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The least cap for removing primes below the square-root twisted level

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Abstract

Let $C_*(h)$ be the least uniform bound, in the sense of an infimum, for a nonnegative prime-free sequence with the ordinary divisor estimates of the integers through the square root and their Möbius-twisted estimates through $x^{1/2-h}$. We prove $C_*(h) = h^{-1+o(1)}$ as $h \downarrow 0$. Pinning the smaller divisor in a complementary pair gives a lower bound even for removal of a fixed fraction of the prime main term. A stopped random covering of circles gives complete removal with cap $O(\log(1/h)/(h \log\log(1/h)))$ and least prime factor at least n^{ch^2} on the perturbation. Both bounds apply to a presieved Goldbach baseline. Continuum certificates also give prime-free models of height below 2 for twisted exponents below 0.13, of height below 3.4 for exponents below 0.175, and of height below four for exponents below 0.18125. At every fixed ordinary exponent $\gamma > 1/2$, the ordinary and signed partner-multiple data at γ and $1-\gamma-h$ force a prime proportion tending to one uniformly over finite caps. Near the midpoint, an ordinary surplus exceeding a sufficiently large multiple of h rules out complete removal.

1 The cap and the two estimates

Let $0 < h < 1/2$, $\alpha = 1/2 - h$. For fixed $B \geq 0$ put

$$D_e(w; Q, X) = \sum_{q \leq Q} \tau(q)^B \max_J \left| \sum_{\substack{n \in J \\ q \mid n}} \mu(n)^e w_n \right|,$$

$$U(w; Q, X) = \sum_{q \leq Q} \tau(q)^B \max_{r \bmod q} \max_J \left| \sum_{\substack{n \in J \\ n \equiv r \ (q)}} w_n \right|,$$

where $J \subset [1, X]$ ranges over intervals and $\mu^0 = 1$. In the flat problem define $C_*(h)$ to be the infimum of C for which a single sequence $0 \leq a_n \leq C$, with $a_p = 0$ for every prime, satisfies, for all fixed $A, B \geq 0$,

$$D_0(a - 1; X^{1/2}, X) + U(\mu a; X^\alpha, X) \ll X(\log X)^{-A}. \tag{1}$$

Constants may depend on the fixed parameters. The inequality on a_n is pointwise on every integer. In all limits X tends to infinity at fixed h before h tends to zero.

For the reflected problem take $X = N$ even, $w = \exp(\sqrt{\log N})$, $P_w = \prod_{p \leq w,\, p \nmid 2N} p$, and put, for $1 \leq n < N$,

$$b_n = \mathfrak{S}(N) \prod_{p \mid P_w} \frac{p-1}{p-2} \mathbf{1}_{(n,N)=1} \mathbf{1}_{(N-n,P_w)=1} \prod_{\substack{p \mid n \\ p > w}} \frac{p-1}{p-2}, \qquad \mathfrak{S}(N) = 2 \prod_{p > 2} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{2 < p \mid N} \frac{p-1}{p-2}.$$

For the finitely many small N we may take the rows identically zero. The variable n is the partner and $N-n$ is the source. Define $C_*^{\mathrm{ref}}(h)$ by $0 \le a_{N,n} \le Cb_n$, $a_{N,p}=0$, $D_0(a-b;N^{1/2},N) \ll N(\log N)^{-A}$, and

$$\sum_{q\le N^{\alpha}} \tau(q)^B \max_{(r,q)=1} \max_{J} \left| \sum_{\substack{n\in J \\ N-n\equiv r \ (q)}} \mu(n)a_{N,n} - \frac{1}{\varphi(q)} \sum_{n\in J} \mu(n)a_{N,n} \right| \ll N(\log N)^{-A}. \tag{2}$$

Here and below reflected intervals are intersected with $[1,N)$. These estimates hold along all even N. We use $b_n = \mathfrak{S} = 1$ in the flat case.

Theorem 1. *Write $L=\log(2/h)$. For all sufficiently small $h>0$, both $\mathcal{C}=C_*(h)$ and $\mathcal{C}=C_*^{\mathrm{ref}}(h)$ satisfy*

$$\frac{1}{h}\exp\!\left(-H\frac{L\log\log L}{\log L}\right) \ \le\ \mathcal{C} \ \ll\ \frac{L}{h\log L} \tag{3}$$

for absolute constants. In particular $\lim_{h\downarrow 0}\log\mathcal{C}/\log(1/h)=1$. The upper bound admits squarefree perturbations supported on $n>1$ with $P^-(n)\ge n^{ch^2}$, $c>0$ absolute. The flat construction also satisfies $U(a-1;X^{\beta},X)\ll X(\log X)^{-A}$ for each fixed $\beta<1/2$.

The lower bound uses weaker data. Assume $0\le a_n\le Cb_n$ and

$$\sum_{q\le Q_e} \max_{J\subset(X/2,X)} \left| \sum_{\substack{n\in J \\ q\mid n}} \mu(n)^e (b_n-a_n) \right| \ll_{A,h,C,\beta} X(\log X)^{-A}, \qquad e=0,1, \tag{4}$$

where $Q_1=X^{\alpha}$ and $Q_0=X^{\beta}$ for every fixed $\beta<1/2$. The reflected centre in (2) and the flat raw estimate in (1) imply the respective signed tests in (4). The proof is given below.

Theorem 2. *For each $0<d<1$ there are $K(d)<\infty$, $\kappa(d)>0$ such that (4) at $C\le h^{-d}$ gives, for all sufficiently small h,*

$$\limsup_{X\to\infty} \frac{2\log X}{X\mathfrak{S}} \left| \sum_{X/2<p<X} (b_p-a_p) \right| \le K(d)h^{\kappa(d)}. \tag{5}$$

The lower bound of (3) holds for removal of any fixed positive proportion on the indicated dyad. For any $T\ge 1$ one can leave at most the proportion e^{-T} at the upper cap in (3), with factor floor $\gg \max\{h^2, h\log(L+T)/(L+T)\}$.

Theorem 3 (Fixed caps)**.** *In the flat and reflected problems the caps in the second column are admissible for every fixed twist strictly below the first column. The third column gives the factor floor of the perturbation.*

α'	C	η
11/100	1.961417	11/200
13/100	1.989829	3/100
29/200	3.424837	29/400
3/20	3.438093	3/40
61/400	3.432396	61/800
31/200	3.988346	31/400
33/200	3.395090	7/200
7/40	3.382977	3/100
9/50	3.881802	3/100
29/160	3.952969	3/100

With the same respective floors, independent coefficient tables give the stronger admissible caps 3.2311635 *at* 3/20*,* 3.4017432 *at* 61/400*, and* 3.9331625 *at* 31/200*, each for all strictly smaller exponents. The middle bound before decimal rounding is*

$$\frac{379121133889496782961543}{111449078491416952307712} = 3.401743101166\ldots.$$

There are also models at the exact exponents 3/8*,* 19/50*,* 23/50*,* 49/100 *with caps* 75*,* 79*,* 281*,* 1352*, respectively, and with positive floors.*

The constants in (3) and (5) are asymptotic. The finite cap assertions use separate certificates. These certificates and the proofs cited below as the source supplement are in the `anc/` directory of this paper's record, whose `README.md` describes every file. The earlier crossing and fixed-point construction gives cap $O(h^{-2})$ with the same order of factor floor, and explicitly cap 722 at 19/50 with floor 1/1440 [2]. The divisor-in-an-interval theorem of Ford and the smooth signed divisor weights of Granville, Koukoulopoulos and Maynard give the starting point for the lower estimate [3, 5]. The variation bound alone gives the lower power $1-(1+\log\log 2)/\log 2$. Here the ordinary distribution also bounds the positive conditional mass after a divisor has been selected. For the construction we use the factorization framework of Ford and Maynard [8] and the arithmetic transfer in [1, 2]. In their untwisted Type I/Type II problem near $(\gamma,\theta,\nu) = (1/2, 0, \nu)$, $1/3 \le \nu < 1/2$, Ford and Maynard exhibit weights of order $1/\varepsilon$ and recover a prime lower constant $1+O_\varrho(\varepsilon)$ with a divisor bound τ^ϱ [8, Sections 2.3, 4.5 and 8.2]. The positive mass on semiprimes in their construction contributes with the opposite Möbius sign from the primes it replaces. The additional signed row here constrains that transfer of mass. We discuss finite permutation grids and actual Goldbach rows in Section 14.

2 The two uses of the ordinary rows

For a list u write $|u|$ for its sum and $\#u$ for its count. Set $D_u f(x) = \prod_{v\in u}(1-\tau_v)f(x)$ with $\tau_v f(x) = f(x+v)$. On a list of j coordinates of total $y>0$ use

$$d\nu_{j,y} = \frac{du_1\cdots du_{j-1}}{j!\prod_{i=1}^{j} u_i}.$$

The single-coordinate mass is $1/y$. On unconstrained lists use the factorial measure $dm = \prod du_i/(j!\prod u_i)$ including its empty atom. Sums over list lengths are understood in ν_y and m.

Choose a smooth compactly supported odd B with $B(z) = -z$ on $[-1,1]$, and put $g(t) = hB((t-1/2)/h)$. Write $u(n)$ for the list of normalized prime logarithms $\log p/\log n$ on squarefree $n>1$. The complementary identity on $J=(X/2,X)$ gives

$$\sum_{n\in J}\mu^2(n)(b_n-a_n)D_{u(n)}g(0) = \sum_{p\in J}(b_p-a_p) + O_{A,h,C}(X(\log X)^{-A}). \tag{6}$$

Indeed $1/2-t = f(t)-f(1-t)+g(t)$ where $f(t) = (1/2-t-g(t))\mathbf{1}_{t<1/2}$ vanishes for $t\ge 1/2-h$. The differences of the linear function on a squarefree list are one on the prime and zero on composites. Complementing divisors in the reflected f term costs the total Möbius sign. Fixed-divisor partial summation uses the interval maxima in (4). For the ordinary term insert μ^2 by summing $\mu(l)$ on $l^2 \mid n$. Truncate l at a large power of $\log X$ and use the ordinary level with a strict exponent margin. The discarded tail costs arbitrary log powers by divisor moments, $b_n \ll \sqrt{\log X}$ and Cauchy's inequality. Fixed divisor weights in errors can be inserted by the same Cauchy argument against the trivial multiple bound.

In the flat case the raw bound also holds for the baseline by Bombieri–Vinogradov for the Möbius function at every strict sub-half exponent, including nonunit classes as in [2, Section 5]. Taking the class zero and subtracting gives the signed part of (4).

For the centred reflected variant choose a prime $r \nmid N$ of size $O(\log N)$. The reduced source class $N \bmod r^2$ has zero signed mass by squarefreeness of the partner. Its centred estimate gives an arbitrary log saving for the total signed mass, with at most a $\varphi(r^2)$ loss. The restriction of the centre to units at a general modulus differs negligibly by the presieve below w and a trivial bound at prime divisors above w. Now take the source class $N \bmod q$ if $(q, N) = 1$. Otherwise the partner multiple vanishes. The signed baseline estimate follows from the Möbius distribution in progressions and the presieve [2].

Normalize by $2 \log X/(X\mathfrak{S})$. At any fixed positive factor floor η, on passage to a subsequence the a_n shape measure on squarefree partners has density $A(u)$ against ν_1, with $0 \le A \le C$. The baseline has density one, including the prime atom. In the flat case this follows by the prime number theorem on each fixed rough simplex. For the reflected case use the rough presieve comparison in [1, 2]; singling out the largest residual prime permits Bombieri–Vinogradov at every presieve modulus $e^{o(\log X)}$, and the prime-to-prime local factors give exactly $\mathfrak{S}(N)$ in this normalization.

For fixed h, C the contribution to (6) from smaller factors tends to zero as $\eta \downarrow 0$ after the integer limit. To recall the majorant, the shifted multiplicative bounds of Henriot in their corrected form [6] give $\sum_{n<X} b_n F(n) \ll (X\mathfrak{S}/\log X) \exp(\sum_{p \le X} F(p)/p)$ for the bounded nonnegative multiplicative factors here. The source polynomial is sifted to w; the omitted prime factors of its sieve contribute $\log(\log X/\log w) + O(1)$ in the upper-bound exponent and the normalization of b restores $\log w$. Use $F(p^k) = |1 - \exp(it \log p/\log X)|$; its Euler exponent is at most $2\log(2 + |t|) + O(1)$. Inflating the factors below X^η by $\max(1, 1/(\eta(1 + |t|)))$ costs $O(1)$ in this exponent, and imposing at least one such factor gains $O(\min(1, \eta(1 + |t|)))$. Integrate against the rapidly decreasing Fourier transform. Changing $\log n$ to $\log X$ in this bound changes the rescaled Fourier envelope by $O_{h,m}((\log X)^{-1}(1 + |t|)^{-m})$ on the dyad. This argument also shows absolute integrability for all the fixed smooth profiles used below.

