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ON THE STONE–WEIERSTRASS PROBLEM

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Abstract. Let A be a C^*-algebra and let B be a C^*-subalgebra of A. Suppose that B separates the points of $P(A) \cup \{0\}$, where $P(A)$ denotes the pure state space of A. We prove that $B = A$ whenever either B or the quotient Banach space A/B is separable. In particular, this gives an affirmative answer to the noncommutative Stone–Weierstrass problem for separable C^*-algebras.

1. Introduction

The classical Stone–Weierstrass theorem asserts that if A is a commutative C^*-algebra and B is a C^*-subalgebra of A that separates the points of $P(A) \cup \{0\}$, where $P(A)$ denotes the pure state space of A, then $B = A$. In 1951, Kaplansky [Kap51] posed the following noncommutative analogue, now known as *the Stone–Weierstrass problem*.

Problem 1.1 (Kaplansky)**.** *Let A be a C^*-algebra and let B be a C^*-subalgebra of A. If B separates the points of $P(A) \cup \{0\}$, must $B = A$?*

For more than seventy years, Problem 1.1 has remained one of the central open problems in the theory of operator algebras. Some of the known special cases and related results on Problem 1.1 are summarized in Table 1.

Hypotheses	Reference
A or B is of type I	Kaplansky [Kap51]
$1_A \in B$ and there exists a norm-one projection from A onto B	Akemann [Ake69]
A is separable, $1_A \in B$, and there exists a completely positive map $A \to B^{**}$ fixing B (in particular, B has the weak expectation property)	Sakai [Sak70]
A is separable and B is a UHF algebra	Sakai [Sak70]
$A = C^*(B, x_0)$, where x_0 belongs to the self-adjoint part A_{sa} and $[x_0, B] \subseteq B$	Elliott [Ell72]
$1_A \in B$ and there exists a sequence of norm-one linear maps $L_n : A \to B$ such that $L_n(b) \to b$ weakly for every $b \in B$	Bunce [Bun80]
$1_A \in B$ and B is separable and nuclear	Bunce [Bun80]
$B = C_r^*(\mathbb{F}_2) \subseteq A \subseteq L(\mathbb{F}_2)$	Bunce [Bun80]
A is separable, $1_A \in B$, and every cyclic representation of B admits a unique extension to A on the same Hilbert space	Anderson–Bunce [AB81]
A is separable, $1_A \in B$, and the von Neumann algebra generated by the GNS representation of every state of B contains a regular maximal abelian self-adjoint subalgebra (MASA)	Anderson–Bunce [AB81]
$1_A \in B$ and B is perfect	Shultz [Shu82]
$1_A \in B$ and B is separable and semi-perfect	Batty [Bat86]
$1_A \in B$ and B is quasi-perfect	Fujimoto [Fuj98]
$1_A \in B$ and B separates the points of $\overline{P(A)}^{w^*}$	Glimm [Gli60]
A is separable, $1_A \in B$, and B separates the factorial states of A	Longo [Lon84], Popa [Pop84]

Table 1. Known special cases and related results on Problem 1.1. The upper part lists additional hypotheses for the pure-state version, and the lower part lists results under stronger separation hypotheses.

In this paper, we establish the following result, which gives an affirmative answer to Problem 1.1 in the separable case and also covers certain nonseparable cases.

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Theorem 1.2. *Let A be a C^*-algebra and let B be a C^*-subalgebra of A that separates the points of $P(A)\cup\{0\}$. If either B or the quotient Banach space A/B is separable*[1]*, then $A = B$.*

Since a closed subspace of a separable Banach space is separable, Theorem 1.2 includes the case where A is separable. The hypothesis on A/B allows both A and B to be nonseparable.

The following corollary gives a criterion for a self-adjoint element $a \in A$ to belong to B, in terms of the right action of $\mathcal{U}(B)$ on A/B.

Corollary 1.3. *Let $B \subseteq A$ be C^*-algebras with a common identity and suppose that B separates $P(A)$. A self-adjoint $a \in A$ belongs to B if and only if $\{au + B : u \in \mathcal{U}(B)\}$ is separable in A/B, where $\mathcal{U}(B)$ denotes the unitary group of B.*

1.1. **High-level description of the proof.** We first assume that A is separable, that A and B have a common identity, and that B separates $P(A)$. The main idea is to study extensions of a factor representation of B. If $B \neq A$, we seek an irreducible representation of A whose restriction to B is reducible. Such a representation would give two distinct pure states of A agreeing on B. Figure 1 summarizes the proof of Theorem 1.2.

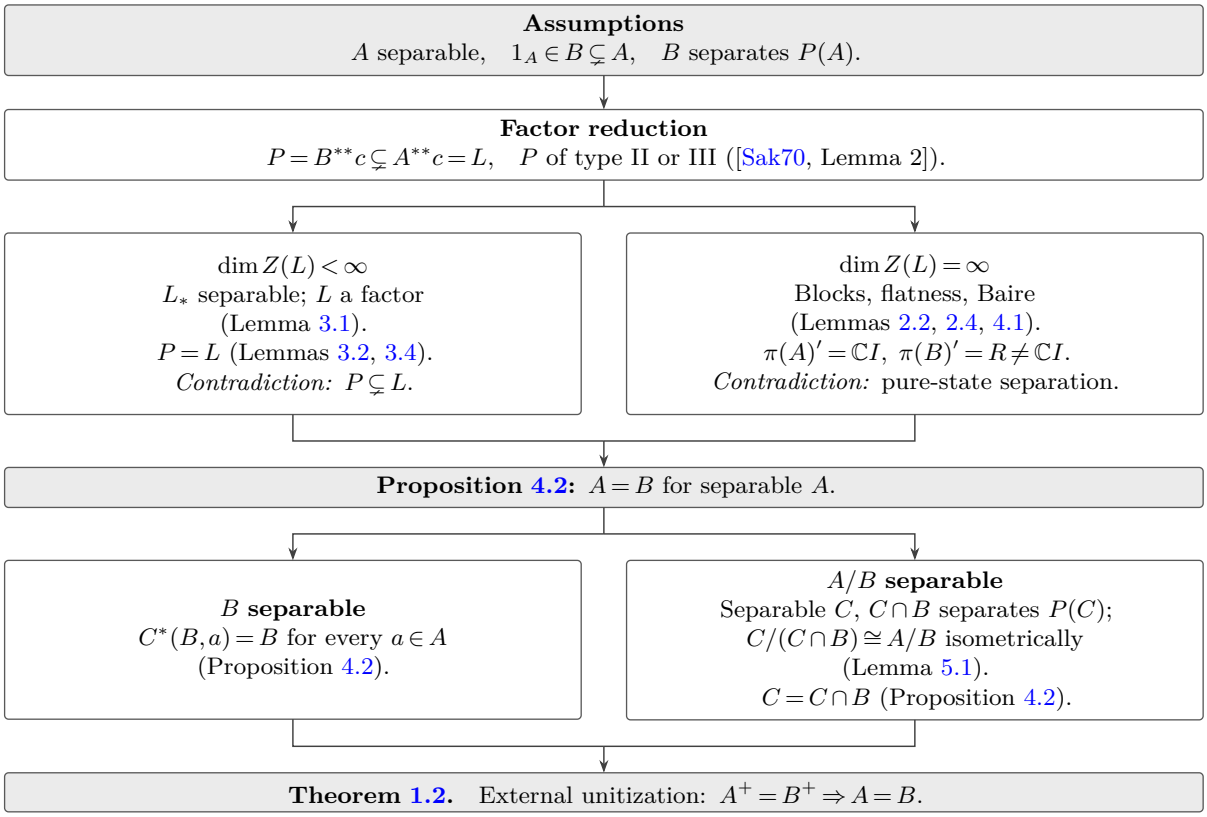

FIGURE 1. Outline of the proof of Theorem 1.2.

• **Step I. Reduction to a factor inclusion.** Suppose that $B \subsetneq A$. By the factorial Stone–Weierstrass theorem of Longo and Popa [Lon84, Pop84], there are two distinct factor states of A agreeing on B. Anderson and Bunce [AB81] show that the common restriction is a factor state of B and that $Z(B^{**}) \subseteq Z(A^{**})$, where $Z(M)$ denotes the center of a von Neumann algebra M. We identify B^{**} with the weak-$*$ closure of B in A^{**}. The central support c of the GNS representation of the common restriction therefore determines an inclusion

$$P = B^{**}c \subsetneq A^{**}c = L,$$

where P is a factor with separable predual. The possibility that P is of type I is excluded by Sakai's uniqueness theorem for atomic restrictions [Sak70, Lemma 2]; thus P is of type II or III.

• **Step II. Approximation of representation extensions.** Let $\rho : P \to \mathcal{B}(H)$ be an arbitrary faithful normal representation on a separable Hilbert space H. Write $R = \rho(P)'$ for its commutant and $\mathcal{X}$ for the

[1]The quotient norm on A/B is defined by $\|a + B\| = \inf_{b\in B} \|a - b\|$ for every $a \in A$.

space of unital representations $\pi : A \to \mathcal{B}(H)$ satisfying $\pi(b) = \rho(bc)$ for every $b \in B$. All representations are assumed to be nondegenerate. Equipped with the pointwise strong-$*$ topology, $\mathcal{X}$ is Polish, and every $\pi \in \mathcal{X}$ factors normally through L. If D is a MASA of $\pi(A)'$, then D is also a MASA of R by [AB81]. Using Sakai's uniqueness theorem [Sak70], Lemma 2.1 establishes uniqueness of the unital completely positive extension with range in D'. Lemma 2.2 gives a finite-dimensional version of the uniqueness assertion: if D_n are increasing finite-dimensional subalgebras generating D, then for any neighborhood of π, all members of $\mathcal{X}$ commuting with D_n lie in that neighborhood for sufficiently large n. For the orbit approximation, we take ρ to be the countably infinite amplification of the standard representation of P (see Haagerup [Haa75]), with H, R and $\mathcal{X}$ defined accordingly. Lemma 2.4 establishes the density of the corresponding unitary orbits in $\mathcal{X}$.

• **Step III. Exclusion of a finite-dimensional center.** If $Z(L)$ is finite dimensional, the separability of A implies that the predual L_* of L is separable. Lemma 3.1 then shows that L is a factor and provides a common point of norm continuity for the vector-state maps $\sigma \mapsto \omega_\eta \circ \sigma$ on any nonempty extension space $\mathcal{X}$ associated with a faithful normal representation of P on a separable Hilbert space H, where $\omega_\eta(T) = \langle T\eta, \eta\rangle$ for $T \in \mathcal{B}(H)$ and $\eta \in H$. Inner products are linear in the first variable. We treat the possible types of P separately.

If P is semifinite or of type III_λ with $0 \le \lambda < 1$, we construct a unital completely positive map $F : A \to P$ satisfying $F(b) = bc$ for every $b \in B$. The construction uses invariant state extensions and, in the type III case, Connes' discrete decomposition [Con73] and Fejér approximation. The resulting equality $P = L$ is established in Lemma 3.2 by applying Lemma 2.3.

For a type III_1 factor P, the argument is different. Using the Connes–Størmer homogeneity theorem [CS78] and Araki's natural-cone estimate [Ara74] (see also [Haa75]), we show that the proper inclusion $P \subsetneq L$ forces $\operatorname{diam} F_\gamma = 2$ in norm, for every normal state γ of P, where F_γ denotes the set of its normal state extensions to L (Lemma 3.3). On the other hand, the bicentralizer theorem recently established by Houdayer and Marrakchi [HM26], together with the results of Haagerup [Haa87] and Popa [Pop81], implies that every type III_1 factor with separable predual contains a MASA onto which there exists a faithful normal conditional expectation. For the type III_1 argument, we use the standard representation of P on $H = L^2(P)$ in place of its amplification, and define R and $\mathcal{X}$ accordingly. Using a MASA with expectation in R, Lemma 3.4 constructs two sequences $\pi_n^+, \pi_n^- \in \mathcal{X}$ converging to the same continuity point $\pi \in \mathcal{X}$ in the pointwise strong-$*$ topology, while

$$\|\omega_\eta \circ \pi_n^+ - \omega_\eta \circ \pi_n^-\| > 1$$

for a fixed unit vector $\eta \in H$. This contradicts norm continuity. Consequently, the center of L must be infinite dimensional.

• **Step IV. Construction of an irreducible extension.** The infinite dimensionality of $Z(L)$ permits us to construct extensions by taking direct sums of representations supported on disjoint central summands of L. Returning to the countably amplified standard representation from Step II, with the corresponding H, R and $\mathcal{X}$, Lemma 4.1 allows us to prescribe a finite partition of the identity in R, whose nonzero projections are all equivalent to the identity, as central projections of the commutant of an extension. For $g \in R_*$ with $g(1) = 0$ and $\epsilon > 0$, consider

$$\{\pi \in \mathcal{X} : \|g|_{\pi(A)'}\| < \epsilon\}.$$

The set is open by weak operator compactness. Its density follows from the block construction, Popa's flatness theorem [Pop21] and the approximation in Lemma 2.2. Since R_* is separable, Baire's theorem gives an extension $\pi \in \mathcal{X}$ for which every such g vanishes on $\pi(A)'$. Thus

$$\pi(A)' = \mathbb{C}I, \qquad \pi(B)' = R \neq \mathbb{C}I.$$

This contradicts pure-state separation and proves the separable case with a common identity (Proposition 4.2).

