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Degree-two extremizers at maximum degree six

A counterexample to Liu's eventual degree-sequence conjecture

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Abstract

Let ρ_n be the largest adjacency spectral radius among connected nonregular graphs of order n and maximum degree six. We prove that there are a residue class a modulo seven and an integer n_0 such that, for every $n \geq n_0$ in that class, some graph attaining ρ_n has degree sequence $(6^{n-1}, 2)$. This disproves the degree-six case of Liu's eventual degree-sequence conjecture, which prescribes $(6^{n-1}, 4)$ for every sufficiently large extremizer. The argument uses Zhu's ordered-vector interpolation theorem and Abdi–Ghorbani's regular-graph stability and resistance estimates. An exact Fiedler completion and adjacency replacements reduce all large extremizers to canonical chains with bounded terminal heads. A chain-invariant torsion score permits compressions that change the head order. Every recurrent degree-four head either admits a strict score improvement or belongs to a class with an exact degree-two partner. Finiteness of the head family yields the arithmetic progression. The proof is nonconstructive and includes exact rational computations for the finite local inequalities.

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1 Introduction

For an integer $n \geq 8$, let $\mathcal{G}_n$ be the set of connected, simple, nonregular graphs of order n and maximum degree exactly six. For a graph G, write $\rho(G)$ for the largest eigenvalue of its adjacency matrix, and set

$$\rho_n = \rho(n, 6) = \max_{G \in \mathcal{G}_n} \rho(G), \qquad \mathcal{M}_n = \{G \in \mathcal{G}_n : \rho(G) = \rho_n\}.$$

We call members of $\mathcal{M}_n$ *extremizers*. The maximum exists because the class is finite up to isomorphism and nonempty. For example, deleting one edge from a connected six-regular circulant gives a member for every $n \geq 8$. The notation $(6^{n-1}, k)$ denotes a degree sequence with $n-1$ entries six and one entry k.

Liu conjectured that, for each fixed maximum degree Δ and all sufficiently large orders, every connected nonregular graph of maximum adjacency spectral radius has precisely one deficient vertex, whose degree is prescribed by the parities of Δ and the order [15, Conjecture 7.2]. For even Δ, the prescribed degree is $\Delta - 2$. This is Conjecture 11 in the survey of Liu and Ning [16]. At $\Delta = 6$ it asserts that every sufficiently large extremizer has degree sequence $(6^{n-1}, 4)$. Our main result contradicts that assertion.

Theorem 1.1. *There exist integers $a \in \{0, 1, \ldots, 6\}$ and n_0 such that, for every $n \geq n_0$ with $n \equiv a \pmod 7$, the set $\mathcal{M}_n$ contains a graph with degree sequence $(6^{n-1}, 2)$.*

The assertion concerns global maximizers in $\mathcal{G}_n$, rather than maximizers within a specified family of chains. It allows a degree-four extremizer to tie a degree-two extremizer. Such a tie already contradicts the universal quantifier in Liu's conjecture. The proof does not identify a, n_0, or a particular infinite family that is globally optimal. It proves existence on an eventual arithmetic progression without enumerating all possible terminal graphs.

1.1 Background and ingredients

The quantitative study of the gap between maximum degree and adjacency spectral radius includes the bounds of Stevanović [17], Cioabă, Gregory, and Nikiforov [6], and Liu, Shen, and Wang [14]. These works provide historical context for the nonregular extremal problem. The related regular-graph minimum-gap problem was studied by Abdi, Ghorbani, and Imrich [2]. The particular structural inputs used here are stated separately below.

Liu determined the extremal structure in degrees three and four [15]. Fan, Kang, and Chen treat fixed odd maximum degree [8]. The dense-degree results of Huang, Liu, and Yang [12] and Huang and Yang [13] concern a different regime, where the degree grows with the order. In particular, an equality between two eight-vertex graphs of deficient degrees two and four is already present in [12]. We use that equality and prove that an appropriate rooted-response identity propagates it through arbitrarily long chains.

The leading scale $6 - \rho_n \sim 2\pi^2/n^2$ is known [5, 19], but does not determine the deficient degree. We use Zhu's ordered-vector interpolation theorem [19, Proposition 3.15] for the Perron mass profile, and Abdi–Ghorbani's regular-graph interval, resistance and stability estimates [1] for the bounded-head reduction. Sections 3 and 4 state the exact inputs and verify their application hypotheses, using the arXiv versions specified in the bibliography. The passage to adjacency extremizers is proved here; the regular completion is not assumed to be an exact Laplacian minimizer.

1.2 Proof strategy

First we reduce the deficient degrees to a single two or four (Theorem 3.17). The Perron mass profile gives a common budget for replacing bounded deficient caps and restoring their order near a Perron maximum. An exact eight-vertex completion then turns the Perron vector into a Fiedler vector of a six-regular graph (Lemma 4.4). Its ordered separating vertices, together with the external interval estimates and adjacency replacements, yield a standard chain with a uniformly bounded head (Theorem 4.10).

A normalized rooted response gives an invariant head score; larger scores win after long extensions to the same order (Lemma 5.3). Local replacements show that every recurrent degree-four head either belongs to an exact equality class or admits a head of degree two or four with larger score (Lemma 7.1). The replacement may change the head order. Maximizing the score over the finite recurrent family then either excludes degree-four extremizers in a whole residue class, or supplies an exact degree-two partner for the maximal degree-four chains. Section 7 makes the order alignment and uniform thresholds explicit.

The finite inequalities are accompanied by exact rational certificates, with domains and graph realizations specified in their proofs. Section 8 describes the checks; they supplement, and do not replace, the structural arguments.

2 Notation and variational preliminaries

For the linear algebra underlying the argument, Horn and Johnson [11] provide background on the Rayleigh–Ritz principle, Perron–Frobenius theory, interlacing, and positive definite matrices. Brouwer and Haemers [4] develop these tools for graph spectra; Haemers [10] surveys graph applications of interlacing. Bapat [3] treats graph Laplacians and resistance, while Doyle and Snell [7] explain the Dirichlet-energy interpretation of networks. Zhang's edited volume [18] is a reference for block elimination and Schur complements. These are background references; the specific identities and comparison inequalities needed below are derived in the text.

All graphs are finite, undirected, and simple. The graphs being optimized are connected; induced subgraphs, including head cores, need not be connected unless this is stated or proved. For a graph G, its vertex and edge sets are $V(G)$ and $E(G)$, its adjacency matrix is $A(G)$, and its Laplacian is $L(G)$. We write $G[U]$ for the induced subgraph on $U \subseteq V(G)$, $G - U$ for vertex deletion, and $N_G(v)$ and $d_G(v)$ for a vertex's neighborhood and degree. The subscript is omitted when the ambient graph is fixed. Complements are taken on the indicated vertex set. The vector $\mathbf{1}$ denotes the all-one column vector of the dimension in use; e_v is the coordinate vector indexed by v. All vectors and matrices in the proof are real, and transposes are denoted by a superscript T.

For $G \in \mathcal{M}_n$, put

$$\lambda = \rho(G), \qquad \mu = 6 - \lambda > 0, \qquad S = \{v \in V(G) : d_G(v) < 6\}, \quad T = V(G) \setminus S.$$

Write $\delta_v = 6 - d_G(v)$ and $D = \sum_v \delta_v$ for the vertex and total deficiencies. Let $x \in \mathbb{R}^{V(G)}$ be a positive Perron vector, so $A(G)x = \lambda x$, and put $M = \max_v x_v$. We normalize $\|x\|_2 = 1$ when a statement uses absolute sizes; local homogeneous calculations may rescale it. Limits in the order always refer to sequences of extremizers with $n \to \infty$. An $O(1)$ bound asserted for all extremizers is uniform over that class. Passing to a fixed bounded graph along a subsequence will always be stated when used.

For any graph G of maximum degree at most six and $z \in \mathbb{R}^{V(G)}$,

$$z^T(6I - A(G))z = \sum_{uv \in E(G)} (z_u - z_v)^2 + \sum_{v \in V(G)} (6 - d_G(v))z_v^2. \tag{1}$$

This follows by expanding each square and collecting the coefficient of each z_v^2. For an extremizer the left side at $z = x$ equals $\mu\|x\|_2^2$. Therefore a graph $G' \in \mathcal{G}_n$ and a nonzero real vector y satisfying

$$y^T(6I - A(G'))y - \mu\|y\|_2^2 < 0 \tag{2}$$

contradict extremality, by the Rayleigh principle [11, 4]. A trial vector need not be positive. When a replacement changes the number of vertices temporarily, the graph compared in (2) is the final graph after the original order has been restored.

2.1 Local structure and switches

We use the following published facts with $\Delta = 6$. For an extremizer of order greater than seven, Liu's deficient-set lemmas [15, Lemmas 3.1–3.3 and Corollary 3.1] give

$$G[S] \text{ is complete}, \quad 1 \le |S| \le 5, \quad \max_{s\in S} x_s < \min_{t\in T} x_t.$$

If $u, v \in S$ and $x_u \le x_v$, then $N_G(u) \cap T \subseteq N_G(v) \cap T$. These conclusions apply to global extremizers in $\mathcal{G}_n$; we do not assume them for an arbitrary comparison graph. Summing the Perron equations gives the useful identity

$$\mu \sum_v x_v = \sum_v \delta_v x_v.$$

A *local switch* deletes disjoint edges ab, cd and adds ac, bd, where the latter pairs are distinct nonedges. It preserves every degree and changes the adjacency quadratic form on x by

$$2(x_a x_c + x_b x_d - x_a x_b - x_c x_d) = 2(x_a - x_d)(x_c - x_b).$$

Liu's connected-switch proposition [15, Proposition 3.1, with Lemma 2.2] applies to the nondecreasing Perron switches of an extremizer used below: their resulting graph is connected. Whenever connectedness is shown directly, no such appeal is needed. If the displayed gain is zero but a vertex equation changes, the old positive vector is not an eigenvector of the new adjacency matrix. Equality in the Rayleigh principle is then impossible, so the spectral improvement is strict. We specify the changed equation whenever this equality case is used.

2.2 Rooted heads and standard blocks

A *k-rooted head*, for $k \in \{2, 4\}$, is a connected simple graph H with a distinguished vertex r of degree k, all other vertices having degree six. Its *core* is $H - r$. The root is not part of the core. A *standard block* C_k has four successive clique layers of sizes $1, k, 6-k, 1$, with all edges between consecutive layers and no other edges. The first and last singleton vertices are its terminals, of degrees k and $6-k$. Each of its six internal vertices has degree six. The two orientations have the same underlying eight-vertex graph, also denoted by M_6.

Given a k-rooted head H, identify the last terminal of a fresh C_k with the root of H and take the first terminal as the new root. Denote the resulting head by $E_k(H)$. Thus E_k adds seven vertices, preserves all old nonroot degrees, and changes the old root's degree to six. Repeating this construction $q \ge 0$ times gives $E_k^q(H)$, of order $|V(H)| + 7q$ and sole deficient degree k.

A *block* of a connected graph means a maximal subgraph without a cut vertex, with a bridge counted as a two-vertex block. The *block tree* is the bipartite incidence tree of blocks and cut vertices. Statements that this tree is a path refer to this incidence tree, not to the vertex set of the graph itself.

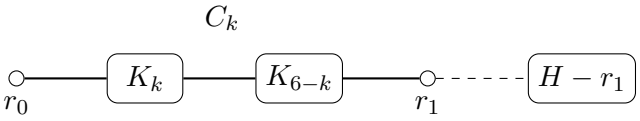


Figure 1: One extension $E_k(H)$. A solid line between layers denotes all possible edges between them. The dashed attachment represents the k edges from the old root r_1 into $H - r_1$. The terminal r_1 has degree $(6-k)+k=6$, while r_0 has degree k.

2.3 An upper comparison at every order

The next construction is included to make the asymptotic upper bound used later independent of any structural conclusion about extremizers.

Lemma 2.1 (An explicit comparison at every order)**.** *For every $n \geq 15$,*

$$6 - \rho(n,6) \leq \frac{2\pi^2}{(n-14)^2}.$$

In particular $6 - \rho(n,6) \leq 2\pi^2 n^{-2} + O(n^{-3})$.

Proof. For $7 \leq m \leq 13$, take the circulant on $\mathbb{Z}/m\mathbb{Z}$ with differences $\pm 1, \pm 2, \pm 3$. Delete its disjoint edges $02, 35$ and add a fresh vertex joined to $0, 2, 3, 5$. This connected cap has $m+1$ vertices, its new root has degree four, and every other vertex has degree six; the difference-one cycle guarantees connectedness.

Write $n = 7q + m + 1$ with $q \geq 1$ and $7 \leq m \leq 13$. Attach q oriented blocks with successive layers of sizes $1, 4, 2, 1$: each layer is a clique and consecutive layers are completely joined. Identify the degree-two terminal of each block with the degree-four terminal of the next, and identify the last terminal with the cap root. Every shared terminal has degree six. Only the first terminal is deficient, of degree four.

Put $z_i = \sin(\pi i/(2q))$, $0 \leq i \leq q$, at successive terminals. Inside block i, put $(5z_i + 2z_{i+1})/7$ on its four-vertex layer and $(4z_i + 3z_{i+1})/7$ on its two-vertex layer. Put value one throughout the terminal cap. The block energy is $8(z_{i+1} - z_i)^2/7$, and the deficiency contribution vanishes because $z_0 = 0$. Thus

$$x^T(6I - A)x = \frac{16q}{7}\sin^2\frac{\pi}{4q}.$$

All six internal coordinates of block i are at least z_i, so

$$\|x\|^2 \geq 7\sum_{i=0}^{q-1} z_i^2 + m + 1 = \frac{7(q-1)}{2} + m + 1 \geq \frac{7q}{2}.$$

The Rayleigh principle and $\sin t \leq t$ give an upper bound $2\pi^2/(7q)^2$. Since $7q = n - m - 1 \geq n - 14$, this proves the lemma. □

3 The deficient vertices of an extremizer

3.1 Nested deficiency profiles

Use the extremizer notation and Liu's nested-neighborhood facts from Section 2, with $n > 7$. In particular,

$$\mu \sum_v x_v = \sum_v (6 - d(v))x_v. \tag{3}$$

If $s = |S|$, then $B = N_T(S)$ is the outside neighborhood of a largest-Perron vertex of S, so $|B| \le 6 - s$. Thus $|S \cup B| \le 6$. Writing $r_1 \le \cdots \le r_s$ for the nested outside-neighborhood sizes, the conditions

$$0 \le r_1 \le \cdots \le r_s \le 6 - s, \qquad r_s \ge 1, \qquad D = s(7-s) - \sum_i r_i \equiv 0 \pmod 2$$

leave 29 profiles. In particular $D \le 10$, and the only profiles with $D = 2$ are one degree-4 vertex, or two adjacent degree-5 vertices with the same four regular neighbors.

3.2 The Perron mass profile

The following interface records the part of Zhu's interpolation theorem used here. For a graph Q and a real vector f on its vertices, write $\mathcal{E}_Q(f) = \sum_{vw \in E(Q)} (f_v - f_w)^2$.

Lemma 3.1 (Regular-graph interpolation, Zhu)**.** *Let Q be a connected simple 6-regular graph on N vertices, and order a real vector on its vertices as $f_1 \le \cdots \le f_N$. There are constants depending only on the degree, an interval $[A, B]$ with $B - A = N + O(1)$, and a continuous nondecreasing piecewise affine function g on this interval such that*

$$g(A) = f_1, \qquad g(B) = f_N, \qquad 8 \int_A^B |g'|^2 \le \mathcal{E}_Q(f).$$

Let F take value f_i on the unit interval centred at $i + 2$, restricted to $[A, B]$ and extended by the extreme values where necessary. The endpoints A, B differ by bounded amounts from the ends of these unit intervals, and

$$\int_A^B |g - F|^2 \le C \sum_{i=1}^{N-1} (f_{i+1} - f_i)^2 \le C \mathcal{E}_Q(f).$$

The assertion applies to arbitrary real vectors; no eigenvector hypothesis is imposed.

Proof. Apply [19, Proposition 3.15] with $d = 6$ and $c = 2(d-2) = 8$. Discard its additional nonnegative strict-piece energy term. The stated endpoint values and the bounded endpoint displacement follow from the endpoint construction in the proof of that proposition. The unit intervals there are centred at $i + c/4 = i + 2$. □

Proposition 3.2 (Perron mass profile)**.** *For degree-6 extremizers of orders tending to infinity, normalize the positive Perron vector by $\sum_v x_v^2 = 1$, and put $M = \max_v x_v$. Then*

$$M = \sqrt{\frac{2}{n}}(1 + o(1)), \qquad \sum_v x_v = \frac{2\sqrt{2}}{\pi}\sqrt{n}\,(1 + o(1)).$$

Consequently $\sum_v x_v/(nM) \to 2/\pi$.

Proof. The total deficiency D is bounded. Complete two disjoint copies of G to a connected simple 6-regular graph $\widehat{G}$ of order $N = 2n + O(1)$, using a bounded auxiliary graph as in [19, proof of Theorem 1.7]. For completeness, one may take a circulant graph on $4D + 14$ vertices with differences $\pm 1, \pm 2, \pm 3$, delete the D disjoint edges $(4j, 4j+2)$, $0 \le j < D$, and attach the $2D$ resulting degree-5 vertices to the deficiency stubs of the two copies. The difference-one cycle ensures that the auxiliary graph remains connected.

Assign values $x/\sqrt{2}$ and $-x/\sqrt{2}$ to the two copies and zero to the auxiliary vertices. The resulting vector f has mean zero, unit norm, and Laplacian energy $\mathcal{E}_{\widehat{G}}(f) = \mu$. Lemma 2.1 gives

$$\mu \le (2\pi^2 + o(1))n^{-2} = (8\pi^2 + o(1))N^{-2}.$$

Order f increasingly. Connectedness and Cauchy–Schwarz along a path give $\max_i |f_i| \le \sqrt{N\mathcal{E}(f)} = O(N^{-1/2})$.

Lemma 3.1 applies because the completion is connected, simple, and 6-regular, and f is a real vector ordered by its values. It supplies an increasing piecewise affine function g on $[A, B]$, of length $L = N + O(1)$, and its associated unit-step function F, such that

$$8\int_A^B |g'|^2 \le \mu, \qquad \|g - F\|_2 = O(N^{-1}).$$

The construction preserves the extreme values: $g(A) = f_1 = -M/\sqrt{2}$ and $g(B) = f_N = M/\sqrt{2}$. The bounded displacement of the endpoints, the bound on $\max |f_i|$, and the step comparison give

$$\int_A^B F^2 = 1 + O(N^{-1}), \quad \int_A^B F = O(N^{-1/2}), \quad \overline{g} = O(N^{-3/2}), \quad \|g - \overline{g}\|_2^2 = 1 + O(N^{-1}).$$

Set $h(t) = \sqrt{L}\,[g(A + Lt) - \overline{g}]$ on $[0, 1]$. It is increasing, mean zero, satisfies $\|h\|_2 \to 1$, and

$$\int_0^1 |h'|^2 = L^2 \int_A^B |g'|^2 \le \pi^2 + o(1).$$

The Neumann Poincare inequality supplies the matching lower bound. Expansion in the Neumann cosine basis, whose next nonconstant eigenvalue is $4\pi^2$, now gives

$$h \longrightarrow -\sqrt{2}\cos(\pi t) \quad \text{strongly in } H^1(0, 1),$$

with this sign selected by monotonicity. The convergence is uniform by the one-dimensional Sobolev inequality. Evaluating at the endpoints yields $\sqrt{L}\,M/\sqrt{2} \to \sqrt{2}$, hence $M \sim \sqrt{2/n}$. Also,

$$\sqrt{2}\sum_v x_v = \int_A^B |F| + O(N^{-1/2}) = \int_A^B |g| + o(\sqrt{N}) = \frac{2\sqrt{2L}}{\pi} + o(\sqrt{N}),$$

which proves the asserted sum. The same Poincare comparison also recovers $n^2\mu \to 2\pi^2$; only the upper bound from Lemma 2.1 was needed as an asymptotic eigenvalue input. □

3.3 Removing deficient vertices by bounded replacements

Lemma 3.3 (Insertion budget)**.** *For any fixed integer $k > 0$, one can insert k degree-six vertices near a Perron maximum, preserving all old degrees and connectivity, with gain $(k + o(1))\mu M^2$ in $y^{\mathsf{T}}(A - \lambda I)y$. The insertions avoid any fixed cap whose coordinates are $o(M)$. They have the same leading gain if all retained old coordinates have first been shifted by a common $h = o(M)$.*

Proof. For fixed R, propagation of the Perron equations gives $x_v = M(1 - O_R(\mu))$ within distance R of a maximum. No deficient vertex occurs there for large n, since $x_s \le 5M/\lambda$. Take $R = 36k$. This ball has at least $36k$ vertices once $n \ge 36k$. A maximal matching in a graph of maximum degree six covers or neighbors all vertices, so a matching of size m accounts for at most $12m$ vertices. Choose $3k$ disjoint edges and divide them into triples. For each triple, remove its edges and join

a fresh vertex to their six endpoints. Every removed edge has a replacement path through this vertex; connectivity and all old degrees are preserved. If the endpoint values are $z_1, \dots, z_6$, paired consecutively, optimizing the fresh coordinate gives gain

$$\frac{(z_1 + \cdots + z_6)^2}{\lambda} - 2(z_1z_2 + z_3z_4 + z_5z_6) = \mu M^2 + O_R(\mu^2 M^2).$$

This follows by substituting $z_i = M + O_R(\mu M)$; its linear error has the additional factor μ. After a common shift the same expansion has baseline $M + h$ and unchanged endpoint errors, giving the same leading term. A fixed $o(M)$ cap is disjoint from these balls. □

We shall repeatedly use the following consequences of the mass profile:

$$W := \sum_{s \in S} (6 - d(s))x_s = \mu \sum_v x_v, \qquad \frac{W^2}{\mu M^2} \longrightarrow 8. \tag{4}$$

Every deficient coordinate and every coordinate a bounded distance from S is $O(W) = o(M)$: first use $x_s \le W$, then $x_w \le \lambda x_v$ across an edge vw. Thus every bounded cap surgery below is disjoint from the insertion region. If a surgery deletes k more vertices than it adds, has loss at most $(c + o(1))\mu M^2$ with $c < k$, and leaves a connected simple nonregular graph of maximum degree at most six, the insertion budget contradicts extremality. The inserted vertices ensure that the final maximum degree is exactly six. Trial coordinates need not be positive.

All subsequent limiting inequalities are interpreted as follows. Suppose the forbidden profile occurs along an unbounded sequence; normalize its deficient coordinates as stated and take a convergent subsequence in the indicated compact domain. A positive uniform margin on that closed domain persists for the original sequence because $\lambda \to 6$. There are only finitely many local incidence patterns, so the resulting threshold is uniform.

Lemma 3.4 (Local quadratic calculus)**.** *All gains in this lemma refer to $Q_\lambda(z) = z^T(A - \lambda I)z$. Deleting a vertex set C from a Perron trial vector has loss $L = x_C^T(\lambda I - A(G[C]))x_C$. Adding an edge vw adds $2x_vx_w$. For a set U of retained vertices and a retained trial vector z, put $\xi = ((A_{\text{new}} - \lambda I)z)_U$. Allowing changes on U adds $\xi^T(\lambda I - A_{\text{new}}[U])^{-1}\xi$, whenever that matrix is positive definite.*

Suppose C contains all deficient vertices and has $q > 0$ edges to its complement. Put $S = \sum_{vw \in E(C,\bar{C})} x_v$, with $v \in C$. Replace C by a graph with r vertices, e internal edges and q' edges to retained vertices, leaving all retained–retained edges unchanged. Let E be the sum of retained coordinates at the latter incidences, with multiplicities. Giving every new vertex coordinate y and shifting every retained coordinate by h gives gain at least

$$-L + 2Ey - (\lambda r - 2e)y^2 - 2Sh + 2q'yh - qh^2. \tag{5}$$

In particular $y = 0, h = -S/q$ adds S^2/q to the deletion gain. At $\lambda = 6$, if $A = 6r - 2e$ and $Aq > (q')^2$, optimizing gives

$$-L + \frac{qE^2 - 2q'ES + AS^2}{Aq - (q')^2}. \tag{6}$$

For a fixed bounded cap and incidence pattern whose relevant coordinates are $O(W)$, evaluation at limiting normalized coordinates has $o(W^2)$ error, uniformly on compact domains. Its optimizing y, h are then $O(W)$.

Proof. Sum the Perron equations on C for the deletion identity; an added edge changes the adjacency quadratic form by its two symmetric entries. Finite-coordinate optimization is completion of the

square. On the retained old graph the residual sum is $-S$, and the all-ones quadratic form is $\mu|\bar{C}|-q$, because all retained vertices were regular. Expand after the common shift and drop the nonnegative term $\mu|\bar{C}|h^2$ to obtain (5). Inverting its two-by-two positive definite negative Hessian gives (6). □

Lemma 3.5 (Exterior order from deficient twins)**.** *Suppose $u, v \in S$ are true twins (their closed neighborhoods coincide) of degree at most five and every vertex of $B = N_T(S)$ is adjacent to both. Then each coordinate outside $S \cup B$ is greater than every coordinate on B.*

Proof. Write $x_u = x_v = a$. If $w \in B$, $z \notin S \cup B$, and $x_z \le x_w$, then z has six regular neighbors whereas w has at most four. Choose $t \in N(z) \setminus (N(w) \cup \{w\})$ and replace uw, zt by vz, wt. The five vertices u, v, w, z, t are distinct, and the added pairs are nonedges. The degree of u falls by one, that of v rises by one, and all others are unchanged. The path $u - v - w$ survives the first deletion; if the second deletion disconnects the graph, one of the added edges reconnects its two sides. The Rayleigh gain is $2(x_w - x_z)(x_t - a) \ge 0$, using regular–deficient separation. In the equality case the old vector fails the changed equation at u, so equality in the Rayleigh principle is impossible. The resulting strict spectral improvement contradicts extremality. □

Proposition 3.6 (Two degree-five twins)**.** *For all sufficiently large orders, if $S = \{u, v\}$ consists of degree-five vertices, the missing edges of their four common regular neighbors B have matching number at most one. Thus they form an empty graph, one edge, a two- or three-edge star, or a triangle, with isolated vertices allowed.*

Proof. Write $x_u = x_v = a$ and $\sum_{b \in B} x_b = (\lambda - 1)a$. A minimum regular coordinate m occurs in B; otherwise its six neighbors are regular and $\lambda m \ge 6m$. Its equation gives $m \ge m_0 := 2a/(\lambda - 4)$. If two disjoint boundary edges are missing, delete v and add them. Paths through v reroute through its true twin; only u is deficient, of degree four. For the paired boundary values, $(z_1 - m_0)(z_2 - m_0) \ge 0$ and its analogue give gain at least

$$2m_0(\lambda - 1)a - 4m_0^2 - \lambda a^2 = -\frac{\lambda(\lambda - 6)^2 a^2}{(\lambda - 4)^2} = o(\mu M^2).$$

One head insertion contradicts extremality. The list is exactly the simple graphs on four vertices containing no two disjoint edges. □

Proposition 3.7 (Initial deletions)**.** *For all sufficiently large extremizers, every deficient vertex has a regular neighbor, $|S| \le 4$, $D \le 6$, and the profile $(s, r) = (3, (2, 2, 2))$ is impossible.*

Proof. If $u \in S$ has no regular neighbor, write $a = x_u$, $s = |S|$. Its equation and the other deficiencies, all at least one, give $W \ge (\lambda + 7 - s)a$. Deleting u preserves connectivity through the remaining deficient clique and costs

$$\frac{\lambda a^2}{\mu M^2} \le \frac{48}{(13 - s)^2} + o(1) \le \frac{3}{4} + o(1) < 1.$$

One insertion is a contradiction. If $s = 5$, positive $r_i \le 1$ would make all five deficient degrees five, contradicting degree-sum parity.