Lemma 4. *Pin any list V of distinct rough prime factors, allowing the empty list, and exclude every further factor up to X^b. If $|V| \le 1/2 - \sigma$ and b/σ is sufficiently small, then in the limiting measure for almost every pinned list*

$$\int_{\min U > b} A(V, U)\, d\nu_{1-|V|}(U) \ll 1/b. \tag{7}$$

The implicit constant is absolute, independent of the cap. All pinned factors are at least the initial floor. They may exceed b.

Proof. Integrate against a nonnegative test of the pins and let q be their squarefree product. Upper-sieve the residual integer, omitting primes of q, at sifting exponent b and divisor level $\sigma/2$. Its positive upper polynomial is valid for a. Each divisor term has modulus qe in a fixed strict sub-half range, so (4) replaces a by b_n with negligible total error. Fixed divisor weights absorb the multiplicities of such decompositions. The flat upper main is $O(X/(2qb \log X))$. For the reflected baseline it is $O(X\mathfrak{S}/(2\varphi(q)b \log X))$ when $(q, N) = 1$, and zero otherwise, by the dimension-one upper sieve. In detail the local multiple density is $1/(p-1)$ for $p \nmid N$ and zero at $p \mid N$, with a relative $1 + o(1)$ change on the polynomial pins. The product of the factors $1 - 1/(p-1)$ for $p \nmid N$ up to X^b, with the factor at two accounted for by coprimality, is $O(\mathfrak{S}/(b \log X))$. The baseline multiple estimates used to evaluate the upper polynomial hold at every strict subunit exponent, by the subpower presieve and elementary counting in classes (see the calculation following the lemma). Omitting prime factors of q from the local product costs $1 + o(1)$ at fixed floor. The upper-sieve constant is absolute for σ/b sufficiently large. The prime number theorem gives $dm(V)$ from the remaining harmonic weights. Nonnegative tests imply the asserted domination, with separate treatment of the empty atom. □

Here are details of the comparison used in the lemma. The large-prime factor in b_n is $1 + O((\log X)/w)$ uniformly, since each prime exceeding w contributes $1 + O(1/w)$. Impose $(n, N) = 1$ by divisors of $\operatorname{rad} N$ and use upper and lower fundamental-lemma weights for $(N - n, P_w) = 1$ at level X^ξ, $\xi > 0$ small and fixed relative to the strict exponent margin. When a fixed multiple q is coprime to N, the additional congruence $n \equiv 0 \bmod q$ automatically excludes the forbidden source residue at primes dividing (q, P_w). Chinese remaindering the remaining congruences costs at most one per interval endpoint and compatible tuple. In the sum over $q \le X^\theta$ the total cost is $X^{\theta+\xi+o(1)}2^{\omega(N)} = o(X(\log X)^{-A})$ if $\theta + \xi < 1$, including any fixed divisor weights. The sieve parameter $\xi \log X/\log w$ tends to infinity faster than any fixed multiple of $\log\log X$. The local main term is $|J|/q$ times $\prod_{p|q,p\le w}(1 - 1/p)^{-1}$ for $(q, N) = 1$, up to $1 + O((\log X)/w)$, and zero otherwise. Indeed for $p > 2$, $p \nmid N$, $p \le w$ the baseline normalization is $(1-1/(p-1)^2)(p-1)/(p-2) = (1-1/p)^{-1}$; at $p \mid N$ coprimality cancels the corresponding normalization. The tail of the singular product beyond w is $1+O(1/w)$. This proves the required baseline estimates with main $\mathbf{1}_{(q,N)=1}|J|/\varphi(q)$ and permits the absolute-constant upper sieve after pinning.

3 A signed large-deviation estimate

Apply a smooth dyadic partition to g on both sides of its central zero. Reflection preserves the absolute full difference. Each resulting lower-side test f is supported in $[1/2 - a_2\Delta, 1/2 - a_1\Delta]$, where $0 < a_1 < a_2$ are fixed, $\Delta \ll h$, and

$$|\widehat{f}(t)| \ll_m \Delta^2(1 + \Delta|t|)^{-m}. \tag{8}$$

Use the Fourier convention e^{itx}. Put $L = \log(1/\Delta)$, $z = \Delta/L^2$, $b = \sqrt{\log L}/L$, and let k count all parts in $[z, b]$. In this section V is the entire list of parts at most b.

Lemma 5. *For every $\epsilon, D > 0$ and $C \ll \Delta^{-D}$ there exists $\sigma' > 0$ such that*

$$\int_{\substack{|V|\le 1/3\\ k>(1+\epsilon)L}} A(u)|D_u f(0)|\, d\nu_1(u) \ll_{\epsilon,D} \Delta^{\sigma'}. \tag{9}$$

Proof. Expand the differences only in the large parts. The selected group and its complement have totals $x, 1 - |V| - x$. Both must be nonempty, since otherwise the compact support gives zero before any Fourier decomposition. With the factorials, disintegration of these two groups in the unweighted measure gives a density at most $b^{-2}dx$, using the Buchstab density $b^{-1}\omega(t/b) \le b^{-1}$. There are at most $2^{1/b}$ large selections at fixed physical list.

Fix $t_0 > 0$ and a positive even integer p, and set $v_0 = e^{t_0/p}$. Bound the indicator in (9) by $v_0^{k-(1+\epsilon)L}$. Split f into real symmetric Fourier bands $F \le |t| \le 2F$ and a bounded low band. For a fixed integer K, condition on the counts n_j of V in $[F^{-1+j/K}, F^{-1+(j+1)/K}) \cap [\eta, b]$, $0 \le j < K$, and put $\ell = \log F/K$. For the measure on lists with a chosen nonempty large sublist, apply Hölder with common density Av_0^k, inserting $W = \prod_{v\in V}\min(1, Fv)$ in the first factor and $W^{-(p-1)}$ in the p-th power factor. By (7) the first integral, before the power $1 - 1/p$, is bounded by

$$O_{p,t_0}(2^{1/b}/b)\prod_j (v_0\ell)^{n_j}/n_j!.$$

In the second integral insert $A \le C$, bound the two large-group densities by b^{-2}, drop the total-size restriction on V, and extend the translation variable x to the real line.

In the integral of $|D_V f_F(x)|^p$ expand the even real power in frequencies $t_1, \dots, t_p$ of sum zero. For a part v the multiplier is $Q(v) = \prod_i(1 - e^{it_i v})$. The unrestricted coordinates below $1/F$ give an

exponential-factorial integral of bounded modulus, since $|Q(v)|(Fv)^{-(p-1)} \le 2^p Fv$. In the other bins the exact factors are

$$\frac{1}{n_j!}\left(\int_{I_j} v_0^{\mathbf{1}_{[z,b]}(v)} Q(v)\,\frac{dv}{v}\right)^{n_j}.$$

At a fixed initial floor all Fourier convolutions and factorial interchanges are justified by absolute convergence. The bounds are uniform in the floor. Call a subset active at j if $|\sum_{i\in S} t_i| \le F^{1-j/K}$, and let r_j be the dimension of the span of its indicator vectors modulo the all-ones vector. There are at most 2^{1+r_j} active subsets by projection onto independent coordinates. For $j \ge 1$ and sufficiently large F, there are at most $\binom{p}{p/2}$ as well. Flip all negative frequency indices in the subset, obtaining sums of positive terms at least F in an interval of length less than F. Their subsets are an antichain. The bound follows by counting maximal chains in the Boolean lattice. Write $m_r = \min(2^{1+r}, \binom{p}{p/2})$. Each active term costs at most $v_0\ell$. Every other term costs $O(v_0)$ by integrating its single exponential by parts from the bin's lower endpoint; there is at most one jump at z. Hence the integral in a strict bin has modulus at most $v_0 m_{r_j}\ell + O_p(v_0)$.

A given nested chain of active sets has frequency-hyperplane volume

$$O_{p,K}\left(F^{p-1-\sum_{j=1}^{K-1} r_j/K}\right).$$

Indeed select independent forms at the strictest scale, extend their basis towards looser scales and complete to coordinates of the hyperplane. Each independent form first selected at j ranges over $O(F^{1-j/K})$, and the other coordinates range over $O_p(F)$. The nonzero coordinate determinant is bounded below in terms of p. Also $r_j \le p-2$ for $j \ge 1$ and large F, since otherwise the same inverse matrix gives $|t_i| = O_p(F^{1-1/K})$ for every i. The first bin uses 2^p with no volume restriction.

There are only finitely many chains at fixed p, K. Taking roots, multiplying the two Hölder factors and summing independent counts costs per strict bin

$$\exp\{(v_0\ell)^{1-1/p}(v_0 m_{r_j}\ell + O_p(v_0))^{1/p}\} \ll_{p,t_0} \exp(v_0 m_{r_j}^{1/p}\ell).$$

Combine the volume penalty, (8), the factor $v_0^{-(1+\epsilon)L}$, and the cap $C^{1/p}$. Sum dyadically, taking the differentiability index in (8) sufficiently large. The geometric costs $2^{1/b}b^{-O(1)}$ are $\Delta^{-o(1)}$. The resulting power of Δ, multiplied by p, is at least

$$\min_{0\le r\le p-2}\{p+1+r-pe^{t_0/p}m_r^{1/p}+(1+\epsilon)t_0-D\} - O_{p,t_0}(1/K) - o(1). \tag{10}$$

The first bin accounts for the $1/K$ error. At bounded frequency the tiny-factor estimate gives $O_{p,K,t_0}(C^{1/p}\Delta^2 2^{1/b}b^{-O(1)})$, with the same count prefactor.

For $r \le p - B_0$ use 2^{1+r}. Concavity puts the lower bound at $r = 0$ or $r = p - B_0$. Their limits before subtracting D are $1 - \log 2 + \epsilon t_0$ and $(2\log 2 - 1)(B_0 - 1) - (1-\epsilon)t_0$. Choose t_0, then B_0, to make both greater than $D + 2$. On $p - B_0 \le r \le p - 2$ the central binomial coefficient bound $O(2^p/\sqrt{p})$ gives a lower bound $1 - B_0 + \log(\pi p/2) - (1-\epsilon)t_0 - D + o(1)$ after subtracting D, for fixed B_0, t_0, D. An even p with $\log p > B_0 + t_0 + D + O(1)$ suffices. Take p large, then K large, and finally let Δ tend to zero. This proves (9). □

4 Pinning the lower divisor

Write T for the parts smaller than z, retaining only the initial rough region. We may discard $|V| > 1/3$ and $|T| > a_1\Delta/2$ at cost $O_D(\Delta^3)$ even with the full cap. For the first assertion use the Fourier product of the small factors and bound the factors above b by $2^{1/b}$. Their nonempty residual

density is at most $1/b$. Insert the exponential weight $e^{\theta v/b}$ for each small part, with $\theta = \frac{1}{2}\log(1/b)$. The indicator costs $e^{-\theta/(3b)}$, while the additional Euler exponent is $O(e^{\theta})$, and the untilted integral is at most $2\log(1+b|t|)+O(1)$. On the all-small lists mark one part by its fraction of the total, cancel its reciprocal, and integrate out its length. The remaining small mass is at least $1-b$. The negative exponent has order $L\sqrt{\log L}$. For the second assertion the density of the nonempty parts above z, weighted by $2^{\#}$ at residual total $y \ge z$, is at most $2z^{-3}$. Mark one part by its share of y and integrate the rest without the total constraint, paying $(2/y)\exp(2\int_z^1 dv/v)$. Tilt the tiny points by $e^{v/z}$ and use the gain $e^{-a_1\Delta/(2z)}$. Mark one part on the all-tiny lists as before. The extra Euler exponent is $O(1)$ and $\Delta/z = L^2$. This beats all the polynomial costs of Fourier integration using (8).

Fix $0 < \epsilon < 1/2$ below. By Lemma 5 it remains to handle $k \le (1+\epsilon)L$. Expand the differences in the parts above z, with selected group P of mass x and complementary group Q of mass $1-|T|-x$. Their factorials arise by marking the selection. A surviving term $D_T f(x)$ requires

$$1/2 - a_2\Delta - |T| \le x \le 1/2 - a_1\Delta, \qquad |T| + x \le 1/2 - a_1\Delta/2. \tag{11}$$

Both groups are nonempty. Also $|D_T f(x)| \le \Delta V_\Delta(T)$ with $\int V_\Delta\, dm \ll 1$: use (8) and

$$V_\Delta(T) = c_m \int_{\mathbb{R}} (1+|t|)^{-m} \prod_{v\in T} \min(2, |t|v/\Delta)\, dt.$$

The logarithm of the associated positive Euler integral is bounded by $2\log(1+|t|z/\Delta)+O(1)$.