• **Step V. Separable reduction and unitization.** If B is separable, we use the one-generator reduction of Elliott [Ell72]: for each $a \in A$, the algebra $C^*(B, a)$ is separable. Since its pure states extend to pure states of A, Proposition 4.2 applies to $B \subseteq C^*(B, a)$, giving $a \in B$.

If only A/B is separable, the main difficulty is to choose a separable subalgebra of A for which pure-state separation is preserved. In Lemma 5.1, we use the agreement of the atomic summands of B^{**} and A^{**} (Bunce–Chu [BC98, Remark 4.8(a)]) and the separability of A/B to construct a separable unital C^*-subalgebra C of A. The natural map $C/(C \cap B) \to A/B$ is a surjective linear isometry. Using this isometry and the Jordan decomposition of self-adjoint functionals, we show that a failure of pure-state separation on C would produce a normal pure state on a summand of A^{**} having no minimal projections. Hence $C \cap B$ separates $P(C)$. Proposition 4.2 gives $C = C \cap B$, so $A/B = 0$. Passing to unitizations removes the common-identity assumption and completes the proof of Theorem 1.2.

1.2. **Organization of the paper.** Section 2 carries out Steps I and II. In Section 3, we exclude a finite-dimensional center as in Step III. Section 4 uses Popa's flatness theorem and Baire's theorem to construct the irreducible extension in Step IV. Section 5 proves the separable reduction in Step V, completes the proof of Theorem 1.2, and establishes the local criterion in Corollary 1.3.

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2. Extending a factor representation

Throughout Sections 2–4, we assume that A is separable, that A and B have a common identity, and that B separates $P(A)$.

By Anderson and Bunce [AB81, Corollaries 1–3], we have

$$Z(B^{**}) \subseteq Z(A^{**}). \tag{1}$$

Moreover, the restriction of any factor state of A to B is a factor state of B, and, for every representation π of A on a separable Hilbert space, every MASA of $\pi(A)'$ is also a MASA of $\pi(B)'$. We shall also use the representation extension theorem of Anderson and Bunce [AB81, Theorem 6]: if σ is a cyclic representation of B and D is a MASA of $\sigma(B)'$, then there exists a unique representation of A extending σ whose range is contained in D'.

We first establish two elementary consequences of pure-state separation that do not require separability. Every irreducible representation of A remains irreducible upon restriction to B. In other words, for every representation π of A,

$$\pi(A)' = \mathbb{C}I \quad \Longrightarrow \quad \pi(B)' = \mathbb{C}I. \tag{2}$$

Indeed, suppose that π is irreducible but $\pi|_B$ is reducible. Choose unit vectors ξ and η in two complementary nonzero reducing subspaces for $\pi(B)$. Since π is irreducible, the vector states associated with $(\xi+\eta)/\sqrt{2}$ and $(\xi-\eta)/\sqrt{2}$ are pure. The two vectors are orthogonal, so their vector states are distinct on the algebra of bounded operators on the representation space. By the bicommutant theorem, $\pi(A)$ is weak operator dense in that algebra, and hence the corresponding pure states of A are distinct. On the other hand, for every $b \in B$, we have $\langle \pi(b)\xi, \eta\rangle = \langle \pi(b)\eta, \xi\rangle = 0$. Consequently, the two pure states agree on B, contradicting the hypothesis that B separates $P(A)$.

Every pure state of B also has a unique state extension to A. Indeed, the Hahn–Banach theorem guarantees the existence of a state extension. Since A and B have a common identity, the set of state extensions of a pure state of B is a nonempty weak-$*$ compact face of the state space of A. Every extreme point of the face is therefore a pure state of A. The separation hypothesis implies that the face contains at most one extreme point, so the Krein–Milman theorem gives uniqueness. Thus, for any two states φ and ψ of A,

$$\varphi|_B = \psi|_B \in P(B) \quad \Longrightarrow \quad \varphi = \psi. \tag{3}$$

Suppose, towards a contradiction, that $B \neq A$. By the factorial Stone–Weierstrass theorem of Longo and Popa [Lon84, Pop84], there exist two distinct factor states of A with the same restriction to B. By [AB81, Corollary 3], the common restriction is a factor state of B. Let c be the central support in B^{**} of the normal extension of the GNS representation associated with the common restriction. Since the GNS representation is factorial, c is a central atom of B^{**}. It follows from (1) that c is also central in A^{**}.

The normal extensions of the two factor states to A^{**} both take the value 1 at c. They are therefore supported on $A^{**}c$. Since the two normal states agree on $B^{**}c$ but remain distinct on $A^{**}c$, we obtain a proper inclusion

$$P = B^{**}c \subsetneq A^{**}c = L, \tag{4}$$

where P is a factor and P and L have common identity c. The GNS representation associated with the common restriction induces a faithful normal representation of P on a separable Hilbert space. Consequently, the predual P_* is separable. Furthermore, the GNS representation of either of the two factor states of A induces a nonzero normal factor representation of L. We regard representations of P and L as representations of B and A via the maps $b \mapsto bc$ and $a \mapsto ac$, respectively.

Let $\rho : P \to \mathcal{B}(H)$ be an arbitrary faithful normal representation of P on a separable Hilbert space H. Set $R = \rho(P)'$, and let $\mathcal{X}$ consist of all unital representations $\pi : A \to \mathcal{B}(H)$ satisfying

$$\pi(b) = \rho(bc), \qquad b \in B. \tag{5}$$

Whenever a faithful normal representation ρ of P is specified, the notation H, R and $\mathcal{X}$ refers to the Hilbert space, commutant and representation extension space associated with ρ.

By (5) and the uniqueness of normal extensions to the bidual, the normal extension of $\pi|_B$ to B^{**} is given by $x \mapsto \rho(xc)$. In particular, the normal extension of π to A^{**} sends c to I. Hence every $\pi \in \mathcal{X}$ factors normally through L. Since Bc is weak-$*$ dense in P and ρ is normal, we also have

$$\pi(B)' = \rho(P)' = R, \qquad \pi \in \mathcal{X}. \tag{6}$$

We equip $\mathcal{X}$ with the topology of pointwise strong-$*$ convergence. To see that $\mathcal{X}$ is Polish, choose a countable norm-dense unital $\mathbb{Q}(i)$-subalgebra of A, closed under involution, whose intersection with B is norm dense in B. Every representation of A is determined by its values on the chosen subalgebra. Since H is separable, every closed operator norm ball in $\mathcal{B}(H)$ is Polish in the strong-$*$ topology. The conditions of linearity, multiplicativity, preservation of the involution, unitality and contractivity, together with the restriction condition (5), define a closed subset of the corresponding countable product of operator balls. Here we use the strong-$*$ continuity of multiplication and involution on norm-bounded sets. Contractivity ensures that every map in the closed subset extends uniquely to a representation of A. It also ensures that pointwise strong-$*$ convergence on the dense subalgebra is equivalent to pointwise strong-$*$ convergence on A. Thus $\mathcal{X}$ is Polish.

The following consequence of Sakai's uniqueness theorem [Sak70, Theorem 1] will be used repeatedly.

Lemma 2.1. *Let ρ be a faithful normal representation of P on a separable Hilbert space, with associated extension space $\mathcal{X}$. If $\pi \in \mathcal{X}$ and D is a MASA of $\pi(A)'$, then π is the unique unital completely positive map $A \to D'$ extending $b \mapsto \rho(bc)$.*

Proof. Choose a faithful unital representation τ of A on a separable Hilbert space and a MASA D_τ of $\tau(A)'$. Set $\Pi = \pi \oplus \tau$ and $E = D \oplus D_\tau$. Since E contains the projections onto the two summands, we have $\Pi(A)' \cap E' = E$. Thus E is a MASA of $\Pi(A)'$ and $(\Pi(A) \cup E)'' = E'$.

Let $\Phi : A \to D'$ be a unital completely positive map extending $b \mapsto \rho(bc)$. Since Π is faithful, the formula

$$T(\Pi(a)) = \Phi(a) \oplus \tau(a), \qquad a \in A,$$

defines a linear contraction $T : \Pi(A) \to E'$ satisfying $T(\Pi(b)) = \Pi(b)$ for every $b \in B$. By Sakai's uniqueness theorem [Sak70], we have $T(\Pi(a)) = \Pi(a)$ for every $a \in A$. Comparing the first summands gives $\Phi = \pi$. □

Lemma 2.1 allows us to approximate representation extensions by imposing commutation with finite-dimensional subalgebras of a MASA.

Lemma 2.2. *Let ρ be a faithful normal representation of P on a separable Hilbert space, with associated extension space $\mathcal{X}$. Let $\pi \in \mathcal{X}$, let D be a MASA of $\pi(A)'$, and let $(D_n)_{n\geq 1}$ be an increasing sequence of finite-dimensional unital subalgebras generating D as a von Neumann algebra. For every neighborhood V of π in $\mathcal{X}$, there exists $n \geq 1$ such that every $\sigma \in \mathcal{X}$ satisfying $D_n \subseteq \sigma(A)'$ belongs to V.*

Proof. Suppose that the assertion fails for some open neighborhood V of π. For every $n \geq 1$, choose $\sigma_n \in \mathcal{X} \setminus V$ such that $D_n \subseteq \sigma_n(A)'$.

The set of unital completely positive maps $\Phi : A \to \mathcal{B}(H)$ satisfying $\Phi(b) = \rho(bc)$ for every $b \in B$ is compact in the topology of pointwise weak operator convergence. Indeed, every such map satisfies $\|\Phi(a)\| \leq \|a\|$ for every $a \in A$. The closed operator balls are weak operator compact, while linearity, unitality, complete positivity and the prescribed restriction to B are closed conditions in the product topology.

Let Φ be a cluster point of $(\sigma_n)_{n\geq 1}$ in the topology of pointwise weak operator convergence. For each fixed $m \geq 1$, the inclusion $D_m \subseteq D_n$ for $n \geq m$ implies that $\sigma_n(A) \subseteq D_m'$. Since D_m' is weak operator closed, passing to a convergent subnet gives $\Phi(A) \subseteq D_m'$. As the union of the D_m generates D as a von Neumann algebra, we obtain $\Phi(A) \subseteq D'$. Lemma 2.1 therefore implies that $\Phi = \pi$.

Since every cluster point of $(\sigma_n)_{n\geq 1}$ equals π, compactness gives pointwise weak operator convergence of σ_n to π. For every $a \in A$ and $\xi \in H$, we have

$$\begin{aligned} \|\sigma_n(a)\xi\|^2 &= \langle \sigma_n(a^*a)\xi, \xi\rangle \\ &\longrightarrow \langle \pi(a^*a)\xi, \xi\rangle = \|\pi(a)\xi\|^2. \end{aligned} \tag{7}$$

Pointwise weak operator convergence implies that $\sigma_n(a)\xi$ converges weakly to $\pi(a)\xi$. Together with (7), we obtain $\|\sigma_n(a)\xi - \pi(a)\xi\| \to 0$. Applying the same argument to a^* gives $\sigma_n(a) \to \pi(a)$ in the strong-$*$ topology for every $a \in A$. Hence $\sigma_n \to \pi$ in $\mathcal{X}$, contradicting the choice of $\sigma_n \in \mathcal{X} \setminus V$ for every $n \geq 1$. □

We also use Lemma 2.1 to obtain a criterion for equality in the factor inclusion (4).

Lemma 2.3. *If there exists a unital completely positive map $F : A \to P$ satisfying $F(b) = bc$ for every $b \in B$, then $P = L$.*

Proof. Let σ be a nonzero normal cyclic representation of L. Since Ac is weak-$*$ dense in L, normality of σ implies that a cyclic vector for $\sigma(L)$ is also cyclic for $\sigma(Ac)$. The separability of A therefore implies that the representation space of σ is separable. Moreover, since P is a factor, the nonzero normal representation $\sigma|_P$ is faithful.

Consider the representation $a \mapsto \sigma(ac)$ of A and the unital completely positive map $a \mapsto \sigma(F(a))$. Both maps extend the representation $b \mapsto \sigma(bc)$ of B, because $F(b) = bc$ for every $b \in B$. Let D be a MASA of $\sigma(L)'$. Since $\sigma(P) \subseteq \sigma(L)$, the ranges of both maps are contained in D'. Applying Lemma 2.1 with $\rho = \sigma|_P$ and $\pi(a) = \sigma(ac)$ for $a \in A$, we obtain $\sigma(ac) = \sigma(F(a))$ for every $a \in A$.

