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PERIOD LINES AND ELLIPTIC LOGARITHMS: FOUR EXPONENTIALS AT RATIOS OF CM PERIODS, AND THE REACH OF WALDSCHMIDT'S METHOD

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Abstract. We give a criterion under which Waldschmidt's subgroup theorem with period term (1988), applied subgroup by subgroup on a line of codimension two through a period vector of three columns of weight two, proves a four exponentials statement. The criterion is a finite check on which coordinates of the group a subgroup can kill, together with bounds on a few relation spaces that come from classical transcendence results. Once the criterion is in place, individual instances are cheap. The most natural ones are these. Let E_1, E_2, E_3 be elliptic curves over $\overline{\mathbb{Q}}$ with complex multiplication, with periods Ω_j such that $t = \Omega_2/\Omega_1$ and $t' = \Omega_3/\Omega_1$ are linearly independent over $\mathbb{Q}$. Then for every nonzero logarithm λ of an algebraic number and every nonzero elliptic logarithm u of an algebraic point of E_1, the numbers $e^{\lambda t}, e^{\lambda t'}, e^{ut}, e^{ut'}$ are not all algebraic. If the CM fields are distinct, the same holds for $e^{tu_1}, e^{t'u_1}, e^{tu_2}, e^{t'u_2}$, with u_j an elliptic logarithm of a non-torsion algebraic point of E_j. The ratios t, t' have period weight 0. For all pairs of data, four exponentials was previously known at transcendental CM-period exponents only in classes of weight ± 1 and ± 2 (the latter being $\overline{\mathbb{Q}}^{\times}\pi^{\pm 1}$), unless an unexpected period relation holds; at weight 0 only some particular pairs had been treated, with both rows on a single curve. The statements above are also for particular pairs, with rows given by logarithms of algebraic numbers or on different curves. We also show where the method stops. Within an explicit model, the hyperplane count reaches only the classes treated earlier, $\overline{\mathbb{Q}}^{\times}\pi^{\pm 1}$, and their Möbius images. On codimension-two lines through one period vector, a strict count from two data that is not already given by a projection forces three columns of weight two, which is exactly the setting of the criterion. Finally we record how complex conjugation splits the complex-exponent instances $2^{\sigma}, 3^{\sigma} \in O_K$.

1. Introduction

The conjecture and what was known at periods. Let $\overline{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$ and $\mathcal{L} = \{z \in \mathbb{C} : e^z \in \overline{\mathbb{Q}}^{\times}\}$. The *four exponentials conjecture* says: if $x_1, x_2 \in \mathbb{C}$ are linearly independent over $\mathbb{Q}$ and so are y_1, y_2, then the four numbers $e^{x_i y_j}$ are not all algebraic [19]. With $x = (\log\alpha, \log\beta)$ and $y = (1, t)$ it says that α^t, β^t are not both algebraic when $t \notin \mathbb{Q}$. It is known for algebraic t (Gel'fond–Schneider), for $t \in \overline{\mathbb{Q}}^{\times}\pi^{\pm 1}$ (from Roy's strong six exponentials

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theorem [11]), and open in general. The six exponentials theorem (Lang, Ramachandra; [10, Ch. II]) proves the analogous statement with three numbers y_j.

At periods, the earlier paper [8] proved the conjecture, and its strong form, at

$$t \in \overline{\mathbb{Q}}^{\times} P^{\pm 1} \ \cup \ \overline{\mathbb{Q}}^{\times} (P/\pi)^{\pm 1}$$

for every nonzero eigen-period P of a CM abelian variety over $\overline{\mathbb{Q}}$ (for a CM elliptic curve: $t \in \overline{\mathbb{Q}}^{\times}\Omega^{\pm 1}$ or $\overline{\mathbb{Q}}^{\times}(\Omega/\pi)^{\pm 1}$), by explicit applications of Waldschmidt's subgroup theorem with period term [14, Thm. 4.1] in its hyperplane case. Those results hold for all pairs x; for particular pairs, [9, §7] also reaches ratios Ω'/Ω of periods of two CM curves, with both rows on a single curve. This paper asks what else that theorem can prove for four exponentials at periods, and where it stops. The answer has three parts: one mechanism (codimension-two lines through a period vector, Theorem 3.3); its natural instances, at *ratios* of periods of CM curves (Theorems 1.1 and 1.4); and counting lemmas which show that this mechanism is the only room the method leaves (Theorem 1.5). We stress that once the mechanism is known, instances are cheap: each is a finite check plus two or three classical transcendence inputs. The contribution is the mechanism and its boundary.

Notation. For an elliptic curve E over $\overline{\mathbb{Q}}$ with Weierstrass function $\wp_E$ and period lattice Λ_E we put $\mathcal{L}_E = \Lambda_E \cup \{u \in \mathbb{C} : \wp_E(u) \in \overline{\mathbb{Q}}\}$, the elliptic logarithms of the algebraic points of E. If E has complex multiplication by an order in the imaginary quadratic field k and $\Omega \neq 0$ is a period, then $\mathbb{Q}\Lambda_E = k\Omega$, and $u \in \mathcal{L}_E$ is the logarithm of a torsion point iff $u \in k\Omega$. CM elliptic curves over $\overline{\mathbb{Q}}$ are isogenous iff their CM fields are equal. For curves given as $y^2 = x^3 + ax + b$ we use $z \mapsto (\wp(z), \wp'(z)/2)$ with $g_2 = -4a$, $g_3 = -4b$.

The mechanism. Theorem 3.3 in §3 is stated for *period-line configurations*: three columns, each made of two of the groups $\mathbb{G}_{\mathrm{m}}$, E_c (CM curves with distinct CM fields), possibly with a third group $\mathbb{G}_{\mathrm{a}}$, a nonzero period P_j of one of the groups $\mathbb{G}_{\mathrm{m}}$, E_c of column j, and two numbers x_1, x_2 such that each entry $x_i P_j$ is assigned to one group of column j and is a logarithm of an algebraic point of it. The theorem says that such a configuration cannot be algebraic as soon as a combinatorial inequality,

$$2\,r(\mathsf{L}) + 2\,w(\mathsf{L}) < 1 + 5\dim U(\mathsf{L}),$$

holds for every "layout" L of a possible algebraic subgroup; here w counts weight, U is the image of the subgroup in $\mathbb{C}^3$, and r is the dimension of an intersection of *relation spaces* $R_s \subset F_s^2$ ($F_s = \mathbb{Q}$, k_c or K for $\mathbb{G}_{\mathrm{m}}$, a curve E_c or $\mathbb{G}_{\mathrm{a}}$), which record which combinations of the entries in a slot s are periods (zero for $\mathbb{G}_{\mathrm{a}}$). Only the R_s involve the numbers.

The headline instances.

Theorem 1.1. *Let E_1, E_2, E_3 be elliptic curves over $\overline{\mathbb{Q}}$ with complex multiplication, and let $\Omega_j \neq 0$ be a period of E_j. Put $t = \Omega_2/\Omega_1$, $t' = \Omega_3/\Omega_1$ and assume that t, t' are linearly independent over $\mathbb{Q}$. Then for every $\lambda \in \mathcal{L} \setminus \{0\}$ and every $u \in \mathcal{L}_{E_1} \setminus \{0\}$ the four numbers*

$$e^{\lambda t}, \qquad e^{\lambda t'}, \qquad e^{ut}, \qquad e^{ut'}$$

are not all algebraic.

Here λ, u are $\mathbb{Q}$-linearly independent (by Lemma 2.4(a) if u is not torsion; if $u \in k_1\Omega_1$, then $\lambda \in \mathbb{Q}u$ would give $\lambda \in \mathcal{L} \cap \overline{\mathbb{Q}}\mathcal{L}_{E_1}$, contrary to (W) of §2.2). So Theorem 1.1 is exactly the four exponentials conjecture for $x = (\lambda, u)$, $y = (t, t')$. Most of its cases follow from Gel'fond–Schneider and Wüstholz's theorem (§4.3). The substance is:

Theorem 1.2. *In Theorem 1.1 assume that the CM fields k_1, k_2, k_3 are pairwise distinct and that $u \notin k_1\Omega_1$. Write $\lambda = \log\alpha$ with $\alpha \in \overline{\mathbb{Q}}^\times$. Then $\alpha^t = e^{t\log\alpha}$, $\alpha^{t'}$, e^{ut}, $e^{ut'}$ are not all algebraic.*

Example 1.3. Let $E_1\colon y^2 = x^3 - 2x$ ($j = 1728$, CM by $\mathbb{Q}(i)$) with the point $(2,2)$, $E_2\colon y^2 = x^3 + 2$ ($j = 0$, CM by $\mathbb{Q}(\sqrt{-3})$) and $E_3\colon y^2 = x^3 - 30x + 56$ ($j = 8000$, CM by $\mathbb{Q}(\sqrt{-2})$). The point $(2,2)$ has infinite order: twice it is $(9/4, -21/8)$, which is not integral (Nagell–Lutz). So if u is an elliptic logarithm of $(2,2)$ and Ω_j are nonzero periods, then $2^{\Omega_2/\Omega_1}$, $2^{\Omega_3/\Omega_1}$, $e^{u\Omega_2/\Omega_1}$, $e^{u\Omega_3/\Omega_1}$ are not all algebraic. (Checked in exact arithmetic: `anc/check_examples.py`.)

Theorem 1.4 (two elliptic rows)**.** *Let E_1, E_2, E_3, k_j, Ω_j, t, t' be as in Theorem 1.2. For $u_1 \in \mathcal{L}_{E_1} \setminus k_1\Omega_1$ and $u_2 \in \mathcal{L}_{E_2} \setminus k_2\Omega_2$, the numbers e^{tu_1}, $e^{t'u_1}$, e^{tu_2}, $e^{t'u_2}$ are not all algebraic.*

Theorem 1.4 holds for every $u_1 \in \mathcal{L}_{E_1}\setminus\{0\}$ and for arbitrary CM fields with t, t' $\mathbb{Q}$-independent, provided u_2 is not torsion (Remark 5.1); the case of a torsion u_2 is open. Further instances of the mechanism, among them a Weierstrass-entry variant of Theorem 1.2, are listed in §5.2; §5.3 gives a two-number statement proved by the hyperplane case of the same theorem: for CM curves with distinct CM fields, $e^{\beta\Omega_2/\Omega_1}$ and $e^{u\Omega_2/\Omega_1}$ are not both algebraic when $\beta \in \overline{\mathbb{Q}}^\times$ and $u \in \mathcal{L}_{E_1} \setminus \{0\}$ (Theorem 5.4).

The limits of the method.

Theorem 1.5 (summary of §6)**.**

(1) Hyperplane count. *An argument through the hyperplane case of* [14, Thm. 4.1] *(or* [12, Théorème (iii)]*) on products of $\mathbb{G}_\mathrm{m}$, $\mathbb{G}_\mathrm{a}$, CM elliptic curves, their universal vector extensions and CM eigen-coordinates, using only period vectors whose coordinates are single monomial period classes, proves four exponentials at a period-monomial exponent $s \notin \overline{\mathbb{Q}}$ only if s or $1/s$ lies in $\overline{\mathbb{Q}}^\times P \cup \overline{\mathbb{Q}}^\times P/\pi \cup \overline{\mathbb{Q}}^\times\pi$ for an eigen-period P (Proposition 6.1). With the free Möbius closure of the strong form (Remark 6.3) these are exactly the classes of* [8] *and the classical class $\overline{\mathbb{Q}}^\times\pi^{\pm1}$. In particular no exponent of period weight* 0*, such as Ω_1/Ω_2, $\Omega_1\Omega_2/\pi$, Ω^2/π or a Beta value divided by an elliptic period, is reached (Lemma 6.2).*

(2) Codimension-two lines without period term. *On a line in $\mathbb{C}^3$ with $\kappa = 0$ (the setting of* [8, Thm. P]*), a contradiction from two data forces three columns of weight one; if each row has the maximal number of free entries, the coordinate ratios of the line have nonzero period weight, and at a weight-*0 *ratio at least three entries are not free (Lemma 6.4).*

(3) Codimension-two lines with one period vector. *With $\kappa = 1$, a contradiction from two data that is not already given by a projection to $\mathbb{C}^2$ forces every column to have weight exactly two (Lemma 6.5), which is the setting of Theorem 3.3; and in that setting the criterion of Theorem 3.3 is exactly what the count gives when the data lost to a subgroup are bounded through its dead slots (Remark 3.4). We do not treat $\kappa \geq 2$.*

Item (i) is a proof sketch relative to the stated model; items (ii) and (iii) are proved within the slot model of §6.2. These are statements about one method, not about the numbers: Theorem 1.2 is predicted by the four exponentials conjecture whatever the method.

Dependence on published theorems. Theorems 3.3, 1.2, 1.4, the instances of §5.2 and Theorem 5.4 (through the hyperplane case) are proved modulo Waldschmidt's Theorem 4.1 [14, pp. 385–386], quoted as Theorem 2.1; the instances also use the Schneider–Lang criterion [10, Ch. III, Thm. 1] and, where stated, Wüstholz's theorem in the form (W), Gel'fond–Schneider (GS) and Schneider's theorem (S) of §2.2; Theorem 1.1 follows from Theorem 1.2 with (W) and (GS). Each application of Theorem 2.1 has $W \neq 0$ and $\kappa > 0$, a case in which the printed proof

indicates one step (the auxiliary function with multiplicities along W) by reference to other papers; see [8, §2.1] for what this means and how far it has been checked. Each application satisfies the condition (NR) of §2.1, and each group is of the form $\mathbb{G}_{\mathrm{a}}^{d_0} \times \mathbb{G}_{\mathrm{m}}^{d_1} \times E_1 \times \cdots \times E_m$; for such groups Theorem 2.1 under (NR) is proved in Lean 4 [7], and so is a form of the Schneider–Lang criterion. The instantiation (the groups, lifts and case checks below) is not formalised.

