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hexagon:2611.01236v1

The Rhythm of Primes

On the Formally Verified Proof of the Generalized Riemann Hypothesis for Dirichlet L-functions

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Abstract. We give a complete account of a proof, due to L. Alpöge working with Claude, that no Dirichlet L-function vanishes in the half-plane $\operatorname{Re} s > 1/2$; in particular every nontrivial zero of the Riemann zeta function has real part $1/2$. The proof was found and formalised in Lean 4; we follow the shorter formalisation `grh-lean`, which we rebuilt and checked, and each statement of the paper is tied to the Lean declarations that prove it. The argument is a descendant of Stepanov's method for curves over finite fields. A one-sided bound for a weighted sum of the von Mangoldt function over a residue class, whose weights $1-\chi_K(n)$ come from an imaginary quadratic field, excludes zeros by Landau's theorem on Dirichlet integrals of one sign. The bound is proved by comparing two estimates for chains of $m \asymp \sqrt{x}$ theta lifts between definite unitary groups over a CM field: an upper bound, in which the divisibility of derivatives of theta functions of CM elliptic curves at each prime reproduces the prime sums, and a lower bound from a product formula for Gram determinants, in which each prime reads the integral structure through its own twist and the norms of the lifts are Hecke L-values controlled by the Rallis inner product formula and by congruences, bounded through Selmer groups of anticyclotomic twists. A second-moment estimate shows that few translates of a Poincaré series span the space of target forms. The main term cancels between the two estimates, and the remainder is bounded by $x^{-1/2+\varepsilon}$.

2020 Mathematics Subject Classification. 11M26 (primary), 11M06, 11F27, 11F55, 11R39, 11G15, 11F80, 03B35, 68V15 (secondary).

Key words and phrases. Generalized Riemann Hypothesis, Dirichlet L-functions, theta correspondence, Stepanov's method, heights, formal verification, Lean.

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The proof expounded in this paper is due to Levent Alpöge, working with Claude [Alp26]. The Lean text referred to throughout is the development `grh-lean` [Ant26], Copyright 2026 Anthropic, PBC, released under the Apache License 2.0; Appendix E identifies the version used.

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1. Introduction

1.1. The theorems

For a Dirichlet character χ modulo an integer $N \geq 1$ let $L(s, \chi)$ be its L-function, continued meromorphically to $\mathbb{C}$; it is entire unless χ is principal, in which case it has a single simple pole, at $s = 1$. Let $\zeta(s)$ be the Riemann zeta function. This paper gives a complete account of a proof of the following two theorems.

Theorem 1.1 (The Generalized Riemann Hypothesis for Dirichlet L-functions). *Let $N \geq 1$, let χ be a Dirichlet character modulo N, and let $s \in \mathbb{C}$ with $\operatorname{Re} s > 1/2$. Then $L(s, \chi) \neq 0$, unless χ is principal and $s = 1$.*

Theorem 1.2. (a) *Every zero s of $L(s, \chi)$ with $0 < \operatorname{Re} s < 1$ has $\operatorname{Re} s = 1/2$, for every Dirichlet character χ, primitive or not.*

(b) (The Riemann Hypothesis.) *Every zero of $\zeta(s)$ other than the trivial zeros $-2, -4, -6, \ldots$ has real part $1/2$.*

The proof is due to Levent Alpöge, working with the AI system Claude [Alp26]; it was announced together with a complete formalisation in the proof assistant Lean 4 [dMU21], on top of its mathematical library Mathlib [mC20]. A shorter formalisation following the same method, the development `grh-lean` [Ant26], was produced at Anthropic in collaboration with Claude and released under the Apache License 2.0; many of its steps are proved by other arguments than in the original formalisation. This paper is an exposition of the proof as it is carried out in `grh-lean`. It introduces no new step: every theorem, proposition and lemma below corresponds to declarations of that development; Appendix C lists, section by section, the Lean modules in which they lie.

In `grh-lean` the five statements are proved in the following form, in the language of Mathlib. The variable N ranges over all integers $N \geq 1$, χ over all Dirichlet characters modulo N, and $L(s, \chi)$ is Mathlib's L-function, the meromorphic continuation of the Dirichlet series $\sum_{n \geq 1} \chi(n) n^{-s}$, regarded as a function on all of $\mathbb{C}$.

(1) For all $s \in \mathbb{C}$: if $\zeta(s) = 0$, $s \notin \{-2, -4, -6, \ldots\}$ and $s \neq 1$, then $\operatorname{Re} s = 1/2$. This is Mathlib's definition of the Riemann Hypothesis, proved twice, once under that name and once in the form stated here.

(2) For all N, χ, s: if $\operatorname{Re} s > 1/2$ and ($\chi \neq 1$ or $s \neq 1$), then $L(s, \chi) \neq 0$.

(3) For all N, χ, s: if $0 < \operatorname{Re} s < 1$ and $L(s, \chi) = 0$, then $\operatorname{Re} s = 1/2$.

(4) The case of (3) in which χ is primitive.

Statement (2) is Theorem 1.1; statements (1), (3) and (4) together are Theorem 1.2. The condition $s \neq 1$ in (1) and (2) appears because Mathlib assigns a value to ζ and to $L(\cdot, 1)$ at their pole. Statement (3) says nothing about zeros with $\operatorname{Re} s \leq 0$: for a character that is not primitive, $L(s, \chi)$ differs from the L-function of the primitive character that induces it by finitely many Euler factors $1 - \chi_0(p) p^{-s}$, whose zeros lie on $\operatorname{Re} s = 0$.

1.2. The Riemann Hypothesis

Riemann's memoir of 1859 [Rie59] continued $\zeta(s) = \sum_{n\geq 1} n^{-s}$ to the plane, proved the functional equation relating s and $1-s$, and expressed the number of primes below x by a sum over the zeros of ζ. In the form proved by von Mangoldt [Man95], for $x > 1$ not a prime power,

$$\psi(x) = \sum_{n\leq x} \Lambda(n) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \log 2\pi - \tfrac{1}{2}\log\bigl(1 - x^{-2}\bigr), \tag{1.1}$$

where ρ runs over the zeros of ζ in the strip $0 < \operatorname{Re} s < 1$, summed symmetrically. Each zero contributes an oscillation x^{ρ}/ρ of amplitude $x^{\operatorname{Re}\rho}$ and frequency $\operatorname{Im}\rho$; these oscillations are the rhythm of the primes in our title. Riemann wrote that it is "sehr wahrscheinlich" that all roots of his function $\xi(t)$, the completed zeta function on the line $s = \frac{1}{2} + it$, are real, that is, that all the zeros ρ have real part $1/2$. If so, then every term has amplitude $\sqrt{x}$, and the error in the prime number theorem is as small as the square root of the main term, up to logarithms; and conversely a bound $\psi(x) = x + O(x^{\beta+\varepsilon})$ for every $\varepsilon > 0$ is equivalent to the absence of zeros in $\operatorname{Re} s > \beta$. Hadamard [Had96] and de la Vallée Poussin [dlVP96] proved the prime number theorem in 1896 by showing that ζ has no zeros on the line $\operatorname{Re} s = 1$.

The hypothesis concerns all zeros, while every unconditional result controls a subset of them. Hardy [Har14] proved that infinitely many zeros lie on the critical line; Selberg [Sel42] that a positive proportion of them do; Levinson [Lev74] that more than one third and Conrey [Con89] that more than two fifths of them do. All zeros with imaginary part between 0 and $3 \cdot 10^{12}$ lie on the line [PT21]. For the history and the classical theory see [Edw74, Tit86], and for the place of the problem in mathematics [Bom00, Con03].

For Dirichlet characters the same picture holds with primes in arithmetic progressions in place of primes: the zeros of $L(s,\chi)$, χ modulo q, govern the distribution of the primes in the residue classes modulo q, and the Generalized Riemann Hypothesis, the statement of Theorem 1.1, gives the error term $O(x^{1/2+\varepsilon})$ in the prime number theorem for every progression; see [Dav80, MV07, IK04].

Over a finite field the analogue of the hypothesis is a theorem. For the zeta function of a curve over a finite field it was proved by Weil [Wei48], who formulated the corresponding conjectures for varieties of any dimension [Wei49]; these were proved by Deligne [Del74, Del80]. Weil's proof rests on a positivity, the Castelnuovo–Severi inequality on the surface $C \times C$. In 1952 Weil showed that the hypothesis for ζ itself is equivalent to the positivity of a quadratic form built from his explicit formula [Wei52]. A different, elementary proof was found by Stepanov [Ste69] for hyperelliptic curves and extended to all curves by Bombieri [Bom74]. It constructs an auxiliary function that vanishes to high order at the rational points of the curve; the order of vanishing, against the degree of the function, bounds the number of points, and the bound for all finite extensions gives the hypothesis.

In the monograph [Liu26] the author analysed the obstructions to transporting these proofs from function fields to $\mathbb{Q}$, organised around the positivity objects that the known proofs use. Of the two proofs for curves it took Weil's as its model: in its words, "it is Weil's proof, not Stepanov's, that this book takes as the paradigm, because Weil's proof is the one in which the positivity object is visible" [Liu26, p. 279]; and of Stepanov's method it recorded that "it

produces no operator, no cohomology and no positive form; and it has resisted every attempt at generalisation to higher dimension" [Liu26, p. 281]. The proof expounded here takes the other route. It is a descendant of Stepanov's method, and the positivity it uses is of a different kind: the nonnegativity of Petersson norms and of a Laplace transform with positive coefficients.

1.3. The method

We outline the argument; Section 2 gives a more detailed plan, and the reading guide at the end of this section says where each step is proved.

A one-sided bound suffices. Fix a prime $d \equiv 7 \pmod 8$, let $\chi_K = (\cdot/d)$ be the character of $K = \mathbb{Q}(\sqrt{-d})$, and consider for a modulus q the weighted sum

$$\sum_{\substack{n \le x \\ n \equiv 1 \ (q)}} \Lambda(n)\big(1 - \chi_K(n)\big)\Big(\frac{1}{n} - \frac{1}{x}\Big) = \frac{\log x}{\varphi(q)} + c_q + R_q(x).$$

Its weights are nonnegative, and they vanish on the primes that split in K. The Laplace transform of the remainder R_q has a pole at every zero of an L-function modulo q, with a residue whose sign on the real axis is fixed once the zeros of the twisted functions $L(s, \psi\chi_K)$ are kept away; a prime d for which this holds in any given box exists by a zero-density estimate for quadratic twists. By Landau's theorem on Dirichlet integrals with coefficients of one sign, a bound $R_q(x) \le x^{\beta-1}$ from *above* only then excludes every zero of every $L(s, \psi)$, ψ modulo q, in $\operatorname{Re} s > \beta$ (Theorem 5.7). It therefore suffices to prove, for one modulus q_0 divisible by any given N and for all primes d in a suitable class,

$$R_{q_0}(x) \le x^{-1/2+\varepsilon} \qquad (x \text{ large}). \tag{1.2}$$

A Stepanov argument with theta lifts. In Stepanov's method an auxiliary function on a curve vanishes to high order at the rational points, and its degree bounds the number of points. Over $\mathbb{Z}$ there is no differentiation of integers, and the proof replaces the parts of this argument as follows.

Stepanov's method over $\mathbb{F}_q$	the proof over $\mathbb{Q}$
auxiliary function	a chain of $m \asymp \sqrt{x}$ theta lifts between definite unitary groups
high order of vanishing	divisibility, at each prime, of derivatives of theta functions
degree bound	archimedean size and height (covolume and index of a lattice)
bound for the number of points	the upper bound (1.2) for the weighted prime sum

The divisibility is where the primes enter. At an inert prime p of the CM field E used below, the derivatives of order a of theta functions of CM elliptic curves are divisible by a power of p that is computed by a sum of the shape $\sum_\nu \Lambda_K(\nu)\,(y/\nu - \kappa)_+$ (Part II); summed over the places of E these divisibilities reproduce the weighted prime sum, with its main term.

Three theorems and a main inequality. For each large x the argument fixes data: scales $x \le x_1 \le b$ with $x_1 \ge x^{10^6}$ and $b \ge x_1^{10^6}$, a CM field E with between x^{11} and x^{12} infinite places, ranks of size x^5, and three large primes (Section 8). It then considers a space $\mathcal{T}$ of automorphic forms on a definite unitary group, the target,

and a family of maps from it, chains of m theta lifts twisted by anticyclotomic Hecke characters ξ. Theorem A says that a large space of integral forms has short images, with sizes given by the main amount less the remainder:

$$\log\|\Theta_1 y\|^2 \le h_{\mathrm{main}} - m\,\mathrm{rem} + O(f\,a\log^4 b).$$

Theorem B says that few translates of these images span almost all of $\mathcal{T}$, by a second-moment computation (Part IV). Theorem C says that integral forms cannot all be short: the covolume of a spanning family of integral forms, corrected by its index, is at least the main amount (Parts V and VI). It rests on a product formula in which the archimedean sizes are read through the untwisted chain Θ_1 while the divisibility at each prime ℓ is read through a chain twisted by a character ξ_ℓ chosen for that prime, together with a height bound for the ratio of the two Gram determinants obtained from the Rallis inner product formula and congruences between theta lifts. Comparing the two bounds, the main amount cancels and

$$m\,\mathrm{rem} \le f\,a\log^5 b$$

(Theorem 9.4). Since $m \ge \sqrt{x}$ and the total remainder rem is, up to negligible terms, $a\,\#\Phi_b\,R_{q_0}(x)$, this is (1.2). The exponent $1/2$ is the reciprocal of the exponent in $m \asymp \sqrt{x}$: a chain of m lifts plays the part of vanishing to order m.

1.4. Formalisation and the status of the proof

The development `grh-lean` [Ant26] consists of a readable edition of about 300,000 lines of Lean code and a compact edition of about 76,000 lines, which prove the same five statements. Its authors report that every file compiles with Lean v4.35.0 and Mathlib at commit `56a00585`, without `sorry`; that the five theorems depend only on the three axioms `propext`, `Classical.choice` and `Quot.sound`; and that an export of the five theorems with everything they depend on is accepted by three independent proof checkers. They state that no mathematician has read the proof through and that no paper accompanies it.

We rebuilt the readable edition at the pinned versions and checked the axioms and the statements ourselves; Appendix E records how, with the hashes of the sources. Our own reading of the Lean text found the argument presented below. Its mathematical correctness is what the formal verification establishes; the present exposition has not been refereed, and where it and the Lean text differ the Lean text is authoritative and the difference is an error of this paper.

The development is organised in two parts. One part, about 177,000 lines, proves results from the literature that Mathlib lacks, often in a weaker form that suffices here and with its own definitions: class field theory, ℓ-adic and p-adic cohomology, Galois representations attached to automorphic forms, the Siegel–Weil and Rallis formulas, and others. The other part, about 122,000 lines, carries out the argument. This paper proves the second part in full. For the first, Appendix B states precisely the form in which each result is proved and used, with its sources; we do not reproduce those proofs.

1.5. Organisation

Section 2 sets out the chain of the proof and the place of each step in the paper (Figure 1 and Table 1), and explains the argument at the level of the main objects and their sizes, and Section 3 fixes the notation. Part I reduces the theorems to the three theorems A, B and C and proves the main inequality. Part II computes the divisibility of derivatives of theta functions. Part III constructs the composite lift and proves Theorem A. Part IV proves Theorem B. Parts V and VI prove Theorem C: the first treats determinants of chains of theta lifts and the cost of a single lift, the second the congruences and Selmer groups that bound the denominators. Appendix A contains an estimate for discrete series coefficients used in Parts IV and V; Appendix B the background results; Appendix C the correspondence with the Lean declarations; Appendix D the constants and exponents; and Appendix E the record of the formal verification.

The names of the Lean declarations that state and prove each statement are listed, statement by statement, in the supplement to the paper (Appendix C). A reader who wishes to check a step can therefore read the corresponding Lean statement, which is complete and precise, next to the proof given here, which explains it.

2. The argument in outline

This section sets out the chain of the proof and the part of the paper in which each step is proved, and describes the objects of the argument and the way the three theorems combine; precise definitions and statements follow in the later parts, to which we refer. Throughout, the parameters q_0, d and the data of Definition 8.7 are fixed: scales $x \le x_1 \le b$, the number $m \asymp \sqrt{x}$ of lifts, ranks $r \asymp x^5$, $k = r + 1 - m$ and $t = 5r$, the CM field $E = K(\zeta_{q_1})$ with its group Φ of f infinite places, the weights $\kappa(\tau)$, and the primes p_0, p_1, N.

2.1. The chain of the proof

The proof is a chain of implications, each of which is a theorem proved in a definite part of the paper. Figure 1 shows the chain, and Table 1 the parts of the paper and their roles.

(S1) *The main theorems.* Theorems 1.1 and 1.2 are Theorems 9.7 and 9.8. The second follows from the first: by the functional equation (Appendix B.7) a zero s of $L(\cdot, \chi)$ with $0 < \operatorname{Re} s < 1$ gives the zero $1 - s$ of $L(\cdot, \bar{\chi})$, both have real part at most $1/2$, and so $\operatorname{Re} s = 1/2$; ζ is the L-function of the character modulo 1.

(S2) *Nonvanishing from a one-sided bound.* Theorem 9.7 combines the criterion Theorem 5.7, by which $L(\rho, \psi) \neq 0$ for every character ψ modulo q_0 and every $\rho \neq 1$ with $\operatorname{Re} \rho > \beta$ as soon as $R_{q_0}(x) \le x^{\beta-1}$ for large x and for every prime d in a residue class, with the one-sided bound Theorem 9.6; passing from q_0 to the modulus of χ changes $L(s, \chi)$ by Euler factors only, and for $\operatorname{Re} s \ge 1$ the nonvanishing is classical. The criterion rests on Landau's theorem for Dirichlet series with nonnegative coefficients and on the choice of d (Sections 4 and 5).

(S3) *The bound at one scale.* Theorem 9.6 follows from Proposition 9.5, which bounds $R_{q_0}(x)$ by $\varphi(q_0) \log^6 b/m$ and the remainders at the second scale

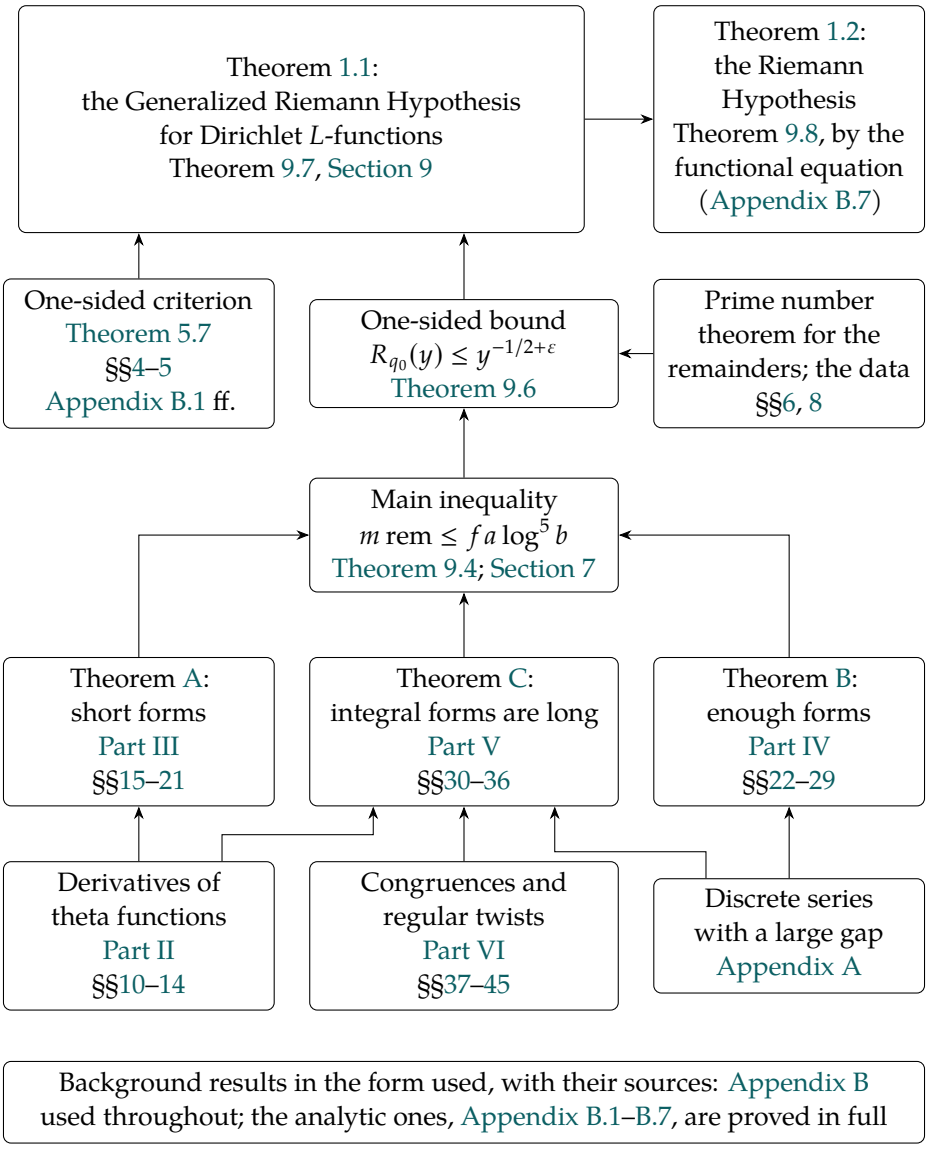

FIGURE 1. The chain of the proof. Arrows lead from the results used to the result proved; boxes name the sections containing the proofs.

x_1. These are small by the prime number theorem in progressions (Theorem 6.6); data with $m \ge \sqrt{x}$ exist for every large x (Theorem 8.14); and Corollary 6.2 passes from integers to all real y.

(S4) *The main inequality.* Proposition 9.5 is the expression of $R_{q_0}(x)$ through the total remainder (Proposition 9.2(c)) together with the main inequality m rem $\le f a \log^5 b$ (Theorem 9.4).

(S5) *Three theorems.* Theorem B provides a family of short forms spanning almost all of the target; Theorem A bounds its covolume and its index from above, and Theorem C bounds the same quantities from below. In the comparison the main amount h_{main} cancels, and m rem remains, bounded by the errors (Section 9.3).

(S6) *Theorem A.* The seesaw identity (Theorem 18.15) writes the chain of lifts of a Poincaré image as a composite lift of a source form. The size of the composite lift (Theorem 20.4) and its divisibility (Theorem 19.13),

computed from the derivatives of theta functions of Part II, give part (a); part (b) is the divisibility read as an index (Section 20).

(S7) *Theorem B.* By Theorems 22.2 and 22.3 it suffices that the projected Poincaré map be a near isometry and that its translates have a small second moment; both are Theorem 22.6, which rests on the unfolding of Poincaré series into Bessel periods (Theorem 25.6), the filling theorem (Theorem 26.8) with its main term and remaining terms (Sections 27 and 28), and the equality of multiplicities (Theorem 23.2).

(S8) *Theorem C.* It is Theorem 7.14 applied to the twisted lattice of the target. The inputs are auxiliary forms (Theorem 33.19); the factorisation of the volume of the chain into the volumes of single lifts and ratios of Hecke values (Theorem 32.14); and the height of each single-lift ratio (Theorem 36.4) for a regular family of twists (Theorem 36.2), which exists by the congruences of Part VI (Theorems 40.7 and 45.2).

Closure. Every theorem, proposition, lemma and corollary of the paper is used in the proof of Theorem 1.1 or Theorem 1.2: it is reached from their proofs by following the references made in proofs and statements. Each of them is proved in the paper, except the background results, which are stated with their sources: those of Appendix B, the analytic ones among them being proved in full there, and the facts on CM curves collected in Theorem 11.3. The references between proofs, followed item by item, form no cycle, so the statements can be proved in an order in which each proof uses only statements already proved. In `grh-lean` each statement corresponds to declarations listed, section by section, in Appendix C, among them declarations used by the five theorems of Section 1.1.

2.2. Spaces and forms

Let E^+ be the maximal totally real subfield of E. All hermitian spaces are over E and *definite*: their unitary groups are compact at every infinite place of E^+. Automorphic forms on such a group are *algebraic modular forms* in the sense of Gross: functions on a finite set of double cosets with values in a finite-dimensional representation, the weight. They carry a natural integral structure at every prime, and their spaces carry Petersson inner products given by finite sums.

- The spaces $X_0, X_1, \dots, X_m$ have ranks $n_j = k + j(t+1)$. The *source* $\mathcal{S}$ is a space of forms on $\mathrm{U}(X_0)$ of a weight concentrated at the a-places, ramified at the level N.
- The spaces $Y_0, Y_1, \dots, Y_m$ have ranks $r + jt$, with $Y_m \subset X_m$ of corank one. The *target* $\mathcal{T}$ is the space of *good* forms on $\mathrm{U}(Y_0)$: forms of principal level at the chosen places above p_0, p_1 cut out by an idempotent e_0 of the group algebra of $\bar{G} = \prod_w \mathrm{GL}_r(\mathbb{F}_w)$, built from Green's characters of finite general linear groups.
- For $1 \le j \le m$, the pair (W_j, V_j) of spaces of ranks r and $r + t$ is a *descent pair*: W_j differs from Y_0 only in its signs at the real places, and V_j is definite.

TABLE 1. The parts of the paper.

part	results	used for
Part I, §§4–9	criterion, prime number theorem for the remainders, twisted lattices, data, main inequality	Theorems 1.1 and 1.2
Part II, §§10–14	divisibility of derivatives of theta functions of CM curves, and its sharpness	sizes and divisibility in Theorems A and C
Part III, §§15–21	spaces and chains, seesaw identity, divisibility and size of the composite lift	Theorem A; the spaces in Theorems B and C
Part IV, §§22–29	equal multiplicities, unfolding, filling theorem, main and remaining terms	Theorem B
Part V, §§30–36	Rallis' formula, exchange identity, norm of the chain, auxiliary forms, cost of one lift	Theorem C
Part VI, §§37–45	Galois representations of lifts, Ribet's method, denominators, Selmer data, twisting	regular families of twists in Theorem C
Appendix A	discrete series with a large gap	Parts IV and V
Appendix B	background results with their sources	throughout

2.3. Maps

- The *Poincaré map* $\mathcal{P}\colon \mathcal{S} \to \{\text{forms on } \mathrm{U}(Y_0)\}$ sends a source form to the form that represents its Bessel period in the Petersson product. After projection by e_0, translation by $g \in \bar{G}$ and a normalisation, it gives the maps $\mathfrak{p}_g\colon \mathcal{S} \to \mathcal{T}$ of Theorems A and B.
- For each *twist*, an anticyclotomic Hecke character ξ of E of weight 0 whose tame part has nontrivial powers $\xi^1, \dots, \xi^m$ at the places of its modulus, the *chain* $\Theta_\xi\colon \mathcal{T} \to \{\text{forms on } \mathrm{U}(Y_m)\}$ is the composite of the m theta lifts $Y_{j-1} \to Y_j$ with the characters $\mu_j^0 \xi^j$ of $\mathrm{U}(1)$, of weights $-\kappa_j$.
- The *composite lift* B_ξ on $\mathcal{S}$ is a finite combination of chains of theta lifts $X_0 \to X_1 \to \cdots \to X_m$, restricted to Y_m. The *seesaw identity* (Part III) states that on ramified source forms

$$\Theta_\xi \circ \mathcal{P} = c_{\mathrm{ss}}\, B_\xi$$

  for a constant $c_{\mathrm{ss}} \neq 0$ independent of ξ.

The two sides of the seesaw identity are read in different ways. The right side is built from theta functions attached to the vectors of lattices of X_j; its values are combinations of derivatives (*jets*) of theta functions of CM elliptic curves, whose sizes and divisibilities can be computed. The left side is a chain of lifts of a target form, whose norms are governed by L-values.

2.4. Theorem A

Each of the m lifts of the composite lift differentiates theta functions: to order a along the a-places, and to order $\kappa(\tau)$ in each of about k further directions, at every place τ. Two estimates are made for the values of B_1 on a source form.

- *Size.* At every conjugate, a derivative of order j costs at most $s_\infty(j)$, with s_∞ as in (9.2).
- *Divisibility.* At each prime ℓ the values are divisible by ℓ to the power g_ℓ, the *gain*, up to an error whose total, weighted by $\log \ell$, is at most the error unit. The gain is the sum over the lifts and the places of the local parts of the prime sums s_{fin}, so that $\sum_\ell g_\ell \log \ell$ is computed by prime sums.

Dividing B_1 by the rational number $\prod_\ell \ell^{g_\ell}$ gives integral forms, and the gains cancel between the two estimates. In the k further directions the size $s_\infty(\kappa(\tau))$ is repaid only by $s_{\mathrm{fin}}(\kappa(\tau), \tau)$, which leaves the main amount $h_{\mathrm{main}} = mk \sum_\tau (s_\infty - s_{\mathrm{fin}})$. Along the a-places the size is matched by the expected value s_{main} of the prime sums, while the divisibility is given by the actual prime sums $s_{\mathrm{fin}}(a, \tau)$; their difference, summed over the a-places and the m lifts, is m rem. The result is the bound $\exp(h_{\mathrm{main}} - m\,\mathrm{rem} + f\,a\log^4 b)$ of Theorem A(a). Part (b), the index bound, records that the divisibility holds on a submodule of small colength in the integral source forms. The bookkeeping is carried out in Part III.

2.5. Theorem B

Theorem B asserts that a few translates $\mathfrak{p}_g(\mathcal{S})$, $g \in \Sigma$, already span almost all of $\mathcal{T}$. It follows from a second-moment bound for the operators $\mathcal{P}^* g \mathcal{P}$: the mean over $g \in \bar{G}$ is close to a multiple of the projection to $\mathcal{T}$, by an unfolding of Poincaré series into Bessel periods, an evaluation of the main term by Siegel's formula and a sieve over the places, and a bound for the remaining terms by Schur's test (Part IV).

2.6. Theorem C

Theorem C is an application of Theorem 7.14 to the twisted lattice $\mathcal{T}$. Its inputs are:

(i) a factorisation of $Z(1)/Z(\xi)$, the ratio of the Gram determinants of the untwisted and the twisted chain, into the ratios $Z_j(1)/Z_j(\xi^j)$ for the single lifts $W_j \to V_j$, times a power of a scalar given by ratios of Hecke L-values (Part V);

(ii) for each prime place ι, a twist ξ_ι allowed at the prime of ι such that each ratio $Z_j(1)/Z_j(\xi_\iota^j)$ has height at most $d_T\left(f\,a\log^4 b/m - k\,h_{\mathrm{dir}}\right)$. The norm of a single lift is a Hecke L-value by the Rallis inner product formula; the height bound combines the archimedean size of this value with a bound for its denominators, obtained from congruences between theta lifts and the smallness of certain anticyclotomic Selmer groups (Part VI);

(iii) auxiliary forms that complete a family of integral forms to a basis at a controlled cost.

Multiplying the m single-lift bounds gives a height of about $-d_T\,h_{\mathrm{main}}$ for $Z(1)/Z(\xi_\iota)$, and Theorem 7.14 turns it into the lower bound $c + 2\delta \geq d_T(h_{\mathrm{main}} - O(f\,a\log^4 b))$.

2.7. Sizes

The exponents are chosen so that every error is smaller than the main amount by a power of x, and so that the main inequality yields the exponent $1/2$. Proposition 8.9 lists the sizes. The choices interact as follows.

- The main inequality has the form $m \operatorname{rem} \leq f a \log^5 b$, and rem is about $a \#\Phi_b R_{q_0}(x)$ with $\#\Phi_b \geq f/(\varphi(q_0)\log b)$. Hence $R_{q_0}(x) \ll \log^6 b/m$, and $m \geq \sqrt{x}$ with $\log b \leq 2x^{\varepsilon}$ gives the exponent $-1/2 + 6\varepsilon$.
- The number of lifts cannot exceed about $\sqrt{x}$: a chain of m lifts adds m blocks of $t+1$ dimensions, and the gap $\Delta = 2r^8$ between the weights of consecutive lifts must dominate the ranks, while all lift weights stay within x^{42} of κ (Section 8.4). The relation between m and x is the quantitative form of the remark, recalled in Section 1.3, that a chain of m lifts plays the part of vanishing to order m.
- The second scale $x_1 \geq x^{10^6}$ makes the prime sums at the places outside Φ_b run up to x_1, where the prime number theorem applies; the third scale $b \geq x_1^{10^6}$ makes the weights large against all other sizes, so that the errors are measured in the unit $\epsilon_{\mathrm{u}} = f a \log^3 b$.

3. Notation and conventions

3.1. General

Λ is the von Mangoldt function, φ Euler's function, $t_+ = \max(t, 0)$, and log the natural logarithm. For a finite set S, $\#S$ is its cardinality. Inner products are linear in the second variable. A *prime place* is a pair of a prime ℓ and a field isomorphism $\iota\colon \mathbb{C} \to \overline{\mathbb{Q}}_\ell$ (Definition 7.1); automorphisms of $\mathbb{C}$ are written σ. A statement holds *for large* x if it holds for all x beyond some bound that depends only on the parameters and on the other fixed quantities named in the statement.

3.2. The letters of the argument

The following letters are fixed by the parameters (Definition 8.1) and the data (Definition 8.7); the second column gives the place where each is defined.

letter	meaning	defined in
q_0, d	the modulus, a multiple of 18480, and the prime $d \equiv q_0/2 - 1 \pmod{q_0}$	Definition 8.1
K, χ_K	$\mathbb{Q}(\sqrt{-d})$ and its quadratic character $(\cdot/d)$	Definition 8.1
s, l_0, $T(s)$	the exponent with $2^s = m^{22}$, the least power of 3 above $27s$, and $3l_0 2^s$	Section 8.1
q_1, q_2	$q_0 T(s)$, the order of ζ; and $(q_0 d)^{s+6}$	Section 8.1
E, Φ, f	$K(\zeta_{q_1})$, its group of infinite places, and their number	Section 8.2
Φ_a, Φ_b	the a-places (class 1 modulo q_0) and the b-places	Sections 8.2 and 8.4
x, x_1, b	the three scales	Definition 8.7
a	the long order xb	Definition 8.7

letter	meaning	defined in
m, r, k, t	the number of lifts, the rank of the target, $r+1-m$, and $5r$	Definition 8.7
n_j, Δ	$k+j(t+1)$, and the gap $2r^8$	Definition 8.7
$\kappa(\tau)$, $\kappa_j(\tau)$	the weight of a place and the lift weights $\kappa+(j+\frac{1}{2})\Delta$	Definition 8.7
p_0, p_1, M, N	the primes of the principal level, $M=p_0p_1$, and the level	Definition 8.7
Ω, rd_E	the period constant (8.1) and the root discriminant	Section 8.2
C_0	the bound for constants	Remark 8.8
Λ_K	$\Lambda(n)(1-\chi_K(n))$	Section 9.1
λ_q, R_q, c_q, G_q	the weights, the remainder, its constant and the function of Definition 4.1	Definition 4.1
s_{fin}, s_∞	the prime sum and the size of a derivative	Section 9.1
h_{dir}, h_{main}	the direction amount and the main amount mkh_{dir}	Section 9.1
s_{main}, rem	the expected value of the prime sums and the total remainder	Section 9.1
ϵ_{u}	the error unit $fa\log^3 b$	Section 9.1
$\mathcal{S}$, $\mathcal{T}$, d_T	the source, the target and its dimension	Section 9.2
$\bar{G}$, $\mathfrak{p}_g$	$\prod_w \mathrm{GL}_r(\mathbb{F}_w)$ and the normalised projected Poincaré maps	Section 9.2
Θ_ξ, $Z(\xi)$	the chain of m lifts twisted by ξ, and $\det(\Theta_\xi^*\Theta_\xi)$	Section 9.2

3.3. Sizes

Inequalities between quantities attached to the data are proved by comparing logarithmic bounds, as explained in Remark 8.10. When an inequality between such quantities is stated without proof, it is of this kind: both sides are products of powers of x, x_1, b, $\log b$, $\log x_1$ and constants, and the inequality follows from the scale inequalities (D1) by comparing exponents.

3.4. The Lean development

Statements are anchored to the Lean development `grh-lean` as described in Appendix C. A name such as `New.star_inequality` refers to a declaration of that development; all declarations of the argument are in the namespace `New`. The development uses its own names for the objects above; the supplement described in Appendix C lists, statement by statement, the declarations that correspond to each statement. When a statement below is a weaker or differently arranged form of the corresponding Lean statement, the difference is noted after it.

Part I

The reduction

4. A weighted prime sum and its Laplace transform

The argument does not bound a sum over primes with sign changes. It bounds from above a sum with nonnegative weights, and the zeros of the Dirichlet L-functions are seen in the poles of its Laplace transform. This section defines the sum, computes the transform and its residues, and records the elementary bounds that the later sections use.

4.1. The weights

Throughout this section d is a prime and $\chi_K = (\cdot/d)$ is the Legendre symbol, extended to $\mathbb{Z}$ by $\chi_K(n) = 0$ for $d \mid n$. For $d \equiv 3 \pmod 4$ it is the quadratic character of $K = \mathbb{Q}(\sqrt{-d})$; the letter K anticipates this case. Let $q \geq 1$ be an integer. For a Dirichlet character ψ modulo q we write $\psi\chi_K$ for the character modulo qd given by $n \mapsto \psi(n)\chi_K(n)$, and we put

$$E_\psi(s) = \begin{cases} L(s,\psi) & \text{if } \psi \text{ is not principal,} \\ (s-1)L(s,\psi) & \text{if } \psi \text{ is principal.} \end{cases}$$

In both cases E_ψ is entire, and $E_\psi(s) \neq 0$ for $\operatorname{Re} s \geq 1$.

Definition 4.1. For $n \geq 1$ let

$$\lambda_q(n) = \begin{cases} \Lambda(n)\big(1 - \chi_K(n)\big) & \text{if } n \equiv 1 \pmod q, \\ 0 & \text{otherwise,} \end{cases}$$

where Λ is the von Mangoldt function. Let

$$G_q(s) = \frac{1}{\varphi(q)}\Big(\sum_{\psi \bmod q} \Big(\frac{E'_{\psi\chi_K}}{E_{\psi\chi_K}}(s) - \frac{E'_\psi}{E_\psi}(s) \Big) - 1 \Big),$$

the sum running over all Dirichlet characters modulo q, and let

$$R_q(x) = \sum_{1 \leq n \leq x} \lambda_q(n)\Big(\frac{1}{n} - \frac{1}{x}\Big) - \frac{\log x}{\varphi(q)} - c_q, \qquad c_q = \operatorname{Re} G_q(1), \tag{4.1}$$

for real $x > 0$. We call R_q the *remainder* modulo q.

The weight $\lambda_q(n)$ vanishes at the prime powers $n = p^k$ with $\chi_K(p^k) = 1$; in particular, for $d \equiv 3 \pmod 4$, at all powers of primes that split in K and at the even powers of inert primes. The function G_q is meromorphic, with poles at most at the zeros of the E_ψ and the $E_{\psi\chi_K}$. By Lemma 4.6, $G_q(1)$ is real when d is odd and prime to q, so that $c_q = G_q(1)$; and in Theorem 6.6 that c_q is then the constant for which $R_q(x) \to 0$.

Lemma 4.2. *For all $n \geq 1$ one has $0 \leq \lambda_q(n) \leq 2\Lambda(n) \leq 2\log n \leq 2n$, and for every integer $N \geq 1$*

$$\sum_{1 \leq n \leq N} \frac{\Lambda(n)}{n} \leq 2\log N. \tag{4.2}$$

Proof. Since $\chi_K(n) \in \{-1, 0, 1\}$, the factor $1 - \chi_K(n)$ lies in $[0, 2]$, which gives the first two inequalities; the others are $\Lambda(n) \leq \log n \leq n$. For (4.2) we use

$N/n \le 2\lfloor N/n \rfloor$ for $1 \le n \le N$ and the identity $\sum_{n\le N} \Lambda(n)\lfloor N/n\rfloor = \log N!$, which follows from $\sum_{e\mid k} \Lambda(e) = \log k$. Hence

$$N \sum_{n\le N} \frac{\Lambda(n)}{n} \le 2 \sum_{n\le N} \Lambda(n) \Big\lfloor \frac{N}{n} \Big\rfloor = 2 \log N! \le 2N \log N. \qquad \square$$

For $r \in \{0,1\}$ and $y > 0$ put

$$A_r(y) = y^{-r} \sum_{1\le n\le y} \lambda_q(n)\, n^{r-1}.$$

Then $R_q(x) = A_0(x) - A_1(x) - \log x/\varphi(q) - c_q$. The sum defining $A_r(y)$ has at most y terms, each at most $2y^r$ by Lemma 4.2, so

$$|A_r(e^v)| \le 2e^v \qquad (v \in \mathbb{R},\ r \in \{0,1\}). \tag{4.3}$$

Since $v/\varphi(q) \le v \le e^v$ for $v \ge 0$, it follows that

$$|R_q(e^v)| \le (|c_q| + 5)\, e^v \qquad (v \ge 0). \tag{4.4}$$

We shall often use the following form of the definition. For $0 < z \le A$ with A an integer,

$$R_q(z) = \frac{1}{A} \sum_{1\le n\le A} \lambda_q(n) \Big(\frac{A}{n} - \frac{A}{z}\Big)_+ - \frac{\log z}{\varphi(q)} - c_q, \tag{4.5}$$

where $t_+ = \max(t, 0)$: the terms with $n > z$ vanish, and for $n \le z$ the positive part is the number itself.

4.2. The Dirichlet series and the transform

Lemma 4.3. *Let d be an odd prime not dividing q. For every character ψ modulo q the character $\psi\chi_K$ modulo qd is not principal.*

Proof. Let a be a quadratic non-residue modulo d, and choose n with $n \equiv 1 \pmod q$ and $n \equiv a \pmod d$. Then $(\psi\chi_K)(n) = \psi(1)\chi_K(a) = -1$, while a principal character takes only the values 0 and 1. $\square$

Lemma 4.4. *Let d be an odd prime not dividing q. For* $\operatorname{Re} s > 1$,

$$\sum_{n\ge 1} \lambda_q(n)\, n^{-s} = \frac{s}{\varphi(q)(s-1)} + G_q(s).$$

Proof. An integer $n \equiv 1 \pmod q$ is prime to q, and for n prime to q orthogonality gives $\sum_{\psi \bmod q} \psi(n) = \varphi(q)$ or 0 according as $n \equiv 1 \pmod q$ or not. Hence

$$\lambda_q(n) = \frac{1}{\varphi(q)} \sum_{\psi \bmod q} \big(\psi(n)\Lambda(n) - (\psi\chi_K)(n)\Lambda(n)\big) \qquad (n \ge 1).$$

For a character χ and $\operatorname{Re} s > 1$ we have $\sum_n \chi(n)\Lambda(n)n^{-s} = -L'(s,\chi)/L(s,\chi)$, which is $-E_\chi'/E_\chi(s)$ if χ is not principal and $-E_\chi'/E_\chi(s) + 1/(s-1)$ if it is. Among the characters ψ and $\psi\chi_K$ only the principal character modulo q is principal, by Lemma 4.3. Summing,

$$\sum_{n\ge 1} \lambda_q(n) n^{-s} = \frac{1}{\varphi(q)} \Big(\frac{1}{s-1} + \sum_{\psi} \Big(\frac{E'_{\psi\chi_K}}{E_{\psi\chi_K}}(s) - \frac{E'_\psi}{E_\psi}(s) \Big) \Big) = \frac{1}{\varphi(q)} \Big(\frac{1}{s-1} + 1 \Big) + G_q(s),$$

and $1/(s-1)+1 = s/(s-1)$. □

Lemma 4.5. *Let d be an odd prime not dividing q. For* $\operatorname{Re} s > 2$,

$$\int_0^\infty R_q(e^v)\, e^{(1-s)v}\,\mathrm{d}v = \frac{G_q(s)/s - c_q}{s-1}.$$

Proof. Write $D(s)$ for the Dirichlet series of Lemma 4.4. Partial summation applied to the coefficients $\lambda_q(n)n^{r-1}$ at the point $s+r-1$, followed by the substitution $t = e^v$, gives

$$D(s) = (s+r-1)\int_0^\infty A_r(e^v)\, e^{(1-s)v}\,\mathrm{d}v \qquad (r \in \{0,1\},\ \operatorname{Re} s > 2);$$

the integrals converge absolutely by (4.3), and the partial sums of the coefficients are $O(y^{1+r})$, which is what partial summation needs on $\operatorname{Re}(s+r-1) > 1+r$. Since $\int_0^\infty v^k e^{(1-s)v}\,\mathrm{d}v = k!/(s-1)^{k+1}$, we obtain

$$\int_0^\infty R_q(e^v) e^{(1-s)v}\,\mathrm{d}v = \frac{D(s)}{s-1} - \frac{D(s)}{s} - \frac{1}{\varphi(q)(s-1)^2} - \frac{c_q}{s-1}.$$

By Lemma 4.4, $D(s)\bigl(\frac{1}{s-1} - \frac{1}{s}\bigr) = \frac{D(s)}{s(s-1)} = \frac{1}{\varphi(q)(s-1)^2} + \frac{G_q(s)}{s(s-1)}$, and the assertion follows. □

Lemma 4.6. *Let d be an odd prime not dividing q. Then G_q is holomorphic on* $\operatorname{Re} s \geq 1$, *and $G_q(1)$ is real; thus $c_q = G_q(1)$.*

Proof. The functions E_ψ and $E_{\psi\chi_K}$ have no zeros on $\operatorname{Re} s \geq 1$, so G_q is holomorphic there, and $F(s) = G_q(s)/s - c_q$ is holomorphic on a neighbourhood of $[1,\infty)$. For real $x > 2$, Lemma 4.5 expresses $F(x)$ as $(x-1)$ times the integral of a real function, so F is real on $(2,\infty)$. The function $x \mapsto \operatorname{Im} F(x)$ is real analytic on $[1,\infty)$ and vanishes on $(2,\infty)$, hence on $[1,\infty)$. At $x = 1$ this says $\operatorname{Im} G_q(1) = 0$. □

4.3. Residues

For a function f meromorphic at ρ we write $\operatorname{ord}_\rho f$ for the order of vanishing of f at ρ when f is holomorphic there; all orders below are of entire functions, hence nonnegative.

Lemma 4.7. *For every $\rho \in \mathbb{C}$,*

$$\lim_{z\to\rho}(z-\rho)\,G_q(z) = \frac{1}{\varphi(q)}\sum_{\psi \bmod q}\bigl(\operatorname{ord}_\rho E_{\psi\chi_K} - \operatorname{ord}_\rho E_\psi\bigr).$$

Consequently, for every constant κ and every $\rho \neq 0, 1$,

$$\lim_{z\to\rho}(z-\rho)\,\frac{G_q(z)/z - \kappa}{z-1} = \frac{1}{\varphi(q)\,\rho(1-\rho)}\sum_{\psi \bmod q}\bigl(\operatorname{ord}_\rho E_\psi - \operatorname{ord}_\rho E_{\psi\chi_K}\bigr).$$

Proof. For an entire function $f \not\equiv 0$, $(z-\rho)f'(z)/f(z) \to \operatorname{ord}_\rho f$ as $z \to \rho$. Summing over ψ gives the first limit; the term $-1/\varphi(q)$ contributes nothing since $z - \rho \to 0$. The second limit is the first divided by $\rho(\rho-1)$, the constant κ again contributing nothing. □

The transform of R_q thus continues to a meromorphic function whose poles in the right half-plane sit at zeros of the E_ψ and the $E_{\psi\chi_K}$. We record exactly what will be used.

Definition 4.8. Let $E\colon \mathbb{R}\to\mathbb{R}$ and $r\colon \mathbb{C}\to\mathbb{C}$. We say that E is *polar with residues* r if E is measurable, there are a constant C with $|E(v)|\le Ce^{v}$ for $v\ge 0$ and a meromorphic function Φ on $\mathbb{C}$ with

$$\Phi(s)=\int_0^\infty E(v)\,e^{(1-s)v}\,\mathrm{d}v \qquad (\operatorname{Re}s>2),$$

and $(z-\rho)\Phi(z)\to r(\rho)$ as $z\to\rho$, $z\neq\rho$, for every ρ with $\operatorname{Re}\rho>0$.

In particular Φ has at most simple poles in $\operatorname{Re}s>0$, and $r(\rho)=0$ where Φ is holomorphic. For $E(v)=R(e^v)$ the integral is $\int_1^\infty R(y)y^{-s}\,\mathrm{d}y$.

Proposition 4.9. *Let $d\equiv 7 \pmod 8$ be a prime not dividing q. Then $v\mapsto R_q(e^v)$ is polar with residues*

$$r_q(s)=\frac{1}{\varphi(q)\,s(1-s)}\sum_{\psi \bmod q}\bigl(\operatorname{ord}_s E_\psi-\operatorname{ord}_s E_{\psi\chi_K}\bigr)\quad (s\neq 0,1),\qquad r_q(1)=0.$$

Proof. Measurability is clear, and the growth bound is (4.4). Put $\Phi(s)=(G_q(s)/s-c_q)/(s-1)$, a meromorphic function on $\mathbb{C}$; by Lemma 4.5 it is the transform for $\operatorname{Re}s>2$. At $\rho\neq 1$ with $\operatorname{Re}\rho>0$ the residue is given by Lemma 4.7. At $\rho=1$ the numerator $F(s)=G_q(s)/s-c_q$ is holomorphic and vanishes, by Lemma 4.6; so $(z-1)\Phi(z)=F(z)\to 0$. □

The sign of r_q on the real axis is the key to the next section: for real $x\in(0,1)$ the factor $x(1-x)$ is positive, and if no twist $E_{\psi\chi_K}$ vanishes at x, then $r_q(x)\ge 0$.

5. Landau's lemma and the one-sided criterion

A Laplace transform of a nonnegative measure cannot be continued analytically across its abscissa of convergence. This is Landau's lemma [Lan05]; see also [MV07, Lemma 15.1] and [Wid41, Ch. II, Thm. 5b]. Applied to the remainder R_q of Section 4 it shows that a bound $R_q(x)\le x^{\beta-1}$ from *one side only* excludes all poles of the transform to the right of β. In the special case at hand the poles are the zeros of Dirichlet L-functions, provided that the twisted functions $E_{\psi\chi_K}$ have no zeros in a box; a prime d with this property is found in Theorem 5.6. The criterion is Theorem 5.7.

5.1. Landau's lemma

Let μ be a measure on $\mathbb{R}$ and $X\colon\mathbb{R}\to(-\infty,0]$ a measurable function. Let $I\subseteq\mathbb{R}$ be the set of real σ for which $e^{\sigma X}$ is μ-integrable. Since $X\le 0$, the set I is an interval unbounded above. On $\{s:\operatorname{Re}s\in\operatorname{int}I\}$ the function $F(s)=\int e^{sX}\,\mathrm{d}\mu$ is holomorphic, and its derivatives are $F^{(k)}(s)=\int X^k e^{sX}\,\mathrm{d}\mu$.

Lemma 5.1. *Let $z\in\operatorname{int}I$, let g be holomorphic on a neighbourhood of z and equal to F near z, and let $\sigma\le z$ be real. If the Taylor series of g at z converges at σ, then $\sigma\in I$.*

Proof. For each y,

$$e^{\sigma X(y)} = \sum_{k\geq 0} \frac{\big((\sigma - z)X(y)\big)^k}{k!}\, e^{zX(y)},$$

and every term is nonnegative, because $\sigma - z \leq 0$ and $X(y) \leq 0$. By monotone convergence,

$$\int e^{\sigma X}\,\mathrm{d}\mu = \sum_{k\geq 0} \frac{(\sigma - z)^k}{k!} \int X^k e^{zX}\,\mathrm{d}\mu = \sum_{k\geq 0} \frac{(\sigma - z)^k}{k!}\, g^{(k)}(z),$$

which is finite by hypothesis. □

Lemma 5.2 (Landau). *Let $a \in \mathbb{R}$ with $(a, \infty) \subseteq \operatorname{int} I$. Suppose that a function g holomorphic at a agrees with F on a neighbourhood of every real $x > a$ close to a. Then there is $b < a$ with $(b, \infty) \subseteq I$.*

Proof. Let g be holomorphic on the disc of radius ε about a. Choose a real $z \in (a, a + \varepsilon/2)$ near which $g = F$. The Taylor series of g at z converges on the disc of radius $\varepsilon/2$ about z. For $\sigma \in (z - \varepsilon/2, a]$, Lemma 5.1 gives $\sigma \in I$; for $\sigma > a$ this holds by hypothesis. Hence $b = z - \varepsilon/2 < a$ serves. □

Lemma 5.3. *Let a be as in Lemma 5.2. Let Ψ be meromorphic at a, equal to F on a punctured neighbourhood of every real $x > a$, and suppose that $(z - a)\Psi(z) \to c$ as $z \to a$, with $\operatorname{Re} c \leq 0$. Then there is $b < a$ with $(b, \infty) \subseteq I$.*

Proof. Near a real $x > a$ the functions Ψ and F are both holomorphic (for x close to a) and agree on a punctured neighbourhood, hence on a neighbourhood. For real $x > a$ we have $F(x) \geq 0$, so $(x - a)\Psi(x) \geq 0$, and letting $x \downarrow a$ gives $c \geq 0$. With $\operatorname{Re} c \leq 0$ this forces $c = 0$. Then $(z - a)\Psi(z)$ extends holomorphically to a with value 0, and Ψ itself extends to a function g holomorphic at a. Lemma 5.2 applies to g. □

Lemma 5.4. *Let $E\colon \mathbb{R} \to \mathbb{R}$ be measurable with $|E(v)| \leq Ce^v$ for $v \geq 0$, let $\beta > 0$, and suppose $E(v) \leq e^{(\beta-1)v}$ for all large v. Then there are a constant K and a measure μ on $\mathbb{R}$ such that, with $X(v) = -|v|$, every s with $\operatorname{Re} s > \beta + 2$ satisfies $\operatorname{Re} s \in I$ and*

$$\int e^{sX}\,\mathrm{d}\mu = \frac{K}{s - \beta} - \int_0^\infty E(v)\, e^{(1-s)v}\,\mathrm{d}v.$$

Proof. Let $E(v) \leq e^{(\beta-1)v}$ for $v \geq Y$. Then $E(v)e^v \leq Ke^{\beta v}$ for all $v > 0$, with $K = \max(1, |C|e^{2Y})$: for $v \geq Y$ the bound is immediate, and for $0 < v \leq Y$ we have $E(v)e^v \leq |C|e^{2v} \leq |C|e^{2Y}$. Let μ be the measure on $(0, \infty)$ with density $Ke^{\beta v} - E(v)e^v \geq 0$. For $\operatorname{Re} s > \beta + 2$ both $Ke^{(\beta-s)v}$ and $E(v)e^{(1-s)v}$ are integrable on $(0, \infty)$, and $e^{sX(v)} \cdot (Ke^{\beta v} - E(v)e^v) = Ke^{(\beta-s)v} - E(v)e^{(1-s)v}$ for $v > 0$. Integrating gives the identity, with $\int_0^\infty Ke^{(\beta-s)v}\,\mathrm{d}v = K/(s - \beta)$. □

Proposition 5.5. *Let E be polar with residues r (Definition 4.8), let $\beta > 0$, and suppose that*

(i) $\operatorname{Re} r(x) \geq 0$ *for every real $x > \beta$, and*
(ii) $E(v) \leq e^{(\beta-1)v}$ *for all large v.*

Then $r(\rho) = 0$ for every ρ with $\operatorname{Re}\rho > \beta$.

Proof. Let C and Φ be as in Definition 4.8, and let μ, K be as in Lemma 5.4, with $X(v) = -|v|$, I the set of convergence and $F(s) = \int e^{sX}\,\mathrm{d}\mu$. Put $\Psi(s) = K/(s-\beta) - \Phi(s)$, a meromorphic function equal to F for $\operatorname{Re} s > \beta + 2$. For $\operatorname{Re}\rho > \beta$,

$$(z-\rho)\Psi(z) \to -r(\rho) \qquad (z \to \rho), \tag{5.1}$$

since $K/(s-\beta)$ is holomorphic at ρ. The complement of I is bounded above by $\beta + 2$. Let a be the least upper bound of $\{\beta\} \cup (\mathbb{R} \setminus I)$. Then $(a, \infty) \subseteq \operatorname{int} I$, so F is holomorphic on $\operatorname{Re} s > a$, and by the identity theorem $\Psi = F$ on a punctured neighbourhood of every point of $\operatorname{Re} s > a$ (both are meromorphic on the half-plane and agree far to the right).

If $a = \beta$, then for $\operatorname{Re}\rho > \beta$ the function Ψ agrees with the holomorphic function F near ρ, so $(z-\rho)\Psi(z) \to 0$, and (5.1) gives $r(\rho) = 0$.

Suppose $a > \beta$. By (i) and (5.1) at $\rho = a$, the limit $c = -r(a)$ has $\operatorname{Re} c \le 0$. Lemma 5.3 gives $b < a$ with $(b, \infty) \subseteq I$. By the definition of a there is $\sigma \notin I$ with $\max(\beta, b) < \sigma \le a$; but $\sigma > b$ forces $\sigma \in I$. This contradiction shows $a = \beta$. □

5.2. The choice of the prime d

The residues r_q of Proposition 4.9 have the right sign on the real axis only where the twisted functions $E_{\psi\chi_K}$ do not vanish. The following theorem provides a prime d for which this holds in any prescribed box. Its proof, a zero-density estimate for the quadratic twists of the characters of a fixed modulus obtained from theta series and the Pólya–Vinogradov inequality, is given in Appendix B.6. Estimates of this kind are classical; see Jutila [Jut75] and Heath-Brown [HB95] for stronger forms obtained from mean values of real character sums.

Theorem 5.6. *Let $N \ge 1$, let $\beta > 1/2$ and $H \ge 0$, and let a be an invertible residue class modulo N such that every prime d in the class satisfies $d \equiv 7 \pmod 8$ and $d \nmid N$. Then there is a prime $d \equiv a \pmod N$ such that $L(s, \xi\chi_d) \ne 0$ for every character ξ modulo N and every s with $\operatorname{Re} s \ge \beta$ and $|\operatorname{Im} s| \le H$, where $\chi_d = (\cdot/d)$.*

By Lemma 4.3 the characters $\xi\chi_d$ are not principal, so $E_{\xi\chi_d} = L(s, \xi\chi_d)$.

5.3. The one-sided criterion

Theorem 5.7. *Let q_0 be a positive multiple of 8, let $d_0 \equiv 7 \pmod 8$ be an integer prime to q_0, and let $\beta > 1/2$. Suppose that for every prime $d \equiv d_0 \pmod{q_0}$ the remainder R_{q_0} formed with $\chi_K = (\cdot/d)$ satisfies*

$$R_{q_0}(x) \le x^{\beta-1} \qquad \text{for all large } x.$$

Then $L(\rho, \psi) \ne 0$ for every Dirichlet character ψ modulo q_0 and every $\rho \ne 1$ with $\operatorname{Re}\rho > \beta$.

Proof. Fix ψ and ρ. Every prime $d \equiv d_0 \pmod{q_0}$ satisfies $d \equiv d_0 \equiv 7 \pmod 8$, since $8 \mid q_0$, and is prime to q_0. Theorem 5.6, with $N = q_0$, $a = d_0$ and $H = |\operatorname{Im}\rho|$, gives such a prime d for which no $E_{\xi\chi_K}$ with ξ modulo q_0 vanishes in the

box $\operatorname{Re} s \geq \beta$, $|\operatorname{Im} s| \leq |\operatorname{Im} \rho|$. For s in the box with $s \neq 0, 1$, the residue of Proposition 4.9 is

$$r(s) = \frac{n(s)}{\varphi(q_0)\, s(1-s)}, \qquad n(s) = \sum_{\xi \bmod q_0} \operatorname{ord}_s E_\xi \geq 0,$$

because the orders of the twisted functions vanish there.

We check hypothesis (i) of Proposition 5.5. For real $x > \beta$ with $x < 1$ the factor $x(1-x)$ is positive and $r(x) \geq 0$. For $x \geq 1$ the functions E_ξ do not vanish, so $n(x) = 0$; at $x = 1$ we have $r(1) = 0$ by definition. Hypothesis (ii) is the assumption, for $E(v) = R_{q_0}(e^v)$. Hence $r(\rho) = 0$, that is $n(\rho) = 0$, as $\rho \neq 0$ and $\rho \neq 1$. In particular $E_\psi(\rho) \neq 0$. Since $\rho \neq 1$, the function E_ψ is $L(s, \psi)$ or $(s-1)L(s, \psi)$ at ρ, and in both cases $L(\rho, \psi) \neq 0$. □

Remark 5.8. The hypothesis concerns all primes d of one class modulo q_0, and only an upper bound for R_{q_0}. No information on real zeros is assumed: the role of the familiar hypothesis that there are no real zeros to the right of β, in one-sided criteria of this kind (compare [MV07, §§15.1–15.2] and Suzuki [Suz24]) is played here by the choice of d. The weights $1 - \chi_K(n)$ make the twisted functions $E_{\psi\chi_K}$ enter the residues with the opposite sign, and Theorem 5.6 removes their zeros from the box in question; what remains on the real axis is a sum of orders of zeros, which is nonnegative.

6. The prime number theorem for the remainders

The one-sided bound of Section 9 compares the remainder R_{q_0} at a point x with the remainders R_{q_0} and R_1 at a much larger point x_1. At x_1 the classical prime number theorem in arithmetic progressions suffices: we need $|R_q(y)| \leq (\log y)^{-B}$ for every B. This is a case of the Siegel–Walfisz theorem [MV07, Cor. 11.19]. We derive it here in the form needed, from the zero-free region of a fixed Dirichlet L-function and a Tauberian theorem, both proved in Appendices B.2 and B.3. Throughout, d is a prime and $q \geq 1$.

6.1. Slow variation

Let

$$\mathcal{E}_q(t) = \sum_{0 \leq n \leq t} \lambda_q(n) - \frac{t}{\varphi(q)} \qquad (t \geq 0)$$

be the error in the weighted count, with $\lambda_q(0) = 0$.

Lemma 6.1. *For $t \geq 0$ one has $|\mathcal{E}_q(t)| \leq 13t$, and for $y \geq 1$*

$$R_q(y) = \int_1^y \frac{\mathcal{E}_q(t)}{t^2}\, \mathrm{d}t - c_q. \tag{6.1}$$

Proof. Chebyshev's bound $\sum_{n \leq t} \Lambda(n) \leq (\log 4 + 4)t \leq 6t$ and $\lambda_q \leq 2\Lambda$ give $0 \leq \sum_{n \leq t} \lambda_q(n) \leq 12t$, while $t/\varphi(q) \leq t$. For (6.1), Abel summation with the function $1/t$ gives

$$\sum_{n \leq y} \frac{\lambda_q(n)}{n} = \frac{1}{y} \sum_{n \leq y} \lambda_q(n) + \int_1^y \frac{1}{t^2} \sum_{n \leq t} \lambda_q(n)\, \mathrm{d}t,$$

and $\int_1^y t\,(\varphi(q)t^2)^{-1}\,\mathrm{d}t = \log y/\varphi(q)$. Comparing with the definition (4.1) gives the identity. □

Corollary 6.2. *For $1 \le y \le z$ one has $|R_q(z) - R_q(y)| \le 13(z-y)/y$. In the variable $v = \log y$: $|R_q(e^v) - R_q(e^u)| \le 26(v-u)$ for $0 \le u \le v \le u+1$. For $y \ge 2$, $R_q(y) - R_q(\lfloor y \rfloor) \le 26/y$.*

Proof. By (6.1) the difference is $\int_y^z \mathcal{E}_q(t)t^{-2}\,\mathrm{d}t$, and the integrand is at most $13/t \le 13/y$ in absolute value. For the second statement put $y = e^u$, $z = e^v$ and use $e^{v-u} - 1 \le 2(v-u)$ for $0 \le v-u \le 1$. For the third, $(y - \lfloor y \rfloor)/\lfloor y \rfloor \le 1/(y/2)$ when $y \ge 2$. □

6.2. The transform in a zero-free region

We use the following form of the classical zero-free region, proved in Appendix B.2.

Theorem 6.3. *Let χ be a Dirichlet character. For all sufficiently large l, the function E_χ does not vanish on the region $\operatorname{Re} s \ge 1 - l^{-25}$, $|s| \le e^l$, and satisfies $|E_\chi'(s)/E_\chi(s)| \le 2l^{25}$ there.*

Lemma 6.4. *For all sufficiently large l, the function $G_q(s)/s$ is holomorphic and of absolute value at most l^{26} on the region $\operatorname{Re} s \ge 1 - l^{-26}$, $|s| \le e^l$.*

Proof. There are finitely many characters ψ and $\psi\chi_K$, say ν characters ψ. By Theorem 6.3, for large l none of the functions E_ψ, $E_{\psi\chi_K}$ vanishes on the region $\operatorname{Re} s \ge 1 - l^{-25}$, $|s| \le e^l$, which contains the region of the lemma, and their logarithmic derivatives are at most $2l^{25}$ there. Then G_q is holomorphic there, with $|G_q(s)| \le \nu \cdot 4l^{25} + 1$. On the region $\operatorname{Re} s \ge 1/2$, so $|G_q(s)/s| \le (8\nu+2)l^{25} \le l^{26}$ once $l \ge 8\nu + 2$. □

6.3. The Tauberian theorem and its consequence

The Tauberian theorem below is proved in Appendix B.3. Its proof reads one Taylor coefficient of the transform at a far point in two ways: as a mean of R against a Gamma density concentrated near a given point, and, by Cauchy's estimate on a disc that reaches across the line $\operatorname{Re} s = 1$, as a small number. Compare [MV07, §6.2], where a contour integral is used instead.

Theorem 6.5. *Let $R\colon \mathbb{R} \to \mathbb{R}$ be measurable, with $|R(v) - R(u)| \le L(v-u)$ for $0 \le u \le v \le u+1$, and let F be a function such that*

$$\int_0^\infty R(v)\,e^{(1-s)v}\,\mathrm{d}v = \frac{F(s) - F(1)}{s-1} \qquad (\operatorname{Re} s > 2).$$

Suppose that there is an integer $A \ge 0$ such that for all large l the function F is holomorphic and of absolute value at most l^A on the region $\operatorname{Re} s \ge 1 - l^{-A}$, $|s| \le e^l$. Then for every integer $B \ge 0$ one has $|R(u)| \le u^{-B}$ for all large u.

Theorem 6.6. *Let d be an odd prime not dividing q. For every integer $B \geq 0$,*

$$|R_q(y)| \leq (\log y)^{-B} \qquad \text{for all large } y.$$

In particular $R_q(y) \to 0$ as $y \to \infty$.

Proof. Apply Theorem 6.5 to $R(v) = R_q(e^v)$ and $F(s) = G_q(s)/s$. The Lipschitz condition is Corollary 6.2. By Lemmas 4.5 and 4.6, $(F(s) - F(1))/(s-1) = (G_q(s)/s - c_q)/(s-1)$ is the transform for $\operatorname{Re} s > 2$. The bound on F, with $A = 26$, is Lemma 6.4. Hence $|R_q(e^v)| \leq v^{-B}$ for large v, which is the assertion at $y = e^v$. □

7. Heights and twisted lattices

The lower bound of the argument is a size-versus-divisibility inequality for a Gram determinant, in the tradition of transcendence methods and of slope inequalities in Arakelov geometry; see Bost [Bos01], Laurent [Lau94], Gaudron [Gau08] and [BG06, Ch. 1]. Its one unusual feature is that the integral structure at each prime ℓ may be read through a different map, a *twist* allowed at ℓ, while the sizes at the archimedean places are read through a single untwisted map. This section sets up the language (heights of families indexed by ℓ-adic places, colengths, twisted lattices) and proves the inequality, Theorem 7.14.

7.1. Places of $\mathbb{C}$ and heights

For a prime ℓ let $\overline{\mathbb{Q}}_\ell$ be an algebraic closure of $\mathbb{Q}_\ell$, with its absolute value extending $|\ell|_\ell = \ell^{-1}$. Since $\mathbb{C}$ and $\overline{\mathbb{Q}}_\ell$ are algebraically closed fields of characteristic zero of the same uncountable cardinality, field isomorphisms $\mathbb{C} \to \overline{\mathbb{Q}}_\ell$ exist.

Definition 7.1. A *prime place* is a pair $\iota = (\ell, \iota)$ of a prime ℓ and a field isomorphism $\iota\colon \mathbb{C} \to \overline{\mathbb{Q}}_\ell$; we write $\iota \mid \ell$. For $z \in \mathbb{C}$ we put $\|z\|_\iota = |\iota(z)|_\ell$. The *conjugate* prime place is $\bar\iota\colon z \mapsto \iota(\bar z)$. The ring $\mathcal{O}_\iota = \{z \in \mathbb{C} : \|z\|_\iota \leq 1\}$ is a valuation ring of $\mathbb{C}$.

Definition 7.2. Let $y = (y_\iota)$ be a family of complex numbers indexed by the prime places, and let $c \in \mathbb{R}$. We say that y has *height at most* c, and write $h(y) \leq c$, if there are $H \in \mathbb{C}^\times$, a real number α and real numbers $h_\ell \geq 0$, one for each prime ℓ, such that

(i) $|\sigma(H)| \leq e^\alpha$ for every field automorphism σ of $\mathbb{C}$;
(ii) $\|H y_\iota\|_\iota \leq \ell^{h_\ell}$ for every prime place $\iota \mid \ell$;
(iii) $\alpha + \sum_{\ell \in T} h_\ell \log \ell \leq c$ for every finite set T of primes.

The multiplier H, and with it the archimedean bound, is the same for all ι, while y_ι may depend on ι; this is what allows each prime its own twist. Heights may be negative.

Lemma 7.3. *Let y, y' be families indexed by the prime places.*

(a) *If $h(y) \leq c \leq c'$ and $\|y'_\iota\|_\iota \leq \|y_\iota\|_\iota$ for all ι, then $h(y') \leq c'$.*
(b) *If $h(y) \leq c$ and $h(y') \leq c'$, then $h(yy') \leq c + c'$; hence $h(\prod_{i \in S} y^{(i)}) \leq \sum_{i \in S} c_i$ for finitely many families with $h(y^{(i)}) \leq c_i$, and $h(y^n) \leq nc$.*

(c) *If $h(y) \le c \le c'$, N_ι are positive integers prime to the prime of ι and $e_\iota \in \mathbb{Q}$, then the family $N_\iota^{e_\iota} y_\iota$ has height at most c'.*
(d) *If there are $H \neq 0$ and reals α, e_ℓ of any sign with $|\sigma H| \le e^{\alpha}$ for all σ, $\|H y_\iota\|_\iota \le e^{e_\ell}$ for all $\iota \mid \ell$ and $\alpha + \sum_{\ell \in T} e_\ell \le c$ for all finite sets T of primes, then $h(y) \le c$.*
(e) *If $H \neq 0$ and $|\sigma H| \le e^{c}$ for all σ, then the constant family H^{-1} has height at most c.*

Proof. (a) Keep H, α, h. (b) Multiply the multipliers and add α and the h_ℓ. (c) A positive integral power $(N_\iota^{e_\iota})^{\mathrm{den}(e_\iota)}$ is an integral power of N_ι, a unit at ℓ, so $\|N_\iota^{e_\iota}\|_\iota = 1$; apply (a). (d) Take $h_\ell = (e_\ell)_+ / \log \ell$; then $\sum_{\ell \in T} h_\ell \log \ell$ is the sum of the e_ℓ over the $\ell \in T$ with $e_\ell \ge 0$, which is at most $c - \alpha$ by hypothesis. (e) Apply (d) with $e_\ell = 0$. □

Proposition 7.4 (Product formula). *If the constant family 1 has height at most c, then $c \ge 0$. Consequently, if $z \in \mathbb{C}^\times$, the constant family z has height at most c, and $|\sigma(z)| \le e^{c'}$ for every automorphism σ of $\mathbb{C}$, then $-c \le c'$.*

Proof. Let H, α, h witness $h(1) \le c$. For every ℓ and every isomorphism $\iota \colon \mathbb{C} \to \overline{\mathbb{Q}}_\ell$ we have $|\iota H|_\ell \le \ell^{h_\ell}$. Applied with $\ell = 2$ this bound is uniform in ι. Hence H is algebraic: otherwise, fixing $t \in \overline{\mathbb{Q}}_2$ transcendental over $\mathbb{Q}$, for each k the field isomorphism $\mathbb{Q}(H) \to \mathbb{Q}(2^{-k}t)$ with $H \mapsto 2^{-k}t$ extends, by extending transcendence bases and passing to algebraic closures, to an isomorphism $\iota \colon \mathbb{C} \to \overline{\mathbb{Q}}_2$, and $|\iota H|_2 = 2^k |t|_2$ is unbounded in k. Let $F = \mathbb{Q}(H)$, of degree n, and let $q = \mathrm{N}_{F/\mathbb{Q}}(H) \in \mathbb{Q}^\times$. Every embedding of F into $\mathbb{C}$ or into $\overline{\mathbb{Q}}_\ell$ extends to an automorphism of $\mathbb{C}$, respectively to an isomorphism $\mathbb{C} \to \overline{\mathbb{Q}}_\ell$, so

$$\log|q| \le n\alpha, \qquad \log|q|_\ell \le n h_\ell \log \ell \quad \text{for every prime } \ell,$$

the norm being the product of the n conjugates. By the product formula for $\mathbb{Q}$, taken over a finite set T of primes containing those that divide the numerator or the denominator of q,

$$0 = \log|q| + \sum_{\ell \in T} \log|q|_\ell \le n\Big(\alpha + \sum_{\ell \in T} h_\ell \log \ell\Big) \le nc.$$

For the second statement, Lemma 7.3(e) gives $h(z^{-1}) \le c'$, so $h(1) = h(z^{-1} z) \le c' + c$ by part (b), and the first statement applies. □

7.2. Colength

Let O be a commutative ring and V an O-module.

Definition 7.5. Let $M, N \subseteq V$ be submodules and $z \in O$. We say that *the colength of N in M divides z* if for every finite family $\Phi_1, \dots, \Phi_n$ of elements of M there is a matrix $u \in M_n(O)$ with $\det u \mid z$ such that $\sum_k u_{ik} \Phi_k \in N$ for every i.

No inclusion $N \subseteq M$ is assumed. Over a valuation ring the condition says that $N \cap M$ has colength at most $v(z)$ in M. The following properties follow directly from the definition.

Lemma 7.6. (a) *If $M \subseteq N$, the colength of N in M divides every z; if it divides z and $z \mid z'$, it divides z'.*

(b) *If the colength of N in M divides z and that of P in N divides a, then that of P in M divides za.*

(c) *Let $f\colon U \to V$ be linear and $M', N' \subseteq U$ with $f(M') \subseteq M$ and $\{x \in M' : f(x) \in N\} \subseteq N'$. If the colength of N in M divides z, so does that of N' in M'.*

(d) *Let $f\colon V \to U$ be linear and $M', N' \subseteq U$ with $M' \subseteq f(M) + N'$ and $f(N) \subseteq N'$. If the colength of N in M divides z, so does that of N' in M'.*

Proof. (a) If $M \subseteq N$, the identity matrix carries every family of M into N, and its determinant 1 divides every z. If $\det u \mid z$ and $z \mid z'$, then $\det u \mid z'$.

(b) Let $\Phi_1, \dots, \Phi_n \in M$, and let u with $\det u \mid z$ carry them into N: $\Psi_i = \sum_k u_{ik}\Phi_k \in N$. Let u' with $\det u' \mid a$ carry the family Ψ into P. Then $u'u$ carries Φ into P, and $\det(u'u) = \det u' \det u$ divides az.

(c) Let $\Phi_1, \dots, \Phi_n \in M'$. Their images $f(\Phi_k)$ lie in M, so some u with $\det u \mid z$ has $\sum_k u_{ik} f(\Phi_k) \in N$. By linearity this is $f(\sum_k u_{ik}\Phi_k)$, and $\sum_k u_{ik}\Phi_k \in M'$; by the hypothesis it lies in N'.

(d) Let $\Phi'_1, \dots, \Phi'_n \in M'$ and write $\Phi'_k = f(\Phi_k) + \Psi_k$ with $\Phi_k \in M$, $\Psi_k \in N'$. Some u with $\det u \mid z$ has $\sum_k u_{ik}\Phi_k \in N$; then $\sum_k u_{ik}\Phi'_k = f(\sum_k u_{ik}\Phi_k) + \sum_k u_{ik}\Psi_k$ lies in $f(N) + N' \subseteq N'$. □

7.3. Twisted lattices

For a family $w_1, \dots, w_n$ of vectors of a complex inner product space we write $\mathrm{Gram}(w) = \det(\langle w_i, w_k\rangle)_{i,k}$. All inner products are linear in the second variable.

Definition 7.7. Let V be a complex vector space and T a set with a distinguished element 1, whose elements we call *twists*. A *twisted lattice* L on V consists of

(i) for each $\xi \in T$ a complex inner product space V_ξ and a linear map $\Theta_\xi\colon V \to V_\xi$;

(ii) for each prime ℓ a subset of T, the twists *allowed* at ℓ;

(iii) for each $\xi \in T$ and each prime place ι an O_ι-submodule $\Lambda_{\xi,\iota} \subseteq V_\xi$, the vectors *integral at* ι, such that $\|\langle\varphi, \varphi'\rangle\|_\iota \le 1$ whenever ξ is allowed at the prime of ι, $\varphi \in \Lambda_{\xi,\bar\iota}$ and $\varphi' \in \Lambda_{\xi,\iota}$;

(iv) a finite set C, a linear map $\mathrm{coord}\colon V_1 \to \mathbb{C}^C$ and a positive definite rational matrix $\Gamma \in M_C(\mathbb{Q})$ with $\langle\varphi, \varphi'\rangle = {}^t\overline{\mathrm{coord}(\varphi)}\,\Gamma\,\mathrm{coord}(\varphi')$ for $\varphi, \varphi' \in V_1$.

A vector $y \in V$ is *integral at* $\iota \mid \ell$ if $\Theta_\xi(y) \in \Lambda_{\xi,\iota}$ for every twist ξ allowed at ℓ; these vectors form an O_ι-submodule V_ι^{int} of V.

For an automorphism σ of $\mathbb{C}$ and $y \in V$ let $\sigma\Theta_1 y \in \mathbb{C}^C$ be the coordinate vector of $\Theta_1 y$ with σ applied to each entry, and $\|\Theta_1 y\|_\sigma^2 = {}^t\overline{\sigma\Theta_1 y}\,\Gamma\,\sigma\Theta_1 y$. Since Γ is rational, these are the norms of the Galois conjugates of $\Theta_1 y$ for the rational structure given by the coordinates.

Definition 7.8. Let $y = (y_i)_{i\in S}$ be a finite family in V.

(i) The family has *index at most* δ if there are reals $\delta_\ell \ge 0$, one for each prime, with $\sum_{\ell\in U} \delta_\ell \log \ell \le \delta$ for every finite set U of primes, and for

every prime place $\iota \mid \ell$ a matrix $u \in M_S(O_\iota)$ with $\|\det u\|_\iota \geq \ell^{-\delta_\ell}$ and $\sum_k u_{ik} y_k \in V_\iota^{\mathrm{int}}$ for all i.

(ii) The family has *covolume at most* c if for every automorphism σ of $\mathbb{C}$,

$$\left|\det\left({}^t\overline{\sigma\Theta_1 y_i}\,\Gamma\,\sigma\Theta_1 y_k\right)_{i,k}\right| \leq e^c.$$

(iii) For $\xi \in T$, when V is a finite-dimensional inner product space, the *volume* $Z(\xi) = \det(\Theta_\xi^* \Theta_\xi)$ is the Gram determinant of the images under Θ_ξ of an orthonormal basis of V.

Note that the matrix u in (i) is the same for all twists allowed at ℓ: the index measures one integral structure, read through all allowed twists at once, and the covolume refers to the trivial twist alone. If Θ_ξ is injective then $Z(\xi) \neq 0$, and for any basis w of V

$$\mathrm{Gram}(\Theta_\xi w) = Z(\xi)\,\mathrm{Gram}(w). \tag{7.1}$$

Lemma 7.9. *Let L be a twisted lattice on V.*

(a) *If y has index at most δ and y' has index at most δ', the union of the two families has index at most $\delta + \delta'$.*

(b) *Let Σ be a finite set, and for $s \in \Sigma$ let $A_s \subseteq V$ be a set every finite family of which has index at most δ. A finite family each member of which lies in some A_s has index at most $\#\Sigma \cdot \delta$.*

(c) *Let $\delta_\ell \geq 0$ with $\sum_{\ell \in U} \delta_\ell \log \ell \leq \delta$ for all finite U. If for every prime place $\iota \mid \ell$ the family y lies in an O_ι-submodule M such that the colength of V_ι^{int} in M divides ℓ^j for some integer $j \leq \delta_\ell$, then y has index at most δ.*

Proof. (a) Add the numbers δ_ℓ and put the two matrices in diagonal blocks. (b) Sort the family by the first s whose set contains the member and apply (a) $\#\Sigma - 1$ times. (c) The colength condition provides, at each $\iota \mid \ell$, a matrix u over O_ι with $\det u \mid \ell^j$, carrying y into V_ι^{int}; then $\|\det u\|_\iota \geq \ell^{-j} \geq \ell^{-\delta_\ell}$. □

Lemma 7.10 (Hadamard and Fischer). *If $\|\Theta_1 z_i\|_\sigma^2 \leq e^h$ for $i = 1, \ldots, n$ and every automorphism σ, then $z_1, \ldots, z_n$ has covolume at most nh. If moreover y has covolume at most c, then the union of y and z has covolume at most $c + nh$.*

Proof. The matrices in Definition 7.8(ii) are Gram matrices for the positive definite form Γ. Hadamard's inequality bounds such a determinant by the product of the diagonal entries, and Fischer's inequality bounds the determinant of a block matrix by the product of the determinants of its diagonal blocks. □

Lemma 7.11. *If y has covolume at most c, then $|\sigma\,\mathrm{Gram}(\Theta_1 y)| \leq e^c$ for every automorphism σ of $\mathbb{C}$.*

Proof. Let $\sigma'(z) = \overline{\sigma(\bar{z})}$. As Γ is rational, $\sigma\langle \Theta_1 v, \Theta_1 w\rangle = {}^t\overline{\sigma'\Theta_1 v}\,\Gamma\,\sigma\Theta_1 w$. Hence $\sigma\,\mathrm{Gram}(\Theta_1 y)$ is the determinant of a pairing between the families $\sigma'\Theta_1 y$ and $\sigma\Theta_1 y$ under Γ. By the Cauchy–Schwarz inequality in the exterior power, its square is at most the product of the Gram determinants of the two families, each at most e^c. □

Lemma 7.12. *Let $\iota \mid \ell$ and let ξ be allowed at ℓ. Let $y_1, \dots, y_n$ have index at most δ, witnessed by δ_ℓ and matrices u at ι and u' at $\bar\iota$. Let $\gamma \in \mathbb{C}$ be such that $\gamma\,\Theta_\xi z_i$ is integral both at ι and at $\bar\iota$ for $i = 1, \dots, n'$. Then*

$$\big\| |\gamma|^{2n'} \operatorname{Gram}\big(\Theta_\xi(y,z)\big)\big\|_\iota \le \ell^{2\delta_\ell},$$

where (y,z) is the family $y_1, \dots, y_n, z_1, \dots, z_{n'}$.

Proof. Let v be the block diagonal matrix with blocks u and $\gamma \cdot 1_{n'}$, and v' the one with blocks u' and $\gamma \cdot 1_{n'}$. Then v carries $\Theta_\xi(y,z)$ into $\Lambda_{\xi,\iota}$ and v' carries it into $\Lambda_{\xi,\bar\iota}$. By Definition 7.7(iii), the entries of ${}^t\bar v' \big(\langle \Theta_\xi w_i, \Theta_\xi w_k\rangle\big)\, v$ are integral at ι, so $\|\overline{\det v'}\|_\iota \|\mathrm{Gram}\|_\iota \|\det v\|_\iota \le 1$. Here $\|\overline{\det v'}\|_\iota = \|\det v'\|_{\bar\iota} \ge \ell^{-\delta_\ell}\|\gamma\|_{\bar\iota}^{n'}$ and $\|\det v\|_\iota \ge \ell^{-\delta_\ell}\|\gamma\|_\iota^{n'}$, while $\||\gamma|^2\|_\iota = \|\bar\gamma\|_\iota\|\gamma\|_\iota = \|\gamma\|_{\bar\iota}\|\gamma\|_\iota$. □

Corollary 7.13. *Let $y_1, \dots, y_n$ have index at most δ. For each prime place ι let ξ_ι be a twist allowed at the prime of ι and $\gamma_\iota \in \mathbb{C}$ such that $\gamma_\iota \Theta_{\xi_\iota} z_i$ is integral at ι and at $\bar\iota$ for all i. Then the family $|\gamma_\iota|^{2n'} \operatorname{Gram}(\Theta_{\xi_\iota}(y,z))$ has height at most 2δ.*

Proof. The matrices of the index at ι and at $\bar\iota$ both serve for the twist ξ_ι, since ι and $\bar\iota$ lie over the same prime and these matrices do not depend on the twist. Apply Lemma 7.12 and Lemma 7.3(d) with $H = 1$, $\alpha = 0$ and $e_\ell = 2\delta_\ell \log \ell$. □

7.4. A twist for each prime

Theorem 7.14. *Let L be a twisted lattice on a finite-dimensional complex inner product space V, with all maps Θ_ξ injective. Let $y_1, \dots, y_n, z_1, \dots, z_{n'}$ be a basis of V, and for each prime place ι let ξ_ι be a twist allowed at the prime of ι. Suppose that*

(i) *y has covolume at most c and index at most δ;*
(ii) *for each twist ξ there is $\gamma(\xi) \neq 0$ such that $\gamma(\xi)\Theta_\xi z_i$ is integral at every prime place $\iota \mid \ell$ for which ξ is allowed at ℓ;*
(iii) *$\|\Theta_1 z_i\|_\sigma^2 \le e^{b_{\mathrm{short}}}$ for all i and all automorphisms σ of $\mathbb{C}$;*
(iv) *the families $|\gamma(\xi_\iota)|^{-2}$ and $Z(1)/Z(\xi_\iota)$ have heights at most b_{den} and b_{vol} respectively.*

Then

$$c + 2\delta + n'(b_{\mathrm{short}} + b_{\mathrm{den}}) + b_{\mathrm{vol}} \ge 0.$$

Proof. Write $w = (y,z)$ and $\mathrm{Gram}(\xi) = \mathrm{Gram}(\Theta_\xi w)$. By (7.1), for every ι,

$$\mathrm{Gram}(1) = |\gamma(\xi_\iota)|^{2n'} \mathrm{Gram}(\xi_\iota) \cdot \big(|\gamma(\xi_\iota)|^{-2}\big)^{n'} \cdot \frac{Z(1)}{Z(\xi_\iota)}.$$

By Corollary 7.13, (iv) and Lemma 7.3(b), the constant family $\mathrm{Gram}(1)$ has height at most $2\delta + n' b_{\mathrm{den}} + b_{\mathrm{vol}}$. It is nonzero, since Θ_1 is injective and w is a basis. By (i), (iii) and Lemma 7.10, the family w has covolume at most $c + n' b_{\mathrm{short}}$, so $|\sigma \,\mathrm{Gram}(1)| \le e^{c + n' b_{\mathrm{short}}}$ for every σ by Lemma 7.11. Proposition 7.4 gives $-(2\delta + n' b_{\mathrm{den}} + b_{\mathrm{vol}}) \le c + n' b_{\mathrm{short}}$. □

In the application (Theorem C) the space V is the space of target forms, Θ_ξ is a chain of m theta lifts twisted by ξ, the family y consists of the images of short integral source forms, the z_i are auxiliary forms completing it to a basis, and ξ_ι is a twist chosen for each prime so that the height of $Z(1)/Z(\xi_\iota)$ is at most about $-d_T\, h_{\mathrm{main}}$.

8. The parameters and the data

The argument works at one length x of the prime sum at a time. For each large integer x it fixes a family of objects, the *data*: three scales $x \le x_1 \le b$, ranks, a CM field E, two sets of infinite places of E with weights, and three primes. This section defines them, records their sizes and proves that they exist for every large x.

8.1. The parameters

Definition 8.1. The *parameters* are an integer q_0 divisible by $18480 = 2^4 \cdot 3 \cdot 5 \cdot 7 \cdot 11$ and a prime d with $d \equiv q_0/2 - 1 \pmod{q_0}$. We put $K = \mathbb{Q}(\sqrt{-d}) \subset \mathbb{C}$ and let $\chi_K = (\cdot/d)$.

The congruence says that $q_0 \mid 2(d+1)$ and $q_0 \nmid d+1$, that is, $d+1$ is an odd multiple of $q_0/2$. The parameters are fixed throughout; constants that depend on them only are called *constants*. Since $16 \mid q_0$, we have $8 \mid d+1$, so $d \equiv 7 \pmod 8$; in particular $d \equiv 3 \pmod 4$ and χ_K is the character of K. Moreover $2(d+1) \ge q_0 \ge 18480$, so $d > 64$.

For $s \ge 1$ let $l_0(s) = 3^{\lfloor \log_3 s \rfloor + 4}$, the least power of 3 exceeding $27s$; then $27s < l_0(s) \le 81s$. Put

$$T(s) = 3\, l_0(s)\, 2^s, \qquad q_1 = q_1(s) = q_0\, T(s), \qquad q_2 = q_2(s) = (q_0 d)^{s+6}.$$

Lemma 8.2. *Let p be a prime dividing q_1. Then p divides $(d+1)/4$ and $\chi_K(p) = 1$; that is, p splits in K. Consequently every prime p with $\chi_K(p) \ne 1$ satisfies $p \ge 13$. Moreover q_1 is prime to d, and $q_1 d \mid q_2$.*

Proof. The primes dividing $T(s)$ are 2 and 3, which divide q_0, so the primes dividing q_1 are those dividing q_0. Since $d+1$ is an odd multiple of $q_0/2$ and $8 \mid q_0/2$, every prime dividing q_0 divides $(d+1)/4$ (for $p = 2$ because $16 \mid q_0$). For $p = 2$, $\chi_K(2) = 1$ since $d \equiv 7 \pmod 8$. For odd p, quadratic reciprocity and $d \equiv 3 \pmod 4$ give $\left(\frac{p}{d}\right) = \left(\frac{-d}{p}\right)$, and $-d \equiv 1 \pmod p$. The primes below 13 divide 18480, hence q_1, and split. If $d \mid q_1$ then $\chi_K(d) = 1$, which is false. □

8.2. The field E and its infinite places

Fix $s \ge 1$. Let $E = K(\zeta)$, where ζ is a primitive root of unity of order q_1.

Proposition 8.3. (a) *The cyclotomic polynomial of order q_1 is irreducible over K; thus $\mathrm{Gal}(E/K) \cong (\mathbb{Z}/q_1\mathbb{Z})^\times$, $a \mapsto (\zeta \mapsto \zeta^a)$. The field E embeds into $\mathbb{Q}(\zeta_{q_1 d})$, so it is abelian over $\mathbb{Q}$, and it is a CM field.*

(b) *The infinite places of E form a group Φ isomorphic to $\mathrm{Gal}(E/K) \cong (\mathbb{Z}/q_1\mathbb{Z})^\times$: fix an embedding $\iota_0 \colon E \to \mathbb{C}$ over K; the place attached to $g \in \mathrm{Gal}(E/K)$ is that of $\iota_0 \circ g$. The embeddings $\Phi_v = \iota_0 \circ g_v$, one for each place v, form a CM type. For an integer ν prime to q_1 we write $\nu \in \Phi$ for the place of the class of ν; thus the automorphism attached to ν maps ζ to ζ^ν.*

(c) *E has $f = T(s)\varphi(q_0)$ infinite places, and exactly $T(s)$ of them have class 1 modulo q_0.*

(d) *Every root of unity of E is a power of ζ.*

(e) *For* $y \in E$, $\prod_{v\in\Phi} \Phi_v(y) = \mathrm{N}_{E/K}(y)$.

Proof. (a) Since q_1 is prime to d, the field $\mathbb{Q}(\zeta_d, \zeta_{q_1})$ has degree $\varphi(d)\varphi(q_1)$ over $\mathbb{Q}$, so the cyclotomic polynomial of order q_1 is irreducible over $\mathbb{Q}(\zeta_d)$, which contains K (by the Gauss sum $\sqrt{-d} = \sum_{a \bmod d} \chi_K(a)\zeta_d^a$, as $d \equiv 3 \pmod 4$). Irreducibility over K follows. The element $\zeta_{q_1 d}^{d}$ is a primitive q_1-th root of unity and $\zeta_{q_1 d}^{q_1}$ a primitive d-th root of unity, which gives the embedding. A totally imaginary quadratic extension of the totally real subfield of an abelian field is CM. (b) is the transport of the group structure. (c) The number of places is $[E : K] = \varphi(q_1)$, and $\varphi(q_1) = T(s)\varphi(q_0)$ because every prime dividing $T(s)$ divides q_0; equivalently $\Phi_{q_1}(X) = \Phi_{q_0}(X^{T(s)})$ for the cyclotomic polynomials. The reduction $(\mathbb{Z}/q_1)^\times \to (\mathbb{Z}/q_0)^\times$ is onto, with kernel of order $T(s)$. (d) A root of unity $y \in E$ satisfies $y^{q_1 d} = 1$ through the embedding of (a). If $y^{q_1} \neq 1$, then y^{q_1} has order d, and E would contain $\mathbb{Q}(\zeta_d, \zeta_{q_1})$, of degree $\varphi(d)\varphi(q_1) > 2\varphi(q_1) = [E : \mathbb{Q}]$ as $d > 64$. (e) The norm is the product of the conjugates $g(y)$, $g \in \mathrm{Gal}(E/K)$, and $\iota_0(g_v(y)) = \Phi_v(y)$. □

We write $\Phi_a \subseteq \Phi$ for the set of places of class 1 modulo q_0, the *a-places*; by (c), $\#\Phi_a = T(s)$ and $f = \#\Phi_a \cdot \varphi(q_0)$. For $\nu \in \mathbb{Z}$ prime to q_1 and $\tau \in \Phi$ we write $-\nu\tau$ for the product in Φ of the place $-\nu$ and τ; for ν not prime to q_1 we let $-\nu\tau = \tau$ (the convention plays no role, since such ν will carry the weight zero).

Let γ be Euler's constant, $\mathrm{rd}_E = |\mathrm{disc}\, E|^{1/[E:\mathbb{Q}]}$ the root discriminant, and

$$\Omega = \exp\Big(\gamma - \mathrm{Re}\,\frac{L'}{L}(1, \chi_K)\Big)\frac{2\pi}{\mathrm{rd}_E}. \tag{8.1}$$

This number enters through the Chowla–Selberg formula for the periods of the CM elliptic curves used in Part II; the constant $\gamma - \mathrm{Re}\,\frac{L'}{L}(1, \chi_K)$ is $-1 - c_1$ by Section 4, with $c_1 = \mathrm{Re}\, G_1(1)$ formed with this d.

8.3. Sparse sets of places

The places of weight b will be chosen so that no prime ℓ links three of them by powers of ℓ. For a prime ℓ write $q_1^{(\ell)}$ for the part of q_1 prime to ℓ.

Definition 8.4. A set $S \subseteq \Phi$ is *sparse for the primes up to* L if there are no prime $\ell \le L$, integers $0 < i < j < l_0(s)$, signs $\epsilon, \epsilon' \in \{\pm 1\}$ and places $\sigma, \tau, \rho \in S$ (not necessarily distinct) with

$$\tau/\sigma \equiv \epsilon\, \ell^i \quad \text{and} \quad \rho/\sigma \equiv \epsilon'\, \ell^j \pmod{q_1^{(\ell)}}.$$

Lemma 8.5 (Behrend's sets). *Let* $n, H, L \ge 1$ *and let* o *be an integer with* $(LH)^n \le o$. *There is a set* S *of at least* $H^n/(nH^2)$ *integers in* $[0, o)$ *such that a congruence* $(\alpha+\alpha')e_2 \equiv \alpha e_1 + \alpha' e_3 \pmod o$ *with* $e_1, e_2, e_3 \in S$, *integers* $\alpha, \alpha' > 0$ *and* $\alpha + \alpha' \le L$ *forces* $e_1 = e_2 = e_3$.

Proof. The vectors of $\{0, \dots, H-1\}^n$ have squared lengths in $[0, nH^2)$, so some sphere $\sum_l v_l^2 = k$ contains at least $H^n/(nH^2)$ of them. Let S be the set of integers whose digits in base LH form such a vector v: $e = \sum_l v_l (LH)^l$. These lie in $[0, o)$. In a congruence as in the statement both sides, $(\alpha + \alpha')e_2$ and $\alpha e_1 + \alpha' e_3$, have

digits below LH in every position (their digit vectors are $(\alpha+\alpha')v$ and $\alpha u+\alpha' w$, with entries $< LH$) and lie in $[0, o)$; so the congruence is an equality of digit vectors, $(\alpha+\alpha')v = \alpha u+\alpha' w$. Then v is a proper convex combination of u and w, and all three have the same length; strict convexity of the sphere, in the form $\alpha\alpha'|u-w|^2 = (\alpha+\alpha')(\alpha|u|^2+\alpha'|w|^2) - |\alpha u+\alpha' w|^2 = 0$, gives $u = v = w$. □

Proposition 8.6. *Let $n \geq 1$ and $W \geq 1$ with $W \leq 2^s \leq W^n$, $(2l_0)^n \leq W$ and $3l_0 n 2^n \leq W^3$, where $l_0 = l_0(s)$, and let L satisfy $2l_0(L+1) < 2^s$. There is a set $S \subseteq \Phi$ of places of class -1 modulo q_0 with $T(s) \leq \#S \cdot W^6$ that is sparse for the primes up to L.*

Proof. Write $q_0 = 2^v u$ with u odd, so $v \geq 4$. The integer $g = 1+3l_0 q_0 = 1+2^v(3l_0 u)$ has order exactly 2^s modulo 2^{s+v}, which divides q_1. Choose $H = 2^k$ with $H^n \leq 2^s/W < (2H)^n$; then $H \leq W$ and $(2l_0 H)^n \leq 2^s$. Let S_0 be a set as in Lemma 8.5 with $L = 2l_0$ and $o = 2^s$, the order of g modulo 2^{s+v}, and let S consist of the places $-g^e$, $e \in S_0$. These are distinct modulo 2^{s+v}, and of class -1 modulo $3l_0 q_0$, hence modulo q_0. The count $\#S \geq H^n/(nH^2) \geq 2^s/(W \cdot 2^n n W^2)$ gives $T(s) = 3l_0 2^s \leq \#S \cdot W^6$ by the hypotheses.

Suppose $\sigma = -g^{e_1}$, $\tau = -g^{e_2}$, $\rho = -g^{e_3}$ violate sparseness for ℓ, i, j. If $\ell = 2$, then modulo $3^{\lfloor \log_3 s \rfloor + 5} = 3l_0$, which divides $q_1^{(2)}$, all places of S are $\equiv -1$, so $\tau/\sigma \equiv 1$, and squaring $\tau/\sigma \equiv \pm 2^i$ gives $4^i \equiv 1$; but $4 = 1+3$ has order $l_0 > i$ modulo $3l_0$. If ℓ is odd, squaring the two congruences modulo 2^{s+v}, which divides $q_1^{(\ell)}$, gives

$$(g^{e_2-e_1})^{2j} \equiv \ell^{2ij} \equiv (g^{e_3-e_1})^{2i} \pmod{2^{s+v}},$$

hence $(2a+2i)e_2 \equiv 2ae_1+2ie_3 \pmod{2^s}$ with $a = j-i > 0$, and $2a+2i = 2j \leq 2l_0$. By Lemma 8.5, $e_1 = e_2 = e_3$, so $\tau = \sigma$ and $\ell^{2i} \equiv 1 \pmod{2^{s+1}}$. By the lifting-the-exponent lemma at the prime 2, if ℓ is odd and $\ell^o \equiv 1 \pmod{2^{s+1}}$ with $o \geq 1$, then 2^s divides $(\ell-1)o$ or $(\ell+1)o$, and hence $2^s \leq (\ell+1)o$. With $o = 2i$ this gives $2^s \leq (\ell+1)\,2i \leq (L+1)\,2l_0$, against the hypothesis. □

8.4. The data

Definition 8.7. The *data* for the parameters consist of the following objects.

(D1) *Scales:* integers $x \leq x_1 \leq b$ such that

$$(\log b)^{100} \leq x, \quad x^{10^6} \leq x_1, \quad x_1^{10^6} \leq b, \quad (\log x_1)^6 \leq \log b, \quad \log\log b \geq 10^{12} C_0,$$

where C_0 is the constant of Remark 8.8; and $q_1 x_1 \mid b$.

(D2) *Ranks:* a power $m = 2^{e+1}$ of 2 with $m \in [\sqrt{x}, 2\sqrt{x}]$, a power r of 3 with $r \in [x^5, 3x^5]$, and the exponent $s = 22(e+1)$, so that $2^s = m^{22}$; we require $s \leq 32 \log x$. We put $k = r+1-m$, $t = 5r$, $n_j = k+j(t+1)$, $\Delta = 2r^8$ (the *gap*) and $a = xb$ (the *long order*).

(D3) *Places and weights:* the field E of Section 8.2 for this s, and a set $\Phi_b \subseteq \Phi$ of places of class -1 modulo q_0, the *b-places*, with $T(s) \leq \#\Phi_b \cdot \log b$, sparse for the primes up to kt. The *weight* of $\tau \in \Phi$ is $\kappa(\tau) = b$ if $\tau \in \Phi_b$ and $\kappa(\tau) = a/x_1$ otherwise. For $j \geq 0$ the *lift weights* are $\kappa_j(\tau) = \kappa(\tau) + (j+\frac{1}{2})\Delta$.

(D4) *Primes:* distinct primes $p_0, p_1 \equiv 1 \pmod{q_2^2}$ in $[b^{10}, b^{11}]$ with $M = p_0p_1 = \pi_M\bar{\pi}_M$ for some $\bar{\pi}_M \in O_K$; and a prime $N \equiv 1 \pmod{q_2^2}$, the *level*, with $N = \pi_N\bar{\pi}_N$ for some $\pi_N \in O_E$ and

$$\log N_- \le \log N \le \log N_- + \sqrt{\log b}, \qquad N_- = \bar{\kappa}^m \Big(\frac{M \operatorname{rd}_E}{2\pi}\Big)^{m-1} b^{-15/2},$$

where $\bar{\kappa}$ is the geometric mean of the weights $\kappa(\tau)$ over all f places.

The scales in (D1) are recorded with their logarithms; the inequalities between them are the only facts about the scales that the argument uses. The weight a/x_1 is an integer since $q_1x_1 \mid b$.

Remark 8.8. The constant C_0 is the sum of q_0, $2d+6$, $|\gamma - \operatorname{Re}\frac{L'}{L}(1,\chi_K)|$, the sum of $|\log|\Phi'_{q_0}(w)||$ over the complex roots w of the cyclotomic polynomial Φ_{q_0}, and a bound for the conjugates of the integral vacuum of Part II. It bounds every constant that the argument meets. In particular $C_0 \ge \max(64, q_0, d, \varphi(q_0))$.

Proposition 8.9 (Sizes). *For data as in Definition 8.7:*

quantity	*size*	*order in x*
s, l_0	$s \le 32\log x$, $27s < l_0 \le 81s$	0
$\log x_1$, $\log b$	$(\log x_1)^6 \le \log b$, $(\log b)^{100} \le x$	$\le 1/100$
m	$x \le m^2 \le 4x$	$1/2$
k, r, t	$k \le r \le 2k$, $t = 5r$	5
2^s, f, q_1	$x^{11} \le m^{22} = 2^s \le f \le q_1 \le x^{12}$	11
rd_E, Ω^{-1}	$x^{10} \le \operatorname{rd}_E \le x^{12}$, $1 \le \Omega^{-1} \le x^{13}$	≈ 11
Δ	$2r^8$	40
weights	$a/x_1 \le \kappa(\tau) \le b$; $x^{999} \le a/x_1 \le b/x^{999}$	
M	$b^{20} \le M \le b^{22}$	
N	$m\log b \le \log N \le 30m\log b$; $N \ge b^{101}$	

Moreover $\log h_E \le x^{30}$ *for the class number* h_E *of* E*; the ranks* m, k *are even and* r, t *odd;* $q_1d \mid N-1$ *and* $q_1d \mid p_i - 1$.

Proof. We go through the table; throughout, constants are at most C_0, and C_0 is negligible against $\log\log b$ by (D1).

Exponents and scales. $s \le 32\log x$ is required in (D2), and $27s < l_0 \le 81s$ holds by the choice of l_0 as the least power of 3 above $27s$. The inequalities between x, x_1 and b are those of (D1).

Ranks. $m \in [\sqrt{x}, 2\sqrt{x}]$ gives $x \le m^2 \le 4x$. As $m \ge 2$, $k = r+1-m \le r$; as $r \ge x^5 \ge 4\sqrt{x} \ge 2m$, $2k = 2r+2-2m \ge r$. The number $t = 5r$ is a definition. The ranks are even or odd as stated: $m = 2^{e+1}$ is even, r is a power of 3 and odd, $k = (r+1) - m$ is even and $t = 5r$ is odd.

The field. By (D2), $2^s = m^{22}$, and $m^{22} \ge (\sqrt{x})^{22} = x^{11}$. By Proposition 8.3(c), $f = T(s)\varphi(q_0) \ge T(s) = 3l_02^s \ge 2^s$, and $f = \varphi(q_1) \le q_1$. Finally $q_1 = 3q_0l_02^s \le 3q_0 \cdot 81s \cdot 2^{22}x^{11} \le x^{12}$, because $q_0 \le C_0$ and $s \le 32\log x$. The different of E is generated by δ_E with $|\delta_E|_v = A_vT(s)\sqrt{d}$ at every infinite place, $|\log A_v| \le C_0$ (Remark 10.3), and $\operatorname{rd}_E^f = \prod_v |\delta_E|_v$ (Proposition 10.2); hence $|\log \operatorname{rd}_E - \log(T(s)\sqrt{d})| \le C_0$, and $x^{11} \le T(s) \le x^{11.5}$ gives $x^{10} \le \operatorname{rd}_E \le x^{12}$. By (8.1), $\Omega^{-1} =$

$\mathrm{rd}_E \exp(-\gamma + \operatorname{Re}\frac{L'}{L}(1,\chi_K))/(2\pi)$, and the exponent is at most C_0 in absolute value; so $1 \le \Omega^{-1} \le x^{13}$.

Class number. By Minkowski's bound and a count of the ideals of bounded norm, $h_E \le (2^n|\mathrm{disc}\,E|)^{n+1}$ with $n = [E:\mathbb{Q}] = 2f$, and $|\mathrm{disc}\,E| = \mathrm{rd}_E^{2f}$; with $f \le x^{12}$ and $\log \mathrm{rd}_E \le 12\log x$ this gives $\log h_E \le (2f+1)\,2f(\log 2 + 12\log x) \le x^{30}$.

Gap and weights. $\Delta = 2r^8$ is a definition, and $r^8 \in [x^{40}, 3^8x^{40}]$. The weights are b and $a/x_1 = xb/x_1$; as $x \le x_1$, $a/x_1 \le b$. In logarithms, $\log(a/x_1) = \log x + \log b - \log x_1 \ge (1-10^{-6})\log b \ge 999\log x$, and $\log b - 999\log x - \log(a/x_1) = \log x_1 - 1000\log x \ge 0$, as $x^{10^6} \le x_1$; so $x^{999} \le a/x_1 \le b/x^{999}$.

Primes. The primes p_0, p_1 lie in $[b^{10}, b^{11}]$ by (D4), so $b^{20} \le M \le b^{22}$. Since $p_i \equiv 1$ and $N \equiv 1$ modulo q_2^2, and $q_1 d \mid q_2$ (Lemma 8.2), $q_1 d$ divides $p_i - 1$ and $N-1$.

The level. By definition $\log N_- = m\log\bar\kappa + (m-1)\log(M\mathrm{rd}_E/2\pi) - \frac{15}{2}\log b$. Here $\log\bar\kappa \in [0, \log b]$, as every weight lies in $[1,b]$; and $\log(M\mathrm{rd}_E/2\pi) \in [19\log b, 23\log b]$ by the bounds for M and rd_E. Hence $\log N_- \ge 19(m-1)\log b - \frac{15}{2}\log b \ge m\log b$ for $m \ge 2$, and $\log N_- \le m\log b + 23(m-1)\log b \le 24m\log b$. By (D4), $\log N \in [\log N_-, \log N_- + \sqrt{\log b}]$, and $\sqrt{\log b} \le 6m\log b$; so $m\log b \le \log N \le 30m\log b$. As $m \ge \sqrt{x} \ge 101$, $N \ge b^{101}$. □

Remark 8.10 (*Comparison of sizes*). Every quantity F attached to the data that occurs in an inequality below has bounds $e^l \le F \le e^{l'}$ in which l and l' are linear combinations of $\log x$, $\log x_1$, $\log b$, $\log\log b$, $\log\log x_1$ and C_0; the bounds of products are the sums of the bounds, and those of sums are the maxima (plus 1 for upper bounds). An inequality $F \le G$ is then proved by checking that the upper exponent of F is below the lower exponent of G, a linear inequality in these six quantities that follows from (D1). In the Lean development this comparison is automatic; in the text we state the resulting inequalities and leave the arithmetic to the reader, indicating the exponents where they are not evident.

8.5. Existence of the data

The primes in (D4) come from short windows in which the logarithm of a prime $\equiv 1$ modulo a large modulus is guaranteed. The following theorem, a consequence of the Siegel–Walfisz theorem [MV07, Cor. 11.19], is proved in Appendix B.4 from the zero-free region by Newman's contour argument.

Theorem 8.11. *Let M_0 be a positive multiple of 4 and $A \ge 1$. For all large W the following holds: for every $k > 1$ with $M_0^k \le W^A$ and every $U \in [W, W^A]$ there is a prime $p \equiv 1 \pmod{M_0^k}$ with $\log p \in [U, U+W]$.*

Lemma 8.12. *Let $W > 0$ and L with $(2h_K+2)W \le L$, where h_K is the class number of K. Suppose that every window $[U, U+W]$ with $10L \le U \le 11L$ contains $\log p$ for a prime $p \equiv 1 \pmod{q_2^2}$. Then there are distinct such primes p_0, p_1 in $[e^{10L}, e^{11L}]$ and $\bar\pi_M \in \mathcal{O}_K$ with $\pi_M\bar\pi_M = p_0p_1$.*

Proof. Take one prime p_i from each of the h_K+1 disjoint windows starting at $10L + 2iW$, $0 \le i \le h_K$. Each $p_i \equiv 1 \pmod{q_1 d}$ splits in K (by Lemma 8.2

$\chi_K(p_i) = \chi_K(1) = 1$, as χ_K has conductor d), say $p_i\mathcal{O}_K = I_iJ_i$ with $J_i = \bar{I}_i$. Among the $h_K + 1$ classes of the I_i two coincide, $[I_i] = [I_j]$ with $i \neq j$; then $I_iJ_j = I_i\bar{I}_j$ is principal, since $[J_j] = [I_j]^{-1}$. A generator $\bar{\pi}_M$ has norm p_ip_j. □

Lemma 8.13. *Let $b = e^L$ with $L \geq 7$ and $W^2 \leq L$; let $m \geq 2$, let weights $\kappa(\tau) \in [1, b]$ be given, let $M \in [b^{20}, b^{22}]$, and let $q_1 d \leq b$. Suppose that every window $[U, U + W]$ with $mL \leq U \leq 24mL$ contains $\log N$ for a prime $N \equiv 1 \pmod{q_2^2}$. Then there are such a prime N with $\log N_- \leq \log N \leq \log N_- + \sqrt{L}$ and $\pi_N \in \mathcal{O}_E$ with $\pi_N\bar{\pi}_N = N$.*

Proof. By the proof of Proposition 8.9, $\log N_- \in [mL, 24mL]$, and $W \leq \sqrt{L}$. A prime $N \equiv 1 \pmod 4$ is a sum of two squares $u^2 + v^2$ by Fermat's theorem, and E contains $i = \zeta^{q_1/4}$, so $\pi_N = u + vi$ serves. □

Theorem 8.14 (Existence of the data). *Let $0 < \varepsilon < 1/100$. For every sufficiently large integer x there are data with this x, with $x_1 = \lfloor \exp(x^{\varepsilon/6}) \rfloor$, with b the least multiple of q_1x_1 exceeding $\exp(x^\varepsilon)$, with m the least power of 2 with $m \geq \sqrt{x}$ and r the least power of 3 with $r \geq x^5$. For them $x^{\varepsilon/6} - 1 \leq \log x_1$ and $x^\varepsilon \leq \log b \leq x^\varepsilon + 1$.*

Proof. Put $y = \log x$ and $W = x^{\varepsilon/6}$. Fix an integer $n \geq 1$ with $99q_0d \leq n\varepsilon/6$; then $q_2^2 \leq W^n$ whenever $s \leq 32y$. For large x the following hold: $100(6\log W + 1) \leq y$; $\log W \geq 10^{12}C_0$; $W \geq 2 \cdot 10^6 y$; $W \geq 2h_K + 2$; $n + 1 \leq W$; $(5184y)^n \leq W$; and $7776\, y\, n\, 2^n \leq W$. These are statements that a fixed power of y is small against $W = e^{\varepsilon y/6}$.

Ranks. Let $m = 2^{e+1}$ and r be the least powers of 2 and 3 with $m \geq \sqrt{x}$, $r \geq x^5$; then $m \leq 2\sqrt{x}$, $r \leq 3x^5$, and $s = 22(e + 1) \leq 32y$. Hence $W \leq \sqrt{x} \leq m \leq 2^s \leq W^n$.

Places. Since $l_0 \leq 81s \leq 2592y$, we have $(2l_0)^n \leq W$, $3l_0n2^n \leq W^3$ and $2l_0(kt + 1) < 2^s = m^{22}$, as $kt \leq 15x^{10}$ and $m^{22} \geq x^{11}$. Proposition 8.6 with $L = kt$ gives a sparse set Φ_b of places of class -1 with $T(s) \leq \#\Phi_b \cdot W^6$.

Scales. Let $x_1 = \lfloor e^W \rfloor$ and b the least multiple of q_1x_1 above $\exp(W^6)$. Since $q_1 \leq W^n$, $\log b \in [W^6, W^6 + 1]$ and $\log\log b \in [6\log W, 6\log W + 1]$. Then $\log b \geq W^6$ gives $T(s) \leq \#\Phi_b \log b$; and the inequalities of (D1) reduce to those listed above, for instance $(\log b)^{100} \leq e^{100(6\log W+1)} \leq x$ and $x^{10^6} \leq x_1$ because $10^6 y \leq W - 1$.