Since σ was an arbitrary nonzero normal cyclic representation of L, and normal cyclic representations separate the points of L, it follows that $ac = F(a) \in P$ for every $a \in A$. The weak-$*$ density of Ac in L, together with the weak-$*$ closedness of P in L, yields $P = L$. □

For the orbit approximation, we now specialize ρ to the countably infinite amplification of the standard representation of P in the sense of Haagerup [Haa75], acting on $H = L^2(P) \otimes \ell^2$. Since P is a factor with separable predual, H is separable and the commutant $R = \rho(P)'$ is a properly infinite factor. The notation $\mathcal{X}$ now refers to the extension space associated with the amplified representation.

The countably infinite amplification of every nonzero normal representation of P on a separable Hilbert space is unitarily equivalent to ρ. Indeed, every normal cyclic representation of P can be realized as the restriction of the standard representation to the cyclic subspace generated by a vector in the natural cone. Every normal representation on a separable Hilbert space is a direct sum of at most countably many normal cyclic representations. Consequently, every such representation can be realized as a corner of ρ by a projection in R.

If the representation is nonzero, the corresponding corner projection is nonzero. After countably infinite amplification, the corner projection becomes properly infinite and has central support 1 in the amplified commutant. Since the amplified commutant is a factor acting on a separable Hilbert space, the amplified corner projection is Murray–von Neumann equivalent to the identity. The equivalence of the projections gives unitary equivalence of the corresponding representations. Finally, the countably infinite amplification of ρ is itself unitarily equivalent to ρ, since $\ell^2 \otimes \ell^2$ is unitarily equivalent to ℓ^2.

In particular, the restriction to P of the countably infinite amplification of any nonzero normal representation of L on a separable Hilbert space is unitarily equivalent to ρ. Transporting the amplified representation of L to H by an intertwining unitary and composing it with $a \mapsto ac$ gives a member of $\mathcal{X}$. The existence of a nonzero normal factor representation of L, established in the construction of (4), therefore ensures that $\mathcal{X} \neq \varnothing$.

Lemma 2.4. *Let ρ be the countably infinite amplification of the standard representation of P, acting on $H = L^2(P) \otimes \ell^2$. Let $\sigma \in \mathcal{X}$ be obtained from the countably infinite amplification of a nonzero normal representation of L on a separable Hilbert space by identifying its restriction to P with ρ. Then the unitary conjugates $u\sigma(\cdot)u^*$, $u \in \mathcal{U}(R)$, are dense in $\mathcal{X}$ in the topology of pointwise strong-$*$ convergence.*

Proof. Step 1. Density of amplified representations. We first show that the representations obtained by transporting countably infinite amplifications of members of $\mathcal{X}$ back to H are dense in $\mathcal{X}$.

Fix $\pi \in \mathcal{X}$. Since ρ is the countably infinite amplification of the standard representation of P, the commutant R contains the algebra $I \otimes \mathcal{B}(\ell^2)$. We may therefore choose isometries $v_n \in R$ such that $v_n \to I$ in the strong-$*$ topology and $I - v_n v_n^*$ is Murray–von Neumann equivalent to I in R. Indeed, on the multiplicity space ℓ^2, one can choose an isometry fixing the first n basis vectors and mapping the remaining basis vectors onto every other basis vector in the tail. Its range has infinite-dimensional orthogonal complement.

For each $n \geq 1$, choose isometries $w_{n,j} \in R$, $j \geq 1$, with mutually orthogonal range projections such that $\sum_{j=1}^{\infty} w_{n,j} w_{n,j}^* = I - v_n v_n^*$, where the sum converges in the strong operator topology. Define a representation π_n of A on H by

$$\pi_n(a) = v_n \pi(a) v_n^* + \sum_{j=1}^{\infty} w_{n,j} \pi(a) w_{n,j}^*, \qquad a \in A, \tag{8}$$

where the series converges in the strong operator topology. By construction, π_n is unitarily equivalent to the countably infinite amplification of π.

Since $v_n, w_{n,j} \in R = \rho(P)'$, it follows from (8) that

$$\begin{aligned}\pi_n(b) &= \rho(bc)\left(v_n v_n^* + \sum_{j=1}^{\infty} w_{n,j} w_{n,j}^*\right)\\ &= \rho(bc), \qquad b \in B.\end{aligned}$$

Thus $\pi_n \in \mathcal{X}$. Moreover, for every $a \in A$ and $\xi \in H$, the orthogonality of the range projections in (8) gives

$$\|(\pi_n(a) - v_n\pi(a)v_n^*)\xi\| \le \|a\|\|(I - v_n v_n^*)\xi\|. \tag{9}$$

Since $v_n \to I$ in the strong-$*$ topology, $v_n\pi(a)v_n^* \to \pi(a)$ strongly and $I - v_n v_n^* \to 0$ strongly. By (9), we obtain $\pi_n(a) \to \pi(a)$ strongly for every $a \in A$. Applying the same argument to a^* yields pointwise strong-$*$ convergence of π_n to π. Thus the amplified representations are dense in $\mathcal{X}$.

Step 2. Approximation by unitary conjugates. Fix $\pi \in \mathcal{X}$ and consider its countably infinite amplification $a \mapsto \pi(a) \otimes I$ on $H \otimes L^2[0,1]$. We shall show that every representation obtained by transporting this amplification back to H, through a unitary intertwining the corresponding representations of P, belongs to the closure of the unitary orbit of σ.

Let D_0 be a MASA of $\pi(A)'$. Since the commutant of the amplified representation is $\pi(A)'\bar{\otimes}\mathcal{B}(L^2[0,1])$, the algebra

$$D = D_0\bar{\otimes}L^\infty[0,1]$$

is a MASA of the amplified commutant, where $L^\infty[0,1]$ acts on $L^2[0,1]$ by multiplication. Here $\bar{\otimes}$ denotes the spatial tensor product of von Neumann algebras.

We claim that every nonzero projection $p \in D$ is Murray–von Neumann equivalent to the identity in

$$R\bar{\otimes}\mathcal{B}(L^2[0,1]). \tag{10}$$

If R is of type III, the claim follows from the equivalence of all nonzero projections in a type III factor acting on a separable Hilbert space.

Suppose that R is semifinite, and let τ be a faithful normal semifinite trace on R. Denote by Tr the usual trace on $\mathcal{B}(L^2[0,1])$. Since the finite-trace projections of R form an increasing net with supremum I, there exists a projection $f \in R$ with $\tau(f) < \infty$ such that

$$(f \otimes I)p(f \otimes I) \neq 0.$$

Applying the finite trace $\tau|_{fRf}$ to the first tensor factor of $(f \otimes I)p(f \otimes I)$ gives a nonzero positive element of $L^\infty[0,1]$. Indeed, the slice map associated with the faithful trace $\tau|_{fRf}$ is faithful on positive elements.

As a multiplication operator on $L^2[0,1]$, the resulting positive element is bounded below by a positive scalar on $L^2(E)$ for some measurable set $E \subseteq [0,1]$ of positive Lebesgue measure. Since $L^2(E)$ is infinite dimensional, the trace of the multiplication operator is infinite. Consequently,

$$\begin{aligned}(\tau\bar{\otimes}\,\mathrm{Tr})(p) &\ge (\tau\bar{\otimes}\,\mathrm{Tr})\big((f \otimes I)p(f \otimes I)\big)\\ &= \infty.\end{aligned}$$

The first inequality follows from the tracial property and the inequality $p(f \otimes I)p \le p$. Since the algebra in (10) is a properly infinite semifinite factor acting on a separable Hilbert space, comparison of projections by the trace shows that p is equivalent to the identity. This proves the claim.

Choose a unitary $W : H \otimes L^2[0,1] \to H$ satisfying

$$W(\rho(x) \otimes I)W^* = \rho(x), \qquad x \in P. \tag{11}$$

The existence of W follows from the unitary equivalence of ρ and its countably infinite amplification.

Let $p_1, \dots, p_m$ be an arbitrary partition of the identity into nonzero projections in D. Since σ is obtained from a countably infinite amplification, the commutant of the representation $W^*\sigma(\cdot)W$ contains a partition of the identity into m nonzero projections $q_1, \dots, q_m$, each equivalent to the identity in that commutant. Such a partition is obtained by decomposing the infinite multiplicity space of σ into m infinite-dimensional subspaces.

By (11), the commutant of $W^*\sigma(B)W$ is precisely the algebra in (10). The projections p_i and q_i are therefore equivalent in $R\bar{\otimes}\mathcal{B}(L^2[0,1])$ for every $1 \le i \le m$. Choose partial isometries $s_1, \dots, s_m$ in this algebra satisfying $s_i^* s_i = q_i$ and $s_i s_i^* = p_i$. Since both families of projections are partitions of the identity, $u = \sum_{i=1}^m s_i$ is a unitary satisfying $uq_iu^* = p_i$ for every $1 \le i \le m$. It follows that

$$[uW^*\sigma(a)Wu^*, p_i] = 0, \qquad a \in A, \quad 1 \le i \le m. \tag{12}$$

Since D acts on a separable Hilbert space, there exists an increasing sequence of finite-dimensional unital subalgebras $D_n \subseteq D$ whose union generates D as a von Neumann algebra. Applying (12) to the partition of the identity determined by the minimal projections of D_n, we obtain a unitary $u_n \in \mathcal{U}(R\bar{\otimes}\mathcal{B}(L^2[0,1]))$ such that

$$u_n W^* \sigma(a) W u_n^* \in D_n', \qquad a \in A.$$

For every $b \in B$, (11) gives

$$u_n W^* \sigma(b) W u_n^* = \rho(bc) \otimes I.$$

Hence the representations $u_n W^* \sigma(\cdot) W u_n^*$ belong to the extension space associated with the faithful normal representation $x \mapsto \rho(x) \otimes I$ of P. We may thus apply Lemma 2.2 to the amplified target $a \mapsto \pi(a) \otimes I$ and the sequence $(D_n)_{n\geq 1}$. It follows that

$$u_n W^* \sigma(a) W u_n^* \longrightarrow \pi(a) \otimes I, \qquad a \in A,$$

in the strong-$*$ topology.

Conjugating by W, we obtain

$$(W u_n W^*)\sigma(a)(W u_n^* W^*) \longrightarrow W(\pi(a) \otimes I)W^*, \qquad a \in A,$$

in the strong-$*$ topology. By (11), $W u_n W^* \in \mathcal{U}(R)$. Therefore $W(\pi(\cdot) \otimes I)W^*$ belongs to the closure of the unitary orbit of σ.

Since $\pi \in \mathcal{X}$ and the intertwining unitary W were arbitrary, Step 1 implies that the closure of the unitary orbit of σ is all of $\mathcal{X}$. □

3. The center above a factor representation

If the factor P in (4) were of type I, the common restriction of the two factor states used to define c would be atomic. Sakai's uniqueness theorem for atomic restrictions [Sak70, Lemma 2] would then imply that the two states coincide, a contradiction. Hence P is of type II or III. We shall prove that $Z(L)$ is infinite dimensional.

Recall that c is the common identity of P and L. For any normal functional φ on L, we have

$$\|\varphi\| = \sup_{\substack{a\in A \\ \|a\|\leq 1}} |\varphi(ac)|. \tag{13}$$

Indeed, by Kaplansky's density theorem, the image of the unit ball of A under the map $a \mapsto ac$ is weak-$*$ dense in the unit ball of L.

Let γ be a normal state of P, and recall that F_γ denotes the set of its normal state extensions to L. The map $\alpha \mapsto (a \mapsto \alpha(ac))$ identifies F_γ with the set of all state extensions of $b \mapsto \gamma(bc)$ to A. To see surjectivity, let φ be a state extension to A. By normality and weak-$*$ density, the normal extension of φ to A^{**} agrees on B^{**} with the map $x \mapsto \gamma(xc)$. In particular, it takes the value 1 at c and is therefore supported on $L = A^{**}c$. The identification is isometric by (13). Since the set of state extensions of a state of B to A is weak-$*$ compact, F_γ is compact in the topology of pointwise convergence on A. We use this compact topology on F_γ throughout the section.

Lemma 3.1. *Suppose that L_* is separable. Then L is a factor. Moreover, let ρ be a faithful normal representation of P on a separable Hilbert space H with nonempty extension space $\mathcal{X}$. There exists $\pi \in \mathcal{X}$ at which the map $\sigma \mapsto \omega_\eta \circ \sigma$ is norm continuous for every $\eta \in H$.*

Proof. We first establish the continuity assertion for an arbitrary faithful normal representation ρ as in the statement. Fix $\eta \in H$. Since every representation in $\mathcal{X}$ factors normally through L, the map $\sigma \mapsto \omega_\eta \circ \sigma$ takes values in L_*, where the normal functionals on L are identified with their restrictions to A.

