Moving Frames in Euclidean Affine Spaces: Rigidity, Reconstruction and Rotational Kinematics

Abdellah Azdoud* and Abdelhakim Aitlamine*

*Hassan II University of Casablanca, Ben M'Sik, Morocco

27 November 2026

hexagon:2611.01240v1

Abstract

This work presents the mathematical foundations of moving frames in oriented Euclidean affine spaces. After introducing affine frames, affine transformations and isometries, we study moving frames depending on several parameters, with particular emphasis on rigid and orthonormal frames. For one-parameter frames, skew-symmetric endomorphisms characterize rigidity, and a system of differential equations determines the frame from its prescribed velocity coefficients. A multiparameter compatibility criterion is also derived from the Maurer–Cartan equation. In dimensions two and three, instantaneous rotation is expressed through angular velocity vectors. Applications to Frenet and Darboux frames, Euler angles, spherical coordinates and the addition of partial rotation vectors illustrate the connection between differential geometry and kinematics.

Keywords: Moving frames; affine geometry; rigid frames; angular velocity; Euler angles; Frenet frame; Darboux frame.

Introduction

Moving frames offer a unified language for describing geometry and motion in Euclidean spaces. Their classical role in the study of curves and surfaces is exemplified by the Frenet and Darboux systems, in which geometric invariants appear as coefficients of differential equations for an orthonormal basis [2, 10]. The choice of frame is not unique: Bishop's construction, for example, replaces the Frenet frame by a relatively parallel frame that can remain meaningful at points of vanishing curvature [1]. These constructions demonstrate the importance of separating the geometry of an object from the particular frame used to describe it.

The modern method of moving frames extends beyond Euclidean kinematics. Fels and Olver developed a systematic theory of moving coframes and equivariant normalization for Lie group actions [3, 4]; Mansfield subsequently presented the associated invariant calculus in a unified computational framework [8]. The present paper uses the more elementary notion of a parameter-dependent affine frame, rather than claiming an extension of the general equivariant moving-frame theory. The Lie group viewpoint nevertheless clarifies the distinction between arbitrary affine motions and rigid motions, since the latter are governed by the Euclidean group and its Lie algebra [5, 9].

In mechanical systems and robotics, the reconstruction of a rigid configuration from translational and angular velocity is a basic kinematic problem [9, 7]. The differential equations governing orientation evolve on the orthogonal group; numerical methods that respect this group structure are discussed in the literature on Lie-group integration [6]. In several parameters, reconstruction additionally requires compatibility between the prescribed infinitesimal motions. This condition is naturally expressed by the Maurer–Cartan equation [5, 11].

Our objective is to present a self-contained synthesis of these classical ideas. We establish the constant-Gram-matrix criterion for rigidity, derive the infinitesimal characterization in arbitrary dimension, prove global reconstruction on an interval, and formulate the three-dimensional rotation and composition laws. We also develop a compatibility criterion for multiparameter reconstruction and illustrate the framework through Frenet and Darboux frames, planar rotations and Euler angles. The contribution is expository and integrative: the results are not claimed to be new research theorems. The manuscript revises an earlier French exposition dated 17 May 2016, drawing on classical geometry and kinematics [12].

1 Mathematical Framework

Definition 1.1. Let V be a vector space over a field K. An affine space with direction V is a nonempty set E equipped with a mapping $(M, N) \mapsto \overrightarrow{MN}$ from $E \times E$ to V satisfying:

1. For every triple (M, N, P) of points of E, $\overrightarrow{MN} + \overrightarrow{NP} = \overrightarrow{MP}$.
2. For every point $O \in E$, the mapping $M \mapsto \overrightarrow{OM}$ from E to V is bijective.

The elements of E are called points, and those of V are called vectors. The dimension of the affine space E is the dimension of V.

Definition 1.2. An affine space E is called Euclidean when its direction is equipped with a Euclidean structure. If this direction is also oriented, E is called an oriented Euclidean affine space.

Remark 1.3. Throughout this paper, E is an oriented Euclidean affine space of dimension $n \geq 2$.

Definition 1.4. An affine transformation of an affine space E is any bijective affine mapping from E onto itself. Under composition, the affine transformations form a subgroup of the group of permutations of E. This group is called the affine group of E, denoted by $GA(E)$.

Remark 1.5. Let $\mathcal{B}(E_n)$ denote the set of ordered bases of the Euclidean vector space E_n.

Remark and Definition 1.6. A basis of E_n is a system $(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n)$ of n linearly independent vectors. It may therefore be identified with a point of the product space $(E_n)^n$. The set $\mathcal{B}(E_n)$ is open in $(E_n)^n$.

Definition 1.7. An affine frame of E is a tuple $(A; \boldsymbol{e}_1, \ldots, \boldsymbol{e}_n)$ consisting of a point $A \in E$ and a basis of E_n; equivalently, it is a point of $E \times \mathcal{B}(E_n)$.

Property 1.8. Let φ be an affine bijection of E with linear part f. If $R = (A; \boldsymbol{e}_1, \ldots, \boldsymbol{e}_n)$ is a frame, its image is the frame

$$\varphi(R) = (\varphi(A); f(\boldsymbol{e}_1), \ldots, f(\boldsymbol{e}_n)) = R'.$$

Conversely, given two frames $R = (A; \boldsymbol{e}_1, \ldots, \boldsymbol{e}_n)$ and $R' = (A'; \boldsymbol{e}'_1, \ldots, \boldsymbol{e}'_n)$, there is a unique affine bijection sending R to R'. It is defined by

$$\varphi\left(A + \sum_{i=1}^{n} x_i \boldsymbol{e}_i\right) = A' + \sum_{i=1}^{n} x_i \boldsymbol{e}'_i, \qquad x_i \in \mathbb{R}. \tag{*}$$

Property 1.9. An affine bijection φ sending R to R' is an isometry if and only if

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \boldsymbol{e}'_i \cdot \boldsymbol{e}'_j \quad (1 \leq i, j \leq n).$$

Proof. Let $M = (x_i)$ and $P = (y_i)$ in the frame R, and let M', P' be their images under φ. By (*),

$$\overrightarrow{M'P'} = \sum_{i=1}^{n} (y_i - x_i)\boldsymbol{e}_i'.$$

Consequently,

$$\|\overrightarrow{M'P'}\|^2 = \sum_{i,j=1}^{n} (y_i - x_i)(y_j - x_j)\boldsymbol{e}_i' \cdot \boldsymbol{e}_j', \tag{1}$$

$$\|\overrightarrow{MP}\|^2 = \sum_{i,j=1}^{n} (y_i - x_i)(y_j - x_j)\boldsymbol{e}_i \cdot \boldsymbol{e}_j. \tag{2}$$

If φ is an isometry, (1) and (2) are equal. Conversely, equality of the scalar products gives equality of the distances. The converse implication also follows by polarization of the quadratic form. □

Lemma 1.10. For an endomorphism $f \in \mathcal{L}(E_n)$ to be skew-symmetric, it suffices that there exist a basis $(\boldsymbol{e}_i)$ such that

$$f(\boldsymbol{e}_i) \cdot \boldsymbol{e}_j + f(\boldsymbol{e}_j) \cdot \boldsymbol{e}_i = 0 \quad \text{for all } i, j.$$

Conversely, if f is skew-symmetric, this property holds for every basis.