If $s \ge 2$ and $D \ge 8$, delete a minimum-Perron deficient vertex of value a. Nestedness reroutes every path through another deficient vertex, which remains deficient. Since $W \ge Da$, the loss coefficient is at most $48/D^2 + o(1) \le 3/4 + o(1) < 1$, again a contradiction. For $s = 1$, its positive even degree gives $D \le 4$.

For $(3, (2, 2, 2))$, the deficient triangle has common value a, $W = 6a$, and deletion loss $3(\lambda - 2)a^2$. Join its two regular neighbors if their edge is absent; they then connect all remaining components

and have degrees three or four. The added edge improves the form. The loss coefficient is at most $8/3 + o(1) < 3$, less than the three insertions restoring order. □

Proposition 3.8 (Four deficient vertices are impossible)**.** *A sufficiently large degree-6 extremizer has* $|S| \le 3$.

Proof. Only $(1,1,2,2)$ and $(2,2,2,2)$ survive the initial deletions. Write their regular boundary as $B = \{b_1, b_2\}$. For the first profile the deficient twin pairs have values a, a, b, b, and the boundary values satisfy

$$z_1 = (\lambda - 1)a - 2b, \quad z_2 = (\lambda + 1)(b - a), \quad 7/6 \le \lim b/a \le 5/3, \quad 7/16 \le \lim z_2/z_1 \le 14/5.$$

The bounds follow from $z_1, z_2 > b$. Eliminating the deficient clique at $\lambda = 6$ gives deletion response and deficiency-flux vector

$$R = \begin{pmatrix} 4/3 & 2/3 \\ 2/3 & 10/21 \end{pmatrix}, \qquad f = \begin{pmatrix} 2 \\ 6/7 \end{pmatrix} = (4,2)^T - R\mathbf{1}.$$

Thus deletion loss is $z^T R z + o(W^2)$ and $W = f^T z + o(W)$. After normalizing $f^T z = 1$, the ratio interval is a line segment and $z^T R z$ is convex. Its endpoint bounds give loss coefficient at most $\max(1028/361, 40/11) = 40/11 < 4$. Delete the four vertices and add $b_1 b_2$ if absent. The resulting boundary joins all remaining components, remains deficient, and the added edge only helps.

For $(2,2,2,2)$, all deficient values are a; put $b = x_{b_1}$, $c = x_{b_2}$. Then $b + c = (\lambda - 3)a$, $W = 4a$, and all exterior values exceed $\max(b, c)$ by Lemma 3.5. If $b_1 b_2$ is absent and $b \le c$, the equation $\lambda b > 4a + 2c$ gives $b > 2(\lambda - 1)a/(\lambda + 2)$, hence $bc \ge (35/16 + o(1))a^2$. Delete the deficient clique and add $b_1 b_2$. The connected remainder has boundary degrees three; its loss is $4(\lambda - 3)a^2 - 2bc$, with coefficient at most $61/16 < 4$.

If $b_1 b_2$ is present, the six-vertex cap is K_6. Its two exterior endpoints p, q satisfy $x_p + x_q = (\lambda + 1)(\lambda - 5)a$, and its deletion gain is $Q = (\lambda + 1)((\lambda - 3)a^2 - b^2 - c^2)$. When $p \ne q$, replace the cap by a degree-two connector at coordinate $(x_p + x_q)/\lambda$. This connects all remaining components and preserves their degrees. Since $bc \ge (2 + o(1))a^2$, its net loss is at most $(14 - 49/6 + o(1))a^2$, with coefficient $35/12 < 5$. When $p = q$, no connector is needed: the remainder is connected, only p is deficient, of degree four, and $b = c$. The loss coefficient is $21/4 < 6$. Each coefficient is below the number of insertions restoring the original order, a contradiction in all cases. □

The conditions $r_i \ge 1$ and $D \le 6$ leave only the two single-vertex profiles, the five two-vertex profiles, the three-vertex profiles $(1,2,3), (2,2,2), (2,3,3)$, and the four-vertex profiles $(1,1,2,2), (2,2,2,2)$. This list follows by substituting $s = 1, 2, 3, 4$ in the integer inequalities at the start of the section. The preceding two propositions remove $(3,(2,2,2))$ and both four-vertex profiles. It remains to exclude

$$(s,r) = (2,(1,3)), (2,(2,2)), (2,(2,4)), (2,(3,3)), (2,(4,4)), (3,(1,2,3)), (3,(2,3,3)).$$

3.4 Three elementary profile exclusions

Proposition 3.9. *The profiles* $(2,(1,3))$, $(2,(2,2))$, *and* $(3,(1,2,3))$ *are impossible for all sufficiently large extremizers.*

Proof. We give the deletions, their losses, and the coordinate bounds proving that the losses are below their insertion budgets.

Profile $(1,3)$. Write deficient values $a \le b$ and regular boundary vertices z_1, z_2, z_3, also denoting their coordinates, the first common to both deficient vertices. Then

$$z_1 = \lambda a - b, \quad z_2 + z_3 = (\lambda+1)(b-a), \quad W = 4a + 2b, \quad 2/\lambda < t := a/b < (\lambda-1)/(\lambda+1).$$

If exactly $k \in \{1,2\}$ common-to-extra boundary edges are present, and $e \in \{0,1\}$ records the extra-to-extra edge, their regular equations and the lower bound b on all regular coordinates give

$$\lambda(z_2+z_3) \ge 12b + k(z_1 - b) + e(z_2 + z_3 - 2b), \qquad t \le \frac{(\lambda-e)(\lambda+1) - 12 + 2k + 2e}{(\lambda-e)(\lambda+1) + k\lambda} \le \frac{2}{3} + o(1).$$

Delete the lower deficient vertex; its neighbors are adjacent and another deficiency remains. The loss coefficient is at most $48t^2/(4t+2)^2 + o(1) \le 48/49 + o(1) < 1$. If both common-to-extra edges are absent, delete instead the higher vertex and add the better of the matchings $\{uz_2, z_1z_3\}$ and $\{uz_3, z_1z_2\}$, where u is the lower vertex. The retained edge uz_1 and the matching connect the entire boundary. All regular degrees remain six and $d(u) = 2$. Averaging the two matching gains bounds the limiting loss, in units b^2, by $13 - 56t + 49t^2$. This convex polynomial is at most $-2/9$ on $[1/3, 5/7]$, by its endpoint values $-2/9, -2$. Thus either operation improves after one insertion.

Profile $(2,2)$. The two deficient vertices are twins of value a, and their boundary coordinates z_1, z_2 sum to $(\lambda-1)a$, and $W = 6a$. Delete both twins. If the boundary edge is absent, add it; since $z_1z_2 \ge a(z_1 + z_2 - a) = (\lambda-2)a^2$, the loss is at most $2a^2$. If the edge is present, the deletion loss is $2(\lambda-1)a^2$; lowering both boundary coordinates by $2a/(\lambda-1)$ adds $8a^2/(\lambda-1)$. In either case the boundary edge connects all remaining components, and the boundary degrees are five or four. The two limiting loss coefficients are $4/9$ and $28/15$, both below the two-insertion budget.

Profile $(1,2,3)$. Write deficient values $a \le b \le c$. The extra boundary coordinates from successive nested neighborhoods are $(\lambda+1)(b-a)$ and $(\lambda+1)(c-b)$, both greater than c. Hence

$$a < \frac{\lambda-1}{\lambda+1}c, \qquad b > a + \frac{c}{\lambda+1}, \qquad W/a = (3a + 2b + c)/a > \frac{6\lambda - 2}{\lambda - 1}.$$

Delete the vertex of value a; its three neighbors form a triangle, so connectivity and a deficiency remain. The residual is $-a$ on that triangle. Lowering all three coordinates by $a/(\lambda-2)$ reduces the loss from λa^2 to $(\lambda - 3/(\lambda-2))a^2$. Its limiting coefficient is at most $525/578 < 1$, below one insertion. These calculations use Lemma 3.4 and (4). □

3.5 Two deficient vertices of degree four

Proposition 3.10. *The profile* $(2,(3,3))$ *is impossible for all sufficiently large orders.*

Proof. The deficient twins have value a, their three regular boundary coordinates have sum $Z = (\lambda-1)a$, and $W = 4a$. Thus $a^2/(\mu M^2) \to 1/2$. Put $C = \max_B x$ and $m = \min_B x$. Lemma 3.5 puts every exterior coordinate above C. Each boundary vertex has two deficient and four regular neighbors, at least two outside B, so $(\lambda-2)m \ge 2(a + C)$. Consequently the limiting quantities $c = C/a$ and m/a satisfy

$$5/3 \le c \le 2, \qquad m/a \ge (1+c)/2.$$

The two smallest normalized boundary values have sum $5 - c$ and each is at least $(1+c)/2$, so their product is at least $3(1+c)(3-c)/4 \ge 9/4$. If a boundary edge is missing, delete one twin and add that edge; the other twin preserves connectivity and remains deficient. The loss is at most $(3/2 + o(1))a^2$, or $(3/4 + o(1))\mu M^2$, below one insertion. Thus B is a triangle.

The five-vertex cap is now K_5. Write its boundary coordinates z_i and their two exterior-neighbor sums Y_i. Its equations give

$$Y_i = (\lambda+1)(z_i - a) > 2C, \qquad Y := \sum_i Y_i = (\lambda+1)(\lambda-4)a.$$

In the limit $5/3 \le c \le 21/11$, and $z_i/a \in [\ell, c]$ for $\ell = 1 + 2c/7$. Summing $(z_i/a - \ell)(z_i/a - c) \le 0$ yields

$$\limsup \frac{\sum_i z_i^2}{a^2} \le \max_{5/3 \le c \le 21/11} (5 + 24c/7 - 6c^2/7) = 1019/121.$$

The quadratic increases on that interval. Deleting the cap therefore has loss

$$L = \sum_i z_i Y_i = (\lambda+1)(\sum_i z_i^2 - aZ) \le (2898/121 + o(1))a^2. \tag{7}$$

If its six exterior endpoints are distinct, replace it by two adjacent vertices, each attached to three endpoints. All old exterior degrees remain six and the two new degrees are four. Giving both new vertices coordinate $Y/[2(\lambda-1)]$ adds $Y^2/[2(\lambda-1)] = (98/5 + o(1))a^2$ to the deletion gain. The net loss coefficient is at most $1316/605 < 3$, below the three insertions needed to restore order.

If an endpoint repeats, there are $k \le 5$ distinct endpoints. Replace the cap by one vertex adjacent once to each. Its degree is $k \le 5$, and an old endpoint of multiplicity j has degree $7 - j$. Both replacements connect every component left after cap deletion. For the latter, use $y = 0$ in Lemma 3.4: $q = 6$, $S = 2Z$, so the gain beyond deletion is $S^2/6$. The loss coefficient is at most $1322/363 < 4$, below the four-insertion budget. The common shift is $O(a) = o(M)$, as required by the insertion lemma. Thus both endpoint patterns contradict extremality. □

3.6 The three-vertex profile $(2, 3, 3)$

Proposition 3.11 (Exclusion of the $(2, 3, 3)$ three-vertex profile)**.** *The profile $s = 3$, $(r_1, r_2, r_3) = (2, 3, 3)$ cannot occur in a sufficiently large degree-6 extremizer.*

Proof. Let u, v, w be the deficient vertices, with values a, b, b; the latter pair are true twins. Label the regular boundary vertices z_1, z_2, z_3, using the same symbols for their coordinates, with the first two adjacent to all three deficient vertices. Then

$$z_1 + z_2 = \lambda a - 2b, \quad z_3 = (\lambda+1)(b-a), \quad W = 2(a+b).$$

We use the limiting normalization $b = 1$, put $t = a/b$, and, when needed, write $x = z_2/b$. Every limit point satisfies

$$\frac{2}{3} \le t \le \frac{6}{7}, \qquad z_1 = 6t - 2 - x, \quad z_2 = x, \quad z_3 = 7(1-t), \quad z_i \ge 1. \tag{8}$$

All inequalities below have uniform strict margins on compact limiting domains. Consequently they persist for $\lambda \to 6$ along the sequence. Moreover $W^2/(\mu M^2) \to 8$, so one head insertion has limiting gain $(t+1)^2/2$ in units of b^2.

Lemma 3.5, applied to the twins v, w, places every exterior coordinate strictly above the whole boundary.

If $z_1 z_2$ is absent, delete v and add $uz_3, z_1 z_2$. All added edges were absent, the remaining high deficient vertex w keeps the graph connected, and the two remaining deficient degrees are four. Since $z_1 z_2 \ge z_1 + z_2 - 1 = 6t - 3$ in the coordinate normalization, the limiting quadratic loss is at most

$$12 - 26t + 14t^2 \le \frac{8}{9}.$$

Here the last inequality follows by convexity and evaluation at $2/3, 6/7$. Dividing by $\mu M^2/b^2 = (t+1)^2/2$ bounds the loss coefficient by $16/25 < 1$. One head insertion gives a contradiction. We may therefore assume that the edge z_1z_2 is present.

If this is the only edge in B, delete all three deficient vertices and add z_1z_3, z_2z_3. The boundary is now a triangle, so the retained graph is connected; its boundary degrees are four, four, and six. The limiting loss is

$$F(t) = 90t^2 - 116t + 38.$$

The function $2F(t)/(1+t)^2$ is increasing on $[2/3, 6/7]$, because its derivative is $16(37t-24)/(1+t)^3$. Its maximum is $460/169 < 3$, so three head insertions contradict extremality.

The boundary path. Suppose z_1z_2, z_1z_3 are present and z_2z_3 is absent. Set $C = \max\{z_1, z_2, z_3\}$. The stronger ordering fact and the three regular boundary equations imply

$$C \le 42t - 21 - 7x, \qquad C \le \frac{7}{2}(x-t), \qquad C \le 14 - 16t + \frac{x}{3}. \tag{9}$$

Together with $C \ge z_i$ and (8), these inequalities define a rational hexagon in (t, x). Delete S and add z_2z_3, making B a triangle. The graph is connected, nonregular, and has boundary degrees three, four, and five. The residual on B of the retained trial vector is

$$r = (-t-2,\ 7(1-t)-t-2,\ x-2)^{\mathsf{T}}.$$

Optimizing the three boundary coordinates gives additional gain

$$r^{\mathsf{T}}(6I - A_{K_3})^{-1}r = \frac{\|r\|^2}{7} + \frac{(\mathbf{1}^{\mathsf{T}}r)^2}{28}.$$

Hence the loss is

$$F(t,x) = \frac{21}{4} + \frac{15}{2}t - \frac{27}{2}x - \frac{173}{28}t^2 + \frac{205}{14}tx - \frac{5}{28}x^2.$$

For an exact finite comparison, normalize $W = 1$ instead, and write $A = a/W$, $Y = z_2/W$. Then $b/W = 1/2 - A$, and the loss is

$$\widetilde{F}(A,Y) = \frac{21}{16} - \frac{3}{2}A - \frac{27}{4}Y - \frac{59}{7}A^2 + \frac{197}{7}AY - \frac{5}{28}Y^2.$$

Its Hessian has determinant -786, so there is no interior local maximum. The projective change $(t,x) \mapsto (t/[2(t+1)], x/[2(t+1)])$ preserves line segments and sends the hexagon to a convex hexagon. Its six vertices and their exact normalized loss coefficients are

t	x	$8\widetilde{F}$
3/4	5/4	890/343
11/14	17/14	364/125
11/14	3/2	10796/4375
208/259	324/259	1296421/436178
21/26	21/13	11109/4418
22/27	13/9	47156/16807

On each edge the quadratic either attains its maximum at an endpoint or has its critical point outside the edge; direct rational substitution checks this. Thus the largest coefficient is $1296421/436178 < 3$. Three head insertions contradict extremality. The complete exact vertex and edge calculation is reproduced by `computation/round2-r233.py` using rational arithmetic only.

The boundary triangle, including coincident exterior endpoints. It remains to treat $G[B] = K_3$. The cap $C_0 = S \cup B$ is K_6 minus uz_3 and has four exterior incidences: one from each of z_1, z_2, two from z_3. Delete it and add one vertex adjacent once to each distinct endpoint. There are $k \in \{2,3,4\}$ endpoints; the new degree is k, and old endpoint multiplicity j becomes degree $7-j$. This connected nonregular replacement loses five vertices. In the limiting normalization the port-side sum is $s = z_1 + z_2 + 2z_3 = 12 - 8t$. Lemma 3.4, with $q = 4, y = 0$, gives gain $-L + s^2/4$ before insertions, independent of endpoint repetitions. The five insertions add $\frac{5}{2}(t+1)^2$ in units of b^2.

Set $P = 6t - 2$ and $q = 7(1-t)$, so that $z_1 = P - x, z_2 = x, z_3 = q$. The original cap loss is

$$L(t,x) = 6(P^2 + q^2) - 14x(P-x) - 2Pq - (t+2)P - 2q.$$

The stronger ordering fact, applied to the exterior endpoints, gives

$$\max\{P - x, x, q\} \le \min\{42t - 21 - 7x,\ 7x - 7,\ 21 - 24t\}. \tag{10}$$

Together with $P - x, x, q \ge 1$, this defines a rational hexagon. The total limiting gain has the lower bound

$$\begin{aligned} H(t,x) &= \frac{(12-8t)^2}{4} - L(t,x) + \frac{5}{2}(t+1)^2 \\ &= -\frac{595}{2} + 797t - 28x - \frac{1139}{2}t^2 + 84tx - 14x^2. \end{aligned}$$

Its Hessian has negative diagonal entries and determinant $24836 > 0$, so H is strictly concave. Its minimum over the hexagon is attained at one of the following six vertices:

t	x	$H(t,x)$
3/4	5/4	57/32
11/14	17/14	1033/392
11/14	3/2	1033/392
21/26	16/13	3437/1352
21/26	21/13	3437/1352
22/27	13/9	4393/1458

Thus $H \ge 57/32 > 0$, with no restriction on repeated exterior endpoints. The hexagon enumeration and all displayed rational values are reproduced by `computation/round2-r233-compression.py`; its exact certificate is `computation/round2-r233-compression.json`.

The positive margin contradicts extremality, completing all cases. □

3.7 Deficient degrees three and five

Lemma 3.12. *In a sufficiently large extremizer with deficient degrees three and five, write their vertices as u, v and coordinates as a, b, respectively. Let $C = \{c_1, c_2\}$ be their common regular neighbors and $E = \{e_1, e_2\}$ the other two neighbors of v. Then c_1c_2 is an edge, the complement of $G[C \cup E]$ has no matching of size two, and every vertex outside $\{u, v\} \cup C \cup E$ has coordinate greater than both common-neighbor coordinates.*

Proof. In this proof a vertex name inside an algebraic expression denotes its Perron coordinate. Put $t = a/b$. The deficient equations and separation give

$$c_1 + c_2 = \lambda a - b, \quad e_1 + e_2 = (\lambda+1)(b-a), \quad W = 3a + b, \quad \frac{3}{\lambda} < t < \frac{\lambda - 1}{\lambda + 1}.$$

First suppose an exterior vertex z has coordinate at most that of a common neighbor w. Vertex z has six regular neighbors, while w has only four regular neighbors. Choose $r \in N(z)$ with $r \neq w$ and rw a nonedge. Delete uw, zr and add vz, wr. The changed deficient degrees are two at u and six at v; all other degrees are unchanged. The Rayleigh change divided by two is

$$(w - z)(r - a) + (b - a)z > 0.$$

Here $b > a$, as follows from the displayed bound on t. The path $u - v - w$ survives deletion of uw. If deletion of zr then disconnects the graph, v, w lie in the same component and one of vz, wr reconnects the two sides. Thus the replacement is admissible, proving the exterior-coordinate assertion.

If the boundary complement has a perfect matching, delete v and add its two edges. For a matching $(p, q), (r, s)$, all four coordinates exceed b, so

$$pq + rs \geq b(p + q + r + s) - 2b^2 = (\lambda - t - 2)b^2.$$

The quadratic gain is at least $(\lambda - 2t - 4)b^2$, whose limiting coefficient is at least $4/7$. The matching can be chosen to give a connected graph. A cross matching joins both extra neighbors to common neighbors; for the other matching, an existing cross edge suffices. If no cross edge exists, all cross pairs are missing and a cross matching is available instead. Every component after deleting v meets a former neighbor of v, and u joins the common neighbors. One head insertion restores the order, contradicting extremality.

Finally, if c_1c_2 is missing, delete v and add ue_1, ue_2, c_1c_2. The result is connected, has u of degree four, and every other vertex of degree six. Since $c_1c_2 \geq b(c_1 + c_2) - b^2$, its limiting quadratic gain divided by b^2 is at least

$$-10 + 26t - 14t^2 \geq -\frac{1}{2} \qquad \left(\frac{1}{2} \leq t \leq \frac{5}{7}\right).$$

The mass profile gives $b^2/(\mu M^2) = 8/(3t + 1)^2 + o(1) \leq 32/25 + o(1)$. The loss is therefore at most $(16/25 + o(1))\mu M^2$, which one head insertion more than restores. All margins are uniform on the compact limiting interval, so the assertions hold for sufficiently large order. □

Proposition 3.13 (The extra-boundary edge in the $(2, 4)$ profile)**.** *For every sufficiently large extremizer with profile $s = 2$, $(r_1, r_2) = (2, 4)$, the two regular neighbors of the higher deficient vertex that are not neighbors of the lower deficient vertex are adjacent.*

Proof. Use Lemma 3.12, with deficient coordinates a, b, common boundary c_1, c_2 and extra boundary e_1, e_2. If e_1e_2 is absent, normalize by b and pass to the limit:

$$c_1 + c_2 = 6t - 1, \quad e_1 + e_2 = 7(1 - t), \quad W/b = 3t + 1, \quad t = a/b \geq 1/2.$$

Let $C = \max(c_1, c_2)$. Each extra boundary vertex has one deficient neighbor, of value 1, and five regular neighbors. Among the latter, zero, one, or two can be common boundary vertices, and all the others are exterior vertices with values at least C. In each case their sum is at least $\frac{5}{2}(c_1 + c_2)$: with two common neighbors use $c_1 + c_2 + 3C$; with one use $c_1 + c_2 - C + 4C$; with none use $5C$. Thus

$$e_i \geq \frac{10t - 1}{4} =: \ell(t), \qquad 7(1 - t) \geq 2\ell(t), \qquad \frac{1}{2} \leq t \leq \frac{5}{8}.$$

In particular $e_1e_2 \geq \ell(t)\big(7(1 - t) - \ell(t)\big)$.

Delete both deficient vertices and add the missing edge between the extra boundary vertices. The common boundary edge is present, and at least one common-to-extra edge is present, since

otherwise the complement of the boundary would contain a matching of size two. The boundary is therefore connected after the addition. Every component after deletion meets the boundary, so the whole graph is connected. Its common boundary degrees are four and four; all other degrees are six.

Deletion and the added edge give gain $-(6t^2+6-2t)+2e_1e_2$. The retained residual sum is $-(9t-3)b$ and total deficiency four. A constant shift by $-(9t-3)b/4$ adds at least $(9t-3)^2b^2/4$; two insertions add $(3t+1)^2b^2/4+o(b^2)$. The total limiting gain divided by b^2 is at least

$$\begin{aligned} H(t,e_1,e_2) &= \frac{33}{2}t^2-10t-\frac{7}{2}+2e_1e_2 \\ &\geq \frac{-248t^2+248t-57}{8} \geq \frac{9}{64} > 0. \end{aligned}$$

The last quadratic is concave on $[1/2, 5/8]$ and has endpoint values $5/8$ and $9/64$. This uniform strict margin persists for $\lambda \to 6$, contradicting extremality and proving the proposition. □

3.7.1 The remaining boundary configurations

Proposition 3.14. *Every sufficiently large extremizer has degree sequence different from* $(6^{n-2},3,5)$.

Proof. Use Lemma 3.12 and Proposition 3.13. Both c_1c_2 and e_1e_2 are present. The missing boundary edges therefore form, up to interchanging the two common or the two extra vertices, one of

$$H=\varnothing, \quad \{c_1e_1\}, \quad \{c_1e_1,c_1e_2\}, \quad \{c_1e_1,c_2e_1\}.$$

Normalize by $b=1$ and pass to a convergent subsequence with $\lambda \to 6$. All the normalized boundary coordinates belong to a compact set. Put

$$a=t, \quad (c_1,c_2,e_1,e_2)=(c,6t-1-c,e,7-7t-e)=:z.$$

Let F be the boundary edge set and let $p_i=4-d_F(i)$ for common vertices and $p_i=5-d_F(i)$ for extra vertices. The exterior-coordinate assertion supplies the following closed polytope of necessary limiting constraints:

$$z_i \geq 1, \qquad 6z_i-\sigma_i-\sum_{ij\in F} z_j \geq p_iz_k \quad (i=1,\dots,4;\ k=1,2), \qquad \sigma=(t+1,t+1,1,1). \tag{11}$$

Remove e_1e_2 and add ue_1, ue_2. Both deficient vertices now have degree five, all other degrees are unchanged, and the graph is connected. Set the trial coordinates at u,v equal to $y=(6-t)/5$. This is the optimal common coordinate with the boundary fixed. The initial limiting quadratic gain is

$$g_0=\frac{14}{5}(1-t)(2t+3)-2e_1e_2.$$

The residual at the four boundary vertices is

$$r=(2y-t-1,2y-t-1,2y-1-e_2,2y-1-e_1)^T.$$

Let $A_{F'}$ be the adjacency matrix of $F'=F\setminus\{e_1e_2\}$. Changing the four boundary trial coordinates by $(6I-A_{F'})^{-1}r$ gives the additional gain $r^T(6I-A_{F'})^{-1}r$. This inverse exists since the boundary has only four vertices. Define the resulting quadratic polynomial by

$$g_H(t,c,e)=g_0+r^T(6I-A_{F'})^{-1}r.$$

For the first three boundary types, exact minimization over (11) gives strictly positive values:

H	$\min g_H$
$\varnothing$	$461/1053$
$\{c_1e_1\}$	$122073707/392165832$
$\{c_1e_1, c_1e_2\}$	$9121/49500.$

The certificate and a complete finite verification method are described below. These replacements already contradict extremality.