For each $a \in A$, the scalar-valued map $\sigma \mapsto \langle \sigma(a)\eta, \eta\rangle$ is continuous on $\mathcal{X}$. By (13), the inverse image of every closed norm ball in L_* under the map $\sigma \mapsto \omega_\eta \circ \sigma$ is closed in $\mathcal{X}$.

Fix a positive integer k. Since L_* is separable, it can be covered by countably many closed norm balls of radius $1/k$. Consequently, $\mathcal{X}$ is a countable union of the closed inverse images of the balls. As $\mathcal{X}$ is Polish, Baire's category theorem implies that every nonempty open subset of $\mathcal{X}$ contains a nonempty open subset on which the image of the map $\sigma \mapsto \omega_\eta \circ \sigma$ has norm diameter at most $2/k$.

For each $k \geq 1$, the union of all open subsets with this diameter property is therefore open and dense in $\mathcal{X}$. Taking the intersection over all positive integers k, we obtain a dense G_δ subset of $\mathcal{X}$ at every point of which the map $\sigma \mapsto \omega_\eta \circ \sigma$ is norm continuous.

Since H is separable, another application of Baire's category theorem gives a point $\pi \in \mathcal{X}$ at which the vector-functional maps associated with all vectors in a fixed countable dense subset of H are norm continuous. For arbitrary $\eta, \zeta \in H$ and $\sigma \in \mathcal{X}$, we have

$$\|\omega_\eta \circ \sigma - \omega_\zeta \circ \sigma\| \leq (\|\eta\| + \|\zeta\|)\|\eta - \zeta\|. \tag{14}$$

The estimate follows from the contractivity of σ and the Cauchy–Schwarz inequality. Since the right-hand side of (14) is independent of σ, approximation by vectors in the countable dense subset shows that $\sigma \mapsto \omega_\eta \circ \sigma$ is norm continuous at π for every $\eta \in H$.

We next prove that L is a factor. For this part of the proof, take ρ to be the countably infinite amplification of the standard representation of P, as in Lemma 2.4, with H, R and $\mathcal{X}$ defined accordingly. The construction in Section 2 ensures that $\mathcal{X}$ is nonempty.

Suppose that L is not a factor. Then there exists a central projection $z \in Z(L)$ such that $z \neq 0$ and $c - z \neq 0$. Choose nonzero normal cyclic representations of Lz and $L(c-z)$, respectively. By composing with the central compression maps, regard both as representations of L. Their representation spaces are separable because A is separable and Ac is weak-$*$ dense in L.

After countably infinite amplification and unitary identification of their restrictions to P with ρ, we obtain two representations $\sigma_1, \sigma_2 \in \mathcal{X}$. Their normal extensions to L are supported on z and $c - z$, respectively. By Lemma 2.4, both unitary orbits

$$\{u\sigma_1(\cdot)u^* : u \in \mathcal{U}(R)\} \quad \text{and} \quad \{u\sigma_2(\cdot)u^* : u \in \mathcal{U}(R)\}$$

are dense in $\mathcal{X}$.

Let $\pi \in \mathcal{X}$ be a common point of norm continuity of the vector-functional maps, and fix a unit vector $\eta \in H$. The density of the first orbit and norm continuity at π imply that $\omega_\eta \circ \pi$, regarded as a normal state of L, is a norm limit of normal states supported on z. Similarly, the density of the second orbit implies that the same state is a norm limit of normal states supported on $c - z$.

However, any normal state supported on z and any normal state supported on $c - z$ have norm distance 2, as is seen by evaluating their difference at the self-adjoint unitary $2z - c$ of L. This contradiction proves that L is a factor. □

We now consider the possible types of P, beginning with semifinite factors and factors admitting a semifinite discrete decomposition. By Connes' decomposition theorem [Con73, Théorèmes 4.4.1(b) and 5.3.1], every factor with separable predual of type III_λ, $0 \leq \lambda < 1$, can be written as a crossed product $T \rtimes \mathbb{Z}$, where T is a semifinite von Neumann algebra.

Lemma 3.2. *Suppose that L_* is separable. If P is semifinite or if $P = T \rtimes \mathbb{Z}$ for a semifinite von Neumann algebra T, then $L = P$.*

Proof. Step 1. Existence of invariant state extensions. Let $T \subseteq P$ be a semifinite von Neumann subalgebra admitting a faithful normal conditional expectation $E : P \to T$. We first construct a normal unital completely positive map $\Psi : L \to T$ extending E.

Fix a faithful normal semifinite trace on T. For a nonzero projection $e \in T$ of finite trace, let τ_e denote the normalized trace on eTe. Define a normal state γ of P by $\gamma(x) = \tau_e(eE(x)e)$ for $x \in P$. Since E is T-bimodular and τ_e is tracial, γ is invariant under conjugation by every unitary of the form $u + c - e$, where $u \in \mathcal{U}(eTe)$.

Conjugation by the unitaries $u + c - e$ therefore induces an action on F_γ by norm isometries. We claim that each of the corresponding transformations is also continuous in the compact topology of F_γ.

Fix $u \in \mathcal{U}(eTe)$ and write $v = u + c - e$. Since v belongs to the unit ball of $B^{**}c$, Kaplansky's density theorem gives a net of contractions $b_i \in B$ such that $b_i c \to v$ in the σ-strong topology of P. For every $\alpha \in F_\gamma$ and $a \in A$, the Cauchy–Schwarz inequality gives

$$|\alpha(v^*acv) - \alpha(b_i^*ab_ic)| \leq 2\|a\|\gamma\big((v - b_ic)^*(v - b_ic)\big)^{1/2}. \tag{15}$$

Indeed, writing

$$v^*acv - b_i^*ab_ic = (v - b_ic)^*acv + (b_ic)^*ac(v - b_ic),$$

we apply the Cauchy–Schwarz inequality to each summand and use $\alpha|_P = \gamma$, together with the contractivity of v and b_ic.

By σ-strong convergence, the right-hand side of (15) tends to zero independently of $\alpha \in F_\gamma$. For each i, the map $\alpha \mapsto \alpha(b_i^*ab_ic)$ is continuous in the compact topology of F_γ. Thus $\alpha \mapsto \alpha(v^*acv)$ is a uniform limit of continuous functions and is therefore continuous. Since $a \in A$ was arbitrary, conjugation by v acts continuously on F_γ.

We next show that F_γ contains a state invariant under conjugation by $u + c - e$ for every $u \in \mathcal{U}(eTe)$. By compactness and Zorn's lemma, there exists a minimal nonempty compact invariant subset K of F_γ.

Since L_* is separable, K is separable. Moreover, by (13), the intersection of a closed norm ball in L_* with K is closed in the compact topology of K. For every $\varepsilon > 0$, cover K by countably many closed norm balls of radius $\varepsilon/2$. Baire's category theorem implies that at least one of the balls contains a nonempty relatively open subset of K. In particular, K contains a nonempty relatively open subset of norm diameter at most ε.

By minimality of K, the translates of the relatively open subset under the action of $\mathcal{U}(eTe)$ cover K. Compactness gives a finite subcover. Since the action is by norm isometries, each translate has norm diameter at most ε. Hence K is norm totally bounded. The set K is also norm closed in L_*, because norm convergence implies convergence in the compact topology of F_γ. Therefore K is norm compact.

By the Arzelà–Ascoli theorem, the group of surjective isometries of the compact metric space K is compact in the topology of uniform convergence. In particular, the closure of the group induced by $\mathcal{U}(eTe)$ is a compact group. Averaging the orbit of any point of K with respect to the normalized Haar measure on this compact group gives an invariant state $\alpha \in F_\gamma$. The average belongs to F_γ because F_γ is norm closed and convex.

Step 2. Extension of the conditional expectation. Let $\alpha \in F_\gamma$ be an invariant state constructed in Step 1. Since $\alpha(e) = \gamma(e) = 1$, the state α is supported on e. Invariance under conjugation by $u + c - e$, for every $u \in \mathcal{U}(eTe)$, implies that eTe is contained in the centralizer of $\alpha|_{eLe}$. Moreover, $\alpha|_{eTe} = \tau_e$.

Consider the normal GNS representation of eLe associated with $\alpha|_{eLe}$, and let ξ be its cyclic unit vector. The cyclic subspace generated by eTe is naturally identified with $L^2(eTe, \tau_e)$.

For each unitary $u \in \mathcal{U}(eTe)$, right multiplication by u on the dense subspace of vectors $x\xi$, $x \in eLe$, is isometric. Indeed,

$$\|xu\xi\|^2 = \alpha(u^*x^*xu) = \alpha(x^*x) = \|x\xi\|^2.$$

Right multiplication by u therefore extends to a unitary on the GNS space. The resulting right multiplication operators commute with the left representation of eLe and leave $L^2(eTe, \tau_e)$ invariant.

Since the unitaries of eTe linearly span eTe, the compression of the left representation of eLe to $L^2(eTe, \tau_e)$ commutes with the right representation of eTe. The commutant of the right representation is precisely the left representation of eTe. Identifying eTe with its left representation, the compression defines a normal unital completely positive map $\Psi_e : eLe \to eTe$.

For $x \in ePe$ and $d_1, d_2 \in eTe$, we have

$$\begin{aligned}\langle \Psi_e(x)d_1, d_2\rangle &= \alpha(d_2^* x d_1)\\ &= \tau_e\big(E(d_2^* x d_1)\big)\\ &= \tau_e(d_2^* E(x) d_1),\end{aligned}$$

where the last equality follows from the bimodularity of E. Since eTe is dense in $L^2(eTe, \tau_e)$, it follows that $\Psi_e(x) = E(x)$ for every $x \in ePe$. Equivalently,

$$\Psi_e(exe) = eE(x)e, \qquad x \in P. \tag{16}$$

Let e range over an increasing net of finite-trace projections in T converging strongly to c. For each e, the map $a \mapsto \Psi_e(eace)$ is a completely positive contraction from A into T. By (16),

$$\Psi_e(ebce) = eE(bc)e, \qquad b \in B. \tag{17}$$

Moreover, $\Psi_e(e) = e$.

The set of completely positive contractions from A into T is compact in the topology of pointwise ultraweak convergence. Passing to a convergent subnet, we obtain a completely positive contraction from A into T. Since $e \to c$ strongly, the limiting map takes 1_A to c. It is therefore unital. By (17), its restriction to B is $b \mapsto E(bc)$.

By duality with the predual T_*, the limiting map extends uniquely to a normal unital completely positive map from A^{**} into T. The restriction of the normal extension to B^{**} agrees with $x \mapsto E(xc)$, by weak-$*$ density of B in B^{**} and normality of E.

In particular, the normal extension sends the projection c to the identity of T. Thus c belongs to its multiplicative domain, and the normal extension vanishes on $A^{**}(1_A - c)$. It consequently factors through $L = A^{**}c$. We have therefore constructed a normal unital completely positive map $\Psi : L \to T$ satisfying

$$\Psi(x) = E(x), \qquad x \in P. \tag{18}$$

Step 3. Construction of a retraction onto P. If P is semifinite, take $T = P$ and let E be the identity map on P. By (18), the map $a \mapsto \Psi(ac)$ is a unital completely positive map from A into P taking b to bc for every $b \in B$. Lemma 2.3 gives $L = P$.

It remains to consider the crossed-product case $P = T \rtimes \mathbb{Z}$. Let $E : P \to T$ be the canonical faithful normal conditional expectation, and let $u \in P$ denote the implementing unitary. For each positive integer n, define

$$\Phi_n(x) = \frac{1}{n} \sum_{i,j=0}^{n-1} u^i \Psi(u^{-i} x u^j) u^{-j}, \qquad x \in L. \tag{19}$$

The map $\Phi_n : L \to P$ is completely positive. Indeed, the map

$$x \mapsto [u^{-i} x u^j]_{i,j=0}^{n-1}$$

is completely positive from L into $M_n(L)$. Applying the matrix amplification of Ψ and compressing by the row operator $\frac{1}{\sqrt{n}}(c, u, \dots, u^{n-1})$ gives precisely the map in (19).

By (18), $\Psi(u^{j-i}) = E(u^{j-i})$. Since $E(u^{j-i}) = 0$ for $i \neq j$ and $E(c) = c$, we obtain $\Phi_n(c) = c$. Hence Φ_n is unital and completely positive. It is also normal because Ψ is normal.

For every $t \in T$ and $k \in \mathbb{Z}$, a direct computation using (19) and (18) gives

$$\Phi_n(tu^k) = \max\left\{1 - \frac{|k|}{n}, 0\right\} tu^k. \tag{20}$$

Indeed, the conditional expectation of $u^{-i} t u^{k+j}$ vanishes unless $i = k + j$. For each pair (i, j) satisfying this relation, the corresponding summand in (19) equals tu^k. The number of such pairs is $\max\{n - |k|, 0\}$.

