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LIEB'S PERMANENTAL DOMINANCE CONJECTURE FOR ALL SUBGROUPS OF S_5

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Abstract. We prove Lieb's General Subgroup Permanental Dominance Conjecture (1966) at order $n = 5$: for every subgroup $H \leq S_5$ (all 156 subgroups in 19 conjugacy classes), every irreducible complex character $\chi \in \mathrm{Irr}(H)$ (78 irreducible characters summed across the 19 subgroup conjugacy classes, corresponding to 617 labelled subgroup–character pairs in 65 simultaneous S_5-conjugacy orbits), and every 5×5 complex Hermitian positive semidefinite matrix A, the normalized generalized matrix function satisfies $d_\chi^H(A) \leq \mathrm{per}(A)$. This advances the general-subgroup frontier beyond $n \leq 3$ (Tabata, 2015) and $n = 4$ (Zeng, 2026), resolving all 16 transitive proper-subgroup characters outside prior S_n immanants across C_5 (5), D_5 (4), $F_{20} = \mathrm{AGL}(1,5)$ (5), and $A_5(\phi_+), A_5(\phi_-)$ (2): three of these ($D_5(\rho_1, \rho_2)$ and $F_{20}(\theta_4)$) follow directly from Lieb's block inequality and AM–GM, while the remaining 13 characters (C_5 (5), $D_5(\mathbf{1}, \mathrm{sgn})$ (2), degree-1 characters of F_{20} (4), and the real-valued Galois-conjugate three-dimensional pair $A_5(\phi_+), A_5(\phi_-)$ over $\mathbb{Q}(\sqrt{5})$ (2)) are settled here via exact degree-5 Walled-Brauer sum-of-squares certificates and elementary proofs.

Our primary mechanism is a dimension-independent, zero-constant-term vector-norm sum-of-squares (SOS) representation in partial-transpose contraction sectors of the walled Brauer algebra $B_{n-|J|,|J|}$. Using exact arithmetic over $\mathbb{Q}$, $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{5})$, and $\mathbb{Q}(\zeta_5)$ (zero floating point), we construct compact positive-weight SOS identities covering all 11 master cases of the transitive subgroups $C_5, D_5, F_{20}, A_5, S_5$ (settling all 26 transitive subgroup characters without external theorems) as well as 8 rational master certificates for S_4 and D_4. We also give short paper-and-pencil elementary proofs for C_5 and $A_5(\phi_+)$, self-contained proofs of Lieb's $(1,4)$ and $(2,3)$ block permanent inequalities, and a self-contained reduction for $S_3 \times S_2$ (including the subdirect product S_3^{diag}). In total, 76 of the 78 irreducible characters across the 19 subgroup conjugacy classes of S_5 are proved without external theorems, while the remaining complex-conjugate pair $A_4(\omega), A_4(\bar{\omega})$ in $A_4 \times S_1$ reduces in one line to Zeng's order-4 theorem. All symbolic certificates and lattice enumerations are verified by a standalone standard-library Python script.

1. Introduction and Main Results

For a subgroup $H \leq S_n$ of the symmetric group on $\{0, 1, \ldots, n-1\}$, an irreducible complex character $\chi \in \mathrm{Irr}(H)$, and an $n \times n$ complex Hermitian positive semidefinite (PSD) matrix $A \succeq 0$, the *normalized generalized matrix function* (or subgroup immanant) is defined by

$$d_\chi^H(A) := \frac{1}{\chi(1)} \sum_{\sigma \in H} \chi(\sigma) \prod_{i=0}^{n-1} A_{i,\sigma(i)}.$$

Schur [17] proved that $d_\chi^H(A) \geq 0$ for all $A \succeq 0$ (and hence $\det(A) \leq d_\chi^H(A)$ for $H = S_n$). At the opposite extreme lies the permanent $\mathrm{per}(A) := d_{\mathbf{1}}^{S_n}(A)$, given by $\sum_{\sigma \in S_n} \prod_{i=0}^{n-1} A_{i,\sigma(i)}$. In

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1966, Lieb [9] proved the block permanent inequality $\operatorname{per}(A) \geq \operatorname{per}(B)\operatorname{per}(D)$ for partitioned Hermitian PSD matrices $A = \left(\begin{smallmatrix} B & C \\ C^* & D \end{smallmatrix}\right) \succeq 0$ and conjectured that the permanent dominates every normalized generalized matrix function:

$$d_\chi^H(A) \leq \operatorname{per}(A) \qquad \text{for all } n \geq 1,\ H \leq S_n,\ \chi \in \operatorname{Irr}(H),\ A \succeq 0. \tag{1}$$

The main result of this paper is the complete resolution of (1) at order $n = 5$.

Theorem 1.1 (Order-5 General Subgroup Permanental Dominance Conjecture)**.** *For every subgroup $H \leq S_5$ (all 156 subgroups across 19 conjugacy classes), every irreducible complex character $\chi \in \operatorname{Irr}(H)$ (78 irreducible characters across the 19 subgroup conjugacy classes, comprising 617 labelled pairs (H, χ) in 65 simultaneous S_5-conjugacy orbits), and every 5×5 complex Hermitian positive semidefinite matrix $A \succeq 0$,*

$$d_\chi^H(A) := \frac{1}{\chi(1)} \sum_{\sigma \in H} \chi(\sigma) \prod_{i=0}^{4} A_{i,\sigma(i)} \ \leq\ \operatorname{per}(A) := \sum_{\sigma \in S_5} \prod_{i=0}^{4} A_{i,\sigma(i)}.$$

1.1. **Literature Context and New vs. Known Classes.** Work on Lieb's permanental dominance conjecture (1) has progressed along two distinct frontiers:

- **Ordinary immanants ($H = S_n$).** Following Merris and Watkins [12], Merris [11], Heyfron [3], Soules [18], and Barrett, Hall, and Loewy [1], Pate [13, 14, 15] established $d_\lambda^{S_n}(A) \leq \operatorname{per}(A)$ for all partitions of $n \leq 13$. Recently, Y. Li [8] extended ordinary S_n immanants through $n \leq 15$, explicitly noting that proper subgroups $H < S_n$ are not covered. Y. Lavi [6] also studied a quadratic permanental dominance conjecture for ordinary S_n characters (proving it through $n \leq 5$), which does not subsume (1) for proper subgroups $H < S_5$. For surveys and related matrix-class or weight-space approaches, see Cheon and Wanless [2], Zhang [22], Wanless [20], Rodtes [16], and Jing, Liu, and Zhang [5].
- **General subgroups ($H \leq S_n$).** For arbitrary subgroups $H \leq S_n$, (1) was previously established through order $n \leq 3$ by Tabata [19] (cf. [20]) and resolved in order $n = 4$ (all 11 subgroup classes and 37 subgroup characters) by Zeng [21] (with Lean 4 formal verification). For degree-one characters (Minc's Problem 2), Lavi's public status tracker [7] reports 38 of 45 order-5 symmetry cases without a dated complete proof of those 38; outside ordinary S_n immanants (which also cover the A_5 characters $\mathbf{1}, \chi_4, \chi_5$ as S_5 averages), Lieb-induced intransitive reductions ($S_4 \times S_1$, $S_3 \times S_2$), and elementary Lieb + AM–GM bounds (for $D_5(\rho_1, \rho_2)$ and $F_{20}(\theta_4)$), the remaining 11 transitive degree-1 characters (C_5, $D_5(\mathbf{1}, \operatorname{sgn})$, F_{20}) and the three-dimensional Galois-conjugate pair $A_5(\phi_+, \phi_-)$ lacked a published proof. Order $n = 5$ is therefore the first open general-subgroup order.

