 4^6 coordinates for $|S| = 1$, and 7424 among 4^{12} for $|S| = 2$. All 30 sign/set cases were also checked over each of $\mathbb{F}_{113}$ and $\mathbb{F}_{257}$. □

Lemma 4.2. *Every A^{ϵ}_S satisfies $\|A^{\epsilon}_S v\| \le \|v\|$.*

Proof. For $|S| \le 2$ this is Lemma 4.1. The weights are real and symmetric, so $A^{-\epsilon}_{\bar{S}} = (A^{\epsilon}_S)^{\dagger}$, and for $|S| \ge 3$ the operator is the adjoint of an isometry. If $A^{\dagger}A = 1$ then $\|y\|^2 - \|A^{\dagger}y\|^2 = \|y - AA^{\dagger}y\|^2 \ge 0$. □

4.3. **Norms of facet balls.** For the ordered 5-simplex 012345 with facets $F_j = \Delta^5 \setminus \{j\}$ and a nonempty proper subset $I \subset \{0, \dots, 5\}$, let $B_I = \bigcup_{j \in I} F_j$, with the orientations induced from $\partial\Delta^5$. Then $B_I = \partial\Delta(I) * \Delta(I^c)$ is a 4-ball with $\partial B_I = \partial\Delta(I) * \partial\Delta(I^c) = \partial B_{I^c}$.

Lemma 4.3. $\|\Psi_{B_I}\| \le 1$ *for every nonempty proper I.*

Proof. For one pentachoron, choose a single facet as source. By (4.2),

$$\|\Psi_{\Delta^4}\|^2 = D^{-6} \sum_{a,b,c,d,z} d_a d_b d_c d_d\, N^z_{ab} N^z_{cd}.$$

This equals 1 by the identities $\sum_{a,b} d_a d_b N^z_{ab} = d_z D^2$ and $\sum_z d_z^2 = D^2$, which follow from the fusion rules and $d^2 = 1 + 2d$ for $d = u^2$. Adding the facets of I one at a time, the $(r+1)$-st facet meets the union of the previous ones in exactly r of its own facets, $1 \le r \le 4$. So each step is an attachment, and Lemma 4.2 applies. □

Lemma 4.4. *The closed sum on the six facets of $\partial\Delta^5$, with f-vector $(6, 15, 20, 15, 6)$, is*

$$C_6 = D^{-20} \sum_{\text{all labels}} \frac{\prod_t d_{a_t}}{\prod_\tau d_{b_\tau}} \prod_{\sigma \subset \partial\Delta^5} W^{\epsilon(\sigma)}_{\sigma} = 1. \tag{4.3}$$

In the basis of $\mathcal{R}$ the raw sum equals $79792439296\, u^2 + 33051115520 = (4 + 4u^2)^{10} = D^{20}$.

Proof. The raw sum was computed by a contraction in exact arithmetic in $\mathcal{R}$. □

Theorem 4.5. *For every nonempty proper I, $\Psi_{B_I} = \overline{\Psi_{B_{I^c}}}$ in every boundary coordinate. Equivalently, every ordered 1–5, 2–4 and 3–3 Pachner move preserves the boundary tensor, for both orientations and all labels on the boundary. All 62 ordered facet partitions are covered.*

Proof. Put $L = \Psi_{B_I}$ and $R = \overline{\Psi_{B_{I^c}}}$. Gluing gives $\langle R, L\rangle = C_6 = 1$, and $\|L\|, \|R\| \le 1$ by Lemma 4.3. Hence

$$0 \le \|L - R\|^2 = \|L\|^2 + \|R\|^2 - 2\,\mathrm{Re}\langle R, L\rangle \le 0.$$

After the common positive boundary weights are cancelled, the identities have coefficients in $\mathcal{R}$. The remaining factors are D^8 for $1 \to 5$, D^2 for $2 \to 4$ and 1 for $3 \to 3$, and they are exactly compensated by the global normalisation in (3.3). □

Being identities in $\mathcal{R}$, these specialise to every $\mathbb{F}_p$ of Section 2.

Remark 4.6. Some cases of Theorem 4.5 were also verified without the positivity argument. For $I = \{5\}$, $\{4,5\}$ and $\{3,4,5\}$ the two boundary tensors were computed separately over $\mathbb{F}_{113}$ and compared in all coordinates. They agree: for the 1–5 move in both orientations (15 boundary labels, 411904 nonzero entries), and with the orientation induced from $\partial\Delta^5$ for the 2–4 move (24 boundary labels, 914798592 nonzero entries) and for the 3–3 move (27 boundary labels, 14338617344 nonzero entries).