Let $\widehat{\theta}$ denote the dual action of $\mathbb{T}$ on P, so that $\widehat{\theta}_z(tu^k) = z^k tu^k$. Set $K_n(z) = n^{-1}|1 + z + \cdots + z^{n-1}|^2$. By (20),

$$\Phi_n(x) = \int_{\mathbb{T}} K_n(z) \widehat{\theta}_z(x)\, dz, \qquad x \in P,$$

where dz denotes normalized Haar measure. Indeed, both sides define normal maps and agree on the algebraic crossed product. For every $x \in P$ and $\omega \in P_*$, the function $z \mapsto \omega(\widehat{\theta}_z(x))$ is continuous. Fejér's theorem therefore gives $\omega(\Phi_n(x)) \to \omega(x)$. Thus $\Phi_n(x) \to x$ ultraweakly for every $x \in P$.

Finally, consider the unital completely positive maps $a \mapsto \Phi_n(ac)$ from A into P. By pointwise ultraweak compactness, there is a cluster point $F : A \to P$, which is again unital and completely positive. Since $bc \in P$ for every $b \in B$, the ultraweak convergence of $\Phi_n(bc)$ to bc gives $F(b) = bc$. Lemma 2.3 therefore yields $L = P$. □

For type III_1 factors, we use the homogeneity theorem of Connes and Størmer [CS78], which asserts that the inner orbit of every faithful normal state of a type III_1 factor with separable predual is norm dense in its faithful normal state space. We shall also use the Araki–Powers–Størmer inequality [Ara74, Theorem 4(8)] (see also [Haa75, Lemma 2.10(2)]):

$$\|\xi_\gamma - \xi_\delta\|^2 \leq \|\gamma - \delta\|, \tag{21}$$

where γ and δ are normal states of a von Neumann algebra and ξ_γ and ξ_δ are their representing vectors in the natural cone of its standard form.

Lemma 3.3. *If P is of type III_1 and $P \subsetneq L$, then $\operatorname{diam} F_\delta = 2$ for every normal state δ of P, where the diameter is taken with respect to the norm topology.*

Proof. Since P is a proper ultraweakly closed subalgebra of L, the Hahn–Banach theorem gives a nonzero self-adjoint normal functional on L vanishing on P. The positive and negative parts of its Jordan decomposition have orthogonal supports and equal norms, because P and L have a common identity. Normalizing both parts, we obtain two normal states α_+ and α_- of L satisfying $\alpha_+|_P = \alpha_-|_P$ and $\|\alpha_+ - \alpha_-\| = 2$.

Choose a faithful normal state γ of P. The state $b \mapsto \gamma(bc)$ admits a state extension to A. By the identification of F_γ established above, there exists a normal state $\alpha \in F_\gamma$. For each $0 < \varepsilon < 1$, the normal states $(1 - \varepsilon)\alpha_+ + \varepsilon\alpha$ and $(1 - \varepsilon)\alpha_- + \varepsilon\alpha$ have a common faithful restriction to P, and their norm distance is $2(1 - \varepsilon)$. Consequently,

$$\sup_{\substack{\gamma \text{ faithful normal} \\ \text{state of } P}} \operatorname{diam} F_\gamma = 2. \tag{22}$$

We next compare the diameters of the extension sets corresponding to different normal states of P. Let γ be a faithful normal state of P, let δ be an arbitrary normal state of P, and fix $\varphi \in F_\gamma$.

Consider the normal GNS representation of L associated with φ. Since $\varphi|_P = \gamma$, the cyclic subspace generated by P and the GNS vector of φ is unitarily equivalent to the GNS representation space of γ. As γ is faithful, its natural cone vector ξ_γ is cyclic for the standard representation of P. We may therefore identify the cyclic subspace with $L^2(P)$.

Compressing the GNS representation of L to this subspace gives a normal unital completely positive map $\Phi : L \to \mathcal{B}(L^2(P))$ whose restriction to P is the standard representation of P. Moreover, $\varphi(x) = \langle \Phi(x)\xi_\gamma, \xi_\gamma \rangle$ for every $x \in L$.

Define a normal state ψ of L by $\psi(x) = \langle \Phi(x)\xi_\delta, \xi_\delta \rangle$ for $x \in L$. Since the restriction of Φ to P is the standard representation, we have $\psi|_P = \delta$, so $\psi \in F_\delta$. By the contractivity of Φ, the Cauchy–Schwarz inequality and (21), we obtain

$$\begin{aligned} \|\varphi - \psi\| &\le (\|\xi_\gamma\| + \|\xi_\delta\|)\|\xi_\gamma - \xi_\delta\| \\ &\le 2\|\gamma - \delta\|^{1/2}. \end{aligned} \tag{23}$$

Applying (23) separately to two arbitrary members of F_γ and using the triangle inequality, we obtain

$$\operatorname{diam} F_\delta \ge \operatorname{diam} F_\gamma - 4\|\gamma - \delta\|^{1/2}. \tag{24}$$

For every $u \in \mathcal{U}(P)$, conjugation by u induces a surjective isometry from F_γ onto the extension set corresponding to the state $x \mapsto \gamma(u^* x u)$ of P. In particular,

$$\operatorname{diam} F_{\gamma(u^* \cdot u)} = \operatorname{diam} F_\gamma. \tag{25}$$

Suppose first that δ is faithful. For any faithful normal state γ of P, the Connes–Størmer homogeneity theorem [CS78] provides a sequence $u_n \in \mathcal{U}(P)$ such that $\gamma(u_n^* \cdot u_n) \to \delta$ in norm. Combining (24) with (25) and letting $n \to \infty$, we obtain $\operatorname{diam} F_\delta \ge \operatorname{diam} F_\gamma$. Since γ was arbitrary, (22) gives $\operatorname{diam} F_\delta = 2$.

Finally, every normal state of P is a norm limit of faithful normal states, as one sees by taking convex combinations with a fixed faithful normal state. Applying (24) to such faithful normal states gives $\operatorname{diam} F_\delta = 2$ for every normal state δ of P. □

We recall a consequence of the bicentralizer theorem that will be used in the proof of Lemma 3.4. Let M be a type III_1 factor with separable predual. By Houdayer and Marrakchi [HM26, Theorem C], M has trivial bicentralizer. Haagerup's theorem [Haa87, Theorem 3.1] then gives a faithful normal state φ such that $M_\varphi' \cap M = \mathbb{C}1$. The centralizer M_φ is a finite factor and admits a φ-preserving faithful normal conditional expectation from M. By Popa's theorem [Pop81, Theorem 3.2], there exists a MASA D of M contained in M_φ. Composing the conditional expectation onto M_φ with the trace-preserving conditional expectation from M_φ onto D, we obtain a faithful normal conditional expectation from M onto D. For the relationship between the bicentralizer and the existence of subalgebras with expectation, see also [AHHM20, Introduction].

Lemma 3.4. *If P is of type III_1 and L_* is separable, then $L = P$.*

Proof. Suppose that $P \subsetneq L$. Throughout this proof, take $\rho : P \to \mathcal{B}(H)$ to be the standard representation on $H = L^2(P)$, and let $R = \rho(P)'$ and $\mathcal{X}$ be the corresponding commutant and extension space.

Every nonzero normal representation of P on a separable Hilbert space is unitarily equivalent to ρ. Indeed, every such representation can be realized as a corner of the countably infinite amplification of the standard representation. Since the commutant of the amplified standard representation is a type III factor acting on a separable Hilbert space, all its nonzero projections are Murray–von Neumann equivalent. In particular, the projection determining the given representation is equivalent to the projection determining a single copy of the standard representation.

Step 1. A MASA with a faithful normal conditional expectation at a continuity point. Since R is also a type III_1 factor with separable predual, the consequence of the bicentralizer theorem established before Lemma 3.4 gives a MASA $D_0 \subseteq R$ admitting a faithful normal conditional expectation from R.

The standard representation ρ is cyclic. Since Bc is weak-$*$ dense in P, every cyclic vector for $\rho(P)$ is also cyclic for the representation $b \mapsto \rho(bc)$ of B. By the representation extension theorem of Anderson and Bunce [AB81, Theorem 6], there exists $\pi_0 \in \mathcal{X}$ such that $\pi_0(A) \subseteq D_0'$. In particular, $\mathcal{X}$ is nonempty.

By Lemma 3.1, L is a factor, and there exists a representation $\pi \in \mathcal{X}$ at which the map $\sigma \mapsto \omega_\eta \circ \sigma$ is norm continuous for every $\eta \in H$. We claim that $\pi(A)'$ contains a MASA D of R admitting a faithful normal conditional expectation from R.

Since both π and π_0 factor normally through L, we use the same notation for the induced normal representations of L. Both representations are faithful because L is a factor.

The countably infinite amplification of π_0 is unitarily equivalent to the countably infinite amplification of the standard representation of L. On the other hand, every normal representation of L on a separable

Hilbert space can be realized as a corner of the amplified standard representation. Consequently, π is unitarily equivalent to the restriction of the countably infinite amplification of π_0 to $q(H \otimes \ell^2)$, where q is a nonzero projection in

$$Q = \pi_0(A)' \bar{\otimes} \mathcal{B}(\ell^2).$$

The algebra Q is a properly infinite factor.

Let D_∞ be the spatial tensor product of D_0 and the diagonal MASA of $\mathcal{B}(\ell^2)$. By (6), we have $\pi_0(A)' \subseteq \pi_0(B)' = R$. Since $D_0 \subseteq \pi_0(A)'$ and D_0 is a MASA of R, it is also a MASA of $\pi_0(A)'$. It follows that D_∞ is a MASA of both Q and $R \bar{\otimes} \mathcal{B}(\ell^2)$.

Taking the spatial tensor product of the faithful normal conditional expectation from R onto D_0 and the canonical diagonal expectation on $\mathcal{B}(\ell^2)$ gives a faithful normal conditional expectation from $R \bar{\otimes} \mathcal{B}(\ell^2)$ onto D_∞. Its restriction to Q is a faithful normal conditional expectation from Q onto D_∞.

We now show that there exists a projection $d \in D_\infty$ Murray–von Neumann equivalent to q in Q.

If Q is of type III, any nonzero projection in D_∞ is equivalent to q.

The factor Q cannot be of type I. Indeed, if Q were of type I, then $\pi_0(A)''$ would also be a type I factor. Thus π_0 would be a multiple of an irreducible representation of A. By (2), its restriction to B would be a multiple of an irreducible representation of B, and hence $\pi_0(B)''$ would be of type I. This contradicts the equality $\pi_0(B)'' = \rho(P)$, since P is of type III.

It remains to consider the case where Q is of type II_∞. Let τ be a faithful normal semifinite trace on Q. Composing the conditional expectation from Q onto D_∞ with a faithful normal state of D_∞ gives a faithful normal state of Q whose centralizer contains D_∞.

Let h be the density of this state with respect to τ. Since the state centralizes D_∞, the uniqueness of the trace density implies that h commutes with every unitary in D_∞. As D_∞ is a MASA of Q, the spectral projections of h belong to D_∞. Furthermore, faithfulness of the state implies that the spectral projections $1_{[1/n,\infty)}(h)$, $n \geq 1$, increase strongly to the identity. They have finite trace, because

$$\tau\big(1_{[1/n,\infty)}(h)\big) \leq n\tau(h) = n.$$

Thus the restriction of τ to D_∞ is semifinite.

Moreover, D_∞ is diffuse. Otherwise, a nonzero minimal projection of D_∞ would also be minimal in Q, by the maximal abelianness of D_∞, contrary to the fact that Q is of type II. The restriction of τ to D_∞ is therefore a diffuse faithful normal semifinite trace.

If $\tau(q) < \infty$, the diffuse trace on D_∞ provides a projection $d \in D_\infty$ satisfying $\tau(d) = \tau(q)$. If $\tau(q) = \infty$, we take $d = I$. In both cases, comparison of projections in the semifinite factor Q shows that d and q are Murray–von Neumann equivalent.

Choose a partial isometry $w \in Q$ such that $w^*w = q$ and $ww^* = d$. Since w commutes with the amplified representation of A, it implements a unitary equivalence between the restrictions of that representation to $q(H \otimes \ell^2)$ and $d(H \otimes \ell^2)$. The equivalence also intertwines the corresponding representations of P.

Since $d \in D_\infty$, compressing the conditional expectation from $R \bar{\otimes} \mathcal{B}(\ell^2)$ onto D_∞ gives a faithful normal conditional expectation

$$d(R \bar{\otimes} \mathcal{B}(\ell^2))d \longrightarrow dD_\infty d.$$

The algebra $dD_\infty d$ is a MASA of the domain of the compressed expectation. Moreover, it is contained in the commutant of the restriction of the amplified representation of A to $d(H \otimes \ell^2)$.

Transporting the compressed expectation to H through the unitary equivalence with π, we obtain a MASA D of R satisfying

$$D \subseteq \pi(A)', \qquad \text{there exists a faithful normal conditional expectation } R \to D. \tag{26}$$

The claim is proved.

Step 2. Approximation of normal state extensions by vector states. Let δ be a normal state of P, and let π_δ be its GNS representation on the Hilbert space H_δ, with cyclic unit vector ζ. We show that the vector states at ζ of normal representations of L extending π_δ are norm dense in F_δ.

