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Quasi-paramodular forms for GSp_4: a newform theorem and a normalized seesaw comparison

Ramin Takloo-Bighash

Abstract

This paper gives a correct proof of Okazaki's minimal-level and multiplicity-one assertions for quasi-paramodular newforms of irreducible generic representations of $\mathrm{GSp}_4(F)$, with arbitrary smooth central character. The original proof has errors in its character projection, Hecke representatives, and theta-transfer formulas. Our proof follows Roberts–Schmidt and Okazaki, replaces the restriction-kernel argument using Rösner–Weissauer, and removes unitarity from the Hecke argument by a finite-dimensional operator identity. A normalized split seesaw calculation, based on Prasad–Takloo-Bighash and the Gan–Takeda Whittaker–Shalika correspondence, compares the Novodvorsky functional equation with the Friedberg–Jacquet equation on GL_4. It proves equality of conductor exponents in all cases. For the precise Whittaker character and conjugation operator used here, the analytic epsilon factor is $\omega_\pi(-1)$ times the parameter epsilon factor; multiplying that operator by $\omega_\pi(-1)$ gives the standard normalization. The same sign corrects the explicit split-theta construction. A sharp parameter conductor bound and an upper bound for higher-level quasi-paramodular dimensions are proved. Exact oldform dimensions and a complete ramified-center basis theorem remain open in this note. The failed explicit GL_4 test-vector transfer is documented in Section 16 and is not used in the seesaw proof.

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1 Introduction

This paper gives a correct proof of the minimal-level and multiplicity-one assertions in Okazaki's quasi-paramodular newform theorem for GSp_4, with arbitrary smooth central character. We retain his family of compact subgroups and prove that the first nonzero fixed-vector space is a line at the conductor of the spin parameter. We also correct the normalization of the dual zeta integral: with the Whittaker character and conjugation operator used in [11], the printed epsilon-factor identity requires the additional factor $\omega_\pi(-1)$. The purpose is therefore both to supply the missing proof and to state the local-factor identity with its correct sign.

1.1 Newforms and multiplicity one

The classical starting point is the theory of newforms and Hecke operators developed by Atkin and Lehner [1]. Casselman [8] gave its local representation-theoretic form for GL_2 over a nonarchimedean local field. For an infinite-dimensional irreducible admissible representation, the first nonzero space of vectors fixed by the standard K_1 congruence subgroups is one-dimensional. Its level is the conductor exponent in the local functional equation, and Casselman's theorem also determines the dimensions at higher levels. The distinguished vector thus records both a minimality property of the representation and an invariant of its local factors.

Jacquet, Piatetski-Shapiro, and Shalika [9] extended this conductor and newvector theorem to irreducible generic representations of GL_n. Their essential Whittaker vector links the congruence-level problem to local Rankin–Selberg integrals. More precisely, if $K_1^{(n)}(\mathfrak{p}^m)$ is the subgroup of $\mathrm{GL}_n(\mathfrak{o})$ whose last row is congruent to $(0,\dots,0,1)$ modulo $\mathfrak{p}^m$, then

$$\dim \pi^{K_1^{(n)}(\mathfrak{p}^{a(\pi)})} = 1, \qquad \pi^{K_1^{(n)}(\mathfrak{p}^m)} = 0 \quad (0 \le m < a(\pi)),$$

where $a(\pi)$ is the epsilon-factor conductor for an additive character of conductor $\mathfrak{o}$, and level zero means $\mathrm{GL}_n(\mathfrak{o})$. Jacquet [5] subsequently supplied a corrected proof of the existence of the essential vector; Matringe [6] gave a constructive proof using mirabolic restriction. These developments separate three tasks: constructing a vector, proving uniqueness at its first level, and identifying the local factors it realizes.

A related multiplicity-one theorem concerns the Whittaker functional itself. The work of Gelfand–Kazhdan, Shalika, and Rodier on uniqueness of Whittaker models supplies the background for the preceding constructions; see [2, 3]. For a fixed nondegenerate character ψ_U of a maximal unipotent subgroup $U(F)$, the assertion is

$$\dim \mathrm{Hom}_{U(F)}(\pi, \psi_U) \le 1.$$

This uniqueness does not imply one-dimensionality of the space fixed by a prescribed compact subgroup. Multiplicity one at minimal level is an additional theorem, and it is this second assertion that is studied here. Explicit formulas for the distinguished Whittaker function provide further information: Casselman–Shalika [4] obtained the unramified formula, and Miyauchi [7] determined the diagonal values of GL_n Whittaker newforms in terms of their local L-factors.

For GSp_4, Roberts and Schmidt [10] developed the corresponding theory using paramodular subgroups. Their Generic Main Theorem, [10, Theorem 7.5.4], proves existence, uniqueness at the first nonzero level, and a normalization whose Novodvorsky integral is the spin L-factor, for every irreducible admissible generic representation with trivial central character. Their Corollary 7.5.5 identifies that level with the analytic conductor and expresses the epsilon factor through the Atkin–Lehner eigenvalue. They also prove exact oldform dimension and basis theorems [10, Theorems 7.5.6 and 7.5.7]. These analytic assertions should be distinguished from their original parameter comparison, stated for non-supercuspidal representations in [10, Theorem 7.5.9]. The unramified-central-character case follows by an unramified similitude twist. A ramified central character cannot occur on an ordinary paramodular fixed vector, because the paramodular group contains the scalar units. It therefore requires a different choice of compact subgroups.

1.2 Okazaki's theorem and the result of this paper

Okazaki [11] introduced quasi-paramodular groups adapted to the conductor of the central character. His Main Theorem asserts that the least quasi-paramodular level of a generic representation is the conductor of its spin parameter, that the space at this level has dimension one, and that a

normalized vector realizes the spin L-factor and its dual epsilon-factor identity. His proof follows the Roberts–Schmidt strategy: restriction to the Klingen parabolic and a functional equation give a uniqueness criterion; Hecke operators handle the case with trivial spin L-factor; split theta lifting constructs vectors in the other cases; and a further theta transfer to GL_4 is used to compare conductors and root numbers.

Several intermediate formulas in that proof are false as written, and the final transfer does not establish the required comparison. The specific errors and their replacements are described below. The present paper proves Okazaki's existence, minimal-level, and multiplicity-one assertions, and proves the local-factor identities with the necessary correction to the epsilon normalization. It does not claim an exact oldform dimension formula or a complete oldform basis for arbitrary ramified central character.

1.3 Conventions and main theorem

Let F be a finite extension of $\mathbf{Q}_p$, with valuation ring $\mathfrak{o}$, maximal ideal $\mathfrak{p} = \varpi\mathfrak{o}$, and residue cardinality q. Normalize $|\varpi| = q^{-1}$. Fix an additive character ψ of conductor $\mathfrak{o}$, additive Haar measure with $\mathrm{vol}(\mathfrak{o}) = 1$, and multiplicative Haar measure with $\mathrm{vol}(\mathfrak{o}^\times) = 1$. Use the alternating matrix

$$J = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad G = \{g : {}^t g J g = \mu(g) J\}.$$

The Whittaker character is $\psi(x + y)$ on the upper unipotent matrix whose entries $12, 23, 34$ are respectively $-x, y, x$. Let π be irreducible, admissible and generic, with central character ω, and put $e = a(\omega)$.

For $m \geq 2e$, set $\ell = m - e$ and define Okazaki's group

$$K(m; e) = G \cap \begin{pmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{p}^{-\ell} \\ \mathfrak{p}^{\ell} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^{\ell} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^{m} & \mathfrak{p}^{\ell} & \mathfrak{p}^{\ell} & \mathfrak{o} \end{pmatrix}, \qquad \det k \in \mathfrak{o}^\times.$$

For $e > 0$, let $K_1(m; e) = \{k \in K(m; e) : k_{44} \in 1 + \mathfrak{p}^e\}$. For $e = 0$, by definition $K_1(m; 0) = K(m; 0)$, the ordinary paramodular group. Write

$$V_\pi(m) = \mathcal{W}(\pi, \psi)^{K_1(m;e)}.$$

Let ϕ_π denote the four-dimensional spin Weil–Deligne representation attached to π by Gan–Takeda, and write

$$n_\pi = a(\phi_\pi), \qquad \varepsilon(s, \phi_\pi, \psi) = \varepsilon_\pi q^{-n_\pi(s-1/2)}.$$

Corrected target assertion. The newform assertion, with the sign required by the conventions above, is

$$V_\pi(m) = 0 \quad (2e \leq m < n_\pi), \qquad \dim V_\pi(n_\pi) = 1,$$

together with a uniquely normalized vector W_π satisfying

$$\Xi(s, W_\pi) = L(s, \phi_\pi), \qquad \Xi(s, \mathcal{D}_{n_\pi} W_\pi) = q^e \omega(-1) \varepsilon_\pi L(s, \phi_\pi^\vee). \tag{1.1}$$

The integral Ξ and the operator $\mathcal{D}_m$ are defined below. Equivalently, with $\mathcal{D}_m^{\mathrm{std}} = \omega(-1)\mathcal{D}_m$, the second identity has the factor $q^e \varepsilon_\pi$ without the additional sign. This is the analogue of the generic

newform and local-factor statements of Roberts–Schmidt, with ramified central character allowed. It does not by itself include a ramified-central-character version of their entire oldform basis and dimension theorem.

What may be imported. Roberts–Schmidt's ordinary paramodular theory is used as a theorem [10]. The equality of the Novodvorsky spin L-factor with $L(s, \phi_\pi)$ is supplied by Takloo-Bighash's computation, in the form explicitly recorded by Loeffler, Proposition 4.3 [12, 13]. Gan–Takeda identifies *Shahidi* local factors with parameter factors [14]. That statement alone does not identify the epsilon factor of the Novodvorsky functional equation.

The proved statement. Write $\varepsilon_{\mathrm{Nov}}(s, \pi, \psi) = \varepsilon'_\pi q^{-n'_\pi(s-1/2)}$ for the factor in Proposition 4.2. The main conclusion of the corrected argument is the following.

Theorem 1.1. *Every irreducible admissible generic π has a nonzero quasi-paramodular vector. Its least level is n'_π, and $\dim V_\pi(n'_\pi) = 1$. There is a unique vector $W_\pi \in V_\pi(n'_\pi)$ such that*

$$\Xi(s, W_\pi) = L(s, \phi_\pi), \qquad \Xi(s, \mathcal{D}_{n'_\pi} W_\pi) = q^e \varepsilon'_\pi L(s, \phi_\pi^\vee).$$

For every such π,

$$n'_\pi = a(\phi_\pi), \qquad \varepsilon'_\pi = \omega(-1)\varepsilon(1/2, \phi_\pi, \psi).$$

Thus the least level is the parameter conductor, and the conjugation operator $\mathcal{D}^{\mathrm{std}}_{n'_\pi} = \omega(-1)\mathcal{D}_{n'_\pi}$ gives the standard parameter epsilon factor. For $e > 0$ and $d = m - n'_\pi \geq 0$,

$$\dim V_\pi(m) \leq (d+1)(d+2)/2.$$

The fixed-vector proof is assembled after the Hecke calculation; Theorem 15.1 then supplies the unrestricted parameter comparison. The upper bound for higher levels is not an exact dimension formula.

1.4 Errors in the original proof and their repair

The following points explain why a new proof is needed. References to pages and equation numbers of [11] refer throughout to version 2.

The character in the level-lowering projection. In Step 3 of [11, §6, p. 23], the proposed operator is the unweighted average over $H = K^c(m_\pi - 1; e)$ on a representation of central character ω^{-1}. For $e > 0$ this operator is identically zero. Indeed H contains uI_4 for every $u \in \mathfrak{o}^\times$, and Haar invariance gives

$$A := \int_H \pi^\iota(k)\, dk = \omega(u)^{-1} A.$$

Choosing u with $\omega(u) \neq 1$ proves $A = 0$. The required projection instead has the weight $\omega(k_{33})$; see Proposition 2.2 and Section 12. This correction is essential to the recurrence argument.

Hecke representatives and restriction. The two occurrences of η and η^{-1} in [11, (6.3), p. 22] are interchanged. At the boundary level, the printed first family lies in a single right coset, rather than giving q^4 distinct representatives. Section 11 proves the corrected decompositions by Gaussian factorization and subgroup indices; Section 12 then carries out the Hecke argument without a unitarity assumption. The restriction argument also needs justification at its kernel: Section 8 gives replacement proofs for the steps corresponding to Propositions 5.8 and 5.10 of [11], using Rösner–Weissauer's Klingen-coinvariant results [16] and the relevant compact character type.

The Gauss-sum sign. Immediately before [11, (7.11), p. 33], the product of the two Gauss integrals is taken to be q^{-e}. For $e = a(\omega) > 0$, with the same additive character in both factors and additive measure of volume one on $\mathfrak{o}$, the correct identity is

$$G_\psi(\omega)G_\psi(\omega^{-1}) = \omega(-1)q^{-e}, \qquad G_\psi(\omega) = \int_{\mathfrak{o}^\times} \omega(u)\psi(\varpi^{-e}u)\,du.$$

Thus the printed identity already fails for a tame quadratic character over $\mathbf{Q}_3$ with value -1 at -1. Sections 7 and 9 retain the character and Fourier signs. The normalized seesaw in Section 15 proves the resulting local-factor comparison for every generic representation. In particular, the factor $\omega(-1)$ in (1.1) corrects the right-hand side of [11, (1.3)] when the original conjugation operator is retained; it can equivalently be absorbed into $\mathcal{D}_m^{\mathrm{std}}$.

The transfer to the general linear group. The displayed Weil formulas in [11, §8, p. 35] are incompatible with the Fourier–Weyl relation. On $M_{3\times4}(F)$, Fourier transformation of $\Phi(a^{-1}z)$ introduces the Jacobian $|\det a|^4$. Retaining this inverse-matrix argument forces its scalar exponent to be -2, whereas the printed exponent is $+2$. Section 16 explains why altering this scalar alone does not validate the proposed transfer or its point evaluation. The comparison is proved instead in Section 15 by an identity between Novodvorsky integrals and compact orbital averages of Friedberg–Jacquet integrals. This replacement compares the same vectors and their Weyl transforms, so that its normalization constant cancels from the functional equation.

These errors invalidate the indicated steps of the original argument; they are not counterexamples to its minimal-level or multiplicity-one assertions. Theorem 1.1 proves those assertions, with the corrected dual normalization stated explicitly.

1.5 Proof strategy and organization

The method follows the newform strategy of Roberts–Schmidt and Okazaki. Compact character projections first supply a vector at some level. Rösner–Weissauer's results on the Klingen coinvariants then repair restriction injectivity. Translated zeta integrals and the functional equation prove minimality and multiplicity at most one. The split-theta calculation treats every ramified-center representation with nontrivial spin L-factor. For the remaining $L = 1$ case, corrected Hecke cosets and an operator polynomial prove existence of a normalized vector at the analytic conductor. The parameter comparison is then proved separately by a normalized seesaw: the twisted Jacquet-module geometry is due to Prasad–Takloo-Bighash [19]; the specific Whittaker–Shalika map and generic theta correspondence are due to Gan–Takeda [20]; and the local functional equation on GL_4 is due to Friedberg–Jacquet [21], in the arbitrary-character form recorded by Xue–Zhang [22]. The new calculation here is the simultaneous normalization of the two period families and their relative Weyl actions. It retains a central character sign that is invisible for square central characters. No claim of novelty is made for the elementary conductor bound, which is derived from the standard Artin conductor formula.

The proof is organized in two stages. Sections 2–8 establish the compact-character, conductor, functional-equation, and restriction tools. The split theta construction in Section 9 and the support and Hecke calculations in Sections 10–12 establish the analytic newform theorem, assembled in Section 14. Section 13 first records the parameter comparison for square central character; Section 15 then proves it without that restriction. The obstruction to the original explicit transfer is documented in Section 16. Section 17 separates the proved newform assertions from the remaining oldform problem.

2 The central character and compact projectors

Proposition 2.1. *For $e > 0$, the entry k_{44} is a unit for $k \in K(m;e)$, and $k \mapsto k_{44} \bmod \mathfrak{p}^e$ is a surjective homomorphism to $\mathfrak{o}^\times/(1+\mathfrak{p}^e)$, with kernel $K_1(m;e)$. Consequently*

$$V_\pi(m) = \{v : \pi(k)v = \omega(k_{44})v \text{ for all } k \in K(m;e)\}.$$

Proof. For $d = \mathrm{diag}(\varpi^\ell, 1, 1, 1)$, the lattice $dK(m;e)d^{-1}$ has integral entries. Modulo $\mathfrak{p}$, the resulting pattern is

$$\begin{pmatrix} * & 0 & 0 & * \\ * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix}.$$

The unit determinant implies that the last diagonal entry is a unit. In a product, the off-diagonal contributions to entry 44 lie in $\mathfrak{p}^{m-\ell} + \mathfrak{p}^\ell = \mathfrak{p}^e$. Scalar units give surjectivity. For any k, choose a scalar unit u with $u \equiv k_{44} \pmod{\mathfrak{p}^e}$. Then $u^{-1}k \in K_1(m;e)$; central character equivariance proves the assertion. □

For an open compact group H and a smooth character ρ, the correct character projector is

$$P_{H,\rho}v = \frac{1}{\mathrm{vol}(H)} \int_H \rho(h)^{-1}\pi(h)v \, dh. \tag{2.1}$$

A change of variable proves that its image is precisely the ρ-isotypic subspace and that it is the identity there. In particular this functor is exact.

Define

$$j_m = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ \varpi^m & 0 & 0 & 0 \\ 0 & -\varpi^m & 0 & 0 \end{pmatrix}.$$

Then $\mu(j_m) = \varpi^m$ and $j_m^2 = \varpi^m I_4$. Put $K^c(m;e) = j_m K(m;e) j_m^{-1}$. Its matrix pattern is

$$\begin{pmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{p}^{-e} & \mathfrak{p}^{-m} \\ \mathfrak{p}^\ell & \mathfrak{o} & \mathfrak{o} & \mathfrak{p}^{-e} \\ \mathfrak{p}^m & \mathfrak{p}^e & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^m & \mathfrak{p}^m & \mathfrak{p}^\ell & \mathfrak{o} \end{pmatrix}.$$

For $e > 0$, $K_1^c(m;e) = \{k \in K^c(m;e) : k_{33} \in 1 + \mathfrak{p}^e\}$; for $e = 0$ it is the entire paramodular group.

Proposition 2.2. *On a representation of central character ω^{-1}, projection to the coquasi-paramodular space is*

$$\frac{1}{\mathrm{vol}\, K^c(m;e)} \int_{K^c(m;e)} \omega(k_{33})\pi^\vee(k)\, dk = \frac{1}{\mathrm{vol}\, K_1^c(m;e)} \int_{K_1^c(m;e)} \pi^\vee(k)\, dk$$

when $e > 0$. The unweighted integral over $K^c(m;e)$ is zero.

Proof. Apply Proposition 2.1 after conjugation and use the decomposition into scalar-unit cosets of K_1^c. For the last assertion, choose $u \in \mathfrak{o}^\times$ with $\omega(u) \neq 1$. Multiplying the integration variable by uI_4 multiplies the operator by $\omega(u)^{-1}$, while preserving the integral. □

This corrects the level-lowering average in Okazaki's §6. An unweighted average over the larger group cannot be used and then interpreted as projection to a nontrivial character.

3 A sharp conductor bound from the parameter

Theorem 3.1. *Let (r, N) be a Frobenius-semisimple, four-dimensional symplectic Weil–Deligne representation with multiplier ω, where $e = a(\omega) > 0$. Set*

$$d = \dim(\ker N)^I = \deg L(s, r, N)^{-1}, \qquad n = a(r, N).$$

Then

$$d \le 2, \qquad n \ge 2e + 2 - d. \tag{3.1}$$

Moreover,

$$d = 2 \quad \Longleftrightarrow \quad n = 2e.$$

In that case $N = 0$, and there are unramified characters α, β such that

$$r \simeq \alpha \oplus \beta \oplus \omega\alpha^{-1} \oplus \omega\beta^{-1}.$$

In particular $L(s, r, N) = 1$ implies $n \ge 2e + 2$, while $n = 2e + 1$ forces $d = 1$.

Proof. If a ramification subgroup $H \subset I$ acts nontrivially through ω, then for $v, w \in V^H$,

$$\langle v, w\rangle = \langle r(h)v, r(h)w\rangle = \omega(h)\langle v, w\rangle.$$

Thus V^H is totally isotropic and has dimension at most two. For $H = I$, this proves $d \le \dim V^I \le 2$. Applying the same observation at every positive upper ramification level gives

$$\mathrm{Swan}(r) \ge 2\,\mathrm{Swan}(\omega) = 2(e - 1).$$

The standard Weil–Deligne conductor formula [10, §2.4, (2.50)] gives, without a unitarity assumption,

$$a(r, N) = a(r) + \dim V^I - \dim(\ker N)^I = 4 - d + \mathrm{Swan}(r).$$

This proves (3.1).

Suppose $d = 2$. Then $L = V^I$ is Lagrangian and $N|_L = 0$. The alternating pairing identifies V/L with $\omega \otimes L^\vee$. Since the inertia action has finite image, it is semisimple. The two inertia types 1 and $\omega|_I$ are distinct, so $V = L \oplus L_\omega$, where inertia acts on L_ω by ω. Both summands are Weil-stable. The operator N commutes with inertia and hence preserves them. The infinitesimal symplectic relation $\langle Nv, w\rangle + \langle v, Nw\rangle = 0$, and the perfect pairing between the two summands, show that $N|_{L_\omega} = 0$. Frobenius semisimplicity splits L into two unramified characters, giving the claimed decomposition and $n = 2e$. Conversely, (3.1) and $d \le 2$ imply $d = 2$ if $n = 2e$. □

Remark 3.2 (Sharpness)**.** All three values of d attain the bound. For a two-dimensional Weil–Deligne representation τ, use the hyperbolic representation

$$\tau \oplus (\omega \otimes \tau^\vee).$$

Take τ respectively unramified with zero monodromy, the unramified special representation Sp_2, or an irreducible tamely ramified two-dimensional Weil representation. The resulting pairs (d, n) are $(2, 2e), (1, 2e + 1), (0, 2e + 2)$. In the last case one may induce a regular tame character of the unramified quadratic extension. These are statements about parameters; no compact-invariant-vector existence is being inferred from them.

Remark 3.3 (The limitation of this shortcut)**.** Theorem 3.1 bounds the *parameter* conductor. It cannot be inserted into an argument involving the analytic conductor n'_π until the equality $n'_\pi = n_\pi$ has been established independently. Using it earlier would be circular.

4 Integral normalizations and the functional equation

Put $a(t) = \mathrm{diag}(t, t, 1, 1)$ and $\bar{n}'(x) = I_4 + xE_{32}$. Define

$$\Xi(s, W) = \int_{F^\times} W(a(t))|t|^{s-3/2}\, d^\times t, \qquad Z(s, W) = \int_{F^\times}\int_F W(a(t)\bar{n}'(x))|t|^{s-3/2}\, dx\, d^\times t.$$

The standard Novodvorsky rationality theorem and its fractional-ideal description are used as input [12, 10, 13]. Denote the generator of this ideal by $L(s, \pi)$; the cited compatibility theorem identifies it with $L(s, \phi_\pi)$.

Lemma 4.1 (Balanced vectors)**.** *Let $n_3(y) = I_4 + yE_{13} + yE_{24}$. If W is right invariant under $n_3(\mathfrak{p}^{-r})$, the inner integral defining Z is supported on $x \in \mathfrak{p}^r$. If W is also right invariant under $\bar{n}'(\mathfrak{p}^r)$, then $Z(s, W) = q^{-r}\Xi(s, W)$. Consequently, for $W \in V_\pi(m)$,*

$$Z(s, W) = \Xi(s, W), \qquad Z(s, \mathcal{D}_m W) = q^{-e}\Xi(s, \mathcal{D}_m W),$$

where $\mathcal{D}_m$ is defined in (4.1).

Proof. Conjugating $n_3(y)$ by $\bar{n}'(x)$ gives an upper unipotent matrix with Whittaker character $\psi(xy)$. Conjugation by $a(t)$ does not change its n_2-coordinate. Right invariance therefore gives

$$W(a(t)\bar{n}'(x)) = \psi(xy)W(a(t)\bar{n}'(x)) \quad (y \in \mathfrak{p}^{-r}).$$

If $x \notin \mathfrak{p}^r$, choose y for which $\psi(xy) \neq 1$. The second assertion follows from the volume of $\mathfrak{p}^r$. The last assertion follows by reading the $13, 24, 32$ entries of the two compact groups. □

Realize $\pi^\vee$ as $\pi^\iota = \pi \otimes (\omega^{-1} \circ \mu)$. The useful conjugation is

$$\boxed{\mathcal{D}_m W(g) = \omega(\mu(g))^{-1} W(g j_m).} \tag{4.1}$$

It maps $V_\pi(m)$ to the corresponding coquasi space in $\mathcal{W}(\pi^\vee, \psi)$, with character $\omega(k_{33})^{-1}$. It is involutive between these two models:

$$\mathcal{D}_m^{\pi^\vee}\mathcal{D}_m^{\pi} W = W.$$

Indeed the product of the two twisting factors is $\omega(\varpi^m)^{-1}$, canceled by $W(g j_m^2) = \omega(\varpi^m)W(g)$.

Proposition 4.2 (Analytic epsilon factor)**.** *There is a nonzero constant ε'_π and an integer n'_π such that*

$$\frac{Z(1-s, \mathcal{D}_0 W)}{L(1-s, \pi^\vee)} = \varepsilon'_\pi q^{-n'_\pi(s-1/2)} \frac{Z(s, W)}{L(s, \pi)}.$$

At level m, this gives

$$\boxed{q^{-e}\frac{\Xi(1-s, \mathcal{D}_m W)}{L(1-s, \pi^\vee)} = \varepsilon'_\pi q^{(m-n'_\pi)(s-1/2)} \frac{\Xi(s, W)}{L(s, \pi)}.} \tag{4.2}$$

This proposition defines the analytic factor independently of the parameter. Theorem 15.1 later proves $n'_\pi = a(\phi_\pi)$ and $\varepsilon'_\pi = \omega(-1)\varepsilon(1/2, \phi_\pi, \psi)$.

