hexagon:2611.01250v1

SMALLER, BIPARTITE, AND LEAN 4-VERIFIED COUNTEREXAMPLES TO THE LEVIT–MANDRESCU CONJECTURES FOR VERY WELL-COVERED GRAPHS

JON MCBEE

Abstract. A finite simple graph G without isolated vertices is *very well-covered* (VWC) if every maximal independent set of G has cardinality $\alpha(G) = |V(G)|/2$. Levit and Mandrescu conjectured (2003–2007) that the independence polynomial $I(G;x) = \sum_{k=0}^{\alpha(G)} s_k x^k$ is unimodal and log-concave for every very well-covered graph G, and in particular for every corona graph $H \circ K_1$. In July 2026, Lucas Baker publicly announced a first counterexample refuting both conjectures: a 4,074-vertex connected corona graph ($\alpha = 2037$) obtained by whiskering the join of $\overline{K_{588}}$ and $39K_{11} \cup 85K_{12}$.

In this paper we present five complementary advances. First, we construct explicit *connected bipartite* VWC (Ravindra) counterexamples nearly three times smaller than Baker's corona—$\alpha = 694$ ($|V| = 1388$) failing log-concavity and $\alpha = 704$ ($|V| = 1408$) failing unimodality—together with smaller connected corona counterexamples at $\alpha = 1721$ ($|V| = 3442$) for log-concavity and $\alpha = 1973$ ($|V| = 3946$) for unimodality (reducing Baker's corona by 64 in α and 128 in $|V|$). Second, we provide the first end-to-end formalization in zero-Mathlib Lean 4 (`leanprover/lean4:v4.27.0`) proving both bipartite VWC disproofs directly from `SimpleGraph` with standard axioms only and zero axioms on the numerical witness inequalities. Third, building on Favaron's matching characterization and semidirected-graph precedents in commutative algebra, we prove a Universal VWC Independence Polynomial Theorem characterizing all polynomials realized by VWC graphs. Fourth, using preorder chains and multipartite corona joins augmented by a two-chain Ferrers tie-breaking layer, we prove that for every $L \geq 1$ there exist connected bipartite VWC graphs and connected corona graphs whose independence polynomials have at least L strict interior local maxima and $L - 1$ strict interior local minima, exhibiting an explicit verified strict trimodal witness at $\alpha = 7602$ ($|V| = 15204$). Fifth, on the positive side, we extend the known strict log-concavity threshold for all VWC graphs from $\alpha \leq 5$ to $\alpha \leq 7$ ($|V| \leq 14$) by exact zero-float enumeration over all 3,104,967 canonical representations at $\alpha = 7$.

2020 *Mathematics Subject Classification.* 05C31, 05C69, 05A20.

Key words and phrases. Independence polynomial, very well-covered graph, bipartite graph, corona graph, unimodality, log-concavity, Favaron characterization, Lean 4.

1. Introduction

Let $G = (V, E)$ be a finite simple graph with independence number $\alpha(G)$. Its *independence polynomial* is

$$I(G; x) = \sum_{k=0}^{\alpha(G)} s_k(G)\, x^k,$$

where $s_k(G)$ denotes the number of independent sets of cardinality k in G. A nonnegative sequence $(s_0, s_1, \ldots, s_m)$ (and its generating polynomial) is *unimodal* if there exists an index $p \in \{0, \ldots, m\}$ such that

$$s_0 \leq s_1 \leq \cdots \leq s_p \geq s_{p+1} \geq \cdots \geq s_m,$$

log-concave if $s_k^2 \geq s_{k-1}s_{k+1}$ for all $1 \leq k \leq m-1$, and *strictly log-concave* if $s_k^2 > s_{k-1}s_{k+1}$ for all $1 \leq k \leq m-1$. Every positive log-concave sequence is unimodal.

A graph G is *well-covered* if every maximal independent set of G has cardinality $\alpha(G)$. It is *very well-covered* (VWC) if it is well-covered, has no isolated vertices, and satisfies $|V(G)| = 2\alpha(G)$. Two distinguished subclasses of VWC graphs are:

- *bipartite VWC graphs*, which coincide with bipartite well-covered graphs without isolated vertices and were characterized by Ravindra [19];
- *corona graphs* $H \circ K_1$, obtained from an arbitrary finite simple graph H (the *skeleton*) by attaching a pendant vertex of degree 1 to each vertex of H; note that $\alpha(H \circ K_1) = |V(H)|$.

Brown, Dilcher, and Nowakowski [4] conjectured that $I(G; x)$ is unimodal for every well-covered graph G. That general conjecture was refuted by Michael and Traves [18] (for $\alpha \geq 4$) and Levit and Mandrescu [15]. Levit and Mandrescu consequently restricted the unimodality and log-concavity conjectures to very well-covered graphs and corona graphs: unimodality of $I(H \circ K_1; x)$ was conjectured in [11, Conjecture 3.3], log-concavity for VWC graphs in [13], and both unimodality and log-concavity for all VWC graphs in [12, 14, 16]. Positive theorems established:

- unimodality for all VWC graphs with $\alpha(G) \leq 9$ [12, 16] (specifically [16, Theorem 2.4(v)]);
- unimodality for corona graphs $H \circ K_1$ whenever the *skeleton* H satisfies $\alpha(H) \leq 4$ [11] or $5 \leq \alpha(H) \leq 8$ [5] (while $\alpha(H \circ K_1) = |V(H)|$ is unrestricted);
- log-concavity for all VWC graphs with $\alpha(G) \leq 5$ [16, 7];
- strict log-concavity for forest coronas $F \circ K_1$ (and variance bounds for large forest coronas) in recent work of Levit [10].

Brown and Cameron [3] also showed that every finite graph is an induced subgraph of a VWC graph with unimodal independence polynomial, and Kadrawi and Levit [8] showed that general (non-VWC) tree independence

polynomials fail log-concavity starting at order 26. As of late 2025, both the VWC and general corona $H \circ K_1$ unimodality and log-concavity conjectures remained open in the published literature [7].

In July 2026, Lucas Baker [1] publicly announced a first counterexample refuting both conjectures: the corona graph

$$\big((39K_{11} \cup 85K_{12}) \vee \overline{K_{588}}\big) \circ K_1,$$

a connected corona on 4,074 vertices ($\alpha = 2037$) whose independence polynomial has strict local maxima at $k \in \{1086, 1101\}$ separated by a strict local minimum at $k = 1095$.

Building upon this development, the present paper establishes five complementary results: (i) significantly smaller *connected bipartite* VWC counterexamples and smaller connected corona counterexamples, (ii) the first end-to-end Lean 4 kernel-verified disproof directly from `SimpleGraph`, (iii) a Universal VWC Independence Polynomial Theorem connecting Favaron's characterization with earlier commutative-algebra precedents [9, 2], (iv) an arbitrary L-modality theorem with strict interior local maxima and minima (together with an explicit verified trimodal witness), and (v) exact exhaustive certification of strict log-concavity for all VWC graphs with $\alpha(G) \leq 7$.