5. The orientation background

Lemma 5.1 (untwisting)**.** *On an ordered pentachoron with flat g, choose h with $g_{ij} = h_i + h_j$, and put $A_{ijk} = T^{h_k} a_{ijk}$ and $B_{ijkl} = T^{h_l} b_{ijkl}$. Then for every fusion-admissible labelling*

$$\big(Z_g^+(a,b)\big)^{*h_4} = (\delta q_h)_{01234}\; Z_0^+(A,B),$$

where q_h is the following unit-phase tetrahedral cochain

$$q_\tau = \begin{cases} 1, & h_l = 0, \\ e_{a_{ijk}}^{g_{kl}}\, \dfrac{U(a_{ikl}, T^{g_{kl}} a_{ijk}; b_{ijkl})}{U(a_{ijl}, a_{jkl}; b_{ijkl})}, & h_l = 1, \end{cases} \qquad \tau = ijkl,$$

and $(\delta q)_{01234} = q_{1234} q_{0234}^{-1} q_{0134} q_{0124}^{-1} q_{0123}$.

Proof. For $h_4 = 0$, substitute $g = \delta h$ in (3.2). Every argument becomes its untwisted counterpart on A, B, the four tetrahedra containing vertex 4 have $q = 1$, and q_{0123} is precisely Φ. Thus the identity holds term by term.

For the other case we first show covariance under a uniform T. Recall $\widetilde{U}(a,b;c) = U(Ta, Tb; Tc)$ from Section 2. Applying (2.1)–(2.2) to all six F factors and the R factor of $\overline{Z_0^+(Ta,Tb)}$, and relabelling the summed labels by T, cancels every U factor involving a summed label. The residual monomial is

$$\frac{\widetilde{U}(a_{024}, a_{234}; b_{0234})}{\widetilde{U}(a_{034}, a_{023}; b_{0234})} \cdot \frac{\widetilde{U}(a_{014}, a_{124}; b_{0124})}{\widetilde{U}(a_{024}, a_{012}; b_{0124})} \cdot \frac{\widetilde{U}(a_{134}, a_{123}; b_{1234})}{\widetilde{U}(a_{124}, a_{234}; b_{1234})} \cdot \frac{\widetilde{U}(a_{034}, a_{013}; b_{0134})}{\widetilde{U}(a_{014}, a_{134}; b_{0134})} \cdot \frac{\widetilde{U}(a_{023}, a_{012}; b_{0123})}{\widetilde{U}(a_{013}, a_{123}; b_{0123})}.$$

Define

$$\alpha_U(a,b)_{ijkl} = \frac{U(a_{ijl}, a_{jkl}; b_{ijkl})}{U(a_{ikl}, a_{ijk}; b_{ijkl})}.$$

The monomial is $(\delta\alpha_U(Ta, Tb))^{-1}$. Therefore

$$\overline{Z_0^+(Ta,Tb)} = \frac{Z_0^+(a,b)}{\delta\alpha_U(Ta,Tb)}, \qquad \overline{Z_0^+(a,b)} = \frac{Z_0^+(Ta,Tb)}{\delta\alpha_U(a,b)}. \tag{5.1}$$

All factors divided here are unit phases.

If $h_4 = 1$, put $\bar{h} = h + 1$ and apply the proved case to $\bar{h}$, whose untwisted labels are TA, TB. Set $r_{ijk} = e_{a_{ijk}}^{1-h_k}$. Equation (2.3) gives on each tetrahedron

$$q_\tau(h)\, q_\tau(\bar{h})\, \alpha_U(TA, TB)_\tau = (\delta r)_\tau.$$

To verify this, put

$$e = e_{a_{ijk}}, \qquad H = \frac{e_{a_{ijl}} e_{a_{jkl}}}{e_{a_{ikl}} e_{a_{ijk}}}, \qquad \gamma = g_{kl}, \qquad \alpha = \frac{U(a_{ijl}, a_{jkl}; b_{ijkl})}{U(a_{ikl}, T^{\gamma} a_{ijk}; b_{ijkl})}. \tag{5.2}$$

The product of the two q's is e^{γ}/α. For $h_l = 0$ the other two expressions are $\alpha_U(TA, TB) = H\alpha$ and $\delta r = He^{\gamma}$; for $h_l = 1$ they are α and e^{γ}. Applying δ eliminates δr, since $\delta^2 r = 1$. Conjugating the $\bar{h}$ identity and using (5.1) now proves the claim. □

Corollary 5.2. *Every ordered Pachner identity holds with a flat background g.*

Proof. The initial ball $B_I = \partial\Delta(I) * \Delta(I^c)$ is a cone on any $v \in I^c$. Set $h_v = 0$ and $h_i = g_{vi}$ for its other vertices. Flatness on triangles vij gives $g_{ij} = h_i + h_j$ on every existing edge. Assign an arbitrary h to a new vertex, if present, and give all new edges these same differences. This supplies one primitive for both sides.

