hexagon:2611.01249v1

ORDER-COMPLEX h-POLYNOMIALS OF GEOMETRIC LATTICES ARE NOT UNIMODAL

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ABSTRACT. For a finite geometric lattice L of rank r with proper part $\bar{L} = L \setminus \{\hat{0}, \hat{1}\}$, let $c_{\bar{L}}(t) = \sum_{k=0}^{r-1} c_k(\bar{L})t^k$ be its proper chain polynomial, $c_L(t) = (1+t)^2 c_{\bar{L}}(t)$ its full chain polynomial, and $h_L(t) = (1-t)^{r-1} c_{\bar{L}}\big(t/(1-t)\big) = \sum_{j=0}^{r-1} h_j(L)t^j$ the h-polynomial of the Cohen–Macaulay order complex $\Delta(\bar{L})$. Following the disproof of real-rootedness and log-concavity of chain and h-polynomials by Bérczi, Coron, Eur, and Li (2026), unimodality of $h_L(t)$ remained open beyond the strong flawlessness inequalities of Nyman–Swartz (2004) (recently extended to arbitrary nested set complexes by Coron–Ferroni–Li, 2026).

We prove that $h_L(t)$ is *not* unimodal in general. Analyzing both the projective Gabidulin absorption family $\mathcal{G}_{r,q}$ of Bérczi–Coron–Eur–Li and a smaller affine Gabidulin counterpart $\mathcal{A}_{r,q}$ on $q^{2(r-2)} + q^{r-2}$ elements, we establish the exact leading asymptotics as $q \to \infty$ for every rank $r \geq 4$:

$$h_{r-3} = \frac{(r-3)^2}{2}\, q^{E_0-1} + O(q^{E_0-2}),$$
$$h_{r-2} = \frac{7r-18}{2}\, q^{E_0-1} + O(q^{E_0-2}),$$
$$h_{r-1} = q^{E_0} + O(q^{E_0-1}).$$

Consequently, $h_L(t)$ exhibits a strict upper-half valley $h_{r-4} < h_{r-3} > h_{r-2} < h_{r-1}$ (with valley ratio $O(1/r) \to 0$) for every rank $r \geq 11$ and all sufficiently large prime powers q. At rank $r = 10$, exact rational polynomial arithmetic and a global Taylor shift at $q = 75$ certify that both $\mathcal{A}_{10,q}$ and $\mathcal{G}_{10,q}$ fail unimodality precisely on the finite set of 16 prime powers $q \in \{27, 29, 31, 32, \ldots, 73\}$, giving the smallest witness $\mathcal{A}_{10,27}$ within the two complete families $\{\mathcal{G}_{r,q}, \mathcal{A}_{r,q}\}$.

We complement this disproof with three structural theorems on the remaining frontier: (i) a universal double-counting inequality proving that $c_{\bar{L}}(t)$ is unimodal for all geometric lattices of rank $r \leq 6$ and $c_L(t)$ is unimodal for all $r \leq 8$; (ii) strict log-concavity $c_2(\bar{L})^2 > 2c_1(\bar{L})c_3(\bar{L})$ for every rank-4 orientable (in particular $\mathbb{R}$-realizable, regular, or graphic) matroid via Melchior's inequality and for every constant-line-size rank-4 geometric lattice; and (iii) an exact multichain zeta-polynomial identity for Whittle's q-lift $L_q(M)$ factoring termwise under direct sums, accompanied by exact integer/rational certifications of real-rootedness and ultra-log-concavity across graphic, binary, ternary, Coxeter, and Whittle-lift matroids up to rank 153.

2020 *Mathematics Subject Classification.* 05B35, 05A20, 06C10, 52B05.

Key words and phrases. Geometric lattice, matroid, chain polynomial, order complex, h-polynomial, unimodality, log-concavity, hyperplane absorption, Gabidulin code, Whittle q-lift.

1. Introduction

Let L be a finite geometric lattice of rank $r \geq 2$ (equivalently, the lattice of flats of a finite simple matroid M of rank r) with bottom element $\hat{0}$, top element $\hat{1}$, and proper part $\bar{L} = L \setminus \{\hat{0}, \hat{1}\}$. The *proper chain polynomial* and *full chain polynomial* of L are

$$c_{\bar{L}}(t) = \sum_{k=0}^{r-1} c_k(\bar{L})\, t^k, \qquad c_L(t) = (1+t)^2 c_{\bar{L}}(t),$$

where $c_k(\bar{L})$ counts k-element chains $\hat{0} < x_1 < \cdots < x_k < \hat{1}$ in $\bar{L}$ (with $c_0(\bar{L}) = 1$). By Björner [4], the $(r-2)$-dimensional order complex $\Delta(\bar{L})$ is EL-shellable and hence Cohen–Macaulay; its *h-polynomial*

$$h_L(t) = \sum_{j=0}^{r-1} h_j(L)\, t^j = (1-t)^{r-1} c_{\bar{L}}\left(\frac{t}{1-t}\right),$$

$$h_j(L) = \sum_{k=0}^{j} (-1)^{j-k} \binom{r-1-k}{j-k} c_k(\bar{L}),$$

has strictly positive integer coefficients $h_j(L) > 0$ for $0 \leq j \leq r-1$. The three polynomials $c_{\bar{L}}(t)$, $c_L(t)$, and $h_L(t)$ are simultaneously real-rooted or not.

Athanasiadis and Kalampogia-Evangelinou [1] conjectured that $c_L(t)$ is real-rooted for every geometric lattice L, proving the conjecture for partition lattices and subspace lattices. Brändén and Saud Maia Leite [6] subsequently proved real-rootedness for all paving matroids (and hence all geometric lattices of rank $r \leq 3$), Dowling lattices, and perfect matroid designs. On the unimodality side, Nyman and Swartz [16] proved that $\Delta(\bar{L})$ admits a convex ear decomposition and hence that $h_L(t)$ is *strongly flawless*,

$$h_0 \leq h_1 \leq \cdots \leq h_{\lfloor (r-1)/2 \rfloor} \qquad \text{and} \qquad h_i \leq h_{r-1-i} \quad \big(0 \leq i \leq \lfloor (r-1)/2 \rfloor\big),$$

and Coron, Ferroni, and Li [7] recently extended convex ear decompositions and strong flawlessness to nested set complexes of matroids with arbitrary building sets. Notice that strong flawlessness places no monotonicity constraint on the upper half $j > (r-1)/2$.

In October 2026, Bérczi, Coron, Eur, and Li [3] disproved the real-rootedness conjecture of [1]—and indeed *log-concavity* of Chow, augmented Chow, chain, and h-polynomials—by introducing *hyperplane coalescence* and *hyperplane absorption* ([3, §3.2]) on finite geometries, including the affine packing matroids $\mathcal{M}_q^{\mathrm{pk}}$ and the projective Gabidulin absorption family $\mathrm{Gab}_{k,q}$ ([3, §4.2], denoted $\mathcal{G}_{r,q}$ below with $r = k$). While [3] defines $\mathrm{Gab}_{k,q}$ for general $k \geq 4$, their analysis evaluates rank 4 in Theorem B(c) (with log-concavity data through ranks $5, 6$) and addresses *log-concavity* only; unimodality of $h_L(t)$ and of $c_{\bar{L}}(t), c_L(t)$, as well as real-rootedness for representable and graphic matroids, remained open.

The purpose of this note is to resolve the unimodality question for $h_L(t)$ in the negative and to map out exact structural barriers and computational frontiers for the remaining open problems.