Primes. Let $L = \log b$. Then $W^2 \leq L$ and $24mL \leq W^{n+7}$, and the windows $[U, U + W]$ with $W \leq U \leq W^{n+7}$ contain logarithms of primes $\equiv 1 \pmod{(q_0d)^k}$ for every $k > 1$ with $(q_0d)^k \leq W^{n+7}$, by Theorem 8.11 (with $M_0 = q_0d$, a multiple of 4, and $A = n + 7$), for large x. Since $q_2^2 = (q_0d)^{2(s+6)} \leq W^n$, Lemma 8.12 with $(2h_K + 2)W \leq W^2 \leq L$ gives $p_0, p_1, \bar{\pi}_M$, and Lemma 8.13 gives N and π_N. □

9. Prime sums, the three theorems and the main inequality

This section states the three theorems on which the argument rests, Theorems A, B and C, derives from them the main inequality Theorem 9.4, and deduces the main theorems of the introduction. The three theorems are proved in Parts III to VI. We fix parameters as in Definition 8.1 and data as in Definition 8.7.

9.1. Prime sums

Let $\Lambda_K = \lambda_1$ be the weight of Definition 4.1 for $q = 1$, that is $\Lambda_K(n) = \Lambda(n)(1 - \chi_K(n))$. For real y and $\tau \in \Phi$ put

$$s_{\mathrm{fin}}(y, \tau) = \sum_{1 \le \nu \le a} \Lambda_K(\nu)\Big(\frac{y}{\nu} - \kappa(-\nu\tau)\Big)_+, \tag{9.1}$$

the *prime sum* at the order y and the place τ, and for an integer $j \ge 0$

$$s_\infty(j) = \log j! + j \log \frac{M}{\Omega}, \tag{9.2}$$

the *size* charged to a derivative of order j. The *direction amount* and the *main amount* are

$$h_{\mathrm{dir}} = \sum_{\tau \in \Phi} \big(s_\infty(\kappa(\tau)) - s_{\mathrm{fin}}(\kappa(\tau), \tau)\big), \qquad h_{\mathrm{main}} = mk\, h_{\mathrm{dir}}.$$

The *expected value* of the prime sums is $s_{\mathrm{main}} = a\big(\log(a/\bar\kappa) + c_1\big)$, where $\log \bar\kappa$ is the mean of $\log \kappa(\tau)$ over all f places and $c_1 = \mathrm{Re}\, G_1(1)$, and the *total remainder* is

$$\mathrm{rem} = \sum_{\tau \in \Phi_a} \big(s_{\mathrm{fin}}(a, \tau) - s_{\mathrm{main}}\big).$$

Finally, the *unit of error* is $\epsilon_{\mathrm{u}} = f\, a \log^3 b$.

The prime sum $s_{\mathrm{fin}}(y, \tau)$ will measure a divisibility, guaranteed by Part II, of a derivative of order y of a theta function at the place τ; s_∞ is the archimedean size of such a derivative. In the direction τ the difference $s_\infty - s_{\mathrm{fin}}$ is the cost of one derivative that is not repaid by divisibility.

Lemma 9.1. *For each $n \ge 1$,*

$$\sum_{\tau \in \Phi_a} \Lambda_K(n)\Big(\frac{a}{n} - \kappa(-n\tau)\Big)_+ = \#\Phi_a\, \Lambda_K(n)\Big(\frac{a}{n} - \frac{a}{x_1}\Big)_+ + \#\Phi_b\, \lambda_{q_0}(n)\Big[\Big(\frac{a}{n} - \frac{a}{x}\Big)_+ - \Big(\frac{a}{n} - \frac{a}{x_1}\Big)_+\Big].$$

Proof. If n is not prime to q_1, then $\Lambda_K(n) = 0$ unless n is a power of a prime $p \mid q_1$; such p split in K (Lemma 8.2), so $\Lambda_K(n) = 0$ in all cases, and both sides vanish. Let n be prime to q_1. Every τ gives the term with weight a/x_1, except those with $-n\tau \in \Phi_b$, which have weight $b = a/x$. Since Φ_b consists of places of class -1 and $\tau \in \Phi_a$ has class 1 modulo q_0, the place $-n\tau$ has class $-n$, so $-n\tau \in \Phi_b$ is possible only if $n \equiv 1 \pmod{q_0}$; and then $\tau \mapsto -n\tau$ maps Φ_a bijectively onto the places of class -1, so exactly $\#\Phi_b$ places $\tau \in \Phi_a$ have $-n\tau \in \Phi_b$. For $n \equiv 1 \pmod{q_0}$ we have $\Lambda_K(n) = \lambda_{q_0}(n)$. □

Proposition 9.2. (a) *The sum of the prime sums over the a-places is*

$$\sum_{\tau \in \Phi_a} s_{\mathrm{fin}}(a, \tau) = a\big(\#\Phi_b(R_{q_0}(x) - R_{q_0}(x_1)) + \#\Phi_a R_1(x_1)\big) + a\Big(\#\Phi_a(\log x_1 + c_1) + \#\Phi_b \frac{\log x - \log x_1}{\varphi(q_0)}\Big).$$

(b) $\#\Phi_a\, s_{\mathrm{main}} = a\Big(\#\Phi_a(\log x_1 + c_1) + \#\Phi_b \dfrac{\log x - \log x_1}{\varphi(q_0)}\Big)$.

(c) $\mathrm{rem} = a\big(\#\Phi_b(R_{q_0}(x) - R_{q_0}(x_1)) + \#\Phi_a R_1(x_1)\big)$.

Proof. (a) Sum Lemma 9.1 over $n \le a$ and apply (4.5) with $A = a$: for $0 < z \le a$,

$$\sum_{n\le a} \lambda_q(n)\Big(\frac{a}{n} - \frac{a}{z}\Big)_+ = a\Big(R_q(z) + \frac{\log z}{\varphi(q)} + c_q\Big),$$

with $q = 1$, $z = x_1$ for the first term and $q = q_0$, $z = x, x_1$ for the bracket; the constants c_{q_0} cancel, and $\varphi(1) = 1$. (b) Since $\kappa = a/x$ on Φ_b and a/x_1 elsewhere, $\sum_\tau \log\kappa(\tau) = f(\log a - \log x_1) - \#\Phi_b(\log x - \log x_1)$; hence $\log a - \log\bar\kappa = \log x_1 + \#\Phi_b(\log x - \log x_1)/f$, and $f = \#\Phi_a\varphi(q_0)$. (c) is (a) minus (b). □

Lemma 9.3. *Let $0 \le y \le a$ and $\tau \in \Phi$.*

(a) *Only the $n \le x_1$ contribute to $s_{\mathrm{fin}}(y,\tau)$, and $0 \le s_{\mathrm{fin}}(y,\tau) \le 4y\log x_1 \le y\log b$.*
(b) *For an integer $y \le a$, $20y\log b \le s_\infty(y) \le 25y\log b$.*
(c) $\kappa(\tau)\log b \le s_\infty(\kappa(\tau)) - s_{\mathrm{fin}}(\kappa(\tau),\tau) \le 25\kappa(\tau)\log b$.
(d) *$b\,\#\Phi_a \le h_{\mathrm{dir}} \le 25 f b\log b$, and $h_{\mathrm{main}} \ge x^4 f a$; in particular $h_{\mathrm{main}} \ge 0$.*

Proof. (a) Since $\kappa \ge a/x_1 \ge y/x_1$, the term of $n > x_1$ vanishes, and each term is at most $2y\Lambda(n)/n$; by (4.2) the sum is at most $4y\log x_1$, and $4\log x_1 \le \log b$. (b) $\log y! \le y\log y \le y\log a \le 2y\log b$, and $20\log b \le \log(M/\Omega) \le 23\log b$ by Proposition 8.9. (c) follows from (a) and (b). (d) Sum (c) over τ. For the lower bound keep only the b-places: they contribute at least $\#\Phi_b \cdot b\log b \ge b\,\#\Phi_a$, by (D3). For the upper bound all weights are at most b. Finally $h_{\mathrm{main}} = mk\,h_{\mathrm{dir}} \ge mk\,b\,f/\varphi(q_0) = mk\,a\,f/(x\varphi(q_0))$, and $mk \ge \sqrt{x}\cdot x^5/2 \ge 2C_0x^5 \ge 2\varphi(q_0)x^5$. □

9.2. The three theorems

The objects in the following statements are constructed in Part III; we describe them briefly and give exact references there.

- The *target* $\mathcal{T}$ is a finite-dimensional complex inner product space of automorphic forms on a definite unitary group $\mathrm{U}(Y_0)$ over the totally real subfield of E; its dimension is $d_T \ge 1$.
- The *twists* are certain anticyclotomic Hecke characters ξ of E (the m-fold twists), with modulus of norm $\mathrm{N}\,Q(\xi)$; the trivial twist is 1. For each twist, Θ_ξ is a chain of m theta lifts from $\mathcal{T}$ to forms on $\mathrm{U}(Y_m)$, twisted by ξ. With the integral structures of algebraic modular forms and ξ allowed at ℓ when $\ell \nmid \mathrm{N}\,Q(\xi)$, this makes $\mathcal{T}$ a twisted lattice $L_{\mathcal{T}}$ in the sense of Definition 7.7.
- The *source* $\mathcal{S}$ is a space of ramified algebraic modular forms on a definite unitary group $\mathrm{U}(X_0)$; $\mathcal{S}^{\mathrm{int}} \subseteq \mathcal{S}$ is the set of forms integral at every prime place.
- $\bar G = \prod_w \mathrm{GL}_r(\mathbb{F}_w)$ is a finite group, the product over the chosen places w of E above p_0 and p_1, acting on forms on $\mathrm{U}(Y_0)$; for $g \in \bar G$, the map $\mathfrak{p}_g\colon \mathcal{S} \to \mathcal{T}$ is the translate by g of a normalised, projected Poincaré map $\mathfrak{p}$.

Theorem A (Short forms). *The following hold.*

(a) *The space $\mathcal{S}$ is spanned by the forms $\varphi \in \mathcal{S}^{\mathrm{int}}$ such that*

$$\|\Theta_1\mathfrak{p}_g\varphi\|_\sigma^2 \le \exp\big(h_{\mathrm{main}} - m\,\mathrm{rem} + f a\log^4 b\big)$$

for all $g \in \bar G$ and all automorphisms σ of $\mathbb{C}$.

(b) *For every finite set $\Sigma \subseteq \bar{G}$, every finite family of target forms each lying in $\mathfrak{p}_g(\mathcal{S}^{\mathrm{int}})$ for some $g \in \Sigma$ has index at most $\#\Sigma \cdot \dim \mathcal{S} \cdot 2\epsilon_{\mathrm{u}}$ in $L_{\mathcal{T}}$.*

Theorem B (Enough forms). *There is a set $\Sigma \subseteq \bar{G}$ with*

$$\#\Sigma \cdot \dim \mathcal{S} \le 5\, d_T \log(1+2b^2) \quad \textit{and} \quad \dim \sum_{g\in\Sigma} \mathfrak{p}_g(\mathcal{S}) \ge \Big(1 - \frac{4}{b^2}\Big) d_T.$$

Theorem C (Integral forms are long). *Let $y_1, \dots, y_n \in \mathcal{T}$ be linearly independent with $n \ge (1 - 4/b^2)d_T$. If y has index at most δ and covolume at most c in $L_{\mathcal{T}}$, then*

$$c + 2\delta \ge d_T \big(h_{\mathrm{main}} - 3 f a \log^4 b\big).$$

Theorem A(a) is an upper bound for archimedean sizes in which the main amount appears with the remainder subtracted; Theorem C is a lower bound in which the main amount appears alone. The divisibility that produces the term $-m$ rem comes from Part II; the lower bound rests on Theorem 7.14.

9.3. The main inequality

Theorem 9.4 (Main inequality). $m \operatorname{rem} \le f a \log^5 b$.

Proof. By Theorem B there is $\Sigma \subseteq \bar{G}$ with $\#\Sigma \dim \mathcal{S} \le 5 d_T \log(1+2b^2)$ whose images span at least $(1 - 4/b^2)d_T$ dimensions. Put

$$\beta = h_{\mathrm{main}} - m \operatorname{rem} + \ell_0, \qquad \ell_0 = f a \log^4 b,$$

and let $\mathcal{S}_{\mathrm{short}}$ be the set of $\varphi \in \mathcal{S}^{\mathrm{int}}$ with $\|\Theta_1 \mathfrak{p}_g \varphi\|_\sigma^2 \le e^\beta$ for all g, σ. By Theorem A(a) these span $\mathcal{S}$, so the sets $\mathfrak{p}_g(\mathcal{S}_{\mathrm{short}})$, $g \in \Sigma$, span $\sum_{g\in\Sigma} \mathfrak{p}_g(\mathcal{S})$. Choose among them a basis $y_1, \dots, y_n$ of their span; thus $(1 - 4/b^2)d_T \le n \le d_T$.

Covolume. Each y_i has $\|\Theta_1 y_i\|_\sigma^2 \le e^\beta$, so y has covolume at most $n\beta$ by Lemma 7.10.

Index. By Theorem A(b) and the bound for $\#\Sigma$, y has index at most $\#\Sigma \dim \mathcal{S} \cdot 2\epsilon_{\mathrm{u}} \le d_T \cdot \iota_0$ with $\iota_0 = 10 \log(1+2b^2)\epsilon_{\mathrm{u}}$.

Comparison. Theorem C gives

$$d_T(h_{\mathrm{main}} - 3\ell_0) \le n\beta + 2 d_T \iota_0.$$

Since $h_{\mathrm{main}} + \ell_0 \ge 0$ (Lemma 9.3(d)) and $n \le d_T$, we have $n(h_{\mathrm{main}} + \ell_0) \le d_T(h_{\mathrm{main}} + \ell_0)$. Subtracting,

$$n\, m \operatorname{rem} \le d_T\, (4\ell_0 + 2\iota_0).$$

Since $4/b^2 \le 1/2$, $d_T \le 2n$; dividing by $n > 0$, $m \operatorname{rem} \le 2(4\ell_0 + 2\iota_0) = 8 f a \log^4 b + 40 \log(1+2b^2) f a \log^3 b \le f a \log^5 b$, as $\log b$ is large. □

The main amount, of size at least $x^4 f a$, cancels between the two bounds, and only the errors remain; they are smaller than the main amount by a factor $x^4/\log^5 b$. The table summarises the errors per dimension of the target.

source	where	amount
size	Theorem A(a)	$f a \log^4 b$
index (counted twice)	Theorems A(b), B	$20 \log(1+2b^2) \cdot f a \log^3 b$
covolume	Theorem C	$3 f a \log^4 b$

9.4. The one-sided bound

Proposition 9.5. $R_{q_0}(x) \le \varphi(q_0) \log^6 b/m + |R_{q_0}(x_1)| + \log b \cdot |R_1(x_1)|$.

Proof. By Proposition 9.2(c),

$$R_{q_0}(x) = \frac{\mathrm{rem}}{a \# \Phi_b} + R_{q_0}(x_1) - \frac{\# \Phi_a}{\# \Phi_b} R_1(x_1).$$

By Theorem 9.4, $\mathrm{rem}/(a\#\Phi_b) \le f a \log^5 b/(ma\#\Phi_b) = \varphi(q_0) \log^5 b \cdot \#\Phi_a/(m\#\Phi_b)$, and $\#\Phi_a = T(s) \le \#\Phi_b \log b$ by (D3). □

Theorem 9.6 (One-sided bound). *Let q_0 be a positive multiple of* 18480, *let $d \equiv q_0/2 - 1 \pmod{q_0}$ be a prime, and let $\varepsilon > 0$. Then $R_{q_0}(y) \le y^{-1/2+\varepsilon}$ for all large real y, where R_{q_0} is formed with $\chi_K = (\cdot/d)$.*

Proof. It suffices to prove the bound with 7δ in place of ε for $0 < \delta < 1/100$. Let x be a large integer and take the data of Theorem 8.14 with $\varepsilon = \delta$; then $\log b \le 2x^{\delta}$, $m \ge \sqrt{x}$ and $\log x_1 \ge x^{\delta/6} - 1 \ge x^{\delta/12}$. By Theorem 6.6 with an integer $B \ge 24/\delta$, for large x both $|R_{q_0}(x_1)|$ and $|R_1(x_1)|$ are at most $(\log x_1)^{-B} \le x^{-2}$; the primes d here are odd and prime to q_0. Proposition 9.5 gives

$$R_{q_0}(x) \le \varphi(q_0) \frac{(2x^{\delta})^6}{\sqrt{x}} + x^{-2} + 2x^{\delta} x^{-2} \le (64\varphi(q_0) + 3)\, x^{-1/2+6\delta}.$$

For real $y \ge 2$, Corollary 6.2 gives $R_{q_0}(y) \le R_{q_0}(\lfloor y \rfloor) + 26/y$, and $\lfloor y \rfloor^{-1/2+6\delta} \le 2y^{-1/2+6\delta}$. Hence $R_{q_0}(y) \le (128\varphi(q_0) + 32)\, y^{-1/2+6\delta} \le y^{-1/2+7\delta}$ for large y. □

9.5. Proof of the main theorems

Theorem 9.7. *For every integer $N \ge 1$, every Dirichlet character χ modulo N and every $s \in \mathbb{C}$ with $\mathrm{Re}\, s > 1/2$, one has $L(s,\chi) \ne 0$, unless χ is principal and $s = 1$.*

Proof. For $s = 1$ and χ not principal this is Dirichlet's theorem $L(1,\chi) \ne 0$; more generally $L(s,\chi) \ne 0$ for $\mathrm{Re}\, s \ge 1$, $s \ne 1$. Let $\rho \ne 1$ with $\mathrm{Re}\, \rho > 1/2$, and choose β with $1/2 < \beta < \mathrm{Re}\, \rho$. Put $q_0 = 18480N$ and $d_0 = 9240N - 1 = q_0/2 - 1$; then $d_0 \equiv 7 \pmod 8$ is odd and prime to $q_0 = 2(d_0+1)$. Every prime $d \equiv d_0 \pmod{q_0}$ satisfies $d \equiv q_0/2 - 1 \pmod{q_0}$, so Theorem 9.6, with $\varepsilon = \beta - 1/2$, gives $R_{q_0}(x) \le x^{\beta-1}$ for all large x. By Theorem 5.7, $L(\rho,\psi) \ne 0$ for every character ψ modulo q_0; in particular for the character ψ modulo q_0 induced by χ. Since $L(s,\psi) = L(s,\chi) \prod_{p \mid q_0,\, p \nmid N} (1 - \chi(p) p^{-s})$, also $L(\rho,\chi) \ne 0$. □

Theorem 9.8.
(a) *Every zero s of a Dirichlet L-function $L(s,\chi)$ with $0 < \mathrm{Re}\, s < 1$ has $\mathrm{Re}\, s = 1/2$.*
(b) *Every zero $s \ne 1$ of the Riemann zeta function other than $-2, -4, -6, \ldots$ has $\mathrm{Re}\, s = 1/2$.*

Proof. (a) Let $L(s, \chi) = 0$ with $0 < \operatorname{Re} s < 1$. By Theorem 9.7, $\operatorname{Re} s \leq 1/2$. The functional equation, extended from primitive to all characters by comparing Euler factors, shows that s is a pole of the gamma factor of χ or that $L(1-s, \bar{\chi}) = 0$; see Appendix B.7. The gamma factor has no poles in $\operatorname{Re} s > 0$. Hence $L(1-s, \bar{\chi}) = 0$, and Theorem 9.7 gives $\operatorname{Re}(1-s) \leq 1/2$.

(b) Let $\zeta(s) = 0$ with $s \neq 1$ and $s \notin \{-2, -4, \dots\}$. Then $s \neq 0$, since $\zeta(0) = -1/2$, and s is not a pole of $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$. Hence s is a zero of the completed zeta function $\Lambda(s) = \Gamma_{\mathbb{R}}(s)\zeta(s)$, and so is $1-s$, by $\Lambda(1-s) = \Lambda(s)$; thus $\zeta(1-s) = 0$. Both s and $1-s$ are zeros of the L-function of the character modulo 1, so Theorem 9.7 gives $\operatorname{Re} s \leq 1/2$ and $\operatorname{Re}(1-s) \leq 1/2$. □

Theorems 9.7 and 9.8 are Theorems 1.1 and 1.2 of the introduction.

Part II

Derivatives of theta functions

10. The field E at a prime

This section collects the arithmetic of the CM field $E = K(\zeta)$ of Section 8.2 that the divisibility arguments of Part II use: a generator of the different, the decomposition of primes, the Weil data of E and the places chosen by the data. Throughout, $s \geq 1$ is fixed, $\zeta \in E$ is the chosen primitive root of unity of order $q_1 = q_1(s)$, and $P = \Phi_{q_1}$ denotes the cyclotomic polynomial of order q_1; the letter Φ alone denotes the group of infinite places. We put

$$\beta = \frac{1+\sqrt{-d}}{2} \in \mathcal{O}_K, \qquad \beta\bar{\beta} = \frac{d+1}{4}, \qquad \beta + \bar{\beta} = 1,$$

and we write $g_v \in \mathrm{Gal}(E/K)$ for the automorphism attached to the place $v \in \Phi$, so that $\Phi_v = \Phi_1 \circ g_v$. For a finite place w of E and $v \in \Phi$ we write $v \cdot w$ for the place $g_v^{-1}(w)$, that is, $x \in v \cdot w$ if and only if $g_v(x) \in w$, and $\bar{w}$ for the complex conjugate place. The residue field of w has q_w elements.

10.1. The different

Proposition 10.1. *Let $z \in \mathcal{O}_E$ be a primitive root of unity of order q_1.*

(a) *$P'(z)$ divides q_1 in $\mathcal{O}_E$.*
(b) *$\mathcal{O}_E = \mathcal{O}_K[z]$, and the different of E/K is the principal ideal $(P'(z))$.*
(c) *$\mathcal{O}_E = \mathbb{Z}[\beta, \zeta]$, and the different of $E/\mathbb{Q}$ is generated by $P'(\zeta)\sqrt{-d}$. It contains $q_1 d$.*

Proof. (a) Write $X^{q_1} - 1 = P(X)Q(X)$ in $\mathbb{Z}[X]$. Differentiating at the root z of P gives $q_1 z^{q_1-1} = P'(z)Q(z)$, and multiplying by z gives $q_1 = P'(z)\, zQ(z)$.

(b) The ring of integers of the cyclotomic field $\mathbb{Q}(z)$ is $\mathbb{Z}[z]$, and P is the minimal polynomial of z; by the formula of Euler and Dedekind for monogenic orders, the different of $\mathbb{Q}(z)/\mathbb{Q}$ is generated by $P'(z)$. The different is transitive in towers, so in $\mathcal{O}_E$

$$\mathfrak{D}_{E/\mathbb{Q}(z)} \cdot (P'(z)) = \mathfrak{D}_{E/\mathbb{Q}} = \mathfrak{D}_{E/K} \cdot \mathfrak{D}_{K/\mathbb{Q}}\mathcal{O}_E = \mathfrak{D}_{E/K} \cdot (\sqrt{-d}).$$

The ideal $(P'(z))$ divides q_1 by (a), and q_1 is prime to d, hence to $(\sqrt{-d})$; so $(P'(z))$ divides $\mathfrak{D}_{E/K}$. On the other hand z is integral over $\mathcal{O}_K$ and generates E over K with minimal polynomial P (Proposition 8.3(a)), so $\mathfrak{D}_{E/K}$ divides $P'(z)\,\mathfrak{f}$, where $\mathfrak{f}$ is the conductor of $\mathcal{O}_K[z]$ in $\mathcal{O}_E$, and in particular $\mathfrak{D}_{E/K} \mid (P'(z))$. Hence $\mathfrak{D}_{E/K} = (P'(z))$, and equality in the Euler–Dedekind divisibility forces $\mathfrak{f} = \mathcal{O}_E$, that is, $\mathcal{O}_E = \mathcal{O}_K[z]$.

(c) $\mathcal{O}_K = \mathbb{Z}[\beta]$ as $d \equiv 3 \pmod 4$, so $\mathcal{O}_E = \mathcal{O}_K[\zeta] = \mathbb{Z}[\beta, \zeta]$. The extension of $\mathfrak{D}_{K/\mathbb{Q}} = (\sqrt{-d})$ to $\mathcal{O}_E$ is generated by $\sqrt{-d}$, and transitivity gives $\mathfrak{D}_{E/\mathbb{Q}} = (P'(\zeta)\sqrt{-d})$. Finally $P'(\zeta) \mid q_1$ and $\sqrt{-d} \mid d$. □

Proposition 10.2. *There is $c \in \mathcal{O}_E$ with $c\bar{c} = 1$ such that $cP'(\zeta)$ is fixed by complex conjugation. The element*

$$\delta_E = c\, P'(\zeta)\sqrt{-d} \in \mathcal{O}_E$$

generates the different of $E/\mathbb{Q}$, satisfies $\bar{\delta}_E = -\delta_E$, and

$$\prod_{v \in \Phi} |\delta_E|_v = \mathrm{rd}_E^f, \qquad |\delta_E|_v = |P'(\zeta)|_v \sqrt{d} \quad (v \in \Phi).$$

Proof. Since $P = \prod_a (X - a)$ over the primitive roots of unity a of order q_1, we have $P'(\zeta) = \prod_{a \neq \zeta}(\zeta - a)$. As q_1 is even, every such a is ζ^{2e+1} for some $e \geq 0$. Let $i = \zeta^{q_1/4}$, a square root of -1 (recall $4 \mid q_1$), and $u = i\,\bar{\zeta}^{\,e+1}$. Then $u\bar{u} = 1$ and

$$u(\zeta - \zeta^{2e+1}) = i\big(\zeta^{-e} - \zeta^{e}\big)$$

is the product of the two purely imaginary elements i and $\bar{\zeta}^{\,e} - \zeta^{e}$, so it is real. The product c of these units over $a \neq \zeta$ satisfies $c\bar{c} = 1$, and $cP'(\zeta)$ is real. Hence δ_E is real times $\sqrt{-d}$, and $\bar{\delta}_E = -\delta_E$. It generates the different by Proposition 10.1(c), since c is a unit. As $c\bar{c} = 1$, $|c|_v = 1$ at every infinite place, and $|\delta_E|_v = |P'(\zeta)|_v\sqrt{d}$. Finally E is totally imaginary of degree $2f$, so $|\mathrm{N}_{E/\mathbb{Q}}(\delta_E)| = \prod_v |\delta_E|_v^2$, and the norm of the different is $|\operatorname{disc} E| = \mathrm{rd}_E^{2f}$. □

Remark 10.3. At every $v \in \Phi$ one has $|\delta_E|_v = A_v\,T(s)\sqrt{d}$ with $A_v > 0$ and $|\log A_v| \leq C_0$: indeed $P(X) = P_0(X^{T(s)})$ for the cyclotomic polynomial P_0 of order q_0, so $P'(\zeta) = T(s)\zeta^{T(s)-1}P_0'(\xi)$ with $\xi = \zeta^{T(s)}$ a primitive root of order q_0, and $|\log|P_0'(\xi)||$ is one of the terms of C_0 in Remark 8.8. Consequently $|\log|\delta_E|_v - \log \mathrm{rd}_E| \leq 2C_0$, and for data as in Definition 8.7 both $|\delta_E|_v$ and rd_E lie in $[x^{10}, x^{12}]$.

10.2. Decomposition of primes

For a prime $p \nmid q_1 d$ the *Frobenius element* at p is the pair

$$\mathrm{Fr}_p = \big(\chi_K(p)p,\ \chi_K(p)\big) \in \Phi \times \{\pm 1\},$$

where the first entry is the place of the class of $\chi_K(p)p$ modulo q_1. The group $\Phi \times \{\pm 1\}$ acts on O_E: $(v, 1)$ acts by g_v and $(v, -1)$ by g_v composed with complex conjugation.

Lemma 10.4. *Let $p \nmid q_1 d$ be prime, $\varphi\colon O_E \to R$ a ring homomorphism to a domain of characteristic p, $n \geq 0$, and (c, e) either* (id, 1) *or* (*complex conjugation*, -1). *For $v \in \Phi$ the following are equivalent:*

(i) $\varphi(g_v x) = \varphi(cx)^{p^n}$ *for all $x \in O_E$;*
(ii) $(v, e) = \mathrm{Fr}_p^n$.

Proof. Both sides of (i) are ring homomorphisms in x, and $O_E = \mathbb{Z}[\beta, \zeta]$, so (i) holds if and only if it holds at ζ and at $\sqrt{-d} = 2\beta - 1$ (p is odd, since $2 \mid q_1$). At ζ, $g_v\zeta = \zeta^{a_v}$ with a_v the class of v, and $c\zeta = \zeta^e$, so the condition is $\varphi(\zeta)^{a_v} = \varphi(\zeta)^{ep^n}$. Since $p \nmid q_1$, $\varphi(\zeta)$ is a primitive root of unity of order q_1 in R, and the condition says $a_v \equiv ep^n \pmod{q_1}$. At $\sqrt{-d}$, g_v fixes $\sqrt{-d}$, $c\sqrt{-d} = e\sqrt{-d}$, and $(\sqrt{-d})^p = (-d)^{(p-1)/2}\sqrt{-d} \equiv \big(\frac{-d}{p}\big)\sqrt{-d} = \chi_K(p)\sqrt{-d}$ modulo p, by Euler's criterion and quadratic reciprocity ($d \equiv 3 \pmod 4$). Thus the condition is $\chi_K(p)^n e\,\varphi(\sqrt{-d}) = \varphi(\sqrt{-d})$. Since $\varphi(\sqrt{-d})^2 = -d \neq 0$ in R and $1 \neq -1$ in R, the condition reads $e = \chi_K(p)^n$. The two conditions together say $(v, e) = \big((\chi_K(p)p)^n, \chi_K(p)^n\big) = \mathrm{Fr}_p^n$. □

Proposition 10.5 (Decomposition). *Let $p \nmid q_1 d$ be prime and w a place of E above p.*

(a) *For $v \in \Phi$: $v \cdot w = w$ if and only if $(v, 1)$ is a power of Fr_p, and $v \cdot w = \bar{w}$ if and only if $(v, -1)$ is a power of Fr_p.*
(b) *$q_w = p^o$, where o is the order of Fr_p; and p is unramified at w.*

(c) *Every place of E above p is $v \cdot w$ or $\overline{v \cdot w}$ for some $v \in \Phi$.*

Proof. (a) Let c be the identity or complex conjugation, with sign e, and let φ be the reduction modulo w. If $(v, e) = \mathrm{Fr}_p^n$, then by Lemma 10.4 $g_v x \equiv (cx)^{p^n}$ modulo w, so $g_v x \in w$ if and only if $cx \in w$; that is, $v \cdot w = c(w)$. Conversely, if $v \cdot w = c(w)$, then $g_v \circ c^{-1}$ maps w to itself and induces an automorphism of the residue field O_E/w, which is a power $y \mapsto y^{p^n}$ of the Frobenius automorphism; so (i) of Lemma 10.4 holds and $(v, e) = \mathrm{Fr}_p^n$.

(b) The residue degree is the least $n \geq 1$ with $y^{p^n} = y$ for all $y \in O_E/w$, which by Lemma 10.4 with $v = 1$, $e = 1$ is the least $n \geq 1$ with $\mathrm{Fr}_p^n = 1$. If $p \in w^2$, then w divides the different of $E/\mathbb{Q}$, which contains $q_1 d$ (Proposition 10.1(c)); then $p \mid q_1 d$, which is excluded. Hence $\mathrm{ord}_w(p) = 1$.

(c) $\mathrm{Gal}(E/\mathbb{Q})$ acts transitively on the places above p, and every element of $\mathrm{Gal}(E/\mathbb{Q})$ is some g_v or some g_v followed by complex conjugation, since $\mathrm{Gal}(E/K)$ has index two and is normal. □

Corollary 10.6 (Inert primes). *Let p be prime with $\chi_K(p) = -1$ and w a place above p. Then $p \nmid q_1 d$, $\mathrm{Fr}_p = (-p, -1)$, and:*

(a) *$v \cdot w = w$ if and only if $v = (-p)^n$ with n even, and $v \cdot w = \bar{w}$ if and only if $v = (-p)^n$ with n odd;*
(b) *$q_w = p^{\mathrm{lcm}(o,2)}$, where o is the order of $-p$ in $(\mathbb{Z}/q_1)^\times$;*
(c) *for odd n and every $y \in O_E$, $y^{p^n} \equiv g_{-p^n}(\bar{y}) \pmod{pO_E}$.*

Proof. Primes dividing q_1 split in K (Lemma 8.2) and $\chi_K(d) = 0$, so $p \nmid q_1 d$, and $\mathrm{Fr}_p^n = ((-p)^n, (-1)^n)$. Then (a) and (b) are Proposition 10.5(a),(b): the order of Fr_p is $\mathrm{lcm}(o, 2)$. For (c) apply Lemma 10.4 to the reduction modulo each place above p, with $v = -p^n$, c complex conjugation and $e = -1 = \chi_K(p)^n$: $g_{-p^n}(y) \equiv \bar{y}^{p^n}$ modulo every place above p, hence modulo p, which is unramified and so the product of these places. Replacing y by $\bar{y}$ gives (c). □

Proposition 10.7 (Primes $\equiv 1$ modulo $q_1 d$). *Let $p \equiv 1 \pmod{q_1 d}$ be prime and w a place above p. Then $\chi_K(p) = 1$, $\mathrm{Fr}_p = 1$, $q_w = p$, $v \cdot w = w$ only for $v = 1$, and $v \cdot w \neq \bar{w}$ for all v. If $a \in O_E$ satisfies $a\bar{a} = pc$ with $c \in \mathbb{N}$ prime to p, then w contains exactly one of a and $\bar{a}$, with $\mathrm{ord}_w(a) = 1$ if $a \in w$, and $v \mapsto$ (the one of $v \cdot w$, $\overline{v \cdot w}$ that contains a) is a bijection from Φ onto the places above p that contain a.*

Proof. $\chi_K(p) = \chi_K(1) = 1$ since χ_K has conductor d, and $p \equiv 1 \pmod{q_1}$, so $\mathrm{Fr}_p = (1, 1)$. Proposition 10.5 gives the statements on $v \cdot w$ and $q_w = p$. If a and $\bar{a}$ both lay in w, then $pc \in w^2$; but $\mathrm{ord}_w(p) = 1$ and c is a unit at w. If neither did, $pc \notin w$. Hence exactly one lies in w, and $\mathrm{ord}_w(a) + \mathrm{ord}_w(\bar{a}) = \mathrm{ord}_w(pc) = 1$. For the bijection: the map is defined, since exactly one of $v \cdot w$ and its conjugate contains a (apply the previous statement to the place $v \cdot w$). It is injective, since $v \cdot w \in \{u \cdot w, \overline{u \cdot w}\}$ forces $u^{-1}v$ to fix w or to conjugate it, hence $u = v$. It is surjective by Proposition 10.5(c). □

An automorphism σ of $\mathbb{C}$ over K permutes the embeddings Φ_v: $\sigma \circ \Phi_v = \Phi_{\sigma \cdot v}$, where $\sigma \cdot v$ is the product in Φ of v and the place of the embedding $\sigma \circ \Phi_1$. We write $v \mapsto \sigma \cdot v$ for this permutation of Φ.

10.3. The Weil data of E

The Weil representation of a hermitian space over E (Section 13.1) depends on *Weil data*: a purely imaginary $\jmath \in E^\times$, a finite idele α of E, and a unitary Hecke character v of E (a *splitting character*) whose restriction to K is the quadratic character attached to E/K times the prescribed archimedean factors. We now fix them.

Definition 10.8. The finite idele α of E has component $P'(\zeta)^{-1}$ at the places w with $\beta \in w$ and 1 at all other places. The lattice $\mathfrak{a}_0 = E \cap \alpha\widehat{\mathcal{O}}_E$ is the set of $y \in E$ with $\alpha^{-1}y$ integral at all finite places.

Lemma 10.9. (a) *A place w with $P'(\zeta) \in w$ contains β or $\bar\beta$; and* $\operatorname{ord}_w P'(\zeta) = \operatorname{ord}_{\bar w} P'(\zeta)$.

(b) *α^{-1} and $q_1\alpha$ are integral; $\alpha\bar\alpha P'(\zeta)$ is a unit at every finite place.*

(c) *With $\jmath = -1/\sqrt{-d}$, the idele $\jmath\alpha\bar\alpha$ has order $-\operatorname{ord}_w \mathfrak{D}_{E/\mathbb{Q}}$ at every w; that is, $\jmath\,\mathfrak{a}_0\bar{\mathfrak{a}}_0$ is the inverse different of E. Moreover $\Phi_\tau(\jmath) = i/\sqrt{d}$ for every τ, and $(1-\jmath^{-1})/2 = \beta$.*

(d) *Every root of unity $y \in E$ is $\delta/\bar\delta$ for a unit $\delta \in \mathcal{O}_E^\times$ with $\prod_v \Phi_v(\delta) = 1$.*

Proof. (a) $P'(\zeta)$ divides q_1, all of whose prime factors divide $\beta\bar\beta = (d+1)/4$; so a prime w containing $P'(\zeta)$ contains β or $\bar\beta$. Any injective endomorphism of $\mathcal{O}_E$ sends ζ to a primitive root of order q_1, so it sends $P'(\zeta)$ to another generator of $\mathfrak{D}_{E/K}$ (Proposition 10.1(b)); applied to complex conjugation this gives $\operatorname{ord}_{\bar w} P'(\zeta) = \operatorname{ord}_w P'(\zeta)$.

(b) α^{-1} has components $P'(\zeta)$ or 1, which are integral; and $q_1\alpha$ has components $q_1/P'(\zeta)$ or q_1, integral by Proposition 10.1(a). No place contains both β and $\bar\beta$, since $\beta + \bar\beta = 1$. If $\beta \in w$, then $\alpha_w = P'(\zeta)^{-1}$ and $\bar\alpha_w = 1$ (because $\bar\alpha_w$ is the conjugate of $\alpha_{\bar w}$ and $\beta \notin \bar w$); if $\bar\beta \in w$, then $\alpha_w = 1$ and $\bar\alpha_w$ has the order of $P'(\zeta)^{-1}$ at w by (a); otherwise $\alpha_w = \bar\alpha_w = 1$ and $P'(\zeta)$ is a unit at w by (a).

(c) By (b), $\jmath\alpha\bar\alpha$ has the order of $(P'(\zeta)\sqrt{-d})^{-1}$, which is the order of $\mathfrak{D}_{E/\mathbb{Q}}^{-1}$ by Proposition 10.1(c). The embedding Φ_τ is over $K \subset \mathbb{C}$, so $\Phi_\tau(\sqrt{-d}) = i\sqrt{d}$ and $\Phi_\tau(\jmath) = -1/(i\sqrt{d}) = i/\sqrt{d}$; and $(1-\jmath^{-1})/2 = (1+\sqrt{-d})/2$.

(d) Let $q_1 = 4m$ and $i = \zeta^m$. The conjugates of ζ over K are the ζ^a with a a unit modulo q_1, so $\mathrm{N}_{E/K}(\zeta^k) = \zeta^{k\sum_a a} = 1$, because $a \mapsto -a$ pairs the units and $\sum_a a \equiv 0 \pmod{q_1}$. Likewise $\mathrm{N}_{E/K}(\zeta - 1) = \prod_a(\zeta^a - 1) = P(1)$, as $\varphi(q_1)$ is even, and $P(1) = 1$ because q_1, divisible by 2 and by 3, is not a prime power; so $\zeta - 1$ is a unit of norm one. Put $\eta = i(\zeta - 1)$; then

$$\frac{\eta}{\bar\eta} = \frac{i(\zeta-1)}{-i(\zeta^{-1}-1)} = -\frac{\zeta-1}{-\zeta^{-1}(\zeta-1)} = \zeta.$$

By Proposition 8.3(d) every root of unity is ζ^k, and $\delta = \eta^k$ serves; by Proposition 8.3(e) the product of the $\Phi_v(\delta)$ is $\mathrm{N}_{E/K}(\delta) = 1$. □

Proposition 10.10. *The element $\jmath = -1/\sqrt{-d}$, the idele α of Definition 10.8 and a splitting character v are Weil data for E; such a character exists. We fix it.*

Proof. The conditions on $\jmath$ and α are that $\jmath$ is purely imaginary and that $\jmath\alpha\bar{\alpha}$ generates the inverse different locally at every finite place; both hold by Lemma 10.9(c). A splitting character exists as soon as every root of unity of E has the form $\delta/\bar{\delta}$ with δ a unit of norm one to K; this is the criterion of Theorem B.18, and the hypothesis is Lemma 10.9(d). □

Definition 10.11. Let $\chi_{\mathrm{f}}(x) = \upsilon(x)|x|^{-1/2}$ for finite ideles x (extended by 1 at the infinite places). The *vacuum root* $s_0 \in \mathbb{C}$ is the principal f-th root of $\chi_{\mathrm{f}}(\alpha)^{-1} = \upsilon(\alpha)^{-1}|\alpha|^{1/2}$.

Proposition 10.12. *A positive power of s_0 lies in K, and for every automorphism σ of $\mathbb{C}$ over K*

$$|\sigma(s_0)|^2 = \frac{\mathrm{rd}_E}{\sqrt{d}}.$$

Proof. A positive power of $\chi_{\mathrm{f}}(\alpha)$ lies in K: υ has finite order on the unit ideles, and a power of α is, up to such units, the idele of a principal ideal (y), $y \in E$, at which the value of υ is the archimedean factor, a norm from E to K (Theorem B.18). Hence $s_0^n \in K$ for some $n \geq 1$, and $|\sigma s_0|^n = |\sigma(s_0^n)| = |s_0^n|$, so $|\sigma s_0| = |s_0|$. Next, $|s_0|^{2f} = |\alpha|$ as υ is unitary. Since $\alpha\bar{\alpha}P'(\zeta)$ is a unit at every finite place (Lemma 10.9(b)) and $|\bar{\alpha}| = |\alpha|$, we have $|\alpha|^2\,|P'(\zeta)|_{\mathrm{f}} = 1$, and the product formula gives $|P'(\zeta)|_{\mathrm{f}}^{-1} = \prod_v |P'(\zeta)|_v^2$. Hence $|s_0|^{2f} = \prod_v |P'(\zeta)|_v = \prod_v |\delta_E|_v/\sqrt{d}^{\,f} = (\mathrm{rd}_E/\sqrt{d})^f$ by Proposition 10.2. □

10.4. The chosen places and characters of given weight

For the rest of Section 10 the data of Definition 8.7 are fixed. Recall that $M = p_0p_1 = \pi_M\bar{\pi}_M$ with $\bar{\pi}_M \in O_K$. The *chosen places* are the finite places w of E with $\pi_M \in w$.

Lemma 10.13. *Every chosen place lies above p_0 or p_1 and has norm $q_w = p_i$; there are finitely many. For $i = 0, 1$, $\pi_M\bar{\pi}_M = p_ic_i$ with $c_i \in \mathbb{N}$ prime to p_i, so by Proposition 10.7 the chosen places above p_i are in bijection with Φ.*

Proof. A chosen w contains $\pi_M\bar{\pi}_M = p_0p_1$, hence p_0 or p_1. The primes p_i are $\equiv 1$ modulo q_2^2, hence modulo q_1d (Lemma 8.2), so Proposition 10.7 applies with $a = \pi_M$ and $c_i = p_{1-i}$. □

At a chosen place w the residue field has p_i elements and $q_2^2 \mid p_i - 1$, so it carries a multiplicative character of order q_2^2; we fix one, $\omega_w^\flat$, and put $\omega_w = (\omega_w^\flat)^{q_2}$, of order q_2. At the other places $\omega_w = 1$. Since $q_1 \mid q_2$, $\omega_w(y) = 1$ for every y in the residue field with $y^{q_1} = 1$.

Let $\mathrm{U}(1)(\mathbb{A})$ be the group of ideles u of E with $u\bar{u} = 1$. A continuous character μ of $\mathrm{U}(1)(\mathbb{A})$, trivial on $\mathrm{U}(1)(E)$, *has weight* $e\colon \Phi \to \mathbb{Z}$ *and tame part* $\psi = (\psi_w)_w$ if almost all ψ_w are trivial and

$$\mu(u) = \prod_{v\in\Phi} u_v^{e(v)} \prod_w \psi_w(u \bmod w)$$

for every u with integral finite part, where $u_v = \Phi_v(u)$. Let $w_0(v) = \kappa_0(v) - \Delta/2 = \kappa(v) + r^8$ be the lift weight κ_0 of Definition 8.7 shifted by half the gap.

Proposition 10.14. *For all $a, n \in \mathbb{N}$ there is a character of $\mathrm{U}(1)(\mathbb{A})$, trivial on $\mathrm{U}(1)(E)$, of weight $-a$ w_0 and tame part $(\omega_w^n)_w$. We fix such characters χ_0 (for $a = 1$, $n = 0$) and χ_M (for $a = 0$, $n = 1$).*

Proof. By Theorem B.19 such a character exists if the formula for $\mu(u)$ gives 1 at every $\gamma \in \mathrm{U}(1)(E)$ that is integral. Such γ is an algebraic integer all of whose conjugates have absolute value 1, hence a root of unity, hence $\gamma = \zeta^k$ (Proposition 8.3(d)). At a finite place the residue of ζ^k is a q_1-th root of unity, so $\omega_w(\zeta^k) = 1$. At the infinite places, $\prod_v \Phi_v(\zeta^k)^{w_0(v)} = 1$: the weight $\kappa(v)$ is a multiple of q_1 (both b and a/x_1 are, since $q_1 x_1 \mid b$ and $a = xb$), so $w_0(v) \equiv r^8 \pmod{q_1}$ for all v, and $\prod_v \Phi_v(\zeta^k)^{r^8} = \mathrm{N}_{E/K}(\zeta^k)^{r^8} = 1$. □

Definition 10.15. A *pair* for the data is an integer z with $1 \le z \le t$ together with a character ν of $\mathrm{U}(1)(\mathbb{A})$, trivial on $\mathrm{U}(1)(E)$, of weight $-c$ with $z \le c(v) \le 4b$ for all $v \in \Phi$ and tame part trivial off the chosen places.

Proposition 10.16. *There are finitely many pairs.*

Proof. The weight c and the integer z range over finite sets; the tame part is a family of characters of the residue fields at the finitely many chosen places; and for a given weight and tame part there are finitely many characters, since two of them differ by a character of the compact group $\mathrm{U}(1)(E)\backslash \mathrm{U}(1)(\mathbb{A})$ that is trivial on the open subgroup of u with $u_v = 1$ for $v \in \Phi$ and $u_w \equiv 1$ modulo w for all w, a subgroup of finite index (Theorem B.19). □

11. The theta function of a CM curve at a non-split prime

The divisibility of the derivatives of theta functions of hermitian spaces over E is reduced, in Section 12, to the behaviour of one theta function of one elliptic curve with complex multiplication by O_K, read in $\mathbb{C}_p$. This section treats that curve. Its main result, Theorem 11.8, writes $\theta(\varpi z)$, for a suitable ϖ of positive valuation, as an integral series times the exponential of a large multiple of an explicit series, the exponent.

11.1. Conventions at a prime

Fix a prime p and a ring homomorphism $\iota_p \colon \mathbb{C} \to \mathbb{C}_p$ (a *p-adic place* of $\mathbb{C}$); complex numbers are read in $\mathbb{C}_p$ through ι_p, usually without notation. Let v be the valuation of $\mathbb{C}_p$ with $v(p) = 1$, so that $|x| = p^{-v(x)}$, let O be its valuation ring and $\mathfrak{m}$ the maximal ideal. A power series over $\mathbb{C}_p$ is *integral* if its coefficients lie in O, and *integral in degree n* if its coefficients of total degree n do. For a series G without constant term, $\exp G$ is the exponential series evaluated at G.

Definition 11.1. For a prime p put $q_p = p^2$ if $\chi_K(p) = -1$ and $q_p = p$ otherwise, and $\pi_p = \sqrt{-d}$ if $p = d$, $\pi_p = p$ otherwise. Let $v(\pi_p)$ be $1/2$ if $p = d$ and 1 otherwise, and

$$\rho_p = \frac{v(\pi_p)}{q_p - 1}, \qquad \gamma_p = \frac{1}{p-1} - \rho_p.$$

The set I_p° of *partner exponents* is the set of odd integers $e \geq 1$ if $\chi_K(p) = -1$, the set of all $e \geq 0$ if $p = d$, and empty if $\chi_K(p) = 1$; $I_p = I_p^\circ \setminus \{0\}$. The prime p is *non-split* if $\chi_K(p) \neq 1$.

Lemma 11.2. (a) *ρ_p is $1/(p-1), 1/(p^2-1), 1/(2p-2)$ and γ_p is $0, p/(p^2-1), 1/(2p-2)$ for p split, inert, ramified in K; in all cases $0 \leq \gamma_p \leq 1/(p-1)$ and*

$$\gamma_p = v(\pi_p) \sum_{e \in I_p} p^{-e}.$$

(b) *For non-split p: $p \geq 13$, $\pi_p \bar{\pi}_p = q_p$, $|\pi_p| = p^{-v(\pi_p)} = \left(p^{-\rho_p}\right)^{q_p - 1}$, $\pi_p \mid p$ in $\mathcal{O}$, and $|\pi_p| \leq |n|$ for $0 < n < q_p$.*

Proof. (a) If p splits, $q_p = p$ and $v(\pi_p) = 1$, so $\rho_p = 1/(p-1)$, $\gamma_p = 0$, and I_p is empty. If p is inert, $q_p = p^2$ and $v(\pi_p) = 1$, so $\rho_p = 1/(p^2-1)$ and $\gamma_p = 1/(p-1) - 1/(p^2-1) = p/(p^2-1) = \sum_{e \text{ odd}} p^{-e}$. If $p = d$, $q_p = p$ and $v(\pi_p) = 1/2$, so $\rho_p = 1/(2p-2)$ and $\gamma_p = 1/(2p-2) = \frac{1}{2}\sum_{e \geq 1} p^{-e}$, the sum over I_p. In all cases $0 \leq \gamma_p \leq 1/(p-1)$. (b) $p \geq 13$ is Lemma 8.2. If p is inert, $\pi_p \bar{\pi}_p = p^2$; if $p = d$, $\sqrt{-d} \cdot \overline{\sqrt{-d}} = d$. For $0 < n < q_p$ we have $v(n) \leq 1$ if $q_p = p^2$ and $v(n) = 0$ if $q_p = p$, and $v(n) \leq v(\pi_p)$ in both cases except when $p = d$ and $p \mid n$, which cannot happen for $n < p$. □

11.2. The curve and its functions

A *CM lattice* for d is a lattice $\Lambda \subset \mathbb{C}$ stable under multiplication by $\mathcal{O}_K$, with a point t of order two modulo Λ such that neither βt nor $\bar{\beta} t$ lies in Λ, scaled so that $\wp''(t) = 2$, together with a square root a_1 of $12\wp(t)$. Let A be the area of a fundamental parallelogram and $H(l, z) = \pi \bar{l} z / A$. The sigma function σ of Λ is the odd theta function for (H, χ_σ) with $\sigma'(0) = 1$, where $\chi_\sigma(l) = 1$ if $l/2 \in \Lambda$ and -1 otherwise. The *theta function* θ of the lattice is the twisted translate of σ by t, normalised by $\theta(0) = 1$:

$$\theta(z) = \frac{\exp\left(-\pi(z\bar{t} + |t|^2/2)/A\right)\sigma(z+t)}{\exp\left(-\pi|t|^2/(2A)\right)\sigma(t)}.$$

It is even, a theta function for (H, χ) for a multiplier χ of modulus one, and its zeros are the points of $t + \Lambda$, all simple. Put

$$x(z) = \wp(z+t) - \wp(t), \qquad y(z) = \tfrac{1}{2}\left(\wp'(z+t) - a_1 x(z)\right), \qquad g = -\log\theta,$$

the last one as a power series at 0 with $g(0) = 0$. More generally, the *twisted translate* of a function h by $u \in \mathbb{C}$ in degree m is

$$(T_u^m h)(z) = \exp\left(-m\pi(z\bar{u} + |u|^2/2)/A\right) h(z+u),$$

and a *section of degree m* is an entire function h with $T_l^m h = \chi(l)^m h$ for $l \in \Lambda$.

The following facts are classical; their proofs are recalled in Appendix B.9.

Theorem 11.3 (Background on the curve). *Let C be a CM lattice for d.*

(i) *The point (x, y) lies on the cubic $y^2 + a_1 xy = x^3 + x$; moreover $x' = 2y + a_1 x$, $y' = 1 + 3x^2 - a_1 y$, $x'^2 = 4x^3 + a_1^2 x^2 + 4x$ and $x'' = 6x^2 + a_1^2 x + 2$. As power series, $x = z^2 + O(z^4)$, $y = z + O(z^2)$, and $g'' = g''(0) + x$; in particular g is even and $g(z) = g_2 z^2 + O(z^4)$.*

(ii) *For $a \in O_K \setminus \{0\}$, $z \mapsto \theta(az)$ is a section of degree $a\bar{a}$. The monomials $\theta^r \sigma^{2t'} G^u$ with $u \le 1$ and $r + 2t' + 3u = m$ span the sections of degree m, where G is the entire function equal to $y\theta^3$ away from the poles of y. If $\delta \in O_K \setminus \{0\}$ and $\delta u \in \Lambda$, then $z \mapsto (T_u^1 \theta)(\bar{\delta} z)$ is a section of degree $\delta\bar{\delta}$.*

Let now p be a prime and $O[[y]]$ the ring of series $G(y)$ with $G \in O[[X]]$, where y also denotes the Taylor series of y at 0 read in $\mathbb{C}_p$.

(iii) (*Integral group law.*) *$x \in O[[y]]$, and $y(z_0 + z_1)$ is an integral series in $y(z_0)$ and $y(z_1)$.*

(iv) (*Integrality along isogenies.*) *Let $(\delta, s) \in \mathbb{C}_p^2$ be a limit of pairs $(\alpha, \alpha\bar{\alpha})$ with $\alpha \in O_K$. Then $y(\delta z) \in O[[y]]$ and $\exp\big(s\, g(z) - g(\delta z)\big) \in O[[y]]$. If $\zeta^{q_p - 1} = 1$, then (ζ, ζ^{p+1}) is such a limit. Moreover $\sqrt{-d}\, g''(0) \in O$.*

(v) (*The formal parameter.*) *Let T be the part of y in the degrees $\equiv 1$ modulo $q_p - 1$. Then $T = z + \cdots$, $O[[y]] = O[[T]]$, and the coefficients of every element of $O[[y]]$ in degrees below q_p are integral. There are $\varpi \in \mathbb{C}_p$ with $|\varpi| = p^{-\rho_p}$ and an integral series Y with $T(\varpi z) = \varpi Y(z)$, $Y - Y^{q_p} \equiv z$ and $Y \equiv \sum_{j \ge 0} z^{q_p^j}$ modulo π_p. For every ϖ' with $|\varpi'| \le p^{-\rho_p}$ and every $G \in O[[y]]$, the series $G(\varpi' z)$ is integral.*

Sources. These are classical facts on the curve with complex multiplication by O_K attached to C: the equation and the expansions in (i) come from the Laurent series of the Weierstrass function [Sil09, Ch. VI]; (ii) is the description of theta functions with complex multiplication as sections [Shi76]; (iii) is the integrality of the formal group law [Sil09, Ch. IV]; and (iv), (v) come from the theory of complex multiplication [Sil94, Ch. II] and of formal groups with complex multiplication [LT65]. All five are proved in the Lean development. For non-split p the curve has supersingular reduction at p if p is inert, and additive reduction becoming supersingular over a ramified extension if $p = d$; (v) is the statement that multiplication by π_p on the formal group is a Lubin–Tate endomorphism of height two (inert p) or one (over the ramified extension), and ϖ is a root of $\varpi^{q_p - 1} = -\pi_p / b_1$ for the unit b_1 in $T(\pi_p z) = \pi_p T + b_1 T^{q_p} + \cdots$. □

11.3. The Frobenius congruence

Theorem 11.4 (Frobenius congruence). *Let p be non-split and $\pi = \pi_p$, $q = q_p$. There are $H, H' \in O[[y]]$ with*

$$\theta(\pi z) = \theta(z)^q\,(1 + \pi H), \qquad q\, g(z) - g(\pi z) = \pi H'.$$

Proof. Step 1: a polynomial in x. By Theorem 11.3(ii), $z \mapsto \theta(\pi z)$ is a section of degree $\pi\bar{\pi} = q$, as is θ^q. Their quotient is therefore an elliptic function for Λ; it is even, since θ is, and its poles lie at the zeros $t + \Lambda$ of θ, with order at most q. An even elliptic function whose only poles lie on $t + \Lambda$ is a polynomial in $x(z) = \wp(z + t) - \wp(t)$, of degree at most half the order of the pole. Indeed, let h be such a function with a pole of order $2D'$ at t (even, as h is even about the point t of order two: $h(t + u) = h(-t - u) = h(t - u)$ since $2t \in \Lambda$); as x has a double pole at t and no other pole modulo Λ, subtracting a multiple of $x^{D'}$ lowers the order of the pole, and an even elliptic function without poles is constant. As q is odd ($p \neq 2$ by Lemma 11.2(b)), the order of the pole of $\theta(\pi z)/\theta(z)^q$ at t, an even number at most q, is at most $q - 1$, and we get $\theta(\pi z) = \theta(z)^q R(x(z))$

with a polynomial R over $\mathbb{C}$ of degree D, $2D < q$. The constant term of R is 1: at $z = 0$, $x(0) = 0$ and $\theta(0) = 1$. Comparing logarithms,

$$R(x) = \exp\big(qg(z) - g(\pi z)\big).$$

Step 2: the coefficients of g below q. Write $g = \sum_n g_n z^n$, so $g_0 = g_1 = 0$. We claim $|\pi g_n| \le 1$ for $n < q$. For $n = 2$, $|\sqrt{-d}\, g''(0)| \le 1$ by Theorem 11.3(iv), $g''(0) = 2g_2$ with 2 a unit, and $|\pi| \le |\sqrt{-d}|$. For $3 \le n < q$, comparing coefficients in $g'' = g''(0) + x$ gives $n(n-1)g_n = x_{n-2}$, where x_{n-2} is integral because $x \in \mathcal{O}[[y]]$ and $n - 2 < q$ (Theorem 11.3(iii),(v)); and one of n, $n-1$ is prime to p while the other is below q, so $|n(n-1)| \ge |\pi|$ by Lemma 11.2(b). Next, for $2 \le n < q$ the coefficient $(q - \pi^n)g_n$ of $qg(z) - g(\pi z)$ satisfies $|(q - \pi^n)g_n| \le |\pi|^2 |\pi|^{-1} = |\pi|$, since $v(q) = 2v(\pi)$ and $v(\pi^n) \ge 2v(\pi)$.

Step 3: the congruence for θ. Let $G = qg - g(\pi z)$. In degrees below q, G has coefficients in $\pi\mathcal{O}$ and no terms of degree below two. Hence in degrees below q the series $\exp G = \sum_k G^k/k!$ involves only $k < q/2$, and $\pi^k/k! \in \pi\mathcal{O}$ for $k \ge 1$ because $v(k!) \le (k-1)/(p-1) \le (k-1)v(\pi)$. Hence $R(x) - 1$ has coefficients in $\pi\mathcal{O}$ in degrees below q. Write $R(X) - 1 = \sum_{j \le D} r_j X^j$. The coefficient of z^{2j} in $R(x) - 1$ is $r_j + \sum_{i<j} r_i [z^{2j}] x^i$, because $x^i = z^{2i} + \cdots$; the numbers $[z^{2j}]x^i$ are integral as $2j < q$. By induction on j every r_j lies in $\pi\mathcal{O}$. Then $H = \sum_j (r_j/\pi) x^j \in \mathcal{O}[[y]]$, and $\theta(\pi z) = \theta^q (1 + \pi H)$.

Step 4: the logarithm. $qg(z) - g(\pi z) = \log(1 + \pi H) = \sum_{k \ge 1} (-1)^{k-1} \pi^k H^k / k$, and $\pi^{k-1}/k \in \mathcal{O}$ because $v(k) \le \log_p k \le (k-1)/2 \le (k-1)v(\pi)$. The ring $\mathcal{O}[[y]] = \mathcal{O}[[T]]$ is closed under such convergent sums, so $H' = \sum_k (-1)^{k-1} \pi^{k-1} H^k / k \in \mathcal{O}[[y]]$. □

Lemma 11.5 (The supersingular coefficient). *Let p be inert. The coefficient of z^{p-1} in x is a unit of $\mathcal{O}$.*

Proof. Let $h_n = [z^{p-1}]x^n$; these are integral, since $x \in \mathcal{O}[[y]]$ and $p - 1 < q_p = p^2$. From the identities of Theorem 11.3(i),

$$(x^n x')' = (4n+6)x^{n+2} + (n+1)a_1^2 x^{n+1} + (4n+2)x^n.$$

The coefficient of z^{p-1} in a derivative F' is $p\,[z^p]F$, and $x^n x'$ is integral in degree $p < q_p$; so $(4n+6)h_{n+2} + (n+1)a_1^2 h_{n+1} + (4n+2)h_n \in \mathfrak{m}$ for all n. Write $p = 2m+1$; then $h_0 = 0$, and $h_m = [z^{2m}]x^m = 1$ since $x = z^2 + \cdots$. Suppose $h_1 \in \mathfrak{m}$. For $n + 2 \le m$ the number $4n + 6 = 2(2n+3)$ is a unit, as $2n + 3 < p$; so by induction $h_n \in \mathfrak{m}$ for all $n \le m$, contradicting $h_m = 1$. Hence $h_1 = [z^{p-1}]x$ is a unit. □

11.4. The exponent

Let p be non-split, $q = q_p$, $\pi = \pi_p$, and let i_0 be the least positive integer $\equiv p + 1 \pmod{q-1}$: $i_0 = p + 1$ if p is inert and $i_0 = 2$ if $p = d$. In both cases $i_0 < q - 1$. For $\omega \in \mathbb{C}_p^\times$ put

$$c_\omega = \frac{\omega^{p+1}}{p}.$$

Let $g^\flat$ be the part of g in the degrees $\equiv p+1 \pmod{q-1}$, and $\Xi_1 = -g^\flat(\omega z)/c_\omega$, the *exponent*.

Lemma 11.6 (Averaging). $\exp(g^\flat - g) \in O[[y]]$. *Consequently, if* $|\omega| \le p^{-\rho_p}$, *then* $\theta(\omega z) = W \exp(c_\omega \Xi_1)$ *for an integral series* W.

Proof. Let ζ run over the $(q-1)$-th roots of unity in $\mathbb{C}_p$. By orthogonality, $g^\flat(z) = \frac{1}{q-1}\sum_\zeta \zeta^{-(p+1)} g(\zeta z)$, so

$$\exp(g^\flat - g)^{q-1} = \prod_\zeta \exp\bigl(\zeta^{-(p+1)} g(\zeta z) - g(z)\bigr).$$

In the variable $w = \zeta z$ a factor reads $\exp(s\, g(w) - g(\delta w))$ with $\delta = \zeta^{-1}$ and $s = \delta^{p+1}$, which lies in $O[[y]]$ by Theorem 11.3(iv), since $\delta^{q-1} = 1$; and the substitution $w = \zeta z$ preserves $O[[y]]$, because $y(\zeta z) \in O[[y]]$ by the same item. Hence every factor lies in $O[[y]]$. The series $F = \exp(g^\flat - g)$ has constant term 1 and $F^{q-1} \in O[[y]] = O[[T]]$. Since $q-1$ is prime to p, the binomial series $(1+X)^{1/(q-1)} = \sum_k \binom{1/(q-1)}{k} X^k$ has coefficients in $\mathbb{Z}_p$: the binomial coefficients $\binom{a}{k}$ are continuous in a and integral for $a \in \mathbb{N}$, and $1/(q-1)$ is a p-adic integer, a limit of natural numbers. Write $F^{q-1} = 1 + h$ with $h \in O[[T]]$ without constant term; then $(1+h)^{1/(q-1)}$ is a convergent series in $O[[T]]$, and it equals F, because both have constant term 1 and the same $(q-1)$-th power, and a series $1+u$ with $(1+u)^{q-1} = 1$ and $u(0) = 0$ has $u = 0$ ($(q-1)u + \cdots = 0$ forces the lowest term of u to vanish, as $q - 1 \neq 0$). Hence F lies in $O[[T]]$. For the second statement write $\theta = \exp(-g) = \exp(g^\flat - g)\exp(-g^\flat)$ and substitute ωz: $\exp(g^\flat - g)(\omega z)$ is integral by Theorem 11.3(v), and $\exp(-g^\flat(\omega z)) = \exp(c_\omega \Xi_1)$. □

Lemma 11.7 (The functional equation). *Let* $|\omega| = p^{-\rho_p}$ *and let* Y *be integral with* $T(\omega z) = \omega Y$. *There is* $\lambda \in O$ *such that for all* n

$$\Bigl(1 - \frac{\pi^n}{q}\Bigr)[z^n]\Xi_1 \equiv \lambda\,[z^n]Y^{i_0} \pmod{\pi O}.$$

Proof. By Theorem 11.4, $q g(z) - g(\pi z) = \pi G$ with $G \in O[[T]]$. Taking parts in the degrees $\equiv p+1 \equiv i_0 \pmod{q-1}$ commutes with $z \mapsto \pi z$, which multiplies the coefficient of degree n by π^n. Write $G = \sum_k c_k T^k$ with $c_k \in O$; since T has terms only in degrees $\equiv 1$, the power T^k has terms only in degrees $\equiv k$, and the part of G in the degrees $\equiv i_0$ is $\sum_{k \equiv i_0} c_k T^k = \lambda' T^{i_0} + T^{i_0+q-1} H$ with $\lambda' = c_{i_0}$ and $H \in O[[T]]$ (here $i_0 < q-1$ is used). Thus

$$q g^\flat(z) - g^\flat(\pi z) = \pi\bigl(\lambda' T^{i_0} + T^{i_0+q-1} H\bigr).$$

Substitute ωz and use $T(\omega z) = \omega Y$: with $f(z) = g^\flat(\omega z)$,

$$q f(z) - f(\pi z) = \pi \omega^{i_0}\bigl(\lambda' Y^{i_0} + \omega^{q-1} R\bigr), \qquad R = Y^{i_0+q-1} H(\omega z) \text{ integral.}$$

Comparing coefficients of z^n and dividing by $-q c_\omega$ gives

$$\Bigl(1 - \frac{\pi^n}{q}\Bigr)[z^n]\Xi_1 = u\bigl(\lambda'[z^n]Y^{i_0} + \omega^{q-1}[z^n]R\bigr), \qquad u = -\frac{\pi \omega^{i_0}}{q c_\omega}.$$

Here $v(u) = v(\pi) + i_0\rho_p - v(q) - (p+1)\rho_p + 1$. For inert p, $i_0 = p+1$, $v(\pi) = 1$ and $v(q) = 2$, so $v(u) = 1 + 0 - 2 + 1 = 0$; for $p = d$, $i_0 = 2$, $v(\pi) = \frac{1}{2}$, $v(q) = 1$ and $\rho_p = 1/(2p-2)$, so $v(u) = \frac{1}{2} + 2\rho_p - 1 - (p+1)\rho_p + 1 = \frac{1}{2} - (p-1)\rho_p = 0$. Moreover $|\omega^{q-1}| = |\pi|$ by Lemma 11.2(b). Hence $\lambda = u\lambda'$ serves. □

For a set S of exponents put

$$\Pi_S(z) = \sum_{i \in S} z^{p^i}.$$

Over a ring of characteristic p the following identities hold, where the sums run over $j \geq 0$:

$$\Pi_{\text{odd}}\,\Pi_{\text{even}} = \sum_{e \text{ odd}} \sum_j \left(z^{p^e} z\right)^{p^j}, \tag{11.1}$$

$$\Pi_{\mathbb{N}}^2 - \tfrac{1}{2}z^2 = \tfrac{1}{2}z^2 + \sum_{j \geq 1} (zz)^{p^j} + 2\sum_{e \geq 1} \sum_j \left(z^{p^e} z\right)^{p^j}. \tag{11.2}$$

Indeed, the products $z^{p^a} z^{p^b}$ with a odd and b even are the terms $(z^{p^e} z)^{p^j}$ with $e = |a-b|$ and $j = \min(a,b)$, and each pair (e,j) with e odd arises exactly once (from $a = j+e$ if j is even, from $b = j+e$ if j is odd); and $\Pi_{\mathbb{N}}^2 = \sum_i z^{2p^i} + 2\sum_{i<i'} z^{p^i+p^{i'}}$.

Theorem 11.8 (The exponent on one curve). *Let C be a CM lattice for d and p a non-split prime. There are $\varpi \in \mathbb{C}_p$ with $|\varpi| = p^{-\rho_p}$, integral series W and Ξ with $\Xi(0) = 0$, and numbers $\lambda_{ej} \in \mathcal{O}$ $(e \in I_p^\circ,\ j \geq 0)$ such that*

$$\theta(\varpi z) = W \exp(c_\varpi \Xi), \qquad \Xi \equiv \sum_{e \in I_p^\circ} \sum_{j \geq 0} \lambda_{ej} \left(z^{p^e} z\right)^{p^j} \pmod{\pi_p}.$$

If p is inert, the λ_{e0} with e odd are units.

Proof. Take ϖ and Y as in Theorem 11.3(v). By Lemma 11.6 we have $\theta(\varpi z) = W\exp(c_\varpi \Xi_1)$ with W integral, and Ξ_1 has no terms below degree i_0, since its terms have degrees $\equiv i_0$ modulo $q-1 > i_0$. Let λ be as in Lemma 11.7. For $n > 2$ we have $|\pi^n/q| \leq |\pi|$, so the factor $1 - \pi^n/q$ is $\equiv 1$; for $n < i_0$ both sides of the functional equation vanish (also $[z^n]Y^{i_0} = 0$, as $Y = z + \cdots$).

Inert p. Then $i_0 = p+1 > 2$, so every n satisfies $n < i_0$ or $n > 2$, and $[z^n]\Xi_1 \equiv \lambda [z^n] Y^{p+1}$ for all n. In particular $\Xi_1 = \Xi$ is integral, and $\Xi \equiv \lambda Y^{p+1}$. Since $Y \equiv \Pi_{\text{even}}$ modulo $\pi = p$ (as $q = p^2$), $Y^{p+1} = Y^p Y \equiv \Pi_{\text{odd}}\Pi_{\text{even}}$, and (11.1) gives the congruence with $\lambda_{ej} = \lambda$. The coefficient of z^{p+1} in $\Pi_{\text{odd}}\Pi_{\text{even}}$ is 1 (the term $e = 1$, $j = 0$), so $\lambda \equiv [z^{p+1}]\Xi$. Finally $[z^{p+1}]\Xi_1 = -(p/\varpi^{p+1})\varpi^{p+1} g_{p+1} = -p\, g_{p+1}$, and comparing coefficients of z^{p-1} in $g'' = g''(0) + x$ gives $[z^{p-1}]x = (p+1)p\, g_{p+1}$; as $p+1$ is a unit, Lemma 11.5 shows that $[z^{p+1}]\Xi$ is a unit, hence so is λ.

$p = d$. Then $q = p$, $\pi^2 = -p$, $i_0 = 2$. For $n \neq 2$ we get $[z^n]\Xi_1 \equiv \lambda [z^n] Y^2$ as before. For $n = 2$ the factor is $1 - \pi^2/p = 2$, so $2[z^2]\Xi_1 \equiv \lambda c$ with $c = [z^2]Y^2$; since $Y \equiv \Pi_{\mathbb{N}}$ we have $c \equiv 1$, and hence $[z^2]\Xi_1 \equiv \lambda(c - \frac{1}{2})$. Thus $\Xi_1 = \Xi$ is integral and $\Xi \equiv \lambda(Y^2 - \frac{1}{2}z^2) \equiv \lambda(\Pi_{\mathbb{N}}^2 - \frac{1}{2}z^2)$, and (11.2) gives the congruence with $\lambda_{00} = \lambda/2$, $\lambda_{0j} = \lambda$ for $j \geq 1$ and $\lambda_{ej} = 2\lambda$ for $e \geq 1$. The integrality of Ξ in both cases uses that a series congruent modulo π to an integral one is integral: its coefficients differ from integral numbers by multiples of π, which are integral. In both cases $\Xi(0) = 0$, as Ξ_1 has no terms below degree $i_0 \geq 2$. □

Remark 11.9. The factor c_ϖ has valuation $(p+1)\rho_p - 1 < 0$, and $c_\varpi \pi_p$ has valuation $(p+1)\rho_p - 1 + v(\pi_p) = 1/(p-1)$ in both cases. This is the threshold below which the exponential series is integral: if $|c\pi| \leq p^{-1/(p-1)}$ and G has coefficients in $\pi\mathcal{O}$, then $\exp(cG)$ is integral. Theorem 11.8 therefore says that the obstruction to the

integrality of $\theta(\varpi z)$ is carried by the reduction of Ξ modulo π_p, which is the explicit series of couplings $z^{p^e} z$.

12. Couplings, the normal form and the count

We pass from one curve to products of theta functions in many variables. The variables are indexed by pairs (τ, i) of a place τ in a finite set Φ and a coordinate i in a finite set; a *partner rule* assigns to each exponent $e \geq 0$ and place τ a place $\mathrm{pr}_e \tau$. In the application Φ is the group of infinite places of E and $\mathrm{pr}_e \tau = -p^e \tau$. The main result, Theorem 12.12, bounds from below the valuation of the derivatives at 0 of a series in normal form along products of minors.

12.1. Couplings and the normal form

Let c be a finite set and $z = (z_{\tau,i})_{(\tau,i)\in\Phi\times c}$. For $e \geq 0$ the *coupling* is

$$\Gamma_e = \sum_{(\tau,i)} z_{\tau,i}^{p^e} z_{\mathrm{pr}_e\tau,i},$$

and for a set S of exponents and numbers $\lambda = (\lambda_{ej})$ in $\boldsymbol{O}$

$$\Xi_{S,\lambda} = \sum_{e\in S} \sum_{j\geq 0} \lambda_{ej}\, \Gamma_e^{p^j}.$$

The sum converges coefficientwise, since $\Gamma_e^{p^j}$ is homogeneous of degree $(p^e+1)p^j$. A series F over $\mathbb{C}_p$ in the variables z *has the normal form* $(S, \mathrm{pr}, \varpi, \lambda)$ if

$$F(\varpi z) = W \exp\bigl(c_\varpi\, \Xi_{S,\lambda}\bigr)$$

for an integral series W, where $c_\varpi = \varpi^{p+1}/p$.

Lemma 12.1. *Let $a \in \mathbb{C}_p$ and $\pi \in \boldsymbol{O}$ with $|a\pi| \leq p^{-1/(p-1)}$, and let G be an integral series without constant term whose coefficients of degree at most B are divisible by π. Then $\exp(aG)$ is integral in all degrees at most B.*

Proof. In degrees at most B the coefficients of G^k are divisible by π^k, and only $k \leq B$ occurs. The coefficients of $a^k G^k/k!$ in these degrees have valuation at least $k\, v(a\pi) - v(k!) \geq k/(p-1) - (k-1)/(p-1) > 0$, by Legendre's formula $v(k!) = (k - s_p(k))/(p-1)$ with $s_p(k) \geq 1$ the sum of the p-adic digits. □

Proposition 12.2. *Let F have the normal form $(S, \mathrm{pr}, \varpi, \lambda)$, and let $\pi \in \boldsymbol{O}$ with $|c_\varpi \pi| \leq p^{-1/(p-1)}$.*

(a) (Reduction modulo π.) *If $F(\varpi z) = W \exp(c_\varpi \Xi)$ with W, Ξ integral, $\Xi(0) = 0$ and $\Xi \equiv \Xi_{S,\lambda} \pmod{\pi}$, then F has the normal form $(S, \mathrm{pr}, \varpi, \lambda)$.*

(b) (Scaling.) *For $\alpha \in \boldsymbol{O}$ the series $F(\alpha z)$ has the normal form $(S, \mathrm{pr}, \varpi, \lambda')$ with $\lambda'_{ej} = \alpha^{(p^e+1)p^j} \lambda_{ej}$. If $\varpi \neq 0$ and $|\varpi'| = |\varpi|$, then F has a normal form $(S, \mathrm{pr}, \varpi', \lambda'')$ for some λ''.*

(c) (Substitution.) *If $Z = (Z_x)$ are integral series without constant term in other variables, then $F(\varpi Z) = W' \exp\bigl(c_\varpi\, \Xi_{S,\lambda}(Z)\bigr)$ with W' integral.*

(d) (Integrality.) *If moreover π divides $\Gamma_e(Z)$ for all $e \in S$ with $p^e \leq B$, then $F(\varpi Z)$ is integral in all degrees at most B.*

Proof. (a) We have

$$F(\varpi z) = W \exp(c_\varpi \Xi_{S,\lambda}) \exp\big(c_\varpi(\Xi - \Xi_{S,\lambda})\big),$$

and the last factor is integral by Lemma 12.1 applied in all degrees; we absorb it into W. (b) The series $F(\alpha\varpi z) = W(\alpha z)\exp(c_\varpi \Xi_{S,\lambda}(\alpha z))$ has the integral factor $W(\alpha z)$, and $\Gamma_e(\alpha z) = \alpha^{p^e+1}\Gamma_e(z)$. For the second statement write $\varpi' = u\varpi$ with $|u| = 1$; then $F(\varpi' z) = W(uz)\exp(c_{\varpi'} u^{-(p+1)}\Xi_{S,\lambda}(uz))$, which is the normal form with $\lambda''_{ej} = u^{(p^e+1)p^j-(p+1)}\lambda_{ej}$. (c) Substitute Z into $F(\varpi z) = W\exp(c_\varpi \Xi_{S,\lambda})$; substitution of series without constant term commutes with products, with convergent sums and with the exponential. (d) The series $\Gamma_e(Z)^{p^j}$ has order at least $p^e + 1$, so for $p^e > B$ it does not contribute in degrees at most B; for the other $e \in S$ it is divisible by π. Hence $\Xi_{S,\lambda}(Z)$ has coefficients divisible by π in degrees at most B, and (c) and Lemma 12.1 give the claim. □

For the sharpness of the count (Section 14) we also need the derivatives of $\Xi_{S,\lambda}$ modulo p. For a variable $y = (\tau', i)$ let $\Pi_{e,y} = \sum z_x^{p^e}$, the sum over the $x = (\tau, i)$ with $(\mathrm{pr}_e\tau, i) = y$.

Lemma 12.3. *If $0 \notin S$, then $\partial\Xi_{S,\lambda}/\partial z_y \equiv \sum_{e\in S}\lambda_{e0}\Pi_{e,y} \pmod p$. In particular, modulo $\mathfrak{m}$ the only monomials of $\partial\Xi_{S,\lambda}/\partial z_y$ are the $z_x^{p^e}$ with $e \in S$ and $(\mathrm{pr}_e\tau, i) = y$, with coefficient λ_{e0}.*

Proof. For $j \geq 1$ the derivative of G^{p^j} is $p^j G^{p^j-1}\partial G$, divisible by p. For $e \geq 1$ the derivative of $z_x^{p^e} z_{x'}$ is $z_x^{p^e}$ in $z_{x'}$ and divisible by p in z_x. Hence $\partial\Gamma_e/\partial z_y \equiv \Pi_{e,y}$ for $e \geq 1$, and only the terms $j = 0$ count. □

12.2. Resolutions of the identity and theta products

Let now Φ be the group of infinite places of $E = K(\zeta)$ (for the parameters and some s), and let n be a finite set. For $y \in E^n$ the linear form $y^\dagger z = \sum_{(\tau,i)}\overline{\Phi_\tau(y_i)}\, z_{\tau,i}$ has coefficients $y^\dagger_{\tau,i} = \overline{\Phi_\tau(y_i)}$. A family of vectors $u_j \in E^n$, $j \in J$, with multiplicities $m_j \in \mathbb{N}$ is a *resolution of the identity* if

$$z = \sum_j m_j\,(u_j^\dagger z)\,u_j \qquad \text{for all } z \in \mathbb{C}^{\Phi\times n},$$

where u_j is read as the vector $(\Phi_\tau(u_{j,i}))_{\tau,i}$. Equivalently $\sum_j m_j\overline{(u_j^\dagger z')}(u_j^\dagger z) = \sum_x \overline{z'_x} z_x$ (Parseval). An element $y \in E$ is *integral above p* if it is integral at every place of E above p; then $y = a/N$ with $a \in O_E$ and $N \in \mathbb{N}$ prime to p, and $|\sigma(y)| \leq 1$ for every embedding $\sigma\colon E \to \mathbb{C}_p$.

Let C be a CM lattice for d and θ, y its functions (Section 11.2). For linear forms μ_j over $\mathbb{C}_p$ and multiplicities m_j put

$$\Theta_{\mu,m} = \prod_j \theta(\mu_j z)^{m_j},$$

and let $O[[y(\mu z)]]$ be the ring of series $G\big((y(\mu_j z))_j\big)$ with G integral. If the μ_j are integral, $O[[y(\mu z)]]$ is stable under partial derivatives, since by Theorem 11.3(i) $\partial y(\mu_j z)/\partial z_x = \mu_{jx}(1 + 3x(\mu_j z)^2 - a_1 y(\mu_j z))$ and $x \in O[[y]]$; and scaling $z \mapsto az$ maps $\Theta_{\mu,m}$ to $\Theta_{a\mu,m}$ and $O[[y(\mu z)]]$ to $O[[y(a\mu z)]]$.

For a prime $p \nmid q_1$, the *partner* of $\tau \in \Phi$ at the exponent e is $\mathrm{pr}_e\tau = (-p^e)\,\tau$, the product of τ with the place of the class of $-p^e$ modulo q_1.

Lemma 12.4 (Partners). *Let p be non-split, $e \in I_p^\circ$, $\tau \in \Phi$ and $\tau' = \mathrm{pr}_e\tau$.*

(a) $\Phi_\tau(a)^{p^e} \equiv \overline{\Phi_{\tau'}(a)} \pmod{\pi_p}$ *for $a \in O_E$, and* $\overline{\Phi_\tau(y)}^{p^e} \equiv \Phi_{\tau'}(y) \pmod{\pi_p}$ *for $y \in E$ integral above p.*

(b) *For $y \in E^n$ integral above p,* $(y^\dagger z)^{p^e} \equiv \sum_{(\tau,i)} \Phi_{\mathrm{pr}_e\tau}(y_i)\, z_{\tau,i}^{p^e} \pmod{\pi_p}$.

(c) *For a resolution of the identity (u_j, m_j) by vectors integral above p,*

$$\sum_j m_j\,(u_j^\dagger z)^{p^e}(u_j^\dagger z) \equiv \Gamma_e \pmod{\pi_p}.$$

Proof. (a) Let $\varphi\colon O_E \to O/\pi_p$ be the reduction of $\iota_p \circ \Phi_1$, a ring homomorphism to a ring of characteristic p. Since $\Phi_v = \Phi_1 \circ g_v$ and complex conjugation commutes with $\Phi_{\tau'}$, the two sides of the first congruence are the ring homomorphisms $\psi_1(a) = \varphi(g_\tau a)^{p^e}$ and $\psi_2(a) = \varphi(g_{\tau'}\bar{a})$. As $O_E = \mathbb{Z}[\zeta, \beta]$ (Proposition 10.1), it suffices to compare them at ζ and at $\sqrt{-d}$. At ζ: $\psi_1(\zeta) = \varphi(\zeta)^{a_\tau p^e}$ and $\psi_2(\zeta) = \varphi(\zeta)^{-a_{\tau'}}$, equal since $a_{\tau'} \equiv -p^e a_\tau \pmod{q_1}$ (here $p \nmid q_1$). At $\sqrt{-d}$: $\psi_2(\sqrt{-d}) = -\varphi(\sqrt{-d})$. If p is inert, $\varphi(\sqrt{-d})^p = \chi_K(p)\varphi(\sqrt{-d}) = -\varphi(\sqrt{-d})$ as in the proof of Lemma 10.4, so $\psi_1(\sqrt{-d}) = (-1)^e\varphi(\sqrt{-d})$, which is $\psi_2(\sqrt{-d})$ since e is odd. If $p = d$, then $\varphi(\sqrt{-d}) = 0$ as $\pi_d = \sqrt{-d}$, and both sides vanish. For the second congruence apply the first to $\bar{a}$, where $y = a/N$ with N prime to p, and use $N^{p^e} \equiv N \pmod{p}$. (b) Modulo p the p^e-th power is additive; apply (a) to each coefficient. (c) By (b) the left side is congruent to $\sum_{(\tau,i)} z_{\tau,i}^{p^e} \sum_j m_j \Phi_{\mathrm{pr}_e\tau}(u_{j,i})(u_j^\dagger z)$, and the inner sum is the coordinate $(\mathrm{pr}_e\tau, i)$ of $\sum_j m_j (u_j^\dagger z) u_j = z$. □

Lemma 12.5. *Let $\pi \mid p$ in O, let Ξ be an integral series in one variable with $\Xi \equiv \sum_{e\in S}\sum_j \lambda_{ej}(X^{p^e}X)^{p^j} \pmod{\pi}$, and let L_i be integral series without constant term, with multiplicities m_i, such that $\sum_i m_i L_i^{p^e} L_i \equiv \Gamma_e \pmod{\pi}$ for $e \in S$. Then $\sum_i m_i \Xi(L_i) \equiv \Xi_{S,\lambda} \pmod{\pi}$.*

Proof. Substitution is continuous, so $\Xi(L_i) \equiv \sum_{e,j} \lambda_{ej}(L_i^{p^e} L_i)^{p^j}$. Modulo p, and hence modulo π, p^j-th powers are additive and $m^{p^j} \equiv m$ for integers m, so $\sum_i m_i (L_i^{p^e} L_i)^{p^j} \equiv (\sum_i m_i L_i^{p^e} L_i)^{p^j} \equiv \Gamma_e^{p^j}$. □

Theorem 12.6 (Normal form of theta products). *Let p be non-split and $C = \mathcal{E}_0$ the CM curve fixed in Section 13.2. There are $\omega \in \mathbb{C}_p$ with $|\omega| = p^{-\rho_p}$ and $\lambda_{ej} \in O$, with λ_{e0} a unit for odd e if p is inert, such that the following holds. For every resolution of the identity (u_j, m_j) by vectors of E^n integral above p and every $H \in O[[y(u^\dagger z)]]$, the series $\Theta_{u^\dagger,m} H$ has the normal form $(I_p^\circ, \mathrm{pr}, \omega, \lambda)$.*

Proof. Take ω, W, Ξ, λ from Theorem 11.8. The forms $\mu_j = u_j^\dagger$ are integral by Lemma 12.4. Then

$$\Theta_{\mu,m}(\omega z) = \prod_j W(\mu_j z)^{m_j} \cdot \exp\Big(c_\omega \sum_j m_j \Xi(\mu_j z)\Big),$$

and $H(\varpi z)$ is integral, because $y(\varpi w) = \varpi Y'(w)$ for an integral Y' (write y as an integral series in T and use $T(\varpi w) = \varpi Y(w)$). By Lemma 12.4(c) and Lemma 12.5 with $\pi = \pi_p$, $\sum_j m_j \Xi(\mu_j z) \equiv \Xi_{I_p^\circ,\lambda}$, and Proposition 12.2(a) applies, since $|c_\varpi \pi_p| = p^{-1/(p-1)}$ (Remark 11.9). □

12.3. Products of minors and the predicted divisibility

Let n, n' be finite sets of columns and rows, $N = \#n'$, and consider variables $z_{\tau,b,a}$, the entries $(b,a) \in n' \times n$ of matrices z_τ, one for each place τ. A *diagram* at the place τ is given by the numbers $\kappa(\tau,h)$ of its columns of height h; a *filling* i' puts a row $i'(\tau,h,j,b) \in n'$ into the box $b < h$ of the column j of height h at τ. For matrices y_τ with rows indexed by n and columns by $\mathbb{N}$, the *product of minors* $P_{i',y}$ is the polynomial

$$P_{i',y}(z) = \prod_\tau \prod_{h<N} \prod_{j<\kappa(\tau,h)} \det\Big(\big(z_\tau y_\tau\big)_{i'(\tau,h,j,b),k} \Big)_{b,k<h},$$

the product over the columns of height below N of the minor of $z_\tau y_\tau$ on the rows of the column and its first h columns. It is homogeneous of degree $B = \sum_\tau \sum_{h<N} h\kappa(\tau,h)$, the number of boxes in columns of height below N. For a polynomial Q and a series F we write $Q(\partial)F(0)$ for the value at 0 of $Q(\partial)F$, the *jet* of F along Q.

Among the heights below N let a_τ be the number of columns of height one at τ, a'_τ the number of the other columns and l_τ the number of their boxes. The *predicted divisibility* at a non-split prime p is

$$D_p(\kappa) = v(\pi_p) \sum_{e\in I_p} \sum_\tau \Big(\Big(\frac{a_\tau}{p^e} - a_{\tau'} - a'_{\tau'} \Big)_+ + \Big(\frac{l_\tau}{p^e} - l_{\tau'} \Big)_+ - \frac{a'_\tau}{p^e} \Big) \\ - [p=d] \frac{1}{2} \sum_\tau \big(\min(a_\tau, a_{\tau''} + a'_{\tau''}) + a'_\tau + \min(l_\tau, l_{\tau''}) \big),$$

where $\tau' = \mathrm{pr}_e \tau$ and $\tau'' = \mathrm{pr}_0 \tau$. For a set $\mathcal{M}$ of *marked* places let κ° be κ without its columns of height one at the marked places, and for places σ, σ' and $e \geq 0$ let the *deduction term* be

$$\delta_e(\sigma,\sigma') = \min\Big(\frac{t_\sigma}{p^e},\, r_{\sigma'} + t_{\sigma'} \Big) + \frac{r_\sigma}{p^e} + \min\Big(\frac{B_\sigma}{p^e},\, B_{\sigma'} \Big),$$

where t_σ is the number of columns of height one at σ (0 if σ is marked), $r_\sigma = a'_\sigma$ and $B_\sigma = l_\sigma$.

Lemma 12.7. *Without marked places,* $v(\pi_p) \sum_{e\in I_p^\circ} \sum_\tau \delta_e(\tau, \mathrm{pr}_e \tau) = \gamma_p B - D_p(\kappa)$, *the series converging. With marked places and for every* e_0,

$$v(\pi_p) \sum_{e\in I_p^\circ,\, e\leq e_0} \sum_\tau \delta_e(\tau, \mathrm{pr}_e \tau) \leq \gamma_p B^\circ - D_p(\kappa^\circ),$$

where B° *is the number of boxes of* κ° *in columns of height below* N.

Proof. $B = \sum_\tau (a_\tau + l_\tau)$, so by Lemma 11.2(a) $\gamma_p B = v(\pi_p) \sum_{e\in I_p} p^{-e} \sum_\tau (a_\tau + l_\tau)$. For $e \in I_p$ and without marks, $p^{-e}(a_\tau + l_\tau) - \delta_e(\tau,\tau')$ is the summand of D_p, by $u - \min(u,u') = (u-u')_+$; and for $p = d$ the term $e = 0$ of $v(\pi_p) \sum \delta_e$ is the subtracted half sum. The terms are nonnegative and $O(p^{-e})$, which gives convergence and the identity. Marking the places of $\mathcal{M}$ changes δ_e exactly

as removing their columns of height one does, and truncating at e_0 drops nonnegative terms. □

12.4. Changing the directions

Two rows of matrices at the places τ and $\tau' = \mathrm{pr}_e\tau$ are paired by $[\beta, \beta']_e = \sum_a \beta_a^{p^e}\beta'_a$. A pair of heights, h at σ and h' at σ', is *exempt* for $\mathcal{M}$ if $h = 1$ and either $\sigma \in \mathcal{M}$, or $h' = 1$ and $\sigma' \in \mathcal{M}$.

Lemma 12.8 (Elimination). *Let $A_i \in \mathcal{O}^m$ $(i \in I, I$ finite$)$ and $\alpha_i \in \mathcal{O}$. There is $g \in M_m(\mathcal{O})$ with $\det g \mid \prod_i \alpha_i$ such that α_i divides $(gA_i)_k$ for all i and k.*

Proof. For one form A with values $a_k = A_k$: choose k_0 with $v(a_{k_0})$ least; subtracting multiples of row k_0 from the other rows makes their values 0, and multiplying row k_0 by α makes it divisible by α; the determinant of these row operations is α. For several forms compose such operations; later operations are integral row operations and keep the earlier divisibilities, and the determinants multiply. □

Proposition 12.9 (New directions). *Let $\pi \mid p$ in $\mathcal{O}$, S a finite set of exponents, $\mathcal{M}$ a set of places, and $y_\tau \in \mathcal{O}^{n\times\mathbb{N}}$ directions at each place. There are matrices $g_{\tau,h} \in M_h(\mathcal{O})$, one for each place and height, such that the new directions $\beta_{\tau,h,k} = \sum_{k'<h}(g_{\tau,h})_{kk'}y_{\tau,\cdot,k'}$ satisfy:*

(i) *π divides $[\beta_{\tau,h,k}, \beta_{\mathrm{pr}_e\tau,h',k'}]_e$ for all $e \in S$, all τ and all pairs of heights (h, h') that are not exempt;*
(ii) $|\pi|^{\sum_{e\in S}\sum_\tau \delta_e(\tau,\mathrm{pr}_e\tau)} \le \prod_\tau \prod_{h<N} |\det g_{\tau,h}|^{\kappa(\tau,h)}$.

Proof. Let $\omega_e \in \mathcal{O}$ be a p^e-th root of π. Four kinds of congruence can be imposed on the rows of $g_{\tau,h}$, for a source σ and its partner $\sigma' = \mathrm{pr}_e\sigma$:

(1) at σ, all h entries of each row $\equiv 0 \pmod{\omega_e}$; then $\beta \equiv 0 \pmod{\omega_e}$ and $\beta^{p^e} \equiv 0 \pmod{\pi}$;
(2) at σ, the linear form $\sum_a \omega_e(y'_{a,0})\beta_a \equiv 0 \pmod{\omega_e}$, where $\omega_e(\cdot)$ denotes a chosen p^e-th root and $y'_{\cdot,0}$ is the direction 0 at σ'; then $\pi \mid [\beta, y'_{\cdot,0}]_e$, because p^e-th powers are additive modulo p;
(3) at σ', all entries of each row $\equiv 0 \pmod{\pi}$;
(4) at σ', the form $\sum_a y_{a,0}^{p^e}\beta'_a \equiv 0 \pmod{\pi}$, where $y_{\cdot,0}$ is the direction 0 at σ.

A row of height one is a multiple of the direction 0, so a pair of heights (h, h') is *covered*, that is $\pi \mid [\beta, \beta']_e$ for all rows of these heights, if (1) is imposed at height h, or (3) at height h', or (2) at h and $h' = 1$, or (4) at h' and $h = 1$. Each congruence modulo ω_e costs $v(\pi)/p^e$ and each congruence modulo π costs $v(\pi)$ in the determinant.

For fixed (e, σ) impose: (2) at every height $h \neq 1$; (1) at $h = 1$ if $\sigma \notin \mathcal{M}$ and $t_\sigma/p^e \le r_{\sigma'} + t_{\sigma'}$, and at $h \neq 1$ if $B_\sigma/p^e \le B_{\sigma'}$; (4) at h' if $t_\sigma/p^e > r_{\sigma'} + t_{\sigma'}$ and, for $h' = 1$, $\sigma' \notin \mathcal{M}$; (3) at $h' \neq 1$ if $B_\sigma/p^e > B_{\sigma'}$. Counting the congruences over all columns of height below N, those at σ weighted by $1/p^e$, gives exactly $\delta_e(\sigma, \sigma')$: the first minimum chooses between the height-one rows at σ and the pairings at σ', the term r_σ/p^e pays for the pairings at σ, and the second minimum chooses between the entries of the other rows at σ and at σ'. Every pair that is not exempt is covered: if $h = 1$, then $\sigma \notin \mathcal{M}$, and either (1) is imposed at height one or (4)

is imposed at h' (when $h' = 1$ the pair is not exempt only if $\sigma' \notin \mathcal{M}$); if $h \neq 1$ and $h' = 1$, (2) covers it; if $h, h' \neq 1$, the second minimum imposes (1) or (3).

All congruences are linear conditions on the rows of $g_{\tau,h}$ that depend on y only. Collect, for each (τ, h), those imposed from all (e, σ) with $\tau = \sigma$ or $\tau = \mathrm{pr}_e \sigma$, and apply Lemma 12.8: $|\det g_{\tau,h}|$ is at least the absolute value of the product of the moduli. Taking the product over (τ, h), once for each of the $\kappa(\tau, h)$ columns, the absolute value of the product of all moduli is $|\pi|$ raised to $\sum_{e \in S} \sum_{\sigma} \delta_e(\sigma, \mathrm{pr}_e \sigma)$, which is (ii). □

12.5. The count

A family of *charges* is a family of exponents $w(\tau, h, b) \geq 0$ (place, height, row); it *covers the exempt pairs* for $\mathcal{M}$ if for all $e \in I_p^\circ$, τ, h and rows b, whenever $(1, h)$ at $(\tau, \mathrm{pr}_e \tau)$ is exempt, $w(\tau, 1, b) \neq 0$ or $w(\mathrm{pr}_e \tau, h, b) \neq 0$. The *charge* of a filling is $|w|_{i'} = \sum w(\tau, h, i'(\tau, h, j, b))$ over its boxes in columns of height below N.

Lemma 12.10 (Legendre's bound). *Let G be a series in r variables over $\mathbb{C}_p$ that is integral in degree l, and Q a homogeneous integral polynomial of degree l. Then $v\big(Q(\partial)G(0)\big) \geq l/(p-1) - r(1 + \lfloor \log_p l \rfloor)$.*

Proof. $Q(\partial)G(0) = \sum_{|d|=l} Q_d \, d! \, G_d$ with $d! = \prod_x d_x!$. By Legendre's formula $v(m!) = (m - s_p(m))/(p-1) \geq m/(p-1) - (1 + \lfloor \log_p m \rfloor)$, since m has at most $1 + \lfloor \log_p m \rfloor$ digits. Sum over the r variables. □

Lemma 12.11 (Chain rule). *Let $c_{\tau,h,b}$ be row factors and $g_{\tau,h}$ matrices, β the new directions of Proposition 12.9, and consider variables $s_{\tau,h,k,b}$ $(h < N, k < h, b \in n')$ and the linear substitution*

$$z_{\tau,b,a} = \sum_{h,k} c_{\tau,h,b} \, \beta_{\tau,h,k,a} \, s_{\tau,h,k,b}.$$

Let $P_*(s) = \prod_{columns} \det(s_{\tau,h,k,i'(\tau,h,j,b)})_{b,k<h}$. Then for every series F

$$P_*(\partial_s) F(z(s))\big|_0 = \prod_{\tau} \prod_{h<N} \prod_{j<\kappa(\tau,h)} \Big(\det g_{\tau,h} \prod_{b<h} c_{\tau,h,i'(\tau,h,j,b)} \Big) \cdot P_{i',y}(\partial_z) F(0).$$

Proof. For a linear substitution the derivative in $s_{\tau,h,k,b}$ acts as

$$\frac{\partial}{\partial s_{\tau,h,k,b}} = c_{\tau,h,b} \sum_a \beta_{\tau,h,k,a} \frac{\partial}{\partial z_{\tau,b,a}}.$$

For one column with rows $b \in R$, the matrix $\big(c \sum_a \beta_{k,a} \partial_{z_{b,a}}\big)_{b,k}$ is the matrix $\big(\sum_a y_{a,k'} \partial_{z_{b,a}}\big)_{b,k'}$ multiplied on the right by g^T and with its rows scaled by the $c_{\tau,h,b}$. Taking determinants and the product over the columns gives the formula, since the operators commute. □

Theorem 12.12 (The count). *Let p be non-split, $|\varpi| = p^{-\rho_p}$, and let F have the normal form $(I_p^\circ, \mathrm{pr}, \varpi, \lambda)$ in the variables $z_{\tau,b,a}$. Let $\mathcal{M}$ be a set of places and w charges that cover the exempt pairs for every $e \in I_p^\circ$ for which the heights concerned occur in κ. Then for every filling i' and integral directions y,*

$$v\big(P_{i',y}(\partial) F(0)\big) + |w|_{i'} \;\geq\; D_p(\kappa^\circ) + \gamma_p(B - B^\circ) - f N^3 \big(1 + \log_p \max(B, 1)\big),$$

where $f = \#\Phi$. In particular, without marked places and charges, $v\big(P_{i',y}(\partial)F(0)\big) \geq D_p(\kappa) - fN^3(1 + \log_p \max(B,1))$.

Proof. Charges at heights that do not occur may be changed to 1 without changing $|w|_{i'}$, so we may assume that w covers all exempt pairs. Let $\pi = \pi_p$, S the set of $e \in I_p^\circ$ with $e \leq \log_p B$, and $D = \sum_{e\in S}\sum_\tau \delta_e(\tau, \mathrm{pr}_e\tau)$.

Step 1. Take g and β from Proposition 12.9, and the substitution of Lemma 12.11 with $c_{\tau,h,b} = p^{w(\tau,h,b)}$. Let $G(s) = F(\varpi z(s))$. By Lemma 12.11, with the factor ϖ for each box,

$$|P_*(\partial)G(0)| = |\varpi|^B p^{-|w|_{i'}} \prod_{\tau,h} |\det g_{\tau,h}|^{\kappa(\tau,h)} |P_{i',y}(\partial)F(0)|.$$

Step 2: G is integral in degree B. By Proposition 12.2(d) it suffices that $\pi \mid \Gamma_e(z(s))$ for $e \in S$. Modulo p,

$$\Gamma_e(z(s)) \equiv \sum_{o,o'} s_o^{p^e} s_{o'} \sum_x z_{x,o}^{p^e} z_{\mathrm{pr}x,o'},$$

where $z_{x,o}$ is the coefficient of s_o in z_x. Let $o = (\tau,h,k,b)$. The inner sum vanishes unless $o' = (\mathrm{pr}_e\tau, h', k', b)$ for some h', k', and then it equals

$$c_{\tau,h,b}^{p^e}\, c_{\mathrm{pr}_e\tau,h',b}\, [\beta_{\tau,h,k}, \beta_{\mathrm{pr}_e\tau,h',k'}]_e.$$

For pairs that are not exempt the pairing is divisible by π; for exempt pairs one of the charges is a positive power of p.

Step 3. P_* is homogeneous of degree B in at most fN^3 variables, so Lemma 12.10 gives $v(P_*(\partial)G(0)) \geq B/(p-1) - fN^3(1 + \lfloor \log_p B \rfloor)$. By Proposition 12.9(ii), $v(\prod \det g^\kappa) \leq v(\pi)D$. Combining with Step 1,

$$\begin{aligned} v\big(P_{i',y}(\partial)F(0)\big) + |w|_{i'} &\geq \frac{B}{p-1} - \rho_p B - v(\pi)D - fN^3(1 + \log_p B) \\ &= \gamma_p B - v(\pi)D - fN^3(1 + \log_p B). \end{aligned}$$

Finally Lemma 12.7 with $e_0 = \lfloor \log_p B \rfloor$ gives $\gamma_p B - v(\pi)D \geq \gamma_p B - \gamma_p B^\circ + D_p(\kappa^\circ)$. □

12.6. Entries of Schur matrices

For each place τ the products of minors span the module of polynomials in an $n \times \mathbb{N}$ array that are semi-invariant under upper triangular matrices acting on the columns, with the weight given by $\kappa(\tau,\cdot)$; a basis is formed by the products of minors of chosen standard fillings (Appendix B.11). For a family $z = (z_\tau)$ of $n' \times n$ matrices, left substitution $f \mapsto f(zy)$ maps the module for n to that for n', and its matrix in the chosen bases is the *Schur matrix* $\mathrm{Sch}_\kappa(z)$, with rows indexed by the standard fillings i' for n' and columns by those i for n. The diagram κ *fits the ranks* if $\#n \leq \#n'$ and κ has no column of height strictly between $\#n$ and $\#n'$. For a series F in the entries of z we write $\mathrm{Sch}_\kappa(F)_{i'i}$ for the jet of F along the polynomial entry (i', i) of $\mathrm{Sch}_\kappa(z)$.

Proposition 12.13. *Let κ fit the ranks. For every series F over $\mathbb{C}_p$ in the entries of z and standard fillings i', i there are integral directions y with $|\mathrm{Sch}_\kappa(F)_{i'i}| \leq |P_{i',y}(\partial)F(0)|$.*

Proof. By the definition of the Schur matrix, $P_{i',y} = \sum_i \big(b_i \cdot \Delta\big)(y)\,\mathrm{Sch}_\kappa(z)_{i'i}$, where b_i is the basis element of the filling i and Δ is the product over τ of the determinant of the first $\#n$ columns of y_τ raised to $\kappa(\tau,\#n)$ if $\#n < \#n'$, and 1 otherwise. Hence $P_{i',y}(\partial)F(0) = \sum_i \alpha_i\,(b_i\Delta)(y)$ with $\alpha_i = \mathrm{Sch}_\kappa(F)_{i'i}$. The polynomials $b_i\Delta$ have integral coefficients and stay linearly independent over the residue field, since the basis is defined over $\mathbb{Z}$. Dividing by an α_{i_0} of largest absolute value and reducing, we get a nonzero polynomial over the residue field, which is algebraically closed and hence infinite; at an integral point y where the reduction does not vanish, $|\sum_i \alpha_i(b_i\Delta)(y)| = |\alpha_{i_0}| \geq |\alpha_i|$ for the given i. □

13. The integral vacuum and standard data

The theta functions whose derivatives enter the argument are those of vectors of the adelic Weil representation of hermitian spaces over E. This section fixes the scale at which their Taylor series are taken, constructs an integral multiple θ° of the vacuum, defines the *standard data* (integral combinations of translates of θ°), and proves that their derivatives are integral (Theorem 13.19) and have the divisibility of Theorem 12.12 at the non-split primes (Theorem 13.21). We fix the parameters and s, and write $E = E(s)$.

13.1. The Weil representation

We recall the objects of the adelic Weil representation in the form in which they are used; the construction is reviewed in Appendix B.10. Let $\mathbb{A} = \mathbb{C}^\Phi \times \mathbb{A}_\mathrm{f}$, one copy of $\mathbb{C}$ for each infinite place, so that $\mathbb{A}$ is the adele ring of E. A *hermitian space with a point* V of rank n over E consists of a hermitian matrix $J \in \mathrm{GL}_n(E)$ and $x \in \mathrm{GL}_n(\mathbb{A})$ such that x^*Jx is the diagonal matrix of signs $\varepsilon \in \{\pm 1\}^n$ at the infinite places and the identity at the finite places. It is *standard* if all signs are $+1$ and $J = x = 1$.

(W1) The Weil data of Proposition 10.10 define a Heisenberg bicharacter on $\mathbb{A}^n$ and a *vacuum function*: a Gaussian at the infinite places times a function of modulus one on the lattice $\alpha\widehat{\mathcal{O}}_E^n$ (zero off it). The space $\mathcal{S}(V)$ of *data* is the corresponding reproducing model, a Banach space on which $\mathbb{A}^n$ acts by Heisenberg operators $\varrho(x)$ and the adelic unitary group of V acts by the Weil representation ω; it carries an inner product $\langle\cdot,\cdot\rangle$, and the vacuum $\mathrm{vac} \in \mathcal{S}(V)$ is a unit vector.

(W2) A *theta functional* is a bounded linear form on $\mathcal{S}(V)$ invariant under the Heisenberg operators of rational vectors and normalised so that $|\theta(\varrho(x)\mathrm{vac})|^2$ has mean 1 over $\mathbb{A}^n$ modulo rational vectors. It exists and is unique up to a factor of modulus one; we fix one, θ_V, compatibly with orthogonal sums. For $f \in \mathcal{S}(V)$ and $x \in \mathbb{A}^n$ we write $f(x) = \theta_V(\varrho(x)f)$, the *theta function* of f. The inner product $\langle f, f\rangle$ is the mean of $|f(x)|^2$.

(W3) For a finite adelic vector x let $\mathrm{fin}(x)$ be the adelic vector with finite part αx and archimedean part 0, and for $\gamma \in E^n$ let $\gamma^\mathrm{f} = \alpha^{-1}\gamma$; thus $\mathrm{fin}(\gamma^\mathrm{f})$ is the finite part of the rational vector γ. The *finite Heisenberg operator* of x is $\varrho(\mathrm{fin}(x))$ times a root of unity, the value of a semicharacter of the lattice. The *finite part* $\mathcal{S}(V)^\mathrm{f}$ is the span of the images of the vacuum

under the finite Heisenberg operators. The lattice $\mathfrak{a}_0^n$ consists of the $\gamma \in E^n$ with γ^{f} integral.

(W4) For standard V and $f \in \mathcal{S}(V)$, the function $\vartheta_f(w) = f(w)\exp(|w|^2/2)$ of $w \in \mathbb{C}^{\Phi\times n}$ (the archimedean coordinates of $\mathbb{A}^n$ in Fock's normalisation) is entire if $f \in \mathcal{S}(V)^{\mathrm{f}}$. A function F on $\mathbb{C}^{\Phi\times n}$ is a *theta function for a lattice* $Y \subseteq E^n$ if it is entire and fixed by the signed translations along the real images of the elements of Y. Translation of f by the finite image of $\gamma \in E^n$ acts on ϑ_f as the twisted translation by minus the real image of γ, whose (τ, i)-coordinate is $\sqrt{2\pi/\sqrt{d}}\,\Phi_\tau(\gamma_i)$.

(W5) The *jet series* of f at the scale $a \in \mathbb{C}$ is the series $J_a(f)$ in variables $z_{\tau,i}$ whose coefficient of z^k is $a^{|k|}/k!$ times $\theta_V(v_{z^k} \otimes f)$, where v_{z^k} is the Fock vector of the monomial z^k and $\otimes$ integrates the real Heisenberg action. For a homogeneous polynomial Q of degree l, $Q(\partial)J_a(f)(0) = a^l\,\theta_V(v_Q \otimes f)$; and $J_{ab}(f)(z) = J_b(f)(az)$.

(W6) (*Fourier inversion.*) Let $\delta \in \widehat{O}_E$ with $\delta\bar{\delta} = N \neq 0$. If V is standard, every theta function for $\bar{\delta}\mathfrak{a}_0^n$ is ϑ_F for some $F \in \mathcal{S}(V)^{\mathrm{f}}$; and if $\theta_V(F) = 1$, a nonzero multiple of the vacuum is a sum of $N^{f\cdot n}$ finite Heisenberg translates of one translate $\varrho(\mathrm{fin}(b))F$. If the values of $\varphi \in \mathcal{S}(V)$ at the points $\mathrm{fin}(x)$ are integral at p, so are those of such a multiple of the vacuum.

13.2. The scale

Fix a nonzero $\Omega_0 \in \mathbb{C}$ and a CM lattice $\mathcal{E}_0$ for d whose lattice is $\Omega_0 O_K$, the *CM curve*. By the formula of Chowla and Selberg (Theorem B.21) there is $u \in \mathbb{C}^\times$, a unit at every p-adic place of $\mathbb{C}$, such that for every automorphism σ of $\mathbb{C}$ over K the conjugate lattice $\mathcal{E}_0^\sigma$ has area

$$\mathrm{area}(\mathcal{E}_0^\sigma)\,|\sigma(u)|^2 = \pi\, e^{c_K}\,\frac{2\pi}{\sqrt{d}}, \qquad c_K = \gamma - \mathrm{Re}\,\frac{L'}{L}(1,\chi_K).$$

Put

$$a_{\mathrm{ar}} = \sqrt{2\pi/\sqrt{d}}, \qquad \varrho_0 = \frac{s_0}{u}, \qquad \mathsf{a} = \frac{\varrho_0\, a_{\mathrm{ar}}}{\Omega_0},$$

the *area scale*, the *period factor* and the *jet scale*; s_0 is the vacuum root of Definition 10.11.

Proposition 13.1. (a) $|\mathsf{a}|^2 = \Omega^{-1}$, *with* Ω *as in* (8.1). *More generally, if* $\mathrm{area}(\mathcal{E}_0^\sigma) = |\Omega_\sigma|^2\sqrt{d}/2$, *then* $|\sigma(\varrho_0)a_{\mathrm{ar}}/\Omega_\sigma|^2 = \Omega^{-1}$.

(b) *At a* p*-adic place* ι_p *and* $\tau \in \Phi$, $|\varrho_0| = |s_0|$ *equals the inverse of the absolute value of* α *at the place of the embedding* $y \mapsto \iota_p(\Phi_\tau(\bar{y}))$. *In particular* $|\varrho_0| = 1$ *if* $p \nmid q_1$.

(c) *For* $y \in \mathfrak{a}_0$, $|\varrho_0\overline{\Phi_\tau(y)}| \le 1$ *at every* p*-adic place. More generally, if* $y \in E^n$ *has* y^{f} *integral at the places above* p, *then the coefficients of the linear form* $\varrho_0\, y^\dagger$ *are integral at* ι_p.