Fix $\alpha \in F_\delta$ and consider its normal GNS representation. Since A is separable and Ac is weak-$*$ dense in L, the GNS representation space of α is separable. Its restriction to P is a nonzero normal representation of P, and hence is unitarily equivalent to π_δ.

Transporting the GNS representation of α to H_δ, we obtain a normal representation σ of L extending π_δ and a unit vector $\xi \in H_\delta$ such that

$$\alpha(x) = \langle \sigma(x)\xi, \xi \rangle, \qquad x \in L.$$

Since $\alpha|_P = \delta$, the vectors ξ and ζ give the same vector state on $\pi_\delta(P)$.

The linear map

$$\pi_\delta(x)\zeta \longmapsto \pi_\delta(x)\xi, \qquad x \in P,$$

is therefore well defined and isometric. By cyclicity of ζ, it extends to an isometry $v \in \pi_\delta(P)'$ satisfying $v\zeta = \xi$.

Since $\pi_\delta(P)'$ is a type III factor, v is a strong operator limit of unitaries in $\pi_\delta(P)'$. Indeed, choose an increasing sequence of proper projections $e_n \in \pi_\delta(P)'$ converging strongly to I. The projections $I - e_n$ and $I - ve_nv^*$ are nonzero and hence equivalent in $\pi_\delta(P)'$. Consequently, the partial isometry ve_n extends to a unitary $u_n \in \pi_\delta(P)'$. Since $u_ne_n = ve_n$, we have $\|(u_n - v)\eta\| \le 2\|(I - e_n)\eta\|$ for every $\eta \in H_\delta$. Hence $u_n \to v$ strongly.

For each $n \ge 1$, the normal representation $x \mapsto u_n^*\sigma(x)u_n$ of L extends π_δ, because $u_n \in \pi_\delta(P)'$. Moreover,

$$\begin{aligned}\|\omega_\zeta \circ (u_n^*\sigma(\cdot)u_n) - \alpha\| &= \|\omega_{u_n\zeta} \circ \sigma - \omega_\xi \circ \sigma\| \\ &\le 2\|u_n\zeta - \xi\| \longrightarrow 0.\end{aligned}$$

Since $\alpha \in F_\delta$ was arbitrary, the desired norm density follows.

Step 3. A contradiction to norm continuity. By (26), there exists a faithful normal conditional expectation from R onto D. Composing the expectation with a faithful normal state of D gives a faithful normal state of R whose centralizer contains D.

Let $\eta \in H$ be the unit vector representing this state in the natural cone of the standard form of R. Since the state is faithful, η is cyclic and separating for both R and $\rho(P)$.

Choose an increasing sequence of finite-dimensional unital subalgebras $D_n \subseteq D$ whose union generates D as a von Neumann algebra. At a fixed level n, let $p_1, \dots, p_m$ be the minimal projections of D_n, so that $\sum_{i=1}^m p_i = I$. The dependence of m and the projections p_i on n is suppressed while the level n is fixed.

Let J be the modular conjugation in the standard form. Since $JRJ = \rho(P)$, there exist mutually orthogonal projections $q_1, \dots, q_m \in P$ determined by $\rho(q_i) = Jp_iJ$ for $1 \le i \le m$. Moreover, $\sum_{i=1}^m q_i = c$.

Let Δ be the modular operator associated with R and the cyclic separating vector η. Since p_i belongs to the centralizer of the state represented by η, we have $\Delta^{it}p_i\eta = p_i\eta$ for every $t \in \mathbb{R}$. Consequently, $\Delta^{1/2}p_i\eta = p_i\eta$. The Tomita identity $J\Delta^{1/2}p_i\eta = p_i\eta$, together with $J\eta = \eta$, gives

$$p_i\eta = Jp_iJ\eta = \rho(q_i)\eta, \qquad 1 \le i \le m. \tag{27}$$

Since η is separating for R, we have $p_i\eta \neq 0$ for every $1 \le i \le m$. On p_iH, consider the unit vector $\zeta_i = p_i\eta/\|p_i\eta\|$. Since $\rho(P)\eta$ is dense in H and $p_i \in R$, the vector ζ_i is cyclic for the restriction of ρ to p_iH. By (27), we also have $\rho(q_i)\zeta_i = \zeta_i$. Consequently, the normal state $\omega_{\zeta_i} \circ \rho$ of P is supported on q_i.

By Lemma 3.3, the set of normal state extensions of $\omega_{\zeta_i} \circ \rho$ to L has norm diameter 2. Since ζ_i is cyclic for the restriction of ρ to p_iH, Step 2 allows us to choose two normal representations σ_i^+ and σ_i^- of L on p_iH, both extending the restriction of ρ, such that their vector states $\alpha_i^\pm = \omega_{\zeta_i} \circ \sigma_i^\pm$ satisfy

$$\|\alpha_i^+ - \alpha_i^-\| > 1, \qquad 1 \le i \le m. \tag{28}$$

For each $n \ge 1$, form the direct sums of the representations σ_i^+ and σ_i^- corresponding to the partition determined by D_n. Since $H = \bigoplus_{i=1}^m p_iH$, the direct sums act on H, and both restrict to ρ on P. Composing them with the map $a \mapsto ac$ gives two representations $\pi_n^+, \pi_n^- \in \mathcal{X}$.

By construction, $D_n \subseteq (\pi_n^\pm(A))'$. Since D is a MASA of $\pi(A)'$ by (26), Lemma 2.2 implies that

$$\pi_n^+ \longrightarrow \pi, \qquad \pi_n^- \longrightarrow \pi \tag{29}$$

in the topology of pointwise strong-$*$ convergence.

The orthogonal decomposition of H associated with D_n gives

$$\omega_\eta \circ \pi_n^\pm = \sum_{i=1}^m \|p_i\eta\|^2 \alpha_i^\pm,$$

where the normal states on L are identified with their restrictions to A.

We claim that

$$\|\omega_\eta \circ \pi_n^+ - \omega_\eta \circ \pi_n^-\| = \sum_{i=1}^m \|p_i\eta\|^2 \|\alpha_i^+ - \alpha_i^-\|. \tag{30}$$

Indeed, since α_i^+ and α_i^- extend the normal state $\omega_{\zeta_i} \circ \rho$ of P, we have $\alpha_i^\pm(q_i) = 1$. Consequently, both normal states are supported on q_i, so their difference is supported on the corner q_iLq_i.

The triangle inequality gives the upper bound in (30). For the reverse inequality, for each $1 \le i \le m$, let $s_i \in q_i L q_i$ be the sign of the self-adjoint normal functional $\alpha_i^+ - \alpha_i^-$. Then $\|s_i\| \le 1$ and

$$(\alpha_i^+ - \alpha_i^-)(s_i) = \|\alpha_i^+ - \alpha_i^-\|.$$

Since the projections q_i are mutually orthogonal, the operator $\sum_{i=1}^m s_i$ is a self-adjoint contraction in L. Evaluating the difference of the two vector states at this contraction and using (13), we obtain the reverse inequality in (30).

Combining (28) with (30), and using $\sum_{i=1}^m \|p_i\eta\|^2 = 1$, we obtain

$$\|\omega_\eta \circ \pi_n^+ - \omega_\eta \circ \pi_n^-\| > 1, \qquad n \ge 1. \tag{31}$$

On the other hand, since π is a point of norm continuity of the map $\sigma \mapsto \omega_\eta \circ \sigma$, (29) gives

$$\|\omega_\eta \circ \pi_n^\pm - \omega_\eta \circ \pi\| \longrightarrow 0.$$

The triangle inequality therefore implies that $\|\omega_\eta \circ \pi_n^+ - \omega_\eta \circ \pi_n^-\| \to 0$, contradicting (31). Hence $L = P$. □

We are now in a position to exclude a finite-dimensional center of L. Suppose that $Z(L)$ is finite dimensional. Then L is a finite direct sum of factors. Each factor summand admits a faithful normal cyclic representation: the normal GNS representation associated with any normal state of a factor is faithful. Moreover, the corresponding GNS representation space is separable, because A is separable and Ac is weak-$*$ dense in L.

Taking the direct sum of the faithful normal cyclic representations of the finitely many factor summands, we obtain a faithful normal representation of L on a separable Hilbert space. It follows that L_* is separable.

Since P is a factor of type II or III, Connes' discrete decomposition theorem [Con73, Théorèmes 4.4.1(b) and 5.3.1] and Lemmas 3.2 and 3.4 give $P = L$, contradicting (4). We have therefore proved that

$$\dim Z(L) = \infty. \tag{32}$$

In particular, by (32), the center of L contains a partition of its identity into arbitrarily many nonzero projections.

4. An irreducible extension

Throughout this section, we work with the countably infinite amplification ρ of the standard representation of P on $H = L^2(P) \otimes \ell^2$, as in Lemma 2.4. Let $R = \rho(P)'$ and let $\mathcal{X}$ be the associated representation extension space. Since P is of type II or III, R is a properly infinite continuous factor. Our aim is to construct an irreducible representation in $\mathcal{X}$, whose restriction to B necessarily has commutant R by (6).

By Popa's flatness theorem [Pop21, Lemma 5.2(2) and Theorem 5.5(1)], if Q is a continuous factor with separable predual and D is a MASA of Q, then for every $g \in Q_*$ with $g(1) = 0$ and every $\epsilon > 0$, there exists $u \in \mathcal{U}(Q)$ such that

$$\|g|_{uDu^*}\| < \epsilon. \tag{33}$$

The infinite dimensionality of $Z(L)$, established in (32), allows us to construct representation extensions with prescribed central projections in their commutants.

Lemma 4.1. *Let $1 = \sum_{i=1}^m p_i$ be a partition of the identity in R such that each p_i is Murray–von Neumann equivalent to 1 in R. If $Z(L)$ contains m mutually orthogonal nonzero projections, then there exists $\sigma \in \mathcal{X}$ such that every p_i is central in $\sigma(A)'$.*

Proof. Choose mutually orthogonal nonzero projections $z_1, \dots, z_m \in Z(L)$. For each $1 \le i \le m$, choose a nonzero normal cyclic representation of Lz_i. Its representation space is separable because A is separable and Acz_i is weak-$*$ dense in Lz_i. Composing with the central compression $x \mapsto xz_i$ gives a nonzero normal representation of L.

Since P is a factor, the restriction of this representation to P is faithful. By the amplification property established in Section 2, the restriction to P of its countably infinite amplification is unitarily equivalent to ρ. Moreover, since p_i is equivalent to 1 in $R = \rho(P)'$, the restriction of ρ to p_iH is unitarily equivalent to ρ.

We may therefore transport the amplified representation of Lz_i to a normal representation σ_i of L on p_iH satisfying $\sigma_i(x) = \rho(x)|_{p_iH}$ for every $x \in P$. By construction, $\sigma_i(z_i)$ is the identity on p_iH, whereas $\sigma_i(z_j) = 0$ whenever $j \ne i$.

Define a representation σ of A on $H = \bigoplus_{i=1}^m p_iH$ by

$$\sigma(a) = \bigoplus_{i=1}^m \sigma_i(ac), \qquad a \in A.$$

For every $b \in B$, we have

$$\sigma(b) = \bigoplus_{i=1}^{m} \rho(bc)|_{p_i H} = \rho(bc).$$

Hence $\sigma \in \mathcal{X}$.

Let $\widetilde{\sigma} = \bigoplus_{i=1}^{m} \sigma_i$ be the resulting normal representation of L. Since Ac is ultraweakly dense in L, we have $\widetilde{\sigma}(L) = \sigma(A)''$. By construction, $\widetilde{\sigma}(z_i) = p_i$ for every $1 \le i \le m$. Since $z_i \in Z(L)$, it follows that

$$p_i \in Z(\sigma(A)'') = Z(\sigma(A)'), \qquad 1 \le i \le m.$$

□

Proposition 4.2. *Let $B \subseteq A$ be C^*-algebras with a common identity. If A is separable and B separates $P(A)$, then $A = B$.*

Proof. Suppose, towards a contradiction, that $B \subsetneq A$. Let $P \subsetneq L$ be the factor inclusion in (4). We work with the countably amplified standard representation ρ of P fixed at the beginning of this section.

Step 1. Openness of the commutant condition. We show that, for every $g \in R_*$ satisfying $g(1) = 0$ and every $\epsilon > 0$, the set

$$\{\pi \in \mathcal{X} : \|g|_{\pi(A)'}\| < \epsilon\} \tag{34}$$

is open in $\mathcal{X}$.

Since $\mathcal{X}$ is Polish, it suffices to show that the complement of (34) is sequentially closed. Suppose that $\pi_j \to \pi$ in $\mathcal{X}$ and that $\|g|_{\pi_j(A)'}\| \ge \epsilon$ for every $j \ge 1$. By (6), we have $\pi_j(A)' \subseteq R$. The unit ball of $\pi_j(A)'$ is weak operator compact, and g, being normal, is weak operator continuous on norm-bounded subsets of R. Hence there exists a contraction $x_j \in \pi_j(A)'$ satisfying $|g(x_j)| \ge \epsilon$.