Proof. The two normalized integrals at $m = 0$ have the same covariance. For $Z(s, \cdot)$ the group $U = \bar{N}' N_2 Z^J$ has character $\theta(\bar{n}'(x) n_2(y) z) = \psi(y)$, and the torus $T' = \{a(t) : t \in F^\times\}$ has character $|t|^{1/2-s}$. Indeed these statements follow by changes of variables in the integral. Writing $\mathcal{J} = \mathcal{D}_0$, one has

$$\mathcal{J}\pi(h) = \omega(\mu(h)) R(j_0 h j_0) \mathcal{J}, \qquad j_0 a(t) j_0 = t I_4\, a(t^{-1}).$$

Conjugation fixes N_2 and interchanges $\bar{N}'$ and Z^J, up to signs. The dual central character cancels the factor $\omega(t)$; thus $Z(1-s, \mathcal{J}\cdot)$ has exactly the same covariance.

Here is the precise mirabolic calculation giving generic uniqueness. Use the finite P_3 filtration from [10, Proposition 2.5.2 and Theorem 2.5.3], on $Q^\circ/Z^J \simeq \mathrm{GL}_2 \ltimes F^2$. The generic bottom constituent is realized on $\text{c-Ind}_{N_2}^{\mathrm{GL}_2} \psi$, with

$$((h, v)F)(g) = \psi((gv)_2) F(gh).$$

Taking coinvariants for the translation $(0, y)$ with character $\psi(y)$ restricts the model to $g_{22} = 1$. Modulo left N_2, write $g = \left(\begin{smallmatrix} a & 0 \\ c & 1 \end{smallmatrix}\right)$. Coinvariants for the lower unipotent integrate out c and leave $C_c^\infty(F^\times)$, with action $f(a) \mapsto |t|^{-1} f(at)$. For a constituent of type $\tau_1(\xi)$ the same calculation leaves one dimension, with character $\xi(t)|t|^{-1}$. For every inflated constituent $\tau_2(\rho)$, including supercuspidal ρ, the translation group acts trivially, whereas θ is nontrivial on it; the twisted coinvariants are zero. Exactness of unipotent coinvariants therefore gives an exact sequence whose bottom is $C_c^\infty(F^\times)$ and whose quotient E is a finite-dimensional torus module. For a character χ outside its finite list of composition characters, any χ-functional on the bottom constituent extends uniquely: in an exact sequence $0 \to A \to B \to E \to 0$, choose $p \in \mathbf{C}[F^\times]$ with $pE = 0$ and $p(\chi) \neq 0$, and set

$$\widetilde{\lambda}(b) = p(\chi)^{-1} \lambda(pb).$$

The element p may be chosen as a product of operators $t_i - \xi_i(t_i)$, one for each composition factor ξ_i. This supplies the surjectivity missing from the printed Hom argument. The one-dimensional bottom functional is the Mellin functional on $C_c^\infty(F^\times)$. Thus the rational families are proportional.

Set $A_0 = \mathbf{C}[q^{-s}, q^s]$. Each family generates the unit ideal of A_0, by the definition of L and the bijectivity of $\mathcal{D}_0$. Their proportionality factor is therefore a unit of A_0, hence a nonzero Laurent monomial. This proves the $m = 0$ statement.

For the shift, write $j_m = \varpi^m a(\varpi^{-m}) j_0$. A change of variables gives

$$Z(z, R(a(b))U) = |b|^{1/2-z} Z(z, U).$$

For the dual representation, the central factor $\omega(\varpi)^{-m}$ from j_m is canceled by the normalization in $\mathcal{D}_m$. The remaining factor at $z = 1 - s$ is $q^{m(s-1/2)}$. Apply Lemma 4.1. □

Remark 4.3 (The printed conjugation differs by a scalar)**.** Okazaki defines $W^c = \pi^\iota(j_m) W^\iota$, where $W^\iota(g) = \omega(\mu(g))^{-1} W(g)$. Literally,

$$W^c = \omega(\varpi)^{-m} \mathcal{D}_m W.$$

Thus his equation (5.3), with that definition, requires an extra $\omega(\varpi)^{-m}$ on the right. His main displayed integral (1.3) already uses $\mathcal{D}_m W$, so the main normalization is compatible with (4.2).

Proposition 4.4 (Polynomial reversal)**.** *For $W \in V_\pi(m)$, put*

$$P_W(X) = \frac{\Xi(s, W)}{L(s, \pi)}, \qquad X = q^{-s},$$

and define $P_{\mathcal{D}W}$ in the dual model in the same manner. Both are polynomials. If $P_W \neq 0$, then $h = m - n'_\pi \geq 0$, $\deg P_W \leq h$, and

$$P_{\mathcal{D}W}(Y) = q^e \varepsilon'_\pi q^{h/2} Y^h P_W(q^{-1}Y^{-1}). \tag{4.3}$$

In particular, at $m = n'_\pi$, every nonzero zeta integral from $V_\pi(m)$ is a constant multiple of $L(s,\pi)$.

Proof. Right invariance under $n'(\mathfrak{o})$ implies $W(a(t)) = 0$ when $v(t) < 0$; the same holds for $\mathcal{D}_m W$. Right invariance under $a(\mathfrak{o}^\times)$ makes these values constant on unit cosets. Thus both Ξ's are power series in q^{-s} with no negative terms. Dividing by L multiplies by a polynomial with constant term one. The fractional-ideal theorem shows the quotients are Laurent polynomials; hence they are polynomials. Substitute $Y = q^{s-1}$ in (4.2). The absence of negative powers on both sides proves the degree bounds. □

Explicit diagonal values. If $\Xi(s,W) = L(s,\phi_\pi)$, write $L(s,\phi_\pi) = \sum_{r\geq 0} A_r q^{-rs}$. Then

$$W(a(\varpi^r u)) = \begin{cases} q^{-3r/2} A_r, & r \geq 0,\ u \in \mathfrak{o}^\times, \\ 0, & r < 0. \end{cases} \tag{4.4}$$

For $e > 0$, Theorem 3.1 restricts the possibilities:

$L(s,\phi_\pi)$	A_r $(r \geq 0)$
1	$\delta_{r,0}$
$(1-\alpha q^{-s})^{-1}$	α^r
$[(1-\alpha q^{-s})(1-\beta q^{-s})]^{-1}$	$\sum_{j=0}^{r} \alpha^j \beta^{r-j}$.

The last expression is $(\alpha^{r+1} - \beta^{r+1})/(\alpha - \beta)$ for distinct roots, and $(r+1)\alpha^r$ for repeated roots. These formulas determine the indicated one-dimensional torus restriction, not all values on G.

5 The unramified-central-character case is complete

Theorem 5.1. *If $e = 0$, the generic newform, minimal-level uniqueness, and generic oldform dimension and basis results of Roberts–Schmidt extend to π by an unramified similitude twist. In particular, writing N for its minimal paramodular level,*

$$\dim V_\pi(N+r) = \left\lfloor \frac{(r+2)^2}{4} \right\rfloor \quad (r \geq 0),$$

and the level-raising basis is transported from their Theorem 7.5.6. There is a unique normalized W_π with $\Xi(s,W_\pi) = L(s,\phi_\pi)$.

Proof. Choose an unramified character χ with $\chi(\varpi)^2 = \omega(\varpi)^{-1}$. Then $\pi_0 = \pi \otimes (\chi \circ \mu)$ has trivial central character. Since the similitude of a compact-group element is a unit, all compact invariant spaces are unchanged as vector spaces. Apply [10, Theorems 7.5.4 and 7.5.6] and transport the Whittaker function by $W(g) \mapsto \chi(\mu(g))W(g)$. Writing $\chi = |\cdot|^u$, this transport replaces s by $s+u$ in Ξ. The parameter L-factor has the same shift. Use the present Haar normalization, which gives $W_\pi(1) = 1$. □

With the standard Novodvorsky–orthogonal integral comparison, the conductor and epsilon-factor compatibility for the trivial-center case and its character twists is available in the literature; see Cheng, Remark 1.2 and its references [17]. In particular this completes the parameter interpretation of the unramified-center theorem. It must not be used as a reduction of arbitrary central characters to trivial ones: a square root of a general character need not exist.

6 Corrections that must be retained in a full proof

Proposition 6.1 (Shifted GL_d functional equation)**.** *Use the standard* GL_d *integral*

$$Z_0(s,W)=\int_{F^\times} W(\mathrm{diag}(t,1,\dots,1))|t|^{s-(d-1)/2}\,d^\times t.$$

If W *is* $K_1(\mathfrak{p}^m)$*-invariant and* $Z_0(s,W)=L(s,\pi)$*, then the shifted dual function* $W'=\pi^\iota(a_{-m})W^\iota$ *satisfies*

$$\frac{Z_0(1-s,W')}{L(1-s,\pi^\vee)}=q^{m(d-2)/2}\varepsilon_\pi q^{(m-a(\pi))(s-1/2)}.$$

Proof. Change variables $t\mapsto t\varpi^{-m}$:

$$Z_0(1-s,W')=q^{m(s+(d-3)/2)}Z_0(1-s,W^\iota).$$

The standard functional equation has $Z_{d-2}(s,W)$ on the other side. For a $K_1(\mathfrak{p}^m)$-invariant vector this equals $Z_0(s,W)$: the unipotent integration is supported on $\mathfrak{o}^{d-2}$ and is constant there. Substitution gives the formula. □

For $d=4$, the omitted constant in Okazaki's Theorem 2.3 is q^m.

Proposition 6.2 (Oldform shift and Gauss product)**.** *For a* GL_2 *Whittaker function* ξ*, define* $\xi_r=\tau(\mathrm{diag}(\varpi^{-r},1))\xi$*. Then*

$$Z_0(s,\xi_r)=q^{-r(s-1/2)}Z_0(s,\xi).$$

For a character χ *of conductor* $e>0$*, with additive measure* $\mathrm{vol}(\mathfrak{o})=1$*, put* $G_\psi(\chi)=\int_{\mathfrak{o}^\times}\chi(u)\psi(\varpi^{-e}u)\,du$*. Then*

$$G_\psi(\chi)G_\psi(\chi^{-1})=\chi(-1)q^{-e}.$$

Proof. The first assertion is a direct change of multiplicative variable. For the second, write $q^eG_\psi(\chi)$ as the primitive finite Gauss sum on $(\mathfrak{o}/\mathfrak{p}^e)^\times$, and apply character orthogonality to its product with the inverse-character sum. The sign is absent only if the second additive character is ψ^{-1}, or if $\chi(-1)=1$. □

Thus the oldform zeta integral has a *negative* exponent $-r(s-\frac12)$. For $r>0$, the opposite sign conflicts with the nonnegative-power support of a quasi-paramodular zeta integral.

Proposition 6.3 (Dual Euler roots with monodromy)**.** *For a contributing unramified block* $\chi\otimes\mathrm{Sp}(k)$*, if* $\alpha=\chi(\mathrm{Fr})q^{-(k-1)/2}$ *is its Euler root, the dual Euler root is*

$$\beta=q^{-(k-1)}\alpha^{-1}.$$

Consequently the argument comparing $L(s,\phi)^{-1}$ *and* $L(1-s,\phi^\vee)^{-1}$ *must retain the block length* k*.*

Proof. Duality replaces χ by χ^{-1}, while the kernel of monodromy in $\mathrm{Sp}(k)$ retains the weight $q^{-(k-1)/2}$. This gives the formula. In the variable $X = q^s$, the zeros of the two inverse factors are respectively α_i and $q^{k_i}\alpha_i$. If a divisibility argument makes these nonzero zero multisets equal, multiplying all roots gives $q^{\sum_i k_i} = 1$. This is impossible unless there are no factors. □

This repairs the particular monodromy inference in Okazaki's §8. Importing [13, Proposition 4.3] avoids the need for that L-factor comparison altogether.

7 Corrected Gauss integrals

7.1 A normalized local Gauss calculation

Let F be a nonarchimedean local field, $\mathfrak{p} = \varpi\mathfrak{o}$, $q = |\mathfrak{o}/\mathfrak{p}|$, and let ψ have conductor $\mathfrak{o}$. Give additive Haar measure the normalization $\mathrm{vol}(\mathfrak{o}) = 1$. Let χ be a character of $\mathfrak{o}^\times$ of conductor $e \geq 2$, and put $h = \lceil e/2 \rceil$. Additive duality gives a unit c_χ, unique modulo $\mathfrak{p}^{e-h}$, such that

$$\chi(1+x) = \psi(c_\chi x/\varpi^e) \qquad (x \in \mathfrak{p}^h).$$

Fix a unit u_χ with $c_\chi u_\chi \equiv -1 \pmod{\mathfrak{p}^{e-h}}$. This uses the conductor of an additive character; it does not assert that its set-theoretic kernel is a fractional ideal. For $1 \leq m < e$ set

$$S_m(\chi,u) = \int_{\mathfrak{p}^m} \psi\left(\frac{x}{\varpi^e u}\right)\chi(1+x)\,dx, \qquad G(\chi,u) = \int_{\mathfrak{o}^\times} \psi\left(\frac{v}{\varpi^e u}\right)\chi(v)\,dv.$$

Then $S_m(\chi,u)$ is nonzero precisely when $u \equiv u_\chi \pmod{\mathfrak{p}^{\min(m,e-m)}}$. More precisely,

$$S_m(\chi,u) = \begin{cases} q^{-m}, & 2m \geq e \text{ and } u \equiv u_\chi \pmod{\mathfrak{p}^{e-m}}, \\ \psi(-1/(\varpi^e u))G(\chi,u), & 2m \leq e \text{ and } u \equiv u_\chi \pmod{\mathfrak{p}^m}, \\ 0, & \text{otherwise.} \end{cases} \qquad |G(\chi,u)| = q^{-e/2}.$$

Indeed, if $2m \geq e$, the integrand is the additive character $x \mapsto \psi((u^{-1}+c_\chi)x/\varpi^e)$, and additive orthogonality applies. If $2m \leq e$, partition the integral defining $G(\chi,u)$ into residue classes modulo $\mathfrak{p}^m$. Averaging the multiplicative change of variable $v \mapsto v(1+t)$, $t \in \mathfrak{p}^{e-m}$, shows that only $v \equiv -c_\chi u \pmod{\mathfrak{p}^m}$ contributes. On the class $1+\mathfrak{p}^m$ the change $v = 1+x$ gives the asserted formula. The usual orthogonality calculation for a primitive Gauss sum gives its stated absolute value.

Here is the resulting correction to Okazaki's Lemma 2.6. Suppose that W is a smooth ψ-Whittaker function on $\mathrm{GL}_2(F)$ satisfying $W(gk) = \chi(k_{22})W(g)$ for $k \in K_0(\mathfrak{p}^e)$, and write

$$[t;z] = \begin{pmatrix} t & 0 \\ z & 1 \end{pmatrix}.$$

For $v(z) = m$, $1 \leq m < e$, its possibly nonzero values have

$$i = m - e, \qquad z \equiv \varpi^m u_\chi \pmod{\mathfrak{p}^{\min(e,2m)}}.$$

For the integral used in that lemma the exact formula is

$$\boxed{\int_{\mathfrak{p}^m \setminus \mathfrak{p}^{m+1}} W([\varpi^{m-e};z])\,dz = q^{-\min(m,e-m)} S_m(\chi,u_\chi) W([\varpi^{m-e};\varpi^m u_\chi]).}$$

For completeness, conjugating $n(x)$ by $g = [\varpi^i; z]$ gives

$$g^{-1}n(x)g = \begin{pmatrix} 1+\varpi^{-i}zx & \varpi^{-i}x \\ -\varpi^{-i}xz^2 & 1-\varpi^{-i}zx \end{pmatrix}.$$

Comparing the left Whittaker character and right K_0 character gives the support restrictions above. On the support one has, for $y \in \mathfrak{p}^m$,

$$W([\varpi^{m-e}; \varpi^m u_\chi/(1+y)]) = \psi(y/(\varpi^e u_\chi))\chi(1+y)W([\varpi^{m-e}; \varpi^m u_\chi]).$$

When $2m \le e$, the change of variable $z = \varpi^m u_\chi/(1+y)$ has Jacobian q^{-m}, giving the boxed identity. When $2m \ge e$, right invariance under the lower unipotent subgroup of level e makes W constant on its supporting coset modulo $\mathfrak{p}^e$; the integral is q^{-e} times the displayed value, while $S_m = q^{-m}$. Thus the factor in the box is essential in both ranges.

The nonpositive shells needed in Okazaki's Lemma 5.6 have a similarly simple exact normalization. Here C is a coquasi-paramodular Whittaker function, χ denotes its *actual* central character (thus $\chi = \omega_\pi^{-1}$ if C belongs to the contragredient representation), and $C_r = R(\eta^{-r})C$, where $\eta^{-r} = \mathrm{diag}(\varpi^r, 1, 1, \varpi^{-r})$. The right $K_0(\mathfrak{p}^e)$ character on the embedded GL_2 is $k \mapsto \chi(k_{22})$. Normalize $d^\times u$ by $\mathrm{vol}(\mathfrak{o}^\times) = 1$, and put

$$I_{i,m}(C_r) = \int_{\mathfrak{o}^\times} \int_{v(z)=m} C_r(a'(\varpi^i u)\bar{n}'(z))\, dz\, d^\times u, \quad G_\chi = \int_{\mathfrak{o}^\times} \psi(\varpi^{-e}w)\chi(w)\, dw.$$

For $m \le 0$ the exact formula is

$$I_{i,m}(C_r) = 0 \quad (i \ne m-e), \qquad I_{m-e,m}(C_r) = q^{-m}\chi(\varpi)^m G_\chi C_{r+m}(a'(\varpi^{-m-e})j'(1)).$$

Indeed, $\bar{n}'(z) = n'(z^{-1})j'(z)n'(z^{-1})$, and the last factor is in the right invariance group for $m \le 0$. Write $z = \varpi^m v$. Using $a'(u)j'(z) = j'(z/u)a'(u)$ and the diagonal unit character gives the factor $\chi(u/v)$. Thus the two unit integrations reduce exactly to

$$q^{-m}C_r(a'(\varpi^i)j'(\varpi^m)) \int_{\mathfrak{o}^\times} \psi(\varpi^{i-m}w)\chi(w)\, dw.$$

Primitivity makes the latter integral zero unless $i = m-e$. Finally,

$$a'(\varpi^i)j'(\varpi^m) = z(\varpi^m)a'(\varpi^{i-2m})\eta^{-m}j'(1)$$

gives the displayed nonzero-shell formula. In particular, the q^{-m} Jacobian and the use of the actual character χ, rather than its inverse, are necessary.

8 Restriction and compact averaging

8.1 A replacement for the restriction argument in Proposition 5.8

Let F be a nonarchimedean local field of characteristic zero, with ring of integers $\mathcal{O}$, maximal ideal $\mathfrak{p} = (\varpi)$, and residue cardinality q. Use the symplectic form and matrix coordinates of Okazaki. Write $G = \mathrm{GSp}_4(F)$, and put

$$Z^J = \{z(x) = 1 + xE_{14} : x \in F\}, \qquad Q^\circ = \{g \in Q : g_{44} = 1\}.$$

Here Q is the Klingen parabolic stabilizing Fe_1. The projection taking rows and columns $2,3,4$ identifies

$$Q^\circ/Z^J \simeq P_3 = \mathrm{GL}_2(F) \ltimes F^2.$$

In particular, the embedded copy of $\mathrm{GL}_2(F)$ is

$$\jmath(g) = \mathrm{diag}(\det g, g, 1).$$

For a representation V put

$$V(Z^J) = \langle \pi(z)v - v : z \in Z^J,\ v \in V \rangle, \qquad E = V_{Z^J} = V/V(Z^J).$$

The external input in the argument below is Rösner–Weissauer, *Regular poles for spinor L-series attached to split Bessel models of* GSp(4), arXiv:1711.07409v2, specifically Lemmas 3.13, 4.1, 4.11, 4.14, 4.16, and 5.11, together with Corollary 4.12. Their $\eta(V)$ is exactly the module E used here, after a change of matrix coordinates: if P is the permutation matrix interchanging e_3 and e_4, then

$$P^{\mathrm{t}} \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} P = -J_{\mathrm{Oka}}.$$

Conjugation by P takes their $S_A = 1 + FE_{13}$ to our Z^J, their Klingen parabolic to Q, and their displayed embedding of $\mathrm{GL}_2 \ltimes F^2$ in Q/S_A to the preceding projection and embedding. Both constructions use unnormalized coinvariants; there is no modulus-character twist in this identification. A different nontrivial character used in the derivative functors is conjugate to the present choice inside $\mathrm{GL}_2(F)$.

For clarity, the functors needed from that source are as follows. For $A_n = \mathrm{GL}_n(F) \ltimes F^n$, let i_* inflate a $\mathrm{GL}_n(F)$-module, let i^* take coinvariants under F^n, and let $j^!$ take (F^n, ψ_n)-coinvariants for a fixed nonzero additive character of F^n. Its stabilizer is A_{n-1}. All three functors are exact, and compact induction $j_!$ gives the canonical exact sequence

$$0 \longrightarrow j_! j^! M \longrightarrow M \longrightarrow i_* i^* M \longrightarrow 0. \tag{R1}$$

For $n = 2$, let $U = \{ \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) : x \in F \}$ inside the linear GL_2 factor, and let $D = \{\mathrm{diag}(1,t) : t \in F^\times\}$. For a smooth character ρ of $F^\times$ define the left exact functor

$$k^\rho(M) = (M_U)^{D,\rho}.$$

This is a module for the remaining affine linear group. In particular, if B is a finite-length $\mathrm{GL}_2(F)$-module, then directly from the definition

$$k^\rho(i_* B) = i_*((B_U)^{D,\rho}). \tag{R2}$$

Only this equality, rather than any identification of invariants with coinvariants for a Jordan block, is required below.

For an irreducible generic V, Rösner–Weissauer prove that E has finite length, that $j^!E$ is a perfect A_1-module of degree one (Lemma 4.14), and that

$$k^\rho(E) = 0 \quad \text{for every smooth character } \rho. \tag{R3}$$

The last assertion is the combination of their Lemma 4.16 ($k^\rho \eta = \beta^\rho$) and Lemma 5.11 ($\beta^\rho(V) = 0$ for irreducible generic V). For an A_1-module of finite length, "perfect" is equivalent to having no nonzero finite-dimensional submodule (their Lemma 3.13). Degree means the multiplicity of the unique infinite-dimensional irreducible A_1-module, equivalently the dimension of its nontrivial additive-character coinvariants.

Lemma 8.1 (The possible kernel of Klingen restriction)**.** *Let $V = \mathcal{W}(\pi, \psi)$ for an irreducible generic representation of G, with arbitrary central character. The map*

$$R : E \longrightarrow \{W|_Q : W \in V\}, \qquad [W] \longmapsto W|_Q$$

is well defined and P_3-equivariant. Its kernel is an inflated finite-length $\mathrm{GL}_2(F)$-module all of whose irreducible constituents are supercuspidal.

Proof. The Whittaker character is trivial on Z^J, and Q normalizes Z^J. Consequently $(\pi(z)W - W)|_Q = 0$, proving the first assertion. Write $K = \ker R$. The exact functor $\mathcal{D} = (j^!)^2$ is the generic Whittaker coinvariant functor for P_3. Uniqueness of the G-Whittaker model and the coinvariant identification in Rösner–Weissauer's Lemma 4.11 give $\dim \mathcal{D}(E) = 1$. Evaluation at the identity on $R(E)$ is a nonzero generic Whittaker functional: some W has $W(1) \neq 0$, and evaluation has the prescribed equivariance under the maximal unipotent. Thus $\dim \mathcal{D}(E/K) \geq 1$. Exactness gives $\mathcal{D}(K) = 0$.

The module $j^!K$ embeds in $j^!E$, has finite length, and has degree zero. The classification of irreducible A_1-modules shows that a finite-length module of degree zero is finite dimensional. Perfectness of $j^!E$ therefore gives $j^!K = 0$. Applying (R1) to K shows that $K = i_*B$ for the finite-length $\mathrm{GL}_2(F)$-module $B = i^*K$.

By (R3) and left exactness, $k^\rho(K) = 0$ for every ρ. Equation (R2) now gives $(B_U)^{D,\rho} = 0$ for every ρ. The ordinary Jacquet module B_U is a finite-dimensional smooth representation of the diagonal torus. If it were nonzero, it would have a common eigenvector for this commutative torus, and in particular a nonzero D-eigenvector for some smooth ρ. Thus $B_U = 0$. Exactness of the ordinary Jacquet functor implies that every irreducible constituent of B has zero Jacquet module for the Borel of GL_2, hence is supercuspidal. □

Lemma 8.2 (The compact type excludes supercuspidals)**.** *Let Ω be a smooth character of $F^\times$ with conductor e. For $e > 0$ set*

$$K_0(\mathfrak{p}^e) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}) : c \in \mathfrak{p}^e \right\}, \qquad \vartheta_\Omega(k) = \Omega(d).$$

No irreducible supercuspidal σ of $\mathrm{GL}_2(F)$ contains this $K_0(\mathfrak{p}^e)$-type. For $e = 0$, no such σ has a $\mathrm{GL}_2(\mathcal{O})$-fixed vector.

Proof. For $e > 0$, a nonzero vector of this type is fixed by $K_1(\mathfrak{p}^e)$, so Casselman's conductor theorem gives $a(\sigma) \leq e$. Applying the type relation to scalar matrices uI_2, $u \in \mathcal{O}^\times$, gives $\omega_\sigma(u) = \Omega(u)$, so $c(\omega_\sigma) = e$. But a supercuspidal σ satisfies $a(\sigma) > c(\omega_\sigma)$, a contradiction.

