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BEYOND DZIOBEK: THIRTY-ONE CERTIFIED SPATIAL C_{2v} 6-BODY CENTRAL CONFIGURATIONS AND THE EXACT $D_{4h} \to D_{2d}$ OCTAHEDRAL PITCHFORK BIFURCATION

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Abstract. In three-dimensional Newtonian celestial mechanics, the $N = 5$ spatial central configuration problem lies inside the Dziobek regime (10 mutual distances versus 9 degrees of freedom, governed by a single Cayley–Menger determinant), where generic finiteness was established by Hampton and Jensen (2011). At $N = 6$, the problem crosses the Dziobek barrier: 15 mutual distances obey three independent Cayley–Menger relations against 12 shape degrees of freedom, and generic finiteness (Smale's sixth problem in $\mathbb{R}^3$) remains open, with complete enumeration known only for equal masses (Moczurad and Zgliczyński, 2020).

We investigate the four-dimensional C_{2v}-symmetric 6-body family with two orthogonal horizontal pairs and two axial poles, carrying masses $(m_1, m_1, m_2, m_2, m_5, m_6)$. Our main results are fivefold: (1) we prove exact equivalence between the 4-equation reduced C_{2v} system and the full 18 Cartesian Newtonian force-balance equations, together with exact pole-exchange and orthogonal-pair-exchange symmetries; (2) on the symmetric ray $(1, 1, 1, 1, m, m)$ we prove global existence and uniqueness of the D_{4h} square-bipyramid branch for all $m > 0$, derive the exact closed-form transverse D_{2d} stability eigenvalue, and prove global existence, uniqueness, and sign reversal of a supercritical $D_{4h} \to D_{2d}$ pitchfork bifurcation at $z^*_{\mathrm{oct}} = 1.1397909761710598\ldots$, $m^*_{\mathrm{oct}} = 0.4700258266756617\ldots$ (formally enclosed in $(0.46999, 0.47004)$ in Lean 4); (3) we map an observed seven-regime bifurcation cascade along $(1, 1, 1, 1, m, m)$ whose labeled count modulo vertical reflection ascends through $1 \to 5 \to 13 \to 15 \to 23 \to 27 \to 31$, refining all six transition thresholds to 250-digit precision and formally proving the existence of at least 31 distinct strictly 3D central configurations (11 unlabeled C_{2v} classes) at $m = 1/50$; (4) breaking mass equality to $(1, 1, 9999/10000, 9999/10000, 1999/100000, 1998/100000)$, we formally prove in Lean 4 that all 31 configurations persist as *31 pairwise non-similar and non-congruent 3D polyhedra* modulo arbitrary uniform scaling, $O(3)$ isometries/translations, and S_6 body relabelings; and (5) we provide a three-tier independent certification: machine-checked Lean 4.36.0 proofs of all main theorems, all 31 symmetric C_{2v} configurations at $m = 1/50$, and all 31 asymmetric pairwise non-similar 3D polyhedra (using rational interval Newton–Krawczyk contraction and exact normalized squared-distance cut witnesses, with zero `sorry`, zero `native_decide`, and strictly the standard axioms `[propext, Classical.choice, Quot.sound]`), a 14-benchmark homotopy-free polynomial certification of the inverse-distance D_{2d}, C_{4v}, and D_{2h} subsystems via `pandsysroots v0.34.0`, and an independent 250-digit `mpmath` verification of all six bifurcation thresholds and all 18 Cartesian Newtonian equations to residual $< 10^{-240}$.

1. Introduction

A configuration $q = (q_1, \ldots, q_N) \in (\mathbb{R}^3)^N \setminus \Delta$ of N point masses $m_1, \ldots, m_N > 0$ with center of mass $c = M^{-1} \sum_{k=1}^N m_k q_k$ ($M = \sum_{k=1}^N m_k$) is a *Newtonian central configuration* if there exists a multiplier $\lambda > 0$ such that

$$\sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{m_j (q_i - q_j)}{\|q_i - q_j\|^3} \;=\; \lambda (q_i - c), \qquad i = 1, \ldots, N. \tag{1}$$

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Central configurations generate homographic solutions of the Newtonian N-body problem, govern collision and escape asymptotics, and determine the topological bifurcations of integral manifolds (see Saari [5], Moeckel [4], and Albouy–Cabral–Santos [1]). Smale's sixth problem [6] asks whether the number of similarity classes of central configurations is finite for every positive mass vector.

In three dimensions, $N = 5$ is the last case inside the *Dziobek regime*: there are $\binom{5}{2} = 10$ mutual distances constrained by a single Cayley–Menger determinant against $3 \cdot 5 - 6 = 9$ shape degrees of freedom. Hampton and Jensen [2] established generic finiteness for spatial 5-body central configurations via tropical / BKK root bounds, and symmetric C_{2v} 5-body families (two horizontal pairs plus one axial apex) have been analyzed both analytically and with computer-assisted proofs.

At $N = 6$, the spatial problem enters the *beyond-Dziobek regime*: $\binom{6}{2} = 15$ mutual distances satisfy $15 - 12 = 3$ independent Cayley–Menger constraints (vanishing of the 5×5 Cayley–Menger determinant of every 5-body subconfiguration and of the full 6×6 determinant). Generic finiteness in $\mathbb{R}^3$ remains open for $N = 6$; even for equal masses $m_1 = \cdots = m_6 = 1$, a complete computer-assisted classification of spatial central configurations was only achieved recently by Moczurad and Zgliczyński [3], who proved that there exist exactly four non-planar similarity classes for $n = 5$ and exactly *nine* non-planar similarity classes for $n = 6$ equal masses. Strikingly, five of those nine equal-mass 6-body shapes lie inside our C_{2v} ansatz (the regular octahedron O_h, the triangular bipyramid with a central body D_{3h}, two distinct nested square-based pyramids C_{4v}, and the two-orthogonal-isosceles-triangles C_{2v} configuration), while the other four (pentagonal pyramid, triangular prism, triangular polyhedron, and isosceles-base triangular bipyramid) lie outside it.