For the last type, the direct minimum is negative, so an additional edge transfer is needed. The four exterior edge incidences from the common vertices have average exterior coordinate

$$\bar{q} = \frac{42t + 2e - 21}{4}.$$

Choose an endpoint q with coordinate at most this average. If $y > \bar{q}$, choose a neighbor r_0 of q outside the six-vertex cap. Such a neighbor exists because q has degree six and has no original deficient neighbor. Transfer qr_0 to ur_0. The new degrees are six at u and five at q, v; the graph stays connected because the path from q through its common neighbor to u survives. Both endpoints q, r_0 retain their original trial coordinates, and $x_{r_0} \geq 1$. The extra gain is at least $2(y - \bar{q})$. When $y \leq \bar{q}$, skip this transfer. Thus the available limiting gain is at least

$$g_H + 2\max\{0, D\}, \qquad D = y - \bar{q} = \frac{129 - 214t - 10e}{20}.$$

For this type, symmetry allows averaging the two common coordinates. The projected polytope is exactly

$$t \geq \tfrac{1}{2}, \quad e \geq 1, \quad 7 - 7t - e \geq 1, \quad 30t + 2e \geq 19, \quad 7e - 5t \geq 6, \quad 54t + 7e \leq 43.$$

Exact two-dimensional minimization on the two regions $D \geq 0$ and $D \leq 0$ gives in both cases

$$\min\big(g_H + 2\max\{0, D\}\big) = \frac{95017}{12139960} > 0.$$

It is attained at $t = 44657/86714$, $e = 407377/216785$, on $D = 0$. All replacements are at the original order. The fixed positive margins, compact normalized coordinate sets, and continuous dependence on λ make them valid for all sufficiently large orders, completing the exclusion. □

Exact finite verification. The standard-library script `root-round2/r24-present-certificate.py` constructs every inequality in (11), inverts the four-by-four rational matrices, and verifies the stated rational minima. For the first three types it enumerates every subset of at most three active linear constraints and solves the associated rational stationary equations, retaining all feasible solutions. This includes the global minimum: take a face of smallest dimension containing a minimizer in its relative interior. If its restricted quadratic Hessian is singular, a stationary null direction preserves the value until a smaller face is reached. Otherwise the stationary system on that face is nonsingular and is explicitly checked. Vertices are included as zero-dimensional faces. The feasible sets are bounded by $z_i \geq 1$ and the displayed sum identities. No floating-point comparisons enter this verification. For the final type, the checker `computation/round2-r24-transfer-audit.py` enumerates polygon vertices and all edge stationary points on each side of $D = 0$. The Hessian is indefinite, excluding interior minima; both checks give the same strictly positive rational margin.

3.8 Two deficient vertices of degree five

Proposition 3.15 (The clique boundary in the $(4,4)$ profile is excluded)**.** *Suppose a sufficiently large degree-6 extremizer has two deficient degree-5 vertices u, v with their four common regular neighbors B. Then $G[B]$ is not complete.*

Proof. Suppose otherwise. The cap $C = \{u, v\} \cup B$ is K_6. Write $x_u = x_v = a$. Every $b \in B$ has one neighbor outside C. Let the distinct outside endpoints be $p_1, \dots, p_k$, with multiplicities $m_1, \dots, m_k$, so $\sum_i m_i = 4$ and $1 \le k \le 4$. Put $P_i = x_{p_i}/a$. Lemma 3.5 places every exterior coordinate strictly above the whole boundary.

The equations on the clique give, for a boundary vertex attached to p_i,

$$x_b = a + \frac{x_{p_i}}{\lambda + 1}, \qquad \sum_i m_i P_i = (\lambda + 1)(\lambda - 5), \qquad P_i > 1 + \frac{P_j}{\lambda + 1} \quad \text{for all } i, j.$$

Hence every limiting port vector belongs to the compact polytope

$$\mathcal{P}_m = \left\{ P : \sum_i m_i P_i = 7,\ P_i \ge 1 + P_j/7 \text{ for all } i, j \right\}. \tag{12}$$

Delete the cap, introduce one fresh vertex adjacent once to each distinct p_i, and later restore the remaining five vertices by head insertions. The graph is connected, since the fresh vertex joins all components left after cap deletion. Its degree is $k \le 4$, and the old endpoint p_i has degree $7 - m_i \le 6$, so the graph is admissible and nonregular.

Set $E = \sum_i x_{p_i}$, $s = (\lambda - 1)a$. The cap-deletion loss is

$$L = (\lambda + 1)(\lambda - 5)a^2 + \frac{\sum_i m_i x_{p_i}^2}{\lambda + 1}.$$

Lemma 3.4, with $r = 1, e = 0, q = 4, q' = k$, adds at the limit $(4E^2 - 2kEs + 6s^2)/(24 - k^2)$ to the deletion gain. The five insertions add $(5/2 + o(1))a^2$, since $W = 2a$ implies $a^2/(\mu M^2) \to 2$. Thus it suffices to show positivity on (12) of

$$F_m(P) = \frac{4(\sum_i P_i)^2 - 10k \sum_i P_i + 150}{24 - k^2} + \frac{5}{2} - 7 - \frac{\sum_i m_i P_i^2}{7}.$$

This function is concave on $\sum m_i P_i = 7$. In fact, for a tangent vector z with $\sum m_i z_i = 0$, weighted Cauchy–Schwarz gives

$$(\sum_i z_i)^2 \le \left(\sum_i \frac{1}{m_i} - \frac{k^2}{4} \right) \sum_i m_i z_i^2,$$

and the quadratic coefficient of F_m is bounded above by

$$\left[\frac{4}{24 - k^2} \left(\sum_i \frac{1}{m_i} - \frac{k^2}{4} \right) - \frac{1}{7} \right] \sum_i m_i z_i^2 < 0.$$

The bracket is $-1/7$ or $-8/105$ for the possible multiplicities.

For $k \ge 2$, every vertex of the polytope has two distinct coordinate levels H, L with $L = 1 + H/7$. Indeed, an active inequality can join only a minimum coordinate to a maximum coordinate; an

intermediate coordinate would leave a nontrivial feasible direction. If the high coordinates have total multiplicity w, then

$$H = \frac{7(w+3)}{6w+4}, \qquad L = \frac{7(w+1)}{6w+4}.$$

Evaluation at these finitely many vertices gives

(m_i)	(4)	$(3,1)$	$(2,2)$	$(2,1,1)$	$(1,1,1,1)$
$\min_{\mathcal{P}_m} F_m$	$1/23$	$6/121$	$29/320$	$23/120$	$179/100.$

For $k = 1$ the polytope is the singleton $P_1 = 7/4$. All margins are strictly positive, contradicting extremality. □

The finite rational calculations are reproduced by `background/round2-r44-clique.py`; the accompanying JSON records every vertex and the positive minimum for each collision pattern.

Proposition 3.16 (Exclusion of two deficient degree-five twins)**.** *For all sufficiently large orders, a degree-*6 *extremizer cannot have two deficient vertices with outside-neighborhood profile* $(4,4)$*.*

Proof. Suppose otherwise along a sequence of unbounded orders. Let u, v be the deficient twins, with common Perron coordinate a, and let B be their four common regular neighbors. The deficiency identity and Perron mass profile give

$$\mu \sum_w x_w = 2a, \qquad \frac{a^2}{\mu M^2} \longrightarrow 2.$$

Every coordinate outside $C = \{u, v\} \cup B$ exceeds every coordinate in B, by Lemma 3.5. Proposition 3.6 says that the missing-edge graph $H = \overline{G[B]}$ has matching number at most one. The complete-boundary case is excluded by Proposition 3.15. Thus H is one edge, a two-edge star, a three-edge star, or a triangle.

Put $h_i = d_H(i)$, $q_i = 1 + h_i$, and $q = \sum_i q_i$; these are the exterior port counts at the four boundary vertices and their sum. The possibilities are $q = 6, 8, 10, 10$, respectively. Pass to a subsequence with fixed exterior incidence pattern. All cap and exterior-endpoint coordinates are $O(a)$, by their Perron equations. Normalize by a and pass to limits, denoting the four boundary coordinates by b_i and the distinct exterior coordinates by p_j. All comparisons below are on these compact limiting sets; a fixed strict margin therefore persists for the original sequence as an $o(a^2)$ error.

The limiting equations and exterior-coordinate separation give

$$b_i \geq 1, \qquad \sum_i b_i = 5, \qquad Q_i := 7(b_i - 1) + \sum_{j:\, ij \in H} b_j \geq q_i b_\ell \quad \text{for all } i, \ell. \tag{13}$$

Write

$$L = 7 \sum_i b_i^2 - 35 + 2 \sum_{ij \in H} b_i b_j, \qquad s = \sum_i q_i b_i.$$

Deleting the cap has quadratic loss $(L + o(1))a^2$. The sum of its port-side coordinates is $(s + o(1))a$, and the sum of the exterior coordinates counted once per port is $(s + 2 + o(1))a$, since $\sum_i Q_i = s + 2$.

We first record the finite boundary bounds used below:

H	K_2	$K_{1,2}$	$K_{1,3}$	K_3
$\max\big(L - s^2/q\big)$	$202/75$	$107/40$	$332/125$	$2773/1058.$

(14)

Isolated boundary vertices are implicit. These are exact rational optimizations on (13). To reproduce them, eliminate b_4 using $\sum b_i = 5$. The quadratic-form matrices for $L - s^2/q$ have positive leading principal minors

$$(83/6, 130, 1232), \quad (27/2, 509/4, 1120), \quad (58/5, 567/5, 5096/5), \quad (68/5, 636/5, 5184/5),$$

respectively. Thus each objective is convex and attains its maximum at a vertex. Enumerating three independent tight inequalities from (13) gives fourteen vertices in each case; exact evaluation gives (14). The complete vertices and arithmetic are in `background/round2-r44-boundary.py` and its JSON output.

Replacements preserving every exterior port. Delete C and introduce a clique K_r, assigning each old exterior port to a new vertex while retaining every old exterior degree. Whenever an exterior endpoint occurs at most r times, assign its incidences to distinct currently least-loaded new vertices. Inductively the loads differ by at most one, so the final maximum load is $\lceil q/r \rceil$. We use the following cases:

q	maximum endpoint multiplicity	r	$D_{\text{new}} = r(7-r) - q$	$(6-r)/2 + 4/D_{\text{new}}$
6	2	2	4	3
8	2	2	2	4
10	4	4	2	3.

The new degrees are at most 6 and $D_{\text{new}} > 0$, so each replacement is nonregular. The new clique joins every component left after cap deletion, and the graph is simple. In particular the $q = 10$ row covers *all* endpoint coincidences, since no outside vertex can meet more than the four vertices of B.

For this port-preserving clique, Lemma 3.4 has $q' = q$, $E = (s+2)a$, and $A = r(7-r) = q + D_{\text{new}}$. Its optimized gain beyond deletion is $(s^2/q + 4/D_{\text{new}})a^2$, attained at $y = 2a/D_{\text{new}}$, $h = y - sa/q$. The $6 - r$ insertions add $((6-r)/2 + o(1))a^2$. Each budget in the final column strictly exceeds the corresponding bound in (14). This excludes all three rows, including every case with three missing boundary edges.

High endpoint multiplicity when $q = 6$ or $q = 8$. The only cases left have an exterior endpoint occurring at least three times. Let k be the number of distinct exterior endpoints. Then $2 \le k \le 4$ for $q = 6$, and $3 \le k \le 6$ for $q = 8$. Delete C and add one new vertex adjacent once to each distinct exterior endpoint. Its degree is at most 6. An old endpoint of multiplicity m now has degree $7 - m \le 6$, and the total deficiency is $6 + q - 2k > 0$. The new star connects every remaining component.

Put $E = \sum_{j=1}^{k} p_j$, counting distinct endpoints once. Lemma 3.4, with $r = 1, e = 0, q' = k$, and five insertions give normalized gain

$$F = \frac{qE^2 - 2kEs + 6s^2}{6q - k^2} + \frac{5}{2} - L. \tag{15}$$

The denominator is positive in all the remaining cases.

Here is an exact finite certificate of $F > 0$ for all possible endpoint incidences; its construction also explains completeness of the enumeration. Let N be the $4 \times k$ binary incidence matrix between B and the distinct exterior endpoints. Its row sums are q_i, its columns are nonzero, and at least one column has sum at least three. Put $c = \max_i b_i$, $\delta_i = c - b_i \ge 0$, and $w_j = p_j - c \ge 0$. With A_H the adjacency matrix of H, the equations become

$$\left[7I + A_H - \frac{3}{2}J \;\middle|\; N\right] \begin{pmatrix} \delta \\ w \end{pmatrix} = \frac{1}{2}\mathbf{1}, \qquad \delta, w \ge 0. \tag{16}$$

The reconstruction is $c = (5 + \sum_i \delta_i)/4$, $b_i = c - \delta_i$, $p_j = c + w_j$. Summing the four equations gives $\sum_i q_i\delta_i + \sum_j m_j w_j = 2$, where m_j is the positive column sum. Thus this polytope is compact. The first four columns are strictly diagonally dominant and nonsingular, so its constraint rank is four. Every vertex is therefore found by choosing four basic variables, solving the corresponding rational linear system, and retaining the nonnegative solutions.

Column permutations and automorphisms of H leave exactly thirty-eight incidence orbits. For every one, evaluate (15) at all vertices v_i of (16), and also evaluate

$$B_{ij} = 2F\left(\frac{v_i + v_j}{2}\right) - \frac{F(v_i) + F(v_j)}{2}.$$

The exact least coefficients, including the diagonal coefficients $F(v_i)$, are

H	k	incidence orbits	least coefficient
K_2	2	2	19/484
K_2	3	4	1/18
K_2	4	2	123/1210
$K_{1,2}$	3	4	41/858
$K_{1,2}$	4	13	1/16
$K_{1,2}$	5	10	2/23
$K_{1,2}$	6	3	104/363.

These are rational computations, reproduced by `background/round2-r44-dedup-certificate.py`; its JSON records the incidence representatives and every basic feasible vertex. For $z = \sum_i \theta_i v_i$, $\theta_i \geq 0$, $\sum_i \theta_i = 1$, the quadratic identity

$$F(z) = \sum_i \theta_i^2 F(v_i) + 2\sum_{i<j} \theta_i\theta_j B_{ij}$$

shows that the positive coefficient table proves $F(z) \geq 19/484$ on every relevant polytope, not merely at its vertices.

Every coefficient is positive, so every remaining pattern contradicts extremality by the common quadratic and insertion lemmas. □

3.9 The two remaining degree sequences

Theorem 3.17 (Two remaining degree sequences)**.** *There is n_0 such that, for every $n \geq n_0$, a connected nonregular graph of order n and maximum degree six maximizing its adjacency spectral radius has one of the degree sequences*

$$(6^{n-1}, 2), \qquad (6^{n-1}, 4).$$

Here 6^m denotes m repetitions of the degree six. In particular, there is exactly one deficient vertex, and its degree is two or four. The total degree deficiency is therefore either four or two.

Proof. The conditions $r_i \geq 1$ and $D \leq 6$ leave the twelve profiles listed above. Proposition 3.8 removes the two four-vertex profiles. Propositions 3.9, 3.10, 3.11, 3.14, and 3.16 remove all remaining profiles except $(s, r) = (1, (2)), (1, (4))$. (The initial deletion also removed $(3, (2, 2, 2))$.) Since $d_i = s - 1 + r_i$, these are precisely the two asserted sequences. Each of the finitely many exclusions has a fixed strict margin, so their thresholds have a common upper bound. □

4 Canonical structure of large extremizers

By Theorem 3.17, a sufficiently large extremizer has one deficient vertex u, of degree two or four. It is the unique Perron minimum: a regular vertex at a global minimum would give $\rho(G)x_v = \sum_{w\sim v} x_w \geq 6x_v$. We show that an uninterrupted standard chain starts at u and leaves only a bounded terminal head. We use ordered separating vertices, not a classification of the whole block tree; the intervening induced graphs may have internal branches.

4.1 Ordered intervals and the regular-graph inputs

Write $\mathcal{E}_H(z) = \sum_{vw\in E(H)}(z_v - z_w)^2$, also denoted by $E_H(z)$. The trial criterion is (2). For a connected graph J, a *Fiedler vector* is a nonzero vector for the second Laplacian eigenvalue $\lambda_2(L(J))$; it is orthogonal to the constant vector. This terminology comes from Fiedler's algebraic connectivity [9].

Fix an ordering $v_1, \dots, v_N$ in which a vector z is nondecreasing, and put $a_i = z_{v_{i+1}} - z_{v_i}$ for $1 \leq i < N$. Its cut current is

$$c_i = \sum_{\substack{v_j v_h \in E(J) \\ j \leq i < h}} (z_{v_h} - z_{v_j}).$$

Each edge contributes its total difference times the sum of the a_i across the cuts it crosses. Hence $\mathcal{E}_J(z) = \sum_{i=1}^{N-1} a_i c_i$. If $L(J)z = \mu z$ and $\sum_v z_v = 0$, summing the eigen-equations on a prefix gives

$$c_i = -\mu \sum_{j\leq i} z_{v_j} = \mu \sum_{j>i} z_{v_j}.$$

In particular every proper cut has positive current when $\mu > 0$ and z is a nonconstant increasing mean-zero vector. For a Perron ordering of a graph with its only deficient vertex below the cut, the same suffix identity holds because all vertices in that suffix are regular.

An *ordered separating singleton* is a vertex v_i with nonempty sets on both sides such that no edge joins an earlier vertex to a later vertex after v_i is removed. In a six-regular graph its type is $k \in \{2, 4\}$ when it has k neighbors on its right and $6 - k$ on its left. These are the two types called P, Q in [1]. A *standard gap* is an interval of seven differences whose two endpoint vertices are ordered separating singletons of the same type. The graph M_6 denotes the underlying unoriented graph of C_2 or C_4; both have clique layers $1, 2, 4, 1$, up to reversal.

Select all standard gaps. Their difference intervals have disjoint interiors: their induced canonical blocks, identified below, cannot have overlapping interiors. The nonempty intervals of differences remaining between them, and at the two ends, are called *exceptional intervals*. An interior exceptional subgraph includes both endpoint vertices; an end subgraph includes its attachment vertex. Thus consecutive pieces may share a singleton. An exceptional interval contains no standard gap. We use these definitions in the exact regular completion and then restrict them to its positive copy. Every induced interval between consecutive separating vertices is connected: a simple path between two of its vertices cannot leave and return through the same boundary, and cannot pass from one boundary to the other outside the interval, since either boundary separates earlier from later vertices. The same argument applies to an initial or terminal interval. Thus these pieces are connected even when their interiors have branches. Their only possible exterior edges are incident with the boundary singletons.

A connected simple two-terminal graph (H, s, t) is *admissible of type* (k, l), for $k, l \in \{2, 4\}$, if its terminal degrees are $d_H(s) = k$ and $d_H(t) = 6 - l$, and all its other degrees are six. Using the Dirichlet principle for unit-conductance networks [7, 3], its effective resistance R_H between s and t is defined by

$$R_H^{-1} = \min\{\mathcal{E}_H(z) : z_s = 0,\ z_t = 1\}.$$

The minimizing vector is harmonic at all nonterminal vertices and lies in $[0,1]$. An *admissible end* is a connected simple rooted graph (H,r) whose root has degree $p \in \{2,4\}$ and whose b other vertices have degree six. Define

$$A_H = L(H)[V(H) \setminus \{r\}] = 6I - A(H-r), \qquad T_H(t) = \mathbf{1}^T (A_H - tI)^{-1}\mathbf{1}$$

whenever the inverse exists. Finally, put

$$\phi(k) = \frac{1}{2}\left(\frac{1}{k} - \frac{1}{6-k}\right) = \frac{1}{k} - \frac{3}{8} \quad (k \in \{2,4\}).$$

The following statements specify the external estimates used in this section. They are the degree-six instances of the indicated results; none assumes that a regular completion of an adjacency extremizer is itself an exact regular minimizer.

Lemma 4.1 (Regular stability and ordered intervals)**.** *The results of Abdi–Ghorbani give the following estimates.*

1. *The minimum of $\lambda_2(L(J))$ over connected six-regular graphs of order N is $8\pi^2 N^{-2} + O(N^{-3})$. There are constants C, N_0 such that, if $N \geq N_0$, $0 \leq \epsilon \leq 1$, and $N^2\lambda_2(L(J)) \leq 8\pi^2 + \epsilon$, standard gaps in a Fiedler ordering cover all but $C(\epsilon + N^{-1})^{1/3}N$ vertices. Their induced subgraphs are actual M_6 blocks on one path of the block tree.*

2. *For an increasing vector on a six-regular graph, an interval whose boundaries are ordering ends or separating singletons of type 2 or 4 has a partition into intervals I of bounded positive length. There are bounded positive numbers R_I and bounded endpoint corrections ψ_I^-, ψ_I^+ such that*

$$\Delta_I := \sum_{i \in I} a_i \leq R_I \max_{i \in I} c_i, \qquad R_I \leq |I|/8 + \psi_I^- - \psi_I^+.$$

   Corrections agree at shared endpoints and hence telescope. On an interval between terminals of type (k,l) their total is $\phi(k) - \phi(l)$. Each partition interval is either strict, with a fixed positive saving in the latter inequality, or has length seven and ends at a separating singleton of type 2 or 4. A nonstrict interval starting at such a singleton is a standard gap. In an initial segment of L differences containing no standard gap, or in an entire interval between separating singletons containing none, at least $cL - C$ partition intervals are strict, for constants $c > 0, C$. The same statements hold for an admissible end after completing its root on the other side.

The first part is [1, Theorem 4.1, its proof, and Corollary 6.2]. The second part is [1, Lemmas 3.17–3.18]. In particular it applies to arbitrary increasing vectors, before eigen-equations are imposed.

Lemma 4.2 (Resistance and end comparisons)**.** *For the admissible graphs defined above the following hold.*

1. *If $|V(H)| = m$, then*

$$R_H \leq (m-1)/8 + \phi(k) - \phi(l).$$

   Equality holds precisely for $k = l$ and a consistently oriented chain of copies of C_k with the specified outer terminals. For each type (k,l) and every sufficiently large m there is an admissible comparison graph J_m of that order satisfying $R_{J_m} \geq (m-1)/8 - C$.

2. *For an admissible end with b nonroot vertices, $\lambda_{\min}(A_H) \ge b^{-2}$. For $0 \le tb^2 \le 1/2$, the vector $(A_H - tI)^{-1}\mathbf{1}$ is positive and $T_H(0) \le T_H(t) \le T_H(0)/(1-tb^2)$. For fixed $0 < \xi \le 1/2$, if its increasing ordering, with root value zero, has no standard gap in its first $\lfloor \xi b \rfloor$ differences, then*

$$T_H(t) \le b^3/24 - c_\xi b^3 + C_\xi b^2 + Ctb^5, \qquad c_\xi > 0.$$

For each root degree $p \in \{2,4\}$ and all sufficiently large b, there is an admissible end E_b with that root degree and exactly b nonroot vertices satisfying $T_{E_b}(0) \ge b^3/24 - Cb^2$.

These are [1, Theorem 5.1 and Lemmas 5.4, 5.6–5.7]. The comparison orders have no residue restriction. The constants are uniform over the four terminal types and the two root degrees; c_ξ, C_ξ may depend on ξ.

We also record explicitly the fixed-size interchange formula. For an admissible H of type (k,l) and order m, set

$$r_H = R_H, \quad r_C = 7/8, \quad \nu_H = m-1, \quad \delta_H = \nu_H r_C - 7\{r_H + \phi(l) - \phi(k)\}.$$

For a fixed two-terminal piece P and fixed outside values a, b, restrict $t \ge 0$ to values for which its free-vertex Dirichlet matrix minus tI is positive definite. In particular this holds for all sufficiently small t, because P is connected. Let $q_P(t; a, b)$ be the minimum of $\mathcal{E}_P - t\|z\|_*^2$ subject to those endpoint values, where

$$\|z\|_*^2 = \sum_{v \text{ nonterminal}} z_v^2 + (a^2 + b^2)/2.$$

The half weights count a shared terminal exactly once when pieces are joined.

Lemma 4.3 (Bounded interchange formula)**.** *There is a real constant W_H, depending only on H and its ordered terminals, such that for $r = r_H + r_C$, $c = (a+b)/2$, and $J = (b-a)/r$,*

$$q_{HC_l}(t; a, b) - q_{C_k H}(t; a, b) = 2t\delta_H cJ + 2tr_C W_H J^2 + O(t^2(a^2+b^2)).$$

For any fixed bound on m, the coefficient W_H, the remainder, and the interval of sufficiently small $t \ge 0$ are uniform over the admissible graphs and over $a, b \in \mathbb{R}$.

This is [1, Lemma 5.11]. The two products mean identification of their adjacent singleton terminals. Their common terminal has degree six, and both products have type (k,l) and the same order. The accompanying perturbation estimate [1, Lemma 5.3] follows by completing the square: if B is the positive definite matrix for the free vertices at $t = 0$, f_0 the harmonic minimizing vector on them, and ω the fixed boundary norm, then

$$q(t) = q(0) - t(\|f_0\|^2 + \omega) - t^2 f_0^T (B - tI)^{-1} f_0, \qquad f_t - f_0 = t(B - tI)^{-1} f_0.$$

We use it only when $B - tI$ is positive definite.

4.2 An exact Fiedler completion for a sole deficient vertex

Lemma 4.4 (Eight-vertex symmetric completion)**.** *Let G be a connected simple graph whose degrees are six except at a single vertex u, whose degree is $6 - D$, where $D \in \{2,4\}$. Put $\lambda = \rho(G)$ and $\mu = 6 - \lambda$. There is a connected simple six-regular graph $\widehat{G}$ of order $2|V(G)| + 8$ with the following property. If*

$$\mu < \begin{cases} 3 - \sqrt{7}, & D = 2, \\ 5 - \sqrt{17}, & D = 4, \end{cases}$$

then $\lambda_2(L(\widehat{G})) = \mu$ is simple. Its Fiedler vector is exactly $(x, -x, 0)$, up to a scalar, where x is the Perron vector of G and the zero entries are on the eight new vertices.

Proof. Take copies G_+, G_- and an eight-vertex gadget H. For $D = 2$, let H be the complement of two disjoint $K_{1,3}$'s and designate their centers; for $D = 4$, use the complement of $K_4 \cup 2K_2$ and designate the K_4 vertices. Join every designated vertex to both copies of u. Designated gadget vertices originally have degree four and the others have degree six, so the result is simple, connected, and six-regular. Put $K_G = 6I - A(G)$, $K_H = 6I - A(H)$, and let b indicate the designated vertices. Their equitable partitions have quotients

$$\begin{pmatrix} 1 & 3 \\ 1 & 5 \end{pmatrix} \ (D = 2), \qquad \begin{pmatrix} 0 & 4 \\ 4 & 2 \end{pmatrix} \ (D = 4).$$

Their positive Perron vectors lift to H, giving $\rho(H) = 3 + \sqrt{7}$ and $1 + \sqrt{17}$, respectively. Thus the stated bound is exactly $\mu < \lambda_{\min}(K_H)$.

Copy interchange fixes H and splits the Laplacian into invariant antisymmetric and symmetric subspaces. On the first, it is K_G, whose simple smallest eigenvalue is μ with vector $(x, -x, 0)$. On the second, in orthonormal coordinates, it is

$$\mathcal{L}_+ = \begin{pmatrix} K_G & -\sqrt{2}e_u b^T \\ -\sqrt{2}be_u^T & K_H \end{pmatrix}.$$

For $0 < t < \mu$, the first block of $\mathcal{L}_+ - tI$ is positive definite; its Schur complement is

$$K_H - tI - 2[(K_G - tI)^{-1}]_{uu} bb^T.$$

It is positive definite minus a rank-one nonnegative matrix, so has at most one negative eigenvalue. The constant vector supplies one, since its Laplacian eigenvalue is zero. Thus no positive symmetric eigenvalue lies below μ.