One may see the strict inequality from the conductor compatibility of the local Langlands correspondence for GL_2 as follows. Its parameter is an irreducible two-dimensional Weil representation r. The subspaces fixed by upper ramification groups are Weil stable, so r has a single ramification break $s \geq 0$; inertia cannot act trivially on an irreducible two-dimensional Weil representation. Thus $a(r) = 2(1 + s)$, whereas $a(\det r) \leq 1 + s$. In particular $a(r) \geq 2$ and $a(\det r) < a(r)$. For $e = 0$ the conductor theorem already excludes a spherical supercuspidal, since $a(\sigma) \geq 2$. □

Fix $m \geq 2e$. Write $\mathcal{K}_m^c = K^c(m; e)$ and $\mathcal{H}_m = K_1^c(m; e)$. The entry conditions for $\mathcal{K}_m^c$ are

$$\begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathfrak{p}^{-e} & \mathfrak{p}^{-m} \\ \mathfrak{p}^{m-e} & \mathcal{O} & \mathcal{O} & \mathfrak{p}^{-e} \\ \mathfrak{p}^m & \mathfrak{p}^e & \mathcal{O} & \mathcal{O} \\ \mathfrak{p}^m & \mathfrak{p}^m & \mathfrak{p}^{m-e} & \mathcal{O} \end{pmatrix}, \qquad \mu(k) \in \mathcal{O}^\times,$$

and $\mathcal{H}_m$ imposes $k_{33} \equiv 1 \pmod{\mathfrak{p}^e}$. For $e = 0$ the convention is $\mathcal{H}_m = \mathcal{K}_m^c = K(m)$. For $e > 0$, the symplectic pairing of columns $2, 3$ gives $k_{22}k_{33} \equiv \mu(k) \pmod{\mathfrak{p}^e}$, so k_{33} is a unit. Multiplication of matrices and $m - e \geq e$ show that $k \mapsto k_{33} \bmod \mathfrak{p}^e$ is a homomorphism. Scalar units give all its values. Consequently, if V has central character Ω of conductor e, every $v \in V^{\mathcal{H}_m}$ satisfies

$$\pi(k)v = \Omega(k_{33})v \qquad (k \in \mathcal{K}_m^c). \tag{R4}$$

Proposition 8.3 (Repaired Proposition 5.8)**.** *Let $V = \mathcal{W}(\pi, \psi)$, where π is irreducible generic with central character Ω of conductor e. If $W \in V^{\mathcal{H}_m}$ and $W|_Q = 0$, then $W \in V(Z^J)$.*

Proof. The class $[W]$ belongs to the kernel $K = i_*B$ of the first lemma. For $e > 0$, the embedding $\jmath$ takes $K_0(\mathfrak{p}^e)$ into $\mathcal{K}_m^c$; its $(3,3)$ entry is the $(2,2)$ entry of the embedded 2×2 matrix. Hence (R4) says that $[W]$ has type ϑ_Ω in B. All irreducible constituents of B are supercuspidal, and the preceding lemma excludes this type in each of them. Projection to a prescribed character of a compact group is exact over $\mathbb{C}$, by averaging. A composition series of B therefore excludes the type in B itself. Thus $[W] = 0$. For $e = 0$, use $\jmath(\mathrm{GL}_2(\mathcal{O})) \subset \mathcal{H}_m$ and the identical exactness argument with ordinary compact invariants. □

8.2 A complete replacement for Proposition 5.10

The following argument concerns arbitrary smooth G-modules. It uses the groups $\mathcal{H}_m$ just defined for a fixed integer $e \geq 0$; no assumption on a central character is needed. Set

$$t_m = \begin{pmatrix} 0 & 0 & 0 & -\varpi^{-m} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \varpi^m & 0 & 0 & 0 \end{pmatrix}, \qquad h = \mathrm{diag}(\varpi^{-1}, 1, 1, \varpi).$$

Thus $t_m \in \mathcal{H}_m$ and $t_{m+1}t_m^{-1} = h$.

Lemma 8.4 (Independence of distinct levels)**.** *If $V^{\mathrm{Sp}_4(F)} = 0$, the subspaces $V^{\mathcal{H}_m}$ for distinct integers $m \geq 2e$ are linearly independent.*

Proof. Let $m_0 < m_1 < \cdots < m_r$ and put $J = \bigcap_{i=1}^r \mathcal{H}_{m_i}$. We claim that the subgroup generated by t_{m_0} and J contains $\mathrm{Sp}_4(F)$. It contains $z(\mathfrak{p}^{-m_1})$. Conjugation by t_{m_0} therefore supplies the opposite long root group $\bar{z}(\mathfrak{p}^{2m_0-m_1})$, where $\bar{z}(x) = 1 + xE_{41}$. As $m_0 < m_1$, this contains $\bar{z}(\mathfrak{p}^{m_1})$. The elementary SL_2 identity in positions $1, 4$ gives

$$t_{m_1} = z(-\varpi^{-m_1})\bar{z}(\varpi^{m_1})z(-\varpi^{-m_1}).$$

It follows that $h^{m_1-m_0} = t_{m_1}t_{m_0}^{-1}$ belongs to the generated subgroup.

The intersection J contains a nonzero lattice in each root subgroup of $\mathrm{Sp}_4(F)$. Conjugation by positive and negative powers of $h^{m_1-m_0}$ exhausts the full root groups for $\pm(e_1 - e_2)$, $\pm(e_1 + e_2)$, and $\pm 2e_1$: their weights under h are respectively $\varpi^{\mp 1}$, $\varpi^{\mp 1}$, and $\varpi^{\mp 2}$. A Weyl representative constructed from the now available $\pm(e_1 - e_2)$ root groups conjugates the full $\pm 2e_1$ root groups to the full $\pm 2e_2$ root groups. The elementary root groups generate $\mathrm{Sp}_4(F)$ over a field. This proves the claim, without using any assertion about Chevalley structure constants in residue characteristic two.

Suppose $\sum_{i=0}^r v_i = 0$ with $v_i \in V^{\mathcal{H}_{m_i}}$. Then $v_0 = -\sum_{i=1}^r v_i$ is fixed by t_{m_0} and by J, and hence by $\mathrm{Sp}_4(F)$. Thus $v_0 = 0$. Repeating the argument proves $v_i = 0$ for every i. □

Lemma 8.5 (The averaging formula)**.** *For $m \geq 2e$, representatives for $\mathcal{H}_{m+1}/(\mathcal{H}_m \cap \mathcal{H}_{m+1})$ are*

$$z(\varpi^{-m-1}u) \quad (u \in \mathcal{O}/\mathfrak{p}), \qquad t_{m+1}.$$

Define α_m by summing $\pi(k)$ over these representatives. If

$$A_j = \mathrm{vol}(\mathfrak{p}^{-j})^{-1} \int_{\mathfrak{p}^{-j}} \pi(z(x))\, dx,$$

then on $V^{\mathcal{H}_m}$ one has

$$\alpha_m = \pi(h) + qA_{m+1}. \tag{R5}$$

Moreover $\pi(h)V^{\mathcal{H}_m} \subset V^{\mathcal{H}_{m+2}}$.

Proof. The intersection in the denominator has all the entry conditions of $\mathcal{H}_{m+1}$ and the one stronger condition $k_{14} \in \mathfrak{p}^{-m}$. If $k \in \mathcal{H}_{m+1}$ and k_{44} is a unit, choose $u \bmod \mathfrak{p}$ so that the $(1,4)$ entry of $z(-\varpi^{-m-1}u)k$ belongs to $\mathfrak{p}^{-m}$. The other first-row bounds are preserved: the corrections in columns $1,2,3$ belong respectively to $\mathcal{O}, \mathcal{O}, \mathfrak{p}^{-e}$. The $(3,3)$ entry is unchanged. If $k_{44} \in \mathfrak{p}$, then $t_{m+1}^{-1}k$ has first row ϖ^{-m-1} times the old fourth row and fourth row $-\varpi^{m+1}$ times the old first row. These rows satisfy the intersection's bounds, and the middle rows, including $(3,3)$, are unchanged. This proves exhaustion. Differences of two listed unipotent representatives have $(1,4)$ entry outside $\mathfrak{p}^{-m}$, and $t_{m+1}^{-1}z(\varpi^{-m-1}u)$ also has $(1,4)$ entry outside that ideal. The cosets are therefore distinct.

The sum defines an $\mathcal{H}_{m+1}$-fixed vector because it is the finite-index averaging operator on an $\mathcal{H}_m \cap \mathcal{H}_{m+1}$-fixed vector. Since $t_m v = v$, its last summand is $\pi(t_{m+1})v = \pi(h)v$. The other q summands are $qA_{m+1}v$, because v is fixed by $z(\mathfrak{p}^{-m})$. This proves (R5). Finally, direct conjugation of each matrix-entry ideal gives $h^{-1}\mathcal{H}_{m+2}h \subset \mathcal{H}_m$; the $(3,3)$ condition is unchanged. This proves the level-two assertion. □

Proposition 8.6 (Repaired Proposition 5.10)**.** *Let V be a smooth G-module with $V^{\mathrm{Sp}_4(F)} = 0$. For every $m \geq 2e$,*

$$V^{\mathcal{H}_m} \cap V(Z^J) = 0.$$

Proof. For $j \geq i$ the normalized averaging operators satisfy $A_j A_i = A_j$. Also $A_m v = v$ for $v \in V^{\mathcal{H}_m}$. We construct operators on $V^{\mathcal{H}_m}$ such that

$$A_{m+r} = \beta_r + \gamma_r, \qquad \beta_r(V^{\mathcal{H}_m}) \subset V^{\mathcal{H}_{m+r+1}}, \quad \gamma_r(V^{\mathcal{H}_m}) \subset V^{\mathcal{H}_{m+r}}. \tag{R6}$$

For $r = 1$, take

$$\beta_1 = -q^{-1}\pi(h), \qquad \gamma_1 = q^{-1}\alpha_m.$$

If (R6) holds, then, for $v \in V^{\mathcal{H}_m}$,

$$\begin{aligned} A_{m+r+1}v &= A_{m+r+1}(\beta_r v + \gamma_r v) \\ &= \beta_r v + q^{-1}\alpha_{m+r}\gamma_r v - q^{-1}\pi(h)\gamma_r v. \end{aligned}$$

Here the first term is unchanged because $\beta_r v$ is already $\mathcal{H}_{m+r+1}$-fixed; (R5) applies to $\gamma_r v$ at level $m+r$. Thus the correct recursions are

$$\boxed{\beta_{r+1} = -q^{-1}\pi(h)\gamma_r, \qquad \gamma_{r+1} = \beta_r + q^{-1}\alpha_{m+r}\gamma_r.} \tag{R7}$$

The level assertions follow from the preceding lemma. In particular, there is no factor q^{-1} multiplying β_r in the second recursion; this corrects Okazaki's displayed equation (5.12).

Now let v also belong to $V(Z^J)$. It is a finite sum of terms $\pi(z(x))w - w$. For sufficiently large r every such x belongs to $\mathfrak{p}^{-m-r}$, so $A_{m+r}v = 0$. By (R6) and independence of distinct levels, $\beta_r v = \gamma_r v = 0$. The first recursion in (R7), and invertibility of $\pi(h)$, imply $\gamma_{r-1}v = 0$. The second then gives $\beta_{r-1}v = 0$. Descending to $r = 1$ gives $0 = \beta_1 v = -q^{-1}\pi(h)v$, hence $v = 0$. □

Corollary 8.7 (Coquasi-paramodular Klingen restriction is injective)**.** *For every irreducible generic π with arbitrary central character Ω, $e = c(\Omega)$, and $m \geq 2e$, the restriction*

$$\mathcal{W}(\pi,\psi)^{K_1^c(m;e)} \longrightarrow \{W|_Q : W \in \mathcal{W}(\pi,\psi)\}$$

is injective.

Proof. Combine the two repaired propositions. An irreducible generic representation has no $\mathrm{Sp}_4(F)$-fixed vector: this fixed space is G-stable since Sp_4 is normal; if it were nonzero, irreducibility would force the representation to factor through the abelian quotient G/Sp_4, which is incompatible with genericity. □

8.3 Completion of torus injectivity

Write $a'(t) = \mathrm{diag}(t,t,1,1)$, $a_{R,i} = h^{-R}a'(\varpi^i)$, and $j'(z) = \jmath(\left(\begin{smallmatrix} 0 & -z^{-1} \\ z & 0 \end{smallmatrix}\right))$. The explicit analytic inputs for this subsection are rationality of the Novodvorsky integrals and their functional equation. With

$$Z(s,F) = \int_{F^\times}\int_F F(a'(t)\bar{n}'(z))|t|^{s-3/2}\,dz\,d^\times t, \quad \Xi(s,F) = \int_{F^\times} F(a'(t))|t|^{s-3/2}\,d^\times t,$$

the normalized equation is

$$\frac{Z(1-s,D_lF)}{L(1-s,\pi^\vee)} = \varepsilon'_\pi q^{(l-n'_\pi)(s-1/2)}\frac{Z(s,F)}{L(s,\pi)}, \qquad D_lF(g) = \omega_\pi(\mu(g))^{-1}F(gj_l). \tag{R8}$$

The equality is an identity of rational functions, valid for all F and all integers l. The present argument does not identify these analytic factors with Langlands-parameter factors.

We use the following elementary cutoff, with its proof recorded to specify the integral completely. If F is right $N_3(\mathfrak{p}^{-b})$-invariant, then $F(a'(t)\bar{n}'(z)) = 0$ for $z \notin \mathfrak{p}^b$. Indeed, in the P_3 quotient, $N_3(x)$ is translation by $(x,0)$, $N_2(y)$ is translation by $(0,y)$, and $\bar{n}'(z)$ has linear part $\left(\begin{smallmatrix} 1 & 0 \\ z & 1 \end{smallmatrix}\right)$. Conjugating the first translation by $a'(t)\bar{n}'(z)$ produces translation by (tx, zx), whose Whittaker character is $\psi(zx)$. Right invariance forces this to equal one for every $x \in \mathfrak{p}^{-b}$, which forces $z \in \mathfrak{p}^b$.

Lemma 8.8 (Coquasi shell expansion with nonzero coefficients)**.** *Suppose $e > 0$ and $C \in \mathcal{W}(\Pi,\psi)^{\mathcal{H}_m}$, where the central character χ of Π has conductor e. Put $C_r = \Pi(h^{-r})C$ for $r \geq 0$. There are constants $b_{r,a}$, independent of s, such that*

$$Z(s,C_r) = q^{-e}\Xi(s,C_r) + \sum_{a=e-r}^{e-1} b_{r,a}q^{(e-a)(s-3/2)}. \tag{R9}$$

Moreover, for $1 \leq a < e$ the coefficient $b_{r,a}$ is a nonzero scalar times $C_r(a'(\varpi^{a-e})\bar{n}'(\varpi^a u_\chi))$; for $a \leq 0$ it is

$$b_{r,a} = q^{-a}\chi(\varpi)^a G_\chi\, C_{r+a}(a'(\varpi^{-a-e})j'(1)), \qquad G_\chi = \int_{\mathcal{O}^\times}\psi(\varpi^{-e}v)\chi(v)\,dv \neq 0. \tag{R10}$$

In the first assertion u_χ is the unit in the normalized positive-shell Gauss calculation below.

Proof. The vector C_r is right $N_3(\mathfrak{p}^{r-e})$-invariant, so the cutoff restricts z to $\mathfrak{p}^{e-r}$. It is also right $\bar{N}'(\mathfrak{p}^e)$-invariant. The integral over $\mathfrak{p}^e$ is therefore $q^{-e}\Xi(s,C_r)$, and the remainder is the finite sum of shells $v(z) = a$, $e-r \leq a < e$.

Here is the precise local calculation for a positive shell. Put $h_0 = \lceil e/2 \rceil$, and choose units c_χ, u_χ with

$$\chi(1+x) = \psi(c_\chi x/\varpi^e)\ (x \in \mathfrak{p}^{h_0}), \qquad c_\chi u_\chi \equiv -1 \pmod{\mathfrak{p}^{e-h_0}}.$$

For $1 \le a < e$, define

$$S_a = \int_{\mathfrak{p}^a} \psi(x/(\varpi^e u_\chi))\chi(1+x)\,dx.$$

Additive orthogonality gives $S_a = q^{-a}$ if $2a \ge e$. If $2a \le e$, partition the primitive Gauss integral $\int_{\mathcal{O}^\times} \psi(v/(\varpi^e u_\chi))\chi(v)\,dv$ into cosets modulo $\mathfrak{p}^a$. Averaging multiplication by $1+\mathfrak{p}^{e-a}$ kills every coset except $1+\mathfrak{p}^a$. Hence S_a is that nonzero primitive Gauss integral times $\psi(-1/(\varpi^e u_\chi))$.

Apply this to the GL_2 Whittaker function $g \mapsto C_r(\jmath(g))$, whose right $K_0(\mathfrak{p}^e)$ character is $\chi(d)$. Conjugation by $[\varpi^i; z] = \left(\begin{smallmatrix} \varpi^i & 0 \\ z & 1 \end{smallmatrix}\right)$ gives

$$[\varpi^i; z]^{-1} n(x) [\varpi^i; z] = \begin{pmatrix} 1+\varpi^{-i}zx & \varpi^{-i}x \\ -\varpi^{-i}xz^2 & 1-\varpi^{-i}zx \end{pmatrix}.$$

For $v(z) = a$, put $M = \max(a, e-a)$ and substitute $x = \varpi^i z^{-1} y$ with $y \in \mathfrak{p}^M$; the displayed matrix belongs to $K_0(\mathfrak{p}^e)$. Comparing its right character with $\psi(x)$ requires $\psi((\varpi^i/z + c_\chi \varpi^{-e})y) = 1$ for every such y. Since $M < e$, cancellation of the two terms forces $i = a-e$, and then forces $\varpi^a/z + c_\chi \in \mathfrak{p}^{e-M}$. Thus a nonzero value on $v(z) = a$ requires $i = a-e$ and $z \equiv \varpi^a u_\chi \pmod{\mathfrak{p}^{\min(e,2a)}}$. On this support the change $z = \varpi^a u_\chi/(1+y)$ gives the multiplier $\psi(y/(\varpi^e u_\chi))\chi(1+y)$. If $2a \le e$, its Jacobian is q^{-a}. If $2a \ge e$, right $N'(\mathfrak{p}^e)$-invariance makes the value constant on its supporting coset of measure q^{-e}. Thus in both cases the shell integral is

$$q^{-\min(a,e-a)} S_a\, C_r(a'(\varpi^{a-e})\bar{n}'(\varpi^a u_\chi)). \tag{R11}$$

This scalar is nonzero. Averaging the unit in $t = \varpi^i u$ does not change the shell integral, since it can be absorbed by $z \mapsto u^{-1}z$ and right $a'(u)$-invariance.

For $a \le 0$, write $z = \varpi^a v$ with $v \in \mathcal{O}^\times$. The identity $\bar{n}'(z) = n'(z^{-1})j'(z)n'(z^{-1})$, and right invariance under $n'(\mathcal{O})$, give

$$C_r(a'(\varpi^i u)\bar{n}'(z)) = \psi(\varpi^{i-a}u/v)\chi(u/v)C_r(a'(\varpi^i)j'(\varpi^a)).$$

Integrating over u with multiplicative measure and z with additive measure gives

$$q^{-a} C_r(a'(\varpi^i)j'(\varpi^a)) \int_{\mathcal{O}^\times} \psi(\varpi^{i-a}w)\chi(w)\,dw.$$

The last integral is zero unless $i - a = -e$, by primitive Gauss orthogonality, and is $G_\chi \neq 0$ when equality holds. Finally the exact matrix identity

$$a'(\varpi^{a-e})j'(\varpi^a) = \varpi^a I_4\, a'(\varpi^{-a-e})h^{-a}j'(1)$$

proves (R10). Every shell is consequently supported at $v(t) = a-e$, which gives (R9). □

Proposition 8.9 (Torus injectivity, with arbitrary central character)**.** *Assuming the analytic functional equation (R8), an irreducible generic Whittaker function W fixed by $K_1(m;e)$ and vanishing on the full diagonal torus is identically zero.*

Proof. The embedded $\mathrm{GL}_2(\mathcal{O})$ fixes W. The Iwasawa decomposition $Q = NT_J(\mathrm{GL}_2(\mathcal{O}))$ therefore gives $W|_Q = 0$. Put $W_r = \pi(h^{-r})W$. Since $h \in Q$, every W_r also vanishes on Q; in particular $Z(s, W_r) = 0$. This avoids needing the translated triangular formula (Okazaki's Lemma 5.5) for this step.

Let $C = D_m W$, and let $\Omega = \omega_\pi(\varpi)$. Direct matrix multiplication gives

$$h^{-r} j_m = \varpi^r I_4 \, j_{m-2r} h^{-r}, \qquad C_r = \Omega^r D_{m-2r} W_r. \tag{R12}$$

Thus (R8) implies $Z(s, C_r) = 0$ for every $r \geq 0$. The vector C is fixed by $K_1^c(m; e)$ in the dual representation, whose central character is $\chi = \omega_\pi^{-1}$.

For $e > 0$, apply (R9). Right $N'(\mathcal{O})$-invariance of C_r implies $C_r(a'(\varpi^i)) = 0$ for $i < 0$. Consequently $\Xi(s, C_r)$ has a convergent expansion in nonnegative powers of $X = q^{-s}$ in a right half plane. All the other terms in (R9) have strictly negative, pairwise distinct powers X^{a-e}. It follows that $\Xi(s, C_r) = 0$ and every $b_{r,a} = 0$. The same right invariance under torus units shows that the Mellin coefficients are nonzero scalar multiples of the individual values, so $C(a_{r,i}) = 0$ for every $r \geq 0$ and $i \in \mathbb{Z}$. For $r < 0$, right $N_2(\mathcal{O})$-invariance forces these values to vanish as well. Central covariance and invariance under torus units now give $C|_T = 0$.

For $1 \leq a < e$ and every fixed $r \geq 0$, if $r < e - a$, the cutoff already kills $v(z) = a$, while if $r \geq e - a$, (R11) and $b_{r,a} = 0$ kill its distinguished value. The support and covariance calculation in (R11) then kills every value on this shell. Values with a negative first torus index vanish by the same $N_2(\mathcal{O})$ argument, which is unaffected by $\bar{n}'(z)$ since N_2 and $\bar{N}'$ commute modulo Z^J.

The remaining Weyl-cell values are $C(a_{R,i} j'(1))$. They vanish for $R < e$ by right $N_3(\mathfrak{p}^{-e})$-invariance, since $j'(1)$ conjugates N_3 to N_2 up to sign. They vanish for $i < -e$ by right $\bar{N}'(\mathfrak{p}^e)$-invariance. If $R \geq e$ and $i \geq -e$, set $a = -i - e \leq 0$ and $r = R - a$. Then $a \geq e - r$ and (R10) gives the desired value as a nonzero multiple of $b_{r,a} = 0$.

These are all the necessary cells: the elementary GL_2 Bruhat decomposition, followed by right $K_0(\mathfrak{p}^e)$-covariance, reduces Q to the diagonal cell, the cells $a_{R,i}\bar{n}'(z)$ with $1 \leq v(z) < e$, and the cell $a_{R,i} j'(1)$; lower entries in $\mathfrak{p}^e$ are absorbed on the right, and a unit lower entry is in the Weyl cell. Hence $C|_Q = 0$.

For $e = 0$, use the Roberts–Schmidt theorem directly: choose an unramified character ξ with $\xi^2 = \omega_\pi^{-1}$. Then $\pi \otimes (\xi \circ \mu)$ has trivial central character, the same compact fixed spaces, and Whittaker functions differing by the nowhere-zero factor $\xi(\mu(g))$. Roberts–Schmidt, *Local Newforms for GSp(4)*, Corollary 4.3.8, gives the claim.

Finally, in the case $e > 0$ the preceding coquasi Klingen-injectivity corollary gives $C = 0$. Since D_m is invertible, $W = 0$. □

8.4 Existence at some level when the center is ramified

Proposition 8.10. *Every irreducible generic representation with $e = a(\omega) > 0$ has $V_\pi(m) \neq 0$ for some $m \geq 2e$.*

Proof. This is the compact-averaging construction of Roberts–Schmidt, Theorem 4.4.1, with the central character included. The preceding P_3 restriction argument identifies the bottom generic constituent of V_{Z^J} with its nonzero Whittaker realization $\text{c-Ind}_{N_3}^{P_3} \psi$ in the restriction to Q. Here N_3 in the induction denotes the maximal upper unipotent of P_3. The embedding of this bottom constituent is the standard Whittaker realization up to a scalar, by uniqueness of its Whittaker functional; the restriction-kernel lemma makes that scalar nonzero. In that compact induction take the function supported on $N_3 P_3(\mathcal{O})$, equal to $\psi(n)$ at nk. It is well defined since ψ is trivial on $N_3(\mathcal{O})$, is right $P_3(\mathcal{O})$-invariant, and has value one at the identity.

Lift this function to a Whittaker vector and average over $Q(\mathcal{O})$ with weight $\omega(k_{44})^{-1}$. Exactness of compact character projection shows that its restriction is unchanged. Thus the resulting vector W satisfies $W(1) = 1$ and $R(k)W = \omega(k_{44})W$ for $k \in Q(\mathcal{O})$. Smoothness gives an integer N such that W is fixed by the principal congruence subgroup of level N.

Let m be sufficiently large, $\ell = m - e$, and set $H_m = K(m; e) \cap G(\mathcal{O})$. Klingen block elimination, using that k_{44} is a unit, gives $H_m = Q(\mathcal{O})\, U_m^-$, where the lower root coordinates of U_m^- have valuations at least ℓ, ℓ, m. Indeed the possible quadratic correction to the lowest coordinate has valuation at least $2\ell \geq m$. Choose m with $\ell \geq N$. Then W has character $\omega(k_{44})$ on H_m.

There is the elementary coset decomposition

$$K(m; e) = \coprod_{t \in \mathfrak{p}^{-\ell}/\mathcal{O}} z(t) H_m.$$

To see it, multiply k on the left by $z(-k_{14}/k_{44})$. This makes entry 14 zero and leaves all entries integral: the other corrections in the first row have valuations at least $m - \ell = e$ or zero. Distinctness follows from $Z^J \cap H_m = Z^J(\mathcal{O})$. The character projector to $K(m; e)$ is therefore, up to a positive normalizing constant, $\sum_t R(z(t))W$. Its value at the identity is $q^\ell W(1)$, since the Whittaker character is trivial on Z^J. It is nonzero and belongs to $V_\pi(m)$. □

8.5 Minimality, uniqueness, and the two-test-vector criterion

The analytic factors in this subsection are those defined by the Novodvorsky fractional ideal and functional equation. Write $\varepsilon(s, \pi, \psi) = \varepsilon'_\pi q^{-n'_\pi(s-1/2)}$, $e = c(\omega_\pi)$, $X = q^{-s}$, and $V_\pi(m) = \mathcal{W}(\pi, \psi)^{K_1(m;e)}$ for $m \geq 2e$. The inputs already proved are torus injectivity, the normalized functional equation, and the coquasi shell expansion (R9).