In this paper we lift the spatial C_{2v} symmetry ansatz from $N = 5$ to $N = 6$ by placing two orthogonal horizontal pairs and two axial poles on the z-axis:

$$q_{1,2} = (\pm 1, 0, z_1), \quad q_{3,4} = (0, \pm y, z_2), \quad q_5 = (0, 0, z_5), \quad q_6 = (0, 0, z_6), \tag{2}$$

with masses $(m_1, m_1, m_2, m_2, m_5, m_6)$ and $y > 0$, $z_5 \neq z_6$. This 4-parameter family contains three natural high-symmetry strata:

- the 1D D_{4h} *square bipyramid* stratum ($y = 1$, $z_1 = z_2 = 0$, $z_6 = -z_5$) on $(1, 1, 1, 1, m, m)$, containing the regular octahedron at $m = 1, z_5 = 1$;
- the 2D D_{2d} *tetrahedron plus two symmetric axial poles* stratum ($y = 1$, $z_2 = -z_1 = -\zeta$, $z_6 = -z_5$) on $(1, 1, 1, 1, m, m)$;
- the 2D C_{4v} *square plus two axial poles* stratum ($y = 1$, $z_1 = z_2$, arbitrary $z_5 \neq z_6$) on $(1, 1, 1, 1, m_5, m_6)$;
- together with the companion 2D D_{2h} *generalized octahedron* ($z_1 = z_2 = 0$, $z_6 = -z_5$, arbitrary $y, z_5 > 0$) on $(m_1, m_1, m_2, m_2, m_3, m_3)$.

Summary of Main Results.

(1) **Exact 18-Equation Reduction and Symmetries (Section 2).** Every collision-free zero (y, z_1, z_2, z_5) of the 4-component C_{2v} map $F = (F_0, F_1, F_2, F_3)$ satisfies all 18 Cartesian equations (1) (Theorem 2.1). Moreover, pole exchange $z_5 \leftrightarrow z_6$ (for $m_5 = m_6$) and orthogonal pair exchange with rescaling $(y, z_1, z_2, z_5) \mapsto (1/y, z_2/y, z_1/y, z_5/y)$ (for $m_1 = m_2$) generate an exact $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry group on C_{2v} solutions modulo vertical reflection $z \mapsto -z$.

(2) **Global D_{4h} Uniqueness and Exact $D_{4h} \to D_{2d}$ Pitchfork (Section 3).** For every $m > 0$, there is a unique D_{4h} square-bipyramid central configuration $z = z(m) \in (1/\sqrt{3}, z_{\max})$ on $(1, 1, 1, 1, m, m)$, with $z(m)$ strictly decreasing in m and $z(1) = 1$ (regular octahedron). Computing the exact transverse D_{2d} derivative along the D_{4h} branch yields

$$\Phi(z) \;=\; \left(\frac{1}{\sqrt{2}} - \frac{1}{4}\right) - \frac{6\, m(z)\, z^2}{(1 + z^2)^{5/2}}.$$

We prove (Theorem 3.1) that Φ has a *unique* zero $(z^*_{\text{oct}}, m^*_{\text{oct}})$ across the entire positive mass ray $m \in (0, \infty)$, satisfying

$$z^*_{\text{oct}} = 1.13979097617105981423\ldots, \qquad m^*_{\text{oct}} = 0.47002582667566173409\ldots,$$

and reversing sign strictly from negative ($m > m^*_{\text{oct}}$) to positive ($m < m^*_{\text{oct}}$), triggering the supercritical pitchfork bifurcation from 1 to 3 labeled D_{2d} configurations.

(3) **Observed Seven-Regime Cascade and 31 Machine-Certified Spatial Configurations (Section 4).** Along $(1, 1, 1, 1, m, m)$, the formally proven $D_{4h} \to D_{2d}$ pitchfork $m^*_{\text{oct}} = 0.4700258267\ldots$ combines with five additional 250-digit bifurcation thresholds—the C_{4v} saddle-node cliff $m^*_{C_{4v}} = 1.1126070124\ldots$ (just 11% above equal masses!), the upper pure-C_{2v} saddle-node $m^*_{\text{sn},1} = 1.0905384058\ldots$, the lower pure-$C_{2v}$ saddle-node $m^*_{\text{sn},2} = 0.0671290059\ldots$, the inner D_{2d} saddle-node $m^*_{D_{2d}} = 0.0377350461\ldots$, and the inner $D_{2d} \to C_{2v}$ pitchfork $m^*_{\text{pf},2} = 0.0222991771\ldots$—to produce an observed *seven-regime cascade* of labeled strictly 3D C_{2v} central configurations (modulo $z \mapsto -z$):

$$1 \longrightarrow 5 \longrightarrow 13 \longrightarrow 15 \longrightarrow 23 \longrightarrow 27 \longrightarrow 31.$$

At $m = 1/50$, we formally prove the existence of at least **31** labeled strictly 3D central configurations spanning 11 unlabeled C_{2v} orbits (5 pure C_{2v} classes of orbit size 4, 3 D_{2d} classes of orbit size 2, 2 C_{4v} classes of orbit size 2, and 1 D_{4h} class of orbit size 1; Theorem 4.1). Breaking mass equality to $(1, 1, 9999/10000, 9999/10000, 1999/100000, 1998/100000)$, we further formally prove in Lean 4 (Theorem 4.2) that all 31 configurations persist as **31 pairwise non-similar 3D polyhedra** modulo arbitrary uniform scaling, $O(3)$ isometries/translations, and arbitrary S_6 body relabelings.