Theorem 1.1 (Disproof of unimodality of $h_L(t)$)**.** *Let $r \geq 4$, let q be a prime power, and let $L_{r,q}$ denote either the projective Gabidulin absorption geometric lattice $\mathcal{G}_{r,q} = \mathrm{Gab}_{r,q}$ of* [3, §4.2] *(on $\frac{q^{2(r-1)}-1}{q-1} + q^{r-1}$ elements, with $E_0 = \frac{(r-1)(3r-4)}{2}$) or the smaller affine Gabidulin absorption geometric lattice $\mathcal{A}_{r,q}$ defined in Lemma 2.2 (on $q^{2(r-2)} + q^{r-2}$ elements, with $E_0 = \frac{3(r-2)(r-1)}{2}$). As $q \to \infty$ over prime powers,*

$$\begin{aligned} c_k(\overline{L_{r,q}}) &= O(q^{E_0-2}) \qquad (0 \leq k \leq r-4), \\ c_{r-3}(\overline{L_{r,q}}) &= \frac{(r-3)^2}{2}\, q^{E_0-1} + O(q^{E_0-2}), \\ c_{r-2}(\overline{L_{r,q}}) &= \frac{r(2r-5)}{2}\, q^{E_0-1} + O(q^{E_0-2}), \\ c_{r-1}(\overline{L_{r,q}}) &= q^{E_0} + O(q^{E_0-1}), \end{aligned}$$

and therefore the order-complex h-polynomial satisfies

$$\begin{aligned} h_{r-3}(L_{r,q}) &= \frac{(r-3)^2}{2}\, q^{E_0-1} + O(q^{E_0-2}), \\ h_{r-2}(L_{r,q}) &= \frac{7r-18}{2}\, q^{E_0-1} + O(q^{E_0-2}), \\ h_{r-1}(L_{r,q}) &= q^{E_0} + O(q^{E_0-1}). \end{aligned}$$

In particular:

1. *For* every rank $r \geq 11$ *and all sufficiently large prime powers q, since $(r-3)^2 - (7r-18) = r^2 - 13r + 27 > 0$, $h_{L_{r,q}}(t)$ is strictly non-unimodal with dip $h_{r-4} < h_{r-3} > h_{r-2} < h_{r-1}$, valley ratio $h_{r-2}/h_{r-3} \to \frac{7r-18}{(r-3)^2} = O(1/r)$, and final jump $h_{r-1}/h_{r-2} \sim \frac{2q}{7r-18} \to \infty$.*
2. *At rank $r = 10$, both $\mathcal{A}_{10,q}$ and $\mathcal{G}_{10,q}$ have strictly non-unimodal h-polynomials ($h_6 < h_7 > h_8 < h_9$) if and only if q belongs to the exact finite set of* 16 *prime powers*

$$\{27, 29, 31, 32, 37, 41, 43, 47, 49, 53, 59, 61, 64, 67, 71, 73\}.$$

The valley in Theorem 1.1 occurs at index $r-2 > (r-1)/2$, precisely in the upper half left unconstrained by strong flawlessness [16, 7]. By contrast, the chain polynomials $c_{\bar{L}}(t)$ and $c_L(t)$ obey a universal low-rank unimodality barrier.

Theorem 1.2 (Universal low-rank unimodality of chain polynomials)**.** *Every finite geometric lattice L of rank $r \geq 2$ satisfies*

$$(k+1)\, c_{k+1}(\bar{L}) \geq 2(r-1-k)\, c_k(\bar{L}) \qquad (0 \leq k \leq r-2).$$

Consequently, $c_{\bar{L}}(t)$ is unimodal for every geometric lattice of rank $r \leq 6$, and $c_L(t) = (1+t)^2 c_{\bar{L}}(t)$ is unimodal for every geometric lattice of rank $r \leq 8$.

At rank 4, all counterexamples in [3] (and $\mathcal{A}_{4,q}$) are non-orientable and have variable line sizes. Our third theorem proves that both features are strictly necessary.

Theorem 1.3 (Rank-4 strict log-concavity barriers)**.** *Let L be a rank-4 geometric lattice (simple rank-4 matroid M). Assume that either:*

1. *M is orientable (in particular, $\mathbb{R}$-realizable, $\mathbb{Q}$-representable, regular, or graphic); or*
2. *every line (rank-2 flat) of L has the same cardinality $a \geq 2$.*

Then $c_1(\bar{L})^2 > 2c_0(\bar{L})c_2(\bar{L})$ and $c_2(\bar{L})^2 > 2c_1(\bar{L})c_3(\bar{L})$. In particular, both $c_{\bar{L}}(t)$ and $c_L(t)$ are strictly log-concave.

Finally, to probe the representable frontier—specifically the $\mathbb{F}_q$-representable Whittle q-lifts [18] of Matt Larson's generalized theta graphs used by Divoux, Lowen, and Wang [9] and Divoux, Larson, Lowen, and Wang [8] to disprove Rota's unimodality conjecture for Whitney numbers W_k—we build on the maximal-flag (catenary-data) framework of Bonin and Kung [5] to prove an exact multichain identity for $L_q(M)$ covering chains of all lengths.

Theorem 1.4 (Exact Whittle q-lift zeta-polynomial identity)**.** *Let M be a simple $\mathbb{F}_q$-representable matroid of rank r realized by $\{v_e : e \in E(M)\} \subset \mathbb{F}_q^r$, and let $L_q(M)$ be Whittle's q-lift on $E(L_q(M)) = \{e_0\} \cup \{(\lambda, v_e) : \lambda \in \mathbb{F}_q,\ e \in E(M)\} \subset \mathbb{F}_q \oplus \mathbb{F}_q^r$ with $e_0 = (1,0)$. The zeta polynomial $Z_{L_q(M)}(m)$ counting m-multichains $\hat{0} \leq F_1 \leq \cdots \leq F_m \leq \hat{1}$ in the full lattice $L(L_q(M))$ is*

$$Z_{L_q(M)}(m) = \sum_{j=0}^{m} S_M(m, j; q),$$

$$S_M(m, j; q) := \sum_{\hat{0} \leq G_1 \leq \cdots \leq G_m \leq E(M)} q^{\mathrm{rk}_M(G_j)}$$

(with $G_0 := \hat{0}$). Moreover, for any direct sum $M_1 \oplus M_2$,

$$S_{M_1 \oplus M_2}(m, j; q) = S_{M_1}(m, j; q)\, S_{M_2}(m, j; q).$$

Section 2 proves Lemmas 2.1–2.2 and Theorem 1.1. Sections 3, 4, and 5 prove Theorems 1.2, 1.3, and 1.4, respectively. Section 6 records our zero-floating-point computational certifications and exact frontier tables, and Section 7 summarizes reproducibility and verification scope.

2. Non-Unimodal Order-Complex h-Polynomials

2.1. **Sequential Crapo modular-cut absorption and Gabidulin families.** We recall the flat structure of sequential hyperplane absorption via Crapo single-element extensions (cf. [3, §3.2] and [17, Prop. 7.2.3]).

Lemma 2.1 (Sequential Crapo modular-cut absorption)**.** *Let M_0 be a simple rank-r matroid ($r \geq 3$) on ground set E_0, and let $\mathcal{H}_1, \ldots, \mathcal{H}_m$ be pairwise disjoint collections of hyperplanes (rank-$(r-1)$ flats) of M_0 such that for every $1 \leq i \leq m$ and all distinct $H, H' \in \mathcal{H}_i$,*

$$\mathrm{rk}_{M_0}(H \cap H') \leq r - 3.$$

Define M_i inductively on $E_i = E_0 \cup \{p_1, \ldots, p_i\}$ ($p_1, \ldots, p_m$ distinct new elements) as the Crapo extension of M_{i-1} along $\widetilde{\mathcal{M}}_i = \mathcal{H}_i \cup \{E_{i-1}\}$. Then:

1. *For every $1 \leq i \leq m$, $\widetilde{\mathcal{M}}_i$ is a valid modular cut of M_{i-1}, so M_m is a simple rank-r matroid on $E_0 \cup P$ ($P = \{p_1, \ldots, p_m\}$), independent of the ordering of $\mathcal{H}_1, \ldots, \mathcal{H}_m$.*
2. *The proper flats (rank $0 \leq \mathrm{rk} \leq r-1$) of M_m are the disjoint union of:*
   - Type I: *$F \cup S$, where $F \in L(M_0)$ has rank $d \in \{0, \ldots, r-2\}$, $S \subseteq P$ has size $s = |S|$ with $d + s \leq r - 1$, excluding precisely the pairs $(F, \{p_j\})$ with $d = r-2$, $s = 1$ such that $F \subset H$ for some (necessarily unique) $H \in \mathcal{H}_j$. Each valid Type I flat has rank $d+s$ and lower interval $[\hat{0}, F \cup S]_{L(M_m)} \cong [\hat{0}, F]_{L(M_0)} \times 2^S$.*
   - Type II: *$H \in \mathcal{F}_{r-1}(M_0) \setminus \bigcup_{j=1}^{m} \mathcal{H}_j$, of rank $r-1$, with lower interval $[\hat{0}, H]_{L(M_0)}$.*
   - Type III: *$H \cup \{p_j\}$ for $1 \leq j \leq m$ and $H \in \mathcal{H}_j$, of rank $r-1$, whose proper subflats in $L(M_m)$ are all $F \subsetneq H$ ($\mathrm{rk}(F) \in \{0, \ldots, r-2\}$) together with all $F \cup \{p_j\}$ for $F \subsetneq H$ with $\mathrm{rk}(F) \in \{0, \ldots, r-3\}$.*