Choose an ordinary orientation of the ball and write its incidence signs as ϵ^0_σ. Compatibility with the local frames says $\epsilon_\sigma = \epsilon^0_\sigma (-1)^{h_{m_\sigma}}$, after a possible uniform change of h. Lemma 5.1, conjugated when $\epsilon^0_\sigma = -1$, replaces each actual tensor by the ordinary untwisted tensor times $((\delta q_h)_\sigma)^{\epsilon^0_\sigma}$. Interior tetrahedral phases cancel in opposite incidence signs; the boundary phase is identical for the two balls. Relabelling by T^h is a bijection and preserves dimensions, so Theorem 4.5 gives the result. □

Lemma 5.3 (frames)**.** *Changing the frames by a vertex function t replaces g by $g + \delta t$, the labels by $a'_{ijk} = T^{t_k} a_{ijk}$ and $b'_{ijkl} = T^{t_l} b_{ijkl}$, and the signs by $\epsilon'_\sigma = \epsilon_\sigma (-1)^{t_{m_\sigma}}$. This does not change the closed sum (3.3). In particular, for $g = 0$ the closed sum is invariant under reversal of the orientation.*

Proof. For $\tau = ijkl$ let e, H, γ and α be as in (5.2), and put $\mu = t_k$ and $\nu = t_l$. On fusion-admissible labellings define the unit phase

$$\beta_\tau = \begin{cases} e^{\gamma\mu}, & \nu = 0, \\ e^{\gamma\mu+\gamma+\mu+1}/(H\alpha), & \nu = 1. \end{cases}$$

It depends only on the data of the tetrahedron. The local identity we need is

$$\begin{aligned} \big(Z^+_{g'}(a', b')\big)^{*t_4} &= (\delta_g \beta)_{01234}\, Z^+_g(a, b), \\ (\delta_g \beta)_{01234} &= \beta_{1234} \beta^{-1}_{0234} \beta_{0134} \beta^{-1}_{0124} \beta^{*g_{34}}_{0123}. \end{aligned} \tag{5.3}$$

To verify it, choose h on this pentachoron with $g = \delta h$ and put $c_{ijk} = e^{h_k t_k}_{a_{ijk}}$. Equation (2.3) gives $\alpha' = H^{\nu}\alpha$, while substitution in the definition of q gives

$$\frac{q_\tau(h + t; a', b')}{q_\tau(h; a, b)} = \beta^{*h_l}_\tau (\delta c)_\tau, \qquad (\delta c)_\tau = H^{h_l \nu} e^{h_l \nu + h_k \mu}.$$

For $(h_l, \nu) = (0,0), (0,1), (1,0), (1,1)$, the quotient on the left is, respectively,

$$1, \qquad e^{\gamma+\mu+1}/(H\alpha), \qquad e^{\mu}, \qquad e^{\gamma}\alpha,$$

which agrees with the right-hand side in all four cases. Apply δ, use $\delta^2 c = 1$, and apply Lemma 5.1 for h and $h + t$. The untwisted labels agree. Transport from the last vertex of each tetrahedron to vertex 4 gives the conjugation of β_{0123} in (5.3).

For the actual signed tensors, use $\epsilon'_\sigma = \epsilon_\sigma (-1)^{t_{m_\sigma}}$. The total phase is

$$\prod_\sigma \prod_{\tau \prec \sigma} \beta_\tau^{\epsilon_\sigma (-1)^{p(\sigma,\tau) + g_{l_\tau m_\sigma}}} = 1,$$

because the two exponents at every tetrahedron are opposite by (3.1). The label substitution is a bijection preserving all dimensions. This proves the independence of the frames; no global primitive of g was needed. Taking $t \equiv 1$ leaves $g = 0$ fixed and reverses all simplex signs, which proves the orientation-reversal assertion. □

6. Branchings and generalised triangulations

Pachner's theorem [12] applies to combinatorial manifolds, that is, to genuine simplicial complexes. Our triangulations are generalised triangulations, and the ordered identities do not yet give independence of the branching. The following lemma supplies both connections.

Lemma 6.1 (refinement)**.** *Let K be a generalised triangulation as in Section 3. Perform two rounds of stellar subdivisions of all faces in descending dimensions $4, 3, 2, 1$. The first round yields a regular cell complex, and the second a genuine simplicial complex K''.*

(a) *Every stellar subdivision is a composite of the ordered Pachner moves of Theorem 4.5. Hence $B(K) = B(K'')$ for every branched K, where the branching is transported through the ordered moves to the canonical branching by face dimension, and the background g is extended by flatness.*

(b) *On a simplicial complex, any two branchings are connected through the canonical branching of the barycentric subdivision. Hence B is independent of the branching.*

Proof. Retain face occurrences and their incidences throughout: cells with the same global vertex set need not be the same cell. After all original faces of dimension greater than k have been subdivided, each remaining pentachoron consists of one original k-face occurrence followed by a strict flag of centres of higher coface occurrences. Thus different occurrences of the face being subdivided have been separated. Its star is an abstract join $\sigma * L$, with remaining identifications only on the boundary. Here L is the barycentric subdivision of the original normal link; its simplex closures are embedded.

For $k = 3, 2, 1$, the link is respectively a 0-, 1-, or 2-sphere. Order its facets by the two-point order, a cyclic edge order, or a shelling of the disk obtained by deleting one triangle from a 2-sphere, followed by that triangle. Such disk shellings exist by induction: a boundary triangle whose boundary intersection has an isolated vertex splits the disk into smaller disks, in one of which a removable boundary triangle can be chosen. This also applies to incidence complexes with parallel cells and embedded simplex closures. Every nonempty proper prefix and complementary suffix of the resulting link order is a ball.

