The permuton limit of uniquely sorted permutations avoiding 231

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Abstract

We prove that the permuton of a uniform uniquely sorted 231-avoiding permutation of odd length converges in probability to the diagonal permuton. The mean number of inversions is subquadratic. We obtain this estimate by combining a stack-reading recurrence with Lagrange inversion and a telescoping bound for its binomial weights. An accompanying Lean development verifies the result on actual permutations.

hexagon:2611.01254v1

1 Statement

West's stack-sorting map s sends the empty word to itself and satisfies $s(LMR) = s(L)s(R)M$ when M is the largest letter. A permutation is uniquely sorted when it has exactly one preimage under s. Write $\mathcal{U}_m$ for the uniquely sorted permutations of length $N = 2m + 1$ with no triple $i < j < k$ satisfying $\pi_k < \pi_i < \pi_j$.

For a permutation π of length N, its permuton μ_π places mass $1/N$ uniformly in each cell $[(i-1)/N, i/N] \times [(\pi_i - 1)/N, \pi_i/N]$. Let Δ be the law of (X, X) for X uniform on $[0, 1]$. Defant posed a limit question for the permutations with exactly one preimage, including pattern-restricted classes, at the Oberwolfach workshop [1, Problem 8, p. 287]. We treat the restriction to 231-avoidance.

Defant gave a bijection between this class and Tamari intervals of semilength m [2, Section 6]. In a uniform Tamari interval, Chapuy showed that the height at a uniformly chosen vertex of either path has scale $m^{3/4}$ [3]. Here we estimate the inversion statistic directly on the permutations.

Theorem 1. *Let Π_m be uniform on $\mathcal{U}_m$. Then*

$$\frac{\mathbb{E}\,\mathrm{inv}(\Pi_m)}{(2m+1)^2} \longrightarrow 0, \qquad \mu_{\Pi_m} \longrightarrow \Delta$$

with the second convergence in probability for the weak topology on probability measures.

2 Stack reading and generating functions

Decreasing plane binary trees give the fibers of s. The inorder word determines the tree by splitting at the maximum. The postorder word of the tree is the image under s of its inorder word.

Read a word of distinct letters from left to right using a stack of completed decreasing full binary trees. Push a leaf at the first letter and after each descent. At any other letter x, pop the two top trees, make them the right and left children of x in that order, and push the result. The reading fails if two trees are unavailable or either root is larger than x. The empty reading has height zero. A word is *admissible* when the reading succeeds.

A nonempty permutation has a unique preimage exactly when the reading ends at height one. To see one direction, use the characterization by Defant, Engen and Miller [4, Proposition 3.3], restated in [2, Theorem 1.1]. A uniquely sorted word has length $2m + 1$ and m descents. In any tree with this postorder, the first letter and each letter after a descent must be leaves, as an internal node follows a smaller child root. There are $m + 1$ forced leaves. A binary tree with $2m + 1$ vertices has at most $m + 1$ leaves, with equality only when each internal node has two children. The reading reconstructs the resulting tree from these fixed arities. Conversely, if a reading ends with one tree and has m joins, then it had $m + 1$ pushes, so the word has length $2m + 1$ and m descents. The same leaf count forces

the arities of every tree in its fiber. The reconstruction therefore gives the only preimage. Lean proves the stack-reading criterion directly for every positive size in `Defant231.lean`.

In a 231-avoiding word LMR split at its maximum, all labels in L are smaller than all labels in R. Conversely, two standardized avoiders give exactly one such word at fixed block sizes, using the smallest labels on the left. The inversions satisfy

$$\operatorname{inv}(LMR) = \operatorname{inv}(L) + \operatorname{inv}(R) + |R|.$$

The reading at M pushes a leaf when L is empty. Otherwise it joins the top two trees of L, which requires a successful left reading of height at least two. If R is nonempty, its first letter follows a descent from M. The root M cannot be consumed by a subsequent join below M in label. Thus R must be readable on its own and adds its height to the stack.

Let $F(z,u)$ count admissible standardized avoiders by length and stack height, with $f = [u]F$. Let $G(z,u)$ denote the same sum weighted by the number of inversions, with $g = [u]G$. The split gives

$$A = zu + (z/u)(F - 1 - uf), \qquad F = 1 + AF, \tag{1}$$

$$G = AG + (z/u)(G - ug)F + AzF_z. \tag{2}$$

The three terms in (2) count inversions inside R, inside L, and from M to R. Here F_z is the partial derivative at fixed u. All expressions represent formal series with polynomial coefficients in u after the indicated cancellations.

Multiplying the counting equation by u gives $P(z,u,F) = 0$, where

$$P(z,u,y) = zy^2 - (u + z + zfu - zu^2)y + u.$$

The series $P_y(z,u,F(z,u))$ equals $-u$ at $z = 0$. It determines a unique formal branch $U(0) = 0$ on which this expression vanishes. Put $Y = F(z,U)$. Differentiating $P(z,u,F) = 0$ in u gives $P_u(z,U,Y) = 0$. The three identities yield

$$U = zY^2, \qquad 1 + zf - 2zU = Y^{-1}, \qquad Y = 1 + U^2.$$

In particular $U = z(1+U^2)^2$ and $f = U(1-U^2)$. Set $t = z^2$ and $W = U^2$, viewing W as a series in t. Then

$$W = t(1+W)^4, \qquad T(t) = \sum_{m \geq 0} |\mathcal{U}_m| t^m = (1+W)^2(1-W).$$