Proof. We proceed by induction on $i = 0, \ldots, m$, the base case $i = 0$ being immediate. Assume the flat classification holds for M_{i-1}. Because $\mathcal{H}_i$ is disjoint from $\mathcal{H}_1, \ldots, \mathcal{H}_{i-1}$, every $H \in \mathcal{H}_i$ is a Type II hyperplane of M_{i-1}; thus $\widetilde{\mathcal{M}}_i = \mathcal{H}_i \cup \{E_{i-1}\}$ is an upper order ideal in $L(M_{i-1})$. For any distinct $H, H' \in \mathcal{H}_i$, their intersection in $L(M_{i-1})$ lies in E_0 and coincides with $H \cap H'$ in M_0 (since $M_{i-1}|_{E_0} = M_0$), so $\mathrm{rk}_{M_{i-1}}(H \cap H') \leq r-3 < (r-1)+(r-1)-r = r-2$. Hence no two distinct hyperplanes in $\mathcal{H}_i$ form a modular pair in $L(M_{i-1})$, proving that $\widetilde{\mathcal{M}}_i$ is a modular cut of M_{i-1}.

By Crapo's theorem [17, Prop. 7.2.3], the flats of $M_i = M_{i-1} +_{p_i} \widetilde{\mathcal{M}}_i$ are: (a) $X \cup \{p_i\}$ for $X \in \widetilde{\mathcal{M}}_i$; (b) X for $X \notin \widetilde{\mathcal{M}}_i$; and (c) $X \cup \{p_i\}$ (of rank $\mathrm{rk}(X) + 1$) for $X \notin \widetilde{\mathcal{M}}_i$ not covered by any member of $\widetilde{\mathcal{M}}_i$. Rule (a) turns each $H \in \mathcal{H}_i$ into a Type III hyperplane $H \cup \{p_i\}$ and sends E_{i-1} to E_i. A proper flat $X \notin \widetilde{\mathcal{M}}_i$ of rank $r-1$ is covered by $E_{i-1} \in \widetilde{\mathcal{M}}_i$, so $X \cup \{p_i\}$ is not a flat. A proper flat X of rank $\leq r-2$ is covered by some $H \in \mathcal{H}_i$ if and only if $\mathrm{rk}(X) = r-2$ and $X \subset H \subseteq E_0$, which forces $X \cap P_{i-1} = \emptyset$, i.e., $X = F \in \mathcal{F}_{r-2}(M_0)$ with $F \subset H \in \mathcal{H}_i$ (with H unique since distinct hyperplanes in $\mathcal{H}_i$ meet in rank $\leq r-3$). Every other flat $X = F \cup S$ of rank $d + |S| \leq r-2$ (none of which was excluded at steps $< i$, as exclusions occur only at rank $r-1$) yields $F \cup (S \cup \{p_i\})$ of rank $d + |S| + 1 \leq r - 1$.

For any valid Type I flat $F \cup S$, every proper subflat has rank $\leq r-2$ and thus cannot be excluded, giving $[\hat{0}, F \cup S]_{L(M_m)} \cong [\hat{0}, F]_{L(M_0)} \times 2^S$. For a Type III flat $H \cup \{p_j\}$, H itself is no longer a flat, and $F \cup \{p_j\}$ is excluded precisely when $F \subsetneq H$ has rank $r-2$. □

Lemma 2.2 (Projective and affine Gabidulin modular-cut families)**.** *Let $r \geq 4$ and let q be any prime power.*

(1) Projective family $\mathcal{G}_{r,q} = \mathrm{Gab}_{r,q}$ ([3, §4.2]): *Set $n = r-1 \geq 3$ and $M_0 = T_r(\mathrm{PG}(2n-1, q))$ on $E_0 = \mathbb{P}(\mathbb{F}_{q^n} \oplus \mathbb{F}_{q^n})$ ($|E_0| = \frac{q^{2n}-1}{q-1}$). Let $C = \left\{\sum_{j=0}^{n-2} c_j x^{q^j} : c_j \in \mathbb{F}_{q^n}\right\} \subset \mathrm{End}_{\mathbb{F}_q}(\mathbb{F}_{q^n})$. For each $\alpha \in \mathbb{F}_{q^n}$, define $\mathcal{H}_\alpha = \{H_f : f \in \alpha x^{q^{n-1}} + C\}$, where $H_f = \{(x, f(x)) : x \in \mathbb{F}_{q^n}\}$ is the graph of f. Then $\{\mathcal{H}_\alpha\}_{\alpha \in \mathbb{F}_{q^n}}$ consists of $m = q^n$ pairwise disjoint hyperplane families of size $s_{\mathrm{cut}} = q^{n(n-1)}$ each ($ms_{\mathrm{cut}} = q^{n^2}$), satisfying $\mathrm{rk}_{M_0}(H_f \cap H_g) = \dim_{\mathbb{F}_q} \ker(f-g) \leq n-2 = r-3$ for distinct $H_f, H_g \in \mathcal{H}_\alpha$. Lemma 2.1 yields the simple rank-r geometric lattice $\mathcal{G}_{r,q}$ on $\frac{q^{2(r-1)}-1}{q-1} + q^{r-1}$ elements.*

(2) Affine family $\mathcal{A}_{r,q}$: *Set $n_{\mathrm{aff}} = r-2 \geq 2$ and $M_0 = T_r(\mathrm{AG}(2n_{\mathrm{aff}}, q))$ on $E_0 = \mathbb{F}_{q^{n_{\mathrm{aff}}}} \oplus \mathbb{F}_{q^{n_{\mathrm{aff}}}}$ ($|E_0| = q^{2n_{\mathrm{aff}}}$), where rank-$d$ flats ($1 \leq d \leq r-1$) are $(d-1)$-dimensional affine $\mathbb{F}_q$-subspaces. Let $C_{\mathrm{aff}} = \left\{\sum_{j=0}^{n_{\mathrm{aff}}-2} c_j x^{q^j} : c_j \in \mathbb{F}_{q^{n_{\mathrm{aff}}}}\right\}$. For each $\alpha \in \mathbb{F}_{q^{n_{\mathrm{aff}}}}$, define*

$$\mathcal{H}_\alpha = \left\{H_{f,b} : f \in \alpha x^{q^{n_{\mathrm{aff}}-1}} + C_{\mathrm{aff}},\ b \in \mathbb{F}_{q^{n_{\mathrm{aff}}}}\right\},$$
$$H_{f,b} = \left\{(x, f(x)+b) : x \in \mathbb{F}_{q^{n_{\mathrm{aff}}}}\right\}.$$

Then $\{\mathcal{H}_\alpha\}$ consists of $m = q^{n_{\mathrm{aff}}}$ pairwise disjoint hyperplane families of size $s_{\mathrm{cut}} = q^{n_{\mathrm{aff}}^2}$ ($ms_{\mathrm{cut}} = q^{n_{\mathrm{aff}}(n_{\mathrm{aff}}+1)}$). For distinct $H_{f,b}, H_{g,b'} \in \mathcal{H}_\alpha$, the intersection is empty (rank 0) if $f = g$, and otherwise either empty or a coset of $\ker(f-g)$ of affine dimension $\leq n_{\mathrm{aff}} - 2$, hence matroid rank $\leq (n_{\mathrm{aff}}-2)+1 = r-3$. Lemma 2.1 yields the simple rank-r geometric lattice $\mathcal{A}_{r,q}$ on $q^{2(r-2)} + q^{r-2}$ elements.

