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Color-Avoiding Paths in Three-Colored Tournaments and 2-Comparable Sets

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Abstract

We prove that every three-colored tournament on $m \geq 1$ vertices contains a simple directed path on at least $m^{1/2+\varepsilon}$ vertices using at most two colors, for an absolute $\varepsilon > 0$. The proof transfers a bound for 2-comparable sets to tournaments by controlling the number of incomparable pairs. A set of triples is pairwise 2-comparable if any two distinct triples admit two strict coordinate comparisons in the same direction. We show that such a subset of $[N]^3$ has size at most $N^{2-\delta}$ for an absolute $\delta > 0$, answering the grid question in Problem 88 of Ben Green. Rectangular induction and random coordinate sampling upgrade the qualitative $o(N^2)$ bound to this fixed power saving. We also obtain higher-dimensional bounds, an exact grid correspondence when one color deletion is acyclic, and an exact maximum of p^2 for Green's modular condition over every positive odd modulus p. The main theorems are verified in Lean 4.

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1 Introduction

In a tournament whose edges have three colors, how many vertices can a directed path visit while avoiding one of the colors? Fox, Sudakov, and Wigderson reported no asymptotic improvement over the lower bound $\sqrt{m}$ for arbitrary m-vertex tournaments [2, Section 1.1]. Theorem 5 improves that bound to $m^{1/2+\varepsilon}$ for a fixed $\varepsilon > 0$. We obtain this improvement from a grid bound. The tournament gives a set of triples with relatively few incomparable pairs, and a sampling argument on a coarse grid controls the size of that set.

Write $[N] = \{1, \ldots, N\}$. Two distinct triples are *2-comparable* if their difference has at least two strictly positive or at least two strictly negative coordinates. A set is pairwise 2-comparable if every pair of distinct elements is 2-comparable. Tiskin obtained the qualitative subquadratic bound in the equivalent tripod-packing problem [6]. Gowers and Long observed that Loh's triangle-removal argument for 2-increasing sequences also applies to 2-comparable sets, giving the bound $n^2/\exp(\Omega(\log^* n))$ [3, Section 1.2 and Proposition 1.10]. Fox, Sudakov, and Wigderson discuss the earlier removal-lemma argument [2]. Green asked whether a fixed power saving holds [4, Problem 88].

Theorem 1. *There exists an absolute constant $\delta > 0$ such that every pairwise 2-comparable set $S \subseteq [N]^3$ satisfies $|S| \leq N^{2-\delta}$.*

For positive integers r, s, t, write $M(r, s, t)$ for the maximum over $[r] \times [s] \times [t]$, and let $G(n) = M(n, n, n)$. Fixed-coordinate sections form strict chains in the other two coordinates. Each two-coordinate projection is injective. The induction starts from $G(n) = o(n^2)$, which we derive in Section 4. The resulting exponent depends on the scale at which that qualitative bound supplies the base of the induction and can be very small.

2 Rectangular induction

Gowers and Long proved a power saving for 2-increasing sequences using rectangular induction [3, Theorems 1.13 and 2.1 and Section 2.3]. Here we use random sampling of shared coordinate values and delete aligned boxes to control the coarse configurations without a common ordering of the triples.

Take $\eta = 2^{-36}$, $J = 2^{54}$ and choose a fixed $k \geq J$ so that $(3J/\eta)G(k) \leq k^2/32$. Put $\delta = \log 2/\log k$ and $\theta = (2-\delta)/3$. Thus $1/2 \leq \theta < 2/3$, $k^{2-3\theta} = 2$,

$$2\eta^{\theta-1/3} \leq 1/32, \qquad 2\eta^{-1/3}J^{\theta-1} \leq 1/32.$$

We prove $M(r,s,t) \leq 2(rst)^\theta$ by strong induction on rst. Coordinates can be replaced by their relative ranks, preserving all strict comparisons. By symmetry, suppose $r \leq s \leq t$. If $r \leq k$, then $M \leq rs$ and

$$rs/(rst)^\theta \leq r^{1-\theta}s^{1-2\theta} \leq r^{2-3\theta} \leq 2.$$

Otherwise divide each axis into k consecutive intervals differing in length by at most one. Let a, b, c be the maximum interval sizes and $V = abc$. For each occupied box e, let u_e, v_e, w_e be the numbers of coordinate values used there, divided by a, b, c respectively, and put $p_e = u_e v_e w_e$. Induction gives $|S_e| \leq 2V^\theta p_e^\theta$.