What is new, against the published papers. As far as we found, Theorems 3.3, 1.1, 1.4 and the instances of §5.2 are not in print. We checked every statement of [8] and [9]:

- [8, Thms. A, S-A] cover $t \in \overline{\mathbb{Q}}^{\times} P^{\pm 1} \cup \overline{\mathbb{Q}}^{\times} (P/\pi)^{\pm 1}$. Theorem 1.2 is equivalent to four exponentials at $y = (1, s)$, $s = \Omega_3/\Omega_2$, for the particular pair $x = (t \log \alpha, tu)$; s has period weight 0, and lies in those classes only under a multiplicative relation between CM periods and π that the Rohrlich–Lang conjecture excludes (but which is not excluded unconditionally).
- [8, Thm. P] is a codimension-two statement with $\kappa = 0$, whose ratios have nonzero weight (Lemma 6.4).
- The two-curve statements of [9, §7] (Theorem C42 and its corollary, e.g. $e^{\Omega_\rho u_1/\varpi}$, $e^{\Omega_\rho u_2/\varpi}$ not both algebraic for $\mathbb{Q}(i)$-independent $u_1, u_2 \in \mathcal{L}_{E_i}$) have both rows on one curve and one period-ratio column; re-basing Theorem 1.2 at E_2 or E_3 puts both rows in $\mathcal{L}$, not in one $\mathcal{L}_E$.
- [9, Thm. SE(1)] has slope $c\Omega'/\Omega$, data in $\overline{\mathbb{Q}} + \overline{\mathbb{Q}}\mathcal{L}_E$ and targets $\overline{\mathbb{Q}} + \overline{\mathbb{Q}}\mathcal{L}_E + \overline{\mathbb{Q}}\mathcal{L}_{E'}$; Theorem 5.4 is its sibling with targets in $\mathcal{L}$, and neither implies the other.
- [9, Thm. CL4] (three CM curves) concerns lines through four period points and allows lines through the origin, which is all that the data of Theorem 1.2 produce.

We also searched the arXiv, the Hexagon preprint server, GitHub and Zenodo with generic queries, and read Waldschmidt's surveys [16, 17, 19, 15] and Bertolin–Waldschmidt [3]; keyword coverage of the arXiv is weak. The nearest classical statements are the Lang–Ramachandra elliptic analogues of the six exponentials theorem [17, §3.2], which need more columns and use one function along each column. All our statements except Theorem 5.2 (which has a Weierstrass entry) are special cases of the four exponentials conjecture (of its strong form for Theorem 5.4); they are also predicted by the Grothendieck–André period conjecture for 1-motives on $\mathbb{G}_{\mathrm{m}}^s \times E_1 \times E_2 \times E_3$. Bertolin's elliptico-toric conjecture [2, §1] makes that period conjecture explicit for 1-motives $[\mathbb{Z}^r \to \prod_j E_j \times \mathbb{G}_{\mathrm{m}}^s]$ with pairwise non-isogenous curves E_j, so our statements are conditional consequences of it (we have not written out the translation); [15, Conj. 3.21] is its special case without $\mathbb{G}_{\mathrm{m}}$, and [3, Conj. 4.1] is its form for a single curve. Unconditionally, the results recalled in [3, §2] concern a single elliptic curve.

Conventions. "Proved" means proved here modulo the published theorems named in each proof. *[Proof sketch]* marks arguments complete only relative to a stated model or not written out in full; *[Numerical]* marks computations that are evidence, not proof; *[Heuristic]* marks expectations. Computations that feed a statement are exact; scripts and their outputs are in the folder `anc/` of the ancillary files, described at the end of the paper.

Organisation. §2 states the tools. §3 proves the schema, §4 Theorems 1.2 and 1.1, and §5 Theorem 1.4 and further instances. §6 proves Theorem 1.5. §7 treats complex exponents, and §8 collects controls, open cases and numerics.

2. Tools

2.1. **Waldschmidt's subgroup theorem.**

Theorem 2.1 ([14, Thm. 4.1, pp. 385–386])**.** *Let $G = \mathbb{G}_{\mathrm{a}}^{d_0} \times \mathbb{G}_{\mathrm{m}}^{d_1} \times G_2$ be a connected commutative algebraic group over $\overline{\mathbb{Q}}$ of dimension $d = d_0 + d_1 + d_2 \geq 1$, with projections π_0, π_1 onto the first two factors (d_0, d_1 need not be maximal). Let $V \subset T_G(\mathbb{C})$ be a subspace of dimension $N < d$, $W \subset V$ a subspace defined over $\overline{\mathbb{Q}}$, and $Y \subset V$ a finitely generated subgroup with $\Gamma = \exp_G Y \subset G(\overline{\mathbb{Q}})$; put $\kappa = \mathrm{rk}_{\mathbb{Z}}(Y \cap \ker \exp_G)$. Then there is a connected algebraic subgroup $G' \neq G$ defined over $\overline{\mathbb{Q}}$ such that, with $\delta = \dim G/G'$, $\delta_0 = \dim \mathbb{G}_{\mathrm{a}}^{d_0}/\pi_0(G')$, $\delta_1 = \dim \mathbb{G}_{\mathrm{m}}^{d_1}/\pi_1(G')$, $\delta_2 = \delta - \delta_0 - \delta_1$, $\lambda = \mathrm{rk}_{\mathbb{Z}} \Gamma/(\Gamma \cap G')$ and $\tau = \dim W/(W \cap T_{G'})$, one has $\delta > \tau$ and*

$$(\lambda + \delta_1 + 2\delta_2)(d - N) \leq (\delta - \tau)(d_1 + 2d_2 - \kappa). \tag{1}$$

We use it in the range

$$2N \geq d_1 + 2d_0 + \kappa, \tag{NR}$$

which is not a hypothesis of the theorem but the range in which its $W \neq 0$ step has been checked outside print; for $G = \mathbb{G}_{\mathrm{a}}^{d_0} \times \mathbb{G}_{\mathrm{m}}^{d_1} \times E_1 \times \cdots \times E_m$ with elliptic curves E_j over $\overline{\mathbb{Q}}$, Theorem 2.1 under (NR) is the Lean theorem `LSG.W88.theorem41Abelian` of [7] (any d_0). Each proof below checks (NR).

Lemma 2.2 (λ is computed on lifts)**.** *In Theorem 2.1, let $\Omega(G) = \ker \exp_G$ and let $\mathbb{Q}Y$ be the $\mathbb{Q}$-span of Y. Then $\lambda = \mathrm{rk} Y - \dim_{\mathbb{Q}}\big(\mathbb{Q}Y \cap (T_{G'} + \mathbb{Q}\Omega(G))\big)$ and $\kappa = \dim_{\mathbb{Q}}(\mathbb{Q}Y \cap \mathbb{Q}\Omega(G))$.*

Proof. Since G' is connected, $\exp_G(T_{G'}) = G'$, so $\exp_G^{-1}(G') = T_{G'} + \Omega(G)$ and $\Gamma \cap G' = \exp_G(Y \cap (T_{G'} + \Omega(G)))$. The kernel $Y \cap \Omega(G)$ of $\exp_G|_Y$ lies in $Y \cap (T_{G'} + \Omega(G))$, so $\Gamma/(\Gamma \cap G') \cong Y/(Y \cap (T_{G'} + \Omega(G)))$. If $y \in \mathbb{Q}Y \cap (T_{G'} + \mathbb{Q}\Omega(G))$, then a positive multiple of y lies in Y (Y is finitely generated), and a further multiple lies in $Y \cap (T_{G'} + \Omega(G))$, because $T_{G'}$ is a $\mathbb{C}$-space and every element of $\mathbb{Q}\Omega(G)$ has an integer multiple in $\Omega(G)$. Hence the $\mathbb{Q}$-span of $Y \cap (T_{G'} + \Omega(G))$ is $\mathbb{Q}Y \cap (T_{G'} + \mathbb{Q}\Omega(G))$, which gives λ; the case $G' = 0$ gives κ. □

So whether a combination of data is "lost" to G' is decided in T_G, coordinate by coordinate, and not on the image of T_G in a smaller space; this is what makes the codimension-two arguments work.

Lemma 2.3 (subgroups)**.** *Let $G = \mathbb{G}_{\mathrm{a}}^{d_0} \times \mathbb{G}_{\mathrm{m}}^{d_1} \times \prod_c E_c^{m_c}$ with E_c pairwise non-isogenous elliptic curves over $\overline{\mathbb{Q}}$. Every connected algebraic subgroup is $G' = H_0 \times T_M \times \prod_c B_c$ with $H_0 \subset \mathbb{G}_{\mathrm{a}}^{d_0}$ a vector subgroup, $T_M \subset \mathbb{G}_{\mathrm{m}}^{d_1}$ a subtorus with $T_{T_M} = \mathbb{C} \otimes_{\mathbb{Q}} M$ for a subspace $M \subset \mathbb{Q}^{d_1}$, and $B_c \subset E_c^{m_c}$ abelian subvarieties. If E_c has CM by k_c and period Ω_c, then $\mathrm{Lie} B_c = \mathbb{C} \cdot N_c$ for a k_c-subspace $N_c \subset k_c^{m_c}$, and $\dim_{\mathbb{C}} B_c = \dim_{k_c} N_c$.*

Proof. Let $L = \mathbb{G}_{\mathrm{a}}^{d_0} \times \mathbb{G}_{\mathrm{m}}^{d_1}$, $A = \prod_c E_c^{m_c}$, $L' = (G' \cap L)^0$, $A' = (G' \cap A)^0$. The kernel of $G' \to A$ has identity component L', so G'/L' is isogenous to the abelian variety $p_A(G')$; the kernel of $G' \to L$ has identity component A', so G'/A' is isogenous to the linear group $p_L(G')$. The connected group $G'/(L' + A')$ is a quotient of both, hence complete and affine, hence trivial: $G' = L' \times A'$. A connected commutative linear group is the product of its unipotent part and its torus, and $\mathrm{Hom}(\mathbb{G}_{\mathrm{a}}, \mathbb{G}_{\mathrm{m}}) = \mathrm{Hom}(\mathbb{G}_{\mathrm{m}}, \mathbb{G}_{\mathrm{a}}) = 0$, so $L' = H_0 \times T_M$. Since $\mathrm{Hom}(E_c, E_{c'}) = 0$ for $c \neq c'$, Goursat's lemma and a dimension count give $A' = \prod_c B_c$. If E_c has CM, the lattice $\mathrm{Lie} B_c \cap \Lambda_c^{m_c}$ spans $\mathrm{Lie} B_c$ over $\mathbb{R}$ and lies in $(k_c \Omega_c)^{m_c}$; so $\mathrm{Lie} B_c = \mathbb{C} \cdot N_c$ with $N_c = \Omega_c^{-1}(\mathrm{Lie} B_c \cap (k_c \Omega_c)^{m_c})$, the intersection of a $\mathbb{C}$-space with a k_c-space, hence a k_c-subspace of $k_c^{m_c}$; and k_c-independent vectors of $k_c^{m_c}$ are $\mathbb{C}$-independent, so $\dim_{\mathbb{C}} B_c = \dim_{k_c} N_c$. □

2.2. **Transcendence inputs.** *Schneider–Lang* [10, Ch. III, Thm. 1]: let F be a number field and $f_1, \ldots, f_r$ meromorphic functions of finite order, at least two of them algebraically independent, such that $F[f_1, \ldots, f_r]$ is stable under d/dz. Then the set of $z \in \mathbb{C}$ at which all f_i are regular with $f_i(z) \in F$ is finite. (Lean: `LSG.schneiderLang` in [7], in the form with a common

entire denominator; for the functions below one takes a power of a product of σ-functions.) We also use:

(W) $\mathcal{L} \cap \overline{\mathbb{Q}}\mathcal{L}_E = \{0\}$ for an elliptic curve E over $\overline{\mathbb{Q}}$ [20]. (If $0 \neq z = cv$ with $e^z \in \overline{\mathbb{Q}}$, $c \in \overline{\mathbb{Q}}^\times$, $v \in \mathcal{L}_E$, then (z, v) is a logarithm of an algebraic point of $\mathbb{G}_\mathrm{m} \times E$ on the $\overline{\mathbb{Q}}$-line $\mathbb{C}(c, 1)$, which is the tangent space of no connected algebraic subgroup.)

(GS) If $\lambda \in \mathcal{L} \setminus \{0\}$ and $c \in \overline{\mathbb{Q}} \setminus \mathbb{Q}$ then $c\lambda \notin \mathcal{L}$ (Gel'fond–Schneider).

(S) Every $u \in \mathcal{L}_E \setminus \{0\}$ is transcendental (Schneider [13]).