Let P, Q be the processed and unprocessed parts of L, and $F = P \cap Q$. The intermediate star has the form

$$\sigma * Q \ \cup \ b * \partial\sigma * P \ \cup \ b * \sigma * F.$$

Use augmented-complex conventions: the complex consisting of the empty face is different from the void complex. In particular, the frontier between the two points of a 0-sphere contributes $b * \sigma$. For the next link facet λ, write $C = P \cap \lambda$ as the union of r facets of $\partial\lambda$, and let D be the union of the remaining facets. The replacement

$$\sigma * \lambda \ \cup \ b * \sigma * C \quad \longleftrightarrow \quad b * \partial\sigma * \lambda \ \cup \ b * \sigma * D$$

is the $(r + 1) \to (5 - r)$ Pachner move on the six abstract vertex occurrences of σ, λ, b. The first step has $r = 0$ and inserts b by a $1 \to 5$ move; the last has $C = \partial\lambda$.

The old common face has exactly the displayed incident pentachora. In the two-dimensional link, a frontier edge has one unprocessed incident triangle. If two frontier edges meet at a vertex of λ, its unprocessed link is an interval whose endpoints are joined by the edge supplied by λ, so it has no further edge. The lower-dimensional cases are immediate. The new common face is absent from the processed incidence complex. Thus all interior cells of the move are distinct and have no exterior incidences. Impose any ambient boundary identifications only after the full open tensor identity, and then contract with the exterior. Equality is preserved, as is the prescribed normalisation: only the standard distinct interior cells are inserted or removed. The replacement moves λ from Q to P, and at completion gives $b * \partial\sigma * L$. For $k = 4$ there is only the initial $1 \to 5$ move. Only shellings of 2-spheres are used.

One descending round is barycentric subdivision. Its flag simplices have vertices of distinct original face dimensions; a boundary identification preserving those dimensions and their barycentric coefficients fixes every point. The first round therefore has embedded simplex

closures and is a regular simplicial cell complex. Its next barycentric subdivision is the order complex of its face poset: inclusions have no incidence multiplicity, and a chain is determined by its vertices. This is a genuine simplicial complex.

Put b after all vertices of σ and before the higher-dimensional face centres in L. Each local six-vertex model then has a compatible order, and the endpoint has the canonical order by face dimension. For a flat background, choose a vertex occurrence v of σ and set $g(v,b)=0$ and $g(b,w)=g(v,w)$ on each new edge occurrence. Old triangle flatness verifies new triangle flatness; trivial face stabilisers ensure agreement on matching occurrences, while parallel edges remain distinct. Other choices of the new frame are covered by Lemma 5.3.

Finally, for two initial branchings use the same descending construction and choose v by one fixed auxiliary enumeration, independent of the branching. Carry the same old frames. The resulting canonical branching and background agree at the endpoint. Reversing one sequence therefore proves branching independence. This proves both assertions. □

Proof of Theorem 3.2. By Lemma 6.1, $B(K)$ equals B of a branched simplicial complex, independently of the branching. Pachner's theorem connects any two simplicial triangulations of the same PL manifold by bistellar moves. An unordered move can be given a total order of its six vertices, which extends to a global vertex order, and Theorem 4.5 with Corollary 5.2 applies. Lemma 5.3 removes the dependence on frames. □

7. Evaluation without a branching

The K3 triangulation X_K admits no branching. Twelve of its triangles have all three edges equal to one loop edge, identified cyclically, so every orientation of that edge makes them cyclic. To evaluate the oriented sum ($g=0$) on X_K, we transport tetrahedron labels between vertex orders. In this section labels are spin indices $j=0,1,2,3$ (doubled $0,2,4,6$): F, N and d denote the data of Section 2 with every label argument doubled. The tetrahedral legs are normalised by

$$\widehat{W} = W \prod_{\tau\prec\sigma} \frac{(-1)^{j_\tau}}{\sqrt{d_{j_\tau}}}, \qquad (\sqrt{d_j}) = (1,u,u,1), \tag{7.1}$$

so that the two factors on an interior tetrahedron multiply to its weight $1/d_{j_\tau}$.

7.1. **The re-ordering law.** For a tetrahedron with ordered vertices, let t_k be the label of the triangle opposite position k. Its channel space has basis $\mathcal{H}(t)=\{f : N^f_{t_1t_3}=N^f_{t_2t_0}=1\}$, and $s=\pm1$ compares the order with the orientation induced from the adjacent pentachoron. Put $\theta_j=\zeta_8^{j(j+1)}$ (so $\theta=(1,i,-i,-1)$) and $\omega_j=(-1)^j\zeta_8^{-j(j+1)/2}$. Swapping positions $k,k+1$ maps $\mathcal{H}(t)$ to $\mathcal{H}(s_kt)$ by the matrix with entries

$$\begin{aligned}
C_0(s,t)_{f'f} &= (\theta_{t_3}/\theta_{t_2})^s\, F(t_0,t_3,t_1;t_2;f',f),\\
C_1(s,t)_{f'f} &= \theta_{t_0}^{-s}(\omega_{t_1}\omega_{t_2})^{-s}\omega_{f'}^s\omega_f^s\, F(t_0,t_1,t_3;t_2;f',f),\\
C_2(s,t)_{f'f} &= F(t_0,t_3,t_1;t_2;f',f),
\end{aligned}$$

and replaces s by $-s$. The F-parts and the half-twists were found from the pentachoron tensor. Without the scalar prefactors the action is only projective: $s_0s_1s_0$ and $s_1s_0s_1$ differ by the full twist $\theta_{t_3}^s$ of the reversed triangle. The prefactors are a solution of the $\mathbb{Z}/8$ linear system expressing an honest action compatible with all pentachoron identities.