Note two further immediate reductions beyond the direct products in $S_4 \times S_1$ and $S_3 \times S_2$. First, among intransitive subgroups, the unique non-direct-product class in $S_3 \times S_2$ — namely $S_3^{\mathrm{diag}} := \{(\sigma, \operatorname{sgn}(\sigma)) : \sigma \in S_3\} = \ker(\operatorname{sgn} \otimes \operatorname{sgn})$ (Class #11, 3 characters) — follows directly from prior order-≤ 3 results and Lieb's $(3, 2)$ block inequality: multiplying by its subgroup indicator $\frac{1}{2}(1 + \operatorname{sgn} \otimes \operatorname{sgn})$ expresses each normalized character function on S_3^{diag} as the average $\frac{1}{2}\big(d_{\tilde{\chi}}^{S_3}(A_1)\operatorname{per}(A_2) + d_{\tilde{\chi}\cdot\operatorname{sgn}}^{S_3}(A_1)\det(A_2)\big)$ of two products of known order-3 and order-2 character functions (see Remark 6.3). Second, among the 16 transitive proper-subgroup characters outside prior S_n immanants, three higher-dimensional characters — $F_{20}(\theta_4)$ (degree 4) and $D_5(\rho_1), D_5(\rho_2)$ (degree 2) — are supported exclusively on C_5 and already follow

directly from Lieb's block inequality and AM–GM: after diagonal scaling ($A_{ii} = 1$, $|A_{ij}| \le 1$), averaging Lieb's $(2,1,1,1)$ inequality $\mathrm{per}(A) \ge 1 + |A_{ij}|^2$ over the 5 edges of a 5-cycle σ yields $|a_\sigma| \le |a_\sigma|^{2/5} \le \mathrm{per}(A) - 1$, giving $d_{\theta_4}^{F_{20}}(A) \le 1 + \frac{4}{4}(\mathrm{per}(A) - 1) = \mathrm{per}(A)$, while averaging Lieb's $(2,2,1)$ inequality $\mathrm{per}(A) \ge (1 + |A_{ij}|^2)(1 + |A_{kl}|^2)$ over the 5 cyclic shifts gives $\mathrm{per}(A) - 1 \ge 2|z|^{2/5} + |z|^{4/5} \ge 3|z|$ for both 5-cycle products z_1, z_2, whence $d_{\rho_k}^{D_5}(A) \le 1+2|\cos(2\pi k/5)|\,|z_1|+2|\cos(4\pi k/5)|\,|z_2| \le 1+\sqrt{5}\max(|z_1|,|z_2|) \le 1+\frac{\sqrt{5}}{3}(\mathrm{per}(A)-1) \le \mathrm{per}(A)$ for both $k \in \{1,2\}$ (since $2|\cos(2\pi k/5)| + 2|\cos(4\pi k/5)| = \sqrt{5}$). Table 1 summarizes the resulting exact status across all 19 conjugacy classes (78 subgroup characters; note that S_5 itself has 7 irreducible characters).

TABLE 1. Exact known-vs.-new status of all 19 subgroup conjugacy classes (78 irreducible subgroup characters) of S_5.

Literature Status	Conjugacy Classes of S_5	Classes	Chars	Attribution / Contribution
Already known (S_n immanants)	#19 S_5 (7); plus 3 of 5 chars of #18 A_5 ($\mathbf{1}, \chi_4, \chi_5$, averages of S_5)	1 + part.	10	Merris–Watkins [12], Heyfron [3], Pate [15] ($n \le 13$), Li [8] ($n \le 15$). Re-proved here via 1–6 term SOS.
Immediate (Lieb [9] + order ≤ 3)	#1 $\{e\}$ (1), #2 C_2^{t} (2), #3 C_2^{d} (2), #4 C_3 (3), #6 C_2^2 (4), #9 S_3 (3), #10 $C_3{\times}C_2$ (6), #11 S_3^{diag} (3), #15 $S_3{\times}C_2$ (6)	9	30	Lieb's block inequality [9] + Schur [17] + order ≤ 3 (Tabata [19]); #11 via indicator $\frac{1}{2}(1 + \mathrm{sgn} \otimes \mathrm{sgn})$ (and Lemma 6.2).
Immediate (Lieb [9] + Zeng [21])	#5 V_4 (4), #7 C_4 (4), #12 D_4 (5), #14 A_4 (4), #17 S_4 (5) inside $S_4{\times}S_1$	5	22	$\mathrm{per}(A) \ge A_{00}\,\mathrm{per}(A_{\{0\}^c})$ + Zeng [21] (S_4 also [1, 15]); 20/22 also re-proved via exact rational SOS.
Elementary (Lieb [9] + AM–GM)	#13 $D_5(\rho_1,\rho_2)$ (2); #16 $F_{20}(\theta_4)$ (1)	part.	3	Direct 5-cycle Lieb $(2,2,1)/(2,1,1,1)$ + AM–GM ($\sum_{\sigma\ne e}|\chi(\sigma)|/\chi(1) \le \sqrt{5} < 3$); also certified by SOS / C_5 average here.
Non-trivial transitive (this work)	#8 C_5 (5), #13 $D_5(\mathbf{1},\mathrm{sgn})$ (2), #16 F_{20} deg. 1 (4), #18 $A_5(\phi_+,\phi_-)$ (2)	**1 + 3 comp.**	**13**	**First complete proof of all 13 remaining transitive proper-subgroup chars:** degree-5 Walled-Brauer SOS + elementary $C_5, A_5(\phi_+)$ proofs.
Total	**All 156 subgroups of S_5**	**19**	**78**	76/78 zero external theorems; 2/78 $(A_4(\omega,\bar\omega){\times}S_1)$ reduced to [21].

2. Complete Classification of Subgroups of S_5

Direct enumeration of the subgroup lattice of S_5 (recomputed from scratch in the companion script `verify_order5_lieb.py`) yields exactly 156 subgroups partitioned into 19 conjugacy classes. Summing $|\mathrm{Irr}(H)|$ over one representative per class gives 78 irreducible subgroup

characters (corresponding to 617 labelled subgroup–character pairs (H, χ) falling into 65 simultaneous S_5-conjugacy orbits; Table 2).

TABLE 2. Complete classification of all 19 subgroup conjugacy classes of S_5 (156 subgroups, 78 irreducible subgroup characters). Newly proved/completed classes are marked with $*$.

#	$\lvert H\rvert$	Conj	$\lvert$Irr$\rvert$	Orbits	Structure	Prior Status	Proof Route
1	1	1	1	(1^5)	$\{e\} \le S_4{\times}S_1$	Lieb + $n \le 3$	Intrans. $(S_4{\times}S_1)$
2	2	10	2	$(1^3,2)$	$C_2^{\mathrm{t}} \le S_4{\times}S_1$	Lieb + $n \le 3$	Intrans. $(S_4{\times}S_1)$
3	2	15	2	$(1,2^2)$	$C_2^{\mathrm{d}} \le S_4{\times}S_1$	Lieb + $n \le 3$	Intrans. $(S_4{\times}S_1)$
4	3	10	3	$(1^2,3)$	$C_3 \le S_4{\times}S_1$	Lieb + Tabata [19]	Intrans. $(S_4{\times}S_1)$
5	4	5	4	$(1,4)$	$V_4 \le S_4{\times}S_1$	Lieb + Zeng [21]	Intrans. $(S_4{\times}S_1)$
6	4	15	4	$(1,2^2)$	$C_2^2 \le S_4{\times}S_1$	Lieb + $n \le 3$	Intrans. $(S_4{\times}S_1)$
7	4	15	4	$(1,4)$	$C_4 \le S_4{\times}S_1$	Lieb + Zeng [21]	Intrans. $(S_4{\times}S_1)$
9	6	10	3	$(1^2,3)$	$S_3 \le S_4{\times}S_1$	Lieb + Tabata [19]	Intrans. $(S_4{\times}S_1)$
12	8	15	5	$(1,4)$	$D_4 \le S_4{\times}S_1$	Lieb + Zeng [21]	Intrans. $(S_4{\times}S_1)$
14	12	5	4	$(1,4)$	$A_4 \le S_4{\times}S_1$	Lieb + Zeng [21]	Intrans. $(S_4{\times}S_1)$
17	24	5	5	$(1,4)$	$S_4 \le S_4{\times}S_1$	Lieb + Pate/Zeng	Intrans. $(S_4{\times}S_1)$
10	6	10	6	$(2,3)$	$C_3{\times}C_2 \le S_3{\times}S_2$	Lieb + Tabata [19]	Direct prod. $(S_3{\times}S_2)$
11	6	10	3	$(2,3)$	$S_3^{\mathrm{diag}} \le S_3{\times}S_2$	Lieb + Tabata (indic.)	Subdirect Lemma (§6.2)
15	12	10	6	$(2,3)$	$S_3{\times}C_2 \le S_3{\times}S_2$	Lieb + Tabata [19]	Direct prod. $(S_3{\times}S_2)$
8*	5	6	5	(5)	C_5	**NEW (5 deg-1)**	WB SOS ($\mathbb{Q}(\zeta_5)$) + §5
13*	10	6	4	(5)	D_5	2 Lieb+AMGM + **2 NEW**	Avg. of F_{20} and C_5
16*	20	6	5	(5)	$F_{20} = \mathrm{AGL}(1,5)$	1 Lieb+AMGM + **4 NEW**	WB SOS ($\mathbb{Q}, \mathbb{Q}(i)$)
18*	60	1	5	(5)	A_5	3 known + **2 NEW** ($\phi_\pm$)	WB SOS ($\mathbb{Q}(\sqrt{5})$) + §5
19	120	1	7	(5)	S_5	Known [15, 8]	WB SOS ($\mathbb{Q}$, 1–6 terms)

For any permutation $\pi \in S_n$, let P_π be the permutation matrix acting by $(P_\pi A P_\pi^T)_{i,j} = A_{\pi(i),\pi(j)}$. Then $\mathrm{per}(P_\pi A P_\pi^T) = \mathrm{per}(A)$ and

$$d_\chi^H(P_\pi A P_\pi^T) = d_{\chi\circ \mathrm{Ad}_{\pi^{-1}}}^{\pi H \pi^{-1}}(A),$$

where $\mathrm{Ad}_{\pi^{-1}}(\tau) = \pi^{-1}\tau\pi \in H$ for $\tau \in \pi H \pi^{-1}$. Moreover, for Hermitian A we have $A^T = \bar{A} \succeq 0$ and $d_{\bar\chi}^H(A) = d_\chi^H(A^T) = d_\chi^H(\bar A)$. Consequently, it suffices to prove (1) for one representative subgroup per conjugacy class and one character per $N_{S_5}(H)$-orbit and complex-conjugate pair.

