THE DUAL SMALE MEAN VALUE CONJECTURE FOR DEGREE EIGHT: A COMPLETE PURE LEAN 4 KERNEL PROOF VIA TWO-VARIABLE GRACE–WALSH–SZEGO COLLAPSE

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ABSTRACT. Let $P(z) = z + a_2 z^2 + \cdots + a_n z^n \in \mathbb{C}[z]$ be a complex polynomial of degree $n \geq 2$. The Dual Smale Mean Value Conjecture (Dubinin–Sugawa and Ng, 2009) asserts that

$$\max_{P'(\zeta)=0} \left| \frac{P(\zeta)}{\zeta} \right| \geq \frac{1}{n},$$

with equality if and only if $P(z) = \frac{1-(1-cz)^n}{nc}$ for some $c \in \mathbb{C} \setminus \{0\}$. While the primal Smale conjecture was disproved in 2026 and the previous dual record $n \leq 7$ (Hinkkanen–Kayumov–Khammatova, 2025) relied on a single minimal-modulus critical point, degree $n = 8$ is the first threshold where the single-point bound fails abruptly ($\min_{\zeta_1} 8|P(\zeta_1)/\zeta_1| \approx 0.6928 < 1$).

In an earlier version of this work, the conjecture at $n = 8$ was proved by combining a Lean 4 formalization of the analytic framework with an external outward-rounded Rust branch-and-bound certifier over 6,852,609 nodes in six complex variables (12 real dimensions). In the present paper, we *eliminate the external Rust verifier completely* and give a 100% self-contained, unconditional proof verified entirely inside the Lean 4.36.0 kernel (`DualSmale.dual_smale_degree_8_complete`), with zero `sorry`, zero `admit`, zero `native_decide`, and strictly the three standard axioms `[propext, Classical.choice, Quot.sound]`.

The elimination of external code rests on four mathematical reductions: (i) a *Two-Variable Grace–Walsh–Szego Collapse* that fixes the three largest-reciprocal critical points $(1, w_2, w_3)$ and collapses the remaining four critical points simultaneously across arbitrary complex linear combinations $\alpha_1 Q_1 + \alpha_2 Q_2 + \alpha_3 Q_3$, reducing the global search space from 12D to 4D and cutting the certificate size by a factor of 579× (from 6,852,609 nodes to 11,825 exact-integer leaves); (ii) an expanded analytic Stage 0 local theorem proving $8|H_8(a, v)| > 1$ on the asymmetric ball $|a - 1| \leq 1/20$, $|v - 1| \leq 1/4$, coupled with an affine homotopy and the strict Maximum Modulus Principle that closes the entire neighborhood $|a - 1| \leq 1/20$, $|v| \leq 1$ in just 2 certificate leaves; (iii) a zero-trigonometry unit-circle covering built from five exact Pythagorean-triple grids (16 to 256 points), a curvature-aware tangent lower bound, and an exact-integer Schur–Cohn zero-freeness test proved sound via clopen homotopy; and (iv) an exact fixed-point ($S = 2^{20}$) approximate-inverse defect identity certifying the secondary critical-point forms Q_2, Q_3 using purely native integer arithmetic inside the Lean kernel.

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1. INTRODUCTION

In 1981, Smale [7] posed his celebrated mean value conjecture: for every complex polynomial $P(z) = z + a_2 z^2 + \cdots + a_n z^n$ of degree $n \geq 2$, does there exist a critical point ζ with $|P(\zeta)/\zeta| \leq 1 - 1/n$ (or $\leq K$ for a universal constant)? In 2026, the primal conjecture was disproved with sharp global supremum $K^* = 4$ [1, 3].

By contrast, the *Dual Smale Mean Value Conjecture*, formulated independently by Dubinin–Sugawa [2] and Ng [6], asserts the reverse sharp lower bound on the *maximum* over all critical points. Prior to degree $n = 8$, the conjecture was proved for $n \leq 7$ by Hinkkanen, Kayumov, and Khammatova [4], for odd polynomials of arbitrary degree by Tang [8], and up to the general asymptotic lower bound $(n - 1/2)^{1/n}/n^2$ by Jatar and Ng [5].

2020 *Mathematics Subject Classification.* Primary 30C10, 30C15; Secondary 68V20, 65G20.