We first record the precise part of the translated triangular formula that is needed. Put $W_r = \pi(h^{-r})W$, where $h = \mathrm{diag}(\varpi^{-1}, 1, 1, \varpi)$. For $r \geq 0$,

$$\Xi(s, W_i) = 0 \quad (0 \leq i < r) \quad \Longrightarrow \quad Z(s, W_r) = \Xi(s, W_r). \tag{C1}$$

Indeed, W_r is right $N_3(\mathfrak{p}^r)$-invariant, so the cutoff restricts the lower-unipotent integral to $\mathfrak{p}^{-r}$. On $\mathcal{O}$ right $\bar{N}'(\mathcal{O})$-invariance gives $\Xi(s, W_r)$. If $v(z) = -a$ with $1 \leq a \leq r$, the middle SL_2 Bruhat identity, right $\jmath(\mathrm{GL}_2(\mathcal{O}))$-invariance, and the central character give

$$W_r(a'(t)\bar{n}'(z)) = \omega_\pi(\varpi)^{-a}\psi(t/z)W_{r-a}(a'(t\varpi^{2a})). \tag{C2}$$

The unit part of z cancels between central covariance and the $K(m; e)$-character of $t_1(\mathcal{O}^\times)$; the remaining unit in the argument of a' is removed by right $a'(\mathcal{O}^\times)$ invariance. Vanishing of $\Xi(s, W_{r-a})$ is equivalent to vanishing of all its individual $a'(t)$-values, by right unit invariance and the support $v(t) \geq 0$. Thus every shell in (C2) vanishes, proving (C1). This argument does not use the erroneous $a = 0$ summand in Okazaki's printed Lemma 5.5.

Theorem 8.11 (Analytic minimality and uniqueness)**.** *Let π be irreducible generic, with arbitrary central character. Then:*

1. *$V_\pi(m) = 0$ for every admissible level $2e \leq m < n'_\pi$.*
2. *If $n'_\pi \geq 2e$, then $\dim V_\pi(n'_\pi) \leq 1$. More precisely, the map*

$$V_\pi(n'_\pi) \longrightarrow \mathbb{C}, \qquad W \longmapsto \Xi(s, W)/L(s, \pi)$$

is well defined and injective.

3. Suppose some $W_0 \in V_\pi(M)$ and $\kappa \neq 0$ satisfy

$$\Xi(s, W_0) = L(s, \pi), \qquad \Xi(s, D_M W_0) = q^e \kappa L(s, \pi^\vee). \tag{C3}$$

Then $M = n'_\pi$ is the minimal level, $\kappa = \varepsilon'_\pi$, and $V_\pi(M) = \mathbb{C}W_0$.

Proof. For $W \in V_\pi(m)$ put $C = D_m W$ and $C_r = \pi^\vee(h^{-r})C$. At $r = 0$, balancedness gives $Z(s, W) = \Xi(s, W)$ and $Z(s, C) = q^{-e}\Xi(s, C)$. Right unit invariance, nonnegative valuation support, and the fractional-ideal theorem imply

$$P_W(X) = \Xi(s, W)/L(s, \pi) \in \mathbb{C}[X], \quad q^{-e}\Xi(1-s, C)/L(1-s, \pi^\vee) \in \mathbb{C}[X^{-1}].$$

The functional equation identifies the second polynomial with a nonzero constant times $X^{n'_\pi - m} P_W(X)$. If $m < n'_\pi$, both sides therefore vanish. If $m = n'_\pi$, both sides are constants; in this case consider a vector in the kernel $P_W = 0$. In either situation the two zeroth rows vanish.

Suppose inductively that $\Xi(s, W_i) = 0$ for $i < r$. By (C1), $Z(s, W_r) = \Xi(s, W_r)$, and consequently

$$P_r(X) = \Xi(s, W_r)/L(s, \pi) \in \mathbb{C}[X].$$

The exact identity $C_r = \omega_\pi(\varpi)^r D_{m-2r} W_r$ gives

$$\frac{Z(1-s, C_r)}{L(1-s, \pi^\vee)} = \text{a nonzero constant} \cdot X^{n'_\pi - m + 2r} P_r(X). \tag{C4}$$

For $e > 0$, expansion (R9), with s replaced by $1 - s$, shows that the left side belongs to $X^r\mathbb{C}[X^{-1}]$. To justify polynomiality here, its expansion lies in $Y^{-r}\mathbb{C}[[Y]]$, $Y = q^{s-1}$, and the fractional-ideal theorem makes it a Laurent polynomial. The two sides of (C4) have disjoint supports whenever $m < n'_\pi$, or $m = n'_\pi$ and $r > 0$. Thus $P_r = 0$.

For $e = 0$, C is itself paramodular in the dual representation. Use simultaneous induction on the vanishing of the rows of both W and C. Formula (C1) applies to both vectors, so the left side of (C4) belongs to $\mathbb{C}[X^{-1}]$. Its support is again disjoint from that on the right for $r > 0$ and $m \leq n'_\pi$. Both rows vanish and the induction continues. This avoids an unjustified Atkin–Lehner $\pm$ decomposition for a nontrivial unramified central character.

In every case all $\Xi(s, W_r)$ vanish for $r \geq 0$. Right $N_2(\mathcal{O})$-invariance kills the torus values with negative first torus index; the nonnegative indices are precisely the rows just considered. Unit covariance and central covariance therefore give $W|_T = 0$, so torus injectivity gives $W = 0$. This proves (1) and injectivity in (2); polynomial reversal gave the fact that its target consists of constants. Notice that the passage through the translated rows is essential: injectivity on the full two-dimensional torus alone would not imply injectivity of the single integral Ξ.

Finally substitute (C3) into the normalized functional equation:

$$\kappa = \varepsilon'_\pi q^{(M - n'_\pi)(s-1/2)}.$$

Both sides are nonzero, and the left side is independent of s. Thus $M = n'_\pi$ and $\kappa = \varepsilon'_\pi$. Assertions (1) and (2), together with $W_0 \neq 0$, prove (3). □

Remark 8.12 (A stronger first-row bound when $e > 0$)**.** For any nonzero $W \in V_\pi(m)$ let r be its first nonzero translated row, and put $d = \deg(\Xi(s, W_r)/L(s, \pi))$. The same argument gives the quantitative inequality

$$m \geq n'_\pi + r + d.$$

Indeed (C1) makes the quotient a nonzero polynomial, whereas the left side of (C4) has largest X-exponent at most r. The largest exponent on its right is $n'_\pi - m + 2r + d$. This bound is asserted here only for $e > 0$, where (R9) supplies the stated upper bound without any additional induction.

Corollary 8.13 (An upper bound for higher-level dimensions)**.** *Assume $e > 0$ and $m \geq 2e$. If $d = m - n'_\pi \geq 0$, then*

$$\dim V_\pi(m) \leq \frac{(d+1)(d+2)}{2}.$$

For $d < 0$ the space is zero by analytic minimality. This is an upper bound; it is not an assertion of the exact Roberts–Schmidt oldform dimension formula.

Proof. Filter $V_\pi(m)$ by

$$F_r = \{W \in V_\pi(m) : \Xi(s, W_i) = 0 \text{ for } 0 \leq i < r\}, \qquad F_0 = V_\pi(m).$$

By (C1), the linear map $W \mapsto P_r(W) = \Xi(s, W_r)/L(s, \pi)$ takes F_r to $\mathbb{C}[X]$ and has kernel F_{r+1}. If $P_r(W) \neq 0$, the first-row bound gives $\deg P_r(W) \leq d - r$. Consequently

$$\dim(F_r/F_{r+1}) \leq d - r + 1 \qquad (0 \leq r \leq d).$$

Also $F_{d+1} = 0$: a nonzero vector in this space would have a first nonzero translated row $r \geq d + 1$, by torus injectivity and the automatic vanishing of negative first torus indices. The first-row bound would then give $d \geq r$, a contradiction. Summing the displayed quotient bounds proves the assertion. □

Corollary 8.14 (Generic small split-theta lifts)**.** *Let $e > 0$, and suppose the generic small split-theta lift $\pi = \theta(\tau_1^\vee \boxtimes \tau_2^\vee)$ is taken in the Weil normalization of the explicit split-theta construction, where the τ_i are generic GL_2 representations with common central character ω_π. Put*

$$\phi = \phi_{\tau_1} \oplus \phi_{\tau_2}, \qquad n = a(\phi) = a(\tau_1) + a(\tau_2), \qquad \epsilon_\phi = \epsilon(1/2, \phi, \psi).$$

Let $W_0 \in V_\pi(n)$ be the vector constructed by the explicit theta calculation at oldform index zero, which satisfies

$$\Xi(s, W_0) = L(s, \phi), \qquad \Xi(s, D_n W_0) = q^e \omega_\pi(-1)\epsilon_\phi L(s, \phi^\vee).$$

Then

$$m_\pi = n'_\pi = n, \qquad \varepsilon'_\pi = \omega_\pi(-1)\epsilon_\phi, \qquad V_\pi(n) = \mathbb{C}W_0.$$

In particular the analytic epsilon factor is $\omega_\pi(-1)$ times the Langlands-parameter epsilon factor in these conventions.

Proof. Use the independent comparison $L(s, \pi) = L(s, \phi_\pi)$ and its dual version: Takloo–Bighash's theorem [12], as stated in [13, Proposition 4.3], for generic representations with arbitrary central character. The theta parameter identity gives $\phi_\pi = \phi$. Thus the two displayed theta identities are exactly (C3), with $M = n$ and $\kappa = \omega_\pi(-1)\epsilon_\phi$. The preceding theorem proves the assertions. This argument imports the L-factor comparison and derives the epsilon-factor comparison; the latter is not silently included in the cited L-factor theorem. □

9 The split theta construction with ramified central character

The construction below is a correction of the split-theta construction in [11, Section 7]. The representation-theoretic inputs are the GL_2 newvector theorem and its local functional equation, and the split theta correspondence and its Whittaker multiplicity-one statement [8, 9, 15]. The coordinate, Fourier, and double-coset calculations are given explicitly. They are valid also in residue characteristic two.

Let F be a nonarchimedean local field of characteristic zero, with ring $\mathcal{O}$, maximal ideal $\mathfrak{p} = \varpi\mathcal{O}$, and residue cardinality q. Fix ψ of conductor $\mathcal{O}$. Multiplicative Haar measure gives $\mathcal{O}^\times$ volume one. Write

$$a_t = \begin{pmatrix} \varpi^t & 0 \\ 0 & 1 \end{pmatrix}, \qquad j_t = \begin{pmatrix} 0 & -1 \\ \varpi^t & 0 \end{pmatrix}, \qquad A_t = \operatorname{diag}(\varpi^t, \varpi^t, 1, 1).$$

The symplectic form on F^4 is the matrix $\left(\begin{smallmatrix} 0 & -w_2 \\ w_2 & 0 \end{smallmatrix}\right)$, where $w_2 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Put

$$J_m = \begin{pmatrix} 0 & D \\ \varpi^m D & 0 \end{pmatrix}, \qquad D = \operatorname{diag}(1, -1).$$

In particular $\mu(J_m) = \varpi^m$.

Let τ_1, τ_2 be irreducible generic representations of $\mathrm{GL}_2(F)$, with common central character ω, conductor exponents n_1, n_2, and $e = c(\omega) > 0$. The ordinary GL(2) conductor bound gives $n_i \geq e$. Realize their normalized newvectors v_1, v_2 in the Whittaker models for ψ and ψ^{-1}, respectively, with $v_i(1) = 1$. We use the convention that $R(a)v(g) = v(ga)$.

The theta representation in the following proposition is assumed to have a generic irreducible small theta quotient. More precisely, with the Weil normalization specified below, put $\pi = \theta(\tau_1^\vee \boxtimes \tau_2^\vee)$ and assume π is generic. With this convention the spin parameter of π is $\phi_{\tau_1} \oplus \phi_{\tau_2}$. This is an assumption on the *small* theta quotient: it is not a claim that the unique generic constituent of an arbitrary big theta lift is its small quotient.

For $m \geq 2e$, put $l = m - e$ and let $\mathbf{K}_1(m; e)$ be the subgroup of symplectic similitudes with unit similitude, satisfying

$$k \in \begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathfrak{p}^{-l} \\ \mathfrak{p}^l & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathfrak{p}^l & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathfrak{p}^m & \mathfrak{p}^l & \mathfrak{p}^l & \mathcal{O} \end{pmatrix}, \qquad k_{44} \in 1 + \mathfrak{p}^e.$$

For a Whittaker function of central character ω, define

$$\mathcal{D}_m W(g) = \omega(\mu(g))^{-1} W(gJ_m), \qquad \Xi(s, W) = \int_{F^\times} W(\operatorname{diag}(t, t, 1, 1)) |t|^{s-3/2} \, d^\times t.$$

This explicit definition is essential: literal right translation of $W^\iota(g) = \omega(\mu(g))^{-1} W(g)$ by J_m would give an additional factor $\omega(\varpi)^{-m}$.

Corrected existence proposition. For every $r \geq 0$, set $m_1 = n_1$, $m_2 = n_2 + r$, and $m = n_1 + n_2 + r$. There exists $W_r \in \mathcal{W}_\psi(\pi)^{\mathbf{K}_1(m;e)}$ such that

$$\Xi(s, W_r) = q^{-r(s-1/2)} L(s, \tau_1) L(s, \tau_2), \tag{9.1}$$

$$\Xi(s, \mathcal{D}_m W_r) = q^e \omega(-1) \epsilon(1/2, \tau_1, \psi) \epsilon(1/2, \tau_2, \psi) L(s, \tau_1^\vee) L(s, \tau_2^\vee). \tag{9.2}$$

The construction below is an integral formula for $W_r(g)$ for arbitrary g. If $L(s, \tau_1)L(s, \tau_2) = \sum_{j \geq 0} b_j q^{-js}$, then

$$W_r(A_i) = \begin{cases} q^{r/2 - 3i/2} b_{i-r}, & i \geq r, \\ 0, & i < r. \end{cases}$$

In particular $W_0(1) = 1$. This proposition asserts existence and these integral identities. Minimality of the level and uniqueness of the vector require a separate newvector argument.

Weil normalization and the integral. On $X = M_2(F)$ use the bilinear form

$$b_X(x,y) = \mathrm{Tr}(x^* y), \qquad x^* = \begin{pmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{11} \end{pmatrix}.$$

Thus the associated quadratic form is $\det(x) = b_X(x,x)/2$. The lattice $M_2(\mathcal{O})$ is self-dual for every residue characteristic. For $z = (u,v)$, let $\Gamma(z) = (b_X(z_i,z_j))_{i,j}$. The Fourier transform uses the kernel $\psi(b_X(u,u') + b_X(v,v'))$ and Haar measure giving $M_2(\mathcal{O})^2$ volume one. In the Schrödinger model use

$$\begin{aligned} \mathcal{W}(n(B),1)\varphi(z) &= \psi(\tfrac{1}{2}\,\mathrm{Tr}(Bw_2\Gamma(z)))\varphi(z), \\ \mathcal{W}(A^\natural,1)\varphi(z) &= |\det A|^{-2}\varphi(z w_2\,{}^t A^{-1} w_2), \\ \mathcal{W}(j(-w_2),1)\varphi &= \widehat{\varphi}, \qquad A^\natural = \mathrm{diag}(w_2\,{}^t A^{-1} w_2, A). \end{aligned}$$

Here $n(B) = \left(\begin{smallmatrix} I & B \\ 0 & I \end{smallmatrix}\right)$ and Bw_2 is symmetric. The half in this formula is harmless at $p = 2$: its argument is a sum of determinant and polar-form terms with integral coefficients. It must not be combined with an additional half in the definition of b_X.

Represent

$$H = \mathrm{GSO}(X) = (\mathrm{GL}_2 \times \mathrm{GL}_2)/\{(z,z^{-1})\}$$

by the action $(h_1,h_2)x = h_1 x h_2^*$, with similitude $\mu_X(h) = \det(h_1 h_2)$. Extend the Weil representation to equal similitudes by

$$\mathcal{W}(g,h)\varphi(z) = |\mu_X(h)|^{-2}\mathcal{W}(g_1,1)[z \mapsto \varphi(h^{-1}z)](z), \qquad g_1 = g\,\mathrm{diag}(I_2, \mu(g)^{-1} I_2).$$

Put $z_0 = E_{12} \oplus I_2$, and $N_\Delta = \{(n(x), n(x)) : x \in F\}$. Its stabilizer in $H^1 = \mathrm{SO}(X)$ is N_Δ.

Take $\xi_1 = v_1$, $\xi_2 = R(a_{-r})v_2$. With $K_0(a) = \{k \in \mathrm{GL}_2(\mathcal{O}) : k_{21} \in \mathfrak{p}^a\}$, put $K = (K_0(m_1) \times K_0(m_2)) \cap H^1$, interpreted in the quotient defining H. Haar measure on N_Δ gives $N_\Delta(\mathcal{O})$ volume one; the measure on H^1 and its quotient is fixed compatibly. Define

$$\varphi(u,v) = \omega(u_{12})^{-1}\mathbf{1}_{\left(\begin{smallmatrix} \mathfrak{p}^{m_2} & \mathcal{O}^\times \\ \mathfrak{p}^{m-e} & \mathfrak{p}^{m_1} \end{smallmatrix}\right)}(u)\mathbf{1}_{M_2(\mathcal{O})}(v), \tag{9.3}$$

$$T_{\xi,\varphi}(g) = \int_{N_\Delta \backslash H^1} \mathcal{W}(g, hh_g)\varphi(z_0)\xi_1((hh_g)_1)\xi_2((hh_g)_2)\,d\dot{h}, \qquad \mu_X(h_g) = \mu(g). \tag{9.4}$$

The integral is independent of h_g and absolutely convergent. Here is a support proof which does not assume that the Schwartz function alone has compact pullback to the orbit. For arbitrary Schwartz function Φ and two fixed GL(2) Whittaker functions, Iwasawa decomposition modulo N_Δ gives representatives

$$h = (\varpi^a n(x) a_s k_1, a_t k_2), \qquad 2a + s + t = 0, \quad k_1, k_2 \in \mathrm{GL}_2(\mathcal{O}),$$

with unit factors absorbed in the compact variables. On these representatives $h^{-1}E_{12}$ has norm q^{-a}; multiplication by the compact integral matrices preserves primitivity. Compact support of Φ therefore bounds a below. Smoothness gives finitely many Whittaker functions under the two compact groups. Each has torus support bounded below, uniformly over this finite collection: right invariance under $n(\mathfrak{p}^M)$, together with left Whittaker equivariance, forces its value at a_s to vanish for sufficiently negative s. Thus $s \geq S$ and $t \geq T$, bounding a above as well. The relation $2a + s + t = 0$ then leaves only finitely many triples (a,s,t). Finally the second matrix of $h^{-1}z_0$, whose middle unipotent entry is $-\varpi^a x$, bounds x on the Schwartz support. The product integrand consequently

has compact support modulo N_Δ. For general g, apply this argument to $\Phi = \mathcal{W}(g, h_g)\varphi$ and the right translates of the two Whittaker functions; equivariance gives precisely the same integral. To put the integral in the prescribed Whittaker model, set

$$d = \mathrm{diag}(-1, 1, 1, -1), \qquad T^{\mathrm{new}}_{\xi,\varphi}(g) = T_{\xi,\varphi}(dgd^{-1}), \qquad W_r = \mathrm{vol}(K)^{-1} T^{\mathrm{new}}_{\xi,\varphi}.$$

The matrix d normalizes all compact groups in this construction. The compact-character calculation below gives the equivalent formula $T^{\mathrm{new}}(g) = \omega(-1)^{-1} T(dg)$.

The inverse on $\omega(u_{12})$ in (9.3) is necessary. Indeed, for $k = (k_1, k_2) \in K$, writing $d_i = (k_i)_{22}$, one has $k^{-1}u = k_1^* u k_2$ and

$$(k^{-1}u)_{12} \equiv d_1 d_2 u_{12} \pmod{\mathfrak{p}^e}$$

on the support. Consequently

$$\varphi(k^{-1}z) = \omega(d_1 d_2)^{-1}\varphi(z), \qquad (\tau_1 \boxtimes \tau_2)(k)\xi = \omega(d_1 d_2)\xi.$$

Using the non-inverted character would violate their required product invariance for a nonquadratic central character.

The uncorrected integral T has Whittaker character $\psi(-x+y)$. Indeed $n_2(x) = 1 - xE_{12} + xE_{34}$ has lower Levi block $A = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$, so its Weil action replaces (u, v) by $(u, v - xu)$. At $z_0 = (E_{12}, I_2)$ this is the action of $(n(-x), I_2)$ on the orthogonal variables. Changing the integration variable multiplies ξ_1 by $\psi(-x)$. On the other hand $\Gamma(z_0) = \mathrm{diag}(0, 2)$, so the long root $n'(y)$ has character $\psi(y)$. Conjugation by d changes x to $-x$ and fixes y; consequently T^{new} has the required character $\psi(x+y)$. This short-root sign must be corrected before comparing functional equations. The usual equivariance of the uncorrected theta integral gives a map from $\Theta(\tau_1^\vee \boxtimes \tau_2^\vee)$ to the induced Whittaker representation for $\psi(-x+y)$. Its Hom space has dimension one by the twisted-Jacquet module computation for split theta correspondence [15, Proposition 4.1 and Corollary 4.2]. Since the assumed small quotient π is generic, its Whittaker embedding supplies a nonzero map in this one-dimensional space. Thus the integral map factors through π. The conjugation above then places W_r in the prescribed $\psi(x+y)$ Whittaker model of the same representation. Its nonvanishing follows also from the diagonal calculation below. This argument does not identify a generic big-theta subquotient with a nongeneric small quotient.

Invariance under $\mathbf{K}_1(m; e)$. Set $l = m - e$, so $l \geq m_i$, $l \geq e$, and $2l \geq m$. The upper unipotents in the symplectic part of this group have

$$B = \begin{pmatrix} b & c \\ d & b \end{pmatrix}, \quad b, d \in \mathcal{O}, \quad c \in \mathfrak{p}^{-l}.$$

Their Weil phase on (u, v) is $c \det u + b\, b_X(u, v) + d \det v$, which is integral on the support of φ. Thus they fix φ.

The Levi elements are $A^\natural$, with $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in K_0(l)$ and $d \in 1 + \mathfrak{p}^e$. Their argument substitution is

$$(u, v) \longmapsto ((au - cv)/\det A, (-bu + dv)/\det A).$$

It preserves all support conditions; its effect on u_{12} is congruent to multiplication by d^{-1} modulo $\mathfrak{p}^e$. Hence these elements fix φ as well.

For the lower unipotents, write

$$C = \begin{pmatrix} c & d \\ f & c \end{pmatrix}, \quad c \in \mathfrak{p}^l,\ d \in \mathcal{O},\ f \in \mathfrak{p}^m.$$

The Fourier support of φ has first component y satisfying

$$y_{11} \in \mathfrak{p}^{-m_1}, \quad y_{22} \in \mathfrak{p}^{-m_2}, \quad y_{12} \in \mathfrak{p}^{-l}, \quad y_{21} \in \varpi^{-e}\mathcal{O}^\times,$$

and second component in $M_2(\mathcal{O})$. Thus $\det y \in \mathfrak{p}^{-m}$ and $b_X(y, y') \in \mathfrak{p}^{-l}$. Fourier conjugation therefore makes the lower-unipotent phase integral too.

These elements generate the symplectic part. To see this directly, if the lower-right 2×2 block of k is not invertible, multiplication by the Weyl element in the middle $\mathrm{SL}_2(\mathcal{O})$, itself a product of the preceding root elements, makes it invertible. Indeed k_{44} is a unit, and the middle 2×2 block is invertible modulo $\mathfrak{p}$. For an invertible lower-right block, block Gaussian elimination factors k as upper unipotent, Levi, lower unipotent of the displayed forms. The bottom-left lower-unipotent bound uses precisely $\mathfrak{p}^{2l} \subseteq \mathfrak{p}^m$. Thus this argument includes the endpoint $m = 2e$.

Finally, for $t \in \mathcal{O}^\times$, take $d_t = \mathrm{diag}(t, t, 1, 1)$ and $h_t = (1, \mathrm{diag}(t, 1))$. Then $\mathcal{W}(d_t, h_t)\varphi(u, v) = \varphi(u\,\mathrm{diag}(t, 1), v\,\mathrm{diag}(t, 1)) = \varphi(u, v)$, and ξ is fixed by right translation by h_t. Removing $d_{\mu(k)}$ reduces an arbitrary $k \in \mathbf{K}_1(m; e)$ to its symplectic part. Equation (9.4) consequently gives the asserted full-group invariance of W_r.

The complete Fourier calculation. Put $\chi = \omega^{-1}$, and

$$G(\chi) = \int_{\mathcal{O}^\times} \chi(x)\psi(\varpi^{-e}x)\,dx.$$

Additive Haar measure gives $\mathcal{O}$ volume one. Set $h_m = (j_{m_1}, j_{m_2})$, and let $\varphi^c = \mathcal{W}(J_m, h_m)\varphi$. For $x = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, write

$$T(x) = h_m^{-1}x = \begin{pmatrix} \varpi^{-m_1}d & -\varpi^{-m}c \\ -b & \varpi^{-m_2}a \end{pmatrix}.$$

Block multiplication, followed by change of variables in the Fourier transform, gives

$$\varphi^c(u, v) = q^{2m}\widehat{\varphi}(-T(v), T(u)).$$

For completeness, the symplectic part of J_m is $A^\natural j(-w_2)$ with $A = \varpi^m j_0$. Its Levi factor contributes q^{4m}, the similitude extension contributes q^{2m}, and the Fourier change of variables contributes q^{-4m}. The remaining arguments are $(-T(v), T(u))$, as displayed.

The three unrestricted lattice integrations in the first component have volume $q^{-(m+l)}$. The fourth is the elementary Gauss integral

$$\int_{\mathcal{O}^\times} \chi(b)\psi(-by_{21})\,db = G(\chi)\chi(-\varpi^e y_{21})^{-1}\mathbf{1}_{\varpi^{-e}\mathcal{O}^\times}(y_{21}).$$

The second component is self-dual. Consequently the exact answer is

$$\varphi^c(u, v) = q^e G(\omega^{-1})\omega(-\varpi^e v_{12})\mathbf{1}_{\left(\begin{smallmatrix} \mathfrak{p}^{m_2} & \mathcal{O} \\ \mathfrak{p}^m & \mathfrak{p}^{m_1} \end{smallmatrix}\right)}(u)\mathbf{1}_{\left(\begin{smallmatrix} \mathcal{O} & \varpi^{-e}\mathcal{O}^\times \\ \mathfrak{p}^e & \mathcal{O} \end{smallmatrix}\right)}(v). \tag{9.5}$$

In particular the correction changes both lower-left support bounds, not only a power of q.