Proposition 7.1. *Over $\mathbb{F}_{113}$ and over $\mathbb{F}_{257}$ the following identities hold:*

(a) *$C_k(-s,s_kt)\,C_k(s,t)=1$ and $C_k(-s,t)^{\mathsf T}C_k(s,t)=1$; the braid relations $s_0s_1s_0=s_1s_0s_1$ and $s_1s_2s_1=s_2s_1s_2$; the relation $s_0s_2=s_2s_0$; and the support condition. These are 3264 checks over all label tuples and both signs.*

(b) *For both orientations and each adjacent swap of the five vertices,*

$$\widehat{W}_{new} = \Big(\bigotimes_{3\text{ affected facets}} C\Big)\widehat{W}_{old}$$

as complete tables. There are 411904 *nonzero entries at* $p = 113$ *and* 411584 *at* $p = 257$.

By (a), transport depends only on the permutation, so (b) extends to all 120 orders.

Definition 7.2. Fix $p \in \{113, 257\}$. Let K be an oriented generalised triangulation satisfying the hypotheses of Section 3. Give every pentachoron an arbitrary local order and every tetrahedron a reference order. Transport each of the five legs of $\widehat{W}_\sigma$ to the reference order, multiply one d_a for each triangle, contract, and multiply by $D^{2(N_0-N_1)-\chi}$. The result is denoted $B'_p(K)$.

Proposition 7.3. *For* $p \in \{113, 257\}$, $B'_p(K)$ *is independent of all these choices. If* K *is branched, then* $B'_p(K) = B_p(K)$. *Moreover* $B'_p(K) = B_p(M)$ *for the closed PL manifold* $M = |K|$.

Proof. Independence of the pentachoron orders follows from Proposition 7.1(b) and path independence. Independence of the reference orders follows from the pairing identity in (a): the two legs of a tetrahedron carry opposite signs, and their transports cancel. On a branched triangulation choose the branching orders throughout; then all transports are trivial and (7.1) reproduces (3.3). For the last statement, run the construction of Lemma 6.1. The local orders needed for each move may be chosen freely, by order independence. Hence $B'_p(K) = B'_p(K'') = B_p(K'')$, and Theorem 3.2 applies. □

As a test, the transported pentachoron tensors of X_K computed from Regina's vertex labels and from two random choices of local orders coincide. On small triangulations without branching, B'_{113} takes the values 1 on S^4 and 2 on $S^2 \times S^2$ for random local orders.

8. Connected sums

Throughout this section the coefficients are in $\mathbb{F}_{113}$, and B denotes B_{113}. Let S be the double of a tetrahedron, a PL 3-sphere with f-vector $(4, 6, 4, 2)$, with coordinates $\lambda = (a_{012}, a_{013}, a_{023}, a_{123}, b_0, b_1)$. Use the boundary vectors (4.1), defined in the same way for the tensors Z^ϵ_g. The cap and the cylinder below carry $g = 0$ and the ordinary orientation signs.

Lemma 8.1. *The two-pentachoron cone on* S *has boundary vector*

$$v(\lambda) = D^{-3}\sqrt{d_{a_{012}} d_{a_{013}} d_{a_{023}} d_{a_{123}}}\ \delta_{b_0 b_1}$$

on admissible labellings, with $v^{\mathsf{T}} v = 1$. *The staircase triangulation of* $S \times [0, 1]$, *with* 8 *pentachora and* f*-vector* $(8, 22, 28, 22, 8)$, *has matrix* $P = vv^{\mathsf{T}}$. *Both identities were verified in all coordinates over* $\mathbb{F}_{113}$ *(*4096 *coordinates for* v*;* 160×160 *nonzero block, all other entries zero, for* P*).*

Proposition 8.2. $B_{113}(M \# N) = B_{113}(M)\, B_{113}(N)$ *for closed connected PL 4-manifolds* M, N.

Proof. Inside a pentachoron of a triangulation of M, perform four nested $1 \to 5$ moves, each time choosing the next pentachoron to contain all previously inserted centres. The tetrahedron on the four new centres has embedded closure contained in the original open pentachoron. Choose their frames to agree in that ball, so all its edges have $g = 0$. Extend the branchings by putting each new centre last. Cutting along it gives a piece X with $\partial X = S$. Capping X with the cone on S gives back a triangulation of M, so $B(M) = \Psi_X^{\mathsf{T}} v$ by Lemma 8.1. Do the same for N and glue in the cylinder. Then $B(M \# N) = \Psi_X^{\mathsf{T}} P \Psi_Y = (\Psi_X^{\mathsf{T}} v)(v^{\mathsf{T}} \Psi_Y) = B(M)B(N)$. The orientation of a summand does not matter by Lemma 5.3. □

8.1. **Triangulations of M_0 and M_1.** The proof of Proposition 8.2 also gives triangulations of connected sums. In each summand perform the four nested $1 \to 5$ moves, cut along the tetrahedron on the four new vertices, which produces a boundary equal to the double of a tetrahedron, and glue the boundary spheres of two summands. For a further summand, cut along one of the two tetrahedra of the seam.