Proof. For $\boldsymbol{V} = \sum_i \lambda_i \boldsymbol{e}_i$,

$$f(\boldsymbol{V}) \cdot \boldsymbol{V} = \sum_{i,j=1}^{n} \lambda_i \lambda_j f(\boldsymbol{e}_i) \cdot \boldsymbol{e}_j.$$

An endomorphism is skew-symmetric if and only if this expression vanishes for every $\boldsymbol{V}$. The stated condition yields the result. □

2 Moving Frames Depending on k Parameters

Definition 2.1. A moving frame of E depending on k parameters is a mapping

$$\rho : \Delta \longrightarrow E \times (E_n)^n, \qquad u \longmapsto (A(u); \boldsymbol{e}_1(u), \ldots, \boldsymbol{e}_n(u)),$$

where $\Delta \subset \mathbb{R}^k$ is a domain, $(\boldsymbol{e}_1(u), \ldots, \boldsymbol{e}_n(u))$ is a basis of E_n for every $u \in \Delta$, and the functions $A : \Delta \to E$ and $\boldsymbol{e}_i : \Delta \to E_n$ are continuous on Δ.

Remark 2.2. (1) The moving frame ρ is continuous on Δ. (2) If A and the $\boldsymbol{e}_i$ are of class C^p (respectively, p times differentiable), then so is the moving frame.

Definition 2.3. A moving basis defined on Δ is a continuous mapping $\Delta \to \mathcal{B}(E_n)$. In particular, it is a mapping $\beta : u \mapsto (\boldsymbol{e}_1(u), \ldots, \boldsymbol{e}_n(u))$ whose components are continuous and form a basis at each point.

Remark 2.4. β is of class C^p if its component functions are of class C^p.

Theorem 2.5. Let E be oriented and let $\rho : \Delta \to E \times \mathcal{B}(E_n)$ be a moving frame on a domain $\Delta \subset \mathbb{R}^k$. For every $u, v \in \Delta$, the frames $\rho(u)$ and $\rho(v)$ have the same orientation.

Proof. Fix $u \in \Delta$. The components of $\boldsymbol{e}_1(v), \ldots, \boldsymbol{e}_n(v)$ in the fixed basis $H = (\boldsymbol{e}_1(u), \ldots, \boldsymbol{e}_n(u))$ are continuous in v. Their determinant $\delta_u(v)$ is therefore continuous on Δ. Since $\delta_u(u) = 1$ and the vectors always form a basis, the determinant never vanishes. By connectedness it is positive everywhere. Hence the orientations coincide. □

Definition 2.6. A moving frame is called positively oriented (respectively, negatively oriented) if $\rho(u)$ is positively oriented (respectively, negatively oriented) for every $u \in \Delta$.

Example 2.7. Let $\Delta \subset \mathbb{R}^2$ and $\dim E = 3$. Consider a C^1 mapping $f : (u, v) \mapsto M(u, v)$ defining a regular surface in E. If

$$\boldsymbol{N}(u, v) = \frac{\partial M}{\partial u} \times \frac{\partial M}{\partial v}$$

is the normal vector associated with the parametrization, then

$$\rho(u, v) = \left(M(u, v); \frac{\partial M}{\partial u}, \frac{\partial M}{\partial v}, \boldsymbol{N}(u, v) \right).$$

This frame plays a role in the theory of surfaces.

2.1 Affine Bijections Associated with a Moving Frame

Definition 2.8. Let $\rho : \Delta \to E \times \mathcal{B}(E_n)$ be a moving frame. For each $u, v \in \Delta$, define the affine bijection $\Psi_{u,v}$ of E sending $\rho(u)$ to $\rho(v)$, so that $\Psi_{u,v}(\rho(u)) = \rho(v)$.

Property 2.9. (1) All the affine bijections $\Psi_{u,v}$ preserve orientation, by the preceding theorem. (2) For all $u, v, w \in \Delta$,

$$\Psi_{u,w} = \Psi_{v,w} \circ \Psi_{u,v}, \qquad \Psi_{v,u} = \Psi_{u,v}^{-1}, \qquad \Psi_{u,u} = \mathrm{Id}_E \,.$$

2.2 Rigid Moving Frames

Definition 2.10. A moving frame ρ of E is called rigid (non-deformable) if its associated affine bijections are isometries.

Remark and Definition 2.11. An isometry of the plane is a direct isometry if it preserves oriented angles. Direct isometries are precisely translations and rotations. A moving frame $\rho : u \mapsto (A(u); \boldsymbol{e}_1(u), \ldots, \boldsymbol{e}_n(u))$ is rigid if and only if the Gram matrix

$$G(u) = (\boldsymbol{e}_i(u) \cdot \boldsymbol{e}_j(u))_{1 \le i,j \le n}$$

is independent of $u \in \Delta$.

Definition 2.12. A moving frame is orthonormal if $\rho(u)$ is orthonormal for every $u \in \Delta$. In particular, it is rigid.

2.3 Attached Frames

Definition 2.13. Let $\rho = (A; \boldsymbol{e}_1, \ldots, \boldsymbol{e}_n)$ and $\sigma = (B; \boldsymbol{f}_1, \ldots, \boldsymbol{f}_n)$ be two frames defined on the same domain $\Delta \subset \mathbb{R}^k$. We say that σ is attached to ρ if the coordinates of $B(u)$ and the components of the vectors $\boldsymbol{f}_i(u)$ in the frame $\rho(u)$ are independent of u.