Nor can it equal μ: for a corresponding vector (y, z), multiplication of the first block equation by x^T gives $b^T z = 0$. Hence $y = \alpha x$, and the second equation gives $z = \sqrt{2}\alpha x_u (K_H - \mu I)^{-1} b$. Positive definiteness and $b \neq 0$ imply $\alpha = 0$ and then $y = z = 0$, a contradiction. The smallest positive eigenvalue is therefore μ, simple and antisymmetric. □

For large extremizers $\mu \to 0$, so the lemma applies. The Fiedler ordering consists of the negative copy in reverse Perron order, the zero gadget, and the positive copy in Perron order. The completion is nearly minimal in the regular class; exact regular minimality is not asserted or used.

4.3 Ordered canonical blocks and exceptional intervals

Proposition 4.5 (Canonical-block coverage)**.** *For every sufficiently large extremizer G of order n, standard gaps in its Perron ordering are actual M_6 blocks and cover all but $O(n^{2/3})$ vertices. Moreover $6 - \rho(G) = 2\pi^2 n^{-2} + O(n^{-3})$.*

Proof. Theorem 3.17 and Lemma 2.1 permit the exact completion of Lemma 4.4: its order is $N = 2n + 8$ and its Laplacian gap is $\mu = 6 - \rho(G)$. The lower bound in Lemma 4.1(1) gives $\mu \geq 8\pi^2/N^2 - O(N^{-3}) = 2\pi^2/n^2 - O(n^{-3})$; the reverse inequality follows from Lemma 2.1. Hence $N^2\mu \leq 8\pi^2 + O(N^{-1})$, and the stability assertion leaves $O(N^{2/3})$ vertices outside its standard gaps. The positive copy is a consecutive interval of the exact Fiedler ordering, in precisely the Perron order. Except for boundedly many blocks meeting the connector, every selected block in that copy is an actual block of G: any larger two-connected subgraph in G would remain one in the completion. Its boundaries are ordered separating singletons. Restricting the coverage to this copy proves the assertion, without any condition on $G - u$ or its block tree. □

We will use the following consequence of the ordered-interval input. Let an interval H contain no standard gap, have m vertices and endpoint values $a < b$, and have only regular interior vertices. Suppose its values have absolute value at most M and its current satisfies $c_i - c_{i-1} = -\mu x_{v_i}$ at interior cuts, as in the exact completion. Set $\Delta = b - a$. For all sufficiently large m,

$$E_H(x) \geq \frac{8+\eta}{m}\Delta^2 - C\mu M\Delta, \qquad \eta > 0. \tag{17}$$

Indeed currents vary by at most $C\mu M$ on each bounded partition interval I of Lemma 4.1(2), so

$$E_I \geq \Delta_I \max_{i\in I} c_i - C\mu M\Delta_I \geq \Delta_I^2/R_I - C\mu M\Delta_I.$$

At least $cm - C$ intervals have a fixed resistance saving. Telescoping the endpoint corrections gives $\sum_I R_I \leq m/(8+\eta)$ for large m, and Cauchy–Schwarz proves (17). All constants are uniform over the endpoint types.

4.4 Bounded regular exceptions and their location

Work in the increasing Perron ordering supplied above. Include shared boundary singletons in each exceptional region. The initial region contains u; all other regions have regular interior vertices and are admissible ends or two-terminal graphs. Their orders are $O(n^{2/3}) = o(n)$.

Proposition 4.6. *For every sufficiently large adjacency extremizer of maximum degree six, every regular exceptional interval has bounded order, uniformly over the extremizer and its order. This includes the terminal exception at the maximum end of the Perron ordering.*

Proof. We argue by contradiction along a sequence of exceptional intervals whose orders tend to infinity. An interval far from the upper end is replaced using its resistance; an interval close to that end is absorbed into an end subgraph and compared through T_H. Write $\mu = 6 - \rho(G) = O(n^{-2})$, and use the increasing Perron order. Every interval in question avoids the deficient vertex. The separating-singleton property of standard gaps implies that an interior regular exception H is an admissible two-terminal graph with boundary degrees in $\{2, 4\}$; an upper end is an admissible one-terminal graph. All its nonterminal vertices have degree six. The exceptional orders are $O(n^{2/3}) = o(n)$.

For an interior exception write $m = |V(H)|$, $a < b$ for its endpoint coordinates, $\Delta = b - a$, and $M = b > 0$. All its values lie in $[a, b]$. Summing the regular equations over an upper suffix gives its cut current as $c_i = \mu \sum_{v>i} x_v > 0$. In particular, if there are t vertices beyond H, then

$$E_H(x) \geq \mu t M\Delta. \tag{18}$$

The second lower bound is (17).

Suppose that a sequence of such intervals has $m \to \infty$. First assume $t \geq Km$, where K is a large fixed constant. By (18), the error in (17) is at most CE_H/t, whence, for large m, $E_H \geq (8+\eta/2)\Delta^2/m$. The same-order, same-boundary-degree comparison graph of Lemma 4.2 has effective resistance at least $(m-1)/8 - C$, and therefore a harmonic vector with the same endpoints saving at least a fixed fraction χE_H of the energy. Every changed coordinate remains in $[a, b]$. The loss of squared norm is at most $2mM\Delta$, while the defect contribution to $x^T(6I - A)x$ is unchanged. Taking $K > 2/\chi$ makes the energy saving larger than μ times this possible norm loss. The resulting connected graph has the same order and degrees, contradicting adjacency extremality.

If $t < Km$, include the whole upper end up to the lower endpoint of H, and use that endpoint as root r. The resulting admissible end has b nonroot vertices with $m - 1 \leq b \leq (K+1)m$ and $b = o(n)$.

For a terminal exception take that end itself. Denote the resulting end again by H, retaining m for the order of the original exceptional interval. With $A_H = 6I - A(G[H - r])$, the regular equations give

$$x_{H-r} = x_r\big(\mathbf{1} + \mu(A_H - \mu I)^{-1}\mathbf{1}\big). \tag{19}$$

The root coordinate is positive. The first $m - 1$ differences in this resolvent ordering have no standard gap. Choose fixed $\xi = 1/[4(K + 1)]$. Since $\mu b^2 = o(1)$, Lemma 4.2 gives $T_H(\mu) \le b^3/24 - c_\xi b^3 + O_\xi(b^2) + O(\mu b^5)$, whereas $T_{E_b}(0) \ge b^3/24 - O(b^2)$. The negative cubic term dominates both errors as m, and hence b, tend to infinity. Thus

$$T_H(\mu) < T_{E_b}(0) \le T_{E_b}(\mu), \qquad T_J(t) = \mathbf{1}^T(A_J - tI)^{-1}\mathbf{1},$$

for sufficiently large m. Both end Dirichlet matrices minus μI are positive definite, by their lower bound b^{-2}. For prescribed root value x_r, the minimum of the end energy minus μ times its nonroot squared norm is exactly $-x_r^2(b\mu + \mu^2 T_J(\mu))$, by completing the square. Replacing H by E_b therefore strictly lowers $x^T(6I - A)x - \mu\|x\|^2$. It preserves all degrees, order and connectedness, and avoids the deficient vertex. This is again a contradiction. Hence no regular exceptional order is unbounded. □

Proposition 4.7. *For every sufficiently large adjacency extremizer of maximum degree six, every regular exceptional interval lies within a bounded number of vertices of the maximum end of the Perron ordering.*

Proof. The terminal exception is bounded by Proposition 4.6. Every interior regular exception H has standard blocks immediately on both sides. Its bounded order lets us use the finite family of two-terminal interchanges of [1, Lemmas 5.11–5.12]. We explain the positive-ground-state modifications, including which end matters.

Put $r_C = 7/8$, $r_H = R_H$, $\nu_H = |H| - 1$, and

$$\delta_H = \nu_H r_C - 7\{r_H + \phi(l) - \phi(k)\}.$$

Here $k, l \in \{2, 4\}$ are the boundary types. The resistance equality case gives $\delta_H > 0$: if H were a standard chain, the regular equations would make its natural interface currents positive and monotone along this positive interval, so its natural order would be the inherited order and it would contain standard gaps. The finite family therefore has a uniform positive lower bound for δ_H.

Let c_H be the midpoint of the two endpoint coordinates, and $I > 0$ their difference divided by r_H. Let M be the maximum of the coordinates on the three consecutive pieces C_k, H, C_l. The Dirichlet perturbation estimate gives $M \le C(c_H + I)$ and an $O(\mu M)$ error between actual and harmonic currents. For the unions on the left and right, respectively, these estimates give

$$c_L = c_H - r_C I/2 + O(\mu M), \quad c_R = c_H + r_C I/2 + O(\mu M), \quad J_L = I + O(\mu M), \quad J_R = I + O(\mu M).$$

For sufficiently small μ, each fixed-boundary Dirichlet matrix minus μI is positive definite, uniformly over the finite family. The old Perron restriction therefore minimizes the local quadratic form with its endpoints fixed. Lemma 4.3 applies to both unions. The sum of the changes in energy minus μ times squared norm for the two interchanges

$$C_k H \longmapsto H C_l, \qquad H C_l \longmapsto C_k H$$

is

$$-2\mu\delta_H r_C I^2 + O(\mu^2 M^2). \tag{20}$$

Indeed the terms containing c_H and W_H cancel between the two directions, while $c_L - c_R = -r_C I + O(\mu M)$ gives the displayed leading term. These are same-order degree-preserving replacements; the defect term stays fixed. Unlike the regular Laplacian problem, the adjacency variational principle has no mean-zero constraint, so no centering correction is needed.

Choose a large fixed K. If $c_H \le KI$, then $M = O_K(I)$ and (20) is negative for large n, a contradiction. Otherwise $c_H > KI$. Let t be the number of vertices above these three pieces. Their last singleton has coordinate at least $c_H/2$ for large K, so its outgoing current is at least $\mu t c_H/2$. It also equals $I + O(\mu M)$. Consequently

$$\mu M/I \le C/t$$

if t exceeds a fixed constant. The right interchange $HC_l \mapsto C_k H$ has leading change $-2\mu\delta_H c_R J_R$, where $c_R = c_H + r_C I/2 + O(\mu M)$ and $J_R = I + O(\mu M)$. Its other first-order term is $O(\mu I^2)$ and its remainder is $O(\mu^2 M^2)$. Choose K large to absorb the former, then t large to absorb the latter. The change is strictly negative, again a contradiction. Thus t is uniformly bounded. Adding the bounded sizes of H and its two neighboring blocks proves the assertion. □

4.5 The initial deficient region

Proposition 4.8. *The initial region from the deficient vertex to the first standard block has bounded order, uniformly over all sufficiently large extremizers.*

Proof. In the exact completion the deficient vertex u is an ordered separating singleton: its $\delta = 6 - d_G(u)$ gadget neighbors precede it, all its $d_G(u)$ positive-copy neighbors follow it, and no edge bypasses u between these sets. Thus the initial region H from u to the first standard block's lower singleton r is an admissible ordered interval. If $u = r$ there is nothing to prove. Otherwise put $k = d_G(u) \in \{2, 4\}$, $p = d_H(r) \in \{2, 4\}$, $m = |H| - 1$, $h = x_u$ and $a = x_r$. Every value on H lies in $[h, a]$, and H contains no standard gap.

Suppose $m \to \infty$ along a sequence. Coverage gives $m = o(n)$, while $\mu = O(n^{-2})$. Applying (17), and decreasing its fixed positive constant slightly to replace $m + 1$ by m, yields

$$E_H(x) + \delta h^2 \ge \frac{8+\eta}{m}(a-h)^2 + \delta h^2 - C\mu a^2 \ge \frac{a^2}{m/(8+\eta) + 1/\delta} - C\mu a^2.$$

The second inequality minimizes the quadratic in h. Since $\mu m = o(1)$, its right side is at least $(8 + \eta/2)a^2/m$ for all sufficiently large m.

Lemma 4.2(1) provides, at this same order $m + 1$, a connected simple comparison of type (p, δ), with terminal degrees p, k, all other degrees six, and resistance $R \ge m/8 - C_0$. Attach its degree-p terminal at the unchanged r and make its degree-k terminal the new deficient vertex. Keeping the root value a, choose harmonic interior values and optimize the deficient value z. The new end energy is

$$\min_z\{(a-z)^2/R + \delta z^2\} = \frac{a^2}{R + 1/\delta} \le \frac{8a^2}{m} + O(a^2/m^2).$$

All new coordinates lie in $[0, a]$. The replacement saves at least ca^2/m in the gap energy, whereas loss of squared norm is at most ma^2. Since $\mu m^2 = o(1)$, the saving exceeds μ times this loss. The new graph has the original order and unique deficient degree, is simple and connected, and contradicts (2). Thus the initial orders are uniformly bounded. □

4.6 Eliminating the deficient end before the canonical chain

Proposition 4.9 (The canonical chain starts at the deficient vertex)**.** *For all sufficiently large degree-six adjacency extremizers, the sole deficient vertex is the outer singleton of the first standard M_6 block in its increasing Perron ordering. There is no nonempty exceptional end preceding that block.*

Proof. If nonempty initial regions persist at unbounded orders, Proposition 4.8 lets us pass to a fixed rooted region H. Write u for its deficient vertex, r for its attachment, $k = d(u)$, $\delta = 6 - k$, $p = d_H(r) \in \{2, 4\}$, $m = |H| - 1 > 0$, and $a = x_r$. The only exterior edges meet r; all other vertices except u have degree six. Put $W = \delta x_u = \mu \sum_v x_v$. The mass profile gives

$$W^2/(\mu M^2) \longrightarrow 8, \qquad W = o(M). \tag{21}$$

Let R_H be the resistance between r, u and $R = R_H + 1/\delta$, the resistance after attaching a zero ground to u with conductance δ. The grounded Dirichlet matrix on $H - r$ is positive definite. Its analytic resolvent, applied to $(6I - A(H - r) - \mu I)x_{H-r} = a\, b$ with b the indicator of $N_H(r)$, gives the zero-frequency voltage–current relation and its error:

$$a = RW + O(\mu W), \qquad x_v = O(W) \quad (v \in H). \tag{22}$$

Summing the grounded equations gives current $J = \sum_{v \in N_H(r)} (a - x_v) = W - \mu \sum_{H-r} x_v = W + O(\mu W)$.

Delete $H - r$, leaving a connected graph G_0 whose sole deficient vertex is r, of degree $6 - p$. For the restricted vector z and $F_0 = \lambda I - A(G_0)$, only row r of $F_0 z$ is nonzero, with value $pa - J$. Hence, shifting all retained coordinates by $t = -a + W/p = O(W) = o(M)$ gives

$$\begin{aligned}(z + t\mathbf{1})^T F_0 (z + t\mathbf{1}) &= pa^2 - aJ + 2t(pa - J) + (p - \mu(n - m))t^2 \\ &= aW - W^2/p + o(W^2) = \big(R - 1/p + o(1)\big)W^2.\end{aligned}$$

Here (22) and $\mu n \to 0$ control all errors. Lemma 3.3 restores the m deleted vertices, avoiding the fixed $O(W)$ cap, and lowers this form by $(m + o(1))\mu M^2 = (m/8 + o(1))W^2$. It applies after the shift because $t = o(M)$. The final graph is connected, simple, of the original order, and has only r deficient. Extremality consequently requires

$$R \geq m/8 + 1/p. \tag{23}$$

But H, ordered from r to u, is admissible of type (p, δ). Lemma 4.2(1) gives $R_H \leq m/8 + \phi(p) - \phi(\delta)$, so $R \leq m/8 + 1/p$. Equality requires $p = \delta$ and a consistently oriented standard chain. In that chain, clique twins have equal Perron values and summing the regular equations above each interface gives a strictly positive current. Successive layer values therefore increase away from u. Since u is a separating singleton in the completion, the first block would be a standard gap, contrary to the definition of H. Thus the resistance bound is strict. Its fixed gap makes the final trial form negative at large orders, contradicting (23). Finiteness of bounded rooted regions makes the exclusion uniform. □

4.7 The canonical-chain conclusion

Theorem 4.10 (Global canonical-chain reduction)**.** *There are constants n_0, B such that every extremizer G of order $n \geq n_0$ is $E_k^q(F)$ for some $k \in \{2, 4\}$, $q \geq 1$, and a connected simple k-rooted head F with $|F| \leq B$. Thus $n = 7q + |F|$, the outer singleton is the unique deficient vertex of degree k, and the possible heads form a finite family independent of n.*

Proof. The two-profile theorem and Proposition 4.5 give ordered standard blocks with $O(n^{2/3})$ exceptional vertices. Propositions 4.6 and 4.7 put every regular exception within bounded distance in the ordering from its maximum end. Absorb that entire bounded end, including its preceding singleton, into F. Proposition 4.9 removes the initial exception. An uninterrupted standard chain therefore starts at the unique deficient vertex and ends at F.

At every shared singleton, degree six requires contributions $2+4$; reversing orientation would instead give $2+2$ or $4+4$. All blocks are therefore C_k, where k is the first terminal's degree. The final block contributes $6-k$ to the root of F, so its degree inside F is k. Ordered separation guarantees no other exterior edges from F; its nonroot degrees are six, and it is connected by the interval argument above. Each block adds seven vertices. The uniform order bound gives finitely many rooted isomorphism classes. □

5 Rooted responses and the head score

The eliminations in this section use Schur complements [18]; the equitable quotient calculations use the standard constant-on-cells subspace of a graph partition [4, 10]. We give the resulting matrices and identities explicitly.

We compare fixed heads after extending them to a common large order. The two block orientations have conjugate response recurrences. Normalizing by that conjugacy gives one comparison argument for both orientations.

5.1 Grounded torsion and a common response

Let H be a k-rooted head of order m, with root r, $K = 6I - A(H-r)$, and b the indicator of its root neighbors. Since every core vertex has degree six in H, $K\mathbf{1} = b$ and $\mathbf{1}^T b = k$. The matrix K is positive definite: the connected nonregular graph H has spectral radius less than six, as does its proper induced core. Define the grounded torsion, Poisson vector, and rooted response by

$$\tau(H) = \mathbf{1}^T K^{-1}\mathbf{1}, \qquad y = K^{-1}\mathbf{1}, \qquad r_H(z) = e_r^T (zI - A(H))^{-1} e_r.$$

Here r_H is initially defined for $z > \rho(H)$ and extended as a rational function. The nonnegative resolvent series shows $y > 0$. Completing the square gives

$$\tau(H) = \max_{w \in \mathbb{R}^{m-1}} (2\mathbf{1}^T w - w^T K w), \quad \tau(H) - (2\mathbf{1}^T w - w^T K w) = (w-y)^T K (w-y). \tag{24}$$

For small real t, put $\tau_H(t) = \mathbf{1}^T (K - tI)^{-1}\mathbf{1}$. The identities $(K-tI)^{-1}b = \mathbf{1} + t(K-tI)^{-1}\mathbf{1}$ and $b^T(K-tI)^{-1}b = k + (m-1)t + t^2\tau_H(t)$ give, by elimination at the root,

$$r_H(6-t) = \frac{1}{6 - k - mt - t^2\tau_H(t)}. \tag{25}$$

Thus

$$r_H(6-t) = \frac{1}{6-k} + \frac{mt}{(6-k)^2} + \frac{\tau(H) + m^2/(6-k)}{(6-k)^2} t^2 + O_H(t^3). \tag{26}$$

Solving the full Poisson equation similarly gives

$$\mathbf{1}^T (6I - A(H))^{-1}\mathbf{1} = \tau(H) + \frac{m^2}{6-k}. \tag{27}$$

When H occurs at the end of an extremizer and its root coordinate is normalized to one, its core Perron coordinates satisfy

$$x_{H-r} = \mathbf{1} + \mu(K - \mu I)^{-1}\mathbf{1} = \mathbf{1} + \mu y + O_H(\mu^2). \tag{28}$$

Consequently finite-Perron ordering inequalities for a fixed recurrent head imply weak Poisson ordering inequalities. A strict limit inequality requires a separate proof.

Lemma 5.1 (Transfer and normalization)**.** *For a scalar $z > 5$, define*

$$D = (z+1)(z-5), \quad B = \frac{8}{D}, \quad A_k = \frac{k(z-5+k)}{D}, \quad C_k = (z - A_k)A_{6-k} + B^2.$$

Then $C_2 = 4(z-2)/(z-5)$, $C_4 = 2(z-4)/(z-5)$, and

$$r_{E_kH}(z) = \Phi_k(r_H(z)), \qquad \Phi_k(s) = \frac{1 - A_{6-k}s}{z - A_k - C_k s}. \tag{29}$$

Put $h_z(s) = [2(z-2)s - 1]/(z-4)$ and $\mathcal{D}_k(s) = z - A_k - C_k s$. As rational identities,

$$\Phi_4(h_z(s)) = h_z(\Phi_2(s)), \tag{30}$$

$$\mathcal{D}_4(h_z(s)) = \mathcal{D}_2(s). \tag{31}$$

For $z > \max\{5, \rho(H)\}$, the inequality $\mathcal{D}_k(r_H(z)) > 0$ implies $z > \rho(E_kH)$. At $z = \rho(E_kH) > \max\{5, \rho(H)\}$ that denominator is zero.

Proof. The clique inverse

$$(zI - A(K_6))^{-1} = \frac{I}{z+1} + \frac{J}{(z+1)(z-5)}$$

(J is the all-one matrix) contributes A_k, A_{6-k} to the new and old root diagonals and B between them. Eliminating the old head gives

$$r_{E_kH}(z) = \frac{1}{z - A_k - B^2 r_H(z)/(1 - A_{6-k}r_H(z))},$$

which is (29). Substitution of the displayed rational coefficients verifies both conjugacy identities. For the last assertions, write $s = r_H(z) > 0$. Since $C_k = (z - A_k)A_{6-k} + B^2$, positivity of $\mathcal{D}_k(s)$ implies

$$0 < s < \frac{z - A_k}{C_k} < \frac{1}{A_{6-k}}.$$

The old-head rank-one update is therefore positive definite, and the last scalar Schur complement is $\mathcal{D}_k(s)/(1 - A_{6-k}s) > 0$. At the extension's Perron root, the preceding blocks are positive definite (they are proper induced subgraphs), while the last Schur complement vanishes. Here $1 - A_{6-k}s = B^2/C_k > 0$. □

Define the *normalized response* by $R_H(z) = h_z(r_H(z))$ for $k = 2$ and $R_H(z) = r_H(z)$ for $k = 4$. Both orientations now satisfy

$$R_{E_kH} = \Phi_4(R_H), \qquad \mathcal{D}_4(R_H) = \mathcal{D}_k(r_H), \qquad \Phi_4'(s) = \frac{B^2}{\mathcal{D}_4(s)^2} > 0.$$

All uses of monotonicity below take place on intervals where the denominator is positive; thus no pole is crossed.

5.2 An exact tie and the comparison invariant

Theorem 5.2 (An exact Perron tie)**.** *Let S_k be obtained from K_7 by deleting a matching of size $k/2$ and adjoining a root to its k endpoints, for $k \in \{2,4\}$. For every $q \geq 0$, the heads $G_{k,q} = E_k^q S_k$ have order $8+7q$, degree sequences $(6^{7+7q}, k)$, and equal spectral radii. In fact,*

$$(z-4)r_{G_{4,q}}(z) = 2(z-2)r_{G_{2,q}}(z) - 1. \tag{32}$$

Proof. The partitions into the root, matching endpoints, and remaining vertices are equitable, with adjacency quotients

$$Q_2 = \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 5 \\ 0 & 2 & 4 \end{pmatrix}, \qquad Q_4 = \begin{pmatrix} 0 & 4 & 0 \\ 1 & 2 & 3 \\ 0 & 4 & 2 \end{pmatrix}.$$

Their positive Perron vectors lift to the seeds. The root coordinate vector is in the invariant cell-constant subspace, so the root response is the first diagonal entry of $(zI - Q_k)^{-1}$. Cofactors give

$$r_{S_k}(z) = \frac{z^2 - 4z + k - 12}{z^3 - 4z^2 - 12z + 8}.$$

Thus $R_{S_2} = R_{S_4}$, and their common recurrence proves (32) for every q. The seed Perron roots exceed five (the degree-two seed has average degree $11/2$), and extension increases them. A connected graph's root resolvent has a pole with positive residue at its Perron root. Since h_z has positive finite slope for $z > 5$, the identity equates the largest poles. The degrees and orders follow from the construction. □

The seed equality is due to Huang–Liu–Yang [12]; the recurrence proves its persistence. The rooted responses are conjugate, not equal, so this does not assert invariance under arbitrary common root grafts.

For a k-rooted head H of order m, define

$$P_2(m) = \frac{m^3}{24} + \frac{m^2}{16} + \frac{127m}{336}, \qquad P_4(m) = \frac{m^3}{24} - \frac{3m^2}{16} + \frac{211m}{336}, \qquad J_k(H) = \tau(H) - P_k(m).$$

Lemma 5.3 (Score invariance and eventual comparison)**.** *The operation E_k preserves J_k. For any two heads U, V whose orders are congruent modulo seven, of root degrees $k, l \in \{2,4\}$, if $J_k(U) > J_l(V)$, their extensions to the same sufficiently large order satisfy $\rho(E_k^a U) > \rho(E_l^b V)$.*

Proof. Substitute (26) into (29) and compare coefficients through degree two:

$$\tau(E_2 H) - \tau(H) = \frac{7}{8}m^2 + 7m + 20, \qquad \tau(E_4 H) - \tau(H) = \frac{7}{8}m^2 + \frac{7}{2}m + \frac{19}{2}.$$

These are $P_k(m+7) - P_k(m)$, proving invariance. The same expansion and the definition of h give the common formula

$$R_H(6-t) = \frac{1}{2} + \frac{m}{4}t + \left\{ \frac{J_k(H)}{4} + \frac{P_4(m)}{4} + \frac{m^2}{8} \right\} t^2 + O_H(t^3).$$

Extend the smaller head to align orders; invariance preserves the score inequality. The displayed expansion gives $R_U(6-t) > R_V(6-t)$ for all sufficiently small positive t.

Here is the pole control needed to propagate that inequality. Write $U_j = E_k^j U$, $V_j = E_l^j V$ for the aligned heads and choose $z = \rho(U_q)$ with q large. The average-degree bound $\rho(U_q) \geq 6 - (6-k)/|U_q|$

ensures $z \to 6$; hence $z > 5$, $z > \rho(U), \rho(V)$, and $R_U(z) > R_V(z)$. For $0 \le j < q$, U_j is a proper induced subgraph of U_q. The transfer lemma gives $\mathcal{D}_4(R_{U_j}(z)) > 0$ for $j < q-1$ and zero for $j = q-1$. Inductively suppose $z > \rho(V_j)$ and $R_{U_j}(z) > R_{V_j}(z)$. Because $C_4 > 0$,

$$\mathcal{D}_4(R_{V_j}(z)) > \mathcal{D}_4(R_{U_j}(z)) \ge 0.$$

The transfer lemma implies $z > \rho(V_{j+1})$. For $j < q-1$ the denominator is positive on the entire interval between the two responses; the positive derivative of Φ_4 therefore gives $R_{U_{j+1}}(z) > R_{V_{j+1}}(z)$, completing the induction. At $j = q-1$ the same denominator comparison directly yields $z > \rho(V_q)$, as required. The argument works for either orientation. □

For S_2, the core Poisson values on the matching endpoints and the other vertices are $7/2$ and 4; for S_4 they are $7/4$ and 2. Thus $\tau(S_2) = 27$, $\tau(S_4) = 13$, and

$$J_2(S_2) = J_4(S_4) = -\frac{19}{14}. \tag{33}$$

Score equality alone would not prove the exact tie above. A head is *recurrent* if it represents an unbounded sequence of extremizers as $E_k^q H$. Comparison heads need not be recurrent. The finite family from Theorem 4.10 enters only in the final maximal-score argument.