Proof. In both constructions, distinct cosets of C (resp. C_{aff}) are disjoint and the graph map $f \mapsto H_f$ (resp. $(f,b) \mapsto H_{f,b}$) is injective. A nonzero q-linearized polynomial of q-degree $\leq d-2$ has ordinary degree $\leq q^{d-2}$ in $\mathbb{F}_{q^d}[x]$, hence at most q^{d-2} roots in $\mathbb{F}_{q^d}$, so $\dim_{\mathbb{F}_q} \ker(f-g) \leq d-2$. Both families therefore satisfy the hypotheses of Lemma 2.1. □

2.2. **Exact closed-form multichain formula.** In the setting of Lemma 2.1, let $\mathcal{H} := \bigsqcup_{j=1}^m \mathcal{H}_j$ denote the set of all absorbed hyperplanes (of total size $M := |\mathcal{H}| = \sum_{j=1}^m |\mathcal{H}_j|$) and let $\mathcal{U} := \mathcal{F}_{r-1}(M_0) \setminus \mathcal{H}$ denote the set of unabsorbed hyperplanes. Let $A(t)$ count t-multichains $\hat{0} \leq X_1 \leq \cdots \leq X_t < \hat{1}$ in $L(M_m) \setminus \{\hat{1}\}$ ($t \geq 1$, $A(0) = 1$). Let $N_0(d,t)$ be the number of t-multichains in $L(M_0)$ ending at rank d, and for each hyperplane $H \in \mathcal{F}_{r-1}(M_0)$ let $N_H(d,t)$ be the number of t-multichains in $[\hat{0}, H]_{L(M_0)}$ ending at rank d.

Partitioning by the top flat X_t according to Lemma 2.1(2) yields the unconditional identity (valid for every simple rank-r matroid M_0 and arbitrary cut sizes $|\mathcal{H}_j|$):

$$\begin{aligned} A(t) = &\sum_{d=0}^{r-2}\sum_{s=0}^{r-1-d} N_0(d,t)\binom{m}{s}t^s - \sum_{H\in\mathcal{H}} N_H(r-2,t)\,t \\ &+ \sum_{H\in\mathcal{U}} N_H(r-1,t) + \sum_{H\in\mathcal{H}} B_H^{\mathrm{abs}}(t), \end{aligned} \tag{1}$$

where $B_H^{\mathrm{abs}}(1) = 1$ and, for $t \geq 2$,

$$B_H^{\mathrm{abs}}(t) = B_H^{\mathrm{abs}}(t-1) + \sum_{d=0}^{r-2} N_H(d,t-1) + (t-1)\sum_{d=0}^{r-3} N_H(d,t-1).$$

Indeed, each excluded Type I flat $(F,\{p_j\})$ corresponds uniquely to a pair (H,F) with $H \in \mathcal{H}_j$ and $F \subset H$ of rank $r-2$, contributing $B_{M_0}(F,t)\,t$ to the unrestricted Type I sum; summing $B_{M_0}(F,t)$ over all rank-$(r-2)$ flats $F \subset H$ equals $N_H(r-2,t)$ identically. (Replacing $N_H(r-2,t)$ by a global average $\kappa_H N_0(r-2,t)/W_{r-2}(M_0)$ requires ridge-homogeneity and fails otherwise, e.g., on $M_0 = U_{2,3}\oplus U_{2,3}$ with a singleton cut, where $A(2) = 208$ matches (1).)

When all hyperplanes $H \in \mathcal{F}_{r-1}(M_0)$ have isomorphic lower intervals $[\hat{0},H]_{L(M_0)}$, the quantities $N_H(d,t)$ and $B_H^{\mathrm{abs}}(t)$ are independent of H, and (1) collapses with $u = W_{r-1}(M_0) - M$ to

$$\begin{aligned} A(t) = &\sum_{d=0}^{r-2}\sum_{s=0}^{r-1-d} N_0(d,t)\binom{m}{s}t^s - M\,N_H(r-2,t)\,t \\ &+ u\,N_H(r-1,t) + M\,B_H^{\mathrm{abs}}(t), \end{aligned} \tag{2}$$

with $M = ms_{\mathrm{cut}}$ whenever $|\mathcal{H}_j| = s_{\mathrm{cut}}$ for all j. In both $T_r(\mathrm{PG}(D-1,q))$ and $T_r(\mathrm{AG}(D,q))$, the groups $\mathrm{GL}_D(\mathbb{F}_q)$ and $\mathrm{AGL}_D(\mathbb{F}_q)$ act transitively on flats of every fixed rank $d \in \{1,\dots,r-1\}$, so both families are hyperplane-homogeneous (and in fact ridge-homogeneous, satisfying $N_H(r-2,t) = \kappa_H N_0(r-2,t)/W_{r-2}(M_0)$ where κ_H is the number of rank-$(r-2)$ subflats of H). Binomial inversion yields $c_k(\overline{M_m}) = \sum_{t=0}^{k}(-1)^{k-t}\binom{k}{t}A(t)$ $(0 \leq k \leq r-1)$.

2.3. **Proof of Theorem 1.1.**

Proof of Theorem 1.1. We first prove the asymptotic formulas for $\mathcal{G}_{r,q}$ ($n = r-1 \geq 3$, $D = 2n$, $E_0 = n^2 + \binom{n}{2} = \frac{(r-1)(3r-4)}{2}$).

Step 1: Flags in $M_0 = T_r(\mathrm{PG}(2n-1,q))$. For any strict rank sequence $0 < d_1 < \cdots < d_a \leq n$, the number of strict flags $F_1 < \cdots < F_a$ in M_0 is

$\begin{bmatrix} 2n \\ d_a \end{bmatrix}_q \prod_{i=1}^{a-1} \begin{bmatrix} d_{i+1} \\ d_i \end{bmatrix}_q$, a monic polynomial in q of degree

$$\begin{aligned}\Phi(d_1,\dots,d_a) &= d_a(2n-d_a) + \sum_{i=1}^{a-1} d_i(d_{i+1}-d_i) \\ &= d_a(2n-d_a) + \binom{d_a}{2} - \Delta(d_1,\dots,d_a),\end{aligned}$$

where $\Delta(d_1,\dots,d_a) := \frac{1}{2}\sum_{u=1}^{a} \delta_u(\delta_u - 1) \geq 0$ with $\delta_1 = d_1$ and $\delta_u = d_u - d_{u-1}$ $(u \geq 2)$. Thus:

- $\Delta = 0$ if and only if $(d_1,\dots,d_a) = (1,2,\dots,d_a)$ (the complete flag up to d_a);
- $\Delta = 1$ if and only if $(d_1,\dots,d_a)$ omits precisely one rank $j \in \{1,\dots,d_a - 1\}$;
- $\Delta \geq 2$ otherwise.

Step 2: Strict chains $\hat{0} < X_1 < \cdots < X_k$ in $\overline{\mathcal{G}_{r,q}}$ by top flat X_k.

Case A: $X_k = F \cup S$ is Type I ($\mathrm{rk}(F) = d \in \{0,\dots,n-1\}$, $|S| = s \geq 0$, $1 \leq d + s \leq n$). If $d = 0$, the number of subsets $S \subseteq P$ is $\binom{q^n}{s} = O(q^{n^2})$, and $E_0 - n^2 = \binom{n}{2} \geq 3$ since $n \geq 3$, so $d = 0$ contributes $O(q^{E_0-3})$. For $1 \leq d \leq n-1$, $\binom{q^n}{s} = \frac{1}{s!}q^{ns}(1 + O(q^{-n}))$, and the maximum q-degree of chains ending at (d,s) is

$$E_{\mathrm{I}}(d,s) = d(2n-d) + \binom{d}{2} + ns = E_0 - \binom{n-d}{2} - n(n-d-s).$$

Whenever $d+s \leq n-1$, $E_0 - E_{\mathrm{I}}(d,s) \geq n \geq 3$. Thus only Type I hyperplanes ($s = n - d \geq 1$) can reach q^{E_0} or q^{E_0-1}:

- For $s \geq 3$, $\binom{n-d}{2} = \binom{s}{2} \geq 3$, giving $O(q^{E_0-3})$.
- For $s = 1$, $d = n-1$: although $E_{\mathrm{I}}(n-1,1) = E_0$, absorption excludes $q^{n^2}\begin{bmatrix} n \\ n-1 \end{bmatrix}_q$ pairs $(F_{n-1}, \{p_\alpha\})$. Using $\begin{bmatrix} 2n \\ n-1 \end{bmatrix}_q = q^{n^2-1}(1 + q^{-1} + 2q^{-2} + O(q^{-3}))$ for $n \geq 3$ and $\begin{bmatrix} n \\ n-1 \end{bmatrix}_q = q^{n-1}(1 + q^{-1} + q^{-2} + \dots)$, the count of *valid* $(n-1,1)$ Type I hyperplanes is

  $$q^n \begin{bmatrix} 2n \\ n-1 \end{bmatrix}_q - q^{n^2} \begin{bmatrix} n \\ n-1 \end{bmatrix}_q = q^{n^2+n-3}\big(1 + O(q^{-1})\big).$$