Property 2.14. (1) The relation "σ is attached to ρ" is an equivalence relation. (2) The frame σ is attached to ρ if and only if the two frames have the same family of associated affine bijections $(\Psi_{u,v})_{u,v \in \Delta}$. Consequently, any moving frame attached to a rigid frame is rigid.

Remark 2.15. In the language of kinematics, two attached rigid moving frames are said to be fixed relative to one another.

3 One-Parameter Moving Frames

Let $I \subset \mathbb{R}$ be an interval and let $A : I \to E$ and $\boldsymbol{e}_i : I \to E_n$ $(1 \le i \le n)$ be $n+1$ continuous functions such that $(\boldsymbol{e}_1(t), \dots, \boldsymbol{e}_n(t))$ forms a basis for every $t \in I$.

Definition 3.1. (1) The frame is of class C^k if A and the $\boldsymbol{e}_i$ are of class C^k. (2) The frame is rigid if and only if the matrix $(\boldsymbol{e}_i(t) \cdot \boldsymbol{e}_j(t))_{i,j}$ is constant. (3) If $\rho = (A; \boldsymbol{e}_1, \dots, \boldsymbol{e}_n)$ is differentiable, then for each $t \in I$ there exists a unique $D_t \in \mathcal{L}(E_n)$ such that

$$D_t(\boldsymbol{e}_i(t)) = \frac{d\boldsymbol{e}_i}{dt}(t).$$

Proposition 3.2. A differentiable moving frame ρ is rigid if and only if D_t is skew-symmetric for every $t \in I$.

Proof. By Lemma 1.10, D_t is skew-symmetric if and only if

$$\frac{d\boldsymbol{e}_i}{dt} \cdot \boldsymbol{e}_j + \frac{d\boldsymbol{e}_j}{dt} \cdot \boldsymbol{e}_i = 0,$$

that is, $\frac{d}{dt}(\boldsymbol{e}_i \cdot \boldsymbol{e}_j) = 0$. This is precisely the rigidity condition. □

Corollary 3.3. If ρ is orthonormal, it is rigid, and the matrix of D_t in the basis $(\boldsymbol{e}_1(t), \dots, \boldsymbol{e}_n(t))$ is skew-symmetric.

Remark and Definition 3.4. Consider the matrix $W(t) = (w_{ij}(t))$ of coefficients determined by the derivatives of the basis vectors. We use the column convention throughout:

$$\frac{d\boldsymbol{e}_i}{dt} = \sum_{j=1}^{n} w_{ji}(t)\boldsymbol{e}_j(t).$$

Theorem 3.5. Let $\beta : I \to \mathcal{B}(E_n)$ be a differentiable moving basis. It is orthonormal if and only if there exists $t_0 \in I$ such that $\beta(t_0)$ is orthonormal and $W(t)$ is skew-symmetric for all $t \in I$.

Proof. It suffices to establish sufficiency. Let $(\boldsymbol{\varepsilon}_1, \dots, \boldsymbol{\varepsilon}_n)$ be a fixed orthonormal basis of E_n, and let $B(t) = (b_{ji}(t))$ be the change-of-basis matrix, so that $\boldsymbol{e}_i(t) = \sum_j b_{ji}(t)\boldsymbol{\varepsilon}_j$. By the definition of w_{ji},

$$\frac{d\boldsymbol{e}_i}{dt} = \sum_{j,k=1}^{n} w_{ji}(t)b_{kj}(t)\boldsymbol{\varepsilon}_k, \qquad \frac{db_{ki}}{dt} = \sum_{j} w_{ji}(t)b_{kj}(t).$$

In matrix form, $B' = BW$, and $(B^T)' = W^T B^T$. Since $W + W^T = 0$,

$$\frac{d}{dt}(BB^T) = B(W + W^T)B^T = 0.$$

Thus BB^T is constant. Since $B(t_0)$ is orthogonal, $B(t)B(t)^T = I_n$ for all $t \in I$. Hence the moving basis is orthonormal. □

3.1 Functions Attached to a Rigid Moving Frame

Let $\rho = (A; \boldsymbol{e}_1, \ldots, \boldsymbol{e}_n)$ be a rigid moving frame defined on I.

Definition 3.6. A vector-valued function $\boldsymbol{V} : I \to E_n$ is attached to ρ if it can be written as $\boldsymbol{V}(t) = \sum_i \lambda_i \boldsymbol{e}_i(t)$, where the λ_i are constants. If ρ is differentiable, then

$$\frac{d\boldsymbol{V}}{dt} = \sum_i \lambda_i \frac{d\boldsymbol{e}_i}{dt} = \sum_i \lambda_i D_t(\boldsymbol{e}_i) = D_t(\boldsymbol{V}(t)).$$

Definition 3.7 (Intrinsic definition of the operator D_t). Consider the affine bijection $\Psi_{t,u}$ sending $\rho(t)$ to $\rho(u)$, and let $\overline{\Psi}_{t,u} \in \mathcal{L}(E_n)$ denote its linear part. Fix $t \in I$. Every $\boldsymbol{W} \in E_n$ can be written as $\boldsymbol{W} = \sum_i \lambda_i \boldsymbol{e}_i(t)$, with constant coefficients. Clearly,

$$\overline{\Psi}_{t,u}(\boldsymbol{W}) = \sum_i \lambda_i \boldsymbol{e}_i(u).$$

If the frame is differentiable, then

$$\frac{\partial}{\partial u}\overline{\Psi}_{t,u}(\boldsymbol{W}) = \sum_i \lambda_i \frac{d\boldsymbol{e}_i}{du}(u) = D_u\left(\sum_i \lambda_i \boldsymbol{e}_i(u)\right).$$

Evaluating at $u = t$ gives

$$\frac{\partial}{\partial u}\overline{\Psi}_{t,u}(\boldsymbol{W})\Big|_{u=t} = D_t(\boldsymbol{W}).$$

Since $\boldsymbol{W}$ is arbitrary, the mapping $u \mapsto \overline{\Psi}_{t,u}$ is differentiable and

$$D_t = \frac{\partial}{\partial u}\overline{\Psi}_{t,u}\Big|_{u=t}.$$

Thus D_t depends only on t and the family of linear parts of the affine bijections associated with ρ.