Key words and phrases. Dual Smale mean value conjecture, critical points of polynomials, Grace–Walsh–Szego coincidence theorem, Schur–Cohn criterion, Lean 4 formalization.

Theorem 1.1 (Main Theorem — Unconditional Pure Lean 4 Proof for $n = 8$)**.** *Let* $P(z) = z + a_2z^2 + \cdots + a_8z^8 \in \mathbb{C}[z]$ *be any complex polynomial of degree* 8 *with* $a_8 \neq 0$*. Then*

$$\max_{P'(\zeta)=0} \left| \frac{P(\zeta)}{\zeta} \right| \geq \frac{1}{8}. \tag{1}$$

Moreover, equality holds in (1) *if and only if* $P(z) = \frac{1-(1-cz)^8}{8c}$ *for some* $c \in \mathbb{C} \setminus \{0\}$.

Equivalently, writing the seven nonzero critical points of P in reciprocal form $w_j = 1/\zeta_j \in \mathbb{C} \setminus \{0\}$ ($j \in \{0, \ldots, 6\}$) so that $P'(z) = \prod_{k=0}^{6}(1 - w_kz)$, Theorem 1.1 is formalized verbatim in Lean 4.36.0 (`DualSmale/DualSmaleEightComplete.lean`) as:

```
theorem dual_smale_degree_8_complete
    (w : Fin 7 -> C) (hw : forall j, w j != 0) :
    (exists j : Fin 7, 1 < ||(8 : C) * dualSmaleQuotient 8 w j||) \/
    (forall j : Fin 7, w j = w 0)
```

This theorem is checked *entirely* by the Lean 4.36.0 kernel, without any external C, Rust, or Python verifier in the trusted computing base, without `native_decide` (`Lean.ofReduceBool`), and depending strictly on `[propext, Classical.choice, Quot.sound]`.

1.1. **Why Degree $n = 8$ Breaks Single-Point Methods.** Normalizing by a critical point of minimal modulus $\zeta_1 = 1$, the remaining six reciprocal critical points $w = (w_2, \ldots, w_7)$ lie in the closed unit disk $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$, and

$$P(1) = \int_0^1 (1-u) \prod_{k=2}^{7} (1 - uw_k)\, du = \sum_{m=0}^{6} \frac{(-1)^m}{(m+1)(m+2)}\, e_m(w_2, \ldots, w_7).$$

By the Grace–Walsh–Szego (GWS) coincidence theorem on $\overline{\mathbb{D}}$, there exists $v \in \overline{\mathbb{D}}$ such that $P(1) = G_8(v) := \int_0^1 (1-u)(1-uv)^6\, du$. While $n|G_n(v)| \geq 1$ on all of $\overline{\mathbb{D}}$ for $2 \leq n \leq 7$ (culminating in the exact unit-circle double tangency $|42G_7(e^{i\theta})|^2 - 36 = 14u(2u-1)^2(12u^2+4u+1)$ at $\theta = \pm 60^\circ$), at degree $n = 8$ the quantity $8|G_8(v)|$ drops to $\approx 0.6928 < 1$ on two symmetric lunes $L_8^{\pm} \subset \overline{\mathbb{D}}$. Hence any proof for $n = 8$ must couple $P(1)$ with secondary critical-point quotients $P(\zeta_j)/\zeta_j$.

1.2. **From Version 1 (Hybrid Lean + Rust) to Version 2 (100% Pure Lean 4).** In our Version 1 proof, the global 12-dimensional domain $\overline{\mathbb{D}}^6$ required 6,852,609 branch-and-bound nodes in an external outward-rounded Rust program (with Lean 4 formalizing the analytic foundations and soundness of the pruning rules). Directly replaying 6.85M floating-point nodes inside a dependent type theory kernel is infeasible. Table 1 summarizes the structural leap achieved in Version 2.

TABLE 1. Comparison between Version 1 (hybrid Lean 4 + Rust) and Version 2 (100% pure Lean 4.36.0 kernel).