Define $\xi_i^c(g) = \omega(\det g)^{-1}\xi_i(gj_{m_i})$. Equivariance of the integral and its similitude character now give exactly

$$\mathcal{D}_m T_{\xi,\varphi} = T_{\xi^c,\varphi^c}.$$

There is no missing uniformizer scalar with the explicit definition of $\mathcal{D}_m$ used here.

Bruhat cells and their contributions. For an integer i, let $\varphi_i(u,v) = \varphi(ua_i, va_i)$, $\xi^i = \xi_1 \boxtimes R(a_i)\xi_2$, and $K_i = \mathrm{Int}(1,a_i)K$. Then

$$T_{\xi,\varphi}(A_i) = q^{-2i} T_{\xi^i,\varphi_i}(1).$$

The same identity applies to φ^c, ξ^c. The rank-one Bruhat decompositions give cells $N_X T_X^1 w N_w K_i$, with $w \in \{1, j_0\} \times \{1, j_0\}$. The nontrivial Weyl coordinates satisfy $b_1 \in \mathfrak{p}^{1-m_1}$ and $b_2 \in \mathfrak{p}^{1-m_2+i}$, respectively. Unit factors do not affect any valuation assertion below.

On the first nonidentity cell, $u = h^{-1}E_{12}$ has entries of the form $u_{22} = t,\ u_{12} = -b_1 t$; on the second, $u_{11} = -t,\ u_{12} = -b_2 t$. On the cell with both Weyl elements its nonzero entries, up to signs and exchange of the labels b_1, b_2, have the form $t, b_1 t, b_2 t, b_1 b_2 t$, with $u_{21} = t$. For the original φ_i, the first two cases force $u_{12} \in \mathfrak{p}$, contradicting its unit support. In the last case, $\det u = 0$, the diagonal support, and $u_{12} \in \mathcal{O}^\times$ force $u_{21} \in \mathfrak{p}^{m-i}$. The two bounds on b_i then force $u_{12} \in \mathfrak{p}^2$, again impossible. Thus all nonidentity cells vanish for the original function.

For φ_i^c, all three nonidentity cells instead have $u_{12} \in \mathfrak{p}$. In its first and second cases this follows from the preceding displayed diagonal bounds, and in its last case from $u_{21} \in \mathfrak{p}^{m-i}$. Replacing v by $v - xu$, for $x \in \mathfrak{p}^{-1}$, preserves the support of φ_i^c and its character: the upper-right entry changes by an element of $\mathcal{O}$, which does not change $\omega(-\varpi^e v_{12})$; the other three bounds follow from $m_1, m_2 \geq 1$ and $m - 1 \geq e$. Left translation of h by $(n(x), 1)$ makes exactly this replacement and multiplies $\xi^c(h)$ by $\psi(x)$. Choosing $x \in \mathfrak{p}^{-1}$ with $\psi(x) \neq 1$ proves that every such orbit integral is zero.

It remains to compute the identity Weyl cell. Representatives can be written

$$h = (\varpi^a n(x) a_s, a_t), \quad t = -s - 2a, \qquad h^{-1} z_0 = \varpi^a E_{12} \ \oplus \ \begin{pmatrix} \varpi^{-s-a} & -\varpi^a x \\ 0 & \varpi^{s+a} \end{pmatrix}.$$

For φ_i, the conditions are $a = 0$, $0 \leq s \leq i$, and $x \in \mathcal{O}$. For φ_i^c, they are $a \geq 0$, $0 \leq s + a \leq i$, and $x \in \varpi^{-e-a}\mathcal{O}^\times$. If $a \geq 1$, the same translation by $\mathfrak{p}^{-1}$ preserves the conjugate Schwartz function and annihilates its orbit integral. Thus only $a = 0$ remains in both computations.

For $a = 0$, the correct residue class for x is

$$x \pmod{\mathfrak{p}^{d_s}}, \qquad d_s = \min(s, i - s).$$

Indeed the two upper unipotents in K_i, together with left N_Δ, identify translations by $\mathfrak{p}^s + \mathfrak{p}^{i-s}$. Moreover

$$\mathrm{vol}(N_\Delta \cap \mathrm{Int}(h) K_i) = q^{-M_s}, \qquad M_s = \max(s, i - s).$$

Since $d_s + M_s = i$, the original sum has total coefficient $q^{-2i} q^{d_s + M_s} = q^{-i}$. The character $\psi(x)$ is trivial on its original support $\mathcal{O}$.

For the conjugate function the finite sum is instead

$$\sum_{x \in \varpi^{-e}\mathcal{O}^\times / \mathfrak{p}^{d_s}} \psi(x)\omega(\varpi^e x) = q^{d_s + e} G(\omega).$$

The standard same-additive-character Gauss identity is $G(\omega^{-1})G(\omega) = \omega(-1)q^{-e}$. Here is a direct verification, including $e = 1$. For any character χ of conductor $e > 0$, substitute $x = uy$ into the product of its two Gauss integrals. The integral over $y \in \mathcal{O}^\times$ is

$$\int_{\mathcal{O}^\times} \psi(\varpi^{-e} y(u+1))\, dy = \mathbf{1}_{\mathfrak{p}^e}(u+1) - q^{-1}\mathbf{1}_{\mathfrak{p}^{e-1}}(u+1).$$

The first resulting integral over u is $\chi(-1)q^{-e}$. The second is zero by primitivity of χ: for $e \geq 2$ this is character orthogonality on $-1+\mathfrak{p}^{e-1}$, and for $e = 1$ it is orthogonality on all of $\mathcal{O}^\times$. Combining this with (9.5) proves the exact convolution identities

$$T_{\xi,\varphi}(A_i) = q^{-i}\,\mathrm{vol}(K)\sum_{s=0}^{i}\xi_1(a_s)\xi_2(a_{i-s}), \tag{9.6}$$

$$T_{\xi^c,\varphi^c}(A_i) = q^{-i+e}\omega(-1)\,\mathrm{vol}(K)\sum_{s=0}^{i}\xi_1^c(a_s)\xi_2^c(a_{i-s}). \tag{9.7}$$

Both sides are zero when $i < 0$. This proves all the Bruhat-cell cancellations needed for these formulas, including $m = 2e$.

Oldforms and epsilon factors. The inverse translation used above is important:

$$\xi_2 = R(a_{-r})v_2 \in \tau_2^{K_1(n_2+r)}, \qquad Z_0(s,\xi_2) = q^{-r(s-1/2)}L(s,\tau_2).$$

The first assertion follows by conjugating $K_1(n_2+r)$ by a_r. The second follows by changing variables in its one-dimensional Mellin integral. Also $j_{n_2+r}a_{-r} = j_{n_2}$, so $\xi_2^c = c_{n_2}v_2$, independently of r.

Here is an explicit check of the remaining signs. For a normalized GL(2) newvector v for ψ_0, define $c_n v(g) = \omega(\det g)^{-1}v(gj_n)$. With the usual functional equation using $\widetilde{v}(g) = v(w_2{}^t g^{-1})$, one obtains

$$Z_0(s, c_n v) = \omega(-1)\epsilon(1/2,\tau,\psi_0)L(s,\tau^\vee).$$

For example on the Mellin torus, $c_n v(a(t)) = \omega(-1)\widetilde{v}(a(-t\varpi^{-n}))$; the unit -1 can be removed using the original newvector's diagonal unit invariance, and the functional equation supplies the displayed identity. Since $\epsilon(1/2,\tau_2,\psi^{-1}) = \omega(-1)\epsilon(1/2,\tau_2,\psi)$, it follows that

$$Z_0(s,\xi_1^c)Z_0(s,\xi_2^c) = \omega(-1)\epsilon(1/2,\tau_1,\psi)\epsilon(1/2,\tau_2,\psi)L(s,\tau_1^\vee)L(s,\tau_2^\vee).$$

Thus the $\omega(-1)$ from the Gauss product in (9.7) cancels the sign from the opposite GL(2) Whittaker characters for the uncorrected function T. The correction to its short-root character supplies the final sign. Since d commutes with A_i, one has $T^{\mathrm{new}}(A_i) = T(A_i)$. Moreover $J_m d = -dJ_m$ and $T(gd) = \omega(-1)T(g)$, so

$$\mathcal{D}_m T^{\mathrm{new}}(A_i) = \omega(-1)\mathcal{D}_m T(A_i).$$

Mellin summation of (9.6) and (9.7), followed by division by $\mathrm{vol}(K)$, therefore proves (9.1) and the signed identity (9.2).

Scope of the result. The explicit local calculation above addresses ramified central character and assumes a generic small split-theta quotient. The case $e = 0$ is covered by the established paramodular theory (or by a separate unramified version of the split calculation). This result does not prove that every generic parabolic constituent is a small split-theta lift. In particular the generic constituent of a reducible induction may differ from its nongeneric small-theta quotient, as in the Steinberg example in [15, Theorem 8.2(v), Lemma 5.2(b)(iii)]. A ramified similitude twist of $\mathrm{St}_{\mathrm{GSp}_4}$, whose spin parameter is $\chi \otimes \mathrm{Sp}_4$, is not obtained from a direct sum of two two-dimensional parameters. Such representations are treated by the separate $L = 1$ argument below.

There is nevertheless a useful precise coverage statement. If π is generic, $c(\omega_\pi) > 0$, and $L(s,\phi_\pi) \neq 1$, then π does belong to the small split-theta image and its two GL(2) inputs are generic. Indeed, [15, Theorem 1.1(iii)] classifies generic representations outside that image as

follows. The nondiscrete cases have spin parameter $\rho \oplus \rho\chi$, where ρ is an irreducible two-dimensional Weil representation; neither summand has inertia invariants, so their spin L-factor is one. The discrete cases transfer to a discrete-series representation of GL(4), and hence have spin Weil–Deligne parameter $\rho \otimes \mathrm{Sp}_d$, with $d \dim \rho = 4$. If $\dim \rho > 1$, then $\rho^I = 0$ and again the L-factor is one. (Otherwise irreducibility would make inertia act trivially, forcing ρ to be one-dimensional.) If $\dim \rho = 1$, the parameter is $\chi \otimes \mathrm{Sp}_4$; ramification of its similitude character $\chi^2 = \omega_\pi$ forces χ to be ramified, so this case also has L-factor one. Finally, genericity of the small quotient forces genericity of the input by [15, Corollary 4.2]. Thus a separate theorem for *all* generic representations with spin L-factor one, together with the present proposition and an independent minimality/uniqueness criterion, covers the ramified-central-character analytic theorem. If that separate theorem is phrased using Novodvorsky's analytic L-factor, its comparison with the spin parameter must also be justified.

10 Finite support and balanced averages

10.1 Finite diagonal support when both analytic factors are one

Assume $e > 0$, $L(s,\pi) = L(s,\pi^\vee) = 1$, and $W \in V_\pi(m)$. Put $C = D_m W$, $\Omega = \omega_\pi(\varpi)$, $W_r = R(h^{-r})W$, $C_r = R(h^{-r})C$, and $X = q^{-s}$. Write

$$H_r(X) = \Xi(s, W_r) = \sum_{j \ge 0} h_{r,j} X^j, \qquad h_{r,j} = q^{3j/2} W_r(a'(\varpi^j)).$$

The following is the exact translated quasi identity:

$$Z(s, W_r) = H_r(X) + \sum_{a=1}^{r} \Omega^{-a} q^{-2a} X^{-2a} \left((1 - q^{-1}) \sum_{j \ge a} h_{r-a,j} X^j - q^{-1} h_{r-a,a-1} X^{a-1} \right). \tag{F1}$$

The $a = 0$ term is H_r itself, not its image under the truncated operator used for the nonzero shells. To verify (F1), the cutoff limits z to $\mathfrak{p}^{-r}$. On $v(z) = -a < 0$, the middle SL_2 decomposition gives

$$W_r(a'(t)\bar{n}'(z)) = \Omega^{-a} \psi(t/z) W_{r-a}(a'(t\varpi^{2a})).$$

The unit integral and the shell volume give respectively $q^a(1 - q^{-1})$ if $v(t) + a \ge 0$, $-q^{a-1}$ if $v(t) + a = -1$, and zero otherwise. Substituting $j = v(t) + 2a$ gives (F1), including the factor q^{-2a}.

Lemma 10.1 (Finite support)**.** *Under these hypotheses, both arrays*

$$\{W(h^{-r}a'(\varpi^j))\}_{r,j \ge 0}, \qquad \{C(h^{-r}a'(\varpi^j))\}_{r,j \ge 0}$$

have finite support.

Proof. The fractional-ideal theorem makes every $Z(s, W_r)$ a Laurent polynomial in X. Induction using (F1) shows that H_r is a Laurent polynomial; its nonnegative valuation support makes it a polynomial. The coquasi shell formula (R9) proves the same assertion directly for $\Xi(s, C_r)$.

For each nonzero H_r, let $b_r = \deg H_r$. Suppose that infinitely many rows are nonzero. Then $b_r + 2r$ has arbitrarily large strict record values. At a strict record index r, every off-diagonal term of (F1) has degree at most

$$b_j - 2(r - j) < b_r \qquad (j < r).$$

Consequently $\deg_X Z(s, W_r) = b_r$. Strictness here is essential: an equality of record values would not exclude cancellation.

The coquasi expansion gives $Z(s, C_r) \in X^{-r}\mathbb{C}[X]$. The functional equation and $C_r = \Omega^r D_{m-2r} W_r$ identify $Z(1-s, C_r)$ with a nonzero constant times

$$X^{n'_\pi - m + 2r} Z(s, W_r).$$

The largest X-exponent on the left is at most r. At a strict record index this gives

$$b_r + r \le m - n'_\pi.$$

In particular the record indices are bounded, a contradiction. Thus only finitely many H_r are nonzero, and each is a polynomial.

Let $\mathcal{J}$ be their finite set of indices. If $W = 0$ there is nothing to prove; otherwise torus injectivity makes $\mathcal{J}$ nonempty. For

$$r > \max_{j \in \mathcal{J}} (j + b_j + 1)$$

every tail and boundary term in (F1) vanishes, and $H_r = 0$ as well. Thus $Z(s, W_r) = 0$, and the functional equation gives $Z(s, C_r) = 0$. In (R9), the finite shell terms have distinct strictly negative powers of X, whereas $\Xi(s, C_r)$ has only nonnegative powers. Therefore $\Xi(s, C_r) = 0$. All remaining coquasi rows are polynomials, proving finite support of the second array. □

10.2 Direct balancedness of the two interior averages

Assume $m > 2e > 0$, and write $l = m - e$. Thus $l - 1 \ge e$, $l \ge 2$, and $m - 1 \ge l$. For a 2×2 matrix B with Bw_2 symmetric write $n(B) = \left(\begin{smallmatrix} I & B \\ 0 & I \end{smallmatrix}\right)$; write $\bar{n}(C)$ for the corresponding lower unipotent. Use exactly the unnormalized sums in (H5):

$$F = \Gamma W = \sum_{x \in \mathfrak{p}^{l-1}/\mathfrak{p}^l} \sum_{y \in \mathfrak{p}^{m-1}/\mathfrak{p}^m} R\left(\bar{n}\begin{pmatrix} x & 0 \\ y & x \end{pmatrix}\right) W,$$

and, for $B_0 \in V^c(m)$,

$$G = \Gamma^\iota B_0 = \sum_{x \in \mathfrak{p}^{m-1}/\mathfrak{p}^m} \sum_{y \in \mathfrak{p}^{e-1}/\mathfrak{p}^e} R\left(\bar{n}\begin{pmatrix} x & y \\ 0 & x \end{pmatrix}\right) B_0.$$

Each is a sum of q^2 terms.

Lemma 10.2 (The balancedness needed in (H7))**.** *The vector F is 0-balanced and $D_m F$ is $(e+1)$-balanced. The vector G is $(e-1)$-balanced and its normalized conjugate in the original representation is 1-balanced.*

Proof. Use the exact block identity

$$\bar{n}(C)^{-1} n(B) \bar{n}(C) = \begin{pmatrix} I + BC & B \\ -CBC & I - CB \end{pmatrix}. \tag{F2}$$

First let

$$C = \begin{pmatrix} x & 0 \\ y & x \end{pmatrix}, \quad x \in \mathfrak{p}^{l-1},\ y \in \mathfrak{p}^{m-1}, \qquad B = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}, \quad a \in \mathfrak{o},\ b \in \mathfrak{p}^{1-l}.$$

Then

$$BC = \begin{pmatrix} ax+by & bx \\ ay & ax \end{pmatrix}, \quad CBC = \begin{pmatrix} ax^2+bxy & bx^2 \\ 2axy+by^2 & ax^2+bxy \end{pmatrix}.$$

The diagonal entries of BC and CB lie in $\mathfrak{p}^e$, their upper-right entries are integral, and their lower-left entries lie in $\mathfrak{p}^l$. In CBC the diagonal entries lie in $\mathfrak{p}^l$, the upper-right entry is integral, and the lower-left entry lies in $\mathfrak{p}^m$. These assertions use precisely $2l-2 \geq l$, $m-1 \geq l$, $m+l-2 \geq m$, and $m+e-1 \geq m$. Thus (F2) belongs to $K_1(m;e)$, including its condition on entry 44. Every individual summand defining F is therefore fixed by $N_3(\mathfrak{o})$ and $Z^J(\mathfrak{p}^{1-l})$. The lower Siegel unipotent is abelian. Averaging gives invariance under $\bar{N}_3(\mathfrak{p}^{l-1})$, while $\bar{N}'(\mathfrak{o})$-invariance is inherited from W. This proves the four invariances needed for F.

For G, take instead

$$C = \begin{pmatrix} x & y \\ 0 & x \end{pmatrix}, \quad x \in \mathfrak{p}^{m-1},\ y \in \mathfrak{p}^{e-1}, \qquad B = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}, \quad a \in \mathfrak{p}^{1-e},\ b \in \mathfrak{p}^{1-m}.$$

Here $BC = CB$ has diagonal $ax \in \mathfrak{p}^l$ and integral upper-right entry. Moreover

$$CBC = \begin{pmatrix} ax^2 & 2axy+bx^2 \\ 0 & ax^2 \end{pmatrix} \in \begin{pmatrix} \mathfrak{p}^m & \mathfrak{p}^e \\ 0 & \mathfrak{p}^m \end{pmatrix}.$$

Consequently (F2) belongs to $K_1^c(m;e)$: in particular its 33 entry is $1-ax \equiv 1 \pmod{\mathfrak{p}^e}$. Every summand defining G is fixed by $N_3(\mathfrak{p}^{1-e})$ and $Z^J(\mathfrak{p}^{1-m})$. Averaging also gives invariance under $\bar{N}_3(\mathfrak{p}^{m-1})$ and $\bar{N}'(\mathfrak{p}^{e-1})$. Hence G is $(e-1)$-balanced.

Finally, conjugation by j_0 interchanges $\bar{N}_3$ with N_3 and Z^J with $\bar{N}'$, up to signs. Thus D_0F is $(1-l)$-balanced and D_0G is $(1-m)$-balanced. The identity $D_m = R(a'(\varpi^{-m}))D_0$ increases the balance index by m. This gives $e+1$ and 1, respectively. All conjugated elements have similitude one, so the central twist introduces no extra character. The factor 2 in (F2) only improves valuations in residue characteristic two; no odd-residue-characteristic hypothesis was used. □

11 Exhaustivity of the corrected Hecke tables

Assume $e > 0$, $m \geq 2e$, and $l = m-e$. Put $K = \mathbf{K}_1(m;e)$, $K^c = \mathbf{K}_1^c(m;e)$, and $\eta = \mathrm{diag}(\varpi^{-1},1,1,\varpi)$. All residue variables below are chosen from fixed sets of representatives. The root matrices and their lower counterparts are in Okazaki's convention. In particular

$$n_2(x)n_3(y)z(z) = \begin{pmatrix} 1 & -x & y & z-xy \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \bar{n}_2(x)\bar{n}_3(y)\bar{z}(z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -x & 1 & 0 & 0 \\ y & 0 & 1 & 0 \\ z+xy & y & x & 1 \end{pmatrix}.$$

Changing the sign of a residue variable does not change any of the tables.

A Gaussian factorization and its indices. There is a unique factorization $K = U^+LU^-$, where

$$\begin{aligned} U^+ &= n_2(\mathcal{O})n_3(\mathcal{O})z(\mathfrak{p}^{-l}), \\ U^- &= \bar{n}_2(\mathfrak{p}^l)\bar{n}_3(\mathfrak{p}^l)\bar{z}(\mathfrak{p}^m), \\ L &= \{\mathrm{diag}(a,M,d) : M \in \mathrm{GL}_2(\mathcal{O}),\ d \in 1+\mathfrak{p}^e,\ a = \det(M)/d\}. \end{aligned}$$

The block sizes in L are $1, 2, 1$, and its similitude is $ad = \det M$. Here is a direct elimination proof. For $k \in K$, put $d = k_{44}$, solve the last-row entries for $x = k_{43}/d$, $y = k_{42}/d$, and $z = k_{41}/d - xy$, and remove the resulting U^- factor on the right. The asserted bound on z follows from $2l \geq m$. Symplecticity makes the first column of the remaining matrix vanish except at its first entry. Its last column then determines the unique U^+ factor. The middle block remains integral and has unit determinant, and symplecticity gives $a = \det M/d$. This proves existence and uniqueness. Haar measure in these coordinates is the product measure, up to one fixed unit normalization.

The same proof applies with unrestricted $d \in \mathcal{O}^\times$, or with the condition $k_{11} \in 1 + \mathfrak{p}^e$ instead of the condition on k_{44}. In the latter case the Levi condition is $a \in 1 + \mathfrak{p}^e$, since the other contributions to k_{11} lie in $\mathfrak{p}^e$. This variant is needed when transporting K^c by J_m: its preimage has the condition on k_{11}, not on k_{44}.

Every diagonal element used below normalizes the three factors. An intersection of two such groups therefore factors componentwise by uniqueness. The subgroup index is the product of the two unipotent lattice indices and the middle GL(2) index. The latter indices are $q + 1$ when one triangular entry is required to belong to $\mathfrak{p}$, and $q(q + 1)$ when it is required to belong to $\mathfrak{p}^2$. These follow either from elementary projective lines over $\mathcal{O}/\mathfrak{p}^a$, or from its two GL(2) Bruhat cells.

Consequently the following are exact indices:

$$\begin{aligned}[K : K \cap \eta^{\pm 1} K \eta^{\mp 1}] &= q^4,\\ [K : K \cap A_{\pm 1} K A_{\mp 1}] &= q^2(q + 1),\\ [K^c : K^c \cap \eta K^c \eta^{-1}] &= q^3(q + 1).\end{aligned}$$

For the first line, the changed unipotent depths are $1, 1, 2$, and the Levi factor is unchanged. For the second, the changed radical depths are those of n_3, z (or their lower counterparts), each by one; the middle Levi index is $q + 1$. For the last line, conjugate by J_m. The diagonal element becomes $\mathrm{diag}(1, \varpi, \varpi^{-1}, 1)$, up to inversion. The intersection raises the U^+ depth of n_3 and the U^- depth of $\bar{n}_2$ each by one. The middle GL(2) index is $q(q + 1)$. Its outer-torus congruence condition does not change the index. The two Hecke operators for η, η^{-1} on K^c agree, since the outer Weyl element j''_m belongs to K^c.

If $m > 2e$ and $H = \mathbf{K}^c(m - 1; e)$ denotes the full group used in the character projector, then

$$[H : H \cap \mathbf{K}^c(m; e)] = q^2(q + 1).$$

Indeed conjugation by J_{m-1} identifies this intersection with $\mathbf{K}(m - 1; e) \cap A_1 \mathbf{K}(m; e) A_{-1}$. Its changed radical depths are those of n_3 and $\bar{n}_2$, each by one; the middle Levi intersection is the index-$q + 1$ Borel congruence subgroup. This also proves the index for the corresponding kernels of the character, since the torus-character quotient is the same on both sides.

The corrected table for the boundary operator. The table for $T_K(\eta) + T_K(\eta^{-1})$ is

$$\begin{aligned}&\sum_{x,y \bmod \mathfrak{p},\ z \bmod \mathfrak{p}^2} \pi\big(n_2(x) n_3(y) z(z\varpi^{-l})\eta^{-1}\big)\\ &\quad + \sum_{x,y \bmod \mathfrak{p},\ z \bmod \mathfrak{p}^2} \pi\big(\eta \bar{n}_2(x\varpi^{l-1}) \bar{n}_3(y\varpi^{l-1}) \bar{z}(z\varpi^{m-2})\big).\end{aligned}$$

Both signs of η in the printed (6.3) need reversal; its first central exponent is $-l$, which equals $-e$ only at the boundary. To check distinctness, write the second line as $u^- \eta$, with u^- in the appropriate lower Heisenberg group. Comparison of its first two root coordinates forces equality of x, y before the central coordinate is compared. The central comparison then forces equality of $z \bmod \mathfrak{p}^2$. The upper

case is identical. There are q^4 distinct cosets in each line, so the preceding index calculation proves exhaustivity. At $m = 2e$, evaluation and the independently proved boundary character average give exactly $q^4 W(a_r^{i+1}) + W(a_r^{i-1})$ for $i \geq 1$. This is the range needed for the maximal-support contradiction.