Theorem 1.1 (Smaller bipartite and corona counterexamples)**.** *Both the log-concavity and unimodality conjectures fail within connected bipartite very well-covered graphs and within connected corona graphs $H \circ K_1$:*

1. *There exists a connected bipartite VWC graph $G_{\mathrm{bip,lc}}$ with $\alpha = 694$ ($|V| = 1388$) whose independence polynomial fails log-concavity at $k = 351$.*
2. *There exists a connected bipartite VWC graph $G_{\mathrm{bip,uni}}$ with $\alpha = 704$ ($|V| = 1408$) whose independence polynomial is non-unimodal, with strict interior dip $s_{373} > s_{374} < s_{375}$ (strict global peaks at $k \in \{370, 375\}$) and log-concavity failures at $k \in \{372, 373, 374\}$.*
3. *There exists a connected corona graph $G_{\mathrm{cor,lc}} = H_1 \circ K_1$ with $\alpha = 1721$ ($|V| = 3442$) whose independence polynomial fails log-concavity at $k = 645$.*
4. *There exists a connected corona graph $G_{\mathrm{cor,uni}} = H_2 \circ K_1$ with $\alpha = 1973$ ($|V| = 3946$, improving Baker's 4,074-vertex corona by 64 in α and 128 in $|V|$) whose independence polynomial is non-unimodal, with strict local maxima at $k \in \{1061, 1069\}$ and a strict local minimum at $k = 1066$.*

Theorem 1.2 (Universal VWC independence polynomial representation)**.** *A polynomial $P(x)$ of degree m is the independence polynomial of some very well-covered graph on $2m$ vertices if and only if there exist a finite simple graph W on r vertices and positive integers $c_1, \ldots, c_r$ with $\sum_{i=1}^{r} c_i = m$ such that*

$$P(x) = (1+x)^m \sum_{U \in \mathcal{I}(W)} \prod_{i \in U} \big(1 - (1+x)^{-c_i}\big),$$

where $\mathcal{I}(W)$ denotes the collection of independent sets of W.

Theorem 1.3 (Arbitrarily many strict interior modes)**.** *For every integer $L \geq 1$, there exist a connected bipartite very well-covered graph G_L and a connected corona graph $H_L \circ K_1$ whose independence polynomials each have at least L strict interior local maxima ($s_{k-1} < s_k > s_{k+1}$) separated by at least $L-1$ strict interior local minima ($s_{\ell-1} > s_\ell < s_{\ell+1}$).*

Theorem 1.4 (Strict log-concavity for $\alpha \leq 7$)**.** *Every very well-covered graph G with $\alpha(G) \leq 7$ ($|V(G)| \leq 14$) has a strictly log-concave independence polynomial, with sharp minimum log-concavity ratio*

$$\min_{1 \leq k \leq m-1} \frac{s_k(G)^2}{s_{k-1}(G)\, s_{k+1}(G)} \geq \frac{(\lfloor m/2 \rfloor + 1)(\lceil m/2 \rceil + 1)}{\lfloor m/2 \rfloor \lceil m/2 \rceil}$$

for $m = \alpha(G) \in \{2, \ldots, 7\}$, attained by mK_2 (and tied by $K_{m,m}$ for $3 \leq m \leq 7$).

The remainder of the paper is organized as follows. Section 2 reviews Favaron's characterization and proves Theorem 1.2 together with exact join and chain identities. Section 3 develops the asymptotic separation analysis, proves corona clique optimality, and establishes Theorem 1.1. Section 4 proves Theorem 1.3 using a two-chain Ferrers tie-breaking layer and presents an explicit verified strict trimodal witness. Section 5 documents the zero-float exact certification proving Theorem 1.4. Section 6 details the Lean 4 formalization of the bipartite VWC disproofs and the exact verification scope boundaries.

2. Structural Characterization and Universal Representation

We recall the structural characterization of bipartite VWC graphs due to Ravindra [19] and of general VWC graphs due to Favaron [6].

Theorem 2.1 (Ravindra [19]; Favaron [6])**.** *A finite simple graph G with no isolated vertices and $|V(G)| = 2m$ is very well-covered with $\alpha(G) = m$ if and only if G admits a perfect matching $M = \{x_1y_1, \ldots, x_my_m\}$ satisfying:*

- *(P1) $N(x_i) \cap N(y_i) = \emptyset$ for every $1 \leq i \leq m$;*
- *(P2) for every $1 \leq i \leq m$, every $u \in N(x_i) \setminus \{y_i\}$ is adjacent to every $v \in N(y_i) \setminus \{x_i\}$.*

Proof of sufficiency ($\Leftarrow$). Since M is a perfect matching, G has no isolated vertices and every independent set S satisfies $|S \cap \{x_i, y_i\}| \leq 1$, so $|S| \leq m$. Suppose S is an independent set with $|S| < m$; then some matched edge $x_iy_i \in M$ is disjoint from S. If S were maximal, there would exist $u, v \in S$ with $u \in N(x_i) \setminus \{y_i\}$ and $v \in N(y_i) \setminus \{x_i\}$. By (P1), $u \neq v$, and by (P2), $uv \in E(G)$, contradicting that S is independent. Hence every maximal independent set meets each edge of M exactly once and has cardinality m. □

Definition 2.2 (Canonical (D, H) representation)**.** Given a VWC graph G on $2m$ vertices with Favaron matching $M = \{x_1y_1, \ldots, x_my_m\}$, choose any maximal independent set Y (which meets each edge of M once) and orient the matching labels so that $Y = \{y_1, \ldots, y_m\}$ and $X = V(G) \setminus Y = \{x_1, \ldots, x_m\}$. Define the directed relation $D = \{(i, j) : i \neq j,\ y_ix_j \in E(G)\}$ on $[m] = \{1, \ldots, m\}$ and the induced graph $H = G[X]$ on $[m]$.

Because Y is independent, (P1) and (P2) translate directly into three axioms on (D, H):

(1) $D \cup \mathrm{id}_{[m]}$ is a *preorder* (reflexive and transitive);
(A) if $(i, j) \in D$, then $ij \notin E(H)$;
(B) if $ki \in E(H)$ and $(i, j) \in D$, then $kj \in E(H)$.

Conversely, every pair (D, H) on $[m]$ satisfying these three conditions defines a unique VWC graph $G(D, H)$ on $2m$ vertices with edge set $\{x_iy_i : i \in [m]\} \cup \{y_ix_j : (i, j) \in D\} \cup \{x_ix_j : ij \in E(H)\}$. Closely related directed/semidirected and matching-based reductions for edge ideals and Hilbert series of very well-covered graphs appear in commutative algebra, notably in Kimura, Terai, and Yassemi [9] and Brennan and Morey [2].

Lemma 2.3 (Master Formula)**.** *For every VWC graph $G = G(D, H)$ on $2m$ vertices,*

$$I(G(D, H); x) = \sum_{S \in \mathcal{I}(H)} x^{|S|} (1 + x)^{m - |D^-[S]|},$$

where $D^-[S] = \{i \in [m] : i \in S$ or $\exists j \in S$ with $(i, j) \in D\}$.

Proof. Every independent set of $G(D, H)$ decomposes uniquely as $S_X \cup T_Y$, where $S_X = \{x_j : j \in S\}$ for some $S \in \mathcal{I}(H)$ and $T_Y \subseteq Y$ has no neighbor in S_X. Since Y is independent and y_i has a neighbor in S_X if and only if $i \in D^-[S]$, the subset T_Y ranges freely over all $2^{m - |D^-[S]|}$ subsets of $\{y_i : i \notin D^-[S]\}$, contributing $(1 + x)^{m - |D^-[S]|}$. □

We now prove Theorem 1.2, expressing the independence polynomial of an arbitrary VWC graph directly in terms of twin-class weights over a single simple quotient graph W.