  Both the q^{n^2+n-1} and q^{n^2+n-2} terms cancel identically. Multiplying by $q^{\binom{n-1}{2}}$ yields degree $(n^2+n-3) + \binom{n-1}{2} = E_0 - 2$, so all valid $(n-1,1)$ flats contribute $O(q^{E_0-2})$.
- For $s = 2$, $d = n-2$: $\binom{2}{2} = 1$, so $E_{\mathrm{I}}(n-2,2) = E_0 - 1$. Choices of $(F_{n-2}, \{a,b\})$ together with a complete flag $F_1 < \cdots < F_{n-2}$ inside F_{n-2} have count $\frac{1}{2}q^{E_0-1} + O(q^{E_0-2})$ (any incomplete flag drops to $O(q^{E_0-2})$). Fixing $F_1 < \cdots < F_{n-2}$ and $\{a,b\}$, a strict chain $X_1 < \cdots < X_k = F_{n-2} \cup \{a,b\}$ in $\{0,1,\dots,n-2\} \times 2^{\{a,b\}}$ whose M_0-ranks cover $\{1,\dots,n-2\}$ has length $k \in \{n-2, n-1, n\}$:

- $k = n-2 = r-3$: no repeated M_0-ranks, so $X_j = (j, S_j)$ with $\emptyset \subseteq S_1 \subseteq \cdots \subseteq S_{n-2} = \{a, b\}$. Each of a, b independently picks its first appearance in $\{1, \ldots, n-2\}$, giving $(n-2)^2 = (r-3)^2$ chains and contributing $\frac{(r-3)^2}{2} q^{E_0-1}$ to c_{r-3}.
- $k = n-1 = r-2$: either (i) $X_1 = (0, S_1)$ ($S_1 \neq \emptyset$) followed by M_0-ranks $1, \ldots, n-2$, giving $(n-1)^2 - (n-2)^2 = 2n-3$ chains; or (ii) one rank $j \in \{1, \ldots, n-2\}$ repeats as $(j, S') < (j, S'')$, giving $(n-1)^2 - (n-2)^2 = 2n-3$ chains for each of the $n-2$ choices of j. Total: $(n-1)(2n-3) = (r-2)(2r-5)$ chains, contributing $\frac{(r-2)(2r-5)}{2} q^{E_0-1}$ to c_{r-2}.
- $k = n = r-1$: contributes $O(q^{E_0-1})$ to c_{r-1} (subleading to q^{E_0}).

Case B: $X_k = H$ is Type II (unabsorbed hyperplane). There are $u = \begin{bmatrix} 2n \\ n \end{bmatrix}_q - q^{n^2} = q^{n^2-1} + O(q^{n^2-2})$ unabsorbed hyperplanes. Inside H, a strict flag reaches degree $\binom{n}{2}$ only when complete ($k = n = r-1$). Thus Type II contributes $O(q^{E_0-1})$ to c_{r-1} and $O(q^{E_0-2})$ to c_k for all $k \leq r-2$.

Case C: $X_k = H \cup \{p_\alpha\}$ is Type III (absorbed hyperplane, count q^{n^2}). Let the distinct nonzero M_0-ranks among $X_1, \ldots, X_{k-1}$ be $0 < d_1 < \cdots < d_a \leq n-1$; the flag count in H has degree $\binom{n}{2} - \Delta(d_1, \ldots, d_a, n)$.

- $\Delta = 0$ (q^{E_0}): $(d_1, \ldots, d_a) = (1, \ldots, n-1)$. Since $(F_{n-1}, \{p_\alpha\})$ is excluded, $X_{k-1} = F_{n-1}$ cannot contain p_α, forcing $X_1, \ldots, X_{k-1}$ all to avoid p_α. This gives the unique chain $F_1 < \cdots < F_{n-1} < H \cup \{p_\alpha\}$ (length $k = r-1$), contributing q^{E_0} to c_{r-1} and 0 at q^{E_0} to c_k ($k \leq r-2$).
- $\Delta = 1$ (q^{E_0-1}): one rank $j \in \{1, \ldots, n-1\}$ is omitted. If $j \leq n-2$, rank $n-1$ is present and forces all X_i ($i < k$) to avoid p_α, yielding 1 chain of length $r-2$ for each of the $n-2 = r-3$ choices of j. If $j = n-1$, then $(d_1, \ldots, d_a) = (1, \ldots, n-2)$ and p_α is permitted; for $k = r-2$ (no repeated M_0-ranks), the indicator of p_α is any weak binary sequence $0 \leq b_1 \leq \cdots \leq b_{n-2} \leq 1$, giving $(n-2)+1 = r-2$ chains (while $k = r-1$ contributes $O(q^{E_0-1})$ to c_{r-1}). Thus Type III contributes $(r-3) + (r-2) = 2r-5$ at q^{E_0-1} to c_{r-2}, and 0 to c_k for $k \leq r-3$.

Step 3: Summation and h-vector transformation. Combining Cases A, B, and C gives $c_k = O(q^{E_0-2})$ for $k \leq r-4$, $c_{r-3} = \frac{(r-3)^2}{2} q^{E_0-1} + O(q^{E_0-2})$, $c_{r-2} = \big(\frac{(r-2)(2r-5)}{2} + (2r-5)\big) q^{E_0-1} + O(q^{E_0-2}) = \frac{r(2r-5)}{2} q^{E_0-1} + O(q^{E_0-2})$,

and $c_{r-1} = q^{E_0} + O(q^{E_0-1})$. Applying $h_j = \sum_{k=0}^{j}(-1)^{j-k}\binom{r-1-k}{j-k}c_k$ yields

$$\begin{aligned}
h_{r-3} &= c_{r-3} + O(q^{E_0-2}) = \frac{(r-3)^2}{2}\,q^{E_0-1} + O(q^{E_0-2}),\\
h_{r-2} &= c_{r-2} - 2c_{r-3} + O(q^{E_0-2})\\
&= \frac{r(2r-5) - 2(r-3)^2}{2}\,q^{E_0-1} + O(q^{E_0-2})\\
&= \frac{7r-18}{2}\,q^{E_0-1} + O(q^{E_0-2}),\\
h_{r-1} &= c_{r-1} + O(q^{E_0-1}) = q^{E_0} + O(q^{E_0-1}).
\end{aligned}$$

For the affine family $\mathcal{A}_{r,q}$ ($n_{\mathrm{aff}} = r-2 \geq 2$, $D = 2n_{\mathrm{aff}}$), affine flags of dimensions $0 \leq a_1 < \cdots < a_k \leq n_{\mathrm{aff}}$ (ranks $d_i = a_i + 1$) have count $q^{2n_{\mathrm{aff}}-a_1}\begin{bmatrix}2n_{\mathrm{aff}}\\ a_k\end{bmatrix}_q \prod_{i=1}^{k-1}\begin{bmatrix}a_{i+1}\\ a_i\end{bmatrix}_q$, with maximum degree $E_0^{\mathrm{aff}} = 2n_{\mathrm{aff}} + n_{\mathrm{aff}}^2 + \binom{n_{\mathrm{aff}}}{2} = \frac{3(r-2)(r-1)}{2}$ at the complete flag, deficit 1 for omitting any single rank $j \in \{1,\ldots,d-1\}$, Type I exponent ($1 \leq d \leq r-2$, $a = d-1$)

$$\begin{aligned}
E_{\mathrm{I}}^{\mathrm{aff}}(d,s) &= 2n_{\mathrm{aff}} + a(2n_{\mathrm{aff}} - a) + \binom{a}{2} + n_{\mathrm{aff}}s\\
&= E_0^{\mathrm{aff}} - \binom{r-1-d}{2} - n_{\mathrm{aff}}(r-1-d-s)
\end{aligned}$$

(and deficit $\binom{r-1}{2} \geq 3$ for $d = 0$), and the identical double cancellation

$$\begin{aligned}
q^{n_{\mathrm{aff}}} \cdot q^{n_{\mathrm{aff}}+1}\begin{bmatrix}2n_{\mathrm{aff}}\\ n_{\mathrm{aff}}-1\end{bmatrix}_q - q^{n_{\mathrm{aff}}(n_{\mathrm{aff}}+1)} \cdot q\begin{bmatrix}n_{\mathrm{aff}}\\ n_{\mathrm{aff}}-1\end{bmatrix}_q\\
= O\big(q^{n_{\mathrm{aff}}^2+2n_{\mathrm{aff}}-2}\big)
\end{aligned}$$

at $(r-2,1)$. All chain counts agree identically, giving the same formulas with $E_0 = E_0^{\mathrm{aff}}$.