Proof. (a) The area of $\Omega_\sigma \mathcal{O}_K$ is $|\Omega_\sigma|^2\sqrt{d}/2$, since $\mathcal{O}_K$ has covolume $\sqrt{d}/2$. By Proposition 10.12 and the Chowla–Selberg formula,

$$\left|\frac{\sigma(\varrho_0)a_{\mathrm{ar}}}{\Omega_\sigma}\right|^2 = \frac{\mathrm{rd}_E}{\sqrt{d}}\cdot\frac{2\pi}{\sqrt{d}}\cdot\frac{\sqrt{d}/2}{\pi e^{c_K}2\pi/\sqrt{d}} = \frac{\mathrm{rd}_E}{2\pi e^{c_K}} = \Omega^{-1}.$$

For $\sigma = 1$ use $\Omega_1 = \Omega_0$. (b) Let $\varphi_\tau(y) = \iota_p(\Phi_\tau(\bar{y}))$. All conjugates of $P'(\zeta)$ over K have the same absolute value under ι_p (Lemma 10.9(a)), so $|\varphi_\tau(\alpha)|$ does not depend on τ. The number $\chi_{\mathfrak{f}}(\alpha)$ is algebraic, and by the description of the algebraic values of the splitting character (Theorem B.18(c)) its absolute value at ι_p is $\prod_\tau|\varphi_\tau(\alpha)|$. Hence $|s_0|^f = |\chi_{\mathfrak{f}}(\alpha)|^{-1} = \prod_\tau|\varphi_\tau(\alpha)|^{-1} = |\varphi_\tau(\alpha)|^{-f}$. As u is a unit, $|\varrho_0| = |s_0|$. If $p \nmid q_1$, then α is a unit above p. (c) $|\varphi_\tau(y)| = |\varphi_\tau(\alpha^{-1}y)|\,|\varphi_\tau(\alpha)|$, and $|\varphi_\tau(\alpha^{-1}y)| \le 1$ when $\alpha^{-1}y$ is integral at the place of φ_τ. □

Lemma 13.2. *For standard V, $f \in \mathcal{S}(V)^{\mathrm{f}}$ and $a \in \mathbb{C}$: the Taylor series at 0 of $t \mapsto \vartheta_f(a\,t\,v)$ is $J_a(f)$ restricted to the line spanned by v; $J_a(f) = 0$ with $a \neq 0$ implies $f = 0$; and if $\vartheta_f(cz) = \prod_i G_i(\mu_i \cdot z)$ for entire G_i and linear forms μ_i, then $\theta_V(f) = \prod_i G_i(0)$ and $J_c(f) = \prod_i T_0(G_i)(\mu_i z)$, where T_0 is the Taylor series at 0.*

Proof. The line. In Fock's model the theta function of f is an integral of f against the real Gaussian kernel. On the line through av this gives

$$\vartheta_f(a\,t\,v) = \int \exp\bigl(t\,L(w)\bigr)\,g(w)\,dw$$

for a conjugate-linear form L on the archimedean coordinates and an integrable function g built from f and the Gaussian vacuum. Expand the exponential. The partial sums of $\sum_k t^k L(w)^k g(w)/k!$ are dominated by

$$\exp\bigl(|t|\,\|L\|\,\|w\|\bigr)\,|g(w)|,$$

which is integrable because g decays like a Gaussian. By dominated convergence the integral is the power series $\sum_k t^k \int L(w)^k g(w)\,dw/k!$, so its k-th derivative at 0 is $\int L(w)^k g(w)\,dw$. By (W5) this is $a^k\,\theta_V(v_{Q_k} \otimes f)$ for the polynomial $Q_k = (v \cdot z)^k$, that is, the jet of $J_a(f)$ along Q_k. Hence the Taylor series on the line is the restriction of $J_a(f)$ to the line.

Vanishing. If $J_a(f) = 0$ with $a \neq 0$, then on every line through 0 the entire function ϑ_f has Taylor series 0, so it vanishes on the line, hence everywhere; and f is determined by its theta function, hence by its values at the real points.

Products. A power series is determined by its restrictions to the lines through 0: the homogeneous component of degree n evaluated at v is the coefficient of t^n of the restriction to the line of v, and a polynomial over $\mathbb{C}$ is determined by its values. If $\vartheta_f(cz) = \prod_i G_i(\mu_i \cdot z)$, then on the line spanned by v both $J_c(f)$ and $\prod_i T_0(G_i)(\mu_i z)$ restrict to the Taylor series of $t \mapsto \prod_i G_i((\mu_i \cdot v)t)$, by the first part; so they agree. At $z = 0$, $\theta_V(f) = \vartheta_f(0) = \prod_i G_i(0)$. □

13.3. Tight frames

Definition 13.3. A *tight frame of length M* is a finite family $w_i \in \mathfrak{a}_0$ with multiplicities $m_i \in \mathbb{N}$ such that

$$\sum_i m_i\,\Phi_\sigma(w_i)\overline{\Phi_\tau(w_i)} = M\,\delta_{\sigma\tau} \qquad (\sigma, \tau \in \Phi).$$

A vector $y \in E$ has a *unit linear form* at p if $|\iota_p(\varrho_0\overline{\Phi_\tau(y)})| = 1$ for all τ.

Lemma 13.4. (a) *$\mathfrak{a}_0$ is an O_E-module stable under $\mathrm{Gal}(E/K)$, with $O_E \subseteq \mathfrak{a}_0$ and $q_1\mathfrak{a}_0 \subseteq O_E$; $y \in \mathfrak{a}_0$ if and only if $P'(\zeta)y$ is integral at the places containing β and y is integral elsewhere.*

(b) *For $x, y \in \mathfrak{a}_0$, $\sum_\tau \Phi_\tau(x)\overline{\Phi_\tau(y)} \in O_K$; in particular $\sum_\tau|\Phi_\tau(y)|^2 \in \mathbb{N}$. If $zP'(\zeta)$ is integral, then $\mathrm{Tr}_{E/K}(z) \in O_K$.*

(c) *For each embedding $\varphi\colon E \to \mathbb{C}_p$ there is y with αy integral and $|\varphi(\alpha y)| = 1$.*

Proof. (a) By Definition 10.8, $y \in \mathfrak{a}_0$ means that $\alpha_w^{-1}y$ is integral at every finite w, where α_w^{-1} is $P'(\zeta)$ at the places containing β and 1 elsewhere; this is the stated criterion, and it is preserved by sums and by multiplication by O_E. As α^{-1} is integral (Lemma 10.9(b)), $O_E \subseteq \mathfrak{a}_0$; as $q_1\alpha$ is integral, $q_1 y = (q_1\alpha)(\alpha^{-1}y)$ is integral for $y \in \mathfrak{a}_0$. Stability under g_v: g_v permutes the places containing β, because $\beta \in K$ is fixed by g_v, and it maps $P'(\zeta)$ to another generator of the different of E/K (Proposition 10.1(b)), so it preserves the order of $P'(\zeta)$ at these places. (b) The trace maps the inverse different $(P'(\zeta))^{-1}$ of E/K into O_K. For $x, y \in \mathfrak{a}_0$, $x\bar{y}P'(\zeta) = (\alpha\bar{\alpha}P'(\zeta))(\alpha^{-1}x)\overline{(\alpha^{-1}y)}$ is integral by Lemma 10.9(b), so $\mathrm{Tr}_{E/K}(x\bar{y}) = \sum_\tau \Phi_\tau(x)\overline{\Phi_\tau(y)}$ lies in O_K. For $x = y$ it is real and nonnegative; a real element of O_K is a rational integer, since the real subfield of K is $\mathbb{Q}$. For the last statement apply the first argument to z: if $zP'(\zeta)$ is integral, z lies in the inverse different of E/K, which the trace maps into O_K. (c) Let w be the place of φ. As E is dense in the finite adeles, some $y \in E$ lies in the open set of the x with $\alpha x \in 1 + \mathfrak{m}_w$ at w and αx integral at all other places. Then αy is integral and is a unit at w, so $|\varphi(\alpha y)| = 1$. □

Lemma 13.5 (Frames from roots of unity). (a) *The powers ζ^i, $0 \le i < q_1$, form a tight frame of length q_1; taken k times each, of length kq_1.*

(b) *For $y \in \mathfrak{a}_0$ with $\nu = \sum_\tau|\Phi_\tau(y)|^2$, the vectors $\zeta^i g(y)$, $i < q_1$, $g \in \mathrm{Gal}(E/K)$, form a tight frame of length $q_1\nu$.*

(c) *Disjoint unions of tight frames are tight frames whose lengths add. If w is a tight frame of length M with multiplicities one, and disjoint pairs of members $(w_j, w_{j'})$ are chosen whose cross terms satisfy $\sum_{pairs}\big(\Phi_\sigma(w_j)\overline{\Phi_\tau(w_{j'})} + \Phi_\sigma(w_{j'})\overline{\Phi_\tau(w_j)}\big) = t\,\delta_{\sigma\tau}$ with $t \in \mathbb{Z}$, then replacing each pair by its sum gives a tight frame of length $M + t$.*

Proof. (a) $\sum_i \Phi_\sigma(\zeta)^i\overline{\Phi_\tau(\zeta)}^i = \sum_i(\Phi_\sigma(\zeta)/\Phi_\tau(\zeta))^i$ is q_1 if $\Phi_\sigma(\zeta) = \Phi_\tau(\zeta)$, which happens only for $\sigma = \tau$, and 0 otherwise. The members lie in $O_E \subseteq \mathfrak{a}_0$. (b) By (a) the sum over i leaves $q_1\delta_{\sigma\tau}\sum_g|\Phi_\sigma(gy)|^2 = q_1\nu\,\delta_{\sigma\tau}$, and $g(y) \in \mathfrak{a}_0$ by Lemma 13.4. (c) Expand $(w_j + w_{j'})(\cdots)$. □

Let $j_1 = \lceil \log_2 q_1 \rceil$, and let b_i $(i < \varphi(q_1))$ be the coefficients of the monic polynomial $P(X)/(X - \zeta)$ over O_E. Put

$$e_i = \bar{\beta}^{j_1}\,\frac{b_i}{P'(\zeta)} \qquad (0 \le i < \varphi(q_1)).$$

Lemma 13.6. (a) *(Euler)* $\mathrm{Tr}_{E/K}\big(\zeta^k b_i/P'(\zeta)\big) = \delta_{ik}$ *for* $i,k < \varphi(q_1)$*; hence* $\sum_i \Phi_\sigma(e_i)\overline{\Phi_\tau(\zeta^{-i})} = \bar\beta^{j_1}\delta_{\sigma\tau}$ *and* $\sum_i \Phi_\sigma(\zeta^{-i})\overline{\Phi_\tau(e_i)} = \beta^{j_1}\delta_{\sigma\tau}$.
(b) $q_1 \mid ((d+1)/4)^{j_1}$*, and* $e_i \in \mathfrak{a}_0$.
(c) $t = \beta^{j_1} + \bar\beta^{j_1}$ *is an integer prime to* q_1.

Proof. (a) Euler's formula for the dual basis of a monogenic extension: the basis $b_i/P'(\zeta)$ is dual to (ζ^k) under the trace form. Writing the trace as $\sum_\sigma \Phi_\sigma$, the matrices $(\Phi_\sigma(b_i/P'(\zeta)))_{i,\sigma}$ and $(\Phi_\sigma(\zeta^k))_{\sigma,k}$ are inverse to each other, hence also in the other order: $\sum_i \Phi_\sigma(\zeta^i)\Phi_\tau(b_i/P'(\zeta)) = \delta_{\sigma\tau}$. Since $\overline{\Phi_\tau(\zeta^{-i})} = \Phi_\tau(\zeta^i)$ and Φ_σ is the identity on K, the two sums follow. (b) Every prime ℓ dividing q_1 divides $(d+1)/4$ (Lemma 8.2) and occurs in q_1 with exponent at most $\log_2 q_1 \le j_1$. At a place containing β, $\alpha^{-1}e_i = \bar\beta^{j_1}b_i$ is integral. At a place w not containing β we have $\mathrm{ord}_w P'(\zeta) \le \mathrm{ord}_w q_1 \le j_1 \,\mathrm{ord}_w(\beta\bar\beta) = j_1 \,\mathrm{ord}_w \bar\beta$, so e_i is integral at w. (c) $t \in \mathbb{Z}$ as it is fixed by conjugation and lies in $\mathcal{O}_K$. Let $\ell \mid q_1$ be prime; then $\ell \mid \beta\bar\beta$ and $\ell \nmid \beta + \bar\beta = 1$, so modulo a prime of K above ℓ one of $\beta, \bar\beta$ is 0 and the other is 1, and $t \equiv 1$. □

Proposition 13.7 (A frame of length prime to q_1). *Let* $\nu_i = \sum_\tau |\Phi_\tau(e_i)|^2$ *and* $t = \beta^{j_1} + \bar\beta^{j_1}$*. There is a tight frame with multiplicities one of length*

$$h = q_1\Big(1 + \sum_i \nu_i\Big) + t,$$

which is prime to q_1 *and contains the vector* $\zeta e_{\varphi(q_1)-1}$*. For every prime* p*, one of* $y = 1$ *and* $y = \zeta e_{\varphi(q_1)-1}$ *has a unit linear form at* p*.*

Proof. By Lemma 13.5 the powers ζ^{-i} $(i < q_1)$ together with the orbits of the e_i form a tight frame of length $q_1(1 + \sum \nu_i)$. Pair ζ^{-i} with the member e_i of the orbit of e_i for $i < \varphi(q_1)$; by Lemma 13.6(a) the cross terms add up to $t\,\delta_{\sigma\tau}$, and Lemma 13.5(c) gives the frame of length $h \equiv t \pmod{q_1}$, prime to q_1 by Lemma 13.6(c). The member $\zeta e_{\varphi(q_1)-1}$ of the orbit of $e_{\varphi(q_1)-1}$ belongs to no pair. For the unit linear form, let $\varphi_\tau(y) = \iota_p(\Phi_\tau(\bar y))$; whether the place of φ_τ contains β does not depend on τ, since $\varphi_\tau(\beta) = \iota_p(\bar\beta)$. If it does not, $|\varrho_0| = 1$ by Proposition 13.1(b) and $y = 1$ serves. If it does, $|\varrho_0| = |\varphi_\tau(P'(\zeta))|$, and since $b_{\varphi(q_1)-1} = 1$ the vector $y = \zeta\bar\beta^{j_1}/P'(\zeta)$ has $|\varphi_\tau(y)| = |\varphi_\tau(P'(\zeta))|^{-1}$, as $\bar\beta$ is a unit at that place. □

Lemma 13.8. *Let* v *be a tight frame with multiplicities one of length* h *prime to* q_1*, and* $R \ge 0$*. For every* $m \ge q_1 h + R(q_1 + h)$ *the powers of* ζ *and the members of* v *form, with suitable multiplicities all at least* R*, a tight frame of length* m*.*

Proof. Every integer $\ge q_1 h$ is $aq_1 + bh$ with $a, b \ge 0$, since q_1 and h are coprime. Write $m - R(q_1 + h) = aq_1 + bh$ and take the powers of ζ with multiplicity $a + R$ and v with multiplicity $b + R$. □

Theorem 13.9 (Frames for the data). *For data as in Definition 8.7:*

(a) *there are* h *prime to* q_1 *with* $1 \le h \le b$ *and a tight frame* v *of length* h *with multiplicities one such that, for every prime* p*, some* y *among the powers of* ζ *and the members of* v *has a unit linear form at* p*;*

(b) *there is a tight frame of length $M = p_0p_1$ with all multiplicities at least $6fb$ that contains, for every prime p, a member with a unit linear form at p.*

Proof. (a) Take the frame of Proposition 13.7. To bound h: the polynomial $P(X) = P_0(X^{T(s)})$ has the coefficients of P_0, which are bounded by 2^{q_0}; the coefficients of $P(X)/(X-\zeta)$ are sums of at most q_1 of them times roots of unity; $|\beta| \le d$; and $|\Phi_\tau(P'(\zeta))| = |\delta_E|_\tau/\sqrt{d} \ge x^{10}/\sqrt{d} \ge 1$ (Remark 10.3). Hence $|\Phi_\tau(e_i)| \le d^{j_1}q_1 2^{q_0}$, $\nu_i \le f(d^{j_1}q_1 2^{q_0})^2$, and $|t| \le 2d^{j_1}$. Since $j_1 \le q_0 + 3s + 10$ we have $d^{j_1} \le b^{1/100}$, and $h \le q_1(1 + f^2(d^{j_1}q_1 2^{q_0})^2) + 2d^{j_1} \le b$ by the comparison of sizes (Remark 8.10). (b) Apply Lemma 13.8 with $R = \lceil 6fb \rceil$ and $m = M \ge b^{20}$, which exceeds $q_1 h + R(q_1 + h)$. The members of positive multiplicity include $1 = \zeta^0$ and all members of v. □

13.4. Products along tight frames

Let C be a CM lattice for d with hermitian form $H_C(l,z) = \pi \bar{l} z/\mathrm{area}(C)$. For a family $w_j \in E$ and $b \in \mathbb{C}$ consider the linear forms

$$A_{j,k}(z) = \frac{b}{a_{\mathrm{ar}}} \sum_\tau \overline{\Phi_\tau(w_j)}\, z_{\tau,k} \qquad (k \in n),$$

and for $\gamma \in E^n$ the shifts $l_{j,k}(\gamma) = -b\sum_\tau \Phi_\tau(\gamma_k)\overline{\Phi_\tau(w_j)}$, so that $A_{j,k}$ at the real image of γ is $-l_{j,k}(\gamma)$. A vector $F \in \mathcal{S}(V)$ is the *resolution product* of entire functions $h_{j,k}$ of degrees m_j along w at the scale b if $F \in \mathcal{S}(V)^{\mathrm{f}}$ and for every $\gamma \in E^n$

$$\vartheta_{\varrho(\mathrm{fin}(\gamma^{\mathrm{f}}))F}(z) = \prod_{j,k} \big(T^{m_j}_{l_{j,k}(\gamma)} h_{j,k}\big)\big(A_{j,k}(z)\big).$$

By Lemma 13.2, the theta value and the jet series of each translate of a resolution product are the corresponding products: $\theta_V(\varrho(\mathrm{fin}(\gamma^{\mathrm{f}}))F) = \prod_{j,k}(T^{m_j}_{l_{j,k}(\gamma)} h_{j,k})(0)$, and its jet series at the scale c is $\prod_{j,k} T_0\big(T^{m_j}_{l_{j,k}(\gamma)} h_{j,k}\big)\big(\frac{bc}{a_{\mathrm{ar}}} w_j^\dagger z\big)$, the form $w_j^\dagger$ acting on the coordinate k.

Theorem 13.10 (Resolution products). *Let V be standard, $\delta \in O_K$ with $\delta\bar{\delta} = N \ne 0$, w a tight frame of length M with multiplicities m_j, and exponents $x_{j,k} = (r, t', u)$ with $r + 2t' + 3u = m_j$. Let C be a CM lattice containing a with $\mathrm{area}(C) = |a|^2 M\sqrt{d}/(2N)$. There is a vector F that is the resolution product of the monomial sections $\theta^r \sigma^{2t'} G^u$ of C along w at the scale $a/\bar{\delta}$; its values at all points $\mathrm{fin}(x)$ are integral at every p-adic place; and ϑ_F is a theta function for $\bar{\delta}\,\mathfrak{a}_0^n$.*

Proof. Isometry. For $u, v \in \mathbb{C}^{\Phi\times n}$, $\sum_{j,k} m_j H_C(A_{j,k}u, A_{j,k}v) = \sum_x u_x^* v_x$: multiply the frame identity at (τ,σ) by $\overline{u_{\tau,k}} v_{\sigma,k}$ and sum; the factor $|a/\bar{\delta}|^2/a_{\mathrm{ar}}^2 \cdot \pi/\mathrm{area}(C) \cdot M$ is 1 by the choice of the area. *Periods.* For $\gamma \in \bar{\delta}\mathfrak{a}_0^n$, $A_{j,k}$ maps the real image of γ into Λ_C: indeed $\frac{a}{\bar{\delta}}\sum_\tau \Phi_\tau(\gamma_k)\overline{\Phi_\tau(w_j)} \in aO_K \subseteq \Lambda_C$ by Lemma 13.4(b). *Invariance.* The product $S(z) = \prod_{j,k} h_{j,k}(A_{j,k}z)$ of sections of degrees m_j is therefore multiplied, under translation by such a period, by the product of the multipliers $\chi_C(A_{j,k}\gamma)^{m_j}$ and the Gaussian factor of the pulled-back form, which by the isometry is the standard one; the multipliers combine to the semicharacter, so S is a theta function for $\bar{\delta}\mathfrak{a}_0^n$. By Fourier inversion (W6),

$S = \vartheta_F$ with $F \in \mathcal{S}(V)^{\mathrm{f}}$, and translating F by the finite image of γ shifts the factor (j,k) by $l_{j,k}(\gamma)$ (W4). *Integrality.* The value of F at $\mathrm{fin}(x)$ is the theta value of a translate by the finite image of a rational γ, hence the product of the values at 0 of twisted translates of monomial sections by torsion points of C ($L\gamma^{\mathrm{f}}$ is integral for some $L \neq 0$, so NL times the shift is a period); these torsion values are integral at every p (Theorem B.20). □

For the frame of the powers of ζ, each taken q_1 times (length q_1^2), the vectors $u_k = \zeta^k/q_1$ give the coordinates $\langle y\rangle_k = -\operatorname{Tr}_{E/K}(y\bar{\zeta}^k)/q_1 \in K$. A *theta product* for a CM lattice C, a period Ω and a level $e \in \mathbb{N}_{>0}$ is a vector F of the finite part whose translates have theta values and jet series

$$\theta_V\big(\varrho(\mathrm{fin}(\gamma^{\mathrm{f}}))F\big) = \prod_{k<q_1}\prod_{i\in n}\big(T^1_{\Omega\langle\gamma_i\rangle_k}\theta\big)(0)^{q_1},$$

the jet series being the corresponding product of Taylor series along the forms $\frac{\Omega c}{q_1 a_{\mathrm{ar}}}\sum_w \overline{\Phi_w(\zeta^k)}z_{w,i}$, whose values at finite points are integral at every prime, and whose theta function is a theta function for $q_1 e\,\mathfrak{a}_0^n$. Then $\theta_V(F) = 1$.

Corollary 13.11. *If V is standard, C has area $|\Omega|^2\sqrt{d}/2$ and $\Omega e \in \Lambda_C$, a theta product exists; in particular for $\mathcal{E}_0$ with $\Omega = \Omega_0$ and $e = 1$. For a theta product of level one, a nonzero multiple of the vacuum is a sum of $(q_1^2)^{fn}$ finite Heisenberg translates of one translate of F.*

Proof. Apply Theorem 13.10 with $\delta = q_1 e$, the frame of the powers of ζ each taken q_1 times, and the monomial θ of degree one; the repeated frame vectors give the powers q_1. The second statement is (W6) with $N = (q_1 e)^2$. □

13.5. The integral vacuum

Proposition 13.12. *There is a constant C_{vac} such that for every s there is $c \neq 0$ with the following properties, for the standard space of rank one: the values $c\,\mathrm{vac}(\mathrm{fin}(x))$ are integral at every p-adic place, and their conjugates over K have absolute value at most $\exp(C_{\mathrm{vac}}q_1^2)$.*

Proof. Let F be the theta product of level one for $\mathcal{E}_0$ (Corollary 13.11) and c the scalar with $c\,\mathrm{vac}$ a sum of $(q_1^2)^f$ translates of F times roots of unity. A value of $c\,\mathrm{vac}$ at a finite point is a sum of $(q_1^2)^f \leq e^{2q_1^2}$ values of F times roots of unity, and a value of F is a product of q_1^2 torsion values $(T^1_{x\Omega_0}\theta)(0)$, $x \in K$. These are integral at every p (Theorem 13.10), and their conjugates over K are bounded by a constant M_0 independent of s: by complex multiplication they are torsion values of the finitely many conjugate lattices $\mathcal{E}_0^\sigma$, and these are bounded uniformly in the torsion point (Theorem B.20). Take $C_{\mathrm{vac}} = 2 + \log^+ M_0$. □

We fix such a scalar $c_{\mathrm{vac}} = c_{\mathrm{vac}}(s)$, the *vacuum scale*, and define the *integral vacuum* of a space of rank n as $\theta^\circ = c_{\mathrm{vac}}^n\,\mathrm{vac}$.

Proposition 13.13. *Let V be standard of rank n.*

(a) $\theta^\circ(\mathrm{fin}(x)) = \prod_{i\in n} c_{\mathrm{vac}}\mathrm{vac}_1(\mathrm{fin}(x_i))$, *with vac_1 the vacuum of rank one; these values are integral at every p-adic place, and for every automorphism σ of $\mathbb{C}$, $|\sigma(\theta^\circ(\mathrm{fin}(x)))| \leq \exp(C_{\mathrm{vac}}q_1^2)^n$.*

(b) *If* $|f(\mathrm{fin}(x))| \le M$ *for all* x, *then* $|\langle f, f\rangle| \le M^2$.
(c) $\theta^\circ(\mathrm{fin}(x)) \neq 0$ *for some* x.

Proof. (a) The vacuum of an orthogonal sum is the product of the vacua, and θ_V is multiplicative on orthogonal sums. Every automorphism σ of $\mathbb{C}$ restricts on K to the identity or to complex conjugation, so σ or σ followed by complex conjugation is over K, with the same absolute values. (b) $\langle f, f\rangle$ is the mean of $|f|^2$; by continuity it suffices to bound f at points whose archimedean part is rational, and translating by a rational vector changes f by a factor of modulus one and moves such a point to one with archimedean part zero. (c) Otherwise (b) with $M = 0$ gives $\langle \theta^\circ, \theta^\circ\rangle = 0$, while vac is a unit vector. □

Proposition 13.14 (Conjugates of jets). *There is a map* $\sigma \mapsto B_\sigma$ *from* $\mathrm{Gal}(\mathbb{C}/K)$ *to the finite adeles of* E *such that every standard space* V *has a vector* $F \in \mathcal{S}(V)^{\mathrm{f}}$ *with* $\theta_V(F) = 1$ *whose translates* $\varrho(\mathrm{fin}(x))F$ *span* $\mathcal{S}(V)^{\mathrm{f}}$, *and for each* σ *a vector* $F'_\sigma \in \mathcal{S}(V)^{\mathrm{f}}$ *and a scale* t_σ *with* $|t_\sigma|^2 = \Omega^{-1}$ *such that, for all* x, σ *maps the jets of* $\varrho(\mathrm{fin}(x))F$ *at the scale* a *to the jets of* $\varrho(\mathrm{fin}(B_\sigma x))F'_\sigma$ *at the scale* t_σ, *with the places permuted by* σ (*Section 10.2*).

Proof. Take F the theta product of level one for $\mathcal{E}_0$; its translates span the finite part by Corollary 13.11. By the main theorem of complex multiplication (Theorem B.22), for each σ there are a period Ω_σ of $\mathcal{E}_0^\sigma$ with a multiple $\Omega_\sigma e_\sigma$ in its lattice and a finite idele β_σ of K such that, for x' near $\beta_\sigma x$, σ maps the Taylor series of $T^1_{x\Omega_0}\theta_{\mathcal{E}_0}$ to that of $T^1_{x'\Omega_\sigma}\theta_{\mathcal{E}_0^\sigma}$. Let F'_σ be the theta product for $\mathcal{E}_0^\sigma$, Ω_σ, e_σ, and B_σ the image of β_σ. Both sides of the claimed identity are locally constant in x, so we may take x the finite image of a rational γ; choosing $x_0 \in K$ close to β_σ, the translates by $B_\sigma\gamma^{\mathrm{f}}$ and by $(x_0\gamma)^{\mathrm{f}}$ of F'_σ agree, and the factors of the two products correspond by complex multiplication. The linear forms correspond up to the permutation of the places, and the scale is $t_\sigma = \sigma(\varrho_0)a_{\mathrm{ar}}/\Omega_\sigma$, of square absolute value Ω^{-1} by Proposition 13.1(a). □

13.6. Standard data

Fix a prime p, a p-adic place ι_p, and let $\mathcal{O}_\iota \subset \mathbb{C}$ be the ring of complex numbers that ι_p maps into $\mathcal{O}$. For $\delta \in \mathcal{O}_K$ let $\mathcal{I}_\delta$ be the group of finite adelic vectors x such that δx is integral at all places of E above p.

Definition 13.15. The *standard data of level* δ *at* p on V are the elements of the $\mathcal{O}_\iota$-module $\mathcal{S}_\delta(V)$ spanned by the finite Heisenberg translates of θ° by the $x \in \mathcal{I}_\delta$.

Proposition 13.16 (Stability). $\mathcal{S}_\delta(V)$ *is stable under multiplication by elements of* $\mathcal{O}_\iota$, *under the finite Heisenberg operators of the* $x \in \mathcal{I}_\delta$, *and under the Weil operators* $\omega(1, g)$ *of finite unitary points* g *that are integral at the places above* p.

Proof. The finite Heisenberg operators of x and y compose to that of $x + y$ up to a root of unity, and $\mathcal{I}_\delta$ is a group. For g integral above p, $\delta g x$ is g applied to δx, so g preserves $\mathcal{I}_\delta$; and $\omega(1, g)$ maps the translate of θ° by x to $(\Delta_{\mathrm{f}}(g^{-1}))^{-1}$ times the translate by gx of the sum of the translates of θ° by gy, where y runs over

the integral vectors modulo those that g keeps integral; the factor is a unit at ι_p since g^{-1} is also integral above p. □

Theorem 13.17 (Standard data as theta products). *Let V be standard, $\delta \in O_K \setminus \{0\}$, $N = \delta\bar{\delta}$, and $f \in \mathcal{S}_\delta(V)$. There is a resolution of the identity (u_j, m_j) by vectors of E^n whose finite images u_j^{f} are integral at the places above p, and $H \in O[[y(u^\dagger z)]]$ for the curve $\mathcal{E}_0$, such that*

$$\iota_p\left(J_{\bar{\delta}a_{\mathrm{ar}}/\Omega_0}(f)\right) = \Theta^N_{u^\dagger,m}\, H.$$

Proof. A frame of square length prime to p. If $p \nmid q_1$ take the powers of ζ (length q_1); otherwise the frame of Proposition 13.7, of length h prime to q_1, hence to p. Repeating each member b times, where b is that length, gives a tight frame w with multiplicities one of length b^2, $p \nmid b$. For $k \neq 0$ the vectors $w_j/(bk)$, placed in each coordinate of E^n, are a resolution of the identity with multiplicities k^2.

Expansion of θ°. By Theorem 13.10 with $\delta = b$ and the monomial θ, there is F of the finite part with $\theta_V(F) = 1$ whose theta function is a theta function for $b\,\mathfrak{a}_0^n$ and whose translates are products of translates of $\theta_{\mathcal{E}_0}$ along the frame. Fourier inversion gives points y_j with by_j integral and coefficients such that a power M of b^2 times F is a combination of translates of the vacuum; applying θ_V and inserting the expression of each multiple of the vacuum as a sum of translates of F, we obtain $\theta^\circ = \sum_j a_j \varrho(\mathrm{fin}(y_j))F$ with $a_j \in O_\iota$, because the values of θ° are integral and M is prime to p.

Translates. Let $x \in \mathcal{I}_\delta$. A translate of F by x (or by $x + y_j$, which lies in $\mathcal{I}_\delta$ as b is prime to p) equals the translate by the finite image of a rational γ such that $m\delta\gamma \in \mathfrak{a}_0^n$ for some m prime to p: approximate x by γ with $\gamma^{\mathrm{f}} - x$ in the standard lattice. Its jet series at $\bar{\delta}a_{\mathrm{ar}}/\Omega_0$ is the product over the frame of the series of $T_t^1\theta$ along the forms $\bar{\delta}(w_j/b)^\dagger z$, with shifts $t = -(\Omega_0/b)\sum_\tau \Phi_\tau(\gamma_i)\overline{\Phi_\tau(w_j)}$, which $kb\delta$ maps into $\Lambda_{\mathcal{E}_0}$ for every multiple k of mb, because $\mathfrak{a}_0$ is self-dual over O_K (Lemma 13.4(b)).

One curve. For $\delta' \in O_K$ with $\delta'\bar{\delta}' = N'$ and $\delta' y_i \in \Lambda$, the product $\prod_i (T^1_{y_i}\theta)(\bar{\delta}'\mu_i z)$ read in $\mathbb{C}_p$ is $\prod_i \theta(\mu_i z)^{N'}$ times an element of $O[[y(\mu z)]]$: by Theorem 11.3(ii), $z \mapsto (T^1_y\theta)(\bar{\delta}'z)$ is a section of degree N', hence $\theta(z)^{N'}$ times an even elliptic function with poles only on $t + \Lambda$, which is integral in y by the integrality of torsion values and of the group law (Theorem B.20). Applied with $\delta' = k\delta$ and rewriting the forms through $w_j/(bk)$, the translate lies in the module of vectors whose jet series is $\Theta^N_{u^\dagger,m}$ times an element of $O[[y(u^\dagger z)]]$, for the resolution $u_j = w_j/(bk)$, whose finite images are integral above p since bk is prime to p times a unit there. This module is an O_ι-module; taking k a common multiple for the finitely many translates in an expression of f proves the claim. □

13.7. Integrality and divisibility of the derivatives

For a CM lattice C and linear forms μ_i over $\mathbb{C}_p$, let $\mathcal{R}_\mu$ be the ring generated by $O[[y(\mu z)]]$, the series $\theta(\mu_i z)$ and the series $g'(\mu_i z)$, where $g = -\log\theta$.

Lemma 13.18. (a) $d\,g'' \in O[[y]]$.

(b) *The constant terms of elements of $\mathcal{R}_\mu$ are integral. If $\mu_i = d\nu_i$ with ν_i integral, then $\mathcal{R}_\mu$ is stable under all partial derivatives.*

(c) *If a subring of power series is stable under partial derivatives and its elements have integral constant terms, then $Q(\partial)F(0)$ is integral for every element F and every polynomial Q with integral coefficients.*
(d) *Consequently, if $\mu = d\nu$ with ν integral and $H \in \mathcal{O}[[y(\mu z)]]$, then $Q(\partial)(\Theta^N_{\mu,m}H)(0)$ is integral for every integral polynomial Q.*

Proof. (a) $g'' = g''(0) + x$ with $x \in \mathcal{O}[[y]]$, and $d\,g''(0) = -\sqrt{-d}\cdot\sqrt{-d}\,g''(0)$ is integral by Theorem 11.3(iv). (b) $y(0) = 0$, $\theta(0) = 1$, $g'(0) = 0$. For the derivatives: $\partial\theta(\mu_i z) = -\mu_{ix}\,g'(\mu_i z)\theta(\mu_i z)$, $\partial g'(\mu_i z) = \nu_{ix}\,(dg'')(\mu_i z) \in \mathcal{O}[[y(\mu z)]]$ by (a), and $\mathcal{O}[[y(\mu z)]]$ is stable since μ is integral. (c) $Q(\partial)F(0)$ is an integral combination of constant terms of derivatives of F. (d) is (b) and (c). □

Theorem 13.19 (Integrality of the derivatives). *Let V be standard, $\delta \in \mathcal{O}_K \setminus \{0\}$, $f \in \mathcal{S}_\delta(V)$ and Q a polynomial in the variables $z_{\tau,i}$ with coefficients in $\mathcal{O}_\iota$. Then $Q(\partial)J_{\bar\delta d\mathsf{a}}(f)(0)$ is integral at ι_p. If Q is homogeneous of degree l and $c \in \mathbb{C}$, then*

$$|\iota_p\big(Q(\partial)J_{c\mathsf{a}}(f)(0)\big)| \le \left|\iota_p\Big(\frac{c}{\bar\delta d}\Big)\right|^l.$$

Proof. By Theorem 13.17, $J_{\bar\delta a_{\mathrm{ar}}/\Omega_0}(f) = \Theta^N_{u^\dagger,m}H$ in $\mathbb{C}_p$. Since $\bar\delta d\mathsf{a} = d\varrho_0\cdot\bar\delta a_{\mathrm{ar}}/\Omega_0$, the series $J_{\bar\delta d\mathsf{a}}(f)$ is this one with z replaced by $d\varrho_0 z$, which turns the forms $u_j^\dagger$ into $d\nu_j$ with $\nu_j = \varrho_0 u_j^\dagger$, integral by Proposition 13.1(c). Apply Lemma 13.18(d). For the second statement, $J_{c\mathsf{a}}(f)(z) = J_{\bar\delta d\mathsf{a}}(f)(\frac{c}{\bar\delta d}z)$, and a homogeneous jet of degree l scales by the l-th power. □

Theorem 13.20 (Normal form of standard data). *Let p be non-split, V standard, $f \in \mathcal{S}_1(V)$ and $c \in \mathcal{O}_\iota$. Then $\iota_p(J_{c\mathsf{a}}(f))$ has the normal form $(I_p^\circ, \mathrm{pr}, \varpi, \lambda)$ for some ϖ with $|\varpi| = p^{-\rho_p}$ and some λ.*

Proof. Non-split primes do not divide q_1, so α is a unit above p and the vectors u_j of Theorem 13.17 (with $\delta = 1$, $N = 1$) are integral above p. Hence $J_{a_{\mathrm{ar}}/\Omega_0}(f) = \Theta_{u^\dagger,m}H$ has the normal form by Theorem 12.6. Moreover $|\varrho_0| = 1$ (Proposition 13.1(b)), so $c\varrho_0$ is integral and Proposition 12.2(b) applies to $J_{c\mathsf{a}}(f)(z) = J_{a_{\mathrm{ar}}/\Omega_0}(f)(c\varrho_0 z)$. □

Theorem 13.21 (Divisibility of standard data). *Let p be non-split, f standard data of level one at p on the space $\mathrm{Hom}(V_n, V_{n'})$ of $n' \times n$ matrices, for standard spaces V_n, $V_{n'}$, and $c \in \mathcal{O}_\iota$; write $F = \iota_p(J_{c\mathsf{a}}(f))$, a series in the entries $z_{\tau,b,a}$, and $N = \#n'$.*

(a) *For every filling i' and integral directions y, $v\big(P_{i',y}(\partial)F(0)\big) \ge D_p(\kappa) - fN^3(1 + \log_p \max(B,1))$.*
(b) *If κ fits the ranks, $\mathcal{M}$ is a set of places and the charges w cover the exempt pairs for $\mathcal{M}$ and every $e \in I_p^\circ$, then for all standard fillings i', i,*

$$v\big(\mathrm{Sch}_\kappa(F)_{i'i}\big) + |w|_{i'} \ge D_p(\kappa^\circ) + \gamma_p(B - B^\circ) - fN^3\big(1 + \log_p \max(B,1)\big).$$

Proof. (a) By Theorem 13.20, F has the normal form $(I_p^\circ, \mathrm{pr}, \varpi, \lambda)$ with $|\varpi| = p^{-\rho_p}$ in the variables $z_{\tau,b,a}$; Theorem 12.12 without marked places and charges gives

the bound. (b) By Proposition 12.13, as κ fits the ranks, there are integral directions y with $|\mathrm{Sch}_\kappa(F)_{i'i}| \le |P_{i',y}(\partial)F(0)|$, that is, $v(\mathrm{Sch}_\kappa(F)_{i'i}) \ge v(P_{i',y}(\partial)F(0))$. The filling i' is the same, so the charge $|w|_{i'}$ is the same, and Theorem 12.12 with the marked places $\mathcal{M}$ and the charges w bounds the right side. □

14. The divisibility is attained

The divisibility of Theorem 12.12 is an upper bound for the size of derivatives at p. The lower bound of Part V needs the opposite: for degrees near the weights of the data some derivative of some datum is no more divisible than predicted, simultaneously at all primes, up to a controlled common factor. This section proves it (Theorem 14.11). The prediction at a prime p is the *local prime sum*

$$s_p(y,\sigma) = v(\pi_p)\sum_{e\in I_p}\Big(\frac{y}{p^e} - \kappa(-p^e\sigma)\Big)_+, \qquad S_p = \sum_{\sigma\in\Phi} s_p(\kappa(\sigma),\sigma),$$

where κ is the weight of Definition 8.7 and $-p^e\sigma = \mathrm{pr}_e\sigma$. The series converges since $\kappa \ge 0$. By Lemma 11.2 and $\Lambda_K(p^e) = \log p\,(1-\chi_K(p)^e) = 2\log p\cdot v(\pi_p)[e \in I_p]$, the prime sum of Section 9 is $s_{\mathrm{fin}}(y,\sigma) = 2\sum_p \log p\ \ s_p(y,\sigma)$, the terms truncated at $p^e \le a$.

14.1. One curve: the limit of divisibility

For a CM lattice C and $n \ge 0$ let $c(n)$ be the largest absolute value of $G^{(n)}(0) = n![z^n]G$ for $G \in \mathcal{O}[[y]]$; the maximum exists and is attained by a power of y, since $G^{(n)}(0)$ is an integral combination of the $(y^i)^{(n)}(0)$, $i \le n$.

Proposition 14.1 (Katz's limit). *$c(0) = 1$, $c(n) \le 1$, and $c(m+n) \le c(m)c(n)$. If $n+1$ is a power of q_p, then $c(n+1) \ge p^{-\gamma_p n}$. Consequently, for every n there is $G \in \mathcal{O}[[y]]$ with $v\big(G^{(n)}(0)\big) \le \gamma_p n$.*

Proof. The bounds $c(n) \le 1$, $c(0) = 1$. By Theorem 11.3(i), $dy/dz = 1 + 3x^2 - a_1 y$ with $x \in \mathcal{O}[[y]]$, so $\mathcal{O}[[y]]$ is stable under d/dz. Hence $G^{(n)} \in \mathcal{O}[[y]]$ for $G \in \mathcal{O}[[y]]$, and $G^{(n)}(0)$, the constant term of an integral series in y with $y(0) = 0$, is integral: $c(n) \le 1$. The series $G = 1$ gives $c(0) = 1$. The maximum exists: since y has no constant term, $G^{(n)}(0) = n![z^n]G$ involves only the coefficients G_i of $G = \sum_i G_i y^i$ with $i \le n$, so it is the integral combination $\sum_{i\le n} G_i\,(y^i)^{(n)}(0)$, and by the ultrametric inequality its absolute value is at most $\max_{i\le n}|(y^i)^{(n)}(0)|$, which is attained by a power of y.

Submultiplicativity. Let $G \in \mathcal{O}[[y]]$. By the integrality of the group law (Theorem 11.3(iii)), $y(z_0+z_1)$ is an integral series in $y(z_0)$ and $y(z_1)$, hence so is $G(z_0+z_1) = \sum_{a,b} h_{ab}\,y(z_0)^a y(z_1)^b$ with $h_{ab} \in \mathcal{O}$. Expanding $G(z_0+z_1) = \sum_n g_n(z_0+z_1)^n$, the coefficient of $z_0^m z_1^n$ is $\binom{m+n}{m} g_{m+n}$, so

$$m!\,n!\,[z_0^m z_1^n]G(z_0+z_1) = (m+n)!\,g_{m+n} = G^{(m+n)}(0).$$

On the other side the coefficient of $z_0^m z_1^n$ in $y(z_0)^a y(z_1)^b$ is $[z^m]y^a \cdot [z^n]y^b$, and only $a \le m$, $b \le n$ contribute. Multiplying by $m!\,n!$ and using the ultrametric inequality,

$$|G^{(m+n)}(0)| \le \max_{a,b}|(y^a)^{(m)}(0)|\,|(y^b)^{(n)}(0)| \le c(m)\,c(n).$$

Taking the maximum over G gives $c(m+n) \le c(m)c(n)$.

Powers of q_p. Let $n+1 = q_p^j$. The formal parameter T of Theorem 11.3(v) lies in $O[[y]]$, and $T(\varpi z) = \varpi Y$ with $|\varpi| = p^{-\rho_p}$ and $Y \equiv \sum_{i\ge 0} z^{q_p^i}$ modulo π_p. Comparing coefficients of z^{n+1}, $\varpi^{n+1}[z^{n+1}]T = \varpi[z^{n+1}]Y$, and $[z^{n+1}]Y \equiv 1$ modulo π_p is a unit, because an integer of $\mathbb{C}_p$ that is 1 modulo the non-unit π_p has absolute value 1. Hence $|[z^{n+1}]T| = |\varpi|^{-n} = p^{\rho_p n}$. Legendre's formula gives $v((n+1)!) = (n+1-s_p(n+1))/(p-1) \le n/(p-1)$, since the digit sum $s_p(n+1)$ is at least 1. Therefore

$$c(n+1) \ge |(n+1)!\,[z^{n+1}]T| \ge p^{-n/(p-1)}\,p^{\rho_p n} = p^{-\gamma_p n}.$$

All n. Put $b_0 = p^{-\gamma_p} \in (0,1]$; then $b_0^q \le b_0^{q-1} \le c(q)$ whenever q is a power of q_p. Fix $n \ge 1$ and suppose $c(n) < b_0^n$. For q a power of q_p write $q = n\lfloor q/n \rfloor + r$ with $0 \le r < n$; submultiplicativity and $c(r) \le 1$ give $b_0^q \le c(q) \le c(n)^{\lfloor q/n \rfloor}$. If $c(n) = 0$ this is impossible. Otherwise put $\epsilon = c(n)/b_0^n < 1$; then $b_0^q \le \epsilon^{\lfloor q/n \rfloor} b_0^{n\lfloor q/n \rfloor}$, that is, $\epsilon^{-\lfloor q/n \rfloor} \le b_0^{-r} \le b_0^{-n}$, which fails for large q. Hence $c(n) \ge b_0^n$, and since the maximum is attained, some $G \in O[[y]]$ has $v(G^{(n)}(0)) \le \gamma_p n$. □

14.2. Least valuations on several copies

Let C be a CM lattice, μ_j linear forms in variables z_x and m_j multiplicities, and $\Theta = \Theta_{\mu,m}$. For $n \ge 1$, a degree ν (a multi-index in the z_x) and $r \in \mathbb{R}$ we write $\mathrm{val}_n(\nu) \le r$ if there are $H_1, \dots, H_n \in O[[y(\mu z)]]$ and multi-indices $\alpha_1 + \cdots + \alpha_n = \nu$ such that $\prod_c v\big([z^{\alpha_c}](H_c\Theta)\big) \le r$; that is, some coefficient of degree ν in some product $\prod_c (H_c\Theta)(z_c)$ over n copies of the variables has valuation at most r, the degree adding up the copies.

Lemma 14.2. (a) $\mathrm{val}_n(0) \le 0$; *if* $\mathrm{val}_n(\nu) \le r$ *and* $\mathrm{val}_{n'}(\nu') \le r'$, *then* $\mathrm{val}_{n+n'}(\nu+\nu') \le r + r'$.

(b) *Let the $A_{c,k}$ be integral and $H_c \in O[[y(\mu z)]]$. If the coefficient of z'^{γ} in $\prod_c (H_c\Theta)\big(\sum_k A_{c,k} z'_k\big)$ has valuation at most r, then $\mathrm{val}_n(\nu) \le r$ for the degree ν obtained by adding up the copies in γ. In particular this holds for $\prod_c (H_c\Theta)(\delta_c z)$ with integral δ_c.*

(c) *If $\mathrm{val}_n(\nu) \le r$, the H_c may be taken to be monomials in the $y(\mu_j z)$ of degree at most $|\nu|$.*

Proof. (a) With all $H_c = 1$ and all $\alpha_c = 0$, the product of the constant terms of the $H_c\Theta$ is 1, of valuation 0, since $\Theta(0) = \prod_j \theta(0)^{m_j} = 1$. If $(H_c, \alpha_c)_{c<n}$ witnesses $\mathrm{val}_n(\nu) \le r$ and $(H'_c, \alpha'_c)_{c<n'}$ witnesses $\mathrm{val}_{n'}(\nu') \le r'$, the concatenated family of $n+n'$ copies has $\sum_c \alpha_c + \sum_c \alpha'_c = \nu + \nu'$ and product of valuations at most $r + r'$.

(b) *One factor.* For a series F and the substitution $z_x \mapsto \sum_k A_k z'_{x,k}$ of copies, the chain rule gives, for an exponent γ in the copies with summed degree δ ($\delta_x = \sum_k \gamma_{x,k}$),

$$[z'^{\gamma}]F\Big(\sum_k A_k z'_k\Big) = A^{\gamma} \prod_x \frac{\delta_x!}{\prod_k \gamma_{x,k}!}\,[z^{\delta}]F,$$

with $A^{\gamma} = \prod_{x,k} A_k^{\gamma_{x,k}}$. The multinomial coefficients are integers and the A_k are integral, so $|[z'^{\gamma}]F(\sum_k A_k z'_k)| \le |[z^{\delta}]F|$. *The product.* The coefficient of z'^{γ} in $\prod_c (H_c\Theta)(\sum_k A_{c,k} z'_k)$ is the sum, over the decompositions $\gamma = \sum_c \gamma_c$, of the products of the coefficients of z'^{γ_c} in the factors. If it has valuation at most r,

then by the ultrametric inequality some term has valuation at most r, and by the bound for one factor $\sum_c v([z^{\delta_c}](H_c\Theta)) \le r$ for the summed degrees δ_c of the γ_c, with $\sum_c \delta_c = \nu$. This is $\mathrm{val}_n(\nu) \le r$. For $\prod_c (H_c\Theta)(\delta_c z)$ take one new copy z' and $A_{c,0} = \delta_c$.

(c) Write $H_c = G_c(y(\mu z))$ with G_c integral. As the $y(\mu_j z)$ have no constant term, a monomial of degree $|e|$ in them has no terms of degree below $|e|$, so only the monomials of degree at most $|\alpha_c|$ contribute to $[z^{\alpha_c}](H_c\Theta) = \sum_e G_{c,e}[z^{\alpha_c}](y(\mu z)^e\Theta)$; by the ultrametric inequality one of these monomials, $y(\mu z)^e$ with $|e| \le |\alpha_c| \le |\nu|$, has a coefficient at least as large. Replacing each H_c by it does not increase the product of the valuations. □

Proposition 14.3 (Six copies). (a) *There are $\delta_0 = 1, \delta_1, \dots, \delta_5 \in \mathbb{Z}_p$ with $\sum_c \delta_c^2 = 0$; for them $\prod_c \Theta(\delta_c z)$ lies in $O[[y(\mu z)]]$, has constant term 1 and is a unit of that ring. If μ is integral and $|\omega| \le p^{-\rho_p}$, then $\prod_c \Theta(\delta_c \omega z)$ is integral with constant term 1.*

(b) *If some μ_{i_0} has unit coefficients, then $\mathrm{val}_6(\nu) \le \gamma_p|\nu| - v(\nu!)$ for all ν, where $\nu! = \prod_x \nu_x!$.*

(c) *If there are integral $a_{x,i}$ and a unit s with $\sum_i a_{x,i}\mu_i = s\, z_x$ for all x, then $\mathrm{val}_6(\nu) \le 0$ for all ν.*

Proof. (a) *The scales.* The set of sums of four squares of p-adic integers is closed, being the image of the compact set $\mathbb{Z}_p^4$, and it contains the natural numbers by Lagrange's theorem, which are dense in $\mathbb{Z}_p$; so it is all of $\mathbb{Z}_p$, and $-1 = \delta_1^2 + \delta_2^2 + \delta_3^2 + \delta_4^2$. With $\delta_0 = 1$ and $\delta_5 = 0$, $\sum_c \delta_c^2 = 0$. *The product in one variable.* Write $\theta = \exp(-g)$. For $\delta \in \mathbb{Z}_p$ the pair (δ, δ^2) is a limit of pairs $(n, n^2) = (\alpha, \alpha\bar{\alpha})$ with $\alpha = n \in \mathbb{Z} \subseteq O_K$, so $\exp(\delta^2 g(z) - g(\delta z)) \in O[[y]]$ by Theorem 11.3(iv). Hence

$$\prod_c \theta(\delta_c z) = \exp\Big(-\sum_c g(\delta_c z)\Big) = \prod_c \exp\big(\delta_c^2 g(z) - g(\delta_c z)\big) \cdot \exp\Big(-\Big(\sum_c \delta_c^2\Big) g(z)\Big)$$

lies in $O[[y]]$, the last factor being 1; call it $u(z)$. Its constant term is $\prod_c \theta(0) = 1$, so u is a unit of $O[[y]]$: the inverse of $1 + h$ with $h(0) = 0$ is $\sum_k (-h)^k$, which converges in $O[[y]]$. *Several forms.* Exchanging the products, $\prod_c \Theta(\delta_c z) = \prod_j u(\mu_j z)^{m_j}$, which lies in $O[[y(\mu z)]]$ with constant term 1, and its inverse $\prod_j u^{-1}(\mu_j z)^{m_j}$ lies there too. For the last statement, an element of $O[[y(\mu z)]]$ with μ integral becomes integral after $z \mapsto \omega z$ when $|\omega| \le p^{-\rho_p}$ (Theorem 11.3(v), applied to each $y(\mu_j \omega z) = y(\omega \mu_j z)$).

(b) Take $G \in O[[y]]$ with $v(G^{(|\nu|)}(0)) \le \gamma_p|\nu|$ (Proposition 14.1) and let V be the inverse of $\prod_c \Theta(\delta_c z)$ in $O[[y(\mu z)]]$. Put $H_0 = G(\mu_{i_0} z)V$ and $H_c = 1$ for $c \ne 0$; they lie in $O[[y(\mu z)]]$. Then

$$\prod_c (H_c\Theta)(\delta_c z) = G(\mu_{i_0} z)\, V(z) \prod_c \Theta(\delta_c z) = G(\mu_{i_0} z),$$

using $\delta_0 = 1$. Its coefficient of z^ν is $\mu_{i_0}^\nu\, [w^{|\nu|}] G \cdot |\nu|!/\nu! = \mu_{i_0}^\nu G^{(|\nu|)}(0)/\nu!$, by the multinomial expansion of $(\mu_{i_0} z)^{|\nu|}$, and $\mu_{i_0}^\nu$ is a unit. Its valuation is at most $\gamma_p|\nu| - v(\nu!)$, and Lemma 14.2(b) gives the claim.

(c) The series $G_x = \sum_i a_{x,i} y(\mu_i z)$ lies in $O[[y(\mu z)]]$, as the ring contains the integral constants. Since $y = z + O(z^2)$, its linear part is $\sum_i a_{x,i}\mu_i z = s z_x$, so G_x has lowest term $s z_x$ for the total degree. Lowest terms multiply, and Θ has

lowest term 1; so $\prod_x G_x^{\nu_x}\Theta$ has lowest term $s^{|\nu|}z^\nu$, and its coefficient of z^ν is the unit $s^{|\nu|}$. This is $\mathrm{val}_1(\nu) \le 0$, and Lemma 14.2(a) adds five copies in degree 0. □

14.3. Couplings

Let Φ be a finite set with a partner rule pr, one coordinate per place, $\Phi_0 \subseteq \Phi$ and $e\colon \Phi_0 \to \mathbb{N}$. The family (Φ_0, e) is a *coupling family* if every $e(\tau)$ is odd and, for $\tau, \tau_1 \in \Phi_0$ and odd $e' \le e(\tau)$, $\mathrm{pr}_{e'}\tau_1 = \mathrm{pr}_{e(\tau)}\tau$ only if $\tau_1 = \tau$ and $e' = e(\tau)$. For $j\colon \Phi_0 \to \mathbb{N}$ let ν' be the degree of $\prod_\tau z_\tau^{p^{e(\tau)}j(\tau)}$ and ν'' that of $\prod_\tau z_{\tau^+}^{j(\tau)}$, where $\tau^+ = \mathrm{pr}_{e(\tau)}\tau$; thus $|\nu''| = |j|$ and $\nu' + \nu''$ is the degree of $\prod_\tau (z_\tau^{p^{e(\tau)}} z_{\tau^+})^{j(\tau)}$.

Lemma 14.4. *Let $\kappa \in \mathbb{C}_p$ with $|\kappa| > 1$, and let $G = W\exp(\kappa\Xi)$ with W, Ξ integral, W a unit and $\Xi(0) = 0$. Then $\partial_x G = (\kappa\,\partial_x\Xi + B_x)G$ with B_x integral, and for a finite family of variables t_i, $i \in T$, and every integral f,*

$$\Big(\prod_{i\in T}\partial_{t_i}\Big)(fG) = \kappa^{\#T}\, g\, G,$$

with g integral and $g \equiv f\prod_{i\in T}\partial_{t_i}\Xi \pmod{\mathfrak{m}}$.

Proof. One derivative. By the chain rule for the exponential series,

$$\partial_x\big(W\exp(\kappa\Xi)\big) = (\kappa\,\partial_x\Xi + \partial_x W/W)\,W\exp(\kappa\Xi),$$

and $B_x = \partial_x W/W$ is integral because W is an integral unit. For integral f,

$$\partial_x(fG) = (\partial_x f + f\kappa\,\partial_x\Xi + fB_x)G = \kappa\big(f\,\partial_x\Xi + \kappa^{-1}(\partial_x f + fB_x)\big)G,$$

and as $|\kappa| > 1$, κ^{-1} lies in $\mathfrak{m}$; so $\partial_x(fG) = \kappa g_1 G$ with $g_1 = f\,\partial_x\Xi + \kappa^{-1}(\partial_x f + fB_x)$ integral and $g_1 \equiv f\,\partial_x\Xi$ modulo $\mathfrak{m}$.

Iteration. Induct on $\#T$. If $(\prod_{i\in T'}\partial_{t_i})(fG) = \kappa^{\#T'}g'G$ with g' integral and $g' \equiv f\prod_{i\in T'}\partial_{t_i}\Xi$, apply the first step to $g'G$ and a further variable t_{i_1}: $\partial_{t_{i_1}}(g'G) = \kappa g''G$ with $g'' \equiv g'\partial_{t_{i_1}}\Xi \equiv f\prod_{i\in T'\cup\{i_1\}}\partial_{t_i}\Xi$. □

Proposition 14.5 (Least valuations along couplings). *Let $p > 2$ be inert, μ integral, and let Θ have the normal form $(I_p^\circ, \mathrm{pr}, \omega, \lambda)$ with $|\omega| = p^{-\rho_p}$ and λ_{e0} a unit for odd e. For a coupling family (Φ_0, e) and j as above, with $\nu = \nu' + \nu''$,*

$$\mathrm{val}_6(\nu) \le -|j|, \qquad v(\nu'!\,\nu''!) + \mathrm{val}_6(\nu) \le \gamma_p|\nu| - |j|.$$

Proof. Setting. Let $F(z) = \Theta(\omega z) = W\exp(\kappa\Xi)$ with $\kappa = c_\omega = \omega^{p+1}/p$ and $\Xi = \Xi_{I_p^\circ,\lambda}$. For inert p, $\rho_p = 1/(p^2-1)$, so $v(\kappa) = (p+1)\rho_p - 1 = 1/(p-1) - 1 < 0$ as $p > 2$, and $|\kappa| > 1$. The series W is a unit because its constant term is $\Theta(0) = 1$. Take δ_c as in Proposition 14.3(a); then $U = \prod_c F(\delta_c z)$ is integral with constant term 1.

The product of copies. In two copies z', z'' of the variables put

$$G(z', z'') = F(z' + z'')\prod_{c\ne 0} F(\delta_c z'),$$

the substitution $z_0 = z' + z''$, $z_c = \delta_c z'$ $(c \ne 0)$ into six copies; as $\delta_0 = 1$, $G(z', 0) = U(z')$. Only the first factor involves z'', so $\partial_{z_x''}G = (\kappa\,\partial_x\Xi + B_x)(z' + z'')\,G$ with B_x integral.

Step 1: the derivatives in z''. Differentiate G $j(\tau)$ times in z''_{τ^+} for each $\tau \in \Phi_0$; by Lemma 14.4 the result is $\kappa^{|j|} g G$ with g integral and $g \equiv \prod_\tau (\partial_{\tau^+}\Xi)^{j(\tau)}(z' + z'')$ modulo $\mathfrak{m}$.

Step 2: $z'' = 0$. At $z'' = 0$ the result is $\kappa^{|j|} g(z', 0) U(z')$ with $g(z', 0) \equiv \prod_\tau (\partial_{\tau^+}\Xi)^{j(\tau)}(z')$.

Step 3: the lowest term. By Lemma 12.3 ($0 \notin I_p^\circ$), modulo $\mathfrak{m}$ the derivative $\partial_{\tau^+}\Xi$ consists of the monomials $\lambda_{a0} z_{\tau_1}^{p^a}$ with $a \in I_p^\circ$ and $\mathrm{pr}_a \tau_1 = \tau^+$. Put $z_{\tau_1} = 0$ for $\tau_1 \notin \Phi_0$. A remaining monomial of degree at most $p^{e(\tau)}$ has $\tau_1 \in \Phi_0$ and odd $a \le e(\tau)$ with $\mathrm{pr}_a \tau_1 = \tau^+$, so $\tau_1 = \tau$ and $a = e(\tau)$ by the coupling condition. Hence, in total degrees up to $p^{e(\tau)}$, $\partial_{\tau^+}\Xi$ agrees with $\lambda_{e(\tau)0} z_\tau^{p^{e(\tau)}}$ modulo $\mathfrak{m}$, and its lowest term is this monomial. Lowest terms multiply, and U has lowest term 1; so the coefficient of $z'^{\nu'}$ in $g(z', 0)U(z')$, after the substitution $z_{\tau_1} = 0$ (which does not change the coefficient of a monomial in the variables of Φ_0), is $\prod_\tau \lambda_{e(\tau)0}^{j(\tau)}$ modulo $\mathfrak{m}$, a unit.

Step 4: coefficients. The derivative $\prod_\tau \partial_{z''_{\tau^+}}^{j(\tau)}$ at $z'' = 0$ turns the coefficient of $z'^{\nu'} z''^{\nu''}$ into $\nu''!$ times it. Hence $|[z'^{\nu'} z''^{\nu''}]G| = |\kappa|^{|j|}/|\nu''!|$. Since $G(z', z'') = \tilde{G}(\varpi z', \varpi z'')$ for $\tilde{G} = \Theta(z' + z'') \prod_{c \neq 0} \Theta(\delta_c z')$, the coefficient of $z'^{\nu'} z''^{\nu''}$ in $\tilde{G}$ is $\varpi^{-|\nu|}$ times that in G, so its valuation V satisfies $V + v(\nu''!) = |j| v(\kappa) - \rho_p |\nu| = |j|(\frac{1}{p-1} - 1) - \rho_p |\nu|$. The substitution has integral coefficients, so Lemma 14.2(b) gives

$$v(\nu''!) + \mathrm{val}_6(\nu) \le |j|\Big(\frac{1}{p-1} - 1\Big) - \rho_p |\nu|.$$

Conclusion. As $|\nu| = \sum_\tau j(\tau)(p^{e(\tau)} + 1) \ge |j|(p+1)$ and $\rho_p(p+1) = 1/(p-1)$, the right side is at most $-|j|$, and $v(\nu''!) \ge 0$ gives $\mathrm{val}_6(\nu) \le -|j|$. For the second bound add $v(\nu'!) \le |\nu'|/(p-1) = (|\nu| - |j|)/(p-1)$ (Legendre): the sum is at most $|\nu|/(p-1) - |j| - \rho_p |\nu| = \gamma_p |\nu| - |j|$. □

14.4. Near pairs and losses

From now on we fix data as in Definition 8.7, and $l_0 = l_0(s)$. A *near pair* at a prime p is a pair (τ, e) with $e \in I_p$, $2e < l_0$, and both τ and $\mathrm{pr}_e \tau$ b-places. Let $n_\sharp = \lceil 6b/k \rceil$ and, for $l \in \mathbb{N}$, let $B(l) = (l^- + 1) \cdots (l^- + n_\sharp)$ with $l^- = (l - n_\sharp)_+$, the *loss product*. Finally, for $\tau \in \Phi$ let $R_\tau = \sum_{e \in I_p} \min\big(\kappa(\tau)/p^e, \kappa(\mathrm{pr}_e \tau)\big)$.

Lemma 14.6. *Let $p \le k$ be non-split.*

(a) *For $0 < i < j < l_0$ the places σ, $(-p)^i\sigma$, $(-p)^j\sigma$ are not all b-places.*
(b) *Near pairs exist only for inert p, with odd exponent; $\mathrm{pr}_e \tau = (-p)^e \tau$ for them.*
(c) *Two near pairs with the same start, or with the same end, coincide; and the end of a near pair is not the start of a near pair.*

Proof. (a) The places form the group $(\mathbb{Z}/q_1)^\times$, and $(-p)^i\sigma/\sigma$ is the class of $(-p)^i = \epsilon\, p^i$ with $\epsilon = (-1)^i$. If σ, $(-p)^i\sigma$, $(-p)^j\sigma$ were all b-places, then with $\tau = (-p)^i\sigma$, $\rho = (-p)^j\sigma$ we would have $\tau/\sigma \equiv \pm p^i$ and $\rho/\sigma \equiv \pm p^j$ modulo q_1, hence modulo its divisor $q_1^{(p)}$, with $0 < i < j < l_0$ and the prime $p \le k \le kt$; this is excluded by the sparseness of Φ_b for the primes up to kt (Definition 8.4, (D3)).

(b) All b-places have class -1 modulo q_0. For $p = d$ the place $\mathrm{pr}_e \tau = -d^e \tau$ has the class of τ times $-d^e$, and $-d^e \not\equiv 1$ modulo q_0: as d is odd and q_0 divides

$2(d+1)$ but not $d+1$, $d^2 \equiv 1$ modulo q_0, so the powers of d are 1 and d, and neither is $\equiv -1$ (for 1 because $q_0 > 2$, for d because $q_0 \nmid d+1$); so τ and $-d^e\tau$ do not have the same class, and there is no near pair for $p = d$. For split p, I_p is empty. Hence a near pair needs p inert, and then I_p consists of the odd e, for which $-p^e = (-p)^e$.

(c) *Same start.* If (τ, e) and (τ, e') are near pairs with $e < e'$, then τ, $(-p)^e\tau$ and $(-p)^{e'}\tau$ are b-places with $0 < e < e' < l_0$, against (a). *Same end.* If (τ, e) and (τ', e') have the same end and $e < e'$, then $(-p)^e\tau = (-p)^{e'}\tau'$, so $\tau = (-p)^{e'-e}\tau'$; the b-places τ', $(-p)^{e'-e}\tau'$ and $(-p)^{e'}\tau'$ contradict (a), as $0 < e' - e < e' < l_0$. If $e = e'$, then $\tau = \tau'$. *End and start.* If $(-p)^e\tau = \tau'$ with (τ', e') a near pair, then τ, $(-p)^e\tau$ and $(-p)^{e+e'}\tau$ are b-places with $0 < e < e + e' < l_0$, as $2e, 2e' < l_0$; this is excluded. □

Lemma 14.7 (Losses). *Let $v = v_p$ on $\mathbb{N}$.*

(a) *$v(n_\sharp!) \le v(B(l))$ and $v(l!) \le v((l^-)!) + v(B(l))$; $n_\sharp \ge x^{42}$.*
(b) *For $p \ge 3$ and $n \le 2b$, $n/(p-1) - (\log b + 2) \le v(n!) \le n/(p-1)$; hence a multinomial coefficient with $n_1 + n_2 + n_3 \le 2b$ has valuation at most $3(\log b + 2)$.*
(c) *$\gamma_p \sum_\tau \kappa(\tau) - S_p = v(\pi_p) \sum_\tau R_\tau$.*
(d) *If near pairs at τ occur only for one exponent e_0, and $\kappa(\tau)/p^{e_0} \le c + 1$ when one occurs, then $R_\tau \le c + 1 + l_0 a/x_1 + 2b/(kp)$.*
(e) *For $p \le k$: $3(\log b + 2) + x^{42} + 1 + l_0 a/x_1 + 2b/(kp) \le v(B(l))$.*

Proof. (a) A product of $n_\sharp$ consecutive integers is divisible by $n_\sharp!$, so $v(n_\sharp!) \le v(B(l))$. For the second claim, $l \le l^- + n_\sharp$, so $l!$ divides $(l^- + n_\sharp)! = (l^-)!\, B(l)$. Finally $n_\sharp \ge 6b/k \ge x^{42}$, since $k \le 3x^5$ and $b \ge x$ to a large power.

(b) By Legendre's formula $v(n!) = (n - s_p(n))/(p-1)$ with $s_p(n)$ the sum of the base-p digits of n. This gives $v(n!) \le n/(p-1)$. As n has at most $\log_p n + 1$ digits, each at most $p-1$, $s_p(n) \le (p-1)(\log_p n + 1)$, so $v(n!) \ge n/(p-1) - \log_p n - 1 \ge n/(p-1) - (\log b + 2)$ for $n \le 2b$ and $p \ge 3$ (then $\log_p n \le \log(2b) \le \log b + 1$). For a multinomial coefficient with $n = n_1 + n_2 + n_3 \le 2b$, $v(n!/(n_1!n_2!n_3!)) \le n/(p-1) - \sum_i (n_i/(p-1) - (\log b + 2)) = 3(\log b + 2)$.

(c) At each place, $\gamma_p \kappa(\tau) = v(\pi_p) \sum_{e \in I_p} \kappa(\tau)/p^e$ by Lemma 11.2(a), and $s_p(\kappa(\tau), \tau) = v(\pi_p) \sum_{e \in I_p} (\kappa(\tau)/p^e - \kappa(\tau'))_+$ with $\tau' = \mathrm{pr}_e \tau$. Since $y - (y - z)_+ = \min(y, z)$ for real y, z, the difference is $v(\pi_p) R_\tau$; sum over τ. All series converge absolutely, the terms being at most b/p^e.

(d) Split R_τ by the exponent. *The near pair*, if it occurs (with $e = e_0$): its term is at most $\kappa(\tau)/p^{e_0} \le c + 1$. *Other terms with $2e < l_0$*: if (τ, e) is not a near pair, then τ or $\mathrm{pr}_e \tau$ is not a b-place and has weight a/x_1, so $\min(\kappa(\tau)/p^e, \kappa(\mathrm{pr}_e \tau)) \le a/x_1$; there are fewer than l_0 such e. *Terms with $2e \ge l_0$*: the term with exponent $e + n$ is at most $\kappa(\tau)/p^{e+n} \le b/(p^n 2^e)$, so the terms with $e \ge n = \lceil l_0/2 \rceil$ add up to at most $2b/p^n$. Since $p^{n-1} \ge 2^{l_0/2-1} \ge 2^{13s} = m^{286} \ge k$, this is at most $2b/(kp)$.

(e) By (a), $v(B(l)) \ge v(n_\sharp!) \ge n_\sharp/p - 1 \ge 6b/(kp) - 1$, the first term of Legendre's formula. On the left, $3(\log b + 2) + x^{42} + 1 \le a/x_1$ and $l_0 a/x_1 \le 81 \cdot 32 \log x \cdot b/x^{999}$, both at most $b/(2kp)$ because $kp \le k^2 \le 9x^{10}$; so the left side is at most $3b/(kp) \le 6b/(kp) - 1$. □

14.5. Twelve copies

Let w be a tight frame of length M with multiplicities c_i (Theorem 13.9(b)), and let $\mu_i = \varrho_0\, w_i^\dagger$ be its linear forms, read in $\mathbb{C}_p$; with π_M the conjugate of $\bar\pi_M$, the vectors $u_i = w_i/\bar\pi_M$ form a resolution of the identity with multiplicities c_i, and $\mu_i = \mathfrak{s}\, u_i^\dagger$ with $\mathfrak{s} = \varrho_0\pi_M$. A degree l (one variable per place) is *near the weights* if $\kappa(\tau) \le l_\tau \le \kappa(\tau) + x^{42}$ for all τ; then $l_\tau \le 2b$. All least valuations below refer to $\Theta = \Theta_{\mu,c}$ for the CM curve $\mathcal{E}_0$.

Lemma 14.8. *Let p be non-split. Then $p \nmid q_1 M$, the u_i are integral above p, and $\mathfrak{s}$ is a unit at p. If p is inert, $\mathrm{val}_6(v) \le 0$ for all v, and Θ has the normal form $(I_p^\circ, \mathrm{pr}, \varpi, \lambda)$ with $|\varpi| = p^{-\rho_p}$ and the λ_{e0} units. The coefficients of the μ_i are integral at every prime. Moreover $|\pi_M \mathsf{a}|^2 = M/\Omega$.*

Proof. Coprimality. Non-split primes do not divide q_1 (Lemma 8.2), and they differ from p_0, p_1, which are $\equiv 1$ modulo $q_1 d$ and therefore split (Proposition 10.7); so $p \nmid q_1 M$. *The vectors.* $q_1 M u_i = (q_1 w_i)\pi_M$, and $q_1 w_i \in \mathcal{O}_E$ because $q_1 \mathfrak{a}_0 \subseteq \mathcal{O}_E$ (Lemma 13.4(a)); as $q_1 M$ is prime to p, u_i is integral at every place above p. *The scale.* π_M and $\bar\pi_M$ are integers with product M, a unit at p, so both are units above p; and $|\varrho_0| = 1$ at $p \nmid q_1$ (Proposition 13.1(b)). Hence $\mathfrak{s} = \varrho_0\pi_M$ is a unit.

Inert p: all degrees. The resolution identity reads $z_\tau = \sum_i c_i \Phi_\tau(u_i)\, u_i^\dagger z$, so $\sum_i c_i \Phi_\tau(u_i)\, \mu_i z = \mathfrak{s}\, z_\tau$ with the integral coefficients $c_i\Phi_\tau(u_i)$ and the unit $\mathfrak{s}$; Proposition 14.3(c) gives $\mathrm{val}_6(v) \le 0$. *Normal form.* By Theorem 12.6, $\Theta_{u^\dagger,c}$ has the normal form $(I_p^\circ, \mathrm{pr}, \varpi, \lambda)$ with $|\varpi| = p^{-\rho_p}$ and the λ_{e0} units; $\Theta = \Theta_{\mu,c}$ is $\Theta_{u^\dagger,c}(\mathfrak{s}z)$, and by Proposition 12.2(b) it has the normal form with $\lambda'_{ej} = \mathfrak{s}^{(p^e+1)p^j}\lambda_{ej}$, again units for $j = 0$. *Integrality.* The coefficients of $\mu_i = \varrho_0 w_i^\dagger$ are integral by Proposition 13.1(c), since $w_i \in \mathfrak{a}_0$. *The scale.* $|\pi_M\mathsf{a}|^2 = |\pi_M|^2|\mathsf{a}|^2 = M\,\Omega^{-1}$ by Proposition 13.1(a). □

Theorem 14.9 (Twelve copies). *Let some μ_{i_0} have unit coefficients at p and let l be near the weights. Then*

$$v(l!) + \mathrm{val}_{12}(l) \le S_p + 2\sum_\tau v(B(l_\tau)).$$

Proof. Write $l! = \prod_\tau l_\tau!$ and $|l| = \sum_\tau l_\tau$. Every l_τ lies in $[\kappa(\tau), \kappa(\tau) + x^{42}]$ and is at most $2b$. Every non-split p is d or inert, and $d \le k$; so the following three cases cover all primes.

Split p. Then $I_p = \emptyset$, $S_p = 0$ and $\gamma_p = 0$. Proposition 14.3(b) gives $\mathrm{val}_6(l) \le -v(l!)$, and six more copies in degree 0 (Lemma 14.2(a)) give $v(l!) + \mathrm{val}_{12}(l) \le 0$.

Inert $p > k$. The coupling family. Take $\Phi_0 = \Phi$ and $e \equiv 1$. The map $\tau \mapsto -p\tau$ is a bijection of Φ, so a partner $t = -p\tau$ has the unique preimage τ, and the only odd $e' \le 1$ is 1: this is a coupling family. Put

$$i(\tau) = \left\lfloor \frac{(l_\tau - n_\sharp)_+}{p} \right\rfloor, \qquad j(\tau) = \min\big(i(\tau), l_{-p\tau}\big).$$

As $l_\tau \le 2b$ and $p > k$, $i(\tau) \le 2b/k \le n_\sharp$. *Degrees.* At a place $t = -p\tau$ the degree of v' is $pj(t) \le pi(t) \le (l_t - n_\sharp)_+$ and that of v'' is $j(\tau) \le \min(n_\sharp, l_t)$; since $(l_t - n_\sharp)_+ + \min(n_\sharp, l_t) = l_t$, we have $v' + v'' \le l$. Write $l = v' + v'' + \eta$. *Least valuation.* By Proposition 14.5, $\mathrm{val}_6(v' + v'') \le -|j|$, and $\mathrm{val}_6(\eta) \le 0$ by

Lemma 14.8; so $\mathrm{val}_{12}(l) \le -|j|$ by Lemma 14.2(a). *One place.* Let $l^- = (l_\tau - n_\sharp)_+$, so $i = i(\tau) = \lfloor l^-/p \rfloor$. By Lemma 14.7(a) and Legendre's recursion $v(m!) = \lfloor m/p \rfloor + v(\lfloor m/p \rfloor !)$,

$$v(l_\tau!) \le v((l^-)!) + v(B(l_\tau)) = i + v(i!) + v(B(l_\tau)) \le i + 2v(B(l_\tau)),$$

as $i \le n_\sharp$ gives $v(i!) \le v(n_\sharp!) \le v(B(l_\tau))$. Moreover $i \le \kappa(\tau)/p$, because $l_\tau - n_\sharp \le \kappa(\tau) + x^{42} - n_\sharp \le \kappa(\tau)$, and $l_{-p\tau} \ge \kappa(-p\tau)$; hence $i - j(\tau) = (i - l_{-p\tau})_+ \le (\kappa(\tau)/p - \kappa(-p\tau))_+$, the term $e = 1$ of $s_p(\kappa(\tau), \tau)$, which is at most $s_p(\kappa(\tau), \tau)$ as all terms are nonnegative and $v(\pi_p) = 1$. Hence $v(l_\tau!) - j(\tau) \le s_p(\kappa(\tau), \tau) + 2v(B(l_\tau))$, and summing over τ, $v(l!) - |j| \le S_p + 2\sum_\tau v(B(l_\tau))$.

Non-split $p \le k$ (so $p \ge 13$). *The coupling family.* Let Φ_0 be the set of starts of near pairs and $e(\tau)$ their exponent, unique by Lemma 14.6(c), and odd by (b). If $\tau_1 \in \Phi_0$ and odd $e' \le e(\tau)$ satisfy $\mathrm{pr}_{e'}\tau_1 = \mathrm{pr}_{e(\tau)}\tau$, then (τ_1, e') is a near pair (its exponent is odd and at most $e(\tau) < l_0/2$, its start is a b-place, and its end is the end of $(\tau, e(\tau))$), with the same end; by Lemma 14.6(c), $\tau_1 = \tau$ and $e' = e(\tau)$. Hence (Φ_0, e) is a coupling family. Put $j(\tau) = \lfloor l_\tau / p^{e(\tau)} \rfloor$. *Degrees.* At a start τ the degree of ν' is $p^{e(\tau)} j(\tau) \le l_\tau$, and that of ν'' is 0, since no end is a start. At an end t of $(\tau, e(\tau))$, the degree of ν'' is $j(\tau) \le 2b/p \le b = \kappa(t) \le l_t$ (t is a b-place), and that of ν' is 0. Hence $\nu' + \nu'' \le l$; write $l = \nu' + \nu'' + \eta$. *Least valuation.* By Proposition 14.5 and Proposition 14.3(b) for η,

$$\mathrm{val}_{12}(l) \le \gamma_p |l| - |j| - v(\nu'!) - v(\nu''!) - v(\eta!).$$

Factorials. At each place $l_\tau = \nu'_\tau + \nu''_\tau + \eta_\tau \le 2b$, so by Lemma 14.7(b) $v(l!) \le v(\nu'!) + v(\nu''!) + v(\eta!) + 3f(\log b + 2)$, and $v(l!) + \mathrm{val}_{12}(l) \le \gamma_p|l| - |j| + 3f(\log b + 2)$. *Comparison with S_p.* As $\gamma_p \le 1$, $\gamma_p|l| \le \gamma_p \sum_\tau \kappa(\tau) + f x^{42}$, and by Lemma 14.7(c), $\gamma_p \sum_\tau \kappa(\tau) = S_p + v(\pi_p) \sum_\tau R_\tau$. Apply Lemma 14.7(d) at each τ with $c = j(\tau)$ for a start (where $\kappa(\tau)/p^{e(\tau)} \le l_\tau/p^{e(\tau)} < j(\tau) + 1$) and $c = 0$ at the other places (where no near pair starts): $R_\tau \le j(\tau) + 1 + l_0 a/x_1 + 2b/(kp)$, with $j(\tau) = 0$ off Φ_0. As $v(\pi_p) \le 1$,

$$\begin{aligned} \gamma_p|l| - |j| + 3f(\log b + 2) &\le S_p + \sum_\tau \left(3(\log b + 2) + x^{42} + 1 + \frac{l_0 a}{x_1} + \frac{2b}{kp} \right) \\ &\le S_p + \sum_\tau v(B(l_\tau)) \end{aligned}$$

by Lemma 14.7(e), which is at most the claim. □

14.6. Sharpness

Let $V_{12} = \mathrm{Hom}(V_1, V_{12})$ be the standard space of 12×1 matrices over E, with variables $z_{\tau,b}$ ($b < 12$) at the place τ, and let $\mathcal{T}(F) = J_{\pi_M \mathfrak{a}}(F)$ for $F \in \mathcal{S}(V_{12})$. For the modulus $\mathfrak{m} = (\pi_M)$ and a class y_0 of 12×1 matrices over $O_E/\mathfrak{m}$, the *indicator datum* $\mathbf{1}_{y_0}$ is the vector whose component at infinity is the Gaussian, whose component at the places dividing $\mathfrak{m}$ is the indicator of y_0, and which agrees with the vacuum at the other finite places.

Proposition 14.10 (Reduction to the curve). *Let $c_i \ge 3|l|$ for all i, and $\mathrm{val}_{12}(l) \le r$ at p. Then for some class y_0 some coefficient of degree l of $c_{\mathrm{vac}}^{12}\, \mathcal{T}(\mathbf{1}_{y_0})$ has valuation at most $r + v(M^{12f} c^\circ)$, where c° is a value of θ° at a point* $\mathrm{fin}(y)$.

Proof. Twelve series. By Lemma 14.2(c) there are twelve series of the form $H_b\Theta$ with $H_b = \prod_i y(\mu_i z)^{e_{b,i}}$, $|e_b| \le |l|$, and exponents α_b with $\sum_b \alpha_b = l$, such that $\sum_b v([z^{\alpha_b}](H_b\Theta)) \le r$. *Monomial sections.* For each b and i, $y^{e_{b,i}}\theta^{c_i} = (y\theta^3)^{e_{b,i}}\theta^{c_i-3e_{b,i}}$ away from the poles of y, a monomial section of degree c_i since $3e_{b,i} \le 3|l| \le c_i$ (Theorem 11.3(ii)). *A vector.* By Theorem 13.10, for the frame w, $\delta = \pi_M$ and the scale $\mathfrak{a}\pi_M$, applied in each of the twelve coordinates, there is $F \in \mathcal{S}(V_{12})$ whose rescaled Taylor series $\iota_p\mathcal{T}(F)$ is the product over the twelve copies b of the series $H_b\Theta$ in the variables of copy b, whose values at the points $\mathrm{fin}(x)$ are integral at ι_p, and whose theta function is a theta function for $\pi_M\mathfrak{a}_0^{12}$. The coefficient of $\mathcal{T}(F)$ at the exponent γ with parts α_b in the copies is $\prod_b [z^{\alpha_b}](H_b\Theta)$, of valuation at most r; its total degree at τ is l_τ.

Fourier inversion. By (W6) with $\delta = \pi_M$, $N = M$, $M^{12f}F = \sum_i c_i\varrho(\mathrm{fin}(u_i))\mathrm{vac}$, a finite sum with $\bar{\pi}_M u_i \in \mathfrak{a}_0^{12}$ (one translate class for each of the M^{12f} residues), and the values of the $c_i\varrho(\mathrm{fin}(u_i))\mathrm{vac}$ are integral at ι_p with those of F. Each translate $\varrho(\mathrm{fin}(u_i))\mathrm{vac}$ is a sum of indicator data $\mathbf{1}_{y_0}$ times roots of unity: at the places dividing $\mathfrak{m}$ the vacuum translated by u_i is a combination of the indicators of the classes modulo $\mathfrak{m}$ that its translates cover, with coefficients of modulus one, and at the other places it is the vacuum. *The linear form.* Apply the linear form L that takes the coefficient at γ of $\mathcal{T}$. Evaluating at a point $\mathrm{fin}(y)$ where the vacuum does not vanish, $c_i\,\mathrm{vac}(\mathrm{fin}(y))$ is a value of $c_i\varrho(\mathrm{fin}(u_i))\mathrm{vac}$ up to a root of unity, hence integral; by the ultrametric inequality some i and some class y_0 satisfy

$$|M^{12f}\,L(F)\,\mathrm{vac}(\mathrm{fin}(y))| \le |L(\mathbf{1}_{y_0})|.$$

Multiplying by c_{vac}^{12} turns $\mathrm{vac}(\mathrm{fin}(y))$ into the value c° of θ°, and gives $v(c_{\mathrm{vac}}^{12}L(\mathbf{1}_{y_0})) \le r + v(M^{12f}c^\circ)$. □

Theorem 14.11 (The divisibility is attained). *Let $l\colon \Phi \to \mathbb{N}$ with $\kappa(\tau) \le l_\tau \le \kappa(\tau) + x^{42}$. There is $t \in \mathbb{C}^\times$ with $|\sigma(t)| \le \exp(fmb/k)$ for every automorphism σ of $\mathbb{C}$ such that, for every prime p and p-adic place, some exponent γ with $\sum_b \gamma_{\tau,b} = l_\tau$ and some class y_0 modulo (π_M) satisfy*

$$v\Big(\frac{c_{\mathrm{vac}}^{12}}{t}\prod_\tau l_\tau!\,[z^\gamma]\mathcal{T}(\mathbf{1}_{y_0})\Big) \le S_p.$$

Proof. The number t. Let $c^\circ \ne 0$ be a value of θ° at a point $\mathrm{fin}(y)$ (Proposition 13.13(c)), and put

$$t = M^{12f}\,c^\circ \prod_\tau B(l_\tau)^2,$$

which is not zero as no $B(l_\tau)$ vanishes. *The frame.* Take the tight frame of Theorem 13.9(b): its multiplicities are at least $6fb \ge 3|l|$, because $|l| \le 2fb$, and at every prime some member has a unit linear form, so some $\mu_{i_0} = \varrho_0 w_{i_0}^\dagger$ has unit coefficients at p. *The valuation.* By Theorem 14.9, $\mathrm{val}_{12}(l) \le r$ with $r = S_p + 2\sum_\tau v(B(l_\tau)) - v(l!)$. By Proposition 14.10 some coefficient of degree l of $c_{\mathrm{vac}}^{12}\mathcal{T}(\mathbf{1}_{y_0})$ has valuation at most $r + v(M^{12f}c^\circ) = S_p + v(t) - v(l!)$. Multiplying by $\prod_\tau l_\tau!$ and dividing by t gives the inequality.

The conjugates. $|\sigma(c^\circ)| \le \exp(12C_{\mathrm{vac}}q_1^2)$ by Proposition 13.13(a), $M^{12f} \le b^{264f}$, and $B(l_\tau) \le (l_\tau + n_\sharp)^{n_\sharp} \le (3b)^{n_\sharp}$, an integer. Hence

$$\log|\sigma(t)| \le 12C_{\mathrm{vac}}q_1^2 + 264f\log b + 2fn_\sharp\log(3b),$$

and with $n_\sharp \le 6b/k + 1$, $q_1 \le x^{12}$, $f \ge x^{11}$ and $m \ge \sqrt{x}$ large against $\log b$, this is at most fmb/k (Proposition 8.9). □

Part III

The upper bound: a composite lift

15. Spaces, weights and forms

This section fixes the hermitian spaces, the weights and the spaces of forms of the upper bound. All spaces are over the CM field $E = E(s)$ of the data, and all are definite except the spaces Y_j $(j < m)$ and W_j, which carry signs at the infinite places. Parameters are doubled throughout, so that they are integers; for a rank n the *block* $[n]$ is the vector $((n-1)/2, (n-3)/2, \dots, (1-n)/2)$, whose doubled entries are $n-1-2p$, $p < n$. We fix data as in Definition 8.7; recall $k = r+1-m$, $t = 5r$, the gap $\Delta = 2r^8$, the lift weights $\kappa_j = \kappa + (j+\frac{1}{2})\Delta$ and $a = xb$. The number m is even and k is even, r and t are odd (Proposition 8.9). Put $\varrho = m/2 - 1$; note $\Delta/2 = r^8$; and let Φ_a be the set of a-places.

15.1. Ranks, weights and the diagrams of the lifts

For $j \geq 0$ let $n_j = k + j(t+1)$; each n_j is even. For an infinite place τ and $h \geq 1$, the *column count* $c_\tau(h)$ is

$$c_\tau(h) = \begin{cases} a_\tau & h = 1, \\ \kappa_1(\tau) - n_1/2 & h = k, \\ \Delta - (t+1) & h > k,\ h \equiv k \pmod{t+1}, \end{cases} \qquad a_\tau = \begin{cases} a & \tau \in \Phi_a, \\ 0 & \text{otherwise,} \end{cases}$$

and $c_\tau(h) = 0$ for all other h. The *weight* on X_j (a definite space of rank n_j) is the highest weight $\mu(X_j)_\tau$ with $c_\tau(h)$ columns of height h for $h < n_j$ and last entry $e_j(\tau) = 0$ for $j = 0$ and $e_j(\tau) = n_{j-1}/2 - \kappa_j(\tau)$ for $j \geq 1$: its entry $p < n_j$ is $e_j(\tau) + \sum_{p<h<n_j} c_\tau(h)$.

Proposition 15.1 (The weights of the X_j). *Let λ^X_τ be the sequence whose first k doubled entries are $[k] + (2a_\tau, 0, \dots, 0)$, followed for $i = 1, 2, \dots$ by the doubled block $-2\kappa_i(\tau) + [t+1]$. Then the first n_j entries are twice the Harish-Chandra parameter of $\mu(X_j)$, and:*

(a) *the diagram $\mu^+_{j+1} = \mu(X_j) + \kappa_{j+1} - n_{j+1}/2$ of lift $j+1$ has exactly $c_\tau(h)$ columns of each height $h < n_{j+1}$; $\mu(X_{j+1}) = (\mu^+_{j+1}, 0, \dots, 0) + n_j/2 - \kappa_{j+1}$; $\mu^+_1 = (a_\tau, 0, \dots, 0) + \kappa_1 - n_1/2$ in its first k rows, and $\mu^+_{j+2} = \mu^+_{j+1} + \Delta - t - 1$ in the rows $p < n_{j+1}$;*

(b) *no column has a height strictly between 1 and k, or strictly between n_j and n_{j+1}; $c_\tau(k) = \kappa_1 - n_1/2 \geq 0$ and $c_\tau(n_{j+1}) = \Delta - t - 1 \geq 0$;*

(c) *for $j < m$, row p of μ^+_{j+1} is at most $a_\tau + \kappa_{j+1}$ for $p = 0$, at most κ_{j+1} for $0 < p < k$, and at most $m\Delta$ otherwise;*

(d) *$e_j(\tau) = n_{j+1}/2 - \kappa_{j+1}(\tau) + c_\tau(n_j)$.*

Proof. Parities. By Proposition 8.9 k is even and t odd, so $n_j = k + j(t+1)$ is even; as Δ is even, each $\kappa_j = \kappa + (j+\frac{1}{2})\Delta$ is an integer. The doubled block $[n_j]$ has the odd entries $n_j - 1 - 2p$. The entries of λ^X_τ are odd as well: $k-1-2p$ (plus $2a_\tau$ at $p = 0$) on $[k]$, and $-2\kappa_i + t - 2b$ on the i-th block. Hence $\mu(X_j)_p = \frac{1}{2}(\lambda^X_p - (n_j - 1 - 2p))$ is an integer: the division by 2 is exact.

Columns and the last entry. The number of columns of height $p+1$ of a weight is $\mu_p - \mu_{p+1} = \frac{1}{2}(\lambda_p - \lambda_{p+1}) - 1$, as consecutive entries of $[n_j]$ differ by 2. Inside $[k]$ and inside a block the doubled entries differ by 2, except $\lambda_0 - \lambda_1 = 2 + 2a_\tau$; this gives a_τ columns of height 1 and none of the other heights below k or strictly between two ranks. Across the end of $[k]$, $\lambda_{k-1} - \lambda_k = (1-k) - (-2\kappa_1 + t)$,

which gives $\kappa_1 - \frac{k+t+1}{2} = \kappa_1 - n_1/2$ columns of height k. Across the end of the i-th block, $i \geq 1$, $\lambda_{n_i-1} - \lambda_{n_i} = (-2\kappa_i - t) - (-2\kappa_{i+1} + t) = 2\Delta - 2t$, which gives $\Delta - t - 1$ columns of height n_i. The last entry is $\frac{1}{2}(\lambda_{n_j-1} - (1 - n_j))$: it is 0 for $j = 0$, and for $j \geq 1$, with $\lambda_{n_j-1} = -2\kappa_j - t$, it is $\frac{1}{2}(n_j - t - 1) - \kappa_j = n_{j-1}/2 - \kappa_j$. Hence $\mu(X_j)_p = e_j(\tau) + \sum_{p<h<n_j} c_\tau(h)$, and the first n_j entries of λ^X are twice the Harish-Chandra parameter of $\mu(X_j)$.

(b) The heights with $c_\tau(h) \neq 0$ are 1, k and the n_i with $i \geq 1$: for $n_j < h < n_{j+1}$, $h - k = j(t+1) + s$ with $0 < s < t+1$, which $t+1$ does not divide. Further $c_\tau(k) = \kappa_1 - n_1/2 \geq 0$ because $n_1 = k + t + 1 \leq \Delta/2 \leq \kappa_1$, and $\Delta - t - 1 \geq 0$ because $\Delta = 2r^8 \geq t + 1$.

(d) For $j = 0$, $e_0 = 0 = n_1/2 - \kappa_1 + c_\tau(k)$. For $j \geq 1$, $n_{j+1}/2 - \kappa_{j+1} + c_\tau(n_j) = \frac{1}{2}n_{j+1} - \kappa_j - \Delta + \Delta - t - 1 = \frac{1}{2}(n_{j+1} - 2t - 2) - \kappa_j = n_{j-1}/2 - \kappa_j = e_j$.

(a) For $p < n_j$, row p of μ^+_{j+1} is $e_j + \kappa_{j+1} - n_{j+1}/2 + \sum_{p<h<n_j} c_\tau(h)$, which by (d) is $\sum_{p<h\leq n_j} c_\tau(h) = \sum_{p<h<n_{j+1}} c_\tau(h)$, by (b); the rows from n_j on vanish. Hence the diagram has exactly $c_\tau(h)$ columns of each height $h < n_{j+1}$. The formula for $\mu(X_{j+1})$ is the same computation with the last entry $e_{j+1} = n_j/2 - \kappa_{j+1}$. The first k rows of μ^+_1 are $a_\tau[p = 0] + c_\tau(k) = a_\tau[p = 0] + \kappa_1 - n_1/2$, and in the rows $p < n_{j+1}$ the diagrams of consecutive lifts differ by the columns of height n_{j+1}, which are $\Delta - t - 1$ in number.

(c) By (a), row p of μ^+_{j+1} is $a_\tau[p = 0] + c_\tau(k)[p < k] + \sum_{1\leq i\leq j,\, n_i>p}(\Delta - t - 1)$. As $c_\tau(k) \leq \kappa_1$ and $\kappa_{j+1} = \kappa_1 + j\Delta$, row 0 is at most $a_\tau + \kappa_{j+1}$ and the rows $0 < p < k$ at most κ_{j+1}; for $p \geq k$ only the heights n_i count, and the row is at most $j\Delta \leq m\Delta$ since $j < m$. □

The weight on the last space Y_m, a definite space of rank $n_m - 1$, is obtained by branching. Let $c^Y_\tau(h)$ be the number of columns of height h of $\mu(X_m)$ that keep their height, namely $\Delta/2 - (t+1)/2$ if $h \geq k$ and $h \equiv k \pmod{t+1}$ and 0 otherwise, plus the columns of height $h+1$ that lose a box. The *doubled parameter of* Y_m is the sequence with first $k - 1$ entries $[k - 1]$, followed for $i = 1, 2, \ldots$ by $-2\kappa_{i-1/2} = \Delta - 2\kappa_i$ and the doubled block $-2\kappa_i + [t]$; its first $k - 1 + i$ entries before the blocks are those of the *target parameter* $\varsigma_\tau = (k/2 - 1, \ldots, 1 - k/2;\ -\kappa_{1/2}, -\kappa_{3/2}, \ldots)$ (Section 15.4).

Proposition 15.2 (Branching from X_m to Y_m). (a) $\mu(Y_m)_p = \mu(X_m)_{p+1} + c^{\mathrm{kept}}(p+1)$, *where* $c^{\mathrm{kept}}(h) \leq c_\tau(h)$; *hence the weights interlace. No column of height* $< k$ *or of height* $n_m - 1$ *keeps its height, and* $\mu(Y_m)$ *has the last entry* e_m *and* $c^Y_\tau(h)$ *columns of height* h.

(b) $c^{\mathrm{kept}}(h) \leq \Delta/2$; *for* $h > k$ *at most* $\Delta/2$ *columns lose a box; and* $c^Y_\tau(h) \leq \Delta$ *for* $h \geq k$.

(c) *The* branching exponent $n_\tau = |\mu(X_m)_\tau| - |\mu(Y_m)_\tau|$ *equals* $a_\tau - m\Delta/2$, *and it is the number of columns of* $\mu(X_m)$ *that lose a box plus* $e_m(\tau)$. *Hence* $\sum_\tau n_\tau = \#\Phi_a \cdot a - f\, m\Delta/2$.

Proof. The two parameters. Write λ for λ^X and λ' for the doubled parameter of Y_m: $\lambda'_p = k - 2 - 2p$ for $p < k - 1$, $\lambda'_{n_j-1} = \Delta - 2\kappa_{j+1}$ (the entry $k - 1 + j$ of 2ς), and $\lambda'_{n_j+b} = -2\kappa_{j+1} + t - 1 - 2b$ for $b < t$. We show $\lambda'_p - \lambda_{p+1} - 1 = 2c^{\mathrm{kept}}(p+1)$. If $p + 1 < k$, then $\lambda'_p - \lambda_{p+1} = (k - 2 - 2p) - (k - 3 - 2p) = 1$ and $c^{\mathrm{kept}}(p+1) = 0$. If $p + 1 = n_j$, then $\lambda'_p - \lambda_{n_j} = \Delta - 2\kappa_{j+1} + 2\kappa_{j+1} - t = \Delta - t$, and $\Delta - t - 1 =$

$2(\Delta/2-(t+1)/2) = 2c^{\mathrm{kept}}(n_j)$. Otherwise $p = n_j + b$ with $b < t$ and $p+1$ inside the block, and both entries are consecutive entries $-2\kappa_{j+1}+t-1-2b$ and $-2\kappa_{j+1}+t-2(b+1)$, of difference 1, while $c^{\mathrm{kept}}(p+1) = 0$.

(a) The entries of $[n_m - 1]$ are those of $[n_m]$ from the second on, raised by 1 (doubled): $n_m - 2 - 2p = (n_m - 1 - 2(p+1)) + 1$. Hence

$$\begin{aligned} 2\mu(Y_m)_p &= \lambda'_p - (n_m - 2 - 2p) \\ &= \lambda_{p+1} + 1 + 2c^{\mathrm{kept}}(p+1) - (n_m - 1 - 2(p+1)) - 1 \\ &= 2\mu(X_m)_{p+1} + 2c^{\mathrm{kept}}(p+1), \end{aligned}$$

the division being exact because all entries of λ' and of $[n_m - 1]$ are even. The kept columns are at most the columns: for $h \geq k$ with $h \equiv k$, $c_\tau(h)$ is $\kappa_1 - n_1/2$ or $\Delta - t - 1$, both at least $\Delta/2 - (t+1)/2$ (as $\kappa_1 \geq \frac{3}{2}\Delta$ and $n_1 \leq \Delta/2$). Hence $\mu(X_m)_{p+1} \leq \mu(Y_m)_p \leq \mu(X_m)_{p+1} + c_\tau(p+1) = \mu(X_m)_p$: the weights interlace. No column of height $h < k$ keeps its height, and $n_m - 1 \equiv k - 1$ modulo $t+1$, so $c^{\mathrm{kept}}(n_m - 1) = 0$ and the last entries agree. Finally $\mu(Y_m)_{h-1} - \mu(Y_m)_h = c^{\mathrm{kept}}(h) + c_\tau(h+1) - c^{\mathrm{kept}}(h+1)$, which is the number $c^Y_\tau(h)$.

(b) $c^{\mathrm{kept}}(h) \leq \Delta/2$ by definition; for $h > k$ the columns that lose a box are $c_\tau(h) - c^{\mathrm{kept}}(h)$, which is $\Delta - t - 1 - (\Delta/2 - (t+1)/2) = \Delta/2 - (t+1)/2 \leq \Delta/2$ at $h = n_i$ and 0 otherwise; so $c^Y_\tau(h) \leq \Delta/2 + \Delta/2 = \Delta$ for $h \geq k$.

(c) The blocks have sum 0, so $2|\mu(X_m)| = \sum_{p<n_m} \lambda_p = 2a_\tau - 2(t+1)\sum_{i\leq m}\kappa_i$ and $2|\mu(Y_m)| = \sum_{p<n_m-1}\lambda'_p = \sum_{i\leq m}(\Delta - 2\kappa_i - 2t\kappa_i) = m\Delta - 2(t+1)\sum_{i\leq m}\kappa_i$. The difference is $2(a_\tau - m\Delta/2)$. Pairing entry p of $\mu(Y_m)$ with entry $p+1$ of $\mu(X_m)$ by (a), $n_\tau = \mu(X_m)_0 - \sum_{1\leq h<n_m} c^{\mathrm{kept}}(h) = e_m + \sum_{0<h<n_m}(c_\tau(h) - c^{\mathrm{kept}}(h))$, the number of columns that lose a box plus e_m. Summing over the f infinite places, of which $\#\Phi_a$ are a-places, gives the last claim. □

15.2. Centre systems and the insertion of a block

All parameters of the chains are built in one way, at one infinite place. A *centre system* consists of a finite set C of centres, each with an integer value c, a length l_c and a kind (*up* or *down*), and a finite set of doubled *stem* entries. For $B \subseteq C$ let λ^+_B be the stem together with the blocks $c + [l_c]$, $c \in B$; its rank is n_B. The sign of a doubled entry y of λ^+_B is (-1) to the number of centres outside B that *count* for y: down centres with $c > y/2$ and up centres with $c < y/2$. The parameter λ_B is λ^+_B lowered by $\frac{1}{2}$ if the stem and B both have odd size, and not lowered otherwise. A vector l on a space with signs s *lists* $2\lambda_B$ *with common sign* σ if the space has rank n_B, the raised entries $l + 2\cdot$(lowering) are the entries of λ^+_B, and $s(x) = \sigma$ times the sign of the entry of x. The system is *separated by* h if all lengths have the parity of the stem size, the stem entries have the parity making λ half-integral or integral as required, and every centre c is at distance at least $(h + l_c)/2$ from the stem and from the blocks of the other centres.

A doubled parameter l on a space with signs s has *gap* h if its entries are distinct, have the parity of the rank minus one, and differ by at least h at coordinates of opposite sign. Two parameters l, l' on spaces of ranks $n < n'$ satisfy the *block rule* for weights γ, γ' if both have gap $500(n+n')^3$, after shifting λ by $n'/2 - \gamma'$ and λ' by $n/2 - \gamma$ the longer is the shorter together with a block centred at 0, and the coordinates of a common shifted entry have equal signs if the entry is positive and opposite signs if it is negative. These are the conditions

under which theta lifts between discrete series of the two groups are nonzero and computable (Appendix B.12).

Proposition 15.3 (Inserting a block). *Let the system be separated by h, $B \subseteq C$, $c \in C \setminus B$, $B' = B \cup \{c\}$, and let l, l' list $2\lambda_B$, $2\lambda_{B'}$ with the same common sign on spaces of ranks n, n'.*

(a) *l has gap h.*
(b) *Raising l by $n' \bmod 2$ and l' by $n \bmod 2$, the entries of l' are those of l together with the block $c + [n' - n]$; at coordinates with equal raised entry the signs differ exactly when c counts for the entry.*
(c) *If $h \geq 500(n + n')^3$ and c is down, then l, l' satisfy the block rule for $\gamma = c + \lfloor n/2 \rfloor$, $\gamma' = c + \lfloor n'/2 \rfloor$; if c is up, then $-l$, $-l'$ satisfy it for $\gamma = -c + \lceil n/2 \rceil$, $\gamma' = -c + \lceil n'/2 \rceil$.*

Proof. Parities and ranks. All lengths l_c have the parity of the stem size s; the doubled stem entries are odd if s is even and even if s is odd, and so are the doubled entries $2c + l_c - 1 - 2b$ of the blocks. Hence every doubled entry of λ_B^+ has the parity of $s - 1$, and $n_B \equiv s(1 + \#B)$ modulo 2. If s is even, n_B is even and the entries are odd, as a parameter on a space of rank n_B requires; if s is odd, the entries are even and $n_B \equiv 1 + \#B$, so a lowering by 1 (doubled) is needed exactly when $\#B$ is odd. This is the definition of λ_B; and the lowering of B is $s\#B \equiv n_{B'}$ modulo 2, that of B' is $s(\#B + 1) \equiv n_B$ modulo 2.

(a) By separation each centre $c \notin B$ is at distance at least $(h + l_c)/2$ from every entry of λ_B^+, and the blocks of the centres in B are disjoint from each other and from the stem; so the entries are distinct and their number is n_B, and the lowering does not change differences. If two coordinates have opposite signs, the numbers of centres outside B that count for their entries y, y' have different parities, so some centre $c \notin B$ counts for exactly one of them. For a down centre this means $y/2 < c \leq y'/2$ or the reverse, for an up centre the same; in both cases c lies between $y/2$ and $y'/2$, and as both are at distance at least $(h + l_c)/2 \geq h/2$ from it, $|y/2 - y'/2| \geq h$.

(b) By the parities above, raising l by $n' \bmod 2$ undoes the lowering of B, and raising l' by $n \bmod 2$ that of B'; so the raised entries are those of λ_B^+ and of $\lambda_{B'}^+ = \lambda_B^+ \cup (c + [l_c])$, with $l_c = n_{B'} - n_B = n' - n$. The set of centres outside B' is that outside B without c, so the sign of an entry in B' is its sign in B times -1 exactly when c counts for it. With the common sign σ on both spaces, the coordinates of a common raised entry have different signs exactly when c counts for the entry.

(c) Both parameters have gap $h \geq 500(n + n')^3$ by (a). Let c be down and $\gamma = c + \lfloor n/2 \rfloor$, $\gamma' = c + \lfloor n'/2 \rfloor$. Shifting λ by $n'/2 - \gamma' = (n' \bmod 2)/2 - c$ and λ' by $n/2 - \gamma = (n \bmod 2)/2 - c$ turns the entries into the raised entries of (b), halved and lowered by c; so the longer shifted parameter is the shorter one together with the block $[n' - n]$ centred at 0. A common shifted entry $y/2 - c$ is positive exactly if $y/2 > c$, that is, exactly if the down centre c does not count for y, and then the two signs agree by (b); it is negative exactly if c counts, and then they differ. This is the block rule. If c is up and $\gamma = -c + \lceil n/2 \rceil$, $\gamma' = -c + \lceil n'/2 \rceil$, shifting $-\lambda$ by $n'/2 - \gamma' = c - (n' \bmod 2)/2$ and $-\lambda'$ by $c - (n \bmod 2)/2$ gives the negatives $-(y/2 - c)$ of the halved raised entries lowered by c, so again the longer is the shorter together with a block centred at 0; a common entry

$-(y/2-c)$ is positive exactly if $y/2 < c$, that is, exactly if the up centre c does not count for y, and the signs on the two spaces, which are not changed, agree; it is negative exactly if c counts, and then they differ. □

15.3. The grids

Two grids of spaces are needed: Y^i_j and $\tilde{Y}^j_i$, $0 \le i, j \le m$, treated together by a flag $z \in \{0,1\}$. The node (i,j) of grid z has a base block of $r+z$ indices, i blocks of length $t+z$ and j blocks of length $t-z$. With an *offset* o (the grids have $o=0$, the *triangles* have $i=0$), the base index numbered n has the sign $(-1)^{o+e+\min(e-o,j)}$ with $e=(n+1-k)_+$, and the added blocks have sign +1. The doubled parameter at τ is: on the base, $2\varsigma_\tau(n)+z$; on the l-th block of the first kind the doubled block of length $t+z$ centred at $\kappa_l(\tau)$; on the l-th block of the second kind the doubled block of length $t-z$ centred at $-\kappa_{o+l}(\tau)$; all lowered by 1 in grid Y if $i+j$ is odd. In grid $\tilde{Y}$ the node (i,j) is the space $\tilde{Y}^j_i$ with the negative of this parameter. The spaces Y_j of the target chain are the nodes $(0,j)$ of grid Y.

Proposition 15.4 (Parameters of the grids). *Let $i, j \le m$ and $\tau \in \Phi$.*

(a) *The node (i,j) has rank $r+z+i(t+z)+j(t-z) \le 2n_m$ and $\lfloor (m-j+z)/2 \rfloor$ negative signs (offset 0); all its signs are +1 for $j \ge m$. At (j,j), the space $\tilde{Y}$ is Y plus one vector of sign $(-1)^j$.*

(b) *The centres $\pm\kappa_{l+1}(\tau)$, $l < m$ (up of length $t+z$, down of length $t-z$), with the stem $\{2\varsigma_\tau(n)+z\}$, form a centre system separated by $\Delta - t - 2 \ge 500(2n_m+2n_m)^3$; the doubled parameter of each node of a grid or triangle lists $2\lambda_B$ for the set B of its blocks, with common sign $(-1)^o$. In particular it has gap Δ, its entries are bounded by $\lceil 20b \rceil$ in absolute value, and the target parameter is strictly decreasing with $2\varsigma_\tau(n) \le k$.*

(c) *Each step $(i,j) \to (i+1,j)$ or $(i,j) \to (i,j+1)$ inserts one block, so the parameters at its two ends correspond (block rule) for the weights $\lfloor N/2 \rfloor - \kappa_{l+1}(\tau)$ or $\lceil N/2 \rceil - \kappa_{l+1}(\tau)$ of the nodes of rank N; and at the node (m,m) the parameters of $\tilde{Y}$ and Y interlace.*

Proof. Write $w_l = \kappa_l(\tau)$; consecutive weights differ by Δ, $w_1 \ge \frac{3}{2}\Delta$, and $k+t+1 \le \Delta/2$. The target parameter is $\varsigma_p = k/2-1-p$ for $p < k-1$ and $\varsigma_{k-1+a} = \Delta/2 - w_{a+1}$ for $a \ge 0$; it is strictly decreasing, its entries are integers with $2\varsigma_p \le k$, and an entry p lies at least $\Delta/2$ above the centre $-w_j$ if $p < k+j-1$ and at least $\Delta/2$ below it otherwise (for $p = k-1+a$ with $a \ge j$, $\varsigma_p + w_j = \Delta/2 - w_{a+1} + w_j \le -\Delta/2$). Entries with different values of $(p+1-k)_+$ differ by at least Δ.

(a) The rank is a count of indices: r (resp. $r+1$) base indices, i blocks of length t (resp. $t+1$) and j of length t (resp. $t-1$); and $r+1+2mt \le 2(k+m(t+1))$ as $r+1 = k+m$. The added blocks have sign +1. On the base, with offset 0, the sign of the index numbered n is $(-1)^{a+\min(a,j)}$ for $a=(n+1-k)_+$, which is +1 for $a \le j$ and $(-1)^{a-j}$ for $a > j$; as n runs over the base, a takes the value 0 for $n < k$ and each of $1, \dots, m-1+z$ once. Hence the negative signs are those with $a = j+1, j+3, \cdots \le m-1+z$, of number $\lfloor (m-j+z)/2 \rfloor$, and none for $j \ge m$. At (j,j) the ranks are $r+2jt$ and $r+2jt+1$ and the numbers of negative signs differ by 1 exactly when $m-j$ is odd, that is, as m is even, when j is odd: $\tilde{Y}$ is Y plus one vector of sign $(-1)^j$.

(b) *Separation.* The lengths $t \pm z$ have the parity of the stem size $r+z$, as t and r are odd; the stem entries $2\varsigma + z$ have the parity of z, opposite to that of $r+z$. The up centre w_{l+1} is at doubled distance at least $2w_1 - (k+1) \geq 2\Delta$ from every stem entry; the down centre $-w_{l+1}$ at least $2 \cdot \Delta/2 - 1 = \Delta - 1$, by the position of the entries; and two centres are at least Δ apart, so a block, whose doubled entries lie within t of twice its centre, is at doubled distance at least $2\Delta - t$ from twice another centre. All these are at least $(\Delta - t - 2) + (t+1)$, which is the separation $\Delta - t - 2$ with the length $l_c \leq t+1$. And $\Delta - t - 2 \geq 500(4n_m)^3$ by Proposition 8.9, as $\Delta = 2r^8$ and $n_m \leq k + m(t+1) \leq 40r^2$; this bounds $500(N+N')^3$ for two nodes of rank at most $2n_m$.

The parameter lists $2\lambda_B$. Undoing the lowering (1 in grid Y for odd $i+j$, which is the case of odd stem size and odd number of blocks), the base entry numbered n is $2\varsigma_n + z$, the entries of an up block are $2w_{l+1} + (t+z) - 1 - 2b$ and those of a down block $-2w_{o+l+1} + (t-z) - 1 - 2b$: this is λ_B^+ for the set B of the up centres $l < i$ and the down centres $o \leq l < o+j$. For the signs, count the centres outside B that count for an entry y. No up centre outside B counts for a base entry (they all lie above it) or for an up block entry (those outside B lie above). A down centre $-w_{l+1}$ counts for a base entry $2\varsigma_n + z$ exactly when $\varsigma_n < -w_{l+1}$, that is, when $l < a = (n+1-k)_+$; so the number is $\#([0,a) \setminus [o, o+j))$, and $\#([0,a) \setminus [o,o+j)) + \min((a-o)_+, j) = a$ gives the base sign $(-1)^{o+a+\min((a-o)_+,j)} = (-1)^o \cdot$(the sign of y). For an entry of the down block of $-w_{o+l+1}$, the down centres outside B that count are those $l' < o$, o in number, so its sign in the system is $(-1)^o$, and the space has +1 there; for an up block both are +1 with $o = 0$. Hence the parameter lists $2\lambda_B$ with common sign $(-1)^o$ (grids $o = 0$, triangles $i = 0$).

Gap Δ. The added blocks have sign +1, so of two coordinates of opposite signs one is a base index x of sign -1, with $a = (n+1-k)_+ > j$. If the other is a base index, its value of a differs, so the entries differ by at least Δ. If it is in an up block, the entries are at least w_1 apart. If it is in the down block of $-w_{o+l+1}$, both neighbours of this centre among the target entries have the sign +1, so ς_n is at least $3\Delta/2$ from it, and the block lies within $(t+1)/2$ of it: the distance is at least Δ. In doubled terms the entries at opposite signs differ by at least 2Δ, and the parameter has gap Δ. Every entry is at most $2w_m + t + 1 \leq 10w_{2m}$ in absolute value, and $w_{2m} \leq 2b$ by Proposition 8.9; this gives $\lceil 20b \rceil$.

(c) The step $(i,j) \to (i+1, j)$ adds the up centre w_{i+1} and the step $(i,j) \to (i, j+1)$ the down centre $-w_{o+j+1}$, so Proposition 15.3(c) applies with the separation of (b): for the down centre with the weights $-w_{l+1} + \lfloor N/2 \rfloor$, for the up centre with the weights $-w_{l+1} + \lceil N/2 \rceil$ on the negated parameters; in grid $\tilde{Y}$ the ranks are even and the two roundings agree.