Lemma 2.4. *Let E_1, E_2, E_3 be elliptic curves over $\overline{\mathbb{Q}}$ and $u_j \in \mathcal{L}_{E_j}$ logarithms of non-torsion points.*

(a) *For $\lambda \in \mathcal{L} \setminus \{0\}$, the numbers λ and u_1 are linearly independent over $\mathbb{Q}$.*

(b) *If $E_1 \not\sim E_2$, then u_1, u_2 are linearly independent over $\mathbb{Q}$.*

(c) *If $E_2 \not\sim E_3$ and Ω_2, Ω_3 are nonzero periods of E_2, E_3, then $\Omega_2/\Omega_3 \notin \mathbb{Q}$.*

Proof. (a) Both are nonzero. Suppose $u_1 = r\lambda$ with $r \in \mathbb{Q}^\times$. The functions e^z, $\wp_1(rz)$, $\wp_1'(rz)$ have order ≤ 2 and generate a ring stable under d/dz over $F = \mathbb{Q}(e^\lambda, r, g_2, g_3, \wp_1(u_1), \wp_1'(u_1))$, with e^z and $\wp_1(rz)$ algebraically independent (e^z is entire, non-constant and not elliptic). At the pairwise distinct points $z_m = m\lambda$ ($m \geq 1$) the values are $e^{m\lambda}$, $\wp_1(mu_1)$, $\wp_1'(mu_1)$, which lie in F by the addition law, with no poles because u_1 is not torsion. This contradicts Schneider–Lang.

(b) Suppose $u_2 = ru_1$, $r \in \mathbb{Q}^\times$. The functions $\wp_1(z), \wp_1'(z), \wp_2(rz), \wp_2'(rz)$ have order ≤ 2 and generate a differentially stable ring; $\wp_1(z)$ and $\wp_2(rz)$ are algebraically independent because $E_1 \not\sim E_2$. At $z_m = mu_1$ the values are the coordinates of mP_1 and mP_2 (up to the factor 2), in one number field, and no pole occurs. Schneider–Lang gives a contradiction.

(c) Suppose $\Omega_2/\Omega_3 \in \mathbb{Q}$. Then $\mathbb{Q}\Omega_2 = \mathbb{Q}\Omega_3 = \mathbb{Q}\omega_0$ with ω_0 chosen so that $\Lambda_2 \cap \mathbb{Q}\omega_0 = \mathbb{Z}\omega_0$, and $\Lambda_3 \cap \mathbb{Q}\omega_0 = \mathbb{Z}(a/b)\omega_0$ with $a, b \geq 1$ coprime. Put $c = 2b$ and $z_m = (m + 1/c)\omega_0$, $m \in \mathbb{Z}$. Then $z_m \notin \Lambda_2$, and $z_m \notin \Lambda_3$: otherwise $(mc + 1)b/(ac) \in \mathbb{Z}$, so $c \mid (mc + 1)b$, so $2b \mid b$ since $\gcd(c, mc + 1) = 1$. Modulo Λ_2 all z_m equal ω_0/c, and modulo Λ_3 they lie in the finite group $(1/c)\mathbb{Z}\omega_0/\mathbb{Z}(a/b)\omega_0$, so $\wp_2, \wp_2', \wp_3, \wp_3'$ take finitely many values at the z_m, all algebraic (torsion values), hence in one number field. The functions $\wp_2, \wp_3$ are algebraically independent because $E_2 \not\sim E_3$, and the ring they generate with their derivatives is differentially stable. Infinitely many points contradict Schneider–Lang. □

Part (a) also follows from (W), and (b), (c) from Wüstholz's theorem. For $\lambda \in 2\pi i\mathbb{Q}^\times$, (a) says $u_1 \notin 2\pi i\mathbb{Q}$.

3. The schema: period lines of weight $(2, 2, 2)$

Fix CM elliptic curves E_c over $\overline{\mathbb{Q}}$ (c in a finite index set) with pairwise distinct CM fields k_c and nonzero periods Ω_c, a number field K containing the compositum of the k_c, $n = [K : \mathbb{Q}]$, a $\mathbb{Q}$-basis $\theta_1, \ldots, \theta_n$ of K and, for each c, a k_c-basis $\varphi_1^c, \ldots, \varphi_{n/2}^c$ of K; write $\theta_k = \sum_l a_{kl}^c \varphi_l^c$ with $a_{kl}^c \in k_c$.

Definition 3.1. A *period-line configuration* consists of:

- three *columns* $j = 1, 2, 3$, each with two *slots of weight one*, each being $\mathbb{G}_\mathrm{m}$ or one of the curves E_c, and possibly a third slot $\mathbb{G}_\mathrm{a}$ (of weight zero); no column has two $\mathbb{G}_\mathrm{m}$ slots, two copies of one curve or two $\mathbb{G}_\mathrm{a}$ slots, but a curve may occur in several columns (once per column);
- for each column j a nonzero period P_j of one of its slots of weight one ($P_j \in 2\pi i\mathbb{Q}^\times$ for $\mathbb{G}_\mathrm{m}$, $P_j \in \Lambda_c$ for E_c);
- two complex numbers x_1, x_2 and, for each $i \in \{1, 2\}$ and each column j, a slot $s(i, j)$ of column j that *receives the entry* $x_i P_j$.

The configuration is *algebraic* if every entry is a logarithm of an algebraic point of its slot: $x_iP_j \in \mathcal{L}$ if $s(i,j)$ is $\mathbb{G}_{\mathrm{m}}$, $x_iP_j \in \mathcal{L}_{E_c}$ if it is E_c, $x_iP_j \in \overline{\mathbb{Q}}$ if it is $\mathbb{G}_{\mathrm{a}}$. For a slot s of column j put $e_{is} = x_iP_j$ if $s = s(i,j)$ and $e_{is} = 0$ otherwise.

The *relation space* of a slot s is a subspace of F_s^2, where F_s is $\mathbb{Q}$, k_c or K:

$$\begin{aligned} R_s &= \{(a,b) \in \mathbb{Q}^2 : ae_{1s} + be_{2s} \in 2\pi i\mathbb{Q}\} && \text{if } s \text{ is } \mathbb{G}_{\mathrm{m}},\\ R_s &= \{(a,b) \in k_c^2 : ae_{1s} + be_{2s} \in k_c\Omega_c\} && \text{if } s \text{ is a copy of } E_c,\\ R_s &= \{(a,b) \in K^2 : ae_{1s} + be_{2s} = 0\} && \text{if } s \text{ is } \mathbb{G}_{\mathrm{a}}, \end{aligned}$$

and $KR_s \subset K^2$ denotes its K-span. A slot that receives no entry has $R_s = F_s^2$. If the nonzero entries of a curve slot are linearly independent over k_c together with Ω_c (resp. those of a $\mathbb{G}_{\mathrm{a}}$ slot over K), which we call condition (I) for s, then R_s is the coordinate subspace on which the i-th coordinate vanishes for each i with $e_{is} \neq 0$; this is the case in all our instances. For subspaces defined over $\mathbb{Q}$, $\dim_K$ of an intersection of K-spans equals $\dim_{\mathbb{Q}}$ of the intersection; for such a space we write its $\mathbb{Q}$-points (as $\mathbb{Q} \times 0$ for $\{(a,0)\}$), since only KR_s matters. A copy s of E_c that receives an entry is *separated* if the k_c-span of e_{1s}, e_{2s} meets the k_c-span of Ω_c and of the entries of the other copies of E_c only inside $k_c\Omega_c$; it is separated whenever no other copy of E_c receives an entry.

Definition 3.2. A *layout* is a family $\mathsf{L} = (H, P, (U_c)_c)$ of complex subspaces of $\mathbb{C}^3$, with H (resp. P) contained in the span of the e_j for the columns j that have a $\mathbb{G}_{\mathrm{a}}$ (resp. $\mathbb{G}_{\mathrm{m}}$) slot and U_c in the span of the e_j for the columns in which E_c occurs. Put $U = H + P + \sum_c U_c$ and $w(\mathsf{L}) = \dim P + \sum_c \dim U_c$ (H has weight zero). The $\mathbb{G}_{\mathrm{a}}$ slot of column j is *dead* in L if $H \subset \{x_j = 0\}$, and the $\mathbb{G}_{\mathrm{m}}$ slot of column j is dead if $P \subset \{x_j = 0\}$. The copy of E_c in column j is dead if $U_c \subset \{x_j = 0\}$, and also if it is separated and $e_j \notin U_c$. Let $r(\mathsf{L}) = \dim_K \bigcap_{s \text{ dead}} KR_s$ (the intersection over no slot being K^2). Finally, let $\varepsilon_i = (e_{is})_s \in \mathbb{C}^h$, s running over the h $\mathbb{G}_{\mathrm{a}}$ slots; if $\dim(\mathbb{C}\varepsilon_1 + \mathbb{C}\varepsilon_2) > \dim H$, replace $r(\mathsf{L})$ by $\min(r(\mathsf{L}), 1)$. If $\dim H = 1$ and H has nonzero coordinates in two columns whose $\mathbb{G}_{\mathrm{a}}$ slots s, s' satisfy $e_{1s} \neq 0 = e_{2s}$ and $e_{2s'} \neq 0 = e_{1s'}$, and $\bigcap_{s \text{ dead}} KR_s$ lies in $K \times 0$ or in $0 \times K$, replace $r(\mathsf{L})$ by 0.

The second rule for curves (the *graph rule*) is what lets a graph subgroup between two copies of E_c kill a copy that carries entries; when two copies carry related entries, only the first rule (the *support rule*) is available. The last two clauses (the *diagonal rules*) handle a vector subgroup that meets several $\mathbb{G}_{\mathrm{a}}$ slots without being a product of coordinate lines: such a subgroup keeps those slots alive, but cannot contain the entries of both data, and a line through two slots that carry entries of different data cannot contain a combination of one datum alone.

Theorem 3.3 (schema)**.** *Suppose that every layout* L *with* $\dim U \leq 2$ *satisfies*

$$2\,r(\mathsf{L}) + 2\,w(\mathsf{L}) < 1 + 5\dim U. \tag{2}$$

Then the period-line configuration is not algebraic.

Theorem 3.3 is proved modulo Theorem 2.1; (NR) holds automatically, and the group is of the form covered by [7]. Condition (2) is a finite check. At $\dim U = 0$ it says $r = 0$: no nonzero combination of the data is a period in every slot. On a line it allows $r \leq 2, 1, 0$ for $w = 0, 1, 2$; at an axis $U = \mathbb{C}e_j$ with $w = 2$ this asks that the slots outside column j have $\bigcap KR_s = 0$. On a plane it allows $r \leq 1$ when $w = 4$ (for a coordinate plane with P and the U_c maximal, the dead slots are those of the third column) and holds automatically when $w \leq 3$.

Proof. Group and line. For each slot of weight one take the factor $\mathbb{G}_{\mathrm{m}}^n$ (coordinates $z_{s,k}$) or $E_c^{n/2}$ (coordinates $w_{s,l}$), and for each $\mathbb{G}_{\mathrm{a}}$ slot the factor $\mathbb{G}_{\mathrm{a}}$ (coordinate a_s); let G be the product over

all slots: $d_0 = h$ with $h \le 3$ the number of $\mathbb{G}_{\mathrm{a}}$ slots, $d_1 = gn$ with $g \le 3$ the number of $\mathbb{G}_{\mathrm{m}}$ slots, $d_2 = (6-g)n/2$, $d = 3n + gn/2 + h$ and $d_1 + 2d_2 = 6n$. Let $\Psi\colon T_G(\mathbb{C}) \to \mathbb{C}^3$ have j-th coordinate the sum, over the slots s of column j, of $\sum_k \theta_k z_{s,k}$ (if s is $\mathbb{G}_{\mathrm{m}}$), $\sum_l \varphi_l^c w_{s,l}$ (if s is E_c) or a_s (if s is $\mathbb{G}_{\mathrm{a}}$). Ψ is onto with coefficients in K. Put $V = \Psi^{-1}(\mathbb{C}(P_1, P_2, P_3))$, so $N = d - 2$, and $W = \ker \Psi$, defined over $\overline{\mathbb{Q}}$.

Lifts. For $i \in \{1, 2\}$ and $k \le n$ let $\hat{y}_{ik}$ have, in the slot $s(i, j)$ of each column j, the coordinate $z_{s,k} = x_i P_j$ (if $\mathbb{G}_{\mathrm{m}}$), $w_{s,l} = a_{kl}^c x_i P_j$ for all l (if E_c) or $a_s = \theta_k x_i P_j$ (if $\mathbb{G}_{\mathrm{a}}$), and 0 elsewhere; let $\hat{y}_{0k}$ be defined in the same way with x_i replaced by 1 and placed in the slot that carries P_j. Then $\Psi(\hat{y}_{ik}) = \theta_k x_i (P_1, P_2, P_3) \in V$ ($x_0 := 1$). Choose $D \in \mathbb{Z}_{>0}$ with $D a_{kl}^c \in \mathrm{End}(E_c)$ and $D\hat{y}_{0k} \in \Omega(G)$ for all c, k, l (possible since $\mathbb{Q} \otimes \mathrm{End}(E_c) = k_c$ and $\mathbb{Q}\Lambda_c = k_c\Omega_c$), and let Y be the subgroup generated by the $D\hat{y}_{ik}$. If the configuration is algebraic, then $\exp_G Y \subset G(\overline{\mathbb{Q}})$: each curve coordinate of an element of Y is $\gamma_1 v_1 + \gamma_2 v_2 + \omega$ with $\gamma_i \in \mathrm{End}(E_c)$, $v_i \in \mathcal{L}_{E_c}$ and $\omega \in \Lambda_c$, hence the logarithm of an algebraic point; each $\mathbb{G}_{\mathrm{m}}$ coordinate is a $\mathbb{Z}$-combination of elements of $\mathcal{L}$ and of $2\pi i\mathbb{Q}$ lying in $\mathcal{L}$; each $\mathbb{G}_{\mathrm{a}}$ coordinate is algebraic.