Proof of Theorem 1.2. Necessity (⇒). Let $G = G(D, H)$ be a VWC graph on $2m$ vertices. Let $C_1, \ldots, C_r$ be the equivalence (twin) classes of the preorder $D \cup \mathrm{id}_{[m]}$, with sizes $c_i = |C_i| \geq 1$ summing to m, and let $P = D/\!\sim$ be the induced strict partial order on $[r]$. By (A), each class C_i is an independent set in H; by (B), any two vertices in the same twin class have identical H-neighborhoods. Thus H descends unambiguously to a simple quotient graph $\overline{H}$ on $[r]$. Define the simple graph $W = \overline{H} \cup \mathrm{comp}(P)$ on $[r]$, where $\mathrm{comp}(P)$ is the comparability graph of P.

For each $S \in \mathcal{I}(H)$, let $\mathrm{supp}(S) = \{i \in [r] : S \cap C_i \neq \emptyset\}$ and let $U = \max_P(\mathrm{supp}(S))$ be the set of P-maximal classes touched by S. Clearly

U is an antichain in P and an independent set in $\overline{H}$, so $U \in \mathcal{I}(W)$. Moreover, $D^-[S] = \bigcup_{k \in P^-[U]} C_k$, where $P^-[U] = \{k \in [r] : k \in U \text{ or } \exists i \in U \text{ with } k <_P i\}$.

We claim that no class $k \in P^-[U] \setminus U$ has any $\overline{H}$-neighbor in $P^-[U]$. Indeed, suppose $k_1k_2 \in E(\overline{H})$ with $k_1, k_2 \in P^-[U]$ and $k_1 \notin U$, so $k_1 <_P i_1$ for some $i_1 \in U$. If $k_2 \in U$, then (B) applied to $k_2k_1 \in E(\overline{H})$ and $k_1 <_P i_1$ yields $k_2i_1 \in E(\overline{H})$, contradicting $U \in \mathcal{I}(\overline{H})$ (if $k_2 \neq i_1$) or looplessness/axiom (A) (if $k_2 = i_1$). If $k_2 \notin U$, so $k_2 <_P i_2 \in U$, applying (B) twice yields $i_1i_2 \in E(\overline{H})$ (if $i_1 \neq i_2$, contradicting $U \in \mathcal{I}(\overline{H})$) or $k_1i_1 \in E(\overline{H})$ (if $i_1 = i_2$, contradicting (A)).

Consequently, an arbitrary subset $S \subseteq [m]$ with $\max_P(\mathrm{supp}(S)) = U$ is automatically in $\mathcal{I}(H)$, and decomposes independently across the classes $C_1, \ldots, C_r$:

- for each $i \in U$, $S \cap C_i$ ranges over all $2^{c_i} - 1$ nonempty subsets of C_i, contributing $(1+x)^{c_i} - 1 = (1+x)^{c_i}\big(1 - (1+x)^{-c_i}\big)$;
- for each $k \in P^-[U] \setminus U$, $S \cap C_k$ ranges freely over all 2^{c_k} subsets of C_k, contributing $(1+x)^{c_k}$;
- for each $k \notin P^-[U]$, $S \cap C_k = \emptyset$, while the corresponding Y-vertices contribute $(1+x)^{c_k}$ in Lemma 2.3.

Summing Lemma 2.3 over all $U \in \mathcal{I}(W)$ yields

$$I(G;x) = (1+x)^m \sum_{U \in \mathcal{I}(W)} \prod_{i \in U} \big(1 - (1+x)^{-c_i}\big).$$

Sufficiency ($\Leftarrow$). Given any simple graph W on $[r]$ and positive integer weights $c_1, \ldots, c_r$ with $\sum_{i=1}^r c_i = m$, let $C_1, \ldots, C_r$ partition $[m]$ with $|C_i| = c_i$. Take $D = \bigcup_{i=1}^r \{(u,v) : u, v \in C_i,\ u \neq v\}$ (so $P = \emptyset$) and put $uv \in E(H)$ if and only if $u \in C_i, v \in C_j$ with $ij \in E(W)$. Then $D \cup \mathrm{id}_{[m]}$ is an equivalence relation, (A) holds because W is loopless, and (B) holds because H-adjacency depends only on twin classes. By the necessity proof, the resulting VWC graph $G(D,H)$ realizes $P(x)$. □

Corollary 2.4. *Every VWC graph $G(D,H)$ whose preorder D is acyclic ($c_i = 1$ for all i) has the exact same independence polynomial as the corona graph $W \circ K_1$, where $W = H \cup \mathrm{comp}(D)$.*

Proof. Setting $c_i = 1$ for all i in Theorem 1.2 gives

$$\begin{aligned} I(G;x) &= (1+x)^m \sum_{U \in \mathcal{I}(W)} \left(\frac{x}{1+x}\right)^{|U|} \\ &= (1+x)^m\, I\left(W; \frac{x}{1+x}\right) = I(W \circ K_1; x). \end{aligned}$$ □

As an independent computational check, Theorem 1.2 was cross-checked against Lemma 2.3 on 2,100 random valid (D,H) graphs ($m = 2, \ldots, 8$) with 100% exact agreement.

We collect three specialized composition rules used in Sections 3 and 4.

Proposition 2.5 (Bipartite X-join, preorder chains, and corona joins)**.** *Let $G_A = G(D_A, \emptyset)$ and $G_B = G(D_B, \emptyset)$ be bipartite VWC graphs on $2m_A$ and $2m_B$ vertices, respectively, with $m_A, m_B \geq 1$.*

(1) Bipartite X-join ($G_A \vee_X G_B$). *Taking $D = D_A \sqcup D_B$ and H to be the complete bipartite graph between X_A and X_B produces a connected bipartite VWC graph on $2(m_A + m_B)$ vertices with bipartition $(X_A \cup Y_B,\ Y_A \cup X_B)$ and*

$$I(G_A \vee_X G_B; x) = (1+x)^{m_B} I(G_A; x) + (1+x)^{m_A} I(G_B; x) - (1+x)^{m_A+m_B}.$$

(2) Bipartite L-layer preorder chain ($G_1 \to \cdots \to G_L$). *For $L \geq 2$, given bipartite VWC graphs $G_j = G(D_j, \emptyset)$ on $2m_j$ vertices ($1 \leq j \leq L$, $m_j \geq 1$, $m = \sum_{j=1}^{L} m_j$), taking $H = \emptyset$ and adding all arcs $Y_i \to X_j$ for $1 \leq i < j \leq L$ yields a connected bipartite VWC graph with bipartition (X, Y) and*

$$I(G_1 \to \cdots \to G_L; x) = (1+x)^m \left[1 - L + \sum_{j=1}^{L} \frac{I(G_j; x)}{(1+x)^{m_j}}\right].$$

(3) Corona join ($(A \vee \overline{K_b}) \circ K_1$). *For any simple graph A on $m - b$ vertices ($m > b \geq 1$), the corona $G = (A \vee \overline{K_b}) \circ K_1$ is a connected VWC graph on $2m$ vertices with*

$$I(G; x) = (1+x)^b I(A \circ K_1; x) + (1+x)^{m-b}(1+2x)^b - (1+x)^m.$$

Proof. In (1), axioms (A) and (B) hold because H has no edges within X_A or X_B and is complete between X_A and X_B. Bipartiteness with parts $(X_A \cup Y_B, Y_A \cup X_B)$ is immediate because D_A connects Y_A to X_A, D_B connects Y_B to X_B, and H connects X_A to X_B; connectedness holds (with diameter ≤ 6) because X_A and X_B form a complete bipartite subgraph and every y-vertex is matched to its x-mate. Every independent set of H lies entirely in X_A or entirely in X_B, with only $\emptyset$ counted twice; Lemma 2.3 gives the formula.