Interlacing at (m,m). The doubled entries of Y_m^m are the base $k-2, k-4, \dots, 2-k$ and $\Delta - 2w_l$ $(1 \leq l \leq m)$, and the blocks $2w_l + [t]$ and $-2w_l + [t]$; those of $\tilde{Y}_m^m$ (the negative of the parameter of grid $\tilde{Y}$) are $k-3, k-5, \dots, 1-k$ and $2w_l - \Delta - 1$ $(1 \leq l \leq m+1)$, and the blocks $2w_l + [t-1]$ and $-2w_l + [t+1]$. In decreasing order they alternate, beginning and ending with $\tilde{Y}$: from the top, $2w_{l+1} - \Delta - 1 = 2w_l + \Delta - 1$ (from $\tilde{Y}$) precedes the interlaced blocks $2w_l + [t]$, $2w_l + [t-1]$, which begin and end with Y, for $l = m, \dots, 1$; then $2w_1 - \Delta - 1$, then the alternating base entries $k-2 > k-3 > \cdots > 2-k > 1-k$; then for $l = 1, \dots, m$ the entry $\Delta - 2w_l$ of Y, far below the preceding entries and at distance Δ above the block around $-2w_l$, followed by the interlaced blocks

$-2w_l + [t+1], -2w_l + [t]$, which begin and end with $\tilde{Y}$. The blocks do not meet the other entries, as $t + 1 < \Delta$. Hence the number of entries above any x is, on $\tilde{Y}$, that on Y or one more: the two parameters interlace. □

15.4. Labels, the finite group $\bar{G}$ and good forms

For a finite field $\Bbbk$ a *label* of degree r is a monic squarefree polynomial $c \in \Bbbk[X]$ of degree r with $c(0) \neq 0$; it is *primitive* if it is irreducible with a root of order $|\Bbbk|^r - 1$, and *allowed* for M if it has no root $x \in \Bbbk$ with $x^M = 1$. A tuple of labels over finite fields $\Bbbk_w$ is *good* for M if every member is allowed and some member is primitive. Labels index the irreducible representations of $\mathrm{GL}_r(\Bbbk)$ that occur in the Deligne–Lusztig decomposition through Green's characters; for each label c there is a central idempotent e_c of $\mathbb{C}[\mathrm{GL}_r(\Bbbk)]$ (Appendix B.13).

Lemma 15.5. *Let c be primitive of degree r over $\Bbbk$, $q = |\Bbbk|$. The product of the roots of c is $\xi^{1+q+\cdots+q^{r-1}}$ for every root ξ, and has order $q - 1$. If $M^2 \mid q - 1$ and $a^M = 1$, then the polynomial with roots multiplied by a is primitive.*

Proof. The roots are the conjugates ξ^{q^i}, so their product is the norm $\nu = \xi^S$ with $S = (q^r - 1)/(q - 1) = 1 + q + \cdots + q^{r-1}$, a generator of $\Bbbk^\times$ since ξ generates $\mathbb{F}_{q^r}^\times$. For the scaling, the roots of the scaled polynomial are the $a\xi^{q^i} = (a\xi)^{q^i}$, the conjugates of $a\xi$; so it suffices that $a\xi$ has order $q^r - 1$. Let d be this order. Then $\xi^d = a^{-d} \in \Bbbk^\times$, so $S \mid d$; write $d = Sd'$ with $d' \mid q - 1$. Then $(\nu a^S)^{d'} = \xi^d a^d = 1$. Since $\Bbbk^\times$ is cyclic and $a^M = 1$, we have $a^S = \nu^{u(q-1)/M}$ for some u, and $\nu a^S = \nu^{1+u(q-1)/M}$. Every prime $\ell \mid q - 1$ divides $(q - 1)/M$: this is clear if $\ell \nmid M$, and if $\ell \mid M$ it follows from $v_\ell(q - 1) \geq 2v_\ell(M)$. Hence $1 + u(q - 1)/M$ is prime to $q - 1$, νa^S generates $\Bbbk^\times$, and $q - 1 \mid d'$. Hence $d = q^r - 1$. □

The basis of the definite space X_j is indexed by $n_j - 1$ indices and one more, the *last* index. The space Y_0 has rank r; its basis is indexed by $e_1, \ldots, e_\varrho$, the k indices of X_0, a *middle* vector z, and $e_1^*, \ldots, e_\varrho^*$, and we number it once and for all by $\{0, \ldots, r - 1\}$. The basis of Y_j is that of Y_0 followed by j blocks of t indices.

Let $\bar{G} = \prod_w \mathrm{GL}_r(\mathbb{F}_w)$, the product over the chosen places w above p_0 and p_1 (Section 10.4). Let $e_0 \in \mathbb{C}[\bar{G}]$ be the sum of the idempotents $\bigotimes_w e_{c_w}$ over the families (c_w) such that, for $i = 0, 1$, the labels at the places above p_i form a good tuple for q_2 of degree r.

Proposition 15.6. *e_0 is a central self-adjoint idempotent. If χ is a character of $\mathbb{F}_w$ with $\chi^{q_2} = 1$ and $l \neq 0$, then e_0 kills every pure tensor that has, at the place w, the* hyperplane sum $\sum \chi(\det h)^{-1} h$ *over the invertible $h = 1 + ul^\mathsf{T}$.*

Proof. The label idempotents e_c of $\mathbb{C}[\mathrm{GL}_r(\mathbb{F}_w)]$ are central, self-adjoint and orthogonal for distinct labels (Appendix B.13). Their tensor products $\bigotimes_w e_{c_w}$ over the families (c_w) are central self-adjoint idempotents of $\mathbb{C}[\bar{G}]$, orthogonal for distinct families, so their sum e_0 over the good families is a central self-adjoint idempotent. For the second claim, let x be a pure tensor whose factor at w is the hyperplane sum $H = \sum_h \chi(\det h)^{-1} h$ over the invertible $h = 1 + ul^\mathsf{T}$. By centrality, xe_0 is the sum over the good families of $\bigotimes_v x_v e_{c_v}$, and it suffices that $He_c = 0$ for every allowed label c at w. The elements h are the transvections and dilations fixing the hyperplane $\ker l$ pointwise, so H lies in the span of the

representations induced from the stabiliser of $\ker l$ with the character $\chi \circ \det$ on its Levi factor; in the Deligne–Lusztig decomposition these have labels with the root x_χ that corresponds to χ under the identification of characters of $\mathbb{F}_w^\times$ with units of $\mathbb{F}_w$, and $x_\chi^{q_2} = 1$ because $\chi^{q_2} = 1$. An allowed label has no such root, so e_c kills them (Appendix B.13). □

15.5. The spaces Y_j and W_j, and the target

For a tame character ϑ (a character of $\mathrm{U}(1)(\mathbb{A})$ of weight 0 whose tame part is supported on a finite set Q of places of norm prime to MN, disjoint from its conjugate), the *level places* are the chosen places and the places of Q. Put $\mathfrak{a} = \mathfrak{a}_0/\pi_M$, generated by the idele $\alpha/\bar{\beta}_M$, where β_M generates $(\bar{\pi}_M)$ and is 1 off the chosen places, and let $\pi_N \in O_E$ with $\pi_N\bar{\pi}_N = N$ (Definition 8.7); the places dividing π_N are split of degree one above N, one in each conjugate pair, and there are f of them.

Definition 15.7. For $j < m$, Y_j is the hermitian space (it exists by Lemma 15.8) of rank $r + jt$ with Gram matrix the identity on the indices of X_j and antidiagonal on a string of $2\varrho + 1 - j$ further vectors, with entries 1 except $-N$ at the middle vector if there is one; its lattice is spanned at the finite places by the graded basis built from $\mathfrak{a}$, π_N and $(\mathfrak{a}/\bar{\mathfrak{a}})^{\lceil (m-j)/2 \rceil}$, up to an integral change of basis that is trivial at the chosen places. For $j \geq m$, Y_j is the standard definite space. The space W_j is the diagonal space of rank r whose i-th sign is -1 when the number of i is at least $k + j - 1$.

Lemma 15.8 (The spaces Y_j exist). *For $j < m$ there is a hermitian space with the signs of Y_j whose Gram matrix is the graded Gram matrix of Definition 15.7 and whose lattice basis is the graded basis times an integral change of basis that reduces to 1 at every chosen place; for $j \geq m$ the standard definite space serves.*

Proof. At an infinite place the graded Gram matrix is the identity on the n_j indices of X_j and an antidiagonal hermitian form on a string of $2\varrho + 1 - j$ vectors, whose number of negative signs is half the length of the string, rounded up (the antidiagonal pairs are hyperbolic, and the middle entry $-N$ is negative); this is the number of minus signs of Y_j, so at each infinite place there is a frame for the Gram matrix with the required signs. At the finite places the lattice with the graded basis is unimodular, as the scalings by $\mathfrak{a}$, π_N and $(\mathfrak{a}/\bar{\mathfrak{a}})^{\lceil (m-j)/2 \rceil}$ are matched by their conjugates on the hyperbolic pairs and $N = \pi_N\bar{\pi}_N$ at the middle vector; a unimodular hermitian lattice over the integers of a local field split over the base has an orthonormal basis Ll with l integral. The chosen places are split, so there the integral unitary group is GL_r of the local integers, which maps onto GL_r of the residue field; correcting l by the integral finite unitary element that lifts the inverse reductions of l at the chosen places and is 1 elsewhere gives a change of basis that reduces to 1 at the chosen places. □

The *type* of forms on Y_j (or W_j) is the product over the infinite places of the discrete series with doubled parameter $2\varsigma_\tau$ on the coordinates of Y_0 (resp. the parameter of the node $(0, j)$), with the signs of the space. Forms of *principal level* are those fixed by the integral points that reduce to 1 at all chosen places.

Definition 15.9 (Target). For a space V of rank r with signs s for which 2ς has gap Δ, let $\mathcal{A}(V)$ be the space of forms on $\mathrm{U}(V)$ of type 2ς and principal level. The group $\bar{G}$ acts on $\mathcal{A}(V)$ through integral lifts. The *good forms* are $e_0\mathcal{A}(V)$, and the *target* is $\mathcal{T} = e_0\mathcal{A}(Y_0)$, of dimension d_T.

Proposition 15.10. *For the signs of Y_0 and of each W_j, the doubled target parameter has gap Δ at every infinite place; the product of the discrete series kernels is a cusp kernel of order $\Delta/2$, and the space of forms of principal level is finite-dimensional.*

Proof. For W_j the sign of the index numbered p is -1 exactly when $p \geq k + j - 1$; by the position of the target entries (proof of Proposition 15.4) the entries of sign $+1$ lie at least $\Delta/2$ above the centre $-\kappa_j$ and those of sign -1 at least $\Delta/2$ below it, so entries at indices of opposite sign differ by at least Δ; the entries are distinct integers of the right parity. For Y_0 the signs are those of the node $(0,0)$ of grid Y, and the gap is Proposition 15.4(b). A product of discrete series kernels whose parameters have gap $500r^3 + \Delta/2$ is a cusp kernel of order $\Delta/2$ (Appendix B.12), and $500r^3 \leq \Delta/2 = r^8$; so gap Δ suffices. Forms of fixed level and of a type whose kernel is a cusp kernel form a finite-dimensional space (Appendix B.12). □

15.6. Descent pairs

A *descent pair* consists of level places M, Q, a hermitian space W of rank r_W with signs, an embedding of its basis indices into those of the definite standard space $V = I_n$ of rank $n > r_W$, a numbering of the basis vectors of W, column counts $k^+(\tau, i)$, $k^-(\tau, i)$, and a character μ of $\mathrm{U}(1)(\mathbb{A})$ with weight and tame part, trivial off MQ. Its forms on V have the highest weight with $k^+(\tau, i)+k^-(\tau, n-i)$ columns of height i and the determinant power $-\sum_i k^-(\tau, i)$, and level the integral points fixing the embedded basis vectors modulo M and preserving the span of the others modulo Q, with the character $\prod_{w\in Q} \psi_w^{-1}(\det g_{00})$. The *block datum* Ψ_{bl} is the function of matrices modulo MQ that is $\prod_{w\in Q} \psi_w(\det X_w|_{\mathrm{incl}})$ if X_w is the inclusion matrix at every $w \in M$, and 0 otherwise; $\Psi_{\mathrm{bl}} \otimes \mathrm{vac}$ is a finite datum on $\mathrm{Hom}(W, V)$. The *normalised descent* of a form Ξ on V is the function $g \mapsto \sum_h \overline{\theta(\omega(g,h)(v_{P_\xi} \otimes \Psi_{\mathrm{bl}} \otimes \mathrm{vac}))}$ on $\mathrm{U}(W)(\mathbb{A})$, the class sum over the level of V applied to $\Xi(h)$, where P_ξ is the Fock polynomial of a weight vector ξ.

Proposition 15.11 (Descent pairs). *Let O be a descent pair and l a doubled parameter on W.*

(a) $\Psi_{\mathrm{bl}} \otimes \mathrm{vac}$ *is the sum over $A \in \prod_{w\in Q} \mathrm{GL}(\mathbb{F}_w)$ of the translates, by lifts of A^{-1}, of $\Psi^1 \otimes \mathrm{vac}$, where Ψ^1 is the indicator of the matrices that are the inclusion at M and whose embedded rows are the identity at Q. Integral points (k, k') whose reductions are related by $k' = \mathrm{diag}(k, 1)j$ at M and $k' = j$ at Q, with j in the residue level, multiply it by $(\prod_M \psi_w(\det k))^{-1}$ times the level character of j.*

(b) *If l* fits *O (the parameters l on W and l' on V satisfy the block rule for the weights r_- and $\lfloor n/2 \rfloor$+the weight of μ, the numbering lists the vectors of each sign by decreasing $\pm l$, and $k^\pm(\tau, i) = 0$ for i above the number of signs $\pm$), then the Fock polynomial of the highest vector of l' is the product over τ of*

products of leading minors (*staircases*)*, each a joint harmonic for the dual pair, and the corresponding Fock vector is an edge between the types l and l'.*

(c) *The normalised descent of* Ξ *is* $\dim \cdot [K_0 : \text{level}]$ *times the conjugate of the lift from* V *to* W*, by the datum of the highest vector, of the conjugate of* Ξ*.*

Proof. (a) For a family X of residue matrices that is the inclusion at M, $\prod_Q \psi_w(\det X_w|_{\text{incl}})$ is the sum over the families A of invertible matrices at Q of $\prod_Q \psi_w(\det A_w)$ times the indicator that the embedded rows of X are A; and the indicator for A is the indicator for the identity moved by a lift of A^{-1}, which reduces to 1 at M and to A^{-1} at Q. The scalar by which the Weil action of the lift multiplies the finite datum is the inverse of the character of det of the lift, which is $\prod_Q \psi_w(\det A_w)$; this gives the sum. For the action of the level, the finite Weil action of an integral point (k, k') is the substitution $X \mapsto k'^{-1} X k$ of the reductions times the values of the characters at the determinants. At M the element j fixes the inclusion matrix and $\operatorname{diag}(k, 1)\,\text{incl} = \text{incl}\, k$, so the support is preserved; at Q the embedded rows of $j^{-1} X k$ are the corner of j^{-1} times those of X times k. The characters at Q cancel against those of the substitution, and the character of $\mathrm{U}(W)$ is trivial on integral points; what remains is $(\prod_M \psi_w(\det k))^{-1}$ times the level character of j.

(b) The highest vector of the weight with $k^{\pm}$ columns goes, under the Fock polynomial map, to the product over τ of the products of the leading $p \times p$ minors of the Fock variables, with the rows ordered by decreasing $\pm l'$ and the columns of each sign by the numbering, to the powers $k^+(\tau, p)$ and $k^-(\tau, p)$: an evaluation vector gives a product of powers of minors, and the Cartan embedding of the mixed weight into the tensor product of the two Weyl modules sends the highest vector to a tensor of two evaluation vectors. Such a staircase is highest for the rows, of the lowest K-type of l', and for the columns, of the staircase weight; it is harmonic on both sides, as its minors are. When l fits O, the block rule produces a joint harmonic for the two lowest K-types (Appendix B.12), which spans the polynomials with these properties, so the lowest K-type of $-l$ is the staircase weight and the staircase is a joint harmonic. A joint harmonic between the two discrete series types is a half edge in both directions, that is, an edge.

(c) The normalised descent is, at g, the class sum over the level of V of the conjugate kernel applied to $\Xi(h)$. By the equivariance of the conjugate kernel under rational points and the level (the real points act on the Fock vector through the conjugate Weyl representation, and the level multiplies the block datum by the level character), the function $h \mapsto$ (kernel at (g, h) applied to $\Xi(h)$) is invariant under the level, so the class sum is $[K_0 : \text{level}]$ times the integral over the finite points. Expanding $\Xi(h)$ in a basis of the Weyl module and using the invariance under the real points of $\mathrm{U}(V)$, the sum of the data $v_{P_{\bar{\xi}}} \otimes \xi$ over the basis collapses to dim times the datum of the highest vector; the result is the conjugate of the lift from V to W of the conjugate form by that datum. □

For $1 \le j \le m$ the j-th *descent pair* (W_j, V_j) has $W = W_j$, $V = I_{r+t}$, the character $\mu_j^0 \vartheta^j$ of weight $-\kappa_j$, the numbering of Section 15.4 (reversed on the minus signs), and the column counts $k^+(\tau, i) = (\varsigma(i-1) - \max(\varsigma(i), \frac{t-1}{2} - \kappa_j) - 1)_+$ and $k^-(\tau, i) = (\min(\varsigma(r-i-1), -\kappa_j - \frac{t-1}{2}) - \varsigma(r-i) - 1)_+$.

Proposition 15.12. *For $1 \le j \le m$, W_j has $m-j$ minus signs, the doubled parameter of forms on V_j at τ is, after the shift by $2(m-j+\kappa_j-\lfloor r/2\rfloor)-1$, the sequence $2\varsigma_p$ $(p < k+j-1)$, the doubled block of length t centred at $-\kappa_j$, and $2\varsigma_{p-t}$; it lists $2\lambda_{\{-\kappa_j\}}$ for the centre system with stem $\{2\varsigma\}$ and the single down centre $-\kappa_j$; and the target parameter fits the j-th descent pair.*

Proof. Write $a = k+j-1$ (the number of plus signs of W_j), $w = \kappa_j(\tau)$ and $\varsigma = \varsigma_\tau$.
Signs. The index numbered p has sign -1 exactly when $p \ge a$, so there are $r-a = m-j$ minus signs.

The parameter on V_j. The doubled Harish-Chandra parameter of a weight of rank $N = r+t$ has the steps $\lambda_p - \lambda_{p+1} = 2 + 2\,\mathrm{cols}(p+1)$, where $\mathrm{cols}(i) = k^+(\tau,i) + k^-(\tau,N-i)$. As $\varsigma_q + w \ge \Delta/2$ for $q < a$, $\varsigma_q + w \le -\Delta/2$ for $q \ge a$, and $t/2 < \Delta/2$, the maxima, minima and positive parts in the column counts are decided: $k^+(\tau,i) = \varsigma_{i-1} - \varsigma_i - 1$ for $i < a$, $k^+(\tau,a) = \varsigma_{a-1} + w - \frac{t-1}{2} - 1$, and $k^+(\tau,i) = 0$ for $i > a$; and $k^-(\tau,N-i)$ is 0 for $a \le i < a+t$, $-w - \frac{t-1}{2} - \varsigma_a - 1$ for $i = a+t$, and $\varsigma_{i-1-t} - \varsigma_{i-t} - 1$ for $i > a+t$. Hence the steps of the parameter are those of the list $L = (\varsigma_0+w, \dots, \varsigma_{a-1}+w, \frac{t-1}{2}, \frac{t-3}{2}, \dots, -\frac{t-1}{2}, \varsigma_a+w, \dots, \varsigma_{r-1}+w)$, doubled: inside the string the steps are 2, which is no column. The level is fixed by the determinant power $-\sum_i k^-(\tau,i)$: the columns above height a, which all come from k^-, cancel it, and the entry p of the parameter is $r - 2a + 2L_p$. Lowering by $2u-1$ with $u = m-j+w-\lfloor r/2\rfloor$, and using $r-2a = 2(m-j)-r$, the entries become $2\varsigma_p$ for $p < a$, the doubled block $-2w+t-1-2(p-a)$ of length t centred at $-w$ for $a \le p < a+t$, and $2\varsigma_{p-t}$ for $p \ge a+t$.

Centre systems. Take the stem $\{2\varsigma_p\}$ and the single down centre $-w$ of length t. Without the block, the sign of an entry is -1 exactly when the centre counts for it, that is, when $\varsigma_p < -w$, i.e. $p \ge a$: this is the sign of W_j. With the block every sign is $+1$, as on the definite V_j. Hence 2ς on W_j lists $2\lambda_\emptyset$, and the parameter on V_j lowered by $2u$, whose entries raised by the lowering 1 are those of the stem and the block, lists $2\lambda_{\{-w\}}$, both with common sign 1; the separation is that of Proposition 15.4(b). Proposition 15.3(c) gives the block rule for the down centre $-w$, with the weights $-w+\lfloor r/2\rfloor$ and $-w+\lfloor (r+t)/2\rfloor$; raising the first weight by u undoes the shift and gives the weights $m-j$, the number of minus signs, and $\lfloor (r+t)/2\rfloor - \kappa_j$.

Fitting. The numbering lists the plus vectors by decreasing ς and the minus vectors by increasing ς, that is by decreasing $\pm\varsigma$; and $k^+(\tau,i) = 0$ for $i > a$, the number of plus signs, and $k^-(\tau,i) = 0$ for $i > r-a$, the number of minus signs, by the formulas above. □

16. Chains of lifts, the composite lift and the Poincaré map

This section defines the two maps whose comparison is the seesaw identity: the *lower chain* ι_μ, a chain of m theta lifts from the target forms on Y_0 to forms on Y_m, and the *composite lift* B_ϑ, a combination of chains of theta lifts $X_0 \to X_1 \to \cdots \to X_m$ of source forms, restricted to Y_m; and the Poincaré map $\mathcal{P}$ from source forms to forms on Y_0. The theta kernels used here are those of Appendix B.10; for definite spaces they are reviewed in Section 17.

16.1. Chains of theta lifts

Let $n_0, n_1, \dots$ be finite index sets, K_j open compact subgroups of the finite adelic unitary groups $\mathrm{U}_{n_j}(\mathbb{A}_\mathrm{f})$ of the standard definite spaces, and for each j a *theta kernel* k_j from algebraic forms of stage j (weight with $\kappa(\tau, h)$ columns and determinant power $e_j(\tau)$, level K_j) to algebraic forms of stage $j+1$: a space S_j of data, an action ω of $\mathrm{U}_{n_j}(\mathbb{A}_\mathrm{f}) \times \mathrm{U}_{n_{j+1}}(\mathbb{A}_\mathrm{f})$ on S_j, and a linear map jet sending a datum to a linear map between the Weyl modules, such that the lift

$$(k_j(\Phi)\varphi)(g') = \sum_g \mathrm{jet}\big(\omega(g, g')\Phi\big)\, \varphi(g),$$

the sum over the double cosets of the level, is an algebraic form of stage $j+1$. The *chain* with data $\Phi_0, \Phi_1, \dots$ is $\mathrm{chain}_j = k_{j-1}(\Phi_{j-1}) \circ \cdots \circ k_0(\Phi_0)$; it extends to a linear map on the tensor product $S_{j-1} \otimes \cdots \otimes S_0$.

Proposition 16.1 (Chains). (a) (Equivariance.) *Moving each factor j of a tensor of data by $\omega(a_j, a_{j+1})$ and the form by a_0 moves the chain by a_m. If $\omega(z_i, z_{i+1}) = 1$ for all i and $\omega(1, z_{j+1})$ fixes the tensor on factor j, then the chain does not see the translation of the form by z_0.*

(b) (Linearity.) *The value at g of the chain of a form is a linear function of the pure tensor of the data.*

(c) (Plain sums.) *Let $\sum_x L_x \otimes \Phi_x$ be* jointly invariant*: unchanged when $a_j \in K_j$ $(j < m,\ a_m = 1)$ move each $\Phi_{x,j}$ by $\omega(a_j, a_{j+1})$. If the levels K_j, $j < m$, are* free *(the stabilisers of the cosets gK_j in the rational unitary group are trivial), then for a form φ of level K_0*

$$\sum_x L_x\big(\mathrm{chain}(\Phi_x)\varphi(g_m)\big) = \sum_c \sum_x L_x\big(\mathrm{jet}(\Phi_{x,m-1}) \circ \cdots \circ \mathrm{jet}(\Phi_{x,0})\, \varphi\big),$$

the outer sum over tuples of double cosets, each factor moved by the representatives (the kernel product*). Consequently any property of vectors preserved by every kernel and satisfied by the values of φ holds termwise, and a multiplicative size bound on each kernel bounds the kernel product by the product of the bounds.*

Proof. (a) Each lift is equivariant: $k(\omega(a, a')\Phi)(a\varphi) = a'\, k(\Phi)\varphi$, because ω is a representation, the double coset sum is invariant under translation and jet intertwines. For a tensor of length 0 (a scalar) the claim is the translation of φ; for a longer tensor both sides are linear in the tensor, and on a pure tensor $s \otimes r$ the claim follows from the claim for r and the equivariance of the last lift. For the second claim, $\omega(1, z_{j+1})$ on factor j is, as all $\omega(z_i, z_{i+1})$ are trivial, the joint action of the tuple that is z_i^{-1} up to stage j and 1 after it; by the first claim this moves the chain by 1 at the end and the form by z_0^{-1}.

(b) The value at g of the chain is obtained by applying the lifts one after the other; each lift is linear in its datum, so the value is the image of the pure tensor $\Phi_{j-1} \otimes \cdots \otimes \Phi_0$ under the linear map that the chain defines on the tensor product, followed by evaluation at g.

(c) Induct on the length of the chain, integrating one variable at a time. The last lift is the integral over $\mathrm{U}(\mathbb{A}_\mathrm{f})$ of the integrand

$$a \mapsto \sum_x L_x\big(\mathrm{jet}(\omega(a, g_m)\Phi_{x,m-1})\, \mathrm{chain}_{m-1}(\Phi_x)\varphi(a)\big),$$

which is left invariant under the rational points and, by joint invariance for an element $b \in K_{m-1}$ placed in slot $m-1$ alone, right invariant under K_{m-1}. As the level is free, each double coset $\Gamma a K_{m-1}$ has volume 1 for the counting normalisation, so the integral is the sum over $\Gamma \backslash \mathrm{U}(\mathbb{A}_{\mathrm{f}})/K_{m-1}$ of the values at the representatives, and each value is a shorter chain whose last kernel has been absorbed into the coefficients L_x. The induction gives the sum over tuples of double cosets of the kernel products, factor j being jet of $\Phi_{x,j}$ moved by the pair of representatives. If a property of vectors holds for the values of φ and is preserved by every kernel, it holds for every kernel product, by induction on the number of factors; and if each kernel multiplies a size by at most r_j, the kernel product has size at most $\prod_j r_j$ times the size of the values of φ. □

Proposition 16.2 (Free levels). *For an index set of rank at most $2n_m$ and $K \subseteq K_0$ reducing to 1 at the chosen places* (the chosen level)*, all stabilisers $\Gamma \cap gKg^{-1}$ are trivial; the same holds in rank at most $4n_m$ if K reduces to 1 at the chosen places above one of p_0, p_1. For every integral level in rank at most $2n_m$, the stabilisers have order dividing $\#\prod_w \mathrm{GL}_{rt}(\mathbb{F}_w)$, whose logarithm is at most x^{32}.*

Lemma 16.3 (Minkowski for definite unitary groups). *Let E be a CM field with at least two infinite places, V the standard definite hermitian space of rank n, and $w_1, \dots, w_s$ distinct finite places with $\prod_i \mathrm{N}\, w_i > (2n)^{[E:\mathbb{Q}]}$. If K is a subgroup of the integral finite adelic unitary group whose elements reduce to 1 modulo every w_i, then for every g the stabiliser $\Gamma \cap gKg^{-1}$ of the coset gK in the rational unitary group is trivial.*

Proof. Let $\gamma \in \Gamma \cap gKg^{-1}$. Then $g^{-1}\gamma g \equiv 1$ modulo the kernel J of the reduction at all w_i, and J meets $\boldsymbol{O}_E$ in $\prod_i w_i$; hence the number $\mathrm{tr}(\gamma - 1) = \mathrm{tr}(g^{-1}\gamma g - 1)$, an integer of E (γ is integral, lying in a compact open subgroup), lies in $\prod_i w_i$. At each infinite place v the image of γ is unitary for the positive definite form, so its entries have modulus at most 1 and $|v(\mathrm{tr}(\gamma - 1))| \leq 2n$. If $\mathrm{tr}(\gamma - 1) \neq 0$, its norm would be at least $\prod_i \mathrm{N}\, w_i$ and at most $(2n)^{[E:\mathbb{Q}]}$, a contradiction. Hence $\mathrm{tr}\, \gamma = n$, and $\mathrm{tr}\big((\gamma - 1)^{\mathsf{H}}(\gamma - 1)\big) = 2n - 2\,\mathrm{Re}\,\mathrm{tr}\, \gamma = 0$ at any infinite place: $\gamma = 1$. □

Proof of Proposition 16.2. The chosen places above p_i are f in number, each of norm p_i, and $[E : \mathbb{Q}] = 2f$; so Lemma 16.3 applies to an index set of rank $n \leq 4n_m$ as soon as $p_i > (2n)^2$, which holds because $p_i \geq b^{10}$ and $n_m \leq 40r^2$ (Proposition 8.9). Levels reducing to 1 at all chosen places reduce to 1 at those above p_0. For an integral level K and rank at most $2n_m$, conjugate by g^{-1} into $K \subseteq K_0$ and reduce at the chosen places: the kernel of the reduction on $\Gamma \cap gKg^{-1}$ lies in the stabiliser for the chosen level, which is trivial, so the stabiliser embeds into $\prod_w \mathrm{GL}_n(\mathbb{F}_w)$; as $n \leq \mathrm{rt}$, the embedding $h \mapsto \mathrm{diag}(h, 1)$ shows that its order divides $\prod_w \#\,\mathrm{GL}_{\mathrm{rt}}(\mathbb{F}_w)$, the chosen group order. A group GL_l over a field with q elements has at most q^{l^2} elements, so the logarithm of the chosen group order is at most $f(\mathrm{rt})^2 \log M \leq x^{32}$ by Proposition 8.9. □

16.2. Lift laws modulo the level

Let P consist of one place of each conjugate pair above the level N; these places are split of degree one, and N is a uniformiser at each (Lemma 10.13). At P we work in bases adapted to a marked lattice: a lattice L with a vector $\varepsilon \in L/NL$ nonzero at every place, written as the last basis vector. The *mirabolic* group is the stabiliser of (L, ε), and $D = \mathrm{diag}(1, \dots, 1, N^{-1})$ takes L to $L^+ = L + O\varepsilon/N$. For matrices T at P define

- the *direct datum* $\mathbf{1}[T \text{ integral}, \ T\varepsilon \equiv \varepsilon']$;
- the *mirror datum* $\mathbf{1}[T \text{ integral}]\,\psi_N(T_{\text{corner}})$, with $\psi_N(c) = \psi(-c/N)$;
- the *glued datum* of a pair (T, R): ψ_N of the corner entry of RT on integral pairs;
- the operator $\mathrm{Up} = \pi(D)\sum_y \psi_N(y_{\text{last}})\pi(k_y)$, the sum over the vectors y modulo N with unit last entry and k_y an integral matrix with last row and last column y.

A vector is *ramified* if at each $w \in P$ it is $\psi - x\psi$ for an integral unit scalar x that is 1 modulo N off w. A lift $L(\Psi)\colon \pi \to \pi'$, linear in the datum Ψ, obeys the *lift law* with factor γ if integral invertible matrices move freely between the datum and the two representations, and $\pi'(D)L(\text{last-row datum}) = \gamma\, L(\text{direct datum})\pi(D)$.

Theorem 16.4 (All lifts become direct). *Let the lifts obey the lift law and let φ be ramified.*

(a) *Integral matrices and the mirabolic group fix the direct and the mirror data;* Up *preserves the mirabolic invariants, commutes with every endomorphism commuting with the representation, and preserves ramified vectors.*

(b) *A lift by a datum that is invariant under the unit scalars supported at one place annihilates φ; in particular the mirror lift may be restricted to unit corner entries.*

(c) $\pi'(D)L(\textit{mirror})\varphi = \mathrm{Up}\, L(\textit{direct})\varphi = \gamma\, L(\textit{direct})\mathrm{Up}\,\varphi$.

(d) *The glued double lift $\sum_y L'(\psi_N\langle\cdot, y\rangle)L(\mathbf{1}[\textit{last column } y])\varphi$ is the number of primitive vectors times $L'(\textit{mirror})L(\textit{direct})\varphi$; moved by D it is a scalar times two direct lifts of* Up φ.

(e) *For a chain, the composite of p glued double lifts takes φ to*

$$\prod_{i<p}\Big(d_i \cdot \#\{\textit{primitive vectors}\} \cdot \prod_{l<2i+2}\gamma_l\Big)$$

times the composite of $2p$ direct lifts of $\mathrm{Up}^p\varphi$.

Proof. Valuations. An integer at P reduces to 0 (resp. 1) modulo N exactly when N divides it (resp. it minus 1) at every place, and an integral vector is the last basis vector modulo N exactly when N divides its other entries and its last entry minus 1. Hence the direct datum is the indicator of the integral T whose last column is $\equiv \varepsilon$, and the last-row datum that of the integral T whose last row is $\equiv \varepsilon^{\mathsf{T}}$. Conjugation by D multiplies the last row off the corner by N and the last column off the corner by N^{-1}; hence $D^{-1}TD$ is integral with last row $\equiv \varepsilon^{\mathsf{T}}$ exactly if T is integral with last column $\equiv \varepsilon$.

(a) If the lift by $\mathbf{1}[\cdot]$ of a function of matrices modulo N is nonzero at g and g^{-1}, then g is integral invertible. For g, h mirabolic, i.e. integral invertible with

last column $\equiv \varepsilon$, the matrix $h^{-1}Tg$ is integral with last column $\equiv \varepsilon$ exactly when T is; so the direct datum is invariant under $T \mapsto h^{-1}Tg$. For the mirror datum read in the bases of L and L'^{+}, $D^{-1}h^{-1}D$ is integral and fixes the last coordinate form modulo N, so the corner entry modulo N is unchanged. For g mirabolic, $k = D^{-1}gD$ is integral with last row $\equiv \varepsilon^{\mathsf{T}}$; both $k\,k_y$ and k_{ky} have last column ky modulo N, so they differ by a mirabolic element, and g permutes the terms of the sum defining Up by $y \mapsto ky$, with ψ_N of the last entry unchanged; so Up preserves the mirabolic invariants. An endomorphism commuting with all matrices commutes with every term; integral unit scalars commute with Up, and a map commuting with them preserves ramified vectors, since $\varphi = \psi - x\psi$ goes to $T\psi - xT\psi$.

(b) Let the datum be invariant under the unit scalars that are 1 away from w, and $\varphi = \psi - x\psi$ with such an x. By the lift law the scalar x moves through the datum, so the lifts of ψ and $x\psi$ agree and the lift of φ vanishes. If r maps matrices modulo N to vectors and commutes with scalars, and y vanishes at w, the datum $\mathbf{1}[r(T) = y]$ is invariant under these scalars. The mirror datum is ψ_N of the corner entry t; sorting by t modulo N, the terms with t vanishing at some place die, so it may be restricted to unit corner entries.

(c) Sort the mirror datum, moved by D, by the last column y of T modulo N (a datum is a sum over the values of a function with finitely many values of the indicators of its fibres). The terms whose y vanishes at some place die by (b). For the others, y has unit last entry (by (b), restricting to unit corner entries), and the term is the translate by k_y, an integral invertible matrix with last column and last row y modulo N, of the direct lift, times $\psi_N(y_{\text{last}})$: this is Up $L(\text{direct})\varphi$. Sorting instead by the last row y, the term is the lift by the last-row datum after k_y, and the lift law, which moves D from the target to the source at the cost γ and turns the last-row datum into the direct datum, gives $\gamma\, L(\text{direct})\text{Up}\,\varphi$.

(d) The glued datum is $\sum_y \mathbf{1}[\text{last column of } T \equiv y] \cdot \psi_N(\langle r(R), y\rangle)$. A term whose y vanishes at some place dies by (b). The integral invertible matrices act transitively on the primitive vectors (each is the image of ε under an invertible matrix modulo N, assembled place by place), and an integral k moves the term of y to that of ky by cancelling between the two lifts (it moves through the first datum by $\mathbf{1}[\text{last column } ky]$ and through the second by $\psi_N\langle \cdot, k^{-1}\cdot\rangle$); so all primitive terms equal that of ε, which is $L'(\text{mirror})L(\text{direct})\varphi$. Moved by D, (c) replaces the mirror lift by a direct lift after Up at the cost γ', and (c) again moves Up through the first direct lift at the cost γ.

(e) Induct on p. The composite of direct lifts preserves ramified vectors, and Up moves through a composite of j direct lifts at the cost $\prod_{l<j}\gamma_l$ by (c). The last glued double lift is, by (d), d_p times the number of primitive vectors times $\gamma_{2p+1}\gamma_{2p}$ times two direct lifts after Up; moving this Up through the earlier $2p$ direct lifts to the source costs $\prod_{l<2p}\gamma_l$, and the product of these factors is the stated scalar. □

16.3. Source forms, chain data and the composite lift

The level of stage j is the subgroup of the chosen level of $\mathrm{U}(X_j)$ whose elements have last row $\equiv (0,\dots,0,1)$ modulo π_N; it contains the principal congruence subgroup of level $\pi_M\pi_N$ and is free in stage 0. The *source forms* $\mathcal{S}$ are the algebraic forms on X_0 of weight (c_τ, e_0) and this level; $\mathcal{F}(X_j)$ denotes the smooth algebraic forms on X_j of weight (c_τ, e_j). A *ramified* source form is one whose translates by z_v^k, $0 \le k < N-1$, sum to 0 for each place $v \mid \pi_N$, where z_v is a fixed scaling at v: an integral norm-one idele, 1 off v and $\bar{v}$, of order $N-1$ modulo v.

Proposition 16.5 (Branching map). *There is an integral matrix* br *from the Weyl module of $\mu(X_m)$ to that of $\mu(Y_m)$ which, over every field of characteristic zero, intertwines* $\mathrm{diag}(g,z)$ *with g up to the character $\prod_\tau z_\tau^{n_\tau}$ (Proposition 15.2(c)), spans the equivariant maps for* $\mathrm{diag}(g,1)$ *over $\mathbb{C}$, and has squared operator norm at most* $\exp(x^{70})$.

Proof. Apply Theorem B.23 at each infinite place to $\kappa = c_\tau$, the columns of $\mu(X_m)$, with $\kappa' = c^{\mathrm{kept}}$, $\gamma = c_\tau - c^{\mathrm{kept}}$ and $\kappa_0 = c_\tau^\gamma$ (Proposition 15.2(a)); the determinant exponents are both e_m, as no column of height $n_m - 1$ keeps its height, and all columns of height below k lose a box, so $q = k$. The character of $\mathrm{diag}(g,z)$ has the exponent $|\gamma|_\tau + e_m(\tau) = n_\tau$ by Proposition 15.2(c). The boxes of κ_0 in the columns of heights $k \le h < n_m - 1$, summed over τ, number at most $l = f n_m^2 \Delta$, since there are at most Δ columns of each such height (Proposition 15.2(b)) and at most n_m heights of at most n_m boxes each. The squared norm is at most $((l^l)^3)^2$, and log of this is at most $6l \log l \le x^{70}$ by Proposition 8.9. □

The *restriction* from X_m to Y_m restricts a form to the points $\mathrm{diag}(h,1)$, applies br, and multiplies by $\prod_\tau \Phi_\tau(\bar{\pi}_N)^{-n_\tau}$. The kernel of lift $j+1$, from X_j to X_{j+1}, is the theta kernel of the pair of standard definite spaces with the characters given by the twists of $\mu_{j+1}^0 \vartheta^{j+1}$ by the ranks, and with derivatives scaled by $\pi_M \mathsf{a}$; its weight is a lift weight because no column has height strictly between n_j and n_{j+1} (Proposition 15.1).

For a tuple $x = (x_j)$ of finite adelic matrices, the j-th translated datum is the integral vacuum of $\mathrm{Hom}(X_j, X_{j+1})$ translated by the finite Heisenberg operator of x_j. For a double lift $X_{2i} \to X_{2i+1} \to X_{2i+2}$ the *gluing* replaces $v_{2i} \otimes v_{2i+1}$ by the sum over y modulo N of the translates of v_{2i} by the last column $-y/N$ at the places $w \mid \pi_N$ and of v_{2i+1} by the last row $-\bar{y}/N$ at their conjugates.

Definition 16.6 (Chain data). Let ϑ be a twist with modulus Q and $q = \prod_{w \in Q} \mathrm{N}\, w$. A *chain datum* for ϑ is a finitely supported function Ψ on tuples of translations such that: the translations in its support vanish at the places prime to Mq and, multiplied by π_M, become integral at the places prime to q; its values are integral away from q; if $Q = \emptyset$, then $\sum_x |\sigma\Psi(x)| \le M^{3fmn_m^2}$ for every automorphism σ; and the glued tensor $\sum_x \Psi(x) \bigotimes_j \mathrm{Glue}(\text{translated data})$ is jointly invariant under the levels of $X_0, \dots, X_{m-1}$. The *composite lift* $B_\vartheta = B_{\vartheta,\Psi}$ is the chain of double lifts with the glued tensor as data, followed by the restriction to Y_m. A *weak chain datum* has the same conditions without gluing, with denominators allowed above N and size $\exp(x^{60})$.

Proposition 16.7 (The glued datum). *Let P, S each contain one place of every conjugate pair above N. In the Schrödinger model at P, the gluing of vacuum-averaged data is the sum over z modulo N of the data $\mathbf{1}[\text{last column of } T \operatorname{diag}(1,\dots,N)_S \equiv z] \otimes \psi_N\langle r(\operatorname{diag}(1,\dots,N)_{P\setminus S}R), z\rangle$; the sum of translates equals the glued datum, with no constant. The Weil action of elements supported above N with components g, k at P moves a Schrödinger datum Ψ to $T \mapsto \Psi(k^{-1}Tg)$ times a quotient of determinant characters, which are trivial on integral matrices if the characters are unramified at P.*

Proof. Integrality at P. Let p_S be N at the places of S and 1 elsewhere. An adele z is integral at P if $\mathbf{1}_P z$ is integral; if Nz is integral at P, then $z - y/N$ is integral at P for exactly one class y modulo N (the residue of Nz), and for none otherwise. Combining a place of P with the conjugate place: if tp_S and $(N+1-p_S)s$ are integral at P, then $t - y/N$ and $s - \bar{y}/N$ are both integral at P for exactly one class y, and otherwise for none, because integrality of an adele and of its conjugate at P is integrality at all places above N, and $(y/N)^*(y/N) = 0$.

The sum of translates. A translate of the indicator of the integral matrices by the last column $-y/N$ at S is the indicator of the matrices integral up to the pole y/N in the last column; the translate of the second datum by the conjugate last row contributes, by the Heisenberg cocycle, ψ_0 of the pairing with y/N (Appendix B.10). For a pair (T, R) let t be the last column of T and r the last row of R; by the previous paragraph at most one class y makes $t - y/N$ and $r - \bar{y}/N$ integral at P, after the multiplications by $\operatorname{diag}(1,\dots,N)_S$ and $\operatorname{diag}(1,\dots,N)_{P\setminus S}$, and its term is $\psi_0(-r \cdot t)$, because $(t - y/N)\cdot(r - \bar{y}/N) - (r \cdot t - c/N)$ is integral at P for the residue c of the corner. Hence the sum is ψ_N of the corner of $R'T'$ on the integral pairs and zero elsewhere: the glued datum, with no constant. Averaging over the Heisenberg operators of the integral vectors supported above N gives the projection to the vacua, which is the same for P and S; this gives the stated sum over z in any model.

Weil action. Read the elements supported above N through their components at P; the Weil action of an element supported at P on a Schrödinger datum is the substitution $T \mapsto k^{-1}Tg$ times the values of the determinant characters $x \mapsto \chi(\det x)|\det x|^{-l/2}$ (Appendix B.10); if the characters are unramified at P, these are trivial on integral invertible matrices, whose determinants are units. □

Proposition 16.8 (Characters above the level). *For a twist ϑ and $j \geq 1$, the character $\mu_j^0\vartheta^j$ twisted by the rank is unramified at every place above N, and its base change through the rank twist is $x \mapsto \mu(x/\bar{x})$. Each lift commutes with the scalings at the places $w \mid \pi_N$, and if $\omega(1, z_w)$ fixes the data of one lift in a chain, the chain annihilates the ramified source forms; likewise for chains of double lifts.*

Proof. The tame part of the character of a lift is supported on the chosen places and on Q, and a place above N is neither chosen nor in Q, nor is its conjugate; the rank twists multiply by powers of the splitting character υ, unramified above N; so the base change of the twisted character is trivial on the local units above N. As the rank $\#X_j$ is even, the rank twist followed by the space character in rank $\#X_j$ is $x \mapsto \nu(x/\bar{x})$. A scaling z at $w \mid \pi_N$ is a scalar of norm one; the pair (z, z) fixes every linear map $X_j \to X_{j+1}$ and acts by the quotient of the characters at $\det z$, which is 1, so $\omega(z, z) = 1$ and each lift commutes with z

by Proposition 16.1(a); hence the lift of $\psi - z\psi$ is $\chi - z\chi$ for the lift χ of ψ. If $\omega(1, z)$ fixes a datum Φ, it translates the lift by z and fixes it, so the lift of $\psi - z\psi$ vanishes. For a chain, $\omega(1, z)$ on factor j inserts z after lift j, and it can be moved to the source; so the chain sends $\psi - z\psi$ to 0. A ramified source form is a sum of such differences for $z = z_w$, as $\sum_k z_w^k \varphi = 0$; for chains of double lifts regroup the data into single factors. □

16.4. The lower chain

For $j < m$ the lift from Y_j to Y_{j+1} is the theta lift with the datum $\varphi_{\mathrm{harm}} \otimes \Upsilon_{j+1}$: at infinity the product over τ of the joint harmonics of the parameters of Y_j and Y_{j+1} (Proposition 15.4(c)), and at the finite places the indicator of the inclusion matrix modulo the level places and the vacuum elsewhere; it is multiplied by the index in K_0 of the level. The *lower chain* $\iota_\mu = \iota_\vartheta$ is the composite of the m lifts from forms of principal level on Y_0 to forms on Y_m, read as algebraic forms of weight $\mu(Y_m)$.

Proposition 16.9 (The lower chain). (a) *Each datum $\varphi_{\mathrm{harm}} \otimes \Upsilon_{j+1}$ is an* arrow *in both directions: its lifts take forms of the type of Y_j to forms of the type of Y_{j+1}, and conversely for the conjugate types.*

(b) *If $k \in K_0$ on Y_m has block upper triangular reductions at the level places with first block $s \in \bar{G}$ at the chosen places, then $k\,\iota_\mu(y) = c\,\iota_\mu(sy)$, where c is the product over the level places and $l < m$ of the tame characters of lift $l + 1$ at the determinant of the diagonal block $l + 1$.*

(c) *Hence ι_μ takes values in the forms of level J_m (the elements of K_0 that are 1 at the chosen places and block upper triangular at Q) with the character $k \mapsto \prod_w \prod_{l<m} \psi_{l,w}(\det k_w^{(l+1)})$; and $\iota_\mu(sy)$ is the translate of $\iota_\mu(y)$ by an element of K_0 reducing to* $\mathrm{diag}(s, 1)$ *at the chosen places and to* 1 *at Q.*

Proof. (a) By Proposition 15.4(c), with the node $(0, j)$ of grid Y as Y_j, the parameters of Y_j and Y_{j+1} correspond for the weights of the characters of lift $j + 1$; so the joint harmonic is a nonzero vector of the two lowest K-types, harmonic on both sides, that is, an edge (Appendix B.12), and an edge tensored with a finite datum is an arrow in both directions, the second for the swapped Weil representation.

(b) Let (k, k') be integral points of $\mathrm{U}(Y_{j+1})$, $\mathrm{U}(Y_j)$ whose reductions g, g' at the level places satisfy $g\iota = \iota g'$, ι the inclusion matrix. They fix the inclusion matrix modulo the product of the level places, so the Weil action multiplies the finite datum Υ_{j+1} by the quotient of the tame characters of the lift at the determinants, the tame parts being trivial off the level places; hence the translate by k of the lift of z is the lift of the translate of z by k', times $\prod_w \psi(\det g_w)/\psi(\det g'_w)$. Induct on j: for k block upper triangular with first block s at the chosen places, the upper left block A of g has the diagonal blocks of g except the last, and the lower left block vanishes, so $g\iota = \iota A$; the new factor is the tame character of lift j at the determinant of the last diagonal block, as the determinant of a block triangular matrix is that of its upper left corner times that of its last diagonal block. On Y_0 the element reduces to s and acts as the chosen lift of s. Passing from forms to algebraic forms commutes with translation.

(c) For k in J_m the reduction is 1 at the chosen places and block triangular at Q; (b) with $s = 1$ gives the character $\prod_w \prod_{l<m} \psi_{l,w}(\det k_w^{(l+1)})$. For $s \in \bar{G}$ apply

(b) to the chosen element reducing to $\operatorname{diag}(s,1)$ at the chosen places and to 1 at Q: an entry below the diagonal blocks lies in a row outside block 0, so all diagonal blocks after the first are 1 and all character values are 1. □

16.5. The Poincaré map

The space Y_0 carries a grading whose Bessel subgroup $R_0 = U \rtimes \mathrm{U}(X_0)$ is the stabiliser of the flag $\langle e_1\rangle \subset \cdots \subset \langle e_1,\dots,e_\varrho\rangle$ together with the generic character ψ_U of its unipotent radical (Appendix B.14). For a source form φ and a linear form t on its values, the function $\mathcal{W}_\varphi(uh) = \psi_U(u)\, t(\rho(h_\infty^{-1})\varphi(h_{\mathrm{f}}))$ on $R_0(\mathbb{A})$ is continuous, left invariant under $R_0(E^+)$ and ψ_U-equivariant; for forms y on Y_0 the *Bessel period* $\int_{[R_0]} \mathcal{W}_\varphi(r)\overline{y(rg)}\,dr$ converges.

Definition 16.10 (Poincaré map). For an archimedean point g of $\mathrm{U}(Y_0)$ and a linear form t, $\mathcal{P} = \mathcal{P}_{g,t} \colon \mathcal{S} \to \mathcal{A}(Y_0)$ is the map such that $\langle y, \mathcal{P}\varphi\rangle$ is the Bessel period of φ against y, for all $y \in \mathcal{A}(Y_0)$.

Proposition 16.11. *$\mathcal{P}$ exists and is linear in φ and in t. For a form z of level J and conjugate type,*

$$[J_M : J \cap J_M] \int_{[\mathrm{U}(Y_0)]} z \cdot \mathcal{P}\varphi = \sum_{\gamma \in J_M/(J\cap J_M)} \int_{[R_0]} \mathcal{W}_\varphi(r)\, z(rg\gamma)\,dr.$$

Proof. On a finite-dimensional inner product space every conjugate linear form is $y \mapsto \langle y, z\rangle$ for a unique z (Riesz). The space $\mathcal{A}(Y_0)$ is finite-dimensional (Proposition 15.10), and for fixed φ the Bessel period is conjugate linear in y; it converges because $r \mapsto \mathcal{W}_\varphi(r)\overline{y(rg)}$ is continuous on the compact quotient $[R_0]$ and invariant under the rational points. Hence there is a unique $\mathcal{P}\varphi$, linear in φ; scaling t scales $\mathcal{W}_\varphi$ and hence $\mathcal{P}\varphi$, so a change of Haar measure is absorbed in t. For the formula, let z be of level J and let $\operatorname{tr} z = \sum_\gamma z(\cdot\gamma)$ over representatives of $J_M/(J \cap J_M)$; it has level J_M, as left multiplication by an element of J_M permutes the cosets, and its integral against a form of level J_M is $[J_M : J \cap J_M]$ times that of z, by the invariance of the integral under right translation. Its conjugate lies in $\mathcal{A}(Y_0)$; the definition of $\mathcal{P}$, applied to it, gives the formula. □

17. Theta kernels between definite spaces

For standard definite spaces I_n, $I_{n'}$ over E with $n < n'$ and a *lift weight* (κ, e) (no column of height strictly between n and n', and $e(\tau) = e_b(\tau) + \kappa(\tau, n)$), the *theta kernel* of scale c attaches to a finite datum f on $\operatorname{Hom}(I_n, I_{n'})$ the *jet matrix* $\mathrm{Sch}_\kappa(J_c(f))$: the matrix of jets of the Taylor series of the theta function of f at scale c along the entries of the Schur matrix (Section 12.6). The pair of finite adelic groups acts on data by the Weil representation twisted by characters a, b of $\mathrm{U}(1)(\mathbb{A})$, and the kernel lifts algebraic forms of weight (κ, e) on $\mathrm{U}(I_n)$ to forms of weight (κ, e_a) on $\mathrm{U}(I_{n'})$ (Section 16.1). This section proves the three facts on single kernels that the composite lift needs: their Galois conjugates, archimedean bounds for their sizes, and p-adic bounds for their gauges.

17.1. Galois conjugation of jets

The theorem rests on a Galois law for maps that scale the finite translations, a form of the Galois action on theta functions with complex multiplication [Shi76]; it is stated here with its proof. A σ-semilinear map L on $\mathcal{S}(V)$ *scales finite translations by* b if it preserves the finite part, $L\varrho(\mathrm{fin}(x)) = \varrho(\mathrm{fin}(bx))L$ there, and $\sigma(\psi(z)) = \psi(b\bar{b}z)$ for the additive character ψ of the finite adeles for which $\hat{O}$ is self-dual. Let $\Delta(g)$ be the multiplier of the Weil representation at a finite point g.

Lemma 17.1 (The Galois law of the Weil representation). *Let L scale finite translations by b.*

(a) *$b\bar{b}$ is a unit at every finite place, and for a set R of places disjoint from its conjugate, $b = ua$ with u of norm one and a a unit of $\hat{O}$ equal to 1 on R; $\omega(1,u)^{-1}L$ scales finite translations by a.*

(b) *For the splitting character v, the number $\sigma(\Delta(g))/\Delta(g)$ has modulus 1 and square $v(\det g)/\sigma(v(\det g))$.*

(c) *If b is a unit of $\hat{O}$, then $L\,\mathrm{vac} = c\,\mathrm{vac}$ for some c, and on the finite part $L\omega(g) = (\sigma(\Delta(g))/\Delta(g))\,\omega(g)L$.*

(d) *In general $L\,\mathrm{vac} = c\,\omega(1,u)\mathrm{vac}$ and $L(\Psi\otimes\mathrm{vac}) = c\,\omega(1,u)((\sigma\circ\Psi)\otimes\mathrm{vac})$ for every Ψ Schwartz–Bruhat above R.*

Proof. (a) As $\hat{O}$ is self-dual under ψ, tz is integral exactly when z is, if $\sigma(\psi(z)) = \psi(tz)$ for all z; testing with 1 and with the inverse of a uniformiser shows that $t = b\bar{b}$ is a unit at every finite place. A unit x of $\hat{O}$ has the norm of b, and it can be changed into a unit a with the same norm that is 1 on R: replace x by 1 at R, by $x\bar{x}$ at $\bar{R}$ and keep it elsewhere, which keeps the norm. Then $u = b/a$ has norm one. Since $\omega(1,u)$ is the scalar u, which is central, $\omega(1,u)^{-1}\varrho(\mathrm{fin}(bx)) = \varrho(\mathrm{fin}(ax))\omega(1,u)^{-1}$.

(b) $\Delta(g)\overline{\Delta(g)} = |\Delta(g)|^2$ is a lattice index, a natural number, which σ fixes; and $\overline{\Delta(g)}/\Delta(g) = v(\det g)$. Hence $(\sigma\Delta/\Delta)^2 = v(\det g)/\sigma v(\det g)$, a quotient of two numbers of modulus 1.

(c) The translations by the lattice fix the vacuum, and L turns them into translations by a times lattice vectors, which are again lattice vectors since a is a unit; so L vac is fixed by the lattice translations, and in the finite part such vectors are multiples of the vacuum. For $\omega(g)$, expand $\omega(g)$vac as $\Delta(g)/|\Delta(g)|^2$ times the sum of the translates of the vacuum by gq, q over the cosets of $\Lambda\cap g^{-1}\Lambda$ in Λ; a unit a of $\hat{O}$ permutes these cosets, so L multiplies the sum by c, and σ multiplies the constant by $\sigma\Delta/\Delta$. Both sides turn the translation by x into that by agx, and the translates of the vacuum span the finite part.

(d) By (a) apply (c) to $\omega(1,u)^{-1}L$, which scales by a. A Schwartz–Bruhat function above R is a combination of translates of the box function (the indicator of the vectors integral at R), which name the translates of the vacuum; since $a = 1$ on R, the vectors y and ay agree above R and only the character sees the change, so σ takes the translate by y of the box function to its translate by ay. Hence $\omega(1,u)^{-1}L(\Psi\otimes\mathrm{vac}) = c\,(\sigma\circ\Psi)\otimes\mathrm{vac}$. □

Theorem 17.2 (Conjugation of jets). *Let V be a standard space over E and σ an automorphism of $\mathbb{C}$ over K. There are a scale t with $|t|^2 = \Omega^{-1}$, a finite adele b and a*

σ-semilinear map L on $\mathcal{S}(V)$ which preserves the finite part, turns finite translation by x into translation by bx, satisfies $\sigma(\psi(z)) = \psi(b\bar{b}z)$ for the additive character, and maps the jets of f at scale a to the σ-conjugates, with the places permuted by σ, of the jets of Lf at scale t. Moreover:

(a) *L is injective on the finite part, and $\sigma(f(\mathrm{fin}(x))) = (Lf)(\mathrm{fin}(bx))$;*

(b) *there are $c \neq 0$ and a norm-one adele u with $L\,\mathrm{vac} = c\,\omega(u)\mathrm{vac}$ and $L(\Psi \otimes \mathrm{vac}) = c\,\omega(u)((\sigma \circ \Psi) \otimes \mathrm{vac})$ for all Schwartz–Bruhat Ψ supported above a set of places disjoint from its conjugate;*

(c) *every value of $L\varrho(\mathrm{fin}(x))\omega(1, g)\theta^{\circ}$ has absolute value at most $\exp(C_0(q_1 n + 1)^2)$.*

Proof. Construction. By Proposition 13.14 there are a vector F whose finite translates span the finite part, a scale t_σ with $|t_\sigma|^2 = \Omega^{-1}$, an adele B_σ and a vector F'_σ such that σ maps the jets of $\varrho(\mathrm{fin}(x))F$ at scale a to those of $\varrho(\mathrm{fin}(B_\sigma x))F'_\sigma$ at scale t_σ, the places permuted by σ. Define L on the finite part by

$$L\Big(\sum_x c_x \varrho(\mathrm{fin}(x))F\Big) = \sum_x \sigma(c_x)\varrho(\mathrm{fin}(B_\sigma x))F'_\sigma;$$

it is well defined, because a vector of the finite part is determined by its jets (Lemma 13.2) and $t_\sigma \neq 0$; it is σ-semilinear and the jets of Lf are the conjugates of those of f. The laws for translations hold on the translates of F, hence on their span. *The additive character.* In rank one, take $N \neq 0$ such that translation by N fixes F and translation by NB_σ fixes F'_σ, and write $z = Nx$. By the Heisenberg relation the translate by $2x + N$ is ψ of a multiple of z times the translate by $2x$ moved by N; comparing the jets of the two sides, which do not vanish, before and after σ gives $\sigma(\psi(z)) = \psi(B_\sigma \bar{B}_\sigma z)$.

(a) If $Lf = 0$, the conjugates of the jets of f vanish, so its jets vanish and $f = 0$. The value of the theta function of f at $\mathrm{fin}(x)$ is the constant term of the jets of $\varrho(\mathrm{fin}(x))f$, and $L\varrho(\mathrm{fin}(x))f = \varrho(\mathrm{fin}(bx))Lf$; so σ takes it to the value of Lf at $\mathrm{fin}(bx)$.

(b) L scales finite translations by b, so Lemma 17.1(d) gives c and u; $c \neq 0$, because L is injective on the finite part by (a) and the vacuum is a unit vector.

(c) L applied to the integral vacuum is a times a unit vacuum, by (b); so $|a|^2$ is the squared norm of the image, which its values at finite points bound, and the value of $L\varrho(\mathrm{fin}(x))\omega(1, g)\theta^{\circ}$ is a value of a translate of a unit vacuum moved by ω, times a. The values of a unit vacuum moved by $\omega(g)$ are bounded by $\exp((2d+6)(q_1 n+1)^2)$: the squared modulus of a value is a theta series of matrix coefficients of the vacuum, which are Gaussians supported on a lattice of rational vectors; at a lattice point γ the vacuum has modulus at most $\exp(-\pi\|\gamma\|^2/\sqrt{d})$, the squared lengths are integers (they are traces of values of the hermitian form, which lie in the inverse different of E/K), and distinct points are at distance at least 1. A packing bound in real dimension $2nf \leq 2nq_1$ estimates the series, completing the square $tD \leq \pi t^2/a + aD^2$ and $\sqrt{d}(\delta+1)^2 + 3\delta \leq 2(2d+6)(M+1)^2$ for $\delta \leq 2M$. With the bound for the vacuum scale (Proposition 13.12) this gives the constant C_0. □

17.2. Archimedean size

A polynomial Q in variables z_i has *conjugate growth* (T, L, w, A) at σ if $|\sigma Q(z)| \le A\exp(w + LR)$ whenever $\sum_i |\sigma z_i|^2 \le T(2L+2)R$. A finite datum f is a *vacuum combination of size B* if $f = \sum_x r_x\, \varrho(\mathrm{fin}(x))\theta^\circ$ with $\sum_x |\sigma r_x| \le B$ for every σ.

Theorem 17.3 (Size of jets). *Let f be a vacuum combination of size B on $\mathrm{Hom}(I_n, I_{n'})$, moved by finite points g, h under the twisted Weil representation, and let Q have conjugate growth (T, L, w, A) at σ with $|\sigma c|^2/\Omega \le T$. Then*

$$|\sigma\big(Q(\partial) J_{c\mathsf{a}}(\omega(g,h)f)(0)\big)| \le B\, A\, e^{w} \exp\big(C_0(q_1 n n' + 1)^2\big)(2L+2)^{f n n'}.$$

For a lift weight κ and every $L \ge 1$, the jet matrix multiplies the Gram norm of the conjugate of a vector by at most the same factor with $w = \frac{1}{2}\sum(l/L + \log l! + l\log(|\sigma c|^2/\Omega))$, the sum over the lengths l of the first n rows of the diagram.

Proof. One translate. Write f as a combination of translates $\varrho(\mathrm{fin}(x))\omega(g)\theta^\circ$ of the integral vacuum, with coefficients of total conjugate size at most B; the twist by the characters contributes the values $a(\det h)b(\det g)^{-1}$, whose conjugates over K have modulus 1, and σ may be replaced by an automorphism over K that gives the same norms. A factor c in the scale moves into the polynomial: the jet along Q at scale $c\mathsf{a}$ is the jet along $Q(cz)$ at scale a. By Theorem 17.2 the conjugate of the jet of a translate F is the Gaussian integral

$$\int Q^\sigma(t\bar{y})\, F'(y)\, e^{-|y|^2/2}\, d\mu(y)$$

of the conjugate polynomial at $t\bar{y}$ against $F' = L(F)$, with μ Lebesgue measure divided by π in each complex coordinate, so that $e^{-|y|^2}$ has integral 1. By Theorem 17.2(c), $|F'| \le \exp(C_0(q_1 n n' + 1)^2)$. With $R = |y|^2/(2L+2)$ the hypothesis $|\sigma c|^2/\Omega \le T$ gives $\sum |t\bar{y}\,\sigma c|^2 \le T(2L+2)R$, so $|Q^\sigma(t\bar{y})| \le A e^{w+LR}$, and $e^{LR}e^{-|y|^2/2} = e^{-R} = e^{-|y|^2/(2L+2)}$. A Gaussian $e^{-|y|^2/a}$ has integral $a^{\#\text{coordinates}}$ for μ, and there are $f n n'$ complex coordinates; this gives $(2L+2)^{f n n'}$. Summing over the translates gives the first bound.

Schur matrices. By duality it suffices to bound $\sigma(y^{\mathsf{T}}\mathrm{Sch}_\kappa(z)\xi)$ for covectors y whose conjugates have dual Gram norm at most 1. The Schur matrix of σz has squared norm at most $\prod \rho^l$, the product over the first n rows of the diagram at every place of a power ρ^l, l the row length, with $\sum \rho \le \sum_i |\sigma z_i|^2$; the row lengths add up to the number of boxes, as no column has height strictly between n and n'. If $\sum \rho \le e^u(2L+2)R$, then $\sqrt{\prod \rho^l} \le \exp(w + LR)$ with $2w = \sum(l/L + \log l! + lu)$: with $v = u + 1/L$, $\rho^l \le l!\, e^{lv} e^{\rho e^{-v}}$, and $\sum \rho e^{-v} \le e^{-1/L}(2L+2)R \le 2LR$ as $e^{1/L} \ge 1 + 1/L$. With $u = \log(|\sigma c|^2/\Omega)$ this is the growth condition, and the first part gives the bound for the matrix. □

Proposition 17.4 (Normalised kernels). *For index sets of size at most Δ and degrees $D \le M$ there is $c \ne 0$ such that, for every datum built from a function Ψ on matrices modulo a modulus $\mathfrak{m} \ni qM^2$ with values integral away from q and a homogeneous polynomial Q of degree D: c times the kernel is integral at every $p \nmid q$ at points integral above p when Q is p-integral; and its σ-conjugates are at most $\exp(x_1)AB$, where A bounds the conjugates of Ψ and B bounds σQ on the polydisc of radius M^4. Moving the points by rational elements substitutes the variables of Q on both sides.*

Proof. Rational points. The kernel is invariant under the rational points; translating g, h by rational γ, γ' leaves the real point, which lies in the maximal compact subgroups because the spaces are definite, and acts on the Fock vector by the substitution $x_\tau \mapsto (\Phi_\tau \gamma'^{-1})^{\mathsf{T}} x_\tau (\Phi_\tau \gamma)^{\mathsf{T}}$ and by the scalars $\prod_\tau \Phi_\tau(\det \gamma')^{-e(\tau)} \Phi_\tau(\det \gamma)^{e'(\tau)}$.

Integrality. Let c be the product of a power of M that clears the denominators of the datum (Proposition 17.6), a power of the vacuum scale and the factor a^{-D}; it depends only on the ranks, the data and D, and $c \neq 0$. The normalised kernel is a jet along Q of the Taylor series of $\omega(g, h)$ applied to the datum, which is standard of level δ at p once its denominator is cleared; the action of points integral above p keeps it standard (Proposition 13.16), and the scale of its series is $\bar{\delta} d \mathsf{a}$, so Theorem 13.19 gives integrality when Q is p-integral.

Size. For Q homogeneous of degree D, a bound B for $|\sigma Q|$ on the polydisc of radius M^4 gives the bound Bt^D on the polydisc of radius tM^4. With $T = |\sigma(M^2 d)|^2/\Omega$ one has $T(2D+2) \leq (M^4)^2$, as $\Omega^{-1} \leq x^{13}$ and $D \leq M$; so $\sum |\sigma z_i|^2 \leq T(2D+2)R$ puts σz in the polydisc of radius $e^R M^4$, and Q has conjugate growth $(T, D, 0, B)$. Theorem 17.3, applied to the datum, a vacuum combination of size A times the number of matrices modulo $\mathfrak{m}$ (Proposition 17.6), bounds the conjugates; the three factors (the cleared denominator, the number of matrices over $\mathcal{O}_E/\mathfrak{m}$, and the exponential of Theorem 17.3) are at most $\exp(x_1)$ by Proposition 8.9, as the ranks are at most Δ and $D \leq M$. □

17.3. p-adic gauges

For a prime p and a p-adic place, the *gauge* of a vector ξ of a Weyl module of weight (κ, e) at a finite point g is the sup norm of $\rho(g^{-1})(\iota_p \circ \xi)$, where g^{-1} acts through the embeddings $y \mapsto \iota_p \Phi_\tau(\bar{y})$ extended to finite adeles. A datum Φ *raises gauges by at most* c (from base points o, o') if for all g, h, ξ the gauge at ho' of $\mathrm{jet}(\omega(g, h)\Phi)\xi$ is at most $\max(c, 0)$ times the gauge at go of ξ; these data form an additive monoid, because the gauge is ultrametric.

Theorem 17.5 (Gauges of one lift). *Let Φ be standard of level $\delta \neq 0$ at p.*

(a) *Standard data are preserved by the Weil action of points integral above p.*

(b) *If D_u is diagonal at the places above p with entries u_τ, x, y are integral above p, and the weighted entries $\prod_\tau |\det u_\tau|^{e_a(\tau)} \prod |u_{\tau,b}| \cdot |\mathrm{Sch}_\kappa(J_c(f))_{i'i}|$ of the jet matrices of all standard data are at most B, then $\omega(1, xD_u y)\Phi$ raises gauges by at most B; the same holds with the jets along products of minors.*

(c) *Φ raises gauges by at most $(|c|/|\bar{\delta} d|)^L$ for the kernel of scale $c\mathsf{a}$, L the number of boxes.*

(d) *If p is non-split, $\delta = 1$ and c is integral, Φ raises gauges by at most $p^{-(D_p(\kappa) - fN^3(1+\log_p \max(L,1)))}$.*

(e) *If $A \cdot f$ is standard of level N for an integer A and $A(Nd)^L$ divides M, then Mf does not increase gauges.*

Proof. (a) The Weil action of a pair of points integral above p is ω at a finite point integral above p times values of the characters, which are p-adic units; it maps the translates of the integral vacuum by vectors that δ makes integral above p to multiples of such translates (Proposition 13.16). The kernel is linear in the datum, and a scalar z scales the gauge by $\|\iota_p z\|$; translating the datum

by (g_0, h_0) amounts to translating the base points, as $(g, h) \mapsto (gg_0, hh_0)$ is a bijection.

(b) Gauges do not change under rational points acting on both the point and the vector, nor under points integral above p acting on the right. By weak approximation choose rational γ, ε such that γg, $\varepsilon g' x D_u$ and $D_u \varepsilon g' x D_u D_u^{-1}$ are integral above p; this moves the two gauges to the points D_u^{-1} and 1, where the claim is an entrywise bound: at the embedding $\iota_p \Phi_\tau(\bar{\cdot})$, D_u acts by the diagonal matrix u_τ, so the operator from the gauge at 1 to the gauge at D_u^{-1} is the jet matrix with its rows scaled by the diagonal weights. Each entry of the jet matrix is bounded by the jet along a product of minors with the same rows (Proposition 12.13), which gives the second form.

(c) With $x = D_u = y = 1$, the entries of the Schur matrix are homogeneous polynomials with integral coefficients, and Theorem 13.19 bounds the jets of standard data of level δ by $(|c|/|\bar{\delta} d|)^L$.

(d) With $x = D_u = y = 1$, jets along products of minors of standard data of level 1 at a non-split p have the predicted divisibility (Theorem 13.21).

(e) Write $M = A(Nd)^L q$ with q an integer. As Af is standard of level N, (c) shows that it raises gauges by at most $(|c|/|Nd|)^L$, which $(Nd)^L$ cancels; with $|c| \le 1$ there remains $\|\iota_p q\| \le 1$. □

Proposition 17.6 (Data from functions modulo an ideal). *Let $\mathfrak{m}$ be a modulus with $qN \in \mathfrak{m}$ and Ψ a function on matrices modulo $\mathfrak{m}$ with values integral away from q. Then for every prime $l \nmid q$, $N^{2fnn'} c_{\mathrm{vac}}^{nn'}$ times the datum that is Ψ at the places dividing $\mathfrak{m}$ and the vacuum elsewhere is standard of level N at l. If $|\sigma\Psi| \le A$ for all σ, this datum is a vacuum combination of size A times the number of matrices modulo $\mathfrak{m}$; the indicator of one matrix gives a combination of size* 1.

Proof. Sizes. The ideal of O_E generated by an integer k has norm k^{2f}, so if $k \in \mathfrak{m}$ the residue ring has at most k^{2f} elements; for the matrices of size $n' \times n$ there are at most $k^{2fnn'}$ of them.

Standardness. By Fourier inversion modulo $\mathfrak{m}$ (Appendix B.10, (W6)), the datum that is Ψ at the places dividing $\mathfrak{m}$ and the vacuum elsewhere, applied to the integral vacuum, is a combination of translates of the integral vacuum by vectors y that N makes integral at the places not dividing q, with the denominator the norm of N to the power nn', that is $N^{2fnn'}$; the coefficients are values of Ψ, integral at every $l \nmid q$. A place above l does not divide q, so these translates are standard data of level N at l. The factor $c_{\mathrm{vac}}^{nn'}$ turns the vacuum into the integral vacuum.

Size. With $q = 0$ no integrality is asked, and Fourier inversion writes the datum as a combination of translates of the integral vacuum whose coefficients, for each σ, have total absolute value at most $\sum_x |\sigma\Psi(x)|$, which is at most A times the number of matrices modulo $\mathfrak{m}$; enlarging the set of translations does not change this. For the indicator of one matrix y_0 only the vector of the entries of y_0 contributes, with a coefficient of absolute value 1. □

18. The seesaw identity

This section proves that the lower chain of the Poincaré series of a ramified source form is a fixed nonzero multiple of a composite lift of the form itself (Theorem 18.15). The mechanism is Kudla's seesaw [Kud84]: one pairs $\iota_\vartheta(\mathcal{P}\varphi)$ with a form on Y_m, moves the m lifts to the other side as theta descents, and unfolds the Bessel period of φ against them one lift at a time (Appendix B.15). Each unfolding step introduces an *outer variable*, a point of $\mathrm{U}(Y_j)(\mathbb{A})$; its archimedean part is integrated once and for all, and its finite part runs over a finite set of classes, the *cut*, with weights. Two consecutive unfolding steps combine into one double lift between the definite parts $X_{2i} \to X_{2i+1} \to X_{2i+2}$, and the finite part of the datum of a double lift, once evaluated, is a combination of the glued data of Section 16.3. Three points need control, and each has a subsection: the support of the unfolded functions (Section 18.4), which makes the cuts finite with multiplicities dividing powers of the norm q of the modulus of ϑ; the evaluation of the vacuum along a double lift (Section 18.6), which produces the gluing above N; and the denominators and sizes of all coefficients (Section 18.7), which are those of a chain datum.

Throughout, ϑ is a tame character with modulus Q, $q = \prod_{w\in Q} \mathrm{N}\, w$, and $\mathfrak{q}$ is a finite idele generating the ideal $\prod_{w\in Q} w$ (the generator of $\pi_M Q$ divided by that of π_M). The levels $J_j = J_\vartheta(Y_j)$ are the elements of the lattice stabiliser J_j° of Y_j that reduce to 1 at the level places, and $J_M \subseteq J_0^\circ$ is the chosen level of Y_0.

18.1. Graded spaces and their lattices

A *grading* of a hermitian space V with a point is a splitting of its basis into a part X, on which the Gram matrix is the identity, and a *string* $b_0, \dots, b_{s-1}$ orthogonal to X, with $\langle b_i, b_{s-1-i}\rangle = 1$ for $2i \neq s-1$ and $\langle b_i, b_i\rangle = -N$ for the middle vector when s is odd. The *grade* is 0 on X and $s-1-2i$ on b_i; a matrix *raises grades* if its entry (p, p') vanishes unless the grade of p exceeds that of p'. The *unipotent group* of the grading consists of the points whose matrix minus 1 raises grades, and the *Bessel group* $R_V = U_V \rtimes \mathrm{U}(X)$ is generated by it and by the embedded unitary group of X; it is the group of Appendix B.14, and taking the X-block is a homomorphism onto $\mathrm{U}(X)$ with kernel U_V. For a grading Y' of V' with a string of length $s-1$, the *shift* $\mathrm{sh}\colon V \to V'$ kills X and sends b_i to a fixed multiple of b'_i (to 0 for $i = s-1$); it lowers grades by one, and $\mathrm{sh}^\dagger\mathrm{sh}$ and $\mathrm{sh}\,\mathrm{sh}^\dagger$ are the shifts of V and V' when the constants of the two gradings agree.

For finite ideles a, π and a norm-one adele u, the *graded lattice basis* is obtained from the basis of V as follows. If s is odd, multiply the string vectors by u and replace the last vector e_0 of $X \oplus \langle e_0\rangle$ by $e_0/\bar\pi + uz/(\pi\bar\pi)$, z the middle vector; if s is even, multiply the first half of the string by u/a and the second half by $u\bar a$. The point of V *spans the graded lattice* if its frame and this basis span the same $\hat{O}_E$-lattice Λ. For a finite point x write $\Lambda_x = x\Lambda$, Λ_x^+ for its projection to $X^\perp$ and $\Lambda_x^- = \Lambda_x \cap X^\perp$, and drop the index when $x = 1$. A lattice is *graded* if it contains the part in each grade of each of its vectors.

Proposition 18.1 (The graded lattice). *Let $N = \pi\bar\pi$, with π a uniformiser at the places of a set P containing no conjugate pair and a unit elsewhere.*

(a) *The Gram matrix of the graded lattice basis is hermitian and invertible over every subring; over* $\mathbb{C}$ *the form is congruent to a diagonal of signs,* $\lfloor (s+1)/2 \rfloor$ *of them negative.*
(b) *If the point spans the graded lattice,* Λ *is graded, and* Λ^-, Λ^+ *are sums of lines on the string vectors whose generators differ only by* N^{-1} *at the middle vector. A vector of* Λ^+ *outside* Λ^- *exists only for odd* s*, and it generates* Λ^+ *over* Λ^- *at some place above* N*.*
(c) *If the constants of* Y *and* Y' *have the valuations of* $a\bar{a}$ *and* $u = (a/\bar{a})^{\lfloor (s+2)/2 \rfloor}$*, the shift maps* Λ^+ *onto* $a\Lambda'^-$*.*
(d) *If an integral* $h \in \mathrm{U}(X \oplus \langle e_0 \rangle)$ *has last row* $\equiv (0, \dots, 0, 1)$ *modulo* π*, its image in* $\mathrm{U}(V)$ *stabilises* Λ*; if moreover* $h \equiv 1$ *modulo a place* w*, the image is* $\equiv 1$ *modulo* w*.*

Proof. (a) In the graded basis the block of X is the identity. A string vector b_i pairs only with b_{s-1-i}; for odd s both are multiplied by u, and $\langle ub_i, ub_{s-1-i} \rangle = u\bar{u} = 1$, while for even s one of them is multiplied by u/a and the other by $u\bar{a}$, and the factors a^{-1} and $\bar{a}$ of partners cancel, so the pairing is again 1. For odd s put $v = e_0/\bar{\pi} + uz/N$. Then

$$\langle v, v \rangle = \frac{1}{\pi\bar{\pi}} + u\bar{u}\,\frac{-N}{N^2} = 0, \qquad \langle v, uz \rangle = u\bar{u}\,\frac{-N}{N} = -1, \qquad \langle uz, uz \rangle = -N,$$

and v is orthogonal to X and to the other string vectors. Hence the Gram matrix of the graded basis is that of the glued form: the identity on X, the antidiagonal 1 on the string with $-N$ in the middle, and for odd s the block $\left(\begin{smallmatrix} 0 & -1 \\ -1 & -N \end{smallmatrix}\right)$ on (v, uz). Its inverse is the glued form with the parameters N and 0 exchanged, whose entries lie in every subring containing N; hence the Gram matrix is hermitian and invertible over every subring. Over $\mathbb{C}$, the complex frame whose columns are $(\sigma_b b_b + b_{s-1-b})/\sqrt{2}$, with $\sigma_b = -1$ for the first $\lfloor (s+1)/2 \rfloor$ indices and $+1$ for the others, and $b_m/\sqrt{N}$ for the middle index m, has the Gram matrix $\mathrm{diag}(\sigma)$: away from the middle, the columns b and $s-1-b$ span a hyperbolic plane, are orthogonal because their signs are opposite, and have norms ± 1, and the middle column has norm $-N/N = -1$. With the identity on X this gives the signature.

(b) Every vector of the graded basis is homogeneous: the vectors of X and e_0 have grade 0, the string vectors their own grades, and for odd s the vector v combines e_0 with the middle vector z, both of grade 0. Hence the part in each grade of a lattice vector is again a lattice vector, and Λ is graded. A vector of Λ lies in $X^\perp$ exactly when its coordinates on $X \oplus \langle e_0 \rangle$ vanish; as v is the only basis vector with a component on e_0, its coefficient must vanish, and the vector is an $\hat{O}$-combination of the scaled string vectors. Hence Λ^- is the sum of the lines $\hat{O}\ell_i b_i$, with ℓ_i the scale of b_i in the graded basis. The projection to $X^\perp$ kills the vectors of X and sends v to uz/N, so Λ^+ is the same sum with the line of the middle vector replaced by $\hat{O}uz/N$, i.e. its generator multiplied by N^{-1}. Let $y \in \Lambda^+ \setminus \Lambda^-$. Its coordinates agree with those of an element of Λ^- except at the middle vector, so s is odd, and its middle coordinate divided by u lies in $N^{-1}\hat{O}$ but not in $\hat{O}$. Since $N = \pi\bar{\pi}$ with π a uniformiser on P, $\bar{\pi}$ a uniformiser on $\bar{P}$ and $P \cap \bar{P} = \emptyset$, N has valuation 1 at the places of $P \cup \bar{P}$ and 0 elsewhere; hence the middle coordinate of y has a simple pole at some place w above N. Every

$y' \in \Lambda^+$ has at most a simple pole there, so $y' - cy$ has none at w for the quotient $c \in \hat{O}_w$ of the two middle coordinates: y generates Λ^+ over Λ^- at w.

(c) Let $\mathfrak{e}^i$ be the string vector b_i times the element of E (a power of the constant of the grading) for which the shift sends $\mathfrak{e}^i$ to the corresponding vector $\mathfrak{e}'^i$ of Y' when $i < s-1$, and kills $\mathfrak{e}^{s-1}$. As the constant has the valuations of $a\bar{a}$ and $u = (a/\bar{a})^{\lfloor (s+2)/2 \rfloor}$, the scale ℓ_i of the graded basis is $a^{s-i}\bar{a}^{-1-i}$ times a power of $a\bar{a}$, and the powers of $a\bar{a}$ cancel place by place. Writing $\mathfrak{l}_i$ for the ideal of $a^{s-i}\bar{a}^{-1-i}$, one finds: $\sum_i t_i \mathfrak{e}^i \in \Lambda^-$ if and only if $t_i \in \mathfrak{l}_i$ in grade at least 0 and $t_i \in N^{-1}\mathfrak{l}_i$ in negative grade; and $\sum_i t_i \mathfrak{e}^i \in \Lambda^+$ if and only if $t_i \in \mathfrak{l}_i$ in positive grade and $t_i \in N^{-1}\mathfrak{l}_i$ in grade at most 0. For Y', with $s-1$ in place of s, the ideals are $\mathfrak{l}'_i = (a^{s-1-i}\bar{a}^{-1-i})$, so $a\mathfrak{l}'_i = \mathfrak{l}_i$; the grade of $\mathfrak{e}'^i$ in Y' is $s-2-2i$, which is at least 0 exactly when the grade $s-1-2i$ of $\mathfrak{e}^i$ in Y is positive. Hence, for every $i < s-1$, the condition on t_i for Λ^+ is the condition for $a\Lambda'^-$, and the shift, which forgets the coordinate t_{s-1}, maps Λ^+ onto $a\Lambda'^-$.

(d) In rational coordinates the image of h acts as h on $X \oplus \langle e_0 \rangle$ and as 1 on the string, so its matrix in the frame of the graded basis is $1 + R^{\mathsf{H}} M R$, with R the integral rows of the frame at $X \oplus \langle e_0 \rangle$ and $M = h - 1$ for even s; for odd s, because the graded basis replaces e_0 by $e_0/\bar{\pi} + uz/N$, M is $h-1$ with its last row divided by π and its last column divided by $\bar{\pi}$. The integral matrices whose last row is $\equiv (0, \dots, 0, 1)$ modulo π form a group, so $h^{-1} = h^{\mathsf{H}}$ has the same property, and its last row read in h says that the last column of h is $\equiv e_0$ modulo $\bar{\pi}$. Hence the last row of $h-1$ is divisible by π and its last column by $\bar{\pi}$; the corner is divisible by both, and at every place one of π, $\bar{\pi}$ is a unit, so the corner divided by $\pi\bar{\pi}$ is integral. Hence M is integral, the image of h is integral with integral inverse, and it stabilises Λ. If moreover $h \equiv 1$ modulo w, for the places w used here, which with their conjugates lie outside P, then π and $\bar{\pi}$ are units at w, $M \equiv 0$ modulo w, and each entry of $R^{\mathsf{H}} M R$ is a sum of products of an entry of M with two integral elements, hence $\equiv 0$ modulo w. □

Proposition 18.2 (The lattices of the spaces Y_k). *For $k < m$ the grading of Y_k carries the constant $-(\jmath M \delta_E)^{-1}$, whose valuations are those of $a\bar{a}$ with $a = \alpha/\bar{\beta}_M$; and the point of Y_k spans the graded lattice of a, π_N and $u = (a/\bar{a})^{\lceil (m-k)/2 \rceil}$. The idele generating $(\bar{\pi}_M)$ has the valuations of $\bar{\pi}_M$, and $\mathrm{N}(\bar{\pi}_M) = M^f$.*

Proof. The chosen idele generates the ideal of $\bar{\pi}_M$ and is 1 away from the chosen places, so it has the valuations of $\bar{\pi}_M$; and $\bar{\pi}_M$ is an integer of K of norm M, viewed in E, so the ideal it generates in E has norm $M^{[E:K]} = M^f$. The self-duality of $\hat{O}_E$ under $(z,u) \mapsto \psi_f(\jmath\alpha\bar{\alpha}zu)$ says that the order of $\jmath\alpha\bar{\alpha}$ at every place is minus that of the different, which δ_E generates; so $\jmath\alpha\bar{\alpha}\delta_E$ is a unit at every finite place. Since $a = \alpha/\bar{\beta}_M$ and $\beta_M\bar{\beta}_M$ has the valuations of M, $\alpha\bar{\alpha}$ has the valuations of $a\bar{a}M$, and therefore $\jmath M\delta_E a\bar{a}$ is a unit at every finite place: the constant $-(\jmath M\delta_E)^{-1}$ has the valuations of $a\bar{a}$. The frame of Y_k was chosen (Definition 15.7) to be the graded basis for a, π_N and $u = (a/\bar{a})^{\lceil (m-k)/2 \rceil}$ up to $\mathrm{GL}(\hat{O})$, which is the assertion that the point spans the graded lattice; note that u gains one factor $a/\bar{a}$ from $k = 2i+2$ to $k = 2i$. □

18.2. Unfolding a pair of lifts

For data f on $\mathrm{Hom}(V, V')$ the *constant term* $C_{\mathrm{sh}}(f)(g, h)$ is the mean, over the adelic maps $V \to V'$ of positive degree modulo the rational ones, of the theta function of the Weil translate of f by (g, h), Heisenberg translated by sh. One unfolding step (Proposition B.27) writes the period of a ψ_U-equivariant function W on R_V against a theta descent as the integral over $R_{V'}(E^+)\backslash \mathrm{U}(V')(\mathbb{A})$ of the form against

$$\mathcal{W}'(h) = \int_{[R_V]} W(r)\, C_{\mathrm{sh}}(f)(r, h)\, dr,$$

and $\gamma \mapsto \mathcal{W}'(\gamma h)$ is again ψ_U-equivariant.

A *pair of edges* $Y \to Y' \to Y''$ consists of gradings of V, V', V'' with strings of lengths s, $s-1$ and $s-2$ (or 0 if $s = 1$), and an identification of the part of grade zero of $V^- \oplus V''$, where V^- is V with the form negated, with $X^- \oplus X''$ plus p hyperbolic planes. It gives a splitting of $\mathrm{Hom}(V', V^- \oplus V'')$ down to $\mathrm{Hom}(X', X^- \oplus X'')$ and an embedding of $\mathrm{U}(X_0)$ into $\mathrm{U}(X'')$. The *middle* pairs have one plane, the *last* pair none. The *join* of data f, f' on the two edges is the datum $f \star f'$ on $\mathrm{Hom}(V', V^- \oplus V'')$ whose theta value is $z \neq 0$ times the product of those of f and f'.

Theorem 18.3 (Two unfolding steps are one double lift). *Let $Y \to Y' \to Y''$ be a pair of edges with definite parts X, X', $X'' \supseteq X_0$. There is $z \in \mathbb{C}$ such that for all characters a, b of $\mathrm{U}(1)(\mathbb{A})$, data f, f' on the two edges, points $k \in \mathrm{U}(X_0)(\mathbb{A})$ and continuous, rationally invariant, ψ_U-equivariant W on R_V,*

$$\int_{[R_V]} \int_{[R_{V'}]} W(g)\, C_{\mathrm{sh}}(f)(g, r)\, C_{\mathrm{sh}'}(f')(r, k)\, dr\, dg$$
$$= \int_{[\mathrm{U}(X)]} W(g)\, z \int_{[\mathrm{U}(X')]} \theta\Big(\omega_{a,b}(g, r, k)\, Z\big(\varrho(\mathrm{sh}^*, \mathrm{sh}')(f \star f')\big)\Big)\, dr\, dg,$$

where $\varrho(\mathrm{sh}^*, \mathrm{sh}')$ is the Heisenberg translation by the stacked map $V' \to V^- \oplus V''$, Z is the zero term along the splitting, and $\omega_{a,b}$ is the joint Weil action of $\mathrm{U}(X) \times \mathrm{U}(X') \times \mathrm{U}(X'')$. The zero term intertwines the joint Weil actions of the graded and the definite spaces with no further scalar.

Proof. The product of two constant terms. The constant term of f at w is the mean of $\theta(\varrho(w + a)f)$ over the maps $a\colon V \to V'$ of positive degree, adelic modulo rational, and similarly for f' at w' with $a'\colon V' \to V''$. A map $V' \to V^- \oplus V''$ of positive degree for the grading of $V^- \oplus V''$ is a pair (a^*, a') with a, a' of positive degree, so the mean over such maps is the iterated mean over a and a'; and the theta value of the join is z times the product of the theta values. Hence the product of the two constant terms is z^{-1} times the value at 0 of the zero term, along the splitting that removes the maps of nonzero degree, of the join of the two Heisenberg translates.

The stacked map. Write $T = \mathrm{sh}\colon V \to V'$ and $T' = \mathrm{sh}'\colon V' \to V''$ and $S = (T^*, T')$. Its adjoint for $V^- \oplus V''$ is $(-T, T'^*)$, so $S^\dagger S = -TT^* + T'^*T'$; both products depend only on the grading of V' and the common constant, and they agree, so $S^\dagger S = 0$: S is isotropic, and translations by S and by any multiple of S compose without a phase.

The unipotent variables. Let n be a unipotent point of $\mathrm{U}(V')$. In rational coordinates $S(n-1)$ is the sum of a map of positive degree and a map of degree zero that sends X' into an isotropic half of the hyperbolic planes of grade zero; the matrix of the latter is the constant of the grading times the block of $n-1$ with rows the basis vectors of grade one and columns X'. The zero term ignores translations by maps of positive degree, and translating by the stacked map and then by $S(n-1)$ is translating by Sn; so z^{-1} times the zero term of the translate by S of the join, evaluated at the map of degree zero just described, is the product of the constant terms of f at $n^{-1}T$ and of f' at $T'n$. A unipotent element of R_V in the first argument of the constant term moves into the map on the right, and one of $R_{V'}$ in the second argument moves into it on the left; this is what the last sentence says.

The integrals. Write $g = u\,\iota_X(g_0)$ and $r = u'\iota_{X'}(r_0)$ with unipotent u, u' and integrate first over the unipotent groups, whose quotients by the rational points have volume 1. Over $[U_V]$ the integrand is invariant: W transforms by ψ_U, and the product of the constant term with the characters of the pair transforms by ψ_U^{-1}, because the string of V has odd length and the constants of the two gradings agree. Over $[U_{V'}]$, by the previous paragraph, the integrand is the zero term evaluated at the degree-zero map attached to u'; as u' runs over $[U_{V'}]$, these maps run over the adelic maps into the isotropic half of the planes modulo the rational ones, with a fixed constant Jacobian, so the mean over $[U_{V'}]$ turns the zero term along the maps of nonzero degree into the zero term Z along the splitting that removes also the hyperbolic planes. Collecting the constants of the two unfoldings and z^{-1} into one constant z gives the formula, with the remaining action of (g_0, r_0, k) on the translated datum.

No further scalar. The zero term is taken in two stages: first to the part of grade zero of $\mathrm{Hom}(V', V^- \oplus V'')$, which is $\mathrm{Hom}(X', Z_0)$ with Z_0 the grade-zero part of $V^- \oplus V''$, and then off the hyperbolic planes of Z_0. At each stage the law of the zero term along a splitting (Appendix B.18) carries the Weil operator of the Levi component to that of the smaller space, times a power of the splitting character v at the determinant. The characters of the pair are the determinant characters a, b, twisted by powers of v depending on the ranks; on the graded spaces the twists of the target characters are $v^{\pm 1}$ according to the parities of the ranks, on the definite spaces they are trivial, and in the product of the two target characters v cancels against the powers produced by the two stages, because the length of the string of V' and the sum of the ranks of V and V'' are even. Hence the zero term carries the joint Weil operator of $(\iota g_0, \iota r_0, \iota k)$ to that of (g_0, r_0, k) with no scalar. □

The *finite evaluation* of a pair of edges is the linear map on joined finite data which translates by the finite part of the stacked map, takes the finite part Z_f of the zero term along the splitting, and splits the result into a tensor of finite data for $\mathrm{Hom}(X, X')$ and $\mathrm{Hom}(X', X'')$ (a left inverse of the join). For the pair $Y_{2i} \to Y_{2i+1} \to Y_{2i+2}$ it is the evaluation of the middle pair for $i < \varrho$ and of the last pair for $i = \varrho$. The finite part Z_f is not zero at the vacuum, obeys the law of the zero term for finite points of the parabolic subgroup, and is equivariant for the finite joint Weil action.

These properties of Z_f are proved as follows. The finite part is $f \mapsto Z(v \otimes f)$ for the Fock vacuum v moved by the real part of a point adapted to the splitting, and the law of Z for the parabolic subgroup restricts to finite points, which do

not move v. At the image $\omega(1, g_f)$vac the zero term is that of $\omega(g)$vac, a nonzero multiple of the vacuum; so Z_f does not vanish on the finite part, and since the finite translates of the vacuum span the finite part and Z_f commutes with them up to a root of unity, it does not vanish at the vacuum. For the equivariance, the images ι_X, $\iota_{X'}$, $\iota_{X''}$ fix the shifts (they act on X, X', X'', which the shifts kill or avoid), so the joint action passes the translation by the stacked map, and by the last part of the proof of Theorem 18.3 it passes the zero term with no scalar; it then splits over the two factors of the join.

18.3. The unfolded functions

Fix a source form φ, a linear form t on its values and an archimedean point g of $\mathrm{U}(Y_0)$. The *outer variables* are points $g_j = (\xi_j, y_j)$ of $\mathrm{U}(Y_j)(\mathbb{A})$, ξ_j archimedean and y_j finite. The datum of lift $j+1$ moved by g_j is $\omega(g_j, 1)(\varphi_{\mathrm{harm}} \otimes \Upsilon_{j+1})$, and the *unfolded functions* are defined by $\mathcal{W}_0 = \mathcal{W}_\varphi$ on R_0 and

$$\mathcal{W}_{k+1}(h) = \int_{[R_k]} \mathcal{W}_k(r\, g_k)\, C_{\mathrm{sh}}\big(\omega(g_k, 1)(\varphi_{\mathrm{harm}} \otimes \Upsilon_{k+1})\big)(r, h)\, dr$$

for $k \le 2\varrho + 1$, where $\mathcal{W}_k(r\, g_k)$ is read as $\mathcal{W}_0(r)$ for $k = 0$. Each $\mathcal{W}_{k+1}$ is continuous, left invariant under $R_{k+1}(E^+)$, and $\gamma \mapsto \mathcal{W}_{k+1}(\gamma\, g_{k+1})$ is ψ_U-equivariant on R_{k+1}. The *translated data* of lift $k+1$ are the finite data $\omega_f(y_k, y_{k+1})\Upsilon_{k+1}$, and the *evaluated datum* $\phi_i(y)$ of the pair i is the finite evaluation of the two translated data of lifts $2i+1$, $2i+2$.

Each of these properties is the conclusion of Proposition B.27 at the step $k+1$, applied to the function $\gamma \mapsto \mathcal{W}_k(\gamma\, g_k)$, which is ψ_U-equivariant by the previous step; the constants of the gradings of all Y_k are the same number $-(\jmath M \delta_E)^{-1}$, as the proposition requires. By construction $\mathcal{W}_{k+1}$ depends only on the outer variables g_j with $j \le k$.

Theorem 18.4 (The unfolded functions are upper chains). *Let $\Theta_i(y)$ be the composite of the first i double lifts $X_0 \to \cdots \to X_{2i}$ with kernels of Section 16.3 and finite data $\phi_0(y), \dots, \phi_{i-1}(y)$. For every i with $2i \le 2\varrho + 1$ and all archimedean ξ there is a linear form l, independent of ϑ, φ and y, such that $\mathcal{W}_{2i}$ restricted to R_{2i} is the Bessel function $\mathcal{W}$ of $\Theta_i(y)\varphi$ at l; and for $2i < 2\varrho + 1$ and every archimedean x there is l with*

$$\mathcal{W}_{2i+2}\big(\iota_X(g)\,(x, y_{2i+2})\big) = l\big((\Theta_{i+1}(y)\varphi)(\iota g)\big) \qquad (g \in \mathrm{U}(X_{2i+2})(\mathbb{A})).$$

Proof. Induction on i. For $i = 0$ the function on R_0 is $\mathcal{W}_\varphi$, the Bessel function of $\varphi = \Theta_0(y)\varphi$ at t. Suppose that the function that $2i$ steps leave on R_{2i} is the Bessel function of $\psi = \Theta_i(y)\varphi$ at a linear form l, that is, the ψ_U-equivariant function equal to the coefficient $l(\psi(g))$ on $\iota\,\mathrm{U}(X_{2i})(\mathbb{A})$.

Two steps. Steps $2i+1$ and $2i+2$ are the left side of Theorem 18.3 for the pair $Y_{2i} \to Y_{2i+1} \to Y_{2i+2}$, with W this function, f and f' the data of lifts $2i+1$ and $2i+2$ moved by the outer variables, and the third point (x, y_{2i+2}). By the theorem they equal z times a double theta integral over $[\mathrm{U}(X_{2i})] \times [\mathrm{U}(X_{2i+1})]$ of the datum $Z(\varrho(\mathrm{sh}^*, \mathrm{sh}')(f \star f'))$ moved by the joint Weil action.

The datum is a pure tensor. Each of the operations (moving by the outer variables, the join, the translation by the stacked map and the zero term) acts separately on the real and on the finite factor of a pure tensor, and on the real factor the characters act through their weights, which do not depend on ϑ.

Hence the datum is $v \otimes \phi_i(y)$, carried to $\mathrm{Hom}(X_{2i+1}, X_{2i}^- \oplus X_{2i+2})$ by the join, with a Fock vector v that depends on the archimedean outer variables and on x but not on ϑ, φ or y; the finite factor is the finite evaluation of the two translated data, by definition the evaluated datum $\phi_i(y)$.

Double lifts. It remains to see that, for every Fock vector u and linear form l, there is l'' such that the double integral of θ of $u \otimes \Phi$ moved by (g_0, r, k) against the coefficient $l(\psi(g_0))$ is the coefficient at l'' of the composite of the two lifts with finite data Φ applied to ψ. For $u = PQ$ a product of a polynomial on $\mathrm{Hom}(X_{2i}, X_{2i+1})$ and one on $\mathrm{Hom}(X_{2i+1}, X_{2i+2})$ (the variables of P transposed into the block of X_{2i}), the datum $PQ \otimes \Phi$ is the join of $P \otimes \Phi_1$ and $Q \otimes \Phi_2$ for $\Phi = \Phi_1 \otimes \Phi_2$, and its kernel is z times the product of the two theta kernels; so the double integral is an iterated theta lift between definite spaces, which Section 17 computes as a class sum of jets, and l'' exists. Sums of such products are the Fock polynomials; these are dense in the Fock space, the integral is continuous in u, and the families given by linear forms l'' make up a finite-dimensional, hence closed, space; so the claim holds for every u, and with $u = v$ it gives the second display. As $\mathcal{W}_{2i+2}$ is ψ_U-equivariant on R_{2i+2}, it is determined by its values on $\iota\,\mathrm{U}(X_{2i+2})(\mathbb{A})$, and the first statement for $i+1$ follows with l'' in place of l. □

18.4. The support of the unfolded functions

A finite point x of $\mathrm{U}(Y_j)$ *has the lattice bounds* for $(\mathfrak{q}, k)$ if $\bar{\mathfrak{q}}^k \Lambda^- \subseteq \Lambda_x^-$ and $\mathfrak{q}^k \Lambda_x^- \subseteq \Lambda^-$. It has *bounded orbit* if some point rx, $r \in R_j(\mathbb{A}_{\mathrm{f}})$, has a graded lattice and the bounds. The lattice of x *splits* along X at a place w if there $\Lambda_x^+ \subseteq \Lambda_x^-$.

Lemma 18.5 (The shift at a nonzero constant term). *Let the constant term at* sh *of the Weil translate by (x, x') of a datum fixed by the Heisenberg translations by b $\hat{O}^{\iota' \times \iota}$ be nonzero. Then some matrix T with $T\Lambda_x \subseteq \Lambda_{x'}$ agrees with $(\bar{b}/\alpha)\,\mathrm{sh}$ up to a matrix that does not lower grades; and if r, r' are finite points of the Bessel groups for which Λ_{rx}, $\Lambda_{r'x'}$ are graded, then $(\bar{b}/\alpha)\,\mathrm{sh}\,\Lambda_{rx}^+ \subseteq \Lambda_{r'x'}'^-$.*

Proof. The finite zero term. The constant term at sh of the translate of $t \otimes f$ by (x, x') is the theta value of the zero term, along the splitting D that removes the maps of positive degree, of the translate by sh (Appendix B.15); each operation acts separately on the real and finite factors, and the characters contribute a scalar. Hence if the constant term does not vanish, the finite zero term Z_f does not vanish at $\varrho(\mathrm{sh}_f)\omega_f(x, x')f$.

The support of Z_f. Let k be the finite isotropic frame of D and $k^\perp$ the kernel of k^{H}. Suppose g is fixed by the Heisenberg operators of $\mathcal{M}\hat{O}^n$, with $\mathcal{M}\mathcal{M}'^{\mathsf{H}} = 1$, and $Z_f(\varrho(y)g) \neq 0$. Rescale k by a t so that $\mathcal{M}'^{\mathsf{H}} kt$ is integral with an integral left inverse L. A vector ζ of the isotropic subspace is a period of Z_f, and if it is also a period of g, then the commutator of the translations by ζ and by y, the scalar $\psi(\langle \zeta, y \rangle)$, fixes the nonzero vector $Z_f(\varrho(y)g)$, hence equals 1. For ξ integral, $\zeta = kt\,\xi$ is such a vector: it is $\mathcal{M}(\mathcal{M}'^{\mathsf{H}} kt\,\xi)$ with $\mathcal{M}'^{\mathsf{H}} kt\,\xi$ integral. Hence $\psi(\langle \xi, (kt)^{\mathsf{H}} y \rangle) = 1$ for all integral ξ, and $v = (kt)^{\mathsf{H}} y$ is integral, since the standard lattice is its own dual. With $l = L^{\mathsf{H}} v$ one has $(kt)^{\mathsf{H}} \mathcal{M}' l = v$, so $y - \mathcal{M}' l \in k^\perp$.

The matrix T. Apply this to $g = \omega_f(x,x')f$, which is fixed by the translations by $b\,\omega_f(x,x')\hat{O}^{\iota'\times\iota}$, so that $\mathcal{M}'$ is $\bar{b}^{-1}$ times the action of (x,x'); and $k^\perp$ consists of the maps of degree at least 0. Hence the finite coordinates of sh, that is sh_f/α, are $\bar{b}^{-1}$ times the image under (x,x') of an integral matrix l, up to a map of degree at least 0. In rational coordinates the image of l is $T = x'lx^{-1}$, which maps Λ_x into $\Lambda_{x'}$, and $T - (\bar{b}/\alpha)\mathrm{sh}$ does not lower grades.

Graded points. For r, r' in the finite Bessel groups, $r'Tr$ maps Λ_{rx} into $\Lambda_{r'x'}$; the matrices of r and r' in rational coordinates do not lower grades (they fix the string vectors of grade ≥ 0 up to vectors of higher grade), degrees add under multiplication, and sh is homogeneous of degree -1, so $r'Tr - (\bar{b}/\alpha)\mathrm{sh}$ again does not lower grades. Its part of degree -1 is therefore $(\bar{b}/\alpha)\mathrm{sh}$. If Λ_{rx} and $\Lambda_{r'x'}$ are graded, the part of degree -1 of a map between them maps the first into the second, being the sum over the grades d of the parts from grade d to grade $d-1$, each of which maps the graded pieces correctly. Finally sh kills X, so $\mathrm{sh}\,y = \mathrm{sh}\,y^+$ for the projection y^+ of y to $X^\perp$, and sh has image in $X'^\perp$. Hence $(\bar{b}/\alpha)\mathrm{sh}\,\Lambda^+_{rx} = (\bar{b}/\alpha)\mathrm{sh}\,\Lambda_{rx} \subseteq \Lambda_{r'x'} \cap X'^\perp = \Lambda'^-_{r'x'}$. □

Proposition 18.6 (The sandwich). *Let Y, Y' be gradings with strings of lengths s, $s-1$ whose points span graded lattices, with $\mathrm{sh}\Lambda^+ = a\Lambda'^-$.*

(a) *If $\mathfrak{q}^k\Lambda^-_x \subseteq \Lambda^-$ and $(\bar{\mathfrak{q}}/a)\,\mathrm{sh}\,\Lambda^+_x \subseteq \Lambda'^-_{x'}$, then $\bar{\mathfrak{q}}^{k+1}\Lambda'^- \subseteq \Lambda'^-_{x'}$.*

(b) *Let $\mathfrak{q}$ be a unit above N. If $\bar{\mathfrak{q}}^k\Lambda^- \subseteq \Lambda^-_x$, then either x has the lattice bounds for $(\mathfrak{q},k)$, or s is odd and the lattice of x splits at some place dividing π_N.*

(c) *If the lattice of x splits at w, the integral scalars of X supported at w and $\bar{w}$ preserve Λ_x.*

(d) *If every point rx with graded lattice has the bounds, x has bounded orbit; if some rx is integral, x has bounded orbit for $k=0$.*

Proof. Duality. The lattice Λ_x is self-dual and $X^\perp$ is nondegenerate, so the dual of $\Lambda^-_x = \Lambda_x \cap X^\perp$ in $X^\perp$ is the projection Λ^+_x, and conversely. The dual of cL is $\bar{c}^{-1}L^\vee$, and duality reverses inclusions. Hence an inclusion $\mathfrak{q}^k\Lambda^-_x \subseteq \Lambda^-$ is equivalent to $\bar{\mathfrak{q}}^k\Lambda^+ \subseteq \Lambda^+_x$, and $\bar{\mathfrak{q}}^k\Lambda^- \subseteq \Lambda^-_x$ is equivalent to $\mathfrak{q}^k\Lambda^+_x \subseteq \Lambda^+$.

(a) By duality $\bar{\mathfrak{q}}^k\Lambda^+ \subseteq \Lambda^+_x$. Apply $(\bar{\mathfrak{q}}/a)\mathrm{sh}$ and use $\mathrm{sh}\Lambda^+ = a\Lambda'^-$:

$$\bar{\mathfrak{q}}^{k+1}\Lambda'^- = (\bar{\mathfrak{q}}/a)\,\mathrm{sh}\,\bar{\mathfrak{q}}^k\Lambda^+ \subseteq (\bar{\mathfrak{q}}/a)\,\mathrm{sh}\,\Lambda^+_x \subseteq \Lambda'^-_{x'}.$$

(b) By duality $\mathfrak{q}^k\Lambda^+_x \subseteq \Lambda^+$, hence $\mathfrak{q}^k\Lambda^-_x \subseteq \mathfrak{q}^k\Lambda^+_x \subseteq \Lambda^+$. If $\mathfrak{q}^k\Lambda^-_x \subseteq \Lambda^-$, the bounds hold. Otherwise take $y \in \Lambda^-_x$ with $\mathfrak{q}^k y \in \Lambda^+\backslash\Lambda^-$. By Proposition 18.1(b), s is odd and $\mathfrak{q}^k y$ generates Λ^+ over Λ^- at a place w above N. There $\mathfrak{q}$ and $\bar{\mathfrak{q}}$ are units, so the two inclusions read $\Lambda^- \subseteq \Lambda^-_x$ and $\Lambda^+_x \subseteq \Lambda^+$ at w, and

$$\Lambda^+_x \subseteq \Lambda^+ = \Lambda^- + \hat{O}_w\,\mathfrak{q}^k y \subseteq \Lambda^-_x \qquad \text{at } w,$$

because $y \in \Lambda^-_x$: the lattice of x splits at w. Since Λ_x is self-dual, splitting at w is equivalent to splitting at $\bar{w}$, so the place may be taken in P; and it divides π_N.

(c) The point of $\mathrm{U}(X^\perp \oplus X)$ that scales X by z sends y to $y+(z-1)(y-y^+)$, with y^+ the projection to $X^\perp$. If the lattice splits at w, then for $y \in \Lambda_x$ the component at w of y^+ lies in $\Lambda^-_x \subseteq \Lambda_x$, hence so does that of $y - y^+$; the same holds at $\bar{w}$, as splitting at w implies splitting at $\bar{w}$. For z integral, of norm one and equal to 1 off w and $\bar{w}$, the vector $(z-1)(y-y^+)$ is supported at w, $\bar{w}$ and is there an integral multiple of a lattice vector; so the scaling maps Λ_x into itself, and so

does its inverse, the scaling by $\bar{z}$. In other words, its conjugate by x is integral and trivial off w and $\bar{w}$.

(d) Every orbit of the finite Bessel group contains a point whose lattice is graded, by the Iwasawa decomposition for the flag of the grades (Appendix B.15); if all such points have the bounds, one of them has both properties, which is the definition of a bounded orbit. If rx is integral, its lattice is Λ, which is graded by Proposition 18.1(b) and has the bounds for $k = 0$. □

Theorem 18.7 (Support of the unfolded functions). *Let φ be ramified and $y_0 \in J_M$.*

(a) *If the lattice of x splits along X_{2j+2} at a place $w \mid \pi_N$, or if $x^{-1} z_w x \in J_{2j+2}$ for the scaling z_w of Section 16.3 acting on X_{2j+2}, then $\mathcal{W}_{2j+2}(\gamma(a,x)) = 0$ for all $\gamma \in R_{2j+2}(\mathbb{A})$ and archimedean a.*

(b) *If $\mathcal{W}_j(a,x) \neq 0$ for some $j = k+1 \leq 2\varrho + 1$ and $r' \in R_j(\mathbb{A}_{\mathrm{f}})$ makes $\Lambda_{r'x}$ graded, then $r'x$ has the lattice bounds for $(\mathfrak{q}, j)$. In particular x has bounded orbit.*

Proof. (a) *The scaling case.* Put x in the last outer variable: by Theorem 18.4, with $y_{2j+2} = x$, the value $\mathcal{W}_{2j+2}(\gamma(a,x))$ is a coefficient of $(\Theta_{j+1}(y)\varphi)(g)$ for a point g. The image Z of the scalar z_w on X_{2j+2} lies in the Bessel group, and $x^{-1} Z x \in J_{2j+2}$ means that $(1, Z)$ fixes the translated datum of lift $2j+2$, since the level fixes the datum Υ_{2j+2} and the tame characters are trivial on it. The finite evaluation carries $\omega(1, Z)$ on the data of the Y_k to $\omega(1, z_w)$ on the second factor of $\phi_j(y)$ (Section 18.2), so $\omega(1, z_w)$ on the last factor fixes the tensor of the chain $\Theta_{j+1}(y)$. By the argument of Proposition 16.8, the chain then sends $\psi - z_w \psi$ to 0 for every ψ, and since φ is ramified, φ is a sum of such differences; so the value vanishes.

The splitting case. If the lattice of x splits at $w \mid \pi_N$, Proposition 18.6(c) shows that $x^{-1} z_w x$ is integral and trivial off w and $\bar{w}$. The places $w, \bar{w}$ lie above N, so they are neither chosen places nor places of Q (all places above N are split and lie in $P \cup \bar{P}$, which Q avoids); so $x^{-1} z_w x$ reduces to 1 at all level places and lies in J_{2j+2}, and the first case applies.

(b) Induction on k, for all points and all graded translates. Suppose $\mathcal{W}_{k+1}(a,x) \neq 0$. By the definition of $\mathcal{W}_{k+1}$ as an integral over $[R_k]$, some $r \in R_k$ gives a nonzero integrand: the function of the previous stage is nonzero at $r g_k$ and the constant term of the moved datum of lift $k+1$ is nonzero at $(r g_k, (a,x))$. Let r_0 in the finite Bessel group make the lattice of $r_0 r y_k$ graded. For $k = 0$ take $r_0 = r^{-1}$: the point $y_0 \in J_M$ is integral, so its lattice is Λ, which is graded and has the bounds for 0 (Proposition 18.6(d)); for $k > 0$ the function of the previous stage at $r g_k$ is a value of $\mathcal{W}_k$, so the induction hypothesis gives the bounds for $(\mathfrak{q}, k)$ at $r_0 r y_k$.

Frequency. Let $b = \mathfrak{q} c$, with c the chosen idele, generate the modulus of the twist, the product of the chosen places and the places of Q. The datum of lift $k+1$ is fixed by the Heisenberg translations by $b\hat{O}$: Υ_{k+1} is the vacuum away from the level places and the indicator of the inclusion matrix modulo the modulus there, a combination of translates $\varrho(t)\mathrm{vac}$ with $\bar{b}t$ integral, and the translations by $b\hat{O}$ commute with these and fix the vacuum. With $a = \alpha/\bar{c}$ one has $\bar{b}/\alpha = \bar{\mathfrak{q}}/a$. Hence Lemma 18.5 gives $(\bar{\mathfrak{q}}/a)\,\mathrm{sh}\,\Lambda^+_{r_0 r y_k} \subseteq \Lambda'^-_{r' x}$.