6 Local structure and compression of terminal heads

A *recurrent terminal head* is a fixed connected rooted simple graph H occurring as the terminal head of adjacency extremizers of unbounded orders in Theorem 4.10. Its root is denoted by r; every vertex other than r has degree six in H. The root degree is $k \in \{2, 4\}$, the full head order is m, and the core order is $N = m - 1$. All exterior vertices mentioned below belong to the core $H - r$; the root is never included in an exterior layer. Write $B = N_H(r)$. The Perron vector x always belongs to a full extremizer containing the head, whereas $y = (6I - A(H-r))^{-1}\mathbf{1}$ is the grounded Poisson vector of the fixed head. We use vertex names for Poisson coordinates only after explicitly passing to this limiting vector.

All code and rational certificate paths below are relative to the accompanying supplementary archive directory `research/c11-d6`.

The bounded-head theorem leaves finitely many rooted heads. Thus an exclusion proved for every recurrent head excludes that configuration from all sufficiently large extremizers. Our order-preserving replacements yield such exclusions. An order-changing replacement instead yields a head of larger score J_k; its use across residue classes is deferred to the final theorem.

We use the following part of Liu's switching result [15, Proposition 3.1]: a legal degree-preserving two-edge switch with nondecreasing adjacency quadratic form at the positive Perron vector of an adjacency extremizer yields a connected graph. In each application the switch is performed in the full extremizer, the four endpoints are distinct, and the added pairs are nonedges. If its quadratic gain is zero but a switched row no longer satisfies the old eigen-equation, the Rayleigh principle still gives strict spectral improvement. Indeed equality with the old spectral radius would require the old unit vector to be a maximizing eigenvector. The arguments below either check connectedness directly or invoke this result with these hypotheses. No optimality of the isolated head is assumed.

For completeness, put $\mu = 6 - \rho(G) > 0$ for an occurrence in a full extremizer and normalize its root coordinate to $a > 0$. The core equations give

$$x_{H-r} = a\big(\mathbf{1} + \mu(6I - A(H-r) - \mu I)^{-1}\mathbf{1}\big) = a\big(\mathbf{1} + \mu y + O(\mu^2)\big). \tag{34}$$

The inverse exists at the actual Perron parameter because the core is a proper principal subgraph of the connected full graph. Its entries are nonnegative, and all core coordinates exceed a. Along recurrent occurrences $\mu \to 0$; consequently every strict finite-coordinate comparison proved below implies the corresponding weak comparison of Poisson coordinates. A finite number of choices of vertices can always be fixed by a subsequence.

Lemma 6.1 (Finite quadratic certificates)**.** *Let P be a bounded rational polytope, expressed in coordinates on its affine hull after eliminating its linear equalities. Its vertices can be enumerated by solving each full-rank set of $\dim P$ active inequalities and retaining the feasible solutions. For a quadratic F and vertices v_i, set*

$$B_{ii} = F(v_i), \qquad B_{ij} = 2F((v_i + v_j)/2) - \tfrac{1}{2}(F(v_i) + F(v_j)).$$

Then $F \geq \min_{i,j} B_{ij}$ on P. If F is concave, its least vertex value alone is a lower bound. Alternatively, an identity $F = \eta + \sum_{i \leq j} c_{ij}\ell_i\ell_j$, with $c_{ij} \geq 0$ and each $\ell_i \geq 0$ on P, proves $F \geq \eta$ there.

Proof. Every vertex has full-rank active constraints, proving completeness of the stated enumeration. Every point is $v = \sum_i \alpha_i v_i$ with $\alpha_i \geq 0$ and $\sum_i \alpha_i = 1$, and expansion gives $F(v) = \sum_i \alpha_i^2 B_{ii} + 2\sum_{i<j} \alpha_i\alpha_j B_{ij}$. The weights sum to one. Concavity gives the second assertion, and nonnegativity of the products gives the last. □

The finite certificates below specify their domains, score quadratics, and exact positive margins. Vertex enumeration and coefficient identities use rational arithmetic; supplementary programs reproduce these finite calculations. The graph reductions and domain inequalities are proved here.

Lemma 6.2 (Uniform-coordinate cap replacement)**.** *Let H be a connected head of root degree four and core order N, with all other degrees six. Let $W \subseteq V(H) \setminus \{r\}$ contain all four root neighbors, let $a = |W|$, and suppose $Q > 0$ edges join W to the retained core T. Let ℓ_v count those edges at $v \in W$, put $S = \sum_{v \in W} \ell_v y_v$, and put $L = y_W^T(6I - A(H[W]))y_W$. Suppose a connected simple replacement head is obtained by deleting W, inserting a connected s-vertex core cap, joining its root to $k \in \{2, 4\}$ cap vertices, and restoring every old exterior incidence, with all nonroot degrees six. Then*

$$\tau(H') - \tau(H) \geq -L + \frac{S^2}{Q} + \frac{(N - a + s)^2}{k}.$$

Its score gain is at least this expression plus $P_4(N + 1) - P_k(N + 1 - a + s)$. If the new cap has six vertices, the root meets two adjacent vertices that are complete to the other four, and all exterior incidences use those other four, the bound improves by $(N - a + 4)^2/8$.

Proof. The matrices $6I - A$ of the old and new cores are positive definite: each core component meets a root neighbor, so its grounded degree form has trivial kernel. Their torsions therefore maximize $2\mathbf{1}^T v - v^T(6I - A)v$ over all real core vectors. Let Z be the sum of old retained endpoint coordinates over the Q exterior incidences, with multiplicity. Summing the old Poisson equations on W gives $Z - S = N - a$, because the root-boundary flux is $\sum_{b \in B} y_b = N$. Removing the old cap lowers the variational objective by L: the old cap equations identify its linear and cross terms with twice its quadratic energy. Shift every retained coordinate by $h \in \mathbb{R}$ and give every new cap vertex the value $w \in \mathbb{R}$. The objective difference is

$$-L - 2Sh - Qh^2 + 2(s + Z)w + 2Qwh - (Q + k)w^2.$$

Here the retained constant-direction coefficient is Q, and the new cap coefficient is $Q + k$, obtained by summing its degree deficits. Optimizing first at $h = w - S/Q$ and then at $w = (N - a + s)/k$

gives the claimed bound. These are unrestricted real trial vectors, so no positivity of the shifted coordinates is required. The score correction follows from the new full order $N+1-a+s$. For the final assertion let $R = N - a + 6$ and give the two root neighbors value z and the other four cap vertices value w. Eliminating the retained shift leaves S^2/Q plus the quadratic optimization with matrix $\left(\begin{smallmatrix} 10 & -8 \\ -8 & 8 \end{smallmatrix}\right)$ and linear coefficient $(2, R-2)^T$. Its value is $5R^2/8 - R/2 + 1/2 = R^2/2 + (R-2)^2/8$, proving the improvement. □

Lemma 6.3 (Balanced exterior incidences)**.** *Suppose a connected replacement cap has s vertices, of which a designated set of t has positive exterior capacities differing by at most one, with total Q. If each old exterior endpoint occurs at most t times, its incidences can be reassigned simply to those capacities. If all retained degrees were six and every retained component met an old incidence, the replacement head is connected and restores all retained degrees.*

Proof. Process one endpoint at a time, assigning its $m \le t$ copies to distinct currently least-loaded vertices of the designated set. If loads initially differ by at most one, raising the m smallest loads preserves that property. Induction gives the unique balanced final load multiset of sum Q; label the cap vertices to match it. Distinct copies of one endpoint never use the same vertex. The connected cap touches every retained component, proving connectedness. □

Lemma 6.4 (Ordered frontiers)**.** *In an actual extremizer, let S be a set of head vertices containing its root, such that every core vertex outside S has coordinate larger than every vertex of S. Put $A = N_H(S) \setminus S$. Then every core vertex outside $S \cup A$ has coordinate larger than every vertex of A. All corresponding weak inequalities hold for the Poisson limit.*

Proof. For $u \in A$ choose an owner $b \in S \cap N(u)$. If a vertex $z \notin S \cup A$ satisfies $x_z \le x_u$, equal degrees six and $b \in N(u) \setminus N(z)$ give $t \in N(z) \setminus (N(u) \cup \{u\})$. The count remains valid when uz is an edge, by deleting the opposite endpoint from each neighborhood. Since z has no neighbor in S, $x_t > x_b$. The legal switch $bu, zt \mapsto bz, ut$ has gain $2(x_u - x_z)(x_t - x_b) \ge 0$ and is connected by Liu's switching result. At zero gain its row at u changes by $x_t - x_b > 0$, so equality in the Rayleigh principle is impossible. Taking the limit proves the Poisson assertion. □

6.1 Additional exact restrictions on the regular head

We first consider a head H attached to the rest of an actual adjacency extremizer at its root r. The root has degree $k \in \{2, 4\}$ inside H, and every other vertex of H has degree six. The sole deficient vertex of the whole graph is outside H. These statements concern the head; they do not assert that H is itself an adjacency extremizer.

Lemma 6.5 (Head-neighbor order)**.** *Let H be a terminal head in an actual adjacency extremizer, with root degree $k \in \{2, 4\}$ and all other head degrees six. Every vertex of $H - N_H[r]$ has Perron coordinate strictly greater than every vertex of $N_H(r)$.*

Proof. The exact resolvent identity (34) puts all core coordinates above x_r. Apply Lemma 6.4 with $S = \{r\}$. □

Lemma 6.6 (Cross-neighborhood closure)**.** *Let H be a terminal head in an actual adjacency extremizer, with root degree $k \in \{2, 4\}$ and all other head degrees six. Let $i, j \in N_H(r)$ be nonadjacent. Put $A_i = N_H(i) \setminus (N_H[r])$ and $A_j = N_H(j) \setminus (N_H[r])$. Every pair of distinct vertices $p \in A_i$, $q \in A_j$ is adjacent.*

Proof. If pq is absent, the switch $ip, jq \mapsto ij, pq$ has gain

$$2(x_i - x_q)(x_j - x_p) > 0$$

by Lemma 6.5. All added edges are new, every degree is preserved, and Liu's Proposition 3.1 guarantees connectedness. The strict gain is impossible. □

Proposition 6.7 (Two forbidden degree-four boundary complements)**.** *For a degree-four terminal head H in an actual adjacency extremizer G, the complement of the graph induced by $N_H(r)$ cannot be a three-edge star. If $\rho(G) \geq 119/20$, it cannot be a triangle and an isolated vertex either.*

Proof. Write $N_H(r) = \{b_0, b_1, b_2, b_3\}$, and let A_i be the neighbors of b_i outside $N_H[r]$.

First suppose that precisely the pairs $b_0 b_i$, $1 \leq i \leq 3$, are missing. Then $|A_0| = 5$ and $|A_i| = 3$ for $i \geq 1$. If $A_0 \cap A_i \neq \varnothing$, a shared vertex has at least $2 + |A_0 \cup A_i| - 1 = 9 - |A_0 \cap A_i|$ neighbors. Its degree six forces $A_i \subseteq A_0$. Each vertex of A_i is then saturated by b_0, b_i and the other four vertices of A_0. For $j \neq i$, if $A_j \subseteq A_0$, then $A_i \cap A_j$ is nonempty and a shared vertex would acquire the extra neighbor b_j. If A_j is disjoint from A_0, cross closure forces its three vertices to be adjacent to every saturated vertex of A_i. Both alternatives are impossible. Thus all three A_i, $i \geq 1$, are disjoint from A_0. Their vertices are saturated by b_i and the five vertices of A_0, so these three sets are pairwise disjoint. Each vertex of A_0 would then be adjacent to b_0 and their nine vertices, again impossible.

Now suppose the missing pairs form a triangle on b_1, b_2, b_3. The sets A_1, A_2, A_3 all have size four. For two of them with nonempty intersection, the same degree count gives intersection size at least three. Intersection size three is impossible: every shared vertex is already saturated by the two corresponding b_i and the four other vertices of their union. It cannot belong to the third set, and cross closure forces the third set into their union minus all three shared vertices, a set of size two. Hence any two intersecting sets are equal. They cannot all be pairwise disjoint, since a vertex in one would have its b_i neighbor and all eight vertices of the other two as neighbors. If two sets are equal, their four vertices form a clique and already have five neighbors; the third set cannot be disjoint, by cross closure. Therefore

$$A_1 = A_2 = A_3 =: C, \qquad |C| = 4.$$

Every vertex of C is saturated by its three neighbors in C and b_1, b_2, b_3. The vertices b_1, b_2, b_3 are likewise saturated by r, b_0, C. Thus b_0 is the only connection from this nine-vertex core to the rest of the head, and has two edges onward. The onward rooted graph has root degree two and all other degrees six, so has at least eight vertices. Consequently $H - r$ has at least fifteen vertices.

Let $a = x_r$, let b be the common coordinate of b_1, b_2, b_3, let c be the common coordinate on C, and let $d = x_{b_0}$. Equality within the displayed classes follows by subtracting their eigen-equations. With $\lambda = 6 - \mu$,

$$c = \frac{3b}{\lambda - 3}, \qquad d = \left(\lambda - \frac{12}{\lambda - 3}\right) b - a.$$

The head-neighbor order gives $c > d$, hence $Db < a$, where $D = \lambda - 15/(\lambda - 3) > 0$ for $119/20 \leq \lambda < 6$. Summing the regular equations over $H - r$ and putting $F = \sum_{H-r} x$ gives

$$\mu F = 3b + d - 4a = Ab - 5a, \qquad A = \lambda + 3 - \frac{12}{\lambda - 3}.$$

Since $A - 5D = \mu(4 + 21/(3 - \mu))$,

$$\frac{F}{a} < \frac{4 + 21/(3 - \mu)}{6 - \mu - 15/(3 - \mu)} \leq \frac{13120}{1021} < 15 \qquad (0 < \mu \leq 1/20).$$

For the middle inequality the numerator increases and the positive denominator decreases with μ, and the displayed rational value is its value at $1/20$. But every coordinate of $H - r$ exceeds a, and there are at least fifteen such vertices, so $F > 15a$. This contradiction excludes the triangle complement. □

6.2 Converting a degree-four head through a missing matching

For two heads of the same order $N + 1$, the score criterion of Lemma 5.3 is simply

$$J_2(H_2) - J_4(H_4) = \tau(H_2) - \tau(H_4) - \frac{N(N+1)}{4}. \tag{35}$$

We use the rooted-response conjugacy $\mathscr{H}_\lambda(r) = [2(\lambda-2)r - 1]/(\lambda - 4)$ from the preceding section only for the equality seed.

6.2.1 A variational torsion bound

Let R be a connected simple six-regular graph on N vertices, and let e_A, e_B be two disjoint edges. Remove both edges and attach a new root to their four endpoints, obtaining H_4. Let $H_{2,A}$ instead remove only e_A and attach its root to the two endpoints of e_A; define $H_{2,B}$ similarly. These heads are connected: an even-regular connected graph has no bridge, so removing either one edge leaves R connected.

Write $K = 6I - A(R - e_A - e_B)$, $y = K^{-1}\mathbf{1}$, and $\tau = \mathbf{1}^T y$. The matrix is positive definite, because each component after the edge deletions has a grounded endpoint. For $J \in \{A, B\}$, put $S_J = \sum_{z \in e_J} y_z$ and let D_J be the difference of the two endpoint values. Summing $Ky = \mathbf{1}$ gives

$$S_A + S_B = N. \tag{36}$$

Restoring e_B gives the core matrix $K_A = K - E_B$, where E_B has two off-diagonal ones at that edge. The Dirichlet variational identity is

$$\tau_A = \max_z \{2\mathbf{1}^T z - z^T K_A z\}.$$

At $z_0 = y + (S_B/2)\mathbf{1}$, its value is

$$\tau + S_B^2 - D_B^2/2.$$

The residual $w = \mathbf{1} - K_A z_0$ is $-S_B/2$ at each endpoint of e_A, the two reversed endpoint values of y at e_B, and zero elsewhere. In particular $\mathbf{1}^T w = 0$.

Choose h equal to one on e_A, zero on e_B, and $1/2$ on every other vertex, and set $\widetilde{h} = h - \mathbf{1}$. Its values on the only grounded vertices, the endpoints of e_A, are zero. Therefore

$$w^T \widetilde{h} = -S_B, \qquad \widetilde{h}^T K_A \widetilde{h} = \mathcal{E}_R(h).$$

Let t count the edges of R between the two endpoint pairs. The four endpoints have degree six, contain their two original matching edges, and have $20 - 2t$ edges to the remaining vertices. Consequently

$$\mathcal{E}_R(h) = t + \frac{20 - 2t}{4} = 5 + \frac{t}{2} \le 7.$$

Optimizing the further change $\alpha\widetilde{h}$ now proves

$$\tau_A - \tau \ge \left(1 + \frac{1}{5 + t/2}\right) S_B^2 - \frac{D_B^2}{2} \ge \frac{8}{7} S_B^2 - \frac{D_B^2}{2}. \tag{37}$$

The analogous bound holds with A, B reversed. No numerical optimization is used in this inequality.

6.2.2 The extremal-head separation closes the imbalance gap

Proposition 6.8. *Let H_4 be a connected degree-four head with root u, boundary $B = N(u)$, and core order N. Suppose:*

1. *the complement of $H_4[B]$ contains a perfect matching;*
2. *for $y = (6I - A(H_4 - u))^{-1}\mathbf{1}$, every coordinate outside B is at least every coordinate on B.*

If $N \geq 8$, some smoothing of a missing boundary pair gives a connected degree-two head H_2 of the same order with

$$\tau(H_2) - \tau(H_4) - \frac{N(N+1)}{4} \geq \frac{155N^2 - 1071N - 224}{4732} > 0.$$

Hence its conjugated response strictly dominates the original head response sufficiently close to six from below. If $N = 7$, the heads are the known eight-vertex pair with exact conjugate-response equality.

Proof. First choose a missing perfect matching whose restoration makes a connected regular core R. If $H_4 - u$ is connected, any such matching works. Otherwise every component meets B in a positive even number of vertices: summing core degrees modulo two gives that parity condition. Thus there are exactly two components, each meeting B twice. All cross pairs are missing, so choose a perfect matching across those components. Its restoration is connected. The preceding variational bound therefore applies, and both degree-two smoothing candidates are connected and simple. Write $\tau_i = \tau(H_i)$ for their torsions.

Put $L = \min_{b \in B} y_b$ and $U = \max_{b \in B} y_b$. Each boundary vertex has core degree five and has at most two neighbors inside B, because the missing graph contains a perfect matching. At a vertex with value L, the equation $6y_i = 1 + \sum_{j \sim i} y_j$ and the separation assumption give

$$6L \geq 1 + 2L + 3U, \qquad 4L \geq 1 + 3U.$$

Also $\sum_{b \in B} y_b = N$ by (36). Hence

$$U \leq \frac{4N - 3}{13}, \qquad U - L \leq \frac{U - 1}{4} \leq \frac{N - 4}{13}.$$

Choose the matching pair with sum at least $N/2$. Its endpoint difference has absolute value at most $(N - 4)/13$. Applying (37) to the smoothing of this pair yields

$$\tau_2 - \tau_4 \geq \frac{2N^2}{7} - \frac{(N-4)^2}{338}.$$

Subtracting $N(N+1)/4$ gives the claimed polynomial divided by 4732. The numerator is increasing for $N \geq 8$, and at $N = 8$ its value is $1128 > 0$. The smallest stated margin is therefore $282/1183$.

If $N = 7$, the connected six-regular core is K_7. With $P = \lambda^3 - 4\lambda^2 - 12\lambda + 8$,

$$r_2(\lambda) = \frac{\lambda^2 - 4\lambda - 10}{P}, \qquad r_4(\lambda) = \frac{\lambda^2 - 4\lambda - 8}{P}.$$

Direct substitution gives $\mathscr{H}_\lambda(r_2) = r_4$ identically where defined. This is the equal-radius seed of [12]. □

Corollary 6.9 (Application to globally extremal chains)**.** *For sufficiently large extremizers in Theorem 4.10, a degree-four head whose root-neighborhood complement has a perfect matching must be the eight-vertex seed obtained from K_7 by deleting two disjoint edges and adjoining a root to their endpoints. Whenever that seed occurs, a degree-two extremizer of the same order also exists.*

Proof. The Poisson limit (34) and Lemma 6.5 give the separation hypothesis of Proposition 6.8. Its positive score gain excludes every recurrent head of order at least nine. A head has order at least eight by its degree requirements. The remaining seed has exact response conjugacy, hence an equal-radius degree-two chain at every corresponding order. Finiteness of the head family makes the exclusion uniform for sufficiently large orders. □

6.3 Converting a head with two adjacent missing boundary edges

Proposition 6.10. *Let H_4 be a recurrent terminal head, with root degree four. Suppose the complement on its four root neighbors consists of two adjacent edges. Then a degree-two head of the same order has strictly larger conjugated response sufficiently close to six from below. Consequently this pattern cannot occur in sufficiently large extremizers.*

Proof. Index the boundary vertices by $0, 1, 2, 3$, with missing pairs $01, 02$. Let $N = |H_4| - 1$ be the core order, $K = 6I - A(H_4 - r)$, and $y = K^{-1}\mathbf{1}$. Write y_i for its four boundary coordinates. The head-neighbor ordering, after expansion at zero parameter, implies every exterior coordinate is at least every y_i. Summing $Ky = \mathbf{1}$ gives $\sum_{i=0}^{3} y_i = N$.

Smoothing a missing pair means removing its two root edges and adding the edge between its endpoints. The root degree becomes two and every other degree stays six. The core is connected: the graph on the four root neighbors is connected, and every component after deleting the root meets that graph. Thus either smoothing produces a simple connected head.

We first derive a response bound for smoothing 01. Put $K' = K - E_{01}$, where E_{01} has two unit off-diagonal entries, and let $\tau = \mathbf{1}^T K^{-1}\mathbf{1}$, $\tau' = \mathbf{1}^T (K')^{-1}\mathbf{1}$. For a perturbation h, the torsion variational principle gives

$$\tau' - \tau \geq 2y_0 y_1 + 2h^T E_{01} y - h^T K' h.$$

Optimize over the five-dimensional space spanned by the four boundary coordinate vectors and the constant vector on the core. In this basis the matrix of the quadratic term and the linear coefficient are

$$M = \begin{pmatrix} 6I - A(K_4 - 02) & d \\ d^T & 2 \end{pmatrix}, \quad d = (0, 0, 1, 1)^T, \quad w = (y_1, y_0, 0, 0, y_0 + y_1)^T.$$

This matrix is positive definite, as the restriction of the positive definite matrix K' to an independent family of vectors. Direct rational inversion gives

$$2y_0 y_1 + w^T M^{-1} w = P(y_0, y_1), \qquad P(s, t) = \frac{1237}{1700} s^2 + \frac{1358}{425} st + \frac{322}{425} t^2. \tag{38}$$

The other smoothing similarly gains at least $P(y_0, y_2)$.

Suppose $N \geq 10$. Normalize $b_i = y_i/N$ and $\theta = 1/N$. At a boundary vertex with d_i neighbors inside the boundary, its other $5 - d_i$ core neighbors are exterior. Their coordinates are at least each y_j. The following linear inequalities are therefore necessary:

$$b_i \geq 0, \quad \sum_i b_i = 1, \quad 0 \leq \theta \leq \frac{1}{10}, \qquad 6b_i - \sum_{\substack{j \sim i \\ j \text{ in boundary}}} b_j - (5 - d_i) b_l \geq \theta \quad (0 \leq i, l \leq 3). \tag{39}$$

This is a compact rational polytope in four affine coordinates. The exact certificate `computation/2026-11-27/head-star2-bernstein.py` enumerates all its twenty-eight vertices, checks all quadratic Bernstein coefficients at vertex pairs, and proves

$$\frac{P(b_0,b_1)+P(b_0,b_2)}{2}-\frac{1+\theta}{4}\geq\frac{1239087}{1803530000}>0. \tag{40}$$

Lemma 6.1 turns these coefficient bounds into the asserted inequality. At least one smoothing consequently satisfies

$$\tau'-\tau>\frac{N(N+1)}{4}.$$

This is the strict conjugated-response criterion.

It remains to check $N<10$. Let A_i denote the exterior neighbor sets of the four boundary vertices. Their sizes are $4,3,3,2$. Thus $N\geq 8$. If $N=8$, there are four exterior vertices and A_0 is their whole set. Cross-neighborhood closure makes the exterior graph a clique: at least three vertices in $A_1\cup A_2$ are joined to every other exterior vertex, which suffices. Each exterior vertex must then have exactly three boundary neighbors. Its incidence columns, up to permutation, are

$$\{0,1,2\},\ \{0,1,2\},\ \{0,1,3\},\ \{0,2,3\}.$$

This is a unique nine-vertex head, with grounded torsion $7155/412$. The complement of the rose with cycle lengths $3,4,4$ is a degree-two head of the same order with grounded torsion $183/5$. The exact linear-system certificates in the same script give

$$\frac{183}{5}-\frac{7155}{412}-18=\frac{2541}{2060}>0.$$

Finally $N=9$ is impossible. There are five exterior vertices; write z for the unique one outside A_0. Since it has degree six and only four other exterior vertices, it must have at least two boundary neighbors, drawn from $1,2,3$. If $z\in A_i$ for $i=1$ or 2, the two vertices of $A_i\cap A_0$ are saturated by boundary neighbors $0,i$ and all four other exterior vertices, using cross closure. If z belongs to both A_1,A_2, these two saturated pairs must be disjoint and partition A_0. Then neither they nor z can belong to A_3, contradicting $|A_3|=2$. If z belongs to exactly one of A_1,A_2, it must also belong to A_3. The saturated pair in A_0 cannot belong to the other three-element set A_j, but only two vertices of A_0 remain. This is again impossible.

We have covered every order. Strict torsion domination gives strict conjugated-response domination in a neighborhood below six. The family of bounded heads is finite, so a uniform neighborhood can be chosen, and Lemma 5.3 yields the assertion for sufficiently large extremizers. □

6.4 One missing boundary edge: a triangle compression

Proposition 6.11. *Let H be a recurrent degree-four terminal head with precisely one missing edge in its root neighborhood. There is an admissible head of root degree two and strictly larger score. If no exterior vertex meets all four root neighbors, its order is one smaller; otherwise it has the same order.*

Proof. Write $B=\{0,1,2,3\}$ with 01 missing, $N=|H|-1$, and $y=(6I-A(H-r))^{-1}\mathbf{1}$. Put $b_i=y_i/N$ and $\theta=1/N$. The boundary flux is $\sum_i b_i=1$. There are ten exterior incidences. With $h=N-4$ exterior vertices, their degree requirements imply $10\geq h(7-h)$; $h\leq 2$ is impossible because a vertex would need more than four boundary neighbors. Thus $h\geq 5$ and $N\geq 9$.