For any $0 < \lambda < 1$ the density at $y \in [2/5, 3/5]$ of lists of parts above z having at most λL parts in $[z, b]$ is

$$\ll \Delta^{-1+I(\lambda)-o(1)}, \qquad I(\lambda) = 1 - \lambda + \lambda\log\lambda. \tag{12}$$

Majorize by $\lambda^{k-\lambda L}$, mark a part proportionally to its share of y, and extend the residual factorial integral. The result is at most $y^{-1}\lambda^{-\lambda L}\exp(\lambda\log(b/z) + \log(1/b))$. The unrestricted density is at most $1/z$.

If P has at most $(1-\epsilon)L$ medium parts, pin the entire lists T and P in (7), now sifting the remaining part to z. By (11) one may use the positive sieve level $a_1\Delta/4$ with parameter $a_1\Delta/(4z) \to \infty$. Its upper is $O(1/z)$, uniformly in the shell. The x-interval has length $O(\Delta)$ and (12) applies to P. This contribution is

$$\ll \Delta^2 z^{-1}\Delta^{-1+I(1-\epsilon)-o(1)} = \Delta^{I(1-\epsilon)-o(1)}. \tag{13}$$

If P has more medium parts, Q has at most $2\epsilon L$. Insert $A \le C$, use the unrestricted density on P, and apply (12) to Q at $1-|T|-x$. The resulting upper is

$$C\Delta^{I(2\epsilon)-o(1)}. \tag{14}$$

Since $I(0+) = 1$, choose ϵ with $I(2\epsilon) > d$. Equations (9), (13), and (14) give $\int A|D_u f|\, d\nu_1 \ll_d \Delta^{\kappa(d)}$ on each shell, and the same holds for the baseline. Sum the geometric series. One may first use finitely many shells, with $\eta < z$ on every one. The smooth central remainder of width δ is a rescaling of a fixed bump, since g is linear there. Its baseline variation tends to zero with δ, uniformly in the floor. For instance the second-moment estimate below with $p = 2$ and $C = 1$ suffices. This also bounds the model remainder by the fixed cap times a vanishing quantity. Remove the initial floor and let $\delta \downarrow 0$ at fixed h, C, before decreasing h. Theorem 2's estimate (5) follows from (6) and the baseline prime mass asymptotic of one.

For the cited second-moment estimate omit the count condition, set $t_0 = 0$, and use $p = 2$ in the proof of Lemma 5. Only $r = 0$ is possible on strict scales. Formula (10) before extracting the cap gives the upper $O((1+\sqrt{C})\delta^{3/2-\sqrt{2}-O(1/K)-o(1)})$ on lists of small total at most $1/3$; the exceptional argument above treats the other lists. This proof applies directly to a central bump meeting the midpoint, since the omission of the two empty large selections only uses its distance from zero and one. Choose K fixed large enough for a strictly positive exponent.

5 A quantitative lower exponent

For some absolute H the preceding proof gives, with $L_h = \log(2/h)$,

$$C_*(h) \geq h^{-1} \exp(-HL_h \log\log L_h / \log L_h) \tag{15}$$

for all sufficiently small h. We record the uniformity in the Fourier moment underlying this refinement. Write $L = \log(1/\Delta)$ on a shell and set $b = 1/L$, $z = \Delta/L^2$. The exceptional $|V| > 1/3$ tilt now costs $\exp(-cL\log L + O(L))$, including its factor $2^{1/b}$.

In the first Hölder factor the integral of $2^{\#U}$ over the large parts, conditional on V of mass at most $1/3$, is $O(b^{-1-2^{25}})$. At most 24 parts exceed $1/24$. Greedily pack the remaining large parts into at most 25 bins of mass at most $1/12$. Each completed bin has mass at least $1/24$. Some bin contains a twenty-fifth of their count. Majorize by $2^{24}\sum_{P\subset U,\,|P|\leq 1/12} 2^{25\#P}$. Pin V and P in (7), then integrate P without its mass constraint. This gives the asserted polynomial cost.

In a Fourier band use $W(v) = \min(1, 2Fv)$. For $v < 1/(2F)$ we have $|Q(v)|W(v)^{-(p-1)} \leq 2Fv$. Treat $[1/(2F), 1/F)$ as an additional counted bin of harmonic length at most $\log 2$. The bound 2^p on its multiplier costs $\exp(O(1))$ after summing its counts following Hölder. Put $K = p^2$ and enlarge the threshold for an active subset at j to $4^p F^{1-j/K}$. All inactive terms together now cost $O(1)$ by parts, provided $v_0 \leq 2$. In the bin estimate the error after taking the root and multiplying the first factor contributes $O(1)$ to the logarithm, by concavity. For $F \geq \exp(p^4)$ the strict-bin antichain argument applies, and rank $p-1$ is excluded by the integer inverse matrix bound with cofactors at most $p!$. For $F < \exp(p^4)$ count the whole region $[1/F, b]$ with the crude multiplier 2^p and sum counts after Hölder. Its cost is a fixed power of $1+F$, giving total $\ll \Delta^2 C^{1/p} b^{-O(1)} \exp(O(p^4))$ on these bands.

Enumerate just the successive basis enlargements for the strict-bin active spans, with deterministic choices in a fixed order. At most $p-1$ vectors and their scales are chosen, each in at most $2^p K$ ways. On every resulting region all the ranks are fixed. Bounding its volume by imposing just those independent restrictions costs at most $(4^p)^{p-1}$ beyond the previous power of F. The inverse Jacobian in integer frequency coordinates has absolute value at most one. Thus the basis enumeration, bin errors, first Hölder factor and fixed Fourier constants together cost at most $\exp(O(p^4 + \log L))$. The unconstrained first bin loses $O(p/K)$ in (10). Twelve derivatives in (8) suffice since the growth exponent in F after taking roots is at most five.

Take even $p \asymp L^{1/20}$, $\epsilon = H_1/\log L$ for a sufficiently large absolute H_1, $t_0 = 8/\epsilon$ and $B_0 = \lceil 10(t_0+5)\rceil$. At $r = 0$ and $p - B_0$ the endpoint estimates following (10) have error $O((B_0+t_0+1)^2/p)$. For $r \geq p - B_0$ the central binomial estimate gives before subtracting D the lower $\log p - O(B_0+t_0+1) - O((t_0+\log p)^2/p)$. All three bounds exceed four for large L and large fixed H_1. Here $D = 1+o(1)$ suffices. The log cost $O(p^4 + \log L)$ is $o(L/p)$, so the high-count contribution is $O(e^{-L/p})$. In (13) and (14) the errors in the logarithms of the density bounds are $O(\log L)$ directly from (12), since $\log(b/z) = L + O(\log L)$. We have $I(1-\epsilon) \gg (\log L)^{-2}$ and $I(2\epsilon) \geq 1 - O(\log\log L/\log L)$. Under the cap on the right of (15) with H large, the shell bounds are summable and their total tends to zero with h. The cap allowed by this bound increases as the shell shrinks, up to a harmless absolute adjustment on the outermost shell. This proves (15), also for removal of a fixed positive prime fraction.

6 An ordinary exponent above one half

The location of the ordinary cutoff matters even at arbitrary finite height. We use partner-multiple tests on the difference from either baseline. The finite cap may vary with the two levels.

Theorem 6. *Let $1/2 < \gamma < 1$, $r = \min(1-\gamma, 2\gamma - 1)$ and $\kappa = 3/2 - \sqrt{2}$. Suppose $0 \le a_n \le Cb_n$ for some finite C, with (4) at every strict ordinary exponent below γ and every strict signed exponent below $1 - \gamma - h$. For $0 < h < cr$ the left side of (5) is*

$$\ll (h/r)^\kappa$$

with absolute constants, uniformly in the finite cap. For any $0 < d \le 1$, removal of a fraction d is impossible when $r > K_d h$. In particular near the half endpoint an ordinary surplus $\gamma - 1/2 > K'_d h$ suffices.

Proof. Write $\beta = 1-\gamma,\ \theta = \gamma + h/2$. Choose a fixed smooth cutoff equal to one on $[-1,1]$, supported on $[-2,2]$, and put $g(t) = (\theta - t)\chi((t-\theta)/h)$. Split the two tails of $\theta - t - g(t)$ at θ. The lower tail lies at least $h/2$ inside the ordinary range. After reflection the other lies at least $h/2$ inside the signed range. The complementary identity (6) therefore applies. Reflection gives $|D_u g(0)| = |D_u f(0)|$ for $f(t) = g(1-t)$, supported in $[\beta - 3h, \beta + 3h]$, with $|\widehat{f}(t)| \ll_m h^2(1 + h|t|)^{-m}$. At fixed h, C pass to shapes and then remove the initial factor floor as in Section 2. We bound the rough integrals against both A and one uniformly in this floor and in C.

Set $E = r/32,\ b_0 = E/B_0,\ b_j = 2^{-j}b_0$, where B_0 is a sufficiently large absolute constant, and assume $h < b_0$. For each finite list take the first j with $S_{b_j} = \sum_{u_i \le b_j} u_i \le E$. Write $b = b_j$ and let V contain all these small parts. Expand the differences of the parts exceeding b by choosing a list P, leaving a list Q. The integrand to be bounded is $|D_V f(x)|$, $x = |P|$. Its support requires $\beta - 3h - E \le x \le \beta + 3h$. Thus P, Q are nonempty. The pins V, P have total at most $\beta + 3h + E$, and adding any further pins of total at most $E/3$ still leaves ordinary margin at least $2E$. Lemma 4 with ordinary level γ gives an $O(1/b)$ upper density for the remaining list with all parts $> b$. Its proof uses only this spare margin and a sufficiently large ratio of margin to b. The baseline density of a nonempty list of parts $> b$ at prescribed total is at most $1/b$ by the Buchstab recurrence. Factorial disintegration, first in the residual Q and then in P at fixed x, bounds the contribution at stage zero by $O(b^{-2})J_b$, where

$$J_b = \int_{v \le b} dm(V) \int_{\mathbb{R}} |D_V f(x)|\, dx.$$

An empty V is included. The condition $|V| \le E$ may be dropped after the ordinary upper sieve.

We need the following estimate, also with weight $e^{|V|/b}$ in J_b:

$$J_b \ll (h+b)^{1/2} h^{3/2} (1 + b/h)^{\sqrt{2}}. \tag{16}$$

First take the unweighted integral. For bands of upper frequency $T = 2^k/h,\ k \ge 0$, including the lowest interval $|t| \le 1/h$, use the triangle inequality on the support of the original difference. Its length is at most $6h + |V|$. Use Cauchy's inequality with $\prod_{v \in V} w(v)$ in the first factor, $w(v) = \lambda \min(Tv, 1)$, $\lambda = \sqrt{2}$. The first squared integral is $O((h+b)(1+bT)^\lambda)$ by the factorial exponential and its first moment. Plancherel bounds the second square by $O((1+bT)^{2/\lambda}) \int_{\text{band}} |\widehat{f}(t)|^2 dt$, since

$$\int_0^b \frac{|1 - e^{itv}|^2}{w(v)} \frac{dv}{v} \le (2/\lambda) \log(1 + bT) + O(1), \qquad |t| \le T.$$

Below $1/T$ use $|t|v$, and above it use the absolute lower bound for $\int_s^t \cos y\, dy/y$ on positive intervals. Summing the bands with the rapid Fourier decay proves (16). For the exponential weight, expand $\prod_{v \in V}[1 + (e^{v/b} - 1)]$. Each additional part v in that expansion costs at most two in the L^1 norm of a difference. The total cost is at most $\exp(2\int_0^b (e^{v/b} - 1) dv/v) = O(1)$, proving the weighted form.

For $j \ge 1$ the preceding stage failed, so $S_{2b} > E$. Either $|V| > E/3$, or the parts in $(b, 2b]$ of P or of Q have mass exceeding $E/3$. The first possibility gains $e^{-E/(3b)}$ by the weighted estimate. In

the other two majorize the indicator by the number of ways to distinguish $n = \lfloor E/(6b) \rfloor$ parts in that shell in the corresponding list. Their combined mass is at most $E/3$. If they belong to P, the remaining P has positive total $x - \sum v \geq \beta - 3h - 4E/3$ and density at most $1/b$. If they belong to Q, adjoin them to the ordinary pins V, P. The residual still has positive total and ordinary margin at least $2E$, so the same $O(1/b)$ upper sieve applies. The distinguished parts contribute $(\log 2)^n/n!$ in both cases. The total bound at stage $j \geq 1$ is therefore $O(e^{-c_1 E/b} b^{-2})$ times the right side of (16). The constants are absolute; initial roughness restrictions may always be retained up to applying an upper bound.