3. The Walled-Brauer Partial-Transpose Vector-Norm SOS Method

Let $A \succeq 0$ be any $n \times n$ complex Hermitian PSD matrix. Choose a Gram factorization $A = U^*U$ with $U = [u_0, \dots, u_{n-1}] \in \mathbb{C}^{r\times n}$ $(r \ge 1)$, so that

$$A_{ij} = \langle u_i, u_j\rangle := \sum_{k=1}^{r} \overline{U_{ki}} U_{kj}, \qquad P_\sigma(A) := \prod_{i=0}^{n-1} A_{i,\sigma(i)} \qquad (\sigma \in S_n).$$

In $(\mathbb{C}^r)^{\otimes n}$, each permutation product satisfies $P_\sigma(A) = \mathrm{tr}\big(P_\sigma(u_0u_0^* \otimes \cdots \otimes u_{n-1}u_{n-1}^*)\big)$. While $\mathrm{per} - d_\chi^H$ is never positive semidefinite as an element of the group algebra $\mathbb{C}[S_n]$ for nontrivial χ (by Frobenius reciprocity, χ has a negative multiplicity on its own isotypic component), each rank-1 factor $u_ju_j^*$ remains PSD under transposition. Taking partial transposes on a subset $J \subset \{0, \dots, n-1\}$ produces contraction tensors in the walled Brauer algebra $B_{n-|J|,|J|}$ whose

Gram matrices expand purely into linear combinations of the permutation monomials $P_\sigma(A)$, with *zero dependence on the ambient rank r* and *zero coefficient on the identity permutation e* in every positive-contraction sector.

3.1. **Explicit Contraction Tensor Families for $n = 5$ and $n = 4$.** For $n = 5$, we use three contraction sectors:

(1) **Sector $B^{(2)}_{3,2}(j_0, j_1)$ (6 vectors in $\mathbb{C}^r$).** Fix $J = \{j_0, j_1\}$ and $J^c = \{0, 1, 2, 3, 4\} \setminus J$. For each of the 6 orderings (i, j, k) of J^c, set
$$v_{i,j,k}(U) := A_{j_0,i} A_{j_1,j}\, u_k \in \mathbb{C}^r.$$
Then $\langle v_{i,j,k}, v_{i',j',k'}\rangle = A_{i,j_0} A_{j,j_1} A_{j_0,i'} A_{j_1,j'} A_{k,k'} = P_\sigma(A)$, where $\sigma \in S_5$ is the unique permutation mapping $i \mapsto j_0$, $j \mapsto j_1$, $j_0 \mapsto i'$, $j_1 \mapsto j'$, $k \mapsto k'$.

(2) **Sector $B^{(1)}_{3,2}(j_0, j_1)$ (12 vectors in $(\mathbb{C}^r)^{\otimes 2} \otimes \overline{\mathbb{C}^r}$).** For $J = \{j_0, j_1\}$, $a \in \{0, 1\}$, $x \in J^c$, and each ordering (c, d) of $J^c \setminus \{x\}$, set
$$v_{a,x,c,d}(U) := A_{j_a,x}\big(u_c \otimes u_d \otimes \overline{u_{j_{1-a}}}\big) \in (\mathbb{C}^r)^{\otimes 2} \otimes \overline{\mathbb{C}^r}.$$
Since $\langle \overline{u}, \overline{v}\rangle = \langle v, u\rangle$, we have
$$\langle v_{a,x,c,d}, v_{a',x',c',d'}\rangle = A_{x,j_a} A_{j_{a'},x'} A_{c,c'} A_{d,d'} A_{j_{1-a'},j_{1-a}} = P_\sigma(A),$$
where σ maps $x \mapsto j_a$, $j_{a'} \mapsto x'$, $c \mapsto c'$, $d \mapsto d'$, $j_{1-a'} \mapsto j_{1-a}$.

(3) **Sector $B^{(1)}_{4,1}(j_0)$ (24 vectors in $(\mathbb{C}^r)^{\otimes 3}$).** Fix $j_0 \in \{0, \dots, 4\}$ and $J^c = \{0, \dots, 4\} \setminus \{j_0\}$. For each $x \in J^c$ and each ordering (c, d, e) of $J^c \setminus \{x\}$, set
$$v_{x,c,d,e}(U) := A_{j_0,x}\big(u_c \otimes u_d \otimes u_e\big) \in (\mathbb{C}^r)^{\otimes 3}.$$
Then $\langle v_{x,c,d,e}, v_{x',c',d',e'}\rangle = A_{x,j_0} A_{j_0,x'} A_{c,c'} A_{d,d'} A_{e,e'} = P_\sigma(A)$, where σ maps $x \mapsto j_0$, $j_0 \mapsto x'$, $c \mapsto c'$, $d \mapsto d'$, $e \mapsto e'$.

Analogously, for the self-contained order-4 certificates in Section 6.3, we use the $n = 4$ sectors (with $J^c := \{0, 1, 2, 3\} \setminus J$):

- $B^{(2)}_{2,2}(j_0, j_1)$ (2 scalars in $\mathbb{C}$): $v_{i,j}(U) := A_{j_0,i} A_{j_1,j}$ for (i, j) an ordering of J^c;
- $B^{(1)}_{2,2}(j_0, j_1)$ (4 vectors in $\mathbb{C}^r \otimes \overline{\mathbb{C}^r}$): $v_{a,x,y}(U) := A_{j_a,x}(u_y \otimes \overline{u_{j_{1-a}}})$ for $a \in \{0, 1\}$ and $\{x, y\} = J^c$;
- $B^{(1)}_{3,1}(j_0)$ (6 vectors in $(\mathbb{C}^r)^{\otimes 2}$): $v_{x,c,d}(U) := A_{j_0,x}(u_c \otimes u_d)$ for $x \in J^c$ and (c, d) an ordering of $J^c \setminus \{x\}$.

For each sector B with m basis vectors $\{v_\alpha(U)\}_{\alpha=1}^m$, denote by $\mathrm{tbl}_B(\alpha, \beta) \in S_n$ the unique permutation determined above such that $\langle v_\alpha(U), v_\beta(U)\rangle = P_{\mathrm{tbl}_B(\alpha,\beta)}(A)$.

Proposition 3.1 (Vector-Norm SOS Positivity)**.** *For any contraction sector B of size m, any $c \in \mathbb{C}^m$, any real $w > 0$, and any $g \in S_n$, setting $U_g := [u_{g(0)}, \dots, u_{g(n-1)}]$, we have*
$$w \left\| \sum_{\alpha=1}^m c_\alpha\, v_\alpha^{(B)}(U_g) \right\|^2 = w \sum_{\alpha,\beta=1}^m \overline{c_\alpha} c_\beta\, P_{g\cdot \mathrm{tbl}_B(\alpha,\beta)\cdot g^{-1}}(A) \;\geq\; 0.$$

Proof. Replacing u_k by $u_{g(k)}$ replaces $A_{i,j} = \langle u_i, u_j\rangle$ by $A_{g(i),g(j)}$. For any $\pi \in S_n$, re-indexing $m = g(i)$ gives
$$\prod_{i=0}^{n-1} A_{g(i),\, g(\pi(i))} = \prod_{m=0}^{n-1} A_{m,\,(g\pi g^{-1})(m)} = P_{g\pi g^{-1}}(A).$$

Expanding the squared Hilbert-space norm yields the identity immediately. □

4. Complete Self-Contained Proof for All Five Transitive Subgroups of S_5

Fix the canonical transitive subgroups on $\{0,1,2,3,4\} = \mathbb{Z}/5\mathbb{Z}$ generated by

$$a(x) := x+1 \pmod 5, \qquad b(x) := 2x \pmod 5, \qquad b^2(x) = -x \pmod 5,$$

namely $C_5 := \langle a \rangle$ (order 5), $D_5 := \langle a, b^2 \rangle$ (order 10), $F_{20} := \langle a, b \rangle = \mathrm{AGL}(1,5)$ (order 20), A_5 (order 60), and S_5 (order 120), which form the two chains $C_5 < D_5 < F_{20} < S_5$ and $C_5 < D_5 < A_5 < S_5$ with $F_{20} \cap A_5 = D_5$ (note b is an odd 4-cycle, so $F_{20} \not\subset A_5$). Every element of F_{20} is written uniquely as $p_{r,s}(x) = 2^s x + r \pmod 5$ ($r \in \{0,\dots,4\}$, $s \in \{0,\dots,3\}$). With $\zeta_5 := e^{2\pi i/5}$, the 26 irreducible characters of the five transitive subgroups are:

- C_5 (5 characters): $\chi_k(p_{r,0}) = \zeta_5^{kr}$ $(k = 0,1,2,3,4)$;
- D_5 (4 characters): $\mathbf{1}_{D_5}$, $\mathrm{sgn}_{D_5}(p_{r,2t}) = (-1)^t$, and the two 2-dimensional characters ρ_1, ρ_2 with $\rho_k(p_{r,0}) = 2\cos(2\pi kr/5)$ and $\rho_k(p_{r,2}) = 0$;
- F_{20} (5 characters): $\varepsilon_m(p_{r,s}) = i^{ms}$ ($m = 0,1,2,3$; with $\varepsilon_3 = \overline{\varepsilon_1}$ complex-conjugate), and the 4-dimensional character ψ_4 taking $\psi_4(e) = 4$, $\psi_4(p_{r,0}) = -1$ ($r \neq 0$), and $\psi_4 = 0$ outside C_5;
- A_5 (5 characters): $\mathbf{1}_{A_5}$, χ_4 (4D), χ_5 (5D), and the real-valued, $\mathbb{Q}(\sqrt{5})$-Galois-conjugate (and S_5-conjugate) 3-dimensional pair ϕ_+, ϕ_- taking values $3, -1, 0, \frac{1\pm\sqrt{5}}{2}, \frac{1\mp\sqrt{5}}{2}$ on the conjugacy classes $(1^5), (2^2,1), (3,1^2), C_{5a}, C_{5b}$;
- S_5 (7 characters): the ordinary symmetric-group characters indexed by partitions (5), (1^5), $(4,1)$, $(2,1^3)$, $(3,2)$, $(2^2,1)$, $(3,1^2)$.

Theorem 4.1 (Exact Walled-Brauer SOS Certificates for All 11 Master Cases)**.** *For each of the* 11 *master irreducible characters* (H,χ) *listed in Table 3, the certificate file* `certificates.json` *specifies an explicit finite list of triples* $(B_t, w_t, c_t)_{t=1}^K$ *with strictly positive exact weights* w_t *(in* $\mathbb{Q}_{>0}$ *or* $\mathbb{Q}(\sqrt{5})_{>0}$*) and exact coefficient vectors* c_t *(in* $\mathbb{Z}^m$*,* $\mathbb{Z}[i]^m$*, or* $\mathbb{Z}[\zeta_5]^m$*) such that the polynomial identity*

$$\mathrm{per}(A) - d_\chi^H(A) \;\equiv\; \frac{1}{|H|}\sum_{g\in H}\sum_{t=1}^{K} w_t \left\| \sum_{\alpha} c_{t,\alpha}\, v_\alpha^{(B_t)}(U_g) \right\|^2 \;\geq\; 0 \tag{2}$$

holds identically on all 120 *permutations* $\sigma \in S_5$.

Proof. By Proposition 3.1, expanding the right-hand side of (2) yields a linear combination $\sum_{\sigma\in S_5} \gamma_\sigma P_\sigma(A)$ whose coefficients γ_σ lie in the exact number field ($\mathbb{Q}$, $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{5})$, or $\mathbb{Q}(\zeta_5)$) generated by the certificate entries. The standalone verifier `verify_order5_lieb.py` checks with zero floating-point arithmetic that: (i) every weight w_t is strictly positive (in $\mathbb{Q}$, or via exact integer inequalities for $a+b\sqrt{5} > 0$ in $\mathbb{Q}(\sqrt{5})$), and (ii) $\gamma_\sigma = 1 - \frac{1}{\chi(1)}\chi(\sigma)\mathbf{1}_{\sigma\in H}$ identically for all 120 permutations $\sigma \in S_5$. Nonnegativity for every $A = U^*U \succeq 0$ follows immediately from Proposition 3.1. □

TABLE 3. Summary of the 11 master exact Walled-Brauer SOS certificates for S_5 in `certificates.json` (zero floating point).

Master Case	Key	H	Field	K	Sectors / Highlights
$F_{20}(\varepsilon_0)$	`F20_eps0`	F_{20}	$\mathbb{Q}$	9	$B^{(2)}_{3,2}, B^{(1)}_{3,2}, B^{(1)}_{4,1}$; rational SOS
$F_{20}(\varepsilon_2)$	`F20_eps2`	F_{20}	$\mathbb{Q}$	9	$B^{(2)}_{3,2}, B^{(1)}_{3,2}, B^{(1)}_{4,1}$; rational SOS
$F_{20}(\varepsilon_1)$	`F20_eps1`	F_{20}	$\mathbb{Q}(i)$	10	$B^{(2)}_{3,2}, B^{(1)}_{3,2}, B^{(1)}_{4,1}$; Gaussian-rational SOS
$A_5(\phi_+)$	`A5_phi+`	A_5	$\mathbb{Q}(\sqrt{5})$	7	$B^{(2)}_{3,2}, B^{(1)}_{3,2}, B^{(1)}_{4,1}$ (+ §5 10-line proof)
$C_5(\chi_1)$	`C5_chi1`	C_5	$\mathbb{Q}(\zeta_5)$	27	$B^{(2)}_{3,2}, B^{(1)}_{3,2}, B^{(1)}_{4,1}$ (+ §5 5-line proof)
$S_5(\det)$	`S5_det`	S_5	$\mathbb{Q}$	6	$B^{(1)}_{4,1}$; rational SOS
$S_5(4,1)$	`S5_41`	S_5	$\mathbb{Q}$	1	Closed-form 1-term: $w=\frac{5}{24}$ on $B^{(1)}_{4,1}(0)$, $c=\mathbf{1}_{24}$
$S_5(2,1^3)$	`S5_2111`	S_5	$\mathbb{Q}$	6	$B^{(2)}_{3,2}, B^{(1)}_{4,1}$; rational SOS
$S_5(3,2)$	`S5_32`	S_5	$\mathbb{Q}$	2	Closed-form 2-term: $\frac{1}{2}$ on $B^{(1)}_{3,2}(0,1)$ + $\frac{1}{12}$ on $B^{(1)}_{4,1}(0)$
$S_5(2^2,1)$	`S5_221`	S_5	$\mathbb{Q}$	6	$B^{(2)}_{3,2}, B^{(1)}_{3,2}, B^{(1)}_{4,1}$; rational SOS
$S_5(3,1^2)$	`S5_311`	S_5	$\mathbb{Q}$	6	$B^{(2)}_{3,2}, B^{(1)}_{4,1}$; rational SOS

In particular, the two shortest certificates in Table 3 give explicit closed-form vector-norm identities for the S_5 immanants $(4,1)$ and $(3,2)$:

$$\operatorname{per}(A)-d^{S_5}_{(4,1)}(A)=\frac{1}{24}\sum_{p=0}^{4}\Bigg\|\sum_{x\neq p}A_{p,x}\sum_{\tau\in S_{[5]\setminus\{p,x\}}}u_{\tau(1)}\otimes u_{\tau(2)}\otimes u_{\tau(3)}\Bigg\|^2\geq 0,$$

$$\begin{aligned}\operatorname{per}(A)-d^{S_5}_{(3,2)}(A)=&\frac{1}{20}\sum_{\{a,b\}}\Bigg\|\sum_{j\in\{a,b\}}\sum_{x\notin\{a,b\}}A_{j,x}\sum_{(c,d)}u_c\otimes u_d\otimes\overline{u_{\{a,b\}\setminus\{j\}}}\Bigg\|^2\\&+\frac{1}{60}\sum_{p=0}^{4}\Bigg\|\sum_{x\neq p}A_{p,x}\sum_{\tau\in S_{[5]\setminus\{p,x\}}}u_{\tau(1)}\otimes u_{\tau(2)}\otimes u_{\tau(3)}\Bigg\|^2\geq 0,\end{aligned}$$

where $[5]:=\{0,1,2,3,4\}$ and (c,d) ranges over ordered pairs of distinct elements in $[5]\setminus\{a,b,x\}$.

Remark 4.2 (Certificate discovery vs. proof verification). The certificates in `certificates.json` were discovered by locating strictly positive-definite interior points of the H-invariant Walled-Brauer SDP cone via accelerated proximal gradient (FISTA), extracting small-support extreme rays via Carathéodory compression, and solving the resulting sparse linear systems exactly over the relevant number fields. The mathematical proof of Theorem 4.1 is completely independent of this numerical discovery step: it relies solely on exact verification of (2) and $w_t>0$.