One $1 \to 5$ move changes the f-vector by $d = (1, 5, 10, 10, 4)$. Cutting along a tetrahedron adds one tetrahedron, and gluing two boundary spheres removes their f-vector $(4, 6, 4, 2)$. Each connected sum therefore changes the f-vector of the disjoint union by $-c = -(4, 6, 4, 0, 0)$, and

$$f(M_0) = f(X_R) + 11f(X_H) + 48d - 11c = (62, 248, 602, 655, 262),$$
$$f(M_1) = f(X_R) + f(X_K) + 8d - c = (31, 69, 196, 225, 90).$$

Both have Euler characteristic 23. The facet pairings and isomorphism signatures of these two triangulations are part of the accompanying files.

9. The three values

9.1. **The contraction.** The closed sums are evaluated as tensor-network contractions over $\mathbb{F}_p$. Each pentachoron contributes a tensor in the labels of its triangles and tetrahedra, stored as the table of its nonzero entries. Two tensors are combined by matching their common labels, and a label is summed as soon as it occurs in no other tensor. Fixing the values of some labels splits the sum into slices, which are evaluated separately. All arithmetic is in integers modulo p.

9.2. **Values.**

input	f-vector	E_D	raw sum	B_{113}	remark
X_R	$(3, 8, 12, 10, 4)$	-11	56	99	branched, $g \neq 0$
X_H	$(5, 6, 14, 15, 6)$	-6	—	2	branched, computed in $\mathcal{R}$
X_K	$(24, 27, 108, 135, 54)$	-30	32	99	unbranched, B'_{113}

Here $E_D = 2(N_0 - N_1) - \chi$, and $D^{-11} \cdot 56 \equiv 99$ and $D^{-30} \cdot 32 \equiv 49 \cdot 32 \equiv 99$ in $\mathbb{F}_{113}$. For X_K at $p = 257$ the raw sum is 143, and $D^{-30} \equiv 36$ gives $B'_{257}(X_K) = 8$. By Lemma 5.3, X_K with the opposite orientation has the same value; as a test, the contraction for the reflected triangulation, a different network, again gives the raw sum 32 in $\mathbb{F}_{113}$.

The network of X_K has 54 tensors with 7 to 15 labels each. Along the contraction tree used, no intermediate tensor has more than 18 labels, and the largest has 6.3 million nonzero entries; the contraction takes about a minute on a desktop computer. This is the reason for evaluating on the unbranched X_K by means of Section 7: for the branched 112-pentachoron triangulation obtained from X_K by 130 Pachner moves the contraction trees we found have intermediate tensors with 35 or more labels, and for the 16-vertex triangulation of Casella and Kühnel [3], with f-vector $(16, 120, 560, 720, 288)$, with about 318 labels.

Remark 9.1. The residues $99 \in \mathbb{F}_{113}$ and $8 \in \mathbb{F}_{257}$ of $B(X_K)$ are both the residue of -2^{11}, which suggests $B(X_K) = -2^{11}$ in $\mathcal{R}$; compare $B(X_H) = 2$. Two residues do not determine an element of $\mathcal{R}$, and Theorem 1.1 uses only $99 \neq 14$ in $\mathbb{F}_{113}$.

10. The three triangulations

10.1. **Manifolds.** The isomorphism signatures of X_R and X_H are

X_R `eLMQcbcddcddaaaa8a8a8a8a8a`,
X_H `gALAMQaacdcdeeffPbgaVbaaJafbyaJafbva`,

and that of X_K is the concatenation of the four lines

```
2ALAMMvAwvPLQwvPwLQPMvPQQQQLQPAwwALQQAAQPPzQPPaaddceffggikqpmllsorsquxwutt
vxxFyyzzAABBCCDDJKIIGGHHMKLLNNOOPPQQSSUUTTVVWWXZZZ0011TbgaiaiakaqaaaMbaawa
UbjbabPa5ayaGauara3bmagaNaUbybNa2aRa2aibbawboaraPbJa2aKaya1aqbub2afbPaZaUb
cayaGawaca-aLb+aoatbfaNabagajaya1axbybrbebubgaubFbtbfaFbFbFbqbzaoaPboa
```

Enumeration of the faces and transport of orientations give

	$(f_0, f_1, f_2, f_3, f_4)$	orientable	χ
X_R	$(3, 8, 12, 10, 4)$	no	1
X_H	$(5, 6, 14, 15, 6)$	yes	4
X_K	$(24, 27, 108, 135, 54)$	yes	24.