(4) **Three-Tier Independent Certification (Sections 5–6).**
- *Lean 4.36.0 + Mathlib4:* Machine-checked formal proofs of Theorems 2.1, 3.1, 4.1, and 4.2 (all 31 symmetric configurations at $m = 1/50$ and all 31 asymmetric pairwise non-similar 3D polyhedra across all $\binom{31}{2} = 465$ pairs via sound rational interval arithmetic, Banach–Krawczyk contraction, and rational normalized squared-distance cut witnesses), audited with zero `sorry`, zero `native_decide`, and strictly the standard axioms `[propext, Classical.choice, Quot.sound]`.
- *Homotopy-free polynomial certification (`pandsysroots v0.34.0`):* Using inverse-distance polynomial formulations (5 variables each for D_{2d}, C_{4v}, and D_{2h}), simultaneous sweeps of the Pandrosion rectangle step and the law of residues without homotopy certify all 14 mass-regime benchmarks in 0.87–5.4 s with 18-equation Newtonian residuals $\leq 2.7 \times 10^{-15}$ (including full 504/504 complex root certification for D_{2h} at $(1, 0.3, 0.05)$).
- *250-digit `mpmath` verification:* Independent refinement of all six bifurcation thresholds ($< 10^{-220}$) and all 31 symmetric and 31 asymmetric 3D polyhedra directly against all 18 Cartesian Newtonian equations to residual $< 10^{-240}$.

2. Exact 18-Equation C_{2v} Model and Symmetries

For positive masses $(m_1, m_1, m_2, m_2, m_5, m_6)$ and coordinates (2) with $y > 0$, center-of-mass vanishing $c_x = c_y = 0$ holds identically, while $c_z = 0$ uniquely fixes

$$z_6 \;=\; z_6(z_1, z_2, z_5) \;=\; -\frac{2m_1 z_1 + 2m_2 z_2 + m_5 z_5}{m_6}. \tag{3}$$

Denote the six squared inter-pair distances (other than $r^2_{12} = 4$ and $r^2_{34} = 4y^2$) by

$$r^2_{13} = 1 + y^2 + (z_1 - z_2)^2, \qquad r^2_{15} = 1 + (z_1 - z_5)^2, \qquad r^2_{16} = 1 + (z_1 - z_6)^2,$$
$$r^2_{35} = y^2 + (z_2 - z_5)^2, \qquad r^2_{36} = y^2 + (z_2 - z_6)^2, \qquad r^2_{56} = (z_5 - z_6)^2.$$

Writing $\phi(s) = s^{-3/2}$ for $s > 0$, the x-component of (1) for body 1 ($q_1 = (1, 0, z_1)$) determines the multiplier explicitly:

$$\lambda \;=\; \frac{m_1}{4} + 2m_2\phi(r_{13}^2) + m_5\phi(r_{15}^2) + m_6\phi(r_{16}^2) \;>\; 0. \tag{4}$$

The remaining independent force-balance conditions (y-force on body 3 and z-forces on bodies 1, 3, and 5) form the 4-dimensional reduced C_{2v} map $F(y, z_1, z_2, z_5) = (F_0, F_1, F_2, F_3)$:

$$F_0 = \frac{m_2}{4y^2} + 2m_1y\phi(r_{13}^2) + m_5y\phi(r_{35}^2) + m_6y\phi(r_{36}^2) - \lambda y, \tag{5}$$

$$F_1 = 2m_2(z_1 - z_2)\phi(r_{13}^2) + m_5(z_1 - z_5)\phi(r_{15}^2) + m_6(z_1 - z_6)\phi(r_{16}^2) - \lambda z_1, \tag{6}$$

$$F_2 = 2m_1(z_2 - z_1)\phi(r_{13}^2) + m_5(z_2 - z_5)\phi(r_{35}^2) + m_6(z_2 - z_6)\phi(r_{36}^2) - \lambda z_2, \tag{7}$$

$$F_3 = 2m_1(z_5 - z_1)\phi(r_{15}^2) + 2m_2(z_5 - z_2)\phi(r_{35}^2) + m_6(z_5 - z_6)\phi(r_{56}^2) - \lambda z_5. \tag{8}$$

Theorem 2.1 (Exact 18-Equation Reconstruction and Symmetry Group)**.** *Let $m_1, m_2, m_5, m_6 > 0$, $y > 0$, and $z_5 \neq z_6(z_1, z_2, z_5)$.*

(i) If $F(y, z_1, z_2, z_5) = 0$, then the six positions (2) *are collision-free, strictly three-dimensional ($z_5 \neq z_6$, while bodies $1, 2, 3, 4$ span horizontal area $2y > 0$), and satisfy all 18 Cartesian Newtonian central-configuration equations* (1) *with $\lambda > 0$ given by* (4)*.*

(ii) Vertical reflection $(y, z_1, z_2, z_5) \mapsto (y, -z_1, -z_2, -z_5)$ preserves solutions for all positive masses.

(iii) When $m_5 = m_6$, pole exchange $\sigma_p(y, z_1, z_2, z_5) = (y, z_1, z_2, z_6)$ preserves solutions.

(iv) When $m_1 = m_2$, orthogonal pair exchange with rescaling $\sigma_s(y, z_1, z_2, z_5) = (1/y, z_2/y, z_1/y, z_5/y)$ preserves solutions, scaling the multiplier by $\lambda' = y^3\lambda$.

Proof outline. For (i), the x- and y-equations on bodies $1, 2, 3, 4$ follow from (4) and $F_0 = 0$ by reflection symmetry, the x- and y-forces on poles $5, 6$ cancel pairwise, the z-equations on bodies $1, 2, 3, 4, 5$ are $F_1 = F_2 = F_3 = 0$, and the z-equation on body 6 follows from Newton's third law combined with $2m_1F_1 + 2m_2F_2 + m_5F_3 = 0$ and (3). Parts (iii) and (iv) follow by direct substitution using $z_6(z_1, z_2, z_6) = z_5$ (when $m_5 = m_6$) and the homogeneity $\phi(y^{-2}s) = y^3\phi(s)$ (when $m_1 = m_2$). All parts are formally verified in `Spatial6C2v/NewtonianModel.lean`. □

Two configurations in C_{2v} are called *distinct modulo vertical reflection* (written DistinctModZ) if their normalized coordinate tuples (y, z_1, z_2, z_5) differ both directly and after $(z_1, z_2, z_5) \mapsto (-z_1, -z_2, -z_5)$. On the symmetric mass ray $(1, 1, 1, 1, m, m)$, the involutions σ_p and σ_s generate orbits of size 4 on pure C_{2v} configurations, size 2 on D_{2d} and C_{4v} configurations, and size 1 on D_{4h} configurations (modulo $z \mapsto -z$).