Sandwich. By Proposition 18.2 and Proposition 18.1(c), $\mathrm{sh}\Lambda^+ = a\Lambda'^-$. With the upper bound $\mathfrak{q}^k \Lambda^-_{r_0 r y_k} \subseteq \Lambda^-$, Proposition 18.6(a) gives the lower bound $\bar{\mathfrak{q}}^{k+1} \Lambda'^- \subseteq \Lambda'^-_{r' x}$. The idele $\mathfrak{q}$ is a unit above N, since no place of Q lies above N;

so Proposition 18.6(b) gives the upper bound unless the string of Y_{k+1} has odd length, i.e. $k+1$ is even, and the lattice of $r'x$ splits at a place dividing π_N. In that case (a) gives $\mathcal{W}_{k+1}(\gamma(a, r'x)) = 0$ for all $\gamma \in R_{k+1}(\mathbb{A})$, and with $\gamma = (1, r'^{-1})$ this contradicts $\mathcal{W}_{k+1}(a, x) \neq 0$. Hence $r'x$ has the bounds for $(\mathfrak{q}, k+1)$. As every graded point of the orbit of x has the bounds, x has bounded orbit by Proposition 18.6(d). □

18.5. Cuts

Let $S, A \supseteq J$ be subgroups of a locally compact group H, and T a finite set of cosets of J. The *cover multiplicity* of $t \in H/J$ is the number of classes $r \in S/(S \cap A)$ with $r^{-1}t \in T$ (zero if there are infinitely many). A *cut* for stage $j > 0$ is a finite set Ω_j of classes of $\mathrm{U}(Y_j)(\mathbb{A}_\mathrm{f})$ modulo J_j which is stable under $R_j(\mathbb{A}_\mathrm{f}) \cap J_j^\circ$, whose points are integral at the places prime to q, on which the multiplicity m_j (for $S = R_j(\mathbb{A}_\mathrm{f})$, $A = J_j^\circ$) divides a power of q, and on which $m_j \neq 0$ at every point of bounded orbit for $(\mathfrak{q}, j)$. Put $\Omega_0 = J_M/J_0$ with $m_0 = 1$.

Proposition 18.8 (Integrals as weighted coset sums). (a) *Let $J \leq A \leq B$ be levels of $\mathrm{U}(V)(\mathbb{A}_\mathrm{f})$, D compact with $D \subseteq B$ and, for each $x \in D$, $(xux^{-1})^{q^n} \in A$ for all $u \in J$ and some n; and let W be a property such that each x with $W(x)$ has $rx \in D$ for some $r \in S$. Then there is a finite T, stable under $S \cap A$, with points in B, on which and at every point with W the multiplicity divides a power of q.*

(b) *For a grading of V and a level A there is κ such that: for every level J, cut T stable under $R_V(\mathbb{A}_\mathrm{f}) \cap A$, and continuous, rationally invariant, integrable f on $R_V(E^+)\backslash \mathrm{U}(V)(\mathbb{A})$ with $m \neq 0$ on its support and $\int_{[R_V]} f(rt)\,dr = [J^\circ : J]\, a(t_\infty) b(t_f)$ for $t_f \in T$,*

$$\int_{R_V(E^+)\backslash \mathrm{U}(V)(\mathbb{A})} f = \kappa \sum_{t \in T} m(t)^{-1}\, b(t).$$

In particular cuts exist for every stage.

Proof. (a) *Multiplicities as indices.* Let C be a set of representatives, one in each S-orbit, of the cosets xJ whose S-orbit meets DJ/J, and let T be the union of the orbits of C under $S \cap A$. The multiplicity is constant on S-orbits: left translation by p^{-1} carries the classes counted at pt to those counted at t. At $t = xJ \in C$, the classes $r(S \cap A)$ with $r^{-1}t \in T$ are those with $r^{-1}t$ in the $(S \cap A)$-orbit of t, i.e. the orbit of the trivial class under the stabiliser $Z = S \cap xJx^{-1}$ of t; so the multiplicity is the index $[Z : Z \cap A]$. This index is finite, because Z lies in the compact group xAx^{-1} and A is open.

The index divides a power of q. Every element of Z is xux^{-1} with $u \in J$, and by hypothesis $(xux^{-1})^{q^n} \in A$; so the q^n-th power of every element of Z lies in $Z \cap A$. Let N be the normal core of $Z \cap A$ in Z, a subgroup of finite index. In the finite group Z/N every element has order dividing q^n, so by Cauchy's theorem every prime dividing its order divides q, and its order, hence $[Z : Z \cap A]$, divides a power of q.

The cut. T is stable under $S \cap A$ by construction. It is finite, because it lies in the image of the compact set AD in H/J and J is open; and its points lie in B, because $D \subseteq B$ and $A \subseteq B$. At a point of T the multiplicity is that at a point of C,

and if $W(x)$, then $rx \in D$ for some $r \in S$, so the S-orbit of xJ contains a point of C and the multiplicity is again that at a point of C. In both cases it divides a power of q.

(b) *A weight.* Let η be the Haar measure on $S = R_V(\mathbb{A}_{\mathrm{f}})$ that gives $S \cap A$ the mass 1. The function $\psi = \sum_{t \in T} m(t)^{-1} \mathbf{1}_{tJ}$ on $\mathrm{U}(V)(\mathbb{A}_{\mathrm{f}})$ satisfies

$$\int_S \psi(u^{-1}t)\, d\eta(u) = \begin{cases} 1, & m(t) \neq 0, \\ 0, & m(t) = 0, \end{cases}$$

because the multiplicity is constant on the orbit of t and the integrand is the constant $m(t)^{-1}$ on the $m(t)$ classes of S modulo $S \cap A$ that it counts and 0 elsewhere. Let β be a cutoff for the archimedean Bessel group, of integral one over each of its orbits, and α its archimedean part; then $w(g) = \alpha(g_\infty)\psi(g_f)$ has integral 1 over each orbit of $R_V(\mathbb{A})$ on which $m \neq 0$, and 0 over the others.

Unfolding. The map $(t, r) \mapsto rt$ carries the measure $w(t)\, dt \times dr$ to the measure of $R_V(E^+)\backslash \mathrm{U}(V)(\mathbb{A})$ on the set where $m \neq 0$; as f vanishes off that set,

$$\begin{aligned} \int_{R_V(E^+)\backslash \mathrm{U}(V)(\mathbb{A})} f &= \int_{\mathrm{U}(V)(\mathbb{A})} w(t) \int_{[R_V]} f(rt)\, dr\, dt \\ &= [J^\circ : J] \int_{\mathrm{U}(V)(\mathbb{A})} \alpha(t_\infty) a(t_\infty)\, \psi(t_f) b(t_f)\, dt, \end{aligned}$$

the second equality by the period hypothesis, as $\psi(t_f) = 0$ unless $t_f \in T$. The integral of a product weight against a product function is the product of the two integrals. The finite one is $\sum_{t \in T} m(t)^{-1} b(t) \operatorname{vol}(J)$, and $\operatorname{vol}(J) = [J^\circ : J]^{-1}$ since J° has volume 1; so the index cancels, and $\kappa = \int \alpha a$ depends only on a, the grading and the level A through η.

Existence of cuts. Let $\mathfrak{q}$ be as in Section 18.4. By the finiteness of the lattices with the bounds (Appendix B.15) there is a finite set C of points supported on $Q \cup \bar{Q}$ such that $R_j(\mathbb{A}_{\mathrm{f}})\, x\, J_j^\circ$ meets C for every x with bounded orbit for $(\mathfrak{q}, j)$. Apply (a) with $J = J_j$, $A = J_j^\circ$, B the group of points integral at the places prime to q, $D = CJ_j^\circ$, which is compact and contained in B, and W the bounded orbit property. The power condition holds: for $x \in D$ and $u \in J_j$, at a place outside $Q \cup \bar{Q}$ the point x is integral and xux^{-1} is integral; at a place $w \in Q$, $u \equiv 1$ modulo w, conjugation by x keeps the elements close enough to 1 integral, and high powers of elements $\equiv 1$ are close to 1 by lifting the exponent, so $(xux^{-1})^{q^n}$ is integral for some n. The set T of (a) is then a cut: it is stable, its points are integral at the places prime to q, its multiplicities divide powers of q, and at a point of bounded orbit the multiplicity divides a power of q, hence is not 0. □

For a function S of tuples of classes, the *cut sum* is

$$\Sigma_k S = \sum_{c_0 \in \Omega_0} \cdots \sum_{c_{k-1} \in \Omega_{k-1}} \prod_{j<k} m_j(c_j)^{-1}\, S(c_0, \ldots, c_{k-1}, 1, 1, \ldots).$$

It commutes with linear maps, and if $b \in \mathrm{U}(X_j)(\mathbb{A}_{\mathrm{f}})$ lies in the level of X_j its image in the Bessel group permutes Ω_j and preserves the multiplicities. If $Q = \emptyset$, each Ω_k has at most $M^{f n_m^2}$ elements.

These properties are immediate. The cut sum is a finite sum with scalar coefficients. An element b of the level of X_j maps into $J_M \subseteq J_j^\circ$ (its reduction at the chosen places is 1, and above N the congruence on the last row holds, Proposition 18.1(d)) and into the Bessel group; for $j > 0$ the cut is stable under

$R_j(\mathbb{A}_{\mathrm{f}}) \cap J_j^\circ$ and the multiplicity is constant on $R_j(\mathbb{A}_{\mathrm{f}})$-orbits, and for $j = 0$ left translation by an element of J_M permutes J_M/J_0. If $Q = \emptyset$, the points over Ω_k are integral everywhere, and the principal level of π_M lies in J_k; so Ω_k injects into the integral points modulo π_M, a subgroup of the $n_k \times n_k$ matrices over O_E/π_M, which has $\mathrm{N}(\pi_M)^{n_k^2} = M^{f n_k^2} \le M^{f n_m^2}$ elements.

18.6. The neighbour lattice

Let P consist of one place of each conjugate pair above N, let X, X', X'' be definite with last vectors e, e'' of $X \oplus \langle e\rangle$ and $X'' \oplus \langle e''\rangle$, and let η be the unit of norm -1. For residues y modulo N at P the *neighbour translation* $n(y)$ is the matrix in $\mathrm{Hom}(X', X^- \oplus X'')$ with row $\eta\bar{y}/N$ at e, row $-\bar{y}/N$ at e'', and zero elsewhere; it is supported at the places of $\bar{P}$. Scaling the rows of the standard lattice by factors κ that are 1 or a norm-one adele δ, trivial above N, and applying an explicit unitary gives the *neighbour lattice* L'. Away from N it is the standard lattice with scaled rows; above N it is a neighbour of the standard lattice, and the translations $n(y)$ represent it modulo its integral points. A *coset representative* is $n(y) + t$, with t a representative of a class of the elements of L' that vanish above N.

Proposition 18.9 (Neighbour cosets). (a) *The neighbour translations lie in L', and every element of L' is, modulo the integral lattice, $n(y) + t$ with unique y and t vanishing above N; the elements of L' vanishing above N are the rows scaled by κ, and the number K of their classes divides a power of the norm of the denominator β of κ.*

(b) *A vector (c, x) solving the gluing congruences (a* neighbour vector*) has its entries at e, e'' congruent to the coefficient c modulo π, and the column of the neighbour matrix is such a vector.*

Proof. The unitary. Let j be the adele that is 1 on P and 0 elsewhere, η a unit of $\hat{O}_E$ with $\bar{\eta}\eta = -1$, and $\rho = e + \bar{\eta}\bar{j}e''$: it is $e + \bar{\eta}e''$ at the places of $\bar{P}$ and e elsewhere. Since $j\bar{j} = 0$, $\langle\rho,\rho\rangle = 1$, and $\langle\rho, x\rangle = x_e + \eta j x_{e''}$. Let A be the adele with $1 + A = \pi$ on P, $\bar{\pi}^{-1}$ on $\bar{P}$ and 1 elsewhere; then $1 + A$ has norm one, and $\bar{\pi}A$ is integral. The matrix $\gamma = 1 + A\rho\rho^*$ sends x to $x + A\langle\rho, x\rangle\rho$; it multiplies ρ by $1 + A$ and fixes $\rho^\perp$, and as $\rho\rho^*$ is a hermitian idempotent, $\gamma^{\mathsf{H}}\gamma = 1 + \big((1 + \bar{A})(1 + A) - 1\big)\rho\rho^* = 1$. The neighbour lattice is the image of the standard lattice under $(\gamma\kappa) \otimes 1$, which acts on each column; away from $P \cup \bar{P}$, $\gamma = 1$, and above N the factors κ are 1, so γ commutes with the scaling.

Above N. At a place of P, γ is integral ($A = \pi - 1$ and $\rho = e$) and $n(y) = 0$. At a place of $\bar{P}$, $A = \bar{\pi}^{-1} - 1$, $\langle\rho, x\rangle = x_e$, and $n(y) = (\eta\bar{y}/N)\rho$ because $\bar{\eta}\eta = -1$; so each column of $\gamma x + n(y)$ is

$$x + \rho\left(\bar{\pi}^{-1}x_e + \eta\bar{y}/N - x_e\right).$$

Its entries off e, e'' are those of x; its entry at e is $\bar{\pi}^{-1}(x_e + \eta\bar{y}/\pi)$, and its entry at e'' is $x_{e''} - \bar{\eta}x_e$ plus $\bar{\eta}$ times the entry at e. At a place of $\bar{P}$, π is a unit and $\bar{\pi}$ a uniformiser, so this column is integral exactly when x is integral and $x_e \equiv -\eta\bar{y}/\pi$ modulo $\bar{\pi}$; if x is integral there, exactly one residue y modulo N at the conjugate place of P satisfies this, and if not, none does. Hence: (i) every $n(y)$ lies in L', since with $x = -\gamma^{-1}n(y)$ the column $\gamma x + n(y) = 0$ is integral, so x is integral at $\bar{P}$, and $n(y)$, being supported at $\bar{P}$, is the image of the part of $-x$ at $\bar{P}$; and (ii)

for every element γx_0 of L' (x_0 integral, $\kappa = 1$ above N) there is exactly one y for which $\gamma x_0 + n(y)$ is integral at the places above N.

Away from N. A matrix that vanishes above N lies in L' exactly when $\kappa^{-1}x = \bar{\kappa}x$ is integral, i.e. these are the rows scaled by κ; such a matrix is κ times an integral matrix, so its rows times β in the first block and $\bar{\beta}$ in the second are integral. Reducing the conjugate of $\bar{\kappa}x$ modulo β defines an injective homomorphism from these elements modulo the integral ones into the matrices over $\hat{O}/\beta$: if the reduction is 0, then x is $\bar{\beta}\kappa$ times an integral matrix, hence integral. Hence K divides the number of such matrices, a power of $\mathrm{N}(\beta)$.

Coset representatives. Let $z \in L'$. Its part above N decides y by (ii), and $z + n(y)$ is integral above N; its part away from N is an element of L' vanishing above N, whose class modulo the integral matrices decides t. Hence $z \equiv n(-y) + t$ with unique (y, t); and $n(y) + t \in L'$ for all y and t by (i).

(b) Let x be the column of γ at an index m and c its coefficient: $A\pi^{-1}\bar{j}$ for $m = e$, $A\pi^{-1}\eta j$ for $m = e''$ and 0 otherwise. The coefficient is supported above N, where $\delta = 1$, so $\delta c = c$. The entries of x off e, e'' are those of the identity; all of $\bar{\pi}x$ is integral as $\bar{\pi}A$ is. A direct computation gives $c - \pi^{-1}x_e = -\omega$ for $m = e$ and 0 otherwise, and $\bar{\eta}c - \pi^{-1}x_{e''} = -\omega$ for $m = e''$ and 0 otherwise, with $\omega = j + (1 - j)\pi^{-1}$, which is integral because π is a unit off P. These are the gluing congruences: the entries at e and e'', divided by π, are congruent to c and $\bar{\eta}c$. □

Theorem 18.10 (Finite evaluation of the vacuum). *Let $Y \to Y' \to Y''$ be a pair of edges with $N = \pi\bar{\pi}$ split and squarefree, let the point of Y' span its graded lattice, let every neighbour vector lift to the lattice of $V^- \oplus V''$, and let the denominators match through κ.*

(a) *The image in $I^\perp/I$ of $L \cap I^\perp$, for the standard lattice L of* $\mathrm{Hom}(V', V^- \oplus V'')$ *and the isotropic subspace I of the splitting, is the neighbour lattice; and $Z_f(\mathrm{vac}) = c \sum_{(y,t)} \varrho(n(y) + t)\,\mathrm{vac}$ with $c \neq 0$.*

(b) *There is C such that, for n prime to N and translations x with denominators β on V^- and $\bar{\beta}$ on V'' away from n, the finite evaluation of $\varrho(x)\mathrm{vac}$ is a finite combination of $C \cdot \mathrm{Glue}(\varrho(t_1)\mathrm{vac} \otimes \varrho(t_2)\mathrm{vac})$ with $\bar{\beta}t_1$, $\bar{\beta}t_2$ integral away from n, whose coefficients have a denominator away from n depending only on β and the index sets, and absolute sum at most 1 under every automorphism of $\mathbb{C}$.*

(c) *The hypotheses hold for the middle and the last pairs of the chain $Y_0 \to \cdots \to Y_m$, with κ the last scaling and $\beta = \bar{\pi}_M$.*

Proof. (a) *The zero term of the vacuum.* Let L be the standard lattice of $\mathrm{Hom}(V', V^- \oplus V'')$, I the isotropic subspace of the splitting, with finite frame k, and i the finite embedding of $\mathrm{Hom}(X', X^- \oplus X'')$. The lattice L° of the zero term is the set of w with $iw + kb \in L$ for some b, the image of $L \cap I^\perp$ in $I^\perp/I$. There are a finite unitary point g and a matrix c with $ig + kc$ integral (take a point adapted to the splitting); for such g and x integral, $(ig + kc)x$ is a lattice vector of class gx, and a vector w lies in L° if and only if $g^{-1}w$ is integral. Hence L° is the image of the standard lattice under a unitary point, hence self-dual. The finite zero term of the vacuum is fixed by the translations by L° and is not zero, so it is a nonzero multiple of $\omega(g)\mathrm{vac}$, that is, of the sum of the translates of the vacuum by representatives of L° modulo its integral points.

Identification with the neighbour lattice. Let B be the matrix of $x \mapsto A_\nu x R$, where the columns of A_ν are the lifts to $V^- \oplus V''$ of the columns of the neighbour matrix with their coefficients, and R consists of the rows of the point of V' at $X' \oplus \langle e' \rangle$. The columns of A_ν lie in the lattice by the hypothesis on neighbour vectors and Proposition 18.9(b), and R is integral because the string of Y' has even length, so that the graded basis is the identity on $X' \oplus \langle e' \rangle$; hence B is integral. B^{H} first restricts a map to $X' \oplus \langle e' \rangle$, where the stacked map vanishes, so B^{H} kills the stacked map. In the part of grade zero a lifted column is the column of $\kappa\gamma$ plus an isotropic vector of the planes, so $Bx = i(\gamma\kappa x) + k(\cdot)$: Bx lies in $L \cap I^\perp$ with class $(\gamma\kappa \otimes 1)x$. Hence the neighbour lattice is contained in L°. Both are images of the standard lattice under unitary points, so both are self-dual, and a self-dual lattice contained in another one equals it ($L_1 \subseteq L_2$ gives $L_2 = L_2^\vee \subseteq L_1^\vee = L_1$). Hence L° is the neighbour lattice, and by Proposition 18.9(a) its representatives modulo integral points are the $n(y) + t$.

(b) *Reduction away from N.* The places above N do not divide n, and β is a unit there, so x is integral above N; its part l above N is a lattice vector, and $\varrho(x)\mathrm{vac}$ is a root of unity times $\varrho(x - l)\mathrm{vac}$. Hence we may assume that x vanishes above N.

One translate. The evaluation translates by the stacked map τ and then takes the finite zero term. The translations by τ and x compose to the translation by $\tau + x$ up to a root of unity. If $Z_f(\varrho(\tau + x)\mathrm{vac}) \neq 0$, then $\tau + x = iw + kb + l$ with l integral, and $Z_f(\varrho(\tau + x)\mathrm{vac})$ is a root of unity times $\varrho(w)Z_f(\mathrm{vac})$; the class w differs from $gB^{\mathsf{H}}(\tau + x)$ by an element of L°, so one may take $w = gB^{\mathsf{H}}(\tau + x) = gB^{\mathsf{H}}x$, as $B^{\mathsf{H}}\tau = 0$. Here $g = (\gamma\kappa) \otimes 1$, and since the isotropic part of the lift is supported above N, where x vanishes, $gB^{\mathsf{H}}x = i^{\mathsf{H}}x$ is the orthogonal projection of x. The projection keeps the denominators: B^{H} maps a vector with row denominators $\beta, \bar\beta$ away from n to one with denominators $\beta\kappa$, $\bar\beta\kappa$, because the denominators match through κ (an entry of B joining indices with different valuations of the denominators vanishes at the conjugate place), and g multiplies back by κ. Hence the evaluation of $\varrho(x)\mathrm{vac}$ is 0 or a root of unity times the left inverse of the join applied to $c\,\varrho(x')\sum_{(y,t)}\varrho(n(y) + t)\mathrm{vac}$, with $x' = i^{\mathsf{H}}x$ vanishing above N and of row denominators $\beta, \bar\beta$ away from n.

The sum over the neighbour lattice. The translation $n(y)$ is supported at $\bar P$ and is orthogonal to every vector vanishing above N, so it commutes with the translations by such vectors, and the translation by $n(y) + v$ is the product of the two translations. For each class t, $\varrho(x')\varrho(t)\mathrm{vac}$ is a root of unity times $\varrho(x' + t)\mathrm{vac}$, and the translate of the vacuum by a matrix z is a root of unity times the join of $\varrho(t_1)\mathrm{vac} \otimes \varrho(t_2)\mathrm{vac}$, with t_1 the unit η times the adjoint of the first block of z and t_2 its second block; as the first block of $x' + t$ has denominator β and the second $\bar\beta$ away from n, $\bar\beta t_1$ and $\bar\beta t_2$ are integral away from n. Under the join, $\sum_y \varrho(n(y))$ is the gluing operator of Proposition 16.7: the adjoint times η turns the row $\eta\bar y/N$ into the last column $-y/N$ of the first edge, and the row $-\bar y/N$ is the last row of the second. Hence the evaluation is c times a root of unity times $\sum_t \zeta_t\,\mathrm{Glue}(\varrho(t_1)\mathrm{vac} \otimes \varrho(t_2)\mathrm{vac})$, a sum of K terms with roots of unity ζ_t. Writing it as $\sum_t(\zeta_t/K) \cdot K\,\mathrm{Glue}(\cdots)$ and $C = cK$, the coefficients have absolute sum 1 under every automorphism, and their denominator divides the number of integral matrices modulo β of the shape of $\mathrm{Hom}(X', X^- \oplus X'')$, because K does; this depends only on β and the index sets.

(c) *Middle pairs.* The lift of a neighbour vector (c, x) has the component sx on $X \oplus \langle e \rangle$, with s the idele by which negating the form scales the point, the component x on $X'' \oplus \langle e'' \rangle$, and the coefficient suc at the isotropic vector $z + z''$ formed by the middle vectors. A vector of V with component x on $X \oplus \langle e \rangle$, coordinate $u(\pi^{-1}x_e + c')$ at the middle vector and 0 at the other string vectors lies in the graded lattice as soon as c', $\bar{\pi}x_e$ and the other entries of x are integral, because its coordinates in the graded basis are x off e, $\bar{\pi}x_e$ at v and c' at the middle vector. The congruence $x_e/\pi \equiv c$ makes $c' = c - \pi^{-1}x_e$ integral. On V'' the same test applies with u'' and $\bar{\eta}c$, because $su_{2i} = \bar{\eta}\delta u_{2i+2}$: the scale of Y_{2i} is that of Y_{2i+2} times $a/\bar{a}$ (Proposition 18.2), and $s\, a/\bar{a} = \bar{\eta}\, c/\bar{c} = \bar{\eta}\delta$ for the chosen idele c, the factors α, $\bar{\alpha}$ cancelling. The inclusion of $X^- \oplus X''$ keeps each index in its block, so the denominators match with $\kappa = 1$.

The last pair. Here no plane is removed, and e'' is z/p for the string vector z of V, with $N = p\bar{p}$. The lift of (c, x) has the component sx on $X \oplus \langle e \rangle$, x on X'' and $\delta x_{e''}/p$ at z; in the membership test the terms in c cancel because $\bar{\eta}\eta = -1$, and the remaining entries are integral by the congruences. The only index that changes its block is e'', whose denominator $\bar{\beta}$ is scaled by δ in the last scaling, and $\bar{\beta}\delta$ has the valuations of β since $\delta = c/\bar{c}$ for an idele c with the valuations of β. Finally δ has norm one and is 1 above N, where β is a unit, and the factors 1, δ have the required denominators, so these are scalings. □

18.7. Admissible combinations

For a family of vectors $v(x)$ indexed by a set $\mathcal{X}$, a vector z is an *admissible combination* with data (A, n, d, p, B) if $z = \sum_{x \in A} r(x)v(x)$ with r finitely supported in A, $d\, r(x)$ integral away from n for all x, and, if the proposition p holds, $\sum_x |\sigma r(x)| \le B$ for every automorphism σ of $\mathbb{C}$. The *translations* with denominator d away from q are the finite adelic vectors x with dx integral at the places prime to q.

Proposition 18.11 (Rules for admissible combinations). *Admissible combinations obey the following rules.*

(a) Substitution. *Let z be an admissible combination of the vectors $v(x)$ with data (A, n, d, p, B), and let each $v(x)$, $x \in A$, be one of the vectors $v'(x')$ with data (A', n, d', p, B'). Then z is an admissible combination of the $v'(x')$ with data (A', n, dd', p, BB'). Scalars integral away from n, of absolute value at most 1 under all σ when p holds, linear and bilinear maps, pure tensors and finite sums obey the corresponding rules; a root of unity may be absorbed, and a nonzero scalar may be normalised away.*

(b) Moved vacua. *Let g be a finite unitary point with multiplier c C and A contain all $g(x + z)$, z integral. If $d\, c/|\Delta_f(g)|^2$ is integral away from q and $|\sigma c| \le 1$ when p holds, then $\omega(1, g)\varrho(x)\mathrm{vac}$ is an admissible combination of the $C\varrho(y)\mathrm{vac}$, $y \in A$, with $(A, q, d, p, 1)$. In particular this holds for g integral away from q.*

(c) Functions modulo an ideal. *The datum of a function modulo $\mathfrak{m}$ (Proposition 17.6) is an admissible combination of translates of the vacuum, with denominator the number of residue matrices.*

Proof. (a) Write $z = \sum_x r(x)v(x)$ and $v(x) = \sum_{x'} r'_x(x')v'(x')$. Then

$$z = \sum_{x'} \Big(\sum_x r(x) r'_x(x') \Big) v'(x'),$$

a finite sum over A'. The ring of complex numbers integral away from n (those whose images under all homomorphisms $\mathbb{C} \to \mathbb{C}_l$, $l \nmid n$, are integral) contains $dr(x)$ and $d'r'_x(x')$, hence $dd' \sum_x r(x)r'_x(x')$. If p holds, $\sum_{x'} |\sigma \sum_x r(x)r'_x(x')| \leq \sum_x |\sigma r(x)| \sum_{x'} |\sigma r'_x(x')| \leq BB'$. A linear map sends $\sum r(x)v(x)$ to $\sum r(x)L(v(x))$ with the same coefficients. A scalar c as stated multiplies the coefficients by c, and a single term $cv(x)$ is a combination of size $|\sigma c|$. For a bilinear map, expand z in the first variable and z' in the second: the data are $(A \times A', dd', BB')$; pure tensors follow by induction on the number of factors, the empty tensor being 1. A root of unity ζ is integral and $|\sigma\zeta| = 1$. If $d \neq 0$ and z is a combination of the $Cv(x)$ with data (A, n, d, p, B), then $(d/(aC))\, az = \sum_x dr(x)v(x)$ has denominator 1 and size dB, which is the normalisation (and 0 if $aC = 0$).

(b) Let $\Lambda = \hat{O}^n$ and $H = \Lambda \cap g^{-1}\Lambda$, of index $|\Delta_f(g)|^2$ in Λ. The Weil operator of g sends the vacuum to $\Delta_f(g)|\Delta_f(g)|^{-2} \sum_{z \in \Lambda/H} \varrho(gz)\mathrm{vac}$ (Lemma 17.1), and $\omega(1, g)\varrho(x) = \varrho(gx)\omega(1, g)$; the translations by gx and gz compose to the translation by $g(x + z)$ up to a root of unity ζ_z, a value of the standard character on a rational multiple. Hence

$$\omega(1, g)\varrho(x)\mathrm{vac} = \sum_{z \in \Lambda/H} \zeta_z \frac{c}{|\Delta_f(g)|^2} C\varrho(g(x + z))\mathrm{vac},$$

with $|\Delta_f(g)|^2$ terms; d times each coefficient is integral away from q by hypothesis, and as $|\Delta_f(g)|^2$ is a natural number, the absolute sum of the σ-conjugates is $|\sigma c| \leq 1$ when p holds. If g is integral at the places prime to q, take $C = 1$, $c = \Delta_f(g)$: $\Delta_f(g) = v(\det u)|\det u|^{-1/2}$ for a matrix u with $u\hat{O}^n = \hat{O}^n \cap g^{-1}\hat{O}^n$, and above a prime $l \nmid q$ the point g and its inverse are integral, so u may be taken 1 there and $\det u$ is a unit above l; hence $\Delta_f(g)$ and its conjugates are units away from q, and $c/|\Delta_f(g)|^2 = 1/\overline{\Delta_f(g)}$ is integral away from q. If $q = 1$, then g is integral everywhere and $\Delta_f(g) = 1$.

(c) Expand the function Ψ modulo $\mathfrak{m}$ in indicators of points. The datum of the indicator of y_0 is the mean, over the vectors modulo $\mathfrak{m}$, of roots of unity times the translates of the integral vacuum by the vectors $\bar{g}^{-1}$ times integral vectors, supported at the places of $\mathfrak{m}$ (finite Fourier inversion, Appendix B.10, (W6)). Hence the coefficients are $\Psi(y_0)$ times roots of unity divided by the number of vectors modulo $\mathfrak{m}$, whose absolute sums are at most $\sum|\sigma\Psi|$, and multiplied by that number they are integral away from q. □

Theorem 18.12 (The evaluated data are glued vacua). *There is C such that for every tame character ϑ and all points y_k of $\mathrm{U}(Y_k)(\mathbb{A}_\mathrm{f})$ integral at the places prime to q, the pure tensor $\phi_0(y) \otimes \cdots \otimes \phi_\varrho(y)$ is an admissible combination of the vectors $C\,\mathrm{Glue}(x)$, where x runs over the tuples of translations with denominator $\bar{\pi}_M$ away from q (the admissible translations), with denominator $(M^{fn_\mathfrak{m}^2})^{2m}$ away from q and, if $Q = \emptyset$, size 1.*

Proof. The data. Let $\mathfrak{m}$ be the modulus of the twist, the product of the chosen places and the places of Q; it divides $\pi_M q$. The datum Υ_{k+1} is the Schrödinger datum of the indicator of the class of the inclusion matrix modulo $\mathfrak{m}$, applied

to the vacuum; by Proposition 18.11(c), with the factor π_M in the role of g, it is a combination of translates $\varrho(t)$vac with $\bar{\pi}_M t$ integral away from q, whose coefficients times the number of matrices modulo π_M are integral away from q, and whose size is 1 (the indicator of one class has $\sum|\sigma\Psi| = 1$), whatever the proposition p.

Moving. The pair (y_k, y_{k+1}) acts on $\mathrm{Hom}(Y_k, Y_{k+1})$ as a scalar c times the Weil operator of one finite point g, integral away from q as the y are. The scalar is a quotient of values of the characters of the lift at the determinants of y_{k+1} and y_k; these characters have a weight, so at a finite point integral above $l \nmid q$ their values are l-adic units, and if $Q = \emptyset$ the points are integral everywhere and the values are products of values of the tame parts, roots of unity, of absolute value 1 under every automorphism. Expanding the datum in translates and moving each by g (Proposition 18.11(b)), the translated datum is an admissible combination of translates of the vacuum with the same denominators and, if $Q = \emptyset$, size 1.

The join. The join is bilinear, and the join of $\varrho(t_1)$vac and $\varrho(t_2)$vac is a root of unity times $\varrho(z)$vac, where the first block of z is minus the unit of norm -1 times the adjoint of t_1 and the second block is t_2; if $\bar{\pi}_M t_1$ and $\bar{\pi}_M t_2$ are integral away from q, the rows of z have denominator π_M on the block of Y_{2i} and $\bar{\pi}_M$ on that of Y_{2i+2}. Hence by the bilinear rule the join of the two translated data of the pair i is a combination of translates $\varrho(z)$vac with these denominators, whose denominator is the product of the numbers of matrices modulo π_M of the two shapes, and of size 1 if $Q = \emptyset$.

The evaluation. No place of Q lies above N, so q is prime to N, and Theorem 18.10(b),(c) applies with $n = q$ and $\beta = \bar{\pi}_M$: each translate of the joined vacuum goes to a combination of the vectors $C\,\mathrm{Glue}(t_1, t_2)$ with $\bar{\beta} t_1, \bar{\beta} t_2$ integral away from q, with denominator the number of integral matrices modulo β of the shape of $\mathrm{Hom}(X_{2i+1}, X_{2i}^- \oplus X_{2i+2})$ and size 1. By substitution, $\phi_i(y)$ is a combination of the $C\,\mathrm{Glue}(t_1, t_2)$ whose denominator is the product of the three numbers of residue matrices; all three are powers of $\mathrm{N}(\pi_M) = M^f$, with exponents the numbers of entries, which add up to at most $4n_m^2$ (the third shape has at most $2n_m^2$ entries, as all ranks are at most n_m), so the product divides $(M^{f n_m^2})^4$.

The tensor. By the rule for pure tensors, $\phi_0(y) \otimes \cdots \otimes \phi_\varrho(y)$ is a combination of the tensors of glued translates with denominator $(M^{f n_m^2})^{4(\varrho+1)} = (M^{f n_m^2})^{2m}$ and, if $Q = \emptyset$, size 1. The tensor of the glued translates of the vacua is a fixed constant (a product of the factors relating the integral vacuum of each edge to its vacuum) times $\mathrm{Glue}(x)$ for the tuple x of translations, which gives the constant C of the statement. □

18.8. The seesaw identity

For $j < m$ and a conjugate form z on Y_{j+1}, the *descent* of z is $[J_j^\circ : J_j]$ times the theta integral $g \mapsto \int_{[\mathrm{U}(Y_{j+1})]} \theta(\omega(g,h)(\varphi_{\mathrm{harm}} \otimes \Upsilon_{j+1}))\, z(h)\, dh$; it has level J_j, because the datum is fixed by J_j, and lift and descent are transposes: $\int z \cdot L_j f = \int (\text{descent } z) \cdot f$. Conjugate forms on Y_{j+1}, $j < m$, are rapidly decreasing. At the top, restriction to Y_m is read in automorphic forms.

Proposition 18.13 (The outer integrals). (a) *For $k \le 2\varrho+1$, a grading of Y_{k+1} and a conjugate form z on Y_{k+1}, the period $\Xi_{k+1} = \int_{R_{k+1}(E^+)\backslash \mathrm{U}(Y_{k+1})(\mathbb{A})} \mathcal{W}_{k+1} z$ converges, and $\int_{[R_k]} \mathcal{W}_k(rg_k)\,(\text{descent } z)(rg_k)\,dr = [J_k^\circ : J_k]\,\Xi_{k+1}$.*

(b) $\int_{[\mathrm{U}(Y_1)]} z \cdot L_0(\mathcal{P}\varphi) = [J_0^\circ : J_M] \sum_{c\in\Omega_0} \Xi_1(g,c)$.

(c) *For every function a of the archimedean outer variables there is c such that, whenever $\Xi_{k+1} = a(\xi)S(y)$ for all y with $y_0 \in J_M$ and $y_m = 1$, with S a function of the classes of y and φ ramified,*

$$\int_{[\mathrm{U}(Y_{k+1})]} z \cdot L_k \circ \cdots \circ L_0(\mathcal{P}\varphi) = c\,\Sigma_{k+1} S.$$

(d) *For a linear form t on the values of forms on X_m there is C such that, if $F(\iota k) = t(\psi(\mathrm{diag}(k,1)))$ on $\mathrm{U}(Y_m)$ for a form ψ on X_m, then $\int_{[\mathrm{U}(Y_m)]} F\bar{x} = C\langle x, \mathrm{res}\,\psi\rangle$ for every form x on Y_m.*

Proof. (a) The descent of z is $[J_k^\circ : J_k]$ times the theta integral of z against the datum of lift $k+1$, because this datum is an arrow for the swapped kernel. Conjugate forms on Y_{k+1} are rapidly decreasing; this does not depend on the twist, and is read off from the trivial twist, whose forms are rapidly decreasing on the source. Hence the second identity of Proposition B.27, applied with W the function of stage k at rg_k, the datum of lift $k+1$ moved by g_k and $y = z$, gives

$$\int_{[R_k]} \mathcal{W}_k(rg_k)\,(\text{descent } z)(rg_k)\,dr = [J_k^\circ : J_k] \int_{R_{k+1}(E^+)\backslash \mathrm{U}(Y_{k+1})(\mathbb{A})} \mathcal{W}_{k+1}(h)\,z(h)\,dh,$$

both sides converging absolutely; the function $\mathcal{W}'$ of that proposition is $\mathcal{W}_{k+1}$, since moving the datum by g_k is the same as evaluating the constant term at rg_k.

(b) Lift and descent are transposes, so the left side is $\int_{[\mathrm{U}(Y_0)]} z' \cdot \mathcal{P}\varphi$ with z' the descent of z, a form of level $J_0 \subseteq J_M$. By Proposition 16.11,

$$[J_M : J_0] \int_{[\mathrm{U}(Y_0)]} z' \cdot \mathcal{P}\varphi = \sum_{\gamma \in J_M/J_0} \int_{[R_0]} \mathcal{W}_\varphi(r)\,z'(rg\gamma)\,dr,$$

and by (a) each term is $[J_0^\circ : J_0]\,\Xi_1(g,\gamma)$; the period depends only on the class of γ modulo J_0, as z' has level J_0. The classes of J_M/J_0 are the classes of Ω_0. Since $[J_0^\circ : J_0]/[J_M : J_0] = [J_0^\circ : J_M]$, the claim follows.

(c) Induction on k, for all functions a and S. For $k = 0$, (b) and the hypothesis give $[J_0^\circ : J_M]\,a(\xi) \sum_{c\in\Omega_0} S(c)$, and the multiplicities on Ω_0 are 1. For the step from k to $k+1$, write the pairing of z with $L_{k+1} \circ \cdots \circ L_0(\mathcal{P}\varphi)$ as the pairing of the descent z' of z with $L_k \circ \cdots \circ L_0(\mathcal{P}\varphi)$. To apply the induction hypothesis to z', compute the period $\Xi'_{k+1} = \int \mathcal{W}_{k+1} z'$ by Proposition 18.8(b) for the cut Ω_{k+1} and $f = \mathcal{W}_{k+1} z'$. The function f is continuous, invariant under the rational points of the Bessel group and integrable by (a); it is supported on points of bounded orbit by Theorem 18.7(b), where $m_{k+1} \neq 0$; and by (a), with t as the outer variable of index $k+1$, its period over R_{k+1} at t is $[J_{k+1}^\circ : J_{k+1}]$ times Ξ_{k+2} at $(\xi, t_\infty; y, t_f)$, which by hypothesis is $a(\xi, t_\infty)S(y, t_f)$. Therefore

$$\Xi'_{k+1} = a'(\xi) \sum_{t\in\Omega_{k+1}} m_{k+1}(t)^{-1} S(y,t),$$

with $a'(\xi) = \kappa$ the constant of Proposition 18.8(b) for the function $t_\infty \mapsto a(\xi, t_\infty)$, which does not depend on ϑ, φ or z. The induction hypothesis for z', a' and the weighted sum of S over Ω_{k+1} gives the claim for $k+1$, with the cut sum $\Sigma_{k+2}S$.

(d) The space Y_m is definite and its grading has an empty string, so its Bessel group is all of $\mathrm{U}(Y_m)$ and $F(\iota k) = t(\psi(\mathrm{diag}(k, 1)))$ at every point. The space of forms on Y_m is finite-dimensional, the quotient $[\mathrm{U}(Y_m)]$ is a finite union of quotients of the compact real group, and $\bar{x}$ is the pairing of the algebraic form of x with the translates of a highest weight vector. Hence the integral is the mean over the finitely many finite classes of an integral over the compact real group of a product of matrix coefficients, which by Schur orthogonality is the pairing divided by the dimension. The map $v \mapsto (k \mapsto t(\psi(\mathrm{diag}(k, 1))))$ is equivariant for the block group, so it is a multiple of the branching map by Theorem B.23(c), and the restriction is the branching map divided by a fixed constant. Hence $\int_{[\mathrm{U}(Y_m)]} F\bar{x} = C\langle x, \mathrm{res}\,\psi\rangle$ with C depending only on t. □

The *evaluated tensor* at a tuple c of classes is $\phi_0(y) \otimes \cdots \otimes \phi_\varrho(y)$ for representatives y of c, and the *seesaw datum* is the cut sum $\phi_\vartheta = \Sigma_m$(evaluated tensor), a tensor of finite data for the chain $X_0 \to \cdots \to X_m$.

Proposition 18.14. (a) *ϕ_ϑ is jointly invariant under the levels of $X_0, \ldots, X_{m-1}$.*

(b) *There is c_0 such that $\iota_\vartheta(\mathcal{P}\varphi) = c_0\,\mathrm{res}\big(\mathrm{chain}(\phi_\vartheta)\varphi\big)$ for every twist and every ramified φ.*

(c) *There is C such that for every ϑ, ϕ_ϑ is an admissible combination of the $C\,\mathrm{Glue}(x)$ over the admissible translations, with denominator $(M^{fn_m^2})^{2m}$ away from q and, if $Q = \emptyset$, size $(M^{fn_m^2})^{2\varrho+2}$.*

Proof. (a) Let b be a tuple with b_j in the level of X_j for $j < m$ and $b_m = 1$, and let h_j be the image of b_j in $\mathrm{U}(Y_j)(\mathbb{A}_\mathrm{f})$. By Section 18.5 the translation by h_j permutes Ω_j and keeps the multiplicities, so the cut sum of the evaluated tensor at the translated classes hc equals the cut sum at c. With representatives $h_j y_j$ the translated data of lift $j+1$ are moved by $\omega(h_j, h_{j+1})$, and the finite evaluation carries $\omega(\iota b_{2i}, \iota b_{2i+1}, \iota b_{2i+2})$ on the joined data to the joint action of $(b_{2i}, b_{2i+1}, b_{2i+2})$ on the two factors of ϕ_i (Section 18.2). Hence the evaluated tensor at hc is the joint action of b on the evaluated tensor at c, and ϕ_ϑ is fixed by the joint action of b.

(b) Pair both sides with a form x on Y_m. By Theorem 18.4 for $i = \varrho$, with $y_m = 1$, the function $\mathcal{W}_m$ at $\iota(g)$ is $l(\xi)$ of $(\Theta_{\varrho+1}(y)\varphi)(\iota g)$, where $\Theta_{\varrho+1}(y)\varphi = \mathrm{chain}(\phi(y))\varphi$ for the evaluated tensor $\phi(y)$; by Proposition 18.13(d) its period against $\bar{x}$ is $C(l(\xi))\,\langle x, \mathrm{res}\,\mathrm{chain}(\phi(y))\varphi\rangle$, and this depends on y only through its classes, as the evaluated data do not change when each y_k is replaced by a representative of its class. Hence the hypothesis of Proposition 18.13(c) holds with $k+1 = m$, $a(\xi) = C(l(\xi))$ and $S(y) = \langle x, \mathrm{res}\,\mathrm{chain}(\phi(y))\varphi\rangle$, and

$$\int_{[\mathrm{U}(Y_m)]} \bar{x} \cdot \iota_\vartheta(\mathcal{P}\varphi) = c\,\Sigma_m S = c\,\big\langle x, \mathrm{res}\,\mathrm{chain}(\phi_\vartheta)\varphi\big\rangle,$$

because the cut sum commutes with the linear map $\phi \mapsto \langle x, \mathrm{res}\,\mathrm{chain}(\phi)\varphi\rangle$ (Proposition 16.1(b)). The constant c does not depend on ϑ, φ or x. As the inner product on forms on Y_m is nondegenerate and x is arbitrary, the two forms agree up to the constant c_0.

(c) By Theorem 18.12, the value of the evaluated tensor at a tuple of points integral away from q, in particular at representatives of cut classes, is an admissible combination of the $C\,\mathrm{Glue}(x)$ with denominator $(M^{fn_m^2})^{2m}$ and, if $Q = \emptyset$, size 1. The cut sum is an admissible combination of these values: its coefficients are products of the weights $m_j(c_j)^{-1}$, and since m_j divides a power of q, $1/m_j$ is integral away from q, so the denominator is 1. If $Q = \emptyset$, then $q = 1$, the multiplicities are 1, and each of the m cuts $\Omega_0, \dots, \Omega_{m-1}$ has at most $M^{fn_m^2}$ elements (Section 18.5); the cut sum therefore has size at most $(M^{fn_m^2})^m$, and $m = 2\varrho + 2$. Substitution multiplies the denominators and the sizes. □

Theorem 18.15 (Seesaw identity). *Let t be a linear form on the values of forms on X_0 and g an archimedean point of $\mathrm{U}(Y_0)$. There is $c \neq 0$ such that every twist ϑ has a chain datum Ψ with*

$$\iota_\vartheta(\mathcal{P}_{g,t}\varphi) = c\, B_{\vartheta,\Psi}(\varphi) \qquad \text{for every ramified source form } \varphi.$$

Proof. Let c_0 and C be as in Proposition 18.14, $B = M^{fn_m^2}$, and let $u = B^{2m}/(c_0 C)$ if $c_0 C \neq 0$ and $u = 1$ otherwise. Write $\phi_\vartheta = \sum_x r(x)\, C\,\mathrm{Glue}(x)$ as in Proposition 18.14(c), with $B^{2m} r(x)$ integral away from q. If $c_0 C \neq 0$, then

$$u\, c_0 \phi_\vartheta = \sum_x B^{2m} r(x)\,\mathrm{Glue}(x)$$

is an admissible combination of the $\mathrm{Glue}(x)$ with denominator 1 away from q and, if $Q = \emptyset$, size $B^{2m} B^{2\varrho+2} = M^{3fmn_m^2}$, since $m = 2\varrho + 2$; if $c_0 C = 0$, then $u\, c_0 \phi_\vartheta = 0$, which is the combination with all coefficients 0. Each admissible translation may be replaced by one supported at the places dividing Mq with the same glued vacuum: at the other places the translation is integral, as $\bar{\pi}_M$ is a unit there and the translation is integral away from q, and the vacuum is fixed by the integral translations. Collecting the coefficients of equal translations gives a finitely supported function Ψ on tuples of translations. It is a chain datum (Definition 16.6): its support consists of admissible translations supported at the level, its values are integral away from q and bounded by $M^{3fmn_m^2}$ in total if $Q = \emptyset$, and its glued tensor $u\, c_0 \phi_\vartheta$ is jointly invariant by Proposition 18.14(a). By Proposition 18.14(b) and the linearity of the chain in its tensor, $u\, \iota_\vartheta(\mathcal{P}\varphi) = \mathrm{res}(\mathrm{chain}(u\, c_0 \phi_\vartheta)\varphi) = B_{\vartheta,\Psi}(\varphi)$, so the theorem holds with $c = u^{-1}$, which is not zero and does not depend on ϑ. □

19. Divisibility of the composite lift

The seesaw identity turns the integrality of the target forms $\mathfrak{p}_g\varphi$ into p-adic bounds for the composite lifts $B_\vartheta\varphi$. This section proves those bounds (Theorem 19.13): for every prime p the composite lift of every chain datum is divisible by $p^{g_p - \epsilon_p}$, with the *gain* g_p asked by the prime sums and an error ϵ_p, on a submodule of the integral ramified source forms whose colength is at most $\epsilon_p \dim \mathcal{S}$; and $\sum_p \epsilon_p \log p \leq \epsilon_{\mathrm{u}}$. Four kinds of primes behave differently. A split prime, or a prime $p \geq x_1$, asks for nothing, and the composite lift is integral. A non-split prime $p < x_1$ asks for the local part of the prime sums, which the derivatives of theta functions provide (Part II) up to the *collisions*, pairs of a-places linked by $-p^e$. If p is not *exceptional* these cost little; if it is,

scaled Hecke operators at the primes above p are applied to the source once per lift, and the theorem on elementary divisors (Section 21) bounds the colength of their image. At the level prime N the lifts lose a bounded number of powers of N, after the operator Up of Section 16.2 has been applied $m/2$ times.

Throughout, p is a prime and $\iota = \iota_p : \mathbb{C} \to \mathbb{C}_p$ an embedding; $\mathcal{O}_\iota$ is the ring of complex numbers that ι maps to p-adic integers. The *gauge* of a form φ is the supremum over g of the gauges of $\varphi(g)$ at g (Section 17.3); φ has *valuation at least* v if its gauge is at most p^{-v}, and is *integral* if its gauge is at most 1. A submodule S of an $\mathcal{O}_\iota$-module L has *colength dividing* z if every finite family in L is carried into S by a square matrix over $\mathcal{O}_\iota$ whose determinant divides z. An endomorphism Y of L is *onto modulo* p^j *up to colength* c if $YL \subseteq L$ and $YL + p^j L$ has colength dividing p^l in L for some integer $l \le c$.

19.1. Gains and losses

Recall from Section 8 the a-places Φ_a (the places of class 1 modulo q_0), the weight κ, the partner exponents I_p and the valuation $v(\pi_p)$ of a non-split prime (Section 12.5), and write $u(\nu)\tau$ for the product of τ with the place named by an integer ν. For a prime p, a real y and a place σ, the *local prime sum* and the *gain* are

$$s_p(y,\sigma) = v(\pi_p) \sum_{e \in I_p} \Big(\frac{y}{p^e} - \kappa\big(u(-p^e)\sigma\big) \Big)_+ ,$$

$$g_p = m \Big(\sum_{\tau \in \Phi_a} s_p(a,\tau) + k \sum_{\sigma \in \Phi} s_p(\kappa(\sigma),\sigma) \Big).$$

A prime is *small non-split* if $p < x_1$ and $\chi_K(p) \neq 1$. For such p the *loss* is $\lambda_p = fa(\log b)^2/p$, multiplied by m if $p \equiv \pm 1 \pmod m$. Let r_p be the least integer with $m \le p^{\mathrm{r}_p}$ and $o_0(p)$ the order of $-p$ in $(\mathbb{Z}/q_0)^\times$. A small non-split p is *exceptional* if $\chi_K(p) = -1$, $p \not\equiv \pm 1 \pmod m$, and $o_0(p)$ is odd and at most r_p. A *collision* outside a set S of places is a pair (τ, e) with $e \in I_p$ such that τ and $u(-p^e)\tau$ are a-places outside S; the *collision sum* $\mathrm{col}_p(S)$ is the sum of a/p^e over the collisions outside S. The *allowed loss* is $\epsilon_p = 2\lambda_p$ for small non-split p, $\epsilon_N = 2mfb$, and $\epsilon_p = 0$ otherwise. Finally put $j_p = \lceil 2mfa/p \rceil$ and $\nu_p = \mathrm{r}_p f(n_m + \mathrm{r}_p)$.

Proposition 19.1 (Gains add up to the prime sums). *Let p be a prime.*

(a) $s_p(y,\sigma) = \sum_{e \ge 1} \frac{1-\chi_K(p^e)}{2} \big(y/p^e - \kappa(u(-p^e)\sigma)\big)_+$, *and* $s_p(y,\sigma) = 0$ *for* $p \ge x_1$ *and* $y \le a$. *Hence* $g_p = 0$ *if p splits in K or $p \ge x_1$.*

(b) *For* $y \le a$ *and every* $U \ge x_1$, $s_{\mathrm{fin}}(y,\sigma) = \sum_{p<U} 2 s_p(y,\sigma)\log p$; *consequently*

$$2 \sum_{p \text{ small non-split}} g_p \log p = m \sum_{\tau \in \Phi_a} s_{\mathrm{fin}}(a,\tau) + mk \sum_{\sigma \in \Phi} s_{\mathrm{fin}}(\kappa(\sigma),\sigma).$$

(c) *For every finite set U of integers,* $\sum_{p \in U} \epsilon_p \log p \le \epsilon_{\mathrm{u}}$.

(d) $\lambda_p \ge fa(\log b)^2/p$, *and if p is small non-split and not exceptional,* $m\,\mathrm{col}_p(\emptyset) \le \lambda_p$.

Proof. (a) If p is inert, I_p is the set of odd e, $v(\pi_p) = 1$, and $(1 - \chi_K(p^e))/2$ is 1 for odd and 0 for even e. If $p = d$, I_p is the set of all $e \ge 1$, and $v(\pi_p) = 1/2 = (1-\chi_K(d^e))/2$. If p splits, I_p is empty and $\chi_K(p^e) = 1$ for all e, so both sides vanish.

Every weight is at least a/x_1; so for $p \geq x_1$ and $y \leq a$, $y/p^e \leq a/x_1 \leq \kappa(u(-p^e)\sigma)$, and every term vanishes. As g_p is built from $s_p(a,\cdot)$ and $s_p(\kappa(\sigma),\cdot)$ with $\kappa(\sigma) \leq a$, it vanishes for split p and for $p \geq x_1$.

(b) The terms of $s_{\mathrm{fin}}(y,\sigma)$ are indexed by the integers $\nu < x_1$ with $\Lambda_K(\nu) \neq 0$, which are prime powers $\nu = p^e$, with $\Lambda_K(p^e) = (1 - \chi_K(p^e))\log p$. Grouping them by p and using (a) gives $s_{\mathrm{fin}}(y,\sigma) = \sum_p 2s_p(y,\sigma)\log p$, the primes $p \geq x_1$ contributing 0; so the sum may be taken over $p < U$ for every $U \geq x_1$. Summing over $\tau \in \Phi_a$ with $y = a$ and over all σ with $y = \kappa(\sigma)$ and the factor k, and multiplying by m, gives the identity, since $g_p = 0$ unless p is small non-split.

(c) Put $T = 4fa(\log b)^2 \log x_1$. Each term $\epsilon_p \log p$ is at most T/c for an integer $1 \leq c \leq x_1$ attached to p, and each c is attached to at most four numbers, namely N, $mc-1$, c and $mc+1$. For the level, $c = 1$: $2mfb\log N \leq T$ by the sizes of Proposition 8.9. For a small non-split $p \not\equiv \pm 1 \pmod m$, take $c = p$: $2\lambda_p \log p = 2fa(\log b)^2 \log p/p \leq T/p$, as $\log p \leq \log x_1$. For $p \equiv \pm 1 \pmod m$, write $p = mc \pm 1$ with $1 \leq c \leq p$; then $m \leq 2p/c$, and $2\lambda_p \log p = 2mfa(\log b)^2 \log p/p \leq 4fa(\log b)^2 \log x_1/c = T/c$. For all other p, $\epsilon_p = 0$. Hence

$$\sum_{p\in U} \epsilon_p \log p \leq 4T \sum_{1\leq c\leq x_1} \frac{1}{c} \leq 4T(1+\log x_1) \leq fa(\log b)^3 = \epsilon_{\mathrm{u}},$$

the last inequality because $\log x_1$ is small against $\log b$ (Proposition 8.9).

(d) The first claim holds because the factor m, when present, is at least 1. A collision (τ, e) has $p \neq d$: for $p = d$ the place $u(-d^e)\tau$ is not an a-place when τ is, because -1 and d have classes of opposite type modulo q_0. Hence e is odd, and for odd e the place $u(-p^e)\tau$ is an a-place exactly when $o_0(p) \mid e$, as $u(-p^e)$ preserves the class 1 modulo q_0 exactly then. If every collision has $e \geq o$, then at each of the f places the collision sum is at most a geometric series with first term a/p^o and ratio at most $1/2$, so $\mathrm{col}_p(\emptyset) \leq 2fa/p^o$. If $p \equiv \pm 1 \pmod m$, take $o = 1$: $m\,\mathrm{col}_p(\emptyset) \leq 2mfa/p \leq \lambda_p$, because $(\log b)^2 \geq 2$ and λ_p carries the factor m. Otherwise, as p is not exceptional, either $\chi_K(p) = 0$, so that $p = d$ and there are no collisions, or $o_0(p)$ is even, so that no odd e is divisible by it and there are again no collisions, or $o_0(p) > \mathrm{r}_p$, so that every collision has $e \geq \mathrm{r}_p + 1$. In the last case $\mathrm{col}_p(\emptyset) \leq 2fa/p^{\mathrm{r}_p+1} \leq 2fa/(mp)$, since $p^{\mathrm{r}_p} \geq m$, and $m\,\mathrm{col}_p(\emptyset) \leq 2fa/p \leq \lambda_p$. □

19.2. Chains as plain sums

For a weak chain datum Ψ (Definition 16.6) the *restricted chain* $R_{\vartheta,\Psi}$ is the combination, with coefficients $\Psi(x)$, of the chains of the m single lifts $X_0 \to \cdots \to X_m$ with data the translated vacua of x, followed by the restriction to Y_m.

Proposition 19.2 (Expanding the gluing). *Let Ψ be a chain datum for a twist ϑ.*

(a) *The glued tensor of Ψ is $\sum_x \sum_y \Psi(x) \bigotimes_j \varrho(x_j + c_j(y))\mathrm{vac}_j$, where y runs over the gluing classes (a residue modulo N for each double lift, middle index and place above π_N), at most $N^{fmn_m/2}$ of them, and the tuples $c(y)$ vanish off the places above N.*

(b) *Hence there is a weak chain datum Ψ' with $B_{\vartheta,\Psi} = R_{\vartheta,\Psi'}$ on all source forms.*

(c) *The projection on the vacuum above N takes the glued tensor to the plain tensor; so the plain tensor of Ψ is jointly invariant and Ψ itself is a weak chain datum.*

Proof. (a) The translations of the support vanish at the places prime to Mq, hence above N, because no place above N divides M or q. The gluing of the double lift i is the sum over the vectors y on X_{2i+1}, with a residue modulo N at each place $w \mid \pi_N$, of the translates of $\Phi_{2i} \otimes \Phi_{2i+1}$ by the matrix with last column $-y/N$ and by its conjugate last row (Section 16.3). For a translated vacuum $\varrho(x_j)\mathrm{vac}_j$ with x_j vanishing above N, translating further by a matrix supported above N gives the translate by the sum, with no phase, because the Heisenberg cocycle of two vectors with disjoint supports is 1. Expanding each of the $m/2$ gluings and the tensor product of the sums gives the formula, with $c(y)$ the tuple that has the last column $-y_i/N$ in the factor $2i$, the conjugate last row in the factor $2i+1$, and 0 elsewhere. Each double lift has at most $N^{f n_m}$ classes (a residue modulo N at each of the f places dividing π_N for each of at most n_m indices), so there are at most $N^{f m n_m/2}$ classes in all.

(b) Collect the terms of (a): $\Psi'(z) = \sum_{x+c(y)=z} \Psi(x)$. Its support consists of tuples $x + c(y)$, which are admissible for qN, because $c(y)$ vanishes at the places not above N; its values are sums of values of Ψ, hence integral away from q, and if $Q = \emptyset$, its absolute sum is at most $N^{f m n_m/2}$ times that of Ψ, which is at most $\exp(x^{60})$ by Proposition 8.9. The plain tensor of Ψ' is the glued tensor of Ψ, which is jointly invariant, so Ψ' is a weak chain datum, and $B_{\vartheta,\Psi} = R_{\vartheta,\Psi'}$ because the two chains have the same tensor of data.

(c) The projection on the vacuum above N is the mean of the Heisenberg translations by the integral vectors supported above N. It fixes $\varrho(x_j)\mathrm{vac}_j$ when x_j is integral above N, since these translations then commute with $\varrho(x_j)$ up to the cocycle, which is 1, and fix the vacuum. It kills a translate by a matrix with a pole above N: if the column $-y/N$ is integral at the places above N, then $y = 0$. Hence in the expansion (a) only the term $y = 0$ survives, and the projection takes the glued tensor to the plain tensor. The levels are integral above N, so their actions commute with the projection; hence the plain tensor is jointly invariant, and Ψ is a weak chain datum. □

Proposition 19.3 (Gauges of chains). *Let Ψ be a weak chain datum for ϑ, with $p \nmid \mathrm{N}\, Q(\vartheta)$.*

(a) *If the tensor $\sum_x \Psi(x) \bigotimes_j \Phi_{x,j}$ is jointly invariant, the value at g_m of its chain on a source form φ is the sum, over tuples of classes of the levels and over x, of $\Psi(x)$ times the kernel product (Proposition 16.1(c)).*

(b) *Let t_j be operators on the data commuting with the actions of the levels, and d_j base points. If each $t_j(\varrho(x_j)\mathrm{vac}_j)$ raises gauges from d_j to d_{j+1} by at most p^{r_j} and φ has gauge at most 1 at d_0, the chain with data $t_j(\varrho(x_j)\mathrm{vac}_j)$ has gauge at most $p^{\sum_j r_j}$ at d_m.*

(c) *For $p \neq N$ restriction to Y_m does not increase gauges; for every σ it multiplies squared norms by at most $\exp(x^{70})N^{-\sum_\tau n_\tau}$.*

(d) *For $p \neq N$ the translated vacua of the support are standard data of level $\bar{\pi}_M$ at p (of every level if $p \nmid M$).*

Proof. (a) The levels of the stages $j < m$ are free, so by Proposition 16.1(c) the value of the chain at g_m is the sum over tuples of double cosets of the kernel products applied to the value of φ at the first point; the chain is linear in the tensor, which gives the sum over x with the coefficients $\Psi(x)$.

(b) The changed data $t_j(\varrho(x_j)\mathrm{vac}_j)$ form again a jointly invariant tensor, because the t_j commute with the actions of the levels: apply the invariance of the original tensor to the image under $\bigotimes_j t_j$. Hence (a) applies to the changed chain. In each kernel product the factors raise gauges from d_j to d_{j+1} by at most p^{r_j}, and the value of φ has gauge at most 1 at d_0; so the product has gauge at most $p^{\sum_j r_j}$ at d_m. The coefficients $\Psi(x)$ are integral at p, as $p \nmid \mathrm{N}\,Q(\vartheta)$, and the gauge is ultrametric, so the whole sum obeys the same bound.

(c) The restriction is $\prod_\tau \Phi_\tau(\bar\pi_N)^{-n_\tau}$ times the branching map applied to the values at $\mathrm{diag}(h,1)$. The branching map has integral entries (Theorem B.23), so it does not increase gauges, and it intertwines the action of $\mathrm{diag}(g,1)$ with that of g; for $p \neq N$ the number $\iota\Phi_\tau(\bar\pi_N)$ is a p-adic unit, because $\pi_N\bar\pi_N = N$. Hence a bound on the gauges of a form at every point gives the same bound for its restriction. For the norms, $|\sigma\Phi_\tau(\bar\pi_N)|^2 = N$ for every σ, so the scalar has squared modulus $N^{-\sum_\tau n_\tau}$, and the branching matrix, which commutes with σ as its entries are integers, multiplies squared Gram norms by at most $\exp(x^{70})$ (Proposition 16.5).

(d) The support condition of a weak chain datum says that $\bar\pi_M x_j$ is integral at the places prime to qN; for $p \neq N$ with $p \nmid q$ the places above p are among them. Hence $\varrho(x_j)\mathrm{vac}_j$ is a translate of the integral vacuum by a vector that $\bar\pi_M$ makes integral above p, a standard datum of level $\bar\pi_M$. If $p \nmid M$, then $\bar\pi_M$ is a unit above p, so x_j is integral there and the datum is standard of every level. □

19.3. The divisibility bound of one lift

For $j < m$ and a set S of places, the *divisibility bound* $\beta_p(j,S)$ is the predicted divisibility D_p of the diagrams of stage j (read at heights below n_{j+1}) with the columns of height one at the places of S removed, plus γ_p for each removed box, minus $f n_{j+1}^3(1+\log_p \max(B,1))$, B the number of boxes before the removal. Let $c_j(\tau)$ and $l_j(\tau)$ be the numbers of columns of height below n_{j+1} and different from 1 at τ, and of boxes in them.

Theorem 19.4 (Only collisions lose). *Let p be small non-split and $j < m$.*

(a) *A standard datum of level one at p raises gauges by at most $p^{-\beta_p(j,\emptyset)}$ under the kernel of lift $j+1$.*

(b) *For every set S of places,*

$$\beta_p(j,S) \;\geq\; \frac{g_p}{m} - \mathrm{col}_p(S) + v(\pi_p)\,\#(\Phi_a \cap S)\frac{a}{x_1} - \frac{f a(\log b)^2}{mp}.$$

Proof. (a) The datum is standard of level one at p, and the weight of lift $j+1$ has the diagrams of stage j read at heights below n_{j+1}; Theorem 17.5(d) gives the bound $p^{-D_p + f n_{j+1}^3(1+\log_p \max(B,1))} = p^{-\beta_p(j,\emptyset)}$.

(b) Put $u = a/(pm)$, $v = v(\pi_p) \in (0,1]$, $L = \ell_S$ the long row (a at the a-places outside S, 0 elsewhere), $c = c_j$ and $l = l_j$. By the definition of D_p (Theorem 12.12), the predicted divisibility of the reduced diagrams is $v\sum_\tau \sum_{e\in I_p} t_e(\tau)$, less a correction of at most $2u$ per place when $p = d$, where

$$t_e(\tau) = \Big(\frac{L(\tau)}{p^e} - L(\tau') - c(\tau')\Big)_+ + \Big(\frac{l(\tau)}{p^e} - l(\tau')\Big)_+ - \frac{c(\tau)}{p^e}, \qquad \tau' = u(-p^e)\tau.$$

Two facts on the diagrams are used: $c(\sigma) \le \kappa(\sigma) + x^{60}$ (there are about $\kappa(\sigma)$ columns of height k and at most the gap of every other height, Proposition 15.1(a), (b)), and $k\kappa(\sigma) \le l(\sigma) \le k\kappa(\sigma) + x^{60}$.

One place. Fix τ; its share in g_p/m is $\mathbf{1}_{\Phi_a}(\tau)s_p(a,\tau) + ks_p(\kappa(\tau),\tau)$, and both are v times series over $e \in I_p$. Compare term by term. For e with $p^e \ge x_1$ the terms of the share vanish by Proposition 19.1(a), and $t_e(\tau) \ge -c(\tau)/p^e$. For $p^e < x_1$:

(i) For an a-place τ, $(a/p^e - \kappa(\tau'))_+ \le (L(\tau)/p^e - L(\tau') - c(\tau'))_+ + x^{60}$, plus a/p^e if $\tau \in S$ or (τ, e) is a collision. Indeed, if $\tau \notin S$ and τ' is not an a-place outside S, then $L(\tau) = a$, $L(\tau') = 0$ and $c(\tau') \le \kappa(\tau') + x^{60}$; if τ' is an a-place outside S, the pair is a collision and a/p^e bounds the left side; and if $\tau \in S$, again a/p^e bounds it. For $\tau \in S$ and $e = 1$ the bound holds even with a/x_1 added to the left side, because $\kappa(\tau') \ge a/x_1$ and $p < x_1$.

(ii) $k(\kappa(\tau)/p^e - \kappa(\tau'))_+ \le (l(\tau)/p^e - l(\tau'))_+ + x^{60}$, by the bounds for l.

Summing over e, the terms $-c(\tau)/p^e$ give $-c(\tau)\gamma_p/v$, and $\gamma_p c(\tau) \le u$ because $c(\tau)$ is small against a/m; the terms a/p^e for $\tau \in S$ give $\gamma_p a/v$; the errors x^{60} occur for at most $\lceil \log_p x_1 \rceil$ exponents, and $2v\lceil \log_p x_1 \rceil x^{60} \le u$. Hence

$$\begin{aligned} &\mathbf{1}_{\Phi_a}(\tau)s_p(a,\tau) + ks_p(\kappa(\tau),\tau) + v\,\mathbf{1}_{\Phi_a \cap S}(\tau)\frac{a}{x_1} \\ &\qquad \le v\sum_e t_e(\tau) + \gamma_p a\,\mathbf{1}_{\Phi_a\cap S}(\tau) + \sum_{e\colon (\tau,e)\text{ collision}} \frac{a}{p^e} + 2u, \end{aligned}$$

where a collision costs at most $v \le 1$ times its term.

All places. Summing over τ gives $g_p/m + v\,\#(\Phi_a \cap S)\,a/x_1 \le v\sum_{\tau,e} t_e(\tau) + \gamma_p a\,\#(\Phi_a \cap S) + \mathrm{col}_p(S) + 2fu$. The removed boxes are the a boxes at each a-place of S, so $\beta_p(j,S)$ contains $\gamma_p a\,\#(\Phi_a \cap S)$; its predicted divisibility is at least $v\sum t_e - 2fu$, and its logarithmic term is at most fu, because $1 + \log_p \max(B,1) \le (\log b)^2$ and $n_{j+1}^3(\log b)^2 \le u$ by Proposition 8.9. Therefore

$$\beta_p(j,S) \ge \frac{g_p}{m} - \mathrm{col}_p(S) + v\,\#(\Phi_a \cap S)\frac{a}{x_1} - 5fu,$$

and $5fu \le fa(\log b)^2/(mp)$ because $(\log b)^2 \ge 5$. □

Proposition 19.5 (Non-split primes that are not exceptional). *Let Ψ be a weak chain datum for ϑ with $p \nmid \mathrm{N}\,Q(\vartheta)$, and v an integral source form.*

(a) *If $p \ne N$ and $p \ne d$, then $R_{\vartheta,\Psi}v$ is integral.*

(b) *If p is small non-split, $R_{\vartheta,\Psi}v$ has valuation at least $g_p - \lambda_p - m\,\mathrm{col}_p(\emptyset)$, and $R_{\vartheta,\Psi}(p^{j_p}v)$ has valuation at least $g_p - \lambda_p$.*

Proof. (a) By Proposition 19.3(d) the data are standard of level $\bar{\pi}_M$ at p, and the kernels have the scale $\pi_M\mathsf{a}$. By Theorem 17.5(c), each lift raises gauges by at most $\left(|\iota\pi_M|/|\iota(\pi_M d)|\right)^L = |\iota d|^{-L}$, which is 1 since $p \ne d$. With all $r_j = 0$, Proposition 19.3(b) bounds the chain by 1 at the base point 1, and Proposition 19.3(c) passes the bound to the restriction.

(b) The chosen primes p_0, p_1 split in K, so a small non-split p does not divide M, and by Proposition 19.3(d) the data are standard of level one. By

Theorem 19.4(a) lift $j+1$ raises valuations by $\beta_p(j,\emptyset)$, and by Theorem 19.4(b) with $S=\emptyset$,

$$\sum_{j<m}\beta_p(j,\emptyset)\ge g_p-m\,\mathrm{col}_p(\emptyset)-\frac{fa(\log b)^2}{p}\ge g_p-\lambda_p-m\,\mathrm{col}_p(\emptyset),$$

using Proposition 19.1(d) for the last step. Proposition 19.3(b),(c) give the first claim. For the second, every collision has $e\ge 1$, so $\mathrm{col}_p(\emptyset)\le 2fa/p$ by the geometric bound in the proof of Proposition 19.1(d); hence $m\,\mathrm{col}_p(\emptyset)\le j_p$, and the factor p^{j_p} pays for it. □

19.4. Exceptional primes: marks and Hecke operators

Let $p<x_1$ be exceptional. Each infinite place τ induces through ι a prime w_τ of E above p. A *marked set* is a finite set Λ of primes, disjoint from its conjugate, each induced by an a-place; a *full mark* contains w_τ or $\bar w_\tau$ for every a-place τ. For $\mathfrak{l}\in\Lambda$ put $A_\mathfrak{l}=a\cdot\#\{\tau\in\Phi_a : w_\tau=\mathfrak{l}\}$ and let $T_\mathfrak{l}$ be the Hecke operator at $\mathfrak{l}$ of the double coset of $\mathrm{diag}(\varpi 1_{\mathfrak{r}_p},1)$; the *scaled Hecke operator* is $Y_\mathfrak{l}=p^{A_\mathfrak{l}}T_\mathfrak{l}$ on $\mathcal{S}$, and $Y_\Lambda=\prod_{\mathfrak{l}\in\Lambda}Y_\mathfrak{l}$.

Lemma 19.6 (Orbits at exceptional primes). *Let $p<x_1$ be exceptional and o the order of $u(-p)$.*

(a) *$o_0(p)$ is odd, $o_0(p)\le\mathfrak{r}_p$, $m\mid o$, every prime w above p has norm p^o and $\bar w\ne w$, and for odd e the place $u(-p^e)\tau$ induces $\bar w_\tau$.*

(b) *A full mark exists. A marked set has at most f elements, and its primes do not divide MN.*

(c) *For an a-place τ, the a-places that induce w_τ or $\bar w_\tau$ form one orbit under $u(-p^{o_0(p)})$, which alternates between w_τ and $\bar w_\tau$.*

(d) *For a full mark, no collision lies outside the places inducing Λ; and for $j<m$*

$$\#\Lambda\,\nu_p+\mathfrak{r}_p\sum_{\sigma\ \text{inducing}\ \Lambda}c_j(\sigma)\le\frac{\lambda_p}{m}+\#(\Phi_a\cap\{\text{places inducing }\Lambda\})\frac{a}{x_1}.$$

Proof. (a) That $o_0(p)$ is odd and at most $\mathfrak{r}_p$ is the definition. The number m is a power of 2 with $m^{22}=2^s$, s the tower exponent, and $\mathfrak{r}_p\le s$ since $m\le 2^s$. The group of places is $(\mathbb{Z}/q_1)^\times$ with $2^{s+1}\mid q_1$, so $(-p)^o\equiv 1$ modulo 2^{s+1} for the order o of $u(-p)$, and lifting the exponent gives $2^s\mid(p\pm1)o$ for one of the signs. As $m\nmid p\pm1$, the 2-adic valuation of $p\pm1$ is less than that of m, which is less than $s/2$; hence $m\mid o$. By Proposition 10.5 the primes above the inert prime p are permuted by the group of places, $u(-p)$ acting as the Frobenius composed with complex conjugation; so $w_{u(-p^e)\tau}$ is $\bar w_\tau$ for odd e and w_τ for even e, and the residue field of w has p^o elements. If $\bar w=w$, then the identity would be an odd power of $u(-p)$, so o would be odd; but $m\mid o$ and m is even.

(b) Fix a linear order on the primes above p, and let Λ consist of the primes induced by a-places that are smaller than their conjugates. It is disjoint from its conjugate, and since $\bar w_\tau\ne w_\tau$, for every a-place τ one of w_τ, $\bar w_\tau$ lies in Λ; the conjugate is also induced by an a-place, namely $u(-p^{o_0})\tau$, which is an a-place because o_0 is odd and $u(-p^{o_0})$ preserves the class 1 modulo q_0. A marked set has at most f elements, each being induced by one of the f places, and its primes lie above $p<x_1$, while the prime factors of MN are at least x_1.

(c) Two places induce the same prime or conjugate primes exactly when they differ by a power of $u(-p)$: even powers fix a prime above p and odd powers conjugate it. The place $u(-p)^i\tau$ is an a-place exactly when $o_0 \mid i$. Hence the a-places inducing w_τ or $\bar{w}_\tau$ are the $u(-p)^{o_0 i}\tau = u(-p^{o_0})^i\tau$, one orbit, and since o_0 is odd the induced primes alternate between w_τ and $\bar{w}_\tau$.

(d) Let (τ, e) be a collision outside the places inducing Λ. Then τ is an a-place inducing no prime of Λ, so it induces the conjugate of one, as Λ is full; e is odd, so $u(-p^e)\tau$ induces a prime of Λ, against the assumption. For the inequality, a place σ inducing a prime of Λ is not a b-place: it lies on the orbit of an a-place under $u(-p)$, and $(-p)^i \equiv -1$ modulo q_0 would give $1 \equiv -1$ after raising to the odd power o_0. Hence $\kappa(\sigma) = a/x_1$ and $c_j(\sigma) \le a/x_1 + x^{60}$. If $p \le m$, then $\mathfrak{r}_p c_j(\sigma) \le a/(pm)$ by Proposition 8.9. If $p > m$, then $\mathfrak{r}_p = 1$, hence $o_0 = 1$, and every place on the orbit of an a-place under $u(-p)$ is an a-place; so σ is an a-place and $\mathfrak{r}_p c_j(\sigma) \le a/x_1 + a/(pm)$. Summing over the at most f places σ and adding $\#\Lambda \nu_p \le f\nu_p$,

$$\#\Lambda\nu_p + \mathfrak{r}_p \sum_\sigma c_j(\sigma) \le f\nu_p + \frac{fa}{pm} + \#(\Phi_a \cap \{\text{places inducing } \Lambda\})\frac{a}{x_1},$$

and $f\nu_p + fa/(pm) \le fa(\log b)^2/(mp) \le \lambda_p/m$, because $f^2\mathfrak{r}_p(n_m + \mathfrak{r}_p)p$ is small against $fa(\log b)^2/m$ (Proposition 8.9). □

Theorem 19.7 (Hecke operators pass through the lifts). *Let Ψ be a weak chain datum for a twist ϑ with $p \nmid \mathrm{N}\, Q(\vartheta)$, Λ a marked set, $\mathfrak{l} \in \Lambda$ of norm z^2, and Θ the lift from X_e to X_{e+1} with a datum of the support.*

(a) *The datum is fixed by the maximal compact subgroup at $\mathfrak{l}$, so Θ maps forms spherical at $\mathfrak{l}$ to such forms; and for $i \le n_e$, $z^{(t+1)i}\,\mathrm{N}\,\mathfrak{l}^{\binom{i}{2}}\,\Theta \circ T_{\mathfrak{l},i}$ is a combination of the $T_{\mathfrak{l},i'} \circ \Theta$, $i' \le i$, with p-integral coefficients independent of the datum.*

(b) *For $i \le \mathfrak{r}_p$ the same holds with the factor $p^{f\mathfrak{r}_p t}$, and for the chain C_e of the first e lifts, $p^{ef\mathfrak{r}_p t} C_e \circ T_{\mathfrak{l},\mathfrak{r}_p}$ is a combination of the $T_{\mathfrak{l},i} \circ C_e$, $i \le \mathfrak{r}_p$.*

(c) *There are numbers c_i, p-integral if $e < m$, with $\sum_{i \le \mathfrak{r}_p} c_i T_{\mathfrak{l},i} \circ C_{e+1} = p^{\nu_p} C_{e+1} \circ T_{\mathfrak{l},\mathfrak{r}_p}$.*

Proof. (a) *Sphericity.* A prime $\mathfrak{l}$ of Λ lies above p, so it contains none of $\mathrm{N}\, Q(\vartheta)$, π_M, $\bar{\pi}_M$ and N; the translations of the support are integral at $\mathfrak{l}$ and $\bar{\mathfrak{l}}$, which are neither chosen places nor places of Q, so the datum is the vacuum at $\mathfrak{l}$. The tame parts of the characters are trivial at $\mathfrak{l}$ and $\bar{\mathfrak{l}}$, so the base changes of the characters are unramified there, and an element of the maximal compact subgroup at $\mathfrak{l}$ acts on the vacuum by a character of its determinant that is 1. Hence the datum is fixed by this subgroup, and Θ maps forms spherical at $\mathfrak{l}$ to such forms. The prime $\mathfrak{l}$ is split, so the local unitary groups are general linear groups, and its norm is p^o with o even, so $\mathrm{N}\,\mathfrak{l} = z^2$ with $z = p^{o/2}$.

Hecke polynomials. Let $H(U) = \sum_i (-U)^i\,\mathrm{N}\,\mathfrak{l}^{\binom{i}{2}} T_{\mathfrak{l},i}$ and $H'(U)$ the same for the target. The value of the base change of the character of the lift at a uniformiser at $\mathfrak{l}$ is a p-adic unit: its norm is p to the sum of $-\kappa_j(\tau)$ over the places τ inducing $\mathfrak{l}$ and of $\kappa_j(\tau)$ over those inducing $\bar{\mathfrak{l}}$, and this sum is 0, because multiplication by $u(-p)$ carries the places inducing $\mathfrak{l}$ onto those inducing $\bar{\mathfrak{l}}$ and preserves the lift weights there. On the vacuum the Hecke operators of the two groups count subspaces of the kernels of an integral matrix and of its transpose modulo $\mathfrak{l}$, and

the q-binomial theorem gives the unramified correspondence (Appendix B.16) in the form

$$H'(U)\,\Theta\varphi = R(U)\,\Theta\big(H(z^{t+1}U)\varphi\big)$$

for spherical φ, with a polynomial R, $R(0)=1$, whose coefficients are p-integral and do not depend on the datum; the ranks are even, so both characters are the base change of the character of the lift.

Comparing coefficients. Divide by R in power series; R^{-1} has p-integral coefficients since $R(0)=1$. The coefficient of U^i on the right is $(-z^{t+1})^i\,\mathrm{N}\,\mathfrak{l}^{\binom{i}{2}}\,\Theta T_{\mathfrak{l},i}\varphi$, and on the left it is $\sum_{i'\le i} b_{ii'}(-1)^{i'}\,\mathrm{N}\,\mathfrak{l}^{\binom{i'}{2}}T'_{\mathfrak{l},i'}\Theta\varphi$, where $b_{ii'}$ is the coefficient of degree i of $R^{-1}U^{i'}$. Only $i'\le i$ contribute, and the coefficients are p-integral and independent of the datum; this is (a).

(b) With $z=p^h$ and $2h\le f$, the factor of (a) is $z^{(t+1)i}\,\mathrm{N}\,\mathfrak{l}^{\binom{i}{2}} = p^{hi(t+i)}$, and for $i\le \mathfrak{r}_p\le t$ (Proposition 8.9) the exponent $hi(t+i)\le \frac{f}{2}\mathfrak{r}_p\cdot 2t = f\mathfrak{r}_p t$; multiplying the identity of (a) by $p^{f\mathfrak{r}_p t - hi(t+i)}$ gives the factor $p^{f\mathfrak{r}_p t}$. For the chain, induct on e: apply the next lift Θ to the identity for e lifts, multiply by $p^{f\mathfrak{r}_p t}$, and move each $T_{\mathfrak{l},i}$, $i\le\mathfrak{r}_p$, through Θ by the identity for one lift; the chain of $e+1$ lifts is Θ after the chain of e lifts.

(c) For $e+1\le m$ the exponent satisfies $(e+1)f\mathfrak{r}_p t\le m f\mathfrak{r}_p t\le \mathfrak{r}_p f(n_m+\mathfrak{r}_p)=\nu_p$, as $mt\le n_m$. Multiplying the identity of (b) for $e+1$ lifts by $p^{\nu_p-(e+1)f\mathfrak{r}_p t}$, an integral power of p when $e<m$, gives the numbers c_i. □

Theorem 19.8 (Valuation on the image of the Hecke operators). *Let Ψ, ϑ, p be as in Theorem 19.7, p exceptional, and Λ marked.*

(a) *Let ϕ be a standard datum of level one, and let g lie, at each $\mathfrak{l}\in\Lambda$, in the double coset of* $\mathrm{diag}(\varpi 1_i, 1)$ *for some $i\le\mathfrak{r}_p$. Then the kernel of $\omega(1,g)\phi$ raises gauges by at most $p^{\sum_{\mathfrak{l}\in\Lambda}A_{\mathfrak{l}}+\ell(\Lambda,j)}$, where the* lift loss *is*

$$\ell(\Lambda,j)=\mathfrak{r}_p\sum_{\sigma\ \textit{inducing}\ \Lambda} c_j(\sigma)-\beta_p(j,S_\Lambda),$$

S_Λ denoting the set of places inducing a prime of Λ.

(b) *For an integral source form v, $R_{\vartheta,\Psi}(Y^m_\Lambda v)$ has gauge at most $p^{\sum_{j<m}(\#\Lambda\nu_p+\ell(\Lambda,j))}$.*

(c) *If Λ is a full mark, $R_{\vartheta,\Psi}(Y^m_\Lambda v)$ has valuation at least $g_p-2\lambda_p$; and for a chain datum, $B_{\vartheta,\Psi}\varphi$ has valuation at least $g_p-2\lambda_p$ for every $\varphi\in Y^m_\Lambda\mathcal{S}^{\mathrm{int}}+p^{j_p}\mathcal{S}^{\mathrm{int}}$.*

Proof. (a) *Neighbours.* At each $\mathfrak{l}\in\Lambda$ the component of g lies in the double coset of $\mathrm{diag}(\varpi 1_i,1)$ with $i\le\mathfrak{r}_p$, and p is a uniformiser at $\mathfrak{l}$; so up to integral invertible factors on both sides, g is the diagonal element δ with entry p at a set $J_\mathfrak{l}$ of at most $\mathfrak{r}_p$ indices at $\mathfrak{l}$, entry p^{-1} at the same indices at $\bar{\mathfrak{l}}$, and 1 elsewhere. The integral factors keep the datum standard of level one, so by Theorem 17.5(b) it suffices to bound the jets of ϕ along the Schur matrices, weighted by the entries of δ read at the places. Read at τ, through the component at the conjugate of the prime induced by τ, δ is diagonal with entries $p^{-\epsilon(\tau,b)}$, where the exponent ϵ is 1 on the marked indices at the places inducing a prime of Λ, -1 on those at the places inducing its conjugate, and 0 elsewhere.

Charges. Charge a box of index b in a column of height h at τ if τ induces a prime of Λ, $h=1$ and b is not marked, or if τ induces the conjugate of a prime of Λ and b is marked. These charges cover the exempt pairs for the set S_Λ: if τ induces a prime of Λ, or if $h=1$ and the partner $u(-p^e)\tau$ does, then the partner

induces the conjugate prime and has the same marked indices, so the index is charged at one of the two places. By Theorem 13.21(b), the jets have valuation at least $\beta_p(j, S_\Lambda)$ minus the charges.

Counting. At a place inducing a prime of Λ a box of the long row costs 1, whether its index is marked or it is charged, and a low column costs at most $\mathfrak{r}_p$; at a place inducing the conjugate, the entries p^{-1} of the marked indices pay for their charges. Hence the exponents ϵ plus the charges, summed over the boxes of a filling, are at most $\sum_{\mathfrak{l}\in\Lambda} A_\mathfrak{l} + \mathfrak{r}_p \sum_{\sigma \text{ inducing } \Lambda} c_j(\sigma)$. The determinant characters contribute nothing, because the weights on X_{j+1} balance a prime of Λ against its conjugate. This gives the bound $p^{\sum A_\mathfrak{l} + \ell(\Lambda, j)}$.

(b) *Moving one operator.* For a list of distinct primes of Λ, induction on the list gives a linear map t on the data of lift $j+1$ with: t commutes with the actions of $\mathrm{U}(X_j)$ and of the maximal compact subgroup of $\mathrm{U}(X_{j+1})$ at the datum; the lift by $t(\varrho(x_j)\mathrm{vac}_j)$ of $C_j(y)$ is C_{j+1} of $\prod_\mathfrak{l} Y_\mathfrak{l}\, y$; and for $j < m$ and g trivial at the primes of the list and in a double coset of some $\mathrm{diag}(\varpi 1_i, 1)$, $i \le \mathfrak{r}_p$, at the other primes of Λ, the kernel of $\omega(1, g)t(\varrho(x_j)\mathrm{vac}_j)$ raises gauges by at most p^β with

$$\beta = \sum_{\mathfrak{l} \text{ in the list}} (\nu_p - A_\mathfrak{l}) + \sum_{\mathfrak{l}\in\Lambda} A_\mathfrak{l} + \ell(\Lambda, j).$$

The step adds a prime $\mathfrak{l}$: by Theorem 19.7(c), $C_{j+1} \circ Y_\mathfrak{l} = p^{A_\mathfrak{l} - \nu_p} \sum_i c_i T_{\mathfrak{l},i} \circ C_{j+1}$, and the lift commutes with the Hecke operators of the target acting on the datum through the target group; so t for the longer list is t for the list followed by $p^{A_\mathfrak{l}-\nu_p} \sum_i c_i$ times the Hecke operators $T_{\mathfrak{l},i}$ acting on the datum. A Hecke operator is a sum of translates by elements of its double coset, so by (a) the new datum, moved by g, raises gauges by at most $p^{\nu_p - A_\mathfrak{l}}$ times the old bound, the c_i being integral. With the full list Λ, $\beta = \#\Lambda\nu_p + \ell(\Lambda, j)$.

All lifts. Move one of the m copies of Y_Λ into the datum of each lift: $R_{\vartheta,\Psi}(Y_\Lambda^m v)$ is the restricted chain with the data $t(\varrho(x_j)\mathrm{vac}_j)$ applied to v. The maps t commute with the levels, so Proposition 19.3(b),(c) gives the bound.

(c) By Lemma 19.6(d) there is no collision outside S_Λ, so Theorem 19.4(b) gives $\beta_p(j, S_\Lambda) \ge g_p/m + \#(\Phi_a \cap S_\Lambda)a/x_1 - fa(\log b)^2/(mp)$, as $v(\pi_p) = 1$ for the inert prime p. With the inequality of Lemma 19.6(d),

$$\begin{aligned}\#\Lambda\nu_p + \ell(\Lambda, j) &\le \frac{\lambda_p}{m} + \#(\Phi_a \cap S_\Lambda)\frac{a}{x_1} - \beta_p(j, S_\Lambda) \\ &\le -\frac{g_p}{m} + \frac{\lambda_p}{m} + \frac{fa(\log b)^2}{mp} \le -\frac{g_p}{m} + \frac{2\lambda_p}{m},\end{aligned}$$

and summing over $j < m$ in (b) gives the valuation $g_p - 2\lambda_p$. For a chain datum, expand the gluing (Proposition 19.2(b)) into a weak chain datum Ψ' with $B_{\vartheta,\Psi} = R_{\vartheta,\Psi'}$; for $\varphi = Y_\Lambda^m v + p^{j_p} v'$ with integral v, v', the first part is bounded by what precedes, and the second has valuation at least $g_p - \lambda_p$ by Proposition 19.5(b). □

Theorem 19.9 (Colength of the scaled Hecke operators). *Let $p < x_1$ be exceptional, Λ marked and $\mathfrak{l} \in \Lambda$. For every j_0, $Y_\mathfrak{l}$ is onto modulo p^{j_0} on $\mathcal{S}^{\mathrm{int}}$ up to colength $(12A_\mathfrak{l}/(mp) + j_0/(m^2 f))\dim \mathcal{S}$. Consequently Y_Λ^m is onto modulo p^{j_p} up to colength $\lambda_p \dim \mathcal{S}$.*

Proof. The cycle. By Lemma 19.6(c) the a-places inducing $\mathfrak{l}$ or $\bar{\mathfrak{l}}$ form one orbit $t_0, \dots, t_{N'-1}$ under $u(-p^{o_0})$, with N' the order of $u(-p^{o_0})$, alternately inducing $\mathfrak{l}$ and $\bar{\mathfrak{l}}$. Let ι_i be the embedding Φ_{t_i}, composed with conjugation when t_i induces $\bar{\mathfrak{l}}$, so that every ι_i induces $\mathfrak{l}$; restricted to the valuation ring of $\mathfrak{l}$ it takes values in the p-integral numbers. For x in this ring, $\iota_{i+1}(x) \equiv \iota_i(x)^P$ modulo p with $P = p^{o_0}$: on the integers of E the automorphism named by $-p^{o_0}$ is congruent to the conjugate of the p^{o_0}-th power, and exactly one of t_i, t_{i+1} induces $\mathfrak{l}$, so exactly one embedding is conjugated. Hence the ι_i form a Frobenius cycle (Definition 21.1) for P, the even letters being those inducing $\mathfrak{l}$; the residue field has $p^o = P^{N'}$ elements, as $o = o_0 N'$.

The operator. By Proposition 19.10, $Y_{\mathfrak{l}}$ is the edge Hecke operator of Section 21 on the integral source forms, with k variables, degree a, blocks $\mathfrak{r}_p$ and $k - \mathfrak{r}_p$, and $A_{\mathfrak{l}} = a \cdot \#\{\text{even letters}\}$.

The size conditions. Theorem 21.14 needs $0 < \mathfrak{r}_p$, $2\mathfrak{r}_p \le k$, $0 < \theta \le 1$, $a^4 \le q$ and $(N' k \log q/\theta)^5 P^{3\mathfrak{r}_p} \le a$ for $q = P^{N'}$ and $\theta = 1/(m^2 f)$. Here $13 \le p < x_1$ (a non-split prime is at least 13), $0 < o_0 \le \mathfrak{r}_p$, $N' \le f$ and $m \le o_0 N'$ because $m \mid o$; hence $q = p^o \ge 13^m$, and $P^{\mathfrak{r}_p} \le p^{\mathfrak{r}_p^2}$ with $\mathfrak{r}_p \le s$. These are inequalities between the parameters of the data, which hold by Proposition 8.9.

The colength. The theorem gives the colength $(3c_{\mathrm{div}} + \theta j_0) \dim \mathcal{S}$ with $c_{\mathrm{div}} = N' a (P^{-2\mathfrak{r}_p} + \mathfrak{r}_p^2/(kP))$. Each term of the bracket is at most $1/(mp)$: $P^{2\mathfrak{r}_p} \ge p^{\mathfrak{r}_p} \cdot p \ge mp$, and $\mathfrak{r}_p^2/(kP) \le 1/(mp)$ because $k \ge x^5 \mathfrak{r}_p^2$ and $P \ge p$, k being large against m. With $N' = 2\ell$ letters, of which ℓ are even, $3c_{\mathrm{div}} \le 6 \cdot 2\ell a/(mp) = 12A_{\mathfrak{l}}/(mp)$.

The product. Colengths of a composite of operators that are onto modulo p^{j_p} add, and Y_Λ^m is a product of $m\#\Lambda$ factors $Y_{\mathfrak{l}}$. Since $\sum_{\mathfrak{l}} A_{\mathfrak{l}} \le af$ (each a-place induces one prime), $\#\Lambda \le f$ and $j_p \le 2mfa/p + 1$,

$$m \sum_{\mathfrak{l} \in \Lambda} \left(\frac{12 A_{\mathfrak{l}}}{mp} + \frac{j_p}{m^2 f} \right) \le \frac{12af}{p} + \frac{j_p}{m} \le \frac{14af}{p} + \frac{1}{m} \le \lambda_p .$$

□

19.5. Hecke operators as edge operators

Let $\mathfrak{l}$ be split, ϖ a uniformiser, the level maximal at $\mathfrak{l}$ and free. An *edge* into a class x is a pair (x, b) with b a coset of the parahoric subgroup with blocks r, $k - r$; its *origin* is the class of the representative of x moved at $\mathfrak{l}$ by βd, with β a representative of b and $d = \mathrm{diag}(\varpi, \dots, \varpi, 1, \dots, 1)$. Fixing representatives gives, for each edge, the class of its origin, a basis change g in $\mathrm{GL}_k(O_{\mathfrak{l}})$ and a rational part.

Proposition 19.10 (Hecke operators are edge operators). (a) *An edge is determined by the class of its origin and by the parahoric coset of the transpose of its basis change.*

(b) $(T_{\mathfrak{l},r}\varphi)(x) = \sum_b \rho(\text{rational part of } (x,b))\, \varphi(\text{origin of } (x,b))$.

(c) *At an a-place inducing $\mathfrak{l}$ or $\bar{\mathfrak{l}}$, a basis change acts on the Weyl module through the letter matrices and powers of p; the Weyl module splits as the monomials of degree a in the k variables at these places tensored with a module on which the edges act integrally; and the integral source forms are the families of integral vectors indexed by the classes.*

Hence, for a Frobenius cycle matching the places above $\mathfrak{l}$, $Y_{\mathfrak{l}}$ is an edge Hecke operator on the integral source forms, and the bound of Theorem 21.14 holds for it.

Proof. (a) Let two edges have origins of the same class and parahoric cosets of the transposes of their basis changes that agree. The quotient η of their rational parts is a rational point that takes the representative of one terminus to that of the other modulo the level; so the termini are the same class, and η fixes a coset of the level, hence $\eta = 1$ by freeness. Then $\beta d g^{-1} = \beta' d g'^{-1}$, and the cosets agree, because $d\, g^{-1} g'\, d^{-1}$ is integral at $\mathfrak{l}$ when $g^{\mathsf{T}} g'^{-\mathsf{T}}$ is parahoric (conjugation by d divides one off-diagonal block by ϖ, multiplies the other by ϖ and fixes the diagonal blocks).

(b) The Hecke operator is the operator of the indicator of the double coset $K \operatorname{diag}(\varpi 1_r, 1) K$ at $\mathfrak{l}$. By weak approximation above p, every coset in it is the coset of $g_{\mathfrak{l}}^{-1} \beta d$ for some $\beta \in \mathrm{GL}_k(\mathcal{O}_{\mathfrak{l}})$, where g is the representative of the class; two such elements give the same coset exactly when $\beta^{-1}\beta'$ is parahoric for the blocks $r, k-r$. Hence the cosets are indexed by the parahoric cosets b, the value of $T_{\mathfrak{l},r}\varphi$ at the representative of x is the sum over b of φ at the origin of (x, b), and writing the origin as the rational part times the representative of its class times an element of the level gives the formula.

(c) At a place s of the cycle the embedding is ι_s for even s and $\iota_s \circ \mathrm{conj}$ for odd s. For a rational part γ with $\gamma^{-1} = g\, d^{-1} \beta$, applying $\iota_s \circ \mathrm{conj}$ to the entries of γ, transposing and multiplying by p for even s gives the letter matrix of g, times the diagonal matrix with p at the deep variables, times the letter matrix of β. At an a-place the weight is one row of length a, so the Weyl module there is the a-th symmetric power of the standard representation, on which these matrices act by their symmetric powers; and symmetric powers and tensor products are multiplicative. The Weyl module is the tensor product of these factors at the places of the cycle with a factor for the other places, on which the Schur matrices of rational parts, integral at the conjugates of the primes induced by the a-places there, are invertible over the p-integral numbers (Appendix B.11). Finally, evaluation at representatives of the classes that are integral above p identifies the integral source forms with the families of integral vectors indexed by the classes.

Conclusion. At the representative of a class both $Y_{\mathfrak{l}}$ and the edge Hecke operator are sums over the edges into the class; in each coordinate, the term of an edge in $Y_{\mathfrak{l}}$ is $p^{A_{\mathfrak{l}}}$ times the matrix of the rational part on the Weyl module, and by (c) this is the term of the edge Hecke operator, the factor $p^{A_{\mathfrak{l}}}$ being the factors p of the even letters in the a-th symmetric powers. Hence the bound of Theorem 21.14 holds for $Y_{\mathfrak{l}}$. □

19.6. The level prime

Let Σ_N be the set of conjugates of the primes above N induced by ι; it contains one place of each conjugate pair above N, and these are split of degree one (Lemma 10.13). At Σ_N let δ_j be $\operatorname{diag}(1, \dots, 1, N)$ at the places dividing π_N and 1 at the others, so that δ_j takes the standard lattice to L_j, the integral vectors whose last coordinate is divisible by π_N, and let $\delta_j^{\vee}$ take it to the dual lattice. The representation of $\mathrm{GL}(\mathcal{O}_{\Sigma_N})$ on forms on X_j is right translation after conjugation by δ_j; with it, Up is the operator of Section 16.2. Let $\mathcal{A}'$ be the module of forms φ on X_0 such that $\mathrm{Up}^{\varrho+1}\varphi$ is a source form integral for L_0 at Σ_N.

Proposition 19.11 (Set-up at the level). (a) *The direct datum is nonzero at a matrix T over $\mathcal{O}_{\Sigma_N}$ exactly if T is integral with last column $\equiv$ that of 1 modulo*

each place; for $k \in \mathrm{U}(X_j)(\mathbb{A}_\mathrm{f})$ the matrix $\delta_j^{-1} k \delta_j$ has this property exactly if k is integral at Σ_N and the last rows of k and k^{-1} satisfy the congruences of the level at the places dividing π_N and their conjugates.

(b) *$\delta_j^{-1}\,\mathrm{level}_j\,\delta_j$ lies in the mirabolic group at Σ_N, and conversely a point supported above N with this property lies in the level; the mirabolic group read in a basis of L_j lies in the level.*

(c) *Ramified source forms are ramified vectors of this representation, and* Up *preserves the source forms.*

Proof. (a) A function of matrices modulo N read on integral matrices: the direct datum is the indicator of the integral T whose last column is that of 1 modulo each place of Σ_N. Conjugation by δ_j multiplies the last column by t and divides the last row by t, where $t = N$ at the places dividing π_N and $t = 1$ at the others. At a place w of Σ_N dividing π_N, $\delta_j^{-1} k \delta_j$ is integral with last column $\equiv e$ exactly if k is integral there and its last row is that of 1 modulo w (the corner entry is covered by the congruence on the last column); at a place w of Σ_N whose conjugate divides π_N, $\delta_j = 1$ there, and the condition on the last column of k at w is the condition on the last row of k^{-1} at $\bar{w}$, because the last row of k^{-1} at $\bar{w}$ is the conjugate of the last column of k at w for a unitary k.

(b) An element of the level of X_j is integral and has last row $\equiv (0, \dots, 0, 1)$ modulo π_N, and so does its inverse; by (a), $\delta_j^{-1} k \delta_j$ lies in the mirabolic group at Σ_N, the integral invertible matrices with last column $\equiv e$ modulo N. Conversely, a point supported above N whose conjugate by δ_j is mirabolic at Σ_N satisfies the congruences of the level by (a), and it is integral and 1 at the other places, so it lies in the level; this applies in particular to the mirabolic group read in a basis of L_j.

(c) Let φ be a ramified source form. At each $w \mid \pi_N$ its mean over the powers of z_w vanishes, so $\varphi = \psi - z_w \psi$ with ψ a combination of the translates of φ by powers of z_w. One of each pair w, $\bar{w}$ above N divides π_N, and z_w is an integral unit scalar at that place, 1 at the others; a scalar is unchanged by the passage to a basis of L_0. Hence φ is a ramified vector for the representation at Σ_N. For Up, write an element k of the level of X_0 as its part above N, with matrix a in a basis of L_0, times a part k' away from N; both lie in the level by (b). Up commutes with k', which fixes φ, and it preserves the vectors fixed by the mirabolic group (Theorem 16.4(a)), to which a belongs. Hence Up φ is again fixed by the level. □

Theorem 19.12 (The level prime). (a) *There is $e \le m f b \dim \mathcal{S}$ such that $\mathcal{S}^{\mathrm{int}} \cap \mathcal{A}'$ has colength dividing N^e in $\mathcal{S}^{\mathrm{int}}$.*

(b) *For every chain datum Ψ of a twist ϑ and every ramified $\varphi \in \mathcal{A}'$, $B_{\vartheta,\Psi}\varphi$ has gauge at most N^{2mfb} at N.*

Proof. (a) *Three modules.* Let $\mathcal{A}_\iota$, $\mathcal{A}^L$, $\mathcal{A}^\vee$ be the source forms with values integral for the standard lattice, for L_0 and for $L_0^\vee$ at Σ_N. Then $\mathcal{A}_\iota \subseteq \mathcal{A}^\vee$, because $(\delta_0^\vee)^{-1}$ acts at each place by an integral matrix and the weight is polynomial; and Up, which preserves the source forms (Proposition 19.11(c)), a sum of right translates with roots of unity as coefficients, maps $\mathcal{A}^L$ into $\mathcal{A}^\vee$, because each translate moves δ_0 to $\delta_0^\vee$ up to an element integral above N.

Colength of $\mathcal{A}^L$ in $\mathcal{A}^\vee$. Every class has a representative γc with $\gamma c \delta_0^\vee$ integral above N and $\gamma c \delta_0$ equal to $\mathrm{diag}(1, \dots, 1, N)$ at Σ_N times an element integral above N. Evaluate at such representatives: a form of $\mathcal{A}^\vee$ lies in $\mathcal{A}^L$ exactly if each coordinate of each value is divisible by N to its degree in the last variable, because $\mathrm{diag}(1, \dots, 1, N)^{-1}$ acts on a basis vector by N to minus that degree. The diagonal matrix of these powers of N shows that $\mathcal{A}^L$ has colength dividing N^l in $\mathcal{A}^\vee$, with l the degree in the last variable summed over the pairs of a class and a basis vector of the values.

Iteration. By induction on i, the preimage of $\mathcal{A}^L$ under Up^i has colength dividing $N^{l(i+1)}$ in $\mathcal{A}_\iota$: the colength l is paid once for the passage from $\mathcal{A}_\iota \subseteq \mathcal{A}^\vee$ to $\mathcal{A}^L$, and once for each factor Up, since the preimage of $\mathcal{A}^L$ under Up^i lies in that of $\mathcal{A}^\vee$ under Up^{i+1}, in which the preimage of $\mathcal{A}^L$ under Up^{i+1} has colength dividing N^l. With $i = \varrho + 1$ this is the colength of $\mathcal{S}^{\mathrm{int}} \cap \mathcal{A}'$.

The degree. The summed degree is the same for all k variables of X_0: up to the class number it is the exponent of 2 in the determinant of the action of $\mathrm{diag}(1, \dots, 2, \dots, 1)$, and a transposition moves the entry 2 from one variable to another. At each place a basis vector has total degree at most a, so $kl \le \dim \mathcal{S} \cdot fa$ (in the model case, the mean degree in one of k variables of the monomials of degree a is a/k). Hence $l(\varrho + 2) \le fa(m/2 + 1) \dim \mathcal{S}/k \le mfb \dim \mathcal{S}$, because $a = xb$ and $(m/2 + 1)x \le km$.

(b) *All lifts become direct.* Ramified source forms are ramified vectors at Σ_N (Proposition 19.11(c)). By the lift laws at Σ_N (Theorem 16.4), for each lift the direct datum, in bases of L_j and L_{j+1}, obeys the two rules, with factor γ_j the ratio of the values of the character of the lift at $\mathrm{diag}(1, \dots, 1, N)$ on X_j and on X_{j+1}; the characters are unramified above N, hence trivial on integral matrices. Moving the two matrices $\mathrm{diag}(1, \dots, 1, N)$ out of the glued datum of the double lift i costs one character value each, and the points δ_{2i+1} cancel between the two lifts; each of the f places of Σ_N contributes one value, of N-adic absolute value at most N^{2b}, because the lift weights are at most $2b$. By Theorem 16.4(e), the chain of glued double lifts of a ramified source form φ is therefore c times the chain of the m direct lifts of $\mathrm{Up}^{\varrho+1}\varphi$. Only these character values can have negative valuation: the factors γ_j have N-adic absolute value at most 1, as the value of the character at $\mathrm{diag}(1, \dots, 1, N)$ decreases with the rank, and the numbers of primitive vectors are integers. Hence $|\iota c| \le (N^{2fb})^{m/2} = N^{mfb}$.

Each direct lift loses at most fb. The direct datum on a translate of the integral vacuum, in bases of L_j, L_{j+1}, is the translate by (δ_j, δ_{j+1}) of the direct datum in the standard bases, so it suffices to bound the latter from the base point 1 to 1. By Fourier expansion modulo N, $N^{2fn_{j+1}n_j}$ times this datum is an integral combination of translates of the integral vacuum whose poles above N are simple and lie at the induced places. There is an integer δ of K in all the induced primes above N with $|\iota(\bar{\delta} d)| \ge 1$: the induced primes lie above one prime of K, whose residue field has N elements, so the generator of the integers of K is congruent there to an integer c; take for δ the generator minus c, which is not divisible by the conjugate prime, while $N \nmid d$. These poles are then cleared by δ, so the multiple is a standard datum of level δ, which by Theorem 17.5(c) loses at most a power of $|\iota(\bar{\delta} d)|^{-1} \le 1$. The exponent $2fn_{j+1}n_j$ is at most $2fn_m^2 \le fb$.

Conclusion. The form $\mathrm{Up}^{\varrho+1}\varphi$ has gauge at most 1 at δ_0, so by Proposition 19.3(b) the chain of the m direct lifts has gauge at most N^{mfb} at δ_m, and with c the composite lift has gauge at most N^{2mfb} at δ_m. A bound at δ_m gives

the same bound for the restriction at 1, because the change of lattice at the last coordinate cancels the normalising factor $\prod_\tau \Phi_\tau(\bar\pi_N)^{-n_\tau}$ of the restriction. □

19.7. The divisibility theorem

Theorem 19.13 (Divisibility of the composite lift). *There are numbers $\epsilon_p \geq 0$, zero for $p \neq N$ that split in K or satisfy $p \geq x_1$, with $\sum_{p\in U} \epsilon_p \log p \leq \epsilon_{\mathrm{u}}$ for every finite U, such that for every prime p and embedding $\iota\colon \mathbb{C} \to \mathbb{C}_p$ the integral ramified source forms have an $\mathcal{O}_\iota$-submodule S_p of colength dividing p^l, $l \leq \epsilon_p \dim \mathcal{S}$, on which every composite lift $B_{\vartheta,\Psi}$, for a twist ϑ with $p \nmid \mathrm{N}\,Q(\vartheta)$ and a chain datum Ψ, has valuation at least $g_p - \epsilon_p$.*

Proof. Take ϵ_p the allowed loss; it is nonnegative, zero for $p \neq N$ that split or satisfy $p \geq x_1$, and Proposition 19.1(c) bounds the sums. Four cases exhaust the primes.

$p \neq N$, split or $p \geq x_1$. Then $p \neq d$, because $d < x_1$ and $\chi_K(d) = 0$, and $g_p = 0$ by Proposition 19.1(a). Take $S_p = \mathcal{S}^{\mathrm{int}}$. By Proposition 19.2(b), $B_{\vartheta,\Psi} = R_{\vartheta,\Psi'}$ for a weak chain datum Ψ', and Proposition 19.5(a) shows that it maps integral forms to integral forms.

$p < x_1$ non-split, not exceptional. Take $S_p = \mathcal{S}^{\mathrm{int}}$. By Proposition 19.5(b) for Ψ', the valuation is at least $g_p - \lambda_p - m\,\mathrm{col}_p(\emptyset)$, and $m\,\mathrm{col}_p(\emptyset) \leq \lambda_p$ by Proposition 19.1(d); so it is at least $g_p - 2\lambda_p = g_p - \epsilon_p$.

$p < x_1$ exceptional. A full mark Λ exists (Lemma 19.6(b)). Take $S_p = Y_\Lambda^m \mathcal{S}^{\mathrm{int}} + p^{j_p}\mathcal{S}^{\mathrm{int}}$; its colength divides p^l with $l \leq \lambda_p \dim \mathcal{S} \leq \epsilon_p \dim \mathcal{S}$ (Theorem 19.9), and the valuation on it is at least $g_p - 2\lambda_p$ (Theorem 19.8(c)).

$p = N$. Since $x_1 \leq N$, $g_N = 0$. Take $S_N = \mathcal{S}^{\mathrm{int}} \cap \mathcal{A}'$, of colength dividing N^e with $e \leq m f b \dim \mathcal{S} \leq \epsilon_N \dim \mathcal{S}$; by Theorem 19.12(b) the valuation on its ramified members is at least $-2mfb = g_N - \epsilon_N$.

Finally intersect each S_p with the ramified forms; the intersection has colength in the integral ramified forms at most that of S_p in the integral forms. □

20. Size of the composite lift and the proof of Theorem A

This section bounds the archimedean size of the composite lift for the trivial twist (Theorem 20.4), shows that short forms span the ramified source forms (Proposition 20.1), and combines both with the seesaw identity (Theorem 18.15) and the divisibility theorem (Theorem 19.13) to prove Theorem A. In the size bound the archimedean sizes of the jets and the gains appear with the same prime sums, which cancel in the exponent; and the restriction to Y_m saves $a \log N$ at each a-place, which pays for the excess of the m lifts over the main amount.

For a form φ on a definite group and $\sigma \in \mathrm{Aut}\,\mathbb{C}$, $\|\varphi\|_\sigma^2$ is the sum over the classes c of the squared Gram norms of $\sigma(\varphi(g_c))$, divided by the orders of the stabilisers; at a free level the stabilisers are trivial.

20.1. Short forms

A form φ is *B-short* if each value $\varphi(g)$ is integral for Weyl's lattice at g at every p-adic embedding, and every conjugate of $\varphi(g)$ under $\mathrm{Aut}\,\mathbb{C}$ has squared norm at most B.

Proposition 20.1 (Short forms span). (a) *Let K be a free level, $N \neq 0$ an integer in the different of E, and $M \geq nf + \#\{\text{boxes in columns of height} < n\}$. If $B \geq (6N(fn)^2)^{6M}$, the forms of level K are spanned by the B-short ones.*

(b) *The class number of the standard lattice in rank n is at most $(1+2c^3)^{4fn^2}$, $c = 6N(fn)^2$; at a level containing the principal level of G in rank at most $2n_m$ there are at most $\exp(x^{29}(1+\log G))$ classes, and at most $\exp(x^{30})$ at a level containing that of $\pi_M\pi_N$ in rank at most n_m.*

(c) *The ramified source forms are the image of $\prod_{w|\pi_N}(1-z_w)$; they have dimension at least $(1-f/(N-1))\dim\mathcal{S} \geq (1-b^{-8})\dim\mathcal{S}$, and are spanned by the e^{ϵ_u}-short ones.*

Proof. (a) *Delta forms.* At a free level K a form is the sum of its *delta forms*, one for each class c: the delta form at a representative x made from a vector v is supported on ΓxK, and its value at y is $\gamma^{-1}v$ for the unique rational γ with $x^{-1}\gamma y \in K$. If the coordinates of $x^{-1}\iota(v)$ have absolute value at most 1 for every p-adic embedding ι and every conjugate of v has squared norm at most B, the delta form is B-short, because its values are the moved vectors $\gamma^{-1}v$ and the gauge and the norms are invariant under rational points. Hence it suffices to find, for each class c, a spanning set of vectors with these two properties.

Short bases. A lattice of rank n over $\mathcal{O}_E$ with a unimodular hermitian form has a *short basis*: a basis Y such that the entries of all conjugates of Y are at most $C = 6N(fn)^2$, by Minkowski's theorem and the integrality of the Gram matrix (N in the different makes the dual lattice contain N^{-1} times the lattice). Let Y_c be a short basis of the lattice of c, so that $c^{-1}Y_c$ is integral, and similarly short bases for the powers of c that absorb the determinant part of the weight.

The vectors. Let A be the action of Y_c in the weight, times the determinants of these short bases, and consider $A\,\mathrm{ev}(y)$ for arrays y of roots of unity, $\mathrm{ev}(y)$ being the evaluation vector (Appendix B.11). They are integral at every embedding: c^{-1} moves the vector to the action of the integral matrix $c^{-1}Y_c$ applied to the values at y of the products of minors, which are polynomials with integral coefficients, times determinants of integral matrices; and the entries of y have absolute value 1 in every normed field. They are B-short: with $R = C$, the Schur matrix of a family z multiplies squared norms by at most $\prod \rho^\lambda$ with $\sum \rho$ at most the sum of the squared entries, each at most $(nR)^2$, the exponents adding up to the number of boxes; and $|\det Y| \leq (n^2R)^n \leq (R^3)^n$; so the squared norm is at most $R^{6M} \leq B$. Evaluation vectors at roots of unity span the Weyl module and A is invertible, so these vectors span.