By the strict-chain property, for a fixed first-coordinate value the occupied pairs of coarse indices for the other two coordinates form a nondecreasing sequence, with at most $2k-1$ distinct pairs. Thus $\sum_e u_e \leq r(2k-1)/a \leq 2k^2$, and similarly for the other axes. Hölder's inequality gives $\sum_e p_e^{1/3} \leq 2k^2$.

- Boxes with $p_e \leq \eta$ contribute at most $2V^\theta k^2(2\eta^{\theta-1/3})$.
- There are at most $2\eta^{-1/3}k^2$ boxes with $p_e > \eta$. While an axis-parallel coarse line contains at least J such remaining boxes, remove a group of exactly J. For a line parallel to the first axis, the three projection sizes of the union are at most $\min(Ja, r)$, b, and c. Hence their product is at most JV and strictly less than rst, since $b < s$ and $c < t$. The other directions are analogous. Consequently induction applies to the whole group after rank compression. All groups contribute at most $2V^\theta k^2(2\eta^{-1/3}J^{\theta-1})$.
- Let R be the remaining heavy boxes. Independently for each axis and each coarse interval, choose one coordinate value from the interval uniformly at random. Mark each box in R if all three chosen coordinates occur in its three projections. The interval lengths on the three axes are at most a, b, c respectively. Independence of the axis choices gives marking probability at least $p_e > \eta$. When two marked boxes share precisely one coarse coordinate, there are points in the two boxes sharing the chosen actual coordinate. Their other two coordinates must be strictly ordered in the same direction, and their distinct coarse indices also have that order. Select marked boxes greedily, deleting those sharing two coarse coordinates with a selected box. Each step accounts for at most $3J$ marked boxes, including the chosen box, because every coarse axis-parallel line now contains fewer than J boxes. The selected boxes form a 2-comparable set in $[k]^3$. With one equal coordinate the preceding argument applies. With none, at least two comparisons agree. Therefore at most $3JG(k)$ boxes can be marked. Expectation gives $|R| \leq 3JG(k)/\eta \leq k^2/32$. As $p_e \leq 1$, this last part contributes at most $2V^\theta k^2/32$.

Each coefficient is at most $1/32$. Since $r, s, t > k$, we have $a = \lceil r/k \rceil \leq 2r/k$, and similarly for b, c. Thus $V \leq 8rst/k^3$. Using $8^\theta \leq 4$ and $k^{2-3\theta} = 2$, the three contributions sum to at most $\frac{3}{4} \cdot 2(rst)^\theta$. This completes the induction.

3 Removing the leading constant

Use lexicographic products, as in [3, Lemma 1.7 and the discussion after Proposition 1.10]. Take words of length ℓ over $S \subseteq [r] \times [s] \times [t]$. Encode the three coordinate words by their lexicographic ranks in $[r^\ell]$, $[s^\ell]$, and $[t^\ell]$, respectively. For two different words, their first differing symbol gives at least two strict comparisons in the same direction. The corresponding coordinate words have the same strict comparisons under lexicographic order. Thus

$$M(r,s,t)^\ell \le M(r^\ell, s^\ell, t^\ell) \le 2(rst)^{\ell\theta}.$$

Taking ℓth roots and sending ℓ to infinity gives

$$M(r,s,t) \le (rst)^\theta, \qquad \theta = (2-\delta)/3. \tag{1}$$

Setting $r = s = t = N$ gives Theorem 1.