Suppose first that every exterior endpoint has multiplicity at most three. Delete B and insert a triangle, joining its root to two vertices. The exterior capacities are $(3,3,4)$. Assign each old endpoint's incidences to distinct least-loaded triangle vertices; Lemma 6.3 proves that the final loads are $(3,3,4)$. Join the root to the two vertices of load three. This is a simple connected head of root degree two, all other degrees six, and order one smaller.

Let $A_B = A(H[B])$, $L = b^T(6I - A_B)b$ and $S = 3b_0 + 3b_1 + 2b_2 + 2b_3$. The common cap bound gives the normalized score gain

$$F = -L + S^2/10 + 3/8 - \theta + 55\theta^2/56.$$

The necessary compact domain is

$$b_i \geq 0, \quad \sum_i b_i = 1, \quad 0 \leq \theta \leq 1/9, \quad 6b_i - \sum_{j \in N_B(i)} b_j - (5 - d_B(i))b_l \geq \theta \quad (i,l \in B). \tag{41}$$

The last inequalities are the boundary equations and exterior floor. Eliminate $b_3 = 1 - b_0 - b_1 - b_2$ and apply the exact enumeration in Lemma 6.1: there are 28 vertices, and the 406 quadratic coefficients have minimum $548/31941 > 0$. This proves the first case.

Now let c meet all of B. Write $A_i = N(i) \setminus N[r]$. Cross closure for the missing pair 01 makes c adjacent to $(A_0 \cup A_1) \setminus \{c\}$. Since $|A_0| = |A_1| = 3$ and c has just two neighbors outside B, necessarily $A_0 = A_1 = \{c, p, q\}$, which induces a triangle. Each of $2, 3$ has one further exterior neighbor. Up to swapping $0, 1$ and $2, 3$, this gives precisely the five incidence multisets in the table below. A mask s denotes the subset of B with binary indicator s. The five possibilities follow by assigning the two remaining incidences either to p, q, to one common new vertex, or to distinct new vertices.

Two same-order replacements suffice. Smoothing 01 deletes its two root edges and adds 01. Optimizing the four boundary coordinates and the constant core shift gives

$$\Delta\tau \geq \frac{37}{32}(y_0 + y_1)^2 - \frac{3}{7}(y_0 - y_1)^2. \tag{42}$$

The quadratic matrix is $M_0 = \begin{pmatrix} 6I - A(K_4) & d \\ d^T & 2 \end{pmatrix}$, where $d = (0,0,1,1)^T$, and the residual is $(y_1, y_0, 0, 0, y_0 + y_1)^T$. The bound is $2y_0y_1 + w^T M_0^{-1} w$ by the torsion variational identity.

Alternatively take an exterior edge uv with $2u, 3v$ absent and replace $r2, r3, uv$ by $2u, 3v$. The boundary stays connected, and if deleting uv disconnects the core the new edges reconnect its separated side. Let A' be the new adjacency matrix on $B \cup \{u, v\}$, let $d = (1,1,0,0,0,0)^T$, and put

$$M = \begin{pmatrix} 6I - A' & d \\ d^T & 2 \end{pmatrix}, \qquad w = (0, 0, y_u, y_v, y_2 - y_v, y_3 - y_u, y_2 + y_3)^T.$$

The same variational calculation yields

$$\Delta\tau \geq 2y_2y_u + 2y_3y_v - 2y_uy_v + w^T M^{-1} w. \tag{43}$$

Both matrices are positive definite restrictions of the new core matrix; the coordinate vectors together with the constant vector are independent since $N \geq 9$.

Here is the complete small certificate for these five configurations. For multiplicities m_s, let z_s be the mean normalized Poisson coordinate of type s, and set

$$f(s,t) = \mathbf{1}_{(0 \in s, 1 \in t) \text{ or } (1 \in s, 0 \in t)}, \quad d_s = 6 - |s| - \sum_t (m_t - \mathbf{1}_{s=t}) f(s,t).$$

Use the compact domain

$$\begin{aligned} &b_i, z_s, \theta \ge 0, \quad \sum_i b_i = 1, \quad 0 \le \theta \le 1/\max(9, 4 + \sum_s m_s), \quad z_s \ge b_i, \\ &6b_i - \sum_{j \in N_B(i)} b_j - \sum_{s \ni i} m_s z_s = \theta, \\ &6z_s - \sum_{i \in s} b_i - \sum_t (m_t - \mathbf{1}_{s=t}) f(s,t) z_t - d_s b_i \ge \theta. \end{aligned} \tag{44}$$

The last row is an equality when $d_s = 0$. It is the mean Poisson row after bounding each remaining neighbor below by b_i. Nonnegativity and the boundary equations bound all coordinates. For the second surgery use the ordered pair of types in the table; closure forces its exterior edge. Replacing endpoint coordinates by their type means is valid for the average gain: $w^T M^{-1} w$ is convex, and the product term averages exactly for distinct types. For equal types, sampling distinct endpoints gives $\mathbb{E}[y_u y_v] = \bar{y}^2 - \mathrm{Var}(y)/(m_s - 1)$, which only improves the negative product term.

Subtract $(1 + \theta)/4$ from each normalized gain and average the two candidates with the displayed smoothing weight w_0. Exact vertex enumeration and Lemma 6.1 give the following least quadratic coefficients:

incidence masks	(u, v) types	w_0	$\#V$	$\min B_{ij}$
$3, 3, 4, 8, 15$	$(3, 3)$	$1/2$	20	39329/3428352
$3, 3, 12, 15$	$(3, 3)$	$1/2$	6	39329/3428352
$3, 4, 11, 15$	$(11, 3)$	$1/2$	14	76350660587/7360158243312
$3, 15, 15$	$--$	1	2	310237/10951200
$7, 11, 15$	$--$	1	4	368797/17006112

Every entry is positive, so at least one actual replacement has $\Delta\tau > N(N + 1)/4$, proving its strict score gain. The verifier `computation/2026-11-27/head-oneedge-short-verify.py` reconstructs the triangle domain, the five-pattern census, the quadratic forms, and every listed coefficient using rational arithmetic. □

6.5 An admissible degree-lowering surgery for clique heads

A core vertex outside B adjacent to B is called a *port*; its multiplicity is the number of its neighbors in B. A port of multiplicity four is called a *quadruple port*.

Let H be a rooted head whose root r has degree four and whose other vertices have degree six. Assume $B = N_H(r)$ induces K_4. Put $N = |H| - 1$, $K = 6I - A(H - r)$ and $z = K^{-1}\mathbf{1}$. In applications to an extremal head, its limiting Poisson outside coordinates are at least the boundary coordinates; the formulas below themselves need no such ordering assumption.

Lemma 6.12 (An available outside edge)**.** *Let H be a connected head of root degree four and all other degrees six, with clique boundary $B = N_H(r)$. Let p be an outside neighbor of B with $1 \le |N(p) \cap B| \le 3$, and choose any $x \in B \setminus N(p)$. There is a neighbor $q \notin B$ of p and a vertex $y \in B \setminus \{x\}$ such that yq is missing. The graph*

$$H' = H - rx - ry - pq + xp + yq$$

is simple and connected, has root degree two, and has every other degree six.

Proof. There are eight incidences between B and its outside neighbors. Put $m = |N(p) \cap B|$. For the prescribed x, a neighbor q fails only if it meets all four boundary vertices, or meets precisely the other three. Such a vertex uses at least three boundary incidences. For $m \le 2$, the remaining $8 - m$ incidences permit at most two bad vertices, whereas p has $6 - m \ge 4$ neighbors outside B. For $m = 3$, the remaining five incidences permit at most one bad vertex, whereas p has three outside neighbors. Thus the required choices exist in either case.

All four vertices are distinct and the two added edges are absent. The degree claims follow directly. Since B is a clique, $H - r$ is connected: each of its components must meet B, and the clique joins all these vertices. Deleting its edge pq leaves at most two components, and the vertices of B remain in one component. The new edges xp, yq reconnect both components if necessary. Finally r retains two neighbors in B, proving connectedness. □

Lemma 6.13 (Ordering at a minimum port)**.** *Let H be a recurrent degree-four terminal head with root r and clique boundary $B = N_H(r)$. Along a sequence of its occurrences in adjacency extremizers with $\mu \to 0$, choose an outside port p of minimum Perron coordinate along a subsequence. Every vertex of B missing p has Perron coordinate at least every vertex of B adjacent to p. The same weak inequalities hold for their limiting Poisson coordinates. In particular, if p has multiplicity at most three, one may choose a largest Poisson boundary coordinate as x in the preceding surgery.*

Proof. Suppose $c \in B \cap N(p)$ and $b \in B \setminus N(p)$, but $x_b < x_c$. Both b, c have exactly two outside neighbors, so there is $q \in N(b) \setminus (B \cup \{r\})$ not adjacent to c. Minimality gives $x_p \le x_q$. The simple degree-preserving switch $bq, cp \mapsto bp, cq$ has adjacency gain

$$2(x_b - x_c)(x_p - x_q) \ge 0.$$

The switch remains connected directly: B is intact and connected, and every component separated by deleting bq, cp contains p or q, both of which are reattached to B. If equality holds, $x_p = x_q$ but $x_b \ne x_c$, so the eigen-equation at p changes; equality in the Rayleigh principle is impossible. Thus $x_b < x_c$ cannot occur.

There are finitely many choices of a minimum port in the fixed head, so a subsequence fixes its identity. With the attachment coordinate normalized to one, the regular head equations give

$$x_{H-r} = \mathbf{1} + \mu(6I - A(H - r) - \mu I)^{-1}\mathbf{1} = \mathbf{1} + \mu z + O(\mu^2).$$

Dividing the proved coordinate comparisons by μ and taking limits gives the asserted Poisson inequalities. A boundary maximum is therefore attained among the nonneighbors of p, as claimed. □

Lemma 6.14 (A computable torsion gain)**.** *Let H, H', r, x, y, p, q be the heads and vertices in Lemma 6.12, and let $z = (6I - A(H - r))^{-1}\mathbf{1}$. Put*

$$X = z_x, \quad Y = z_y, \quad P = z_p, \quad Q = z_q, \quad S = X + Y.$$

If $\tau = \mathbf{1}^T K^{-1}\mathbf{1}$ and $\tau' = \mathbf{1}^T(6I - A(H' - r))^{-1}\mathbf{1}$, then

$$\begin{aligned}\tau' - \tau \ &\ge 2(XP + YQ - PQ) + \frac{S^2}{2} \\ &\quad + \frac{4(P+Q)^2 - 4(P+Q)S + 5S^2}{32} + \frac{(P-Q)^2}{14}.\end{aligned}$$

Proof. Use $\tau' = \max_w(2\mathbf{1}^T w - w^T K' w)$, where $K' = 6I - A(H' - r)$. Starting with $z + h\mathbf{1}$, the change is

$$2(XP + YQ - PQ) + 2hS - 2h^2.$$

Its optimal shift is $h = S/2$, giving the first line. At this trial vector the residual $\mathbf{1} - K'(z + h\mathbf{1})$ on B is $(P, Q, -S/2, -S/2)$. Allow simultaneous further changes of all four boundary coordinates and of the uniform shift. Since $K'\mathbf{1}$ is the indicator of the two retained root neighbors, the resulting boundary Schur matrix is

$$7I_4 - \mathbf{J}_4 - \frac{1}{2}dd^T, \qquad d = (0, 0, 1, 1)^T.$$

Here I_4 is the identity matrix and $\mathbf{J}_4$ is the four-by-four matrix with every entry equal to one. The Schur matrix is positive definite. On the two antisymmetric pair directions its eigenvalue is seven; on pair-constant vectors its matrix is $\left(\begin{smallmatrix} 5 & -2 \\ -2 & 4 \end{smallmatrix}\right)$. Inverting these blocks gives the second line of the claimed bound. All these are trial-vector improvements, so optimizing the remaining coordinates can only increase τ'. □

6.5.1 Exact optimization for a minimum non-quadruple port

Let a degree-four head have core order N, let B be its four root-neighbours, and suppose B induces K_4. Write b_i for the core Poisson coordinates; thus $\sum_{i\in B} b_i = N$. Assume that a globally minimum exterior port p has multiplicity at most three, that every boundary coordinate is at most p, and that the admissible surgery can choose $x = \max_{i\in B} b_i$ as its first removed root-neighbour. The available-edge and minimum-port lemmas above justify these graph-choice hypotheses for recurrent heads; the next lemma proves the required numerical lower bound.

Delete the two root edges to x, y and an exterior edge with endpoint coordinates p, q, and add the two edges of coordinate products xp, yq. Put $S = x + y$, $P = p + q$, and $D = p - q$. Optimizing all four boundary coordinates and the uniform core shift gives the torsion lower bound

$$2xp + 2yq - 2pq + \frac{S^2}{2} + \frac{4P^2 - 4PS + 5S^2}{32} + \frac{D^2}{14}. \tag{45}$$

In particular, minimizing freely over $q \in \mathbb{R}$ leaves

$$F(x, y, p) = \frac{7x^2 - 42y^2 + 21xy + 14xp + 98yp - 48p^2}{11}. \tag{46}$$

The exact minimizer of the right side of (45) in q is $(53p - 56y + \frac{7}{2}(x + y))/11$.

Lemma 6.15. *Let H be a degree-four head with clique boundary B, core order $N \geq 10$, and grounded Poisson coordinates. Suppose a minimum exterior port has coordinate p and multiplicity at most three, every exterior coordinate is at least every boundary coordinate, x is the largest boundary coordinate, and y is another boundary coordinate. For F defined in (46), put $\theta = 1/N$. Then*

$$F(x, y, p) - \frac{N(N + 1)}{4} \geq \frac{251}{1163800}N^2 > 0.$$

Proof. Divide every coordinate by N, retaining the same letters. The four boundary equations are

$$7b_i = 1 + \theta + z_i + z_i'.$$

All eight port coordinates are at least p, so each boundary coordinate is at least $\ell = (1 + \theta + 2p)/7$. Their sum is one, and x is their maximum. Consequently

$$\frac{1}{4} \leq x \leq p, \qquad y \geq \max(1 - 3x, \ell), \qquad \frac{1}{4} \leq p \leq \frac{3 - 4\theta}{8} =: p_2.$$

The last inequality follows by summing the eight port coordinates: their sum is $3 - 4\theta$. For $0 < x, y \le p$, differentiation gives

$$11F_y = -84y + 21x + 98p \ge 21x + 14p > 0.$$

Thus we may set $y = \max(1 - 3x, \ell)$ when obtaining a lower bound. On the branch $y = 1 - 3x$, the numerator $11F$ becomes

$$-434x^2 + (273 - 280p)x - 42 + 98p - 48p^2,$$

which is concave in x. On the branch $y = \ell$, F increases in x, since $11F_x = 14x + 21\ell + 14p > 0$. Since $\ell \le 1/4$, the only possible minimizing choices are therefore

$$(x, y) = (1/4, 1/4), \quad (p, 1 - 3p) \quad (p \le p_0), \quad ((1 - \ell)/3, \ell) \quad (p \ge p_0), \qquad p_0 = \frac{6 - \theta}{23}.$$

For the first choice,

$$F = \frac{-7/8 + 28p - 48p^2}{11}.$$

For the second,

$$F = \frac{-762p^2 + 371p - 42}{11}.$$

For the third,

$$F = \frac{36 - 75\theta - 62\theta^2 + (984 + 592\theta)p - 1592p^2}{693}.$$

Every displayed expression is concave in p. Hence it suffices to check $p = 1/4, p_0, p_2$. After subtracting $(1 + \theta)/4$, the respective corner values are

$$\frac{3 - 22\theta}{88}, \qquad \frac{373 - 3375\theta - 3048\theta^2}{23276}, \qquad \frac{1 + 10\theta - 96\theta^2}{88}.$$

For $0 \le \theta \le 1/10$, the first is at least $1/110$, the second is decreasing and at least $251/1163800$, and the third is concave with endpoint values at least $1/88$. This proves the assertion. □

The condition $N \ge 10$ is automatic for a clique boundary. If there are $h = N - 4$ exterior vertices, their total incidence with B is eight. Each has degree six and at most $h - 1$ exterior neighbours, and hence at least $7 - h$ neighbours in B. For $1 \le h \le 5$, either the degree condition is impossible or $h(7 - h) > 8$ (the cases $h = 1, 2$ already require more than four boundary neighbours). The case $h = 0$ is impossible because a boundary vertex needs two exterior neighbours. Thus $h \ge 6$.

Corollary 6.16 (A fully joined exterior vertex is necessary)**.** *In every sufficiently large extremizer with a degree-four head whose root neighborhood is a clique, some exterior vertex is adjacent to all four root neighbors. Moreover, along any sequence with a fixed head, an outside port of minimum Perron coordinate must eventually have this multiplicity four.*

Proof. Otherwise, since the head family is finite, pass to an unbounded sequence with one fixed head and one fixed minimum port p of multiplicity at most three. The minimum-port ordering lemma permits a largest boundary coordinate as the omitted root neighbor x. The available-edge lemma supplies the second omitted neighbor y and the edge pq, with a simple connected degree-two replacement. Pass to the Poisson limit as in Corollary 6.9. Lemma 6.15 and the torsion bound give a strict margin of at least $251N^2/1163800$ over the conjugacy threshold $N(N + 1)/4$. The strict response comparison and Lemma 5.3 then contradict global extremality for sufficiently large orders. Since there are only eight boundary–exterior incidences, there can be at most two vertices of multiplicity four. □

6.6 Reduction to three end configurations

Corollary 6.17 (Three local configurations)**.** *For a sufficiently large degree-four extremizer, choose the canonical chain as long as possible, absorbing any initial standard blocks of its bounded head into the chain. Either there is an admissible head of strictly larger score, or its remaining rooted head is the eight-vertex equality seed, or it has a clique B of four root neighbors and one of the following configurations:*

1. *exactly one exterior vertex is adjacent to all of B;*
2. *exactly two exterior vertices c, d are adjacent to all of B, and c, d are nonadjacent;*
3. *exactly two exterior vertices c, d are adjacent to all of B, they are adjacent, and their respective other neighbors are distinct.*

In the equality-seed case, a degree-two extremizer of the same order exists. That seed is possible only when $n \equiv 1 \pmod 7$.

Proof. A graph on four vertices with matching number at most one is empty, a star with one, two or three edges, or a triangle and an isolated vertex. Apply this to the complement of the head's root neighborhood. Corollary 6.9 treats all complements with a perfect matching. Proposition 6.7 excludes the three-edge star and triangle. Proposition 6.10 excludes the two-edge star, and Proposition 6.11 supplies a strict score improvement for the single edge. Every remaining nonequality head therefore has a clique boundary. Corollary 6.16 supplies at least one exterior vertex adjacent to the whole boundary, and the eight available incidences permit at most two.

With two such vertices c, d, every boundary vertex already has its root neighbor, its three clique neighbors, and c, d as its six neighbors. If c, d are adjacent, each has exactly one remaining neighbor. If these are the same vertex w, the layers $r, B, \{c, d\}, w$ form a standard block C_4, attached to the rest only at w. Absorbing it into the chain contradicts maximality of the chosen chain. This leaves precisely the three configurations listed. Finally the equality seed has eight vertices, so its order formula is $n = 7q + 8$, and its conjugate degree-two chain has the same radius. □

6.7 Two quadruple ports

Proposition 6.18. *Let H be a recurrent degree-four terminal head whose clique boundary B has exactly two common exterior neighbors c, d. Either its first block is standard, or an admissible head has strictly larger score.*

Proof. Apply Lemma 6.4 to the root together with B. Every later finite Perron coordinate is strictly larger than both ports; the Poisson inequalities are weak. Write $b = N/4$. The boundary symmetry and equations give $c + d = 3b - 1$. There are at least seven later vertices: for their number $h \le 4$, degree six is impossible, while for $h = 5, 6$ the required $h(7 - h)$ cap incidences exceed the available four. Thus $N \ge 13$, or $b \ge 13/4$.

Adjacent ports. Each has one later neighbor. If these coincide, the layers form a standard block. Otherwise call them p, q and delete B, c, d, joining the old root to p, q. This is a simple connected root-degree-two head, restoring every retained degree. With $D = c - d$, the equations give

$$p = 7(c - b), \quad q = 7(d - b), \qquad |D| \le b/2 - 3/4,$$

where the inequality follows from $\min(p, q) \ge \max(c, d)$. The old cap is K_6, with energy $L = 21b^2/2 - 7b + 5/2 + 7D^2/2$. After deleting it, shifting all retained coordinates by $-(c + d)/2$ gains

$(c+d)^2/2$ in the torsion objective. The score-polynomial correction therefore gives

$$J_2 - J_4 \geq 2b^2 - 6b + 225/56 - 7D^2/2 \geq \tfrac{9}{8}(b-3/2)^2 - 27/56 \geq 2655/896 > 0.$$

Nonadjacent ports. Write their two later-neighbor sets as P_c, P_d. The row at the smaller port gives $6\min(c,d) - 4b - 1 \geq 2\max(c,d)$, hence again $|D| \leq b/2 - 3/4$. If the sets are disjoint, delete the six-vertex cap and join the root to their four distinct vertices. The new head has root degree four. The old energy is $L = 15b^2 - 10b + 3 + 3D^2$, and the same retained shift now gains $(c+d)^2$. Subtracting the order-six score correction gives

$$J_4' - J_4 \geq 6b^2 - 17b + 645/56 - 3D^2 \geq 21b^2/4 - 59b/4 + 1101/112 \geq 7771/448 > 0.$$

Both direct reconnections preserve connectivity because every retained component meets an old incidence and is joined to the new root.

If $P_c = P_d = \{p, q\}$, then $c = d$. Replace the old $K_6 - cd$ cap by K_6, attach the root to two vertices, and assign its four old incidences one apiece to the remaining four. This is simple even though each exterior endpoint occurs twice. For completeness, retaining the later coordinates plus a shift h and optimizing the clique coordinates adds

$$\frac{2 + 2(1+p+h)^2 + 2(1+q+h)^2 + (6+2p+2q+4h)^2}{7}$$

to the objective after deleting the old cap and shifting the retained part by $-4h(c+d) - 4h^2$. The optimum is $h = b + 1/4$. Using $p + q = 6c - 4b - 1$ and subtracting the old cap energy yields

$$J_2 - J_4 \geq (N-6)/4 + (p-q)^2/7 > 0.$$

Finally suppose $P_c = \{p, s\}$ and $P_d = \{p, t\}$ with distinct p, s, t. Cross closure makes these three vertices a triangle. Applying Lemma 6.4 to the root together with B, c, d orders every later coordinate above this triangle. Delete B, c, d, p, s, t and insert a K_5, with two root neighbors and exterior loads $(1, 1, 2, 2, 2)$. There are eight old incidences, two at p and three each at s, t, so each endpoint occurs at most three times. Lemma 6.3 assigns them to the five balanced capacities, preserving simplicity, degrees, and connectedness.

Put $a = 2p - 5b + 4$ and $D = c - d$. The old cap coordinates are

$$c, d = (3b - 1 \pm D)/2, \qquad p = (5b - 4 + a)/2, \qquad s, t = (5b - 4 - a)/2 \pm 3D.$$

Apply Lemma 6.2 with $(a_{\mathrm{cap}}, s_{\mathrm{cap}}, Q, k) = (9, 5, 8, 2)$. The energy is determined by the stated nine-vertex graph and $S = 2p + 3s + 3t$. Substitution gives

$$J_2 - J_4 \geq 2b^2 - 3b - 23/28 - 123D^2 - 9a^2/2. \tag{47}$$

Normalize $b = 1$, writing $\theta = 1/b$ and replacing every constant one in the coordinate formulas by θ. The necessary domain is

$$0 \leq \theta \leq 4/13, \quad c, d \geq 1, \quad p, s, t \geq \max(c, d),$$

$$6p - c - d - s - t \geq \theta + 2u, \quad 6s - c - p - t \geq \theta + 3u, \quad 6t - d - p - s \geq \theta + 3u \qquad (u \in \{p, s, t\}).$$

The last rows are the exterior-floor bounds. These inequalities bound $|D| \leq 1$ and $|a| \leq 3$, so the domain is compact. It has twelve vertices. The normalized right side of (47) is concave and has minimum $103031904/115897327 > 0$ at $(a, D, \theta) = (-554/4069, \pm 60/4069, 4/13)$. Lemma 6.1 completes this final case. □

6.8 Port ordering and equal-degree exterior twins

Suppose a recurrent degree-four terminal head has four root neighbors B inducing a clique and have a common exterior neighbor c. Then $C = B \cup \{c\}$ induces K_5. Each $b \in B$ has precisely one further neighbor, called its second port. Distinct second ports p, q have disjoint nonempty incidence sets $I_p = N(p) \cap B$, $I_q = N(q) \cap B$. Put $d_C(p) = |I_p| + \mathbf{1}_{pc \in E}$.

Lemma 6.19. *Let H be a recurrent degree-four terminal head whose clique boundary B has a common exterior neighbor c, and put $C = B \cup \{c\}$. For distinct secondary ports p, q, put $d_C(v) = |N(v) \cap C|$. Then:*

1. *if $d_C(p) < d_C(q)$ then $x_p > x_q$;*
2. *if $d_C(p) = d_C(q)$ then*
$$N(p) \setminus (C \cup \{q\}) = N(q) \setminus (C \cup \{p\}).$$

These conclusions concern the exact finite Perron vector.

Proof. Assume $x_p \le x_q$ and choose $b \in I_q$. Define $O_p = N(p) \setminus (C \cup \{q\})$ and O_q analogously. If $d_C(p) \le d_C(q)$, then $|O_p| \ge |O_q|$, since their sizes are $6 - d_C(p) - \mathbf{1}_{pq \in E}$ and $6 - d_C(q) - \mathbf{1}_{pq \in E}$. If $O_p \setminus O_q$ contains t, the switch $bq, pt \mapsto bp, qt$ is simple and degree-preserving. Here bp is absent because I_p, I_q are disjoint. Its gain is

$$2(x_q - x_p)(x_t - x_b) \ge 0,$$

and $x_t > x_b$ by the head-neighbor order. Liu's local-switch proposition gives connectedness. If the gain is zero, the row at p changes because it exchanges neighbors t and b with different coordinates, so the spectral improvement is still strict. This is impossible. Hence $O_p \subseteq O_q$. Strict inequality of the displayed cardinalities is impossible, proving the first claim. For equal cardinalities the sets must agree. Interchanging p, q if necessary proves the second claim for any coordinate order. □

Corollary 6.20. *Under the hypotheses of Lemma 6.19, if three distinct second ports have equal C-degree, they have one common outside neighborhood and induce either a triangle or an independent set.*

Proof. The lemma identifies their neighborhoods outside the three ports and C. It also says that each one of the three is adjacent to either both or neither of the other two; hence their induced graph is complete or empty. □

Lemma 6.21 (Strict assortativity)**.** *Under the hypotheses of Lemma 6.19, write r, s for the two neighbors of c outside B. If a second port z differs from r, s, then $x_z < \min\{x_r, x_s\}$.*

Proof. Choose $b \in I_z$. The switch $cr, bz \mapsto cz, br$ is legal and preserves degrees. The internal K_5 stays intact and every outside component remains attached to it. Its gain is $2(x_c - x_b)(x_z - x_r)$. The first factor is strictly positive by the head-neighbor order, so $x_z > x_r$ is impossible. If $x_z = x_r$, the row at r changes from neighbor c to b, whose coordinates differ; equality in the Rayleigh principle is again impossible. Thus $x_z < x_r$, and similarly $x_z < x_s$. □

In particular, if an unshared second port z is not adjacent to c and another second port r is adjacent to c, the two lemmas imply $|I_z| \ge |I_r| + 1$. These restrictions give the remaining incidence patterns recorded in the main head analysis. All the strict finite-coordinate inequalities pass to weak inequalities for their Poisson limits; the graph-theoretic twin identities are exact and remain unchanged.