For $b \geq h$, (16) divided by b^2 is $O((h/b)^\kappa)$. For $b < h$ it is $O((h/b)^2)$. At every stage it is thus $O(4^j (h/b_0)^\kappa)$. The rare-stage factors give the convergent sum $1 + \sum_{j \geq 1} 4^j e^{-c_1 B_0 2^j}$, proving the assertion. □

The multiplication-table refinement. In Theorem 6 one can replace the bound by

$$\ll (h/r)^\delta (1 + \log(r/h))^{-3/2}, \qquad \delta = 1 - \frac{1 + \log\log 2}{\log 2}. \tag{17}$$

Indeed $b^{-2} J_b$ for $b \geq h$ is the unit-scale translation integral for a smooth bump of height and width $y = h/b$. For such a bump the dyadic first-moment estimate is $O(y^\delta (1 + \log(1/y))^{-3/2})$. The proof, including the one-sided walk bound, is in the source supplement. It restricts to the union of the translates of the original support before splitting into Fourier bands and uses Ford's interval-cover bound [3, Lemma 3.1]. The weighted estimate follows by the same extra-point expansion as in (16). Writing $x = h/b_0$, a loss of at most $O((1+j)^2 4^j)$ relative to $x^\delta (1 + \log(1/x))^{-3/2}$ covers every scale, including $b < h$. This is summable against the rare-stage gains.

Squarefree ordinary information. The same bounds apply without a pointwise height restriction if the ordinary estimates are given on $\mu^2(a-b)$ with unit divisor weights and the signed estimates on $\mu(a-b)$. In particular a polylogarithmic relative height suffices with the original raw data. Here the rough limit may be a measure. The ordinary positive upper sieve dominates precisely the marginals in the proof after summing the unmarked residual. For small-factor localization, distinguishing one prime below X^η and applying the positive upper sieve to a random subset of the other small primes bounds the normalized Fourier integrand by $O(\eta(64 + |t|)^D) + O_A((\log X)^{-A})$. The distribution hypotheses give this uniformly in frequency, with an absolute finite exponent D. The source supplement proves the sieve inequality and the passage to the rough marginals. To absorb divisor multiplicities with unweighted errors, first discard $\omega(n) > B \log\log X$: a positive fraction of all subsets then have product at most $\sqrt{X}$ and at least $\omega(n)/4$ factors. The ordinary data bound their total by any prescribed log saving if B is large enough.

Alternatively, at arbitrary nonnegative height the original raw inputs need only the one additional scalar

$$\sum_{X/2 < n < X} \mu^2(n) a_n \geq \sum_{X/2 < n < X} \mu^2(n) b_n - o(X\mathfrak{S}/\log X).$$

Indeed let $Z = (\log X)^B$ and forbid all $p^2 \mid n$ for $p \leq Z$. Two adjacent Bonferroni polynomials of degree $k \asymp_A \log\log X/\log\log\log X$ use distinct square moduli $d^2 \leq Z^{2k} = X^{o(1)}$ with unit coefficients. Their baseline gap is $O(XC^k/k!)$ up to an arbitrary log saving, since $\sum_p 1/\varphi(p^2) < \infty$. Thus the raw ordinary estimates determine the mass avoiding these small squares. Its excess over the full squarefree baseline is $O(X(\log X)^{O(1)}/Z)$. Subtracting the assumed scalar leaves $o(X\mathfrak{S}/\log X)$ on all nonsquarefree integers at any fixed polynomial rough floor. The Fourier localization above also applies to raw a under the raw ordinary estimates. Each distinct prime enters the Fourier product

once on every integer, including those with square factors. The random sieve has the same survival product, which gives the required localization. If the scalar has normalized deficit e instead, the rate incurs an additional $O_\gamma(e^c)$ for an absolute $c > 0$. On the retained low-factor-count part the same random sieve bounds the squared Fourier integrand by a fixed power of $64 + |t|$: tiny squared factors are still at most $(64+2|t|)\log p/\log X$, and each medium factor costs at most four. Cauchy's inequality at a weaker signed gap $H \geq h$ costs $O(H^{-D}\sqrt{e})$, to be balanced against (17) at H. The high counts only require the original first-moment estimate.

The raw estimates also give the full rate for every $\gamma > 1/2$ if $\sum_{X/2<n<X} a_n^{1+s} \leq X^{1+s\gamma/2+o(1)}$ for some fixed $s > 0$. In particular the pointwise height may reach $X^{\gamma/2+o(1)}$. On any fixed rough floor, repeated primes below $X^{\gamma/2-\xi}$ are seen through p^2. For the collar of width 2ξ in the exponent pin p alone. Above $X^{\gamma/2+\xi}$ there are $O(X^{1-\gamma/2-\xi})$ candidate integers; Hölder saves a power of X. Send ξ to zero after taking the dyadic limit. Small-factor localization then supplies the squarefree bridge. The supplement gives the corresponding stability under a small surplus in the moment exponent.

For $2/3 \leq \gamma < 1$ even the raw ordinary estimates suffice without any cap. At fixed initial floor write a nonsquarefree integer as $n = p^2 m$. Below $X^{1/3-\xi}$ pin p^2. Above $X^{1/3+\xi}$ pin $pm \leq X^{2/3-\xi}$, leaving a rough residual; for each m the marked prime lies in $(\sqrt{X/(2m)}, \sqrt{X/m})$, of reciprocal mass $O(1/\log X)$. The m-sum is over rough integers and costs a constant at fixed floor. In the intervening band pin p alone with fixed spare ordinary level, obtaining $O_\eta(\xi)$. Send ξ to zero at fixed floor. Small prime factors in the ordinary short profile are covered by the same Fourier localization over distinct primes. The full proof and the initial-interval version are in the supplement.

For all γ in a sufficiently small fixed neighbourhood of $2/3$ the original raw estimates, at arbitrary height, give the uniform bound

$$\ll h^\delta (1 + \log(1/h))^{-3/2} + (1 - 3\gamma/2)_+.$$

Below two thirds only $n = p^2 m$ with squarefree m and $u = \log p/\log n \in [\gamma/2, 1-\gamma]$ remains. Subtract the affine function from the ordinary lower tail, then reflect in the sum $1-u$ of the distinct parts. With $a = 1 - 3\gamma/2 + h$, its complementary profile has width and height $O(a)$ and derivative of bounded variation uniformly in h, a. In the stopping argument begin at $b = a$, then halve. The distinguished pin p supplies the translation integral in its one-prime marginal; the other selected parts have total at most $3a$ and the residual upper sieve costs $O(1/b)$. The factorial translation integral is $O(a^2)$ for $b \leq a$ by the Fourier proof above, also with the exponential tilt. The regular stage cost is $O(a^3/b^2)$; later stages gain $O(\exp(-c/b))$ from the failed preceding cutoff. Thus all rough repeated-prime terms together cost $O(a)$. The supplement gives the full argument and the limiting order.

Smooth divisor tests and the complementary line. Put $W_\Phi(n) = \sum_{d|n} \mu(d)\Phi(\log d/\log n)$ for a fixed smooth Φ on $[0,1]$. The same estimates bound the absolute sum of $(a_n - b_n)W_\Phi(n)$ in the dyadic normalization of (5), with constants depending on Φ. For arbitrary height and projected ordinary data the observable is $\mu^2 W_\Phi$; the raw-data versions above give W_Φ itself. Subtract $\Phi(\theta)$, whose divisor difference vanishes identically, and take $g(t) = (\Phi(t) - \Phi(\theta))\chi((t-\theta)/h)$. The rescaled bumps $g(\theta + hz)/h$ have bounded smooth seminorms. The two short tails and the translated first-moment bound therefore apply. On rough $p^2 m$ the outstanding profile has width and height $O_\Phi(a)$ and derivative of uniformly bounded variation on the required range, as above.

For any $0 < \gamma < 1$ and finite relative cap C, put $r_0 = \min(\gamma, 1-\gamma)$. At the ordinary and signed levels $\gamma, 1-\gamma-h$ (every strict subexponent suffices), the bound for $0 < h < cr_0$ is

$$\ll_\Phi (1 + C)(h/r_0)^\delta (1 + \log(r_0/h))^{-3/2}.$$

Indeed dominate the initial rough measures in sum by $(1 + C)\nu_1$. At each stage, including after the shell marks, the nonempty residual has baseline density at most $1/b_j$; this replaces the ordinary pin. Both required positive totals follow with $E = r_0/32$. The remaining translation integral and stopping sum are unchanged. The supplement gives the arithmetic details. For one bounded flat sequence satisfying the ordinary and signed data through all powers strictly below γ and $1-\gamma$, take $\Phi(t) = (1-t)^k$ and then $h \downarrow 0$. This gives $\sum_{n\le x} a_n \Lambda_k(n) \sim kx(\log x)^{k-1}$ for each fixed $k \ge 1$. Flat initial-interval estimates at every large argument suffice: a grid with $(\log X)^D$ points gives the dyadic interval maxima at every smaller exponent by boundedness. The strict supercomplementary question appears in Ford [4], who attributes the signed condition to Hooley. The closed-line moment conclusion with exact-power interval-maximal inputs and examples of deficits strictly below the line were obtained earlier in unpublished work of the author. Here the modulus from below also applies to smooth divisor tests.

A partial converse on the integers. There are flat sequences of height at most two whose deficit is at least $h/2$ at the levels of Theorem 6, for all $0 < h < \min(1-\gamma, 1/2)$. Put $\alpha = 1-\gamma-h$. Choose a nonnegative smooth v compactly supported in $(\alpha, 1-\gamma)$, with integral one and $\|v\|_\infty < 2/h$. Write

$$M(n) = \sum_{p|n} \frac{\log p}{\log n} v\Big(\frac{\log p}{\log n}\Big), \qquad a_n = 1 + (h/2)\lambda(n)(1 - M(n))$$

for $n > 1$, with λ Liouville's function. Then $0 \le a_n \le 2$ and $a_p = 1 - h/2$. For the ordinary discrepancy, each marked prime in a dyad has $[p, d] \le X^{1-c}$ when $d \le X^\gamma$, by the strict upper support. On its multiples complete multiplicativity and the prime number theorem for λ give every logarithmic saving, also with the smooth weight by partial summation. For the twisted discrepancy $\mu\lambda = \mu^2$. At $d \le X^\alpha$ the marked prime cannot divide d on the central dyad. For squarefree d, squarefree multiples of pd have main density $\zeta(2)^{-1}\prod_{\ell|pd}(\ell+1)^{-1}$ and interval error $O(\tau(pd)\sqrt{X/(pd)})$, by square inversion. At each spatial argument z the sum of the marked weights divided by $p+1$ equals $1 + O_A((\log X)^{-A})$ by the prime number theorem. The interval errors save a power on summing over the strict subunit product range; at nonsquarefree d the signed terms vanish. These checks allow every fixed divisor weight and both stated endpoints. With v approaching the constant density on its support, the same proof gives every deficit below $h/(1-h)$ for $h < 1/2$ by rescaling $1 - M(n)$.

Exact determination of smooth tests in the flat case. Fix $0 < \gamma, \alpha < 1$ with $\gamma + \alpha < 1$. Assume the partner-multiple estimates (4) with $b_n = 1$ at every strict ordinary exponent below γ and every strict signed exponent below α. All bounded flat sequences with these estimates have dyadic W_Φ discrepancy $o(X/\log X)$ exactly when Φ is constant on $[\gamma, 1-\alpha]$. Each failure has a witness of height at most two. For sufficiency subtract the constant and approximate by two tails strictly inside the respective levels. Smoothness and flatness at both endpoints give convergence in every fixed smooth norm; the flat absolute Fourier majorant controls the normalized discrepancy in such a norm.

For necessity when $\gamma \ge \alpha$, replace the perturbation in the preceding construction by $\lambda(n)(\int w - \sum_{p|n} u_p w(u_p))$, where $u_p = \log p/\log n$ and w is a small smooth signed bump in $(\alpha, 1-\gamma)$. Its normalized response on W_Φ equals $\int w(u)\Phi'(1-u)\,du$. For $\Phi(t) = (1-t)^k$ expand over ordered positive compositions of k. The signed integral on unmarked squarefree shapes is $k[z^k](1-z)$. With a marked interior prime of exponent u the remaining composition of degree K contributes $[z^K](1-z)$, leaving the marked marginal ku^{k-1} for $k \ge 2$ and zero for $k = 1$. Polynomial approximation gives the smooth identity. When $\gamma < \alpha$, exchange the inputs and multiply the perturbation by $\mu(n)$. Squarefree projection preserves the strict ordinary exponents, and complementing divisors reflects

the test. The response is now $-\int w(u)\Phi'(u)\,du$ for a bump on $(\gamma, 1-\alpha)$. The supplement gives the disintegration and the two strict support checks.

Writing Δ_k for the dyadic normalized discrepancy of $\Lambda_k(n)/(\log n)^k$, the upper argument also gives $\Delta_k - k(1-\theta)^{k-1}\Delta_1$ an additional factor h in the limiting absolute bound. Indeed subtract the affine Taylor polynomial of $(1-t)^k$ at θ before smoothing. More generally, a degree-m Taylor subtraction gains h^m. Conversely, bumps with m vanishing moments preserve the first m generalized Mangoldt means and change the next normalized one by a nonzero constant times h^{m+1} at height two.