4 The qualitative bound via triangle removal

For completeness, we give the triangle-removal proof of the qualitative bound [6, 2, 5]. Fix a pairwise 2-comparable set $S \subseteq [n]^3$. In a tripartite graph with parts of size n, let each triple in S supply a triangle. They are edge-disjoint by injectivity of two-coordinate projections. An extra triangle with vertices (a,b,c) would come from three distinct triples $(a',b,c), (a,b',c), (a,b,c')$. If $a' = a$, the triangle already comes from S. If $a' < a$, comparing the first two triples at fixed c gives $b < b'$. Comparing the second and third triples at fixed a then gives $c > c'$. In the first and third triples the second coordinate is b, so $c > c'$ forces $a' > a$, a contradiction. Reversing the inequalities excludes $a' > a$. Hence there are at most n^2 triangles altogether, and the triangle removal lemma gives a set of $o(n^2)$ edges whose deletion removes all triangles. The supplied edge-disjoint triangles require at least $|S|$ deletions.

5 The modular problem

Let p be a positive odd integer. Put $H = \{1, \dots, (p-1)/2\} \subset \mathbb{Z}/p\mathbb{Z}$, and let D_p consist of all coordinate permutations of $(0, u, -v)$ with $u, v \in H$. For odd primes p, this is the forbidden difference set in Green's comment to Problem 88 [4]. Call a set A modularly admissible if $(A - A) \cap D_p = \varnothing$.

Theorem 2. *For every positive odd integer p, the maximum size of $A \subseteq (\mathbb{Z}/p\mathbb{Z})^3$ with $(A-A)\cap D_p = \varnothing$ is p^2.*

Proof. The plane $\{(x,y,x) : x, y \in \mathbb{Z}/p\mathbb{Z}\}$ has size p^2. Any of its differences has the form (d,e,d). If exactly one entry vanishes, the nonzero entries coincide, so they cannot lie in the disjoint sets H and $-H$.

For the upper bound, consider the map $(x,y,z) \mapsto (x, y+z)$. If distinct points of A have the same image, their difference is $(0,d,-d)$ for some $d \neq 0$. Since H and $-H$ partition the nonzero residues, this difference belongs to D_p. The map is therefore injective on A and has a codomain of size p^2. □

Each element of D_p has exactly one zero coordinate. In particular, the condition allows all nonzero axis-parallel differences. This permits a plane of p^2 points for every odd prime and rules out a power saving in Green's modular question.

5.1 Axis-parallel lines in the modular and weak integer problems

Call a set in $[N]^3$ *weakly admissible* if it contains no two points with one coordinate equal and the other two differences nonzero and of opposite signs. If $W(N)$ is the maximum size of such a set, then

$$N^2 \le W(N) \le N^2 + \left\lfloor \frac{N-1}{3} \right\rfloor. \tag{2}$$

Furthermore, a weakly admissible set with at most $d \ge 1$ points on each axis-parallel line has size at most $d\,G(N)$.

To see this, connect consecutive points on every axis-parallel line by directed edges in the increasing direction. Each vertex has outdegree and indegree at most one. For example, the ends of two outgoing edges in different directions would form a forbidden pair. The coordinate sum increases along edges, so the graph is a disjoint union of directed paths, including isolated vertices. Cyclically coloring each path in order with d colors gives the line-occupancy bound. The points of any one line occur consecutively along its path, so no two points of one color agree in two coordinates. Pairs in one color with one equal coordinate are 2-comparable by the weak condition, and the other pairs are 2-comparable by majority.

For (2), the lower bound is given by $\{(x, y, x) : x, y \in [N]\}$. For the upper bound let m be the size of the set, c the number of directed paths, and P the sum of the sizes of the three two-coordinate projections. Counting the directed edges by lines and by paths gives $3m - P = m - c$, so $P = 2m + c$. Each path has at most $3N - 2$ vertices, which gives $c \ge m/(3N - 2)$.

Build a tripartite graph on three copies of $[N]$ with the projections as its edges. Its triangles are exactly the triples of the set. Indeed, a triangle outside the set would require three supplying points, one for each edge. These points differ from the candidate triple in three different coordinates, with nonzero changes d_1, d_2, d_3. Comparing the supplying points in pairs, the weak condition would require the three changes to have pairwise opposite signs, an impossibility. Each absent triangle contains a missing edge, and each missing edge occurs in N candidate triples. Hence $N^3 - m \le N(3N^2 - 2m - c)$. Substitution gives

$$m \le \frac{N^3(3N-2)}{3N^2 - 3N + 1} < N^2 + \frac{N}{3},$$

which gives (2) since m is an integer.