3.2 Determining a Moving Frame from Its Velocity Field

Theorem 3.8. Let $(\xi_i, w_{ij})_{1\le i,j\le n}$ be a system of $n + n^2$ continuous real-valued functions on an interval $I \subset \mathbb{R}$, and let R_0 be an arbitrary fixed affine frame. For each $t_0 \in I$, there exists a unique moving frame $\rho = (A; \boldsymbol{e}_1, \ldots, \boldsymbol{e}_n)$ on I satisfying $\rho(t_0) = R_0$ and

$$\frac{dA}{dt} = \sum_{i=1}^{n} \xi_i(t)\boldsymbol{e}_i(t), \qquad \frac{d\boldsymbol{e}_i}{dt} = \sum_{j=1}^{n} w_{ji}(t)\boldsymbol{e}_j(t). \tag{*}$$

Moreover, if the coefficients are of class C^p, then ρ is of class C^{p+1}.

Proof. Write $A(t) = (x_i(t))$ and $\boldsymbol{e}_i(t) = (b_{ji}(t))_{j=1}^n$ in the fixed frame R_0. By (*),

$$\frac{dx_i}{dt} = \sum_j \xi_j(t) b_{ij}(t), \qquad \frac{db_{ji}}{dt} = \sum_k w_{ki}(t) b_{jk}(t). \tag{**}$$

This is a linear differential system with continuous coefficients on I. It admits a unique solution defined on I satisfying the initial conditions $x_i(t_0) = 0$ and $b_{ji}(t_0) = \delta_{ij}$. To verify that the solution defines a frame, set $\Delta(t) = \det B(t)$. Then

$$\Delta'(t) = \left(\sum_{i=1}^{n} w_{ii}(t)\right)\Delta(t),$$

so, since $\Delta(t_0) \neq 0$,

$$\Delta(t) = \Delta(t_0) \exp\left(\int_{t_0}^{t} \sum_{i=1}^{n} w_{ii}(\tau)\, d\tau \right).$$

The determinant never vanishes. If the coefficients are of class C^p, the regularity assertion follows from the differential system. □

Remark 3.9. (1) If R_0 is orthonormal and $W(t)$ is skew-symmetric for every $t \in I$, the resulting moving frame is orthonormal and has the same orientation as R_0. (2) If ρ is rigid, $W(t)$ is called the body-coordinate infinitesimal rotation matrix (it need not be skew-symmetric in a nonorthonormal basis). (3) When $n = 3$, the matrix $W(t)$ has a kinematic interpretation, and prescribing the functions (ξ_i, w_{ij}) is equivalent to specifying the velocity field of ρ at each instant relative to $\rho(t)$.

4 Rigid Moving Frames in Dimensions Two and Three

We begin by studying frames in a three-dimensional space E. The case of moving frames in dimension two will be considered subsequently.

4.1 Instantaneous Rotation of a Rigid Moving Frame in Dimension Three

Let

$$\rho : I \to E \times \mathcal{B}(E_3), \quad t \mapsto (A(t); \boldsymbol{e}_1(t), \boldsymbol{e}_2(t), \boldsymbol{e}_3(t))$$

be a one-parameter moving frame, defined and differentiable on an interval $I \subset \mathbb{R}$. For every $t \in I$, let D_t be the endomorphism of E_3 defined by

$$D_t(\boldsymbol{e}_i(t)) = \frac{d\boldsymbol{e}_i}{dt}, \quad i = 1, 2, 3.$$

We know that ρ is rigid if and only if D_t is skew-symmetric for every $t \in I$. The skew-symmetric endomorphisms of oriented E_3 are of the form $\boldsymbol{X} \mapsto \boldsymbol{K} \times \boldsymbol{X}$, where $\boldsymbol{K}$ is a fixed vector of E_3. Therefore, if ρ is rigid, there exists a vector-valued function $\boldsymbol{\Omega} : I \to E_3$ satisfying

$$\frac{d\boldsymbol{e}_i}{dt} = \boldsymbol{\Omega}(t) \times \boldsymbol{e}_i(t), \quad i = 1, 2, 3. \tag{2}$$

Since the basis vectors are linearly independent, $\boldsymbol{\Omega}(t)$ is uniquely determined by this relation.

Definition 4.1. If $\rho = (A; \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ is a rigid moving frame of an oriented Euclidean affine space E, the vector $\boldsymbol{\Omega}(t)$ defined by (2) is called the instantaneous rotation vector of the frame at parameter value t.

4.1.1 The Case of an Orthonormal Frame

Let $\rho = (A; \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ be an orthonormal moving frame of E (positively oriented for the formulas below). Denote the components of $\boldsymbol{\Omega}(t)$ in the moving basis by $(p(t), q(t), r(t))$. Applying (2) yields

$$\begin{aligned} \frac{d\boldsymbol{e}_1}{dt} &= r\boldsymbol{e}_2 - q\boldsymbol{e}_3, \\ \frac{d\boldsymbol{e}_2}{dt} &= p\boldsymbol{e}_3 - r\boldsymbol{e}_1, \\ \frac{d\boldsymbol{e}_3}{dt} &= q\boldsymbol{e}_1 - p\boldsymbol{e}_2. \end{aligned} \tag{3}$$

Hence,

$$\begin{aligned} p &= \boldsymbol{e}_3 \cdot \frac{d\boldsymbol{e}_2}{dt} = -\boldsymbol{e}_2 \cdot \frac{d\boldsymbol{e}_3}{dt}, \\ q &= \boldsymbol{e}_1 \cdot \frac{d\boldsymbol{e}_3}{dt} = -\boldsymbol{e}_3 \cdot \frac{d\boldsymbol{e}_1}{dt}, \\ r &= \boldsymbol{e}_2 \cdot \frac{d\boldsymbol{e}_1}{dt} = -\boldsymbol{e}_1 \cdot \frac{d\boldsymbol{e}_2}{dt}. \end{aligned} \tag{4}$$

Relations (4) make it possible to determine p, q, r by differentiating the basis vectors.

Remark 4.2. Alternatively, one could define the three functions p, q, r directly by (4) and verify that $\boldsymbol{\Omega}(t) = p\boldsymbol{e}_1 + q\boldsymbol{e}_2 + r\boldsymbol{e}_3$ satisfies (2). This provides an elementary proof of the existence of $\boldsymbol{\Omega}$.