7 Necessary factor sizes and counts

These two restrictions are uniform in the height. Consider a hard-floor limiting shape with deviation $F = A - 1 \geq -1$. Let $W_\eta(y)$ denote the density of the lists of total y with all parts at least η, and $W_\eta^o(y)$ its odd-count portion. Then

$$1 - A((1)) \leq 2\int_{1/2-h}^{1/2} (1/2 - x) W_\eta(x) W_\eta^o(1-x)\,dx. \tag{18}$$

Indeed the full divisor difference of $l(t) = 1/2 - t$ is one on the prime and zero on composites. Expand its integral against F. All lower-half sublists contribute zero by their ordinary rows. In every upper-half term pin the complementary list, of mass x. The multiplier becomes $-(1/2 - x)$ times the residual cycle sign. The signed rows remove $x \leq 1/2 - h$. For each remaining pin the ordinary residual integral is zero, so the signed integral equals minus twice its odd part and is at most $2W_\eta^o(1-x)$. This proves (18). Only the negative part of F needs the stated floor, since pinning a smaller part leaves a nonnegative integrand with zero ordinary row.

The exponential convolution for rough parts gives $W_\eta(y) = \eta^{-1}\omega(y/\eta)$ and $2W_\eta^o(y) = W_\eta(y) + \rho(y/\eta - 1)/y$ for $y \geq \eta$, with ω, ρ the Buchstab and Dickman functions. Both densities in (18) vanish below η. The divisor difference of a constant gives the further balance $\int_{\alpha<|P|<1/2} R_1 F(P)\,dm(P) = 0$. Here $dm(P)$ is the factorial measure on the pinned list and $R_1 F(P)$ its signed residual. Its lower bound is $-2W_\eta^e(1 - |P|)$ by the even part of the same ordinary row. Consequently for any $\alpha \leq m \leq 1/2$ the deficit is at most

$$2\int_\alpha^{1/2} W_\eta(x)\big[(m-x)_+ W_\eta^o(1-x) + (x-m)_+ W_\eta^e(1-x)\big]\,dx.$$

At the midpoint $\int |x-m|dx = h^2/4$. Since $2W_\eta^{o,e}(y), W_\eta(y) \sim e^{-\gamma_E}/\eta$ uniformly on compact positive y-intervals, removal of a fixed fraction $d > 0$ requires

$$\eta \leq (e^{-\gamma_E}/(2\sqrt{d}) + o(1))\,h.$$

There is also a restriction on factor counts, even with an arbitrarily small positive floor. Suppose F_- vanishes on lists with more than K parts. For fixed $0 < c < 2$, if $K \leq c\log(1/h)$, the missing prime proportion is $O_c(h^{2I(c/2)-o(1)})$, where $I(t) = 1 - t + t\log t$. To see this use the lower-half shells in (8) and their reflections. Put $z = \Delta/L^2$ and let T comprise all parts at most z. On the support of F, $\#T \leq K$ almost everywhere: pinning $K+1$ such parts has total less than half and leaves a nonnegative integrand with zero row. In (11), after pinning $V = (T, P)$, any further $K+2$ such parts still leave the total below half for small h. Use the even Bonferroni upper polynomial of degree $2\lceil K/2\rceil$ for the event that the residual has no part at most z. Its integral against $F(V, \cdot)$ is zero

by the ordinary rows. It majorizes the indicator on positive deviations and is exact on the negative ones, where at most $K - \#V$ factors remain. Thus

$$\int_{\min Q > z} |F(V,Q)|\, d\nu_{1-|V|}(Q) \le 2 \sum_{j \le K-\#V} \nu_{j,1-|V|}(\min Q > z).$$

For $0 < \lambda < 1$, marking one part by its share gives the $\lambda^{\#}$-tilted baseline density of lists above z at total y at most $\lambda z^{-\lambda}/y$. Apply this to P and the baseline residual; their combined count is at most K. The interval for $|P|$ costs $O(\Delta)$, and the tiny-factor Fourier bound following (11) costs $O(\Delta)$ after integration. A shell is therefore $O(\Delta^{2-2\lambda} L^{4\lambda} \lambda^{-K})$. Sum the dyadic series and choose $\lambda = c/2$. The central remainder vanishes at each fixed floor and height before the uniform estimate is taken. In particular a fixed positive deficit requires negative deviations on lists of at least $(2-o(1))\log(1/h)$ factors. It also requires positive deviations on at least $(1-o(1))\log(1/h)$ factors: if F_+ vanishes above count k, pinning $k+1$ parts of total below half shows that F vanishes above count $2k+1$ almost everywhere.

8 Cycle sizes and the two residual rows

On the simplex $u_1 + \cdots + u_j = y$ use $d\nu_{j,y} = du_1 \cdots du_{j-1}/(j! \prod u_i)$; the one-part mass is $1/y$. For a smooth symmetric shape F of total one the two rows at an observed list v of total t are

$$R_e F(v) = \sum_{j \ge 1} (-1)^{ej} \int F(v,u)\, d\nu_{j,1-t}(u), \qquad e = 0, 1.$$

We require $R_0 = 0$ through $1/2$, $R_1 = 0$ through α, both with a positive margin. On squarefree n insert its normalized prime logarithms in F and use $a_n = 1 - F(n)$; take $a_n = 1$ elsewhere. The arithmetic lifting theorems in [2] apply to smooth finite shapes with a positive factor floor and these row identities, including after multiplication by the reflected presieve. Thus a shape with prime value $F((1)) = 1$ and $1 - C_h \le F \le 1$ suffices.

Given positive labelled lengths of sum one, group them into cycles of an independent uniform permutation. The mass u_C of a cycle is the sum of its labels' lengths a_i. Define

$$H(C) = \#\{i \in C : u_C > 1/2,\ u_C - a_i \le 1/2\}, \qquad f = 1 - \sum_C H(C).$$

For fixed lengths at most b, an observed collection of cycles with mass strictly below $1/2$ has no mark. In a uniform ordering of the remaining labels exactly one prefix first exceeds $1/2$. For a set T of l out of k residual labels the probability of its giving that prefix is $H(T)/(l\binom{k}{l})$, while its probability of being a cycle in the residual permutation is $1/(l\binom{k}{l})$. Thus the conditional mean of f is zero. If the observed mass is less than $1/2 - 2b$, at least two labels remain even after prescribing a marked residual cycle of mass at most $1/2 + b$. The mean of $(-1)^{\#\text{cycles}}$ is zero on a symmetric group of degree at least two. This gives the signed conditional identity. Choosing ordered observed cycles and disintegrating in their sizes gives the two physical row identities as in [2, Section 1]. To include the ordinary endpoint, discard a small open strip about every proper half-subset sum of the labels before the final permutation. This retains an ordinary margin.

9 The last split

Put one circle at each physical size u_i, with arclength of total one. Sample iid uniform points on their union. A configuration succeeds when every circle is occupied and all cyclic gaps are at most

s, where $4s < h$ for the two-pass argument. The explicit construction below uses $3s < h$. Let E_m mean that every circle is occupied at time m, at least one gap exceeds s, and the next point gives success. The events are disjoint. On the single circle their union has probability one, since eventual coverage has probability one. Given occupation, the conditional probability of E_m is the symmetric function

$$J_m(a) = \sum_{i=1}^{m} \mathbf{1}_{a_i > s}(2s - a_i)_+ \prod_{j \neq i} \mathbf{1}_{a_j \leq s}. \tag{19}$$

The new point must split the unique long gap into two pieces of size at most s. The maximum initial atom on the support is $2s$.

Take the uniform simplex law on m labelled lengths, multiply by mJ_m, and group them by a uniform permutation. The resulting unsigned physical density relative to $\nu_{j,1}$ equals $\Pr_u(E_m)$. Indeed for cyclic group sizes $l_1, \ldots, l_j \geq 1$ the factor from the permutation with its cycles in uniform order is $1/(j! \prod l_i)$. The labelled atom density after multiplying by m is $m!$. Each group has fibre volume $u_i^{l_i - 1}/(l_i - 1)!$. Division by the physical reference density gives the multinomial weights $m! \prod_i u_i^{l_i}/l_i!$. Conditional on those occupancies, the gaps rooted at a uniformly named point on each circle are independent uniform simplexes of those totals. For one point the gap is the whole circle. The same calculation holds with any additional statistic of the oriented labelled cycles after uniform labelling and rooting; in particular it holds with independent marks on the sample points. Multiplication by f therefore gives a signed density bounded above by the disjoint-event sum, uniformly on all physical lists. The prime has no mark.

A circle of length $u > s$ can be divided into $\lceil 2u/s \rceil$ equal arcs of length between $s/3$ and $s/2$. A gap longer than s contains one entire such arc. There are at most $3/s$ arcs in total. Hence

$$\sum_{m \geq n} \Pr_u(E_m) \leq \min(1, 3s^{-1} e^{-ns/3}). \tag{20}$$

Since at most m atoms can be marked, (20) alone bounds the expected number of marks by $O(s^{-1} \log(2/s))$, uniformly in u. The first arrival at full coverage on a multi-circle list can instead be a first hit on a small empty circle. Excluding that possibility in (19) supplies a quantitative reserve. If a specified circle has length at most $s/16$, it is empty at $n = \lceil 15 s^{-1} \log(1/s) \rceil$ with probability at least $(1 - s/16)^n \gg s^2$. All long circles are already covered at time n outside an event of probability $O(s^4)$ by the same arc test. On their intersection no counted transition ever occurs. Consequently $\sum_m \Pr_u(E_m) \leq 1 - c_0 s^2$ on such lists, for small s.

10 Ordering the contractions

Take $s = h/32$. Attach two independent continuous uniform priorities to each original outgoing slot, independently of the lengths and permutation. Initially each labelled atom is a path. In a pass at threshold b, visit the current pass's initial outgoing slots in its priority order. When the source path has mass at least b, skip the slot. Otherwise disclose the target. Discard on closure, and on a merger attach the short source before its target. Only the outgoing slot that was queried is removed. A skipped path stays above threshold. A finished pass leaves all paths of mass at least b. On the full cycle a disclosure cannot close a proper path, and the path of mass one would never be queried. At every stop the residual permutation of path heads is uniform conditional on the priorities, lengths and disclosed edges. Indeed every bijection of the remaining slots to heads has one extension and all were equiprobable. This argument only requires that the next query and each stop use disclosed information. All retained histories preserve the physical circle sizes.

Lemma 7. *Suppose the starting paths have mass at most $B \geq 2b$. A path which first exceeds $B + D$ requires at least $r = \lfloor D/b \rfloor$ distinct boundary points in a single arc of length at most $B + D + b$. Read in backward spatial order, their priorities are increasing.*

Proof. Trace the target of the overflowing path backwards to when its persistent constituent first reached b. It was an initial path, or two paths of mass below b had just merged. It then had mass at most B. Since becoming ineligible as a source, it has only grown on its incoming side by paths of mass below b. The successive boundaries erased there occur in the order of the queries. Their spatial order is unchanged by earlier contractions inside incoming paths. □

In the applications $B + D \geq 3s$ and $B + D + b \leq 7s$. On circles of circumference above $B + D$ cover every possible witness arc by one of $O(1/s)$ deterministic arcs of length at most $9s$. Use a mesh of spacing at most s on long circles and each whole short circle when necessary. For n iid points, the probability of a witness among the first $m \leq n$ points is at most

$$O(r/s)\binom{n}{r}(9s)^r/r! \leq O(r/s)(9e^2ns/r^2)^r. \tag{21}$$

In a fixed covering arc, choose the point labels and use independence of positions and priorities. At most r cyclic rotations of their order qualify, including where the arc is the whole circle. Enlarging the set of points preserves the existence of the ordered subset, even when the preceding pass has erased some slots. Assign priorities to the whole infinite iid sample to couple all m.

For the first pass take $b = c_1 s \log L/L$. End the pass immediately after a merger first exceeds $3s$, keeping its result. All its paths are then at most $4s$. On normal completion there are at most $1/b$ paths. An early stop requires (21) with $B = 2s$, $D = s$. For $m \leq n_1 = \lceil 30L/s \rceil$, the logarithm of the power on the right is $-(1/c_1 + o(1))L$; the prefactor has logarithm $(1 + o(1))L$. Take c_1 small. The probability bound is $O(s^4)$ uniformly on every physical list. Since the events E_m are disjoint, the expected number of marks at early stops with $m \leq n_1$ is at most $O(n_1 s^4)$. For $m > n_1$ the maximum cost m has total expectation $o(1)$ by (20) and summation by parts. Altogether the expected number of marks is at most $1/b + O(1)$, also after any subsequent contractions or discarding, since their number is always bounded by the number of paths at the end of the first pass.