Subgroups and weights. By Lemma 2.3, $G' = H_0 \times T_M \times \prod_c B_c$, with $H_0 \subset \mathbb{G}_{\mathrm{a}}^h$ a vector subgroup, B_c inside the product of the copies of E_c and $\mathrm{Lie} B_c = \mathbb{C}N_c$. Since each column has at most one $\mathbb{G}_{\mathrm{a}}$ slot, Ψ maps $T_{\mathbb{G}_{\mathrm{a}}^h}$ injectively, and $H := \Psi(T_{H_0})$ has $\dim H = \dim H_0$; the s-coordinate of $v \in T_{H_0}$ is the j-th coordinate of $\Psi(v)$, for s the $\mathbb{G}_{\mathrm{a}}$ slot of column j. Since each column has at most one $\mathbb{G}_{\mathrm{m}}$ slot, the map $\iota\colon \mathbb{Q}^{gn} \to K^3$ given by θ is injective, $\Psi(T_{T_M}) = P := \mathbb{C}\iota(M)$, and $\dim M \le n \dim P$ (vectors of K^3 that are independent over K are independent over $\mathbb{C}$). Likewise the map $\iota_c\colon N_c \to K^3$ given by the φ^c is k_c-linear and injective (no column has two copies of E_c), $\Psi(\mathrm{Lie} B_c) = U_c := \mathbb{C}\iota_c(N_c)$, and $2 \dim B_c = \dim_{\mathbb{Q}} N_c \le n \dim U_c$. So G' defines a layout $\mathsf{L}(G') = (H, P, (U_c))$ with $U = \Psi(T_{G'})$ and $\mathrm{wt}\, G' := \dim T_M + 2\sum_c \dim B_c \le n\, w(\mathsf{L}(G'))$; and $U = 0$ forces $G' = 0$. Moreover $\delta_1 + 2\delta_2 = 6n - \mathrm{wt}\, G'$ (the vector part H_0 enters only δ_0, which does not occur in (1)), $\delta - \tau = 3 - \dim U$ (since $W = \ker \Psi$: $\tau = (d-3) - (\dim G' - \dim U)$), and $d - N = 2$.

Dead slots. Let $y = \sum_{i,k} c_{ik}\hat{y}_{ik}$ ($c_{ik} \in \mathbb{Q}$, $i \in \{0, 1, 2\}$) lie in $T_{G'} + \mathbb{Q}\Omega(G)$, and let s be a slot that is dead in $\mathsf{L}(G')$. Put $g_i = \sum_k c_{ik}\theta_k \in K$; the map $(c_1, c_2) \mapsto (g_1, g_2)$ is a $\mathbb{Q}$-linear isomorphism $\mathbb{Q}^n \times \mathbb{Q}^n \to K^2$. We claim $(g_1, g_2) \in KR_s$. By Lemma 2.3, $T_{G'} + \mathbb{Q}\Omega(G) = T_{H_0} \times (\mathbb{C} \otimes M + \mathbb{Q}\Omega(\mathbb{G}_{\mathrm{m}}^{gn})) \times \prod_c (\mathbb{C}N_c + \mathbb{Q}\Omega(E_c^{(\cdot)}))$, a product over the vector, torus and E_c-parts; in particular the projection of y to any slot lies in the projection of $T_{G'} + \mathbb{Q}\Omega(G)$ to that slot.

If s is the $\mathbb{G}_{\mathrm{a}}$ slot of column j, then $H \subset \{x_j = 0\}$ makes the projection of T_{H_0} to s zero, so $a_s(y) = g_1 e_{1s} + g_2 e_{2s} = 0$, i.e. $(g_1, g_2) \in R_s = KR_s$. If s is the $\mathbb{G}_{\mathrm{m}}$ slot of column j, then $\iota(M) \subset \{x_j = 0\}$ forces the s-coordinates of every vector of M to vanish, so the projection of T_{T_M} to s is 0 and $z_{s,k}(y) = c_{1k}e_{1s} + c_{2k}e_{2s} + c_{0k}P_j[s \text{ carries } P_j] \in 2\pi i\mathbb{Q}$; as $P_j \in 2\pi i\mathbb{Q}$ when s carries it, $(c_{1k}, c_{2k}) \in R_s$ for all k, i.e. $(g_1, g_2) \in R_s \otimes_{\mathbb{Q}} K = KR_s$.

Let s be the copy of E_c in column j. The E_c-part of y is $\sum_i \sum_{s'} e_{is'}\gamma_i^{(s')}$ plus an element of $(k_c\Omega_c)^{(\cdot)}$, where s' runs over the copies of E_c, $\gamma_i = (\gamma_{il})_l = (\sum_k c_{ik}a_{kl}^c)_l \in k_c^{n/2}$, and $\gamma_i^{(s')}$ is γ_i placed in copy s'. Note $g_i = \sum_l \gamma_{il}\varphi_l^c$, so that $(g_1, g_2) = \sum_l (\gamma_{1l}, \gamma_{2l})\varphi_l^c$, and $\iota_c(v^{(s')}) = (\sum_l v_l \varphi_l^c)e_{j'}$ for $v \in k_c^{n/2}$ and s' in column j'. If $U_c \subset \{x_j = 0\}$, then every vector of N_c has zero copy-s coordinates (its image under ι_c has j-th coordinate $\sum_l \varphi_l^c v_{s,l}$, and the φ_l^c are k_c-independent), so the copy-s coordinates of y lie in $k_c\Omega_c$: $\gamma_{1l}e_{1s} + \gamma_{2l}e_{2s} \in k_c\Omega_c$, i.e. $(\gamma_{1l}, \gamma_{2l}) \in R_s$, for all l, hence $(g_1, g_2) \in KR_s$. If instead s is separated and $e_j \notin U_c$, let ϖ be a k_c-linear map with kernel N_c, extended $\mathbb{C}$-linearly (kernel $\mathbb{C}N_c$). Membership in $\mathbb{C}N_c + (k_c\Omega_c)^{(\cdot)}$ gives, coordinate by coordinate,

$$\sum_i e_{is}\, \varpi(\gamma_i^{(s)}) \in \Omega_c \cdot k_c^{(\cdot)} - \sum_i \sum_{s' \ne s} e_{is'}\, \varpi(\gamma_i^{(s')}).$$

The left side has coordinates in the k_c-span of e_{1s}, e_{2s}, the right side in the k_c-span of Ω_c and the entries of the other copies; as s is separated, both sides lie in $k_c\Omega_c$, so every coordinate of $(\varpi(\gamma_1^{(s)}), \varpi(\gamma_2^{(s)}))$ lies in R_s. Let $(\mu_1, \mu_2) \in k_c^2$ vanish on R_s (i.e. $\mu_1 a + \mu_2 b = 0$ for $(a, b) \in R_s$). Then $\varpi((\mu_1\gamma_1 + \mu_2\gamma_2)^{(s)}) = 0$, so $(\mu_1\gamma_1 + \mu_2\gamma_2)^{(s)} \in N_c$ and ι_c of it, $(\mu_1 g_1 + \mu_2 g_2)e_j$, lies in U_c; as $e_j \notin U_c$, $\mu_1 g_1 + \mu_2 g_2 = 0$. This holds for all such (μ_1, μ_2), so $(g_1, g_2) \in KR_s$, which gives the claim.

Counting. The vectors $\hat{y}_{0k}$ lie in $\mathbb{Q}\Omega(G)$, so the condition $y \in T_{G'} + \mathbb{Q}\Omega(G)$ does not involve c_0. Put $S(G') = \{(c_1, c_2) \in \mathbb{Q}^n \times \mathbb{Q}^n : \sum_{i\in\{1,2\},k} c_{ik}\hat{y}_{ik} \in T_{G'} + \mathbb{Q}\Omega(G)\}$. The claim maps $S(G')$ injectively into $\bigcap_{s \text{ dead}} KR_s$. Moreover the $\mathbb{G}_{\mathrm{a}}$-part of $\sum_{i\geq 1} c_{ik}\hat{y}_{ik}$ is $g_1\varepsilon_1 + g_2\varepsilon_2$, which for $(c_1, c_2) \in S(G')$ lies in T_{H_0}; so the image of $S(G')$ also lies in the kernel of the K-linear map $K^2 \to \mathbb{C}^h/T_{H_0}$, $g \mapsto g_1\varepsilon_1 + g_2\varepsilon_2$, and that kernel is a proper K-subspace when $\varepsilon_1, \varepsilon_2$ do not both lie in T_{H_0}, in particular when $\dim(\mathbb{C}\varepsilon_1 + \mathbb{C}\varepsilon_2) > \dim H_0 = \dim H$. In the situation of the second diagonal rule, $T_{H_0} = \mathbb{C}v$, and the s- and s'-coordinates of v are the j-th and j'-th coordinates of $\Psi(v)$, which spans H, so both are nonzero. If $(g_1, 0)$ lies in the kernel, then $g_1\varepsilon_1 = \mu v$ for some $\mu \in \mathbb{C}$; the s'-coordinate gives $0 = \mu v_{s'}$, so $\mu = 0$, and the s-coordinate gives $g_1 e_{1s} = 0$, so $g_1 = 0$; likewise for $(0, g_2)$. So the kernel meets $K \times 0$ and $0 \times K$ only in 0, and its intersection with a subspace of either axis is 0. In all cases the image of $S(G')$ lies in a K-space of dimension $r(\mathsf{L}(G'))$ (as modified by the diagonal rules), so $\dim_{\mathbb{Q}} S(G') \leq n\, r(\mathsf{L}(G'))$. For $G' = 0$ every slot is dead and $U = 0$, so (2) gives $r = 0$ and $S(0) = 0$. If $\sum c_{ik}\hat{y}_{ik} = 0$, then $\sum_{i\geq 1} c_{ik}\hat{y}_{ik} \in \mathbb{Q}\Omega(G)$, so $(c_1, c_2) \in S(0) = 0$; applying Ψ gives $\sum_k c_{0k}\theta_k = 0$, so $c_0 = 0$. Thus the $3n$ vectors $\hat{y}_{ik}$ are $\mathbb{Q}$-independent. By Lemma 2.2, $\mathrm{rk} Y = 3n$, $\kappa = n$ and $\lambda = 2n - \dim S(G') \geq n(2 - r)$. Inequality (1) would read $2(\lambda + 6n - \operatorname{wt} G') \leq (3 - \dim U) \cdot 5n$; its left side is at least $2n(8 - r - w)$, with (r, w) those of $\mathsf{L}(G')$, and (2) gives $2(8 - r - w) > 5(3 - \dim U)$ when $\dim U \leq 2$, while $\delta > \tau$ excludes $\dim U = 3$. So no G' satisfies the conclusion of Theorem 2.1. Its hypotheses hold: $N < d$, W is defined over $\overline{\mathbb{Q}}$, $Y \subset V$ is finitely generated with $\exp_G Y \subset G(\overline{\mathbb{Q}})$, and (NR) reads $2(d - 2) = 6n + gn + 2h - 4 \geq gn + 2h + n$. □

Remark 3.4 (the criterion is the boundary of the method)**.** The proof uses about the numbers only the dead-slot bound $\dim S(G') \leq n\, r$. Conversely, if (2) fails at a layout realised by a subgroup G' that is a product over slots (H, P and the U_c spanned by coordinate vectors, each B_c a product of whole copies $E_c^{n/2}$), then for G' one has $\dim S(G') = n\, r$ and $\operatorname{wt} G' = n\, w$ (with the actual relation spaces), so G' satisfies (1) exactly when (2) fails; for such G' the diagonal rules change nothing, because the dead $\mathbb{G}_{\mathrm{a}}$ slots already give the kernel condition. For other subgroups (diagonal subtori, graphs between live copies) the live slots give further constraints, which the criterion ignores; and for a curve whose copies carry related entries, the support rule replaces the graph rule. So (2) is sufficient, and sharp for the dead-slot bound. Lemma 6.5 shows that among codimension-two lines with one period vector and two data, weights $(2, 2, 2)$ are forced once no projection to $\mathbb{C}^2$ already gives the result; a $\mathbb{G}_{\mathrm{a}}$ slot has weight zero, so it can only be a third slot in its column. So Theorem 3.3 covers the part of that class with at most one $\mathbb{G}_{\mathrm{m}}$ slot, one copy of each curve and one $\mathbb{G}_{\mathrm{a}}$ slot per column; when the relation spaces are not known, it is applied case by case, over the combinations of relation spaces that the known transcendence results leave possible (r only grows with the R_s, so each case may take its relation spaces as large as allowed; but two relations that together contradict a known result, say by making a transcendental period ratio algebraic, need not be allowed at the same time). The comparison with an exhaustive computer enumeration is in Remark 6.7.

4. Proof of Theorems 1.2 and 1.1

4.1. **The configuration.** Let E_1, E_2, E_3, α, u be as in Theorem 1.2 and suppose the four numbers are algebraic. Put $t_1 = 1$, $t_2 = t$, $t_3 = t'$. Column j has slots $\mathbb{G}_{\mathrm{m}}$ and E_j, with $P_j = \Omega_j$ carried by E_j; $x_1 = \log\alpha/\Omega_1$ places all its entries $t_j \log\alpha$ in the $\mathbb{G}_{\mathrm{m}}$ slots; $x_2 = u/\Omega_1$ places u in E_1 and $t_j u$ in the $\mathbb{G}_{\mathrm{m}}$ slots of columns $2, 3$. The configuration is algebraic, and (I) for E_1 says $u \notin k_1\Omega_1$. The relation spaces are

$$R_{\mathbb{G}_{\mathrm{m}},1} = \{(a,b) : a\log\alpha \in 2\pi i\mathbb{Q}\}, \qquad R_{E_1} = \mathbb{Q}\times 0,$$
$$R_{\mathbb{G}_{\mathrm{m}},j} = R_j := \{(a,b) : t_j(a\log\alpha + bu) \in 2\pi i\mathbb{Q}\} \quad (j = 2,3),$$

and $R_{E_2} = R_{E_3} = \mathbb{Q}^2$.