In (2), $H = \emptyset$ satisfies (A) and (B) vacuously, adding all arcs from layer i to layer $j > i$ preserves transitivity of D, and G is connected bipartite with parts (X, Y). Applying Theorem 1.2, the comparability graph $\mathrm{comp}(P)$ adds a complete L-partite join across the L layers, so every nonempty independent set $U \in \mathcal{I}(W)$ lies inside a single layer j, giving the stated inclusion–exclusion formula.

In (3), $H = A \vee \overline{K_b}$ is connected with $I(H; t) = I(A; t) + (1+t)^b - 1$. Substituting $t = x/(1+x)$ into $I(H \circ K_1; x) = (1+x)^m I(H; t)$ yields the formula. □

3. Counterexamples and Asymptotic Separation Analysis

3.1. **Bipartite VWC family.** Consider the two-layer bipartite X-join

$$G(a, c, b) = (aK_{c,c}) \vee_X (bK_2)$$

for positive integers a, c, b, which has $\alpha(G) = m = ac + b$, $|V(G)| = 2m$, and $|E(G)| = ac^2 + b + abc$. Because $K_{c,c}$ is a single twin class of size c, $I(K_{c,c}; x) = 2(1 + x)^c - 1$, while $I(K_2; x) = 1 + 2x$. By Proposition 2.5(1),

$$I(G(a,c,b);x) = (1+x)^b\big(2(1+x)^c-1\big)^a+(1+x)^{ac}(1+2x)^b-(1+x)^{ac+b}. \tag{1}$$

Expanding the first term by the binomial theorem and the second by Vandermonde's convolution gives the exact coefficient formula for every $0 \le k \le ac + b$:

$$\begin{aligned} s_k\big(G(a,c,b)\big) = &\sum_{j=0}^{a}(-1)^{a-j}\binom{a}{j}2^j\binom{cj+b}{k} \\ &+\sum_{i=0}^{k}\binom{ac}{i}\binom{b}{k-i}2^{k-i}-\binom{ac+b}{k}. \end{aligned} \tag{2}$$

The mechanism producing a bimodal valley is governed by the normalized value $V_F(1) = I(F; 1)/2^{\alpha(F)}$ and mean shift $\delta_F(1) = I'(F; 1)/I(F; 1) - \alpha(F)/2$ of a VWC building block F at $x = 1$. For $K_{c,c}$,

$$V_{K_{c,c}}(1) = 2 - 2^{-c}, \qquad \delta_{K_{c,c}}(1) = \frac{c}{2^{c+2} - 2},$$

whereas K_2 attains the maximum possible shift per vertex among VWC blocks, $\delta_{K_2}(1) = 1/6$, with $V_{K_2}(1) = 3/2$. To separate the two Gaussian peaks of the two positive summands in (1) at small total $\alpha = m$, one matches peak heights ($V_A(1) \approx V_B(1)$) while maximizing the mean-shift gap $\Delta = \delta_B - \delta_A$. Comparing single-block families $G(a, c, b)$ via the Gaussian separation heuristic

$$m_{\mathrm{crit}}(c) \approx \frac{4\bar{\sigma}^2}{\Delta^2}\left(c + \frac{\ln V_{K_{c,c}}(1)}{\ln(3/2)}\right)$$

identifies $c = 4$ ($K_{4,4}$) as the best block size among $K_{c,c}$ ($m_{\mathrm{est}} \approx 699.2$, versus 750.8 for $K_{5,5}$ and 769.8 for $K_{3,3}$). Scanning $G(a, 4, b)$ yields our explicit connected bipartite VWC witnesses $G_{\mathrm{bip,lc}} = G(122, 4, 206)$ ($\alpha = 694$) and $G_{\mathrm{bip,uni}} = G(125, 4, 204)$ ($\alpha = 704$). Because twin classes of size $c = 4$ achieve exponentially smaller mean shift $\delta_{K_{4,4}}(1) = 2/31 \approx 0.0645$ per four vertices than any clique corona block, these bipartite VWC witnesses are nearly three times smaller than the smallest known corona counterexamples. Determining the exact minimum α over all VWC graphs (or all bipartite VWC graphs) remains open.

3.2. **Corona graphs and clique optimality.** For corona graphs $H \circ K_1$ (the $c_i \equiv 1$ specialization of Theorem 1.2), the optimal low-mean building blocks are cliques.

Proposition 3.1. *For every nonempty graph C, the mean shift of $C \circ K_1$ at $x = 1$ satisfies*

$$\delta_{C\circ K_1}(1) = \frac{1}{2}\left.\frac{tI'(C;t)}{I(C;t)}\right|_{t=1/2} \geq \frac{1}{2}\left(1 - \frac{1}{I(C;1/2)}\right),$$

with equality if and only if C is a complete graph K_d.

Proof. Differentiating $I(C \circ K_1; x) = (1+x)^{|V(C)|} I(C; x/(1+x))$ at $x = 1$ gives

$$\frac{I'(C\circ K_1;1)}{I(C\circ K_1;1)} = \frac{|V(C)|}{2} + \frac{1}{4}\frac{I'(C;1/2)}{I(C;1/2)} = \frac{|V(C)|}{2} + \frac{1}{2}\left.\frac{tI'(C;t)}{I(C;t)}\right|_{t=1/2}.$$

Because every nonempty independent set $S \in \mathcal{I}(C)$ has $|S| \geq 1$,

$$tI'(C;t) = \sum_{\emptyset \neq S \in \mathcal{I}(C)} |S|\, t^{|S|} \geq \sum_{\emptyset \neq S \in \mathcal{I}(C)} t^{|S|} = I(C;t) - 1,$$

with equality at $t = 1/2$ if and only if every nonempty independent set of C has size 1, i.e., $\alpha(C) = 1$, which means $C = K_d$. □

Taking $H = \left(\bigcup_r a_r K_{d_r}\right) \vee \overline{K_b}$ with $\sum_r a_r d_r + b = m$, Proposition 2.5(3) yields

$$\begin{aligned} I(H \circ K_1; x) = (1+x)^b \prod_r \left((1+x)^{d_r} + d_r x (1+x)^{d_r - 1}\right)^{a_r} \\ + (1+x)^{m-b}(1+2x)^b - (1+x)^m. \end{aligned} \tag{3}$$

Baker's July 2026 counterexample [1] uses $H = (39K_{11} \cup 85K_{12}) \vee \overline{K_{588}}$ ($\alpha = 2037$, $|V| = 4074$). Using adjacent clique sizes K_{10} and K_{11} yields a smaller connected corona unimodality counterexample $G_{\mathrm{cor,uni}}$ with $H = (126K_{10} \cup 10K_{11}) \vee \overline{K_{603}}$ at $\alpha = 1973$ ($|V| = 3946$), while uniform K_{14} gives a connected corona log-concavity counterexample $G_{\mathrm{cor,lc}}$ at $\alpha = 1721$ ($|V| = 3442$).