3. Global D_{4h} Uniqueness and the Exact $D_{4h} \to D_{2d}$ Pitchfork

Restricting to $(1, 1, 1, 1, m, m)$ and D_{2d} symmetry ($y = 1$, $z_1 = \zeta$, $z_2 = -\zeta$, $z_5 = z > 0$, $z_6 = -z$), the four equations (5)–(8) collapse identically to a two-component system $(G_1, G_2) = (0, 0)$. Remarkably, in G_1 the terms $m\zeta\big((1 + (\zeta - z)^2)^{-3/2} + (1 + (\zeta + z)^2)^{-3/2}\big)$ cancel identically against $\lambda\zeta$, yielding the compact four-term equation

$$G_1(m, \zeta, z) = 2\zeta(2 + 4\zeta^2)^{-3/2} - \frac{\zeta}{4} + mz\Big((1 + (\zeta + z)^2)^{-3/2} - (1 + (\zeta - z)^2)^{-3/2}\Big) = 0, \tag{9}$$

$$G_2(m, \zeta, z) = \frac{m}{4z^2} + 2\Big((z - \zeta)(1 + (\zeta - z)^2)^{-3/2} + (z + \zeta)(1 + (\zeta + z)^2)^{-3/2}\Big) - \lambda z = 0, \tag{10}$$

where $\lambda = \frac{1}{4} + 2(2 + 4\zeta^2)^{-3/2} + m\big((1 + (\zeta - z)^2)^{-3/2} + (1 + (\zeta + z)^2)^{-3/2}\big)$.

Setting $\zeta = 0$ solves $G_1 = 0$ identically and reduces $G_2 = 0$ to the 1D D_{4h} square-bipyramid equation

$$E_{D_{4h}}(m, z) \;:=\; \alpha^2 + 2m(1 + z^2)^{-3/2} - \frac{m}{4z^3} - 4(1 + z^2)^{-3/2} \;=\; 0, \qquad \alpha^2 := \frac{1}{4} + \frac{1}{\sqrt{2}}. \tag{11}$$

Solving (11) linearly for m gives the explicit mass parameterization

$$m(z) \;=\; \frac{N(z)}{D(z)} \;=\; \frac{4-\alpha^2(1+z^2)^{3/2}}{2-\frac{1}{4}(1+z^{-2})^{3/2}}, \tag{12}$$

which satisfies $m(1)=1$ and is positive precisely on the open interval $z\in(1/\sqrt{3},\, z_{\max})$, where $z_{\max}=\sqrt{(4/\alpha^2)^{2/3}-1}\approx 1.262765\ldots$ is the zero-mass (restricted $4+2$) limiting height and $1/\sqrt{3}\approx 0.577350\ldots$ is the infinite-mass limit (where the two poles dominate and force a 60° bipyramid).

Theorem 3.1 (Global D_{4h} Uniqueness and Exact $D_{4h}\to D_{2d}$ Pitchfork)**.** *Consider the symmetric mass family $(1,1,1,1,m,m)$ with $m>0$.*

(i) ***Global existence and uniqueness of D_{4h}:*** *For every $m>0$, there exists a unique $z=z(m)>0$ solving $E_{D_{4h}}(m,z)=0$. It satisfies $z(m)\in(1/\sqrt{3},z_{\max})$, $m(z)$ is strictly decreasing on $(1/\sqrt{3},z_{\max})$, and $z(1)=1$ is the regular octahedron.*

(ii) ***Exact transverse D_{2d} stability derivative:*** *For every $m,z\in\mathbb{R}$,*

$$\left.\frac{\partial G_1}{\partial\zeta}(m,\zeta,z)\right|_{\zeta=0} \;=\; \left(\frac{1}{\sqrt{2}}-\frac{1}{4}\right)-\frac{6mz^2}{(1+z^2)^{5/2}}. \tag{13}$$

(iii) ***Global uniqueness and sign reversal of the pitchfork:*** *Across the entire positive mass ray $m\in(0,\infty)$, there exists a* unique *pair $(z^*_{\rm oct},m^*_{\rm oct})$ with $z^*_{\rm oct}>0$, $m^*_{\rm oct}>0$, $E_{D_{4h}}(m^*_{\rm oct},z^*_{\rm oct})=0$, and $\partial_\zeta G_1(m^*_{\rm oct},0,z^*_{\rm oct})=0$. It satisfies the formally verified enclosure*

$$\frac{113979}{100000}<z^*_{\rm oct}<\frac{113980}{100000},\qquad \frac{46999}{100000}<m^*_{\rm oct}<\frac{47004}{100000},$$

with 50-digit value

$$\begin{aligned} z^*_{\rm oct}&=1.1397909761710598142349260944227467781242800921021\ldots,\\ m^*_{\rm oct}&=0.4700258266756617340916144721048975215152513560388 1\ldots. \end{aligned}$$

Moreover, $\Phi(z)=\partial_\zeta G_1(m(z),0,z)$ is strictly negative for all $z\in(1/\sqrt{3},z^*_{\rm oct})$ (i.e. $m>m^*_{\rm oct}$) and strictly positive for all $z\in(z^*_{\rm oct},z_{\max})$ (i.e. $0<m<m^*_{\rm oct}$).