6.9 Structure and exhaustive census for a unique quadruple port

Throughout this section, exterior vertices belong to the head core $H - r$; the head root r is excluded. We consider a fixed degree-four head occurring along an unbounded sequence of actual adjacency extremizers. Its root-neighbour set B is a clique. A *port* is a core vertex outside B adjacent to B, with multiplicity equal to its number of neighbours in B. Suppose exactly one port c has multiplicity four, and put $C = B \cup \{c\}$, which induces K_5.

Lemma 6.22 (The minimum port and the exterior floor)**.** *Let H be a recurrent degree-four terminal head with clique boundary B and unique quadruple port c, and put $C = B \cup \{c\}$. In every sufficiently large occurrence of this fixed head, c is the unique minimum finite-Perron port. Every core vertex outside C has coordinate strictly greater than x_c. Its limiting Poisson coordinate is therefore at least c.*

Proof. Corollary 6.16 makes every minimum port eventually quadruple. A nonquadruple port tied for the minimum could itself be selected in that same-order improving construction. Thus the unique quadruple port c is the unique minimum among ports.

Minimize the Perron coordinate over core vertices outside B. A minimizer which is not a port has no neighbour in B or at the head root, whose head neighbours are exactly B. All six neighbours are thus in the set being minimized over. Its equation would give $\lambda x_v \geq 6x_v$, contrary to $\lambda < 6$. Hence every minimizer is a port, and c is the only possible one. The Poisson conclusion follows from the established rooted-resolvent expansion. □

Each vertex of B has precisely one further neighbour besides its root neighbour, its three clique neighbours and c. These further neighbours are the *secondary ports*. Their distinct incidence sets $I_p = N(p) \cap B$ are disjoint and nonempty. For a core vertex outside C, write $N_C(p) = N(p) \cap C$.

Lemma 6.23 (Ordered neighbourhood inclusion)**.** *Let H, B, c, C satisfy Lemma 6.22. Suppose p, q are core vertices outside C, $N_C(q) \setminus N_C(p) \neq \varnothing$, and $x_p \leq x_q$. Then*

$$N(p) \setminus (C \cup \{q\}) \subseteq N(q) \setminus (C \cup \{p\}).$$

This applies to any two distinct secondary ports in their coordinate order. Equal cap degrees make the inclusion equality.

Proof. Choose $b \in N_C(q) \setminus N_C(p)$. If $t \in N(p) \setminus (C \cup \{q\})$ is not adjacent to q, the switch $bq, pt \mapsto bp, qt$ is simple and degree-preserving. The exterior floor and head-neighbour order give $x_t > x_c \geq x_b$. The gain is $2(x_q - x_p)(x_t - x_b) \geq 0$; the connected-switch proposition applies. Equality also contradicts extremality, since the row at p exchanges unequal coordinates x_t, x_b. Thus no such t exists. Distinct secondary ports have disjoint boundary incidence sets, so an owner in I_q supplies b. Equal cap degrees imply equal outside cardinalities. □

This is the inclusion step of Lemma 6.19. We also use its cap-degree ordering and Lemma 6.21: if s, t are the two neighbours of c outside B, a secondary port $z \notin \{s, t\}$ satisfies $x_z < \min(x_s, x_t)$.

Lemma 6.24 (Strong closure for an unshared secondary port)**.** *Let H, B, c, C satisfy Lemma 6.22. Let s, t be the two neighbours of c outside B, and let z be a secondary port not adjacent to c. For every $a \in \{s, t\}$ and $w \in N(z) \setminus C$ with $a \neq w$, the edge aw is present.*

Proof. Suppose aw is absent. The switch $ca, zw \mapsto cz, aw$ is simple: cz is absent by the unshared assumption and all four vertices are distinct. Its gain is

$$2(x_a - x_z)(x_w - x_c) > 0.$$

The first factor is positive by strict assortativity and the second by the exterior floor, since w is a core vertex outside C. The connected-switch proposition supplies an admissible strict improvement, contradicting extremality. □

Lemma 6.25 (Ordering beyond the full port layer)**.** *Let H, B, c, C satisfy Lemma 6.22. Put $A = N(C) \setminus (C \cup \{r\})$. Every core vertex beyond $C \cup A$ has finite Perron coordinate larger than every vertex of A. The corresponding Poisson comparison is weak.*

Proof. Apply Lemma 6.4 to $S = \{r\} \cup C$, using the strict exterior floor of Lemma 6.22. □

Lemma 6.26 (Cross-neighbourhood closure in an ordered layer)**.** *In an actual adjacency extremizer containing a terminal head, suppose p, q are nonadjacent vertices in a fixed layer, and all their neighbours in a subsequent layer have coordinates greater than both x_p, x_q. Then every distinct cross-pair $a \in N(p)$, $b \in N(q)$ in that subsequent layer is adjacent.*

Proof. If ab were absent, the switch $pa, qb \mapsto pq, ab$ would have strictly positive gain $2(x_a - x_q)(x_b - x_p)$. The connected-switch proposition gives an admissible improvement, a contradiction. □

Proposition 6.27 (The eight unique-quadruple incidence patterns)**.** *Let H, B, c, C satisfy Lemma 6.22. Let $m_i = |I_{p_i}|$ and let $\varepsilon_i = 1$ if cp_i is an edge. Up to permutation of equal multiplicities, the complete list of necessary possibilities is*

secondary multiplicities	*ports adjacent to c*
$1+1+1+1$	*none*
$2+1+1$	*none; or both singleton ports*
$2+2$	*none; or both ports*
$3+1$	*none; or the singleton alone; or both ports.*

This is a necessary-case census, not an assertion that every listed subcase is realizable.

Proof. The multiplicities partition four and each is at most three. There are at most two shared ports, since c has only two neighbours outside B. For an unshared port z and a shared port p, the inequality $m_z < m_p + 1$ would give $d_C(z) < d_C(p)$ and hence $x_z > x_p$ by port-degree ordering. Strict assortativity gives the opposite inequality. Therefore

$$\min_{\varepsilon_i=0} m_i \geq 1 + \max_{\varepsilon_j=1} m_j$$

when both index sets are nonempty. Enumerating the four partitions with this inequality and at most two shared ports gives exactly the eight cases displayed. □

6.10 Unshared secondary ports: one parametric calculation

Proposition 6.28. *Let H be a recurrent degree-four terminal head with clique boundary B and unique quadruple port c. If neither remaining neighbor of c is a secondary port, and the secondary multiplicities are $1+1+1+1$, $2+1+1$, or $3+1$, there is a smaller admissible degree-two head of strictly larger score.*

Proof. Put $C = B \cup \{c\}$, and let s, t be the remaining neighbors of c. Strong closure and equal outside degrees make every singleton secondary port an exterior twin of s, t. These vertices have one neighbor in C and five outside it. Thus their induced graph is complete or empty, and they have one common neighborhood outside their union and C. The next-layer ordering follows from Lemma 6.25.

For four singleton ports the six twins form either K_6 or an independent set with five common neighbors. Degree saturation makes the entire head have order twelve or seventeen, respectively. Replace its root and C by a root u and vertex w, with uw an edge, joining u to one twin and w to the other five. This preserves every retained degree and connectivity. The old and new grounded Poisson systems give

$\lvert H\rvert$	τ_4	τ_2	$J_2 - J_4$
12	38	27	369/28
17	2053/23	82	30215/644

For the other two partitions let $j \in \{2, 3\}$ be the multiplicity of their heavy port p. The $4 - j$ singleton ports together with s, t form a twin class Q of size $6 - j$. Write $\epsilon = 1$ when Q is complete, and zero otherwise. Its common outside set T has size $5 - (5 - j)\epsilon$. The heavy port meets either all of Q or none; write $\delta = 1$ or zero. In the first case $N(p) \setminus C = Q$; in the second this neighborhood is a $(6 - j)$-subset of T, by strong closure. Hence $(j, \epsilon, \delta) = (2, 1, 0)$ is impossible.

Delete $W = C \cup \{p\} \cup Q$, of size $a = 12 - j$. Its exterior incidence count is $D = (6 - j)[6 - (5 - j)\epsilon - 2\delta]$. Insert six vertices with root degree two, using the following caps:

D	internal graph and root neighbors	exterior assignment
4	K_6; any two vertices	four distinct cap vertices
6	$K_6 - ab$; two vertices off ab	loads $(2, 2, 1, 1)$
12	$K_6 - C_4$; the two off-cycle vertices	four loads of three
16	$K_{2,4}$ plus edge in its two-part; that part	four loads of four
18	0123450 with chords 24, 35; $\{0, 1\}$	six loads of three
24	six-vertex path; its ends	six loads of four

Here C_4 denotes a four-cycle, not a canonical block. For $D = 6$ there are two exterior endpoints, each occurring three times; join each to both double-load vertices and to a different single-load vertex. For $D = 12$ the endpoints have multiplicities either $(4, 4, 4)$ or $(3, 3, 3, 3)$; use the complete bipartite graph or its complement of a matching, respectively. For $D = 16$ join the four port vertices to all four old endpoints. For $D = 4$ one endpoint is allowed to occur four times. For $D = 18, 24$, Lemma 6.3 applies; the maximum endpoint multiplicities are four and five, respectively. These assignments verify simplicity, all degrees, and connectedness. The order decreases by $6 - j$.

Let a_0 be the common boundary coordinate owned by p, and b_0 the other boundary value. Write q for the common coordinate of the singleton ports and z for that of s, t. Equality in each class follows by subtracting twin equations and, for boundary-owned pairs, using $7(b_i - b_l) = q_i - q_l$. The equations needed below are

$$ja_0 + (4 - j)b_0 = N, \qquad 7a_0 = 1 + N + c + p, \qquad 7b_0 = 1 + N + c + q,$$
$$6c = 1 + N + 2z, \qquad (6 + \epsilon)(q - z) = b_0 - c.$$

When $\delta = 1$, also $6p = 1 + ja_0 + (4 - j)q + 2z$. Let $L = y_W^T(6I - A(H[W]))y_W$ and let S be its cap-side exterior incidence sum. Lemma 6.2 bounds the score gain by

$$G = -L + S^2/D + (N - 6 + j)^2/2 + P_4(N + 1) - P_2(N - 5 + j). \tag{48}$$

This single formula treats every row of the following tables.

If $\delta = 1$, the displayed system uniquely specifies all coordinates. For $(j, \delta, \epsilon) = (3, 0, 1)$, the common set T instead gives $6(p - q) = 3a_0 - b_0 - 2z$, or $a_0 - b_0 = (32 - 7N)/585$. Substitution in

(48) gives:

j	δ	ϵ	$[N^2]G$	$[N]G$	$[1]G$	$N_{\min}$
2	1	0	939/3304	−4283/1652	7341/1652	11
2	1	1	29/102	−529/204	120583/27132	11
3	1	0	2441/21864	−5761/10932	−119197/76524	11
3	1	1	179/1602	−845/1602	−69841/44856	11
3	0	1	743/4680	−2923/2340	8369/8190	12

Each polynomial is positive and increasing for $N \geq N_{\min}$. The order bounds count the cap and its required distinct exterior vertices.

For $\delta = \epsilon = 0$, put $d = a_0 - b_0$. The first five displayed equations and this definition uniquely specify all six coordinates in terms of N, d. The $(6 - j)$ neighbors of p form a subset of the five vertices of T, and all those vertices have coordinate at least z. Their sums are $6p - 1 - ja_0$ and $6q - 1 - b_0$. These inequalities give

j	$d_{\min}$	$d_{\max}$	$[d^2]G$	$N_{\min}$
2	$(122 - 15N)/1658$	$(41 - 9N)/1740$	−2787923/12696	15
3	$(163 - 24N)/1371$	$(41 - 9N)/747$	−97825/529	14

Concavity reduces the sign check to the two endpoints. Their polynomials, listed as the coefficients of $(N^2, N, 1)$, are

$(j, \text{endpoint})$	$[N^2]G$	$[N]G$	$[1]G$
$(2, \min)$	4793899/16493784	−7348779/2748964	90180023/19242748
$(2, \max)$	2454877/8073600	−35504489/12110400	2943697411/508636800
$(3, \min)$	1857805/15037128	−1677391/2506188	−19916509/17543316
$(3, \max)$	77101/496008	−909847/744012	15157577/15624252

Each is positive and increasing from its stated $N_{\min}$. This exhausts the possibilities and proves the proposition. □

6.11 Compressing the mixed two–one–one single-quadruple configuration

Proposition 6.29. *Let H_4 be a recurrent degree-four terminal head. Its four root neighbors $B = \{b_1, b_2, b_3, b_4\}$ induce K_4 and have a common outside neighbor c. Suppose their second outside neighbors are p, p, r, s, respectively, where p, r, s are distinct, c is adjacent to r, s, and cp is absent. Suppose also that c is the unique minimum finite-Perron port, as follows in the remaining single-quadruple case after the minimum-nonquadruple-port exclusion. Then there is a degree-two rooted head H_2 of order $|H_4| - 2$ with $J_2(H_2) > J_4(H_4)$.*

Proof. Put $C = B \cup \{c\}$. The ports p, r, s have equal C-degree two and distinct C-neighborhoods. Lemma 6.19 makes them exterior twins. Consequently $\{p, r, s\}$ is either a clique or an independent set, and its vertices have a common further neighbor set T of size two or four, respectively. The head has at least ten nonroot vertices, so, writing $N = |H_4| - 1$ and $b = N/4$, we have $b \geq 5/2$.

We retain the unequal Poisson coordinates of these ports. Symmetry gives values a, a, d, d on B and equal values $r = s$. Put $x = a - d$, so $a = b + x/2$, $d = b - x/2$. The boundary and c equations give

$$p - r = 7x, \qquad c = \frac{10b - 1 - 7x}{8}, \quad r = s = \frac{14b - 7 - 21x}{8}, \quad p = \frac{14b - 7 + 35x}{8}.$$

Since their further neighbors are identical, subtraction of the port equations gives

$$(6+\varepsilon)(p-r) = 2a - d - c, \qquad x = \frac{1-2b}{317+56\varepsilon}, \tag{49}$$

where $\varepsilon = 1$ for the clique and 0 for the independent case. This explicitly accounts for their different owner-coordinate sums; the port coordinates need not be equal.

Delete the eight core vertices $C \cup \{p, r, s\}$. In the clique case the two vertices of T each lose three edges. Insert K_6 with one edge deleted, attach the root to two vertices not incident to that missing edge, attach both missing-edge endpoints to both vertices of T, and attach the two remaining clique vertices one apiece to T. All degrees are restored to six, and the root has degree two. In the independent case the four vertices of T each lose three edges. Insert K_6 with the edges of a four-cycle deleted, attach the root to the two vertices outside that cycle, and attach its four cycle vertices to T by $K_{4,4}$ minus a perfect matching. Again all nonroot degrees are six. These constructions are simple. Every retained component meets T, and the new core is connected and meets all of T, so the new heads are connected. Both replacements reduce the head order by two.

Here is an exact variational bound sufficient to compare their scores. Let v be the old eight core coordinates, let $L = v^T(6I - A(C \cup \{p, r, s\}))v$, and put

$$q = \begin{cases} 6, & \varepsilon = 1, \\ 12, & \varepsilon = 0, \end{cases} \qquad S = (4 - 2\varepsilon)(p + 2r).$$

Lemma 6.2, with $(a, s, Q, k) = (8, 6, q, 2)$, gives

$$\tau(H_2) - \tau(H_4) \geq -L + S^2/q + (4b-2)^2/2.$$

The score-polynomial difference for an order decrease of two is $P_4(N+1) - P_2(N-1) = 47/56$. Substitution of (49) therefore gives

$$J_2(H_2) - J_4(H_4) \geq \begin{cases} \dfrac{1112}{1119}(b^2 - b) - \dfrac{88499}{62664}, & \varepsilon = 1, \\ \dfrac{944}{951}(b^2 - b) - \dfrac{75227}{53256}, & \varepsilon = 0. \end{cases}$$

Both bounds increase for $b \geq 5/2$. At $b = 5/2$ they equal $145021/62664$ and $123013/53256$, respectively, and are positive. This proves the proposition. □

6.12 Compressing the two–two collision at a unique quadruple port

Suppose a recurrent degree-four terminal head has a clique boundary B and a unique quadruple exterior port c. Suppose the four secondary boundary incidences use exactly two vertices p, q, each twice, and both p, q are adjacent to c. Write $\varepsilon = 1$ if pq is an edge, and $\varepsilon = 0$ otherwise.

Proposition 6.30. *Let H_4 be a recurrent degree-four terminal head with clique boundary B, unique quadruple port c, and secondary ports p, q of multiplicity two, both adjacent to c. There is a connected degree-two head of order one smaller, with all other degrees six, satisfying*

$$J_2(H_2) - J_4(H_4) \geq \frac{N^2}{16} - \frac{15}{7} \geq \frac{943}{112} > 0,$$

where N is the old core order.

Proof. First, the exterior-twin lemma for ports of equal cap degree applies to p, q: their neighbours outside the five-vertex clique $B \cup \{c\}$ coincide after removing each other. Let b_p, b_q be the finite Perron coordinates of their two respective boundary neighbours. The equations imply

$$(\lambda + 1)(b_p - b_q) = x_p - x_q, \qquad (\lambda + \varepsilon)(x_p - x_q) = 2(b_p - b_q).$$

Since $\lambda \to 6$, the coefficient $(\lambda + \varepsilon)(\lambda + 1) - 2$ is nonzero. Thus $x_p = x_q$, and the same equality holds for their Poisson coordinates. The boundary Poisson coordinates are all $b = N/4$, and the equations give

$$c = \frac{5N - 2}{16}, \qquad p = q = \frac{7N - 14}{16} =: r.$$

Delete the seven core vertices $B \cup \{c, p, q\}$. Their exterior port count is $Q = 6 - 2\varepsilon$, with $3 - \varepsilon$ incidences at each of p, q. Every exterior endpoint occurs at most twice. If $\varepsilon = 1$, replace the cap by K_6, join the old root to two vertices, and assign the four ports one each to the other four. If $\varepsilon = 0$, use K_6 minus an edge, join the root to two vertices not incident with that missing edge, and assign port loads $(2, 2, 1, 1)$ to the remaining four vertices, with load two at the missing-edge endpoints. Lemma 6.3 assigns the incidences to the four positive-capacity vertices; each endpoint occurs at most twice. The new head is connected, has all required degrees, and is one vertex smaller.

Let L denote the old cap quadratic energy and S its cap-side port sum. The five vertices $B \cup \{c\}$ have no neighbours outside this seven-vertex cap, so their cap equations have right side one. Consequently

$$L = N + c + 2r\big((6 - \varepsilon)r - c - N/2\big), \qquad S = Qr.$$

The enhanced form of Lemma 6.2, with $a = 7$, $s = 6$, and $k = 2$, gives

$$J_2 - J_4 \geq -L + S^2/Q + 5(N - 1)^2/8 - (N - 1)/2 + 1/2 - N^2/8 + 27/56 = N^2/16 - 15/7.$$

Here the last two terms are $P_4(N + 1) - P_2(N)$; substitution of the stated c, r, S gives the equality for either value of ε.

Finally $N \geq 13$. There are $h = N - 7$ vertices outside the old cap, none adjacent to B or c, and each has at most two cap neighbours. For $h \leq 4$ degree six is impossible; for $h = 5$ the required ten cap incidences exceed $Q \leq 6$. Thus $h \geq 6$, proving the stated positive bound. □

6.13 Two–two secondary ports outside the quadruple port

Let H_4 be a recurrent degree-four terminal head with clique boundary B and unique quadruple port c, which is eventually its unique minimum finite-Perron port by Corollary 6.16. Let the second neighbors of the four clique-boundary vertices be p, p, q, q. Assume cp, cq are absent. Lemma 6.19 implies that p, q have the same further neighbor set T, modulo each other. Symmetry then gives equal values on p, q and on all four boundary vertices. Write their core Poisson values as z, z and $b = N/4$, respectively. If the remaining two neighbors of c are r, s, then

$$c + z = 3b - 1, \qquad r + s = 6c - 4b - 1.$$

The single-quadruple cross-closure implies that each of r, s is adjacent to every distinct member of the outside neighborhood of each of p, q.

Proposition 6.31. *Let H_4 be a recurrent degree-four terminal head with clique boundary B, unique quadruple port c, and secondary ports p, q of multiplicity two, neither adjacent to c. If pq is an edge, there is a degree-two head of order $|H_4| - 4$ with score gain at least $(1312027/730303)b^2 > 0$.*

Proof. Here $|T| = 3$. Cross-closure makes r, s adjacent to p, q, so $r, s \in T$. Write $T = \{r, s, t\}$. Closure also makes this set a clique. Thus $\{p, q, r, s, t\}$ induces K_5. The ten old core vertices $B \cup \{c, p, q, r, s, t\}$ have four exterior incidences: one each at r, s and two at t.

Delete those ten vertices and insert a K_6, join the root to two clique vertices, and attach the four old exterior incidences to the other four vertices, one apiece. This is simple even when exterior endpoints repeat, restores their degrees, and is connected. The new root has degree two. Every component after deletion meets an old port, so the reconnection proves connectedness without an extra hypothesis.

Put $D = r - s$. The internal equations give

$$z = 3b - 1 - c, \quad r, s = \frac{6c - 4b - 1 \pm D}{2}, \quad t = 17b - 5 - 11c.$$

Strict assortativity gives $x_r, x_s > x_p = x_q$; the full-port-layer ordering gives $x_t > x_p$. Thus the prefix $S = \{r_{\text{root}}\} \cup B \cup \{c, p, q\}$ satisfies Lemma 6.4, whose frontier is $\{r, s, t\}$. Every later Poisson coordinate is therefore at least $\max(r, s, t)$.

Let $C = B \cup \{c, p, q, r, s, t\}$ be the ten-vertex cap and let $A_C = A(H_4[C])$. For its Poisson vector $v = y_C$, set $L = v^T(6I - A_C)v$ and $S = r + s + 2t$. Retaining old exterior coordinates plus a uniform shift and giving the new K_6 a common coordinate yields

$$\tau_2 - \tau_4 \geq -L + \frac{S^2}{4} + \frac{(4b - 4)^2}{2}.$$

The order decrease of four adds $4b^2 - 3b + 61/28$ to this torsion difference when passing to the scores.

Here is the exact certificate for positivity. Normalize $b = 1$ and write $\theta = 1/b$. In coordinates (c, D, θ) use the formulas above with every constant 1 replaced by θ. The necessary linear constraints are

$$0 \leq \theta \leq 4/17, \quad c \geq 1, \quad z \geq c, \quad r, s, t \geq z,$$

and, for $u \in \{r, s, t\}$,

$$6r - c - 2z - s - t - \theta \geq u, \quad 6s - c - 2z - r - t - \theta \geq u, \quad 6t - 2z - r - s - \theta \geq 2u.$$

The bound $N \geq 17$ follows because the four exterior incidences require at least seven exterior vertices: any smaller set would need more than four incident edges to complete its degrees to six. This polytope is compact: $1 \leq c \leq (3 - \theta)/2$, θ is bounded, and $r, s \geq z > 0$ bound D. Its twelve vertices are enumerated exactly in `background/2026-11-27/singlequad-22-edge-verify.py`. For the quadratic score bound F, every pair of vertices u, v satisfies

$$2F((u + v)/2) - \tfrac{1}{2}(F(u) + F(v)) \geq \frac{1312027}{730303} > 0.$$

Lemma 6.1 proves the bound throughout the polytope, without a concavity assumption. □

Now suppose pq is absent, so $|T| = 4$. Exactly one of r, s cannot belong to T: that vertex would be adjacent to c, p, q, the other three vertices of T, and the other one of r, s, giving degree seven. The remaining possibilities are that both or neither belong to T.

Proposition 6.32. *Let H_4 have the hypotheses of Proposition 6.31, except that the presence of pq is not assumed. Write r, s for the two neighbors of c outside B, and T for the common neighborhood of p, q outside $B \cup \{c, p, q\}$. If pq is absent and $r, s \in T$, a same- or opposite-orientation head of order $|H_4| - 5$ has strictly larger score.*

Proof. Write $T = \{r, s, a, d\}$. Closure makes r, s adjacent to every other vertex of T. The only undecided internal edge is ad. The vertices r, s are saturated; a, d each have one exterior incidence if ad is present and two otherwise. Symmetry and the internal Poisson equations give

$$c = \frac{72b - 9}{56}, \quad z = \frac{96b - 47}{56}, \quad r = s = \frac{104b - 55}{56}, \quad a + d = \frac{64b - 57}{14}.$$

Put $E = a - d$. Every vertex beyond this eleven-vertex core cap has coordinate at least $\max(a, d)$, by the same owner-edge switch. The equations at the smaller of a, d therefore give

$$|E| \le \frac{1}{2}\left(b - \frac{11}{4}\right).$$

There are at least seven exterior vertices, by the same degree count with at most four incidences, so $N \ge 18$ and $b \ge 9/2$.

Delete the eleven core vertices and insert K_6. If ad is present, join the root to four clique vertices and restore the two old incidences on the other two; the new root has degree four. If ad is absent, join the root to two clique vertices and restore the four incidences on the other four; the root has degree two. Repeated exterior endpoints cause no difficulty because each incidence uses a distinct clique vertex. Both graphs are simple, connected, and restore every retained degree.

Applying Lemma 6.2 gives the following exact score lower bounds:

$$\begin{cases} \frac{34}{7}b^2 - \frac{61}{7}b + \frac{11}{28} - \frac{7}{2}E^2, & ad \text{ present}, \\ \frac{34}{7}b^2 - \frac{68}{7}b + \frac{23}{14} - 3E^2, & ad \text{ absent}. \end{cases}$$

Substituting the bound on $|E|$ makes each expression increasing for $b \ge 9/2$. Their values at $9/2$ are $50943/896$ and $24187/448$, respectively, both positive. □

6.14 The remaining two–two outside configuration

Proposition 6.33. *Let H_4, p, q, c, r, s, T satisfy the configuration hypotheses of Proposition 6.32. Suppose pq is absent and both remaining neighbors r, s of c lie outside the common four-vertex neighborhood T of p, q. Then a rooted head of degree two or four, possibly of a different order, has strictly greater chain score.*

Proof. Each of r, s is adjacent to all four vertices of T, by cross-closure. The thirteen core vertices are

$$C = B \cup \{c, p, q, r, s\} \cup T.$$

Each vertex of T already has the four neighbors p, q, r, s. Thus $F = H[T]$ has maximum degree two. Up to isomorphism its seven possibilities are the empty graph, one edge, a two-edge path, two disjoint edges, a three-edge path, a triangle and an isolated vertex, and a four-cycle. Write ε for the indicator of rs. The exterior loads at r, s are $1 - \varepsilon$ each, and at $t_i \in T$ they are $2 - d_F(t_i)$. Hence the total exterior load is

$$q = 10 - 2(|E(F)| + \varepsilon).$$

This gives fourteen cases.