Part (1) follows immediately from $(r-3)^2 - (7r-18) = r^2 - 13r + 27 > 0$ for all $r \geq 11$ (note $h_{r-4} = O(q^{E_0-2}) < h_{r-3}$). Part (2) follows from exact polynomial arithmetic in $\mathbb{Q}[q]$ using (1): at $r = 10$, we have $h_7 - h_8 = -\frac{3}{2}q^{116} + \frac{211}{2}q^{115} + O(q^{114})$ for $\mathcal{G}_{10,q}$ and $-\frac{3}{2}q^{107} + 106q^{106} + O(q^{105})$ for $\mathcal{A}_{10,q}$. Evaluating exact integer h-vectors for all prime powers $q < 75$ gives the 16 prime powers in $[27, 73]$, while expanding $h_{j+1}(75+u) - h_j(75+u) \in \mathbb{Q}[u]$ for all $0 \leq j \leq 8$ verifies strictly positive coefficients in u for every j, proving $h_0 < h_1 < \cdots < h_9$ for all real $q \geq 75$ (Section 6). □

3. Universal Low-Rank Unimodality of Chain Polynomials

Proof of Theorem 1.2. Count pairs (C, y) where $C = (x_1 < \cdots < x_k)$ is a k-chain in $\bar{L}$ and $y \in \bar{L}\backslash C$ is comparable with every element of C. Each $(k+1)$-chain in $\bar{L}$ arises from exactly $k+1$ such pairs, yielding $(k+1)\,c_{k+1}(\bar{L})$. On the other hand, fix C and set $x_0 = \hat{0}$, $x_{k+1} = \hat{1}$, and $g_i = \mathrm{rk}(x_i) - \mathrm{rk}(x_{i-1}) \geq 1$, so $\sum_{i=1}^{k+1} g_i = r$. Each interval $[x_{i-1}, x_i]$ is a geometric lattice of rank

g_i, and the closures of subsets of any basis of atoms in $[x_{i-1}, x_i]$ embed the Boolean lattice B_{g_i} into $[x_{i-1}, x_i]$. Thus $|(x_{i-1}, x_i)| \geq 2^{g_i} - 2 \geq 2(g_i - 1)$ for all $g_i \geq 1$. Summing over $i = 1, \ldots, k+1$ gives at least $2\sum_{i=1}^{k+1}(g_i - 1) = 2(r - 1 - k)$ choices for y, proving

$$(k+1)\, c_{k+1}(\bar{L}) \geq 2(r-1-k)\, c_k(\bar{L}) \qquad (0 \leq k \leq r-2).$$

For $c_{\bar{L}}(t)$, the ratio $\frac{2(r-1-k)}{k+1} \geq 1$ holds whenever $k \leq \lfloor (2r-3)/3 \rfloor$. For $r \leq 6$, this covers all $k \leq r-3$, so $c_0 \leq c_1 \leq \cdots \leq c_{r-2}$, making $(c_0, \ldots, c_{r-1})$ unimodal regardless of c_{r-1}.

For $c_L(t) = (1+t)^2 c_{\bar{L}}(t) = \sum_{j=0}^{r+1} d_j t^j$, where $d_j = c_j + 2c_{j-1} + c_{j-2}$ (setting $c_k = 0$ outside $0 \leq k \leq r-1$): when $r \leq 6$, multiplying a unimodal nonnegative sequence by the log-concave sequence $(1, 2, 1)$ preserves unimodality [13].

- At $r = 7$, the inequality gives $c_0 < c_1 < c_2 < c_3 < c_4$ with $c_3 \geq \frac{8}{3}c_2$, $c_4 \geq \frac{3}{2}c_3$, and $c_5 \geq \frac{4}{5}c_4$. Thus $d_0 < \cdots < d_4$; also $c_5 \geq \frac{4}{5} \cdot \frac{3}{2} \cdot \frac{8}{3}c_2 = \frac{16}{5}c_2 > c_2$, so $d_5 - d_4 = (c_4 - c_3) + (c_5 - c_2) > 0$. Since $d_7 = 2c_6 + c_5 > c_6 = d_8$, the only possible valley is $d_5 > d_6 < d_7$, which is equivalent to $c_4 + c_3 > c_6 + c_5$ and $c_6 > c_5 + c_4$. Adding these forces $c_3 > 2c_5$, contradicting $2c_5 \geq 2 \cdot \frac{4}{5} \cdot \frac{3}{2}c_3 = \frac{12}{5}c_3 > c_3$.
- At $r = 8$, the inequality gives $c_0 < \cdots < c_5$ with $c_4 \geq 2c_3$, $c_5 \geq \frac{6}{5}c_4$, and $c_6 \geq \frac{2}{3}c_5$. Thus $d_0 < \cdots < d_5$; also $c_6 \geq \frac{2}{3} \cdot \frac{6}{5} \cdot 2c_3 = \frac{8}{5}c_3 > c_3$, so $d_6 - d_5 = (c_5 - c_4) + (c_6 - c_3) > 0$. Since $d_8 = 2c_7 + c_6 > c_7 = d_9$, the only possible valley is $d_6 > d_7 < d_8$, i.e., $c_5 + c_4 > c_7 + c_6$ and $c_7 > c_6 + c_5$. Adding these forces $c_4 > 2c_6$, contradicting $2c_6 \geq 2 \cdot \frac{2}{3} \cdot \frac{6}{5}c_4 = \frac{8}{5}c_4 > c_4$. □

4. Strict Log-Concavity of Rank-4 Orientable and Constant-Line-Size Lattices

Proof of Theorem 1.3. Let W_1, W_2, W_3 denote the numbers of points, lines, and planes of L, and $f_{12}, f_{13}, f_{23}, f_{123}$ the flag counts, so $c_0 = 1$, $c_1 = W_1 + W_2 + W_3$, $c_2 = f_{12} + f_{13} + f_{23}$, and $c_3 = f_{123}$. Since $f_{ij} \leq W_i W_j$, we always have $c_1^2 > 2(W_1W_2 + W_1W_3 + W_2W_3) \geq 2c_2 = 2c_0c_2$. Set $x = f_{12} > 0$, $y = f_{13} > 0$, $z = f_{23} > 0$.

Case (1): M is orientable. Extending any point p to a basis $\{p, a, b, c\}$ gives three distinct planes $\mathrm{cl}(p, a, b)$, $\mathrm{cl}(p, a, c)$, $\mathrm{cl}(p, b, c)$ through p, so $f_{13} \geq 3W_1$. Every line contains ≥ 2 points ($f_{12} \geq 2W_2$) and every plane contains ≥ 3 points ($f_{13} \geq 3W_3$), whence $c_1 \leq \frac{1}{2}x + \frac{2}{3}y$.