3.3. **Explicit verified counterexamples.** Table 1 summarizes our six explicit counterexamples proving Theorem 1.1 (and witnessing $L = 3$ in Theorem 1.3), alongside Baker's corona for comparison. Every graph constructed here is materialized explicitly as a full adjacency representation and verified in exact arbitrary-precision integer arithmetic by `verify_counterexamples.py` (verifying simplicity, connectedness, no isolated vertices, Favaron's (P1) and (P2) on all m matching edges, explicit bipartiteness or corona leaf structure, low-degree identities $s_0 = 1$, $s_1 = 2m$, $s_2 = \binom{2m}{2} - |E(G)|$, and exact coefficient inequalities).

Proof of Theorem 1.1. By Proposition 2.5, the bipartite graphs $G_{\mathrm{bip,lc}} = G(122, 4, 206)$ and $G_{\mathrm{bip,uni}} = G(125, 4, 204)$ are connected and VWC with independence polynomials given by (1)–(2), while $G_{\mathrm{cor,lc}}$ and $G_{\mathrm{cor,uni}}$ are connected corona graphs with independence polynomials given by (3). Exact integer evaluation establishes:

TABLE 1. Explicit verified counterexamples (all connected VWC), compared with Baker's July 2026 corona [1]. The rightmost column lists the LC-failure index or the strict local maxima (in braces) and strict dip.

ID	Construction	α	$\lvert V\rvert$	$\lvert E\rvert$ Peaks / LC witness
$G_{\text{bip,lc}}$	$(122K_{4,4}) \vee_X (206K_2)$	694	1388	102686 LC fail $k = 351$
$G_{\text{bip,uni}}$	$(125K_{4,4}) \vee_X (204K_2)$	704	1408	104204 $\{370, 375\}$; dip 374
$G_{\text{cor,lc}}$	$((86K_{14}) \vee \overline{K_{517}}) \circ K_1$	1721	3442	632015 LC fail $k = 645$
$G_{\text{cor,uni}}$	$((126K_{10} \cup 10K_{11}) \vee \overline{K_{603}}) \circ K_1$	1973	3946	834303 $\{1061, 1069\}$; dip 1066
Baker [1]	$((39K_{11} \cup 85K_{12}) \vee \overline{K_{588}}) \circ K_1$	2037	4074	861804 $\{1086, 1101\}$; dip 1095
$G_{\text{cor,pure}}$	$((143K_{10}) \vee \overline{K_{632}}) \circ K_1$	2062	4124	912257 $\{1107, 1124\}$; dip 1110
$G_{\text{bip,tri}}$	$900K_{4,4} \to (450K_{4,4} \cup 734K_2) \to 1468K_2$	7602	15204	18150914 $\{3874, 3952, 4027\}$

(1) For $G_{\text{bip,lc}}$ ($\alpha = 694$), the 243-digit coefficients at $k \in \{350, 351, 352\}$ satisfy $s_{351}^2 < s_{350} s_{352}$ (ratio ≈ 0.999989786589).
(2) For $G_{\text{bip,uni}}$ ($\alpha = 704$), the 247-digit coefficients at $k \in \{371, \ldots, 375\}$ satisfy $s_{374} < s_{373}$ and $s_{374} < s_{375}$ with ratios $s_{373}/s_{374} \approx 1.000039951793$ and $s_{375}/s_{374} \approx 1.000003986427$, refuting unimodality, with log-concavity failures at $k \in \{372, 373, 374\}$ and exact global peaks at $k \in \{370, 375\}$.
(3) For $G_{\text{cor,lc}}$ ($\alpha = 1721$), $s_{645}^2 < s_{644} s_{646}$ (ratio ≈ 0.999999072495).
(4) For $G_{\text{cor,uni}}$ ($\alpha = 1973$), $s_{1066} < s_{1065}$ and $s_{1066} < s_{1067}$ with ratios $s_{1065}/s_{1066} \approx 1.000000796090$ and $s_{1067}/s_{1066} \approx 1.000024623418$, strict local maxima at $k \in \{1061, 1069\}$, and log-concavity failures at $k \in \{1063, \ldots, 1067\}$.

In Section 6, Lean 4 formally proves part (1) and the non-unimodality and non-log-concavity of $G_{\text{bip,uni}}$ in part (2) (via the strict interior dip $s_{373} > s_{374} < s_{375}$ and LC failure at $k = 372$), while the exact global peak indices $\{370, 375\}$ and parts (3)–(4) are certified by `verify_counterexamples.py`. □

4. Arbitrarily Many Strict Interior Modes

To ensure genuine *strict* interior local maxima and minima (ruling out flat plateaus between separated peak regions), we combine L-layer mixtures with a small two-chain Ferrers tie-breaking poset.

Lemma 4.1 (Two-chain Ferrers tie-breaking poset)**.** *For any positive integers d_1, d_2 and any integer $t \in \{0, 1, \ldots, d_1 d_2\}$, there exists a strict partial order P_t on $r = d_1 + d_2$ elements whose comparability graph $W_t = \text{comp}(P_t)$ has independence polynomial $I(W_t; u) = 1 + ru + tu^2$. Consequently, the bipartite VWC graph $H_t = G(P_t, \emptyset)$ and the corona graph $W_t \circ K_1$ both satisfy*

$$I(H_t; x) = I(W_t \circ K_1; x) = (1+x)^r + rx(1+x)^{r-1} + tx^2(1+x)^{r-2}.$$

Proof. Let $C_1 = \{u_1, \ldots, u_{d_1}\}$ and $C_2 = \{v_1, \ldots, v_{d_2}\}$ be disjoint chains with $u_1 < \cdots < u_{d_1}$ and $v_1 < \cdots < v_{d_2}$. Order the $d_1 d_2$ pairs $(i, j) \in [d_1] \times [d_2]$

lexicographically by increasing (i, d_2+1-j), and add the cross-arc $u_i <_{P_t} v_j$ for the first $d_1d_2 - t$ pairs in this order. Because every cross-arc is directed from C_1 to C_2, P_t is irreflexive and antisymmetric. If $u_i <_{P_t} v_j$ and $i' \le i$ and $j \le j'$, then $(i', d_2+1-j') \le_{\mathrm{lex}} (i, d_2+1-j)$, so $u_{i'} <_{P_t} v_{j'}$; hence P_t is transitive. Since P_t is covered by two chains, every antichain of P_t has size at most 2, and the antichains of size 2 are precisely the t unordered pairs $\{u_i, v_j\}$ not chosen as cross-arcs. Thus $I(\mathrm{comp}(P_t); u) = 1 + ru + tu^2$, and Corollary 2.4 gives the stated formula for $I(H_t; x) = I(W_t \circ K_1; x)$. □

Proof of Theorem 1.3. For $L = 1$, the path $P_4 = K_2 \circ K_1$ is simultaneously a connected bipartite VWC graph and a connected corona graph, with $I(P_4; x) = 1+4x+3x^2$ having a strict interior local maximum at $k = 1$ $(1 < 4 > 3)$.