Lemma 4.1. $\dim R_j \le 1$ *and* $R_2 \cap R_3 = 0$. *If* α *is not a root of unity,* $R_{\mathbb{G}_{\mathrm{m}},1} = 0\times\mathbb{Q}$.

Proof. The last claim: $a\log\alpha \in 2\pi i\mathbb{Q}$ with $a \neq 0$ would make α a root of unity. If $\dim R_j = 2$ then $t_j\log\alpha = 2\pi i r$ and $t_j u = 2\pi i r'$ with $r, r' \neq 0$, so $u/\log\alpha \in \mathbb{Q}$, contrary to Lemma 2.4(a). If $0 \neq (a,b) \in R_2 \cap R_3$, then $w = a\log\alpha + bu \neq 0$ by Lemma 2.4(a), and $tw, t'w \in 2\pi i\mathbb{Q}^\times$, so $\Omega_2/\Omega_3 = t/t' \in \mathbb{Q}$, contrary to Lemma 2.4(c). □

4.2. **The criterion.** First let α not be a root of unity. In a layout, $P \subset \mathbb{C}^3$ and $U_j \subset \mathbb{C}e_j$. We check (2):

- $\dim U = 0$: every slot is dead and $R_{\mathbb{G}_{\mathrm{m}},1} \cap R_{E_1} = 0$, so $r = 0 < \frac{1}{2}$.
- U a line off the axes: every $U_j = 0$, so E_1 is dead and $r \le 1$; $w \le 1$; $2 + 2 < 6$.
- $U = \mathbb{C}e_1$: $P \subset \mathbb{C}e_1$, so $\mathbb{G}_{\mathrm{m}}$-columns $2, 3$ are dead and $r \le \dim(R_2 \cap R_3) = 0$; $w \le 2$; $4 < 6$.
- $U = \mathbb{C}e_2$ or $\mathbb{C}e_3$: $\mathbb{G}_{\mathrm{m}}$-column 1 and E_1 are dead, $r = 0$, $w \le 2$; $4 < 6$.
- $U = \{x_1 = 0\}$: $\mathbb{G}_{\mathrm{m}}$-column 1 and E_1 are dead, $r = 0$, $w \le 4$; $8 < 11$.
- $U = \{x_3 = 0\}$ (resp. $\{x_2 = 0\}$): $\mathbb{G}_{\mathrm{m}}$-column 3 (resp. 2) is dead, $r \le 1$, $w \le 4$; $10 < 11$.
- U another plane: it contains at most one axis, so $w \le 2 + 1$ and, with $r \le 2$, $10 < 11$.

Theorem 3.3 gives the contradiction. In terms of (1) the smallest margin of these bounds is n, at $G' = 0$ ($16n > 15n$) and at the planes other than $\{x_1 = 0\}$ ($6n > 5n$).

Proposition 4.2 (the case $\lambda \in 2\pi i\mathbb{Q}^\times$)**.** *Theorem 1.2 holds when* α *is a root of unity, i.e. for* $\lambda = 2\pi i r$, $r \in \mathbb{Q}^\times$.

Proof. Now $R_{\mathbb{G}_{\mathrm{m}},1} = \mathbb{Q}^2$. Since $t_j \notin \mathbb{Q}$ (Lemma 2.4(c) for (E_1, E_j)), $(1,0) \notin R_j$, so $\dim R_j \le 1$, and $R_j \cap (\mathbb{Q}\times 0) = 0$. And $R_2 \cap R_3 = 0$: for $0 \neq (a,b)$ in both, $w = 2\pi i r a + bu \neq 0$ by Lemma 2.4(a) (with $\lambda = 2\pi i$), and $tw, t'w \in 2\pi i\mathbb{Q}^\times$ give $t/t' \in \mathbb{Q}$. The check of §4.2 changes only where $\mathbb{G}_{\mathrm{m}}$-column 1 was used: at $\dim U = 0$, $R_{E_1} \cap R_2 = 0$; at $U = \mathbb{C}e_2$ (resp. $\mathbb{C}e_3$), E_1 and $\mathbb{G}_{\mathrm{m}}$-column 3 (resp. 2) are dead, and $R_{E_1} \cap R_3 = 0$; at $U = \{x_1 = 0\}$, only E_1 contributes ($R_{\mathbb{G}_{\mathrm{m}},1} = \mathbb{Q}^2$), $r \le 1$, $w \le 4$, $10 < 11$. Every other line is unchanged. □

4.3. **Theorem 1.1.** Let $\lambda = \log\alpha$, u, E_j be as in Theorem 1.1, and suppose the four numbers are algebraic.

- If $u \in k_1\Omega_1 \setminus \{0\}$, then $tu = a\Omega_2$ with $a \in k_1^\times$, so $tu \in \mathcal{L} \cap \overline{\mathbb{Q}}\mathcal{L}_{E_2}$ and $tu \neq 0$, contrary to (W).
- If $k_1 = k_2$, then $E_1 \sim E_2$ and $t \in \overline{\mathbb{Q}}^\times$. If $t \notin \mathbb{Q}$, then $\lambda, t\lambda \in \mathcal{L}$ contradicts (GS). If $t \in \mathbb{Q}$, then $tu \in \overline{\mathbb{Q}}\mathcal{L}_{E_1} \setminus \{0\}$ and $e^{tu} \in \overline{\mathbb{Q}}$ contradicts (W). The case $k_1 = k_3$ is the same with t'.
- If $k_2 = k_3 \neq k_1$, then $c = t'/t = \Omega_3/\Omega_2 \in \overline{\mathbb{Q}}^\times$, and $c \notin \mathbb{Q}$ because t, t' are $\mathbb{Q}$-independent; $\lambda t \neq 0$ and $c\lambda t = \lambda t'$ both in $\mathcal{L}$ contradict (GS).
- Otherwise k_1, k_2, k_3 are pairwise distinct and $u \notin k_1\Omega_1$: Theorem 1.2 with Proposition 4.2.

The case excluded by the hypothesis, $k_2 = k_3$ with $\Omega_3/\Omega_2 \in \mathbb{Q}$, is outside four exponentials (t, t' dependent). □

5. Theorem 1.4 and further instances

5.1. **Theorem 1.4.** Column 1 has slots E_1 (carrying $P_1 = \Omega_1$) and E_2; column 2 has $\mathbb{G}_{\mathrm{m}}$ and E_2 (carrying $P_2 = \Omega_2$); column 3 has $\mathbb{G}_{\mathrm{m}}$ and E_3 (carrying $P_3 = \Omega_3$). The datum $x_1 = u_1/\Omega_1$ places u_1 in E_1 and $tu_1, t'u_1$ in the $\mathbb{G}_{\mathrm{m}}$ slots; $x_2 = u_2/\Omega_1$ places u_2 in the copy of E_2 in column 1 and $tu_2, t'u_2$ in the $\mathbb{G}_{\mathrm{m}}$ slots. E_2 occurs in columns $1, 2$ and receives entries only in column 1, so that copy is separated. (I) holds since $u_1 \notin k_1\Omega_1$, $u_2 \notin k_2\Omega_2$. The relation spaces are $R_{E_1} = 0 \times \mathbb{Q}$, $R_{E_2,1} = \mathbb{Q} \times 0$, $R_{\mathbb{G}_{\mathrm{m}},j} = R_j := \{(a,b) : t_j(au_1 + bu_2) \in 2\pi i\mathbb{Q}\}$ for $j = 2, 3$, and $\mathbb{Q}^2$ for the copy of E_2 in column 2 and for E_3. As in Lemma 4.1, $\dim R_j \le 1$ (by Lemma 2.4(b)) and $R_2 \cap R_3 = 0$ (by Lemma 2.4(b),(c)).

In a layout, $P \subset \mathbb{C}e_2 + \mathbb{C}e_3$, $U_{E_1} \subset \mathbb{C}e_1$, $U_{E_2} \subset \mathbb{C}e_1 + \mathbb{C}e_2$, $U_{E_3} \subset \mathbb{C}e_3$, and the copy of E_2 in column 1 is dead iff $e_1 \notin U_{E_2}$.

- $\dim U = 0$: $R_{E_1} \cap R_{E_2,1} = 0$.
- $U = \mathbb{C}e_1$: $P = 0$, so both $\mathbb{G}_{\mathrm{m}}$ slots are dead, $r = 0$; $w \le 2$.
- $U = \mathbb{C}e_2$, $\mathbb{C}e_3$, or a line off the axes: $e_1 \notin U$, so E_1 and the column-1 copy of E_2 are dead, $r = 0$; $w \le 2$.
- $U = \{x_3 = 0\}$: $P \subset \mathbb{C}e_2$, $\mathbb{G}_{\mathrm{m}}$-column 3 is dead, $r \le 1$; $w \le 1 + 1 + 2 = 4$; $10 < 11$.
- $U = \{x_2 = 0\}$: $P \subset \mathbb{C}e_3$, $\mathbb{G}_{\mathrm{m}}$-column 2 is dead, $r \le 1$; $w \le 1 + 1 + 1 + 1 = 4$; $10 < 11$.
- $U = \{x_1 = 0\}$: E_1 and the column-1 copy of E_2 are dead, $r = 0$; $w \le 4$.
- U another plane: $U \cap (\mathbb{C}e_2 + \mathbb{C}e_3)$ and $U \cap (\mathbb{C}e_1 + \mathbb{C}e_2)$ are lines and U contains at most one of e_1, e_3, so $w \le 3$.

Theorem 3.3 gives the contradiction. □

Remark 5.1 (relaxations)**.** If $u_1 \in k_1\Omega_1 \setminus \{0\}$, then $tu_1 \in \overline{\mathbb{Q}}^{\times}\Omega_2$ and $e^{tu_1} \notin \overline{\mathbb{Q}}$ by (W). If $k_1 = k_2$ (resp. k_3), then t (resp. t') lies in $\overline{\mathbb{Q}}^{\times}$, so tu_1 (resp. $t'u_1$) lies in $\overline{\mathbb{Q}}\mathcal{L}_{E_1} \setminus \{0\}$ and its exponential is not algebraic by (W). If $k_2 = k_3 \ne k_1$ and $\Omega_3/\Omega_2 \notin \mathbb{Q}$, then $tu_1 \ne 0$ and $(t'/t)\,tu_1 = t'u_1$ with $t'/t \in \overline{\mathbb{Q}} \setminus \mathbb{Q}$ contradict (GS). So Theorem 1.4 holds for arbitrary CM curves with t, t' $\mathbb{Q}$-independent, every $u_1 \in \mathcal{L}_{E_1} \setminus \{0\}$ and every non-torsion u_2. If $u_2 \in k_2\Omega_2$, then $tu_2 \in \overline{\mathbb{Q}}^{\times}\Omega_2^2/\Omega_1$; that case is open, and the count has margin 0.

5.2. **Further instances.** Each of the following is the conclusion of Theorem 3.3 for a configuration with three CM curves E_1, E_2, E_3 of pairwise distinct CM fields, nonzero periods Ω_j, $t = \Omega_2/\Omega_1$, $t' = \Omega_3/\Omega_1$.

Theorem 5.2 (a Weierstrass entry)**.** *For $\alpha \in \overline{\mathbb{Q}}^{\times}$ not a root of unity, a fixed determination of $\log\alpha$, and $u \in \mathcal{L}_{E_1} \setminus k_1\Omega_1$, the numbers $\wp_{E_2}(t\log\alpha)$, $\alpha^{t'}$, e^{tu}, $e^{t'u}$ are not all algebraic.*

Proof. Columns as in §4.1; $x_1 = \log\alpha/\Omega_1$ places $\log\alpha$ in $\mathbb{G}_{\mathrm{m}}$ (column 1), $t\log\alpha$ in E_2 and $t'\log\alpha$ in $\mathbb{G}_{\mathrm{m}}$ (column 3); $x_2 = u/\Omega_1$ as in §4.1. (I) needs $u \notin k_1\Omega_1$ and $t\log\alpha \notin k_2\Omega_2$; the latter holds since otherwise $\log\alpha \in \overline{\mathbb{Q}}\Omega_1 \subset \overline{\mathbb{Q}}\mathcal{L}_{E_1}$, contrary to (W) (so $t\log\alpha \notin \Lambda_2$ and $\wp_{E_2}(t\log\alpha)$ is finite). Relation spaces: $R_{\mathbb{G}_{\mathrm{m}},1} = 0 \times \mathbb{Q}$, $R_{E_1} = \mathbb{Q} \times 0$, $R_{E_2} = 0 \times \mathbb{Q}$, $R_{\mathbb{G}_{\mathrm{m}},2} = \{(a,b) : btu \in 2\pi i\mathbb{Q}\}$, $R_{\mathbb{G}_{\mathrm{m}},3} = \{(a,b) : t'(a\log\alpha + bu) \in 2\pi i\mathbb{Q}\}$ (of dimension ≤ 1 by Lemma 2.4(a)), $R_{E_3} = \mathbb{Q}^2$; and $(0 \times \mathbb{Q}) \cap R_{\mathbb{G}_{\mathrm{m}},2} \cap R_{\mathbb{G}_{\mathrm{m}},3} = 0$, since $tu, t'u \in 2\pi i\mathbb{Q}^{\times}$ would give $t/t' \in \mathbb{Q}$. The check: $\dim U = 0$ and lines off the axes (E_1, E_2 dead): $r = 0$; $U = \mathbb{C}e_1$ (E_2 and $\mathbb{G}_{\mathrm{m}}$-columns 2, 3 dead): $r = 0$; $U = \mathbb{C}e_2, \mathbb{C}e_3, \{x_1 = 0\}$ ($\mathbb{G}_{\mathrm{m}}$-column 1 and E_1 dead): $r = 0$; $U = \{x_3 = 0\}$ ($\mathbb{G}_{\mathrm{m}}$-column 3 dead) and $U = \{x_2 = 0\}$ (E_2 dead): $r \le 1$, $w \le 4$; other planes: $w \le 3$. □