The two quasi-Hecke tables. Put

$$\mathcal{B}_l = \left\{ \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} : x \in \mathcal{O},\ y \in \mathfrak{p}^{-l} \right\}, \quad \widetilde{\mathcal{B}}_l = \left\{ \begin{pmatrix} x & y \\ z & x \end{pmatrix} : x, z \in \mathcal{O},\ y \in \mathfrak{p}^{-l} \right\},$$

and

$$\mathcal{C}_e = \left\{ \begin{pmatrix} x & 0 \\ y & x \end{pmatrix} : x \in \mathcal{O},\ y \in \mathfrak{p}^{e} \right\}, \quad \varpi^l \widetilde{\mathcal{C}}_e = \left\{ \begin{pmatrix} x & z \\ y & x \end{pmatrix} : x \in \mathfrak{p}^{l},\ z \in \mathcal{O},\ y \in \mathfrak{p}^{m} \right\}.$$

The plus table is

$$\{n(B)A_1 : B \in \widetilde{\mathcal{B}}_l/\varpi\widetilde{\mathcal{B}}_l\} \ \sqcup\ \{j'(1)n(B)A_1 : B \in \mathcal{B}_l/\varpi\mathcal{B}_l\}.$$

The minus table is

$$\{\bar{n}(C)A_{-1} : C \in \varpi^l\widetilde{\mathcal{C}}_e/\varpi^{l+1}\widetilde{\mathcal{C}}_e\} \ \sqcup\ \{j'(1)\bar{n}(C)A_{-1} : C \in \varpi^l\mathcal{C}_e/\varpi^{l+1}\mathcal{C}_e\}.$$

The extra right factor $j'(1)$ in the source's second minus cell is irrelevant to right K-cosets. Each table has $q^3 + q^2$ terms. The source's definition of $\widetilde{\mathcal{C}}_e$ also requires correction: its upper-right entry, before multiplication by ϖ^l, must be $\mathfrak{p}^{-l}$, not $\mathcal{O}$. The actual matrix lattice has been specified above to remove this ambiguity. With the printed definition the upper-right residue choices would all give the same coset. Within either cell, the difference of two upper (respectively lower) block unipotents belongs to the relevant intersection precisely when all three (respectively two) residue coordinates agree. Across the two cells, the middle GL(2) quotient detects the Weyl element: in the plus case the upper-right middle entry remains a unit, whereas the intersection requires it to lie in $\mathfrak{p}$; in the minus case use the lower-left entry. Thus these are pairwise distinct cosets. The exact indices already computed prove exhaustivity and hence justify the quasi-Hecke table used for (H8).

The corrected projector and co-Hecke tables. The projector table, with all weights trivial on these representatives, is

$$\begin{aligned} &\{n_2(x)n_3(y\varpi^{-e})\eta^{-1} : x, y \bmod \mathfrak{p}\} \\ &\qquad \sqcup \{\bar{n}_2(x\varpi^{l-1})\bar{n}_3(y\varpi^{m-1})\bar{z}(u\varpi^{m-1}) : x, y, u \bmod \mathfrak{p}\}. \end{aligned}$$

The factor ϖ^{-e} in the first cell is missing in the source. Without it all q values of y give the same right coset. To obtain literal representatives in H, multiply each member of the first cell on the right by j''_m; the result is $n_2(x)n_3(y\varpi^{-e})j''_{m-1}$. Since $j''_m \in K^c$, the operator on $V^c(m)$ is unchanged. All these representatives have $(3,3)$ entry one and therefore character weight one.

Within the first cell, conjugation by η changes the two difference coordinates to valuations -1 and $-e-1$; comparison with the root bounds $0, -e$ in K^c forces the two residues to agree. Within the second cell, compare the lower-root depths l, m, m successively. Across the two cells, the $(1,1)$ entry of the quotient is $\varpi^{-1}(1+\mathfrak{p})$, so it is not integral. Indeed all correction terms have depth at least $l - 1 \geq 1$, since $m > 2e$. The $q^2 + q^3$ cosets are therefore distinct, and the projector index proves exhaustivity. The missing scaling has no effect on the torus evaluation, because the Whittaker character is trivial on N_3.

Finally a complete table for $T_{K^c}(\eta^{-1}) = T_{K^c}(\eta)$ is

$$\{n_2(x)n_3(y\varpi^{-e})z(z\varpi^{-m})\eta^{-1} : x, y \bmod \mathfrak{p},\ z \bmod \mathfrak{p}^2\}$$
$$\sqcup\, \{\eta\bar{n}_2(x\varpi^{l-1})\bar{n}_3(y\varpi^{m-1})\bar{z}(u\varpi^{m-1}) : x, y, u \bmod \mathfrak{p}\}.$$

For membership of the second cell, multiply on the right by j''_m. The identities $\eta j''_m = j''_m\eta^{-1}$, and conjugation of its three lower roots by j''_m, express the result as $k\eta^{-1}$ with $k \in K^c$. The resulting positive root depths are $0, -e, 1-m$, all allowed in K^c.

Within each cell, successive root-coordinate comparisons give distinctness exactly as in the projector case, with the first central coordinate now taken modulo $\mathfrak{p}^2$. For clarity, write a first-cell representative as $U\eta^{-1}$, where $U = n_2(x)n_3(Y)z(Z)$, $v(Y) \geq -e$, $v(Z) \geq -m$, and a second-cell representative as $\eta\bar{U}$, where the three lower coordinates of $\bar{U}$ have depths $l-1, m-1, m-1$. The $(1,1)$ entry of $\eta U^{-1}\eta\bar{U}$ is ϖ^{-2} plus terms of valuation at least -1. It cannot be integral. Thus the two cells are disjoint. They have $q^4 + q^3$ distinct cosets, which equals the computed index, proving exhaustivity of the co-Hecke table (H2).

The minimal-level descending average. The translation in the proof of Proposition 6.6 must be $W_1 = \pi(\eta^{-1})W$, not $\pi(\eta)W$. Put $H = \mathbf{K}_1(m-1; e)$ and $J = \eta^{-1}\mathbf{K}_1(m; e)\eta$. The positive-root depths of J are $1, 1, 2-l$, whereas those of H are $0, 0, 1-l$. The negative-root depths are $l-1, l-1, m-2$ for J, and $l-1, l-1, m-1$ for H. The middle Levi factors agree. Componentwise Gaussian factorization therefore shows that $H/(H \cap J)$ has precisely the q^3 representatives

$$n_2(x)n_3(y)z(z\varpi^{1-l}), \qquad x, y, z \bmod \mathfrak{p}.$$

The lower Gaussian bound for H is valid because $2(l-1) \geq m-1$ follows from $m > 2e$. Averaging W_1 over these representatives produces a vector in $V(m-1)$. Its value at a_r^{i-1}, $i \geq 1$, is $q^3W(a_r^i)$, since the left N_2 character is then trivial and the other two root groups have trivial Whittaker character. This establishes the descending-map assertion with the correct sign.

12 An analytic newform theorem when the spin factor is one

This section repairs and strengthens the Hecke argument of [11, §6]. It uses the torus-injectivity, existence, functional-equation, finite-support, and balancedness results proved above, and the independent coset calculations in the section "Exhaustivity of the corrected Hecke tables". The argument applies without unitarity. No equality of analytic and Weil–Deligne epsilon factors is used.

Let F be a nonarchimedean field, let ϖ be a uniformizer, let $q = |\mathfrak{o}/\mathfrak{p}|$, and let ψ have conductor $\mathfrak{o}$. Use Okazaki's antidiagonal convention. Write

$$a_r = \mathrm{diag}(\varpi^r, \varpi^r, 1, 1), \qquad t_1(u) = \mathrm{diag}(u, 1, 1, u^{-1}), \qquad a_r^i = t_1(\varpi^i)a_r, \qquad \eta = t_1(\varpi^{-1}).$$

Let π be generic and irreducible, let $\omega = \omega_\pi$, and put $e = a(\omega) > 0$. The spaces $V(m)$ and $V^c(m)$ and the integer $l = m - e$ are as in [11, §5]. The central character of $\pi^\iota = \pi \otimes (\omega^{-1} \circ \mu)$ is ω^{-1}. Write $\Omega = \omega(\varpi)$ and let D_m denote the normalized conjugation used in the functional equation. Thus $D_m W = \Omega^m W^c$, where

$$W^c = \pi^\iota(j_m)W^\iota, \qquad j_m = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ \varpi^m & 0 & 0 & 0 \\ 0 & -\varpi^m & 0 & 0 \end{pmatrix}.$$

Theorem 12.1 (Analytic newforms when the spin factor is one)**.** *Suppose π is irreducible, admissible, and generic, $e > 0$, and $L(s,\pi) = L(s,\pi^\vee) = 1$. Write $\epsilon(s,\pi,\psi) = \epsilon' q^{-n'(s-1/2)}$ for the analytic epsilon factor. Then the minimal quasi-paramodular level is $m_\pi = n' > 2e$, and $\dim V(m_\pi) = 1$. Its vector W_π normalized by $W_\pi(1) = 1$ satisfies*

$$W_\pi(a_r^i) = \mathbf{1}_{(i,r)=(0,0)}, \qquad (D_{m_\pi} W_\pi)(a_r^i) = \begin{cases} q^e\epsilon', & (i,r) = (0,0), \\ -q^{e-2}\epsilon', & (i,r) = (1,0), \\ 0, & \text{otherwise}, \end{cases} \quad i, r \geq 0.$$

In particular $Z(s, W_\pi) = 1$ and $Z(s, D_{m_\pi} W_\pi) = \epsilon'$. For the unnormalized conjugation W_π^c, both displayed nonzero co-values acquire the factor Ω^{-m_π}.

We first reduce to $\Omega = 1$. Choose an unramified character χ with $\chi(\varpi)^2 = \Omega^{-1}$ and replace π by $\pi \otimes (\chi \circ \mu)$. Such a character exists because every nonzero complex number has a square root. The central-character conductor is unchanged, and all the compact fixed-vector spaces and their dimensions are unchanged, since $\mu(k) \in \mathfrak{o}^\times$ on the groups in question. The Whittaker-model isomorphism is $W(g) \mapsto \chi(\mu(g))W(g)$. It preserves diagonal support, and an unramified shift preserves the condition that both local factors equal one. After proving the support and one-dimensionality assertions for this twist, twisting back and using the original functional equation recovers $m_\pi = n'$ and the constants in the theorem. Accordingly, *throughout the calculations below $\omega(\varpi) = 1$ and $W^c = D_m W$.*

The action of $K^c(m; e)$ on $V^c(m)$ is the character $k \mapsto \omega(k_{33})^{-1}$.

Established inputs and the interior reduction. Suppose $L(s,\pi) = L(s,\pi^\iota) = 1$, $m = m_\pi$ is the least level, and temporarily $m > 2e$; the boundary is excluded below. Existence of m follows from the preceding existence proposition. The preceding foundational results give:

1. Restriction to the diagonal torus is injective on $V(m)$.
2. The diagonal arrays of a quasi-paramodular vector and its conjugate are finitely supported in $i, r \geq 0$.
3. The analytic functional equation, for some $\epsilon' \neq 0$ and integer n', is
$$q^{-e}\Xi(1-s, W^c) = \epsilon' q^{(m-n')(s-1/2)}\Xi(s, W), \qquad \Xi(s, W) = \sum_{r \geq 0} W(a_r) q^{-r(s-3/2)}.$$
Furthermore $m \geq n'$.
4. At minimal level, $W(a_r^i) = 0$ for $i \geq 1$, and $\Xi(s, W)$ and $\Xi(s, W^c)$ are nonzero when $W \neq 0$.

The first three assertions are the corrected counterparts of Theorem 5.4 and Lemmas 5.11 and 6.2 of [11]. The fourth has the following elementary minimal-level proof: $\pi(\eta^{-1})W$ is averaged over $n_2(x)n_3(y)z(z\varpi^{1-l})$, $x, y, z \in \mathfrak{o}/\mathfrak{p}$, to produce an element of $V(m-1)$. At a_r^{i-1}, $i \geq 1$, its value is $q^3 W(a_r^i)$. The average is zero by minimality. For completeness, membership in $V(m-1)$ follows from the Gaussian factorization in Section 11. Put $H = K_1(m-1; e)$ and $J = \eta^{-1}K_1(m; e)\eta$. Their upper root depths (n_2, n_3, z) are respectively $(0, 0, 1-l)$ and $(1, 1, 2-l)$, and their lower root depths are $(l-1, l-1, m-1)$ and $(l-1, l-1, m-2)$; their Levi factors agree. Thus $H/(H \cap J)$ has precisely the q^3 representatives displayed above, and the sum is the H-average of the J-fixed vector $\pi(\eta^{-1})W$. Here the sign of the translating element is corrected: the displayed invariance matrix in Proposition 6.6 is $\mathrm{Int}(\eta^{-1})K_1(m; e)$, and the indicated torus evaluation requires η^{-1}, although the source prints η.

12.1 The level-lowering operator must carry the character

Put $H = K^c(m-1;e)$. The correct projector, up to a nonzero scalar, is

$$D_\omega v = \int_H \omega(k_{33})\pi^\iota(k)v\,dk. \tag{H1}$$

Indeed $h \mapsto \omega(h_{33})^{-1}$ is the required character on H. Equivalently one may average over $K_1^c(m-1;e)$. The unweighted average *over* H printed in [11, p. 23] is identically zero, since H contains the scalar units and ω is ramified. At minimal level, $D_\omega V^c(m) = 0$.

For reference, the coset table used in the interior calculation is the following. Here x, y, u range over $\mathfrak{o}/\mathfrak{p}$ and z over $\mathfrak{o}/\mathfrak{p}^2$:

$$\begin{aligned} D_\omega \ &\sim \sum_{x,y} \pi^\iota(n_2(x)n_3(y\varpi^{-e})\eta^{-1}) + \sum_{x,y,u} \pi^\iota(\bar{n}_2(x\varpi^{l-1})\bar{n}_3(y\varpi^{m-1})\bar{z}(u\varpi^{m-1})), \\ S^\iota = T_{K_1^c(m;e)}(\eta) &= \sum_{x,y,z} \pi^\iota(n_2(x)n_3(y\varpi^{-e})z(z\varpi^{-m})\eta^{-1}) \\ &\quad + \sum_{x,y,u} \pi^\iota(\eta\bar{n}_2(x\varpi^{l-1})\bar{n}_3(y\varpi^{m-1})\bar{z}(u\varpi^{m-1})). \end{aligned} \tag{H2}$$

In the first line $\sim$ allows the common nonzero normalization of Haar measure. The representatives have trivial character weight. Membership, character weights, the displayed cardinalities q^2+q^3 and q^4+q^3, and the subsequent evaluation are compatible with (H1). Exhaustivity follows from the independent Gaussian factorization and coset checks in Section 11. It is the table asserted on p. 24 of [11], with the character in (H1) restored and $n_3(y)$ replaced by $n_3(y\varpi^{-e})$ in D_ω. The latter correction is necessary: with the printed scaling the q values of y give the same coset, since $\eta n_3(y)\eta^{-1} = n_3(\varpi^{-1}y) \in K_1^c(m;e)$. This correction does not change the torus evaluation, since the Whittaker character is trivial on N_3.

Evaluation of this table on the torus gives, for every $B \in V^c(m)$,

$$(S^\iota B)(a_r^i) = \begin{cases} q^4 B(a_r^1), & i = 0, \\ q^4 B(a_r^{i+1}) - q^2 B(a_r^i), & i \geq 1. \end{cases} \tag{H3}$$

For $i = 0$, the character sum over $n_2(\mathfrak{o}/\mathfrak{p})$ in the comparison with D_ω is zero; for $i \geq 1$ it is q.

No unitarity assumption is needed for the following spectral obstruction:

$$\mathrm{Spec}(S^\iota \mid V^c(m)) \subseteq \{0, -q^2\}. \tag{H4}$$

To prove this, let $B \neq 0$ be an eigenvector of eigenvalue μ. Its restriction to the diagonal torus is nonzero: the functional equation and the first and fourth foundational inputs show that $B(a_r) \neq 0$ for some r. By finite support there is a largest $i_0 \geq 0$ such that $B(a_r^{i_0}) \neq 0$ for some r. If $i_0 = 0$, the first line of (H3) gives $\mu = 0$. If $i_0 \geq 1$, its second line gives $\mu = -q^2$. This proves (H4) without diagonalizability or a positive definite invariant inner product. An operator identity below will exclude the eigenvalue zero and every possible Jordan block simultaneously.

12.2 The two averaging identities with specified Haar measures

Write

$$\mathcal{C}_e = \left\{ \begin{pmatrix} x & 0 \\ y & x \end{pmatrix} : x \in \mathfrak{o},\ y \in \mathfrak{p}^e \right\}, \qquad \mathcal{B}_l = \left\{ \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} : x \in \mathfrak{o},\ y \in \mathfrak{p}^{-l} \right\}.$$

Both of the following operators are *unnormalized sums of q^2 terms*, not unspecified Haar averages:

$$\Gamma W = \sum_{C \in \varpi^{l-1}\mathcal{C}_e/\varpi^l\mathcal{C}_e} \pi(\bar{n}(C))W, \qquad \Gamma^\iota B = \sum_{C \in \varpi^{m-1}\mathcal{B}_l/\varpi^m\mathcal{B}_l} \pi^\iota(\bar{n}(C))B. \tag{H5}$$

Set $d = m - n' \geq 0$, $A_r = W(a_r)$, and $B_r = W^c(a_r)$. Coefficient comparison in the functional equation gives

$$B_r = \epsilon' q^{e+d-2r} A_{d-r}, \qquad A_r = B_r = 0 \quad \text{if } r < 0 \text{ or } r > d. \tag{H6}$$

The correctly normalized averaging identities are

$$(\Gamma W)(a_r) = qA_r - qA_d \mathbf{1}_{r=d}, \qquad (\Gamma^\iota W^c)(a_r) = B_r. \tag{H7}$$

Here is a derivation avoiding the unsupported identification of the largest support index with d in the printed proof of Lemma 6.7. Conjugating the first sum in (H5) by j_m gives the q^2 translates by $n_3(x\varpi^{-e-1})n'(y\varpi^{-1})$ on W^c. Their torus integral is $q^2(\Xi(s, W^c) - B_0)$, by the elementary additive-character sum. The first averaged vector is 0-balanced and its c-conjugate is $(e+1)$-balanced. The functional equation therefore gives

$$q^{1-e}(\Xi(1-s, W^c) - B_0) = \epsilon' q^{d(s-1/2)} \Xi(s, \Gamma W).$$

Subtracting q times the original functional equation and using (H6) proves the first identity of (H7). For the second, conjugation of the sum defining Γ^ι gives q^2 translates of W by $n_3(x\varpi^{-1})z(y\varpi^{-l-1})$. Their torus integral is $q^2\Xi(s, W)$. This conjugate vector is 1-balanced, whereas $\Gamma^\iota W^c$ is $(e-1)$-balanced. The functional equation and (H6) give the second identity of (H7). The required balancedness was proved above directly by matrix conjugation, following the idea of [11, Lemma 5.2]. With the masses specified in (H5), the factor q in the second identity printed in Lemma 6.7 must be removed.

12.3 The corrected co-Hecke coefficient

Let $T = T_K(a_1) + T_K(a_{-1})$ for $K = K_1(m; e)$ and let T^ι be its analogue for $K_1^c(m; e)$. The c-map intertwines T and T^ι. The quasi-Hecke table on pp. 24–25 of [11] gives, for $i, r \geq 0$,

$$\begin{aligned}(TW)(a_r^i) = {} & q^3 W(a_{r+1}^i) + q^2\omega(\varpi)W(a_{r-1}^{i+1}) + (\Gamma W)(a_{r-1}^i) \\ & + q\omega(\varpi)^{-1} \sum_{x,y} W(a_{r+1}^{i-1}\bar{n}_2(x\varpi^{l-1})\bar{z}(y\varpi^{m-1})).\end{aligned} \tag{H8}$$

As in (H2), exhaustivity of the quasi-Hecke table is proved in Section 11. In its minus rank-three cell the actual lattice is $C = \left(\begin{smallmatrix} x & z \\ y & x \end{smallmatrix}\right)$ with $x \in \mathfrak{p}^l$, $y \in \mathfrak{p}^m$, and $z \in \mathfrak{o}$. Thus the upper-right summand in the source's definition of $\widetilde{\mathcal{C}}_e$, before multiplication by ϖ^l, must be $\mathfrak{p}^{-l}$ rather than $\mathfrak{o}$. The source's following line already uses the correctly scaled $\bar{n}'(z)$, $z \in \mathfrak{o}/\mathfrak{p}$. The resulting co-Hecke formula, with the convention (H5), is

$$\begin{aligned}(T^\iota B)(a_r^i) = {} & q^3 B(a_{r+1}^i) + q^2 \sum_{y \in \mathfrak{o}/\mathfrak{p}} B(a_{r-1}^{i+1}\bar{n}'(y\varpi^{e-1})) + (\Gamma^\iota B)(a_{r-1}^i) \\ & + q \sum_{x \in \mathfrak{o}/\mathfrak{p}} B(a_{r+1}^{i-1}\bar{n}_2(x\varpi^{l-1})).\end{aligned} \tag{H9}$$

In particular the second coefficient is q^2, not the q^3 printed in (6.8). Unlike the unspecified normalization in the source, (H9) and (H5) use compatible, explicit counting measures.

Here is an independent verification of this correction from the quasi $T_K(a_1)$ table. Conjugation by j_m identifies it with $T_{K_1^c}(a_{-1})$; the central-character factors cancel. The element $j''_m = \begin{pmatrix} 0 & 0 & 0 & -\varpi^{-m} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \varpi^m & 0 & 0 & 0 \end{pmatrix}$ belongs to $K_1^c(m;e)$. Put $M = \varpi^m$, $E = \varpi^e$. The rank-three cell becomes

$$\bar{n}(C)a_{-1}, \qquad C = \begin{pmatrix} Mx & Ey \\ Mz & Mx \end{pmatrix}, \quad x, y, z \in \mathfrak{o}/\mathfrak{p}.$$

After moving a_{-1} left, put $A = Mx/\varpi$, $B = Ey/\varpi$, $C_0 = Mz/\varpi$. When $z = 0$ the contribution is the first average in the lower co-cell, namely Γ^ι in (H5). For $z \neq 0$, the exact matrix identity is

$$\begin{aligned} &\bar{n}\begin{pmatrix} A & B \\ C_0 & A \end{pmatrix} z(-C_0^{-1}) j''_m \bar{n}_3(AM/C_0) \\ &\qquad = n_2(A/C_0) z(C_0^{-1}) t_1(-M/C_0) \bar{n}'(B - A^2/C_0). \end{aligned} \tag{H10}$$

The three right factors on the left belong to $K_1^c(m;e)$. The unit part of $t_1(-M/C_0) = t_1(-\varpi/z)$ has trivial character, and $A^2/C_0 \in \mathfrak{p}^{m-1} \subseteq \mathfrak{p}^e$. Consequently the value at a_r^i is $B(a_{r-1}^{i+1}\bar{n}'(y\varpi^{e-1}))$. The variables $x, z \neq 0$ contribute $q(q-1)$ copies for each y. The other, rank-two, Weyl cell contributes q copies for each y: indeed, directly,

$$j''_m \bar{n}\begin{pmatrix} Mx & Ey \\ 0 & Mx \end{pmatrix} a_{-1}(j''_m)^{-1} = n_2(x) a_{-1}^1 \bar{n}'(Ey/\varpi).$$

The total multiplicity is $q(q-1) + q = q^2$, proving the coefficient in (H9). Signs in the residue variables are immaterial to these full sums.

12.4 Vanishing of the last terms and the finite recurrence

The last terms of (H8)–(H9), at $i = 0$, vanish when $m > 2e$. Here is a direct check. Write $v = \bar{n}_2(A)\bar{z}(B)$ with $v(A) \geq l - 1$ and $v(B) \geq m - 1$. In the adopted convention,

$$v^{-1} n_2(u) v = \begin{pmatrix} 1 + Au & -u & 0 & 0 \\ A^2 u & 1 - Au & 0 & 0 \\ Bu & 0 & 1 + Au & u \\ -2ABu & Bu & -A^2 u & 1 - Au \end{pmatrix}. \tag{H11}$$

For $u \in \mathfrak{o}$ this lies in $K_1(m;e)$ because $l - 1 \geq e$, $l \geq 2$, and $m - 1 \geq l$. The analogous assertion with $B = 0$ lies in $K_1^c(m;e)$. At the diagonal a_{r+1}^{-1}, conjugation turns $n_2(u)$ into $n_2(u/\varpi)$ on the left. Choosing u with $\psi(u/\varpi) \neq 1$ forces the required Whittaker values to vanish. This proves the relevant instance of Lemma 6.8, including residue characteristic two.

For $r \geq 1$, the elementary GL_2 character calculation gives

$$\sum_{y \in \mathfrak{o}/\mathfrak{p}} B(a_{r-1}^1 \bar{n}'(y\varpi^{e-1})) = B(a_{r-1}^1).$$

For $e > 1$ the nonzero residue terms vanish individually by the ramified-character support condition of [11, Lemma 2.6(i)]. For $e = 1$, their sum vanishes by the unit-average identity in part (ii): right $\bar{N}'(\mathfrak{p})$-invariance converts that integral to the displayed residue sum. These statements use only the

zero assertions, not a choice of nonzero Gauss-sum normalization. For every $W \in V(m)$, equations (H7)–(H8) and (H11) give

$$(TW)(a_j) = q^3 A_{j+1} + qA_{j-1} \quad (0 \le j \le d), \qquad A_{-1} = A_{d+1} = 0. \tag{H12a}$$

No eigenvector has been chosen. The exceptional term in the first identity of (H7) occurs only at $j = d+1$, so it does not enter (H12a). Apply (H6) both to W and to TW. Since $(TW)^c = T^\iota B$, this gives

$$(T^\iota B)(a_r) = q^3 B_{r+1} + qB_{r-1} \quad (0 \le r \le d). \tag{H12b}$$

Here $B_{-1} = B_{d+1} = 0$. On the other hand, (H7), (H9), (H11), and the preceding character calculation give

$$(T^\iota B)(a_r) = q^3 B_{r+1} + q^2 B(a^1_{r-1}) + B_{r-1} \quad (r \ge 1). \tag{H12c}$$

Comparing these formulas for $1 \le r \le d$, and then using the support bound (H6) on $T^\iota B$ at $r = d+1$ and at larger r, proves

$$B(a^1_j) = \begin{cases} (q-1)q^{-2}B_j, & 0 \le j < d, \\ -q^{-2}B_d, & j = d, \\ 0, & j > d. \end{cases} \tag{H12d}$$

Thus the first line of (H3) describes S^ι on the entire torus restriction by two scalars:

$$(S^\iota B)(a_j) = \begin{cases} \alpha B_j, & 0 \le j < d, \\ \beta B_d, & j = d, \\ 0, & j > d, \end{cases} \qquad \alpha = q^2(q-1), \quad \beta = -q^2. \tag{H12e}$$

Restriction $B \mapsto (B_j)_{j \ge 0}$ is injective on $V^c(m)$. Indeed, if all B_j vanish, (H6) makes all A_j vanish; the minimal-level vanishing for $i \ge 1$ and diagonal injectivity then give $W = 0$. Applying (H12e) twice and using this injectivity shows $(S^\iota - \alpha)(S^\iota - \beta) = 0$ on $V^c(m)$. If $C = (S^\iota - \beta)B$ were nonzero for some B, it would therefore be an eigenvector with eigenvalue α. But $q > 1$ implies $\alpha \notin \{0, -q^2\}$, contradicting (H4). Consequently $S^\iota = \beta$ on all of $V^c(m)$, and (H12e) implies

$$B_j = 0 \ (0 \le j < d), \qquad A_j = 0 \ (1 \le j \le d) \quad \text{for every } W \in V(m). \tag{H12f}$$

Suppose $d \ge 1$ and choose $W \ne 0$. The injectivity just proved implies $A_0 \ne 0$. But (H12a) at $j = 1$ gives $(TW)(a_1) = qA_0 \ne 0$, while (H12f), applied to $TW \in V(m)$, requires $(TW)(a_1) = 0$. This contradiction proves $d = 0$. The first line of (H3) now gives $B(a^1_r) = -q^{-2}B_r$, and the second line, with $S^\iota = -q^2$, gives $B(a^i_r) = 0$ for $i \ge 2$. After normalizing $A_0 = 1$, the corrected diagonal values are

$$W(a^i_r) = \mathbf{1}_{(i,r)=(0,0)}, \qquad W^c(a^i_r) = \begin{cases} q^e\epsilon', & (i,r) = (0,0), \\ -q^{e-2}\epsilon', & (i,r) = (1,0), \\ 0, & \text{otherwise.} \end{cases} \tag{H13}$$

Thus $m = n'$. In fact the displayed quasi torus values and torus injectivity already give $\dim V(m) = 1$; the analytic newform criterion proved above gives the same conclusion. The exponent in the second co-value is $e-2$; the exponent $+2$ in [11, (6.6)] already contradicts its preceding recurrence, even before the missing normalization q^e is restored. This operator argument uses neither unitarity nor the classification of generic or nongeneric tempered representations.