Fix an integer $L \ge 2$. We first construct connected bipartite VWC graphs. Set

$$\gamma = \frac{\ln(31/16)}{\ln(3/2)} \approx 1.63121.$$

For any positive integer N divisible by $L-1$, define for $1 \le j \le L$

$$p_j = \frac{L-j}{L-1}N, \qquad q_j = \left\lfloor \frac{j-1}{L-1}\gamma N \right\rfloor,$$

and set $G_j = p_jK_{4,4} \cup q_jK_2$ ($m_j = 4p_j + q_j \ge 1$, $m = \sum_{j=1}^{L} m_j = \Theta(LN)$). Let $s = \lceil \sqrt{m} \rceil$, set $d_1 = s+2$, and set $d_2 = d_1$ if m is even or $d_2 = d_1 + 1$ if m is odd. Putting $r = d_1 + d_2 = O(\sqrt{LN})$ and $m' = m + r$, we have:

- m' is even, so $M = m' - 2 \ge 2$ is even;
- $d_1d_2 \ge (s+2)^2 = s^2 + 4s + 4 \ge m + 4s + 4 > m + 2s + 5 \ge m'$ for every $m \ge 1$.

For each $t \in \{0, 1, \ldots, d_1d_2\}$, let $H_t = G(P_t, \emptyset)$ be the bipartite VWC graph of Lemma 4.1, and form the $(L+1)$-layer connected bipartite VWC preorder chain

$$G'_L(N, t) = G_1 \to G_2 \to \cdots \to G_L \to H_t$$

of total independence number m'. By Proposition 2.5(2) and Lemma 4.1,

(4)
$$I\big(G'_L(N,t); x\big) = \sum_{j=1}^{L} \widetilde{T}_j(x) - (L-1)(1+x)^{m'} + rx(1+x)^{m'-1} + tx^2(1+x)^M,$$

where $\widetilde{T}_j(x) = (1+x)^{m'-4p_j-q_j}\big(2(1+x)^4 - 1\big)^{p_j}(1+2x)^{q_j}$.

Write $s_k(t) = [x^k]I(G'_L(N,t); x) = A_k + tB_k$, where $B_k = [x^k]\big(x^2(1+x)^M\big) = \binom{M}{k-2}$ (with $\binom{M}{j} = 0$ for $j < 0$). At $k = 1$, $s_0(t) = 1 < 2m' = s_1(t)$ strictly. For $k = 2$, $B_2 - B_1 = 1 \ne 0$. For $3 \le k \le m'$, setting $j = k - 2 \in \{1, \ldots, M\}$, we have $\binom{M}{j} = \binom{M}{j-1}$ if and only if $2j = M + 1$, which is impossible because M is even. Thus $B_k - B_{k-1} \ne 0$ for all $2 \le k \le m'$, so each equation $s_k(t) = s_{k-1}(t)$ eliminates at most one integer t. Since there

are $m'-1$ such indices and $d_1d_2+1 > m'$ integers in $\{0,1,\ldots,d_1d_2\}$, there exists $t^* \in \{0,1,\ldots,d_1d_2\}$ such that

$$s_k(t^*) \neq s_{k-1}(t^*) \qquad \text{for all } 1 \leq k \leq m'. \tag{5}$$

We now show that for sufficiently large N, the coefficients $s_k(t)$ exhibit L separated peak regions uniformly for all $t \in \{0,1,\ldots,d_1d_2\}$. Each factor $(1+x)$, $2(1+x)^4-1 = 1+8x+12x^2+8x^3+2x^4$, and $1+2x$ has strictly positive consecutive coefficients. Hence $\widetilde{T}_j(x)/\widetilde{T}_j(1)$ is the probability generating function of a sum of independent bounded integer-valued random variables with span 1, total mass

$$\widetilde{T}_j(1) = 2^{m'}\left(\frac{31}{16}\right)^{p_j}\left(\frac{3}{2}\right)^{q_j} = \Theta\left(2^{m'}\left(\frac{31}{16}\right)^N\right)$$

uniformly across $1 \leq j \leq L$ (up to a factor at most $3/2$ from rounding q_j), variance $\widetilde{\sigma}_j^2 = \Theta(m') = \Theta(LN)$, and mean

$$\widetilde{\mu}_j = \frac{m'}{2} + \frac{2}{31}p_j + \frac{1}{6}q_j, \quad \widetilde{\mu}_{j+1} - \widetilde{\mu}_j = \left(\frac{\gamma}{6} - \frac{2}{31}\right)\frac{N}{L-1} + O(1) \approx \frac{0.20735\,N}{L-1}.$$

By the local central limit theorem for sums of independent lattice random variables, at $a_j = \lfloor \widetilde{\mu}_j \rfloor$ $(1 \leq j \leq L)$ we have

$$[x^{a_j}]\widetilde{T}_j(x) = \Theta\left(\frac{2^{m'}(31/16)^N}{\sqrt{LN}}\right).$$

At each midpoint $b_j = \lfloor (\widetilde{\mu}_j + \widetilde{\mu}_{j+1})/2 \rfloor$ $(1 \leq j \leq L-1)$, every summand satisfies $|b_j - \widetilde{\mu}_u| \geq c_1 N/L$ for an absolute constant $c_1 > 0$, so Hoeffding's inequality (since the squared ranges sum to $O(m') = O(LN)$) yields

$$\begin{aligned}[x^{b_j}]\,\widetilde{T}_u(x) &\leq \widetilde{T}_u(1)\,\exp\left(-\frac{c_2N^2}{L^2\cdot LN}\right)\\ &= O\left(2^{m'}\left(\frac{31}{16}\right)^N \exp\left(-\frac{c_2N}{L^3}\right)\right)\end{aligned}$$

for an absolute constant $c_2 > 0$ and all $1 \leq u \leq L$. Meanwhile, the last three terms in (4) have total magnitude at most $(L+r+d_1d_2)2^{m'} = O(LN\cdot 2^{m'})$, and at the boundaries $b_0 = 0$ and $b_L = m'$ we have $s_0(t) = 1$ and $s_{m'}(t) \leq \sum_{j=1}^{L} 2^{p_j+q_j} + r + d_1d_2 = O(L\cdot 2^{\gamma N} + LN)$, all exponentially smaller than $2^{m'}(31/16)^N/\sqrt{LN}$. Thus, for sufficiently large N ($N \gg L^3 \ln L$), we have $0 = b_0 < a_1 < b_1 < \cdots < a_L < b_L = m'$ and

$$s_{a_j}(t^*) > \max\bigl\{s_{b_{j-1}}(t^*),\, s_{b_j}(t^*)\bigr\} \qquad (1 \leq j \leq L).$$

For each $1 \leq j \leq L$, the maximum of $s_k(t^*)$ over $k \in [b_{j-1}, b_j]$ is attained at an interior index $k_j^* \in (b_{j-1}, b_j)$, which by (5) is a *strict* interior local maximum ($s_{k_j^*-1}(t^*) < s_{k_j^*}(t^*) > s_{k_j^*+1}(t^*)$). Between each consecutive pair $k_j^* < k_{j+1}^*$ $(1 \leq j \leq L-1)$, the minimum of $s_k(t^*)$ over $[k_j^*, k_{j+1}^*]$ is attained

at an interior index $\ell_j^* \in (k_j^*, k_{j+1}^*)$ (since $s_{b_j}(t^*) < \min(s_{k_j^*}(t^*), s_{k_{j+1}^*}(t^*))$), which by (5) is a *strict* interior local minimum.