For odd $p \ge 3$, partition each residue axis into two consecutive intervals of length at most $h = (p+1)/2$. In each of the resulting eight boxes, the signs of the nonzero integer differences agree with the modular halves H and $-H$. Thus a modularly admissible set with at most d points per axis-parallel line has size at most $8dG(h) \le 8dh^{2-\delta}$. In particular, the power saving holds if nonzero axis-parallel differences are forbidden as well.

6 Higher-dimensional grids

Write $G_{q,r}(N)$ for the maximum size of a subset of $[N]^q$ in which any two distinct points have at least r strict coordinate comparisons in one common direction.

Proposition 3. *For $2 \le r < q$ the same δ as in Theorem 1 satisfies*

$$G_{q,r}(N) \le N^{q-r-1} G(N) \le N^{q-r+1-\delta}.$$

Proof. Fix the values of $q - r - 1$ coordinates. In any pair from this fiber, at least r of the $r + 1$ remaining coordinates compare strictly in the same direction. Projection onto any three of

the remaining coordinates therefore preserves at least two such comparisons. The projection is injective and its image is 2-comparable, so the fiber contains at most $G(N)$ points. There are at most N^{q-r-1} fibers. □

Fox, Sudakov, and Wigderson discuss power savings for these parameters [2, Section 1.2]. For $r \geq 3$, their Proposition 1.5 makes the $(r+1, r)$ comparable-set problem equivalent to the increasing-sequence problem since $r > 2(r+1)/3$. The same projection on three coordinates then permits the use of Gowers and Long's bound for 2-increasing sequences [3, Theorem 1.13]. Thus a fixed power saving for $r \geq 3$ already follows from the earlier results. For $r = 2$, the bound in every dimension $q \geq 3$ is a direct consequence of Theorem 1, by fixing $q - 3$ coordinates.

The correspondence with aligned (q, r)-pod packings [2, Section 1.2] transfers this bound to a packing density of $O_{q,r}(N^{-\delta})$ for pods of side length N. Indeed, one pod consists of the vectors in $\{0, \ldots, N-1\}^q$ with at most $r-1$ positive coordinates, and has $O_{q,r}(N^{r-1})$ points. Two translations with anchors in the same N-box are disjoint exactly when the anchors are r-comparable. This follows by testing the coordinatewise maximum of the two anchors. Partitioning the anchor lattice into N-boxes and applying Proposition 3 gives the claimed density. For $q = 3, r = 2$ this is the tripod bound. The same three-dimensional grid bound applies to the monotone partial matrices of [1, Section 4.3] by sending each filled cell to its column, row, and label.

7 Color-avoiding paths under an acyclicity assumption

Let T be an m-vertex tournament with three edge colors. For $i \in \{1, 2, 3\}$ let D_i be the spanning digraph obtained by deleting the edges of color i, and let L_i be the largest number of vertices in a simple directed path of D_i.

Proposition 4. *If at least one D_i is acyclic, then*

$$m \leq M(L_1, L_2, L_3) \leq (L_1 L_2 L_3)^{\theta}, \qquad \max_i L_i \geq m^{1/(2-\delta)}.$$

For any positive integers n_1, n_2, n_3, the maximum order of such a tournament subject to $L_i \leq n_i$ is exactly $M(n_1, n_2, n_3)$.

Proof. For any digraph D, take an inclusion-maximal acyclic spanning subgraph H. Let the height of a vertex be the maximum number of vertices in a directed path in H ending there. Height increases strictly along edges of H. Also, equal-height vertices have no edge between them in D. An edge $u \to v$ outside H would require a path in H from v to u by maximality, and that would make their heights different.

Relabel the colors so that D_1 is acyclic and take $H_1 = D_1$ with height h_1. Every edge between vertices of the same h_1 has color 1. Inside each level, choose a directed Hamilton path in the induced tournament. Such a path exists by successively inserting vertices into a directed path. Let P be the union of these paths' edges.