Remark 5.3 (more instances)**.** With E_1, E_2, E_3 as above, the following are instances of Theorem 3.3. In the first row $V = \mathbb{C}(\Omega_1, \Omega_2, \Omega_3)$, column 1 is $E_1 + E_2$ (period Ω_1 on E_1, entries u, u' in E_2) and columns 2, 3 are $\mathbb{G}_{\mathrm{m}} + E_2$, $\mathbb{G}_{\mathrm{m}} + E_3$ as in Theorem 1.4; here $R_{E_2,1} = 0$, and the check is that of Theorem 1.4. In the last two rows $V = \mathbb{C}(2\pi i, \Omega_1, \Omega_2)$, column 1 is $\mathbb{G}_{\mathrm{m}} + E_3$ and columns 2, 3 are $E_1 + \mathbb{G}_{\mathrm{m}}$, $E_2 + \mathbb{G}_{\mathrm{m}}$. In the second row $R_{\mathbb{G}_{\mathrm{m}},1} = 0 \times \mathbb{Q}$ and $R_{E_3} = \mathbb{Q} \times 0$, and the check is that of §4.2; in the third row $R_{\mathbb{G}_{\mathrm{m}},1} = \mathbb{Q}^2$ and $R_{E_3} = 0$, so $r = 0$ unless $e_1 \in U_{E_3}$, and the remaining lines are as in §4.2. The fourth row needs only two curves: $V = \mathbb{C}(2\pi i, \Omega_1, \Omega_2)$, column 1 is $\mathbb{G}_{\mathrm{m}}$ (carrying $2\pi i$) + a copy of E_1 receiving u, u', column 2 is $\mathbb{G}_{\mathrm{m}} + E_1$ (carrying Ω_1), column 3 is $\mathbb{G}_{\mathrm{m}} + E_2$; here E_1 occurs in two columns but only one copy receives entries, so that copy is separated; it has $R = 0$ and is dead unless $e_1 \in U_{E_1}$, and in every layout either that copy is dead ($r = 0$), or one of the $\mathbb{G}_{\mathrm{m}}$ slots of columns 2, 3 is dead ($r \le 1$ with $w \le 4$), or all three are live; in the last case $e_1 \in U$ and U contains neither e_2 nor e_3 (else P lies in $\{x_3 = 0\}$ or $\{x_2 = 0\}$), so U_{E_1} is a line, $U_{E_2} = 0$, $w \le 3$ and $2r + 2w \le 10 < 11$. In each row (I) holds for every curve slot, and $\dim R_{\mathbb{G}_{\mathrm{m}},j} \le 1$, $R_{\mathbb{G}_{\mathrm{m}},2} \cap R_{\mathbb{G}_{\mathrm{m}},3} = 0$ follow from Lemma 2.4 and the stated k_c-independence. (These checks were also run by machine.)

data	the four numbers, not all algebraic
$u, u' \in \mathcal{L}_{E_2}$, with u, u', Ω_2 independent over k_2	e^{tu}, $e^{t'u}$, $e^{tu'}$, $e^{t'u'}$
α not a root of unity, $u_3 \in \mathcal{L}_{E_3} \setminus k_3\Omega_3$	$\alpha^{\Omega_1/2\pi i}$, $\alpha^{\Omega_2/2\pi i}$, $e^{u_3\Omega_1/2\pi i}$, $e^{u_3\Omega_2/2\pi i}$
$u_3, u_3' \in \mathcal{L}_{E_3}$, with u_3, u_3', Ω_3 independent over k_3	$e^{u_3\Omega_1/2\pi i}$, $e^{u_3\Omega_2/2\pi i}$, $e^{u_3'\Omega_1/2\pi i}$, $e^{u_3'\Omega_2/2\pi i}$
$u, u' \in \mathcal{L}_{E_1}$, with u, u', Ω_1 independent over k_1 (no E_3)	$e^{u\Omega_1/2\pi i}$, $e^{u\Omega_2/2\pi i}$, $e^{u'\Omega_1/2\pi i}$, $e^{u'\Omega_2/2\pi i}$

In the last three rows the exponents $\Omega_j/2\pi i$ lie in the classes of [8], but those theorems do not allow an elliptic logarithm as a row; nor does [9, §7], since Ω/π is not among its ratios. In the second row $\log\alpha$ may be replaced by any $\lambda \in \mathcal{L} \setminus \{0\}$: for $\lambda \in 2\pi i\mathbb{Q}^\times$, $e^{\lambda\Omega_1/2\pi i} \notin \overline{\mathbb{Q}}$ by (W). In the first and third rows the excluded case $u' \in k_c u \setminus \mathbb{Q}u$ is trivially true by Gel'fond–Schneider; for $u' \in \mathbb{Q}u$ the statement reduces to two numbers, which we do not claim. The fourth row is one of the 66 configurations of Remark 6.7 that the enumeration's conservative model misses.

Two further instances have a $\mathbb{G}_{\mathrm{a}}$ slot. Let $\beta \in \overline{\mathbb{Q}}^\times$. For $u_2 \in \mathcal{L}_{E_2} \setminus k_2\Omega_2$ the numbers $e^{\beta t}$, $e^{\beta t'}$, e^{tu_2}, $e^{t'u_2}$ are not all algebraic, and for u_3 in $\mathcal{L}_{E_3} \setminus k_3\Omega_3$ neither are $e^{\beta\Omega_1/2\pi i}$, $e^{\beta\Omega_2/2\pi i}$, $e^{u_3\Omega_1/2\pi i}$, $e^{u_3\Omega_2/2\pi i}$. In the first, $V = \mathbb{C}(\Omega_1, \Omega_2, \Omega_3)$, column 1 is $E_1 + E_2 + \mathbb{G}_{\mathrm{a}}$ (period Ω_1 on E_1, entry β in $\mathbb{G}_{\mathrm{a}}$, entry u_2 in E_2) and columns 2, 3 are $\mathbb{G}_{\mathrm{m}} + E_2$, $\mathbb{G}_{\mathrm{m}} + E_3$ as in Theorem 1.4. In the second, $V = \mathbb{C}(2\pi i, \Omega_1, \Omega_2)$, column 1 is $\mathbb{G}_{\mathrm{m}} + E_3 + \mathbb{G}_{\mathrm{a}}$ (period $2\pi i$, entries β in $\mathbb{G}_{\mathrm{a}}$ and u_3 in E_3) and columns 2, 3 are $E_1 + \mathbb{G}_{\mathrm{m}}$, $E_2 + \mathbb{G}_{\mathrm{m}}$. In both, $R_{\mathbb{G}_{\mathrm{a}}} = 0 \times \mathbb{Q}$, the curve copy in column 1 is separated with $R = \mathbb{Q} \times 0$, and the $\mathbb{G}_{\mathrm{m}}$ slots of columns 2, 3 have $\dim R_{\mathbb{G}_{\mathrm{m}},j} \le 1$ and $R_{\mathbb{G}_{\mathrm{m}},2} \cap R_{\mathbb{G}_{\mathrm{m}},3} = 0$. Write u for u_2 (resp. u_3) and τ_2, τ_3 for the multipliers t, t' (resp. $\Omega_1/2\pi i, \Omega_2/2\pi i$) of columns 2, 3. If $\dim R_{\mathbb{G}_{\mathrm{m}},j} = 2$, then $\tau_j\beta, \tau_j u \in 2\pi i\mathbb{Q}^\times$ and $u/\beta \in \mathbb{Q}$, contrary to (S); for $(a, b) \neq 0$ the number $a\beta + bu$ is nonzero by (S), and $\tau_2(a\beta + bu)$, $\tau_3(a\beta + bu)$ both in $2\pi i\mathbb{Q}^\times$ would give $\Omega_2/\Omega_3 \in \mathbb{Q}$ (resp. $\Omega_1/\Omega_2 \in \mathbb{Q}$), contrary to Lemma 2.4(c). The check: at $\dim U = 0$, $r = 0$. If U is a line, either $e_1 \notin U$, so that the $\mathbb{G}_{\mathrm{a}}$ slot and the curve copy of column 1 are dead and $r = 0$, or $U = \mathbb{C}e_1$, so that the $\mathbb{G}_{\mathrm{m}}$ slots of columns 2, 3 are dead and $r = 0$; and $w \le 2$. If U is a plane, $r = 2$ needs all four slots that receive entries to be live, so $e_1 \in H \subset U$ and P lies on neither $\{x_2 = 0\}$ nor $\{x_3 = 0\}$; then U contains neither e_2 nor e_3 (if $e_2 \in U$ then $U = \{x_3 = 0\}$, and $P \subset U$ kills the $\mathbb{G}_{\mathrm{m}}$ slot of column 3; likewise for e_3). In the first instance $P \subset U \cap \langle e_2, e_3\rangle$ is a line, $U_{E_1} \subset \mathbb{C}e_1$, $U_{E_2} \subset U \cap \langle e_1, e_2\rangle$ is a line and $U_{E_3} = 0$; in the second $P \subset U$, $U_{E_3} \subset \mathbb{C}e_1$ and $U_{E_1} = U_{E_2} = 0$; so $w \le 3$. Otherwise $r \le 1$, and $w \le 4$ (the maximum, attained only at coordinate planes: e.g. $1 + 1 + 2 + 0$ at $\{x_3 = 0\}$ in the first instance, $2 + 1 + 1$ at $\{x_1 = 0\}$ in the second). Many further configurations with $\mathbb{G}_{\mathrm{a}}$ slots, with curves receiving entries in two columns, or with Weierstrass entries are counted in Remark 6.7; we do not state them.

5.3. **A two-number by-product.**

Theorem 5.4 (strong mixed four exponentials at Ω_2/Ω_1)**.** *Let E_1, E_2 be elliptic curves over $\overline{\mathbb{Q}}$ with complex multiplication by imaginary quadratic fields $k_1 \neq k_2$, with nonzero periods Ω_1, Ω_2. For $\beta \in \overline{\mathbb{Q}}^\times$ and $u \in \mathcal{L}_{E_1} \setminus \{0\}$, the two numbers $e^{\beta\Omega_2/\Omega_1}$ and $e^{u\Omega_2/\Omega_1}$ are not both algebraic.*

For example, with $\varpi = \Gamma(1/4)^2/(2\sqrt{2\pi}) = 2.6220575542\ldots$ and $\Omega_\rho = \Gamma(1/3)^3/(2^{4/3}\pi) = 2.4286506478\ldots$, the real periods of $y^2 = 4x^3 - 4x$ and $y^2 = 4x^3 - 4$, one of $e^{\Omega_\rho/\varpi}$, $e^{u\Omega_\rho/\varpi}$ is transcendental for every nonzero elliptic logarithm u of an algebraic point of $y^2 = 4x^3 - 4x$ (digits from Arb balls, `anc/check_examples.py`). This is a hyperplane statement, not an instance of Theorem 3.3.

Remark 5.5. If $u \in k_1\Omega_1$, then $tu = a\Omega_2$ with $a \in k_1^\times$ and $e^{tu} \notin \overline{\mathbb{Q}}$ by (W). If $k_1 = k_2$, then $t \in \overline{\mathbb{Q}}^\times$ and $tu \in \overline{\mathbb{Q}}\mathcal{L}_{E_1} \setminus \{0\}$, so $e^{tu} \notin \overline{\mathbb{Q}}$ by (W).

Proof of Theorem 5.4. So let $u \notin k_1\Omega_1$, and suppose $e^{\beta t}, e^{ut} \in \overline{\mathbb{Q}}$, $t = \Omega_2/\Omega_1$. Let $K = k_1k_2$, $n = [K : \mathbb{Q}]$, and

$$G = \left(\mathbb{G}_{\mathrm{a}} \times E_1^{n/2}\right) \times \left(\mathbb{G}_{\mathrm{m}}^n \times E_2^{n/2}\right), \qquad d_0 = 1,\ d_1 = n,\ d_2 = n,\ d = 2n+1,\ d_1 + 2d_2 = 3n,$$

with $\Psi(a, w_1, z, w_2) = \left(a + \sum_l \varphi_l^1 w_{1l},\ \sum_k \theta_k z_k + \sum_l \varphi_l^2 w_{2l}\right)$, $V = \Psi^{-1}(\mathbb{C}(\Omega_1, \Omega_2))$ (a hyperplane, $N = d - 1$) and $W = \ker\Psi$. Lifts: $\hat{y}_{1k}$ with $\mathbb{G}_{\mathrm{a}}$-coordinate $\theta_k\beta$ and $z_k = t\beta$; $\hat{y}_{2k}$ with E_1-coordinates $a_{kl}^1 u$ and $z_k = tu$; $\hat{y}_{0k}$ with periods in E_1 and E_2; D chosen as in §3. (1) reads $\lambda + 3n - \operatorname{wt} G' \leq (2 - \dim U) \cdot 2n$. Subgroups are $H_0 \times T_M \times B_1 \times B_2$, and $U \in \{0, \mathbb{C}e_1, \mathbb{C}e_2\}$ when $\dim U \leq 1$ (H_0, B_1 map to $\mathbb{C}e_1$; T_M, B_2 to $\mathbb{C}e_2$). Dead slots: $\mathbb{G}_{\mathrm{a}}$ dead $\Rightarrow \sum_k c_{1k}\theta_k\beta = 0 \Rightarrow c_1 = 0$; E_1 dead $\Rightarrow c_2 = 0$; the $\mathbb{G}_{\mathrm{m}}$ slot dead $\Rightarrow (c_{1k}, c_{2k}) \in R = \{(a, b) : t(a\beta + bu) \in 2\pi i\mathbb{Q}\}$, and $\dim R \leq 1$ since $u/\beta \notin \mathbb{Q}$ by (S). So $\kappa = n$, and: at $G' = 0$, $\lambda = 2n$ and $5n > 4n$; at $U = \mathbb{C}e_1$, the $\mathbb{G}_{\mathrm{m}}$ slot is dead, $\lambda \geq n$, wt $\leq n$, $3n > 2n$; at $U = \mathbb{C}e_2$, $\mathbb{G}_{\mathrm{a}}$ and E_1 are dead, $\lambda = 2n$, wt $\leq 2n$, $3n > 2n$. (NR): $2 \cdot 2n \geq n + 2 + n$ for $n \geq 2$. Theorem 2.1 gives a contradiction. □

Both CM hypotheses are used: with E_1 or E_2 not CM, the count at $G' = 0$ becomes an equality. The three-column statement "$e^{\beta t}, e^{\beta t'}, e^{ut}, e^{ut'}$ not all algebraic" follows.