Keep only $m \leq M = \lceil Kh^{-2}L \rceil$. Carry out a second pass with the fresh independent order, now at threshold $b' = c_2h^2$, discarding if a path exceeds $6s$. A discard on the prime requires (21) with $B = 4s$, $D = 2s$, $r = \lfloor 2s/b' \rfloor$, and $n = M$. Since $Ms/r^2 = O(Kc_2^2hL)$, this loss is bounded by $\exp(-(1/(16c_2) + o(1))L/h)$. The second round uses only a subset of the original boundary points; (21) majorizes by the entire independent second marked sample. The prime loss from $m > M$ is bounded by (20). Given any fixed A_0, choose K large and then c_2 small so that the combined losses are $o(s^2e^{-A_0L/h})$. All retained paths now have lengths in $[c_2h^2, 6s]$. The signed margin follows from $12s < h$. The constants in these two passes are absolute. The explicit moderate-gap caps below use a separate stopping-time estimate.

11 Smoothing, correcting and lifting

For fixed h, M there are finitely many discrete priority orders and disclosure histories. On each retained history approximate its nonnegative atom density from below by a smooth compactly supported density in the interior of its defining length conditions. Insert the smooth proper half-subset mask on the final paths. All excluded comparison and half-subset walls have prime measure zero, since each is a nontrivial affine equality on the original simplex. Symmetrize the labels. At fixed h the additional prime loss can be made arbitrarily small. The cutoffs use only the history, so the

residual permutation is still uniform. Projecting the compactly supported density onto the path lengths and then onto cycle lengths preserves smoothness, by fibre integration for each linear grouping. The conditional identities in Section 8 give exact zero rows on open neighbourhoods of both required ranges. In particular when an observed total is close to half all contributing shapes vanish. The signed density F is bounded above by the original unsigned event density, and its negative part by the estimated mark cost.

Use the fixed-point correction of [2, Section 3], with an odd number $k \asymp h^{-1}$ of nearly equal atoms large enough that $1/k + 1/(10k^2) < s/16$. The inequalities in that proof improve when kh is increased. Its shape H has the same zero rows, prime value one and norm at most $\exp(A_0 L/h)$ for an absolute A_0. All factors are $\gg h$ and every nonprime in its support has a part of size at most $s/16$. Choose K, c_2 for this A_0 and then smooth sufficiently closely. The prime deficit $e = 1 - F((1))$ satisfies $e\|H\|_\infty < c_0 s^2/2$. Thus $F + eH \le 1$ everywhere, has exact prime value one, and its negative bound increases by less than one. This proves the upper bound and factor floor in Theorem 1.

For a prescribed surviving prime fraction e^{-T}, $T \ge 1$, the same cap permits a factor floor

$$\gg \max\{h^2,\ h\log(L+T)/(L+T)\}.$$

Indeed take $M = \lceil K(L+T)/s \rceil$ and use second threshold $cs\log(L+T)/(L+T)$. In (21) we have $Ms = O(L+T)$ and $r \asymp (L+T)/(c\log(L+T))$. The exponent in the power is $-(2/c + o(1))(L+T)$ when $r = \lfloor 2s/b' \rfloor$. With K large and c small the truncation and second-pass losses together are smaller than $e^{-T}/2$. The first pass keeps early stops and smoothing at fixed h, T costs arbitrarily little. No final correction is needed. Use the exact theorem whenever its floor is larger.

Additional divisor ranges and ordinary surplus. At the cap and floor of Theorem 1, one can also suppress the perturbation whenever a nontrivial proper divisor has logarithmic size in any prescribed compact $E \subset [0, 1/2]$ of measure $\le \exp(-K/h^2)$, for a sufficiently large absolute K. This strengthens the cap in the analogous assertion of [2]. Indeed on any stopped history with $k \le 1/\eta$ paths, a designated proper j-element subset forms a residual cycle with probability $1/(j\binom{k}{j}) \ge 1/(k2^k)$; a full cycle has probability $1/k$. If a path subset has total in E, designate one such subset. The unsigned physical image is dominated by the factorial baseline. On lists of floor η, its mass with a marked proper cycle in E is at most $\int_{E\cap[\eta,1-\eta]} W_\eta(1-t)\,dt/t \le \eta^{-2}|E|$. Thus masking all such histories costs the prime at most $2^{1/\eta}\eta^{-2}|E|$, uniformly in the number of pre-contraction points. At $\eta = ch^2$ this loss fits the final correction above. Masking the correction itself costs $O(2^k k^3|E|)$ of its prime value under the conditioned displacement law of [2]. The original margins and smoothing are retained.

In particular the ordinary multiple exponent can reach $1/2 + \exp(-K/h^2)$ at signed exponent $1/2 - h$ with complete removal and the stopped cap. Mask a short interval just below half; complementation also kills the perturbation just above half. For each fixed surviving fraction in $(0, 1)$ the positive ordinary surplus can reach $\exp(-K_d L/(h\log L))$, by using the larger partial-removal floor and the same mask. The full partial formula and limiting order are in the supplement.

The large norm of the fixed-point correction is necessary for its support. Let $H((1)) = 1$, give H zero ordinary rows below γ and floor η, and require every composite in its support to have a part in $I \subset [\eta, b]$. Write Z for the count of such parts and $J = \int_I du/u$. For $mb < \gamma$, $(m+1)b < 1$ all $\int H\binom{Z}{j}\,d\nu_1 = 0$, $0 \le j \le m$, by marking. The alternating sum leaves the prime and the tail $Z > m$. On this tail $\binom{Z-1}{m} \le \binom{Z}{m+1}$, whose integral over the floor-restricted baseline is at most $J^{m+1}/(\eta(m+1)!)$. Thus $\|H_+\|_\infty \ge \eta(m+1)!/J^{m+1}$ for odd m, and the same lower bound holds for $\|H_-\|_\infty$ for even m. For the k almost-equal correction atoms, $J \asymp 1/k$. Consecutive parities at $m = k/2 + O(1)$ give $\log\|H_\pm\|_\infty \ge (1 - o(1))k\log k$ separately. This matches the order of the available logarithmic upper bound.

The limitation of a single positive kernel

For a nonnegative integrable kernel K supported in $[\eta, b]$, $0 < \eta \le b < 1$, put $B(t) = t\sum_{j\ge1} K^{*j}(t)/j$. Suppose $B \le 1$ on $(0,1)$. Even at a single total a product of these unsigned cycle densities has the restriction

$$B(1) \le \big(1 - \rho(b/\eta)\big)^{\lceil 1/b\rceil - 1} < 1,$$

where ρ is the Dickman function. To see this put $S(t) = 1 - \int_0^t K$. Logarithmic differentiation of $\widehat{S} = (1-\widehat{K})/p$ at large real Laplace variable p gives $tS = (1-B)*S$ since $\widehat{B} = -\widehat{K}'/(1-\widehat{K})$. On $[0,\eta]$, $S = 1$. Comparison with $t\rho(t/\eta) = (\mathbf{1}_{[0,\eta]} * \rho(\cdot/\eta))(t)$ gives $S(t) \ge \rho(t/\eta)$ through one. The Volterra operator has nonnegative kernel, and after the initial interval its iterates are bounded by $\eta^{-j}/j!$. Hence the mass deficit $\delta = 1 - \int K$ is at least $\rho(b/\eta)$. Marking one atom gives $B = tK + K*B$; beyond b each iteration costs the factor $1-\delta$. At total one we may iterate at least $\lceil 1/b\rceil - 1$ times, proving the bound.

If $1 - B(1) \le e^{-T}$, $T \ge 1$, then $\rho(b/\eta) \ll be^{-T}$ for small b. Iterating $\rho(x) \ge \rho(x-1/2)/(2x)$ gives $\rho(x) \ge e^{-O(x\log(2+x))}$. Thus $\eta \ll b\log V/V$, where $V = T + \log(2/b)$. The inverse kernel of [2, Section 4] attains this order: use its dimensionless lower cutoff $\log V$, scale $W = DV/b$, and a smooth upper cutoff between $0.9b$ and b. The bound on the lost prime mass there is $\exp(O(\sqrt{V}\log V + \log W) - cWb)$, in addition to e^{-W}. Choose D large. This proves matching orders for the single kernel with this individual-size upper. Qualitatively $B(1) < 1$ also holds with no positive lower floor: $\int_0^a B(t)dt/t$ is finite for small a by summing the convolution series. At a hypothetical first zero t_0 of S the integral $(1-B)*S(t_0)$ would be strictly positive from arbitrarily small positive arguments of $1-B$. Thus $S(b) > 0$.

The exact exponential rate can be expressed using the classical step inverse $k(t) = -\rho'(t/\eta)/\eta$, used in the modified Liouville construction of [8, Section 9]. Its cycle density is $\mathbf{1}_{t\ge\eta}$, and its renewal measure is $\delta_0 + \eta^{-1}\omega(t/\eta)dt$. Set $K_0 = k\mathbf{1}_{t\le b}$, $r = b/\eta > 2$, and define $s > 0$ by $\int_0^b e^{st}k(t)dt = 1$. Then $\int_0^b e^{st}K(t)dt \le 1$ for every admissible K, with equality only for K_0. Indeed put $E = \delta_0 + E_+ = \exp_*(\mathbf{1}_{[\eta,b]}(1-B(t))dt/t)$. On $[0,b]$, the convolution logarithm gives $\delta_0 - K = (\delta_0 - k)*E$, identifying densities with measures. Thus the addition to the tilted k-mass is

$$\int_{(0,b]} e^{sv}\left(\int_0^{b-v} e^{su}k(u)du - 1\right)E_+(dv) \le 0.$$

The integrand is strictly negative. Equality forces $E_+ = 0$ there.

The tilted renewal equation beyond b now gives $B_K(1) \le e^{-s(1-b)}$. For K_0 the tilted kernel has mass one. Its original cycle density equals one on $[\eta, b]$ and $1 - tk(t)$ on $(b, b+\eta)$, since a deleted atom in this latter range leaves a residual shorter than η. Stop the tilted recurrence when the residual first falls below $b+\eta$. For $1 > b+\eta$ this gives

$$(1-\rho(r-1))e^{-s(1-\eta)} \le B_{K_0}(1) \le \sup_K B_K(1) \le e^{-s(1-b)}.$$

Here $s \sim e^{-\gamma_E}\rho(r)/\eta$ as $r\to\infty$: the lost mass is $\rho(r)$, the truncated first moment is $\eta\int_0^r \rho - b\rho(r) \sim e^{\gamma_E}\eta$, and $sb = O(r\rho(r)) = o(1)$.

In fact

$$-\log\sup_K B_K(1) \sim e^{-\gamma_E}\rho(r)/\eta \qquad (b\to 0,\ r\to\infty).$$

The sandwich proves this whenever s stays away from zero. For $s\to 0$, mark one deleted atom at total one to bound $1 - B_{K_0}(1)$ by $k(1) + \int_b^1 k(v)\eta^{-1}\omega((1-v)/\eta)dv$. On $v \le 1/2$ the renewal density is uniformly $(e^{-\gamma_E} + o(1))/\eta$. The rest is negligible by $\omega \le 1$ and $\rho(1/(2\eta)) = o(\rho(r))$, using $x\rho(x) \le \rho(x-1)$. This gives the matching leading constant even for vanishing loss. Smooth

approximations from below retain the upper and approximate the truncated density at one. Since $-\log \rho(r) \sim r \log r$, the largest floor at prime error e^{-T} is $(1+o(1))b \log V/V$, $V = T + \log(1/b)$, uniformly along $T \geq 1$.

The mean length without a hard lower cutoff

The preceding order of magnitude also limits the mean atom length, even allowing lengths arbitrarily close to zero. For $K \geq 0$ supported on $(0, b)$ with $B \leq 1$ through one and $B(1) \geq 1 - e^{-T}$, the supremum of $\int tK(t)\, dt$ is

$$(e^{\gamma_E} + o(1))\, b \log V/V, \qquad V = T + \log(1/b).$$

This holds uniformly along $T \geq 1$, $b \downarrow 0$. Here is a Poisson comparison giving the leading constant. Put $G = 1 - B$. The first-zero argument above gives $\delta = S(b) > 0$. Since $B = tK + K * B$, the subprobability convolution beyond b propagates $B \leq 1$ to every total. Put $\eta_0 = b \exp(-\int_0^b B(t)dt/t)$, $M = e^{\gamma_E}\eta_0$. Let X be the total of the Poisson process with intensity $\mathbf{1}_{(0,b)}G(v)\, dv/v$. As measures on the positive half-line,

$$S(t)\, dt = M\, \mathcal{L}(X) * \exp_*(\mathbf{1}_{v>b} G(v)\, dv/v)\,.$$

Indeed both Laplace transforms have logarithmic derivative $-\widehat{G}(p)$. Their leading terms as $p \to \infty$ are $1/p$, by $\int_0^b (1 - e^{-pv})\, dv/v = \log(pb) + \gamma_E + o(1)$. The measure $M\mathcal{L}(X)$ thus has a density $H = S$ up to b, and $H \leq \delta$ beyond b. The Poisson transform gives $tH = (\mathbf{1}_{(0,b)}G) * H$; hence $\sup_{[jb,(j+1)b]} H \leq \delta/j!$, $j \geq 1$, for essential suprema. For $j \geq 2$ the recurrence bounds this supremum by $1/j$ times the larger of itself and its predecessor.