(b) *The class number.* To a class g attach a short basis Y of its lattice, its Gram matrix $G = Y^{\mathsf{H}}Y$, and a rational matrix X congruent to $(g^{-1}Y)^{-1}$ modulo $\hat{\mathcal{O}}$ with entries at most $(2f)^2N$. Both G and GX have integral entries, because $Y^{\mathsf{H}}Z = (gY)^{\mathsf{H}}(gZ)$ for unitary g; their conjugates are at most C^3. The pair (G, GX) determines the class: if two classes give the same pair, then $\gamma = YY'^{-1}$ is a rational isometry, since the Gram matrices agree, and $g = \gamma g' k$ with $k = A'A^{-1}$, $A = g^{-1}Y$, $A' = g'^{-1}Y'$, which is integral as the sum of the integral matrices

$A'A^{-1}-A'X$ and $A'X'$. The number of algebraic integers of E with all conjugates at most C^3 is at most $(1+2C^3)^{2f}$, and there are $2n^2$ entries; this gives $(1+2C^3)^{4fn^2}$. A level containing a principal level of G has index at most $\#\mathrm{GL}_n(\mathcal{O}/G) \le G^{2fn^2}$ in the integral points, and every class of the integral points splits into at most that many classes; with Proposition 8.9 this gives the two bounds.

(c) *Ramified forms as an image.* The scaling z_w acts on source forms with $z_w^{N-1} = 1$: its component at w reduces to a generator of the units of the residue field, which has N elements, and its power z_w^i lies in the level exactly when $N-1 \mid i$. For an operator T with $T^r = 1$, the translates $T^i\varphi$, $0 \le i < r$, have sum 0 exactly when $\varphi = \psi - T\psi$: the operator $r - \sum_i T^i$ is r on such φ, and $1-T$ divides it. Computing in the commutative group algebra of the z_w, the intersection of the kernels of the sums $\sum_i z_w^i$ is the image of $\prod_w (1-z_w)$.

Dimension. The kernel of $1-z_w$ consists of the forms fixed by the level K generated by the level of X_0 and z_w; both levels are free, so the dimensions are in the ratio $[K : \text{level}]$, which is at least $N-1$ since some power z_w^i with $0 < i \le [K : \text{level}]$ lies in the level. Hence this kernel has dimension at most $\dim \mathcal{S}/(N-1)$, and the kernel of a product of endomorphisms has dimension at most the sum of the dimensions of the kernels; with the f primes $w \mid \pi_N$ this gives $(1 - f/(N-1))\dim \mathcal{S}$, and $f/(N-1) \le b^{-8}$ since $f \le b$ and $N \ge b^9 + 1$.

Short forms. Apply (a) to the free level of X_0 (Proposition 16.2), of rank k, with the integer $q_1 d$ of the different, and $B = e^{\epsilon_u}/4^f$; the condition $B \ge C^{6M}$ holds by Proposition 8.9. If φ is B-short and k is integral, then φ minus its right translate by k is $4B$-short, as $|u-v|^2 \le 2|u|^2 + 2|v|^2$ and integrality is preserved; so $\prod_w (1-z_w)$, a product of f such factors, takes B-short forms to e^{ϵ_u}-short forms. As the short forms span $\mathcal{S}$, their images span the image of the product, which is the space of ramified forms. □

20.2. Archimedean size

For $j < m$ and a place τ, row i of the diagram of lift $j+1$ has *main length* $c_\tau(1)$ for $i = 0$, $\kappa(\tau)$ for $0 < i < k$ and 0 otherwise, and *slack* $b + \lfloor a/\lfloor x_1 \rfloor \rfloor + r^{16}$ for $i = 0$ and r^{16} otherwise. Put $\mathrm{rs}_j = \sum_\tau \sum_{i<n_j} s_\infty(\ell_{\tau,i})$ over the row lengths $\ell_{\tau,i}$ of the diagram of lift $j+1$, and $\mathrm{cs} = m\big(\#\Phi_a\, s_\infty(a) + k\sum_\tau s_\infty(\kappa(\tau))\big)$.

Lemma 20.2 (One kernel). *For $j < m$, a translation t, finite points g, g' and $\sigma \in \mathrm{Aut}\,\mathbb{C}$, the jet matrix of $\omega(g,g')\varrho(t_j)\mathrm{vac}_j$ for the trivial twist multiplies the norm of the σ-conjugate of a vector by at most $\exp(x^{50} + \mathrm{rs}_j/2)$; and $\sum_{j<m} \mathrm{rs}_j \le \mathrm{cs} + \epsilon_u/2$.*

Proof. One kernel. Apply Theorem 17.3 to the translated vacuum, a vacuum combination of size 1, with the scale $\pi_M \mathsf{a}$ and the growth rate $L = a+1$. The conjugate of $\bar{\pi}_M$ that scales the jets has squared absolute value M, so $|\sigma c|^2/\Omega = M/\Omega$, and the term w of the growth condition is

$$w = \tfrac{1}{2}\sum\big(\ell/L + \log \ell! + \ell\log(M/\Omega)\big) = \tfrac{1}{2}\mathrm{rs}_j + \tfrac{1}{2}\sum \ell/L,$$

the sum over the row lengths ℓ of the first n_j rows at all places. Every row has length at most $2a$, because both its main length and its slack are at most a, so $\sum \ell/L \le 2fn_j$. The remaining factors, $\exp(C_0(q_1 n_j n_{j+1}+1)^2)$ and $(2L+2)^{fn_j n_{j+1}}$, are at most $e^{x^{50}}$ by Proposition 8.9, which gives the first claim.

The sum. Each row length is at most its main length plus its slack, and for a slack e,

$$s_\infty(\ell+e) \le s_\infty(\ell) + e\big(\log(\ell+e+1) + \log(M/\Omega)\big) \le s_\infty(\ell) + 25e\log b,$$

since $\ell + e \le 2a$ and $\log(2a+1) + \log(M/\Omega) \le 25\log b$ (Proposition 8.9). The main lengths give, at each place, $s_\infty(c_\tau(1))$ for the first row and $s_\infty(\kappa(\tau))$ for each of the next $k-1$ rows, and $s_\infty(0) = 0$ for the others; as $s_\infty \ge 0$ one may count k rows of main length $\kappa(\tau)$. Here $c_\tau(1) = a$ at the a-places and 0 elsewhere, so the main lengths of the m lifts add up to at most cs. The slacks add up, per lift and place, to $b + \lfloor a/\lfloor x_1 \rfloor \rfloor + n_j r^{16}$, which is $O(b)$; with the factor $25\log b$, over the mf pairs of a lift and a place, and with $mb \le a$, the total is at most $\epsilon_{\mathrm{u}}/2$. □

Proposition 20.3 (The level pays for the excess). *With s_{main} the main prime sum of Section 9, the following hold.*

(a) $s_\infty(a) - s_{\mathrm{main}} \le a\big(\frac{1}{f}\sum_\tau \log\kappa(\tau) + \log\frac{M\,\mathrm{rd}_E}{2\pi}\big) + 1 + \frac{1}{2}\log a$, *and* $m(s_\infty(a) - s_{\mathrm{main}}) - a\log N \le 32a\log b$.

(b) $h_{\mathrm{main}} - m\,\mathrm{rem} + 2\sum_p g_p \log p = m\#\Phi_a\, s_{\mathrm{main}} + mk\sum_\tau s_\infty(\kappa(\tau))$, *the sum over the small non-split primes.*

(c) $\mathrm{cs} - \#\Phi_a\, a\log N \le h_{\mathrm{main}} - m\,\mathrm{rem} + 2\sum_p g_p\log p + 32fa\log b$.

Proof. (a) By Stirling's formula, $\log a! \le a\log a - a + 1 + \frac{1}{2}\log a$, the logarithm of the Stirling sequence being decreasing. By (8.1) and Section 8, $\log\Omega = -1 - c_1 + \log(2\pi/\mathrm{rd}_E)$ with $c_1 = \mathrm{Re}\,G_1(1) = \mathrm{Re}\,\frac{L'}{L}(1,\chi_K) - \gamma - 1$, so $\log(M/\Omega) = \log M + \log\mathrm{rd}_E - \log 2\pi + 1 + c_1$. With $s_{\mathrm{main}} = a(\log(a/\bar\kappa) + c_1)$ and $\log\bar\kappa = \frac{1}{f}\sum_\tau\log\kappa(\tau)$,

$$s_\infty(a) - s_{\mathrm{main}} \le a\log a - a + 1 + \tfrac{1}{2}\log a + a\left(\log\frac{M\,\mathrm{rd}_E}{2\pi} + 1 + c_1\right) - a\log a + a\log\bar\kappa - ac_1,$$

in which $a\log a$ and c_1 cancel; this is the first inequality. The second is the choice of the level: $N \ge N_- = \bar\kappa^m (M\mathrm{rd}_E/2\pi)^{m-1} b^{-15/2}$ (Definition 8.7), so $a\log N \ge am\log\bar\kappa + a(m-1)\log(M\mathrm{rd}_E/2\pi) - \frac{15}{2}a\log b$, and the terms $ma(\log\bar\kappa + \log(M\mathrm{rd}_E/2\pi))$ cancel; what remains is at most $a\log(M\mathrm{rd}_E/2\pi) + \frac{15}{2}a\log b + m(1 + \frac{1}{2}\log a) \le 32a\log b$ by Proposition 8.9.

(b) The gain is zero at split primes, so the sum in Proposition 19.1(b) may run over all primes below x_1, and each prime sum is the sum of its local terms over the same primes. Substitute the definitions $h_{\mathrm{main}} = mk\sum_\tau(s_\infty(\kappa(\tau)) - s_{\mathrm{fin}}(\kappa(\tau),\tau))$ and $\mathrm{rem} = \sum_{\tau\in\Phi_a}(s_{\mathrm{fin}}(a,\tau) - s_{\mathrm{main}})$:

$$\begin{aligned} &h_{\mathrm{main}} - m\,\mathrm{rem} + m\sum_{\tau\in\Phi_a} s_{\mathrm{fin}}(a,\tau) + mk\sum_\tau s_{\mathrm{fin}}(\kappa(\tau),\tau) \\ &\quad = mk\sum_\tau s_\infty(\kappa(\tau)) + m\#\Phi_a\, s_{\mathrm{main}}, \end{aligned}$$

and the left side is $h_{\mathrm{main}} - m\,\mathrm{rem} + 2\sum_p g_p\log p$.

(c) Multiplying (a) by $\#\Phi_a \le f$ gives $\#\Phi_a(m(s_\infty(a) - s_{\mathrm{main}}) - a\log N) \le 32fa\log b$. Adding $m\#\Phi_a\, s_{\mathrm{main}} + mk\sum_\tau s_\infty(\kappa(\tau))$ to both sides and using (b) gives $\mathrm{cs} - \#\Phi_a\, a\log N \le h_{\mathrm{main}} - m\,\mathrm{rem} + 2\sum_p g_p\log p + 32fa\log b$. □

Theorem 20.4 (Size of the composite lift). *Let Ψ be a chain datum for the trivial twist and $\sigma \in \mathrm{Aut}\,\mathbb{C}$. If the values of $\sigma\varphi$ have squared norm at most e^B, then those of the σ-conjugate of $B_{1,\Psi}\varphi$ have squared norm at most $\exp(B+\beta)$, with*

$$\beta = h_{\mathrm{main}} - m\,\mathrm{rem} + 2 \sum_{p \text{ small non-split}} g_p \log p + \epsilon_{\mathrm{u}}.$$

The same holds for $R_{1,\Psi'}$ for every weak chain datum Ψ'.

Proof. By Proposition 19.2(b), $B_{1,\Psi} = R_{1,\Psi'}$ for a weak chain datum Ψ', so it suffices to treat $R_{1,\Psi'}$. By Proposition 19.3(a) a value of the chain at $\mathrm{diag}(h,1)$ is a sum, over tuples of classes and over t, of $\Psi'(t)$ times products of m kernels applied to values of φ. By Lemma 20.2 each product multiplies the norm of the conjugates by at most $\exp(mx^{50} + \frac{1}{2}\sum_j \mathrm{rs}_j)$, and by its second part $\frac{1}{2}\sum_j \mathrm{rs}_j \le \frac{1}{2}\mathrm{cs} + \frac{1}{4}\epsilon_{\mathrm{u}}$. The levels of the stages contain the principal level of $\pi_M\pi_N$, so there are at most $\exp(x^{30})$ classes at each stage and $\exp(mx^{30})$ tuples of classes (Proposition 20.1(b)); and $\sum_t |\sigma\Psi'(t)| \le \exp(x^{60})$, by the size condition of a weak chain datum for the trivial twist. Hence the norm of a value of the conjugate chain is at most $e^{B/2}\exp(mx^{50}+mx^{30}+x^{60}+\frac{1}{2}\mathrm{cs}+\frac{1}{4}\epsilon_{\mathrm{u}})$. Squaring, and restricting to Y_m by Proposition 19.3(c), the squared norm is multiplied by at most

$$\exp\Big(2mx^{50} + 2mx^{30} + 2x^{60} + x^{70} + \mathrm{cs} + \tfrac{1}{2}\epsilon_{\mathrm{u}} - \textstyle\sum_\tau n_\tau \log N\Big).$$

The branching exponent $\sum_\tau n_\tau$ is $\#\Phi_a\, a$, since the restriction removes the long row at each a-place (Proposition 15.2(c)). By Proposition 20.3(c),

$$\mathrm{cs} - \#\Phi_a\, a \log N \le h_{\mathrm{main}} - m\,\mathrm{rem} + 2\sum_p g_p \log p + 32 f a \log b,$$

and $2mx^{50} + 2mx^{30} + 2x^{60} + x^{70} + 32 f a \log b \le \frac{1}{2}\epsilon_{\mathrm{u}}$ by Proposition 8.9; so the exponent is at most β. □

20.3. The good projection

For a form z of principal level on Y_0 and $s \in \bar{G}$, the *good part* of sz is $\#\bar{G}\, e_0 sz$, a target form.

Proposition 20.5. *Let z be a form of principal level on Y_0, $s \in \bar{G}$ and ϑ a tame character.*

(a) *For every g, with $\ell(ts)$ the lift of* $\mathrm{diag}(ts,1)$ *of Proposition 16.9(c),*

$$\iota_\vartheta(\#\bar{G}\, e_0 sz)(g) = \sum_{t\in\bar{G}} \#\bar{G}\, e_0(t)\, \iota_\vartheta(z)\big(g\,\ell(ts)\big).$$

(b) *If $c\,\iota_\vartheta z$ is integral at an embedding, so is $c\,\iota_\vartheta(\#\bar{G}\, e_0 sz)$.*

(c) *If the values of $\sigma\iota_1 z$ have squared norm at most e^B, then $\|\iota_1(\#\bar{G}\, e_0 sz)\|_\sigma^2 \le \exp(x^{99}+B)$.*

Proof. (a) The action of e_0 is a sum over $\bar{G}$, and by Proposition 16.9(c) the lift turns the action of ts on forms of principal level into right translation by $\ell(ts)$; the formula follows by linearity.

(b) The numbers $\#\bar{G}\, e_0(t)$ are algebraic integers: $\#\bar{G}\, e_0$ is a sum of the $\dim\pi\cdot\bar{\chi}_\pi$ over a set of irreducible representations π of $\bar{G}$, and character values are sums of roots of unity. By (a) the value of $c\,\iota_\vartheta(\#\bar{G}\, e_0 sz)$ at g is a combination, with these coefficients, of values of $c\,\iota_\vartheta z$ at the right translates $g\,\ell(ts)$, which are integral

above ℓ; the gauge at g of the value at $g\,\ell(ts)$ is that at $g\,\ell(ts)$, because $\ell(ts)$ is integral, and the gauge is ultrametric.

(c) The coefficients of σe_0 have absolute value at most 1, because $\#\bar{G}\,\sigma e_0$ is the character of a representation of dimension at most $\#\bar{G}$. Hence each value of $\sigma\iota_1(\#\bar{G}\,e_0 sz)$ is a sum of $\#\bar{G}$ terms of norm at most $\#\bar{G}\,e^{B/2}$, and its squared norm is at most $\#\bar{G}^4 e^B$. The level contains the principal level of $\pi_M\pi_N$, so there are at most $\exp(x^{30})$ classes; and $\log\#\bar{G} \le x^{32}$ (Proposition 16.2). As $x^{30} + 4x^{32} < x^{99}$, the class sum is at most $\exp(x^{99} + B)$. □

20.4. The target lattice

The target $\mathcal{T}$ is a twisted lattice (Definition 7.7): its twists are the m-fold twists ϑ (twists whose tame part ψ has $\psi_w^n \ne 1$ for $w \in Q$ and $1 \le n \le m$), the map at ϑ is $\Theta_\vartheta = \iota_\vartheta$ with values in the forms on Y_m of the level and character of Proposition 16.9(c), ϑ is allowed at ℓ if $\ell \nmid \mathrm{N}\,Q(\vartheta)$, the integral vectors at ι are the forms integral at ι, and the coordinates of $\Theta_1 y$ are the values at representatives of the classes in the standard basis of the Weyl module.

Proposition 20.6. *This defines a twisted lattice: the levels of the forms on Y_m are free, and the inner product of two forms integral at $\bar{\iota}$ and ι is integral at ι. A target form is integral at $\iota \mid \ell$ exactly if $\Theta_\vartheta y$ is integral at ι for all ϑ with $\ell \nmid \mathrm{N}\,Q(\vartheta)$; the norms $\|\Theta_1 y\|_\sigma$ of the lattice are those of the forms; the volume at ϑ is $\det(\Theta_\vartheta^*\Theta_\vartheta)$; and each Θ_ϑ is injective on $\mathcal{T}$.*

Proof. Free levels. For a tame character ϑ the level of the forms on Y_m consists of the integral points that reduce to 1 at the chosen places and to block upper triangular matrices at the places of Q; it lies in the chosen level, hence is free (Proposition 16.2): $\Gamma \cap gKg^{-1}$ is trivial for every g.

Inner products. At a free level the inner product is the sum over the classes of the Gram pairings of the values. Weyl's lattice is self-dual up to the integral Gram matrix, so the pairing of a vector integral at $\bar{\iota}$ (the place of ι composed with complex conjugation) with a vector integral at ι is integral at ι.

The remaining statements. Integrality of a target form at ι is defined through its lifts $\Theta_\vartheta y$ for the allowed ϑ, which is the stated criterion. The coordinates of $\Theta_1 y$ are the values at representatives g_c of the classes; the norm of the lattice is the sum over c of the squared Gram norms of $\sigma(\Theta_1 y(g_c))$, and $\|\Theta_1 y\|_\sigma^2$ is the same sum with the term at c divided by $\#(\Gamma \cap g_c K g_c^{-1})$, which is 1. The volume at ϑ is $\det(\Theta_\vartheta^*\Theta_\vartheta)$ by definition. Injectivity of Θ_ϑ follows from the formula for the norms of the chain proved in Part V, which writes $\|\Theta_\vartheta y\|^2$ as a product of nonzero constants times $\|y\|^2$. □

20.5. Proof of Theorem A

Let ϵ_ℓ be the errors of Theorem 19.13 and $g'_\ell = \lfloor g_\ell - \epsilon_\ell \rfloor$; it vanishes unless ℓ is small non-split or $\ell = N$. Put $c_B = \prod_\ell \ell^{-g'_\ell}$, the product over the small non-split primes and N, a positive rational number. Let c_{ss} be the constant of the seesaw identity for the linear form and the archimedean point fixed in Section 16.5, and define

$$\mathfrak{p}_s\varphi = c_B\,\#\bar{G}\,e_0\,s\,c_{\mathrm{ss}}^{-1}\mathcal{P}\varphi \qquad (s \in \bar{G},\ \varphi \in \mathcal{S}^{\mathrm{ram}}).$$

Then $\mathfrak{p}_s = s\,\mathfrak{p}_1$, and its image is $\#\bar{G}\,e_0 s\mathcal{P}(\mathcal{S}^{\mathrm{ram}})$. Let $\mathcal{S}^{\mathrm{int}}$ denote the ramified source forms integral at every prime place.

Lemma 20.7. *$\log c_B \le -\sum_\ell g_\ell \log \ell + \epsilon_{\mathrm{u}} + b$, the sum over the small non-split primes; $|c_B|_\ell = |\ell^{-g'_\ell}|_\ell$; and $\sum_\ell \log \ell \le b$ over the small non-split primes and N.*

Proof. The bound. By definition $g'_\ell = \lfloor g_\ell - \epsilon_\ell \rfloor \ge g_\ell - \epsilon_\ell - 1$, so $\log c_B = -\sum_\ell g'_\ell \log \ell \le \sum_\ell (-g_\ell + \epsilon_\ell + 1)\log \ell$, the sum over the small non-split primes and N. Here $g_N = 0$, because $N \ge x_1$; the errors give at most ϵ_{u} by Theorem 19.13; and the rounding gives at most $\sum_\ell \log \ell$.

The sum of the logarithms. By Chebyshev's bound, $\sum_{\ell < x_1} \log \ell \le 2x_1$, and $2x_1 + \log N \le b$ by Proposition 8.9.

The absolute values. c_B is a product of powers of the distinct primes ℓ of the index set, so $|c_B|_\ell = |\ell^{-g'_\ell}|_\ell$, the other factors being ℓ-adic units. □

Proof of Theorem A(a). By Proposition 20.1(c) it suffices to show that every $e^{\epsilon_{\mathrm{u}}}$-short ramified form φ lies in $\mathcal{S}^{\mathrm{int}}$ and satisfies the bound. Integrality is part of being short. By Theorem 18.15, with $z = c_{\mathrm{ss}}^{-1}\mathcal{P}\varphi$, there is a chain datum Ψ for the trivial twist with $\iota_1 z = B_{1,\Psi}\varphi$. By Theorem 20.4 the values of $\sigma\iota_1 z$ have squared norm at most $\exp(\epsilon_{\mathrm{u}} + \beta)$; by Proposition 20.5(c) the target form $\#\bar{G}\,e_0 s z$ has $\|\Theta_1(\cdot)\|_\sigma^2 \le \exp(x^{99} + \epsilon_{\mathrm{u}} + \beta)$; and multiplying by c_B adds $2\log c_B$. By Lemma 20.7 the gains cancel:

$$\|\Theta_1 \mathfrak{p}_s \varphi\|_\sigma^2 \le \exp\bigl(h_{\mathrm{main}} - m\,\mathrm{rem} + x^{99} + 4\epsilon_{\mathrm{u}} + 2b\bigr) \le \exp\bigl(h_{\mathrm{main}} - m\,\mathrm{rem} + f a \log^4 b\bigr),$$

since $x^{99} + 4 f a \log^3 b + 2b \le f a \log^4 b$. □

Proof of Theorem A(b). Fix s and a prime place $\iota \mid \ell$. Let S_ℓ be the module of Theorem 19.13, of colength dividing ℓ^l in $\mathcal{S}^{\mathrm{int}}$ with $l \le \epsilon_\ell \dim \mathcal{S}$. For $\varphi \in S_\ell$ and a twist ϑ with $\ell \nmid \mathrm{N}\,Q(\vartheta)$, $B_{\vartheta,\Psi}\varphi$ has valuation at least $g_\ell - \epsilon_\ell \ge g'_\ell$, so $c_B \iota_\vartheta z = c_B B_{\vartheta,\Psi}\varphi$ is integral at ι, because $v_\iota(c_B) = -g'_\ell$. By Proposition 20.5(b) so is $c_B \iota_\vartheta(\#\bar{G}\,e_0 s z) = \Theta_\vartheta \mathfrak{p}_s \varphi$, and $\mathfrak{p}_s\varphi$ is an integral vector of the target lattice at ι (Proposition 20.6). Hence a family in $\mathfrak{p}_s(\mathcal{S}^{\mathrm{int}})$ becomes integral at ι after a matrix of determinant dividing ℓ^l, $l \le \epsilon_\ell \dim \mathcal{S} \le 2\epsilon_\ell \dim \mathcal{S}^{\mathrm{ram}}$ (Lemma 7.9(c); as $\dim \mathcal{S} \le 2 \dim \mathcal{S}^{\mathrm{ram}}$ by Proposition 20.1(c)). With $\delta_\ell = 2\epsilon_\ell \dim \mathcal{S}^{\mathrm{ram}}$, $\sum_\ell \delta_\ell \log \ell \le 2\epsilon_{\mathrm{u}} \dim \mathcal{S}^{\mathrm{ram}}$. For a set Σ of translates, a family whose members lie in $\mathfrak{p}_s(\mathcal{S}^{\mathrm{int}})$ for various $s \in \Sigma$ splits into $\#\Sigma$ subfamilies, and the indices add (Lemma 7.9(a), (b)), which gives $\#\Sigma \cdot \dim \mathcal{S}^{\mathrm{ram}} \cdot 2\epsilon_{\mathrm{u}}$. □

Theorem A is stated in Section 9.2 with $\mathcal{S}$ for the ramified source forms; the proofs above use exactly that space.

21. Elementary divisors of edge Hecke operators

This section proves the theorem on elementary divisors used at the exceptional primes (Theorem 19.9). A Hecke operator of an edge type, acting on functions on lattices with values in forms of large degree twisted by a cycle of Frobenius powers, is shown to be onto modulo p^m up to a colength whose mean exponent per coordinate is at most $3c_{\mathrm{div}} + \theta m$, with c_{div} small (Theorem 21.14). The proof factors the operator through functions on edges, $M \subseteq M_{\mathrm{all}} \subseteq M_{\mathrm{edge}} \to \mathcal{N} \subseteq M^\circ$, and bounds three things: the colength of the integral functions on lattices among the functions on edges that depend only

on the origin (*depth*, Section 21.7); the image of the adding map, through a near inverse that satisfies congruences at each edge (*averaging*, Section 21.8); and the number of further generators these congruences leave room for (*defect count*, Section 21.6), which rests on a root lemma of Stepanov type and on the volume of the cone of Pieri's inequalities.

21.1. Frobenius cycles

Definition 21.1 (Frobenius cycle). Let W be a local ring with finite residue field and R a ring. A *Frobenius cycle* with N letters consists of a prime p, an exponent o with $P = p^o \geq 5$, ring maps $\iota_s \colon W \to R$ for the *letters* $s \in \mathbb{Z}/N$, each marked *even* or *odd* with neighbouring letters of opposite parity, and an element $w \in R$ with $w^2 = p$, such that $\iota_{s+1}(x) \equiv \iota_s(x)^P \pmod p$ for all x and the residue field of W has $q = P^N$ elements.

The variables are $x_{b,s}$ with $b < k$ and $s \in \mathbb{Z}/N$, and a *form* of degree n is a polynomial homogeneous of degree n in each letter; its coefficients are indexed by the *exponents*, N-tuples of monomials of degree n in k variables. An element $g \in \mathrm{GL}_k(W)$ acts on the letter s through the *letter matrix* $g_s = \iota_s(g)$ for even s and $\iota_s({}^t g^{-1})$ for odd s, by the linear substitution $x_{c,s} \mapsto \sum_b (g_s)_{bc} x_{b,s}$; on coefficient vectors this is the Kronecker product $\rho(g)$ of the n-th symmetric powers of the g_s. For $0 < r \leq k/2$ a variable $x_{b,s}$ is *deep* if s is even and $b \geq r$ or s is odd and $b < r$; the *depth* of an exponent is its degree in the deep variables. For indices $a \neq a'$, a letter i and $u_s = x_{a',s}$, $\bar u_s = x_{a,s}$ for even s (reversed for odd s), the *defect* of an exponent β at i along the root (a, a') is

$$\mathrm{def}_i(\beta) = \beta(u_i) + \sum_{s \in \mathbb{Z}/N} \frac{P^{(s-i) \bmod N}}{q-1} \big(\beta(u_s) - \beta(\bar u_s)\big),$$

a linear form in β; for β of degree at most d in each letter, $|\mathrm{def}_i(\beta)| \leq 2d$.

21.2. Edges and halfway lattices

An *edge* is a pair of W-lattices $L' \subseteq L$ of rank k with $\varpi L \subseteq L'$ and $\dim L/L' = r$; L' is its *origin*. In a basis of L' adapted to the edge, the stabiliser of the edge is the parahoric subgroup with blocks r, $k - r$. The forms integral for $L' + wL$, times $w^{-nN/2}$, have the basis of the *normalised monomials* $w^{-\mathrm{depth}\,\beta} z^\beta$; this module is the *halfway lattice* of the edge. Let $\mathcal{M}_{\mathrm{all}}$ be the forms that lie in the halfway lattices of all edges from a given lattice, in the normalised coefficients of one edge. The lattices $H \supseteq H^\circ$ spanned by the monomials and by the divided powers $(n!^N/\beta!) z^\beta$ are stable under $\mathrm{GL}_k(W)$, and $n!^N/\beta! = \prod_s \binom{n}{\beta_s}$ has p-adic valuation at most $c_0 = k^2 \log_p q$ when $n < q$.

Proposition 21.2 (Edges). *Let $R = O$ be a domain of characteristic zero.*

(a) *The stabiliser of an edge preserves its halfway lattice: for u parahoric, $\Delta\rho(u) = A\Delta$ with A integral, Δ the diagonal of the $w^{\mathrm{depth}\,\beta}$.*

(b) *If p is not a unit and $n < q$, there are weights b, b' with $\mathrm{diag}(b)\rho({}^t g) = {}^t\rho(g)\mathrm{diag}(b)$ for all g and $b_\beta b'_\beta = p^{Nc_0}$: an* edge pairing, *which pairs the halfway lattices of an edge and of the dual edge.*

(c) *Let $a < r$, $v \in \mathcal{M}_{\mathrm{all}}$ and g block diagonal. If $\mathrm{def}_i(\beta) \geq 3j$ along (a, r), then w^j divides the coefficient of $\rho(g)v$ at β; and if $\mathrm{def}_i(\beta) > 3kc$, then w^c divides the*

pairing of v with $\rho(g)e_\beta$. In particular $\mathcal{M}_{\mathrm{all}}$ satisfies the defect conditions*: its block diagonal translates have no term of defect above $9kc$ modulo w^c.*

(d) *Suppose every edge from every origin occurs among the edges e, with origin* $\mathrm{src}(e)$ *and basis change g_e, and all depths are at most x. The functions on edges that depend only on the origin form a direct summand, the image of a matrix Q with a left inverse; and the restriction of integral functions on origins is Qd with* $\det d \mid w^l$, $l = \#X \sum_{1\le\delta\le x} \dim \mathcal{D}_\delta$, *where $\mathcal{D}_\delta$ is the space over the residue field of forms all of whose translates have depth at least δ.*

Proof. (a) For a letter s let Δ_s be the diagonal matrix with w at the deep variables of s and 1 at the others; then Δ is the Kronecker product over the letters of the n-th symmetric powers of the Δ_s. For u parahoric, the entries of the letter matrix u_s in a shallow row and a deep column are divisible by p: for even s they come from the entries of u in the positions (i, j) with $i < r \le j$, which lie in the maximal ideal and are sent into (p) by ι_s, and for odd s from the corresponding entries of u^{-1}, which is parahoric too. Hence $\Delta_s u_s \Delta_s^{-1} = A_s$ is integral: its entry (i, j) is $w^{d_i - d_j}(u_s)_{ij}$ with d the depth indicator, and only $d_i = 0$, $d_j = 1$ gives a negative power, at an entry divisible by $p = w^2$. Taking symmetric powers and Kronecker products, $\Delta\rho(u) = A\Delta$ with $A = \bigotimes_s \mathrm{Sym}^n(A_s)$ integral.

(b) Over a domain of characteristic zero the n-th symmetric powers of a matrix and of its transpose are transposes of each other up to the multinomial coefficients: $\langle z^\alpha, z^\alpha \rangle = \alpha!$ defines a pairing for which the substitutions by g and ${}^t g$ are adjoint. In one letter take $b_0(\gamma) = p^{k \log_p q} / \binom{n}{\gamma}$, which is integral because a multinomial coefficient of degree $n < q$ has p-adic valuation at most $k \log_p q$; multiply by a fixed power of p to raise the exponent to $c_0 = k^2 \log_p q$, and take the product over the letters. Then $\mathrm{diag}(b)\rho({}^t g) = {}^t\rho(g)\mathrm{diag}(b)$, and $b'_\beta = p^{Nc_0}/b_\beta$ is integral.

(c) *A consequence of the root lemma.* The depth weights rise by 1 from $\bar{u}_s$ to u_s in each letter, and the defect weights of u_s and $\bar{u}_s$ differ by an amount in $[0, \frac{3}{2}]$ (as $5P^{(s-i) \bmod N} \le q$); so the weights $\mathrm{depth} - \frac{1}{3}\mathrm{def}_i$ rise by amounts in $[\frac{1}{2}, 1]$. If in all translates of a polynomial by the $\mathrm{SL}_2(W)$ in the rows and columns a, r every term has w-adic order plus depth at least v_0, then every term of the polynomial itself has order plus depth at least $v_0 + \frac{1}{3}\mathrm{def}_i$: otherwise take, among all terms of all translates, one of least order plus weight for the weights $\mathrm{depth} - \frac{1}{3}\mathrm{def}_i$; by Theorem 21.5 its defect is at most 0, so its order plus depth is below v_0, against the hypothesis.

Application. Write $v = w^{-m}\Delta h$ with w^m dividing the normalised coefficients $w^{\mathrm{depth}\,\beta}(\rho(\gamma)h)_\beta$ for all $\gamma \in \mathrm{GL}_k(W)$. For g block diagonal, $\rho(g)$ commutes with Δ, and the $\mathrm{SL}_2(W)$ of the root is contained in $\mathrm{GL}_k(W)$; so the consequence above, applied to $\rho(g)h$, gives that w^{m+j} divides $w^{\mathrm{depth}\,\beta}(\rho(g)h)_\beta$ when $\mathrm{def}_i(\beta) \ge 3j$, that is, w^j divides the coefficient of $\rho(g)v$ at β. For the pairing, $\langle v, \rho(g)e_\beta \rangle = b_\beta(\rho({}^t g)v)_\beta$ by (b), and ${}^t g$ is again block diagonal; a defect above $3kc \ge 3c$ gives the factor w^c. With $j = c$ and the threshold $9kc \ge 3c$ the defect conditions follow.

(d) *The direct summand.* Let J be the matrix of the restriction $\phi \mapsto (e \mapsto \Delta\rho(g_e)\phi(\mathrm{src}(e)))$. By elementary divisors over the valuation domain (a Hermite form followed by a Smith form) there are an invertible P, a matrix Q with a left inverse R and scalars t_i with $JP^{-1} = Q\,\mathrm{diag}(t)$. A function on edges that

depends only on the origin has a multiple $w^m\psi$ in the image of J, hence in the image of Q, and QR fixes it; so these functions are the image of Q, a direct summand, and $J = Qd$ with $d = \operatorname{diag}(t)P$.

The determinant. Since every edge from every origin occurs, every $\gamma \in \mathrm{GL}_k(W)$ is ug_e with u parahoric and $\operatorname{src}(e)$ the given origin; so by (a), if t_i divides all coefficients of the restriction of the column i of P^{-1}, it divides those of $\Delta\rho(\gamma)$ applied to its value at every origin. With $\gamma = 1$ and all depths at most x, t_i divides w^x times a column of an invertible matrix, hence divides w^x. If t_i does not divide w^δ, then $w^{\delta+1}$ divides $w^{\operatorname{depth}\beta}(\rho(\gamma)P^{-1}e_i)_\beta$ for all γ and β, so the reductions of the values of this column lie in $\mathcal{D}_{\delta+1}$. The columns of P^{-1} are independent modulo the maximal ideal, so there are at most $\#X \dim \mathcal{D}_{\delta+1}$ such i. As $\det d$ is $\prod_i t_i$ up to a unit, and each t_i divides w^x, the product divides w^l with l the number of pairs (i, δ), $\delta < x$, with $t_i \nmid w^\delta$, which is at most $\#X \sum_{1\le\delta\le x} \dim \mathcal{D}_\delta$. □

21.3. Frobenius factors

The twists $\iota_s(g)$ are not algebraic in g, but modulo p they are entrywise P-th powers of one another. Take one generic $k \times k$ matrix for each letter (a *copy*) and let the copy $s - d$, $0 \le d \le \ell$, act on the letter s through its entrywise P^d-th power, or through the cofactor matrix of this power if s is odd, the last factor $d = \ell$ acting on the deep variables only; this is the *substitution operator*. A term of an entry of a product of matrices belongs to a path of indices, and the *weight* of a monomial in the copies is the sum of the scaled index weights along the path.

Lemma 21.3 (Weight and defect). *Let $r = a + 1$, $\ell \le N$ and $Nnk(\ell + 2) \le P^\ell$, and take an entry in row β of the substitution operator at the generic factors after a change of frame by a block diagonal h. If the coefficient of a monomial in it is not divisible by w^c, its weight at the copy i is within $3kc$ of the defect of β at the letter $i + \ell$.*

Proof. Levels. Use the Gauss filtration for the w-adic order, shifted by real weights of the variables: a polynomial has level t if its coefficient at each monomial μ is divisible by $w^{\lceil t - \text{weight}\,\mu\rceil}$. Levels add under products, a cofactor in row b of a matrix whose row b has level $x_b - v$ (with $\sum x_b = 0$, $v \ge 0$) has level $-x_b - kv$, and the n-th symmetric power of a matrix whose row b has level y_b has the row β of level $\sum_b \beta_b y_b$.

Potentials. Give the entry (b, c) of the copy j the weight ϵ times its entry weight: ρ_b/P^ℓ if $j = i$, and $P^{(j-i) \bmod N}(\rho_b - \rho_c)/(qP^\ell)$ otherwise, with $\rho = e_{a'} - e_a$ and $|\epsilon| \le \frac{1}{2}$. The entry (b, c) of the P^d-th power of the copy $s - d$ then has a weight that differs by at most $1/P^\ell$ from the difference of the scaled index weights of b at the exponent $d + 1 + m'$ and of c at the exponent $d + m$, with $s - d = i + m$, $m < N$; so the potentials telescope along the product of at most $\ell + 1$ factors. The change of frame by the block diagonal h does not change the levels: conjugation by $\iota(h)$ preserves each level, and the P^d-th power of the conjugate by $\iota_{s-d}(h)^{-1}$, conjugated back by $\iota_s(h)$, agrees modulo p with the entrywise power, because $\iota_{s-d}^{P^d} \equiv \iota_s$ modulo p; the difference, a multiple of $p = w^2$, is absorbed since all weights are at least -1. Hence row β of the operator has level $\sum_{b,s} \beta_{b,s} x_{b,s} - Nnk(\ell+2)/(2P^\ell)$, where $x_{b,s}$ is $\pm$ the final potential, with

+ for even and − for odd s (the cofactor changes the sign and multiplies the loss by at most k).

The defect. For $j = i + \ell$, $(1 - 1/q)$ times the defect weight of (b, s) is $\pm\rho_b P^{d+m}/(qP^\ell)$, the scaled index weight of the final potential, with $d = \ell + 1$ or ℓ according to whether (b, s) is deep: one of (a, s), (a', s) is deep and the other is not, which gives the two exponents of the defect weight. Hence row β has level $t = \epsilon(1 - 1/q)\mathrm{def}_{i+\ell}(\beta) - L$ with $L = Nnk(\ell + 2)/(2P^\ell) \le \frac{1}{2}$.

Conclusion. Let D be the weight of the monomial and $E = (1 - 1/q)\mathrm{def}_{i+\ell}(\beta)$. For $\epsilon = \pm\frac{1}{2}$ the coefficient is divisible by $w^{\lceil \epsilon(E-D)-L \rceil}$; as it is not divisible by w^c, $\epsilon(E - D) - L \le c$, so $|E - D| \le 2c + 1$. Since $2n \le q$, $|\mathrm{def} - E| = |\mathrm{def}|/q \le 1$. Hence the weight is within $2c + 2 \le 3kc$ of the defect, as $k \ge 2$ and $c \ge 1$. □

21.4. The root lemma

Lemma 21.4 (Stepanov's count). *Let W be local with q residue classes and $\iota\colon W \to R$ a ring map. If a nonzero polynomial $\Phi \in R[X]$ satisfies $X^E \mid \Phi(X + \iota x)$ for every $x \in W$, then $Eq \le \deg \Phi$.*

Proof. Lift the residue classes to elements $\iota(x_1), \dots, \iota(x_q)$ of R; for $x \ne y$ the difference $\iota(x) - \iota(y)$ is the image of a unit of W, hence a unit, so the polynomials $(X - \iota x)^E$ are pairwise coprime. Each divides Φ, so their product, monic of degree Eq, divides $\Phi \ne 0$, and $Eq \le \deg \Phi$ [Ste69, Bom74]. □

Theorem 21.5 (Root lemma). *Let $a \ne a'$, i a letter, and let weights c of the variables rise from $\bar{u}_s$ to u_s by amounts in $[\frac{1}{2}, 1]$. Suppose that in all translates of f by the $\mathrm{SL}_2(W)$ in the rows and columns a, a' every term has w-adic order plus weight at least $l + c \cdot \alpha$, and that w^{l+1} does not divide the coefficient of x^α in f. Then $\mathrm{def}_i(\alpha) \le 0$. Over a field of characteristic p, a term of least height among the terms of all translates, for heights rising by 1, has defect at most 0.*

Proof. Notation. Write $X_s = \bar{u}_s$, $Y_s = u_s$, $P_s = P^{(s-i) \bmod N}$, and let ν be the weight of α for the weights q at Y_i and P_s at Y_s $(s \ne i)$, and $q\nu'$ its weight for the weights qP_s at X_s; all other variables have weight 0. Then $\mathrm{def}_i(\alpha) = (\nu - \nu')/(q - 1)$, so it suffices to show $\nu q \le q\nu'$.

The auxiliary polynomial. With a new variable t, substitute $Y_i \mapsto X_i$, $X_s \mapsto t^{qP_s} X_s - \epsilon_s t^{P_s} Y_s$ and $Y_s \mapsto Y_s - \epsilon_s X_s$ for $s \ne i$, with $\epsilon_s = \pm 1$ according to the parity of s relative to i, and let $\Phi(t)$ be the coefficient, modulo w^{l+1}, of the monomial obtained from x^α by writing X_i for Y_i. In the main part of the substitution the coefficient of $t^{q\nu'}$ in Φ is the coefficient of x^α in f, which is not divisible by w^{l+1}: an exponent is determined by its image under $Y_i \mapsto X_i$ together with its weight for the weights qP_s. Filter the polynomials over $R[t]$ by giving w the weight 1, t the weight $-\rho$ with $\rho = (c(Y_i) - c(X_i))/q > 0$, X_s the weight $c(X_s) + qP_s\rho$ and the other variables their weights c. As $5P_s \le q$ and the increments lie in $[\frac{1}{2}, 1]$, the cross terms change the image of f only in a level higher by $\frac{1}{20}$, and the hypothesis on the terms of f puts the image at level l plus the weight of x^α. Hence $\Phi \ne 0$ has degree at most $q\nu'$: this is the expansion at infinity.

The finite points. For $r \in W$ let γ_r be the element of the root $\mathrm{SL}_2(W)$ that is $\left(\begin{smallmatrix} 0 & 1 \\ -1 & \pm r \end{smallmatrix}\right)$ in the rows and columns a, a'; it is a transposition times a transvection times a diagonal sign. It sends X_s to Y_s and Y_s to $X_s + \epsilon_s \iota_s(r) Y_s$ up to signs.

Acting by γ_r, substituting as above and replacing t by $t + \iota_i(r)$ gives, up to signs, the substitution applied to the variables with X_s and Y_s exchanged, plus the Frobenius errors $(t + \iota_i(r))^{P^n} - t^{P^n} - \iota_{i+n}(r)$, which are multiples of p (as $\iota_{i+n} \equiv \iota_i^{P^n}$ modulo p) and so gain 2 in the filtration. Applying the hypothesis to the translate of f by γ_r^{-1}, in the filtration in which t weighs ρ and a variable z weighs $c(z)$ minus its zero order times ρ, the shifted polynomial $\Phi(t + \iota_i(r))$ has its coefficients of t^j divisible by w^{l+1} for $j < \nu$: it vanishes to order at least ν at $t = \iota_i(r)$.

Stepanov. By Lemma 21.4 with $E = \nu$, $\nu q \le \deg \Phi \le q\nu'$, so $\mathrm{def}_i(\alpha) \le 0$. For the second statement, over a field of characteristic p, take $w = 0$, so that orders are 0 or infinite, and the weights the heights, which rise by 1; a term of least height among the terms of all translates satisfies the hypothesis with l its height minus its weight. □

21.5. The cone of Pieri's inequalities

Vectors ν are real functions of (b, s). The *step map* sends ν to the signed gaps between consecutive letters, twisted by permutations, and the *Pieri cone* is cut out by the inequalities $\mathrm{def}_i \le 0$ along all simple roots $(b, b+1)$ at all letters, one half of the interlacing inequalities between ν and P times its shift. The *slab* of degrees D consists of the $\beta \ge 0$ whose degree in each letter s lies in $[D_s - 1, D_s]$.

Proposition 21.6 (Volume of the cone). *For every slab,* $\prod_{l<k} \frac{q^{l+1}-1}{q-1} \cdot \mathrm{vol}(\textit{Pieri cone} \cap \textit{slab}) = \mathrm{vol}(\textit{slab})$*; that is, the cone fills the inverse of the number of complete flags over* $\mathbb{F}_q$ *of each slab.*

Proof. Interlacing as a determinant. For weakly decreasing $x, y \in \mathbb{R}^k$, the matrix with entry 1 at (b, c) if $y_c < x_b$ and 0 otherwise has determinant 1 if $x_0 > y_0 \ge x_1 > y_1 \ge \dots$ and 0 otherwise, by induction on k: as every $x_{b+1} \le y_0$, interlacing is $y_0 < x_0$ together with interlacing of the tails. Hence the indicator of the cone, read on the decreasing rearrangement ν' of each block of ν, is the signed sum over families σ of permutations of the indicators that all steps $\pm(P\nu(b, s+1) - \nu(\sigma_s b, s))$ are positive.

Volumes. For $P > 1$ the step map with permutations σ is invertible (in its kernel $P\nu$ would only rearrange the entries of ν), and up to a constant independent of σ its determinant is $\det(1 - \bar{\sigma}/q)$, $\bar{\sigma} = \sigma_0 \cdots \sigma_{N-1}$, by factoring it as a signed shift, a triangular map and a permutation of blocks that moves all twists to the last step and using $\det(1 - BA) = \det(1 - AB)$. The set of ν whose image under the step map with permutations lies in the slab and has positive entries has volume $\mathrm{vol}(\mathrm{slab})/|\det|$, the walls being null. Integrating the pointwise identity over ν shows that the cone fills the share

$$c = \frac{|\det(1 - 1/q)|}{k!} \sum_{\tau} \frac{\operatorname{sign} \tau}{|\det(1 - \tau/q)|}$$

of every slab, the sum over the permutations of $\{0, \dots, k-1\}$, after summing over the families with a given product, each product occurring $k!^{N-1}$ times.

One letter. The share depends only on k and q, so it may be computed for one letter with the sign + and q in place of P. There the cone $\{q\beta_{b+1} \le \beta_b\}$ meets the simplex $\{\beta \ge 0,\ \sum \beta_b \le 1\}$ in an orthant for the coordinates $\beta_b - q\beta_{b+1}$;

rescaling these coordinates by suitable factors gives a triangular map that keeps the degree $\sum_b \beta_b$, and its determinant is $\prod_{l<k}(q-1)/(q^{l+1}-1)$. This is the share. The comparison with the Horn problem is in [FZ21]. □

21.6. Counting forms of bounded defect

Theorem 21.7 (Few forms have bounded defect). *Let K be a field of characteristic p, $a' = a+1$, $t \geq 2(k+1)$, $\theta \leq 1$ and $8Nkt\,P^{2a'} \leq \theta n$. Let C be the space of forms whose translates by the Levi subgroup with blocks a', $k-a'$ have no term of defect above $3t$ along (a, a') at any letter. Then*

$$[\mathrm{GL}_k(W) : \textit{parahoric}] \cdot \dim C \leq (1+\theta) \cdot \#\{\textit{exponents}\}.$$

Proof. Power packing. Let m be a power of q. In characteristic p, $\iota_s(x)^m = \iota_s(x)$ on the values of the twists, so the action of $\mathrm{GL}_k(W)$ commutes with the substitution $x \mapsto x^m$. For each class u of exponents modulo m let γ_u be the exponent in the class u whose degree in each letter lies in $[mk, mk+m]$ (add a multiple of m at the index a in each letter). Given forms $F_u \in C$, one for each class, the packed polynomial $\sum_u F_u(x^m)x^{\gamma_u}$ determines them: its coefficient at $m\beta+\gamma_u$ is that of F_u at β. Its translate by g is $\sum_u(\rho(g)F_u)(x^m)\,(g\cdot x^{\gamma_u})$, so every exponent of a translate is $m\delta+\epsilon$ with δ an exponent of some $\rho(g)F_u$ and ϵ of degree in $[mk, mk+m]$ in each letter. For g in the Levi subgroup, the defect of such an exponent along (a, a') is at most $m(3t+2(k+1)) \leq 4tm$, since δ has defect at most $3t$ and the defect of ϵ is at most twice its degree.

Least height. Give the variable (b, s) the height b for even s and $-b$ for odd s. In characteristic p, a matrix that is lower triangular modulo the maximal ideal sends each variable to a multiple of itself plus variables of greater height, so it multiplies the coefficient of a term of least height by a product of diagonal entries. Hence a term of least height among all terms of all translates of a nonzero packed polynomial by the Levi subgroup occurs in the translate by a representative of a coset of the lower Iwahori subgroup in the Levi subgroup (a flag), and by Theorem 21.5 (second form) its exponent α has defect at most 0 along the simple roots inside the blocks.

The shift. Let $\mu = (P^{2a}, \dots, P^2, 1, 0, \dots, 0)$, and let ν be μ at even and $P\mu$ at odd letters; then the step image σ of ν is $(P^2-1)\mu$ at even and 0 at odd letters, an integral vector ≥ 0 of degree at most $P^{2a'}-1$ whose defect is at most -1 along (a, a') and at most 0 along the other simple roots (the defect of a step image is computed by telescoping over the letters). Hence $\alpha/m + 4t\sigma$ lies in the Pieri cone, in the slab of degrees $D_s = n+k+1+\deg_s(4t\sigma) \in [1, n+4tP^{2a'}]$.

The count. Packing m^{kN} elements of C and listing the coefficients of the translates by the flag representatives at the exponents α with $\alpha/m + 4t\sigma$ in the cone and slab gives an injective linear map. Hence $m^{kN}\dim C$ is at most the number of flags of the Levi subgroup times the number of points of $m^{-1}\mathbb{Z}^{kN}$ in the cone cut by the slab; dividing by m^{kN} and letting m grow, the point count tends to the volume of this bounded convex set. By Proposition 21.6 the volume is the volume of the slab divided by the number $\prod_{l<k}(q^{l+1}-1)/(q-1)$ of complete flags, and the parahoric index times the number of flags of the Levi subgroup is at most that number. The degrees satisfy $D_s \leq n+4tP^{2a'}$, as $4t \geq k+1$, so the slab has volume at most $((n+4tP^{2a'})^{k-1}/(k-1)!)^N$, and there are at least

$(n^{k-1}/(k-1)!)^N$ exponents; the ratio is at most $(1+4tP^{2a'}/n)^{(k-1)N} \le e^{\theta/2} \le 1+\theta$ because $8Nk tP^{2a'} \le \theta n$ and $\theta \le 1$. □

21.7. Few forms are deep

Lemma 21.8 (Zeros against cost). *Let K be a field, give the variable u of a polynomial ring over K the cost $1/\kappa_u$ for integers $\kappa_u > 0$, and let z_t be at least $\prod_u \kappa_u$ distinct points at which each $X_u^{\kappa_u}$ agrees with a polynomial g_u whose monomials cost less than 1. Then every $f \neq 0$ that vanishes to order d at all these points has a monomial of cost at least d.*

Proof. Put $\psi_u = X_u^{\kappa_u} - g_u$, which vanishes at all the points, and $b_{\alpha,\gamma} = X^\alpha \psi^\gamma$. *Division.* If $\operatorname{cost}(\alpha) + |\gamma| \le R$, then $b_{\alpha,\gamma}$ is a combination of the $b_{\beta,\gamma'}$ with $\beta < \kappa$ entrywise and $\operatorname{cost}(\beta) + |\gamma'| \le R$: by induction on the cost, if $\alpha_u \ge \kappa_u$ write $X^\alpha = X^{\alpha-\kappa_u e_u}(\psi_u + g_u)$; the first term raises $|\gamma|$ by 1 and lowers the cost by 1, and the second replaces $X_u^{\kappa_u}$ by monomials of cost below 1. *Jets.* Let J send a polynomial to its jets of order below d at the points, the coefficients of degree below d after translating each point to 0. As the points are distinct, J is onto, and $b_{\alpha,\gamma}$ with $|\gamma| \ge d$ lies in its kernel. By division every polynomial lies in the span of the $b_{\beta,\gamma}$ with $\beta < \kappa$, $|\gamma| < d$, plus the kernel of J; so the images under J of these products span the space of jets, whose dimension is the number of points times the number of γ with $|\gamma| < d$, at least their number. Hence they are linearly independent. *Conclusion.* If all monomials of f cost less than d, division keeps f in the span of these products, as costs are nonnegative; and $J(f) = 0$, so $f = 0$. □

Theorem 21.9 (Depth). *Over a field of characteristic p, with $0 < 2r \le k$, for every M*

$$\sum_{1 \le \delta \le M} \dim \mathcal{D}_\delta \le c_{\mathrm{div}} \cdot \#\{\textit{exponents}\}, \qquad c_{\mathrm{div}} = Nn\big(P^{-2r} + r^2/(kP)\big).$$

Proof. Write $k = 2r + m'$. A variable (b, s) is *deep* if s is even and $b \ge r$, or s is odd and $b < r$; even and odd letters alternate, so each index is deep in $N/2$ letters. By definition $F \in \mathcal{D}_\delta$ when every translate of F has only monomials of degree at least δ in the deep variables.

Specialisation. Let L be a field containing the polynomials over K in all variables. The map that keeps the deep variables and reads the constants and the shallow variables in L is injective on polynomials, so a nonzero F stays nonzero as a polynomial f in the deep variables over L, and each monomial of f is the deep part of a monomial of F.

Many zeros. The *positions* are the pairs $t = (a, l)$ with $a < r$ and $l \le m'$, placed in the row a and the column $c_t = a + r$ if $l = 0$ and $c_t = k - l \ge 2r$ otherwise. For residues $y = (y_t)$ let A be the matrix with lifts of the y_t at the positions and 0 elsewhere. Its nonzero entries lie in rows $< r$ and columns $\ge r$, so $A^2 = 0$ and $(1 + A)^{-1} = 1 - A$. The translate of an even variable $X_{(b,s)}$ by $(1 + A)^{-1}$ is $X_{(b,s)} - \sum_c \iota_s(A_{cb}) X_{(c,s)}$, with $c < r$ in the sum, and that of an odd variable is $X_{(b,s)} + \sum_c \iota_s(A_{bc}) X_{(c,s)}$, with $c \ge r$; in both cases a deep variable X_v becomes $X_v - z_v(y)$, where $z_v(y)$ is a linear form in the twists ι_s of the lifts whose coefficients are shallow variables (up to sign). Let G be the translate

of F by $1+A$; then F is the translate of G by $(1+A)^{-1}$, and the specialisation sends every deep variable of this translate into the ideal of the point $z(y)$. As G has only monomials of deep degree at least δ, the specialised F lies in the δ-th power of this ideal: f vanishes to order at least δ at $z(y)$. Distinct residues give distinct points, because each ι_i of a lift is the value at $z(y)$ of a polynomial (next paragraph), and the residue of an element is determined by its image under ι_i.

Small leading powers. Give the deep variable $v=(b,s)$ the degree $\kappa_v=P^{2r}$ if $b\ge 2r$ and $\kappa_v=P$ if $b<2r$. In characteristic p, raising to a power P^j is additive and $\iota_s(w)^{P^j}=\iota_{s+j}(w)$, so $z_v^{P^j}$ is a linear form in the ι_{s+j} of the lifts, with the P^j-th powers of the coefficients of z_v. Fix a letter i and consider the coefficient vectors of the $z_v^{P^j}$ with $s+j=i$ and $P^j<\kappa_v$; they span all vectors indexed by the positions. Indeed, at a position (b,l) with $l\neq 0$ the deep variables $(k-l,s)$ with s even have $\kappa=P^{2r}$, and $z_{(k-l,s)}=\sum_{a<r}\iota_s(y_{(a,l)})X_{(a,s)}$; the r shifts $j<2r$ with $i-j$ even give an $r\times r$ system with matrix $(X_{(a,i-j)}^{P^j})_{j,a}$, whose determinant is a signed sum of distinct monomials, as the exponents P^j are distinct, hence nonzero over L. At a position $(b,0)$ take $j=0$ and a deep variable of the letter i whose form involves $y_{(b,0)}$; its other entries lie at positions with $l\neq 0$, already in the span. Hence every linear form in the $\iota_i(y_t)$ is the value at $z(y)$ of a polynomial, independent of y, whose monomials are products of powers z_v^e with $e<\kappa_v$; with the variable v of cost $1/\kappa_v$ these monomials cost less than 1. In particular $z_v^{\kappa_v}$, a linear form in the ι_{s+J} of the lifts for $\kappa_v=P^J$, agrees at every point with such a polynomial. The product of the κ_v is $(P^{2r})^{m'N/2}P^{2r\cdot N/2}=q^{rm'}q^r$, the number $q^{r(m'+1)}$ of families of residues at the positions. By Lemma 21.8, f has a monomial of cost at least δ; for F this is a monomial of weight at least δ for the costs P^{-2r} of the deep variables with $b\ge 2r$, P^{-1} of the other deep variables and 0 of the shallow ones.

Counting. If every nonzero member of a space of functions is nonzero somewhere on a set S, the dimension is at most $\#S$, since the restriction to S is injective. With S the exponents of weight at least δ, $\dim \mathcal{D}_\delta\le \#\{\beta:\text{weight}(\beta)\ge\delta\}$. An exponent is counted for at most weight(β) values $\delta\ge 1$, so $\sum_{1\le\delta\le M}\dim\mathcal{D}_\delta\le\sum_\beta\text{weight}(\beta)$. Exchanging two indices permutes the exponents, so each variable has mean exponent n/k over all exponents, and $\sum_\beta\text{weight}(\beta)$ is $\#\{\text{exponents}\}\cdot n/k$ times the sum of the costs of all variables. At most k indices have $b\ge 2r$ and $2r$ have $b<2r$, each deep in $N/2$ letters, so the costs add up to at most $\frac{N}{2}(kP^{-2r}+2rP^{-1})\le N(kP^{-2r}+r^2P^{-1})$. Hence

$$\sum_\beta \text{weight}(\beta)\le \#\{\text{exponents}\}\cdot\frac{n}{k}N\Big(\frac{k}{P^{2r}}+\frac{r^2}{P}\Big)=c_{\text{div}}\#\{\text{exponents}\}. \qquad \square$$

21.8. Averaging

A monomial in the entries of a matrix is *bad* if its row weight $\sum\alpha_{ij}\rho_i$ exceeds a threshold.

Proposition 21.10 (An invariant average). *Let O be a valuation domain of characteristic 0 in which p is not a unit, $\iota\colon W\to O$ with residue field of at least $D+k$ elements, $D<p^f$, L a Levi subgroup and ρ, t weights and a threshold. There is a linear form I on polynomials in the entries of a $k\times k$ matrix, invariant under conjugation by*

$L(W)$, vanishing on bad monomials, and congruent modulo any ideal J to $p^{k^2 f}\psi(1)$ on every ψ of degree at most D whose conjugates have their bad coefficients in J.

Proof. Adjoint weights. An integer u prime to p is a unit of O: in the valuation domain one of u, p divides the other, $p \mid u$ is excluded by a relation $1 = au + bp$, and then $u \mid p$ gives $u \mid 1$. A binomial coefficient $\binom{n}{j}$ with $n < p^f$ is p^v times such an integer, and $v \le f$ by Kummer's theorem, v being the number of carries when j and $n - j$ are added in base p; so it divides p^f in O. A multinomial coefficient with at most k^2 parts and total degree below p^f is a product of at most k^2 such binomial coefficients, so it divides $p^{k^2 f}$. Put $\eta_\gamma = p^{k^2 f} / \binom{|\gamma|}{\gamma}$ in degree below p^f and $\eta_\gamma = 0$ otherwise. For the form $\langle x^\alpha, x^\alpha \rangle = \alpha!$ on polynomials, with distinct monomials orthogonal, the linear substitutions by a matrix and by its transpose are adjoint: written out for two monomials this is $\alpha! [x^\alpha] A \cdot x^\beta = \beta! [x^\beta] A^{\mathrm{T}} \cdot x^\alpha$, which follows from the multinomial theorem. Substitutions keep the degree, and $\eta_\gamma = p^{k^2 f} \gamma!/|\gamma|!$ is $\gamma!$ times a constant of degree $|\gamma|$, so the same adjointness holds for the diagonal form $\langle x^\gamma, x^\gamma \rangle = \eta_\gamma$.

The Schur algebra. Let $U = (O^k)^{\otimes d}$, with basis tensors indexed by multi-indices $i \colon \{1, \dots, d\} \to \{1, \dots, k\}$, and let T be the d-th tensor power of the transposed generic block matrix, whose entry at (i, j) is the monomial $\prod_l X_{j_l i_l}$ when i_l and j_l lie in the same block for all l, and 0 otherwise. The *Schur algebra* of L in degree d consists of the matrices obtained by applying a linear form on the monomials of degree d to the entries of T; at the point $a \in L(W)$, evaluation gives $\iota(a)^{\otimes d}$. The points of $L(W)$ span the Schur algebra when the residue field has at least $d + k$ elements. By Nakayama's lemma it suffices that a form Φ of degree d over the residue field that vanishes at all points of L with unit determinant is zero; $\Phi \cdot \det$ is a form of degree $d + k$ that vanishes at all points over the residue field, which has at least $d + k$ elements, so it is zero, and $\det \neq 0$ gives $\Phi = 0$ [Gre80]. Consequently a submodule stable under all $\iota(a)^{\otimes d}$ is stable under the Schur algebra, and products of points are points.

The dual operator. For a polynomial ψ let $\psi_d^\sharp$ be the element of the Schur algebra obtained from the linear form $\alpha \mapsto \eta_\alpha [x^\alpha] \psi$ on the monomials of degree d: its entry at (i, j) is η_α times the coefficient in ψ of $\alpha = \prod_l X_{j_l i_l}$ when the indices lie in the same blocks, and 0 otherwise. By adjointness, moving the substitution onto the entries of T, the conjugate $\psi(\iota(a)^{-1} X \iota(a))$ has the dual operator $\iota(a)^{\otimes d}\, \psi_d^\sharp\, \iota(a^{-1})^{\otimes d}$ for $a \in L(W)$. The trace of $\alpha_d^\sharp$ for a monomial α of degree d is η_α times the number of multi-indices i with $\prod_l X_{i_l i_l} = \alpha$, which is the multinomial coefficient of α when α is diagonal and 0 otherwise; so $\operatorname{tr} \alpha_d^\sharp = p^{k^2 f} \alpha(1)$ for $d < p^f$, and summing over $d \le D$ and the monomials of ψ, $\sum_{d \le D} \operatorname{tr}_U \psi_d^\sharp = p^{k^2 f} \psi(1)$ when ψ has degree at most $D < p^f$.

The average. The weight of a basis tensor j is $\sum_l \rho_{j_l}$. Let $U_+ \subseteq U$ be the submodule generated under the Schur algebra by the basis tensors of weight above t. It is finitely generated, being a finite sum of images of U, and torsion free, hence free over the valuation domain O; by the previous paragraph it is stable under the Schur algebra, and $L(W)$ acts on U and on U_+ by automorphisms. Put

$$I(\psi) = \sum_{d \le D} \left(\operatorname{tr}_U \psi_d^\sharp - \operatorname{tr}_{U_+} \psi_d^\sharp \right).$$

Invariance. The dual operator of a conjugate is the conjugate of $\psi_d^\sharp$ by $\iota(a)^{\otimes d}$, which is an automorphism of U and of U_+; traces are invariant under such conjugation.

Bad monomials. The operator $\alpha_d^\sharp$ kills every basis tensor j whose weight differs from the row weight of α, since a nonzero entry at (i, j) requires $\alpha = \prod_l X_{j_l i_l}$. If α is bad, $\alpha_d^\sharp$ therefore maps U into U_+: the basis tensors it does not kill have weight above t, and their images under an element of the Schur algebra lie in U_+. Over the field of fractions, choose a complement of U_+ in U; in a basis adapted to it, $\alpha_d^\sharp$ is block triangular with zero block on the complement, so its traces on U and on U_+ agree, and $I(\alpha) = 0$.

The congruence. Let the bad coefficients of all conjugates of ψ lie in J. By density, U_+ is spanned by the tensors $\iota(a)^{\otimes d} e_j$ with $a \in L(W)$ and j of weight above t. With φ the conjugate of ψ by a^{-1}, $\psi_d^\sharp \iota(a)^{\otimes d} e_j = \iota(a)^{\otimes d} \varphi_d^\sharp e_j$, and $\varphi_d^\sharp e_j$ is a combination of the $\alpha_d^\sharp e_j$ over the monomials α of row weight equal to the weight of j, which are bad, with coefficients in J; each $\alpha_d^\sharp e_j$ lies in U_+. Hence $\psi_d^\sharp$ maps U_+ into JU_+, as $\iota(a)^{\otimes d}$ keeps JU_+. An endomorphism of a finite free module with values in J times the module has its trace in J, so $\operatorname{tr}_{U_+} \psi_d^\sharp \in J$ and $I(\psi) \equiv \sum_{d \le D} \operatorname{tr}_U \psi_d^\sharp = p^{k^2 f} \psi(1)$ modulo J. □

Proposition 21.11 (The averaging map). *Let p be a non-unit of the valuation domain O of characteristic 0, $r = a+1$, $N = 2\ell$ and $Nk(\ell+2)n + n + k < P^\ell$, and let (b, b') be an edge pairing. For every c there is a matrix* Av *such that no block diagonal translate of* Av h *has a term of defect above $9kc$ modulo w^c, and $\langle \mathrm{Av}\, h, h' \rangle \equiv p^{Nc_0} \langle h, h' \rangle$ (mod w^c) for all h and all $h' \in \mathcal{M}_{\mathrm{all}}$ of the dual edge.*

Proof. Contractions. Substitute N generic matrices, one copy for each letter, in the operator of Section 21.3, and for a set J of copies let Av_J apply the average I_j of Proposition 21.10 in the copies $j \in J$ and p^{c_0} times the evaluation at 1 in the others. At the identity matrices the substitution operator is 1, so $\mathrm{Av}_\emptyset = p^{Nc_0}$, and $\mathrm{Av} = \mathrm{Av}_{\mathrm{all}}$. The averages are taken for the Levi subgroup of the blocks r, $k - r$, the weights $\rho = e_r - e_a$ and the threshold $2c^* P^\ell$, $c^* = 3kc$; they exist because the degree in one copy is at most $Nnk(\ell+2)P^\ell$, which is small against the size of the residue field.

Weight and defect in one copy. For g block diagonal, the translate by g of Av h is the average I_j applied to the polynomial that remains in the copy j after the forms are applied in the other copies, in the frame g; the change of frame conjugates the remaining copy, under which I_j is invariant. Let a coefficient at α of this polynomial in its row β not be divisible by w^c. An invariant form kills the monomials whose row and column weights differ, so the copies other than j contribute nothing to the weight, and by Lemma 21.3 the row weight of α over P^ℓ is within c^* of $\mathrm{def}_{j+\ell}(\beta)$. Hence a defect above $3c^*$ makes α bad, and a bad α forces a defect above c^*.

First property. Modulo w^c, the polynomial that remains in the copy j, in a row β of defect above $3c^*$ at the letter $j + \ell$, has only bad monomials, on which I_j vanishes; so w^c divides the coefficient of $\rho(g)\mathrm{Av}\, h$ at β.

Second property. Adding one copy j to J changes the pairing with $h' \in \mathcal{M}_{\text{all}}$ by $(I_j - p^{c_0}\text{ev}_1)\psi$, where ψ is the pairing of h' with the polynomials that remain in the copy j. By Proposition 21.10 it suffices that the bad coefficients of the conjugates of ψ are divisible by w^c. Such a coefficient is a combination, with the coefficients of the remaining polynomials in the frame g as scalars, of pairings of h' with translates of monomials e_β, and a nonzero scalar at a bad monomial forces $\text{def}(\beta) > c^*$; by Proposition 21.2(c) these pairings are divisible by w^c. Adding the differences along a chain of subsets from $\emptyset$ to all copies gives $\langle \text{Av}\, h, h' \rangle \equiv p^{Nc_0}\langle h, h' \rangle$ modulo w^c. □

Proposition 21.12 (Conditions at an edge). *Let p be a non-unit of O, the degree be large (Definition 21.13), (b, b') an edge pairing and $c = 5Nc_0$. There are a module C containing $\mathcal{M}_{\text{all}}$ and $w^c O^{\text{exp}}$, a matrix* Av *with values in C that paired against $\mathcal{M}_{\text{all}}$ is multiplication by p^{Nc_0} modulo w^c, and a space S over the residue field containing the reductions of the h with $w^{c-1}h \in C$, with* $[\text{GL}_k(W) : \textit{parahoric}] \dim S \leq (1 + \theta)\#\{\textit{exponents}\}$.

Proof. Let C be the defect conditions modulo w^c with threshold $9kc$ along $(r-1, r)$, S the defect conditions over the residue field with threshold $3t$, $t = 3kc$, and Av the averaging map of Proposition 21.11; its hypothesis $Nk(\ell+2)n+n+k < P^\ell$ for $N = 2\ell$ follows from the large degree, as $P^\ell = \sqrt{q} \geq n^2$ and $n \geq (Nk)^2 + 2$. Then $\mathcal{M}_{\text{all}} \subseteq C$ by Proposition 21.2(c), multiples of w^c lie in C, the values of Av lie in C and its pairing property holds by Proposition 21.11. If $w^{c-1}h \in C$, the bad coefficients of the translates of $w^{c-1}h$ are divisible by w^c, so those of h lie in the ideal generated by w, which lies in the maximal ideal since p is not a unit; the reduction of h lies in S. The residue field of O has characteristic p, and the dimension bound is Theorem 21.7 with $t = 3kc \geq 2(k+1)$, using $8Nk(3kc)P^{2r} \leq \theta n$, a consequence of the size conditions: with $u = Nk \log q \geq 27Nk$ the left side is at most $180Nk^3uP^{2r}$, and $(u/\theta)^5 P^{3r} \leq n$. □

Definition 21.13 (Large degree). For a Frobenius cycle, a rank k, a degree n, a block size r and a proportion θ, the degree is *large* if $0 < r$, $2r \leq k$, $0 < \theta \leq 1$, $n^4 \leq q$ and $(Nk \log q/\theta)^5 P^{3r} \leq n$.

Let X represent the classes of pairs of a W-lattice of rank k and a free O-module with basis indexed by H, under a group acting freely. The *edge coordinates* record, for an edge (x, b) into x, the representative of its origin, a basis change g at the origin, a basis change at the terminus and the transition matrix on O^H; an edge is determined by its origin and the parahoric coset of ${}^t g$. The *edge Hecke operator* on functions $F\colon X \to O^I$, $I = \{\text{exponents}\} \times H$, is

$$(AF)(x) = \sum_b \left({}^t(\rho(g_{(x,b)})\Delta^2\rho(g'_b)) \otimes \text{transition}_{(x,b)}\right) F(\text{origin}(x, b)),$$

$p^{Nn/2}$ times the sum of the values at the origins carried to x.

Theorem 21.14 (Elementary divisors). *Let O be a valuation domain of characteristic zero, T a Frobenius cycle with p not a unit of O, and let the degree be large. Let f be a linear map from the functions $X \to O^I$ to a module M and $Y \in \text{End}\, M$ with $Yf = fA$. Then Y keeps $L = f(O^{X \times I})$, and for every m the colength of $YL + p^m L$ in*

L divides p^l for some $l \le (3c_{\mathrm{div}} + \theta m)\#X\,\#I$. If O is a discrete valuation ring, the length of $L/(YL + p^m L)$ is at most l times that of O/p.

Proof. Factorisation. Let Δ be the depth scaling and b the weights of an edge pairing (Proposition 21.2(b)). By the adjointness $(\Delta\rho(\gamma))^{\mathsf{T}}\mathrm{diag}(b)\Delta\rho({}^t g) = (\rho(g)\Delta^2\rho(\gamma))^{\mathsf{T}}\mathrm{diag}(b)$, on vectors multiplied by b the operator A is the composite of the restriction to edges, $\phi \mapsto (e \mapsto \Delta\rho(g_e)\phi(\mathrm{origin}\, e))$, with the adding map Add, which pairs a function on edges with the functions restricted to the dual edges (the edges into x read from the dual lattices): at (x, i) both sides are sums over the edges into x, term by term.

Two ends. Every pair of a point of X and a parahoric coset occurs among the edges, as an edge is determined by its origin and the parahoric coset of its basis change, the map being injective on a finite set; so it does among the dual edges. By Proposition 21.2(d) and Theorem 21.9 (over the residue field of O, of characteristic p), the restriction is Qd and the dual restriction is $Q'd'$, with Q, Q' bases of the functions that depend only on the origin and $\det d$, $\det d'$ dividing w^l, $l \le c_{\mathrm{div}}\,\#X\,\#\exp$ per basis vector of O^H.

Near inverse. Let Inv apply the transpose of the left inverse R' of Q' and then, at each edge, the inverse transition, multiplication by b' and the averaging map. With 1 in place of the averaging map this is a right inverse of Add up to p^{Nc_0}, because $bb' = p^{Nc_0}$; with the averaging map, the entry of Add $\circ$ Inv is a sum over the edges of pairings of Av against the columns of Q', whose values lie in $\mathcal{M}_{\mathrm{all}}$, so by Proposition 21.12 Add $\circ$ Inv $\equiv p^{2Nc_0}$ modulo w^c entrywise; and the values of Inv satisfy the conditions $\mathcal{C}$.

Few generators. By elementary divisors modulo the direct summand $\operatorname{im} Q$ there are an invertible U and scalars t_j with $\operatorname{im} Q + \operatorname{im}\mathrm{Inv} = \operatorname{im} Q + \operatorname{span}(t_j U_j)$, the left inverse of Q killing the U_j with $t_j \neq 0$ (take a Smith form of the part of Inv in the complement $\ker R$). At the indices with $u = w^{c-1} \mid t_j$ the vector $t_j U_j$ is a multiple of u; at the others $0 \neq t_j \mid u$, and $(u/t_j)\, t_j U_j$ has values in $\mathcal{C}$, so U_j reduces into S at every edge. The columns of Q and these U_j have dual rows and reduce into S, so they are independent modulo the maximal ideal and their number is at most $\#\mathrm{edges} \cdot \dim S = \#X[\mathrm{GL}_k(W) : \mathrm{parahoric}] \dim S \le (1 + \theta)\#X\#\exp$; as Q has $\#X\#\exp$ columns, at most $\theta\#X\#\exp$ indices are exceptional. Applying Add, and using Nakayama's lemma with u a multiple of p^{2Nc_0} times the non-unit w (as $c - 1 \ge 4Nc_0 + 1$), $p^{2Nc_0} O^{X\times\exp}$ lies in the span of the image of Add $\circ\, Q$ and of these at most $\theta\#X\#\exp$ further vectors.

Colengths add. If $a\, O^E$ lies in the span of the image of a matrix B and of a finite set s, then the colength of the image of $d'^{\mathsf{T}} B d$ plus zO^E divides $\det d' \cdot a^{\#E} \det d \cdot z^{\#s}$. With $B = \mathrm{Add}\, Q$, $a = p^{2Nc_0}$ and $z = p^m$, the exponent of p is at most $c_{\mathrm{div}}\#E + 2Nc_0\#E + \theta m\#E \le (3c_{\mathrm{div}} + \theta m)\#E$, since $Nc_0 \le c_{\mathrm{div}}$ under the size conditions; here $\#E = \#X\#\exp$ for each basis vector of O^H, so the total is at most $(3c_{\mathrm{div}} + \theta m)\#X\#I$. Carry the colength along f: $Yf = fA$, so $YL + p^m L = f(\operatorname{im} A + p^m O^{X\times I})$, which has colength dividing p^l in L by Lemma 7.6(d), the factored operator being A on vectors multiplied by b. Over a discrete valuation ring this bounds the length of $L/(YL + p^m L)$ by l times that of O/p. □

Part IV

Spanning: a second moment

22. The second moment and the spanning theorem

Theorem B asks for few translates $\mathfrak{p}_s$, $s \in \bar{G}$, whose images span almost all of the target. This part proves it by a second-moment argument. If the Poincaré map, normalised, is nearly an isometry on the good part, and the operators $P^* s P$ have, on average over $s \in \bar{G}$, a second moment nearly as small as the Cauchy–Schwarz inequality allows, then every subspace of the target has a small preimage under some translate (Theorem 22.2), and a greedy choice of translates fills the target (Theorem 22.3). The three inputs are: a bound for the dimension of the target, proved by comparing the good forms on Y_0 with those on the definite space W_m (Section 23); the near isometry; and the bound for the second moment. The last two are the *filling theorem*, whose proof by unfolding the Petersson products of Poincaré series occupies Section 24 to Section 29.

In this section, for finite-dimensional complex inner product spaces, the norm $\|A\|$ of a linear map is its Hilbert–Schmidt norm: $\|A\|^2 = \operatorname{tr}(A^*A) = \sum_i \|Ab_i\|^2$ for an orthonormal basis (b_i), the sum of the squared moduli of the matrix entries in orthonormal bases. For a subspace U, pr_U is the orthogonal projection, and $\|\mathrm{pr}_U\|^2 = \dim U$.

22.1. One translate

A linear map $A\colon S \to T$ is a *near isometry in mean square* with error ε if $\|A \,\mathrm{pr}_U\|^2 \geq \dim U - \varepsilon \dim S$ for every subspace $U \subseteq S$. A family of maps $B_s\colon V \to W$ *spreads* with error η if for every subspace $F \subseteq W$, of dimension $\alpha \dim W$, some B_s has $\dim B_s^{-1}(F) \leq ((1+\alpha)/2 + \eta)\dim V$.

Lemma 22.1. (a) *If $\|A^*A - 1\|^2 \leq \varepsilon^2 \dim S$ with $\varepsilon \geq 0$, then A is a near isometry with error ε.*

(b) *If A is a near isometry with error ε and an orthogonal projection pr satisfies $\|(1-\mathrm{pr})A\|^2 \leq \delta \dim S$, then $\mathrm{pr}\,A$ is a near isometry with error $\varepsilon + \delta$. For an orthogonal projection p and any q, $\|(1-pq)A\|^2 \leq \|(1-p)A\|^2 + \|(1-q)A\|^2$.*

(c) *If $\mathrm{pr}_0 B$ is a near isometry with error $\varepsilon < 1$ and $S \neq 0$, then $\mathrm{pr}_0 \neq 0$.*

Proof. (a) We have

$$\|A \,\mathrm{pr}_U\|^2 = \operatorname{tr}(\mathrm{pr}_U A^* A \,\mathrm{pr}_U) = \dim U + \operatorname{tr}\big(\mathrm{pr}_U (A^*A - 1)\big),$$

and by Cauchy–Schwarz the last trace is at least

$$-\|\mathrm{pr}_U\|\|A^*A - 1\| \geq -\sqrt{\dim U}\,\varepsilon\sqrt{\dim S} \geq -\varepsilon \dim S.$$

(b) The norm splits as

$$\|A \,\mathrm{pr}_U\|^2 = \|\mathrm{pr}\, A \,\mathrm{pr}_U\|^2 + \|(1-\mathrm{pr})A \,\mathrm{pr}_U\|^2,$$

and the second term is at most $\|(1-\mathrm{pr})A\|^2$. For the second claim write $1 - pq = (1-p) + p(1-q)$; the two terms have orthogonal ranges, and $\|p(1-q)A\| \leq \|(1-q)A\|$. (c) Take $U = S$. □

Theorem 22.2 (One translate). *Let a finite group Γ act unitarily on T, let pr_0 be the orthogonal projection on a stable subspace T_0, and $B\colon S \to T$. If $\mathrm{pr}_0 B$ is a near isometry with error $\varepsilon \geq 0$ and*

$$\dim T_0 \cdot \sum_{s \in \Gamma} \|B^* s B\|^2 \leq \#\Gamma\,(1+\varepsilon)(\dim S)^2,$$

then the maps $\mathrm{pr}_0 sB\colon S \to T_0$, $s \in \Gamma$, *spread with error* $3\varepsilon/2$.

Proof. Let $F \subseteq T_0$ have dimension $\alpha \dim T_0$. The vectors sBB^*s^{-1} have Hilbert–Schmidt inner products $\langle sBB^*s^{-1}, tBB^*t^{-1}\rangle = \|B^*s^{-1}tB\|^2$, and $\|\mathrm{pr}_F sB\|^2 = \langle \mathrm{pr}_F, sBB^*s^{-1}\rangle$. By Cauchy–Schwarz,

$$\Big(\sum_s \|\mathrm{pr}_F sB\|^2\Big)^2 \le \|\mathrm{pr}_F\|^2 \Big\|\sum_s sBB^*s^{-1}\Big\|^2 = \dim F \cdot \#\Gamma \sum_s \|B^*sB\|^2,$$

which is at most $\alpha\,(\#\Gamma)^2(1+\varepsilon)(\dim S)^2$. Hence some s has

$$\|\mathrm{pr}_F sB\|^2 \le \sqrt{\alpha(1+\varepsilon)}\dim S \le \tfrac{1+\alpha+\varepsilon}{2}\dim S.$$

Let U be the preimage of F under $\mathrm{pr}_0 sB$. On U the maps $\mathrm{pr}_0 sB$ and $\mathrm{pr}_F sB$ agree, and $\mathrm{pr}_0 sB = s\,\mathrm{pr}_0 B$ is a near isometry with error ε because s is unitary and commutes with pr_0. Hence $\dim U - \varepsilon \dim S \le \|\mathrm{pr}_0 sB\,\mathrm{pr}_U\|^2 \le \|\mathrm{pr}_F sB\|^2$, and $\dim U \le (\frac{1+\alpha}{2} + \frac{3}{2}\varepsilon)\dim S$. □

Theorem 22.3 (Few translates). *Let V, W be nonzero, $0 < \varepsilon < 1/16$, and let maps $B_s\colon V \to W$ spread with error $3\varepsilon/2$. Then the images of a set Σ of them with*

$$\#\Sigma \cdot \dim V \le \tfrac{5}{2}\big(1 + \log(1/(16\varepsilon))\big)\dim W$$

span a subspace of dimension at least $(1-16\varepsilon)\dim W$.

Proof. For $F \subseteq W$, $\dim(F + B_s V) + \dim B_s^{-1}F = \dim F + \dim V$. If F misses more than $16\varepsilon \dim W$ dimensions, choose s by spreading with $\alpha = \dim F/\dim W$: $\dim B_s^{-1}F \le (\frac{1+\alpha}{2} + \frac{3}{2}\varepsilon)\dim V$, so adding $B_s V$ adds at least $(\frac{1-\alpha}{2} - \frac{3}{2}\varepsilon)\dim V \ge \frac{2}{5}(1-\alpha)\dim V$ dimensions, since $\frac{3}{2}\varepsilon < \frac{1}{10}\cdot 16\varepsilon \le \frac{1}{10}(1-\alpha)$. The codimension is multiplied by at most $1 - \frac{2\dim V}{5\dim W}$. After l greedy steps the missing share is at most $\max(16\varepsilon, \exp(-2l\dim V/(5\dim W)))$; take $l = \lceil \frac{5}{2}\log(1/(16\varepsilon))\dim W/\dim V\rceil$ (Wolsey's analysis of the greedy covering [Wol82]). □

Lemma 22.4 (Singular minors). *Let π be a representation of a finite group G by $n \times n$ matrices over a field with q elements, π' its contragredient, A_0, A_1 two $n \times m$ matrices, and $\Theta \ge 0$ of positive type on G. If for every $v \ne 0$ the Θ-masses of the stabilisers of $A_0 v$ under π and of $A_1 v$ under π' are at most $Y \ge 0$, then*

$$q \cdot \Theta\{s \in G \mid \det({}^t A_1 \pi(s) A_0) = 0\} \le 2\Theta(G) + q^{m+n/2}Y.$$

The proof rests on a bound for one character sum. Write $x \cdot z = \sum_i x_i z_i$, let ψ be a primitive additive character of the field k, and let $\pi'(s) = {}^t\pi(s)^{-1}$.

Lemma 22.5 (A character sum of positive type). *Let $\Theta(a^{-1}b) = \langle u_a, u_b\rangle$ for vectors u_a of a complex inner product space, and let $x, z \in k^n$. If the Θ-masses of the stabiliser of x under π' and of z under π are at most Y, then*

$$\mathrm{Re}\sum_{s\in G} \psi\big(x \cdot \pi(s) z\big)\,\Theta(s) \le q^{n/2}\,Y.$$

Proof. In the space $\ell^2(k^n)$ let e_ζ be the indicator of the point ζ, and let $\chi_w(\zeta) = q^{-n/2}\psi(-w\cdot\zeta)$; the e_ζ are orthonormal, and so are the χ_w, since $\sum_\zeta \psi((w'-w)\cdot\zeta)$ is q^n for $w = w'$ and 0 otherwise. Put

$$X_0 = \sum_{a\in G} e_{\pi(a)z}\otimes u_a, \qquad X_1 = \sum_{a\in G}\chi_{\pi'(a)x}\otimes u_a.$$

For an orthonormal family (c_w) and any vector w_0, $\|\sum_a c_{\rho(a)w_0}\otimes u_a\|^2 = \sum_{a,b}[\rho(a)w_0 = \rho(b)w_0]\,\Theta(a^{-1}b)$, and the substitution $b = as$ shows that this is $\#G$ times the Θ-mass of the stabiliser of w_0. Hence $\|X_0\|^2 \le \#G\,Y$ and $\|X_1\|^2 \le \#G\,Y$. On the other hand, the inner product being conjugate linear in the first argument, $\langle\chi_{\pi'(a)x}, e_{\pi(b)z}\rangle = q^{-n/2}\psi\big((\pi'(a)x)\cdot\pi(b)z\big) = q^{-n/2}\psi\big(x\cdot\pi(a^{-1}b)z\big)$, as ${}^t\pi(a)^{-1}x\cdot\pi(b)z = x\cdot\pi(a)^{-1}\pi(b)z$; so the substitution $b = as$ gives

$$\langle X_1, X_0\rangle = q^{-n/2}\,\#G\sum_s \psi\big(x\cdot\pi(s)z\big)\Theta(s).$$

Finally $\mathrm{Re}\langle X_1, X_0\rangle \le \|X_1\|\|X_0\| \le \frac12(\|X_1\|^2+\|X_0\|^2) \le \#G\,Y$. □

Proof of Lemma 22.4. For $s\in G$ let $Z_s = {}^tA_1\pi(s)A_0$, an $m\times m$ matrix. If $\det Z_s = 0$, the left kernel $\{y : {}^tyZ_s = 0\}$ contains the q multiples of a nonzero vector, so it has at least q elements. Since $\Theta\ge 0$,

$$q\cdot\Theta\{s:\det Z_s = 0\} \le \sum_s \Theta(s)\,\#\{y\in k^m : {}^tyZ_s = 0\}.$$

By orthogonality of characters, $q^{-m}\sum_{v\in k^m}\psi(w\cdot v)$ is 1 if $w = 0$ and 0 otherwise; with $w = {}^tZ_s y$ the right side is

$$q^{-m}\sum_{y,v\in k^m}\sum_s \Theta(s)\,\psi\big((A_1y)\cdot\pi(s)(A_0v)\big).$$

On the axes the character is 1: the terms with $y = 0$ give $q^{-m}\cdot q^m\,\Theta(G) = \Theta(G)$, and those with $v = 0\ne y$ give at most the same. For $y\ne 0\ne v$ the inner sum is the sum of Lemma 22.5 with $x = A_1y$ and $z = A_0v$, whose stabilisers have mass at most Y by hypothesis; its real part is at most $q^{n/2}Y$. There are fewer than q^{2m} such pairs. The left side is real, so taking real parts,

$$q\cdot\Theta\{s:\det Z_s = 0\} \le 2\Theta(G) + q^{-m}q^{2m}q^{n/2}Y = 2\Theta(G) + q^{m+n/2}Y.$$ □

22.2. The spanning theorem

Let $\mathcal{P}$ be the Poincaré map of Definition 16.10 for the linear form and archimedean point chosen in Section 24, and let the source $\mathcal{S}$ (all source forms, of dimension D_X) carry the standard inner product of its coordinates (c, i), a class and a coordinate of the Bessel function (Section 25.2). The *scaled Poincaré map* is $P = (\mathrm{vol}\,J\cdot\nu)^{-1/2}\mathcal{P}$, with J the principal level and ν the covolume of its unipotent points. Let $\deg = \prod_v \deg_v$ be the product of the formal degrees of the Bessel systems at the infinite places, mass_r the mass of the genus of the standard lattice of rank r, and $\mathcal{D}_{\mathrm{main}} = D_X^2/(\deg\cdot\mathrm{mass}_r)$.

Theorem 22.6 (Inputs of the spanning theorem). (a) $d_T \le \#\bar G\cdot\deg\cdot\mathrm{mass}_r$.

(b) $e_0 = g_0\otimes g_1$ *is the tensor product of central idempotents* g_i *of the group algebras of the factors of* $\bar G$ *above* p_i*, and the coefficient of* $1-g_i$ *at the identity has real part at most* $b^{-8}/2$.

(c) e_0P *is a near isometry with error* $5b^{-4}$.

(d) $\sum_{s\in\bar{G}}\|P^*sP\|^2 \le (1+9b^{-4})\mathcal{D}_{\mathrm{main}}$; *hence* $d_T\sum_s\|P^*sP\|^2 \le \#\bar{G}(1+9b^{-4})D_X^2$.

Proof. (a) By Theorem 23.2(c) the target has the dimension of the good forms on the definite space W_m, which are among its forms of principal level. There the level is free, so the dimension is the class number at the principal level times the dimension of the weight; the class sum at the maximal level is mass_r, the principal level has index $\#\bar{G}$ in it, and the dimension of the weight is the product of the compact formal degrees, which by Weyl's dimension formula equal the degrees $\deg_v$ of the Bessel systems. (b) e_0 is the product over the two level primes of the sums of the label idempotents of the good tuples (Proposition 15.6); a label idempotent is central. The coefficient at 1 of the complement is the share of the bad tuples, which is at most $b^{-8}/2$ by Green's degree formula and a count of polynomials (Proposition 28.1). (c) and (d) are the filling theorem (Theorem 26.8) for the data of D, with the blocks of P^*sP given by the unfolding of Theorem 25.6 and (b). □

Proof of Theorem B. By Theorem 22.6(c),(d) and Theorem 22.2 with $\varepsilon = 5b^{-4} \le (8b^2)^{-1}$, the maps $s \mapsto \#\bar{G}\, e_0 s\mathcal{P}$, which have the same preimages as e_0sP, spread with error $\frac{3}{2}(8b^2)^{-1}$; in particular $d_T \ge 1$ by Lemma 22.1(c). By Theorem 22.3 with $\varepsilon = (8b^2)^{-1}$ there is Σ with $\#\Sigma\cdot D_X \le \frac{5}{2}(1+\log(b^2/2))d_T \le 5d_T\log(1+2b^2)$ such that the images of all source forms span at least $(1-2/b^2)d_T$ dimensions. Restricted to the ramified source forms, each translate loses at most the codimension $D_X - \dim\mathcal{S}^{\mathrm{ram}} \le b^{-8}D_X$ (Proposition 20.1(c)), so the images of $\mathcal{S}^{\mathrm{ram}}$ span at least $(1-2/b^2)d_T - \#\Sigma\, b^{-8}D_X \ge (1-4/b^2)d_T$ dimensions. The maps $\mathfrak{p}_s$ differ from $\#\bar{G}\, e_0 s\mathcal{P}$ by the nonzero factors c_B, c_{ss}^{-1}, so they have the same images. □

23. Equality of multiplicities

The spaces Y_0 and $W_1,\dots,W_m$ have the same rank, level and parameter and differ only in their signs at the real places (Definition 15.7); W_m is definite. This section proves that their good forms have the same Hecke eigensystems with the same multiplicities (Theorem 23.2). The tool is a comparison of simple trace formulas for a test function that is cuspidal at one split place (Appendix B.17): the good idempotent e_0 is a sum of idempotents of tuples of labels, one of which is primitive, and the idempotent of a primitive label is cuspidal. This is the analogue of the Jacquet–Langlands correspondence for a definite inner form, and is used twice: for the dimension of the target (Theorem 22.6(a)) and, in Part V, for the determinants of Rallis' operators.

23.1. Spherical Hecke operators away from the level

A finite place w is *split off the level* if $\bar{w} \ne w$ and neither w nor $\bar{w}$ is a chosen place. Let G^M be the finite adelic points that are 1 at the chosen places. An element a of the group algebra of G^M is *spherical* if in every representation of G_{f} it maps vectors fixed by the principal level J_M to such vectors. The *Hecke algebra* $\mathcal{H}$ consists of the families of operators on the forms of principal level of all hermitian spaces V of rank r given by one spherical element; it contains the

operators $T_{w,i}$ for w split off the level. For an *eigensystem* p, a function of pairs (w, i), the *eigenpart* of V is the space of forms on which each $T_{w,i}$ acts by $p(w, i)$.

Proposition 23.1 (Hecke algebra and eigensystems). (a) *Forms of principal level are spherical at every place split off the level, and $T_{w,i}$ preserves them. The group algebra of $\bar{G}$ acts through integral lifts and commutes with $\mathcal{H}$; so $\mathcal{H}$ acts on the good forms.*

(b) *The eigenparts are independent, finitely many meet a finite-dimensional space, and they span the forms of principal level when these are finite-dimensional; a form that is an eigenvector of all $T_{w,i}$ lies in an eigenpart, on which the Hecke polynomials act by the corresponding scalars.*

(c) *For an eigensystem q and a finite set F of eigensystems there is $T \in \mathcal{H}$ that is* 1 *on the q-eigenpart and* 0 *on the d-eigenparts, $d \in F \setminus \{q\}$, of all spaces at once; eigenparts are stable under $\bar{G}$.*

Proof. (a) *Sphericity.* Let w be split off the level. The principal level J_M consists of the integral points that reduce to 1 at the chosen places, so it contains every integral point supported at w and $\bar{w}$; hence a form of principal level is fixed by the part of the maximal compact subgroup supported there, that is, it is spherical at w. The operator $T_{w,i}$ is the sum of the translates by representatives g of the cosets gK_w in the double coset of $\mathrm{diag}(\varpi 1_i, 1)$, placed at w and $\bar{w}$; these elements are 1 at the chosen places, so they commute with the parts of J_M at the chosen places, and the part of J_M at w permutes the cosets. Hence $T_{w,i}$ preserves the forms of principal level, and one element of the group algebra of G^M acts as $T_{w,i}$ in every representation: its value at 1 in the regular representation of G^M. Hence the $T_{w,i}$ lie in $\mathcal{H}$.

The group $\bar{G}$. An element g of $\bar{G} = \prod_x \mathrm{GL}_r(\mathbb{F}_x)$ acts on forms of principal level through an integral lift that is 1 off the chosen places; the lift normalises J_M, so it maps vectors fixed by J_M to such vectors, and two lifts differ by an element of J_M. An element of the group algebra of $\bar{G}$ acts through the corresponding combination of lifts. A spherical element is supported on G^M, whose elements are 1 at the chosen places, while the lifts are 1 off them; elements supported at disjoint sets of places commute, so $\mathcal{H}$ commutes with the group algebra of $\bar{G}$, in particular with the idempotent e_0, and preserves its image, the good forms.

(b) *Independence.* Eigenparts for different eigensystems are joint eigenspaces of the commuting family $T_{w,i}$ for different joint eigenvalues, so their sum is direct; in a finite-dimensional subspace only finitely many of them can be nonzero. *Spanning.* The operators $T_{w,i}$ commute (Appendix B.16), and $T_{\bar{w},i}$, the operator of the inverse double coset, is the adjoint of $T_{w,i}$ for the Petersson product; so on a finite-dimensional space of forms of principal level they form a commuting family of normal operators, which by the spectral theorem has a basis of joint eigenvectors. A joint eigenvector y lies in the eigenpart of the system $p(w, i)$ of its eigenvalues. On such a y the Hecke polynomial $\sum_{i \le r} (-X)^i \, \mathrm{N}\, w^{\binom{i}{2}} T_{w,i}$ acts by $\sum_i (-X)^i \, \mathrm{N}\, w^{\binom{i}{2}} p(w, i)$.

(c) For each $d \in F \setminus \{q\}$ choose a pair $x = (w, i)$ with $d(x) \neq q(x)$, and let T be the product over these d of the operators $(T_x - d(x))/(q(x) - d(x))$. On a form with eigensystem e, the factor of d acts by $(e(x) - d(x))/(q(x) - d(x))$, which is 1 for $e = q$ and 0 for $e = d$. The product lies in $\mathcal{H}$, which is an algebra, and

acts in this way on all spaces at once. Eigenparts are stable under $\bar{G}$ because $\bar{G}$ commutes with every $T_{w,i}$. □

23.2. Equal traces

Theorem 23.2 (Equal multiplicities). *Let V, V' be hermitian spaces of rank r over E with signs for which the doubled parameter 2ς has gap Δ at every infinite place.*

(a) *For every $T \in \mathcal{H}$, $\operatorname{tr}(T \mid e_0\mathcal{A}(V)) = \operatorname{tr}(T \mid e_0\mathcal{A}(V'))$; hence the two actions have the same characteristic polynomials and determinants.*

(b) *For every eigensystem q, $\dim(e_0\mathcal{A}(V) \cap$ eigenpart $q) = \dim(e_0\mathcal{A}(V') \cap$ eigenpart $q)$.*

(c) *$d_T = \dim e_0\mathcal{A}(W_j)$ for $1 \le j \le m$.*

(d) *The series of Rallis' inner product formula (Part V) acts on the forms of principal level of V and of V' as one element of $\mathcal{H}$; in particular it preserves the good forms and has the same determinant on $e_0\mathcal{A}(V)$ and on $e_0\mathcal{A}(V')$.*

Proof. (a) *Reduction to the labels.* The idempotent e_0 is the sum of the idempotents e_c of the good tuples of labels (Proposition 15.6); as T commutes with e_0 and e_0 projects onto the good forms, $\operatorname{tr}(T \mid e_0\mathcal{A}(V)) = \operatorname{tr}(Te_0 \mid \mathcal{A}(V)) = \sum_c \operatorname{tr}(Te_c \mid \mathcal{A}(V))$, the traces taken on the forms of principal level.

Cuspidality. A good tuple c has at each of the primes p_0, p_1 monic allowed polynomials of degree r at the chosen places, one of which is primitive: irreducible, with a root that generates the units of the extension of degree r of the residue field (Lemma 15.5). Let x be a chosen place with a primitive label. Green's class function of a monic irreducible polynomial of degree r is cuspidal: it has mean zero over the unipotent radical of every proper parabolic subgroup of $\mathrm{GL}_r(\mathbb{F}_x)$ cut out by coordinates (Theorem B.25). The idempotent e_c is the product of the label idempotents, so the element of the group algebra of the finite points that acts as Te_c on every space, the spherical element of T times the lifts of e_c, is a test function that is cuspidal at the split place x.

Comparison. The archimedean part of the kernel is the product of the discrete series idempotents of the type, whose doubled parameters have the gap Δ at every infinite place, for the signs of V and of V'. The comparison of simple trace formulas (Theorem B.29) shows that the trace of such a test function on the forms of principal level depends neither on the space nor on its signs. Summing over c gives the equality of traces. Applied to the powers T^j, which lie in $\mathcal{H}$, it gives the equality of the traces of all powers, hence of the characteristic polynomials (over a field of characteristic zero the power sums determine the characteristic polynomial), hence of the determinants.

(b) Finitely many eigensystems occur in the good forms of V or of V', which are finite-dimensional; let F be the set of them together with q. By Proposition 23.1(c) one $T \in \mathcal{H}$ is 1 on the q-eigenpart and 0 on the other eigenparts in F of both spaces, and also on the eigenparts outside F, which are zero in the good forms. As the eigenparts span and are stable under $\bar{G}$, T acts on $e_0\mathcal{A}(V)$ as the projection onto $e_0\mathcal{A}(V) \cap$ (eigenpart q), whose trace is the dimension; the same holds for V', and (a) gives the equality.

(c) Apply (a) to the identity, which lies in $\mathcal{H}$: the traces are the dimensions of the good forms of Y_0 and of W_j. The doubled parameter has the gap Δ for the signs of Y_0 and of every W_j (Definition 15.7), and d_T is the dimension of the good forms of Y_0.

(d) *One term.* A term of the series is a vacuum coefficient of the Weil representation times $\langle z, \gamma y\rangle$ for a representative γ of a coset, divided by the number of matrices over the residue ring of the modulus. The representative is supported away from the places of the modulus, so it is 1 at the chosen places. For such γ, let $K = K_0 \cap G^M$ and $H = K \cap \gamma K \gamma^{-1}$, open in the compact group K; the mean of the translations by $k\gamma$ over the cosets kH is a spherical element, and since K acts unitarily and fixes z, all its terms have the same inner product with z: it gives an element $p_\gamma \in \mathcal{H}$ with $\langle z, p_\gamma y\rangle = \langle z, \gamma y\rangle$ on every space. The vacuum coefficient does not depend on the space.

The series. Let the series, times a on V and a' on V', converge to B and B'. The pairs of functions $(z, y) \mapsto (a\langle z, py\rangle, a'\langle z, py\rangle)$ with one $p \in \mathcal{H}$ form a finite-dimensional, hence closed, subspace, since the forms of principal level are finite-dimensional; it contains the pair of every partial sum, hence the pair (B, B'). Hence one element of $\mathcal{H}$ acts as the series on both spaces. It preserves the good forms by Proposition 23.1(a), and (a) gives the equality of the determinants. □

24. Bessel functions at a real place

The Poincaré series of Section 22 are built from Bessel functions at the infinite places: vector-valued functions on the real unitary group G_1 of Y_0 that transform on the left by the generic character of the unipotent radical U of the Bessel subgroup and by an irreducible representation of $H = \mathrm{U}(X_0)$, and that lie in a discrete series. This section constructs them, with their normalisation, Schur's bound for their derivatives, their decay, a cut of bounded support, and the *open cell bound* for integrals of a Bessel function against its translates. The local band limits at one place (Section 29) complete the list of properties that the filling theorem uses (Definition 24.14).

24.1. The group and its frame

Let G_1 be the complex unitary group of the hermitian form J_1 made of ϱ hyperbolic planes $\langle e_a, e_a^*\rangle$, the standard positive form on $X_0 = \mathbb{C}^\kappa$, $\#\kappa = k$, and a vector z with $h(z, z) = -1$; its rank is $r = k+2\varrho+1$. The *frame* is $f = (e_1, \dots, e_\varrho, z)$, of $n = \varrho + 1$ vectors. The *Bessel unipotent group* U consists of the isometries that are block upper unitriangular for the levels of the basis; $U^0 \subseteq U$ fixes the frame; $H = \mathrm{U}(X_0)$ is embedded on X_0; and the *Bessel subgroup* is $R = UH$, a semidirect product, with the Haar measure $du\,dh$. The *length* of $y \in G_1$ is $\ell(y) = \inf_{s\in R}(\|sy\| + 1)$, and the *frame box* of radius ρ consists of the y whose *frame vectors* $y^{-1} f_a$ have norm at most ρ.

Throughout, the Euclidean product is the one for which e_a, x_i, z, e_a^* is an orthonormal basis, norms of vectors are Euclidean, and norms of matrices are operator norms; $h(v, w) = \langle v, J_1 w\rangle$. Since J_1 is a unitary involution, every $g \in G_1$ has $g^{-1} = J_1 g^* J_1$, hence $\|g^{-1}\| = \|g\|$ and $1 \le \|g\|$. An element $s \in R$ maps f_b to f_b plus a combination of the f_a with $a < b$: its block $m(s)$ on the frame is upper unitriangular, and elements of H and of U^0 fix the frame.

Lemma 24.1 (Frame vectors). *Let $y, \gamma \in G_1$.*

(a) $\|y^{-1} f_a\| \le \|y\|$; *for $s \in U$, $y^{-1} f_a = \sum_b m(s)_{ba}\,(sy)^{-1} f_b$; for s fixing the frame, $(sy)^{-1} f_a = y^{-1} f_a$.*

(b) *The matrix $Z(\gamma)$ of the products $h(f_a, \gamma f_b)$ satisfies*

$$Z(\gamma)_{ab} = h\big(y^{-1}f_a,\ y^{-1}\gamma f_b\big), \qquad |Z(\gamma)_{ab}| \le \|y^{-1}f_a\|\,\|y^{-1}\gamma f_b\|.$$

For s fixing the frame, $Z(s\gamma) = Z(\gamma s) = Z(\gamma)$, and for $s \in U$, $Z(\gamma) = m(s)^* Z(s\gamma)$.

Proof. (a) The frame vectors f_a are unit vectors, and y^{-1} applied to a unit vector has norm at most $\|y^{-1}\| = \|y\|$. Next $y^{-1} = (sy)^{-1}s$ and $s f_a = \sum_b m(s)_{ba} f_b$, so $y^{-1}f_a = (sy)^{-1}s f_a = \sum_b m(s)_{ba}(sy)^{-1}f_b$; if s fixes the frame, $m(s) = 1$.

(b) The isometry y^{-1} preserves h, which gives the first identity. For the inequality, $|h(v,w)| = |\langle v, J_1 w\rangle| \le \|v\|\,\|J_1 w\| = \|v\|\,\|w\|$, because J_1 is unitary. If s fixes the frame, then $Z(s\gamma)_{ab} = h(s^{-1}f_a, \gamma f_b) = h(f_a, \gamma f_b)$ and $Z(\gamma s)_{ab} = h(f_a, \gamma s f_b) = h(f_a, \gamma f_b)$. For $s \in U$, the frame block is multiplicative on R, since R preserves the span of the frame, so $s^{-1}f_a = \sum_c m(s)^{-1}_{ca} f_c$; as h is conjugate linear in the first variable,

$$Z(s\gamma)_{ab} = h(s^{-1}f_a, \gamma f_b) = \sum_c \overline{m(s)^{-1}_{ca}}\, Z(\gamma)_{cb},$$

that is $Z(s\gamma) = (m(s)^{-1})^* Z(\gamma)$, or $Z(\gamma) = m(s)^* Z(s\gamma)$. □

Proposition 24.2 (Length and frames). (a) *If the vectors $y^{-1}e_a$, $y^{-1}z$ and $y^{-1}e_a^*$ have norm at most $M \ge 1$, then $\|y\| + 1 \le 8(\varrho+1)M$. The length is continuous, left invariant under R, and $\ell(y) \le \|x^{-1}y\|\,\ell(x)$.*

(b) *The Gram–Schmidt vectors of the frame vectors are left invariant under R; if $\ell(y) \le M$ their norms lie in $[M^{-1}, M]$, and conversely, if they lie in $[M^{-1}, M]$ with $M \ge 1$, then $\ell(y) \le 8(\varrho+1)M$. The triangular change of basis depends continuously and measurably on y.*

(c) *The* discriminant *of a $(\varrho+1)\times(\varrho+1)$ matrix Z with leading minors D_l is $\mathfrak{g}(Z) = |D_{\varrho+1}|^2 - |D_\varrho|^2$; up to sign it is the Gram determinant of the frame and its image under an isometry with frame Gram matrix Z.*

Proof. (a) Put $A = y^{-1}$. The Euclidean product is $\langle v, w\rangle = h(v,w) + \langle v, (1-J_1)w\rangle$, and $AJ_1A^* = J_1$. Hence $AA^* = J_1 + A(1-J_1)A^*$ and

$$\|A\|^2 = \|AA^*\| \le 1 + \|A(1-J_1)\|\,\|A\|.$$

The columns of $1 - J_1$ are 0 at the x_i, $2z$ at z, and $\pm(e_a - e_a^*)$ at e_a and e_a^*; so the columns of $A(1-J_1)$ have norm at most $2M$, and $\|A(1-J_1)\| \le (4\varrho+2)M$. Dividing by $\|A\| \ge 1$ gives $\|A\| \le 1 + (4\varrho+2)M$, and $\|y\| + 1 = \|A\| + 1 \le 2 + (4\varrho+2)M \le 8(\varrho+1)M$. Left invariance holds by definition. For $s \in R$, $\|sy\| \le \|sx\|\|x^{-1}y\|$ and $1 \le \|x^{-1}y\|$, so $\|sy\| + 1 \le (\|sx\| + 1)\|x^{-1}y\|$; take the infimum over s. Applying this with x and y exchanged, $\ell(x)/\|y^{-1}x\| \le \ell(y) \le \|x^{-1}y\|\ell(x)$, and both bounds tend to $\ell(x)$ as $y \to x$.

(b) Let $g_a(y)$ be the Gram–Schmidt vectors of $y^{-1}f_1, \dots, y^{-1}f_n$ for the Euclidean product (not normalised). If two families v, w satisfy $v_a - w_a \in \operatorname{span}(w_b : b < a)$ for all a, they have the same Gram–Schmidt vectors: by induction the earlier ones agree, and the difference of the a-th vectors lies in the span B of the earlier ones and is orthogonal to B, hence 0. For $s \in R$, (a) of Lemma 24.1 applied to s^{-1} shows that the frame vectors of sy differ from those of y by combinations of earlier ones, so $g(sy) = g(y)$. Since g_a is $y^{-1}f_a$ minus its projection to the earlier vectors, $\|g_a\| \le \|y^{-1}f_a\| \le \|y\|$; and $y g_a$ is f_a plus a combination of

f_b, $b < a$, so $1 \le \|yg_a\| \le \|y\|\|g_a\|$. If $\ell(y) \le M$, some $s \in R$ has $\|sy\| < M$, and these bounds at sy, together with $g(sy) = g(y)$, give $\|g_a\| \in [M^{-1}, M]$.

Conversely let $\|g_a\| \in [M^{-1}, M]$. The Gram–Schmidt matrix is upper unitriangular, and is the frame block of an element $u \in U$ with $(uy)^{-1} f_a = g_a$. Put $w_a = J_1 g_a / \|g_a\|^2$ for $a \le \varrho$. As $h(v, J_1 w) = \langle v, w\rangle$, we get $h(g_b, w_a) = \delta_{ab}$; and $h(w_a, w_b) = h(g_a, g_b)/(\|g_a\|^2\|g_b\|^2) = 0$ for $a, b < n$, since the g_a are the images of the isotropic e_a under an isometry. Hence $(w_a)_{a\le\varrho}$ is an isotropic system dual to the frame (g_b), and $\|w_a\| = \|g_a\|^{-1} \le M$. The images of (e_a^*) under $(uy)^{-1}$ form another such system; two such systems differ by an element of U^0 (Witt's theorem with the frame fixed), so $w_a = (uy)^{-1} u_0 e_a^*$ for some $u_0 \in U^0$, and $y' = u_0^{-1} uy$ has frame vectors g_a and $y'^{-1} e_a^* = w_a$. By (a), $\ell(y) = \ell(y') \le \|y'\| + 1 \le 8(\varrho+1)M$. Continuity of g and of the triangular matrix follows along the Gram–Schmidt recursion, which divides only by the nonzero numbers $\|g_a\|^2$.

(c) Let $D = \operatorname{diag}(0, \dots, 0, -1)$ be the Gram matrix of the frame. For h, the Gram matrix of $(f, \gamma f)$ has the diagonal blocks D and the off-diagonal blocks Z and Z^*, $Z = Z(\gamma)$. Exchange the two block columns, at the cost of the sign $(-1)^n$. For square Z, W and the matrix E with the single entry c at (t,t), t the last index, expanding along the two rows of index t gives $\det \begin{pmatrix} Z & E \\ E & W \end{pmatrix} = \det Z \det W - c^2 \operatorname{adj}(Z)_{tt} \operatorname{adj}(W)_{tt}$. With $W = Z^*$, $c = -1$ and $\operatorname{adj}(Z)_{tt} = D_\varrho(Z)$ the determinant is $|\det Z|^2 - |D_\varrho|^2 = \mathfrak{g}(Z)$. □

For invertible Z let $\xi(Z) = (Z^{-1})_{tt}$, the last diagonal entry of the inverse, and $\nu(Z)^2 = 1 - |\xi(Z)|^2$. By Cramer's rule $\xi = D_\varrho / \det Z$, so

$$\mathfrak{g}(Z) = |\det Z|^2 \, \nu(Z)^2. \tag{24.1}$$

The *open cell* Ω is the set of invertible Z with $|\xi(Z)| < 1$; by (24.1) it is the open set $\{\mathfrak{g} > 0\}$.

24.2. Bessel functions

A *Bessel function* of type π, a unitary representation of H on $\mathbb{C}^I$, is a bounded measurable $W\colon G_1 \to \mathbb{C}^I$ with $W(uhy) = \psi(u)\pi(h)W(y)$ for $u \in U$, $h \in H$, where $\psi(u) = \exp(\sigma(u) - \bar\sigma(u))$ and $\sigma(u)$ is the sum of the entries of u just above the diagonal on the frame. For a doubled Harish-Chandra parameter $l = 2\lambda$ of large gap with discrete series coefficient c_λ and formal degree deg, the *Bessel average* at a point x and a unit vector v is $W_x(y) = \int_R c_\lambda(x^{-1}sy)\, \pi(s)^{-1} v \, ds$.

Here π is extended to R by $\pi(uh) = \psi(u)\pi(h)$; this is a unitary representation of R, since σ is additive on U and conjugation by H, which acts only on X_0, does not change the entries above the diagonal on the frame. The coefficient c_λ is written on G_1 through a unitary frame in which the form is diagonal (Appendix A); conjugation by this frame preserves operator norms. We write $F * k$ for the right convolution $(F * k)(y) = \int_{G_1} k(a^{-1}) F(ya)\, da$, and $e = \deg \cdot c_\lambda$. Since $c_\lambda(a^{-1}) = \overline{c_\lambda(a)}$, the condition $F * e = F$ says that F is fixed by the idempotent of the discrete series, and Schur's orthogonality relations give $e * e = e$.