Dimension	Version 1 (Hybrid)	Version 2 (100% Pure Lean 4)
Trusted base	Lean 4 + 1,525-line Rust binary	**Lean 4.36.0 kernel only**
Search space	12 real dims ($w_2, \ldots, w_7 \in \overline{\mathbb{D}}^6$)	**4 real dims** ($(w_2, w_3) \in C_+ \times C_\pm$)
Arithmetic	IEEE-754 `f64` outward rounding	**Exact $\mathbb{Z}$ fixed-point** ($S = 2^{20}$)
Trigonometry	4-ULP outward sin / cos	**Zero trig** (Pythagorean triples)
Stage 0 ball	$\max(\lvert a-1\rvert, \lvert v-1\rvert) \leq 1/100$	$\lvert a-1\rvert \leq 1/20$, $\lvert v-1\rvert \leq 1/4$
Cert. leaves	6,852,609 nodes	11,825 **exact-$\mathbb{Z}$ leaves** (579× **less**)
Proof engine	1,886 lines (24 theorems)	**8,808 lines + 41 cert modules**
Axioms	`propext, Classical.choice, Quot.sound`	`propext, Classical.choice, Quot.sound`

2. The Two-Variable Grace–Walsh–Szego Collapse (12D → 4D)

The centerpiece of Version 2 is the realization that one never needs to branch on all six free critical points $(w_2, \dots, w_7)$: fixing only the two largest remaining reciprocal critical points (w_2, w_3) and applying 4-variable GWS *simultaneously* to any complex linear combination of the three leading critical-point quotients collapses the remaining four complex variables $(z_0, z_1, z_2, z_3) = (w_4, w_5, w_6, w_7)$ into a single complex variable v on the smaller disk $\overline{D}(0, |w_3|)$.

2.1. **Exact Integer Polynomials J_0, J_1, J_2 and Q_1, Q_2, Q_3.** Multiplying by $840 = 105 \cdot 8$ clears all denominators. For $v \in \mathbb{C}$, define the three exact-integer degree-4 polynomials

$$\begin{aligned} J_0(v) &= 420 - 560v + 420v^2 - 168v^3 + 28v^4, \\ J_1(v) &= -140 + 280v - 252v^2 + 112v^3 - 20v^4, \\ J_2(v) &= 70 - 168v + 168v^2 - 80v^3 + 15v^4. \end{aligned} \tag{2}$$

For any $w_2, w_3 \in \mathbb{C} \setminus \{0\}$ and $v \in \mathbb{C}$, define

$$\begin{aligned} Q_1(w_2, w_3, v) &:= J_0(v) + (w_2 + w_3)J_1(v) + w_2 w_3 J_2(v), \\ Q_2(w_2, w_3, v) &:= Q_1\left(\frac{1}{w_2}, \frac{w_3}{w_2}, \frac{v}{w_2}\right), \\ Q_3(w_2, w_3, v) &:= Q_2(w_3, w_2, v). \end{aligned} \tag{3}$$

Observe that $Q_1(w_2, w_3, v)$ is precisely $840 \cdot P(1)$ evaluated at the diagonal specialization $w = (1, w_2, w_3, v, v, v, v)$, while by scale and permutation invariance of the dual Smale quotient $q_j(w) = P(\zeta_j)/\zeta_j$, $Q_2(w_2, w_3, v)$ and $Q_3(w_2, w_3, v)$ are precisely $840 \cdot q_1$ and $840 \cdot q_2$ at $(1, w_2, w_3, v, v, v, v)$ respectively. Crucially, as functions of the four remaining variables $z = (z_0, z_1, z_2, z_3) \in \mathbb{C}^4$, all three quotients $q_0(1, w_2, w_3, z)$, $q_1(1, w_2, w_3, z)$, and $q_2(1, w_2, w_3, z)$ are *symmetric multiaffine* in (z_0, z_1, z_2, z_3)!

Theorem 2.1 (Two-Variable Simultaneous GWS Collapse)**.** *Let $w_2, w_3 \in \mathbb{C} \setminus \{0\}$, $R > 0$, and $z \in \mathbb{C}^4$ satisfy $|z_i| \le R$ for $i = 0, 1, 2, 3$. Suppose there exist complex weights $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$ such that*

$$\left|\alpha_1 Q_1(w_2, w_3, v) + \alpha_2 Q_2(w_2, w_3, v) + \alpha_3 Q_3(w_2, w_3, v)\right| \;>\; 105\left(|\alpha_1| + |\alpha_2| + |\alpha_3|\right) \tag{4}$$

for all $v \in \mathbb{C}$ with $|v| \le R$. Then for $w = (1, w_2, w_3, z_0, z_1, z_2, z_3)$,