For connected corona graphs, set $\gamma_{\mathrm{cor}} = \ln 6/\ln(3/2) \approx 4.41902$, define $p_j = \frac{L-j}{L-1}N$ and $q_j = \lfloor \frac{j-1}{L-1}\gamma_{\mathrm{cor}} N \rfloor$ with $m = \sum_{j=1}^{L}(10p_j + q_j) = \Theta(LN)$, choose $d_1, d_2, r = d_1 + d_2$, and $m' = m + r$ (even) identically as above, and form the connected $(L+1)$-partite join

$$H'_L(N,t) = \left(\bigvee_{j=1}^{L} \left(p_j K_{10} \cup \overline{K_{q_j}} \right) \right) \vee W_t,$$

where $W_t = \mathrm{comp}(P_t)$ is the graph of Lemma 4.1. Then $H'_L(N,t) \circ K_1$ is a connected corona graph of independence number m' satisfying

$$\begin{aligned} I\big(H'_L(N,t) \circ K_1; x\big) = \sum_{j=1}^{L} \widetilde{T}_j^{\mathrm{cor}}(x) - (L-1)(1+x)^{m'} \\ + rx(1+x)^{m'-1} + tx^2(1+x)^M, \end{aligned}$$

where $\widetilde{T}_j^{\mathrm{cor}}(x) = (1+x)^{m'-10p_j-q_j}\big((1+x)^{10} + 10x(1+x)^9\big)^{p_j}(1+2x)^{q_j}$. Because $I(K_{10} \circ K_1; 1) = 2^{10} \cdot 6$ and $\delta_{K_{10}\circ K_1}(1) = \frac{5}{12}$ by Proposition 3.1, each $\widetilde{T}_j^{\mathrm{cor}}$ has total mass $\Theta(2^{m'} \cdot 6^N)$ uniformly in j, variance $\Theta(LN)$, and adjacent mean separation

$$\widetilde{\mu}_{j+1}^{\mathrm{cor}} - \widetilde{\mu}_j^{\mathrm{cor}} = \left(\frac{\gamma_{\mathrm{cor}}}{6} - \frac{5}{12} \right) \frac{N}{L-1} + O(1) \approx \frac{0.31984\, N}{L-1},$$

while $s_{m'}(t) \le \sum_{j=1}^{L} 11^{p_j} 2^{q_j} + r + d_1 d_2 \ll 2^{m'} \cdot 6^N/\sqrt{LN}$. Applying the identical tie-breaking choice of t^*, local central limit theorem, and Hoeffding bound yields at least L strict interior local maxima separated by $L-1$ strict interior local minima. □

Remark 4.2. At $L = 3$, the explicit connected bipartite VWC chain (without needing a tie-breaking layer)

$$G_{\mathrm{bip,tri}} = (900K_{4,4}) \to (450K_{4,4} \cup 734K_2) \to (1468K_2)$$

($\alpha = 7602$, $|V| = 15204$, $|E| = 18{,}150{,}914$; last row of Table 1) is verified in exact 2,545-digit integer arithmetic to be strictly trimodal, with three strict interior local maxima at $k \in \{3874, 3952, 4027\}$ separated by two strict interior local minima at $k \in \{3906, 4005\}$.

5. Exhaustive Strict Log-Concavity Certification for $\alpha \le 7$

As recalled in Section 1, unimodality of $I(G;x)$ was already established for all VWC graphs with $\alpha(G) \le 9$ [12, 16] (and for corona graphs $H \circ K_1$ whose skeleton H satisfies $\alpha(H) \le 8$ [11, 5]), whereas log-concavity for all VWC graphs had previously been proved only for $\alpha(G) \le 5$ [16, 7]. (For the subclass of forest coronas $F \circ K_1$, strict log-concavity was recently proved for arbitrary size by Levit [10].)

TABLE 2. Exhaustive zero-float certification of all VWC graphs for $\alpha = 2, \ldots, 7$ ($|V| \leq 14$). For $3 \leq \alpha \leq 7$, the exact minimum ratio is attained by both αK_2 and $K_{\alpha,\alpha}$; the last column displays an extremal polynomial.

α	$\lvert V\rvert$	Preorders	(D,H) pairs	LC fails	$\min_k \frac{s_k^2}{s_{k-1}s_{k+1}}$	Extremal $I(G;x)$
2	4	3	4	0	4	$2K_2$: $[1,4,4]$
3	6	9	20	0	3	$K_{3,3}$: $[1,6,6,2]$
4	8	33	164	0	9/4	$K_{4,4}$: $[1,8,12,8,2]$
5	10	139	2,260	0	2	$K_{5,5}$: $[1,10,20,20,10,2]$
6	12	718	59,113	0	16/9	$K_{6,6}$: $[1,12,30,40,30,12,2]$
7	14	4,535	3,104,967	0	5/3	$K_{7,7}$: $[1,14,42,70,70,42,14,2]$

To advance the positive log-concavity frontier for general VWC graphs, we exhaustively certified all canonical (D,H) representations (Definition 2.2) for $m = \alpha(G) \in \{2,\ldots,7\}$ ($|V(G)| \leq 14$) in `exhaust_certify_m2_to_m7.cpp`. By Theorem 2.1 and Definition 2.2, every VWC graph on $2m$ vertices arises from at least one unlabeled preorder $D \cup \mathrm{id}_{[m]}$ (counted by OEIS A001930: $3, 9, 33, 139, 718, 4535$ for $m = 2, \ldots, 7$) together with a graph H satisfying axioms (A) and (B). For each unlabeled preorder, we enumerate all admissible quotient graphs $\overline{H}$ (whose edges form upward-closed pairs in $P \times P$ avoiding comparabilities), evaluate $I(G(D,H);x)$ exactly via Lemma 2.3 in 64-bit exact integer arithmetic (each coefficient is bounded above by $2^{14} = 16{,}384$), check *strict* log-concavity $s_k^2 > s_{k-1}s_{k+1}$ (rejecting equality $s_k^2 \leq s_{k-1}s_{k+1}$) for all $1 \leq k \leq m-1$, and determine the exact minimum rational ratio via zero-float `__int128` cross-multiplication.

As an end-to-end validation of the canonical enumeration, at $m = 2$ ($|V| = 4$) and $m = 3$ ($|V| = 6$) we independently enumerated all $2^{\binom{2m}{2}}$ simple graphs by brute force, filtered directly by the definition of very well-coveredness (enumerating all maximal independent sets), and confirmed exact set equality of both unlabeled VWC isomorphism classes and realized independence polynomials.

Table 2 records the complete results, establishing Theorem 1.4.

6. Formal Verification in Lean 4 and Independent Audits

To provide machine-checked certainty for our smallest disproofs, the bipartite VWC counterexamples $G_{\mathrm{bip,lc}}$ ($\alpha = 694$) and $G_{\mathrm{bip,uni}}$ ($\alpha = 704$) are formalized end-to-end in `VWCDisproof.lean` (2,257 lines), using only Lean 4 core `Init` (zero Mathlib or external dependencies). The file formalizes:

(1) standard graph-theoretic definitions (`SimpleGraph`, `IsIndepSet`, `IsMaximalIndepSet`, `HasNoIsolated`, `IsBipartite`, `IsConnected`) and Favaron's sufficiency lemma (`favaron_maximal_hits_pair`);
(2) the explicit vertex type `BVertex a c b` of cardinality $2(ac + b)$ (`cardBVertex_univ`) and graph `bipVWCGraph a c b`, proved bipartite

(`bipVWCGraph_bipartite`), connected for $a, c, b \geq 1$ (`bipVWCGraph_connected`), and very well-covered (`bipVWCGraph_vwc`);

(3) the exact independence polynomial `indepPolyBVertex a c b` defined by summing $x^{|S|}$ over all independent sets S (verified equivalent to `IsIndepSet` via `isIndepBVertex_iff`), proved equal to (1) (`indepPolyBVertex_eq_formula`) and to a chunked linear-time GMP evaluator `coeffGFast` (`indepPolyBVertex_eq_coeffGFast`) using binary-split products and binomial stepping;

(4) pure kernel `by decide` evaluations of all eight required 243- and 247-digit witness coefficients (`val_694_350`–`val_694_352` and `val_704_371`–`val_704_375`), yielding the main theorems `disproof_log_concavity_vwc` ($a = 122$, $c = 4$, $b = 206$, $\alpha = 694$, proving $\neg$ IsLogConcave via $s_{351}^2 < s_{350}s_{352}$) and `disproof_unimodality_vwc` ($a = 125$, $c = 4$, $b = 204$, $\alpha = 704$, proving $\neg$ IsUnimodal via the strict interior dip $s_{373} > s_{374} < s_{375}$ and $\neg$ IsLogConcave via $s_{372}^2 < s_{371}s_{373}$).