For $q > 0$, delete these thirteen core vertices and insert six vertices with the following internal graph and root degree:

q	new core	root neighbors	exterior loads
10	$K_6 -$ (a four-vertex cycle)	the four cycle vertices	$(2,2,2,2,1,1)$
8	$K_6 - P_4$	the four path vertices	$(2,2,1,1,1,1)$
6	$K_6 - e$	the two endpoints of e	$(1,1,1,1,1,1)$
4	K_6	any two vertices	$(1,1,1,1,0,0)$
2	K_6	any four vertices	$(1,1,0,0,0,0)$.

Each old endpoint occurs at most six times. The displayed positive capacities are balanced, so Lemma 6.3 assigns all incidences simply and preserves degrees and connectivity. The new head order is smaller by seven.

We specify fully the necessary-domain and exact sign certificate. Normalize the boundary value to $b = 1$ and put $\theta = 1/b$ before normalization. The other coordinates are $c, r, s, t_1, \dots, t_4$, and the common coordinate of p, q is $z = 3 - \theta - c$. The internal equations are

$$6c - 4 - r - s = \theta, \qquad 6z - 2 - \sum_i t_i = \theta.$$

We have $c \geq 1$, $z \geq c$, and $r, s, t_i \geq z$. Moreover, every vertex beyond this cap has coordinate at least $\max(r, s, t_1, \dots, t_4)$. Indeed, all cap vertices in that last layer have owners in $\{c, p, q\}$; a vertex beyond the cap has no neighbor in the earlier saturated layers. The owner-edge switch applies just as before, and removal of that owner edge leaves the cap connected. The old earlier-layer ordering makes the other exchanged endpoint strictly larger than the owner.

For each last-layer vertex u let

$$R_u = 6x_u - \sum_{v \in N_C(u)} x_v - \theta, \qquad \ell_u = 6 - d_C(u).$$

The exact necessary relations are $R_u = 0$ if $\ell_u = 0$, and $R_u \geq \ell_u x_v$ for every last-layer vertex v otherwise. Finally $0 \leq \theta \leq 4/(13 + h_q)$, where h_q is the smallest positive integer h satisfying $q \leq 6h$ and $h(7 - h) \leq q$. These two inequalities follow from the total degrees and maximum number of internal edges of the retained exterior. Thus the parameter domain is a rational polyhedron specified entirely by the stated graph and inequalities.

Let $A_C = A(H_4[C])$ be the adjacency matrix of the thirteen-vertex cap defined above, and let $v = y_C$ be its Poisson vector. Put $L = v^T(6I - A_C)v$, and $S = \sum_u \ell_u x_u$. For the new root degree k, the common-new-coordinate and uniform-old-shift trial gives, before normalization,

$$\tau_k - \tau_4 \geq -L + \frac{S^2}{q} + \frac{(N-7)^2}{k}.$$

The score correction for the order decrease of seven is

$$P_4(N+1) - P_k(N-6) = \begin{cases} \frac{7}{8}N^2 - 7N + 20, & k = 4, \\ \frac{5}{8}N^2 - \frac{15}{4}N + \frac{19}{2}, & k = 2. \end{cases}$$

Consequently the normalized score lower bound G is an explicitly specified rational quadratic on the necessary domain.

For each of the thirteen cases with $q > 0$, the certificate gives the exact polynomial identity

$$G = \frac{1}{100000} + \sum_{i \leq j} \alpha_{ij} L_i L_j, \qquad \alpha_{ij} \in \mathbb{Q}_{\geq 0},$$

after eliminating the stated linear equalities. Here each L_i is either the constant one or one of the necessary nonnegative linear forms above. This proves a positive score gain at least $b^2/100000$. The JSON contains the rational coefficients and transformations; `background/2026-11-27/singlequad-22-outside-verify.py` independently reconstructs every graph, equality, inequality, quadratic, and replacement load, and verifies every coefficient identity using only integer and rational arithmetic. A floating-point linear program was used only to find a certificate support; it is not part of the sign proof.

The sole $q = 0$ case has F equal to a four-vertex cycle and $\varepsilon = 1$. Connectedness forces the whole head to be this closed fourteen-vertex graph. Its exact Poisson solution gives a score smaller than the canonical eight-vertex seed score $-19/14$ by $7719/280$. That seed is therefore an admissible higher-score competitor, completing all fourteen cases. □

6.15 The three–one collision with nonadjacent secondary ports

Suppose a recurrent degree-four terminal head has clique boundary B, a unique quadruple port c, and two secondary ports p, q, both adjacent to c. Suppose p meets three boundary vertices, q meets the remaining one, and pq is absent. We consider the remaining situation in which c is a minimum port.

Proposition 6.34. *Let H_4 be a recurrent degree-four terminal head with clique boundary B, unique quadruple port c, and secondary ports p, q of multiplicity three and one, respectively. Suppose cp, cq are edges and pq is absent. There is a degree-two head of a possibly different order, with all other degrees six, whose chain-invariant score is strictly larger than that of this degree-four head.*

Proof. The port-degree ordering gives $x_p < x_q$ at the finite Perron parameter. Its switch proof also gives neighbourhood inclusion: if an outside neighbour z of p were not adjacent to q, use the boundary vertex owned by q to obtain a strict improving switch. Thus the two outside neighbours of p, denoted r, s, are among the four outside neighbours r, s, t, w of q. Cross-neighbourhood closure for the missing pair pq forces every distinct cross-pair between $\{r, s\}$ and $\{r, s, t, w\}$ to be adjacent. Consequently the graph on $A = \{r, s, t, w\}$ is K_4, possibly with just tw absent. Write $e = 1$ if tw is present and $e = 0$ otherwise.

Lemma 6.25 orders every vertex beyond $B \cup \{c, p, q\}$ above p, q. Applying Lemma 6.4 to $S = \{r_{\mathrm{root}}\} \cup B \cup \{c, p, q\}$ orders every vertex beyond its frontier A above A, also weakly in the Poisson limit.

Delete the eleven core vertices $B \cup \{c, p, q\} \cup A$. Their exterior loads at (r, s, t, w) are $(1, 1, 2, 2)$ when $e = 1$ and $(1, 1, 3, 3)$ when $e = 0$. If $e = 1$, replace the cap by K_6 minus an edge, with the root adjacent to two vertices off that missing edge; assign the six ports with loads $(2, 2, 1, 1)$ on the other four vertices, the double loads going to the missing-edge endpoints. The head order decreases by five. If $e = 0$, use K_5, with two root-neighbours and port loads $(1, 1, 2, 2, 2)$; the head order decreases by six.

The positive capacity lists are balanced on four and five vertices, respectively. Every old endpoint occurs at most four times, so Lemma 6.3 proves simplicity, restored degrees, and connectivity for both assignments.

Let N be the old core order, $u = q - p \geq 0$, and let b_p, b_q be the boundary values. The cap equations give

$$b_p = \frac{N}{4} - \frac{u}{28}, \quad b_q = \frac{N}{4} + \frac{3u}{28}, \quad c = \frac{5N - 2 + u}{16}, \quad p = \frac{7N - 14 - 5u}{16}, \quad q = \frac{7N - 14 + 11u}{16}.$$

Put $V = 6p - 1 - c - 3b_p$ and $W = 6q - 1 - c - b_q - V$. Then $r + s = V$ and $t + w = W$. These identities parameterize the old cap by $N, u, r - s, t - w$.

Let $C = B \cup \{c, p, q\} \cup A$ denote the eleven-vertex cap, $A_C = A(H_4[C])$ its adjacency matrix, and $v = y_C$ its Poisson vector. Put $L = v^T(6I - A_C)v$ and $S = \sum_{a \in A} \ell_a y_a$, with the exterior loads listed above. Apply the enhanced form of Lemma 6.2 for $e = 1$ and its ordinary form for $e = 0$. Adding the order-minus-five or order-minus-six score correction gives

$$J_2 - J_4 \geq \begin{cases} -L + S^2/6 + N^2 - 33N/4 + 156/7, & e = 1, \\ -L + S^2/8 + N^2 - 17N/2 + 1345/56, & e = 0. \end{cases} \tag{50}$$

These formulas include the exact order-minus-five and order-minus-six score corrections.

We finish by specifying the rational sign certificate. Necessarily $N \geq 17$: there are $h = N - 11$ later vertices, each with at most four cap neighbours. For $h \leq 2$ degree six is impossible; for $h = 3, 4, 5$ the required incidence count $h(7-h)$ exceeds the available six or eight ports. Hence $h \geq 6$. Normalize all coordinates by N and put $\theta = 1/N$. Use variables $(u/N, (r-s)/N, (t-w)/N, \theta)$. The displayed cap formulas, with each constant 1 replaced by θ, determine every normalized cap coordinate. The necessary linear constraints used are

$$u \geq 0, \quad 0 \leq \theta \leq 1/17, \quad p \geq c \geq b_q, \quad r, s, t, w \geq q,$$

and, for every $i, j \in A$,

$$6v_i - \sum_{k \in N(i) \cap C} v_k \geq \theta + \ell_i v_j,$$

where $\ell = (1, 1, 2, 2)$ or $(1, 1, 3, 3)$ is the exterior load vector. These follow from the next-layer ordering. The polytope is bounded: $p \geq c$ bounds u, and the nonnegative fixed sums $r + s = V$, $t + w = W$ bound both differences. It has twenty-eight vertices for each value of e.

The exact certificate enumerates all vertices and every quadratic Bernstein coefficient of the normalized expression (50). Their minima are

$$\frac{1059723546863}{7126821270268} > 0 \quad (e = 1), \qquad \frac{21107087737}{150847972856} > 0 \quad (e = 0).$$

Apply Lemma 6.1. □

6.16 Three–one shared ports with their joining edge present

Proposition 6.35. *Let H_4 be a recurrent degree-four terminal head with clique boundary B, unique minimum quadruple port c, and secondary ports p, p, p, q. Suppose cp, cq, pq are all edges. Then a degree-two head of a possibly different order has strictly greater chain score.*

Proof. Lemmas 6.19 and 6.23 give $x_p < x_q$ and inclusion of the outside neighborhood of p in that of q, modulo the pair. Thus p has one further neighbor z, while q has further neighbors z, s, t, where z, s, t are distinct. No vertex outside this layer can be adjacent to B, c, p, q, all of which are saturated. Write $F = H[\{z, s, t\}]$. Its eight labeled possibilities are all subgraphs of a triangle.

The full port layer outside $B \cup \{c\}$ is $\{p, q\}$. Lemma 6.25, then Lemma 6.4 with $S = \{r_{\text{root}}\} \cup B \cup \{c, p, q\}$, order all later coordinates above $\max(z, s, t)$.

Let $b = N/4$, where $N = |H_4| - 1$. The three majority boundary vertices have a common value a and the fourth has value d. The flux and internal Poisson equations give

$$\begin{aligned} a &= (24b - 1 - 8c)/14, & d &= (24c - 16b + 3)/14, \\ p &= 8b - \tfrac{3}{2} - 5c, & q &= 11c - 12b + \tfrac{1}{2}, \\ z &= 6p - 3a - c - q - 1, & s + t &= 6q - d - c - p - z - 1. \end{aligned}$$

Only c and $D = s - t$, together with the scale b, remain as parameters. The exterior loads of z, s, t are

$$\ell_z = 4 - d_F(z), \quad \ell_s = 5 - d_F(s), \quad \ell_t = 5 - d_F(t), \qquad Q = 14 - 2|E(F)|.$$

Delete the old ten core vertices and use the following new core, always with root degree two:

Q	new core	root neighbors	exterior loads
14	a four-vertex cycle	any two cycle vertices	$(3,3,4,4)$
12	$K_4 - e$	the two endpoints of e	$(3,3,3,3)$
10	K_4	any two vertices	$(2,2,3,3)$
8	K_5	any two vertices	$(1,1,2,2,2)$.

Each old endpoint occurs at most three times. The positive capacities are balanced, so Lemma 6.3 verifies the assignments, degrees, and connectivity. The order decreases by six for $Q = 14, 12, 10$, and by five for $Q = 8$.

Let $C = B \cup \{c, p, q, z, s, t\}$ be the ten-vertex cap, $A_C = A(H_4[C])$, and $v = y_C$. Put $L = v^T(6I - A_C)v$, $S = \ell_z z + \ell_s s + \ell_t t$, and let h be the order decrease. Lemma 6.2 gives

$$\tau_2 - \tau_4 \geq -L + \frac{S^2}{Q} + \frac{(N-h)^2}{2}.$$

Adding $P_4(N+1) - P_2(N+1-h)$ gives an explicit quadratic score lower bound G.

The exact three-dimensional verification is specified as follows. Normalize $b = 1$, set $\theta = 1/b$, and replace each constant 1 in the coordinate formulas by θ. Use coordinates (c, D, θ). Necessary linear inequalities are

$$\theta \geq 0, \quad c \geq a, d, \quad p \geq c, \quad q \geq p, \quad z, s, t \geq q.$$

For $u \in \{z, s, t\}$, its internal equation further gives

$$6x_u - \sum_{v \in N_C(u)} x_v - \theta \geq \ell_u x_w \qquad (w \in \{z, s, t\}).$$

The retained exterior has at least h_Q vertices, where h_Q is the smallest integer $h \geq 4$ with $Q \leq 3h$ and $h(7-h) \leq Q$. Indeed, every exterior vertex has at most three neighbors in the cap; the two displayed bounds follow by counting its incidences and the maximum possible exterior edges. Hence $\theta \leq 4/(10 + h_Q)$.

These constraints define a compact rational polytope: $q \geq p$ and $p \geq c$ bound c, while $s, t \geq q$ bound D and the displayed bounds control θ. For each of the eight choices of F, exact enumeration gives twelve vertices. Every pairwise quadratic Bernstein coefficient of G is at least

$$\frac{277654862}{297045225} > 0.$$

Lemma 6.1 therefore gives $J_2 - J_4 \geq (277654862/297045225)b^2 > 0$. □

6.17 The mixed triple–singleton port configuration

Proposition 6.36. *Let a recurrent degree-four terminal head have a clique root neighborhood B and unique quadruple exterior port c, in the remaining case where c is the minimum exterior port in the finite Perron vectors. Suppose its secondary ports consist of a triple port p and a singleton port q, with pc absent and qc present. Then there is a simple connected degree-two head, of order three,*

four or five less, with strictly larger invariant score. More precisely, if the old head has N nonroot vertices, the score increases by at least

$$\frac{1935411}{673004672}N^2.$$

Proof. Let u be the head root, let a denote its singleton-port neighbor, and put $C = B \cup \{c\}$. Thus C induces K_5. Write s for the other neighbor of c outside B; it is distinct from p, q and has no neighbor in B. The exact finite-coordinate ordering is

$$x_c < x_p < x_q < x_s. \tag{51}$$

The first inequality follows from the unique-minimum quadruple-port case. The second is the port-degree ordering, since the C-degrees of p, q are three and two. For the last, suppose $x_s \le x_q$. After excluding C and the possible edge qs, vertex s has one more neighbor than q. Choose an outside neighbor z of s not adjacent to q. The switch $aq, sz \mapsto as, qz$ has gain $2(x_q - x_s)(x_z - x_a) \ge 0$, and is strictly improving spectrally also at equality because $x_z > x_a$. Liu's local-switch proposition supplies connectedness, a contradiction.

Put $Q = N(p) \setminus C$. It has three vertices. The strict assortativity and cross-closure conclusions imply that every distinct pair from $\{q, s\} \times Q$ is adjacent. To recall the switch directly: if $z \in Q$, $v \in \{q, s\}$, $z \ne v$ and vz is absent, replace cv, pz by cp, vz. The gain is $2(x_v - x_p)(x_z - x_c) > 0$. Here every core vertex outside C has coordinate at least $x_p > x_c$: a minimum outside C occurs at one of its boundary ports p, q, s, since otherwise its regular Perron equation with $\lambda < 6$ is impossible.

Set

$$e = \mathbf{1}_{pq \in E}, \qquad f = \mathbf{1}_{ps \in E}, \qquad g = \mathbf{1}_{qs \in E}, \qquad T = Q \setminus \{q, s\}, \qquad j = |T| = 3 - e - f.$$

Cross-closure says that each member of T meets all of p, q, s, and that $g = 1$ whenever $e + f > 0$. Every $t \in T$ satisfies $x_t > x_s$. Indeed, t has no C-neighbor while s has one; if $x_t \le x_s$, the equal-degree count gives an outside neighbor z of t absent from $N(s)$, and $cs, tz \mapsto ct, sz$ is a strict spectral improvement, including at equality.

A further layer ordering is decisive:

$$x_z > \max_{t \in T} x_t \quad \text{for every core vertex } z \notin C \cup \{p, q, s\} \cup T. \tag{52}$$

Vertex p is saturated by its three B-neighbors and Q, so such a z has no neighbor in C and is not adjacent to p. If $x_z \le x_t$, equal degrees and $p \in N(t) \setminus N(z)$ give $w \in N(z) \setminus (N(t) \cup \{t\})$. This w is outside C and differs from p. The outside minimum argument gives $x_w > x_p$ (a nonboundary vertex cannot tie that minimum). The switch $pt, zw \mapsto pz, tw$ has gain $2(x_t - x_z)(x_w - x_p) \ge 0$, and a strict spectral improvement also when $x_t = x_z$, because the row at t changes from neighbor p to the larger-coordinate vertex w. This proves (52). All switches here preserve the head root and its degree; Liu's proposition applies to the full extremal graph.

Delete the core cap

$$W = C \cup \{p, q, s\} \cup T, \qquad h = |W| = 8 + j.$$

Its graph is completely specified by e, f, g and $H = G[T]$. Let $\ell = |E(H)|$. The exterior port counts at $q, s, t \in T$ are

$$d_q = 1 - g + f, \qquad d_s = 2 - g + e, \qquad d_t = 3 - d_H(t),$$

and all other cap vertices have zero exterior ports. Hence

$$D = \sum_{v \in W} d_v = 12 - 2g - 2(e+f) - 2\ell \in \{4, 6, 8, 10, 12\}.$$

There are precisely thirteen cap types: one with $(e, f) = (1, 1)$, two each with $(1, 0), (0, 1)$, and four each with $(0, 0)$ and $g = 0, 1$. For $j \le 3$ the number of edges determines the isomorphism type of H.

Replace W by six core vertices, retaining root u and joining it to vertices $0, 1$. Start with K_6, and delete the following edges, according to the required exterior port count:

D	deleted edges	exterior capacities, as a multiset
4	$\varnothing$	$1, 1, 1, 1, 0, 0$
6	01	$1, 1, 1, 1, 1, 1$
8	$01, 23$	$2, 2, 1, 1, 1, 1$
10	$01, 23, 45$	$2, 2, 2, 2, 1, 1$
12	$02, 03, 14, 15$	$2, 2, 2, 2, 2, 2$

Only q, s, T have exterior ports, so an old endpoint occurs at most five times (at most four when $D = 4$). The positive capacities are balanced. Lemma 6.3 proves simplicity, connectivity, and all degree claims. The head order changes by

$$d = 6 - h \in \{-3, -4, -5\}.$$

We compare the invariant scores exactly at zero energy. Let y be the old grounded Poisson solution, so $(6I - A_{H-u})y = \mathbf{1}$. The three B-vertices owned by p are twins; write their common coordinate as b, and use a, c, p, q, s, t_i also for the other coordinates. The necessary cap equations are

$$\begin{aligned} 3b + a &= N, & 4b - a - c - p &= 1, \\ 6a - 3b - c - q &= 1, & 6c - 3b - a - q - s &= 1, \\ 6p - 3b - eq - fs - \sum_i t_i &= 1. \end{aligned}$$

All coordinates are nonnegative, and the finite ordering gives

$$b, a \le c \le p \le q \le s \le t_i.$$

For each port owner $v \in \{q, s\} \cup T$ and every $t_i \in T$, (52) gives the necessary linear bound

$$6y_v - \sum_{w \in N(v) \cap W} y_w - 1 \;\ge\; d_v t_i. \tag{53}$$

Let $r = N - h$ be the number of retained core vertices, and let k be the number of cap vertices having positive exterior port count. Then

$$r \ge 7 - k, \qquad D \ge r(7 - r).$$

These follow by counting the at least $7 - r$ cap neighbors of each retained degree-six vertex; also $r \ge 1$. They give the lower bounds on N used in the certificate below.

Put

$$L = y_W^T (6I - A_W) y_W, \qquad A = \sum_{v \in W} d_v y_v.$$

Lemma 6.2, with $a = h$, $s = 6$, $Q = D$ and $k = 2$, gives

$$\tau_2 - \tau_4 \geq -L + \frac{(N+d)^2}{2} + \frac{A^2}{D}. \tag{54}$$

The lower bound for $J_2 - J_4$ is the right side of (54) plus $P_4(N+1) - P_2(N+1+d)$. Normalize every coordinate by N and put $\theta = 1/N$. The displayed equations, order constraints, (53), nonnegativity, and the lower bound on N define a compact rational polytope. The normalized score bound is a quadratic polynomial on it. Compactness also follows directly from $3b + a = 1$, nonnegativity, and the four remaining cap equations, which bound all other coordinates.

Here is the exact certificate summary. In the last column a simple strict lower bound is recorded; the verifier stores all exact coefficients.

e	f	g	ℓ	D	$N_{\min}$	#vertices	least quadratic coefficient
1	1	1	0	6	15	8	1935411/673004672
1	0	1	0	8	16	24	$> 9/100$
1	0	1	1	6	16	24	$> 9/100$
0	1	1	0	8	16	24	$> 1/10$
0	1	1	1	6	16	24	$> 1/10$
0	0	0	0	12	13	58	$> 11/100$
0	0	0	1	10	13	55	$> 11/100$
0	0	0	2	8	17	62	$> 3/20$
0	0	0	3	6	17	59	$> 3/20$
0	0	1	0	10	16	56	$> 7/50$
0	0	1	1	8	17	56	$> 3/20$
0	0	1	2	6	17	56	$> 3/20$
0	0	1	3	4	18	56	$> 3/20$

Lemma 6.1 proves the claimed bound. The exact verifier `computation/2026-11-27/head-singlequad-mixed-verify.py` reconstructs all thirteen domains and coefficients. Its vertex enumeration uses the equivalent complete traversal of the connected vertex–edge graph, with ranks and feasible edge endpoints checked exactly. □

7 A residue class with degree-two global extremizers

Let S_k be the seeds from Theorem 5.2. The *degree-four seed class* is $\{E_4^a S_4 : a \geq 0\}$. Its score is $-19/14$, and each member has an exact degree-two spectral partner of the same order.

Lemma 7.1 (Exhaustive conversion alternative)**.** *For a bounded degree-four head H occurring at infinitely many extremal orders, either H belongs to the degree-four seed class, or there is a connected head U of root degree $k \in \{2, 4\}$, all other degrees six, such that*

$$J_k(U) > J_4(H).$$

The order of U need not equal the order of H.

Proof. Absorb initial exact standard blocks into the chain until the head is irreducible. Its score is unchanged, its bounded order decreases, and it represents the same unbounded sequence of extremizers. Corollary 6.17 either supplies the required strict score improvement immediately, or leaves the equality seed or a clique boundary with one or two quadruple ports. In particular, the one-missing-edge compression is used as a score improvement, not as a same-order exclusion.

For two quadruple ports, Proposition 6.18 gives a removable standard block or a strict score improvement, covering all intersections of the two next-neighbor sets.

For a unique quadruple port, the exhaustive census is Proposition 6.27. The strict score comparisons are as follows; each entry constructs a competitor.

Multiplicities	Shared with c	Comparison
$1+1+1+1$	None	Proposition 6.28
$2+1+1$	None	Proposition 6.28
$2+1+1$	Both singletons	Proposition 6.29
$2+2$	None	Propositions 6.31, 6.32, 6.33
$2+2$	Both	Proposition 6.30
$3+1$	None	Proposition 6.28
$3+1$	Singleton only	Proposition 6.36
$3+1$	Both	Propositions 6.34, 6.35

The exterior-floor and ordering hypotheses follow from Lemma 6.22 and its successors. These comparisons supply strict score increases, not a presumed cross-order exclusion. Every case is covered. □

Proof of Theorem 1.1. Theorem 4.10 gives a finite family of heads representing all sufficiently large extremizers. Include all heads obtained by deleting initial standard blocks; this family remains finite. Let $\mathcal{F}_4$ consist of its degree-four heads recurring at unbounded orders. All other degree-four heads occur only finitely often, so beyond one threshold every degree-four extremizer has a head in $\mathcal{F}_4$. If $\mathcal{F}_4$ is empty, Theorem 3.17 makes all large extremizers degree two, and the claim follows.

Otherwise set $M = \max_{H\in\mathcal{F}_4} J_4(H)$. If some head attaining M is outside the seed class, Lemma 7.1 supplies a k-rooted head U with $J_k(U) > M$. Extend U to orders $n \equiv |U| \pmod 7$. Any degree-four extremizer of sufficiently large such order has a head $H \in \mathcal{F}_4$ with $|H| \equiv |U| \pmod 7$ and $J_4(H) < J_k(U)$. The score comparison lemma, after aligning the head orders by standard blocks, puts its spectral radius below the candidate from U. The comparison threshold is uniform because $\mathcal{F}_4$ is finite. Thus no degree-four extremizer occurs at large orders in this residue; the two-profile theorem makes every extremizer there degree two.

It remains that all heads attaining M belong to the seed class. Consider $n \equiv 1 \pmod 7$. Every seed-class degree-four chain of order n has exactly the radius of $E_2^{(n-8)/7}S_2$. Every other relevant head in $\mathcal{F}_4$ has score less than $M = J_2(S_2)$, so its long chains have strictly smaller radius, uniformly over this finite family. If a degree-four global extremizer exists, the degree-two seed chain therefore ties its global maximum. Otherwise the two-profile theorem supplies a degree-two extremizer directly. This proves the claim in residue one in the remaining case.

The degree-two extremizers at unbounded orders contradict the universal quantifier in Conjecture 11, including when they tie degree-four graphs. □

The argument does not identify the residue, a usable threshold, or an explicit globally optimal family. Classifying all recurring heads and resolving degree-two/degree-four ties at each residue remain open here.

8 Exact computations and reproducibility

The finite calculations use the certificate principles proved in Lemma 6.1: concave vertex bounds, quadratic Bernstein coefficients, and nonnegative products of affine constraints. The preceding proofs specify the graphs, domains and inequalities, and establish admissibility and case completeness.

All final signs and identities use exact rational arithmetic; floating-point searches used to discover certificates are not proof inputs.

Vertex enumeration uses all full-rank active sets or exhaustive vertex-edge traversal of a bounded polytope, after eliminating affine equalities. In the latter method every independent set of one fewer active constraints than the dimension is examined, with exact rank, feasibility and endpoint checks. The connected vertex graph establishes completeness of the traversal.

8.1 Supplement and execution

The supplementary directory `anc/` contains the programs and rational data under `research/c11-d6/`. Unless another prefix is displayed, manuscript paths are relative to that latter directory. With Python 3.10 or later, run

```
python3 anc/verify.py
```

The standard-library runner executes the 31 checks in `anc/checks.json` in a temporary copy, preserving saved data. They cover the deficiency profiles, completion gadgets, response and seed identities, and terminal-head inequalities and censuses. All 31 passed. Full output, exit statuses, interpreter version and script hashes appear in `anc/verification-results.json`. These checks certify the stated finite calculations, not the entire proof in a proof assistant. The external theorems remain cited inputs, and the written graph reductions require separate mathematical assessment.

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