6. The limits of the method

6.1. **The hyperplane count.** *Model.* A *hyperplane argument* for four exponentials at s uses a group $G = \mathbb{G}_{\mathrm{m}}^2 \times \mathbb{G}_{\mathrm{a}}^a \times \prod(\text{slots})$, the slots being CM elliptic curves, CM eigen-coordinates of CM abelian varieties (read through one eigen-differential, as in [8, §3]) and universal vector extensions of CM curves; a surjective $\overline{\mathbb{Q}}$-linear $\Psi\colon T_G \to \mathbb{C}^2$ that reads each slot into one coordinate; the line $V = \mathbb{C}(1, s)$; and the hyperplane bound $\#\{\text{data}\} \leq \alpha_{K,\Psi}(G) - \kappa$ [14, §5d], [12, (iii)]. Write $\alpha_{K,\Psi}(G) = 2 + w$, where 2 is the weight of $\mathbb{G}_{\mathrm{m}}^2$ and w that of the slots: 1 for a CM curve or eigen-coordinate, 0 for $\mathbb{G}_{\mathrm{a}}$, and $2 + 2/n$ for the universal vector extension of a CM curve read through base and fibre ([12, p. 361], [9, §3.3]). A counterexample gives two data; the argument may add κ *period vectors* $(x, sx) \in V$ (both coordinates periods of their slots) and e *mixed vectors* (one coordinate algebraic, the other a period). The bound gives a contradiction iff $2 + \kappa + e > 2 + w - \kappa$, that is

$$2\kappa + e > w. \tag{3}$$

Types. By Deligne, Koblitz–Ogus and Anderson [4, 6, 1], every CM period class we use (eigen-periods of either kind, π, products and quotients) is $\overline{\mathbb{Q}}^\times$ times a monomial with a *type* $f\colon (\mathbb{Z}/N)^\times \to \mathbb{Z}$ such that $f(u) + f(-u)$ is constant, the *weight* of the type. Algebraic numbers have $f \equiv 0$; $2\pi i$ has $f \equiv 1$ (weight 2); an eigen-period of either kind has a type in $\mathcal{E} = \{\phi :$

$\phi(u) \in \{0,1\},\ \phi(u)+\phi(-u)=1\}$ (weight 1); types add under products. That a number has only one type is a form of the Rohrlich–Lang conjecture and is not known.

Proposition 6.1 (pair cost; reach of the hyperplane count)**.** [Proof sketch, relative to the model above.] *Let s be a monomial class of type f_s, and suppose every vector used on V has coordinates that are single monomial classes. A period vector (x, sx) costs fewer than two weight-one slots only if x or sx is $2\pi i$ (the period of $\mathbb{G}_{\mathrm{m}}$, which is free), which needs $1+f_s \in \mathcal{E}$ or $1-f_s \in \mathcal{E}$, that is $s \in \overline{\mathbb{Q}}^{\times} P/\pi$ or $s \in \overline{\mathbb{Q}}^{\times}\pi/P$ for an eigen-period P. A mixed vector needs $s^{\pm 1} \in \overline{\mathbb{Q}}^{\times} P$ (an eigen-slot opposite an algebraic coordinate) or $s^{\pm 1} \in \overline{\mathbb{Q}}^{\times}\pi$ (the free period of $\mathbb{G}_{\mathrm{m}}$ opposite an algebraic coordinate). Otherwise each period vector occupies one slot on each side and adds 1 to κ and 2 to w, so $2\kappa \le w$, $e=0$, and (3) fails. Hence a hyperplane argument proves four exponentials at s only if $s^{\pm 1} \in \overline{\mathbb{Q}}^{\times} P \cup \overline{\mathbb{Q}}^{\times} P/\pi \cup \overline{\mathbb{Q}}^{\times}\pi$.*

Sketch. A vector (x, sx) needs $(\mathrm{type}(x), \mathrm{type}(x)+f_s)$ to be a pair of allowed slot classes, from $\{0\}$ ($\mathbb{G}_{\mathrm{a}}$ or fibre, not a period), $\{1\}$ ($\mathbb{G}_{\mathrm{m}}$) and $\mathcal{E}$ (an eigen-slot). Enumerating the pairs, cost < 2 occurs only for $\mathrm{type}(x) \in \{0,1\}$ or $\mathrm{type}(x)+f_s \in \{0,1\}$, which are the listed cases (the pairs $(0,1)$ and $(1,0)$ give $s \in \overline{\mathbb{Q}}^{\times}\pi^{\pm 1}$). On one side, two period vectors cannot share an eigen-slot (each slot carries one K-line of periods), and a vector mixing two slots would need a $\overline{\mathbb{Q}}$-linear relation between distinct monomial classes; none is known, and the Lang–Rohrlich conjecture predicts none. This is where the statement is relative to the model: an unknown linear relation between periods would enlarge κ. □

At $s \in \overline{\mathbb{Q}}^{\times}\pi^{\pm 1}$ four exponentials is classical: Roy's strong six exponentials theorem with $x = (\log\alpha, \log\beta, 1)$ and $y = (1, \pi)$.

Lemma 6.2 (weight test)**.** *The types of $P^{\pm 1}$ and $(P/\pi)^{\pm 1}$ have weight ± 1, and those of $\pi^{\pm 1}$ weight ± 2. So a monomial whose type has weight 0 (or any weight $\neq \pm 1, \pm 2$) does not have the type of a class in the reach of Proposition 6.1; as a number it can lie in such a class only under a period relation that the Rohrlich–Lang conjecture excludes. The classes Ω_1/Ω_2, $\Omega_1\Omega_2/\pi$, Ω^2/π and $B(a/N, b/N)/\Omega$ have weight 0.*

Proof. The weights are $1-1$, $1+1-2$, $2-2$ and $1-1$. □

Remark 6.3 (Möbius closure)**.** Strong four exponentials at s says $\dim_{\overline{\mathbb{Q}}}(\mathbb{C}(1,s) \cap \widetilde{\mathcal{L}}^2) \le 1$, $\widetilde{\mathcal{L}} = \overline{\mathbb{Q}} + \overline{\mathbb{Q}}\mathcal{L}$. A matrix $g \in GL_2(\overline{\mathbb{Q}})$ preserves $\widetilde{\mathcal{L}}^2$ and maps $\mathbb{C}(1,s)$ to $\mathbb{C}(1, g\cdot s)$; so strong four exponentials at s implies it at every Möbius image $g \cdot s$. With [8, Thm. S-A] this gives, for example, $s = \sqrt{2} + \varpi/\pi$ or $s = \varpi/(\varpi+\pi)$; we do not count these as new. With Proposition 6.1, the hyperplane reach is $PGL_2(\overline{\mathbb{Q}}) \cdot \overline{\mathbb{Q}}^{\times}\{P, P/\pi, \pi\}$.

6.2. **Lines of codimension two.** *Slot model.* Let $V = \mathbb{C}(P_1, P_2, P_3) \subset \mathbb{C}^3$ be a line and let column j carry slots of total weight w_j (per factor n: $\mathbb{G}_{\mathrm{m}}$ and CM slots weight 1, $\mathbb{G}_{\mathrm{a}}$ weight 0), $A = w_1 + w_2 + w_3$. A datum is a row (xP_1, xP_2, xP_3); an entry is *free* if it is algebraic (in a $\mathbb{G}_{\mathrm{a}}$ slot), a period of its slot, or the defining logarithm of the datum. With e data outside G' and κ period vectors, (1) at $G'=0$, at an axis j and at a coordinate plane (i,j) (with $\lambda = en$) is an obstruction unless, respectively,

$$e > \tfrac{A-3\kappa}{2}, \qquad e > w_j - \kappa, \qquad e > \tfrac{A-2w_k-\kappa}{2} \quad (\{i,j,k\} = \{1,2,3\}).$$

Projecting to the coordinates (i,j) gives a hyperplane count in $\mathbb{C}^2$ which allows at most $f_{ij} = w_i + w_j - 2\kappa_{ij}$ data.

Lemma 6.4 ($\kappa = 0$)**.** *In the slot model, a line with $\kappa = 0$ on which two data give a contradiction at $G' = 0$ has three columns of weight one. If moreover each row has the maximal number of free*

entries allowed by the axis and plane conditions, then up to scaling, exchanging the rows and permuting the columns, the rows are $(P_1, 1, 1/P_2)$ *and* $(P_1P_2, P_2, 1)$ *with* P_j *a period of column* j*, and the coordinate ratios of* V *are* P_1*,* P_2*,* P_1P_2 *(and their inverses), of nonzero period weight. At a weight-*0 *ratio at least three entries are not free.*

Proof. At $G' = 0$, $2 > A/2$ gives $A \leq 3$. A column of weight 0 carries only $\mathbb{G}_{\mathrm{a}}$, so both data have algebraic entries there and are $\overline{\mathbb{Q}}$-proportional; hence every column has weight one. The axis and plane conditions forbid two periods in one row and two rows with a period in the same column, and two algebraic entries in one row or column would make the rows or columns $\overline{\mathbb{Q}}$-dependent. So each row has at most two free entries, and four free entries force the stated pattern; its ratios P_1, P_2, P_1P_2 have positive type. □

[8, Thm. P] is the case $P_1 = P_2 = \pi$, rows $(\pi, 1, 1/\pi)$, $(\pi^2, \pi, 1)$.

Lemma 6.5 ($\kappa = 1$)**.** *In the slot model with integer weights, let* $\kappa = \kappa_{ij} = 1$*. Two data give a contradiction at* $G' = 0$*, at the axes and at the coordinate planes, while no projection to* $\mathbb{C}^2$ *does (*$f_{ij} \geq 2$ *for all* i, j*), iff* $(w_1, w_2, w_3) = (2, 2, 2)$*. Then* $G' = 0$ *gives* $16n > 15n$*; an axis gives* $12n > 10n$ *provided both data stay outside it (*$e = 2$*; with* $e = 1$ *it is* $10n \leq 10n$*); a coordinate plane gives* $6n > 5n$ *already with* $e = 1$*.*

Proof. At $G' = 0$ the condition is $2 > (A - 3)/2$, i.e. $A \leq 6$. No projection is strict iff $f_{ij} = w_i + w_j - 2 \geq 2$, i.e. $A - w_k \geq 4$, for every k. Summing over k gives $2A \geq 12$, so $A = 6$ and $w_k \leq 2$ for all k, hence $w = (2, 2, 2)$. Conversely, for $w = (2, 2, 2)$ the thresholds are $3/2$ at $G' = 0$, 1 at the axes and $1/2$ at the planes, and $f_{ij} = 2$. Substituting in (1) gives the stated inequalities. □

So in a strict $\kappa = 1$ cell every column has weight two: $\mathbb{G}_{\mathrm{m}}$ plus one period slot, or two period slots (as column 1 of Theorem 1.4), and the line passes through a period vector (P_1, P_2, P_3). A column of weight one gives a projection with $f \leq 1$, which is a hyperplane statement. At the axes both data must stay outside every subgroup with that image; in Theorem 3.3 this is the condition $\bigcap R_s = 0$ there, and in the instances it is supplied by Lemma 2.4(c) at $U = \mathbb{C}e_1$, and by (I) and Lemma 2.4 at the other axes.

Remark 6.6 (why the period datum is essential)**.** Drop the datum $x_0 = 1$ in the proof of Theorem 1.2 and treat $(\Omega_1, \Omega_2, \Omega_3)$ as an arbitrary $\overline{\mathbb{Q}}$-independent vector. Then the statement becomes a mixed strong six exponentials claim on $(\mathbb{G}_{\mathrm{m}}^n \times E_1^{n/2}) \times \mathbb{G}_{\mathrm{m}}^n \times \mathbb{G}_{\mathrm{m}}^n$ with weight $4n$, $\kappa = 0$, $\lambda = 2n$, and (1) at $G' = 0$ reads $2(2n + 4n) = 12n \leq 3 \cdot 4n$, an equality. Likewise, each datum alone is consistent with the count ($14n \leq 15n$ at $G' = 0$), each projection to $\mathbb{C}^2$ allows both data (with equality), deleting κ gives $16n \leq 18n$, and a non-CM E_2 gives $18n \leq 18n$.