Passing to a subsequence, we may assume that x_j converges in the weak operator topology to a contraction $x \in R$, since H is separable. For every $a \in A$, the definition of the topology of $\mathcal{X}$ gives $\pi_j(a) \to \pi(a)$ in the strong-$*$ topology. Since $x_j \pi_j(a) = \pi_j(a) x_j$, and multiplication of a bounded weakly convergent net by a bounded strong-$*$ convergent net on either side preserves weak operator convergence, we obtain $x\pi(a) = \pi(a)x$. Thus $x \in \pi(A)'$.

Normality of g gives

$$\|g|_{\pi(A)'}\| \ge |g(x)| = \lim_j |g(x_j)| \ge \epsilon.$$

Therefore the complement of (34) is closed, which proves that the set in (34) is open.

Step 2. Density via flatness and disjoint central summands. We now prove that the set in (34) is dense in $\mathcal{X}$. Let V be a nonempty open subset of $\mathcal{X}$.

Recall that the construction of (4) provides a nonzero normal factor representation of L on a separable Hilbert space. By Lemma 2.4, there exists $\pi \in V$ unitarily equivalent to the countably infinite amplification of this factor representation, regarded as a representation of A.

The commutant $Q = \pi(A)'$ is a properly infinite factor with separable predual. Moreover, Q cannot be of type I. Indeed, if Q were of type I, then $\pi(A)''$ would also be a type I factor, and π would be a multiple of an irreducible representation of A. By (2), its restriction to B would be a multiple of an irreducible representation of B. Consequently, $\pi(B)''$ would be of type I, contrary to the fact that $\pi(B)'' = \rho(P)$ is of type II or III. Thus Q is a continuous factor.

Since π is a countably infinite amplification, it is unitarily equivalent to its own countably infinite amplification. The product-MASA construction in the proof of Lemma 2.4 therefore gives a MASA D of Q such that every nonzero projection in D is Murray–von Neumann equivalent to 1 in R. Here the unitary equivalence used to transport the product MASA to H also intertwines the corresponding representations of P.

Applying (33) to the normal functional $g|_Q$, we may replace D by a unitary conjugate inside Q and assume that

$$\|g|_D\| < \epsilon. \tag{35}$$

Since $Q \subseteq R$, conjugation by a unitary in Q preserves the equivalence of every nonzero projection in D to 1 in R.

By Anderson and Bunce [AB81, Corollary 1], D is also a MASA of $\pi(B)' = R$. Choose an increasing sequence of finite-dimensional unital subalgebras $D_n \subseteq D$ whose union generates D as a von Neumann

algebra. Since D is a MASA of R, we have

$$\bigcap_{n=1}^{\infty}(R\cap D_n')=R\cap D'=D.$$

The unit balls of $R\cap D_n'$ form a decreasing sequence of weak operator compact sets whose intersection is the unit ball of D. Since g is weak operator continuous on the unit ball of R, it follows that

$$\lim_{n\to\infty}\|g|_{R\cap D_n'}\|=\|g|_D\|. \tag{36}$$

Indeed, the restriction norms form a decreasing sequence bounded below by $\|g|_D\|$. For each $n\geq 1$, choose a contraction in $R\cap D_n'$ at which the corresponding restriction norm is attained. Every weak operator cluster point of the resulting sequence belongs to the unit ball of D. Evaluating g at such a cluster point shows that the limit cannot exceed $\|g|_D\|$, proving (36).

By (35) and (36), we have $\|g|_{R\cap D_n'}\|<\epsilon$ for all sufficiently large n. Moreover, since $\pi\in V$, Lemma 2.2 ensures that every representation in $\mathcal{X}$ commuting with D_n belongs to V for all sufficiently large n. We may therefore fix $n\geq 1$ such that

$$\begin{aligned}&\|g|_{R\cap D_n'}\|<\epsilon,\\ &\{\sigma\in\mathcal{X}:D_n\subseteq\sigma(A)'\}\subseteq V.\end{aligned} \tag{37}$$

Let $p_1,\dots,p_m$ be the minimal projections of D_n. Every p_i is nonzero and belongs to D, so each p_i is equivalent to 1 in R. By (32), $Z(L)$ contains m mutually orthogonal nonzero projections. Lemma 4.1 therefore provides a representation $\sigma\in\mathcal{X}$ such that $p_i\in Z(\sigma(A)')$ for every $1\leq i\leq m$.

Since D_n is generated by $p_1,\dots,p_m$, we obtain

$$D_n\subseteq Z(\sigma(A)').$$

In particular, $D_n\subseteq\sigma(A)'$, and hence $\sigma\in V$ by (37).

Furthermore, every operator in $\sigma(A)'$ commutes with D_n. Together with $\sigma(A)'\subseteq\sigma(B)'=R$, this gives $\sigma(A)'\subseteq R\cap D_n'$. Consequently, (37) implies that

$$\|g|_{\sigma(A)'}\|\leq\|g|_{R\cap D_n'}\|<\epsilon.$$

Thus $\sigma\in V$ and σ belongs to the set in (34). Since V was arbitrary, the set in (34) is dense in $\mathcal{X}$.

Step 3. Baire category and irreducibility. Since R acts on a separable Hilbert space, its predual R_* is separable. The closed hyperplane $\{g\in R_*:g(1)=0\}$ is therefore separable. Choose a countable norm-dense subset $\{g_j:j\geq 1\}$ of this hyperplane.

By Steps 1 and 2, for every pair of positive integers j,k, the set

$$\{\pi\in\mathcal{X}:\|g_j|_{\pi(A)'}\|<1/k\}$$

is open and dense in $\mathcal{X}$. Since $\mathcal{X}$ is a nonempty Polish space, Baire's category theorem gives a representation $\pi\in\mathcal{X}$ satisfying

$$\|g_j|_{\pi(A)'}\|<1/k,\qquad j,k\geq 1.$$

Letting $k\to\infty$, we obtain $g_j|_{\pi(A)'}=0$ for every $j\geq 1$. By norm density of $\{g_j:j\geq 1\}$, it follows that

$$g|_{\pi(A)'}=0\quad\text{for every } g\in R_* \text{ with } g(1)=0. \tag{38}$$

We now show that $\pi(A)'=\mathbb{C}I$. Fix a normal state φ of R. For any $g\in R_*$, the normal functional $g-g(1)\varphi$ vanishes at the identity. By (38), for every $x\in\pi(A)'$, we have

$$g(x)=g(1)\varphi(x)=g(\varphi(x)I).$$

Since R_* separates the points of R and $\pi(A)'\subseteq R$, we conclude that $x=\varphi(x)I$. Hence $\pi(A)'=\mathbb{C}I$, and π is irreducible.

On the other hand, (6) gives

$$\pi(B)'=R\neq\mathbb{C}I,$$

because R is a continuous factor. This contradicts (2). We conclude that $A=B$. □

5. A separable Banach quotient

We now drop the separability assumptions on A and B, retaining the common identity until the unitization argument. To apply Proposition 4.2, we need a separable unital C^*-subalgebra $C \subseteq A$ such that $C \cap B$ separates $P(C)$. The difficulty is that the restriction of a pure state to a subalgebra need not be pure.

By (3), B has the pure extension property in A. Since $1_A \in B$, the algebra B separates $P(A) \cup \{0\}$. Let z_B and z_A be the atomic central projections of B^{**} and A^{**}, respectively. By Bunce and Chu [BC98, Remark 4.8(a)], specifically the implication (iii)⇒(ii), we have $B^{**}z_B = A^{**}z_A$. Since the identities of these two summands coincide, $z_B = z_A =: z_{\mathrm{at}}$. In particular,

$$B^{**}z_{\mathrm{at}} = A^{**}z_{\mathrm{at}}. \tag{39}$$

Lemma 5.1. *Let $B \subseteq A$ be C^*-algebras with a common identity. Suppose that A/B is separable and that B separates $P(A)$. Then there exists a separable unital C^*-subalgebra $C \subseteq A$ such that $C \cap B$ separates $P(C)$ and the natural map $C/(C \cap B) \to A/B$ is a surjective linear isometry.*

Proof. Let $B^\perp = \{h \in A^* : h|_B = 0\}$ be the annihilator of B. We identify $B^\perp$ with the space of normal functionals on A^{**} vanishing on B^{**}. For every self-adjoint $h \in B^\perp$, write $h = h_+ - h_-$ for its Jordan decomposition and set $|h| = h_+ + h_-$.

Let e be the supremum of the support projections of $|h|$ as h ranges over the self-adjoint elements of $B^\perp$. We first establish that

$$e \in Z(A^{**}) \cap B^{**}, \qquad ez_{\mathrm{at}} = 0. \tag{40}$$

For every $u \in \mathcal{U}(B^{**})$, conjugation by u preserves the self-adjoint part of $B^\perp$. Since the support projection of the absolute value of a self-adjoint normal functional transforms under conjugation by the corresponding unitary, the family of support projections defining e is invariant under conjugation by u. Hence $ueu^* = e$ for every $u \in \mathcal{U}(B^{**})$, which implies that e commutes with B^{**}.

By the definition of e, every self-adjoint element of $B^\perp$ vanishes on both $(1-e)A^{**}$ and $A^{**}(1-e)$. Since every element of $B^\perp$ is a complex linear combination of self-adjoint elements of $B^\perp$, both spaces are annihilated by $B^\perp$. The annihilator of $B^\perp$ in A^{**} is precisely B^{**}. Consequently,

$$(1-e)A^{**} \subseteq B^{**}, \qquad A^{**}(1-e) \subseteq B^{**}.$$

In particular, $1 - e \in B^{**}$, so $e \in B^{**}$.

Since e commutes with B^{**}, the inclusion $(1-e)A^{**} \subseteq B^{**}$ gives $(1-e)xe = e(1-e)x = 0$ for every $x \in A^{**}$. Taking adjoints, we also obtain $ex(1-e) = 0$ for every $x \in A^{**}$. Thus $ex = xe$ for every $x \in A^{**}$, and hence $e \in Z(A^{**})$.

By (39), every functional in $B^\perp$ vanishes on $A^{**}z_{\mathrm{at}}$. Since z_{at} is central, the positive and negative parts of every self-adjoint $h \in B^\perp$ also vanish on $A^{**}z_{\mathrm{at}}$. It follows that the support projection of $|h|$ is orthogonal to z_{at}. Taking the supremum over all self-adjoint $h \in B^\perp$ proves (40). In particular, the von Neumann algebra eA^{**} has no nonzero minimal projection.

Step 1. Countable generation of eB^{**}. For every $u \in \mathcal{U}(B)$, right multiplication by u induces a surjective linear isometry of A/B, given by $x + B \mapsto xu + B$. Indeed, since $Bu = B$, we have

$$\|xu + B\| = \inf_{b \in B} \|xu - b\| = \inf_{b \in B} \|x - bu^*\| = \|x + B\|.$$

The inverse is induced by right multiplication by u^*.

Since A/B is separable, the pointwise norm topology on the set of surjective linear isometries of A/B is second countable and metrizable. To see this, choose a countable norm-dense subset of A/B. Evaluation on the chosen subset identifies the space of isometries with a subspace of a countable product of copies of A/B. The induced topology is the pointwise norm topology on the entire quotient, because all the isometries have operator norm one.

Consequently, there exists a countable set $U_0 \subseteq \mathcal{U}(B)$ such that the right multiplication maps induced by the elements of U_0 are dense, in the pointwise norm topology, among the right multiplication maps induced by $\mathcal{U}(B)$.

We claim that

$$\overline{C^*(eU_0, e)}^{w^*} = eB^{**}. \tag{41}$$

Here $eU_0 = \{eu : u \in U_0\}$, and e is the identity of eB^{**}.

Fix $u \in \mathcal{U}(B)$. By the choice of U_0, there exists a sequence $(u_j)_{j \geq 1}$ in U_0 such that

$$\|xu_j - xu + B\| \longrightarrow 0, \qquad x \in A. \tag{42}$$

Choose an ultraweakly convergent subnet of (u_j) in the unit ball of B^{**}, with limit v. Fix a self-adjoint $h \in B^{\perp}$. By (42),

$$h(x(v-u)) = 0, \qquad x \in A.$$

By normality and the weak-$*$ density of A in A^{**}, the same identity holds for every $x \in A^{**}$. Let s be the sign of h, so that $h(sy) = |h|(y)$ for every $y \in A^{**}$. Taking $x = s(v-u)^*$, we obtain

$$|h|((v-u)^*(v-u)) = 0.$$

Hence $v-u$ annihilates the support projection of $|h|$. Since e is the supremum of these support projections as h ranges over the self-adjoint elements of $B^{\perp}$, it follows that $(v-u)e = 0$. By the centrality of e, we have $eu = ev$. Thus eu is an ultraweak limit of elements of eU_0.