$$\max_{j \in \{0,1,2\}} \left|8\, q_j(w)\right| > 1.$$

Proof. Because every linear combination of symmetric multiaffine forms in (z_0, z_1, z_2, z_3) is itself symmetric multiaffine in (z_0, z_1, z_2, z_3), applying the 4-variable Grace–Walsh–Szego coincidence theorem (`grace_walsh_szego_disk`) on the closed disk $\overline{D}(0, R)$ yields a *single* point $v \in \overline{D}(0, R)$ such that

$$840\left(\alpha_1 q_0(w) + \alpha_2 q_1(w) + \alpha_3 q_2(w)\right) = \alpha_1 Q_1(w_2, w_3, v) + \alpha_2 Q_2(w_2, w_3, v) + \alpha_3 Q_3(w_2, w_3, v).$$

By (4) and the triangle inequality,

$$105\left(|\alpha_1| + |\alpha_2| + |\alpha_3|\right) < 105\left(|\alpha_1|\,|8q_0(w)| + |\alpha_2|\,|8q_1(w)| + |\alpha_3|\,|8q_2(w)|\right),$$

which forces $\max(|8q_0(w)|, |8q_1(w)|, |8q_2(w)|) > 1$. Formalized in `two_var_gws_exceeds_one` (`DualSmale/TwoVarGWS.lean`). □

Notice the dual power of Theorem 2.1:

1. *Modulus ordering*: sorting the seven critical points so that $1 = |w_0| \ge |w_2| \ge |w_3| \ge \max_i |z_i|$ shrinks the GWS disk radius from 1 to $R = |w_3|$, dramatically strengthening lower bounds whenever $|w_3| < 1$.
2. *Phase-aligned linear combinations*: whenever Q_1, Q_2, Q_3 individually dip below 105 on disjoint arcs of $|v| = R$, choosing small Gaussian-integer weights $(\alpha_1, \alpha_2, \alpha_3)$ cancels the dips and yields a single degree-4 polynomial $F_\alpha(v)$ that is zero-free on $|v| \le R$ and satisfies $|F_\alpha(v)| > 105(|\alpha_1| + |\alpha_2| + |\alpha_3|)$ everywhere on $|v| = R$.

3. Expanded Stage 0 Analytic Ball and Two-Leaf Homotopy Closure

To eliminate the need for deep branch-and-bound subdivision near the extremal singularity $(a, v) = (1, 1)$ of the single-variable GWS envelope $H_8(a, v) := 840P(1)\big|_{(a,v,v,v,v)}$, we first enlarge the analytic local ball by a factor of 5 in a and 25 in v.

Theorem 3.1 (Expanded Stage 0 Analytic Ball)**.** *For all* $(a, v) \in \overline{\mathbb{D}}^2 \setminus \{(1, 1)\}$ *satisfying*

$$|a - 1| \le \frac{1}{20}, \qquad |v - 1| \le \frac{1}{4},$$

we have strictly $|H_8(a, v)| > 105$ *(i.e.,* $8|P(1)| > 1$*).*

Proof. Write $a = 1 + d_1$, $v = 1 + d_2$ with $d_1 = x_1 + iy_1$, $d_2 = x_2 + iy_2$, and expand $8H_8(1 + d_1, 1 + d_2)/105 = Q + R$, where $Q = 1 - \frac{1}{7}d_1 - \frac{5}{7}d_2 + \frac{5}{21}d_1d_2 + \frac{10}{21}d_2^2$ and every term of the degree-6 remainder R is divisible by d_2^2. Using $|d_1| \le 1/20$, $|d_2| \le 1/4$, and the unit-disk constraints $2x_i + x_i^2 + y_i^2 \le 0$, we establish $|R| \le \frac{3}{20}|d_2|^2$ and a strict sum-of-squares lower bound on $|Q|^2 - (1 + \frac{3}{20}|d_2|^2)^2$ (`stage0_H8_wide_ball` in `DualSmale/Stage0Expanded.lean`). □

Theorem 3.2 (Complete Stage 0 Closure on $|a - 1| \le 1/20$, $|v| \le 1$)**.** *For all* $a, v \in \overline{\mathbb{D}}$ *with* $|a - 1| \le 1/20$*,* $\mathrm{Im}(a) \ge 0$*, and* $(a, v) \ne (1, 1)$*, we have strictly* $|H_8(a, v)| > 105$*.*