The facet pairings and face identifications are valid, with embedded open faces and trivial face stabilisers. Every vertex link is a PL 3-sphere: each link is connected to the double of a tetrahedron by a sequence of Pachner moves, edge collapses and 2–0 and 2–1 moves, whose legality is checked at each step. The sequences for the 24 links of X_K have about 200 moves in total, and the largest link has 234 tetrahedra. Hence X_R, X_H and X_K are closed PL 4-manifolds.

Proposition 10.1. *X_R is PL homeomorphic to $\mathbb{RP}^4$.*

Proof. The quotient of the boundary of the 5-dimensional cross-polytope by the antipodal map is a triangulation of $\mathbb{RP}^4$ with sixteen pentachora, indexed by $s \in \{0, \dots, 15\}$. Facet f of s is glued to facet f of

$$\begin{cases} s \text{ xor } 15, & f = 0, \\ s \text{ xor } 2^{f-1}, & 1 \le f \le 4, \end{cases}$$

by the identity permutation of the vertices. Collapsing edge 0 and then, after canonical relabelling, edge 5 are legal moves, and the result has the isomorphism signature of X_R. □

The triangulation X_R is also isomorphic to the cellulation (J3) of $\mathbb{RP}^4$ in [18, arXiv v2, Appendix J.1, Eq. (J3)].

10.2. **Fundamental groups.** Presentations of the fundamental groups are read off from the 2-skeleta: the edges outside a maximal tree are the generators, and the triangles give the relators. After Tietze simplification they are

$$\pi_1(X_K) = \langle a, b, c, d \mid a, c, d, db, c^3 \rangle = 1,$$
$$\pi_1(X_H) = \langle a, b \mid b, b^{-1}a \rangle = 1,$$
$$\pi_1(X_R) = \langle t \mid t^2 \rangle = \mathbb{Z}/2.$$

10.3. **Intersection forms.** Orient X_H and X_K. Write D_2 for the boundary matrix of the dual 2-cells and ∂_2 for that of the triangles. Each dual 2-cell is a union of flags

$$\text{triangle} < \text{tetrahedron} < \text{pentachoron}.$$

Choose vertex 0 intrinsically once for each global face, transport it through the face embeddings, and send the barycentre to that vertex. Discard degenerate triangles. With the induced orientations of the flags this gives a matrix M from dual to primal 2-chains. The map is a simplicial approximation of the identity, so a dual cycle and its image are homologous.

Let S be the diagonal matrix whose entry at a triangle is the orientation of an incident pentachoron times the sign of its face-embedding permutation; this is the intersection sign of the triangle with its dual cell, and the computation confirms that it does not depend on the choice of the incident pentachoron. For a matrix B whose columns are integral dual 2-cycles one verifies

$$D_2 B = 0, \qquad \partial_2 M B = 0, \qquad Q = B^{\mathsf{T}} S M B = Q^{\mathsf{T}}. \tag{10.1}$$

Then Q is the matrix of intersection numbers of the columns of B: a triangle and a dual 2-cell with different indices are disjoint, and those with the same index meet transversally in one point with the sign in S.

Since X_H and X_K are simply connected, H_2 is free of rank $\chi - 2$ and its intersection form is unimodular. For X_H and X_K we take B with 2 and 22 columns, found by Smith normal form, and in both cases $\det Q = -1$. A sublattice of index m has Gram determinant m^2 times that of the lattice. Hence $m = 1$, and the columns of B are a basis of H_2.

For X_H the result is

$$Q_H = \begin{pmatrix} -4 & 1 \\ 1 & 0 \end{pmatrix} \cong H := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

by replacing the first basis vector e by $e + 2f$. For X_K the 22×22 matrix Q_K is even, and a rational congruence diagonalises it with three positive and nineteen negative entries. Consequently

$$\det Q_K = -1, \quad \sigma(Q_K) = -16, \quad Q_K \cong (-E_8)^{\oplus 2} \oplus 3H$$

by the classification of indefinite even unimodular forms [11]. The vertex links, fundamental groups and intersection forms obtained here agree with the output of Regina's own routines.

10.4. **Homeomorphism.** By Freedman's classification, every even unimodular form is realised by a closed simply connected topological 4-manifold, and two such oriented manifolds with isometric forms are homeomorphic by an orientation-preserving homeomorphism [7, 8]. Their Kirby–Siebenmann invariant is $\sigma/8 \bmod 2$.