Proof outline. For (i), $N(z)$ is strictly decreasing and $D(z)$ is strictly increasing on $(1/\sqrt{3},z_{\max})$, with $N(z)\to 0^+$ as $z\to z_{\max}^-$ and $D(z)\to 0^+$ as $z\to(1/\sqrt{3})^+$, so $m(z)=N(z)/D(z)$ is a strictly decreasing bijection onto $(0,\infty)$. For (ii), differentiating (9) at $\zeta=0$ gives $2\cdot 2^{-3/2}-1/4+mz\cdot\big(-3(1+z^2)^{-5/2}z-3(1+z^2)^{-5/2}z\big)$, which is (13). For (iii), on $z\in(1/\sqrt{3},1]$ we have $m(z)\ge m(1)=1$ and $6z^2(1+z^2)^{-5/2}>1/2>1/\sqrt{2}-1/4$, so $\Phi(z)<0$. On $[1,z_{\max})$, both $m(z)$ and $b(z)=z^2(1+z^2)^{-5/2}$ are positive and strictly decreasing (for $b(z)$, the polynomial factorization $A^2(1+B)^5-B^2(1+A)^5=(B-A)\big[5AB(2AB-1)+(A+B)(5A^2B^2-1)+A^2B^2(A^2+AB+B^2)\big]>0$ for $1\le A<B$ gives strict monotonicity without derivatives). Hence $\Phi(z)$ is strictly increasing on $[1,z_{\max})$. Rational bounds prove $\Phi(1.13979)<0<\Phi(1.13980)$, yielding uniqueness and the stated enclosure. Every step is formalized in `Spatial6C2v/OctahedronPitchfork.lean`. □

Remark 3.2. Because $m^*_{\rm oct}\approx 0.470026<1$, the equal-mass regular octahedron ($m=1$) lies strictly inside the transverse-stable regime where no D_{2d} bifurcation has yet occurred; this matches the Moczurad–Zgliczyński classification [3] at equal masses. As m decreases through $m^*_{\rm oct}$, two symmetric D_{2d} configurations ($\pm\zeta\neq 0$, a tetrahedral equatorial distortion with two axial poles) emerge continuously from the D_{4h} square bipyramid.

4. The Seven-Regime Cascade and 31 Certified Spatial Configurations

Combining analytical bifurcation tracking, homotopy-free polynomial solving on the D_{2d} and C_{4v} subsystems, and 250-digit continuation on the full 4D C_{2v} system reveals a remarkably rich *seven-regime cascade* along the symmetric ray $(1, 1, 1, 1, m, m)$ as m decreases from ∞ to 0^+:

Table 1. Observed seven-regime bifurcation cascade of strictly 3D C_{2v} 6-body central configurations on $(1, 1, 1, 1, m, m)$ (counts modulo vertical reflection $z \mapsto -z$, obtained by 250-digit continuation and subsystem certification; at $m = 1/50$, the lower bound ≥ 31 is formally proved in Lean 4, while global upper-bound completeness of the full 4D system remains open).

Regime	Mass interval	Labeled	Unlabeled	D_{4h}	C_{4v}	D_{2d} / pure C_{2v}
I	$m > m^*_{C_{4v}} \approx 1.112607$	1	1	1	0	0 D_{2d}, 0 pure C_{2v}
II	$m^*_{\mathrm{sn},1} < m < m^*_{C_{4v}}$	5	3	1	2 (4)	0 D_{2d}, 0 pure C_{2v}
III	$m^*_{\mathrm{oct}} < m < m^*_{\mathrm{sn},1}$	13	5	1	2 (4)	0 D_{2d}, 2 pure C_{2v} (8)
IV	$m^*_{\mathrm{sn},2} < m < m^*_{\mathrm{oct}}$	15	6	1	2 (4)	1 D_{2d} (2), 2 pure C_{2v} (8)
V	$m^*_{D_{2d}} < m < m^*_{\mathrm{sn},2}$	23	8	1	2 (4)	1 D_{2d} (2), 4 pure C_{2v} (16)
VI	$m^*_{\mathrm{pf},2} < m < m^*_{D_{2d}}$	27	10	1	2 (4)	3 D_{2d} (6), 4 pure C_{2v} (16)
VII	$0 < m < m^*_{\mathrm{pf},2}$	**31**	**11**	1	2 (4)	3 D_{2d} (6), 5 pure C_{2v} (20)

All six transition thresholds are refined to 250-digit precision in `verify_250digits.py` (with m^*_{oct} formally proved globally unique in Theorem 3.1, and the other five thresholds determined by 250-digit Newton refinement of the fold/pitchfork equations):

- C_{4v} **saddle-node cliff** ($1 \leftrightarrow 5$)**:** $m^*_{C_{4v}} = 1.112607012447870147566341697282999\ldots$, at $(u^*, v^*) = (1.4257918952\ldots, 0.4353667033\ldots)$ in square-centered pole coordinates (residual 6.99×10^{-221}). Strikingly, equal masses $m = 1$ sit only 10.1% below this cliff and already belong to Regime III (13 labeled C_{2v} configurations forming 5 distinct S_6 spatial classes, matching Items 1–5 of the 9 equal-mass spatial 6-body classes in Moczurad–Zgliczyński [3]).
- **Upper pure-C_{2v} saddle-node** ($5 \leftrightarrow 13$)**:** $m^*_{\mathrm{sn},1} = 1.090538405769300966941302777269263\ldots$ (residual 4.06×10^{-261}), birthing two pure-C_{2v} classes (8 labeled roots), one of which passes through the D_{3h} triangular bipyramid with a central body at $m = 1$.
- $D_{4h} \to D_{2d}$ **supercritical pitchfork** ($13 \leftrightarrow 15$)**:** $m^*_{\mathrm{oct}} = 0.47002582667566173409161447210489 7\ldots$, formally proved globally unique in Theorem 3.1 (residual 2.54×10^{-262}).
- **Lower pure-C_{2v} saddle-node** ($15 \leftrightarrow 23$)**:** $m^*_{\mathrm{sn},2} = 0.067129005941829290188058607044080\ldots$ (residual 2.03×10^{-261}), creating a second pair of pure-C_{2v} classes (8 labeled roots).
- **Inner D_{2d} saddle-node birth** ($23 \leftrightarrow 27$)**:** $m^*_{D_{2d}} = 0.037735046108294076224969792630487\ldots$, at $(\zeta^*, z^*) = (0.6880271435\ldots, 0.3578060756\ldots)$ (residual 2.93×10^{-222}), creating two new inner D_{2d} classes (4 labeled roots, pushing D_{2d} from 3 to 7).
- **Inner $D_{2d} \to C_{2v}$ pitchfork** ($27 \leftrightarrow 31$)**:** $m^*_{\mathrm{pf},2} = 0.022299177060560368737006697368962\ldots$, at $(\zeta^*, z^*) = (0.7016538344\ldots, -0.2463131813\ldots)$ along the innermost D_{2d} branch (Class 8 spawning Class 5, residual 1.52×10^{-261}).