It follows that the von Neumann algebra generated by eU_0, with identity e, contains eu for every $u \in \mathcal{U}(B)$. Since B is linearly spanned by its unitaries and B is weak-$*$ dense in B^{**}, we obtain (41).

Step 2. Construction of a quotient-preserving separable subalgebra. Choose representatives in A of a countable norm-dense subset of A/B. Let C_0 be the unital C^*-subalgebra of A generated by the chosen representatives and U_0. Since the generating set is countable, C_0 is separable.

We construct inductively an increasing sequence of separable unital C^*-subalgebras $C_n \subseteq A$. Suppose that C_n has been constructed. Choose a countable norm-dense subset of C_n. For every element x of the chosen subset, choose a sequence $(b_{x,j})_{j\geq 1}$ in B such that

$$\|x - b_{x,j}\| \longrightarrow \operatorname{dist}(x, B).$$

Let C_{n+1} be the C^*-subalgebra of A generated by C_n and all the elements $b_{x,j}$. The algebra C_{n+1} is separable and contains C_n.

Set

$$C = \overline{\bigcup_{n=0}^{\infty} C_n}^{\|\cdot\|}.$$

Then C is a separable unital C^*-subalgebra of A containing U_0 and the chosen representatives of the dense subset of A/B.

For every element x belonging to one of the countable dense subsets used in the construction, we have $b_{x,j} \in C \cap B$ for every $j \geq 1$. Consequently,

$$\operatorname{dist}(x, B) \leq \operatorname{dist}(x, C \cap B) \leq \lim_{j\to\infty} \|x - b_{x,j}\| = \operatorname{dist}(x, B).$$

Since the union of the chosen countable dense subsets is norm dense in C, and both distance functions are Lipschitz continuous, we obtain

$$\operatorname{dist}(x, C \cap B) = \operatorname{dist}(x, B), \qquad x \in C. \tag{43}$$

By (43), the natural linear map

$$\begin{aligned} C/(C \cap B) &\longrightarrow A/B, \\ x + (C \cap B) &\longmapsto x + B \end{aligned} \tag{44}$$

is an isometry. Its range is dense because C contains the chosen representatives of a norm-dense subset of A/B. Moreover, its range is closed because $C/(C \cap B)$ is a Banach space. Hence the map in (44) is surjective, which gives

$$A = B + C. \tag{45}$$

Since $U_0 \subseteq C \cap B$, it follows from (41) that

$$\overline{e(C \cap B)}^{w^*} = eB^{**}. \tag{46}$$

Indeed, $e(C \cap B)$ is a unital C^*-subalgebra of eB^{**} containing eU_0.

By (45), every element of eA is the sum of an element of eB and an element of eC. On the other hand, (46) gives

$$eB \subseteq eB^{**} = \overline{e(C \cap B)}^{w^*} \subseteq \overline{eC}^{w^*}.$$

Since the weak-$*$ closure of eC is a linear space, we conclude that $eA \subseteq \overline{eC}^{w^*}$. The weak-$*$ density of eA in eA^{**} therefore yields

$$\overline{eC}^{w^*} = eA^{**}. \tag{47}$$

Step 3. Preservation of pure-state separation. It remains to prove that $C \cap B$ separates $P(C)$. Suppose, towards a contradiction, that there exist two distinct pure states $\varphi, \psi \in P(C)$ such that $\varphi|_{C\cap B} = \psi|_{C\cap B}$. Set $h_0 = \varphi - \psi$.

We first observe that the positive part $(h_0)_+$ of the Jordan decomposition of h_0 is a nonzero scalar multiple of a pure state of C.

If the GNS representations of φ and ψ are unitarily inequivalent, then the normal extensions of φ and ψ to C^{**} have orthogonal central supports. In this case, $(h_0)_+ = \varphi$.

If their GNS representations are unitarily equivalent, then φ and ψ can be realized as vector states in the same irreducible representation of C. The von Neumann algebra generated by the irreducible representation is the algebra of all bounded operators on the representation space. In this realization, h_0 is represented by the difference of the rank-one projections onto two distinct one-dimensional subspaces. This difference is a nonzero self-adjoint operator of rank two and trace zero. Its positive part therefore has rank one, so $(h_0)_+$ is a positive scalar multiple of a pure vector state of C.

In either case, we have

$$\frac{(h_0)_+}{\|(h_0)_+\|} \in P(C). \tag{48}$$

Since h_0 vanishes on $C \cap B$, it induces a self-adjoint bounded linear functional on $C/(C \cap B)$ with norm $\|h_0\|$. By the surjective linear isometry in (44), this functional corresponds to a bounded linear functional on A/B with the same norm. Composing with the quotient map from A onto A/B, we obtain a functional $h \in B^\perp$ satisfying

$$h|_C = h_0, \qquad \|h\| = \|h_0\|. \tag{49}$$

Since the map in (44) preserves the involution, h is self-adjoint.

Write $h = h_+ - h_-$ for its Jordan decomposition. By (49), $h_+|_C - h_-|_C = h_0$. Both $h_+|_C$ and $h_-|_C$ are positive functionals on C. Since C contains the identity of A, we also have

$$\begin{aligned} \|h_+|_C\| + \|h_-|_C\| &= h_+(1) + h_-(1) \\ &= \|h\| = \|h_0\|. \end{aligned}$$

By uniqueness of the norm-minimal decomposition of a self-adjoint functional as a difference of positive functionals, it follows that

$$h_+|_C = (h_0)_+, \qquad h_-|_C = (h_0)_-. \tag{50}$$

By the definition of e, the normal positive functional h_+ on A^{**} is supported on e. We may therefore regard h_+ as a normal positive functional on eA^{**}, with no change of norm.

Let θ be a positive functional on eA^{**} satisfying $0 \le \theta \le h_+$. Since θ is dominated by a normal positive functional, it is normal. Moreover, the positive functional $x \mapsto \theta(ex)$ on C satisfies

$$0 \le \theta(e\,\cdot) \le h_+|_C = (h_0)_+.$$

By (48) and (50), there exists $t \in [0,1]$ such that

$$\theta(ex) = t h_+(x) = t h_+(ex), \qquad x \in C.$$

By normality of θ and h_+, together with the weak-$*$ density in (47), we obtain $\theta = t h_+$ on eA^{**}.

Thus every positive functional on eA^{**} dominated by h_+ is a scalar multiple of h_+. By the characterization of pure states in terms of dominated positive functionals, $h_+/\|h_+\|$ is a normal pure state of eA^{**}. The support projection of a normal pure state of a von Neumann algebra is a nonzero minimal projection. This contradicts (40), which implies that eA^{**} has no nonzero minimal projection.

We conclude that $C \cap B$ separates $P(C)$. Together with (44), this completes the proof. □

We now prove Theorem 1.2. For the case where B is separable, we use the one-generator reduction of Elliott [Ell72, p. 355].

Proof of Theorem 1.2. First suppose that A and B have a common identity.

Assume that B is separable, and fix $a \in A$. The C^*-subalgebra $C^*(B, a)$ of A is separable. Every pure state of $C^*(B, a)$ extends to a pure state of A. Indeed, the set of state extensions of a pure state of $C^*(B, a)$ is a nonempty weak-$*$ compact face of the state space of A, and every extreme point of the face is a pure state of A.

Since B separates $P(A)$, it follows that B separates $P(C^*(B, a))$. Otherwise, two distinct pure states of $C^*(B, a)$ agreeing on B would have distinct pure extensions to A agreeing on B. Applying Proposition 4.2 to

the inclusion $B \subseteq C^*(B,a)$, we obtain $C^*(B,a) = B$. In particular, $a \in B$. Since $a \in A$ was arbitrary, we conclude that $A = B$.

Assume now that A/B is separable. By Lemma 5.1, there exists a separable unital C^*-subalgebra $C \subseteq A$ such that $C \cap B$ separates $P(C)$ and the natural map $C/(C \cap B) \to A/B$ is a surjective linear isometry. Since C and $C \cap B$ have a common identity, Proposition 4.2 gives $C = C \cap B$. Consequently, $C/(C \cap B) = 0$, and hence $A/B = 0$. Thus $A = B$.

It remains to remove the common-identity assumption. Pass to the external unitizations $B^+ \subseteq A^+$, adjoining a new identity even when A or B is already unital.

Every irreducible representation of A^+ either vanishes on the ideal A or restricts to a nondegenerate irreducible representation of A. Consequently, the restriction map from $P(A^+)$ to $P(A) \cup \{0\}$ is a bijection. The pure state corresponding to 0 is the character of A^+ annihilating A, whereas the pure state corresponding to a member of $P(A)$ is its unique state extension to A^+.

Since B separates the points of $P(A) \cup \{0\}$, it follows that B^+ separates $P(A^+)$.

If B is separable, then B^+ is separable. If A/B is separable, the natural map

$$\begin{aligned} A/B &\longrightarrow A^+/B^+, \\ a + B &\longmapsto a + B^+ \end{aligned}$$

is a surjective linear contraction, because every element of A^+ differs from an element of A by a scalar multiple of the identity, which belongs to B^+. Hence A^+/B^+ is separable.

Since A^+ and B^+ have a common identity, the common-identity case established in the proof gives $A^+ = B^+$. Intersecting both algebras with A, we obtain $A = B$. □

Proof of Corollary 1.3. If $a \in B$, then $\{au + B : u \in \mathcal{U}(B)\} = \{0\}$, which is clearly separable.

Conversely, suppose that $\{au + B : u \in \mathcal{U}(B)\}$ is separable. Since $a = a^*$, taking adjoints induces an isometry between the right orbit $\{au + B : u \in \mathcal{U}(B)\}$ and the left orbit $\{ua + B : u \in \mathcal{U}(B)\}$. Thus the left orbit is also separable.

Left and right multiplication by unitaries of B induce surjective linear isometries of A/B. Choose countable families $(u_i)_{i\geq 1}$ and $(v_j)_{j\geq 1}$ in $\mathcal{U}(B)$ such that $\{u_i a + B : i \geq 1\}$ and $\{av_j + B : j \geq 1\}$ are norm dense in the left and right orbits, respectively.

For every $u, v \in \mathcal{U}(B)$ and all $i, j \geq 1$, the triangle inequality gives

$$\begin{aligned} \|uav - u_i a v_j + B\| &\leq \|(u - u_i)av + B\| + \|u_i a(v - v_j) + B\| \\ &= \|ua - u_i a + B\| + \|av - av_j + B\|. \end{aligned} \tag{51}$$

The last equality follows from the isometric actions of left and right multiplication by unitaries of B.

By (51), the set $S = \{u_i a v_j : i, j \geq 1\}$ is countable and satisfies

$$\overline{\{s + B : s \in S\}}^{\|\cdot\|} = \overline{\{uav + B : u, v \in \mathcal{U}(B)\}}^{\|\cdot\|}. \tag{52}$$

For each $n \geq 0$, let F_n be the linear span of the words in B and a containing at most n occurrences of a. Let $q : A \to A/B$ be the quotient map. We prove by induction that $\overline{q(F_n)}$ is separable.

For $n = 0$, this follows from $F_0 = B$. Suppose that $n \geq 1$ and that $\overline{q(F_{n-1})}$ is separable. Since B is linearly spanned by its unitaries, every word containing exactly n occurrences of a is a linear combination of words of the form

$$u_0 a u_1 a \cdots a u_n = (u_0 a u_1)(a u_2) \cdots (a u_n), \qquad u_0, \ldots, u_n \in \mathcal{U}(B).$$

By (52), each factor can be approximated in norm by an element of the form $s_i + b_i$, where $s_i \in S$ and $b_i \in B$. In the expansion of $\prod_{i=1}^n (s_i + b_i)$, every term except $\prod_{i=1}^n s_i$ belongs to F_{n-1}. Therefore

$$q(F_n) \subseteq \overline{\operatorname{span}\Big(q(F_{n-1}) \cup \{q(s_1 \cdots s_n) : s_1, \ldots, s_n \in S\}\Big)}^{\|\cdot\|}.$$

The right-hand side is separable by the induction hypothesis and the countability of S. Hence $\overline{q(F_n)}$ is separable.

Let $C = C^*(B,a)$. Since $a = a^*$, the union $\bigcup_{n=0}^\infty F_n$ is norm dense in C. The natural inclusion $C/B \to A/B$ is isometric, and $\bigcup_{n=0}^\infty q(F_n)$ is norm dense in its image. Consequently, C/B is separable.

Every pure state of C has a pure state extension to A, by the compact-face argument used in the proof of Theorem 1.2. Since B separates $P(A)$, it follows that B separates $P(C)$. As B and C have a common identity, B also separates the points of $P(C) \cup \{0\}$.

Applying Theorem 1.2 to the inclusion $B \subseteq C$, we conclude that $C = B$. Since $a \in C$, we obtain $a \in B$. □

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