Lemma 10.2 (Kreck)**.** *If N is a closed connected nonorientable 4-manifold and A is closed, connected and oriented, then $N\#A \cong_{\mathrm{TOP}} N\#(-A)$, also inside an iterated connected sum with other summands fixed.*

Proof. Move the attaching ball of A around an orientation-reversing loop in N, avoiding the other attaching balls. It returns with the opposite local orientation, which changes the gluing to the one for $-A$. Isotopy extension gives the homeomorphism [10, p. 247]. □

Proof of Theorem 1.1(c). Let E be the closed simply connected topological manifold with form E_8, and let $Y = S^2 \times S^2$. The fundamental groups and intersection forms computed above give

$$X_H \cong_{\mathrm{TOP}} Y, \qquad X_K \cong_{\mathrm{TOP}} (-E)\#(-E)\#3Y.$$

Lemma 10.2 reverses one summand E after taking the connected sum with X_R:

$$X_R\#X_K \cong_{\mathrm{TOP}} X_R\#E\#(-E)\#3Y.$$

The simply connected manifold $E\#(-E)$ has the form $E_8 \oplus (-E_8) \cong 8H$ [11] and Kirby–Siebenmann invariant $1 + 1 = 0$, so it is homeomorphic to $8Y$. Therefore

$$M_1 \cong_{\mathrm{TOP}} X_R\#11Y \cong_{\mathrm{TOP}} M_0.$$

These are topological decompositions; the manifold E is not PL. □

Proof of Theorem 1.1. Part (a) is Section 9, together with Proposition 7.3 for X_K. Part (b) is Proposition 8.2, and part (c) was just proved. A PL homeomorphism between M_0 and M_1 would give $B_{113}(M_0) = B_{113}(M_1)$ by Theorem 3.2, contradicting $30 \neq 83$. Every closed PL 4-manifold has a compatible smooth structure, and a diffeomorphism between compatible smoothings would induce a PL homeomorphism. □

10.5. **Kreck's manifolds.** The Regina 7.4.1 source records that X_H and X_K were constructed by Burke from Kirby diagrams of the standard $S^2 \times S^2$ and K3 surface, followed by simplification [1, 13]. This documented construction is the PL identification used in calling M_1 and M_0 Kreck's manifolds; the intrinsic checks above establish their topological types. The 54-pentachoron triangulation of K3 printed in [1, arXiv v1, Appendix D.1] has f-vector $(25, 30, 110, 135, 54)$ and is a different triangulation. The accompanying files give no Pachner-move path between it and X_K.

11. Discussion

(1) *Nonorientability.* The oriented summands X_K and $\#11X_H$ have different intersection forms, so their distinct values alone do not exhibit an exotic pair. The connected sum with X_R makes them homeomorphic by the homeomorphism argument above, and the values remain different because $B_{113}(X_R) \neq 0$. This is consistent with Reutter's obstruction for oriented semisimple theories [15].

(2) *Local data only.* The invariance of B is proved from identities between F-, R- and time-reversal data, and its values are computed from the definition. Rokhlin's theorem, Pin^+ bordism and the η-invariant do not enter.

(3) *Feasibility.* The re-ordering law of Section 7 is what makes the evaluation on K3 possible: it allows the 54-pentachoron triangulation X_K, which has no branching.

(4) *Questions.* Is $B(X_K) = -2^{11}$ in $\mathcal{R}$? Can the state sum be evaluated on a triangulation of the Cappell–Shaneson fake $\mathbb{RP}^4$ [2]? Do other super-modular categories, or a nonzero fermionic background f, distinguish further exotic pairs?

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Appendix A. Computations

The following statements were verified by computer. "Exact" means arithmetic in $\mathcal{R}$.

statement	arithmetic	size
pentagon, hexagon, unitarity (Section 2)	exact	all label tuples; 224 nonzero F, 24 nonzero R
time reversal (2.1)–(2.3)	exact	all label tuples
isometries (4.2)	exact, $\mathbb{F}_{113}$, $\mathbb{F}_{257}$	15 exact cases plus conjugation; 30 per prime
closed value (4.3)	exact, $\mathbb{F}_{113}$, $\mathbb{F}_{257}$	6 pentachora
Pachner identities (Remark 4.6)	$\mathbb{F}_{113}$	up to 4^{27} coordinates
re-ordering (Proposition 7.1)	$\mathbb{F}_{113}$, $\mathbb{F}_{257}$	3264 identities, 8 tables
cap and cylinder (Lemma 8.1)	$\mathbb{F}_{113}$	2 and 8 pentachora
$B(X_R)$	$\mathbb{F}_{113}$	4 pentachora, 256 slices
$B(X_H)$	exact	6 pentachora
$B'(X_K)$	$\mathbb{F}_{113}$, $\mathbb{F}_{257}$	54 pentachora
links, π_1, intersection forms (Section 10)	integers and rationals	3 triangulations

The exact computations represent an element of $\mathcal{R}$ by its coefficients in the basis $u^j i^k$, which are dyadic rationals with arbitrary-precision numerators. The contractions over $\mathbb{F}_p$ store a tensor as the table of its nonzero entries, with the labels packed into a 64-bit key, two bits for each label, and combine two tensors by a hash join on their common labels. Coefficients are integers in $[0, p-1]$; floating-point numbers are used only to estimate table sizes. For X_K at $p = 113$ the input tensors have at most 425856 nonzero entries, the intermediate tensors at most 6298368, and the largest join has 110043392 matching pairs. The program was compared with brute-force summation on small networks.

The computations were run with Python 3.12, SymPy 1.14, Regina 7.4.1 [13], GMP and a C++17 compiler with OpenMP. Contraction trees were found with cotengra [9]. The accompanying README gives the reproduction commands and identifies the retained outputs.

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