In D_2, the edges of P together with all color-3 edges form an acyclic subgraph. Height h_1 strictly increases along the color-3 edges and stays constant along P, which is a disjoint union of paths. Extend this subgraph to an inclusion-maximal acyclic spanning subgraph H_2 with height h_2. Let Q consist of the edges with equal h_2 at both ends. These edges have color 2 by the maximality observation, so they strictly increase h_1. The edges of P strictly increase h_2 and keep h_1 constant. Thus $P \cup Q$ lies in D_3 and is acyclic since $h_1 + h_2$ strictly increases along it. Extend it to an inclusion-maximal acyclic spanning subgraph H_3 with height h_3.

Label each vertex by (h_1, h_2, h_3). Two vertices with equal h_1 lie on the same path of P, so both h_2 and h_3 strictly increase in its order. For equal h_2, the edge in Q strictly increases h_1 and h_3. For equal h_3, their edge has color 3 and lies in H_1 and H_2, so h_1 and h_2 strictly increase. If no coordinate agrees, two comparisons have the same direction. The labeling is injective and its image is 2-comparable, with $h_i \le L_i$. The bounds now follow from (1).

For the converse, orient each pair in a 2-comparable subset of $[n_1] \times [n_2] \times [n_3]$ toward the point larger in at least two coordinates. Color the edge by the sole coordinate that fails to increase, or choose any color if all three increase. This is the standard construction from comparable sets [2, Equation (3)]. Each D_i is acyclic, and coordinate i strictly increases on its paths, giving $L_i \le n_i$. Taking a maximum set proves equality of the extrema. □

More generally, if the vertices of some D_i can be partitioned into b sets inducing acyclic digraphs, then with $L = \max_i L_i$ we have $m \le bG(L) \le bL^{2-\delta}$ by applying the proposition to each induced subtournament. Section 8 gives a power saving for arbitrary 3-colored tournaments.

8 Color-avoiding paths in arbitrary three-colored tournaments

The next theorem applies to arbitrary three-colored tournaments. As in Section 7, paths are simple and their lengths count vertices.

Theorem 5. *There exists an absolute constant $\varepsilon > 0$ such that every m-vertex tournament with three edge colors contains a directed path on at least $m^{1/2+\varepsilon}$ vertices using at most two colors, for every $m \ge 1$.*

Write D_i for the digraph obtained by deleting edges of color i and set $L = \max_i L_i$, where L_i is the largest number of vertices in a simple directed path in D_i. Take $0 < \delta \le 1/2$ so that $G(n) \le n^{2-\delta}$. We first show that

$$m \le CL^{2-\gamma}, \qquad \gamma = \delta^2/96 \tag{3}$$

for an absolute constant C.

8.1 Ordered blocks and incomparable pairs

Let a digraph have no simple directed path longer than L. Run depth-first search until $12L$ vertices have finished, counting all starting components of the search. If the graph has at most $12L$ vertices, use it as a single block. Otherwise keep the finished vertices as a block and discard the active stack, which is a simple directed path and hence contains at most L vertices. No edge goes from the finished block to the undiscovered vertices, since a finished vertex has explored all its outgoing edges. Repeat on the undiscovered induced subgraph. The resulting blocks have size at most $12L$. All edges between retained blocks point from later to earlier blocks in DFS completion order, hence forward in the reverse of that order. The full blocks are disjoint, and each costs at most one discarded stack, so at most $m/12$ vertices are discarded in one digraph.

Apply this construction in each D_i and retain a common set W of at least $3m/4$ vertices. For each i, start with the interblock edges of $D_i[W]$ and extend to an inclusion-maximal acyclic spanning subgraph H_i of $D_i[W]$. These edges form an acyclic digraph by the block order. Let h_i be the height in H_i, defined as in Section 7. Equal heights force the edge between the pair to have color i. Thus the height triples form a set $S \subseteq [L]^3$ of size $|W|$, and each two-coordinate projection is injective.

Let B be the number of ordered pairs of distinct, incomparable points in S. An incomparable pair has some equal height h_i, since three unequal coordinates always give two strict comparisons in the same direction. Its edge has color i. If the pair crossed blocks for both D_j with $j \neq i$, the edge would lie in both H_j, making the two other heights increase in the direction of the edge. Thus every incomparable pair shares a height h_i and a block of some D_j with $j \neq i$.