Compiling `VWCDisproof.lean` under `leanprover/lean4:v4.27.0` succeeds cleanly (exit code 0, 0 errors, 0 warnings, ≈ 19 s) with zero escape tokens (`sorry`, `admit`, `axiom`, `unsafe`, `implemented_by`, `native_decide`, `partial`, `skip KernelTC`). Exact `#print axioms` output reports only standard axioms across all theorems:

- `[propext, Classical.choice, Quot.sound]` for `disproof_log_concavity_vwc`, `disproof_unimodality_vwc`, and `indepPolyBVertex_eq_formula`;
- `[propext, Quot.sound]` for `indepPolyFormula_eq_coeffGFast`;
- no axioms for `lc_witness_694`, `uni_dip_witness_704`, and `uni_lc_witness_704`.

For exact transparency of verification scope: Lean 4 kernel-verifies that $G_{\text{bip,lc}}$ ($\alpha = 694$) fails log-concavity and that $G_{\text{bip,uni}}$ ($\alpha = 704$) fails both unimodality (via the strict interior dip at $k = 374$) and log-concavity (at $k = 372$), end-to-end from `SimpleGraph`. The exact global peak locations $\{370, 375\}$ of $G_{\text{bip,uni}}$, the corona counterexamples ($G_{\text{cor,lc}}$ at $\alpha = 1721$, $G_{\text{cor,uni}}$ at $\alpha = 1973$, $G_{\text{cor,pure}}$ at $\alpha = 2062$), the trimodal witness ($G_{\text{bip,tri}}$ at $\alpha = 7602$), the computational cross-check of Theorem 1.2, and the exhaustive $\alpha \leq 7$ certification (Theorem 1.4) are verified by standalone zero-dependency Python 3.10+ (`verify_counterexamples.py`, calibrated against 2^n brute-force enumeration on 11 explicit graphs up to $n = 18$) and C++20 with `__int128` (`exhaust_certify_m2_to_m7.cpp`).

Data Availability and Methodology

All formal proofs, verification scripts, and machine-readable certificates (`VWCDisproof.lean`, `verify_counterexamples.py`, `exhaust_certify_m2_to_m7.cpp`, and `counterexamples.json`) are publicly archived in the companion repository at `https://github.com/coracle-us/conjectures` (with a companion Zenodo archive). Counterexample discovery and proof drafting were AI-assisted, with every claim independently audited via zero-trust

exact arithmetic, exhaustive cross-checks, cross-model adversarial review, and Lean 4 kernel checking following the structured verification workflow of Loader, Oppenheim, and Osborne [17].

References

[1] L. Baker, *Counterexample to the Levit–Mandrescu conjecture for very well-covered graphs* (4,074-vertex connected corona graph), LinkedIn announcement (July 2026), `https://www.linkedin.com/posts/tesuji_levit-mandrescu-counterexample-activity-7487528446403768320-b1pc`; archived records at `https://aiproofwatch.com/problems/very-well-covered-independence-unimodality` and `https://vibemathed.com`.

[2] J. P. Brennan and S. Morey, *Regular edges, matchings and Hilbert series*, J. Algebraic Combin. **63** (2026), Paper No. 10, `arXiv:2412.10335`.

[3] J. I. Brown and B. Cameron, *On the unimodality of independence polynomials of very well-covered graphs*, Discrete Math. **341** (2018), no. 4, 1138–1143, `arXiv:1709.08236`.

[4] J. I. Brown, K. Dilcher, and R. J. Nowakowski, *Roots of independence polynomials of well covered graphs*, J. Algebraic Combin. **11** (2000), no. 3, 197–210.

[5] S.-Y. Chen and H.-J. Wang, *Unimodality of independence polynomials of very well-covered graphs*, Ars Combin. **97A** (2010), 509–529.

[6] O. Favaron, *Very well covered graphs*, Discrete Math. **42** (1982), 177–187.

[7] D. T. Hoang, V. E. Levit, E. Mandrescu, and M. H. Pham, *Log-concavity of the independence polynomials of $\mathbf{W}_p$ graphs*, `arXiv:2409.00827`, rev. Sep. 2025.

[8] O. Kadrawi and V. E. Levit, *The independence polynomial of trees is not always log-concave starting from order 26*, Ars Math. Contemp. (2025), `doi:10.26493/1855-3974.3207.2ad`.

[9] K. Kimura, N. Terai, and S. Yassemi, *The projective dimension of the edge ideal of a very well-covered graph*, Nagoya Math. J. **230** (2018), 160–179, `doi:10.1017/nmj.2017.7`.

[10] V. E. Levit, *Variance bounds and strict log-concavity for forest coronas*, `arXiv:2610.41102` (Oct. 2026).

[11] V. E. Levit and E. Mandrescu, *A family of well-covered graphs with unimodal independence polynomials*, Congr. Numer. **165** (2003), 195–207.

[12] V. E. Levit and E. Mandrescu, *Very well-covered graphs and the unimodality conjecture*, `arXiv:math/0406623` (2004).

[13] V. E. Levit and E. Mandrescu, *Very well-covered graphs with log-concave independence polynomials*, Carpathian J. Math. **20** (2004), 73–80, `arXiv:math/0411239`.

[14] V. E. Levit and E. Mandrescu, *The independence polynomial of a graph—a survey*, in *Proceedings of the 1st International Conference on Algebraic Informatics* (2005), 233–254.

[15] V. E. Levit and E. Mandrescu, *Independence polynomials of well-covered graphs: generic counterexamples for the unimodality conjecture*, European J. Combin. **27** (2006), 931–939.

[16] V. E. Levit and E. Mandrescu, *Independence polynomials and the unimodality conjecture for very well-covered, quasi-regularizable, and perfect graphs*, in *Graph Theory in Paris*, Trends in Mathematics, Birkhäuser (2007), 243–254.

[17] D. Loader, J. Oppenheim, and T. J. Osborne, *How to train your slop cannon: A short guide to using large language models for research in the natural sciences*, Zenodo (2026), `doi:10.5281/zenodo.22670687`.

[18] T. S. Michael and W. N. Traves, *Independence sequences of well-covered graphs: non-unimodality and the roller-coaster conjecture*, Graphs Combin. **19** (2003), 403–411.

[19] G. Ravindra, *Well-covered graphs*, J. Combin. Inform. System Sci. **2** (1977), no. 1, 20–21.

Independent Researcher
Email address: jon.mcbee@gmail.com
URL: https://orcid.org/0009-0000-7524-4470