Theorem 4.1 (31 Certified Strictly 3D C_{2v} 6-Body Central Configurations at $m = 1/50$)**.** *For the rational mass vector $(m_1, m_1, m_2, m_2, m_5, m_6) = (1, 1, 1, 1, \frac{1}{50}, \frac{1}{50})$, there exist at least **31** pairwise distinct (modulo $z \mapsto -z$) strictly three-dimensional C_{2v} 6-body central configurations, each satisfying all 18 Cartesian Newtonian force-balance equations* (1)*. They partition into 11 geometric orbits under $\langle \sigma_p, \sigma_s \rangle$: 5 pure C_{2v} classes (orbit size 4 each, 20 configurations), 3 D_{2d} classes (orbit size 2*

each, 6 configurations), 2 C_{4v} classes (orbit size 2 each, 4 configurations), and 1 D_{4h} class (orbit size 1, 1 configuration).

Table 2 lists all 11 certified representatives at $m = 1/50$, their orbit sizes, Lean 4 Banach–Krawczyk box radii δ, contraction factors, and 250-digit 18-equation Cartesian Newtonian residuals.

TABLE 2. All 11 certified geometric classes (= 31 labeled strictly 3D C_{2v} central configurations) at $(1, 1, 1, 1, 1/50, 1/50)$.

Class	Stratum (orbit)	y	z_1	z_2	z_5	z_6	Lean δ	Contr.	18-eq err
C1	C_{2v} (4)	0.97759516	+0.66884697	−0.69075190	+1.75416546	+0.43632806	10^{-6}	0.074	$< 10^{-245}$
C2	C_{2v} (4)	0.98649354	−0.66081113	+0.69598349	−1.59206324	−1.92517315	10^{-7}	0.063	$< 10^{-245}$
C3	C_{2v} (4)	0.98999175	+0.69009188	−0.67703727	+0.46656006	−1.77202135	10^{-5}	0.064	$< 10^{-245}$
C4	C_{2v} (4)	0.99269128	+0.68348640	−0.70139913	+1.77950987	+0.01176298	10^{-6}	0.118	$< 10^{-245}$
C5	C_{2v} (4)	0.99413804	+0.69885513	−0.70049190	+0.32812340	−0.16444606	10^{-8}	0.055	$< 10^{-245}$
C6	D_{2d} (2)	1.00000000	+0.68310997	−0.68310997	−1.78812299	+1.78812299	10^{-5}	0.131	$< 10^{-245}$
C7	D_{2d} (2)	1.00000000	+0.69126067	−0.69126067	−0.45969738	+0.45969738	10^{-6}	0.108	$< 10^{-245}$
C8	D_{2d} (2)	1.00000000	+0.70265201	−0.70265201	−0.23493280	+0.23493280	10^{-8}	0.108	$< 10^{-245}$
C9	C_{4v} (2)	1.00000000	−0.00622792	−0.00622792	+1.24954399	−0.00396045	10^{-5}	0.083	$< 10^{-245}$
C10	D_{4h} (1)	1.00000000	0.00000000	0.00000000	+1.25768466	−1.25768466	1D IVT	exact	$< 10^{-245}$
C11	C_{4v} (2)	1.00000000	−0.01250564	−0.01250564	+1.10910935	+1.39201887	10^{-7}	0.065	$< 10^{-245}$

Theorem 4.2 (Machine-Certified Symmetry Breaking to 31 Pairwise Non-Similar 3D Polyhedra)**.** *For the symmetry-breaking rational mass vector*

$$(m_1, m_1, m_2, m_2, m_5, m_6) \;=\; \left(1,\, 1,\, \frac{9999}{10000},\, \frac{9999}{10000},\, \frac{1999}{100000},\, \frac{1998}{100000}\right),$$

*there exist at least **31** strictly three-dimensional C_{2v} 6-body central configurations, each satisfying all 18 Cartesian Newtonian equations* (1)*, that are* pairwise non-similar and non-congruent as unlabeled 3D polyhedra*: for every $i \neq j$, there is no dilation $s > 0$ and no permutation $\sigma \in S_6$ satisfying $\|q^{(i)}_{\sigma(a)} - q^{(i)}_{\sigma(b)}\|^2 = s\,\|q^{(j)}_a - q^{(j)}_b\|^2$ for all $a, b \in \{1, \ldots, 6\}$ (in particular, no similarity in $\mathbb{R}_{>0} \times (O(3) \ltimes \mathbb{R}^3) \times S_6$ maps $q^{(j)}$ to $q^{(i)}$).*

Proof outline. Each of the 31 perturbed configurations is individually certified inside a 4D rational box of radius $\delta = 10^{-8}$ via the mass-parameterized Banach–Krawczyk engine `certify_c2vAsym_root` in `Spatial6C2v/AsymmetricCert.lean`. For any 6-body configuration $p \in (\mathbb{R}^3)^6$, the normalized squared-distance count $\#\{(a, b) \in \{1, \ldots, 6\}^2 : 2\|p_a - p_b\|^2 / \sum_{u,v} \|p_u - p_v\|^2 < t\}$ is strictly invariant under $\mathbb{R}_{>0} \times (O(3) \ltimes \mathbb{R}^3) \times S_6$ for every threshold $t > 0$. Evaluating rational interval enclosures for the eight distinct squared-distance classes of each box, we exhibit an explicit rational cut witness $t_{ij} \in \mathbb{Q}_{>0}$ for each of the $\binom{31}{2} = 465$ pairs $i < j$ whose interval lower count on one polyhedron strictly exceeds the interval upper count on the other. All 31 root certificates and all 465 cut witnesses are verified by kernel reduction (`decide +kernel`) in `Spatial6C2v/AsymmetricThirtyOneCertified.lean`. Independently, all 31 asymmetric polyhedra are refined to 250-digit precision in `verify_250digits.py` (18-equation residual $< 10^{-240}$, minimum ℓ_∞ gap 2.6918×10^{-6} across all 465 normalized 15-distance spectra). □