Lemma 6. *In a three-colored tournament in which every simple path in each D_i has at most L vertices, suppose the vertices are partitioned into ordered blocks and all interblock edges of D_j point forward in the block order. Fix $i \neq j$ and select from each block a set whose internal edges all have color i. The selected sets have total size at most $G(L)$.*

Proof. Restrict to the chosen vertices and relabel so that $j = 1$, $i = 2$. Let P be the union of the edges of a Hamilton path within each block. Such a path exists in every tournament by successive vertex insertion. The digraph formed by P and the interblock edges of D_1 is acyclic, since the interblock edges respect the block order and the edges within a block form a directed path. Let x_1 be its height. Two vertices at the same x_1 are in different blocks, and their edge has color 1. Let Q be the union of Hamilton paths within these height levels.

In D_2, take an inclusion-maximal acyclic spanning subgraph containing Q and all color-3 edges. The color-3 edges go between blocks and strictly increase x_1, so this seed is acyclic. Let its height be x_2. In D_3, extend the seed consisting of P, Q, and the interblock color-2 edges to an inclusion-maximal acyclic spanning subgraph with height x_3. Again all seed edges outside Q strictly increase x_1, and Q is a disjoint union of paths within levels.

If two vertices agree in x_1, their order along Q increases both x_2 and x_3. If they agree in x_2, their edge has color 2. In the same block the order along P increases both x_1 and x_3. In separate blocks the edge itself increases both coordinates. Finally, agreement in x_3 forces a color-3 edge, which increases x_1 and x_2. These labels therefore form a 2-comparable set, with all three heights at most L. □

For each ordered choice $i \neq j$, in each D_j-block let c_{ba} count vertices in block b with $h_i = a$. The largest height class in each block is monochromatic in color i. By Lemma 6, $\sum_b \max_a c_{ba} \leq G(L)$. Also $\sum_a c_{ba} \leq 12L$ for each b. Counting the ordered pairs that share a height and a block gives

$$B \leq \sum_{i \neq j} \sum_{b,a} c_{ba}^2 \leq 72LG(L) \leq 72L^{3-\delta}. \tag{4}$$

8.2 Counting points with few incomparable pairs

Put $n = L$. Divide each coordinate into k consecutive intervals of lengths at most $s = \lceil n/k \rceil$, where $1 \leq k \leq n$. In a section with one actual coordinate fixed, call the pairs of coarse intervals of the other two axes cells. Remove points in cells of size below t. This costs at most $3nk^2t$ points. Also delete a whole section whenever two cells of size at least t have coarse indices in strictly opposite order. The two cells yield at least t^2 unordered incomparable pairs, or $2t^2$ ordered pairs. For each fixed axis the pairs counted in different sections are disjoint, so at most B/t^2 sections are deleted. Each section contains at most n points by injectivity of the two-coordinate projections. Over the three axes this step costs at most $3nB/t^2$ points. Both kinds of deletions are determined from the original set S. In the remaining set X, each section visits a weak chain of coarse cells.

As in Section 2, let $u_{e,i}$ be the fraction of the width s used on axis i by the points of X in a coarse box e, and put $p_e = \prod_i u_{e,i}$. Each section visits at most $2k - 1$ cells, so $\sum_e u_{e,i} \leq 2k^2$ and $\sum_e p_e^{1/3} \leq 2k^2$. Fix $0 < \eta \leq 1$ and a positive integer g. When $p_e \leq \eta$, some ratio $u_{e,j}$ is at most

$\eta^{1/3}$. For any $i \neq j$, injectivity of projection onto axes i, j gives $|X \cap e| \leq s^2 u_{e,i} u_{e,j} \leq s^2 \eta^{1/3} u_{e,i}$. Summing and using $\sum_e \sum_i u_{e,i} \leq 6k^2$ gives the bound $6s^2 k^2 \eta^{1/3}$.