Lemma 24.3 (The Bessel subgroup and unfolding). (a) *The measure $du\, dh$ on R is left and right invariant.*

(b) *For measurable $f \ge 0$ on G_1 and every x, $\int_{G_1} f = \int_{U\backslash G_1} \int_R f(xsy)\, ds\, dy$.*

(c) *If l has gap $500r^3 + A$, then $|c_\lambda(g)| \le (2/\|g\|)^{j+9r^2}$ for $j \le A$, and*

$$\int_R \Big(\frac{2}{\|x^{-1}sy\|}\Big)^{9r^2} ds \le 16^{9r^2}\, V(x), \qquad V(x) = \mathrm{vol}\{s \in R : \|x^{-1}sx - 1\| \le 1\},$$

uniformly in y; and $0 < V(x) < \infty$.

Proof. (a) Write the elements of R as hu with $h \in H$, $u \in U$. Right multiplication by $h'u'$ sends hu to $(hh')(h'^{-1}uh'u')$, that is (h,u) to $(hh', h'^{-1}uh'u')$: a right translation of H, and on U the conjugation by h'^{-1}, which preserves du because H acts on the entries of U by unitary changes of coordinates, followed by a right translation of U. Left multiplication by $h'u'$ sends hu to $(h'h)(h^{-1}u'h\,u)$, a left translation of H and, over each h, a left translation of U. Haar measures of H (compact) and of U (unipotent) are left and right invariant, so $du\,dh$ is.

(b) The quotient measure on $U\backslash G_1$ is defined by $\int_{G_1} f = \int_{U\backslash G_1}\int_U f(uy)\,du\,dy$. Writing $s = hu$ and using this for the function $g \mapsto f(xhg)$,

$$\begin{aligned}\int_{U\backslash G_1}\int_R f(xsy)\,ds\,dy &= \int_H\int_{U\backslash G_1}\int_U f(xhuy)\,du\,dy\,dh \\ &= \int_H\int_{G_1} f(xhg)\,dg\,dh = \int_{G_1} f,\end{aligned}$$

by left invariance of Haar measure on G_1 and because H has mass one.

(c) The first bound is Proposition A.1(a) with $c = j + 9r^2$, which is allowed since $9r^2 \le 500r^3$ and $j \le A$. For the integral put $J(s) = x^{-1}sx$, a homomorphism from R to G_1, viewed in $\mathrm{U}(\varepsilon)$ of size r through the unitary frame, and $y' = x^{-1}y$; then $x^{-1}sy = J(s)y'$. The group R is closed (U is defined by polynomial conditions and H is compact), so J is proper, and ds is right invariant by (a). The last statement of Lemma A.5 with $d = r$ and $c = 9r^2 > 8r^2$ gives $\int_R (2/\|J(s)y'\|)^{9r^2} ds \le 16^{9r^2}\ \mathrm{vol}\{s : \|J(s) - 1\| \le 1\}$, uniformly in y'. The set in $V(x)$ is the preimage of a compact set under the proper map J, hence compact, so $V(x) < \infty$; it contains the open neighbourhood $\{\|J(s) - 1\| < 1\}$ of 1, so $V(x) > 0$. □

Theorem 24.4 (Bessel averages). *Let l have gap at least $500r^3$ plus an excess, and let π be continuous.*

(a) *The integrand is integrable and W_x is a continuous Bessel function of type π, reproduced by the idempotent* $\deg \cdot c_\lambda$; *and* $\deg \|W_x\|^2_{L^2(U\backslash G_1)} = \langle v, W_x(x)\rangle$.

(b) *If $\langle v, \pi(h)v\rangle$ is the coefficient of a parameter u of H and the volume of $\{s \in R : \|x^{-1}sx - 1\| \le 1\}$ is at most P times the modulus of the period $\int_R c_\lambda(x^{-1}sx)\langle v, \pi(s)^{-1}v\rangle ds$, then it is at most P* $\deg \|W_x\|^2$.

(c) (*Stationary point.*) *If l is* steep (*lowest K-type of weight* 0 *on X and decreasing weight in $[-3B, -B]$ on the other coordinates, with alternating signs*) *and the parameters are bounded with excess at least $10^{30}((r+k)^2 \log\beta)^3$, there is x with $\ell(x) \le b^{5n}$ at which this volume is at most $\beta^{20(r+k)^2}$ times the modulus of the period.*

Proof. (a) As π is unitary and $\|v\| = 1$, the integrand has modulus $|c_\lambda(x^{-1}sy)| \le (2/\|x^{-1}sy\|)^{9r^2}$, which by Lemma 24.3(c) is integrable over R with integral at most $16^{9r^2}V(x)$, uniformly in y. Hence W_x is bounded; it is continuous because

the integrand is jointly continuous and dominated. For $s \in R$, the substitution $t \mapsto ts^{-1}$, allowed by right invariance, gives

$$W_x(sy) = \int_R c_\lambda(x^{-1}tsy)\,\pi(t)^{-1}v\,dt = \int_R c_\lambda(x^{-1}t'y)\,\pi(t's^{-1})^{-1}v\,dt' = \pi(s)W_x(y),$$

so W_x is a Bessel function of type π. For the reproduction, exchange the integrals in $(W_x * e)(y) = \int_{G_1} e(a^{-1})W_x(ya)\,da$; the double integral converges absolutely by the covering bound and the integrability of c_λ over G_1 (Lemma A.5). The inner integral is $\int_{G_1} c_\lambda(x^{-1}sya)\,e(a^{-1})\,da = (c_\lambda * e)(x^{-1}sy) = c_\lambda(x^{-1}sy)$.

For the norm, insert the definition of $W_x(y)$ in one factor of $\langle W_x(y), W_x(y)\rangle$ and move $\pi(s)^{-1} = \pi(s)^*$ to the other side:

$$\langle W_x(y), W_x(y)\rangle = \int_R \overline{c_\lambda(x^{-1}sy)}\,\langle v, \pi(s)W_x(y)\rangle\,ds = \int_R \Phi(sy)\,ds,$$

where $\Phi(g) = \overline{c_\lambda(x^{-1}g)}\,\langle v, W_x(g)\rangle$. By Lemma 24.3(b) with $x = 1$, $\|W_x\|^2 = \int_{G_1}\Phi$. Moreover $\overline{c_\lambda(x^{-1}g)} = c_\lambda(g^{-1}x)$, and with $g = xa$, $\int_{G_1} c_\lambda(g^{-1}x)W_x(g)\,dg = \deg^{-1}(W_x * e)(x) = \deg^{-1} W_x(x)$. Hence $\deg\|W_x\|^2 = \langle v, W_x(x)\rangle$.

(b) By definition $\langle v, W_x(x)\rangle = \int_R c_\lambda(x^{-1}sx)\langle v, \pi(s)^{-1}v\rangle\,ds$ is the period, and by (a) it equals $\deg\|W_x\|^2 \geq 0$. The hypothesis on $\langle v, \pi(h)v\rangle$ identifies the integrand with the Laplace integrand of Theorem A.9: for $s = uh$, $\langle v, \pi(s)^{-1}v\rangle = \overline{\psi(u)\,c_u(h)} = \exp(\bar\sigma - \sigma)\,\overline{c_u(h)}$.

(c) Let μ be the lowest weight of l and μ' that of u, which is a at one coordinate and 0 on X. Apply Proposition A.8 with $d = \varrho$, $L = 2n$ coordinates of G_1 outside X and $L' = 1$ coordinate of H outside X; then $L = L' + 2d + 1$ and there is a stationary position $(v_1, \dots, v_n; i_0)$ with v_n of norm -1, whose entries, and those of $\mu^a v_1$ for $a \leq 2\varrho$, are at most $M = (3B + a)^{2n}$. The vectors $w_a = \mu^a v_1$, $a \leq 2\varrho$, continue $v_1, \dots, v_n$, and by the residue computation in the proof of Proposition A.8 their products are 0 for $a + b < 2\varrho$ and -1 for $a + b = 2\varrho$. The columns of i_0 and $v_1, \dots, v_n$ have the Gram matrix of the basis of X_0 and of the frame, and both families are independent; by Witt's theorem there is $x \in G_1$ carrying i_0 to the inclusion of X_0 and v to the frame (after the unitary frame of signs). The vectors x^{-1}-images of the e_a^* are then given by the w_a, of norm at most $\sqrt{r}M$, and the argument of Proposition 24.2(b) gives $\ell(x) \leq 8n\sqrt{r}M \leq b^{5n}$, using $3B + a \leq b^2$ and $8nr \leq b$. For this x, conjugation $s \mapsto x^{-1}sx$ maps R properly onto the group of the position: $x^{-1}sx - 1$ lowers the flag of the v_j, with sum of coefficients $\sigma(s)$, and induces on i_0 the image of s in H; conversely every element of the group of the position arises. Theorem A.9 with $N = r + k$ bounds the volume of the s with $\|x^{-1}sx\| \leq \beta + 1$, which contains the unit ball since H is compact, by $\beta^{20(r+k)^2}$ times the modulus of the period. □

Theorem 24.5 (Schur's bound, decay and cut). *Let F be a Bessel function of type π, integrable over $U\backslash G_1$ and reproduced by* $\deg \cdot c_\lambda$*, for l of gap $500r^3 + A$ with entries bounded by L.*

(a) (*Schur.*) *F has right derivatives V_w along all words w in the matrix units, again Bessel functions of type π; for $|w| \leq 2n^2$ and integrable φ, $\|g \mapsto \int \varphi(y)V_w(gy)dy\|^2_{L^2(U\backslash G_1)} \leq B^{2|w|}\|F\|_2^2\|\varphi\|_2^2/\deg$ whenever $B \geq (r+1)^4(L + r + 1 + 2n^2)$.*

(b) (*Decay.*) *Let x have $\ell(x) \leq b^{5n}$ and $b^m \deg\|W_x\|^2 \geq$ the ball volume, with $b \geq 2000$, $r + L + 1 \leq b^4$, $m \leq 301r^2$, $3r^3 f \leq A$. The normalised $F =$*

cW_x with $\|F\|_2^2 = \#I$ and its cut $F^\flat = \chi(\ell/b^{6n+2})F$, χ a fixed cutoff, have a decay majorant*: a left R-invariant $w \le a < \infty$ with* $\max(1, \ell/b^{6n})^{3r^3 f}|F|$, $\max(1, \ell/b^{6n})^{3r^3 f}|F^\flat| \le w$ *and a* $\int w \le b^{400r^2}\#I$.

(c) (*Cut pair.*) *Then F and $F^\flat$ are Bessel functions, $\|F\|_2^2 = \#I$, $\|F^\flat\|_2^2 \le \#I$, $F^\flat$ vanishes where $\ell > b^{6n+2}$, $F^\flat = F$ where $\ell \le \frac{1}{2}b^{6n+2}$, and $\|F - F^\flat\|_2^2 \le b^{-2r^3 f}\#I$.*

(d) (*Orthonormality.*) *If π is irreducible, the operator $\int_{U\backslash G_1} F \otimes F^*$ is 1 for F and a scalar in $[0,1]$ for $F^\flat$; so the coordinates of F are orthonormal in $L^2(U\backslash G_1)$.*

The proof uses two lemmas on the discrete series at one real place: the derivatives of the coefficient along words in the matrix units, and the decay of the coefficient measured against the formal degree. The coefficient c_λ is a smooth vector, and $U(\mathfrak{g})c_\lambda$ is the space of its right derivatives, on which a matrix X acts by the derivative along X. We norm it by $\|x\| = \deg^{1/2}\|x\|_{L^2(\mathrm{U}(\varepsilon))}$, so that $\|c_\lambda\| = 1$, the formal degree being the inverse of $\int|c_\lambda|^2$. Let M_0 be the least subspace that contains c_λ and is stable under the matrix units E_{ab} with $\varepsilon_a = \varepsilon_b$, and $M_{s+1} = M_s + \sum_{a,b} E_{ab}M_s$.

Lemma 24.6 (Steps of the filtration). *Let l have gap $500r^3 + A$ and entries bounded by L.*

(a) *For $x \in M_s$ and every matrix X, $Xx \in M_{s+1}$ and $\|Xx\| \le (r+1)^4(L + r + 1 + s)\|X\|\|x\|$.*

(b) *Let β be a bump function on $\mathrm{U}(\varepsilon)$ with $c_\lambda * \beta = c_\lambda$, and w a word of length s in the matrix units, conjugated by the unitary frame. Then $c_\lambda * \partial_w\beta$ lies in M_s and*

$$\|c_\lambda * \partial_w\beta\| \le \big((r+1)^4(L + r + 1 + s)\big)^s.$$

Proof. (a) $Xx = \sum_{a,b} X_{ab}E_{ab}x$ lies in M_{s+1} by definition. For the norm we bound $\Sigma(y) = \sum_{a,b}\|E_{ab}y\|^2$ for $y \in M_s$. The L^2 product is invariant, so E_{ab} is adjoint to $\varepsilon_a\varepsilon_b E_{ba}$, and for the Casimir operator $\Omega = \sum_{a,b} E_{ab}E_{ba}$ and its compact part Ω_K, the sum over the pairs with $\varepsilon_a = \varepsilon_b$,

$$\langle y, \Omega y\rangle = \sum_{\varepsilon_a=\varepsilon_b}\|E_{ba}y\|^2 - \sum_{\varepsilon_a\neq\varepsilon_b}\|E_{ba}y\|^2, \qquad \Sigma(y) = 2\langle y, \Omega_K y\rangle - \langle y, \Omega y\rangle.$$

On the discrete series $\langle y, \Omega y\rangle \ge \kappa_\lambda\|y\|^2$ with $\kappa_\lambda = \sum_a(l_a^2 - (2\rho)_a^2)/4$, and $|(2\rho)_a| \le r$, so $-\langle y, \Omega y\rangle \le -\kappa_\lambda\|y\|^2 \le \frac{r^3}{4}\|y\|^2$. On the other side, the generator c_λ is the highest vector of the lowest K-type, of weight Λ with $|\Lambda_a| \le L + r$. A K-type of M_s has a highest vector not orthogonal to some $E_w c_\lambda$ with w a word of length at most s, so its highest weight μ is the weight of $E_w c_\lambda$, which has entries of modulus at most $N = L + r + s$; on that K-type Ω_K acts by $c_K(\mu) = \sum_a \mu_a(\mu_a + (2\rho_c)_a) \le rN(N+r)$. Hence $\Sigma(y) \le C\|y\|^2$ with $C = 2rN(N+r) + r^3/4$. By Cauchy–Schwarz over the r^2 pairs and $|X_{ab}| \le \|X\|$,

$$\|Xx\| \le \|X\|\sum_{a,b}\|E_{ab}x\| \le \|X\|\big(r^2C\big)^{1/2}\|x\| \le (r+1)^4(N+1)\|X\|\|x\|,$$

since $r^2C \le (r+1)^8(N+1)^2$, an inequality between polynomials with nonnegative coefficients.

(b) Induction on w. For the empty word, $c_\lambda * \beta = c_\lambda \in M_0$ has norm 1. Let $w = qw'$ with a first letter $q = (a, b)$, and let M be the matrix unit E_{ab} conjugated by the unitary frame; it has operator norm 1, as conjugation by a unitary matrix keeps the norm. Convolution on the left commutes with right derivatives, so $c_\lambda * \partial_w\beta$ is the derivative along M of $x = c_\lambda * \partial_{w'}\beta$, that is Mx. By induction $x \in M_{s-1}$ with $\|x\| \le ((r+1)^4(L+r+s))^{s-1}$, and (a) gives $Mx \in M_s$ with

$$\|Mx\| \le (r+1)^4(L+r+s)\,\|x\| \le \big((r+1)^4(L+r+1+s)\big)^s. \qquad \square$$

Lemma 24.7 (Formal degree and decay). *Let l have gap $500r^3 + A$ and entries bounded by L, and $K = 2000(r+L+1)^6$.*

(a) *For every $R \ge 1$, $\deg \cdot \mathrm{vol}\{g \in G_1 : \|g\| \le R\} \le 4(KR^2)^{2r^2}$.*

(b) *For every integer $M > 8r^2$ the function $(2/\|g\|)^M$ is integrable on G_1, and $\deg \int_{G_1}(2/\|g\|)^M dg \le 8K^{2r^2}2^M$. In particular*

$$\deg \int_{G_1} \Big(\frac{2}{\|g\|}\Big)^M dg \le b^{50r^2+1}2^M \quad \text{if } b \ge 2000,\ r+L+1 \le b^4, \tag{24.2}$$

and then, for $M = 9r^2$, $\deg \cdot 16^M \int_{G_1}(2/\|g\|)^M dg \le b^{99r^2}$.

(c) *If $b \ge 2000$, $r+L+1 \le b^4$ and $T \ge 1$, then $\deg \cdot \mathrm{vol}\{g : \|g\| \le T\} \le T^{9r^2}b^{99r^2}$.*

Proof. Conjugation by the unitary frame carries G_1 onto $\mathrm{U}(\varepsilon)$, keeps operator norms and carries Haar measure to Haar measure; so we may work on $\mathrm{U}(\varepsilon)$, whose matrices have size r. Put $\beta = r + L + 1$.

(a) By Proposition A.1(c), $|c_\lambda| \ge 1/2$ on the set of g with $\|g - 1\| \le (200\beta^5)^{-1}$, so this set has volume at most $\int 4|c_\lambda|^2 = 4/\deg$. Lemma A.5, applied to the identity of $\mathrm{U}(\varepsilon)$ with $y = 1$, $D = 200\beta^5$ and $\kappa = KR^2 \ge 600\beta^5R^2 = 3DR^2$, gives $\mathrm{vol}\{\|g\| \le R\} \le (KR^2)^{2r^2} \cdot 4/\deg$.

(b) Since $\|g\| \ge 1$, the layer cake formula and (a) with $R = t^{-1/M} \ge 1$ give

$$\begin{aligned}\int \|g\|^{-M} dg &= \int_0^1 \mathrm{vol}\{g : \|g\|^{-M} > t\}\, dt \\ &\le \frac{4K^{2r^2}}{\deg} \int_0^1 t^{-4r^2/M} dt = \frac{4K^{2r^2}}{\deg} \cdot \frac{M}{M - 4r^2}.\end{aligned}$$

As $M > 8r^2$, the last factor is less than 2; this proves the integrability, and multiplication by $\deg \cdot 2^M$ gives the bound. If $b \ge 2000$ and $r + L + 1 \le b^4$, then $K \le 2000b^{24} \le b^{25}$ and $8 \le b$, so $8K^{2r^2} \le b^{50r^2+1}$, which is (24.2). For $M = 9r^2$, $2^M 16^M \le b^{2M}$, and $50r^2 + 1 + 18r^2 \le 99r^2$.

(c) Let $M = 9r^2$. On the ball $\|g\| \le T$, $1 \le (T/\|g\|)^M = (T/2)^M(2/\|g\|)^M$. Integrating over the ball and using (24.2),

$$\deg \cdot \mathrm{vol}\{\|g\| \le T\} \le (T/2)^M \deg \int_{G_1} \Big(\frac{2}{\|g\|}\Big)^M dg \le T^M b^{50r^2+1} \le T^{9r^2}b^{99r^2}. \qquad \square$$

Proof of Theorem 24.5. (a) Let $X_\emptyset = e$ and $X_w = e * \partial_w\beta$ for a word w, where ∂_w is the derivative along the letters of w conjugated by the frame. By Lemma 24.6(b), as $\int |X_w|^2/\deg = \|c_\lambda * \partial_w\beta\|^2$, $\int |X_w|^2/\deg \le ((r+1)^4(L+r+1+|w|))^{2|w|} \le B^{2|w|}$ for $|w| \le 2n^2$, and $X_\emptyset * X_w = X_w$. Put $V_w = F * X_w$; since $F = F * e$, differentiating under the integral shows that along a curve γ with $\gamma' = \gamma Z$ the derivative of $V_w \circ \gamma$ is $\sum_{a,b} Z_{ab} V_{(a,b)w} \circ \gamma$, so the V_w are the right derivatives of F. Convolution

on the right commutes with left translations, so each V_w is again a Bessel function of type π. For the bound, the right translates of F span a space on which G_1 acts by a multiple of the discrete series of l, because $F = F * e$; on it the operator of an integrable kernel k has norm at most $\|k\|_2/\deg^{1/2}$ (Plancherel's inequality for the discrete series, from Schur's orthogonality). The function $g \mapsto \int \varphi(y)V_w(gy)\,dy$ is the image of F under the operators of X_w and of φ; so its squared norm is at most $\|F\|_2^2 \cdot \deg^{-1} \int |X_w|^2 \cdot \deg^{-1} \int |\varphi|^2 \le B^{2|w|}\|F\|_2^2\|\varphi\|_2^2/\deg$.

(b) Put $M = 9r^2$, $\varphi(g) = (2/\|g\|)^M$ and $\Phi(y) = \int_R \varphi(x^{-1}sy)\,ds$, which is left invariant under R by Lemma 24.3(a), and write $\tilde\ell(y) = \max(1, \ell(y)/b^{6n})$. Since $\ell(y) = \ell(sy) \le \|x^{-1}sy\|\,\ell(x) \le \|x^{-1}sy\|\,b^{5n} \le \|x^{-1}sy\|\,b^{6n}/2$ by Proposition 24.2(a) and $b \ge 2$, every s has $\tilde\ell(y) \le \max(1, \|x^{-1}sy\|/2)$. Let $i = 3r^3 f \le A$ and $g = x^{-1}sy$. If $\tilde\ell(y) = 1$, the decay of Lemma 24.3(c) with $j = 0$ gives $\tilde\ell(y)^i|c_\lambda(g)| \le \varphi(g)$. Otherwise $\|g\| \ge 2\tilde\ell(y) > 2$, and the decay with $j = i$ gives

$$\tilde\ell(y)^i|c_\lambda(g)| \le \Big(\frac{\|g\|}{2}\Big)^i \Big(\frac{2}{\|g\|}\Big)^{i+M} = \varphi(g).$$

Integrating over R, $\tilde\ell^i|W_x| \le \Phi$. Take $w = |c|\,\Phi$; as $|F^\flat| \le |F|$, both bounds of the decay majorant hold. By Lemma 24.3(c), $\Phi \le 16^M V(x)$, so $w \le a = |c|\,16^M V(x)$. By Lemma 24.3(b), $\int_{U\backslash G_1} \Phi = \int_{G_1} \varphi$. Using $c^2\|W_x\|^2 = \#I$ and $V(x) \le b^m \deg\|W_x\|^2$,

$$a \int w = c^2\,16^M\,V(x) \int_{G_1} \varphi \le b^m\,\#I \cdot \deg\,16^M \int_{G_1} \varphi \le b^{m+99r^2}\,\#I \le b^{400r^2}\,\#I,$$

where Lemma 24.7(b) gives $\deg\,16^M \int \varphi \le b^{99r^2}$, and $m \le 301r^2$.

(c) The constant c exists: $V(x) > 0$ and $V(x) \le b^m \deg\|W_x\|^2$ give $\|W_x\|_2 > 0$. The cut is $F^\flat(y) = \chi(\ell(y))F(y)$ with $\chi(t) = \max(0, \min(1, 2 - 2t/R))$, $R = b^{6n+2}$; it is continuous, a Bessel function because ℓ is left invariant under R (Proposition 24.2(a)), equal to F where $\ell \le R/2$ and zero where $\ell \ge R$, and $|F^\flat| \le |F|$, so $\|F^\flat\|_2^2 \le \#I$. Where $F^\flat \ne F$ we have $\ell > R/2$, so $\tilde\ell \ge b^2/2 \ge b$ and $b^{3r^3 f}|F^\flat - F| \le \tilde\ell^{3r^3 f}|F| \le w \le a$. Multiplying the two bounds $|F^\flat - F| \le b^{-3r^3 f}a$ and $|F^\flat - F| \le b^{-3r^3 f}w$ and integrating,

$$\|F^\flat - F\|_2^2 \le b^{-6r^3 f}\,a \int w \le b^{400r^2 - 6r^3 f}\,\#I \le b^{-2r^3 f}\,\#I,$$

since $100r^2 \le r^3 \le r^3 f$.

(d) For $s \in H$, left translation by s normalises U and preserves its Haar measure, and the characters cancel in $W(y) \otimes W(y)^*$; so $\int_{U\backslash G_1} W(sy) \otimes W(sy)^* = \int W \otimes W^*$. With $W(sy) = \pi(s)W(y)$ this says that the operator $T = \int W \otimes W^*$ commutes with $\pi(H)$, hence is a scalar by irreducibility, and its trace is $\|W\|_2^2$. For F this is $\#I$, so $T = 1$; for $F^\flat$ the scalar lies in $[0, \|F^\flat\|_2^2/\#I] \subseteq [0, 1]$. □

24.3. The open cell

For $\gamma \in G_1$ let $Z(\gamma)$ be the matrix of products of f with γf. The *open cell* Ω is the set of invertible Z whose inverse has last diagonal entry ξ with $|\xi| < 1$; on it $\nu(Z)^2 = 1 - |\xi|^2$. There is an explicit real-analytic section $Z \mapsto \gamma_Z \in G_1$ with $Z(\gamma_Z) = Z$, with polynomial bounds for it and its derivative where Z, Z^{-1} and ν^{-1} are bounded (*bounded in the cell*); by Witt's theorem $Z(\gamma) = Z \in \Omega$ determines γ up to the fixer U^0H of the frame on both sides.

We say that Z is *bounded in the cell by* P if $Z \in \Omega$ and $\|Z\|$, $\|Z^{-1}\|$ and $\nu(Z)^{-1}$ are at most P. Let F be the matrix with the columns f, and F' that with the columns $e_1^*, \dots, e_\varrho^*, x_{k_0}$ for a fixed index k_0. For $Z \in \Omega$ let ζ be the last row of Z^{-1} and $Y(Z)$ the matrix with last row $(0, \dots, 0, -1)$ and other rows those of $-\zeta^*\zeta$, halved outside the last column. Then $F^\vee(Z) = F\,Y(Z) + F'\,\mathrm{diag}(1, \dots, 1, \nu(Z))$ is a family dual to the frame ($F^* J_1 F^\vee = 1$) with Gram matrix $-\zeta^*\zeta$, the congruence of the Gram matrix D of the frame by Z^{-1}. The section γ_Z is the isometry that maps F to $F^\vee(Z)Z$, maps $F^\vee(Z^*)$ to $F(Z^*)^{-1}$ and fixes the x_i, $i \neq k_0$; its matrix is a polynomial in the entries of Z, Z^{-1}, ν and ν^{-1}, written out in the archive.

Proposition 24.8 (The section of the frame Gram map). (a) *If $Z \in \Omega$, then $\gamma_Z \in G_1$ and $Z(\gamma_Z) = Z$. The map $Z \mapsto \gamma_Z$ is measurable on all matrices (the identity off Ω) and real differentiable on Ω.*

(b) *If $Z(\gamma) = Z \in \Omega$, then $\gamma = s\gamma_Z s'^{-1}$ with s, s' in the fixer $U^0 H$ of the frame.*

(c) *If Z is bounded in the cell by $T \geq 4$, then $\|\gamma_Z\| \leq T^{18}$ and the real derivative of $Y \mapsto \gamma_Y$ at Z has norm at most T^{25}.*

(d) (*Perturbation.*) *Let Z be bounded in the cell by $T \geq 4$ and $\|Y - Z\| < T^{-6}$. Then Y is invertible with $\|Y^{-1}\| \leq 2T$ and $\|Y^{-1} - Z^{-1}\| \leq 2T^{-4}$, $\nu(Y)^2 \geq (1 - 8T^{-2})\nu(Z)^2$, and Y is bounded in the cell by $2T$; if moreover $T \geq 8$ and $T \geq n$, then $\mathfrak{g}(Y) \geq \mathfrak{g}(Z)/2$.*

(e) (*Sizes at and near a point.*) *If $n \leq M$, the entries of Z are at most M^2 and $\mathfrak{g}(Z) \geq M^{-1}$, then Z is bounded in the cell by every $T \geq M^{8n}$. If moreover $M \geq 8n$ and $P \geq M^{9n}$, every Y with $\|Y - Z\| < P^{-6}$ has $\mathfrak{g}(Y) \geq \mathfrak{g}(Z)/2$ and is bounded in the cell by P.*

Proof. (a) The products of F with $\gamma_Z F = F^\vee(Z)Z$ are $F^* J_1 F^\vee(Z)Z = Z$; the products of $F^\vee$ with F are the identity off the last index, because $F^* J_1 F' = \mathrm{diag}(1, \dots, 1, 0)$ and $F^* J_1 F = \mathrm{diag}(0, \dots, 0, -1)$, and the factor $\mathrm{diag}(0, \dots, 0, -1)$ keeps only the last row $(0, \dots, 0, -1)$ of $Y(Z)$. That γ_Z is an isometry is checked in the span of the orthonormal columns $(F \,|\, F')$, where γ_Z acts as yx^{-1} for the matrices x, y whose columns are $(FZ^{-1} \,|\, F^\vee(Z^*))$ and $(F^\vee(Z) \,|\, F(Z^*)^{-1})$: these two families have the same Gram matrix, which reduces to polynomial identities in the entries of Z^{-1} given $\nu^2 = 1 - |\xi|^2$; on the orthogonal complement γ_Z is the identity. The entries are holomorphic in (Z, W) at $W = Z^*$ near $\Omega \times \Omega$, the square root being taken of $1 - \xi(Z)\xi(W)$, which lies in the slit plane; restricting to $W = Z^*$ gives real differentiability, and measurability follows. (b) The families $(F \,|\, \gamma F)$ and $(F \,|\, \gamma_Z F)$ have the same Gram matrix, which is invertible because its determinant is $\pm\mathfrak{g}(Z) \neq 0$ (Proposition 24.2(c)). By Witt's theorem some g carries one to the other; g fixes the frame, and so does $\gamma_Z^{-1} g \gamma$. (c) Write the matrix of γ_Z as the value at $W = Z^*$ of a matrix in two variables (Z, W). If $P \geq 8$, Z is bounded in the cell by P and $W \in \Omega$ has $\|W^{-1}\| \leq P$, the formula gives the norm bound P^{12}: $\|Y\| \leq 2 + 8P^2$ for the coefficient matrix, $|s| \leq 2$ and $|s^{-1}| \leq 2P$ for the root s of $1 - \xi(Z)\xi(W)$, and the factors F, F' have norm one. By (d) every (Y, W) in the ball of radius T^{-6} about (Z, Z^*) satisfies these hypotheses with $P = 2T \geq 8$, so $\|\gamma_Z\| \leq (2T)^{12} \leq T^{18}$ (as $2^{12} \leq T^6$), and the two-variable matrix is holomorphic and bounded by $(2T)^{12}$ on that ball; Schwarz's lemma bounds its derivative at the centre by $2(2T)^{12}T^6 \leq T^{25}$ (as $2^{13} \leq T^7$). (d) $Y = Z(1 + Z^{-1}(Y - Z))$ with $\|Z^{-1}(Y - Z)\| \leq T^{-5} \leq 1/2$ gives the inverse with $\|Y^{-1}\| \leq 2T$, and $Y^{-1} - Z^{-1} = Y^{-1}(Z - Y)Z^{-1}$ has norm at most $2T^{-4}$. Hence $|\xi(Y) - \xi(Z)| \leq 2T^{-4}$, and since $\nu(Z)^2 \geq T^{-2}$, $\nu(Y)^2 \geq$

$\nu(Z)^2 - 4T^{-4} \geq (1 - 8T^{-2})\nu(Z)^2$; the other bounds follow. For $\mathfrak{g} = |\det|^2\nu^2$, write $Z = (1 - E)Y$ with $E = (Y - Z)Y^{-1}$, $\|E\| \leq 2T^{-5}$; then $|\det Z| \leq (1 + 2T^{-5})^n |\det Y|$, and $\mathfrak{g}(Y) \geq (1 + 2T^{-5})^{-2n}(1 - 8T^{-2})\,\mathfrak{g}(Z) \geq e^{-4n/T^5} \cdot \frac{7}{8}\,\mathfrak{g}(Z) \geq \frac{1}{2}\mathfrak{g}(Z)$ for $T \geq 8$ and $T \geq n$. (e) The operator norm of Z is at most $nM^2 \leq M^3$. By (24.1), with $|\det Z| \leq M^{3n}$ and $\nu(Z)^2 \leq 1$, the inequality $M^{-1} \leq |\det Z|^2\nu^2$ bounds each factor below: $|\det Z|^{-1} \leq M^{1/2}$ and $\nu^{-2} \leq M^{6n+1}$, so $\nu^{-1} \leq M^{3n+1}$. Cramer's rule gives $\|Z^{-1}\| \leq |\det Z|^{-1}\|\operatorname{adj} Z\| \leq M^{1/2}\, n!\, M^{2(n-1)} \leq M^{8n}$, as $n! \leq M^n$. The second statement is (d) with $T = P/2$. □

Theorem 24.9 (Mean value inequality). *For measurable $f \geq 0$ on G_1,*

$$\int_\Omega c_G\, \mathfrak{g}(Z)^{k-1} \int_{U^0H} \int_{U^0H} f(s'\gamma_Z s)\, ds\, ds'\, dZ \leq \int_{G_1} f,$$

with $c_G = \operatorname{vol} \mathrm{U}(k)/\operatorname{vol} \mathrm{U}(r)$ times the area of the unit sphere of $\mathbb{C}^k$. The law of $s\gamma_Z$ with density proportional to $\mathfrak{g}(Z)^{k-1}$ is at most the fibre measure of the Gram map $Y \mapsto {}^t\bar{Y}J_1Y$ over the Gram matrix of the frame.

Proof. Dual tuples. The tuples of n vectors dual to the frame are $J_1F + Rp$, where R has the columns e_b and x_i and p runs over matrices. Every such tuple is $s\,y(T)$ for its Gram matrix T and an element s of the fixer U^0H, where $y(T)$ is the standard dual tuple $e_b^* + \frac{1}{2}\sum_a T_{ab}e_a$ $(b < n)$, $-z + \sum_a T_{a,t}e_a + \sqrt{1 + T_{tt}}\, x_{k_0}$ (defined when $1 + T_{tt} \geq 0$). Choosing first the ϱ vectors $e_b^* + \ldots$, which fixes the upper left block of T and the unipotent part of s, and then the last vector, whose part in X_0 runs over a sphere of squared radius $1 + T_{tt}$ (polar coordinates in X_0, with H rotating it), Lebesgue measure is

$$dp = 2^{-\varrho}\, m_k(1 + T_{tt})\, dt\, ds, \qquad m_k(\rho) = \frac{\pi^k \rho^{k-1}}{(k-1)!}\ (\rho > 0),$$

with t the real coordinates of T and ds Lebesgue measure on the unipotent parameters times the Haar probability measure of H.

Frames with given products. For invertible Z, the tuples y whose products with the frame form Z are $y = y'Z$ with y' dual to the frame, of Gram matrix $T' = Z^{-*}TZ^{-1}$ if y has Gram matrix T. The substitution costs $|\det Z|^{2(\varrho+k)}$ in dp and $|\det Z|^{2(\varrho+1)}$ in dT, so $dp = |\det Z|^{2(k-1)}2^{-\varrho}m_k(1 + T'_{tt})\, dt\, ds$. At the Gram matrix $T_0 = D$ of the frame, $T' = -\zeta^*\zeta$ and $1 + T'_{tt} = \nu(Z)^2$, the dual tuple is $F^\vee(Z)$, and $y(T')Z = \gamma_Z F$. By (24.1), $|\det Z|^{2(k-1)}2^{-\varrho}m_k(\nu^2) = 2^{-\varrho-1}\sigma_k\, \mathfrak{g}(Z)^{k-1}$, with $\sigma_k = 2\pi^k/(k-1)!$ the area of the unit sphere of $\mathbb{C}^k$.

Below the fibre measure. For a tuple E of vectors, say that a measure m on G_1 is *below the fibre measure over E* if, for every lower semicontinuous $K \geq 0$ on tuples Y of the size of E with $\int_{T(Y)\in S} K\, dY \leq C \operatorname{vol} S$ for all small open sets S of hermitian matrices near the Gram matrix of E, one has $\int K(gE)\, dm(g) \leq C$; here $T(Y) = {}^t\bar{Y}J_1Y$ in its real coordinates and dY is Lebesgue measure. This is the sense in which the image of m under $g \mapsto gE$ is at most the fibre measure dY/dT of the Gram map [Sie35].

The section. Let λ be the law of $s\gamma_Z$ under $2^{-\varrho-1}\sigma_k\mathfrak{g}(Z)^{k-1}dZ\, ds$, $Z \in \Omega$. It is below the fibre measure over F: in the coordinates (T, Z, s) above, the tuples with $Z \in \Omega$ have Lebesgue measure with the density computed, continuous in T at T_0, and the tuples with $Z \notin \Omega$ only add to $\int_{T(Y)\in S} K\, dY$.

From frames to Haar measure. Three facts on the fibre measure give the comparison. (i) On bases, a measure below the fibre measure over the standard basis is at most $\operatorname{vol} \mathrm{U}(r)/2^r$ times Haar measure of G_1, normalised through its compact dual; this is the computation of the fibre measure of the Gram map on bases. (ii) If m is below the fibre measure over E, and ν fixes E almost everywhere and maps, by $g \mapsto gE'$, to a measure below the fibre measure of $Y' \mapsto ({}^t\bar{E} J_1 Y', T(Y'))$ over E', then $m * \nu$ is below the fibre measure over the joined tuple (E, E'), because Lebesgue measure on (Y, Y') is the product. (iii) For E' the dual vectors e_b^*, the measure $2^{-\varrho} du$ on U^0 satisfies the hypothesis of (ii), by the coordinates of the dual tuples above; and for E' the basis x_i of X_0, the complement of the tuple (f, e^*), which is definite, the measure $\operatorname{vol} \mathrm{U}(k) 2^{-k} dh$ on H does, by (i) for $\mathrm{U}(k)$. Applying (ii) twice to $m = \lambda$ and then (i),

$$\lambda * 2^{-\varrho} du * \frac{\operatorname{vol} \mathrm{U}(k)}{2^k} dh \le \frac{\operatorname{vol} \mathrm{U}(r)}{2^r}\, dg, \qquad \text{so} \qquad \lambda * ds \le \frac{\operatorname{vol} \mathrm{U}(r)}{2^{\varrho+1} \operatorname{vol} \mathrm{U}(k)}\, dg,$$

as $r = k+2\varrho+1$. Integrating f against $\lambda * ds$ gives the triple integral of the theorem with density $2^{-\varrho-1} \sigma_k \mathfrak{g}(Z)^{k-1}$; multiplying by $2^{\varrho+1} \operatorname{vol} \mathrm{U}(k)/\operatorname{vol} \mathrm{U}(r)$ turns it into $c_G\, \mathfrak{g}(Z)^{k-1}$ with $c_G = \sigma_k \operatorname{vol} \mathrm{U}(k)/\operatorname{vol} \mathrm{U}(r)$. The second statement of the theorem is the paragraph on the section. □

Theorem 24.10 (Open cell bound). (a) *Let F satisfy Schur's bound with constants $P \ge 8r$ and D, let every Y within P^{-6} of Z be bounded in the cell by P with $\mathfrak{g}(Y) \ge c > 0$, and let φ be integrable, vanishing where $\|y\| > P^{20}$. Then $\int \| \int \varphi(y) F(\gamma_Z (uh)^{-1} y) dy \|^2$ over $(h, u) \in H \times U^0$ with $\|u\| \le P^{20}$, for Haar measure times $c_G c^{k-1}$, is at most $P^{2000 n^2} D \int |\varphi|^2$.*

(b) *For a Bessel system (Definition 24.14) X of scale B, if $\mathfrak{g}(Z(\gamma)) > 0$ and some y has y, $\gamma^{-1} y$ in the frame box of radius $\rho \ge B$, there are a compact A, shifts in the fixer of the frame averaged by a probability measure, and a measure ν of mass at most*

$$\Big(\frac{\rho^{9n}}{\mathfrak{g}}\Big)^{10^5 n^4} \frac{(\#I)^2}{\deg \cdot c_G \cdot (\mathfrak{g}/2)^{k-1}}$$

such that the mean of $\| \int_A \varphi(z) W(\gamma t z) \otimes \overline{W(tz)} dz \|^2$ is at most $\int |\varphi|^2 d\nu$ for bounded measurable φ.

(c) *For systems X_v at all infinite places and φ of modulus at most 1, invariant under the fixers, with $\varphi \cdot (W^\flat(y) \otimes W^\flat(\gamma^{-1} y)^*)$ supported in the boxes, the integral of this product has compact support and norm at most the root of the product of the local bounds of (b), plus $\int |\varphi| \cdot \| W \otimes W^* - W^\flat \otimes W^{\flat *} \|$.*

The proof of (a) rests on three lemmas.

Lemma 24.11 (Unfolding along the frame). *Let $w, \phi \ge 0$ be measurable on G_1, w left invariant under R and ϕ of integral at most K on every orbit of U^0, and let $R_0 \ge 1$ bound the entries of $m(s)$ for every $s \in U$ with $\phi(y) \ne 0 \ne \phi(sy)$ for some y. Then $\int_{G_1} \phi w \le K (2R_0)^{2n^2} \int_{U \backslash G_1} w$. If Q is a measurable set of matrices bounded in the cell by $P \ge 1$ with $\mathfrak{g} \ge c > 0$ on Q, then*

$$\int_Q \int_{H \times U^0} w\big(\gamma_Z (uh)^{-1}\big)\, c_G c^{k-1}\, du\, dh\, dZ \le (2P^2)^{2n^2} \int_{U \backslash G_1} w.$$

Proof. Unfold along U; it suffices to bound $\int_U \phi(sy)\,ds$. Write $s = u\,\tilde{x}\,u_0$ with $u \in U^0$, $\tilde{x}$ the unitriangular lift of the entries x above the diagonal of $m(s)$, and $\phi(u_0 y) \neq 0$: the integral over u is at most K, and it vanishes unless all coordinates of x lie in the disc of radius R_0; the box of these coordinates has volume at most $(2R_0)^{2n^2}$. For the second statement, let χ be a cutoff function for U^0 and ϕ its product with the indicator of $\{y : Z(y) \in Q\}$; compute $\int \phi w$ once in the coordinates of Theorem 24.9, where on Q the density of the open cell is at least $c_G c^{k-1}$ and along an orbit of the fixer only χ changes, with integral 1; and once by the first statement. If $Z(y)$ and $Z(sy)$ are bounded in the cell by P and $s \in U$, the entries of $m(s)$ are at most P^2, since $Z(y) = m(s)^* Z(sy)$ by Lemma 24.1(b). □

Lemma 24.12 (Sobolev at the section). *Let (V_w) be a family of measurable functions indexed by words in an alphabet $\mathcal{L}$, with norms left invariant under R, such that along a curve with $\gamma' = \gamma X$, $\|X\| \le C$, the derivative of $V_w \circ \gamma$ exists with norm at most $\Lambda C \sum_{\lambda \in \mathcal{L}} \|V_{\lambda w} \circ \gamma\|$, where $\Lambda \ge 0$ and $\#\mathcal{L}\,\Lambda \le P^{44}$. Let $P \ge 8r$, let every Y within P^{-6} of Z be bounded in the cell by P with $\mathfrak{g}(Y) \ge c > 0$, and let $\int_{U \setminus G_1} \|V_w\|^2 \le K$ for $|w| \le 2n^2$. Then*

$$\int_{\|u\| \le P^{20}} \left\| V_\emptyset\big(\gamma_Z(uh)^{-1}\big) \right\|^2 c_G c^{k-1}\,du\,dh \le P^{550n^2} K.$$

Proof. Choose real linear coordinates $Y = Z + e(x)$ on the matrices of size n, preserving Lebesgue measure, whose coordinate vectors have norm at most $P^2/4$, such that the cube $[0, P^{-8}]^{2n^2}$ lies within P^{-6} of Z. Along a coordinate line $t \mapsto \gamma_{p(t)} k$ with $\|k\| \le P^{20}$, the logarithmic derivative of $\gamma_{p(t)} k$ has norm at most P^{85} by Proposition 24.8(c),(d), so the derivatives of $t \mapsto V_w(\gamma_{p(t)} k)$ are bounded by ΛP^{85} times the next members of the family. Sobolev's inequality on the cube (the value at a corner bounded by the mean squares of the mixed derivatives of order at most one in each coordinate) costs at most $2P^8(1 + (\#\mathcal{L}\Lambda P^{85})^2)$ per coordinate; integrating over the cube with Lemma 24.11 costs $2P^2$ per coordinate. The product over the $2n^2$ coordinates is at most $(P^{275})^{2n^2} = P^{550n^2}$. □

Lemma 24.13 (Derivatives of convolutions). *Let V_w, for words w in the matrix units, be bounded continuous functions on G_1 such that the derivative of $V_w \circ \gamma$ along $\gamma' = \gamma X$ is $\sum_{a,b} X_{ab} V_{(a,b)w} \circ \gamma$. Let ψ be integrable weights on G_1, indexed by words in the letters $((a,b), c, d)$, with $\psi_{((a,b),c,d)i}(y) = \psi_i(y)\,(y^{-1})_{ad}\,y_{cb}/R_0^2$. Then the functions $g \mapsto \int \psi_i(y) V_{i'}(gy)\,dy$, with i' the word of the first components of i, satisfy the hypothesis of Lemma 24.12 with $\Lambda = R_0^2$.*

Proof. Let $G_i(g) = \int \psi_i(y) V_{i'}(gy)\,dy$ and let γ be a curve with $\gamma' = \gamma X$, $\|X\| \le C$. For fixed y the curve $\gamma(t)y$ has $(\gamma y)' = (\gamma y)(y^{-1} X y)$, so by hypothesis

$$\frac{d}{dt} V_{i'}(\gamma(t)y) = \sum_{a,b} (y^{-1}Xy)_{ab}\, V_{(a,b)i'}(\gamma(t)y), \qquad (y^{-1}Xy)_{ab} = \sum_{c,d} (y^{-1})_{ad} X_{dc} y_{cb}.$$

Multiplying by $\psi_i(y)$ and writing $\lambda = ((a,b),c,d)$, the factor $\psi_i(y)(y^{-1})_{ad}y_{cb}$ is $R_0^2\,\psi_{\lambda i}(y)$, and the first components of λi form the word $(a,b)i'$. The derivatives are dominated by $R_0^2 C\sum_\lambda|\psi_{\lambda i}|\sup\|V_{(a,b)i'}\|$, which is integrable, so we may differentiate under the integral:

$$\frac{d}{dt}G_i(\gamma(t)) = \sum_\lambda R_0^2 X_{dc}\,G_{\lambda i}(\gamma(t)),$$

whose norm is at most $R_0^2 C\sum_\lambda\|G_{\lambda i}(\gamma(t))\|$, because $|X_{dc}|\le\|X\|\le C$. □

Proof of Theorem 24.10. (a) Let V_w be the right derivatives of F given by Theorem 24.5(a), and ψ_i the weights of Lemma 24.13 built from φ with $R_0 = P^{20}$; as φ vanishes where $\|y\| > P^{20}$, $|\psi_i|\le|\varphi|$. By Schur's bound the convolutions $g\mapsto\int\psi_i(y)V_{i'}(gy)\,dy$ have squared norms at most $P^{2|i|}D\int|\varphi|^2 \le P^{4n^2}D\int|\varphi|^2$ for $|i|\le 2n^2$. There are r^4 letters and $\Lambda = P^{40}$, so $\#\mathcal{L}\Lambda\le P^{44}$, and Lemma 24.12 gives the bound $P^{550n^2}P^{4n^2}D\int|\varphi|^2 \le P^{2000n^2}D\int|\varphi|^2$.

(b) Put $Z = Z(\gamma)\in\Omega$ and $M = \rho^{9n}/\mathfrak{g}(Z)$. The entries of Z are at most ρ^2 by Lemma 24.1(b), so $\mathfrak{g}(Z)\le\rho^{8n}$ and $\rho\le M$, $M^{-1}\le\mathfrak{g}(Z)$; let $M_0 = M^{9n}$ and $P = (M_0^5)^n$. By Proposition 24.8(b), $\gamma t(h,u) = s\gamma_Z(uh)^{-1}$ for some $s\in R$ and a measurable $t\colon H\times U^0\to U^0H$. Take A a compact tube over a set $\Omega_0\subseteq U^0$ of elements of norm at most P^{20}, the probability measure m the fixer measure conditioned on Ω_0, and ν the measure $\|W\|^2\,dz$ on A times $(m(\Omega_0))^{-1}P^{2000n^2}\#I/\deg$. The Bessel system satisfies Schur's bound with $P\ge B^5$ and $D = \#I/\deg$ (Definition 24.14); the near-point bounds of Proposition 24.8(e) give the hypothesis of (a) with $c = \mathfrak{g}(Z)/2$. For each column of the kernel, (a) bounds the mean over m of $\|\int_A\varphi(z)W(s\gamma_Z(uh)^{-1}z)\overline{W_j(tz)}\,dz\|^2$; the norm of $\int\psi(y)W(sgy)\otimes\overline{W(ty)}$ does not depend on $s,t\in R$. The mass of ν is at most $M^{10^5n^4}(\#I)^2/(\deg c_G(\mathfrak{g}/2)^{k-1})$, by $\int_A\|W\|^2\le K(2R^2)^{2n^2}\#I$ for the measure K of the fibres of A along U^0, and the identity $P^{2000n^2}M_0^5(MP^{40})^{2n^2} = M^{93600n^4+2n^2+45n}\le M^{10^5n^4}$. Finally, if z is in the frame box of radius $\rho\le M$ and $\ell(s\gamma_Z(uh)^{-1}z)\le M$, then $\|u^{-1}z\|\le M_0^5$: move $\gamma_Z z$ by an element of U to a point of norm at most M, and express the columns $z^{-1}e_a^*$ through the frames at both points with coefficients from Z^{-1}; so $z\in A$ for the tube of radius M_0^5.

(c) At each place (b) gives a compact A_v, measures ν_v and probability measures m_v on the shifts. The operator bounds multiply over the places, so the mean over the product of the m_v of the squared norm of the pairing is at most $\int|\varphi|^2 d\prod\nu_v \le\prod_v\nu_v(G_1)$, as $|\varphi|\le 1$; hence some shift t does no worse than the mean, and φ, being invariant under the fixers, is unchanged by it. After the translation by t the cut integrand lives on the compact set $\prod A_v$, where the difference with the uncut integrand is bounded by $\int|\varphi|\,\|W\otimes W^* - W^\flat\otimes W^{\flat*}\|$. The kernel $W(y)\otimes W(\gamma^{-1}y)^*$ is the adjoint of $W(\gamma^{-1}y)\otimes W(y)^*$, and γ^{-1} has the same discriminant as γ. □

24.4. Bessel systems

Definition 24.14 (Bessel system). A *Bessel system* of scale $b\ge 8r$ and decay rate f consists of a finite set I, continuous functions $W, W^\flat\colon G_1\to\mathbb{C}^I$, a degree $\deg > 0$ and an irreducible unitary representation π of H on $\mathbb{C}^I$, such that $(W,W^\flat)$ is a cut pair of type π with decay rate f (Theorem 24.5(c)), W satisfies Schur's bound with constants P and $\#I/\deg$ for every $P\ge b^5$, and $W^\flat$ has the

local bounds of Theorem 29.17: a sieve system of forms whose top form is the inner product of normalised orbital integrals of $W^\flat$.

Theorem 24.15 (Bessel systems exist). *For admissible data and each infinite place v there is a Bessel system of scale b and decay rate f whose full function is the Bessel function of the discrete series of parameter $2\varsigma(v)$ (the target parameter, doubled) and whose type has the column numbers of the weight at v. Its degree is the formal degree, so that $\deg_v$ is the dimension of the corresponding representation of the compact form, and $\log \prod_v \deg_v$ lies within explicit bounds; each system has at most b^{2k} coordinates. The filling data of D consist of N, p_0, p_1 with their chosen places, π_N, the generator ω of the different, the level K_0 of X_0 and these systems; the level K_0 is the preimage of the principal congruence subgroup $K(M)$, which includes the congruence modulo π_N.*

Proof. The type. Let π be the representation of $H = \mathrm{U}(X_0)$ given by the Schur matrices of the weight with the column numbers of the weight at v, written in an orthonormal basis for its invariant Gram form (write the positive Gram matrix as B^*B and conjugate by B). It is continuous, unitary and irreducible, and at a highest weight unit vector e its coefficient $\langle e, \pi(h)e\rangle$ is the discrete series coefficient c_u of the parameter u of this weight, a product of principal minors in the decreasing order of u. The lowest weight of u is the number of columns of height one at a single index and 0 elsewhere, and the entries of u are at most $3a$.

The parameter. Let $l = 2\varsigma(v)$ in a unitary frame with the signs of Y_0. Its entries are at most $3a$; it is steep for B (Proposition 15.4) with $3B$ plus the column number at most b^2; its gap exceeds $500r^3$ by at least r^7, and $r^7 \geq 10^{30}((r+k)^2 \log b^3)^3$ by Proposition 8.9. Its formal degree is Weyl's dimension polynomial at l, at most b^{2r^2} since each of the at most r^2 halved differences is at most $3a \leq b^2$.

The point. With $\beta = b^3 \geq r + k + 3a$, Theorem 24.4(c) gives x with $\ell(x) \leq b^{5n}$ at which $V(x) \leq \beta^{20(r+k)^2}|\mathrm{period}|$, and by Theorem 24.4(b) $V(x) \leq b^{60(r+k)^2} \deg\|W_x\|^2$. As $k < r$, $m = 60(r+k)^2 \leq 240r^2 \leq 301r^2$.

The functions. Theorem 24.5(b),(c) give $F = cW_x$, reproduced by e, and its cut, forming a cut pair; the exponents satisfy $3 \cdot 20(r+k)^2 + 99r^2 \leq 400r^2$. Schur's bound for F (Theorem 24.5(a)) holds with every $P \geq b^5 \geq (r+1)^4(3a + r + 1 + 2n^2)$ and $D = \|F\|_2^2/\deg = \#I/\deg$. The local bounds of the cut follow from Theorem 29.17, whose size conditions hold with box radius b^{6n+2} and exponent $k - (3n+6)$; they use the norm of the matrix coefficient of F and the volume of balls. The number of coordinates is the dimension of π, at most b^{2k}, and $\mathrm{vol}\,\mathrm{U}(k)/\mathrm{vol}\,\mathrm{U}(r) \leq b^{r^2}$, so $\deg_v$ times the constant of the local bounds is at most b^{4r^2}.

The data. The filling data record N, the primes p_0, p_1 with the places dividing π_M, the idele of π_N, the generator of the different and these systems. The level $D.\mathrm{level}_0$ of X_0 is the set of k that satisfy the congruence modulo π_N and reduce to 1 at the chosen places; as $p_0 \neq p_1$ are unramified, p_0p_1 divides an integral adele exactly when both primes do, which happens exactly when its components at the chosen places and their conjugates lie in the maximal ideals. Hence the level is the preimage of $K(M)$, $M = p_0p_1$. □

25. Filling data and the unfolding of the second moment

The near isometry and the second moment of Theorem 22.6 are statements about the Petersson products of Poincaré series of Bessel functions. This section sets up the abstract data of the *filling theorem*, shows that the Poincaré map of Definition 16.10 is such a Poincaré series, and unfolds the Petersson products into sums of *amplitudes* over rational points (Theorem 25.6). The amplitude sums are then the subject of Sections 26 to 28.

25.1. Filling data

A *shape* gives ϱ, the index set κ of X_0 and an index $k_0 \in \kappa$; the space V is ϱ hyperbolic planes, X_0 and a line Ez with $h(z,z) = -N$, and its Gram matrix in the basis $e_1, \dots, e_\varrho$, x_i ($i \in \kappa$), z, $e_1^*, \dots, e_\varrho^*$ is the split Gram matrix. *Filling data* over E consist of a prime $N = \pi\bar\pi$ split as in Definition 8.7, two primes $p_0, p_1 \in [b^{10}, b^{11}]$ with f chosen places each, $M = p_0p_1$, a purely imaginary generator ϖ of the different, a free level K_0 of $\mathrm{U}(X_0)$, a parameter b, and a Bessel system of scale b at each infinite place, subject to size conditions ($10^5 \le \varrho$, $1000(2\varrho+2)^9 \le k \le f$, $(200fk\,\mathrm{rd}_E)^{1000} \le b, \dots$). The lattice of V has the basis in which x_{k_0} is replaced by $x_{k_0}/\bar\pi + z/N$. Write K for its stabiliser in the finite adelic group, $\bar G = \prod \mathrm{GL}(\mathbb{F}_w)$ over the chosen places, $K \to \bar G$ the reduction, and $K(M)$ its kernel. A *class* is a double coset c of $\mathrm{U}(X_0)$ for K_0, with representative h_c; the *source* is the Euclidean space with coordinates (c, i), i a tuple of coordinates of the Bessel systems, of dimension D_X. The data are *admissible* if K_0 is the preimage of $K(M)$, the reduction is onto, and

$$b^{11kf}\mathcal{D}_{\mathrm{id}} \le \mathcal{D}_{\mathrm{main}} \le (M/b^6)^{kf}\mathcal{D}_{\mathrm{id}}, \qquad \mathcal{D}_{\mathrm{id}} = \#\bar R_0 \cdot D_X, \quad \mathcal{D}_{\mathrm{main}} = \frac{D_X^2}{\deg \cdot \mathrm{mass}_r},$$

with $\bar R_0$ the reduction of the Bessel subgroup, together with bounds for ϖ, the scales of the frame, the number of Bessel coordinates, an index, and the Siegel constant.

The size conditions in full are: $10^5 \le \varrho$, $2^{-\varrho} \le b^{-6}$, $(\varrho+1)[E:\mathbb{Q}] \le 2^{2\varrho}$, $1000(2\varrho+2)^9 \le k \le f$, $(200fk\,\mathrm{rd}_E)^{1000} \le b$, $f \le b/1000$, $\mathrm{rd}_E \le b$, $\varrho + 1 \le b$, $b^{19} \le M \le b^{23}$, $N \ge 5$, and, with $n = \varrho + 1$, $10^{10}n^4 \le n^6$, $3n^6 + 12 \le n^7 \le k$. The chosen places for p_i are f in number, no place is chosen twice and no conjugate of a chosen place is chosen, and an integral adele is divisible by M exactly when it and its conjugate vanish modulo every chosen place. Admissibility asks moreover that $|\varpi|_v \le b$, $1 \le |\kappa_v| \le b^{23}$ for the numbers κ_v below, $N \le b^{60n}$, that the Bessel coordinates be at most b^{2kf} in number, that the elements of the stabiliser of the standard lattice of X_0 that preserve the lattice of V have index at least $(N^k/b)^f$ in it, and an upper bound for the Siegel constant of Section 25.3.

Scales. At an infinite place v put $\kappa_v = M\,v(\varpi)/(2\pi i)$, a real number because ϖ is purely imaginary. The *normalised basis* at v scales e_a by $\kappa_v^{\varrho-a+1}/\sqrt{N}$, x_i by 1, z by $1/\sqrt{N}$ and e_a^* by the inverse of the scale of e_a; it carries the Gram matrix with middle block $\mathrm{diag}(1,\dots,1,-N)$ to the one with middle block $\mathrm{diag}(1,\dots,1,-1)$, so that the rescaled image $\gamma \mapsto \gamma_v$ maps the rational group $\mathrm{U}(V)(E)$ into G_1 at every infinite place. The square of the scale of the frame vector of index $a = 0, \dots, \varrho$ is $\kappa_v^{2(\varrho-a)}/N$, and Λ denotes the product over v and a of these squares to the powers $k + 2(\varrho - a)$. The *unipotent character* is $\psi_U(u) = \psi_{\mathrm{f}}(-\sigma(u)/(M\varpi))$ on the finite adelic points, with ψ_{f} the finite part of the standard additive character

and σ the sum of the entries just above the diagonal on the frame; the character $x \mapsto \psi_{\mathrm{f}}(-x/\varpi)$ has $\hat{\boldsymbol{O}}_E$ as its own dual, since ϖ generates the different.

Groups. For a class c, $U_c(M)$ is the group of rational $u \in U$ with $h_c^{-1} u h_c \in K(M)$, and $U_c^0(M)$ its subgroup fixing the frame. The reductions of the integral points of U, of H (the copy of $\mathrm{U}(X_0)$) and of the Bessel subgroup $R = UH$ are written $\bar{U}$, $\bar{H}$ and $\bar{R}_0$. A rational γ is a *term over* (c, s, c'), $s \in \bar{G}$, if $h_c^{-1} \gamma h_{c'} \in K$ with reduction s; terms compose like a groupoid: the product of terms over (c, s, c') and (c', t, c'') is a term over (c, st, c''), and the inverse of a term over (c, s, c') is a term over (c', s^{-1}, c).

Lemma 25.1 (The finite groups $\bar{U}$, $\bar{H}$, $\bar{R}_0$). (a) *$\bar{U}$, $\bar{H}$ and $\bar{R}_0$ are the products of their projections to the chosen places. The factor of $\bar{U}$ at w consists of the matrices over $\mathbb{F}_w$ that are block upper unitriangular for the levels of the basis, and $\#\bar{U} = M^{f\varrho(2k+2\varrho+1)}$; the factor of $\bar{H}$ is $\mathrm{GL}_k(\mathbb{F}_w)$.*

(b) *$\bar{U} \cap \bar{H} = 1$, $\bar{R}_0 = \bar{U}\bar{H}$ normalises $\bar{U}$, and $\#\bar{R}_0 = \#\bar{U}\,\#\bar{H}$.*

(c) *For admissible data, $\#\bar{H}$ is the index of K_0 in the group of points of $\mathrm{U}(X_0)(\mathbb{A}_{\mathrm{f}})$ whose image preserves the lattice of V.*

Proof. (a) Let $e \in \hat{\boldsymbol{O}}_E$ be an idempotent with $e\bar{e} = 0$ that reduces to 1 at a set P of chosen places and to 0 at the others; it exists because no chosen place is conjugate to a chosen place. If u, u' are integral with $u^* J u' = J$, then $k = 1 + e(u-1) + \bar{e}(u'-1)$ preserves J (the terms with the factor $e\bar{e}$ vanish), agrees with 1 where u and u' do, and reduces like u at P and like 1 elsewhere. Applied to u and its inverse adjoint $u' = J^{-1}(u^{-1})^* J$, which is again block unitriangular, this shows that the reductions are products over the places. A block unitriangular matrix over $\mathbb{F}_w$ lifts entry by entry to an integral one, completed by its inverse adjoint. Counting the free entries of the level function with block sizes $1, \dots, 1, k+1, 1, \dots, 1$ gives $\varrho(2k + 2\varrho + 1)$ entries; over the $2f$ residue fields of orders p_0, p_1 this gives $M^{f\varrho(2k+2\varrho+1)}$. For $\bar{H}$ lift a family in $\prod_w \mathrm{GL}_k(\mathbb{F}_w)$ to an integral k that is 1 away from the chosen places and their conjugates; its image preserves the lattice because either π is a unit at w or w has norm N and k is 1 there. (b) An element of $\bar{U} \cap \bar{H}$ is unipotent and acts only on X_0, hence is 1. Each element of the Bessel subgroup is a product of an element of U and one of H, and R normalises U. (c) K_0 is the preimage of $K(M)$, the kernel of the reduction onto $\bar{H}$. □

Lemma 25.2 (Strong approximation and the unipotent character). *Let the data be admissible.*

(a) *$U(E)$ is dense in $U(\mathbb{A}_{\mathrm{f}})$. For $x \in U(\mathbb{A}_{\mathrm{f}})$ and $y \in \mathrm{U}(V)(\mathbb{A}_{\mathrm{f}})$ there is a rational $u \in U$ with $y^{-1}x^{-1}uy \in K(M)$, and then $\psi_U(u) = \psi_U(x)$.*

(b) *For rational $\gamma \in U$, $\prod_v \exp\bigl(\sigma(\gamma_v) - \overline{\sigma(\gamma_v)}\bigr) = \overline{\psi_U(\gamma)}$, and the Bessel function satisfies $W(\gamma_\infty y) = \psi_U(\gamma)^{-1} W(y)$; in particular W is invariant under $U_c(M)$.*

(c) *The rational $\gamma \in U$ with $h_c^{-1}\gamma h_c$ integral map onto $\bar{U}$, and on them $\psi_U(\gamma)$ depends only on the reduction.*

(d) *If $\gamma \in R(E)$ and $h_c^{-1}\gamma h_d \in K(M)$, then $c = d$ and $\gamma \in U_c(M)$. Every $s \in \bar{R}_0$ is the reduction of $h_c^{-1}\gamma h_d$ for some class d and rational $\gamma \in R$, for every c.*

(e) (*Freeness.*) *If γ is a rational point of the standard unitary group in at most k variables and $g^{-1}\gamma g$ is integral and reduces to 1 at the chosen places of one of the primes, then $\gamma = 1$.*

Proof. (a) U is built floor by floor from Heisenberg groups $e_a^* \mapsto e_a^* + v + (\beta - h(v,v)/2)e_a$, whose parameters (v,β), a vector and a skew element, can be approximated by rational ones (the skew elements $t - \bar{t}$, $t \in E$, are dense in the skew finite adeles); so each floor, and $U(\mathbb{A}_f)$, lies in the closure of $U(E)$. As $y^{-1}K(M)y$ is open, a rational u in $xyK(M)y^{-1}$ exists. Write $u = x\,k'$ with $k' \in yK(M)y^{-1} \cap U$; the character ψ_U is additive on U through σ, and $\sigma(k')$ is divisible by M with integral quotient at the finite places (Lemma 25.1 with $K(M)$ congruent to 1 modulo M), so $\psi_U(k') = 1$ because $\hat{O}_E$ is self-dual.

(b) In the normalised basis the sum of the entries above the diagonal on the frame of γ_v is $-2\pi i$ times the v-component of $-\sigma(\gamma)/(M\varpi)$, by the choice of κ_v. The product over v of the exponentials is therefore the archimedean factor of the standard character at $-\sigma(\gamma)/(M\varpi)$, which by the product formula (ψ is trivial on E) is the inverse of the finite factor $\psi_U(\gamma)$, that is its conjugate. The local Bessel functions transform under U by exactly these exponentials.

(c) Given $s \in \bar{U}$ lift it to an integral point x of $U(\mathbb{A}_f)$ and apply (a) with $y = h_c$; the reduction of $h_c^{-1}u\,h_c$ is that of x, and ψ_U agrees.

(d) Split $h_c^{-1}\gamma h_d$ into a part in U and a part in H; the part in H lies in $K(M)$ too and relates the representatives of c and d modulo K_0. As K_0 is free, $c = d$ and this part is 1, so γ is unipotent. For the second statement split $h_c k = u\,h$ with $u \in U(\mathbb{A}_f)$, $h \in H(\mathbb{A}_f)$, approximate u by a rational x as in (a), and write $h = \eta\, h_d\, a$ with η rational, d a class and $a \in K_0$; then $\gamma = x\eta$ is a term over (c, s, d), since $h_c^{-1}\gamma h_d$ is k times an element of $K(M)$.

(e) The f chosen places above one prime have norm $p_i \geq b^{10} > (2k)^2$, and a nontrivial element of finite order of $\mathrm{U}(k)$ has eigenvalues whose differences from 1 are algebraic integers of norm less than $(2k)^2$ at some place above p_i; a rational element congruent to 1 modulo all these places, conjugate into an integral group, has finite order (the definite unitary group is compact at infinity) and hence is 1. □

Lemma 25.3 (Eichler elements). *Let E contain a primitive 5040th root of unity. For $a < b$ and an integer x of E, $1 + Mx\,E_{ab}$ is the frame block $m(u)$ of some $u \in U_c(M)$; for every $u \in U_c(M)$ the entries of $m(u)$ above the diagonal are M times integers of E. For a row vector l vanishing at e_b, e_b^* for $b \leq a$ and integral on the lattice of c, and a skew integer s, some $u \in U_c(M)$ has in the rows e_b of $u - 1$, outside the column e_a^*, the row Ml at e_a and zeros elsewhere, and only the entry Ms at (e_a, e_a^*) when $l = 0$.*

Proof. For isotropic columns x orthogonal to each other and to a matrix u, a skew s and $\tau + \bar{\tau} = g$, the matrix $1 + g\big(xu^*J - (u + x(\tau u^*Ju - s))x^*J\big)$ preserves J (the Eichler map): expanding, the products through x^*Ju, x^*Jx and u^*Jx vanish and the rest cancels. Along the columns e_b it is unitriangular for the levels. With $g = M$ and $\tau = M\tau_0$ for an integer τ_0 with $\tau_0 + \bar{\tau}_0 = 1$ (available because 2 is a unit times a norm in E, which contains a 5040th root of unity), and x, u integral on the lattice of c, it is $\equiv 1$ modulo M on that unimodular lattice, hence lies in $U_c(M)$. A row vector integral on the lattice is u^*J for an integral u, because the lattice is unimodular, and the frame vector f_b is part of a basis of the lattice, so an integral form taking the value 1 at f_b and 0 at the other basis vectors off X_0 exists; this gives the element with prescribed frame block. Conversely, in the basis of the lattice $M^{-1}(u - 1)$ is integral and the frame belongs to this basis. □

Proposition 25.4 (Cutoffs, covolume and blocks). *Let the data be admissible and E contain a primitive* 5040*th root of unity.*

(a) *The image of $U_c(M)$ at the infinite places is discrete, a measurable cutoff β_c exists, and for every y the integral of $u \mapsto \beta_c(uy)$ over U_∞ is a number ν, the* unipotent covolume, *independent of c and y; explicitly $\nu = (M^{\varrho(2(\varrho+k)+1)}/N^k)^f \operatorname{vol}(\hat{O})^{-\varrho(\varrho+k)}|D_E|^{\varrho/4}/\Lambda$, with Λ the product of the frame scales.*

(b) *The block sums converge absolutely; for stable T any measurable weight with orbit sums* 1 *may replace $\beta_c(y)\beta_{c'}(\gamma^{-1}y)$; and replacing W by its cut changes the blocks by at most $\nu b^{-k^3 f}/2$ in total.*

(c) *For integral $u, u' \in U$, the block at $\bar{u}s\bar{u}'^{-1}$ is $\psi_U(u)^{-1}\psi_U(u')$ times the block at s; so the block of a stable set vanishes unless s agrees.*

(d) *For the full Bessel function, the terms of the Bessel subgroup over $s = 1$ give ν times the identity in the coordinates (c, i).*

Here the *class cutoff* is $\beta_c = \beta_c^0 \cdot \omega$, the product of a cutoff β_c^0 for $U_c^0(M)$ (whose translates under $U_c^0(M)$ add up to 1) and the indicator ω of the *reduced points* of Proposition 25.5(c); by Proposition 25.5(c) its translates under $U_c(M)$ add up to 1. A set T of rational elements is *stable* if $u^{-1}\gamma u' \in T \iff \gamma \in T$ for rational $u, u' \in U$, and s *agrees* if $\psi_U(u) = \psi_U(u')$ for integral $u, u' \in U$ with $\bar{u}s = s\bar{u}'$.

Proof. (a) *Discreteness.* For a compact set C of matrices at the infinite places and a fixed pair of indices, an entry of $u \in U_c(M)$ with image in C is a rational element lying in a compact set of adeles: at infinity it is bounded by C, and at the finite places $h_c^{-1}uh_c$ is integral in the lattice basis. There are finitely many such elements, so the image is discrete; a measurable cutoff for a countable discrete group acting freely exists, and $U_c^0(M)$ maps injectively since an entry of u is determined by its image under an embedding.

Coordinates. A point of U_∞ is given, row by row, by the free entries of its row e_a (the columns e_b, e_b^* with $b > a$, and the middle block) and the imaginary part of its entry at (e_a, e_a^*), at all places; these *row coordinates* carry the product of the Haar measures to Lebesgue measure. If the free entries of u vanish in the rows e_b, $b > a$, and the entries at (e_b, e_b^*) are real there, then the free entries in the row e_a of xu are the sums of those of x and u, and likewise for the imaginary part at (e_a, e_a^*) once the free entries of the row e_a vanish. Hence $U_c(M)$ translates the coordinates by explicit lattices, triangularly in the order of the rows from the last to the first.

The lattices. The free entries of the row e_a of the images of $U_c(M)$ form the lattice of vectors $(M\, s_j/s_{e_a}\, v(x_j))_{v,j}$, with s the scales, for the rational vectors x on the free columns whose transformed vector under the block of the lattice basis of c is integral (Lemma 25.3 gives the converse inclusion); the block has determinant of generalised index N^{-f}, since the basis of the class differs from the identity only in the middle block and its determinant is that of a unitary element over $\bar{\pi}$. The imaginary parts at (e_a, e_a^*) form the lattice $(M/(\text{scale}^2)\operatorname{Im} v(s))_v$ for skew integers s, because the entry at (e_a, e_a^*) is M times a skew algebraic integer. The covolume of the first lattice is $\left(\prod_v M^2/\text{scale}_{v,a}^2\right)^{\#\text{free}}/\operatorname{vol}(\hat{O})^{\#\text{free}}$ with $2(\varrho - 1 - a) + k + 1$ free columns, and the product over all rows of both covolumes is the stated number ν, independent of c.

The integral. Let $B \subseteq U_\infty$ be the box over measurable fundamental domains of these lattices; each coset $u\Gamma$ of the image Γ of $U_c(M)$ meets B in exactly one point, by the triangular translation. The inverse of B is a fundamental domain for the left translations by Γ, so the integral of $u \mapsto \beta_c(uy)$ over U_∞, whose translates add up to 1, is $\mathrm{vol}(B) = \nu$.

(b) *Layers.* For γ a term over (c, s, d) write $\gamma^{-1}y$ for $\gamma_\infty^{-1}y$, and let $\tilde\ell(x) = \prod_v \max(1, \ell(x_v)/b^{6n})$. The terms whose points $\gamma^{-1}y$ have $\tilde\ell \le R$, weighted by $\beta_d(\gamma^{-1}y)$, have total weight at most $(b^{6n^2}R^2)^{2nrf}$ by the packing of Proposition 25.5(b), applied in the space of tuples of frame vectors, of real dimension $2nrf$. If points with $\tilde\ell \le R$ have total weight at most $(KR^e)^d$ for all $R \ge 1$, the weighted sum of $\tilde\ell^{-i}$ is at most $2K^d$ as soon as $2ed < i$ (layer cake). With $i = r^3 f$, which exceeds $8nrf$ since $8n < r^2$, this gives

$$\sum_d \sum_{\gamma \in T \text{ over } (c,s,d)} \beta_d(\gamma^{-1}y)\, \tilde\ell(\gamma^{-1}y)^{-r^3 f} \le 2b^{12n^3 rf}.$$

Convergence. At each place the Bessel functions lie below a decay majorant $w_v \le a_v$ with the power $3r^3 f$ of the scaled length to spare (Theorem 24.5(b)). Use the power $r^3 f$ at $\gamma^{-1}y$ for the sum above and unfold β_c at y over U: for w_v left invariant under U, $\int \beta_c(y) \prod_v w_v(y_v)\, dy = \nu \prod_v \int_{U\backslash G_1} w_v$, by (a). Hence the sum of the amplitude majorants is at most $\nu \cdot 2b^{12n^3 rf} \cdot b^{400r^2 f} \cdot \#\{\text{Bessel coordinates}\} < \infty$, and the amplitudes are summable; this also proves the uniform bound $\sum_{d,\gamma} \beta_d(\gamma^{-1}y) \prod_v \|W_v((\gamma^{-1}y)_v)\| \le C$.

The cut. Where the kernel of W and that of W^b differ, the cut acts at y or at $\gamma^{-1}y$ at some place, so $\tilde\ell(y)\tilde\ell(\gamma^{-1}y) \ge b$, and the remaining power $2r^3 f$ of the scaled lengths gains $b^{-2r^3 f}$. The kernels differ by at most

$$2b^{-2r^3 f} \prod_v w_v(y_v) \prod_v a_v\, \tilde\ell(\gamma^{-1}y)^{-r^3 f},$$

and the same computation gives the bound

$$2\nu \cdot 2b^{12n^3 rf}\, b^{400r^2 f}\, \#\{\text{coordinates}\}\, b^{-2r^3 f} \le \nu b^{-k^3 f}/2 :$$

the number of coordinates is at most b^{2kf}, and $12n^3 rf + 1 + 400r^2 f + 2kf + k^3 f \le 2r^3 f$ because $12n^3 + 500 \le k < r$.

Change of weight. The group $U_c(M) \times U_{c'}(M)$ acts on the pairs (γ, y), $\gamma \in T$ over (c, s, c'), by $(u\gamma u'^{-1}, uy)$, preserving the measure and the kernel (for T stable, by (b) of Lemma 25.2, the characters at u and u' being trivial). Both $\beta_c(y)\beta_{c'}(\gamma^{-1}y)$ and the given weight are cutoffs for this action. Two cutoffs give every invariant measurable function the same integral, since under both weighted measures the function has the law it has on the quotient; Fubini then identifies the sums of integrals.

(c) Approximate u, u' modulo $K(M)$ by rational x, $x' \in U$ with the same characters (Lemma 25.2(a)); they are terms over $(c, \bar u, c)$ and $(c', \bar u', c')$. Moving the block by them, that is substituting $\gamma \mapsto x\gamma x'^{-1}$ and $y \mapsto x_\infty y$ with the change of weight of (b), multiplies the kernel by $\psi_U(x)^{-1}\psi_U(x')$. If $\bar u s = s\bar u'$ the block at s is multiplied by this scalar without changing, so a nonzero block forces it to be 1.

(d) By Lemma 25.2(d) only $\gamma \in U_c(M)$ with $c = c'$ contribute. By the change of weight the whole weight β_c may be put on the single term $\gamma = 1$; unfolding over U at every place leaves ν times the product of the local inner

products $\int_{U\backslash G_1} W_{v,i}\overline{W_{v,j}}$, which are δ_{ij} by Theorem 24.5(d). The character of U has modulus 1 and cancels. □

Proposition 25.5 (Packing and reduced points). (a) (*Rigidity.*) *Let γ be a term, and let y have length at most $b^{6n+2}R$ and frame vectors of norm at most $\rho = b^{37n}R$. If the frame vectors of $\gamma^{-1}y$ and y are within $(N\rho)^{-1}$ of each other at every place, then γ fixes the frame.*

(b) (*Packing.*) *Terms γ for which the points $\gamma^{-1}y$ have bounded length and frame vectors, and differ by elements of a real subspace V, have at most $(b^{6n^2}R^2)^{\dim V}$ frames γf; weights bounded by cutoffs add up to at most this bound.*

(c) (*Reduced points.*) *For a measurable set D of representatives modulo $M\mathcal{O}_E$ of modulus at most b^{26}, a point is* reduced *if the entries above the diagonal of its triangular matrix lie in D; each point is moved to a reduced point by exactly one coset of unipotent elements, and the indicator of the reduced points is a partition of unity in boxes. The* boxed points *are the reduced points in the cut at every place; their frame vectors have norm at most b^{37n}.*

Proof. (a) Let δ bound the moves of the frame vectors. The products $Z(\gamma)_{ba} = h(f_b, \gamma f_a)$ are algebraic integers, as $h_c^{-1}\gamma h_{c'}$ is integral. At each place, $h(f_b, \gamma f_a) - h(f_b, f_a)$ is at most $N\|y^{-1}f_b\|$ times the distance of the a-th frame vectors of $\gamma^{-1}y$ and y (Lemma 24.1(b)), hence less than $N\rho\delta \le 1$ in modulus; an algebraic integer of modulus less than 1 at every place is 0. Hence $Z(\gamma) = Z(1)$, which says that $\gamma f_a - f_a$ has no components along e_b^* and z; unitarity then makes the middle part of $\gamma f_a - f_a$ isotropic for the middle form, which is anisotropic off z, so it vanishes. The coordinates along the e_b remain. By downward induction on b: the coordinates of γf_a and f_a along e_x agree for $x > b$; the difference along e_b, times the ratio of the scales of f_a and e_b, is at most $\|wy\|$ times the distance of the frame vectors for every $w \in U$, by reading the row b of w; the ratio of scales is at least $b^{-23\varrho}$ and $\inf_w \|wy\| \le \ell(y) \le b^{6n+2}R$, so the difference, an integer of E, has modulus less than 1 at every place and vanishes.

(b) With $\rho = b^{37n}R$, terms with different frames give points of V at distance at least $(N\rho)^{-1}$ by (a), applied to $\gamma'^{-1}\gamma$, which sends $\gamma^{-1}y$ to $\gamma'^{-1}y$. These points lie in a ball of radius 2ρ in V. Comparing volumes, the number of frames is at most $(4N\rho^2+1)^{\dim V} \le (b^{6n^2}R^2)^{\dim V}$, using $N \le b^{60n}$. For weights: terms with the same frame differ by an element of $U_c^0(M)$ (Lemma 25.2(d) applied to $\gamma^{-1}\gamma'$), so their weights, at most x times the terms of a partition of unity, add up to at most x.

(c) Work in the group of unitriangular matrices over the ring of tuples over the infinite places. For $u \in U$, the triangular matrix of $u_\infty y$, read in the rational frame, is that of y times $m(u)^{-1}$ (Lemma 24.1(a), Proposition 24.2(b)), and the kernel of $u \mapsto m(u)$ on $U_c(M)$ is $U_c^0(M)$. By Lemma 25.3 the image of $U_c(M)$ contains the elementary matrices $1 + E_{ab}l$, $a < b$, $l \in M\mathcal{O}_E$, and consists of unitriangular matrices with entries in $M\mathcal{O}_E$ above the diagonal. If every element of the ring has exactly one translate by $M\mathcal{O}_E$ in D, then for every unitriangular x exactly one element of such a group puts all entries of $x\gamma$ above the diagonal in D: the entries are coordinates graded by their distance from the diagonal, and the group translates them one diagonal after the other. Hence the $u \in U_c(M)$ with $u_\infty y$ reduced form exactly one coset of $U_c^0(M)$, and the indicator of the reduced points has orbit sums 1 against every weight with sums

1 on the cosets of $U_c^0(M)$. A measurable D with points of modulus at most b^{26} exists because $M \le b^{23}$ and $\mathrm{rd}_E \le b$ give a fundamental parallelotope of that size. For a reduced point the Gram–Schmidt matrix has entries at most b^{26} in the normalised frame up to the scales, so the frame vectors have norm at most $b^{27}\ell(y)$; in the cut, $\ell \le b^{6n+2}$, and $27 + 6n + 2 \le 37n$. □

25.2. The Poincaré series

A hermitian space V with a grading *fits* filling data if its type, its indexing and its lattice agree with those of the data up to a central element. For $a = (c, i)$ let $\phi_a(g) = \beta_c(y)W_i(y)$ if g_{f} lies in $h_c J$ (with J the principal level) and 0 otherwise, y the image of g_∞; the *Poincaré series* of a source vector x is $\sum_a x_a \sum_{\gamma \in \mathrm{U}(V)(E)} \phi_a(\gamma g)$.

Precisely, V fits the data if the Bessel system at v has type (l_v, kw_v), the grading labels the basis in the standard way, the type kernel is idempotent, and the finite part of the point of V is a finite idele times the image of a central element η of $\mathrm{U}(X_0)(\mathbb{A}_{\mathrm{f}})$ times the lattice basis times an integral matrix reducing to 1 at the chosen places. Then the Gram matrix of V is the rational Gram matrix of the data, and the point of V gives identifications of the rational, adelic, real and finite groups of V with those of the data: the real one, $g_v \mapsto d_v^{-1} x_v g_v p^{-1}$ with d_v the diagonal of scales and p the unitary frame of signs, turns left translation into left translation and preserves Haar measure; the finite one is conjugation by an integral matrix $\equiv 1$ at the chosen places, so it preserves the lattice stabilisers, the reductions and the principal levels.

Theorem 25.6 (Poincaré series and Bessel periods). *Let V fit admissible filling data, with $\jmath$ times the constant of the grading equal to $-(M\varpi)^{-1}$.*

(a) *The Poincaré series converge absolutely to forms of the type and of the principal level, linear in x; the source forms of level K_0, read in the coordinates (c, i) of their values at the class representatives, form a space of dimension $D_X = \#\{\text{classes}\} \cdot \dim(\text{weight}) > 0$.*

(b) *(Folding.) For a form z of the type and principal level, $\int_{[\mathrm{U}(V)]} z \cdot \mathcal{P}_x = \mathrm{vol}\, J \sum_c \int_{G_\infty} \beta_c(y) \sum_i x_{(c,i)} W_i(y)\, z(g, x_c)\, dg$.*

(c) *(Unfolding.) For $a = (c, i)$, $a' = (c', i')$ and $k \in K$ with reduction s, $\langle k\mathcal{P}_{a'}, \mathcal{P}_a \rangle = \mathrm{vol}\, J \cdot A_{\mathrm{all}}(c, s, c')_{ii'}$.*

(d) *(Period.) There are a real point g_0 and a linear form t such that for every source form ψ and every form y of the conjugate type, $\int y \cdot \mathcal{P}\psi = \int_{[R]} \mathcal{W}(r) y(r g_0) dr$: the Poincaré series of the Bessel function represents the Bessel period. In particular the Poincaré map of Definition 16.10 for (g_0, t) is the Poincaré series with the coordinates of the central translate of ψ.*

(e) *Hence, for the scaled map $P = (\mathrm{vol}\, J \cdot \nu)^{-1/2}\mathcal{P}$, the block (c, c') of $P^* s P$ is the conjugate of $A_{\mathrm{all}}(c, s, c')/\nu$* (the blocks are amplitude sums).

Proof. (a) *One class.* For $a = (c, i)$ and adelic g, the sum of $\phi_a(ug)$ over $u \in U_c(M)$ is $W_i(y)$ if $h_c^{-1}\eta\, g_{\mathrm{f}} \in K(M)$ and 0 otherwise: the Bessel function and the condition at the finite places are invariant under $U_c(M)$ (Lemma 25.2(b)), and the cutoffs add up to 1. *Convergence.* All terms of $\sum_\gamma |\phi_a(\gamma g)|$ vanish unless some δ satisfies the level condition at the finite places; writing the terms as $\gamma^{-1}\delta$, those that do not vanish are terms over $(c, 1, c)$, and the uniform bound

in the proof of Proposition 25.4(b) at the point $\delta_\infty y$ bounds the sum uniformly in g. Hence the series converges absolutely and uniformly, ϕ_a is integrable on the adelic group, and the quotient by the rational points has finite volume. *Type and level.* Grouping the series by cosets of $U_c(M)$, each part is a translate of the Bessel function times an indicator at the finite places, which is fixed by right convolution with the type kernel (at each place the kernel fixes the local function, Theorem 24.4(a)). The level is principal because the condition at the finite places is. *Dimension.* At a free level a form is a family of arbitrary values at the class representatives in the weight, read in the coordinates of the Bessel systems by a fixed linear isomorphism intertwining the Schur matrices with the Kronecker product of the local representations; so the dimension is D_X, and $D_X > 0$.

(b) For a bounded automorphic z, z times the series is integrable on the quotient, and unfolding the sum over the rational points gives $\int_{\mathrm{U}(V)(\mathbb{A})} z\,\phi_a$. On the coset $x_c J$ of the finite group, z does not depend on the finite part, so the integral splits into $\operatorname{vol} J$ times $\int_{G_\infty} \beta_c(y) W_i(y) z(g, x_c)\,dg$.

(c) Apply (b) to $z = k\mathcal{P}_{a'}$, which is again a form of principal level because K normalises J. At (g, x_c) the term γ^{-1} of the series of a' survives exactly when $h_{c'}^{-1}\gamma_{\mathrm{f}}^{-1}x_c k \in K(M)$, that is when γ is a term over (c, s, c'); the sum and the integral can be exchanged entrywise by the absolute convergence of Proposition 25.4(b). What remains is $\operatorname{vol} J$ times the (i, i') entry of $A_{\mathrm{all}}(c, s, c')$.

(d) *The Bessel function as one integral.* The group $R(F_\infty)$ is the product of the local groups R, and its Haar measure is a multiple of the product of the measures $du\,dh$. The local Bessel functions are integrals over R of $c_\lambda(x^{-1}ry)\,\pi(r)^{-1}v$ (Theorem 24.4); so their tensor product, paired with a vector ζ of the weight through the fixed isomorphism, is the integral over $s = na \in R(F_\infty)$ of $\overline{e(g_0^{-1}s^{-1}y)}\,\chi(s)^{-1}\,t(a\zeta)$, with e the type kernel, g_0 the point with components x_v, χ the character $r \mapsto \psi(-\sigma(r)/(M\varpi))$ of $R(\mathbb{A})$, which is trivial on $H(\mathbb{A})$, and t a linear form absorbing the constants. The condition on J makes χ equal to the generic character ψ_U on $U(\mathbb{A})$.

Moving the quotient. Insert this in (b): for a form y of the conjugate type and principal level,

$$\int_{U_c\backslash G_\infty}\Big(\int_{R(F_\infty)} e(g_0^{-1}s^{-1}x)\,\mathcal{W}(s)\,ds\Big)\,y(x, h_c)\,dx = \int_{U_c\backslash R(F_\infty)} \mathcal{W}(s)\,y(s\,g_0, h_c)\,ds,$$

because the kernel e reproduces y; everything converges since y is bounded, e integrable and $\mathcal{W}$ integrable over $U_c\backslash R(F_\infty)$.

The period. The weights $r \mapsto \beta_c(r_\infty)[r_{\mathrm{f}} \in h_c J]$, summed over c, form a cutoff for the action of $R(E)$ on $R(\mathbb{A})$: every r can be moved by a rational element of R so that its finite part lies in some $h_c J$ (write $r = na$, $a_{\mathrm{f}}\eta = \delta\, h_c\, k$ with δ rational and $k \in K_0$, and approximate n by a rational unipotent element), and if $g_{\mathrm{f}} \in h_d J$ and $(\gamma g)_{\mathrm{f}} \in h_c J$ for rational $\gamma \in R$ then $c = d$ and $\gamma \in U_c$ (Lemma 25.2(d)). Hence, with $J' = R(\mathbb{A}_{\mathrm{f}}) \cap J$, $[R]/J'$ is the disjoint union of the $U_c\backslash R(F_\infty)$, and $\int_{[R]} F = \operatorname{vol}(J') \sum_c \int_{U_c\backslash R(F_\infty)} F(s, h_c)\,ds$ for F invariant under J' on the right. The source function $\mathcal{W}(ua) = \psi_U(u)\,t(\check{a}_\infty^{-1}\psi(a_{\mathrm{f}}))$ of a form ψ fixed by K_0 is right invariant under J': an element of J' is $u\,a$ with $a \in K_0$ and $\chi(u) = 1$. Both sides of the claimed identity are multiples of $\sum_c \int_{U_c\backslash R(F_\infty)} \mathcal{W}(s, h_c) y(s\,g_0, h_c)\,ds$, and the ratio of the factors goes into t. The Poincaré map of Definition 16.10, which represents the Bessel period, therefore equals the Poincaré series with the

coordinates of the central translate of ψ by η, since both forms have the same inner product with every form.

(e) $\langle k\mathcal{P}_{a'}, \mathcal{P}_a\rangle$ is the (a, a') entry of $\mathcal{P}^* k\mathcal{P}$; divide (c) by $\operatorname{vol} J \cdot \nu$. □

25.3. Sizes of the filling data

Proposition 25.7 (Admissibility and the main second moment). *For admissible data D, the filling data of Theorem 24.15 are admissible, and:*

(a) $\bar{R}_0 = \bar{U}\bar{H}$ *with* $\#\bar{U} = M^{f\varrho(2k+2\varrho+1)}$ *and* $\#\bar{H}$ *the index of* K_0 *in the lattice stabiliser of* X_0.

(b) $\mathcal{D}_{\mathrm{main}} = (\#\bar{H}\,(N^k-1)^f \operatorname{mass}_k d)^2/(\deg\cdot\operatorname{mass}_r)$, *with* d *and* deg *the dimension and degree of the tensor product of the Bessel systems, and* $b^{11kf} \le \mathcal{D}_{\mathrm{main}}/\mathcal{D}_{\mathrm{id}} \le (M/b^6)^{kf}$.

(c) (*Mean values.*) *Let* $\mathrm{fr}_v = N^k \mathrm{sph}_{k+1}\cdots\mathrm{sph}_r$ *be the frame masses and* $c_v = \mathrm{fr}_v N^k d_v^2/(\mathrm{sph}_k \deg_v)$. *The two products*

$$\#\bar{H}\, n_X \prod_v \frac{\mathrm{fr}_v d_v^2}{\deg_v} \qquad \text{and} \qquad \#\bar{H}^2 \operatorname{mass}_k \prod_v \mathrm{sph}_k c_v$$

are at most $(1+b^{-9})\mathcal{D}_{\mathrm{main}}$, *and*

$$\prod_v \mathrm{fr}_v \le (1+b^{-9})(1-N^{-k})^{2f} N^{kf} \operatorname{mass}_k/\operatorname{mass}_r.$$

Proof. Fit and admissibility. The finite frame of Y_0 is the graded lattice basis times an integral matrix $\equiv 1$ at the chosen places, by the choice of Y_0; the graded basis scales only the string by the central unit u = the lattice scale, so it is the lattice basis of the filling data scaled by u on the whole space and by $u^{-1} = \bar{u}$ on X_0: this is the fit. Reduction at the chosen places maps K onto $\bar{G}$: lift the residues and write the lift in the lattice basis.

(a) is Lemma 25.1.

(b) An integral point of $\mathrm{U}(X_0)$ preserves the lattice of V exactly when its row k_0 is the unit vector modulo every place dividing π; the stabiliser of the unit vector in $\mathrm{GL}_k(\mathbb{F}_w)$ under the inverse transpose has index N^k-1, and there are f such places, so the index is $(N^k-1)^f$. The classes have trivial automorphism groups, so their number n_X is the index of K_0 in the stabiliser of the standard lattice times mass_k; the index splits along $K_0 \le \{\text{lattice preserving}\} \le \{\text{standard}\}$, giving $n_X = \#\bar{H}(N^k-1)^f \operatorname{mass}_k$ and $D_X = n_X d$. With $\mathcal{D}_{\mathrm{id}} = \#\bar{R}_0 D_X = \#\bar{U}\#\bar{H} D_X$,

$$\frac{\mathcal{D}_{\mathrm{main}}}{\mathcal{D}_{\mathrm{id}}} = \frac{d\,(N^k-1)^f \operatorname{mass}_k}{\deg\cdot\operatorname{mass}_r \cdot \#\bar{U}}.$$

Insert the size of the degree: from the formula

$$\deg = \exp \sum_v \sum_{k-1\le l<r} \Big(\sum_{i<l} \log(\varsigma_i - \varsigma_l) - \log l! \Big)$$

and the bounds $W_v \le \varsigma_i - \varsigma_l$ for $i < k-1$ and $\varsigma_i - \varsigma_l \le$ the lift weight, the logarithm of deg plus $f\sum_{k-1\le l<r} \log l!$ lies between $\sum_v m(k-1)\log W_v$ and $\sum_v mr(\log W_v + 1/x)$, with $m = r-k+1$. Insert also the ratio of the masses $(\log(\operatorname{mass}_{l+1}/\operatorname{mass}_l)$ has the main term $f((l+\tfrac12)\log \mathrm{rd}_E + \log l! - (l+1)\log 2\pi)$ with error $2^{1-\varrho}$, Theorem B.31(d)) and the range of N. All factorials cancel except $(k-1)!$, the sums of indices telescope, and up to a factor $b^{\pm kf/4}$ the ratio

is $(Mb^{-15/2})^{kf}\, d\,((k-1)!)^f$. Since $1 \le d((k-1)!)^f \le b^{5kf/4}$ and $b^{19} \le M$, the ratio lies in $[b^{11kf}, (M/b^6)^{kf}]$.

(c) Here $\mathrm{sph}_l = (2\pi)^l/(|\varpi|_v^{l-1/2}(l-1)!)$ is the mass of the unit sphere of $\mathbb{C}^l$ for the fibre measure when $\mathbb{C}$ carries $2/|\varpi|_v$ and $\mathbb{R}$ carries $|\varpi|_v^{-1/2}$ times Lebesgue measure; as $2 \le |\varpi|_v \le b$, $\mathrm{sph}_l \le 1$. The logarithm of $\prod_v \mathrm{sph}_{l+1}$ is $-f((l+\frac{1}{2})\log \mathrm{rd}_E + \log l! - (l+1)\log 2\pi)$, with $\prod_v |\varpi|_v = \mathrm{rd}_E^f$. Hence $\prod_v \mathrm{sph}_{l+1}$ is $\mathrm{mass}_l/\mathrm{mass}_{l+1}$ up to a factor $e^{\pm 2^{1-\varrho}}$, and these ratios telescope over $k \le l < r$; the $2\varrho+1$ errors add up to at most $\log(1+b^{-9}) - 4f/N$, and $\log(N^k/(N^k-1)) \le 2/N$ accounts for $(1-N^{-k})^{2f}$. This gives the bound for $\prod \mathrm{fr}_v$. The two products are $\#\bar{H} n_X (\prod \mathrm{fr}_v) d^2/\deg$ and $\#\bar{H}^2 \mathrm{mass}_k N^{kf} (\prod \mathrm{fr}_v) d^2/\deg$ (the masses sph_k cancel), and with $n_X = \#\bar{H}(N^k-1)^f \mathrm{mass}_k$ and the bound for $\prod \mathrm{fr}_v$ both are at most $(1+b^{-9})(n_X d)^2/(\deg \mathrm{mass}_r) = (1+b^{-9})\mathcal{D}_{\mathrm{main}}$. Finally $\prod_v \mathrm{fr}_v \cdot c_G^f |D_E|^{\varrho/2} = c_S N^{2kf} \mathrm{vol}(\hat{\mathcal{O}})^{\varrho+1+2\varrho(\varrho+k)}$ with the Siegel constant $c_S = ((2\pi)^k \mathrm{rd}_E^{1/2-k}/((k-1)!N^k))^f$, both sides being positive with logarithms linear in those of 2, π, N, rd_E and the factorials; with (c) and the definition of ν, in which Λ cancels, this is the bound for the scaled Siegel constant required by admissibility. □

26. Terms, cells and the filling theorem

The blocks of $P^* sP$ are sums of amplitudes over rational terms γ (Theorem 25.6). This section sorts the terms by the position of the frame $f = (e_1, \dots, e_\varrho, z)$ relative to γf, isolates those in the Bessel subgroup, which give the operators $r_\psi(s)$ of a twisted translation, and assembles the *filling theorem* (Theorem 26.8) from three bounds: the terms near the Bessel subgroup (Theorem 26.7), the remaining terms (Section 27) and the main term (Section 28).

26.1. Frames, links and cells

Let $A = \langle e_1, \dots, e_\varrho \rangle$, $A^+ = A + Ez$, and for a rational isometry γ let $Z(\gamma)$ be the matrix of products $h(f_a, \gamma f_b)$, with leading minors D_l. The *discriminant* of γ is $\mathfrak{g}(\gamma) = |D_{\varrho+1}|^2 - N^2 |D_\varrho|^2$. For t orthogonal to A and γA the middle coordinates of t and $\gamma^{-1}t$ in $A^\perp/A = X_0 \perp Ez$ form a pair; these pairs are the *link* of γ. A *norm-one pair occurs from* j if for some ζ with $\zeta\bar{\zeta} = 1$ and every index $a \ge j$ some combination of f equals some combination of γf, both without coefficients above a and with coefficients ζ and 1 at a. The element γ has *cell rank* $j \le \varrho+1$ if a norm-one pair occurs from j, $D_j \ne 0$, and $j = \varrho+1$ exactly when $\mathfrak{g}(\gamma) \ne 0$. It *agrees* if $\psi_U(u) = \psi_U(u')$ for rational $u, u' \in U$ with $u\gamma = \gamma u'$. A term is *near* if some boxed point y (Proposition 25.5(c)) has $\gamma^{-1}y$ boxed.

For a stable set T of terms and $\Omega \subseteq \bar{G}$, the *link second moment* of T over Ω is the sum of $\|A_{T_\Lambda}(c, s, c')\|^2$, for the cut function, over the links Λ, the pairs of classes (c, c') and the $s \in \Omega$; here T_Λ is the set of terms of T with link Λ. An element $s \in \bar{G}$ is *open* if at some chosen place above p_0 no proper leading minor of the Gram matrix of the frame products of s vanishes. For a set S_g of chosen places above p_0 and, at each $w \in S_g$, a nonzero vector d_w of the span of the frame modulo w or modulo $\bar{w}$, the *fixer* $\mathrm{fixer}(S_g, d)$ is the subgroup of the $s \in \bar{G}$ that fix every d_w.

Here $s \in \bar{G}$ acts modulo a place w above M by its component at w, and modulo $\bar{w}$ by the inverse adjoint of that component; s *meets* at a place if, modulo it, the span of the frame and its image under s share a nonzero vector, and the *meeting norm* $\mathrm{mn}(s)$ is the norm of the product of the places where s meets. An element fixing a nonzero vector of the span of the frame modulo a place meets there, and the reduction $\bar{H}$ of $\mathrm{U}(X_0)$ fixes the frame, hence lies in every fixer.

Lemma 26.1 (The link as an isometry). *Let z^+ be a vector of norm N and $X_0 \perp Ez^+$ the positive space with the coordinates of $X_0 \perp Ez$.*

(a) *A vector t is orthogonal to A and γA exactly when the coordinates of t and $\gamma^{-1}t$ at the e_a^* vanish. The link is the image of $(A+\gamma A)^\perp$ under $t \mapsto ([t], [\gamma^{-1}t])$, it has rank $k+1$, and $I_\gamma = A \cap \gamma A$ is the kernel; so the link has the rank of $(A+\gamma A)^\perp / I_\gamma$.*

(b) *For a pair $(y_0+\lambda' z,\ y_0'+\lambda z)$ of the link put $\mathrm{src} = y_0+\lambda z^+$ and $\mathrm{tgt} = y_0'+\lambda' z^+$. Both maps are bijections from the link onto $X_0 \perp Ez^+$, and $\Upsilon_\gamma = \mathrm{tgt} \circ \mathrm{src}^{-1}$ is an isometry of the positive space whose graph is the link; two elements have the same link exactly when they have the same Υ.*

(c) *On the big cell, for σ fixing the hyperbolic planes and the Weyl element $w_p \colon e_a \mapsto p_a e_a^*$, the matrix $Z(w_p\sigma)$ is diagonal with entries p_a and $t = h(z, \sigma z)$, so $\mathfrak{g}(w_p\sigma) = d\bar{d}(t\bar{t} - N^2)$ with $d = \prod_a p_a$; the link of $w_p\sigma$ is the graph of the middle block of σ, so $w_p\sigma$ and $w_q\tau$ have the same link exactly when $\sigma = \tau$.*

Proof. (a) The products of t with e_a read off the coordinate of t at e_a^*; those with γe_a are the products of $\gamma^{-1}t$ with e_a, γ being an isometry. The map to pairs is linear on $(A+\gamma A)^\perp$; its kernel consists of the t whose middle and last blocks vanish together with those of $\gamma^{-1}t$, that is $A \cap \gamma A$. Since γA has rank ϱ (the first ϱ rows of γ^{-1} are a left inverse of its first columns), the ranks of $A+\gamma A$ and I_γ add up to 2ϱ, and those of $A+\gamma A$ and its orthogonal add up to r; so the link has rank $r - 2\varrho = k+1$. (b) For y, y' in $(A+\gamma A)^\perp$ their product is that of their middle coordinates in $X_0 \perp Ez$ (the hyperbolic terms vanish), and the same holds for $\gamma^{-1}y$, $\gamma^{-1}y'$. Comparing the two and moving the terms in z, which have norm $-N$, to the other side gives $\langle \mathrm{src}\, p, \mathrm{src}\, q\rangle = \langle \mathrm{tgt}\, p, \mathrm{tgt}\, q\rangle$ in the positive space. The positive form is anisotropic (at an infinite place it is a sum of squares with positive weights), so a pair vanishes as soon as one of its images does; hence both maps are injective on the link, which has the dimension $k+1$ of the positive space, and bijective. (c) $w_p\sigma$ maps e_a to $p_a e_a^*$ and fixes z up to σ; read off the products. For the link take $a = 0$ in the description by coordinates: the pairs are $(\sigma_{\mathrm{mid}} y, y)$. □

Lemma 26.2 (Common vectors of the two flags). *Let γ agree.*

(a) *If μ, μ' are the coefficients at j of a vector common to the flags of f and γf (both without coefficients above j), they are the coefficients at every $l \geq j$ of such a common vector, and $\mu\bar{\mu} = \mu'\bar{\mu}'$.*

(b) *If a nonzero common vector has top index j (the least index above which its coordinates in both frames vanish), its two coordinates there have the same nonzero norm.*

(c) *If $\mathfrak{g}(\gamma) = 0$, γ has a cell rank $j \leq \varrho$: with j the least index from which one pair $(\zeta, 1)$, $\zeta\bar{\zeta} = 1$, occurs, the leading minor D_j is nonzero.*

Proof. (a) It suffices to go one step up. A common vector at $j+1$ with coefficients μ, μ' exists if every linear form m killing f_b and γf_b for $b \le j$ takes the same value on μf_{j+1} and $\mu'\gamma f_{j+1}$. Let x, y be the coordinates of the common vector at j in the two frames; for $r \in E$ the Eichler maps u along $r \cdot (f\,x)$ and w along $r \cdot (\gamma f\ y)$ with the vector representing m lie in U, and as $f\,x = \gamma f\,y$ they satisfy $u\gamma = \gamma w$. Their characters are ψ_{f} at r times the values of m on the two shifted vectors (Lemma 25.3, the factor r cancelling the constants); agreement gives equality of $\psi_{\mathrm{f}}(rm(\cdot))$ for all r, hence of the two values. For the norms: at the last index the common vector is a multiple of z in one frame and of γz in the other, with norms $-N\mu\bar{\mu}$ and $-N\mu'\bar{\mu}'$. (b) Scale the pair at the top index; by (a) it recurs at the last index, where (a) gives equal norms; the coordinate in the first frame is nonzero because the vector is. (c) A norm-one pair occurs vacuously from $\varrho + 1$; $\mathfrak{g} = 0$ gives a common nonzero vector, so by (b) a pair of nonzero coordinates with equal norms occurs at its top index $l \le \varrho$ and, scaled to $(\zeta, 1)$, at all higher indices; so $j \le \varrho$. If $D_j = 0$, some $y \ne 0$ vanishing from j on is killed by the first j rows of $Z(\gamma)$; the remaining rows kill y too, because for every w some vector shared by f and $\gamma^{-1}f$ has coordinates $\bar{w}$ from j on, and its pairing with y is a multiple of $y_{\mathrm{top}} = 0$. Then $\gamma f\,y$ lies in the span of f although y vanishes from j on, which contradicts the minimality of j by (b). □

Lemma 26.3 (Meeting places). *Let k be integral with agreeing reduction, and let w be a place above M where the spans of f and $k\,f$ share a nonzero vector modulo w.*

(a) *Some common vector modulo w has last coordinates μ, μ' not both zero.*
(b) *Let t be an integral matrix with two columns such that $P = (f \mid k\,f)^* t$ vanishes except in the two rows of the last vectors of f and $k\,f$, whose 2×2 determinant δ is a finite idele. Then* $\mathrm{mn}(k) \le [\hat{O} : \delta\hat{O}]$.

Proof. (a) As in Lemma 26.2(a), modulo w: lift the data to $\hat{O}$ and compare the unipotent character on the Eichler maps u, u' with $uk \equiv ku'$ modulo M; since $\hat{O}$ is self-dual for $x \mapsto \psi_{\mathrm{f}}(-x/\varpi)$, testing against every $r \in \hat{O}$ shows that M divides the difference of the lifted values, so the coefficients move up one index modulo w. Start at the top index of the shared vector and go up to the last index. (b) Modulo w, P^* kills $(x, -y)$ for the common vector of (a), and only the two last columns of P^* take part; so $\delta \equiv 0$ modulo w at every place where k meets. The index $[\hat{O} : \delta\hat{O}]$ is $\#\hat{O}/\delta$, which maps onto the product of the residue fields at those places. □

Proposition 26.4 (Links and cells). (a) *After exchanging the z-coordinates of the members of each pair, the link is the graph of an isometry Υ_γ of the positive space $X_0 \perp Ez^+$, $h(z^+, z^+) = N$. The link, $\mathfrak{g}$, the norm-one pair condition and the cells are unchanged under $\gamma \mapsto u^{-1}\gamma u'$, $u, u' \in U$, and under inversion; so each cell is a stable set.*
(b) *$\mathfrak{g}(\gamma) = 0$ exactly if A^+ and γA^+ meet nonzero; $\mathfrak{g}(\gamma)$ is totally positive or zero, by a norm identity at each infinite place; and for a term over a block the entries of $Z(\gamma)$ are algebraic integers, of absolute value at most b^{134n} when γ is near.*
(c) *If γ agrees and $\mathfrak{g}(\gamma) = 0$, then γ has a cell rank $j \le \varrho$, and $j + \dim(A \cap \gamma A) = \varrho$; then $A = A_{\le j} \oplus I_\gamma$ and $\gamma A = \gamma A_{\le j} \oplus I_\gamma$ with $I_\gamma = A \cap \gamma A$, and $(A_\gamma^+)^\perp / I_\gamma$ is isometric to $X_0 \cap \Upsilon_\gamma^{-1} X_0$.*

(d) *A stable set none of whose elements agrees has vanishing blocks. If the reduction s of an integral k agrees and the spans of f and sf meet modulo a place above M, a common vector with last entries not both zero exists there; and the norm of the product of the places where they meet is at most the generalised index of a 2×2 determinant of products with kf.*

Proof. (a) is Lemma 26.1(b). An element v of U is block unitriangular and fixes the last two blocks of a vector whose last block vanishes; so $t \mapsto vt$ carries the vectors that define the link of γ to those that define the link of $u^{-1}\gamma u'$. It acts on the span of the frame by the unitriangular matrix $m(v)$, so $Z(u^{-1}\gamma u') = m(u)^* Z(\gamma) m(u')$ has the same leading minors and the same $\mathfrak{g}$, and a common vector of the two flags with coefficients vanishing above a keeps its coefficients at a. For inversion, apply γ^{-1} to the shared vector, exchange the two sides and scale by $\bar\zeta = \zeta^{-1}$.

(b) By Proposition 24.2(c) (with the middle block $-N$ at z) the Gram determinant of $(f, \gamma f)$ is $(-1)^n \mathfrak{g}(\gamma)$, and it vanishes exactly when the spans meet. For the sign, Cramer's rule gives y with $Z(\gamma)y = (0,\dots,0,\det Z(\gamma))$ and last coordinate D_ϱ. The vector $t = \gamma f y$ is orthogonal to $e_1,\dots,e_\varrho$ and has product $\det Z$ with z; hence it has no part along the e_a^*, its z-coordinate is $-\det Z/N$, and, γ being an isometry, at each infinite place

$$N\cdot\|t_{X_0}\|^2 = |\det Z|^2 - N^2|D_\varrho|^2,$$

the image of $\mathfrak{g}(\gamma)$. Hence $\mathfrak{g}(\gamma)$ is real and nonnegative at every infinite place, and it is totally positive unless it vanishes (if $\det Z = 0$, the same identity gives $-N^2|D_\varrho|^2 \ge 0$, so $\mathfrak{g} = 0$). Integrality: the Gram matrix of $h_c^{-1}\gamma h_{c'}$ in the lattice basis is that of γ, the frame belongs to the lattice basis, and $h_c^{-1}\gamma h_{c'}$ is integral. For a near term, at an infinite place w the entry (i,j) of $Z(\gamma_w)$ is $\kappa_w^{(\varrho-i)+(\varrho-j)}/N$ times that of $Z(\gamma)$, and it is at most the product of the norms of the frame vectors of y and $\gamma^{-1}y$ (Lemma 24.1(b)), both at most b^{37n}; with $1 \le |\kappa_w|$ and $N \le b^{60n}$ this gives b^{134n}.

(c) The cell rank is Lemma 26.2(c). Let j be the cell rank and $I_\gamma = A \cap \gamma A$. The common vectors with coefficients $(\zeta, 1)$ at the indices $a \ge j$, minus their parts in $A_{\le j}$, lie in I_γ, and they are independent modulo $A_{\le j}$; hence $\dim I_\gamma \ge \varrho - j$, and $D_j \ne 0$ makes $A_{\le j} \cap I_\gamma = 0$, so $A = A_{\le j} \oplus I_\gamma$, and likewise for γA. For the isometry: the map $y \mapsto y_0$ from $(A_\gamma^+)^\perp$ to X_0 has kernel I_γ, sends the form of the whole space to the standard form of X_0 (the z-part of y vanishes), and has image $X_0^\gamma = X_0 \cap \Upsilon_\gamma^{-1} X_0$: every $v \in X_0$ is the X_0-part of some y orthogonal to A and γA with $\lambda = 0$, and the coefficient of $\Upsilon_\gamma v$ at z^+ is λ'. Here $\lambda = \bar\zeta\lambda'$ for every such y, because the pair $(\zeta, 1)$ occurs at the last index; with $\lambda = 0$ this forces $\lambda' = 0$, so $X_0^\gamma = X_0$ when $\mathfrak{g}(\gamma) = 0$. A dimension count then gives $j + \dim I_\gamma = \varrho$.

(d) For a stable set of terms none of which agrees, compute the block with a weight adapted to the stabilisers of the cosets $\gamma\, U_{c'}(M)$ (Proposition 25.4(b)): the term of each γ is a multiple of the integral of the kernel against a cutoff for the stabiliser Γ of the coset. The kernel is invariant under Γ, so this is its integral over the quotient by Γ; and some $u \in \Gamma$ twists the kernel by the scalar $\psi_U(u)^{-1}\psi_U(u') \ne 1$ (by (b) of Lemma 25.2), so the integral vanishes. The rest is Lemma 26.3. □

26.2. The Bessel subgroup and its neighbourhood

For $s \in \bar{G}$ let $r_\psi(s)$ be the operator on the source whose blocks are the conjugates of the amplitude sums of the terms in the Bessel subgroup, divided by ν. For s in the reduction $\bar{R}_0$ of the Bessel subgroup it is the translation by s twisted by the character, unitary, with $r_\psi(1) = 1$ and $r_\psi(st) = r_\psi(s) r_\psi(t)$; it vanishes off $\bar{R}_0$. A set $\Omega \subseteq \bar{G}$ *avoids the scalar cosets* if it meets no coset $\zeta \bar{R}_0$ for an integral $\zeta \neq 1$ with $\zeta\bar{\zeta} = 1$ (a root of unity, by Kronecker's theorem). Let $\psi_U^\natural$ be the mean of the character ψ_U of the integral unipotent elements, read in $\bar{G}$ and extended by 0, over the conjugacy class.