Example 4.3 (Frenet and Darboux Frames). Introducing the rotation vector enables us to give a simple interpretation of the Frenet and Darboux formulas.
(a) Frenet frame. Let γ be an oriented regular arc of class C^3 in E, without inflection points, defined by an arc-length parametrization $s \mapsto M(s)$ on I. Let

$$F : s \mapsto (M(s); \boldsymbol{\tau}(s), \boldsymbol{\nu}(s), \boldsymbol{\beta}(s))$$

be the associated Frenet frame. It is a moving frame of class C^1 on I. Using the standard signed torsion convention, the Frenet equations are

$$\frac{dM}{ds} = \boldsymbol{\tau}, \quad \frac{d\boldsymbol{\tau}}{ds} = \rho\boldsymbol{\nu}, \quad \frac{d\boldsymbol{\nu}}{ds} = -\rho\boldsymbol{\tau} + \theta\boldsymbol{\beta}, \quad \frac{d\boldsymbol{\beta}}{ds} = -\theta\boldsymbol{\nu},$$

where ρ and θ denote curvature and torsion. The corresponding rotation vector is

$$\boldsymbol{\omega}(s) = \theta(s)\boldsymbol{\tau}(s) + \rho(s)\boldsymbol{\beta}(s). \tag{5}$$

(b) Darboux frame. Let Σ be a geometric surface of class C^2 in E, and let γ be a regular arc of class C^2 on Σ, defined by an arc-length parametrization $s \mapsto P(s)$ on I. Associated with this arc is a moving frame called the Darboux frame,

$$s \mapsto (P(s); \boldsymbol{\tau}(s), \boldsymbol{g}(s), \boldsymbol{h}(s)),$$

which satisfies

$$\begin{aligned} \frac{dP}{ds} &= \boldsymbol{\tau}, & \frac{d\boldsymbol{\tau}}{ds} &= \rho_g\boldsymbol{g} + \rho_n\boldsymbol{h}, \\ \frac{d\boldsymbol{g}}{ds} &= -\rho_g\boldsymbol{\tau} - \theta_g\boldsymbol{h}, & \frac{d\boldsymbol{h}}{ds} &= -\rho_n\boldsymbol{\tau} + \theta_g\boldsymbol{g}. \end{aligned}$$

Here ρ_n, ρ_g, θ_g denote the normal curvature, geodesic curvature and geodesic torsion of γ, respectively. Comparing with (3), the instantaneous rotation vector is

$$\boldsymbol{\Omega}(s) = -\theta_g\boldsymbol{\tau} - \rho_n\boldsymbol{g} + \rho_g\boldsymbol{h}.$$

4.1.2 Functions Attached to a Rigid Frame

Return to the general case of a differentiable rigid moving frame $\rho = (A; \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ defined on an interval I, and let $\boldsymbol{\Omega}$ denote its instantaneous rotation vector. By definition, a vector-valued function attached to ρ has the form

$$\boldsymbol{V} : I \to E_3, \qquad t \mapsto X_1\boldsymbol{e}_1(t) + X_2\boldsymbol{e}_2(t) + X_3\boldsymbol{e}_3(t),$$

where X_1, X_2, X_3 are arbitrary constants. Such a function is differentiable on I and satisfies

$$\frac{d\boldsymbol{V}}{dt} = \sum_{i=1}^{3} X_i \frac{d\boldsymbol{e}_i}{dt} = \sum_{i=1}^{3} X_i \boldsymbol{\Omega}(t) \times \boldsymbol{e}_i(t) = \boldsymbol{\Omega}(t) \times \boldsymbol{V}(t).$$

Similarly, a mapping from I into E is said to be attached to ρ if it has the form

$$t \mapsto M(t) = A(t) + \sum_{i=1}^{3} x_i \boldsymbol{e}_i(t),$$

where x_1, x_2, x_3 are arbitrary constants. In kinematics, such a function is called a point motion attached to ρ. It is differentiable and satisfies, for all $t \in I$,

$$\frac{dM}{dt} = \frac{dA}{dt} + \boldsymbol{\Omega}(t) \times \overrightarrow{A(t)M(t)}.$$

To determine the derivatives at t of all functions of this form, it suffices to know the vectors dA/dt and $\boldsymbol{\Omega}(t)$. Generally, these vectors are defined by their components in the moving basis; we write

$$\frac{dA}{dt} = \alpha \boldsymbol{e}_1 + \beta \boldsymbol{e}_2 + \delta \boldsymbol{e}_3, \qquad \boldsymbol{\Omega} = p\boldsymbol{e}_1 + q\boldsymbol{e}_2 + r\boldsymbol{e}_3.$$

4.1.3 Determination of a Three-Dimensional Orthonormal Frame from $\alpha, \beta, \delta, p, q, r$

Theorem 4.4. Let $(\alpha, \beta, \delta, p, q, r)$ be six continuous real-valued functions on an interval $I \subset \mathbb{R}$, and let R_0 be a positively oriented orthonormal frame of E. For every $t_0 \in I$, there exists a unique moving frame $\rho = (A; \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ of class C^1 on I satisfying $\rho(t_0) = R_0$ and

$$\begin{aligned}
\frac{dA}{dt} &= \alpha \boldsymbol{e}_1 + \beta \boldsymbol{e}_2 + \delta \boldsymbol{e}_3, \\
\frac{d\boldsymbol{e}_1}{dt} &= r\boldsymbol{e}_2 - q\boldsymbol{e}_3, \\
\frac{d\boldsymbol{e}_2}{dt} &= p\boldsymbol{e}_3 - r\boldsymbol{e}_1, \\
\frac{d\boldsymbol{e}_3}{dt} &= q\boldsymbol{e}_1 - p\boldsymbol{e}_2.
\end{aligned}$$

Moreover, this frame is orthonormal and positively oriented.

This follows immediately from the preceding existence theorem and the fact that the matrix

$$W(t) = \begin{pmatrix} 0 & -r(t) & q(t) \\ r(t) & 0 & -p(t) \\ -q(t) & p(t) & 0 \end{pmatrix}$$

is skew-symmetric.