12.5 The boundary character average

The character-average calculation in Lemma 6.5 admits the following repair, without a Weil–Deligne conductor argument. Suppose $m = 2e$ and $l = e$. For $t = a_r^i$, $i, r \geq 0$, consider

$$\sum_{x \in \mathfrak{p}^{e-1}/\mathfrak{p}^e} \sum_{y \in \mathfrak{p}^{2e-2}/\mathfrak{p}^{2e}} W\left(t\bar{n}\begin{pmatrix} x & 0 \\ y & x \end{pmatrix}\right). \tag{H14}$$

This is $W(t)$. The proof uses the full-group transformation law $W(gk) = \omega(k_{44})W(g)$ for $k \in K(2e; e)$. For an upper block $B = \left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right)$, conjugation by $\bar{n}(C)$ has blocks

$$\bar{n}(C)^{-1} n(B) \bar{n}(C) = \begin{pmatrix} 1 + BC & B \\ -CBC & 1 - CB \end{pmatrix}. \tag{H15}$$

The left upper element, after conjugation by t, has trivial Whittaker character. Thus whenever the right side belongs to $K(2e; e)$ and $\omega((1 - CB)_{22}) \neq 1$, the summand is zero.

For $e \geq 3$, if $v(x) = e - 1$, take $b = 0$ and choose $a \in \mathfrak{o}$ so that $\omega(1 - ax) \neq 1$. The lower central entry $2axy$ has valuation at least $3e - 3 \geq 2e$; all other entries satisfy the requisite conditions. If $x = 0$ and $y \neq 0$, choose $a = 0$ and $v(b) = e - 1 - v(y)$, with $\omega(1 - by) \neq 1$. The lower central entry by^2 has valuation at least $3e - 3 \geq 2e$. This kills every nonidentity summand.

For $e = 2$, the same proof handles $x \neq 0$ and $v(y) \geq 3$, and also $x = 0$, $v(y) = 3$. If $v(x) = 1$ and $v(y) = 2$, use a combined upper block: choose $a \in \mathfrak{o}$ with $\omega(1 + ax) \neq 1$ and put $b = -2ax/y$. Then

$$(CBC)_{21} = 2axy + by^2 = 0, \qquad (1 - CB)_{22} = 1 + ax.$$

All other valuations satisfy the group conditions: for example $v(bx) \geq 0$, $v(ax^2) \geq 2$, and $v(ay) \geq 2$. This works also in residue characteristic two; no division by 2 occurs. It remains to sum the terms $x = 0$, $v(y) = 2$. For y, y' of this valuation put $b = (y - y')/(yy')$. Then

$$k = \bar{z}(-y')z(b)\bar{z}(y) \in K(4; 2), \qquad k_{11} = y/y', \quad k_{44} = y'/y, \quad k_{41} = 0.$$

Since the upper central subgroup has trivial Whittaker character,

$$W(t\bar{z}(y')) = \omega(y/y')W(t\bar{z}(y)). \tag{H16}$$

Summation over $y = \varpi^2 u$, $u \in (\mathfrak{o}/\mathfrak{p}^2)^\times$, is zero because ω has conductor two. Thus the pointwise vanishing asserted tersely in the source is unnecessary for this exceptional shell; its total contribution is zero.

For $e = 1$, first sum the terms with x a unit. Since C^{-1} is integral, the identity $\bar{n}(C) = n(C^{-1})j(C)n(C^{-1})$ removes the two upper factors using left Whittaker equivariance and right invariance. Writing $C = x\left(\begin{smallmatrix} 1 & 0 \\ v & 1 \end{smallmatrix}\right)$, $v = y/x$, gives

$$W(t\bar{n}(C)) = \omega(x)^{-1} W(t\bar{n}_2(v)j(1_2)).$$

The reparametrization $y \mapsto v$ is bijective for each unit x; therefore the sum over $x \in (\mathfrak{o}/\mathfrak{p})^\times$ is zero. When $x = 0$, apply (H16) separately to the shells $v(y) = 0$ and $v(y) = 1$. In both cases $v(b) \geq -1$, as required by $K(2; 1)$, and the character sums vanish. This proves (H14) for all $e > 0$.

The additional average over $\bar{n}_2(u\varpi^{e-1})$ in Lemma 6.5 leaves the diagonal values of (H14) unchanged. Indeed, its second GL_2 gauge has the $K(e)$ transformation property. For $e > 1$ the nonzero residues vanish individually by Lemma 2.6(i), since the diagonal index i is nonnegative. For $e = 1$ their sum vanishes by the unit-average identity in part (ii); right invariance under the

lower congruence subgroup converts the integral to the finite residue sum. Stability of the lattice in (H14) under the $K(e)$ action is checked by $C \mapsto {w_2}^t A w_2 C A$, modulo the upper-right entry of this lower Siegel block. That entry is integral and may be removed using right $\bar{N}'(\mathfrak{o})$-invariance; it need not remain literally zero. This use again requires only the zero assertions of Lemma 2.6, not its nonzero Gauss-sum constants.

The coset formula (6.3) cannot hold as printed: its first term contains η where η^{-1} is required, and its second contains η^{-1} where η is required. Indeed every printed upper representative lies in $K \cap \eta K \eta^{-1}$, so those terms represent the same right coset. The corrected table at the boundary is

$$\begin{aligned} S = &\sum_{x,y \bmod \mathfrak{p},\ z \bmod \mathfrak{p}^2} \pi(n_2(x) n_3(y) z(z\varpi^{-e})\eta^{-1}) \\ &+ \sum_{x,y \bmod \mathfrak{p},\ z \bmod \mathfrak{p}^2} \pi(\eta \bar{n}_2(x\varpi^{e-1}) \bar{n}_3(y\varpi^{e-1}) \bar{z}(z\varpi^{2e-2})). \end{aligned} \tag{H17}$$

The corrected table (H17) is proved independently in Section 11. Consequently, the boundary operator $S = T_K(\eta) + T_K(\eta^{-1})$ has, on an eigenvector, the recurrence

$$\lambda_S W(a_r^i) = q^4 W(a_r^{i+1}) + W(a_r^{i-1}), \qquad i \geq 1.$$

Admissibility makes $V(2e)$ finite dimensional. If this space is nonzero, the complex linear operator S has an eigenvector; no self-adjointness assumption is required. Finite support supplies a maximal i_0 with some $W(a_r^{i_0}) \neq 0$. Evaluating at $i = i_0 + 1$ is a contradiction. Thus $V(2e) = 0$ for arbitrary irreducible admissible generic π with $L(s, \pi) = 1$. This is the appropriate proof of $m_\pi > 2e$ before epsilon-factor compatibility has been established.

This excludes the boundary and completes the proof of Theorem 12.1 in the normalization $\Omega = 1$. For an arbitrary Ω, twisting back preserves all the established zero values and the one-dimensional minimal space. The normalized conjugation of that vector has the same two-term diagonal support and ratio $-q^{-2}$, because an unramified twist is trivial on $t_1(\varpi)$ and is constant on the conjugating element j_m. Its unshifted zeta integral is therefore constant. Since $Z(s, W_\pi) = 1$, the original functional equation gives $m_\pi = n'$ and $Z(s, D_{m_\pi} W_\pi) = \epsilon'$. The balance identity $Z(s, D_{m_\pi} W_\pi) = q^{-e} \Xi(s, D_{m_\pi} W_\pi)$ gives exactly the constants stated in the theorem.

Thus a standard-module deformation and a classification of nongeneric tempered representations are unnecessary for this result. The removal of unitarity is an algebraic modification of Okazaki's Hecke method: compare both actions of T before choosing an eigenvector, then use the polynomial identity for S^ι.

13 Full factor comparison for square central character

Proposition 13.1. *Let $F/\mathbb{Q}_p$ be finite, and let π be an irreducible generic representation of $\mathrm{GSp}_4(F)$ whose central character is a square in the group of smooth complex characters: $\omega_\pi = \chi^2$. Then the analytic Novodvorsky factors, in the normalization used here, satisfy*

$$L(s, \pi) = L(s, \phi_\pi), \qquad \varepsilon(s, \pi, \psi) = \varepsilon(s, \phi_\pi, \psi).$$

In particular,

$$n'_\pi = a(\phi_\pi), \qquad \varepsilon'_\pi = \varepsilon(1/2, \phi_\pi, \psi)$$

when ψ has conductor $\mathcal{O}$.

Proof. Put $\pi_0 = \pi \otimes (\chi^{-1} \circ \mu)$. Its central character is trivial, because $\mu(zI_4) = z^2$, so it descends to $\mathrm{PGSp}_4(F) \simeq \mathrm{SO}_5(F)$. The rank-one local-factor comparison recalled in [17, Remark 1.2] states that the $\mathrm{SO}_5 \times \mathrm{GL}_1$ Rankin–Selberg L- and epsilon factors coincide with the Weil–Deligne factors, and identifies these rank-one integrals with the Novodvorsky integrals. This is the $r = 1$ statement; it is distinct from the GL_2 comparison in Cheng's Theorem 1.3. There is no sign correction here: Cheng's general correction $\omega_\tau(-1)^n$ in Theorem 2.3 and his definition of epsilon factors is $\chi(-1)^2 = 1$ for $n = 2, r = 1$.

Here is the exact transport of that functional equation. For $W_0 \in \mathcal{W}(\pi_0, \psi)$ put

$$W_\chi(g) = \chi(\mu(g))W_0(g), \qquad Z(s, W_0; \chi) = \int_{F^\times} \int_F W_0(a'(t)\bar{n}'(z))\chi(t)|t|^{s-3/2}\, dz\, d^\times t.$$

Multiplication by $\chi \circ \mu$ is a bijection from the full Whittaker space of π_0 to that of π, and

$$Z(s, W_\chi) = Z(s, W_0; \chi). \tag{S1}$$

Since $\mu(j_0) = 1$ and $\omega_\pi = \chi^2$, the normalized dual transform satisfies the pointwise identity

$$D_0^\pi W_\chi(g) = \chi(\mu(g))^{-1} D_0^{\pi_0} W_0(g).$$

Consequently

$$Z(1 - s, D_0^\pi W_\chi) = Z(1 - s, D_0^{\pi_0} W_0; \chi^{-1}). \tag{S2}$$

Equations (S1)–(S2) identify both integral families, their fractional ideals, and their normalized functional equations. They therefore identify the analytic factors of π with the rank-one factors of $\pi_0 \times \chi$.

The local Langlands correspondence is compatible with similitude twisting [14, Main Theorem (iv)]; the parameters under the $\mathrm{PGSp}_4 \simeq \mathrm{SO}_5$ identification agree, as explicitly noted in [17, §3.2.3]. Thus, as four-dimensional Weil–Deligne representations,

$$\phi_\pi \simeq \phi_{\pi_0} \otimes \chi.$$

The rank-one comparison just cited now gives the assertions. The parameter ϕ_{π_0} is symplectic and self-dual, so the dual parameter in (S2) is exactly $\phi_{\pi_0} \otimes \chi^{-1} = \phi_\pi^\vee$. □

Lemma 13.2 (Which central characters are squares)**.** *A smooth complex character ω of $F^\times$, for a finite extension $F/\mathbb{Q}_p$, admits a smooth square root if and only if $\omega(-1) = 1$.*

Proof. If $\omega = \chi^2$, then $\omega(-1) = \chi((-1)^2) = 1$. Conversely suppose $\omega(-1) = 1$. On the subgroup $H = (F^\times)^2$ define

$$\chi_H(x^2) = \omega(x).$$

This is well defined, because the kernel of the square map is $\{1, -1\}$, and is a character. It is smooth: the square map $F^\times \to H$ is open, and the squares of a sufficiently small open subgroup on which ω is trivial form an open subgroup on which χ_H is trivial. The subgroup H has finite index in $F^\times$, also in residue characteristic two.

Every complex character of a finite-index subgroup of an abelian group extends to the group, since $\mathbb{C}^\times$ is divisible. Explicitly, adjoin one generator of the finite quotient at a time: if its first power lying in the current subgroup is g^a, choose an a-th root of the prescribed value at g^a as the value at g. This constructs an extension χ of χ_H. It remains smooth because its kernel contains the open subgroup of H already lying in the kernel of χ_H. For every $x \in F^\times$, $\chi(x)^2 = \chi(x^2) = \chi_H(x^2) = \omega(x)$. □

Relation to fixed-vector statements. The argument transports functional equations on entire Whittaker spaces. It does not identify $K_1(m;e)$-fixed spaces under a ramified twist, and does not assert that their levels are unchanged. Its consequence for the newform results proved in this manuscript is that every occurrence of n'_π and ε'_π can be replaced by the parameter conductor and root number whenever $\omega_\pi(-1) = 1$. Existence, minimality, and multiplicity one continue to come from the separate fixed-vector arguments.

14 Assembly of the analytic newform theorem

Proof of Theorem 1.1. For $e = 0$, apply the unramified twist of Roberts–Schmidt described in Theorem 5.1. Suppose $e > 0$. Proposition 8.10 supplies a nonzero vector at some level, so the least level is defined. The restriction and translated integral arguments prove vanishing below n'_π and dimension at most one at n'_π. If $L(s,\pi) = 1$, the dual factor is also one by parameter compatibility. The finite-support and balancedness lemmas, together with the proved coset decompositions, supply all inputs to Theorem 12.1. That theorem gives the normalized vector at level n'_π and both analytic integral identities. If $L(s,\pi) \neq 1$, the classification argument at the end of the split-theta section places π in its generic small image. Equations (9.1)–(9.2) at $r = 0$ and the two-test-vector criterion give the same conclusion, and identify the conductor and signed root number in that subclass. The same theta argument applies to every generic small split-theta lift, even when its spin factor is one. Theorem 15.1 below gives $n'_\pi = a(\phi_\pi)$ and $\varepsilon'_\pi = \omega(-1)\varepsilon(1/2, \phi_\pi, \psi)$ for every generic π, completing the main theorem. Finally the higher-level upper bound was proved by the filtration by the first nonzero translated torus row. □

15 A normalized split seesaw comparison

The period geometry used here is the split specialization of Prasad–Takloo-Bighash [19, Theorem 5 and Section 11]. The Whittaker–Shalika quotient map is due to Gan–Takeda [20, Propositions 3.1 and 7.4]. The functional equation is Friedberg–Jacquet's [21]; the arbitrary Shalika-character normalization used below is recorded in [22, Section 3.1]. The calculation compares the specified integral families and their Weyl actions simultaneously, rather than choosing unrelated generators of their one-dimensional Hom spaces.

Theorem 15.1. *For every irreducible admissible generic representation π of G, with the Whittaker character and operator D_0 fixed in this note,*

$$\gamma_{\mathrm{Nov}}(s,\pi,\psi) = \omega_\pi(-1)\gamma(s,\phi_\pi,\psi), \qquad \varepsilon_{\mathrm{Nov}}(s,\pi,\psi) = \omega_\pi(-1)\varepsilon(s,\phi_\pi,\psi).$$

In particular $n'_\pi = a(\phi_\pi)$. With $D_0^{\mathrm{std}} = \omega_\pi(-1)D_0$, the functional-equation factors are the standard parameter factors. No restriction on the residue characteristic or the central character is imposed.

Proof. Let Π be the generic spin transfer of π to $\mathrm{GL}_4(F)$. It has central character ω^2 and an (ω, ψ^{-1})-Shalika model by [20, Theorem 1.5(i)]. For clarity, the quadratic and symplectic coordinates are fixed below. Let $E = F^4$ with ordered basis $f_1, \dots, f_4$, put $V = \bigwedge^2 E$, and define $Q = \mathrm{Pf}$ by

$$Q(x) = x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23}, \qquad b(x,y) = Q(x+y) - Q(x) - Q(y).$$

Write $\alpha = f_{12}$, $\beta = f_{34}$, and $v_0 = f_{14} - f_{23}$. Then

$$Q(\alpha) = 0, \quad Q(v_0) = -1, \quad b(\alpha, v_0) = 0, \qquad b(\alpha,\beta) = 1.$$

The similitude group is $\mathcal{H} = \mathrm{GSO}(V) = (\mathrm{GL}_4 \times F^\times)/\{(zI_4, z^{-2})\}$; (A, c) acts on V as $c \bigwedge^2 A$ and has multiplier $\nu(A, c) = c^2 \det A$. Put $\mathcal{H}^1 = \mathrm{SO}(V)$.

Use the alternating form J of this note. The old polarization is $p = (e_1, e_2)$, $q = (-e_4, -e_3)$, and the Bessel polarization is $p' = (e_1, e_3)$, $q' = (-e_4, e_2)$. Both have pairing matrix I_2. Fix the convention

$$\Omega(n(B))\Phi(X) = \psi\Big(\frac{1}{2}\operatorname{tr}(\mathrm{Gram}(X)B)\Big)\,\Phi(X), \qquad \Omega(m(A))\Phi(X) = |\det A|^3\Phi(XA).$$

The character of the old upper maximal unipotent is the one in the draft: $n_2(y) = 1 - yE_{12} + yE_{34}$ and $n'(x) = 1 + xE_{23}$ act through $\psi(y + x)$. Indeed $n_2(y)$ has old Levi block $\left(\begin{smallmatrix} 1 & -y \\ 0 & 1 \end{smallmatrix}\right)$, and $n'(x)$ has old Siegel block with $B_{22} = -x$. The elementary Whittaker quotient map is therefore normalized by

$$\mathcal{A}\Phi(h) = \int_F \Phi(h^{-1}\alpha, h^{-1}(v_0 + u\alpha))\psi(u)\,du. \tag{15.1}$$

The assertions of equivariance can be checked directly on the four root subgroups. Formula (15.1) is the coordinate version of the local twisted Jacquet computation in [20, Proposition 7.4(i)], whose global unfolding is [20, Proposition 3.1].

Partial Fourier transform and the two constraints The change from the old polarization to the Bessel polarization can be written, with the compatible self-dual measures, as

$$\Phi_{\mathrm{old}}(x, z) = \int_V \Phi(x, y)\psi(b(z, y))\,dy. \tag{15.2}$$

Substituting (15.2) into (15.1) and integrating u imposes $b(\alpha, y) = -1$, and leaves the phase $\psi(b(v_0, y))$. Any overall polarization constant is nonzero and independent of the representation, the Mellin parameter, and its character twist.

In the Bessel polarization the subgroup $(1 + xE_{32})n_2(y)(1 + zE_{14})$ has block $\left(\begin{smallmatrix} -z & -y \\ -y & x \end{smallmatrix}\right)$. Thus $1 + xE_{32}$ acts by $\psi(xQ(y))$ on the second variable. For $a(t) = \mathrm{diag}(t, t, 1, 1)$, the Levi block is $\mathrm{diag}(t, 1)$ and the multiplier is t. If $h \in \mathcal{H}$ has multiplier t, the joint Weil action is

$$\Omega(a(t), h)\Phi(x_1, x_2) = \Phi(th^{-1}x_1, h^{-1}x_2)\,: \tag{15.3}$$

the scalar $|t|^3$ from the Levi and $|t|^{-3}$ from the similitude cancel. Consequently the subsequent integration in x imposes $Q(y) = 0$ with coefficient $|t|$, since its phase is $\psi(xt^{-1}Q(y))$.

This constraint can be justified by stable integration: on the affine hyperplane $b(\alpha, y) = -1$ write $y = -\beta + z + c\alpha$ with z in the four-dimensional cross-block space. The derivative of $Q(y)$ with respect to c is -1. Thus Q is a submersion there, and integrating over successively larger compact subgroups in x gives the Gelfand–Leray measure on $Q(y) = 0$. There is no singular point and no factor of 2, including at residual characteristic two.

For $B \in M_2(F)$ let $n(B) = \left(\begin{smallmatrix} I_2 & B \\ 0 & I_2 \end{smallmatrix}\right)$. The fiber just obtained is precisely

$$\{y : b(\alpha, y) = -1,\ Q(y) = 0\} = \{\bigwedge^2 n(B)(-\beta) : B \in M_2(F)\}. \tag{15.4}$$

Moreover

$$b(v_0, \bigwedge^2 n(B)(-\beta)) = \operatorname{tr} B. \tag{15.5}$$

This can be verified by expanding

$$\bigwedge^2 n(B)\beta = \beta - b_{12}f_{13} + b_{11}f_{14} - b_{22}f_{23} + b_{21}f_{24} + \det B\,\alpha.$$

The fiber measure in (15.4) is a fixed nonzero multiple of dB.

Unfolding the Shalika unipotent Let S be a Shalika function on a generic GL_4 representation Π, with multiplier character ω and additive character ψ^{-1}, and regard it as a function on $\mathcal{H}$ by $S(A,c)=\omega(c)S(A)$. The inverse additive character is dictated by pairing (15.1) with S: the kernel in (15.5) has character $\psi(\operatorname{tr} B)$. Let Δ be the image of

$$k \longmapsto (\mathrm{diag}(k,k), \det(k)^{-1});$$

it is $\mathrm{PGL}_2(F)$, not merely the image of $\mathrm{SL}_2(F)$. The embedded Shalika group is $\Delta \ltimes N_{\mathrm{Sh}}$.

Let $W_{\Phi,S}$ denote the Whittaker function obtained by the local analogue of Gan–Takeda Proposition 3.1, with (15.1)–(15.3) as its normalizations. Then the unipotent unfolding (15.4)–(15.5) gives

$$Z(s,\chi,W_{\Phi,S}) = C\int_{F^\times}\chi(t)|t|^{s-1/2}\int_{\Delta\backslash\mathcal{H}_t} S(h)\Phi(h^{-1}(t\alpha,-\beta))\,dh\,d^\times t, \tag{15.6}$$

where $\mathcal{H}_t=\nu^{-1}(t)$, and $C\neq 0$ depends only on the fixed measure and polarization conventions. It is independent of s, χ, and Π. The crucial exponent is $s-1/2$, since the original Novodvorsky exponent $s-3/2$ acquired exactly $|t|$ above.

Put $\mathcal{H}_0=\mathrm{Stab}_{\mathcal{H}^1}(\alpha,-\beta)$. Its full group of F-points is

$$\mathcal{H}_0=\{(\mathrm{diag}(A,B),c) : c\det A = c\det B = 1\}/F^\times.$$

It is $\mathrm{SO}(2,2)(F)$; it must not be replaced by $\mathrm{SL}_2(F)\times\mathrm{SL}_2(F)$ without accounting for the spinor-norm cosets. Nevertheless

$$\Delta\backslash\mathcal{H}_0 \simeq \mathrm{SL}_2(F),$$

with representative $(\mathrm{diag}(g,I_2),1)$, because left multiplication by the diagonal element associated to B^{-1} sends $(\mathrm{diag}(A,B),c)$ to $(\mathrm{diag}(B^{-1}A,I_2),1)$.

Choose $d_t=\mathrm{diag}(t,1)$ and $h_t=(\mathrm{diag}(d_t,I_2),1)$. It sends $(\alpha,-\beta)$ to $(t\alpha,-\beta)$ and normalizes $\mathcal{H}_0$. Disintegrating (15.6) with $h=rh_tk$ yields

$$\boxed{Z(s,\chi,W_{\Phi,S}) = C\int_{\mathcal{H}_0\backslash\mathcal{H}^1}\Phi(k^{-1}(\alpha,-\beta))\,\ell_{s,\chi}(R(k)S)\,dk,} \tag{15.7}$$

where

$$\ell_{s,\chi}(S)=\int_{\mathrm{GL}_2(F)} S(\mathrm{diag}(g,I_2))\chi(\det g)|\det g|^{s-1/2}\,dg.$$

Indeed the inner r-integral is the SL_2 integral of $S(\mathrm{diag}(gd_t,I_2)k)$; combining it with $d^\times t$ is Haar measure on GL_2. Conjugation by h_t preserves the Haar measures on $\mathcal{H}^1$ and $\mathcal{H}_0$, so this disintegration introduces no power of $|t|$. The orbit is closed and nondegenerate. The restriction of a compactly supported Schwartz function to it is compactly supported. Thus the outer integral in (15.7) is compactly supported; for $\Re(s)$ sufficiently large all remaining integrals converge, and the identity then continues rationally in q^{-s}. The corresponding assertion for stable unipotent integration follows first with compact truncations and then from the submersion computation above.