To pass from $\mathbf{1}_{(0,\eta_0)}dv/v$ to the intensity of X, a finite equal mass is removed from below η_0 and added above it. This is exactly the defining equation for η_0. Couple the common process and use the same Poisson count for the moved jumps. Then X dominates $\eta_0 X_D$, with X_D of density $e^{-\gamma_E}\rho$, the classical scale-invariant Poisson sum [12]. For $r = b/\eta_0$,

$$\delta \geq \frac{1}{(e-1)r} \int_r^\infty \rho(t)\, dt, \qquad \int_0^b tK(t)\, dt + b\delta = \int_0^b S(t)\, dt \leq M.$$

The first inequality follows by integrating H beyond b. The tail integral has negative logarithm asymptotic to $r \log r$ by the Dickman estimate and its unit-step recurrence. As $\delta \ll be^{-T}$, this proves the upper bound. For the truncated Dickman kernel above, $S(t) = \rho(t/\eta)$ through b and the admissible η is $(1 - o(1))b \log V/V$. Using $\int_0^\infty \rho = e^{\gamma_E}$ gives the matching lower bound.

12 An explicit cap without the contraction passes

Let τ be the first time all circles longer than s are occupied and have all gaps at most s. On E_m we have $\tau = m + 1$. At time n an unsuccessful long circle is empty or has a named point followed by an empty arc of length s. At most $1/s$ long circles and n starting points need be considered. For $\rho = 1 - s$ this gives $\Pr(\tau > n) \leq s^{-1}\rho^n + n\rho^{n-1}$, and hence

$$\mathbb{E}\tau \leq T = n_0 + n_0\rho^{n_0-1}/s + 2\rho^{n_0}/s^2 \qquad (n_0 \geq 1).$$

If the only possible marked cycle has size $y > 1/2$, its expected marks are at most $y\mathbb{E}\tau$ by Wald's identity applied to the number of points in that circle. They are also at most $y/(y - 1/2)$ since each marked gap is at least $y - 1/2$. The minimum is always at most $1 + T/2$. For fixed h, finite truncation and smoothing can make the prime deficit arbitrarily small at this fixed mark bound. The strict

composite reserve of the model in [2, Sections 2–3] allows a convex mixture with a slightly enlarged copy of that model, chosen to restore exact prime value at arbitrarily small cost to the cap. Indeed for its shape G_0 there is some $\epsilon > 0$ with $(1+\epsilon)G_0 \le 1$ on all composites, by the product upper there and the strict correction bound on its support. If the deficit is e, use enlarged-component proportion $e/(e+\epsilon)$. In this uncontracted construction only one atom can exceed s. The sum of any two distinct atoms is strictly below h provided $3s < h$. This suffices for the signed conditional proof. A marked cycle leaves either no label, contradicting that its marking atom has mass less than h, or a singleton j, contradicting $a_i + a_j < h$ for its marking atom i and the distinct label j, or at least two labels with zero cycle-sign mean. Every cap strictly above $2 + T/2$ follows, with a positive factor floor depending on h and the cap. Take $s = 999h/3000$ and use the following rational evaluations, rounded upwards.

α	3/8	19/50	23/50	49/100
n_0	118	124	471	2366
$2 + T/2$ (upper)	74.622	78.276	280.074	1351.925
C	75	79	281	1352

The circular spacing law and random covering estimates themselves are classical [10, 11]. Spacing couplings for Ewens permutations and integer factorizations are developed in [9]. The stopping multiplier (19) is used here for its uniform physical upper across all cycle groupings.

Density of sets of composites

In the flat problem the supremal density of a set of composites passing the same tests relative to its density equals $1/C_*(h)$. For one direction normalize the indicator. For the other take independent uniforms U_n and the set $A = \{n > 1 : U_n < a_n/C\}$ at any feasible cap C. Hoeffding's inequality and a union bound over intervals, residues and moduli bound every random row error by $O(\sqrt{(x/q+1)\log x})$ almost surely at all large dyadic endpoints, also for the Möbius-weighted rows. Summing costs $x^{3/4+o(1)}$. The set has density $1/C$. For the constructed model the comparison set $\{n > 1 : U_n < 1/C\}$ agrees term by term with A off the rough support and has relative density $1/C$ on the primes.

13 Continuum certificates at fixed height

In this section f denotes the deviation from the baseline, before normalization at the prime. It takes the opposite sign to the removal shape F in Section 8. Partition (η, α') into small bins and $(\alpha', 1/2)$ into rough bins. The tables with $2\eta = \alpha'$ use a single small bin. The multi-bin tables record all the internal cuts; for example at 33/200 the small lower endpoints are $7/200, 13/200, 1/10, 13/100$. Write r for the vector of all bin counts. Give free value c_r to a shape of total one with no giant and all parts in bins; an omitted coefficient is zero. In each nonzero box the two smallest small-bin lower endpoints, counted with multiplicity when there are at least two, sum to at least α'. Thus a physical pair of small parts with sum below α' has zero free coefficient. When there is exactly one small part s and j rough parts add $w_j H(s)$, where $w_2 = 1,\ w_3 = -1$ and all other $w_j = 0$. A part below η always gives zero deviation. Away from the finitely many bin walls there are two kinds of physical lists, these free lists and lists with a unique giant $y > 1/2$. We first extend the free data to the latter, then determine H and one scalar coefficient by two signed equations.

13.1 Inverting the ordinary row

Write $\|v\| = \sum_i v_i$ for a count vector, e_i for the unit of bin i, and $\chi = \mathbf{1}_{(1/2,\infty)}$. With convolution in the mass variable, put

$$K_v = \underset{i=0}{\overset{d}{*}} \frac{(\mathbf{1}_{I_i}(t)\,dt/t)^{*v_i}}{v_i!}, \qquad K_0 = \delta_0.$$

We use the same symbols for the densities of the nonempty measures. For $y > 1/2$ define

$$L_d(y) = \sum_{v \le d} (-1)^{\|v\|} [K_v * (\chi K_{d-v})](y), \tag{22}$$

$$T_d(y) = (-1)^{\|d\|} K_d(y) + \sum_{v \le d} (-2)^{\|v\|} [K_v * (\chi K_{d-v})](y). \tag{23}$$

Here d as a subscript is a count vector. For the constant-box data the value with a giant y and remaining observed bin counts z is $-y \sum_{r \ge z} c_r L_{r-z}(y)$. In fact imposing the ordinary row first gives the completion with no giant; completing a larger observed list around a giant convolves with K_v. The vector-indexed convolution kernel $(K_v)_v$ has inverse $((-1)^{\|v\|} K_v)_v$, by the binomial theorem in each coordinate. The truncation to giant masses can be placed at the innermost completion: if the last mass exceeds half, all previous masses do too. This proves (22). The signed row adds the sign of each remaining part. Expanding the giant value and combining both selections of unobserved parts gives $\sum_{v' \le v} \prod_i \binom{v_i}{v'_i} = 2^{\|v\|}$, giving (23). These formulas include the empty observation and a one-part completion with mass $1/y$.

For a single pooled rough interval $(\alpha', 1/2)$, denote the corresponding kernels by R_j, L_j^R, T_j^R. Fragmenting a giant leaves every observed small pair in the free list. Hence, at every signed level strictly below α' the only nonempty observation with a nonzero deviation is one small part s. For s in small bin i its equation is

$$H(s) = -\frac{\sum_{r \ge e_i} c_r T_{r-e_i}(1-s)}{\sum_{j=2,3} w_j T_j^R(1-s)}. \tag{24}$$

Pinning s removes one factorial occurrence in its bin. The denominator stays positive on the entire small interval in each certificate. For the contribution with the tag unobserved put $J_0(t) = \mathbf{1}_{(\eta,\alpha')}(t) H(t)/t$, $J_k = J_0 * R_k$. Define

$$P_k = J_k + \sum_{v=1}^{k-1} (-1)^v R_v * (\chi J_{k-v}) - \sum_{v=0}^{k-2} (-1)^v J_v * (\chi R_{k-v}),$$

$$S_k = (-1)^{k+1} J_k + J_k + \sum_{v=1}^{k-1} (-2)^v R_v * (\chi J_{k-v}) - 2 \sum_{v=0}^{k-2} (-2)^v J_v * (\chi R_{k-v}).$$

Here $P_0 = 0$ on $y > 1/2$. In a giant row with b observed rough parts and no observed small part the added value is $-y \sum_{j \ge b} w_j P_{j-b}(y)$. With one observed small part s it is $-yH(s) \sum_{j \ge b} w_j L_{j-b}^R(y)$; with more there is no tagged contribution. These expressions follow by putting the tag in each of the two factors of (22). The corresponding empty signed contribution is $\sum_j w_j S_j(1)$ by (23). Let d_r be a fixed real direction supported on no-small boxes. Vary c_r by $t d_r$, where

$$t = -\frac{\sum_r c_r T_r(1) + \sum_j w_j S_j(1)}{\sum_r d_r T_r(1)}.$$

The one-box certificates use a single $d_g = 1$. Some multi-bin certificates use $d_r = \operatorname{sgn} T_r(1)$ on no-small boxes whose outward interval excludes zero. Every certificate separates its denominator from zero. This preserves (24) and enforces the empty signed equation. In the multi-box verification the common scalar is retained while summing the responses to a giant or a prime.

13.2 Enclosing all physical values

The source supplement contains the rational tables and a verifier for the whole continuous simplex. The following upper bounds are rounded outwards. Here m is the number of mesh cells in $[0,1]$, $d_{\min}$ is a lower bound for the denominator in (24), p_+ bounds $f((1))$ above, and $[f_-, f_+]$ encloses all nonprime values, including free boxes with coefficient zero.

α'	m	$\#c_r$	$d_{\min}$	p_+	f_-	f_+
11/100	480000	161	4.05824	−.980750	−.954415	.942909
13/100	800000	378	3.41527	−.990241	−.988389	.980170
29/200	240000	31	3.45902	−.988205	−.948991	2.396238
3/20	800000	110	3.35837	−.994857	−.953429	2.425556
61/400	800000	192	3.30644	−.993899	−.982502	2.417558
31/200	800000	111	3.25344	−.994902	−.991497	2.973112
33/200	400000	202	2.77297	−.987055	−.942858	2.364087
7/40	800000	536	2.54047	−.991527	−.990276	2.362787
9/50	800000	450	2.43747	−.992919	−.981533	2.861397
29/160	800000	547	2.41152	−.992695	−.981301	2.931399

The verifier computes $1+f_+/(-p_+)$ using its exact rational endpoints. The strict lower reserve is $f_- - p_+ > 0$. The three independently fitted six-bin tables use dyadic coefficients of denominator 2^{24} and mesh 320000. The rough edges after the signed cutoff are $1/5, 11/40, 7/20, 17/40, 1/2$. In particular the 31/200 fit has small bin $(31/400, 31/200)$. Their independent verifier returns positivity reserves exceeding $.01018, .00777, .00461$, respectively in the order $3/20, 61/400, 31/200$. The supplement supplies these coefficients, the independent implementation and the exact rational outputs.

The printed table rounds a second time and is intended to show the size of the reserves. A check at one fixed cap takes the following steps.

Represent each cell of each signed kernel by upper and lower integers in units of 10^{-8}. For $t \in [i/m, (i+1)/m]$ enclose $1/t$ by outward rounding on its active bin. Convolution of two step functions is linear between mesh vertices. Its value at k/m is $1/m$ times coefficient $k-1$ of the product of their arrays. On each new cell take the minimum of the lower endpoints and the maximum of the upper endpoints. For signed kernels separate positive and negative parts with their appropriate rounding. Exact finite field transforms and Chinese remaindering give the integer products. The code checks the product of the active moduli against a deterministic bound on every coefficient. All bins and the cut at $1/2$ meet mesh boundaries. These steps successively enclose the kernels in (22)–(24) and their tagged convolutions.

For a giant cell, intersect the possible small-coordinate interval with the complement of that giant cell and the sum of the other observed bin intervals. A range-minimum and range-maximum query includes every intersecting small-coordinate cell. The final product uses both endpoints of the giant cell. One-sided traces on the bin walls suffice for the integral equations. At a rational bin endpoint a proper prime coordinate could agree with it only if n were a prime power. The uniform bounds include all open cells and their limiting values.