4.1.4 Orthonormal Moving Frames in Dimension Two

In the oriented plane E, choose a fixed positively oriented orthonormal frame $(O; \boldsymbol{i}, \boldsymbol{j})$. Specifying a vector-valued function $\boldsymbol{e} : I \to E_2$ of constant positive norm r and class C^p is equivalent, up to an additive integer multiple of 2π, to specifying a real-valued function θ of class C^p on I satisfying

$$\boldsymbol{e}(t) = r(\cos\theta(t)\boldsymbol{i} + \sin\theta(t)\boldsymbol{j}).$$

The function θ is a continuous determination of the oriented angle $(\boldsymbol{i}, \boldsymbol{e}(t))$. Thus, specifying a positively oriented orthonormal moving frame $(A; \boldsymbol{e}_1, \boldsymbol{e}_2)$ is equivalent to specifying a function $A : I \to E$ defining its origin, and an angle function $\theta : I \to \mathbb{R}$ satisfying

$$\boldsymbol{e}_1 = \cos\theta\, \boldsymbol{i} + \sin\theta\, \boldsymbol{j}, \qquad \boldsymbol{e}_2 = -\sin\theta\, \boldsymbol{i} + \cos\theta\, \boldsymbol{j}. \tag{*}$$

The frame is of class C^p if and only if A and θ are of class C^p. Differentiating (*) gives

$$\frac{d\boldsymbol{e}_1}{dt} = \frac{d\theta}{dt}\boldsymbol{e}_2, \qquad \frac{d\boldsymbol{e}_2}{dt} = -\frac{d\theta}{dt}\boldsymbol{e}_1. \tag{**}$$

To interpret (**), identify E with an affine plane P of an oriented three-dimensional space and choose a unit vector $\boldsymbol{k}$ orthogonal to P such that $(O; \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ is positively oriented. Setting $\boldsymbol{\Omega}(t) = \theta'(t)\boldsymbol{k}$, relations (**) are equivalent to

$$\frac{d\boldsymbol{e}_1}{dt} = \boldsymbol{\Omega} \times \boldsymbol{e}_1, \qquad \frac{d\boldsymbol{e}_2}{dt} = \boldsymbol{\Omega} \times \boldsymbol{e}_2.$$

4.2 Rigid Moving Frames Depending on k Parameters in Dimension Three

With kinematic applications in mind, we restrict our attention to moving frames depending on k parameters in an oriented three-dimensional space E. Let Δ be a domain of $\mathbb{R}^k$ ($k \geq 2$) and let

$$\rho : \Delta \to E \times \mathcal{B}(E_3), \qquad u \mapsto (A(u); \boldsymbol{e}_1(u), \boldsymbol{e}_2(u), \boldsymbol{e}_3(u))$$

be a differentiable rigid moving frame defined on Δ. Denote the coordinates of $u \in \Delta$ by $(u_1, \ldots, u_k)$ and the λth partial derivative of a differentiable function f by $\partial f/\partial u_\lambda$. For each $\lambda = 1, \ldots, k$ and $u \in \Delta$, define the endomorphism $D_{\lambda,u}$ of E_3 by

$$D_{\lambda,u}(\boldsymbol{e}_i(u)) = \frac{\partial \boldsymbol{e}_i}{\partial u_\lambda}(u), \quad i = 1, 2, 3.$$

Since the frame is rigid, this endomorphism is skew-symmetric: the calculations are the same as in the one-parameter case. There therefore exists a vector $\boldsymbol{\Omega}_\lambda(u)$ satisfying

$$\frac{\partial \boldsymbol{e}_i}{\partial u_\lambda}(u) = \boldsymbol{\Omega}_\lambda(u) \times \boldsymbol{e}_i(u), \quad i = 1, 2, 3. \tag{1}$$

This vector is called the virtual instantaneous rotation of ρ at u with respect to u_λ, or the λth partial instantaneous rotation.

Intuitively, $\boldsymbol{\Omega}_\lambda(u)$ is the instantaneous rotation of the one-parameter frame obtained by varying only u_λ. This interpretation often enables us to determine these vectors without calculation, using the following principle. If, for every real a sufficiently close to zero, the direct isometry sending $\rho(u_1, \ldots, u_k)$ to $\rho(u_1, \ldots, u_\lambda + a, \ldots, u_k)$ is a rotation of angle a about an oriented axis, then $\boldsymbol{\Omega}_\lambda(u)$ is the unit direction vector of that axis.

Remark 4.5. Let $\boldsymbol{V}(u) = \sum_{i=1}^3 X_i \boldsymbol{e}_i(u)$ be a vector-valued function attached to ρ, where X_1, X_2, X_3 are constants. It is differentiable, and its partial derivatives satisfy

$$\frac{\partial \boldsymbol{V}}{\partial u_\lambda}(u) = \sum_i X_i \frac{\partial \boldsymbol{e}_i}{\partial u_\lambda}(u) = \boldsymbol{\Omega}_\lambda(u) \times \boldsymbol{V}(u).$$

Its differential at u is therefore

$$d\boldsymbol{V} = \sum_{\lambda=1}^k \frac{\partial \boldsymbol{V}}{\partial u_\lambda}(u)\, du_\lambda = \left(\sum_{\lambda=1}^k \boldsymbol{\Omega}_\lambda(u)\, du_\lambda \right) \times \boldsymbol{V}(u).$$

The expression $\sum_{\lambda=1}^k \boldsymbol{\Omega}_\lambda(u)\, du_\lambda$, which represents a vector-valued differential form, is called the infinitesimal rotation of the moving frame ρ. The following example is fundamental.

4.2.1 A Frame Defined by Euler Angles

Suppose that a positively oriented orthonormal frame $R = (O; \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ of E is given. To each point $(\psi, \theta, \varphi) \in \mathbb{R}^3$, associate the vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}, \boldsymbol{I}, \boldsymbol{J}, \boldsymbol{K}$ defined by

$$\begin{aligned} \boldsymbol{u} &= \cos\psi\, \boldsymbol{i} + \sin\psi\, \boldsymbol{j}, & \boldsymbol{v} &= -\sin\psi\, \boldsymbol{i} + \cos\psi\, \boldsymbol{j}, \\ \boldsymbol{K} &= \cos\theta\, \boldsymbol{k} - \sin\theta\, \boldsymbol{v}, & \boldsymbol{w} &= \boldsymbol{K} \times \boldsymbol{u}, \\ \boldsymbol{I} &= \cos\varphi\, \boldsymbol{u} + \sin\varphi\, \boldsymbol{w}, & \boldsymbol{J} &= -\sin\varphi\, \boldsymbol{u} + \cos\varphi\, \boldsymbol{w}. \end{aligned}$$

The frame $\rho_1(\psi) = (O; \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{k})$ is obtained from R by a rotation of angle ψ about $(O; \boldsymbol{k})$. The frame $\rho_2(\psi, \theta) = (O; \boldsymbol{u}, \boldsymbol{w}, \boldsymbol{K})$ is obtained from ρ_1 by a rotation of angle θ about $(O; \boldsymbol{u})$. Finally, $\rho(\psi, \theta, \varphi) = (O; \boldsymbol{I}, \boldsymbol{J}, \boldsymbol{K})$ is obtained from ρ_2 by a rotation of angle φ about $(O; \boldsymbol{K})$. All these frames are positively oriented and orthonormal.