For each plane H, the restriction $M|_H$ is a simple rank-3 orientable matroid. By the Folkman–Lawrence Topological Representation Theorem [10], $M|_H$ corresponds to a simple essential arrangement of $|H|$ pseudolines in $\mathbb{RP}^2$ whose vertices $V = \ell_H$ correspond to lines $\ell \subset H$, each of degree $2|\ell|$. Thus the number of edges is $E = \frac{1}{2}\sum_{\ell \subset H} 2|\ell| = f_{12}(H)$. Because the arrangement is essential and any two distinct pseudolines meet in a unique

point in $\mathbb{RP}^2$, there are no 1-gons or 2-gons, so $3F \leq 2E$. By Euler's formula $\chi(\mathbb{RP}^2) = 1$ (Melchior's inequality [15]),

$$1 = V - E + F \leq \ell_H - \frac{1}{3} f_{12}(H) \implies f_{12}(H) \leq 3\ell_H - 3.$$

Summing over all planes H gives $c_3 = f_{123} \leq 3f_{23} - 3W_3 < 3z$, so $2c_1c_3 < 3xz + 4yz$ and

$$c_2^2 - 2c_1c_3 > (x+y+z)^2 - (3xz+4yz) = (x+y-z)^2 + xz > 0.$$

Case (2): Constant line size $a \geq 2$. Here $x = aW_2$ and $c_3 = af_{23} = az$. Every plane contains a line plus at least one external point, so $y = f_{13} \geq (a+1)W_3 > aW_3$. By Basterfield–Kelly [2] and Greene [11], $W_1 \leq W_3$, whence $aW_1 \leq aW_3 < y$. Thus $c_1c_3 = (aW_2 + aW_1 + aW_3)z < (x+2y)z$, and

$$c_2^2 - 2c_1c_3 > (x+y+z)^2 - (2xz+4yz) = (x+y-z)^2 + 2xz > 0.$$

In both cases, $c_{\bar{L}}(t)$ is strictly log-concave; multiplication by the strictly log-concave positive polynomial $(1+t)^2$ preserves strict log-concavity [12], so $c_L(t)$ is also strictly log-concave. □

5. Exact Whittle q-Lift Multichain Identity

Bonin and Kung [5] introduced catenary data (counts of maximal flags of flats) and studied their behavior under matroid operations; Theorem 1.4 gives the corresponding exact multichain identity for Whittle's q-lift $L_q(M)$ across all chain lengths $m \geq 1$.

Proof of Theorem 1.4. The flats F of $L_q(M)$ partition into two types:

- *Type A ($e_0 \in F$):* Since $(\mu, v_e) = (\lambda, v_e) + (\mu - \lambda)e_0$, $F = \{e_0\} \cup (\mathbb{F}_q \times H)$ for a unique flat $H \in L(M)$, of rank $\mathrm{rk}_M(H) + 1$.
- *Type B ($e_0 \notin F$):* Because $e_0 = (1,0) \notin \mathrm{span}_{\mathbb{F}_q}(F)$, the projection $\pi\colon \mathbb{F}_q \oplus \mathbb{F}_q^r \to \mathbb{F}_q^r$ restricts to an $\mathbb{F}_q$-linear isomorphism $\mathrm{span}_{\mathbb{F}_q}(F) \xrightarrow{\sim} W_G := \mathrm{span}_{\mathbb{F}_q}\{v_e : e \in G\}$ for a unique flat $G \in L(M)$. Thus $F = F_{G,\varphi} := \{(\varphi(v_e), v_e) : e \in G\}$ for a unique linear functional $\varphi \in W_G^* = \mathrm{Hom}_{\mathbb{F}_q}(W_G, \mathbb{F}_q)$, giving exactly $q^{\mathrm{rk}_M(G)}$ Type B flats above G.

Inclusion in $L(L_q(M))$ holds if and only if: $F_{G,\varphi} \leq F_{G',\varphi'} \iff G \leq G'$ and $\varphi'|_{W_G} = \varphi$; $F_{G,\varphi} \leq \{e_0\} \cup (\mathbb{F}_q \times H) \iff G \leq H$; Type A $\leq$ Type A iff $H \leq H'$; and never Type A $\leq$ Type B.

Split any m-multichain $\hat{0} \leq F_1 \leq \cdots \leq F_m \leq \hat{1}$ in $L(L_q(M))$ by the last index $j \in \{0, \ldots, m\}$ such that F_j is Type B ($j = 0$ if all F_i are Type A). Projecting yields an m-multichain $\hat{0} \leq G_1 \leq \cdots \leq G_m \leq E(M)$ in $L(M)$. Conversely, for any such multichain in $L(M)$ and any $j \in \{0, \ldots, m\}$: there is $1 = q^0$ lift for $j = 0$, and for $j \geq 1$ there are $q^{\mathrm{rk}_M(G_j)}$ choices for $\varphi_j \in W_{G_j}^*$, which uniquely determine $\varphi_i = \varphi_j|_{W_{G_i}}$ $(1 \leq i \leq j)$ as well as $F_{j+1}, \ldots, F_m$. Summing over $j = 0, \ldots, m$ yields $Z_{L_q(M)}(m) = \sum_{j=0}^{m} S_M(m, j; q)$. For $M =$

$M_1 \oplus M_2$, the additive rank $\mathrm{rk}_{M_1\oplus M_2}(G_j^{(1)} \cup G_j^{(2)}) = \mathrm{rk}_{M_1}(G_j^{(1)}) + \mathrm{rk}_{M_2}(G_j^{(2)})$ directly gives $S_{M_1\oplus M_2}(m, j; q) = S_{M_1}(m, j; q) S_{M_2}(m, j; q)$. □

6. Exact Computational Certifications and Frontier Tables

All numerical certifications below use exact integer/rational arithmetic (`int`, `fractions.Fraction`, or C++20 `__int128`), with zero floating-point reliance for any decision.

6.1. **Concrete zero-trust matroid cross-checks.** To eliminate formula or indexing errors, we constructed small concrete absorption matroids explicitly on their ground sets, verified all matroid flat axioms (F1)–(F3) from scratch, and matched brute-force flat-poset DP against (1)–(2); we also cross-checked Theorem 1.4 on three explicit $\mathbb{F}_q$-linear Whittle lifts (Table 1). For the $q = 2$ seven-spread Denniston packing of $\mathrm{PG}(3, 2)$ on $16 + 7 = 23$ elements, note that there are two valid rank-4 geometric lattices: (a) the uncoalesced seven-spread absorption keeping all $\binom{7}{3} = 35$ extension-only triples as separate rank-3 flats ($W_3 = 511$, $c_{\bar{L}}(t) = [1, 787, 4832, 5026]$); and (b) coalescing those 35 triples into a single rank-3 hyperplane P ($W_3 = 477$, $c_{\bar{L}}(t) = [1, 753, 4650, 4858]$, which is the concrete $q = 2$ packing checked in `verify_all.py`).

Table 1. Zero-trust concrete matroid axiom and formula cross-checks.

Concrete Matroid	r	$\lvert E\rvert$	$(W_1, W_2, W_3, \dots)$	Brute-Force $c_{\bar{L}}(t)$ = Formula
$U_{4,6} + 2$ cuts	4	8	$(8, 28, 44)$	$[1, 80, 336, 288]$
$U_{5,8} + 2$ cuts	5	10	$(10, 45, 120, 194)$	$[1, 369, 3570, 7880, 4800]$
$\mathcal{A}_{4,2}$ (4 cosets, $\mathbb{F}_4$)	4	20	$(20, 190, 336)$	$[1, 546, 3276, 3368]$
$q = 2$ packing (uncoal.)	4	23	$(23, 253, 511)$	$[1, 787, 4832, 5026]$
$q = 2$ packing (coal. P)	4	23	$(23, 253, 477)$	$[1, 753, 4650, 4858]$
$\mathcal{G}_{4,2}$ (8 cosets, $\mathbb{F}_8$)	4	71	$(71, 1183, 4839)$	$[1, 6093, 50555, 61999]$
$L_2(K_4)$	4	13	—	$[1, 62, 283, 270]$
$L_3(U_{3,4})$	4	13	—	$[1, 104, 502, 480]$
$L_2(\Theta(1, 2, 2))$	4	11	—	$[1, 54, 231, 210]$

6.2. **Exact non-unimodal $h_L(t)$ witnesses and rank-10 global window.** Exact symbolic arithmetic in $\mathbb{Q}[q]$ confirms the leading coefficients of Theorem 1.1 for both $\mathcal{G}_{r,q}$ and $\mathcal{A}_{r,q}$ across all $4 \le r \le 14$. At rank 10, the full exact polynomials $h_j(\mathcal{G}_{10,q})$ (degree ≤ 117) and $h_j(\mathcal{A}_{10,q})$ (degree ≤ 108) satisfy that every Taylor coefficient of $h_{j+1}(75 + u) - h_j(75 + u)$ $(0 \le j \le 8)$ is strictly positive, proving unimodality globally for all real $q \ge 75$; exact integer evaluation for all prime powers $q < 75$ certifies the exact 16-element non-unimodal window $[27, 73]$. For $11 \le r \le 14$ and $q \le 101$, both $\mathcal{G}_{r,q}$ and $\mathcal{A}_{r,q}$ fail unimodality for every prime power $q \ge q_0(r)$ with $q_0(11) = 31$, $q_0(12) = q_0(13) = 37$, and $q_0(14) = 41$. Representative exact witnesses are

listed in Table 2. Note that $\mathcal{A}_{10,27}$ is minimal strictly within the two complete families $\{\mathcal{G}_{r,q}, \mathcal{A}_{r,q}\}$: deleting even a single auxiliary point from $\mathcal{A}_{10,27}$ (keeping $m = 27^8 - 1$ cuts) still yields a strict valley $h_6 < h_7 > h_8 < h_9$ on $27^{16} + 27^8 - 1 = 79{,}766{,}443{,}077{,}154{,}939{,}399{,}841$ elements.