Rearrange (2) so that G has coefficient $1 - A - zF/u = -P_y(z,u,F)/u$. Multiply by u and set $u = U$. This gives $g = (Y-1)Y^{-2}F_z(z,U)$. Write $B = F_u(z,U)$. A second differentiation in u gives $zB^2 - B/Y + zY = 0$. Its solution with zero constant term is

$$B = \frac{Y}{2U}\left(1 - \sqrt{\frac{1-3U^2}{Y}}\right).$$

Using $U' = Y^3/(1-3U^2)$ and $Y' = 2UU'$, the chain rule gives $F_z(z,U) = (2U - B)U'$. Substituting this and $U = zY^2$ into the formula for g proves the first-moment identity

$$S(t) = \sum_{m \geq 0} t^m \sum_{\pi \in \mathcal{U}_m} \operatorname{inv}(\pi) = \Phi(W), \tag{3}$$

$$\Phi(x) = \frac{(1+x)^{7/2}(1-3x)^{-1/2} - (1+x)^3}{2}.$$

Both binomial series are expanded at the origin with constant term one. Division by a series with zero constant term above is justified by the displayed identities, or by working in formal Laurent series and observing that the resulting expressions are power series.

3 The subquadratic estimate

Write $C_m = [t^m]T$ and $I_m = [t^m]S$. Lagrange inversion applied to $(1+W)^2(1-W)$ gives, by simplification of the binomial coefficients,

$$C_m = \frac{2}{(3m+1)(3m+2)}\binom{4m+1}{m+1}.$$

This also holds at $m=0$, and in particular $\mathcal{U}_m$ is nonempty for every m.

Set $a_k = [v^k]\Phi(v/3)$. We first show $a_k \to 0$. The coefficients of $(1-v)^{-1/2}$ are $c_j = 4^{-j}\binom{2j}{j}$. They lie in $[0,1]$. The ratio $c_j/c_{j-1} = 1 - 1/(2j)$ and the divergence of the harmonic series imply $c_j \to 0$. If $b_j = [v^j](1+v/3)^{7/2}$, then $|b_{j+1}| \le |b_j|/2$ for $j \ge 1$ by the binomial recurrence. Thus $\sum_j |b_j| < \infty$. For each fixed j, $b_j c_{k-j} \to 0$ as $k \to \infty$. The convolution sum tends to zero by splitting it into a fixed initial segment and the tail bounded by $\sum_{j \ge J} |b_j|$. Subtracting the polynomial in (3) leaves $a_k \to 0$.

For $m \ge 1$, Lagrange inversion in (3) gives

$$I_m = \sum_{k=0}^{m} \frac{k}{m} D_{m,k} a_k, \qquad D_{m,k} = 3^k \binom{4m}{m-k}.$$

Set $D_{m,m+1} = 0$. For $0 \le k \le m$ the ratio of successive weights is $3(m-k)/(3m+k+1)$. Hence

$$0 \le (k/m)D_{m,k} \le D_{m,k} - D_{m,k+1}, \qquad \sum_{k=0}^{m} (k/m) D_{m,k} \le D_{m,0}.$$

After division by $D_{m,0}$, the part with $k < K$ is bounded in absolute value by $m^{-1}\sum_{k<K} k|a_k|$, since $D_{m,k} \le D_{m,0}$. For any $\varepsilon > 0$, choose K so that $|a_k| \le \varepsilon$ for $k \ge K$. The remaining part has absolute value at most ε. This proves $I_m/D_{m,0} \to 0$. The explicit counting coefficient gives

$$D_{m,0} = \binom{4m}{m} \le 10m^2 C_m \qquad (m \ge 1).$$

Since $\mathbb{E}\,\mathrm{inv}(\Pi_m) = I_m/C_m$, the first assertion of Theorem 1 follows.

4 From inversions to the permuton

For a fixed permutation of length N, let a'_i count the larger letters to the left of position i and b'_i count the smaller letters to its right. Counting all smaller letters gives $\pi_i - i = b'_i - a'_i$, so $\sum_i |\pi_i - i| \le 2\,\mathrm{inv}(\pi)$. If (X,Y) has law μ_π, then (X,X) has law Δ. Inside the occupied cell of position i, $|X-Y| \le (|\pi_i - i| + 1)/N$, and each cell has mass $1/N$. In the Wasserstein metric with Euclidean cost this coupling gives

$$W_1(\mu_\pi, \Delta) \le \frac{2\,\mathrm{inv}(\pi)}{N^2} + \frac{1}{N}.$$

The right-hand side tends to zero in mean for Π_m, hence also in probability by Markov's inequality. The metric W_1 metrizes weak convergence on the unit square. This completes the proof.

Formalization

The Lean project represents West's map on permutation words and the measure for a permutation by its uniform occupied cells. It checks the counting and moment series identities, the Lagrange coefficient calculation, the subquadratic bound and convergence in probability in `DefantLimit231.lean`. The accompanying `README.md` identifies the two theorems and gives commands to reproduce the build and check their axioms.

Contributions and use of AI. The author directed the work of the AI agents. OpenAI models developed the main argument, wrote the manuscript, and produced the Lean formalization. Claude models (Anthropic) gave a separate derivation of the subquadratic estimate for the mean number of inversions and reviewed the proof and the formalization.

The manuscript is released under CC BY 4.0. The accompanying Lean source is released under Apache 2.0.

References

[1] M. Bóna, M. Bouvel, R. Brignall and J. Pantone (organizers), *Mini-Workshop: Permutation Patterns*, Oberwolfach Reports 6/2024, doi:10.4171/OWR/2024/6.

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[4] C. Defant, M. Engen and J. A. Miller, *Stack-sorting, set partitions, and Lassalle's sequence*, J. Combin. Theory Ser. A **175** (2020), 105275. arXiv:1809.01340.