There are at most $2k^2/\eta^{1/3}$ boxes with $p_e > \eta$. Repeatedly remove g such boxes on a common coarse axis-parallel line. Each group contains at most s^2 points by projection onto its two fixed axes. For the remaining boxes, use the sampling of Section 2. If two marked boxes agree in exactly one coarse index, the sampled value in that interval occurs in both boxes. In the section at that shared value, the two cells lie on a weak chain. Their two other coarse indices are distinct on both axes, so both comparisons are strict and in the same direction. Pairs of boxes with no common coarse index are 2-comparable by majority. In the greedy selection, each step takes one marked box and removes every other box sharing two coarse indices with it. This accounts for at most $3g$ marked boxes per step, including the chosen box. Thus at most $3gG(k)$ boxes are marked in any trial. Each remaining box is marked with probability at least $p_e > \eta$, so at most $3gG(k)/\eta$ boxes remain. Combining these bounds gives

$$|X| \leq s^2 \left(6k^2 \eta^{1/3} + \frac{2k^2}{\eta^{1/3} g} + \frac{3gG(k)}{\eta} \right).$$

Set $k = \lfloor n^{\delta/16} \rfloor$, $t = \lceil n^{1-\delta/4} \rceil$, $\eta = k^{-\delta/2}$ and $g = \lceil k^{\delta/3} \rceil$. Here $1 \leq k \leq n$ for every $n \geq 1$. Since $G(k) \leq k^{2-\delta}$ and $s \leq 2n/k$, the bound for $|X|$ is $O(n^2 k^{-\delta/6}) = O(n^{2-\delta^2/96})$. The discarded terms $nk^2 t$ and nB/t^2 are $O(n^{2-\delta/8})$ and $O(n^{2-\delta/2})$, respectively, by (4). Together with $|S| \geq 3m/4$ this proves (3).

8.3 A bound with coefficient one

For $m > 1$, we also have $m < L^2$. To see this, take the height levels of a maximal acyclic spanning subgraph of D_1. Each level is monochromatic in color 1. Choose a directed Hamilton path in each level. The union of these paths is acyclic, since their vertex sets are disjoint. All their edges have color 1, so this union is a common subgraph of D_2 and D_3. Extend it separately to an inclusion-maximal acyclic spanning subgraph of each digraph. The two resulting heights give an injection of the vertices into $[L]^2$, since a pair agreeing in both would have an edge of both color 2 and color 3. Each original level maps to a chain that is strict in both coordinates. If $m = L^2$, the map is onto, but the last row and last column of $[L]^2$ contain $2L - 1$ points and each chain meets their union in at most one point. There are at most L chains and $L \geq 2$, giving a contradiction.

Equation (3) implies $L \geq cm^{1/(2-\gamma)}$ for an absolute $c > 0$. For all sufficiently large m, this gives $L \geq m^{1/2+\varepsilon_0}$ for some $\varepsilon_0 > 0$. For the remaining finite range with $m > 1$, the strict bound $L > \sqrt{m}$ allows a single smaller $\varepsilon > 0$. When $m = 1$ the result holds with equality. This proves Theorem 5.

Formal verification and availability

Lean 4.19.0 with Mathlib v4.19.0 verifies Theorems 1 and 2, as well as the rectangular estimate (1) with coefficient one. The grid formalization includes triangle removal, rectangular induction, and the lexicographic product argument.

The formalization also verifies the fiber reduction and power bound in Proposition 3. For Proposition 4, it verifies both directions of the grid–tournament equivalence for three path caps and the bounds on the longest color-avoiding paths. It also verifies the bound $m \leq bL^{2-\delta}$ for a partition into b acyclic induced parts. The declaration `general_color_avoiding_power` verifies Theorem 5 in full, with its coefficient-one bound for an actual simple directed path. The proof includes the ordered blocks, the label and defect bounds, and the grid estimates.

The weak integer bound and the line-occupancy estimates in Section 5.1, and the applications to pod packings and monotone matrices in Section 6, are proved in the text.

The Lean source and verification instructions list the theorem declarations and their axiom dependencies, which are limited to `propext`, `Classical.choice`, and `Quot.sound`. The companion note provides supporting notes on the extensions.

AI disclosure

The project maintainer selected the problem, directed the investigation, and assembled the proof and supporting materials. AI reasoning systems generated the proof arguments, produced the Lean formalization, and helped prepare and revise this manuscript.

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