Proof outline. This is formalized as `stage0_full_ball_sound` in `DualSmale/CertSoundness.lean`. Two cases arise:

- **Case** $a = 1$**,** $v \ne 1$: Theorem 3.1 handles $|v - 1| \le 1/4$. For $|v| = 1$ with $|v - 1| > 1/4$, a single 64-point Grid 2 exact-integer certificate (`stage0A1_cert`) proves $|H_8(1, v)| > 105$ and verifies via Schur–Cohn that $H_8(1, \cdot)$ is zero-free on $\overline{\mathbb{D}}$. By the Minimum Modulus Principle, $|H_8(1, v)| \ge 105$ on $\overline{\mathbb{D}}$. If equality held at an interior point $|v| < 1$, then $1/H_8(1, \cdot)$ would attain an interior modulus maximum on the connected open disk $\mathbb{D}$ and hence be constant, contradicting $|H_8(1, 0)| = 280 \ne 105$.
- **Case** $a \ne 1$: The entire intersection $\{a \in \overline{\mathbb{D}} : |a - 1| \le 1/20,\ \mathrm{Im}(a) \ge 0\}$ lies inside the single dyadic box $[15/16, 1] \times [0, 1/16]$ with center $a_0 = \frac{31}{32} + \frac{1}{32}i$. Consider the affine homotopy $H(t, z) := (1 - t)H_8(a_0, z) + tH_8(a, z) = H_8(a_t, z)$ where $a_t = (1 - t)a_0 + ta$ ($t \in [0, 1]$). Because $\mathrm{Im}(a_0) = 1/32 > 0$, $\mathrm{Im}(a) \ge 0$, and $a \ne 1$, we have $a_t \ne 1$ for *every* $t \in [0, 1]$. Thus Theorem 3.1 gives $|H(t, z)| > 105$ for all $|z| = 1$ with $|z - 1| \le 1/4$, while a single 128-point Grid 3 certificate (`stage0Box_cert`) proves $|H(t, z)| > 105$ on the remaining boundary $|z| = 1$, $|z - 1| > 1/4$ and verifies $H(0, \cdot)$ is zero-free on $\overline{\mathbb{D}}$. The clopen homotopy lemma (Section 4) proves $H_8(a, \cdot)$ is zero-free on $\overline{\mathbb{D}}$, and compactness on $\partial\mathbb{D}$ plus Minimum Modulus lifts the pointwise boundary inequality to $|H_8(a, v)| > 105$ for all $|v| \le 1$.

□

4. Zero-Trigonometry Pure-Integer Kernel Certification Engine

Every polynomial lower bound in Stages 0, 1b, Double-Crit, 1c, 2, and 3 is certified by a single universal pure-integer engine (`DualSmale/SchurCohn.lean`, `DualSmale/CertEngine.lean`, `DualSmale/Grids.lean`) operating at fixed-point scale $S = 2^{20} = 1{,}048{,}576$ with zero floating-point operations and zero transcendental functions.

4.1. **Minimum Modulus, Clopen Homotopy, and Exact Schur–Cohn.** To prove $|F(v)| > M$ on the entire closed unit disk $\overline{\mathbb{D}}$ for a polynomial F perturbed around a center polynomial P_0:

(1) **Schur–Cohn at the Center** (`schurCohn4..6`): For $P(z) = \sum_{k=0}^{d} c_k z^k$ ($d \le 6$) with Gaussian-integer coefficients $c_k \in \mathbb{Z}[i]$, if $|c_0|^2 > |c_d|^2$ then the Schur–Cohn transform $T(P)(z) = \overline{c_0}P(z) - c_d z^d \overline{P(1/\overline{z})}$ has degree $\le d - 1$ and strictly fewer roots in $\overline{\mathbb{D}}$ iff P has none (proved via Rouché / boundary comparison in `DualSmale/SchurCohn.lean`). Iterating d exact integer steps certifies $P_0(z) \ne 0$ for all $|z| \le 1$.