The lifting theorem in [1] is stated for class data. Its mask estimate and the four transfer steps use bounded densities on fixed polyhedral bins, with piecewise Lipschitz dependence along the remaining coordinates. Here the minimum observed count parameter is one. To obtain an ordinary level margin, multiply the free data by the proper half-subsum mask from [1, Section 4] and repeat both signed solves with the masked columns. For signed observations of total at most α', a mixed subsum near half uses some and omits some of the integrated variables. Holding their pairwise total fixed, one included coordinate varies across an interval of length $O(\varepsilon)$. The fixed floor bounds the

simplex densities and the number of such choices. A subsum involving only observed coordinates, or their complement, cannot be near half. In the ordinary extension each fragmentation retains existing subsums. Consequently the masked value vanishes on its forbidden half-strip and changes by $O(\varepsilon)$ elsewhere, by the same masking lemma. The denominator and empty direction stay separated from zero, and both corrections move by $O(\varepsilon)$. The nonprime and cap reserves therefore survive the normalization $a = 1 + f/|f((1))|$, including on the strip. The functions are piecewise Lipschitz on fixed polytopes: the simple kernels are smooth inside bins, convolution integrates over finitely many moving linear faces, and division uses the uniform separation in (24). The arithmetic lifting theorem of [2] is stated for smooth shapes. Its proof uses smoothness only to control variation in a prime variable, to evaluate rows at the observed list of a fixed divisor, and to approximate the test on logarithmic cells. Every wall here has the form $\sum_{i\in U} u_i = c$ with $0 < c < 1$, and the row identities hold off the walls. In absolute logarithms the coefficient of each coordinate is either $-c$ or $1 - c$. Keeping the other prime variables fixed, a varying prime crosses each wall at most once. The observed list of a fixed divisor meets each observed-coordinate wall at most at one scale, a null set in the outer integral. In the cell argument the cells of width $(\log X)^{-H}$ meeting a wall fit into an interval of length $O((\log X)^{-H})$ in one residual normalized logarithm when the others are fixed. Apply the cell-wise progression estimate to bound these cells. Their principal terms cost $O((\log X)^{1-H})$ times the trivial dyadic bound. These observations give (1) and the reflected estimates with the asserted floor by the same lifting proof. This completes the certificate part of Theorem 3.

13.3 When two small parts must contribute

There is a limitation on the support restriction used in the multi-bin tables. Write $f = A - 1$ on a free list $u = (u_1, \dots, u_k)$ with $\max u_i < 1/2$. The empty ordinary inverse and the empty signed row give $f((1)) = \int Q(u) f(u)\, d\nu$, where $Q = -\ell + t/3$ and

$$\ell = \sum_{|u_S|>1/2} (-1)^{k-\#S}, \qquad t = (-1)^k + \sum_{|u_S|>1/2} (-2)^{k-\#S}.$$

Here $|u_S|$ is the mass of the indexed sublist. These are the unbinned kernels of (22) and (23), including the full subset. Suppose the free perturbation is supported on $B_k = \{u : \min_{i<j}(u_i + u_j) \geq a,\ \max u_i < 1/2\}$, where $1/3 \leq a < 1/2$. Apart from null sets the possible counts are three, four and five. In count three all pairs exceed half. In count four exactly one pair in each complementary pair of pairs exceeds half. In count five every triple exceeds half, since its three pair-sums are at least a. This gives $Q = 0, 2, 4$, respectively. Thus the deficit is at most $2\nu_4(B_4) + 4\nu_5(B_5)$, using only $f \geq -1$.

For the least coordinate x, put $b = \max(x, a - x)$ and $r = 1 - x - (k-1)b$. The other $k - 1$ coordinates have product at least $b^{k-2}(b + r)$ when $r \geq 0$. Dropping their upper restrictions gives

$$\nu_k(B_k) \leq \frac{1}{(k-1)!(k-2)!} \int_{\max(0,((k-1)a-1)/(k-2))}^{1/k} \frac{r_+^{k-2}}{x b^{k-2}(b+r)}\, dx,$$

with integrand taken as zero for $r < 0$. The supplement encloses this integral by rational boxes on either side of $a/2$. At least 0.355055 of the prime survives at $a = 1/3$ and 0.686197 at $a = 13/36$, at arbitrary height. A model removing the prime at the latter level therefore needs a contribution of magnitude at least 0.686197 from free profiles with an observable pair.

14 Finite grids and the Goldbach comparison

The uniform height in this problem has the same normalization as the whole-shape permutation grids of [1, Section 8]. The exact zeros at mesh orders $36, 72$, floor $1/36$, twisted level $13/36$ and

height four hold at discrete cycle sizes. Verification on the continuous cells above is what permits the arithmetic conclusions in Theorem 3. There is a simple obstruction to using a single grid by polynomial interpolation.

Lemma 8. *Suppose symmetric G_j are real analytic on the open total-one simplex in each dimension j, and $G_j = 0$ for all sufficiently large j. If the ordinary residual rows vanish almost everywhere at every observation of total below some positive level, including the empty observation, all G_j vanish.*

Proof. At the greatest nonzero index $j \geq 2$, observing $j-1$ entries leaves only the one-coordinate completion. The row equals $G_j(s_1, \ldots, s_{j-1}, 1 - \sum s_i)/(1 - \sum s_i)$. It vanishes on an open subset by continuity, hence on the entire connected open simplex by real analyticity. The empty row then gives the result in dimension one. □

For example assign M independent sample points to boxes of masses $u_1, \ldots, u_j$; average the grid perturbation of the occupancies, giving zero if any box is empty. The resulting functions are polynomials, retain the pointwise cap and retain the prime value. Lemma 8 shows that any such finite interpolation of a prime-free grid has an ordinary residual error at every positive level. A cutoff and an exact correction would have to control both that error and the physical cap. The 13/36 grids alone leave the arithmetic height-four question there open.

Positive lower bounds on the label lengths

Another exact interpolation gives explicit physical costs. Give each of the M labels mass $a + \delta U_i$, with $\delta = 1 - Ma > 0$, where U is uniform on the simplex, independent of the permutation. Group the masses along its cycles. For even M, $a > 1/(M+2)$ ensures that observing physical mass at most one half uses at most $M/2$ labels, with an open margin. Conditionally on any specified observed labelled cycles and all atom lengths the residual permutation is uniform, so the discrete zero rows hold after grouping and disintegration. At $M = 72$ this also gives the signed budget 26 through 13/36.

Write g for the discrete deviation, with $g \geq -1$, $g((M)) = -1$. The grouped deviation relative to $\nu_{k,1}$ at k masses x_i is

$$\widetilde{g}(x) = \frac{M! \prod_i x_i}{\delta^{M-1}} \sum_{\substack{l_i \geq 1 \\ \sum l_i = M}} g(l) \prod_i \frac{(x_i - a l_i)_+^{l_i - 1}}{l_i!}.$$

This is the Dirichlet density of the group totals times the cycle index of the uniform permutation, normalized at its full cycle. For $l_i = 1$ use the positive-side indicator. At $x_i = j_i/M$ for a proper partition j, the coefficient of $g(j)$ is $W_j = \delta^{1-k} M! \prod_i (j_i/M)^{j_i}/j_i!$. If $s = \delta/a \leq 2$, any other contributing assignment moves one label from a donor d to a recipient b with $x_b > 1/s$. Its coefficient divided by W_j is

$$\frac{j_b}{j_b + 1} \left(1 - \frac{1}{s x_b}\right)^{j_b} \left(1 + \frac{1}{s x_d}\right)^{j_d - 2} \leq 2^{-M/2}.$$

The product decreases with $1/s$. At $1/s = 1/2$ the two powers are at most 2^{-M} and $2^{M/2}$, respectively, using $x_b \leq 1$ and $x_d \leq 1/2$. Assignments reaching a face with zero slack in the recipient vanish there. Thus a bound $\widetilde{g} \leq C - 1$ gives $g(j) \leq (C-1)/W_j + (k-1)2^{-M/2}$.

The two empty rows sum separately to zero on odd cycle counts. The prime atom has mass $1/M$. Multiplying the centre inequality by the uniform permutation probability and summing over

odd composite partitions gives

$$1 \le (C-1)\frac{M^{M+1}}{M!}\sum_{\substack{k\ge 3\\ k\text{ odd}}}\frac{\delta^{k-1}}{k!}[z^M]\left(\sum_{j\ge j_0}\frac{(j-1)!}{j^j}z^j\right)^k + MH_M\,2^{-M/2}.$$

Here j_0 is the grid floor in labels and $H_M=\sum_{j\le M}1/j$ is the expected cycle count. At $(M,j_0)=(72,2)$ the rational coefficient calculation with $\delta\le 2/(M+2)$ gives $C>5.58181$. At $(60,2)$ it gives $C>5.06957$. This lower bound for the uniform Dirichlet interpolation uses only the empty rows and $g\ge -1$. It concerns every admissible witness on the indicated grid. The grouped-length laws themselves are classical; see [9] for permutation-spacing couplings.

For the particular certified 72-grid witness of [1, Section 8], there is a stronger obstruction covering arbitrary allocations. Consider the ordered tuple $j=(2,2,2,2,2,2,2,58)$. Its grid deviation is below $-.23889$. The deviations on $(2^6,3,57),(2^6,4,56),(2^5,3,3,56)$ are all nonpositive. The source supplement gives the four class values with their certified radii.

Suppose each ordered tuple l distributes its full signed cycle-index mass $Mg(l)/(k!\prod l_i)$ on $R_l=\{x_i\ge al_i,\ \sum x_i=1\}$, with any probability distribution, $1/74\le a\le 1/72$. Such a distribution may depend on the entire permutation. If R_l meets R_j, then $\sum_i(l_i-j_i)_+\le 1/a-M\le 2$. The nonzero grid floor is two, so all possible contributing types are the four just listed. In units $M/(k!\prod j_i)$ the physical baseline mass of R_j is at most

$$\frac{(1-Ma)^{k-1}}{Ma^k(k-1)!}\le\frac{74}{72}\frac{2^7}{7!}=\frac{74}{2835}.$$

Indeed the residual simplex has volume $(1-Ma)^{k-1}/(k-1)!$ and $x_i\ge aj_i$. The negative mass from type j alone is more than nine times this upper bound. Consequently the resulting measure gives negative mass to R_j. This calculation includes possible singular allocations and boundary intersections.

The genuine source

For a genuine Goldbach row the source weight is $\Lambda(N-n)$. Huang and Li [7] study its Möbius twist with the full initial-interval centre divided by $\varphi(q)$ in reduced source classes. For this particular row ordinary distribution follows from Bombieri–Vinogradov, with a logarithmic loss at the square root. A separate conditional estimate [13] treats the signed level $N^{1/2-h}$ by bounding the variation of an odd complementary-divisor profile. On rough odd composite shapes the classical upper sieve bounds the limiting mass by four times the simplex mass. This is an integrated inequality on composite factor boxes. The pointwise inequality $a_{N,n}\le Cb_n$ defining C_*^{ref} is a condition on every integer and includes the prime stratum. In particular the conditional Goldbach estimate of [13] uses the arithmetic upper only on those composite boxes. With the classical coefficient four, the signed hypothesis at $h=2^{-16}$ gives a limiting Goldbach proportion greater than $1/31$. The maximum-conditioned calculation gives a proportion greater than $1/589$ at $h=2^{-15}$ with the same classical coefficient. With the switched coefficient 3.440568 on these boxes, the interval bound of [13] gives a proportion greater than $1/10205$ at $h=1/15900$. With the same coefficient, a spatially weighted estimate, conditioned on the largest small factor, gives relative error below 0.996501886 and a limiting proportion greater than $1/286$ at $h=1/11500$ [13]. Both calculations use the signed hypothesis for the actual source. The switched coefficient uses an additional estimate for the ordinary distribution of products in the complementary range: Proposition 5.2 of [14], with the factorization of its Proposition 6.1, used at the shift N.

At fixed cap the variation estimate is also an explicit obstruction for the models here. If $D(u)=D_ug(0)$ for one of the odd profiles of [13], the prime and all even-count lists have $D=0$. The flat

baseline has $\int D\,d\nu_1 = -g'(0) = 0$, by the Laplace-transform evaluation of the model first mean in [13]. Hence its two signs have equal mass $\mathcal{V}/2$, where $\mathcal{V} = \int |D|\,d\nu_1$. In (6) the limiting shape of the model obeys $0 \le A \le C$, so the relative prime deficit is at most $C\mathcal{V}/2$. For example, for the radius-15 odd quadratic spline of the maximum-conditioned estimate in [13], the unrounded bound at $h = 2^{-15}$ gives $\mathcal{V} < 0.499151$. Monotonicity in the signed level excludes a height-four prime-free model at and above $\alpha = 1/2 - 2^{-15}$. It leaves a wide interval between this obstruction and the constructions in Theorem 3.

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