Theorem 4.3 (Exact Algebraic Implications for the Remaining 15 Transitive Characters)**.** *All remaining* 15 *irreducible characters of* $C_5, D_5, F_{20}, A_5, S_5$ *satisfy* $d^H_\chi(A)\leq\operatorname{per}(A)$ *as direct consequences of Theorem 4.1 via exact identities in* $\mathbb{C}[S_5]$*:*

(1) $S_5(5)$*:* $d^{S_5}_{(5)}(A)=\operatorname{per}(A)$ *identically.*

(2) $A_5(\mathbf{1})$, $A_5(\chi_4)$, $A_5(\chi_5)$*: since* $S_5=A_5\sqcup(01)A_5$,

$$d^{A_5}_{\mathbf{1}}=\tfrac{1}{2}(\operatorname{per}+\det),\qquad d^{A_5}_{\chi_4}=\tfrac{1}{2}\big(d^{S_5}_{(4,1)}+d^{S_5}_{(2,1^3)}\big),\qquad d^{A_5}_{\chi_5}=\tfrac{1}{2}\big(d^{S_5}_{(3,2)}+d^{S_5}_{(2^2,1)}\big).$$

(3) $A_5(\phi_-)$: *ϕ_- is the $\mathbb{Q}(\sqrt{5})$-Galois conjugate of the real character ϕ_+, and conjugation by $(01) \in S_5 \setminus A_5$ swaps $C_{5a} \leftrightarrow C_{5b}$, so* $d^{A_5}_{\phi_-}(A) = d^{A_5}_{\phi_+}(P_{(01)} A P^T_{(01)}) \le \mathrm{per}(A)$.

(4) $F_{20}(\varepsilon_3)$: *since $\varepsilon_3 = \overline{\varepsilon_1}$ (complex conjugation),* $d^{F_{20}}_{\varepsilon_3}(A) = d^{F_{20}}_{\varepsilon_1}(A^T) \le \mathrm{per}(A^T) = \mathrm{per}(A)$.

(5) $D_5(\mathbf{1})$, $D_5(\mathrm{sgn})$, $C_5(\chi_0)$:

$$d^{D_5}_{\mathbf{1}} = \tfrac{1}{2}\big(d^{F_{20}}_{\varepsilon_0} + d^{F_{20}}_{\varepsilon_2}\big), \qquad d^{D_5}_{\mathrm{sgn}} = \tfrac{1}{2}\big(d^{F_{20}}_{\varepsilon_1} + d^{F_{20}}_{\varepsilon_3}\big), \qquad d^{C_5}_{\chi_0} = \tfrac{1}{2}\big(d^{D_5}_{\mathbf{1}} + d^{D_5}_{\mathrm{sgn}}\big).$$

(6) $C_5(\chi_2, \chi_3, \chi_4)$, $D_5(\rho_1, \rho_2)$, $F_{20}(\psi_4)$: *conjugation by powers of $b \in F_{20}$ acts transitively on $\{\chi_1, \chi_2, \chi_4, \chi_3\}$, yielding $d^{C_5}_{\chi_k}(A) \le \mathrm{per}(A)$ for $k \in \{1,2,3,4\}$, and*

$$d^{D_5}_{\rho_1} = \tfrac{1}{2}\big(d^{C_5}_{\chi_1} + d^{C_5}_{\chi_4}\big), \qquad d^{D_5}_{\rho_2} = \tfrac{1}{2}\big(d^{C_5}_{\chi_2} + d^{C_5}_{\chi_3}\big), \qquad d^{F_{20}}_{\psi_4} = \tfrac{1}{2}\big(d^{D_5}_{\rho_1} + d^{D_5}_{\rho_2}\big).$$

Proof. Each convex-combination identity is checked coefficient-by-coefficient on all 120 permutations of S_5 (and independently confirmed in `verify_order5_lieb.py`); each term in the average is at most $\mathrm{per}(A)$ by Theorem 4.1. □

Theorems 4.1 and 4.3 complete a 100% self-contained proof of (1) for all 26 irreducible characters of the five transitive subgroups of S_5.

5. Independent Paper-and-Pencil Elementary Proofs for C_5 and $A_5(\phi_+)$

As an independent human-readable cross-check requiring no computer certificates, both C_5 (hence also $D_5(\rho_1, \rho_2)$ and $F_{20}(\psi_4)$) and $A_5(\phi_\pm)$ admit short elementary proofs. If any diagonal entry $A_{ii} = 0$, then $A \succeq 0$ forces the entire i-th row and column to vanish, so $d^H_\chi(A) = \mathrm{per}(A) = 0$. Otherwise, replacing A by $D^{-1}AD^{-1}$ with $D = \mathrm{diag}(\sqrt{A_{00}}, \dots, \sqrt{A_{44}})$ divides both sides of (1) by $\prod_{i=0}^4 A_{ii} > 0$, so we may assume $A_{ii} = 1$ for all i (and hence $|A_{ij}| \le 1$).

Write $S_2(M) := \sum_{i<j} |M_{ij}|^2$ and $S_{2,2}(M) := \sum_{\{e_1,e_2\},\, e_1 \cap e_2 = \emptyset} |M_{e_1}|^2 |M_{e_2}|^2$, where $M_e := M_{ij}$ for an edge $e = \{i, j\}$.

Lemma 5.1 (Sharp 3×3 Permanent Bound)**.** *For every 3×3 Hermitian PSD matrix B with $B_{ii} = 1$, writing $s := |B_{01}|^2 + |B_{12}|^2 + |B_{20}|^2$, we have*

$$\mathrm{per}(B) \;\ge\; 1 + \frac{2}{3} s.$$

Proof. Since $\det(B) = 1 - s + 2\,\mathrm{Re}(B_{01}B_{12}B_{20}) \ge 0$, we have $\mathrm{per}(B) = \det(B) + 2s \ge 2s$. If $s \ge \frac{3}{4}$, then $2s = 1 + \frac{2}{3}s + \frac{4}{3}(s - \frac{3}{4}) \ge 1 + \frac{2}{3}s$. If $s \le \frac{3}{4}$, AM–GM yields $|B_{01}B_{12}B_{20}| \le (s/3)^{3/2} = \frac{s}{3}\sqrt{s/3} \le \frac{s}{6}$, whence

$$\mathrm{per}(B) = 1 + s + 2\,\mathrm{Re}(B_{01}B_{12}B_{20}) \ge 1 + s - 2|B_{01}B_{12}B_{20}| \ge 1 + \frac{2}{3}s. \qquad \square$$

Lemma 5.2 (Averaged $3 + 2$ Lieb Bound)**.** *For every 5×5 Hermitian PSD matrix A with $A_{ii} = 1$,*

$$\mathrm{per}(A) \;\ge\; 1 + \frac{3}{10} S_2(A) + \frac{2}{15} S_{2,2}(A).$$

Proof. For each of the 10 partitions $\{0,1,2,3,4\} = T \sqcup e$ with $|T| = 3$ and $|e| = 2$, Lieb's block inequality (proved self-contained in Lemma 6.1) and Lemma 5.1 give

$$\operatorname{per}(A) \;\geq\; \operatorname{per}(A_T)\operatorname{per}(A_e) \;\geq\; \left(1 + \frac{2}{3}\sum_{f\subset T}|A_f|^2\right)(1+|A_e|^2).$$

Averaging over all 10 partitions: each of the 10 edges appears once as e and three times inside T, giving coefficient $(1+3\cdot\frac{2}{3})/10 = \frac{3}{10}$; each of the 15 disjoint edge pairs appears twice (once with $e_1 = e$, $e_2 \subset T$ and once reversed), giving coefficient $2\cdot\frac{2}{3}/10 = \frac{2}{15}$. □

Proposition 5.3 (Elementary Proof for C_5)**.** *For every $k \in \{0,1,2,3,4\}$ and every 5×5 Hermitian PSD matrix A with $A_{ii} = 1$, $|d^{C_5}_{\chi_k}(A)| \leq \operatorname{per}(A)$.*

Proof. Let $x_i := |A_{i,i+1}|^2$, $y_i := |A_{i,i+2}|^2$ (indices modulo 5), $z_1 := \prod_{i=0}^4 A_{i,i+1}$, and $z_2 := \prod_{i=0}^4 A_{i,i+2}$. Observe that the 15 vertex-disjoint edge pairs of K_5 decompose into:

- 5 outer-cycle disjoint pairs $\big\{\{i,i+1\},\{i+2,i+3\}\big\}$ with weights x_ix_{i+2} ($i\in\mathbb{Z}/5\mathbb{Z}$),
- 5 inner-star disjoint pairs $\big\{\{i,i+2\},\{i+1,i+3\}\big\}$ with weights y_iy_{i+1} ($i\in\mathbb{Z}/5\mathbb{Z}$),
- 5 mixed disjoint pairs (one outer edge and one inner chord).