(2) **Clopen Homotopy** (`zero_free_and_lower_bound_of_homotopy`): Along $H(t,z) = (1-t)P_0(z) + tF(z)$ $(t \in [0,1])$, if $|H(t,z)| \geq m > 0$ for all $(t,z) \in [0,1] \times \partial\mathbb{D}$, then the set $A = \{t \in [0,1] : \forall z \in \overline{\mathbb{D}},\ H(t,z) \neq 0\}$ is both open (by compactness of $\overline{\mathbb{D}}$) and closed (by the Minimum Modulus Principle, since $t \in A \implies \inf_{\overline{\mathbb{D}}} |H(t,\cdot)| \geq m > 0$). Since $0 \in A$ and $[0,1]$ is connected, $1 \in A$, whence F is zero-free on $\overline{\mathbb{D}}$ and $|F(z)| \geq m$ everywhere on $\overline{\mathbb{D}}$.

4.2. **Zero-Trigonometry Pythagorean Unit-Circle Covering.** To certify $|H(t,z)| > M$ on $\partial\mathbb{D}$ without ever invoking sin or cos, we construct five exact rational grids on $\partial\mathbb{D}$ from primitive Pythagorean triples $(a,b,c) \in \mathbb{Z}^3$ satisfying $a^2 + b^2 = c^2$ $(c > 0)$: Grid 0 (16 pts, $c \leq 100$), Grid 1 (32 pts, $c \leq 200$), Grid 2 (64 pts, $c \leq 400$), Grid 3 (128 pts, $c \leq 1000$), and Grid 4 (256 pts, $c \leq 3500$).

In the first quadrant, consecutive grid points $(a_k/c_k, b_k/c_k)$ advance monotonically ($a_{k+1}c_k \leq a_k c_{k+1}$ and $b_k c_{k+1} \leq b_{k+1}c_k$) from $(1,0)$ to $(0,1)$ and are replicated by 90° rotations across all four quadrants. By the elementary arc-chord lemma (`unit_circle_arc_chord_bound`), any unit-circle point lying on the arc between two grid points at chord distance $\ell \leq \sqrt{2}$ has angular distance at most $0.53\,\ell$ to the nearer endpoint (since $\frac{1}{2}\theta/\sin(\theta/2) \leq \frac{\pi/4}{\sin(\pi/4)} \approx 1.111$, and $0.53 > \frac{1}{2} \cdot 1.055$ on all chords of length ≤ 0.4). Each grid point stores an integer radius $d_u \in \mathbb{Z}$ verified by a pure integer inequality $53^2\big((a'c - ac')^2 + (b'c - bc')^2\big)S^2 \leq 100^2 d_u^2 (cc')^2$. All five grids are proved `ValidUnitGrid` directly inside Lean 4 (`DualSmale/Grids.lean`).

4.3. **Curvature-Aware Tangent Lower Bound and Approximate-Inverse Identity.** At each grid point $u = (a+ib)/c$ and any $z = ue^{i\phi}$ with $|\phi| \leq \delta = d_u/S$, expanding $P_0(ue^{i\phi})$ to second order in ϕ and projecting onto the unit direction $\overline{P_0(u)}/|P_0(u)|$ yields the curvature-aware lower bound (`poly6_arc_tangent_lower_bound`):

$$\operatorname{Re}\left(\frac{\overline{P_0(u)}}{|P_0(u)|}\,P_0(ue^{i\phi})\right) \geq |P_0(u)| - \frac{|\operatorname{Im}(\overline{P_0}uP_0')|}{|P_0|}\,\delta - \frac{\max\big(0,\,\operatorname{Re}(\overline{P_0}(uP_0' + u^2P_0''))\big)}{2|P_0|}\,\delta^2 - \frac{|uP_0'|\,\delta^4}{8} - |u^2P_0''|\left(\frac{\delta^3}{2} + \frac{\delta^4}{8}\right) - \frac{L_3\,\delta^3}{6},$$

where $L_3 = \sum_{k\geq 3} k(k-1)(k-2)|c_k|$. Crucially, the first-order oscillation term uses only the *tangential* component $|\operatorname{Im}(\overline{P_0}uP_0')|$ rather than the full $|uP_0'|$, while outward radial curvature ($\operatorname{Re} < 0$) contributes *zero* penalty. Every square root $|P_0|$ is bounded above by a self-checked 3-step integer Newton bound $r \in \mathbb{Z}$ verifying $r > 0 \wedge r^2 \geq |P_0|^2$, so soundness never depends on Newton convergence.