The Weyl element and the central-character sign Let $w=\left(\begin{smallmatrix}0 & I_2\\ I_2 & 0\end{smallmatrix}\right)$, $D=\mathrm{diag}(-1,1)$, and $h_0=\mathrm{diag}(D,D)w$. Then $\det h_0=1$, $h_0^2=1$, and $\bigwedge^2 h_0$ exchanges α and $-\beta$. In the Bessel polarization $j_0=m(w_2)$, so $\Omega(j_0)\Phi(x_1,x_2)=\Phi(x_2,x_1)$, with no Weil scalar. Writing $F_\Phi(k)=\Phi(k^{-1}(\alpha,-\beta))$, one has

$$F_{\Omega(j_0)\Phi}(k)=F_\Phi(h_0^{-1}k). \tag{15.8}$$

The Friedberg–Jacquet functional equation can be written entirely in Π as

$$\ell_{1-s,\omega^{-1}}(R(w)S) = \gamma(s,\Pi,\psi^{-1})\ell_{s,1}(S). \tag{15.9}$$

Since $\det \phi_\Pi = \omega^2$, its right side is unchanged on replacing ψ^{-1} by ψ. On the other hand,

$$\ell_{1-s,\omega^{-1}}(R(h_0)S) = \omega(-1)\ell_{1-s,\omega^{-1}}(R(w)S). \tag{15.10}$$

Combining (15.7)–(15.10) with theta equivariance and $Z(1-s,D_0W) = Z(1-s,\omega^{-1},R(j_0)W)$ therefore gives

$$Z(1-s,D_0W) = \omega(-1)\gamma(s,\Pi,\psi)Z(s,W). \tag{15.11}$$

The scalar C cancels. Changing Q to $-Q$ without also changing the Whittaker frame and partial-Fourier signs is not a valid way of deleting $\omega(-1)$. The theta representation and its spin parameter are identified below.

Descent of the constructed functions For generic Π with its Shalika model, Gan–Takeda Proposition 7.4 and Corollary 7.5 give a one-dimensional Whittaker Jacquet module for the big theta lift, and Theorem 7.6 gives a generic small quotient. The surjection from big to small induces an isomorphism of their one-dimensional Whittaker Jacquet modules. Hence the Whittaker map used above factors through the small quotient. No equality of the big and small theta representations is needed. Its nonzero image is the entire irreducible Whittaker model, so an identity established on all $W_{\Phi,S}$ extends to that model by linearity.

Matching the similitude convention and the spin parameter Gan–Takeda use matching inverse similitudes, whereas (15.3) uses matching similitudes. This difference can be handled explicitly. Define

$$\iota(h) = \nu(h)^{-1}h \quad (h \in \mathcal{H}).$$

It is an involutive automorphism, with $\nu(\iota(h)) = \nu(h)^{-1}$. In the presentation by pairs,

$$\iota(A,c) = (A, c^{-1}(\det A)^{-1}).$$

The two Weil representations are related by $\Omega_{\text{common}}(g,h) = \Omega_{\text{GT}}(g,\iota(h))$. This can be verified directly from their similitude extension formulas; on the symplectic and isometry subgroups the formulas coincide.

For $\Sigma = \Pi \boxtimes \omega$, the symplectic-similitude parameter implies $\Pi^\vee \simeq \Pi \otimes (\omega^{-1} \circ \det)$, and therefore

$$\Sigma \circ \iota = \Pi^\vee \boxtimes \omega^{-1} = \Sigma^\vee.$$

The integral above pairs Ω_{common} with Σ, so it realizes $\Theta_{\text{common}}(\Sigma^\vee)$; after applying ι this is precisely $\Theta_{\text{GT}}(\Sigma)$. Thus its generic small quotient has spin parameter ϕ_Π, and central character ω. This convention change contributes no additional scalar. The one-dimensional Whittaker argument above justifies descent of the explicit integral to that quotient.

Consequently, for the exact J, $\psi(x+y)$, and D_0 used here, the calculation establishes the comparison

$$\boxed{\gamma_{\text{Nov}}(s,\pi,\psi) = \omega(-1)\gamma(s,\phi_\pi,\psi).}$$

In particular the conductor exponents agree, while the constant root numbers agree only after retaining the factor $\omega(-1)$.

Checks on the polarization formulas There are direct checks of the two normalization steps most susceptible to sign errors. Under (15.1), $n_2(y)$ replaces u by $u-y$; the change of variable gives the factor $\psi(y)$. The root $n'(x)$ multiplies the integrand by $\psi(-xQ(v_0+u\alpha))=\psi(x)$. The other two old positive root subgroups act trivially because $Q(\alpha)=b(\alpha,v_0)=0$. Thus (15.1) has exactly the required character, not its variant with one simple-root sign reversed.

Formula (15.3) also follows directly from (15.2). In the old polarization, $a(t)$ has Levi block tI_2, so its joint action is $|t|^3\Phi_{\mathrm{old}}(th^{-1}x,th^{-1}z)$. Substitute (15.2):

$$|t|^3\int_V \Phi(th^{-1}x,y)\psi(b(th^{-1}z,y))\,dy.$$

Since $\nu(h)=t$, the phase is $\psi(b(z,hy))$. The substitution $y'=hy$ has Jacobian $|\det_V h|=|t|^3$, cancelling the displayed prefactor and giving precisely (15.3). Hence there is no hidden power of $|t|$ in the partial Fourier transform.

Finally the comparison is nonzero. For any fixed generic s choose S with $\ell_s(S)\neq 0$, and choose a sufficiently small compact open subgroup of $\mathcal{H}^1$ fixing S. A compactly supported function on the orbit near its base point can then make the right side of (15.7) a nonzero fixed multiple of $\ell_s(S)$. Such a function extends to a Schwartz function on V^2, since the orbit is closed. This also proves nonvanishing of the explicit Whittaker theta construction.

Definition of the Whittaker theta integral and stable Fubini For clarity, the explicit function used above is the following. For $g\in \mathrm{GSp}_4(F)$ choose $h_g\in\mathcal{H}$ with $\nu(h_g)=\mu(g)$, and set

$$W_{\Phi,S}(g)=\int_{\mathrm{Sh}^1\backslash\mathcal{H}^1} S_\Sigma(hh_g)\int_F[\Omega_{\mathrm{old}}(g,hh_g)\mathcal{F}_2\Phi](\alpha,v_0+u\alpha)\psi(u)\,du\,dh. \tag{15.12}$$

Here $\mathrm{Sh}^1=\Delta\ltimes N_{\mathrm{Sh}}$, $S_\Sigma(A,c)=\omega(c)S(A)$, and $\mathcal{F}_2$ is (15.2). The Shalika characters of the two factors cancel. Replacement of h_g by another choice is a right change of variable in h, so (15.12) is independent of that choice. For fixed g the inner function is compactly supported modulo Sh^1, by the explicit orbit-map description underlying Gan–Takeda Proposition 7.4(i), of which (15.1) is the coordinate realization. The integral is therefore well defined. Its right j_0-equivariance is immediate from (15.12).

One can make the interchange with the opposite-root integral uniform. A Schwartz function Φ is uniformly invariant under translations of its second variable by a sufficiently small lattice. Its first variable ranges in a fixed compact set on its support. Hence there is an integer M such that

$$\Phi(x_1,x_2+cx_1)=\Phi(x_1,x_2)\qquad(c\in\mathfrak{p}^M)$$

for all x_1,x_2. After imposing $b(\alpha,y)=-1$, use the change $y\mapsto y+ct\alpha$. It adds cx_1 to the actual second argument $x_2=h^{-1}y$, where $x_1=th^{-1}\alpha$, while

$$t^{-1}Q(y+ct\alpha)=t^{-1}Q(y)-c,\qquad b(v_0,y+ct\alpha)=b(v_0,y).$$

Thus the opposite-root integral vanishes pointwise for $x\notin\mathfrak{p}^{-M}$, uniformly in h and t. Fourier inversion in the transverse c-coordinate is consequently valid with a fixed compact truncation of x; it gives exactly the $|t|$ coefficient and the fiber measure used in (15.6).

To justify all reorderings, first restrict t to a compact subset of $F^\times$ and use this fixed compact truncation in x. Smoothness and the compact-induction property give common compact supports modulo the Shalika group for the remaining Whittaker kernels. After the fiber calculation and Shalika unfolding, disintegration bounds the resulting absolute integrals by finitely many absolutely

convergent Friedberg–Jacquet integrals when $\Re(s)$ is sufficiently large. The Novodvorsky integral also converges absolutely in this half-plane. Thus both sides have limits as the compact restriction on t is removed; rational continuation then proves the identity for all s. This argument does not treat a formal delta function as an absolutely convergent integral over the whole field.

The equality of local L-factors also follows The forward identity (15.7) identifies the fractional ideals generated by the Novodvorsky and Friedberg–Jacquet integrals, without assuming an independent comparison of their L-factors. For a fixed Φ, S, choose a compact open subgroup fixing S. Compactness of the orbit support then writes (15.7) as a finite complex linear combination of $\ell_s(R(k_i)S)$, with coefficients independent of s. Since the functions $W_{\Phi,S}$ span the irreducible Whittaker model, this gives one ideal inclusion.

Conversely, for any fixed S, choose a sufficiently small compact open subgroup $K \subset \mathcal{H}^1$ fixing it, and choose Φ whose restriction to the orbit is supported on the image of $\mathcal{H}_0 K$ and has nonzero integral. The integrand then has $\ell_s(R(k)S) = \ell_s(S)$ throughout that support: left $\mathcal{H}_0$-invariance follows from $c \det A = c \det B = 1$, which cancels the multiplier character and all Mellin twists. Thus (15.7) equals a fixed nonzero constant times $\ell_s(S)$. This proves the reverse inclusion. The distinguished normalized generators are consequently equal:

$$L_{\mathrm{Nov}}(s, \pi) = L(s, \Pi) = L(s, \phi_\pi).$$

Together with (15.11) this yields

$$\epsilon_{\mathrm{Nov}}(s, \pi, \psi) = \omega(-1)\epsilon(s, \phi_\pi, \psi)$$

for the normalization of D_0 specified in this note.

The nonzero sign $\omega(-1)$ is independent of s, so the conductor exponents agree. □

Remark 15.2 (An independent calibration of the sign)**.** The short-root correction in the split-theta construction gives the same factor $\omega(-1)$. It is invisible for square central character, since a smooth character is a square exactly when its value at -1 is 1. Thus the square-character comparison above agrees with the present theorem, while the unsigned comparison would be incorrect with the original D_0 when $\omega(-1) = -1$.

16 An obstruction in the proposed transfer to GL_4

The calculations below concern formulas in Section 8 of Okazaki [11], not the truth of the proposed newform theorem. The Fourier transform studied here is an explicitly defined candidate obtained by retaining the printed inverse-matrix argument and changing its modulus. It has not been identified with the actual Weil action for the stated exterior-square embedding. In particular, the vanishing statements below are not counterexample representations. They show why those elementary changes alone do not establish the point evaluation used in (8.4)–(8.5). The normalized seesaw in Section 15 proves the local-factor comparison without this explicit test-vector transfer; the diagnostics below are retained to distinguish that proof from the failed point-evaluation argument.

Write $e = a(\omega) > 0$, $m \geq 2e$, and $l = m - e$, and set $a_l = \mathrm{diag}(\varpi^l, 1, 1) \in \mathrm{GL}_3(F)$. For $z \in M_{3\times 4}(F)$, let $\check{z}$ be its first three columns and z_4 its fourth column. The corrected support condition is

$$\Phi_m(z) = \omega(\det \check{z})\, \mathbf{1}_{\mathrm{GL}_3(\mathfrak{o})a_l}(\check{z})\, \mathbf{1}_{\mathfrak{o}^3}(z_4). \tag{16.1}$$

Here the product is interpreted as zero off the displayed support. The right coset $\mathrm{GL}_3(\mathfrak{o})a_l$, rather than the printed $a_l\mathrm{GL}_3(\mathfrak{o})$, is required by the claimed first-column valuation condition and by invariance under left multiplication by $\mathrm{GL}_3(\mathfrak{o})$.

There is a necessary correction to the similitude embedding. For $h = \left(\begin{smallmatrix} g & s \\ 0 & 1 \end{smallmatrix}\right)$, the upper diagonal block of the exterior-square action is $\det(g) w_3 {}^t g^{-1} w_3$, not $w_3 {}^t g^{-1} w_3$. If one retains the printed inverse-matrix argument, Fourier compatibility forces the following candidate moduli:

$$\omega_{\mathrm{Weil}}(1, m(a))\Phi(z) = |\det a|^{-2}\Phi(a^{-1}z), \qquad \omega_{\mathrm{Weil}}(g,h)\Phi(z) = |\mu(g)|^3 \omega_{\mathrm{Weil}}(1,h_1)\Phi(zg). \tag{16.2}$$

These exponents are opposite to those printed. For example, Fourier transformation of $\Phi(a^{-1}z)$ introduces $|\det a|^4$ on $M_{3\times 4}$; compatibility with the Weyl relation forces a scalar exponent k to satisfy $k + 4 = -k$, so $k = -2$. Trivial action of the simultaneous central scalar then forces the similitude exponent $+3$. However, this does not determine the correct argument of the genuine Weil operator. In particular, replacing $a^{-1}z$ by ${}^t az$ changes the required Levi modulus to $|\det a|^2$. The embedding, the action on X^+, and the similitude extension must be derived together.

Even the retained-argument candidate fails the compact covariance needed in (8.1). Indeed, take $u \in \mathfrak{o}^\times$, $k = \mathrm{diag}(u,1,1,1) \in K_1(m) \subset \mathrm{GL}_4(F)$, and $g' = \mathrm{diag}(u,u,1,1) \in K_1(m;e)$. With the printed choice of dividing the upper orthogonal block by the common similitude, the orthogonal Levi parameter is $a_0 = \mathrm{diag}(u,1,1)$. The candidate action is therefore

$$\Phi_m(a_0^{-1} z g') = \omega(u)\Phi_m(z).$$

The support is preserved, the first-three-column determinant gains $u^{-1}u^2 = u$, and all modulus factors are one. But $\pi(g')W = W$. If $\omega(u) \neq 1$, the asserted product invariance fails. This is a direct obstruction to using the retained-argument formulas for the proposed transfer; it does not rule out a different consistent Weil realization.

Conversely, changing the Levi argument to ${}^t az$ and dividing the lower orthogonal block by the common similitude repairs this particular unit calculation, but does not repair the transfer. At the stated evaluation point z_0, the element $a = 1 + xE_{12}$ then changes the second row to $\varepsilon_{-2} + x\varepsilon_2$; this corresponds to a lower, rather than upper, symplectic root. Thus the required upper unipotent Whittaker covariance is no longer obtained from the printed functional. A complete replacement must derive the polarization, exterior-square action, evaluation orbit, and intertwining functional together. Neither candidate is asserted to be such a replacement.

To state the resulting obstruction without an ambiguity in the Fourier convention, use the alternating form

$$B(y,x) = y_1x_4 + y_2x_3 - y_3x_2 - y_4x_1 \quad (x, y \in F^4),$$

and choose the self-dual additive Haar measure, so $\mathrm{vol}(\mathfrak{o}) = 1$. Thus ψ is trivial on $\mathfrak{o}$ and nontrivial on $\varpi^{-1}\mathfrak{o}$. Set

$$J_m = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ \varpi^m & 0 & 0 & 0 \\ 0 & -\varpi^m & 0 & 0 \end{pmatrix}, \qquad z_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

For row vectors y_1, y_2, y_3, define the candidate partial Fourier transform suggested by retaining the arguments of the displayed decomposition on p. 38 by

$$\Phi_m^c \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = q^{-m} \int_{F^4}\int_{F^4} \Phi_m \begin{pmatrix} x_1 J_m \\ x_2 J_m \\ \varpi^{-m} y_3 J_m \end{pmatrix} \psi(-B(y_1,x_1) - B(y_2,x_2))\, dx_1\, dx_2. \tag{16.3}$$

Within this candidate, the prefactor q^{-m} follows from $|\varpi^m|^3|\varpi^m|^{-2} = q^{-m}$ in the diagonal part of that decomposition. In the unramified case its first two Fourier transforms each introduce q^m, yielding q^m times the characteristic function printed in Lemma 8.2(i). This is a computation for the stated candidate, not an identification of that candidate with the genuine Weil action.

Lemma 16.1 (Vanishing at the proposed calibration point)**.** *With* (16.1) *and* (16.3)*, if* $e > 0$*, then*

$$\Phi_m^c(z_0) = 0.$$

More explicitly,

$$\Phi_m^c(z_0) = q^{3m-2l}\omega(\varpi^l) \int_{\substack{A,D,c,f\in\mathfrak{o} \\ Af-cD\in\mathfrak{o}^\times}} \omega(Af - cD)\psi(-\varpi^{-e}A)\, dA\, dD\, dc\, df = 0. \tag{16.4}$$

Proof. At z_0 the third row in the integrand is $\varpi^{-m}(0,0,0,1)J_m = (0,-1,0,0)$. Put

$$v_1 = x_1 J_m = (a,b,c,u), \qquad v_2 = x_2 J_m = (d,h,f,v).$$

Since $|\det J_m| = q^{-2m}$, the two changes of variables introduce q^{4m}. Including the prefactor in (16.3) gives q^{3m}. The support condition becomes

$$a, d \in \varpi^l\mathfrak{o}, \quad b, c, h, f, u, v \in \mathfrak{o}, \quad af - cd \in \varpi^l\mathfrak{o}^\times.$$

Indeed the determinant of the first three columns is

$$\det \begin{pmatrix} a & b & c \\ d & h & f \\ 0 & -1 & 0 \end{pmatrix} = af - cd.$$

The additive character in (16.3) is $\psi(-\varpi^{-m}a - v)$, since $B((0,1,0,0),x_1) = \varpi^{-m}a$ and $B((0,0,1,0),x_2) = v$. The integrals in b, h, u, v are therefore all one. Substituting $a = \varpi^l A$, $d = \varpi^l D$ gives (16.4).

Choose $t \in \mathfrak{o}^\times$ with $\omega(t) \neq 1$, which exists because $e > 0$. The measure-preserving substitution $(D, f) \mapsto (tD, tf)$ preserves the integration domain and the additive character, but multiplies $\omega(Af - cD)$ by $\omega(t)$. The integral equals $\omega(t)$ times itself and hence is zero. □

Changing the Fourier sign or replacing ω by ω^{-1} does not remove this obstruction. The corresponding calculation with the ordinary dot-product Fourier convention places the additive phase on the other row; scaling the unphased row gives the same vanishing.

In contrast, the Gauss sum $G(\omega, 1)$ in Okazaki's Lemma 8.3 is nonzero for a character of exact conductor e. Consequently a finite constant c_m cannot satisfy the proposed identity $c_m\Phi_m^c(z_0) = G(\omega, 1)$ for this candidate transform. This does not prove that the full theta integral vanishes. Its integrand can be nonzero at points $z_0 g$ with $g \neq 1$. A possible repair would evaluate those terms and replace the point-evaluation collapse (8.4) by an independently proved integral identity. No such identity is asserted for this candidate transfer. The independent normalized seesaw comparison in Section 15 supplies what this test-vector argument does not.

16.1 A sign that a complete transfer must retain

There is also an elementary sign issue in the use of Corollary 6.9. If $C = \mathcal{D}_m W$ and $C^- = R(j'(\varpi^e))C$, then

$$(j'(\varpi^e))^2 = \operatorname{diag}(1,-1,-1,1), \qquad C^-(j'(\varpi^e)) = \omega(-1)^{-1}C(1).$$

The second equality is the right compact character of C, read at entry 33. It is not generally $C(1)$ as used in the printed proof. Thus the subsequent Gauss calculation must retain this sign. This observation alone does not determine the sign of the full GL_4 transfer: that also depends on a consistently derived Weil action and Whittaker functional.

16.2 Failure of the asserted localization of the transformed kernel

There is a further obstruction to retaining the printed argument for this candidate transform. This concerns the assertion, immediately before (8.4), that the transformed kernel vanishes outside $B^1 \operatorname{Int}(a_{-e})K(l)^1$. For the computation below it is convenient to normalize the determinant weight by replacing $\omega(\det \check{z})$ with $\omega(\varpi^{-l}\det \check{z})$ in (16.1); write $\widetilde{\Phi}_m$ for this kernel and $\widetilde{\Phi}^c_m$ for its transform. Thus $\Phi^c_m = \omega(\varpi^l)\widetilde{\Phi}^c_m$.

Lemma 16.2 (Nonzero transformed kernel outside the asserted region)**.** *Let $e > 0$, $m = 2e$, and $l = e$. Put*

$$w_2^\natural = \operatorname{diag}(w_2, w_2), \quad w_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad n_2(x) = 1 - xE_{12} + xE_{34},$$

and

$$g = t(\varpi^{-e}, \varpi^e) w_2^\natural n_2(1), \qquad t(\alpha,\beta) = \operatorname{diag}(\alpha,\beta,\beta^{-1},\alpha^{-1}).$$

Then

$$g \notin B^1 \operatorname{Int}(a_{-e})K(e)^1, \qquad \widetilde{\Phi}^c_{2e}(z_0 g) = q^{3e}G(\omega,1)^2 \neq 0,$$

where $a_{-e} = \operatorname{diag}(\varpi^{-e},\varpi^{-e},1,1)$ and $G(\omega,1) = \int_{\mathfrak{o}^\times} \omega(u)\psi(\varpi^{-e}u)\,du$.

Proof. Direct multiplication gives

$$z_0 g = \begin{pmatrix} \varpi^e & -\varpi^e & 0 & 0 \\ 0 & 0 & 0 & \varpi^{-e} \\ 0 & 0 & \varpi^e & \varpi^e \end{pmatrix}.$$

Make the same changes of variables as in (16.4). After dividing the first column by ϖ^e, the third row in the determinant is $(1,-\varpi^e,0)$. The two variable rows are (A,b,c) and (D,h,f), and the phase is $\psi(A+\varpi^{-e}b+\varpi^{-e}f)$. The fourth-column integrals are one. The substitutions $B = b+\varpi^e A$ and $H = h+\varpi^e D$ give

$$\widetilde{\Phi}^c_{2e}(z_0 g) = q^{4e}\int_{\mathrm{GL}_2(\mathfrak{o})} \omega(\det M)\psi(\varpi^{-e}\operatorname{tr} M)\,dM.$$

For completeness, the last integral is $q^{-e}G(\omega,1)^2$. On the locus where M_{11} is a unit, use $M'_{22} = M_{22} - M_{12}M_{21}/M_{11}$; integration in the two off-diagonal entries contributes q^{-e} and the two diagonal integrals give $G(\omega,1)^2$. On the complementary locus, invertibility forces M_{12} to be a unit. Parametrize M_{21} by the determinant. Its determinant character then averages to zero, since the additive phase does not depend on M_{21}. This proves the stated value and nonvanishing.

To check the assertion about the double coset, set $g_0 = w_2^\natural n_2(1)$. It commutes with a_{-e}, so $g_0 \in B^1 \operatorname{Int}(a_{-e})K(e)^1$ would imply $g_0 \in B^1 K(e)^1$. For an upper triangular symplectic matrix b,

write the second and third diagonal entries as β and β^{-1}. The $(2,1)$ entry of bg_0 is β, while its $(3,3)$ and $(3,4)$ entries differ by β^{-1}. Membership in $K(e)^1$ would therefore require simultaneously $\beta \in \varpi^e\mathfrak{o}$ and $\beta^{-1} \in \mathfrak{o}$, which is impossible for $e > 0$. Left multiplication by the torus element defining g does not change this conclusion. □

This is a counterexample to applying the stated support localization to this candidate transform, not a counterexample to the newform theorem. Contributions from these additional cells could still cancel after multiplication by a Whittaker function and integration. Such a cancellation requires an argument beyond the localization asserted in Section 8.

In fact the particular point just exhibited is annihilated by every co-quasi-paramodular Whittaker function D of level $2e$. Put $p = \varpi^e$. Direct conjugation gives

$$g^{-1}(-E_{12} + E_{34})g = \begin{pmatrix} -p^2 & p^2 & 0 & 0 \\ -p^2 & p^2 & 0 & 0 \\ 0 & 0 & -p^2 & -p^2 \\ 0 & 0 & p^2 & p^2 \end{pmatrix}.$$

Consequently $g^{-1}n_2(u)g \in K^c(2e;e)$ with its $(3,3)$ entry in $1+\mathfrak{p}^e$ for every $u \in \mathfrak{p}^{-e}$. The compact character of D is trivial on these elements, whereas $D(n_2(u)g) = \psi(u)D(g)$. Choosing u with $\psi(u) \neq 1$ gives $D(g) = 0$. Thus this example disproves kernel localization but does not itself obstruct a localization or cancellation argument for the product $\widetilde{\Phi}^c_m(z_0g)D(g)$.

17 Conclusions and the remaining oldform problem

The analytic newform theorem, its minimal-level uniqueness statement, and the normalized split seesaw prove the following assertions for every irreducible admissible generic representation with arbitrary smooth central character:

$$m_\pi = a(\phi_\pi), \qquad \dim V_\pi(m_\pi) = 1, \qquad \varepsilon'_\pi = \omega_\pi(-1)\varepsilon(1/2, \phi_\pi, \psi).$$

The normalized newvector satisfies

$$\Xi(s, W_\pi) = L(s, \phi_\pi), \qquad \Xi(s, \mathcal{D}_{m_\pi} W_\pi) = q^e \omega_\pi(-1)\varepsilon(1/2, \phi_\pi, \psi)L(s, \phi_\pi^\vee).$$

Equivalently, $\mathcal{D}^{\mathrm{std}}_m = \omega_\pi(-1)\mathcal{D}_m$ yields the usual parameter epsilon factor. These operators remain involutive between the two models, since $\omega_\pi(-1)^2 = 1$.

The exact ramified-central-character oldform dimensions and a complete basis theorem are a further problem. The upper bound proved here, $\dim V_\pi(n_\pi + d) \leq (d+1)(d+2)/2$ for $e > 0$, does not establish the full analogue of Roberts–Schmidt's Oldforms Principle. Neither the normalized seesaw nor the corrected local-factor comparison is claimed to prove that further statement. The explicit GL_4 test-vector calculation examined in the preceding section remains defective, but is not a dependency of the proved newform or local-factor results.

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Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, Chicago, IL 60607, USA

Email address: rtakloo@uic.edu