5. HOMOTOPY-FREE POLYNOMIAL CERTIFICATION WITH `PANDSYSROOTS V0.34.0`

To certify real solutions across all bifurcation regimes without relying on homotopy continuation or initial guesses, we cast all three 2D high-symmetry 6-body subsystems (D_{2d}, C_{4v}, and D_{2h}) as compact 5-variable sparse polynomial systems by introducing inverse distances as algebraic variables:

- D_{2d} **system in** (ζ, z, q, s, t) with $q = (2+4\zeta^2)^{-1/2}$, $s = (1+(\zeta-z)^2)^{-1/2}$, $t = (1+(\zeta+z)^2)^{-1/2}$:

$$
\begin{aligned}
&2q^2 + 4\zeta^2 q^2 - 1 = 0, \quad s^2(1+(\zeta-z)^2) - 1 = 0, \quad t^2(1+(\zeta+z)^2) - 1 = 0,\\
&2\zeta q^3 - \tfrac{1}{4}\zeta + mz(t^3 - s^3) = 0,\\
&m + (8-4m)z^3(s^3+t^3) + 8\zeta z^2(t^3 - s^3) - z^3 - 8z^3q^3 = 0.
\end{aligned}
$$

- C_{4v} **system in** (u, v, s, t, w) for arbitrary pole masses $(1, 1, 1, 1, m_5, m_6)$ with square at $z = 0$, poles at $u > v$, $s = (1+u^2)^{-1/2}$, $t = (1+v^2)^{-1/2}$, $w = (u-v)^{-1}$: five sparse polynomials of degree ≤ 4 with total mixed-volume / BKK bound 144.
- D_{2h} **generalized octahedron in** (y, z, u, v, w) for masses $(m_1, m_1, m_2, m_2, m_3, m_3)$ with inverse distances $u = (1+y^2)^{-1/2}$, $v = (1+z^2)^{-1/2}$, and $w = (y^2+z^2)^{-1/2}$: five sparse polynomials with bound up to 504.

We solve all 14 benchmarks across these three subsystems using `pandsysroots v0.34.0`, a Rust polynomial system solver based on simultaneous sweeps of the Pandrosion rectangle step and the law of residues without homotopy. Every certified real positive ball is reconstructed into $\mathbb{R}^{18}$ and verified directly against all 18 Cartesian Newtonian equations (1).

TABLE 3. All 14 `pandsysroots v0.34.0` benchmarks across D_{2d}, C_{4v}, and D_{2h} (single-threaded Rust `--release`).

Subsystem	Benchmark	Cert. / Bound	Labeled	Max 18-eq err	Time (s)
D_{2d}	$m = 1.00$ (equal masses)	372 / 456	1	4.44×10^{-16}	4.38
D_{2d}	$m = 0.48$ (above m^*_{oct})	372 / 456	1	4.44×10^{-16}	5.13
D_{2d}	$m = 0.46$ (below m^*_{oct})	372 / 456	3	4.44×10^{-16}	4.64
D_{2d}	$m = 0.04$ (above $m^*_{D_{2d}}$)	372 / 456	3	2.22×10^{-16}	4.65
D_{2d}	$m = 0.035$ (below $m^*_{D_{2d}}$)	372 / 456	7	4.44×10^{-16}	4.83
D_{2d}	$m = 0.02$ (31-root regime)	372 / 456	7	1.33×10^{-15}	5.29
C_{4v}	$m = 1.12$ (above $m^*_{C_{4v}}$)	102 / 144	1	2.22×10^{-16}	3.18
C_{4v}	$m = 1.11$ (below $m^*_{C_{4v}}$)	102 / 144	5	8.88×10^{-16}	4.34
C_{4v}	$m = 1.00$ (equal masses)	102 / 144	5	1.33×10^{-15}	4.81
C_{4v}	$m = 0.02$ (31-root regime)	100 / 144	5	1.33×10^{-15}	5.00
C_{4v}	asym ($m_5 = 0.5, m_6 = 0.2, u > v$)	102 / 144	5	8.88×10^{-16}	4.18
D_{2h}	$(1, 1, 1)$ regular octahedron	286 / 360	1	2.22×10^{-16}	3.33
D_{2h}	$(1, 2, 5)$ unequal masses	474 / 504	1	2.66×10^{-15}	5.43
D_{2h}	$(1, 0.3, 0.05)$ extreme ratio	504 / 504	1	1.11×10^{-16}	0.87

Table 3 cleanly confirms every sharp subsystem jump: $1 \to 3$ across $m^*_{\mathrm{oct}} \in (0.46, 0.48)$, $3 \to 7$ across $m^*_{D_{2d}} \in (0.035, 0.04)$, $1 \to 5$ across $m^*_{C_{4v}} \in (1.11, 1.12)$, persistence of all 5 non-congruent asymmetric C_{4v} roots at $(m_5, m_6) = (0.5, 0.2)$, and uniqueness of the D_{2h} generalized octahedron (reaching full 504/504 complex root certification in 0.87 s at $(1, 0.3, 0.05)$). Note that on D_{2d} (372/456) and C_{4v} (100–102/144), the mixed-volume / BKK bound exceeds the number of isolated finite affine complex roots certified by `pandsysroots` (reflecting roots at infinity or higher-multiplicity degeneracies of the inverse-distance lifting), so those counts certify lower bounds rather than global affine completeness.

6. Machine-Checked Lean 4.36.0 Formalization and Axiom Audit

The analytical proofs, the 31-root symmetric theorem at $m = 1/50$, and the 31-polyhedron asymmetric non-similarity theorem are formalized in Lean 4.36.0 with Mathlib4 across seven modular files in `lean/Spatial6C2v/`:

(1) `NewtonianModel.lean`: Defines the full 18-equation Cartesian Newtonian predicate `IsNewtonian6CC` and proves `c2v_reconstruction_18eq`, `swapPoles`, `swapPairs`, and the exact embeddings `d4hToC2v`, `d2dToC2v`, and `c4vToC2v`.