TABLE 2. Certified non-unimodal order-complex h-polynomial witnesses (exact integer arithmetic; ratios shown to 6 decimal places).

Witness	r	q	Strict Dip	h_{r-2}/h_{r-3}	h_{r-1}/h_{r-2}	Digits
$\mathcal{A}_{10,27}$ (min in fam.)	10	27	$h_6 < h_7 > h_8 < h_9$	0.902044	1.023914	155
$\mathcal{G}_{10,27}$	10	27	$h_6 < h_7 > h_8 < h_9$	0.901838	1.025673	168
$\mathcal{A}_{10,29}$ (min prime, fam.)	10	29	$h_6 < h_7 > h_8 < h_9$	0.912101	1.101025	158
$\mathcal{A}_{11,31}$	11	31	$h_7 < h_8 > h_9 < h_{10}$	0.795498	1.037539	202
$\mathcal{G}_{11,31}$	11	31	$h_7 < h_8 > h_9 < h_{10}$	0.795275	1.038874	217
$\mathcal{G}_{12,37}$	12	37	$h_8 < h_9 > h_{10} < h_{11}$	0.717034	1.110104	276
$\mathcal{G}_{14,41}$	14	41	$h_{10} < h_{11} > h_{12} < h_{13}$	0.586096	1.015128	399
$\mathcal{G}_{20,101}$	20	101	$h_{16} < h_{17} > h_{18} < h_{19}$	0.400493	1.648842	1067

6.3. **Rank-4 thresholds and representable/graphic frontier.** Table 3 records exact rank-4 real-rootedness and log-concavity prime-power thresholds. The named affine packing family $\mathcal{M}_q^{\mathrm{pk}}$ of [3, §4.1] (keeping the extension-only triples separate) is defined for *odd* prime powers q: on that domain ($q \leq 101$ odd), the full chain polynomial $c_L(t)$ log-concavity threshold (79/81) independently reproduces [3, Theorem B(b)], while our exact Sturm and log-concavity calculations from their flag counts yield the real-rootedness threshold 11/13 and the proper chain polynomial $c_{\bar{L}}(t)$ log-concavity threshold 61/67 (evaluating the same formula on even prime powers or coalescing P also passes at $q = 64$, with identical first failure $q = 67$). Likewise, for $\mathcal{G}_{4,q} = \mathrm{Gab}_{4,q}$ (defined for all prime powers q), the $c_L(t)$ log-concavity threshold (79/81) reproduces [3, Theorem B(c)], while the 11/13 real-rootedness threshold, the 64/67 $c_{\bar{L}}(t)$ log-concavity threshold, and the affine family $\mathcal{A}_{4,q}$ are computed here.

TABLE 3. Exact rank-4 prime-power thresholds (last pass / first fail for $q \leq 101$).

Rank-4 Family	Sturm RR	$c_{\bar{L}}$ LC	c_L LC	Attribution
$\mathcal{M}_q^{\mathrm{pk}}$ (odd q, uncoal.)	11/13	61/67	79/81	c_L: [3, Thm B(b)]; RR, $c_{\bar{L}}$: here
$\mathcal{G}_{4,q} = \mathrm{Gab}_{4,q}$ (all q)	11/13	64/67	79/81	c_L: [3, Thm B(c)]; RR, $c_{\bar{L}}$: here
$\mathcal{A}_{4,q}$ (affine, all q)	11/13	67/71	79/81	New family

Table 4 summarizes our exact-arithmetic search across graphic, binary, ternary, Coxeter, and Whittle q-lift representable matroids. Every entry reports zero failures of real-rootedness or ultra-log-concavity (ULC). In particular, Matt Larson's generalized theta graph $\Theta(1, 26, 26, 26)$ (rank 76, which

fails Whitney-number log-concavity at rank 74 while retaining a unique Whitney peak at rank 39) and all seven sampled $\mathbb{F}_q$-representable Whittle lifts $L_q(\Theta(a_1, a_2, a_3, a_4)^{\oplus K})$ up to rank 153 ($q \leq 101$, each likewise having a unique Whitney peak, inside the family whose much larger parameters break Whitney unimodality in [9, 8]) remain strictly ultra-log-concave in both $c_{\bar{L}}(t)$ and $h_L(t)$.

TABLE 4. Exact integer/rational certifications on representable and graphic matroids (zero failures throughout).

Matroid Class	Exact Scope	Exact Certification
Graphic ($n = 5$)	All 1,024 (728 rk-4, exh.)	`__int128` cubic + ULC
Graphic ($n = 6$, rk 5)	All 112 conn. (exhaustive)	`__int128` quartic + ULC ($c_{\bar{L}}, H_M$)
Graphic ($n = 7$–9)	1,708 (1,217/452/39; 1,424 H_M)	`__int128` ULC + `Fraction` Sturm
Binary PG$(3, 2)$	All 31,232 rk-4 (incl. 840 B_4)	`__int128` cubic + ULC
PG$(4, 2)$ / PG$(3, 3)$	2,977 rk-5 + 2,499 rk-4	`__int128` quartic/cubic + ULC
Coxeter / $\mathbb{Q}, \mathbb{R}$ rk 4	A_4–H_4 ($\mathbb{Q}(\sqrt{5})$), Reye, $[k]^3$	`Fraction` Sturm + Melchior
Larson Θ & lifts	$\Theta(1, 26^3)$ & $L_q(\Theta^{\oplus K})$, rk ≤ 153	Exact `int` ULC ($c_{\bar{L}}, h_L$)

Remark 6.1 (Remaining open problems)**.** Two natural questions remain open: (1) whether $c_{\bar{L}}(t)$ (for $r \geq 7$) or $c_L(t)$ (for $r \geq 9$) can fail unimodality (in $\mathcal{G}_{r,q}$ and $\mathcal{A}_{r,q}$, Theorem 1.1 gives $c_{r-3} < c_{r-2} < c_{r-1}$ asymptotically as $q \to \infty$, whereas the subtraction $-2c_{r-3}$ in h_{r-2} creates the dip; at small q, $c_{\bar{L}}$ can peak before the top rank, e.g., $c_{\bar{L}}(\mathcal{A}_{5,2}) = [1, 128266, 2339756, 7052488, 5252688]$); and (2) whether every representable—or in particular graphic, regular, or binary—matroid of rank $r \geq 5$ has a real-rooted or log-concave chain polynomial.

7. Data Availability and Methodology

All standalone Python and C++20 verification scripts and source code are publicly archived at `https://github.com/coracle-us/conjectures` under `results/matroid-order-complex-h-unimodality/`. Running `python3 -W error verify_all.py` (Python 3.10+ standard library plus a C++20 compiler supporting `__int128`, zero third-party dependencies) reproduces Tables 1–3 (with the coalesced $q = 2$ concrete packing in Table 1, and both packing formulas in Table 3) and the graphic/binary rows of Table 4 in approximately 20 seconds; `test_coxeter_and_realizable.py` and `test_whittle_qlift.py` reproduce the remaining rows of Table 4.

This work was carried out using an AI-assisted, error-corrected mathematical research and adversarial refereeing workflow inspired by Loader, Oppenheim, and Osborne [14]. Every theorem and exact numerical witness was audited independently from scratch without code reuse: Theorems 1.1–1.4 (including the global rank-10 Taylor shift at $q = 75$ for all $0 \leq j \leq 8$), concrete matroids $\mathcal{A}_{4,2}$ and $\mathcal{G}_{4,2}$, Table 3, Coxeter H_4, all 1,024 graphs on $n = 5$, all 112 connected graphs on $n = 6$, and all 31,232 rank-4 subsets

of $\mathrm{PG}(3,2)$ were independently re-executed from scratch by an adversarial referee; the rank 6–8 graphic sweep (1,708 chain and 1,424 Chow polynomials) and the $\mathrm{PG}(4,2)$ / $\mathrm{PG}(3,3)$ samples were code-audited and verified end-to-end by `verify_all.py`.

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