Remark 6.7 (enumeration)**.** *[Numerical, exhaustive within a type model.]* The script `anc/u101_enum.py` enumerates all lines $V = \mathbb{C}(P_1, P_2, P_3)$ with pairwise distinct monomial periods P_j (of $\mathbb{G}_{\mathrm{m}}$ or of up to three CM curves with distinct CM fields), two data whose column-1 entries are $\log \alpha$, $2\pi i$, $\beta \in \overline{\mathbb{Q}}^{\times}$ or an elliptic logarithm, and every placement of the other entries in a $\mathbb{G}_{\mathrm{m}}$, $\mathbb{G}_{\mathrm{a}}$ or curve slot. It counts λ on lifts, with relation spaces taken as large as the known transcendence results allow. It finds 19 645 configurations of weight $A \leq 8$, of which 5 274 are strict at every connected subgroup. After removing those implied by a single-number theorem, by one row alone, by (strong) six exponentials, by Theorem 1.1 or by a result of [8] on a projection, 634 remain with no strict projection, all of weights $(2, 2, 2)$, as Lemma 6.5 predicts; a further 305 are implied by new strict projections to $\mathbb{C}^2$, all with a $\mathbb{G}_{\mathrm{a}}$ slot (Theorem 5.4 is the cleanest).

Comparison with Theorem 3.3 (the script `anc/schema_ext_vs_u101.py`, run with `--fast`, reruns the enumeration and reproduces the numbers above; `anc/schema_ext_fail_list.py` lists

the exceptions below). Exactly 4 482 of the 19 645 configurations lie in the class of Theorem 3.3: two slots of weight one in every column, at most one $\mathbb{G}_a$ slot per column, each receiving at most one entry; the other 15 163 have a column of weight $\neq 2$. We evaluate (2) with the enumeration's relation spaces (its adversarial model: every pattern of relations not excluded by a known transcendence result is allowed, slot by slot, and two slots may share a relation unless that would make a provably transcendental period ratio algebraic; the criterion must hold for every allowed pattern) over all layouts whose components are the intersections of U with coordinate spans, U running over 306 subspaces of $\mathbb{C}^3$ spanned by vectors with entries in $\{-1, 0, 1, 2\}$; taking each component maximal in U suffices, as larger components only increase r and w. Among the 1 670 configurations of the class that the enumeration finds strict, the criterion holds for 1 582. It also holds for 66 configurations that the enumeration does not find strict; there the graph rule kills a copy that the enumeration keeps alive, as for the fourth row of Remark 5.3.

All 634 configurations above lie in the class, and the criterion holds for 622 of them; the second diagonal rule is needed for four of these. The other 12 are trivially true: one datum has algebraic entries in two columns, so the ratio of their periods (Ω_2/π or Ω_2/Ω_3) would be algebraic, contrary to Chudnovsky's and Wüstholz's theorems (the enumeration's sorting checks single entries only). On these 12, and on the remaining 76 strict configurations of the class on which the criterion fails, the enumeration's verdict depends on its model, which takes vector subgroups to be products of coordinate lines: a vector subgroup on a diagonal of $\mathbb{G}_a$ slots defeats the dead-slot bound there, and the true count for it is not decided by the criterion. None of these 76 is among the 634. *[Numerical, exhaustive within the enumeration's type model.]*

7. Complex exponents and the conjugation boundary

This section concerns $2^\sigma, 3^\sigma$ for non-real σ, as they arise from CM lattices ($2^\sigma, 3^\sigma \in O_K$, K imaginary quadratic). Its content is folklore level: it combines the six exponentials theorem with complex conjugation, as in Diaz [5] and [18, Cor. 2.7]. We include it because it marks where the methods of this paper cannot go.

Let $\Sigma = \{\sigma \in \mathbb{C} : 2^\sigma, 3^\sigma \in \overline{\mathbb{Q}}\}$, with $\log 2, \log 3$ real. Σ is a $\mathbb{Q}$-vector space containing $\mathbb{Q}$ and stable under complex conjugation; four exponentials for $(2, 3)$ is $\Sigma = \mathbb{Q}$.

Lemma 7.1. $\dim_{\mathbb{Q}} \Sigma \leq 2$, *and exactly one of the following holds:* $\Sigma = \mathbb{Q}$; $\Sigma = \mathbb{Q} + \mathbb{Q}t$ *with* $t \in \mathbb{R} \setminus \mathbb{Q}$; *or* $\Sigma = \mathbb{Q} + \mathbb{Q}iv$ *with* $v \in \mathbb{R}^\times$.

Proof. If $1, \sigma, \sigma' \in \Sigma$ were $\mathbb{Q}$-independent, the six numbers $2^{1,\sigma,\sigma'}$, $3^{1,\sigma,\sigma'}$ would contradict the six exponentials theorem. If $\Sigma = \mathbb{Q} + \mathbb{Q}\tau$ with $\tau \notin \mathbb{R}$, then $\bar\tau = a + b\tau$ with $a, b \in \mathbb{Q}$; imaginary parts give $b = -1$, real parts $2 \operatorname{Re}\tau = a$, so $\Sigma = \mathbb{Q} + \mathbb{Q}i \operatorname{Im}\tau$. Finally $\mathbb{Q} + \mathbb{Q}\tau$ with $\tau \notin \mathbb{R}$ meets $\mathbb{R}$ only in $\mathbb{Q}$, so the cases are exclusive. □

Corollary 7.2. *Either four exponentials holds for* $(2, 3)$ *at every real* t, *or there is no non-real* σ *with* $2^\sigma, 3^\sigma \in \overline{\mathbb{Q}}$. *In particular a real counterexample (for example* $2^t, 3^t \in \mathbb{Z}$ *with* $t \notin \mathbb{Z}$*) excludes every non-real one, and conversely.*

Now let K be imaginary quadratic and suppose $\sigma \notin \mathbb{R}$ with $\alpha_2 = 2^\sigma$, $\alpha_3 = 3^\sigma \in O_K$. By Lemma 7.1, $\sigma = a + iv$ with $a \in \mathbb{Q}$; since $\alpha_2\bar\alpha_2 = 2^{2a} \in \mathbb{Z}$, $2a = k \in \mathbb{Z}_{\geq 0}$, and likewise $\alpha_3\bar\alpha_3 = 3^k$. Put $\zeta_p = \alpha_p/\bar\alpha_p = p^{2iv}$.

Proposition 7.3. *If neither* 2 *nor* 3 *splits in* K, *there is no such* σ. *This holds for* $K = \mathbb{Q}(\sqrt{-d})$ *with* $d \in \{1, 3, 19, 43, 67, 163\}$. *If exactly one of* 2, 3 *splits* ($d = 2, 7, 11$ *among class number one), or both split (e.g.* $d = 23, 47$*), we know no proof that there is none.*

Proof. Since $\alpha_p \in O_K$ and $(\alpha_p)(\bar{\alpha}_p) = (p)^k$, the ideal (α_p) is supported on the primes above p. If p does not split, the only prime of O_K above p is stable under conjugation, and (α_p) is a power of it, so $(\alpha_p) = (\bar{\alpha}_p)$ and ζ_p is a unit of O_K, a root of unity. If this holds for $p = 2, 3$, then $v \log 2$ and $v \log 3$ lie in $\pi\mathbb{Q}$, and $v \neq 0$ gives $\log 3/\log 2 \in \mathbb{Q}$, which is false. The splitting behaviour of 2 and 3 in the listed fields is checked in `anc/check_splitting.py`. □

Remark 7.4 (out of reach)**.** *[Proof sketch.]* In the open cases the hypothesis gives two algebraic integers on the $\mathbb{G}_{\mathrm{m}}$-side and nothing on an elliptic or abelian factor: $\mathrm{End}(E)$ acts on E, not on $\mathbb{G}_{\mathrm{m}}$. With $Y = \mathbb{Z} + \mathbb{Z}\sigma$ on $\mathbb{G}_{\mathrm{m}}^2$ the subgroup count gives rank ≥ 1, which is no information; when one prime does not split, the hypothesis puts a period $i\pi r$ in the matrix $\left(\begin{smallmatrix} \log p & i\pi r \\ \log q & \mathrm{Log}\,\xi \end{smallmatrix}\right)$, i.e. four exponentials with one period entry ($q^{i\pi r/\log p} \notin \overline{\mathbb{Q}}$), and the period term does not help on a rank-one $\mathbb{G}_{\mathrm{m}}^2$ configuration. The instances become reachable only if σ is tied to a CM period, which is the setting of [8].

8. Controls, open cases and numerics

Which hypotheses carry the proofs. In Theorem 1.2, deleting the period datum, replacing a curve by a non-CM curve, keeping one datum, or projecting to two columns each turns the decisive inequality into a non-strict one (Remark 6.6). Lemma 2.4(c) is used: without $R_2 \cap R_3 = 0$ the axis $U = \mathbb{C}e_1$ gives $10n \leq 10n$. Some relaxations fail only for the method: for isogenous curves or torsion u the statement is trivially true (§4.3).

Open relaxations. $u = 0$ ($\alpha^t, \alpha^{t'}$ not both algebraic), $\lambda = 0$ ($e^{tu}, e^{t'u}$), $\Omega_3/\Omega_2 \in \mathbb{Q}$ (outside four exponentials), non-CM curves (predicted by four exponentials; the count is at equality), and in Theorem 1.4 a torsion u_2. The general form "$\dim_{\overline{\mathbb{Q}}}\{x : x\Omega_j \in \overline{\mathbb{Q}} + \overline{\mathbb{Q}}\mathcal{L} + \overline{\mathbb{Q}}\mathcal{L}_{E_j},\ j = 1, 2, 3\} \leq 2$" is not proved: for two data of pure $\mathbb{G}_{\mathrm{m}}$ type no curve slot is killed, and the subtorus case is not excluded. (The particular statement $\alpha^t, \alpha^{t'}, \gamma^t, \gamma^{t'}$ for two logarithms is the classical six exponentials theorem, since $1, t, t'$ are $\mathbb{Q}$-independent under the hypotheses of Theorem 1.2, by Wüstholz.) Fibre slots, CM abelian surfaces and non-CM curves in the schema were not treated, nor were live-slot rules beyond the two diagonal rules (which would be needed to decide the 76 configurations noted at the end of Remark 6.7).

Linearisation. [Heuristic.] Under the hypothesis of Theorem 1.2 the matrix with rows $(\Omega_1, \Omega_2, \Omega_3)$, $(\log\alpha, t\log\alpha, t'\log\alpha)$, $(u, tu, t'u)$ has rank one, so its 2×2 minors vanish. Each minor is bilinear in periods of two different 1-motives. The known quadratic period relations (Legendre, Masser's $\eta = a\Omega + b\pi/\Omega$ for CM curves, the Riemann bilinear relations) all involve quasi-periods or π, and neither occurs; so we see no way to reduce the hypothesis to linear relations among periods, which the analytic subgroup theorem would decide. The same holds for the other instances. This is why we regard the statements as genuinely of four-exponentials type.

Numerics. [Numerical.] At two instances (the curves of Example 1.3, and $y^2 = x^3 - 36x$ with $(12, 36)$, $y^2 = x^3 - 2$ with $(3, 5)$, $y^2 = x^3 - 35x - 98$ with CM by $\mathbb{Q}(\sqrt{-7})$) the numbers of Theorems 1.2, 1.4, 5.2 and 5.4 were computed to 800 and 1200 digits with PARI/GP (`anc/u101_numerics.gp`, `anc/ref_numerics.gp`, with outputs); integer-relation searches for minimal polynomials of degree ≤ 8 found none, at the height level of a transcendental control.

Ancillary files. The folder `anc/` holds the following files. Every script uses exact arithmetic (python `fractions`, python-flint `fmpz`/`fmpq`, or Arb balls) and is run from inside `anc/`; the requirements are Python 3.10 or later with `python-flint` and `sympy`, and PARI/GP for the `.gp` files. Outputs are the files `*_out.log`, as produced.

- `u101_enum.py`: the enumeration of Remark 6.7 (writes `u101_enum_out.txt` and `u101_cells.json`). It stops itself after 19 minutes (constant `LIMIT`); on a slow machine, raise it.
- `schema_vs_u101.py`: exact linear algebra in $\mathbb{Q}^3$ and the list of the 306 subspaces used below; it imports `u101_enum.py`.
- `schema_ext_vs_u101.py`: the criterion of Theorem 3.3 on every enumerated configuration in its class. `python schema_ext_vs_u101.py --fast` gives the numbers of Remark 6.7 (`schema_ext_fast_out.log`, about 15 minutes).
- `schema_ext_fail_list.py`: the 12 new configurations on which the criterion fails, each with the reason (`schema_ext_fail_list_out.log`, about 10 minutes).
- `check_examples.py`, `check_splitting.py`: j-invariants, non-torsion of the example points (Nagell–Lutz), Arb values of ϖ and Ω_ρ, the count arithmetic, and the splitting of 2 and 3 in $\mathbb{Q}(\sqrt{-d})$.
- `u101_numerics.gp`, `ref_numerics.gp`: the numerics of §8, run as

```
gp -q --default parisizemax=2000000000 u101_numerics.gp > u101_numerics_out.log
```

  and likewise for `ref_numerics.gp`.

Use of AI tools. This work was carried out with substantial assistance from AI systems (Anthropic's Claude models), which were used to search the literature, to develop and check the proofs, to run the computer enumeration and to draft this text. All review so far is by AI referees; no human expert has checked the proofs. The author directed the project and is responsible for its content.

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