Since $|z_1|,|z_2|\leq 1$, $\prod_{i=0}^4 x_ix_{i+2} = |z_1|^4$, and $\prod_{i=0}^4 y_iy_{i+1} = |z_2|^4$, AM–GM yields

$$\frac{3}{10}\sum_{i=0}^4 x_i + \frac{2}{15}\sum_{i=0}^4 x_ix_{i+2} \;\geq\; \frac{3}{2}|z_1|^{2/5} + \frac{2}{3}|z_1|^{4/5} \;\geq\; \frac{13}{6}|z_1| \;\geq\; 2|z_1|,$$

$$\frac{3}{10}\sum_{i=0}^4 y_i + \frac{2}{15}\sum_{i=0}^4 y_iy_{i+1} \;\geq\; \frac{3}{2}|z_2|^{2/5} + \frac{2}{3}|z_2|^{4/5} \;\geq\; \frac{13}{6}|z_2| \;\geq\; 2|z_2|.$$

Dropping the 5 nonnegative mixed disjoint edge pairs in Lemma 5.2 gives $\operatorname{per}(A) \geq 1 + 2|z_1| + 2|z_2| \geq |d^{C_5}_{\chi_k}(A)|$. □

Proposition 5.4 (Elementary Proof for $A_5(\phi_+)$)**.** *For every 5×5 Hermitian PSD matrix A with $A_{ii} = 1$, $d^{A_5}_{\phi_+}(A) \leq \operatorname{per}(A)$.*

Proof. The 24 five-cycles of A_5 pair into 6 undirected cycles $C_r^{(a)}$ from C_{5a} (with oriented cycle products z_r) and 6 undirected cycles $C_r^{(b)}$ from C_{5b} (with oriented cycle products w_r). Since ϕ_+ vanishes on 3-cycles, takes value -1 on double transpositions, and takes values $\frac{1\pm\sqrt{5}}{2}$ on C_{5a}, C_{5b},

$$d^{A_5}_{\phi_+}(A) \;\leq\; 1 - \frac{1}{3}S_{2,2}(A) + \frac{1+\sqrt{5}}{3}\sum_{r=1}^6|z_r| + \frac{\sqrt{5}-1}{3}\sum_{r=1}^6|w_r|.$$

By Lemma 5.2, it suffices to show

$$\frac{3}{10}S_2(A) + \frac{7}{15}S_{2,2}(A) \;\geq\; \frac{1+\sqrt{5}}{3}\sum_{r=1}^6|z_r| + \frac{\sqrt{5}-1}{3}\sum_{r=1}^6|w_r|.$$

By A_5-transitivity, every edge of K_5 belongs to 3 cycles in C_{5a} (and 3 in C_{5b}), and every disjoint pair of edges belongs to 2 cycles in C_{5a} (and 2 in C_{5b}). Allocating weight $p = \frac{1}{10}$ of S_2 to C_{5a}, weight $u = \frac{2}{15}$ of $S_{2,2}$ to C_{5a}, and weight $v = \frac{1}{10}$ of $S_{2,2}$ to C_{5b} satisfies $3p = \frac{3}{10}$ and

$2u + 2v = \frac{7}{15}$. Applying AM–GM to the 5 edges and 5 non-adjacent edge pairs of each 5-cycle (and $|z_r|, |w_r| \le 1$) gives

$$\frac{3}{10}S_2(A) + \frac{7}{15}S_{2,2}(A) \;\ge\; (5p+5u)\sum_{r=1}^{6}|z_r| + 5v\sum_{r=1}^{6}|w_r| = \frac{7}{6}\sum_{r=1}^{6}|z_r| + \frac{1}{2}\sum_{r=1}^{6}|w_r|.$$

Since $\frac{7}{6} > \frac{1+\sqrt{5}}{3}$ and $\frac{1}{2} > \frac{\sqrt{5}-1}{3}$ (both equivalent to $25 > 20$), we obtain $d_{\phi_+}^{A_5}(A) \le \operatorname{per}(A)$. $\square$

6. Complete Proof for All Fourteen Intransitive Subgroups of S_5

Every intransitive subgroup $H \le S_5$ is conjugate to a subgroup of $S_4 \times S_1$ (11 classes, 37 subgroup characters) or $S_3 \times S_2$ (3 classes, 15 subgroup characters).

6.1. Self-Contained Proof of Lieb's Block Inequality for $n_1 \in \{1,2\}$.

Lemma 6.1 (Lieb [9], self-contained for $n_1 \in \{1,2\}$)**.** *For any Hermitian PSD block matrix $A = \left(\begin{smallmatrix} B & C \\ C^* & D \end{smallmatrix}\right) \succeq 0$ with B of size $n_1 \times n_1$ ($n_1 \in \{1,2\}$) and D of size $n_2 \times n_2$, we have $\operatorname{per}(A) \ge \operatorname{per}(B)\operatorname{per}(D)$.*

Proof. By continuity we may assume $B \succ 0$. Decompose $A = R + (0 \oplus S)$ where $R = \left(\begin{smallmatrix} B & C \\ C^* & E \end{smallmatrix}\right) \succeq 0$, $E := C^*B^{-1}C \succeq 0$, and $S := D - E \succeq 0$ is the Schur complement. Factor $B = V^*V$ with $V = [v_1, \dots, v_{n_1}] \in \mathbb{C}^{n_1 \times n_1}$ and set $W := (V^*)^{-1}C = [w_1, \dots, w_{n_2}] \in \mathbb{C}^{n_1 \times n_2}$, so that $R = [V,W]^*[V,W]$ and $E = W^*W$.

By Laplace expansion of the permanent of $R + (0 \oplus S)$,

$$\operatorname{per}(A) - \operatorname{per}(B)\operatorname{per}(D) = \sum_{m=0}^{n_2}\sum_{|I|=|J|=m}\Big(\operatorname{per}\big(R_{[n_1]\cup I^c,\,[n_1]\cup J^c}\big) - \operatorname{per}(B)\operatorname{per}(E_{I^c,J^c})\Big)\operatorname{per}(S_{I,J}),$$

where $I, J \subseteq \{1, \dots, n_2\}$ and $I^c := \{1, \dots, n_2\} \setminus I$. Writing $S = Y^*Y$ with $Y = [y_1, \dots, y_{n_2}]$, the $\binom{n_2}{m} \times \binom{n_2}{m}$ matrix $Q_{I,J}^{(m)} := \operatorname{per}(S_{I,J})$ is $\frac{1}{m!}$ times the Gram matrix of the symmetric tensors $\sum_{\pi \in S_m} y_{i_{\pi(1)}} \otimes \cdots \otimes y_{i_{\pi(m)}}$, hence $Q^{(m)} \succeq 0$. By the Schur product theorem, it suffices to show that for every $k = n_2 - m \in \{0, \dots, n_2\}$, the $\binom{n_2}{k} \times \binom{n_2}{k}$ matrix

$$M_{I^c,J^c}^{(k)} := \operatorname{per}\big(R_{[n_1]\cup I^c,\,[n_1]\cup J^c}\big) - \operatorname{per}(B)\operatorname{per}(E_{I^c,J^c})$$

is Hermitian PSD. For any $T \subseteq \{1, \dots, n_2\}$ of cardinality t, define $\psi_T := \sum_{\tau \in S_t} w_{t_{\tau(1)}} \otimes \cdots \otimes w_{t_{\tau(t)}} \in (\mathbb{C}^{n_1})^{\otimes t}$, so that $\langle \psi_S, \psi_T \rangle = t!\,\operatorname{per}(E_{S,T})$.

Case $n_1 = 1$. Here $v_1 \in \mathbb{C}$ is a scalar, so $\operatorname{per}(R_{\{1\}\cup I^c,\,\{1\}\cup J^c}) = (k+1)B_{11}\operatorname{per}(E_{I^c,J^c})$, whence $M_{I^c,J^c}^{(k)} = kB_{11}\operatorname{per}(E_{I^c,J^c}) = \frac{kB_{11}}{k!}\langle \psi_{I^c}, \psi_{J^c} \rangle$, which is PSD.

Case $n_1 = 2$. Partition the $(k+2)!$ bijections $\pi : \{1,2\} \cup I^c \to \{1,2\} \cup J^c$ by the number $s \in \{0,1,2\}$ of elements of $\{1,2\}$ mapped into J^c:

- $s = 0$: π preserves $\{1,2\}$ and maps $I^c \to J^c$, contributing $\operatorname{per}(B)\operatorname{per}(E_{I^c,J^c})$, which cancels the subtracted term identically.
- $s = 1$ (present when $k \ge 1$): π maps some $a' \in \{1,2\}$ to $j \in J^c$, some $i \in I^c$ to $a \in \{1,2\}$, $3 - a'$ to $3 - a$, and bijects $I^c \setminus \{i\} \to J^c \setminus \{j\}$. Defining

$$\Psi_{I^c} := \sum_{a \in \{1,2\}}\sum_{i \in I^c}\langle v_a, w_i \rangle\big(\overline{v_{3-a}} \otimes \psi_{I^c \setminus \{i\}}\big) \in \overline{\mathbb{C}^2} \otimes (\mathbb{C}^2)^{\otimes(k-1)},$$

and using $\langle \overline{v_{3-a}}, \overline{v_{3-a'}} \rangle = \langle v_{3-a'}, v_{3-a} \rangle$, we compute

$$\frac{1}{(k-1)!}\langle \Psi_{I^c}, \Psi_{J^c} \rangle = \sum_{a,a' \in \{1,2\}} \sum_{i \in I^c,\, j \in J^c} \langle w_i, v_a \rangle \langle v_{a'}, w_j \rangle \langle v_{3-a'}, v_{3-a} \rangle \times \operatorname{per}(E_{I^c \setminus \{i\},\, J^c \setminus \{j\}}),$$

which is the exact $s = 1$ contribution to $M^{(k)}_{I^c,J^c}$ and is therefore PSD.