(2) `OctahedronPitchfork.lean`: Proves `regular_octahedron_is_cc`, `d4h_exists_unique`, `d2dG1_hasDerivAt_zero`, `d4h_pitchfork_exists_unique`, and `d4h_pitchfork_enclosure` (Theorem 3.1).

(3) `BanachEngine.lean`: Proves algebraic $r^{-3/2}$ enclosure lemmas (`invCube_lower`, `invCube_upper`, `dInvCube_bounds`) and a reusable d-dimensional Banach–Krawczyk fixed-point theorem `banach_krawczyk_fin` on $(\mathrm{Fin}\, d \to \mathbb{R}, \|\cdot\|_\infty)$ avoiding all determinant calculations: invertibility of the rational preconditioner A is guaranteed by a single Neumann-series bound $\|I - B_0 A\|_\infty \leq 1/2$.

(4) `IntervalCert.lean`: Implements sound rational interval arithmetic `QIvl`, proves the exact 4D directional derivative `c2v50_hasDerivAt_seg` of the C_{2v} map at $m = 1/50$, and establishes the single-call certification theorem `certify_c2v50_root`, converting any finite rational certificate `C : C2v50Cert` satisfying `C.valid` into an exact `C2vCentralConfig6 1 1 (1/50) (1/50)` inside the closed box $\overline{B}_\infty(x_0, \delta)$.

(5) `ThirtyOneRootsCertified.lean`: Certifies Class 10 (D_{4h}) by 1D IVT and Classes 1–9 and 11 by `certify_c2v50_root`, expands all orbits via `swapPoles` and `swapPairs` to 31 explicit roots, verifies pairwise box disjointness modulo $z \mapsto -z$ across all $\binom{31}{2} = 465$ pairs via kernel reduction (`decide +kernel`, not `native_decide`), and proves `thirty_one_distinct_3D_C2v_6body_central_configurations` and `thirty_one_newtonian_18eq` (Theorem 4.1).

(6) `AsymmetricCert.lean`: Generalizes the analytical Jacobian and rational interval certification engine (`certify_c2vAsym_root`) to the symmetry-breaking rational masses $(1, 1, 9999/10000, 9999/10000, 1999/100000, 1998/100000)$, defines full 3D similarity `AreSimilar6` modulo $\mathbb{R}_{>0} \times O(3) \times S_6$, and proves the rational normalized squared-distance cut-witness theorem `not_areSimilar6_of_cut`.

(7) `AsymmetricThirtyOneCertified.lean`: Certifies all 31 asymmetric configurations inside boxes of radius $\delta = 10^{-8}$ and verifies all $\binom{31}{2} = 465$ rational cut witnesses via kernel reduction (`decide +kernel`), proving `thirty_one_asymmetric_non_similar_3D_polyhedra` (Theorem 4.2).

The audit script `Spatial6C2v/Audit.lean` executes `#print axioms` on all eighteen public theorems and definitions. Every single theorem depends strictly on Lean's three foundational axioms `[propext, Classical.choice, Quot.sound]`, with zero `sorry`, zero `admit`, zero custom axioms, and zero `native_decide` (`Lean.ofReduceBool` / `Lean.trustCompiler`).

7. Conclusion and Open Questions

Moving from $N = 5$ to $N = 6$ in three dimensions crosses the Dziobek barrier and triggers an explosive growth in spatial central configuration multiplicity even within the single C_{2v} symmetry class: whereas spatial C_{2v} 5-body configurations exhibit a 3-regime cascade up to a handful of shapes, the 6-body C_{2v} family exhibits an observed seven-regime cascade up to at least 31 machine-certified strictly 3D central configurations (and 31 machine-certified pairwise non-similar 3D polyhedra modulo $\mathbb{R}_{>0} \times O(3) \times S_6$ upon breaking mass equality), anchored by the exact $D_{4h} \to D_{2d}$ pitchfork at $m^*_{\mathrm{oct}} = 0.4700258267\ldots$.

Two natural problems remain open: (1) certifying global *upper* bounds (exact completeness) on the full 4D C_{2v} family (and formally proving the remaining five numerical bifurcation thresholds

$m^*_{C_{4v}}$, $m^*_{\mathrm{sn},1}$, $m^*_{\mathrm{sn},2}$, $m^*_{D_{2d}}$, $m^*_{\mathrm{pf},2}$), as well as checking whether any further bifurcations occur as $m \to 0^+$ (the restricted $4+2$ spatial co-orbital limit); and (2) extending the inverse-distance polynomial formulation to asymmetric non-C_{2v} 6-body families in $\mathbb{R}^3$ to probe Smale's sixth problem beyond Dziobek.

References

[1] A. Albouy, H. E. Cabral, and A. A. Santos, *Some problems on the classical n-body problem*, Celestial Mech. Dynam. Astronom. **113** (2012), 369–375.

[2] M. Hampton and A. Jensen, *Finiteness of spatial central configurations in the five-body problem*, Celestial Mech. Dynam. Astronom. **109** (2011), 321–332.

[3] M. Moczurad and P. Zgliczyński, *Central configurations in spatial n-body problems for $n = 5, 6$ with equal masses*, Celestial Mech. Dynam. Astronom. **132** (2020), Article 56.

[4] R. Moeckel, *Central configurations*, in *Central Configurations, Periodic Orbits, and Hamiltonian Systems*, Adv. Courses Math. CRM Barcelona, Birkhäuser, 2015, pp. 105–167.

[5] D. G. Saari, *Collisions, Rings, and Other Newtonian N-Body Problems*, CBMS Regional Conference Series in Mathematics, vol. 104, AMS, 2005.

[6] S. Smale, *Mathematical problems for the next century*, Math. Intelligencer **20** (1998), no. 2, 7–15.