Finally, to build exact fixed-point Taylor enclosures for the inverse-argument forms $Q_2(w_2, w_3, v)$ and $Q_3(w_2, w_3, v)$ across a box centered at $c_2 \in S^{-1}\mathbb{Z}[i]$, we compute an integer approximate inverse $z_0 \approx c_2^{-1}$ with self-checked defect $\eta \geq |1 - c_2 z_0|$ and apply the exact algebraic identity (`inv_approx_decomp` in `DualSmale/TwoVarSoundness.lean`):

$$w_2^{-1} - z_0 = -(w_2 - c_2)z_0^2 + (1 - c_2 z_0)z_0 + \frac{(1 - w_2 z_0)^2}{w_2}. \tag{5}$$

The linear term $-(w_2 - c_2)z_0^2$ feeds directly into the first-order perturbation polynomial $d_2(u)$ (projected against $\overline{P_0(u)}$ so opposite-box cancellation is preserved!), while the quadratic-and-defect terms accumulate into a rigorous uniform remainder `rem_unif`.

5. Global Case Decomposition and Complete Pure Lean 4 Verification

Define the exact dyadic upper danger cap $C_+ \subset \overline{\mathbb{D}}$ (scaled by $S = 2^{20}$) by

$$C_+ := \Big\{ w = x + iy \in \overline{\mathbb{D}} : y \geq \frac{442368}{S},\ -\frac{901120}{S} \leq x \leq \frac{933888}{S},\ x - y \leq \frac{409600}{S},\ -x - y \leq \frac{311296}{S} \Big\},$$

and its conjugate $C_- := \overline{C_+}$. Note that C_+ fits cleanly inside the single dyadic root rectangle $B_+ := [16384 \pm 917504]/S \times [745472 \pm 303104]/S$.

Given any $w : \mathrm{Fin}\,7 \to \mathbb{C} \setminus \{0\}$, the master theorem `dual_smale_degree_8_complete` (`DualSmale/DualSmaleEightComplete.lean`) permutes and scales w so that $w_0 = 1 \geq |w_1| \geq |w_2| \geq |w_3|, |w_4|, |w_5|, |w_6|$ and (via conjugation invariance) $\mathrm{Im}(w_1) \geq 0$, then exhausts all non-extremal configurations across six pure-kernel stages:

(1) **Stage 1b ($|w_1| \leq 183/200$)**: Applying 6-variable GWS on $\overline{D}(0, 183/200)$ reduces $840P(1)$ to $840G_8(\frac{183}{200}u)$ on $|u| \leq 1$, certified > 105 in **1 leaf** on Grid 3 (`stage1b_cert`).
(2) **Double-Crit ($w_0 = w_1 = 1$, $w_2 \in C_+$)**: Applying 4-variable GWS with $\alpha = (1, 0, 0)$ reduces to $Q_1(1, w_2, v)$, certified > 105 across all of B_+ in **2 leaves** (Grids 1 and 3, `doubleCrit_cert`).
(3) **Stage 0 ($w_1 \notin C_+$, $|w_1 - 1| \leq 1/20$, $w_1 \neq 1$)**: Closed by Theorem 3.2 using `stage0_H8_wide_ball` plus **2 leaves** (`stage0Box_cert`, `stage0A1_cert`).
(4) **Stage 1c ($w_1 \notin C_+$, $|w_1 - 1| > 1/20$, $\mathrm{Im}(w_1) \geq 0$)**: Closed by 5-variable GWS ($H_8(w_1, v)$) over $[-1, 1] \times [0, 1]$ in **158 leaves** (`stage1c_cert`). The same certificate also resolves any configuration with $w_1 \in C_+$ and $w_2 \notin C_+ \cup C_- \cup \{1\}$ after swapping (w_1, w_2) and conjugating if $\mathrm{Im}(w_2) < 0$.
(5) **Stage 2 ($(w_1, w_2) \in C_+ \times C_-$, $|w_2| \leq |w_1|$, $|w_1| > 183/200$)**: Closed purely by Q_1 ($\alpha = (8, 0, 0)$) in **211 leaves** (`stage2_cert`).
(6) **Stage 3 ($(w_1, w_2) \in C_+ \times C_+$, $|w_2| \leq |w_1|$, $|w_1| > 183/200$)**: Closed by Theorem 2.1 using phase-aligned Gaussian-integer combinations $(\alpha_1, \alpha_2, \alpha_3)$ of (Q_1, Q_2, Q_3) in **11,451 leaves** across 16 geometric shards (automatically balanced into 33 parallel pure-kernel modules, zero Grid-4 leaves, `stage3_cert_0`..`stage3_cert_15`).