- $s = 2$ (present when $k \geq 2$): π maps $\{1,2\}$ onto $\{j_1 < j_2\} \subseteq J^c$ and $\{i_1 < i_2\} \subseteq I^c$ onto $\{1,2\}$. Defining

$$\Omega_{I^c} := \sum_{\{i_1 < i_2\} \subseteq I^c} \operatorname{per}\big([v_1, v_2]^*[w_{i_1}, w_{i_2}]\big)\, \psi_{I^c \setminus \{i_1,i_2\}} \in (\mathbb{C}^2)^{\otimes (k-2)},$$

and using $\overline{\operatorname{per}([v_1,v_2]^*[w_{i_1},w_{i_2}])} = \operatorname{per}([w_{i_1},w_{i_2}]^*[v_1,v_2])$, we compute

$$\frac{1}{(k-2)!}\langle \Omega_{I^c}, \Omega_{J^c} \rangle = \sum_{\substack{\{i_1 < i_2\} \subseteq I^c \\ \{j_1 < j_2\} \subseteq J^c}} \operatorname{per}(R_{\{i_1,i_2\},\{1,2\}}) \operatorname{per}(R_{\{1,2\},\{j_1,j_2\}}) \times \operatorname{per}(E_{I^c \setminus \{i_1,i_2\},\, J^c \setminus \{j_1,j_2\}}),$$

which is the exact $s = 2$ contribution to $M^{(k)}_{I^c,J^c}$ and is therefore PSD.

Hence $M^{(k)} \succeq 0$ for every k, and $\operatorname{per}(A) - \operatorname{per}(B)\operatorname{per}(D) = \sum_{m=0}^{n_2} \sum_{I,J} M^{(n_2-m)}_{I^c,J^c} Q^{(m)}_{I,J} \geq 0$. □

6.2. **Reduction for Orbit-Type** $(2,3)$ **Subgroups in** $S_3 \times S_2$**.**

Lemma 6.2 (Subdirect-Product Reduction for Orbit Type $(2,3)$)**.** *For every subgroup $H \leq S_3 \times S_2$ having orbit partition $(2,3)$ (Classes #10, #11, and #15) and every $\chi \in \operatorname{Irr}(H)$, $d^H_\chi(A) \leq \operatorname{per}(A)$ follows from order-≤ 3 subgroup permanental dominance and Lemma 6.1.*

Proof. Write $A_1 := A_{\{0,1,2\}}$ and $A_2 := A_{\{3,4\}} = \left(\begin{smallmatrix} a & b \\ \bar{b} & c \end{smallmatrix}\right) \succeq 0$ (so $ac \geq |b|^2 \geq 0$), and let $\tau := (34)$ generate S_2.

Case 1: Direct products (Classes #10 and #15). Here $(e,\tau) \in H$, so $H = H_1 \times S_2$ with $H_1 \in \{C_3, S_3\}$. Every $\chi \in \operatorname{Irr}(H)$ factors as $\chi_1 \boxtimes \chi_2$, so $d^H_\chi(A) = d^{H_1}_{\chi_1}(A_1)\, d^{S_2}_{\chi_2}(A_2)$. Since $d^K_\psi(M) \geq 0$ for every PSD M by Schur [17], order-3 and order-2 dominance together with Lemma 6.1 give $d^H_\chi(A) \leq \operatorname{per}(A_1)\operatorname{per}(A_2) \leq \operatorname{per}(A)$.

Case 2: Subdirect product (Class #11, $H = S_3^{\mathrm{diag}} = \{(\sigma, \operatorname{sgn}(\sigma)) : \sigma \in S_3\}$). Here $(e,\tau) \notin H$, the projection $\pi_1 : H \xrightarrow{\sim} S_3$ is an isomorphism, and $K := H \cap (S_3 \times \{e\}) = A_3 \times \{e\}$ has index 2 in H. Writing $ac = (ac - |b|^2) + |b|^2$ yields the exact decomposition

$$d^H_\chi(A) = (ac - |b|^2) \left[\frac{1}{\chi(1)} \sum_{\sigma \in A_3} \chi(\sigma, e) P_\sigma(A_1) \right] + |b|^2\, d^{S_3}_{\chi \circ \pi_1^{-1}}(A_1).$$

Restricting χ to A_3 decomposes $\chi|_{A_3} = \sum_j m_j \psi_j$ with $\sum_j m_j \psi_j(1) = \chi(1)$, so the bracketed expression is a convex combination of $d^{A_3}_{\psi_j}(A_1) \leq \operatorname{per}(A_1)$, while the second term satisfies

$d^{S_3}_{\chi\circ\pi_1^{-1}}(A_1) \le \operatorname{per}(A_1)$. Since $ac - |b|^2 \ge 0$ and $|b|^2 \ge 0$,

$$\begin{aligned} d^H_\chi(A) &\le (ac - |b|^2)\operatorname{per}(A_1) + |b|^2 \operatorname{per}(A_1) = ac \operatorname{per}(A_1) \\ &\le (ac + |b|^2)\operatorname{per}(A_1) = \operatorname{per}(A_1)\operatorname{per}(A_2) \le \operatorname{per}(A). \end{aligned}$$ □

Remark 6.3 (Direct indicator reduction for S_3^{diag}). Class #11 ($S_3^{\mathrm{diag}} = \ker(\operatorname{sgn}\otimes\operatorname{sgn}) \le S_3\times S_2$) also follows directly from prior order-≤ 3 results and Lieb's $(3,2)$ block inequality: setting $\tilde\chi := \chi\circ\pi_1^{-1} \in \operatorname{Irr}(S_3)$ and inserting the subgroup indicator $\mathbf{1}_{S_3^{\mathrm{diag}}}(\sigma,\tau) = \frac{1}{2}(1+\operatorname{sgn}(\sigma)\operatorname{sgn}(\tau))$ yields

$$d^{S_3^{\mathrm{diag}}}_\chi(A) = \frac{1}{2}\Big(d^{S_3}_{\tilde\chi}(A_1)\operatorname{per}(A_2) + d^{S_3}_{\tilde\chi\cdot\operatorname{sgn}}(A_1)\det(A_2)\Big) \le \operatorname{per}(A_1)\operatorname{per}(A_2) \le \operatorname{per}(A),$$

since $\tilde\chi\cdot\operatorname{sgn} \in \operatorname{Irr}(S_3)$ has the same degree as $\tilde\chi$ and $0 \le \det(A_2) \le \operatorname{per}(A_2)$. For this reason Class #11 is classified with the prior-implied classes in Table 1.

6.3. **Reduction of $S_4\times S_1$ and Order-≤ 4 Verification.** For any $H \le S_4\times S_1$ (fixing vertex 0), $d^H_\chi(A) = A_{00}\, d^{H_1}_\chi(A_{\{0\}^c}) \le A_{00}\operatorname{per}(A_{\{0\}^c}) \le \operatorname{per}(A)$ by Lemma 6.1, reducing all 11 classes (37 subgroup characters) to order ≤ 4. For completeness, we verify 35 of the 37 order-4 characters (and all order-≤ 3 characters) completely from scratch:

- **Order ≤ 3 (all 4 classes of S_3, 9 subgroup characters, 100% self-contained).** For $\{e\}$ and C_2, $1 \pm |A_{01}|^2 \le 1 + |A_{01}|^2$. For S_3, $\operatorname{per}(B) - \det(B) = 2S_2(B) \ge 0$ and $\operatorname{per}(B) - d^{S_3}_{(2,1)}(B) = \frac{1}{2}\sum_{\mathrm{cyc}}\|B_{20}u_1 + B_{21}u_0\|^2 \ge 0$. For $C_3(\chi_k)$, normalizing $B_{ii} = 1$ gives $|B_{01}B_{12}B_{20}| \le \frac{1}{3}S_2(B)$ by AM–GM, so $d^{C_3}_{\chi_k}(B) \le 1 + 2|B_{01}B_{12}B_{20}| \le 1 + \frac{2}{3}S_2(B) \le \operatorname{per}(B)$ by Lemma 5.1.
- **Order 4, intransitive (6 classes of S_4).** All 6 intransitive classes reduce to order ≤ 3 via Lemma 6.1 (for the subdirect product $C_2^{\mathrm{diag}} = \langle(01)(23)\rangle \le S_2\times S_2$, $1 \pm |B_{01}|^2|B_{23}|^2 \le (1+|B_{01}|^2)(1+|B_{23}|^2) \le \operatorname{per}(B)$).
- **Order 4, S_4 (5 characters) and D_4 (5 characters).** Table 4 lists the 8 exact rational Walled-Brauer SOS certificates in `certificates.json` (verified identically on all 24 permutations of S_4 in `verify_order5_lieb.py`) covering all 4 nontrivial S_4 immanants and all 4 linear characters of D_4. The 2-dimensional character of D_4 is $\frac{1}{2}(d^{C_4}_{\chi_1} + d^{C_4}_{\chi_3})$.
- **Order 4, V_4 and C_4 (8 characters).** By AM–GM on 4-cycles ($2|B_{01}B_{12}B_{23}B_{30}| \le |B_{01}|^2|B_{23}|^2 + |B_{12}|^2|B_{30}|^2$), every irreducible character of V_4 or C_4 satisfies (for $B_{ii} = 1$)

$$d^H_\chi(B) \;\le\; 1 + S_{2,2}(B) \;=\; \frac{1}{6}\operatorname{per}(B) + \frac{1}{6}\det(B) + \frac{2}{3}d^{S_4}_{(2,2)}(B) \;\le\; \operatorname{per}(B),$$

  where the identity holds identically on all 24 permutations of S_4.
- **Order 4, A_4 (4 characters).** The characters $\mathbf{1}_{A_4}$ and χ_3 satisfy $d^{A_4}_{\mathbf{1}} = \frac{1}{2}(\operatorname{per}+\det)$ and $d^{A_4}_{\chi_3} = \frac{1}{2}(d^{S_4}_{(3,1)} + d^{S_4}_{(2,1,1)})$, both $\le$ per from S_4. Finally, for the complex-conjugate pair $A_4(\omega), A_4(\bar\omega)$ inside $A_4\times S_1$, Lemma 6.1 gives

$$d^{A_4\times S_1}_\omega(A) = A_{00}\, d^{A_4}_\omega(A_{\{0\}^c}) \le A_{00}\operatorname{per}(A_{\{0\}^c}) \le \operatorname{per}(A)$$

  directly by Zeng's order-4 theorem [21].

TABLE 4. Summary of the 8 exact rational Walled-Brauer SOS certificates for S_4 and D_4 in `certificates.json` (verified identically on all 24 permutations of S_4).

Master Case	Key	H	Field	K	Sectors / Explicit Structure
$S_4(\mathrm{det})$	`S4_det`	S_4	$\mathbb{Q}$	2	$\frac{3}{2}$ on $B^{(1)}_{2,2}(0,1)$ with $[1,1,1,1]$ and $[1,-1,-1,1]$
$S_4(3,1)$	`S4_31`	S_4	$\mathbb{Q}$	1	$\frac{2}{3}$ on $B^{(1)}_{3,1}(0)$ with $\mathbf{1}_6$
$S_4(2,1,1)$	`S4_211`	S_4	$\mathbb{Q}$	2	$\frac{2}{3}$ on $B^{(1)}_{3,1}(0)$ $(\mathbf{1}_6)$ + 1 on $B^{(1)}_{2,2}(0,1)$ $([1,-1,1,-1])$
$S_4(2,2)$	`S4_22`	S_4	$\mathbb{Q}$	1	$\frac{3}{2}$ on $B^{(1)}_{2,2}(0,1)$ with $\mathbf{1}_4$
$D_4(+,+)$	`D4_++`	D_4	$\mathbb{Q}$	1	$B^{(1)}_{2,2}$; rational SOS
$D_4(+,-)$	`D4_+-`	D_4	$\mathbb{Q}$	7	$B^{(2)}_{2,2}, B^{(1)}_{2,2}, B^{(1)}_{3,1}$; rational SOS
$D_4(-,+)$	`D4_-+`	D_4	$\mathbb{Q}$	2	$B^{(2)}_{2,2}, B^{(1)}_{2,2}$; rational SOS
$D_4(-,-)$	`D4_--`	D_4	$\mathbb{Q}$	6	$B^{(1)}_{2,2}, B^{(1)}_{3,1}$; rational SOS

Remark 6.4 (Structural contrast between $n = 4$ and $n = 5$). At order $n = 4$, the non-real character $A_4(\omega)$ cannot lie in the Walled-Brauer SOS cone generated by $B^{(2)}_{2,2}, B^{(1)}_{2,2}, B^{(1)}_{3,1}$ for a simple coefficient obstruction. In $B^{(2)}_{2,2}$, neither transpositions (T_2) nor 3-cycles (C_3) appear; in both $B^{(1)}_{2,2}$ and $B^{(1)}_{3,1}$, every diagonal entry is a transposition (contributing $w_t|c_{t,\alpha}|^2$ to $\sum_{\tau\in T_2}\gamma_\tau$), no off-diagonal entry is a transposition, and each row α and column β contains exactly two off-diagonal 3-cycle entries. By the triangle inequality and $|c_{t,\alpha}||c_{t,\beta}| \le \frac{1}{2}(|c_{t,\alpha}|^2+|c_{t,\beta}|^2)$, every element $\sum_{\sigma\in S_4}\gamma_\sigma P_\sigma$ in this cone must satisfy

$$\frac{1}{2}\sum_{\sigma\in C_3}|\gamma_\sigma| \;\le\; \sum_{\tau\in T_2}\gamma_\tau.$$

For per $-d^{A_4}_\omega$, however, the left-hand side is $\frac{1}{2}\cdot 8|1-\omega| = 4\sqrt{3}$ while the right-hand side is 6, contradicting $4\sqrt{3} > 6$. (Moreover, as originally observed by James [4] — cf. [2, 20, 21] — on the rank-2 PSD matrix $B = J_4+2(0\oplus vv^*)$ with $v = (1,\omega,\omega^2)^T$, one has $\mathrm{per}(B) = d^{A_4}_\omega(B) = 72$, a nontrivial boundary zero.) By contrast, at order $n = 5$, Theorem 4.1 and Theorem 4.3 show that per $-d^H_\chi$ lies inside the degree-5 Walled-Brauer SOS cone generated by $B^{(2)}_{3,2}, B^{(1)}_{3,2}, B^{(1)}_{4,1}$ for *every* transitive subgroup character of S_5 (with strictly positive exact weights establishing nonnegativity for all complex Hermitian PSD matrices $A \succeq 0$).

Combining Sections 4 and 6 completes the proof of Theorem 1.1.

7. Computational Verification, Data Availability, and Methodology

All exact certificates and verification routines are publicly archived in the companion repository at

https://github.com/coracle-us/conjectures
`results/lieb-permanental-dominance-order-5/`

The verification artifact consists of two files:

(1) `certificates.json` (32,518 bytes): all 19 exact Carathéodory-compressed Walled-Brauer vector-norm SOS certificates (11 master cases for S_5 and 8 master cases for S_4/D_4; 111 strictly positive weights and 1,512 exact coefficients in total), stored purely as exact integers and rational strings with zero floating-point data.

(2) `verify_order5_lieb.py`: a standalone Python 3 (standard library only) verifier that (a) enumerates the complete subgroup lattices of S_4 (30 subgroups, 11 classes, 37 subgroup characters) and S_5 (156 subgroups, 19 classes, 78 subgroup characters) from scratch; (b) checks exact weight positivity $w_t > 0$ and exact permutation-by-permutation polynomial equality for all 11 S_5 certificates (on all 120 permutations) and all 8 S_4/D_4 certificates (on all 24 permutations) in exact arithmetic over $\mathbb{Q}$, $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{5})$, and $\mathbb{Q}(\zeta_5)$; (c) checks all algebraic implication identities of Theorem 4.3 and Section 6.3; and (d) performs an independent numerical Hilbert-space evaluation of the literal vector-norm sums on random complex PSD matrices across ranks 1–5 (maximum relative error 3.45×10^{-15} for S_5 and 6.05×10^{-16} for S_4).

Methodology Note. Discovery of the Walled-Brauer SOS certificates, the elementary proofs in Section 5, and the subdirect-product reduction in Section 6.2 was carried out using an error-corrected, multi-agent AI-assisted mathematical research workflow modeled on Loader, Oppenheim, and Osborne [10], including independent zero-trust referee audits that re-derived the subgroup lattices, character tables, contraction Gram tables, and exact quotient-field certificate checks without code reuse. The validity of Theorem 1.1 depends solely on the self-contained mathematical arguments presented above, exact symbolic verification of `certificates.json` (covering 76 of the 78 subgroup characters directly), and Zeng's order-4 theorem [21] (used in one line via Lemma 6.1 for the remaining complex-conjugate pair $A_4(\omega), A_4(\bar{\omega})$ in $A_4 \times S_1$).

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