The mapping $(\psi, \theta, \varphi) \mapsto \rho(\psi, \theta, \varphi)$ defines a three-parameter moving frame. We compute its virtual rotations by the principles above.

1. If θ, φ remain fixed and ψ increases by $\Delta\psi$, the frame rotates by $\Delta\psi$ about $(O; \boldsymbol{k})$. Hence $\boldsymbol{\Omega}_1 = \boldsymbol{k}$.

2. If ψ, φ remain fixed and θ increases by $\Delta\theta$, the frame rotates by $\Delta\theta$ about $(O; \boldsymbol{u})$. Hence $\boldsymbol{\Omega}_2 = \boldsymbol{u}$.

3. If ψ, θ remain fixed and φ increases by $\Delta\varphi$, the frame rotates by $\Delta\varphi$ about $(O; \boldsymbol{K})$. Hence $\boldsymbol{\Omega}_3 = \boldsymbol{K}$.

In summary, the infinitesimal rotation of ρ at (ψ, θ, φ) is

$$\boldsymbol{k}\, d\psi + \boldsymbol{u}\, d\theta + \boldsymbol{K}\, d\varphi.$$

The infinitesimal rotations of ρ_1 and ρ_2 are respectively $\boldsymbol{k}\, d\psi$ and $\boldsymbol{k}\, d\psi + \boldsymbol{u}\, d\theta$.

Remark 4.6. For any real r, the point M defined by $\overrightarrow{OM} = r\boldsymbol{K}$ has spherical coordinates (r, ψ, θ) in the given frame R. Regarding this point as a function of three real variables, differentiation gives

$$\begin{aligned} \frac{\partial M}{\partial r} &= \boldsymbol{K}, \\ \frac{\partial M}{\partial \psi} &= r\frac{\partial \boldsymbol{K}}{\partial \psi} = r\boldsymbol{\Omega}_1 \times \boldsymbol{K} = r\boldsymbol{k} \times \boldsymbol{K} = r\sin\theta\, \boldsymbol{u}, \\ \frac{\partial M}{\partial \theta} &= r\frac{\partial \boldsymbol{K}}{\partial \theta} = r\boldsymbol{\Omega}_2 \times \boldsymbol{K} = -r\boldsymbol{w}. \end{aligned}$$

Consequently,

$$dM = \boldsymbol{K}\, dr + r\sin\theta\, \boldsymbol{u}\, d\psi - r\boldsymbol{w}\, d\theta.$$

If three differentiable functions $r(t), \psi(t), \theta(t)$ are defined on a common interval I, the composite function $P(t) = M(r(t), \psi(t), \theta(t))$ satisfies the corresponding velocity formula

$$\frac{dP}{dt} = \boldsymbol{K}\frac{dr}{dt} + r\sin\theta\, \boldsymbol{u}\frac{d\psi}{dt} - r\boldsymbol{w}\frac{d\theta}{dt}.$$

This gives the velocity of a point motion defined in spherical coordinates.

4.3 Return to One-Parameter Frames in Dimension Three: The Addition Principle for Rotation Vectors

A one-parameter moving frame of E is often specified by a system of parameters $(u_1(t), \dots, u_k(t))$ defining its position at time t. In this case, the instantaneous rotation of the frame is easily obtained by applying the following theorem.

Theorem 4.7 (Addition Principle for Rotation Vectors). Let Δ be a domain of $\mathbb{R}^k$ $(k \geq 2)$, and let $\rho : \Delta \to E \times \mathcal{B}(E_3)$ be a differentiable rigid moving frame. Let $u : I \to \Delta$, $t \mapsto (u_1(t), \dots, u_k(t))$, be a differentiable function defined on an interval $I \subset \mathbb{R}$. Then the one-parameter moving frame $t \mapsto \rho(u(t))$ is rigid, and its instantaneous rotation vector at time t is

$$\boldsymbol{\omega}(t) = \sum_{\lambda=1}^{k} \boldsymbol{\Omega}_\lambda(u(t)) u'_\lambda(t), \tag{*}$$

where $\boldsymbol{\Omega}_1(u), \dots, \boldsymbol{\Omega}_k(u)$ are the partial rotation vectors of the given frame at u.

Proof. Write $\rho(u) = (A(u); \boldsymbol{e}_1(u), \boldsymbol{e}_2(u), \boldsymbol{e}_3(u))$ and, for all $t \in I$, let $\boldsymbol{\varepsilon}_i(t) = \boldsymbol{e}_i(u(t))$ for $i = 1, 2, 3$. Since ρ is rigid, the moving frame

$$\sigma : t \mapsto (A(u(t)); \boldsymbol{\varepsilon}_1(t), \boldsymbol{\varepsilon}_2(t), \boldsymbol{\varepsilon}_3(t))$$

is also rigid. By the chain rule,

$$\frac{d\boldsymbol{\varepsilon}_i}{dt} = \sum_{\lambda=1}^{k} \frac{\partial \boldsymbol{e}_i}{\partial u_\lambda}(u(t)) u'_\lambda(t).$$

By the definition of partial rotations, this becomes

$$\frac{d\boldsymbol{\varepsilon}_i}{dt} = \sum_{\lambda=1}^{k} \left(\boldsymbol{\Omega}_\lambda(u(t)) \times \boldsymbol{e}_i(u(t))\right) u'_\lambda(t).$$

Thus the vector defined by (*) satisfies

$$\frac{d\boldsymbol{\varepsilon}_i}{dt} = \boldsymbol{\omega}(t) \times \boldsymbol{\varepsilon}_i(t), \quad i = 1, 2, 3.$$