Every certificate theorem is checked directly inside the Lean 4.36.0 C++ kernel via `by decide +kernel` on chunked subtrees ($\leq$ 1,200 grid-point evaluations per kernel theorem, combined in $O(1)$ via the zero-Mathlib module `DualSmale/PureKernel.lean`), keeping memory modest and avoiding `native_decide`. Table 2 details all verified stages.

TABLE 2. Complete 100% pure Lean 4.36.0 kernel verification breakdown for degree $n = 8$ (zero external verifiers, zero `sorry`, zero `native_decide`).

Stage	Certified Region	Leaves	Grid Evals	Lean 4 Module(s)
0	$\lvert a - 1\rvert \leq 1/20,\ \lvert v\rvert \leq 1$	2	192	`Stage0Expanded`, `Stage1Certs`
1b	$\lvert w_k\rvert \leq 183/200$	1	128	`Stage1Certs` (`stage1b_cert`)
DC	$w_0 = w_1 = 1,\ w_2 \in C_+$	2	160	`Stage1Certs` (`doubleCrit_cert`)
1c	$w_1 \in \overline{\mathbb{D}} \setminus C_+,\ \lvert w_1 - 1\rvert > 1/20$	158	25,024	`Stage1Certs` (`stage1c_cert`)
2	$(w_1, w_2) \in C_+ \times C_-$	211	15,424	`Stage2Cert` (`stage2_cert`)
3	$(w_1, w_2) \in C_+ \times C_+$	11,451	602,336	`Stage3Cert0..15` (33 leaf files)
Total	**All of** $(\mathbb{C} \setminus \{0\})^7$	**11,825**	**643,264**	`DualSmaleEightComplete` **(PASS)**

6. Formal Verification Architecture and Reproducibility

Table 3 lists all proof-engine modules in the Lean 4.36.0 library (preserving all six Version 1 modules alongside the new Version 2 pure kernel formalization).

The entire proof can be independently audited from source in Lean 4.36.0 with three commands (the proof engine compiles in $\sim$ 35 s on cached Mathlib, and the 41 zero-Mathlib certificate modules verify all 643,264 grid-point evaluations in $\sim$ 75 min at 2-worker concurrency on a 16 GB Apple Silicon laptop, or $\sim$ 15–20 min on an 8-core 32 GB+ workstation):

```
cd lean
lake exe cache get
lake build DualSmale
lake env lean check_axioms.lean
```

TABLE 3. Summary of Lean 4.36.0 proof-engine modules (zero `sorry`, zero `native_decide`).

Module (`DualSmale/.*`)	Lines	Role
`GraceWalshSzego`, `PolynomialReduction`	933	Laguerre polar deriv., GWS on $\overline{D}(0,r)$, 840-identities
`LocalMinimum`, `DegreesUpToSeven`	605	Universal local min ($n \geq 3$), real w, $n \leq 7$ SOS
`EqualModulusEight`, `IntervalSoundness`	348	Circle-clique ($n = 8$), Version 1 soundness (preserved)
`SchurCohn`	803	Min. modulus, clopen homotopy, Schur–Cohn, arc tangent
`Stage0Expanded`	359	Analytic wide ball $\lvert a-1\rvert \leq 1/20$, $\lvert v-1\rvert \leq 1/4$
`TwoVarGWS`	566	Scale/conj/perm invariance, Q_1, Q_2, Q_3, 2-var GWS
`PureKernel` (zero Mathlib)	726	Pure-`Init` exact-$\mathbb{Z}$ engine, 5 grids, tree combiners
`CertEngine`, `Grids`	1,468	Circle/Schur–Cohn soundness + 5 grid proofs
`CertWrappers`, `TreeCombine`	10	Module re-exports connecting pure kernel and analytic layer
`CertSoundness`	1,031	Soundness for Stage 0, 1b, Double-Crit, Stage 1c
`TwoVarSoundness`	1,528	Soundness for `buildP1Int`, `buildP2Int`, Stages 2–3
`DualSmaleEightComplete`	431	Unconditional theorem `dual_smale_degree_8_complete`
Proof engine total	**8,808**	**+ 41 pure-kernel cert modules (zero Mathlib)**

The axiom audit confirms that `DualSmale.dual_smale_degree_8_complete` depends strictly on `[propext, Classical.choice, Quot.sound]`.

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