It is therefore the instantaneous rotation vector of σ at time t. □

Example 4.8. Suppose that three real-valued functions ψ, θ, φ are differentiable on an interval I. For every $t \in I$, let $\sigma(t) = (O; \boldsymbol{I}(t), \boldsymbol{J}(t), \boldsymbol{K}(t))$ be the positively oriented orthonormal frame whose Euler angles relative to $R = (O; \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ are $\psi(t), \theta(t), \varphi(t)$. Its instantaneous rotation vector is

$$\begin{aligned}
\boldsymbol{\omega}(t) &= \psi' \boldsymbol{k} + \theta' \boldsymbol{u} + \varphi' \boldsymbol{K} \\
&= \theta' \boldsymbol{u} + \psi' \sin\theta \, \boldsymbol{w} + (\psi' \cos\theta + \varphi') \boldsymbol{K} \\
&= (\theta' \cos\varphi + \psi' \sin\theta \sin\varphi) \boldsymbol{I} \\
&\quad + (\psi' \sin\theta \cos\varphi - \theta' \sin\varphi) \boldsymbol{J} \\
&\quad + (\psi' \cos\theta + \varphi') \boldsymbol{K}.
\end{aligned}$$

The components p, q, r in the moving frame are therefore

$$\begin{aligned}
p &= \psi' \sin\theta \sin\varphi + \theta' \cos\varphi, \\
q &= \psi' \sin\theta \cos\varphi - \theta' \sin\varphi, \\
r &= \psi' \cos\theta + \varphi'.
\end{aligned}$$

The components in the fixed frame are

$$\begin{aligned} p_1 &= \varphi' \sin\theta \sin\psi + \theta' \cos\psi, \\ q_1 &= -\varphi' \sin\theta \cos\psi + \theta' \sin\psi, \\ r_1 &= \varphi' \cos\theta + \psi'. \end{aligned}$$

These formulas are useful in rigid-body dynamics.

5 Compatibility and reconstruction in several parameters

Let $U \subset \mathbb{R}^k$ be a simply connected domain. In fixed orthonormal coordinates write a direct rigid frame as (a, Q), with $Q : U \to SO(n)$ and $a : U \to \mathbb{R}^n$. Define the body-coordinate coefficients

$$W_\lambda = Q^T \partial_\lambda Q, \qquad b_\lambda = Q^T \partial_\lambda a.$$

The matrices W_λ are skew-symmetric. The associated homogeneous matrix and its infinitesimal generators are

$$g = \begin{pmatrix} Q & a \\ 0 & 1 \end{pmatrix}, \qquad X_\lambda = \begin{pmatrix} W_\lambda & b_\lambda \\ 0 & 0 \end{pmatrix}.$$

Thus $\partial_\lambda g = g X_\lambda$. This is the standard body-coordinate formulation on the special Euclidean group [9, 5].

Proposition 5.1 (Compatibility and reconstruction). Suppose $W_\lambda : U \to \mathfrak{so}(n)$ and $b_\lambda : U \to \mathbb{R}^n$ are of class C^1. Given an initial direct orthonormal frame at $u_0 \in U$, there exists a unique C^2 frame satisfying the above system if and only if, for all λ, μ,

$$\partial_\lambda W_\mu - \partial_\mu W_\lambda + [W_\lambda, W_\mu] = 0, \tag{1}$$

$$\partial_\lambda b_\mu - \partial_\mu b_\lambda + W_\lambda b_\mu - W_\mu b_\lambda = 0. \tag{2}$$

Proof. Equality of mixed derivatives of g gives $\partial_\lambda X_\mu - \partial_\mu X_\lambda + [X_\lambda, X_\mu] = 0$, whose blocks give the stated equations. Conversely, the equations are the flatness condition for the first-order system $dg = gX$, where $X = \sum_\lambda X_\lambda du_\lambda$. Along every piecewise smooth path from u_0, solve the linear matrix ordinary differential equation obtained by pulling back X. The flatness condition implies invariance of its endpoint under fixed-endpoint homotopies (the standard integrability theorem for flat connections [11]). Since U is simply connected, the solution is independent of the path. Skew-symmetry preserves $Q^T Q = I_n$ and continuity preserves $\det Q = 1$; the last row of g remains $(0, \dots, 0, 1)$. Local differentiation gives the required system and C^2 regularity. Uniqueness follows from uniqueness of the pathwise ordinary differential equation. □

6 Conclusion

The affine and differential viewpoints on moving frames lead to a coherent description of rigid motion. A constant Gram matrix is equivalent to rigidity, and its infinitesimal counterpart is skew-symmetry of the spatial velocity operator. Prescribed continuous body-coordinate velocities reconstruct a unique frame on an interval, while in several parameters the Maurer–Cartan compatibility equations provide the additional integrability requirement. In three dimensions the cross-product representation yields a direct interpretation of Frenet and Darboux invariants, Euler-angle velocities and the addition principle. This synthesis provides a foundation for further investigations of invariant differential equations and structure-preserving numerical reconstruction.

References

[1] R. L. Bishop, There is more than one way to frame a curve, *American Mathematical Monthly* 82 (1975), 246–251. doi:10.2307/2319846.

[2] M. P. do Carmo, *Differential Geometry of Curves and Surfaces*, revised and updated second edition, Dover, 2016.

[3] M. Fels and P. J. Olver, Moving coframes. I. A practical algorithm, *Acta Applicandae Mathematicae* 51 (1998), 161–213.

[4] M. Fels and P. J. Olver, Moving coframes. II. Regularization and theoretical foundations, *Acta Applicandae Mathematicae* 55 (1999), 127–208.

[5] B. C. Hall, *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction*, 2nd ed., Springer, 2015.

[6] A. Iserles, H. Z. Munthe-Kaas, S. P. Nørsett and A. Zanna, Lie-group methods, *Acta Numerica* 9 (2000), 215–365. doi:10.1017/S0962492900002154.

[7] K. M. Lynch and F. C. Park, *Modern Robotics: Mechanics, Planning, and Control*, Cambridge University Press, 2017.

[8] E. L. Mansfield, *A Practical Guide to the Invariant Calculus*, Cambridge University Press, 2010.

[9] R. M. Murray, Z. Li and S. S. Sastry, *A Mathematical Introduction to Robotic Manipulation*, CRC Press, 1994.

[10] B. O'Neill, *Elementary Differential Geometry*, revised 2nd ed., Academic Press, 2006.

[11] R. W. Sharpe, *Differential Geometry: Cartan's Generalization of Klein's Erlangen Program*, Springer, 1997.

[12] J. Lelong-Ferrand and J.-M. Arnaudiès, *Cours de mathématiques, Tome 3: Géométrie et cinématique*, 2nd ed., Dunod, 1993.