Lemma 26.5 (The twisted translation). *Let the data be admissible and E contain a primitive* 5040*th root of unity, and let P have amplitude blocks for a representation ρ of $\bar{G}$.*

(a) *An element γ of $R(E)$ acts on the values of the Bessel function by a unitary matrix M_γ: $W(\gamma_\infty y) = M_\gamma W(y)$, with M_γ a scalar of modulus one times a tensor product of the local representations. If γ is a term over (c, s, c'), the block of the Bessel subgroup over (c, s, c') is νM_γ, and that over (c, s, d) vanishes for $d \neq c'$; the block over (d, s^{-1}, c) is its adjoint.*

(b) *$r_\psi(s) = 0$ for $s \notin \bar{R}_0$; for $s \in \bar{R}_0$, $(r_\psi(s)\phi)_c = \overline{M_\gamma}\, \phi_{c'}$ for any term $\gamma \in R(E)$ over (c, s, c'), which exists for every c (Lemma 25.2(d)). Hence $r_\psi(1) = 1$, $r_\psi(st) = r_\psi(s) r_\psi(t)$, $r_\psi(ts) = r_\psi(t) r_\psi(s)$ for $t \in \bar{R}_0$, $s \in \bar{G}$, and $r_\psi(s)^* = r_\psi(s^{-1})$.*

(c) *For $t \in \bar{R}_0$: $P^* \rho(st) P = (P^* \rho(s) P)\, r_\psi(t)$, $\rho(t) P = P\, r_\psi(t)$ when ρ is unitary, and $P^* \rho(ts) P = r_\psi(t)\, P^* \rho(s) P$.*

(d) *$\| P^* \rho(s) P - r_\psi(s) \|^2$ is the second moment at s of the terms off the Bessel subgroup, and $\operatorname{tr} r_\psi(s) = D_X \psi_U(s)$ when s is* 1 *above one of the primes.*

Proof. (a) Write $\gamma = u\eta$ with $u \in U(E)$ and η_v acting on X_0 by $k_v \in \mathrm{U}(\kappa)$; left translation multiplies the Bessel function by $\otimes_v \pi_v(k_v)$ divided by $\psi_U(u)$ (Lemma 25.2(b)). Translation by γ moves the block $(c', 1, d)$ of the Bessel subgroup to (c, s, d) and acts on the kernel by $M_\gamma \otimes 1$; the block $(c', 1, d)$ is $\nu\, \delta_{c'd}$ times the identity (Proposition 25.4(d)). The block at (d, s^{-1}, c) uses the inverse term, and the kernel of γ^{-1} is the adjoint. (b) If γ is over (c, s, c') and δ over (c', t, c''), then $\gamma\delta$ is over (c, st, c'') and $M_{\gamma\delta} = M_\gamma M_\delta$; the adjoint is (a). (c) A term δ of the Bessel subgroup over (d, t^{-1}, d') gives the column (d, j) of $r_\psi(t)$; the blocks of all terms transform in the same way under right translation by δ, because the set of all terms is stable (Proposition 25.4(b), moving blocks). This gives the first identity; the second follows because the difference $\rho(t)P - P r_\psi(t)$ is orthogonal to all translates of the image of P, hence to itself; the third is the adjoint of the second applied to t^{-1}. (d) The blocks of $P^* \rho(s) P - r_\psi(s)$ are the amplitude sums of the terms off the Bessel subgroup. For the trace, the diagonal blocks of the Bessel subgroup at s that is 1 above one prime come from the unipotent terms over (c, s, c), whose kernels are ψ_U^{-1} times the identity. □

Lemma 26.6 (Scalar cosets). (a) *If a term γ over (c, s, c') has a norm-one pair occurring from* 0*, then $\gamma \in \zeta R(E)$ and $s \in \zeta \bar{R}_0$ for an integral ζ with $\zeta\bar{\zeta} = 1$; if a common vector with coefficients* 1, 1 *occurs at every index, $\gamma \in R(E)$.*

(b) *An integral ζ with $\zeta\bar{\zeta} = 1$ that is 1 modulo every chosen place above one prime is 1. The elements of a scalar coset $\zeta\bar{R}_0$ are not open, and there are at most $M^f \# \bar{R}_0$ of them in all scalar cosets.*

(c) *The set of s that coincide with an element of $\bar{R}_0$ above p_i avoids the scalar cosets, and each such s meets at the f chosen places above p_i and their conjugates.*

Proof. (a) At the index 0 the common vector gives $\gamma e_1 = \zeta e_1$, so ζ is the diagonal entry of γ at e_1, an entry of the integral matrix $h_c^{-1}\gamma h_{c'}$. Then $\zeta^{-1}\gamma$ has common vectors with coefficients 1, 1 at every index; by induction $\zeta^{-1}\gamma e_b$ is e_b plus earlier frame vectors, so $\zeta^{-1}\gamma$ lies in R, and the scalar ζ is an element of the lattice stabiliser whose reduction is the residue of ζ. (b) View ζ as an isometry of the standard lattice of rank one and apply the freeness of Lemma 25.2(e). An element of $\bar{R}_0$ fixes e_1 modulo every chosen place, so for $s \in \zeta\bar{R}_0$ the frame vector e_1 is mapped to a multiple of itself, and the leading minors of positive size of the frame Gram matrix vanish (e_1 is orthogonal to the frame). An element ζt is determined by the residues of ζ at the chosen places and by t. (c) Such an s fixes e_1 above p_i, so $\zeta \equiv 1$ there and $\zeta = 1$ by (b); and e_1 is a common vector. □

Theorem 26.7 (Near the Bessel subgroup). *Let the data be admissible and E contain a primitive 5040th root of unity.*

(a) (*Expansion of the squares.*) *If Ω avoids the scalar cosets and the link second moment of the terms with $\mathfrak{g} \neq 0$ over Ω is at most $C\nu^2$, then $\left(\sum_{s\in\Omega}\|P_s - r_\psi(s)\|^2\right)^{1/2} \leq C^{1/2} + b^{-15}\mathcal{D}_{\mathrm{id}}^{1/2}$.*

(b) *For Ω the set of s that coincide with an element of $\bar{R}_0$ above p_i, $\sum_{s\in\Omega}\|P_s - r_\psi(s)\|^2 \leq b^{-12}\mathcal{D}_{\mathrm{id}}$.*

(c) (*Near isometry.*) $\|P^*P - 1\|^2 \leq b^{-12}D_X$.

(d) (*Induced character.*) $\sum_{s\in\bar{G}}|\psi_U^\natural(s)|^2 \leq 2$; *for a central idempotent x of the factor above p_i with* $\mathrm{Re}\, x(1) \leq b^{-8}/2$, $\mathrm{Re}\,\mathrm{tr}\, r_\psi(x) \leq b^{-4}D_X$ *and* $\|xP\|^2 \leq 2b^{-4}D_X$.

(e) *For $e_0 = g_0 \otimes g_1$ with central idempotents g_i such that* $\mathrm{Re}(1 - g_i)(1) \leq b^{-8}/2$: $\|(1 - e_0)P\|^2 \leq 4b^{-4}D_X$, *and e_0P is a near isometry with error $5b^{-4}$.*

Proof. (a) By Lemma 26.5(d) the left side is the square root of the second moment over Ω of the terms off R. Over $s \in \Omega$ these have the same blocks as the remaining terms Γ^* of Section 27: the terms off R and off Γ^* that do not agree form a stable set with vanishing blocks (Proposition 26.4(d)), and one that agrees, with $\mathfrak{g} = 0$ and $A = \gamma A$, has a norm-one pair from 0 by Proposition 26.4(c) and so lies over no block at $s \in \Omega$ (Lemma 26.6(a)). Split the second moment of Γ^* into the pairs of terms with the same link, which give the link second moment for the cut function, at most $C\nu^2$, and the pairs with different links or with $\mathfrak{g} = 0$, at most $b^{-kf}\mathcal{D}_{\mathrm{id}}\nu^2$ by Theorem 27.26. Passing from the cut to the full function costs $b^{-20}\mathcal{D}_{\mathrm{id}}^{1/2}$ in the square root, by Proposition 25.4(b); and $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$.

(b) Every $s \in \Omega$ meets at the f chosen places above p_i and at their conjugates (Lemma 26.6(c)), so $\mathrm{mn}(s) \geq p_i^{2f} \geq (M/b)^f$, since each level prime is at most b times the other. The link second moment over Ω is then at most $(b^{2f}/\mathrm{mn})^k\mathcal{D}_{\mathrm{main}}\nu^2$ by Theorem 27.26(c), and with $m = kf$ and $\mathcal{D}_{\mathrm{main}} \leq (M/b^6)^m\mathcal{D}_{\mathrm{id}}$ this is at most $(b^3/M)^m(M/b^6)^m\mathcal{D}_{\mathrm{id}} = b^{-3m}\mathcal{D}_{\mathrm{id}} \leq b^{-30}\mathcal{D}_{\mathrm{id}}$. Since Ω avoids the scalar cosets, (a) gives the bound.

(c) For $s \in \bar{R}_0$, $P_s - r_\psi(s) = (P^*P - 1)r_\psi(s)$ by Lemma 26.5(c), which has the norm of $P^*P - 1$ because $r_\psi(s)$ is unitary. Summing (b) over $s \in \bar{R}_0$, which lies in Ω, gives $\#\bar{R}_0\|P^*P - 1\|^2 \le b^{-12}\mathcal{D}_{\mathrm{id}} = b^{-12}\#\bar{R}_0 D_X$.

(d) *The induced character.* Let T be the set of the chosen places and their conjugates, K_T and $K_T(M)$ the integral points and the principal congruence subgroup of level M at T, and χ_M the function equal to $\psi(\sigma(u)/M)$ on $uK_T(M)$, $u \in U$, and 0 off $UK_T(M)$. For l in the lattice stabiliser, $\chi_M(l_T)$ is ψ_U at the reduction of l, and $a(z) = \int_{K_T} \chi_M(yzy^{-1})\,dy$ is $\mathrm{vol}(K_T)$ times $\psi_U^\natural$ at the reduction of z (translate y by lifts of the elements of $\bar{G}$ and sum). The function a is constant on the cosets $\sigma(s)K_T(M)$ of lifts $\sigma(s)$ of the $s \in \bar{G}$, which are disjoint, so $\sum_s |a(\sigma(s))|^2$ is at most $\int |a|^2/\mathrm{vol}\,K_T(M)$. The completions at T carry a large lattice datum of modulus M, with $3\varrho + 6$ hyperbolic pairs and $k - (6\varrho + 13)$ orthonormal vectors, and the mass bound at the finite places (Theorem 29.20 with $Q = 2$) bounds this integral by $2\,\mathrm{vol}(K_T)^2$. Dividing by $\mathrm{vol}(K_T)^2$ gives $\sum|\psi_U^\natural|^2 \le 2$.

The trace. For a central idempotent x of the factor above p_i, each s in its support is 1 above the other prime, so $\mathrm{tr}\, r_\psi(x) = D_X \sum_s x(s)\psi_U(s) = D_X \sum_s x(s)\psi_U^\natural(s)$ (x is a class function). By Cauchy–Schwarz, with $\sum_s |x(s)|^2 = \mathrm{Re}\, x(1) \le b^{-8}/2$ (compare the coefficients of 1 in $x^2 = x$, using $x(s^{-1}) = \overline{x(s)}$), $\mathrm{Re}\,\mathrm{tr}\, r_\psi(x) \le D_X(b^{-8}/2 \cdot 2)^{1/2} = b^{-4}D_X$.

One prime. x acts through ρ as an orthogonal projection, so

$$\|xP\|^2 = \mathrm{tr}(P^*xP) = \mathrm{tr}\, r_\psi(x) + \mathrm{tr}\, P'_x, \qquad P'_x = \sum_t x(t)\big(P_t - r_\psi(t)\big).$$

The map $F \mapsto \sum_t x(t)F(\cdot\, t)$ is a contraction on functions on the group; applied to $P_s - r_\psi(s)$ cut down to $\Omega = \bar{R}_0 G_i$, whose value at $s \in \bar{R}_0$ is $r_\psi(s)P'_x$ of norm $\|P'_x\|$, it gives $\#\bar{R}_0\|P'_x\|^2 \le \sum_{s\in\Omega}\|P_s - r_\psi(s)\|^2 \le b^{-12}\mathcal{D}_{\mathrm{id}}$ by (b). Hence $|\mathrm{tr}\, P'_x| \le \|1\|\|P'_x\| \le D_X^{1/2}(b^{-12}D_X)^{1/2} = b^{-6}D_X$, and $\|xP\|^2 \le 2b^{-4}D_X$.

(e) $1 - e_0 = (1 - g_0) + g_0(1 - g_1)$ with orthogonal ranges, so $\|(1 - e_0)P\|^2 \le \|(1 - g_0)P\|^2 + \|(1 - g_1)P\|^2 \le 4b^{-4}D_X$ by (d) for the central idempotents $1 - g_i$. By (c) and Lemma 22.1(a), P is a near isometry with error b^{-6}, and by Lemma 22.1(b) e_0P is one with error $b^{-6} + 4b^{-4} \le 5b^{-4}$. □

26.3. The filling theorem

Theorem 26.8 (Filling theorem). *Let E carry Weil data and the filling data be admissible. Let ρ be a unitary representation of $\bar{G}$ and P a linear map from the source whose blocks are the amplitude sums (Theorem 25.6(e)). Then*

$$\sum_{s\in\bar{G}} \|P^*\rho(s)P\|^2 \le (1 + 9b^{-4})\,\mathcal{D}_{\mathrm{main}},$$

$\|P^*P - 1\|^2 \le b^{-12}D_X$, and for central idempotents g_i as in Theorem 26.7(e), e_0P is a near isometry with error $5b^{-4}$.

The proof uses the following bounds.

Lemma 26.9 (Fixers). *Let $\beta(s) = \|P^*\rho(s)P\|^2$.*

(a) *β is of positive type, so $\beta(s) \le \beta(1) \le 2D_X$.*
(b) *If $\#S_g = f$, then $\sum_{s\in\mathrm{fixer}(S_g,d)} \beta(s) \le b^{-6kf}\mathcal{D}_{\mathrm{main}}$.*

(c) *Over the open elements of* $\mathrm{fixer}(S_g, d)$, $\sum \beta(s) \le (1 + 4b^{-4})b^{-6k\#S_g}\mathcal{D}_{\mathrm{main}}$.
(d) *Let* Γ *be a subgroup and* w *a chosen place above* p_0 *such that the stabiliser in* Γ *of every nonzero vector of the span of the frame modulo* w *or* $\bar{w}$ *carries at most a share* b^{-6k} *of* $\sum_\Gamma \beta$. *Then the elements of* Γ *that are not open carry at most a share* b^{-4}.

Proof. (a) By the computation in the proof of Theorem 22.2, $\beta(a^{-1}b) = \langle u_a, u_b\rangle$ with $u_a = aPP^*a^{-1}$ in the Hilbert–Schmidt space, and a real function of positive type is largest at 1 by Cauchy–Schwarz. By Theorem 26.7(c), $\|P^*P\| \le \|P^*P - 1\| + \|1\| \le (1 + b^{-6})D_X^{1/2}$, so $\beta(1) \le 2D_X$.

(b) Every s in the fixer meets at all f places above p_0, so $\mathrm{mn}(s) \ge p_0^f$. The scalar cosets of $\bar{R}_0$ contain at most $M^f \# \bar{R}_0$ elements (Lemma 26.6(b)), each with $\beta \le 2D_X$, which is at most $2M^f\mathcal{D}_{\mathrm{id}} \le \frac{1}{4}b^{-6kf}\mathcal{D}_{\mathrm{main}}$ because $8M^f \le b\, b^{23f}$, $23f + 1 \le 5kf$ and $b^{11kf}\mathcal{D}_{\mathrm{id}} \le \mathcal{D}_{\mathrm{main}}$. Off the scalar cosets the expansion of the squares (Theorem 26.7(a)) applies with the link second moment of Theorem 27.26(c) for $z = p_0^f$, $(b^{2f}/p_0^f)^k\mathcal{D}_{\mathrm{main}} \le \frac{1}{8}b^{-6kf}\mathcal{D}_{\mathrm{main}}$ as $b^{10} \le p_0$; with $b^{-15}\mathcal{D}_{\mathrm{id}}^{1/2}$ small, and the contribution $\|r_\psi(s)\|^2 = 0$ off $\bar{R}_0$, the total is at most $b^{-6kf}\mathcal{D}_{\mathrm{main}}$.

(c) No open element lies in a scalar coset (Lemma 26.6(b)), so the expansion of the squares applies with the link second moment of the main term (Theorem 28.17), $C = (1 + b^{-4})b^{-6k\#S_g}\mathcal{D}_{\mathrm{main}}$; with $x = b^{-4}$, $(1 + x)^{1/2} \le 1 + x/2$ and $b^{-15}\mathcal{D}_{\mathrm{id}}^{1/2} \le \frac{x}{2}C^{1/2}$, which follows from $b^{5kf}\mathcal{D}_{\mathrm{id}} \le b^{-6k\#S_g}\mathcal{D}_{\mathrm{main}}$, the square of the bound is at most $(1 + 4x)b^{-6k\#S_g}\mathcal{D}_{\mathrm{main}}$.

(d) Let $l < n$ and δ_l the leading minor of size l of the frame Gram matrix at w; it is the determinant of the pairings of the first l frame vectors with their images, ${}^t(Je)\, s\, e$ with e the first l frame vectors. Apply Lemma 22.4 to Γ acting on the lattice modulo w, a field with p_0 elements, with $A_0 = e$, $A_1 = Je$ and $\Theta = \beta$: the stabiliser of a vector $A_0 v$ under s, and that of the form $A_1 v$ under the contragredient, fix a nonzero vector of the span of the frame modulo w or $\bar{w}$, so their masses are at most $Y = b^{-6k}\sum_\Gamma \beta$. Hence $p_0 \sum_{\delta_l = 0} \beta \le 2\sum_\Gamma \beta + p_0^{l+r/2}Y$, and $p_0^{l+r/2} \le b^{6k}$ (square, and use $p_0 \le b^{11}$, $r = k + 2n - 1$ and $2000n \le k$) gives a share at most $3b^{-10}$. Summing over the n minors, $3n \le 3b \le b^6$, the elements that are not open carry at most b^{-4}. □

Proof of Theorem 26.8. A *choice* is a set S_g of chosen places above p_0 with, at each w, a nonzero vector d_w of the span of the frame modulo w or modulo $\bar{w}$; its *mass* is $\mu(S_g, d) = \sum_{s \in \mathrm{fixer}(S_g, d)} \beta(s)$. Choose (S_g, d) maximising $b^{6k\#S_g}\mu(S_g, d)$. Since the fixer of the empty choice is $\bar{G}$, it suffices to show $\mu(S_g, d) \le (1 + 9b^{-4})b^{-6k\#S_g}\mathcal{D}_{\mathrm{main}}$.

If S_g consists of all f places, this is Lemma 26.9(b). Otherwise take $w \notin S_g$. For every vector d' at w, the stabiliser of d' in the fixer is the fixer of $(S_g \cup \{w\}, d \cup \{d'\})$, so by maximality it carries at most a share b^{-6k} of $\mu(S_g, d)$. By Lemma 26.9(d) the elements of the fixer that are not open carry at most $x\mu$, $x = b^{-4}$, and by Lemma 26.9(c) the open elements contribute at most $(1 + 4x)C$ with $C = b^{-6k\#S_g}\mathcal{D}_{\mathrm{main}}$. Hence $\mu \le (1 + 4x)C + x\mu$, and since $0 \le x \le 1/10$, $(1 - x)(1 + 9x) - (1 + 4x) = x(4 - 9x) \ge 0$ gives $\mu \le (1 + 9x)C$. The other two statements are Theorem 26.7(c),(e). □

27. The remaining terms

The *remaining terms* Γ^* are the rational γ with $\mathfrak{g}(\gamma) \neq 0$, together with those that agree, have $\mathfrak{g}(\gamma) = 0$ and move A. This section bounds their contribution to the second moment (Theorem 27.26). A block of Γ^* is bounded, orbit by orbit under the unipotent elements U_c^0 that fix the frame, by Schur's test with a weight made of volume ratios at the real places (Theorem 27.10); its *size* depends on the cell rank and on a *gauge* of the term. Squaring and summing over pairs of near terms, a pair gains from the discriminant of its *common lattice* in X_0 (Theorem 27.20); and lattices of given corank and small discriminant, and frames over them, are few (Theorem 27.24). Pairs whose links differ are negligible; pairs with equal links gain the norm of the places where the block meets.

Conventions. Throughout the data are admissible (Section 25.1) and E contains a primitive 5040th root of unity. We write f for the number of infinite places of E, $k = \dim X_0$, $n = \varrho + 1$, $\tau = b^{kf/4}$, so that $\tau^4 = b^{kf}$, and $n_p = \#\bar{R}_0\, n_X N^{-kf}$, with n_X the number of classes. Since $k \geq 1000(2\varrho + 2)^9$, every power b^e with $e \leq 10^5 n^9 f$ is at most τ, and so is the number $n^3 + 1$. For a finite idele a the *generalised index* $[a]$ is the inverse of the factor by which multiplication by a scales Haar measure on the finite adeles, and $[a] = 0$ if a is not a unit; for integral a it is $\#\hat{O}/a\hat{O}$, and for $a \in E^\times$ it is $\mathrm{Nm}(a) = |\mathrm{N}_{E/\mathbb{Q}}(a)| = \prod_v |a|_v^2$, the product over the infinite places with the chosen embeddings. For the discriminant $\mathfrak{g}(\gamma)$, which is fixed by complex conjugation, $\mathfrak{g}_{\mathrm{Nm}} = \prod_v |\mathfrak{g}(\gamma)|_v$; it is a positive integer when $\mathfrak{g}(\gamma) \neq 0$. The lattice of a class c is written Λ, and vectors are written in its coordinates, in which $h_c^{-1}\gamma h_{c'}$ is the matrix of a term.

27.1. Lattices over the finite adeles

Let $\hat{O}$ be the ring of integral finite adeles of E. A *lattice of rank r* in $\mathbb{A}_f^\kappa$ is the $\hat{O}$-span L of the columns of xa, where x is a $\kappa \times r$ matrix over E with independent columns and $a \in \mathrm{GL}_r(\mathbb{A}_f)$; its *discriminant* is $\operatorname{disc} L = [\det((xa)^*(xa))]^{1/2}$. For a matrix H over $\mathbb{A}_f$ with a left inverse (the passage to the coordinates of a lattice) and a subspace V of E^κ, the *saturated lattice* Λ_V is the set of points of the adelic span of V that H makes integral; its discriminant is written $d_H(V)$.

Lemma 27.1 (Discriminants). (a) *The discriminant does not depend on the basis xa of L, and it is positive.*

(b) *If $M \subseteq L$ is an $\hat{O}$-submodule of finite index, then M is a lattice of rank r and* $\operatorname{disc} M = [L : M] \operatorname{disc} L$.

(c) *A matrix η over E with a left inverse maps lattices of rank r to lattices of rank r; if $\eta^*\eta = 1$, it keeps the discriminant.*

(d) *Λ_V is a lattice of rank* $\dim V$, *and $d_H(V) \geq 1$ if every adelic matrix m with Hm integral has an integral Gram matrix m^*m. If H maps a lattice L of rank r into integral vectors, then for the span W of L one has $L \subseteq \Lambda_W$ with finite index i, and* $\operatorname{disc} L = i\, d_H(W)$.

(e) *Every ideal of finite index of $\hat{O}$ is principal.*

Proof. (a) Two adelic bases B, C of the same $\hat{O}$-module, B with a left inverse, satisfy $C = BU$ and $B = CU'$ with U, U' integral, so U lies in $\mathrm{GL}_r(\hat{O})$ and

$[\det U] = 1$. Since $\det(C^*C) = \overline{\det U}\det U \det(B^*B)$ and the generalised index is multiplicative, the two Gram determinants have the same index. Positivity: at a complex embedding the Gram matrix of r independent vectors is positive definite, so $\det(x^*x) \neq 0$, and $[\det((xa)^*(xa))] = \mathrm{Nm}(\det x^*x)[\det a]^2 > 0$.

(b) Over each completion $\hat{O}_v$ is a principal ideal domain, and $\hat{O}$ satisfies the following exchange property: any a, b are multiples of $(1-e)a + eb$ for an idempotent e. Column operations over $\hat{O}$ therefore bring a generating matrix $(C \mid R)$ of a module of finite index in $\hat{O}^r$ to a square matrix; hence M has a basis BA with A integral and invertible over $\mathbb{A}_f$, and $[L : M] = [\hat{O}^r : A\hat{O}^r] = [\det A]$. Then $\operatorname{disc} M^2 = [\det A]^2 \operatorname{disc} L^2$.

(c) The image of the span of the columns of xa is the span of the columns of $(\eta x)a$, and ηx has independent columns. If $\eta^*\eta = 1$ the Gram matrix does not change.

(d) There are a basis y of V and $a \in \mathrm{GL}_s(\mathbb{A}_f)$, $s = \dim V$, such that Hya is primitive, that is integral with an integral left inverse p (take a basis of the integral points of the span locally at each place); the columns of $m = ya$ span Λ_V, because a point mt of Λ_V has $t = p(Hmt)$ integral. If Hm integral forces m^*m integral, the Gram determinant of m is a nonzero integral idele, whose index is at least 1. Finally let L have the basis xa and let W be the span of x, with chosen basis $S = yb$ of Λ_W. Then $x = yT$ with T invertible over E, and $xa = SA$ with $A = b^{-1}Ta$; the matrix $A = p\,H(xa)$ is integral, so $L \subseteq \Lambda_W$ with index $[\det A]$, and $\operatorname{disc} L = [\det A]\, d_H(W)$ as in (b).

(e) An ideal I of finite index contains its index, a nonzero integer m. Hence I is determined by its images in the finitely many factors $\hat{O}_v$ with $v \mid m$, each a power of the maximal ideal, and the idele that is a uniformiser to that power at these places and 1 elsewhere generates it. □

For an R-module X let $\zeta_X(s) = \sum_M [X : M]^{-s}$, the sum over the submodules of finite index.

Lemma 27.2 (Sums over sublattices). *Let R be a commutative ring whose ideals of finite index are principal.*

(a) $\zeta_{R\times N}(s+1) \le \zeta_R(s+1)\,\zeta_N(s)$ *for every R-module N.*
(b) $\zeta_{R^r}(r+\mu) \le \zeta_R(\mu)^r$.
(c) *For a lattice L of rank r over $\hat{O}$, $\sum_M [L : M]^{-(r+\mu)} \le \zeta_{\hat{O}}(\mu)^r$. For $\mu = 2\varrho + 4$ one has $\zeta_{\hat{O}}(\mu) \le 2$, so the sum is at most $2^r \le 2^k$ when $r \le k$.*

Proof. (a) Let $M \subseteq R \times N$ have finite index. Its first coordinates form an ideal $I = (g)$ of finite index, and with $M' = M \cap N$ and any n with $(g, n) \in M$, the module M is spanned by (g, n) and M'; the class of n modulo M' determines M. The index is $[R \times N : M] = [R : I]\,[N : M']$, and there are $[N : M']$ classes. Hence

$$\zeta_{R\times N}(s+1) \le \sum_I \sum_{M'} [N : M'] \left([R : I][N : M']\right)^{-(s+1)} = \zeta_R(s+1)\,\zeta_N(s).$$

(b) Induction on r, with $R^r = R \times R^{r-1}$: by (a), $\zeta_{R^r}(r+\mu) \le \zeta_R(r+\mu)\zeta_{R^{r-1}}(r-1+\mu) \le \zeta_R(\mu)\cdot\zeta_R(\mu)^{r-1}$, since ζ_R decreases in s. (c) A lattice of rank r is isomorphic to $\hat{O}^r$ through its basis, and the ideals of finite index of $\hat{O}$ are principal (Lemma 27.1(e)). Contraction to O_E is injective on ideals of finite

index and keeps the index, so $\zeta_{\hat{\mathcal{O}}}(\mu)$ is at most the Dedekind zeta value $\zeta_E(\mu)$, an Euler product; for $\mu = 2\varrho + 4$ and $(\varrho+1)[E:\mathbb{Q}] \le 2^{2\varrho}$ its logarithm is at most $(\log 2)/(\varrho+1)$, so $\zeta_{\hat{\mathcal{O}}}(\mu)^{\varrho+1} \le 2$ and $\zeta_{\hat{\mathcal{O}}}(\mu) \le 2$. □

Fix u in the finite adelic unitary group of $X_0 = E^\kappa$ and consider the unimodular lattice $u\hat{\mathcal{O}}^\kappa$ and the rational lattice $E^\kappa \cap u\hat{\mathcal{O}}^\kappa$. For a subspace T let $d(T)$ be the discriminant of $T \cap u\hat{\mathcal{O}}^\kappa$, a saturated lattice for $H = u^{-1}$, and write $\|X\|$ for the Euclidean length of a vector over all infinite places together; rd_E is the root discriminant.

Proposition 27.3 (Siegel's lemma). (a) *(Jacobi.) $d(T) = d(T^\perp)$ for every subspace T.*

(b) *(Siegel's lemma.) A subspace T of codimension s is the orthogonal of s nonzero vectors $X_1, \dots, X_s$ of $E^\kappa \cap u\hat{\mathcal{O}}^\kappa$ with*

$$\Big(\prod_i \|X_i\|\Big)^{2f} \le (8fs\,\mathrm{rd}_E)^{fs}\, d(T).$$

(c) *$\sum_T d(T)^{-(k+1)} \le (200fk\,\mathrm{rd}_E)^{f(k+1)s}$, the sum over the subspaces T of codimension s.*

Proof. (a) Let xb and ya be the chosen bases of $T \cap u\hat{\mathcal{O}}^\kappa$ and $T^\perp \cap u\hat{\mathcal{O}}^\kappa$; the matrices $u^{-1}xb$ and $u^{-1}ya$ are primitive and $x^*y = 0$. In the unimodular lattice $\hat{\mathcal{O}}^\kappa$ two orthogonal primitive sublattices of complementary ranks have Gram determinants of the same index: complete $u^{-1}xb$ to a basis of $\hat{\mathcal{O}}^\kappa$; the inverse of its Gram matrix is integral and its block on the complement is the inverse of the Gram matrix of the dual basis, which spans the projection of $u^{-1}ya$; this is Jacobi's identity for complementary minors of a unimodular matrix. As u^{-1} is unitary it drops out of the Gram matrices, which gives $[\det a]^2 \,\mathrm{Nm}(\det y^*y) = [\det((xb)^*(xb))]$.

(b) Take y and a as in (a) for $T^\perp$, which has dimension s, and at each infinite place v an $r_v \in \mathrm{GL}_s(\mathbb{C})$ with $(y_v r_v)^*(y_v r_v) = 1$. The adelic matrix (r, a) has $|\det(r,a)| \cdot d(T) \ge 1$: its infinite part cancels the norm of $\det y^*y$, and its finite part the index of $\det a$, by (a). Minkowski's second theorem over E [BV83], in its adelic form for (r,a), gives an invertible B over E with $a^{-1}B^{\mathrm{T}}$ integral and

$$|\det(r,a)| \prod_i \ell(B_i)^{2f} \le (2fs\,\mathrm{rd}_E)^{fs},$$

where $\ell(B_i)$ is the Euclidean length of the vectors $r_v^{-1}B_i$ at all places together. Put $X_i = yB_i$. Then $u^{-1}yB^{\mathrm{T}} = (u^{-1}ya)(a^{-1}B^{\mathrm{T}})$ is integral, so X_i lies in $E^\kappa \cap u\hat{\mathcal{O}}^\kappa$; $X_i \ne 0$ because B is invertible and y has a left inverse; the X_i span $T^\perp$, whose orthogonal is T; and $\|X_i\| = \ell(B_i)$, because $y_v r_v$ is an isometry. Multiplying by $|\det(r,a)| d(T) \ge 1$ gives the bound, with $2 \le 8$.

(c) Raise (b) to the power $k+1$: with $m = f(k+1)$ and $C = (8fk\,\mathrm{rd}_E)^m$, $d(T)^{-(k+1)} \le \prod_i C\|X_i\|^{-2m}$. The subspace T is recovered from the tuple (X_i) as its orthogonal, so the sum over T is at most the sth power of $C\sum_X \|X\|^{-2m}$ over the nonzero lattice vectors. These have length at least 1 (an algebraic integer whose conjugates are all smaller than 1 in modulus is 0), so they are 1-separated, and since $2m$ exceeds the real dimension $2fk$ by at least 2, a packing count over

the shells $2^t \le \|X\| < 2^{t+1}$ gives $\sum_X \|X\|^{-2m} \le 5^{2m}$. As $200 = 5^2 \cdot 8$, this is the bound. □

27.2. Schur's test for one orbit

For $\gamma \in G_1$ and y, let $\nu_\gamma(y)$ be the ratio of the volumes of the projections of γf and of f to the negative space yV_- of J_1. A term of cell rank j has $i = \min(\varrho, j)$, and its *Schur weight* at y is the boxed cutoff at $\gamma^{-1}y$ times $\prod_v \nu_\gamma(y_v)^{-(k+i)}$.

Lemma 27.4 (Boxed points and integral vectors). (a) *If A maps injectively to vectors in E^ι whose entries are algebraic integers of modulus at most r at every infinite place, then $\#A \le (1+2r)^{2\#\iota f}$.*

(b) *The boxed points form a measurable set, invariant under left multiplication by unipotent elements fixing every frame vector at every place. At a boxed point y the entries of the Gram–Schmidt matrix have modulus at most b^{26}, and y_v lies in the frame box of radius $b^{27}\ell(y_v) \le b^{37n}$.*

Proof. (a) Embed A into $\mathbb{C}^{\iota\times\{v\}}$, a real space of dimension $2\#\iota f$. The image lies in the ball of radius r for the maximum norm, and distinct points are at distance at least 1, because a nonzero algebraic integer has a conjugate of modulus at least 1. Balls of radius $1/2$ around the points are disjoint and lie in the ball of radius $r + 1/2$, which gives the count. (b) The reduced points are defined by the entries above the diagonal of the triangular matrix, a measurable function, lying in a measurable set D (Proposition 25.5(c)); such a D exists because in each coset of $M\boldsymbol{O}_E$ a measurable injection to $\mathbb{R}$ picks a least point of a bounded set meeting the coset in a finite nonempty set. A unipotent s fixing the frame leaves the frame vectors $y^{-1}f_a$ and the triangular matrix unchanged, and it does not change the length. The scale of the frame vector of index a is $\kappa_v^{\varrho-a}/\sqrt{N}$ with $|\kappa_v| \ge 1$, so ratios of later to earlier scales are at most 1; a frame vector is a sum of at most $\varrho + 1 \le b$ orthogonal vectors, each not longer than $\ell(y_v)$, with coefficients of modulus at most b^{26}. Hence its norm is at most $b^{27}\ell(y_v)$, and in the cut $b^{27}b^{6n+2} \le b^{37n}$. □

Lemma 27.5 (A triangular basis of the negative space). *Let $\gamma, y \in G_1$, let a norm-one pair occur for γ from j, $i = \min(\varrho, j)$, and let the block B of $Z(\gamma)$ on the subframe $e_1, \dots, e_i, z$ be invertible. There are vectors $\omega_0, \dots, \omega_i$ of yV_- with Gram matrix -1 whose products with the subframe form a lower triangular invertible matrix L and whose products with the frame are those of the basis of the span of $\gamma e_1, \dots, \gamma e_i, \gamma z$ dual to the subframe, times L. For such (ω, L):*

(a) *ω is a basis of $yV_- \cap I_\gamma^\perp$ pairing perfectly with the subframe, $I_\gamma = A \cap \gamma A$;*

(b) *$|\det L| \le \rho^{i+1}$ if the frame vectors of y have norm at most ρ, and $|\det L| = |\det L_1||L_{zz}|$ for the leading block L_1;*

(c) *with $\omega' = \gamma(e_1, \dots, e_i, z)B^{-1}L$ and v the parts in X_0 of the columns of $\omega' - \omega$, the differences $\omega' - \omega$ are orthogonal to the frame and have the Gram matrix of v;*

(d) *if y has length at most θ, then $|L_{zz}|^2 \le \theta^{2(j-i)}(1+|v_z|^2)$, with equality $|L_{zz}|^2 = 1 + |v_z|^2$ when $j \le \varrho$;*

(e) *$|\det B| = |D_j|$ for the leading minor D_j of order j of $Z(\gamma)$.*

Proof. The space V_- is spanned by the columns of $(1 - J_1)f$, and yV_- is negative definite of dimension $\varrho + 1$; a vector of yV_- is determined by its products with the frame f, since the frame pairs perfectly with yV_- (the form is ≤ 0 on the span of f, and $-|x_z|^2$ on fx). Let ω_0 be the vectors of yV_- whose products with the frame are those of the dual basis $\gamma(e_1, \dots, e_i, z)B^{-1}$; Gram–Schmidt for the negative form applied to ω_0 gives $\omega = \omega_0 L$ with L lower triangular, invertible, and Gram matrix -1; the products of the subframe with ω form L, because those with ω_0 form the identity.

(a) A vector of I_γ lies in γA, so it is orthogonal to the frame of γ, hence to ω; and the products with the subframe form the invertible L. As $\dim yV_- \cap I_\gamma^\perp = \varrho + 1 - \dim I_\gamma = i + 1$, ω is a basis. (b) L_{al} is the product of y^{-1} applied to the ath subframe vector with the unit vector $y^{-1}\omega_l$ of V_-, so $|L_{al}| \le \rho$; Hadamard's inequality on the triangular L gives $|\det L| \le \prod_a |L_{aa}| \le \rho^{i+1}$, and the determinant of a triangular matrix is the product of the leading block and the last entry. (c) The products of ω' and of ω with the frame coincide by construction, so $\omega' - \omega$ is orthogonal to the frame; a vector orthogonal to the frame has vanishing coordinates at the e_a^* and at z and its form is that of its part in X_0, up to the isotropic coordinates, which pair to zero. (d) If $j \le \varrho$ the flags of f and γf share a vector $fx = \gamma f x'$ with $x'_z = 1$ and $|x_z| = 1$; it is a combination of the image of the subframe, and ω'_z is a multiple of it, hence lies in the span of f, where it is orthogonal to the difference $\omega'_z - \omega_z$ (which is orthogonal to the frame). Therefore

$$-1 = \langle \omega_z, \omega_z \rangle = \langle \omega'_z, \omega'_z \rangle + |v_z|^2 = -|L_{zz}|^2 + |v_z|^2.$$

At the top level $i = \varrho$ take u unipotent with $\|uy\| \le \theta$; as ω_z is orthogonal to A, its product with z is that with $u^{-1}z$, so $|L_{zz}| \le \|uy\| \le \theta$, and $1 + |v_z|^2 \ge 1$. (e) If $j = \varrho + 1$ the subframe is the whole frame and $B = Z(\gamma)$. Otherwise $i = j$, and the shared vector at the index of z, corrected by the shared vectors at the indices from i to $\varrho - 1$, is a vector $fx = \gamma f x'$ with x' supported on the subframe, $x'_z = 1$ and $|x_z| = 1$; its products with the subframe are $(0, \dots, 0, -x_z)$, because the e_a are isotropic and orthogonal to every frame vector, and z has norm -1 for J_1. By Cramer's rule, replacing the last column of B by Bx' leaves the determinant unchanged, since $x'_z = 1$; expanding along this column, $|\det B|$ is the modulus of the leading minor of order i. □

Proposition 27.6 (The weight at one real place). *Let a norm-one pair occur for γ_0 from j, $i = \min(\varrho, j)$, Z the frame products of γ_0 with $\delta = \det$ of its block on the subframe $e_1, \dots, e_i, z$ nonzero, and let y have length at most θ and frame vectors of norm at most ρ, $k \ge 7$.*

(a) *$\nu_\gamma(y)$ is the modulus of the determinant of the products of a negative basis of $yV_- \cap I_\gamma^\perp$ with $\gamma e_1, \dots, \gamma e_i, \gamma z$, divided by that with $e_1, \dots, e_i, z$; and $\nu_\gamma(y) \ge 2^{-i-1}|\delta| \det(1 + vv^*)/|\det L|^2$ for a triangular basis (ω, L), with v the parts in X_0 of the differences of two bases, which move affinely when y is moved by a unipotent element fixing the frame.*

(b) *There is a measurable ϕ on X_0^i with $\int \phi \le (2\rho^2)^{(i+1)i}(2^{i+1}\theta^{2(j-i)})^k/|\delta|^{k+i}$ and $\nu_\gamma^{-(k+i)} \le \phi(A)$ whenever γ has the same Z and norm-one pairs from j, and its subframe differs in X_0 from that of γ_0 by ${}^t\bar{A}$ times the first i rows of the block.*

Proof. (a) Let N be the matrix of the projection of γf to yV_- in the basis given by the projection of f; then $\nu_\gamma(y) = |\det N|$. Let ω be a negative basis of $yV_- \cap I_\gamma^\perp$ and P the matrix of the products of the frame with ω; then $P(B_\omega)^{-1}$, with B_ω the rows of the subframe, holds the products with the basis dual to the subframe. For an index b outside the subframe the flags of f and γf share a vector at b with leading coefficients ζ and 1, $|\zeta| = 1$, and the part of N at these indices is unitriangular up to these factors; two triangular changes of basis reduce $|\det N|$ to the determinant of the products of ω with $\gamma e_1, \dots, \gamma e_i, \gamma z$ divided by that with $e_1, \dots, e_i, z$. For the lower bound take a triangular basis (Lemma 27.5) and $K = \omega^* J_1 \omega'$. The products of ω with the image of the subframe are $KL^{-1}B$ and those with the subframe are L^*, so

$$\nu_\gamma(y) = |\det K|\,|\delta|/|\det L|^2.$$

Expanding the form on $\omega' - \omega$, whose Gram matrix is v^*v (Lemma 27.5(c)), gives $v^*v = \omega'^* J_1 \omega' - 1 - K - K^* \le -1 - K - K^*$, as the form is ≤ 0 on the span of γf. Thus the hermitian part of $-K$ is at least $\frac{1}{2}(1 + v^*v)$, and the Ostrowski–Taussky inequality [OT51] gives $|\det K| \ge \det(\frac{1}{2}(1 + v^*v)) = 2^{-i-1} \det(1 + vv^*)$. If the part in X_0 of the image of the subframe moves by ${}^t\bar{A}$ times the first i rows of B, then ω' moves accordingly and v moves by ${}^t\bar{A}$ times the first i rows of L; the last column of v does not move, since L is lower triangular. Left multiplication by a unipotent $u = u(a, b)$ fixing the frame adds ${}^t\bar{a}$ times the isotropic rows of Z to the part in X_0 of γf, which is a move of this kind.

(b) Fix a triangular basis at y for Z and write v_0 for the differences at γ_0, L_1 for the leading block of L, c for the conjugate transpose of the first i columns of v_0, $G = 1 + v_{0,z} v_{0,z}^*$ and $C = (2^{i+1}|\det L|^2/|\delta|)^{k+i}$. Put

$$\phi(A) = C\,\left|\det\bigl(G + (L_1^* A + c)^*(L_1^* A + c)\bigr)\right|^{-(k+i)}.$$

At a translate by A the matrix $1 + vv^*$ is $G + (L_1^* A + c)^*(L_1^* A + c)$, because the last column of v is that of v_0 and the others are v_0's plus ${}^t\bar{A}$ times the first i rows of L; so (a) gives $\nu_\gamma^{-(k+i)} \le \phi(A)$. Hua's bound [Hua63] for the integral of a negative power of $|\det(G + Y^*Y)|$ over the rows of $Y = SA + c$, S invertible and at least seven columns, gives

$$\int \phi \le C\,|\det L_1|^{-2k} (\det G)^{-k} = \frac{(2^{i+1}|\det L|^2)^i \left(2^{i+1}|L_{zz}|^2/\det G\right)^k}{|\delta|^{k+i}},$$

with $|\det L| = |\det L_1||L_{zz}|$. Finally $|\det L| \le \rho^{i+1}$ and $|L_{zz}|^2/\det G \le \theta^{2(j-i)}$ (Lemma 27.5(b), (d)), since $\det G = 1 + |v_{0,z}|^2$. □

Lemma 27.7 (The lattice of columns). *For a class c and a finite set ι, the image of the ι-tuples of vectors of $\Lambda \cap X_0$ under the conjugated embeddings is a discrete subgroup spanning $\prod_v (\mathbb{C}^\kappa)^\iota$ over $\mathbb{R}$; it has a measurable fundamental domain of finite measure at least* 1.

Proof. Hermitian products of vectors of $\Lambda \cap X_0$ are algebraic integers, as the basis of X_0 is orthonormal for the integral Gram matrix of the lattice. A nonzero vector has squared length an algebraic integer whose conjugates are not all less than 1, so the image is discrete. Every tuple of vectors of X_0 has a nonzero integer multiple in $\Lambda \cap X_0$ (clear the finitely many denominators), and E is dense in $\prod_v \mathbb{C}$, so the span is dense and closed, hence everything. The Euclidean

product of two lattice points is $\sum_v \operatorname{Re}\Phi_v(x)$ for the hermitian product x of the two tuples, an algebraic integer. Let $\iota \in E$ with $\iota^2 = -1$ and let t_v be the image of x under the embedding at v that sends ι to $\sqrt{-1}$; then $\operatorname{Re} t_v = \operatorname{Re}\Phi_v(x)$, and $t = \sum_v t_v$ is an algebraic integer equal to $(p - q\sqrt{-1})/2$, for the traces p of x and q of ιx. Its squared modulus $(p^2 + q^2)/4$ is an algebraic integer and rational, hence an integer, so p and q are even and $\sum_v \operatorname{Re}\Phi_v(x) = p/2$ is an integer. The lattice therefore has an integral Gram matrix in a basis, and its covolume is the square root of a positive integer. □

Lemma 27.8 (Cauchy–Schwarz and integration in stages). *For classes c, c' and a rational γ write $I(c, c', \gamma, i)$ for the integral of the boxed cutoff of c times the squared norm of the cut Bessel function times the Schur weight for (c', γ) with exponent i.*

(a) *The square of the amplitude majorant of γ for the cut Bessel functions is at most $I(c, c', \gamma, i)\, I(c', c, \gamma^{-1}, i)$.*
(b) *$\int \beta_c \|W^\flat\|^2 \le \nu\, \#I$, with $\#I$ the number of Bessel coordinates.*
(c) *Let F be measurable and invariant under $U_c^0(M)$, let s be measurable with values in the fixer of the frame, and B of finite positive measure. If $\int_B F(us(x)y)\,dx \le |B|C$ for all u in the fixer and all boxed y, then $\int F \cdot (\text{boxed cutoff of } c) \cdot \|W^\flat\|^2 \le C\nu\, \#I$.*

Proof. (a) Where the integrand of the majorant,

$$\beta_c(y)\,\beta_{c'}(\gamma^{-1}y)\,\|W^\flat(y)\|\,\|W^\flat(\gamma^{-1}y)\|,$$

is nonzero, every local factor is nonzero, each vanishes outside the cut, and by the definition of the reduced points y and $\gamma^{-1}y$ are boxed, so the cutoffs equal the boxed cutoffs. Write the integrand as the geometric mean of $\beta_c(y)\|W^\flat(y)\|^2\nu(y)^{-(k+i)}\beta_{c'}(\gamma^{-1}y)$ and $\beta_{c'}(\gamma^{-1}y)\|W^\flat(\gamma^{-1}y)\|^2\nu(y)^{k+i}\beta_c(y)$; since $\nu_{\gamma^{-1}}(\gamma^{-1}y) = \nu_\gamma(y)^{-1}$ (exchanging the frames inverts the ratio), the substitution $y \mapsto \gamma y$ turns the integral of the second into $I(c', c, \gamma^{-1}, i)$, and Cauchy–Schwarz gives (a). (b) The norm of $W^\flat$ is invariant under U_∞, which acts on $W^\flat$ by a unitary character. Integrate over the orbits of U_∞: the integral of β_c over an orbit is ν (Proposition 25.4(a)), and $\int_{U\backslash G}\|W^\flat\|^2 \le \#I$ (Theorem 24.5). (c) Unfold along the fixer G^0 of the frame at the infinite places: the integral is an integral over orbits G^0a with a measure of total mass one on each orbit. On an orbit inside the boxed points the Bessel norms and all factors of the boxed cutoff are constant except the cutoff p for $U_c^0(M)$; for a Γ-invariant F and a cutoff p of a countable group Γ acting on a unimodular group, $\int Fp$ does not change when F is translated on the right (insert the sum of the translates of p and substitute in each term). Averaging over the translations $s(x)$, $x \in B$, bounds the p-weighted integral of F over the orbit by C times that of p; integrating over the orbits and using (b) gives the bound. □

For a term γ of cell rank j over (c, s, c'), $i = \min(\varrho, j)$, a term p of its orbit is a *lattice translate* of p_0 with parameter l, a matrix with columns in $\Lambda \cap X_0$, if modulo the isotropic space the frame of p is that of p_0 moved in X_0 by $M\,l$ times the first i isotropic rows of $Z(\gamma)$. The *Eichler shift* with parameter $x \in \prod_v (\mathbb{C}^\kappa)^i$ is the unipotent element that acts on X_0 by the product of the Eichler maps along $e_1, \dots, e_i$ with the parameters $\overline{M/\mathrm{sc}_v(e_l)}\, x_l$, sc_v the scales of the normalised basis. The *isotropic row index* $\mathrm{iri}(\gamma)$ is the index, in the lattice spanned by the

products of $\Lambda \cap A$ with the frame of $h_c^{-1}\gamma h_{c'}$, of the sublattice coming from the e_a with $a < j$.

Lemma 27.9 (The orbit as lattice translates). *Let γ be a term of cell rank $j \geq 1$ over (c, s, c') and $i = \min(\varrho, j)$.*

(a) *The terms of the orbit $U_c^0 \gamma U_{c'}^0$ lie over (c, s, c'), have the cell rank j and the matrix of products $Z(\gamma)$, and the middle parts of their frames differ from that of γ, row by row, by M times $\hat{\mathbf{O}}$-combinations of the conjugates of the isotropic rows of the products of the frame of $h_c^{-1}\gamma h_{c'}$.*

(b) *The orbit is the union of at most $\mathrm{iri}(\gamma)^k$ sets each consisting of lattice translates of one of its points.*

(c) *At a place v the local kernel of Proposition 27.6(b), in the parameters of the Eichler shift, has integral at most*

$$\mathrm{kb}_v = \left(\frac{(2\rho^2)^{i+1}N^j}{|D_j|_v}\right)^i \left(\frac{(b^{13}/M^2)^i N^{(j-\varrho)_+}}{|D_j|_v}\right)^k, \qquad \rho = b^{37n},$$

and $\prod_v \mathrm{kb}_v \leq \tau\,(b^{13}/M^2)^{ikf}\,\mu_j^{-k/2}$, with $\mu_j = \mathrm{Nm}(D_j)$ divided by N^{2f} at the top cell rank.

Proof. (a) Write $\delta = u\gamma x$ with $u \in U_c^0$, $x \in U_{c'}^0$. Moves fix the frame and keep the products, the cell rank, and the block (a move of c is a term over $(c, 1, c)$). The new frame is $u' = h_c^{-1} u h_c$ times the old one in the lattice basis, and $M^{-1}(u' - 1)$ is integral; off its last block of columns a row of u' agrees with the identity, and the products of e_a with the frame of $h_c^{-1}\gamma h_{c'}$ are the coordinates of that frame along e_a^*. (b) The middle parts of two frames of the orbit differ by k rows, which in a basis of $\Lambda \cap X_0$ lie in M times the lattice spanned by all isotropic rows; colour each term by the classes of these rows modulo M times the sublattice of the rows $a < j$, at most iri^k colours. Two terms of one colour differ by M times that sublattice; the rows of the leading block are independent, so the adelic coefficients come from a rational l with columns in $\Lambda \cap X_0$, and the terms are lattice translates. (c) With δ_u the leading minor in the unit frame, $|\delta_u| = |\kappa_v|^{2w}|D_j|_v/N^j$ for $w = \sum_{a<j}(\varrho - a)$, and the shift scaling has $|\det| = (M\sqrt{N})^i/|\kappa_v|^w$ (Section 25.1). Dividing the bound of Proposition 27.6(b) by the Jacobian $|\det|^{2k}$ of the passage to the shift parameters, the power of $|\kappa_v|$ cancels in the kth power and may be dropped in the ith power ($|\kappa_v| \geq 1$); with $\theta = b^{6n+2}$ one has $2^{i+1}\theta^{2(j-i)} \leq b^{13i}$ (as $i \geq 1$, and $i = \varrho$ is large when $j > i$). For the product, $\prod_v |D_j|_v = \mathrm{Nm}(D_j)^{1/2} = N^{f(j-\varrho)_+}\mu_j^{1/2}$, and D_j is a nonzero algebraic integer, so $\prod_v |D_j|_v^{-i} \leq 1$; the remaining factor $\left((2\rho^2)^{i+1}N^j\right)^{if}$ is at most τ because $N \leq b^{60n}$. □

Theorem 27.10 (Schur's test for one orbit). *Let γ be a term of cell rank $j \geq 1$ over (c, s, c'), $i = \min(\varrho, j)$. Then:*

(a) (*Cauchy–Schwarz.*) *The square of the amplitude majorant of γ (with the cut Bessel functions) is at most the product of the Schur integrals for (c, c', γ) and (c', c, γ^{-1}).*

(b) (*Orbit.*) *The frames of the orbit $U_c^0 \gamma U_c^0$, lie in at most* iri^k *cosets of a lattice,* iri *the isotropic row index of the term; summing the Schur weights over a lattice translate and integrating in stages,*

$$\sum_{\text{orbit}} \text{Schur integral} \le \nu \,\#I \cdot \tau^2 (b^{13}/M^2)^{ikf} \left(\mathrm{iri}/\sqrt{\mu_j}\right)^k, \qquad \tau = b^{kf/4},$$

with μ_j the normalised norm of the leading minor of order j.

(c) *Hence the square of the sum over the orbit of the absolute amplitudes is at most $(\nu \,\#I\, \tau^2 (b^{13}/M^2)^{ikf})^2 (N^f/\mathsf{g}^\sharp)^k$, with $\#I$ the number of Bessel coordinates and $\mathsf{g}^\sharp$ the minor gauge ($N^f \mu_j$ divided by the two isotropic row indices).*

Proof. (a) is Lemma 27.8(a) with $i = \min(\varrho, j)$.

(b) Let F be the sum of the Schur weights (exponent i) over the orbit. It is invariant under U_c^0, which permutes the orbit and fixes the frame, and the sum of the Schur integrals over the orbit is $\int F \cdot (\text{boxed cutoff of } c) \cdot \|W^\flat\|^2$. Fix u in the fixer of the frame and a boxed y, and let $s(x)$ be the Eichler shift with parameter x. By Lemma 27.9(b) the orbit is the union of at most iri^k sets B of lattice translates of a point $p_0 \in B$. *Fibres.* Terms of B with the same parameter l have frames that differ only in the isotropic space, so the points $p^{-1}y$ differ by elements of the space V of n-tuples of isotropic vectors seen from y, of real dimension $2n\varrho f$; by packing (Proposition 25.5(b), with $R = 1$ at boxed points) their weights add up to at most $(b^{6n^2})^{\dim V} \le \tau$ times a common bound for $\prod_v \nu^{-(k+i)}$. *From cosets to the lattice.* Let $\phi = \prod_v \phi_v$ be the product of the local kernels of Lemma 27.9(c) for the base point $u^{-1}p_0$. By the formula for the parts in X_0 of the frames after an Eichler shift (Proposition 27.6(a)), at the point $us(x)y$ a term with parameter l is a translate of $u^{-1}p_0$ by the rescaled point $x + l$, so its ratio of volumes is bounded by $\phi(x + l)$; hence the sum of the weights over B at $us(x)y$ is at most $\tau \sum_{q \in x + (\Lambda \cap X_0)^i} \phi(q)$. *From the lattice to an integral.* The mean of $x \mapsto \sum_q \phi(x + q)$ over a fundamental domain D of the lattice of columns is $\int \phi / |D| \le \prod_v \mathrm{kb}_v$, as $|D| \ge 1$ (Lemma 27.7). Hence the mean of $F(us(x)y)$ over $x \in D$ is at most $C = \mathrm{iri}^k \tau \prod_v \mathrm{kb}_v$, and Lemma 27.8(c) gives the bound $C\nu \,\#I$; insert Lemma 27.9(c).

(c) By (a) the square of the sum over the orbit of the majorants is at most the product of the two sums of Schur integrals over the orbits of γ and γ^{-1}; inversion maps the first orbit onto the second, which has the same cell rank and the same μ_j (the products of γ^{-1} are the conjugate transpose), and its isotropic row index is the other index of γ. Multiplying the two bounds of (b), $\mathrm{iri}(\gamma)\mathrm{iri}(\gamma^{-1})/\mu_j = N^f/\mathsf{g}^\sharp$. □

27.3. The size of a block

For a term γ of cell rank j over (c, s, c') put $i = \min(\varrho, j)$ and $k_\gamma = h_c^{-1} \gamma h_{c'}$. Let $\mu_j(\gamma) = \mathrm{Nm}(D_j)$, divided by N^{2f} if $j = \varrho + 1$; let the *subframe minor* $\delta_*(\gamma)$ be the determinant of the products of the subframe $e_1, \dots, e_j, z$ (the whole frame if $j \ge \varrho$) with its image under γ; let $\mathrm{iri}(k_\gamma)$ be the isotropic row index (Section 27.2) and $t(k_\gamma)$, the *torsion index*, the index of $\Lambda \cap A + k_\gamma(\Lambda \cap A)$ in $\Lambda \cap (A + k_\gamma A)$. The gauges of γ are

$$\mathsf{g}^\sharp = \frac{\mathrm{Nm}(\delta_*)}{\mathrm{iri}(k_\gamma)\,\mathrm{iri}(k_\gamma^{-1})\,N^f}, \qquad \mathsf{g}^{\max} = \max(\mathsf{g}^\sharp, \mathsf{g}_{\mathrm{Nm}}), \qquad \mathsf{g}^* = M^{2if}\,\mathsf{g}^{\max},$$

the minor gauge, the maximal gauge and the *level gauge*. The *cell second moment* $\mathcal{D}_j$ is $\mathcal{D}_{\mathrm{main}}$ at the top cell rank and $\tau^{-16}\mathcal{D}_{\mathrm{id}}$ below; the *orbit bound* at gauge x is $\mathrm{ob}_j(x) = \tau^2(\mathcal{D}_j/(n_p(M^{2if}x)^k))^{1/2}$, and the *size* of the orbit of γ is $\mathrm{size}(\gamma) = \mathrm{ob}_j(\mathfrak{g}^{\max}) = \tau^2(\mathcal{D}_j/(n_p\mathfrak{g}^{*k}))^{1/2}$, in units of v. A *near representative* of a set T over (c, s, c') is a near term of T over the block that has a cell rank, is the chosen element of its orbit $U_c^0\gamma U_{c'}^0$, and lies over an agreeing s.

Lemma 27.11 (Cell rank, minors and the subframe). *Let γ have cell rank j and let P_j be the matrix made of the first j rows of $Z = Z(\gamma)$ and of rows of the identity below them.*

(a) *γ has no other cell rank, γ^{-1} has the cell rank j, and $\mu_j(\gamma^{-1}) = \mu_j(\gamma)$.*
(b) *$\det P_j = D_j \neq 0$, and for $b \geq j$ the combination of γf with the coefficients of column b of P_j^{-1} lies in the span of f.*
(c) *$\mathrm{Nm}(\delta_*) = N^{2f}\mu_j$; in particular $\delta_* \neq 0$.*
(d) *The subframe is a basis of A^+/I_γ, and its image under γ a basis of $\gamma A^+/I_\gamma$.*
(e) *$\mathfrak{g}^\sharp = N^f\mu_j/(\mathrm{iri}(k_\gamma)\mathrm{iri}(k_\gamma^{-1}))$ and $\mathfrak{g}^{\max} = \max(N^f\mu_j, \mathfrak{g}_{\mathrm{Nm}})/(\mathrm{iri}(k_\gamma)\mathrm{iri}(k_\gamma^{-1}))$; at the top cell rank both indices are 1. The gauges are positive.*

Proof. (a) Let $j < j'$ both be cell ranks. Then $j' \leq \varrho$, since the top rank is decided by $\mathfrak{g} \neq 0$. The shared vector at the index $j' - 1 \geq j$ is $fx = \gamma fy$ with y supported on the first j' indices and $y_{j'-1} = 1$; its products with e_a, $a < j'$, vanish because A is totally isotropic and orthogonal to z, so the leading block of order j' of Z kills the nonzero vector y, against $D_{j'} \neq 0$. The products of γ^{-1} form $Z(\gamma)^*$, so its leading minors are the conjugates, $\mathfrak{g}$ is unchanged, and a shared vector $fx = \gamma fy$ gives $\gamma^{-1}fx' = fy'$ after exchanging the roles of the two frames. (b) Expanding along the rows of the identity leaves the leading minor. For $b \geq j$, combining the shared vectors at the indices $j, \dots, \varrho$ (each with coefficient 1 at its top index) gives y with $y_l = \delta_{lb}$ for $l \geq j$ and γfy in the span of f; as in (a) the first j entries of Zy vanish, so P_jy is the bth unit vector. (c) At the top rank $\delta_* = D_{\varrho+1}$. Below it, δ_* is the determinant of P_j with its last row replaced by the last row of Z. The last column y of P_j^{-1} gives a shared vector $\gamma z + \cdots = \zeta z + \dots$ with $\zeta\bar{\zeta} = 1$, and the last row of Z sends y to $h(z, \zeta z) = -N\zeta$; replacing a row of a matrix whose inverse has y as that column multiplies the determinant by the new row applied to y. Hence $\delta_* = -N\zeta D_j$ and $\mathrm{Nm}(\delta_*) = N^{2f}\mathrm{Nm}(D_j)$. (d) Let $\gamma fx' \in I_\gamma$ with x' supported on the subframe. It lies in γA, and the columns of γf are independent, so $x'_z = 0$; it lies in A, so its products with e_a, $a < j$, which are the entries $(Zx')_a$, vanish; as $D_j \neq 0$, $x' = 0$. Coefficients outside the subframe are absorbed by I_γ: if $\mathfrak{g} = 0$, $A = A_{\leq j} + I_\gamma$ and $\gamma A = \gamma A_{\leq j} + I_\gamma$ (Proposition 26.4(c)), and if $\mathfrak{g} \neq 0$ there are none. The statement for A^+ follows by applying this to γ^{-1}, which carries $I_{\gamma^{-1}}$ to I_γ. (e) is (c) and the definitions; at the top rank all isotropic vectors are counted, and $\mathfrak{g} \neq 0$ there. The indices are finite, hence positive (Theorem 27.16 below), and $\mu_j > 0$ by (b). □

Lemma 27.12 (Orbit bounds). (a) *$|\mathfrak{g}(\gamma)|_v \leq |D_{\varrho+1}|_v^2$, and for a near term $\mathfrak{g}_{\mathrm{Nm}} \leq b^{270n^2f}$ and $\mathfrak{g}^* \leq b^{n^3f}$.*
(b) *$\mathcal{D}_j \leq \mathcal{D}_{\mathrm{main}}$, $n_p \leq \mathcal{D}_{\mathrm{main}}$, and $\mathcal{D}_j \geq \tau^{8-32i}\mathcal{D}_{\mathrm{id}}$ for $j \geq 1$.*

(c) *If γ is near and $\mathfrak{g}_{\mathrm{Nm}} \geq 1$, then $\mathrm{ob}_{\varrho+1}(\mathfrak{g}_{\mathrm{Nm}}) \geq b^{-k^3 f}$, and the product over the infinite places of the local bounds of Theorem 24.10(b) at γ is at most $(\nu \,\mathrm{ob}_{\varrho+1}(\mathfrak{g}_{\mathrm{Nm}})/2)^2$.*

(d) *If γ has cell rank $j \geq 1$, lies over (c, s, c') and $\mathfrak{g}_{\mathrm{Nm}} \leq N^f \mu_j$, then the absolute amplitudes of its orbit add up to at most $\nu \,\mathrm{ob}_j(\mathsf{g}^{\max})$.*

Proof. (a) $\mathfrak{g} = |D_{\varrho+1}|^2 - N^2|D_\varrho|^2$ is a difference of two nonnegative numbers at each place, and it is ≥ 0 (Proposition 26.4(b)). For a near term the entries of Z are at most b^{134n} (Proposition 26.4(b)), so by Hadamard's inequality the squares of its minors are at most b^{270n^2} at every place; the indices in the gauges are at least 1, and $M^{2if} N^f \leq b^{(46n+60n)f}$, with $46n + 60n + 270n^2 \leq n^3$. (b) $\tau^{-16}\mathcal{D}_{\mathrm{id}} \leq \mathcal{D}_{\mathrm{id}} \leq \tau^{44}\mathcal{D}_{\mathrm{id}} \leq \mathcal{D}_{\mathrm{main}}$ by admissibility, $n_p \leq \mathcal{D}_{\mathrm{id}}$ as $\#I \geq 1$, and for $i \geq 1$, $\tau^{8-32i} \leq \tau^{-24} \leq \tau^{-16}$, while $\mathcal{D}_{\mathrm{main}} \geq \mathcal{D}_{\mathrm{id}}$. (c) Comparing squares, the pair count n_p cancels against $n_p \leq \mathcal{D}_{\mathrm{main}}$, and it remains that $M^{2\varrho f k}\mathfrak{g}_{\mathrm{Nm}}^k \leq b^{(23\cdot 2\varrho + 270n^2)fk} \leq b^{2k^3 f}$, since $\varrho + 1 \leq n \leq k$. For the local bounds: their product is a constant C_{pb} divided by $\mathfrak{g}_{\mathrm{Nm}}^{10^5 n^4 + k - 1} \geq \mathfrak{g}_{\mathrm{Nm}}^k$. At a place the squared scales $s_a = \kappa_v^{2(\varrho - a)}/N$ of the frame have product between b^{-60n^2} and b^{46n^2}, and $\rho^{9n}/s^2 \leq b^{453n^2}$ for $\rho = b^{37n}$; with $(453 \cdot 10^5 + 92 + 184)n^6 f \leq 10^5 n^9 f$ this gives $C_{\mathrm{pb}} \leq 2^{kf}\tau(\#I)^2/(\deg c_G^f \Lambda^2)$, and with the bound for the scaled Siegel constant of admissibility (Proposition 25.7) and $2^{kf} n_p M^{2\varrho f k} \leq \tau a_* \,\mathrm{vol}(\hat{O})^n$, one gets $C_{\mathrm{pb}} \leq (\nu\tau^2/2)^2 \mathcal{D}_{\mathrm{main}}/(n_p M^{2\varrho f k})$; dividing by $\mathfrak{g}_{\mathrm{Nm}}^k$ gives the claim. (d) By Theorem 27.10(c), and as $\mathsf{g}^{\max} = \mathsf{g}^\sharp$ under the hypothesis, it suffices that $(\#I)^2 (b^{13}/M^2)^{2ikf} N^{kf} \leq \mathcal{D}_j/(n_p M^{2ikf})$. Since $b^{19} \leq M$, $b^{13}/M^2 \leq (b^4 M)^{-1}$, so the left side is at most $(\#I)^2 N^{kf} b^{-8ikf} M^{-2ikf}$. By (b), $\mathcal{D}_j/n_p \geq b^{2kf - 8ikf}\mathcal{D}_{\mathrm{id}}/n_p = b^{2kf-8ikf}\#I\, N^{kf}$, as $\mathcal{D}_{\mathrm{id}} = \#\bar{R}_0 n_X \#I$; and $\#I \leq b^{2kf}$ by admissibility. □

Theorem 27.13 (Size of a block). (a) *The second moment of the difference between the blocks of U and those for the cut functions is at most $b^{-40}\mathcal{D}_{\mathrm{id}}$; $\sqrt{\text{second moment}}$ is subadditive.*

(b) (*Open cell.*) *For a near term with $\mathfrak{g} \neq 0$ and a measurable weight $\omega \leq 1$ invariant under the fixer of the frame and zero at the points in the cut but not in the box, the integral of ω times the cut kernel has norm at most the root of the product of the local bounds of Theorem 24.10(b), plus the integral of ω times the difference of the kernels; and this is at most the orbit bound at the top cell rank.*

(c) *For a stable $T \subseteq \Gamma^*$, the cut block of T over (c, s, c') is at most ν times the sum of the sizes of the near representatives over the block (the near terms that are the chosen element of their orbit and have a cell rank); there are none unless s agrees.*

Proof. (a) For fixed c and s the norms of the differences of the blocks, summed over c', are at most $\eta = \nu b^{-k^3 f}/2$ (Proposition 25.4(b)); so the sum of their squares is at most η^2. Summing over $s \in \bar{G}$ and over the $n_X \leq \mathcal{D}_{\mathrm{id}}$ classes c and dividing by ν^2 gives at most $\#\bar{G}\, b^{-2k^3 f}\mathcal{D}_{\mathrm{id}}/4$. The group $\bar{G}$ is a product of f general linear groups of degree $2\varrho + k + 1$ over each of the two residue fields of order at most p_i, so $\#\bar{G} \leq M^{(2\varrho+k+1)^2 f} \leq b^{23(2\varrho+k+1)^2 f}$, and $23(2\varrho + k + 1)^2 f + 40 \leq$

$2k^3 f$. The square root of the second moment is a multiple of the norm in an ℓ^2 space of blocks, hence subadditive.

(b) The first statement is Theorem 24.10(c) for the weight ω. As $\mathfrak{g}$ is a nonzero algebraic integer, $\mathfrak{g}_{\mathrm{Nm}} \geq 1$; by Lemma 27.12(c) the root of the product of the local bounds is at most $\nu\,\mathrm{ob}_{\varrho+1}(\mathfrak{g}_{\mathrm{Nm}})/2$, and the integral of ω times the difference of the kernels is at most $\nu b^{-k^3 f}/2 \leq \nu\,\mathrm{ob}_{\varrho+1}(\mathfrak{g}_{\mathrm{Nm}})/2$.

(c) *Terms with* $\mathfrak{g} \neq 0$ *throughout.* If $p\gamma = \gamma w$ with $p \in U_c^0$, $w \in U_{c'}$ and $\mathfrak{g}(\gamma) \neq 0$, then w fixes the frame of γ^{-1}, whose matrix of products with f is invertible; so w fixes vectors whose dual coordinates form the identity matrix, $w = 1$ and $p = 1$. Hence in each double coset $U_c^0 \gamma U_{c'}$ exactly one pair (a, b) carries γ to the chosen representative $a\gamma b^{-1}$, where the representative is chosen to be a near representative if the double coset contains one. The weight $\phi_\gamma(y) = [\gamma \text{ is the representative}]\,\omega(y)$, ω the indicator of the reduced points, then has orbit sums 1 (the reduced indicator is a partition of unity in boxes, Proposition 25.5(c)), and Proposition 25.4(b) lets it replace $\beta_c(y)\beta_{c'}(\gamma^{-1}y)$. If the representative γ is a near representative, (b) bounds its integral by $\nu\,\mathrm{ob}_{\varrho+1}(\mathfrak{g}_{\mathrm{Nm}})$. Otherwise the cut kernel of γ vanishes at every reduced point: if it did not vanish at a reduced y, translating $\gamma^{-1}y$ into the boxed points by some $u \in U_{c'}$ (Proposition 25.5(c)) and taking the chosen element of the orbit of γu^{-1} would give a near representative in the double coset.

Terms of one cell rank. Let T be stable with all elements of cell rank j and none of cell rank 0 over the block. On the stable part K where $\mathfrak{g}_{\mathrm{Nm}} > N^f \mu_j$ the cell rank is the top one, the indices are 1 and $\mathfrak{g}^{\max} = \mathfrak{g}_{\mathrm{Nm}}$, so the previous paragraph applies with $\mathrm{ob}_{\varrho+1}(\mathfrak{g}_{\mathrm{Nm}}) = \mathrm{ob}_j(\mathfrak{g}^{\max})$. Off K, sum the absolute amplitudes orbit by orbit: an orbit whose sum is nonzero contains a term with a nonzero majorant, which is near (otherwise the boxed cutoffs at y and $\gamma^{-1}y$ are never both nonzero and the Schur integrals vanish), and its chosen element is a near representative, whose orbit contributes at most $\nu\,\mathrm{ob}_j(\mathfrak{g}^{\max})$ by Lemma 27.12(d). The blocks vanish unless s agrees (Proposition 25.4(c)).

Sorting by the cell rank. Every element of Γ^* has a cell rank (Lemma 26.2(c) for $\mathfrak{g} = 0$; the top rank for $\mathfrak{g} \neq 0$), the parts of T of one cell rank are stable (Proposition 26.4(a)), and none has cell rank 0, since a norm-one pair from 0 would put γ off Γ^*. The norm of the block is at most the sum of the norms of the blocks of these parts. □

27.4. The lattice of a link and the gain

For a term γ over (c, s, c') let Λ be the lattice of c. The vectors of Λ orthogonal to A and γA give a lattice L^1_γ in the link; those whose A-part is divisible by M give L^+_γ; and the *lattice of the term* $L_\gamma \subseteq \Lambda \cap X_0$ consists of the v for which some vector of $v + M(\Lambda \cap A)$ is orthogonal to γf. For two terms, their *common link* is the subspace of X_0 sent by both links to one vector of X_0, and their *common lattice* L_{12} is the part of $L_\gamma \cap L_{\gamma'}$ in it; its corank l is k minus its rank.

On pairs of middle coordinates $X_0 \oplus Ez$ write $T = \mathrm{diag}(1, -N)$ for the middle Gram matrix, $\mathrm{lf} = \mathrm{diag}(T, -T)$ for the *link form* and $Q = \frac{1}{2}\mathrm{diag}(T, T) + N^{-1}L^*L$, with L the two z-rows of T, for the *link metric*. For a matrix v with columns in the link and $e \in \mathrm{GL}(\mathbb{A}_f)$ write $\mathrm{lds}(v, e) = \mathrm{Nm}(\det v^* Q v)[\det e]^2$, the squared discriminant of the lattice with basis ve for Q. The *z-forms* of a pair are the products of its two members with z.

Lemma 27.14 (Link forms and Liouville). (a) *For t orthogonal to A, γA and t' orthogonal to A, $\gamma' A$, the pairs p, p' of t and t' in the links of γ and γ' have $\mathrm{lf}(p,p') = h(t,t') - h(t,\gamma\gamma'^{-1}t')$; in particular the link is isotropic for the link form.*

(b) *Q is the mean of the forms of $X_0 \perp Ez^+$ on the two images* src *and* tgt *of a pair; on the link it is the form of $X_0 \perp Ez^+$ on the source; it is positive definite at every infinite place, and independent columns have a nonzero Gram determinant.*

(c) (*Liouville.*) *Let Q' be positive definite at every infinite place, a, b matrices over E with $\det(a^*Q'b) \neq 0$, v, v' invertible finite adelic matrices, and $0 \le x \le [\det(v^*(a^*Q'b)v')]$. Then*

$$x^2 \le [\det v]^2 \,\mathrm{Nm}(\det a^*Q'a)\,\mathrm{Nm}(\det b^*Q'b)\,[\det v']^2.$$

Proof. (a) The columns of t and $\gamma^{-1}t$ have vanishing coordinates at the e_a^*, and on such vectors the form is T on the middle coordinates; so $\mathrm{lf}(p,p') = h(t,t') - h(\gamma^{-1}t, \gamma'^{-1}t')$, and $h(\gamma^{-1}t,\gamma'^{-1}t') = h(t,\gamma\gamma'^{-1}t')$ since γ is an isometry. For $\gamma' = \gamma$ the two terms cancel. (b) Expand: $\frac{1}{2}(T \oplus T)$ gives the mean of the forms of the two members, and $N^{-1}|z\text{-form}|^2$ on each half turns $-N|\lambda|^2$ into $+N|\lambda|^2$ after the exchange of the z-coordinates that defines src and tgt. On the link src and tgt differ by the isometry Υ_γ (Lemma 26.1). In the coordinates Q is diagonal with entries $1/2$ on X_0 and $N/2$ at z. (c) At each place the Gram matrix of the columns of $(a \mid b)$ for Q' is positive semidefinite, which gives $|\det a^*Q'b|_v^2 \le \det(a^*Q'a)_v \det(b^*Q'b)_v$; multiplying over the places and squaring, $\mathrm{Nm}(\det a^*Q'b)^2 \le \mathrm{Nm}(\det a^*Q'a)\mathrm{Nm}(\det b^*Q'b)$. Since $[\det(v^*Xv')] = [\det v]\mathrm{Nm}(\det X)[\det v']$ for X over E, the hypothesis gives $x \le [\det v]\mathrm{Nm}(\det a^*Q'b)[\det v']$. □

Lemma 27.15 (The vectors z^+ and $\Upsilon_\gamma^{-1}z^+$). *Let γ be a term over (c,s,c') and $X_0^\gamma = X_0 \cap \Upsilon_\gamma^{-1}X_0$.*

(a) *The products of $a \in X_0 \perp Ez^+$ with z^+ and $\Upsilon_\gamma^{-1}z^+$ are N times the z^+-coordinates of a and $\Upsilon_\gamma a$; the vectors orthogonal to both are those of X_0^γ. For y orthogonal to A and γA with source a, these products are minus the products of y with γz and z.*

(b) *The products of z^+ and $\Upsilon_\gamma^{-1}z^+$ with the images of vectors of Λ orthogonal to A and γA are integral.*

(c) *A vector $v \in X_0$ lies in $L_\gamma^+ \cap X_0^\gamma$ if and only if $v \in \Lambda$ and some vector of $v + M(\Lambda \cap A)$ is orthogonal to γf; hence $L_\gamma = L_\gamma^+ \cap X_0^\gamma$.*

(d) *Let γ have cell rank j and let u be the z^+-coordinate of $\Upsilon_\gamma^{-1}z^+$. Then $\mathfrak{g} = |D_j|^2(1-|u|^2)$; the Gram matrix of $z^+, \Upsilon_\gamma^{-1}z^+$ is $N\left(\begin{smallmatrix} \frac{1}{\bar u} & u \\ & 1 \end{smallmatrix}\right)$, with determinant $N^2\mathfrak{g}/|D_j|^2$. If $\mathfrak{g} = 0$, $\Upsilon_\gamma^{-1}z^+$ is a multiple of z^+; if $\mathfrak{g} \neq 0$, the two are independent and the product over the places of the moduli of their Gram determinant is $N^f \mathfrak{g}_{\mathrm{Nm}}/\mathfrak{g}^\sharp$.*

(e) *The pairs ε of the link with sources $-\Upsilon_\gamma^{-1}z^+$ and $-z^+$ represent the z-forms: ε^*Qw is the matrix of the z-forms of the columns w of the link, and the Gram matrix of ε for Q is that of $\Upsilon_\gamma^{-1}z^+, z^+$ in $X_0 \perp Ez^+$.*

Proof. (a) The first statement is the definition of the positive form, in which z^+ has norm N, and of Υ_γ, an isometry. For y in $(A+\gamma A)^\perp$ with pair $([y],[\gamma^{-1}y])$, the z^+-coordinate of the source is the z-coordinate of $\gamma^{-1}y$ and that of the target, $\Upsilon_\gamma a$, is the z-coordinate of y; the products of y with z and of $\gamma^{-1}y$ with z are $-N$ times these. (b) In the lattice basis, where the middle block of the Gram matrix is integral, the linear forms $x \mapsto h(x, k_\gamma z)$ and $x \mapsto h(x,z)$ have integral coefficients when k_γ stabilises the standard lattice. (c) Write the vector as $a+v$ with $a \in A$ and $v \in X_0$; it lies in Λ exactly when a has integral coordinates and $v \in \Lambda \cap X_0$. A vector of X_0 in X_0^γ has vanishing z-forms, and a lift $y = v - Ma$ orthogonal to A and γA with vanishing z-coordinate in $\gamma^{-1}y$ is orthogonal to γf. (d) If $\mathfrak{g} = 0$, then $X_0^\gamma = X_0$ (Proposition 26.4(c)), so $\Upsilon_\gamma^{-1} z^+$ is orthogonal to X_0, hence μz^+, and $|\mu| = 1$ because Υ_γ is an isometry; both sides of the identity vanish. If $\mathfrak{g} \neq 0$, then $j = \varrho + 1$ and $D = \det Z \neq 0$; Cramer's rule gives y with $Zy = De_z$ and $y_z = D_\varrho$, and the combination $\gamma f y$, which is orthogonal to the e_a and has product D with z, shows that the link sends $y_0 + D_\varrho z^+$, for some $y_0 \in X_0$, to $-(D/N)z^+$. Hence $u = -ND_\varrho/D$ and $|D|^2(1-|u|^2) = |D|^2 - N^2|D_\varrho|^2 = \mathfrak{g}$. The Gram matrix follows from (a), and its determinant is $N^2(1-|u|^2)$. If $\mathfrak{g} \neq 0$, $|u| \neq 1$ and the vectors are independent; at the top cell rank $\mathfrak{g}^\sharp = \mathrm{Nm}(D_{\varrho+1})/N^f$, and the product of $N^2|\mathfrak{g}|_v/|D_{\varrho+1}|_v^2$ over the places is $N^{2f}\mathfrak{g}_{\mathrm{Nm}}/\mathrm{Nm}(D_{\varrho+1}) = N^f \mathfrak{g}_{\mathrm{Nm}}/\mathfrak{g}^\sharp$. (e) By (b) of Lemma 27.14 on the link Q is the positive form on the sources, and by (a) the products of the source of a pair with $-\Upsilon_\gamma^{-1}z^+$ and $-z^+$ are its two z-forms. □

An *orthogonal lift* of a basis v of the link of γ at cell rank j is a matrix u whose columns are orthogonal to A and γA, whose first columns, one for each $b \geq j$, are dual to the e_b^* and sent to 0 in the link (they are a basis of I_γ), and whose other columns are orthogonal to the e_b^* and sent to the columns of v.

Theorem 27.16 (The lattice of a link). *Let γ be a term of cell rank j over (c, s, c'), $i = \min(\varrho, j)$.*

(a) *γ has orthogonal lifts of a basis of its link, isometric to it for the link metric after a correction by the products with z and γz; adjoining two vectors of norm N gives isotropic lists $(e_a, z+z')$, $(\gamma e_a, \gamma z + z'')$, $a < j$, whose Gram determinant has norm the square of the norm of the subframe minor $\delta_*(\gamma) \neq 0$ (Jacobi's identity in the quotient).*

(b) *There is a basis of the link in L_γ^+ whose torsion index times discriminant is at most $M^{2if}\mathfrak{g}^\sharp/N^f$.*

(c) *The products of z^+ and $\Upsilon_\gamma^{-1}z^+$ with L_γ^1 are integral; their orthogonal complement is $X_0 \cap \Upsilon_\gamma^{-1}X_0$; they are proportional if $\mathfrak{g} = 0$ and independent otherwise, with an explicit Gram determinant.*

Moreover the indices $\mathrm{iri}(k_\gamma)$, $\mathrm{iri}(k_\gamma^{-1})$ *and* $t(k_\gamma)$ *are finite, and* $\mathrm{iri}(k_\gamma)\mathrm{iri}(k_\gamma^{-1})t(k_\gamma)$ *divides the index of the integral classes of the two lists.*

Proof. (a) *Lifts.* The three conditions on a column of u form one linear system. For a column of $(0 \mid v)$ in the link there is $t \in (A+\gamma A)^\perp$ with that image; a vector of I_γ with prescribed coordinates at the indices $b \geq j$ exists by Proposition 26.4(c) (the shared vectors at these indices lie in I_γ with leading coefficient 1), and subtracting it corrects the products with the e_b^*, which read off the coordinates at e_b. The first columns are orthogonal to both frames and to all columns. *Isometry.*

On vectors orthogonal to A and γA the form factors through the map to pairs; by Lemma 27.14(a) with $\gamma' = \gamma$ and (b), the Gram matrix of the lifts plus N^{-1} times the products of their two z-forms is the Q-Gram matrix of v. *The lists.* Order the vectors e_a, γe_a $(a < j)$, z, γz list by list and add N to the diagonal entries of z and γz, which amounts to adjoining z', z'' of norm N orthogonal to everything. The two lists are isotropic, and exchanging the two halves of the rows turns the Gram matrix into a block antidiagonal matrix whose off-diagonal block is the matrix of products of the subframe with its image; its determinant is $\pm|\delta_*|^2$, and $\delta_* \neq 0$ by Lemma 27.11(c).

(b) Pass to the quotient $(\Lambda \cap I_\gamma^+)/(\Lambda \cap I_\gamma)$, which is unimodular, with a basis B and coordinates q; the classes of rational vectors orthogonal to I_γ have the products of the vectors (the first columns of a lift being a basis of I_γ, orthogonal to the others). Choose $e \in \mathrm{GL}(\mathbb{A}_f)$ so that the classes of the lifts of ve form a primitive matrix, the lattice of the vectors whose lifts have integral classes. In the quotient enlarged by z', z'' of norm N, the classes of the lifts of ve are orthogonal to the two lists, and with the border of (a) their Gram matrix is that of ve for Q; the ranks add up, since the rank of the quotient is twice the number of $a < j$ plus $k+1$. Jacobi's identity, applied in this enlarged lattice to the primitive lattice of the lifted classes of ve and the lattice of the two lists, compares $\mathrm{lds}(v, e)$ with the norm $(N^{2f}\mu_j)^2$ of the Gram determinant of the lists (by (a) and Lemma 27.11(c)); the enlargement contributes the factor $[N]^2 = N^{4f}$, and the lattice of the lists has index ind in the integral vectors of its span, finite because $\delta_* \neq 0$. This gives $\mathrm{lds}(v, e) \cdot \mathrm{ind}^2 = \mu_j^2$. The classes of the e_a, γe_a with $a < j$ span the classes of all of them. The chain of lattices obtained by adding in turn the classes of the e_a, those of the $k_\gamma e_a$, and the integral vectors of their span has indices divisible by $\mathrm{iri}(k_\gamma)$, $\mathrm{iri}(k_\gamma^{-1})$ and $t(k_\gamma)$ respectively: the first two through the products with the two frames, which are defined on classes, and the third because a vector of the span differs from the representative of its class by a vector of I_γ in the lattice of the e_a and $k_\gamma e_a$. Hence the product w of the three indices divides ind, is nonzero, and $\mathrm{lds}(v, e)w^2 \leq \mu_j^2$. *The level.* The lifts tw of ve have integral parts m in A; the vectors x for which mx is congruent modulo $A \cap \gamma A$ to M times a vector of $\Lambda \cap A$ form a submodule of the integral vectors with a basis B' of index dividing the index of ML in L, where L is the lattice of the isotropic rows of $Z(k_\gamma)$, a copy of $(\Lambda \cap A)/(\Lambda \cap I_\gamma)$ generated up to finite index by i vectors; so $[\det B'] \leq M^{2if}$. The lattice of veB' lies in L_γ^+ (adding vectors of I_γ makes the parts in A divisible by M), and

$$t^2 \,\mathrm{lds}(v, eB') = t^2 \mathrm{lds}(v, e)[\det B']^2 \leq \frac{M^{4if}\mu_j^2}{(\mathrm{iri}\,\mathrm{iri}')^2} = \left(\frac{M^{2if}\mathsf{g}^\sharp}{N^f}\right)^2.$$

(c) is Lemma 27.15(b), (a), (d). □

Lemma 27.17 (The cut). *Let γ be a term of cell rank j over (c, s, c') with s agreeing, $x \leq \mathrm{mn}(s)$ a positive integer, and $G = \max(\mathfrak{g}_{\mathrm{Nm}}, N^f\mu_j)$.*

(a) *A principal minor of order $r \leq 1 + (j - \varrho)_+$ of the Gram matrix of z^+, $\Upsilon_\gamma^{-1} z^+$ has $\mathrm{Nm} \leq (G/\mu_j)^2$.*

(b) *If the columns of w have integral lifts (they are images of vectors of Λ that are adelic combinations of rational vectors orthogonal to A and γA), a minor of the matrix of the z-forms of w that is a unit has generalised index at least x, provided both forms are chosen unless $x = 1$.*

(c) *Let the r columns of y lie in the link, $r \le 1 + (j - \varrho)_+$ and $r = 2$ unless $x = 1$, let the z-forms be injective on their span, and let them take on the basis yv the values they take on a matrix with integral lifts. Then $x^2\mu_j^2 \le \operatorname{lds}(y, v)\, G^2$.*

(d) *Let m have a left inverse, columns in the link on which both z-forms vanish, and let ma lie in L_γ^+. Then the span V of the parts in X_0 of m has dimension #columns, the lattice $\Lambda_V \cap L_\gamma$ is a lattice of rank $\dim V$, and its squared discriminant is at most $\operatorname{lds}(m, a)$.*

Proof. (a) The minor is 1, N or the Gram determinant. $1 \le (G/\mu_j)^2$ and $\operatorname{Nm}(N) = N^{2f} \le (G/\mu_j)^2$ because $G \ge N^f \mu_j$. For $r = 2$, $j = \varrho + 1$ and by Lemma 27.15(d) the determinant has modulus $N^2|\mathfrak{g}|_v/|D_j|_v^2$ at each place, so its norm is $N^{4f}\mathfrak{g}_{\mathrm{Nm}}^2/\operatorname{Nm}(D_j)^2 = (\mathfrak{g}_{\mathrm{Nm}}/\mu_j)^2 \le (G/\mu_j)^2$. (b) The values of the z-forms are the products of the integral lifts n with z and $k_\gamma z$ in the lattice basis, which are integral; if both are chosen, the lifts are orthogonal to A and $k_\gamma A$, so their products with $(f \mid k_\gamma f)$ vanish except in the two rows of z and $k_\gamma z$, and Lemma 26.3(b) bounds $\operatorname{mn}(s)$ by the index of the 2×2 determinant. (c) By Lemma 27.15(e) the z-forms are the products with the pairs ε; choose r of them that pair nondegenerately with y, say e. The pairing e^*Qyv is the minor of the z-forms, and its index is at least x by (b); Liouville's inequality (Lemma 27.14(c)) gives $x^2 \le \operatorname{lds}(y, v)\operatorname{Nm}(\det e^*Qe)$, and by (a) and Lemma 27.15(e), $\operatorname{Nm}(\det e^*Qe)\mu_j^2 \le G^2$. (d) Where both z-forms vanish the first halves of the pairs lie in X_0, the two halves have equal Gram matrices, and Q is the standard Gram matrix of the parts in X_0. The integral lifts n of ma show that the columns of the parts pa lie in L_γ: n is orthogonal to both frames, so the products of its middle part with the frame of k_γ are those of its part in A, up to sign, and this part is divisible by M. Hence $\Lambda_V \cap L_\gamma$ contains the lattice of pa with finite index in Λ_V, and its discriminant is at most that of pa (Lemma 27.1(b)). □

Lemma 27.18 (Splitting a link along a second one). *Let γ be as in Lemma 27.17, γ' any rational element, and let $x = 1$ unless $j = \varrho + 1$ and the links of γ and γ' are equal. The dimension of the intersection of the two links exceeds that of the common link by at most $1 + (j - \varrho)_+$, and by exactly 2 if $j = \varrho + 1$ and the links are equal. There are a basis of a complement of the intersection in the link of γ, of rank the corank c^+, and a lattice in the complement whose basis vectors, moved along the intersection, have integral lifts, with squared discriminant D, such that the part of L_γ in the common link is a full lattice there, of discriminant d, and*

$$(t(k_\gamma)\, d\, x)^2 D \le \mathfrak{g}^{*2}.$$

Proof. Dimensions. $\dim X_0^\gamma + (j - \varrho)_+ = k$, and a vector of the intersection of the links whose z-forms vanish is the pair of a vector of the common link; the z-forms are two linear forms, of which only one is independent of the other below the top rank (Lemma 27.15(d)). If the links are equal at the top rank, the intersection is the whole link, of dimension $k + 1$, and the common link is X_0^γ, of dimension $k - 1$. *Splitting.* Take a basis of the link of γ in three parts: m spans the kernel of the two z-forms in the intersection, m and u together span the intersection, and y completes them. Apply Theorem 27.16(b) to this basis: for some ϕ the lattice of $(m \mid u \mid y)\phi$ lies in L_γ^+ with $t^2 \operatorname{lds} \le (M^{2if}\mathfrak{g}^\sharp/N^f)^2$. Two integral changes of

basis make ϕ block triangular, which splits the squared discriminant into the product $\operatorname{lds}(m,a)\operatorname{lds}(u,v_0)\operatorname{lds}(y,v)$ of three graded pieces, each moved along the earlier pieces by shears and still with integral lifts. The part m: its parts in X_0 span the common link, and Lemma 27.17(d) gives $d^2 \le \operatorname{lds}(m,a)$. The part u has rank r at most $1+(j-\varrho)_+$, with $r=2$ when $x\neq 1$, and the z-forms are injective on its span; Lemma 27.17(c) gives $x^2\mu_j^2 \le \operatorname{lds}(u,v_0)G^2$. The part y has rank c^+, by the count of dimensions. Hence

$$(t\,d\,x)^2 D\,\mu_j^2 \le t^2 \operatorname{lds}(m,a)\operatorname{lds}(u,v_0)\operatorname{lds}(y,v)\,G^2 \le \left(\frac{M^{2if}\mathsf{g}^\sharp G}{N^f}\right)^2 = \mathsf{g}^{*2}\mu_j^2,$$

using $\mathsf{g}^\sharp G/(N^f\mu_j) = \mathsf{g}^{\max}$ (Lemma 27.11(e)). □

Lemma 27.19 (The congruence and the index). *Let γ, γ' be terms over one block (c,s,c').*

(a) *If w, w' have integral lifts for γ, γ', then $w^*(M^{-1}\mathrm{lf})w'$ is integral.*
(b) *If b', b are i columns of $X_0 \perp Ez^+$ with the same Gram matrix, then* $\operatorname{Nm}(\det(b-b')^*(b-b')) \le 4^{2if}\operatorname{Nm}(\det b^*b)$.
(c) *If y, y' have columns in the links of γ, γ',* lf *pairs them nondegenerately, and $v^*(M^{-1}y^*\mathrm{lf}\,y')v'$ is integral, then* $1 \le (2/M)^{4if}\operatorname{lds}(y,v)\operatorname{lds}(y',v')$.
(d) *The part of L_γ in the common link contains L_{12} with an index dividing $t(k_{\gamma'})$.*

Proof. (a) The element $\delta = \gamma\gamma'^{-1}$ is a term over $(c,1,c)$, so its matrix k in the lattice basis is congruent to 1 modulo M. By Lemma 27.14(a) the pairing of the lifts n, n' is $M^{-1}(n^*Jn' - n^*Jkn') = -n^*J\big(M^{-1}(k-1)\big)n'$ with the integral Gram matrix J of the lattice, which is integral. (b) The Gram matrices of $b-b'$ and $b+b'$ are positive semidefinite and add up to 4 times that of b, which is $\Theta+\Theta^*$ for Θ twice the Gram matrix of b; so at each place the first is at most $\Theta+\Theta^*$ minus a positive semidefinite matrix, and its determinant is at most $2^i|\det\Theta|$ (Ostrowski–Taussky, [OT51]), that is 4^i times the determinant of the Gram matrix of b; take the product over the places. (c) Let a, b be the sources of y, y' and $u = \Upsilon_\gamma^{-1}\Upsilon_{\gamma'}$. By Lemma 27.14(a), (b) the link form pairs y with y' as a with $(1-u)b$ in the positive space, so $M^{-1}a$ and $(1-u)b$ pair integrally after v, v' and nondegenerately. Liouville's inequality gives $1 \le [\det v]^2\operatorname{Nm}(\det(M^{-1}a)^*(M^{-1}a))\operatorname{Nm}(\det((1-u)b)^*((1-u)b))[\det v']^2$, and with $\operatorname{Nm}(M^{-1})^{2i} = M^{-4if}$ and (b) this is at most $(2/M)^{4if}\operatorname{lds}(y,v)\operatorname{lds}(y',v')$. (d) The common lattice L_{12} is the intersection of $L_{\gamma'}$ with the part of L_γ in the common link. A vector v of that part has a lift $y = v - Ma$, $a \in \Lambda\cap A$, orthogonal to the frame of γ. Let $m = \gamma'\gamma^{-1}$, which is $\equiv 1$ modulo M in the lattice basis. Then my is orthogonal to the frame of γ', since $h(my,\gamma' f) = h(\gamma^{-1}y, f) = h(y,\gamma f) = 0$; and $(m-1)y$ lies in $A+\gamma' A$, because the link isometries of γ and γ' agree on v: if $t' \in v+A$ is orthogonal to A and $\gamma' A$, then $\gamma^{-1}y - \gamma'^{-1}t' \in A$ and $(m-1)y = (t'-y) + \gamma'(\gamma^{-1}y - \gamma'^{-1}t')$. Hence $v = Ma + Mw + my$ with $w = -M^{-1}(m-1)y$ in $\Lambda\cap(A+k_{\gamma'}A)$, and the products of v with the frame of $k_{\gamma'}$ are M times those of $a+w$. The vectors of $k_{\gamma'}(\Lambda\cap A)$ have products 0 with that frame (A is totally isotropic and orthogonal to z). Let $\pi(x)$ be the vector of the products of x with the frame of $k_{\gamma'}$. The homomorphism $v \mapsto M^{-1}\pi(v)$ modulo $\pi(\Lambda\cap A)$ takes values in $\pi(\Lambda\cap(A+k_{\gamma'}A))/\pi(\Lambda\cap A)$, a quotient of $\Lambda\cap(A+k_{\gamma'}A)/(\Lambda\cap A + k_{\gamma'}(\Lambda\cap A))$, whose order is $t(k_{\gamma'})$; its kernel is L_{12}, by

the definition of $L_{\gamma'}$. The index of L_{12} is the order of the image, a divisor of $t(k_{\gamma'})$. Positivity of the gauges: the indices are finite by Theorem 27.16. □

Theorem 27.20 (Gain). *Let γ, γ' be terms of cell ranks j, j' over one block (c, s, c') with s agreeing, and $x \le \mathrm{mn}(s)$. Then*

$$\mathrm{disc}(L_{12})^2 \le (2/M)^{2c^+ f}\, \mathsf{g}^*(\gamma)\mathsf{g}^*(\gamma'),$$

c^+ the codimension of the intersection of the two links in each; if $c^+ = 0$ and γ has the top rank, the bound may be divided by x^2.

Proof. Let $X = x$ if $c^+ = 0$ and $j = \varrho + 1$, and $X = 1$ otherwise. If $X \neq 1$ the links are equal, so $\mathfrak{g}$ vanishes for both or for neither (the link determines X_0^γ, which is X_0 if $\mathfrak{g} = 0$ and a hyperplane otherwise, Lemma 27.15(d)); hence γ' has the top rank as well. *Congruence.* Split both links along their intersection by Lemma 27.18, with complements y, y' of rank c^+ and squared discriminants D, D'. A vector of the complement of γ' orthogonal for lf to the complement of γ is orthogonal to the whole link of γ (it is orthogonal to the intersection as well, since the intersection lies in the link of γ', which is isotropic for lf), and that link is its own orthogonal, so it lies in both links and vanishes; the pairing is nondegenerate, and the terms along the intersection vanish. By Lemma 27.19(a), (c), $1 \le (2/M)^{4c^+ f} DD'$. *Cut.* Lemma 27.18 gives $(t\, d\, X)^2 D \le \mathsf{g}^{*2}$ and $(t'\, d'\, X)^2 D' \le \mathsf{g}^{*\prime 2}$, with d, d' the discriminants of the parts of L_γ and $L_{\gamma'}$ in the common link. Multiplying and using the congruence,

$$(t\, d\, t'\, d'\, X^2)^2 \le \left((2/M)^{2c^+ f} \mathsf{g}^* \mathsf{g}^{*\prime}\right)^2.$$

Index. By Lemma 27.19(d) and Lemma 27.1(b), $\mathrm{disc}(L_{12}) \le d\, t'$ and, exchanging the two terms, $\mathrm{disc}(L_{12}) \le d'\, t$. Hence $(\mathrm{disc}(L_{12})X)^2 \le t\, d\, t'\, d'\, X^2 \le (2/M)^{2c^+ f} \mathsf{g}^* \mathsf{g}^{*\prime}$. □

27.5. Counting lattices and frames

A class c has the lattice $\Lambda \cap X_0$ of the vectors v for which $g^{-1}v$ is integral with entry k_0 divisible by $\bar{\pi}$, g the chosen representative of c. The *lattice stabiliser cosets* are the cosets, in the stabiliser of the standard lattice of X_0, of the subgroup of the elements whose row k_0 is congruent to that of 1 modulo $\bar{\pi}$; by admissibility there are at least $(N^k/b)^f$ of them. Write $H = b^{n^3 f}$.

Proposition 27.21 (Lattices of given corank). *Let $r + l = k$. The sum of $\mathrm{disc}(L)^{-k}$ over all classes c and all lattices $L \subseteq \Lambda \cap X_0$ of rank r with $\mathrm{disc}\, L \le H$ is at most $\tau^{l+1} n_X N^{-kf}$.*

Proof. Means over the classes. Let A be a finite set mapped by $\varphi \mapsto a_\varphi$ into the finite adelic unitary group of X_0, and $F \ge 0$ a function on the classes. If $\sum_{\varphi \in A} F(\text{class of } u a_\varphi) \le \#A \cdot B$ for every u, then $\sum_c F(c) \le n_X B$: the defect $B - F$, a function on the double cosets, has nonnegative sums over the translates, and its integral over the double coset space, a weighted sum of its values with the weights of the classes, is translation invariant; as the level fixes it, the integral is $\sum_c (B - F(c))$ up to the common weight. Apply this with the lattice stabiliser cosets, a_φ an element of φ.

Unimodular lattices. The chosen representative of the class of ua_φ is $\eta u y$ with η rational and y in the coset φ, because the level is the preimage of $K(M)$. If L lies in the lattice of that class, then $\eta^{-1}L$ lies in $u\hat{O}^\kappa$ and the linear form *row* k_0 *of* $y^{-1}u^{-1}$, the *form* of φ, vanishes on it modulo $\bar{\pi}$; the rational isometry η^{-1} keeps discriminants (Lemma 27.1(c)). Hence the sum over the lattices of the class of ua_φ is at most the sum over the lattices $L \subseteq u\hat{O}^\kappa$ of rank r, $\operatorname{disc} L \le H$, on which the form of φ vanishes modulo $\bar{\pi}$.

Few forms vanish on a lattice. Let $L \subseteq u\hat{O}^\kappa$ have rank r. By Lemma 27.1(d) it has a finite index i in the saturated lattice Λ_V of its span V, with basis mA for a basis m of Λ_V, $[\det A] = i$, and $\operatorname{disc} L = i\, d(V)$. Extend $u^{-1}m$ to a basis U of $\hat{O}^\kappa$. Integral forms whose values on mA are divisible by an integral idele d and which are pairwise incongruent modulo d are at most $[\det A]\,[d]^l$ in number: write the values on mA as dW; the class of W modulo the rows of A, of which there are $[\det A]$, fixes the values on m modulo d, and the values on the other l basis vectors have $[d]^l$ classes. Two cosets whose forms are congruent modulo $\bar{\pi}$ are equal (their inverses have congruent rows k_0). With $d = \bar{\pi}$ and $[\bar{\pi}] = N^f$, at most $i(N^f)^l$ cosets φ have a form vanishing on L.

Summing. Each such φ contributes $(d(V)i)^{-k}$, and for $d(V)$, $i \ge 1$ with $d(V)i \le H$ and $r \le k$,

$$i\,(d(V)i)^{-k} \le H^{\mu+2} d(V)^{-(k+1)} i^{-(r+\mu)},$$

since $d(V)\, i^{1+\mu} \le H^{1+\mu}$. Sort the lattices L by their span V; with $\mu = 2\varrho + 4$, Lemma 27.2(c) sums over the sublattices of Λ_V and Proposition 27.3(c) over the subspaces of codimension l:

$$\sum_\varphi \sum_L \operatorname{disc}(L)^{-k} \le (N^f)^l H^{2\varrho+6} \,(200 f k \operatorname{rd}_E)^{f(k+1)l}\, 2^k.$$

Constants. $b^f H^{2\varrho+6} 2^k \le \tau$, as $1+n^3(2\varrho+6)+1 \le 10^5 n^9$; and $N^f (200 f k \operatorname{rd}_E)^{f(k+1)} \le \tau$, comparing thousandth powers with $(200 f k \operatorname{rd}_E)^{1000} \le b$ and $N \le b^{60n}$. Hence the right side is at most τ^{l+1}/b^f. Dividing by the number of cosets, at least $(N^k/b)^f$, the mean is at most $\tau^{l+1} N^{-kf}$, and the first step gives the claim. □

For a term γ of cell rank j let Ξ_0 be the first ϱ rows of $Z = Z(\gamma)$, $\operatorname{fm}_j(Z) = M \det(P_j P_j^*)$ the *frame modulus*, and let the *shared vectors* be the coordinates in A of the columns $b \ge j$ of $\gamma f \cdot \operatorname{adj} P_j$ (the others set to 0). A *move* is an element of $U_c^0(M)$. The *part in* X_0 of the frame of γ is the matrix of the coordinates in X_0 of the vectors γf_a.

Lemma 27.22 (Short vectors and moves). *Let γ, γ' have cell rank j over blocks with first class c.*

(a) *For $b \ge j$ or b the last index, the combination of γf with the coefficients of column b of P_j^{-1} has no coordinates at the e_a^*, and its part in X_0 has squared length at most $1/5$ at every place. A vector of $\Lambda \cap X_0$ of squared length less than 1 at every place is 0.*

(b) *If γ is near, the entries of $\operatorname{adj} P_j$ are at most $b^{(2+134n)\varrho}$ and those of the shared vectors at most b^{182n^2} at every place; $[MD_j] \le b^{2n^3 f}$ and $[\operatorname{fm}_j(Z)] \le b^{2n^3 f}$.*

(c) *Moves fix the frame, keep the products, the cell rank, the shared vectors and the blocks. Near terms with the same label whose frames agree after moves are equal.*

(d) *If $Z(\gamma') = Z(\gamma)$ is integral, the shared vectors agree, the frames agree outside the isotropic space and their isotropic rows differ by* $\mathrm{fm}_j(Z)X$ *with X integral, then a move carries the frame of γ to that of γ'.*

(e) *For a matrix η with columns in $\Lambda \cap X_0$ there is a move subtracting $M\eta\Xi_0$ from the part in X_0 of every frame. If $Z(\gamma') = Z(\gamma)$ is integral and the parts in X_0 are congruent modulo $MD_j(\Lambda \cap X_0)$, they differ by $M\eta\Xi_0$ for such an η, and a move makes them equal.*

(f) *If the lattice L with basis ya lies in L_γ, it has a rational lift: a matrix w with wa integral such that every vector of L plus M times the corresponding combination of the e_a is orthogonal to γf. Two lifts whose rows are congruent modulo the rows of the Gram matrix of ya, a lattice of index* $\mathrm{disc}(L)^2$ *in the integral rows, are joined by a move. The lattice $\Lambda \cap X_0 \cap L^\perp$ has rank $l = k - r$.*

Proof. (a) For $b \geq j$ the vector lies in the span of f (Lemma 27.11(b)), where the coordinates at the e_a^* and in X_0 vanish. For the last index below j we have $j = \varrho + 1$, $P_j = Z$ and $Zx = e_z$; the vector $y = \gamma f x$ is orthogonal to the e_a, so it has no coordinates at the e_a^*, and $h(z, y) = 1$, so its coordinate along z is $-1/N$. Comparing $h(y, y) = |y_0|^2 - 1/N$, with y_0 the part in X_0, with $h(y, y) = h(fx, fx) = -N|x_z|^2$ gives $N|y_0|^2 = 1 - |Nx_z|^2 \leq 1$ at every place, and $N \geq 5$. The squared length of a vector of $\Lambda \cap X_0$ is an algebraic integer; if all its conjugates are less than 1 it is 0, and the form is definite on X_0. (b) Entries of an adjugate are minors of order ϱ up to sign, so they are at most $(n^2B)^n$ if the entries are at most B; the entries of Z are at most b^{134n} for a near term. For the shared vectors, write the vector as $\gamma f x = fw$ with x a column of $\operatorname{adj} P_j$. At the boxed point $\gamma^{-1}y$ its length is bounded because x is: with scales of the frame differing by at most $b^{23\varrho}$ and frame vectors of length at most b^{37n}, it is at most $b^{(2+134n)\varrho}b^{1+23\varrho+37n}$ divided by a scale. At the boxed point y, a coefficient of a combination of the frame vectors is at most b^{46n^2} times its length: take u unipotent with uy small; its unitriangular matrix T on the frame and Tx are small, and $x = \operatorname{adj} T\,(Tx)$. The exponents add up to at most $182n^2$. For the indices: an element of E whose squared modulus is at most b^e at every place has generalised index at most b^{ef}; and $|MD_j|^2$, $|\mathrm{fm}_j|^2 \leq b^{2n^3}$ at every place. (c) A move u fixes every frame vector, so $Z(u\gamma) = Z(\gamma)$, and the columns $b \geq j$ of $\gamma f \operatorname{adj} P_j$ lie in the span of f, which u fixes; $h_c^{-1}uh_c$ is in the principal congruence subgroup, so u is a term over $(c, 1, c)$. If $u\gamma$ and $u'\gamma'$ have equal frames, the reductions of the two terms lie in one coset of $\bar{R}_0$, and their labels are equal, so they have the same reduction; the classes agree, the terms lie in one double coset, and both are its chosen element. (d) With $P = \Xi_0$, $\delta = \det P_j$ and α the first ϱ columns of $\operatorname{adj} P_j$, the difference of the isotropic rows is $\xi = \mathrm{fm}_j X$, and $\xi^* P$ is skew-hermitian because both frames have the Gram matrix of f. The columns $b \geq j$ of $X \operatorname{adj} P_j$ vanish, as the shared vectors agree, and at the top cell rank the last column vanishes too (a matrix μ with $P^*\mu$ skew-hermitian kills the last column of $\operatorname{adj} Z$); hence $X\alpha P = \delta X$. The matrix $\sigma = \bar{\delta}X\alpha - \delta(X\alpha)^* - \alpha^* P^* X\alpha$ is skew-hermitian, integral, and solves $\sigma P = \delta\bar{\delta}X$. An Eichler element of level M with matrix σ and zero part in the lattice of c fixes f and adds $M\sigma P = \xi$ to the isotropic rows of every frame (Lemma 25.3). (e) The Eichler map made from η, orthogonal to the frame, fixes the frame; in the coordinates of X_0 it subtracts $M\eta$ times the products with the e_a, which are the rows Ξ_0. If the difference v of the parts in X_0 is congruent to 0 modulo MD_j, then $vP_j^{-1} = M\eta'$ with columns of η'

in $\Lambda \cap X_0$; the last column and the columns $b \geq j$ of vP_j^{-1} are differences of two vectors of squared length at most $1/5$ by (a), so their squared length is less than 1, and by (a) they vanish. Hence $v = M\eta\Xi_0$ with η the remaining columns, and the move of the first statement makes the parts equal. (f) By the definition of L_γ each basis vector has a lift over the finite adeles; the lifts solve a rational linear system, so there is a rational solution near any adelic one. Two lifts give maps $L \to M(\Lambda \cap A)$ whose difference, if each row is an integral combination of the rows of $(ya)^*(ya)$, is given by the products with a row η of rational vectors of $L \subseteq \Lambda \cap X_0$ ($\eta = y(y^*y)^{-1}w^*$); by (e) a move removes it. The rows of the Gram matrix span the image of L in its dual, of index $\operatorname{disc}(L)^2$. The last statement is the rank of an orthogonal complement in a definite space. □

Proposition 27.23 (Frames over a lattice). *Let $L \subseteq \Lambda \cap X_0$ be a lattice of corank l with $\operatorname{disc} L \leq H$, and B a set of near terms over blocks with first class c, with frames distinct modulo moves, each of some cell rank and with $L \subseteq L_\gamma$. Then $\#B \leq b^{2000n^6(l+1)f}$. In particular at most $b^{2000n^6(l+1)f}$ near representatives of a given label over c have $L \subseteq L_\gamma$.*

Proof. Colour the terms successively; if every set of terms of one colour has at most x elements and there are at most C colours, there are at most Cx terms, and colouring families of vectors of a module K' modulo a module K uses at most $[K' : K \cap K']^{\#\text{family}}$ colours. *Cell rank and products.* There are $\varrho + 2 \leq b^{2f}$ cell ranks. The matrices Z have integral entries of modulus at most b^{134n}, so by Lemma 27.4(a) there are at most $(1 + 2b^{134n})^{2n^2 f} \leq b^{270n^3 f}$ of them. Fix j and Z, with $d = MD_j$ and $d' = \operatorname{fm}_j(Z)$. *First round: the lift of L.* Colour the terms by the rows of their lifts of L (Lemma 27.22(f)) modulo the lattice spanned by the Gram matrix of a basis of L; there are at most $(\operatorname{disc} L^2)^\varrho$ colours, and in one colour a move gives every frame the lift of one of them. *Second round: the part in X_0.* With equal lifts, the parts in X_0 of the frames have the same products with the basis of L, so they differ from that of one of them by vectors of $\Lambda \cap X_0 \cap L^\perp$, of rank l; sort them modulo d times this lattice, $[d]^{ln}$ classes; in one class the parts are congruent modulo $d(\Lambda \cap X_0)$ and a move makes them equal (Lemma 27.22(e)). *Third round: the isotropic part.* The shared vectors are integral with entries at most b^{182n^2}, so they take at most $b^{366n^4 f}$ values; with equal shared vectors and equal parts in X_0, a term is determined by the isotropic rows of its moved frame modulo d' (Lemma 27.22(d)), which have $[d']^{n\varrho}$ classes. With $\operatorname{disc} L \leq H$ and $[d], [d'] \leq b^{2n^3 f}$ (Lemma 27.22(b)) the three rounds give at most $b^{(366n^4 + 2n^3(\varrho + ln + n\varrho))f}$ terms, and $2 + 270n^3 + 366n^4 + 2n^3(\varrho + ln + n\varrho) \leq 2000n^6(l+1)$. Near representatives of one label over c have frames distinct modulo moves by Lemma 27.22(c). □

Theorem 27.24 (Counting). (a) (*Sublattices.*) *Over a ring whose ideals of finite index are principal, $\sum_L [\Lambda : L]^{-(r+\mu)}$ over the sublattices of finite index of a free module of rank r is at most $\zeta(\mu)^r$, a power of the sum over the ideals (an upper bound in the direction of Hey's formula); for subspaces T of codimension s, $\sum_T d(T)^{-(k+1)} \leq (200 f k \operatorname{rd}_E)^{f(k+1)s}$, by Siegel's lemma (a subspace of codimension s is cut out by s lattice vectors whose norms have a bounded product times its discriminant, Bombieri–Vaaler).*

(b) (*Lattices of corank l.*) $\sum_c \sum_L \operatorname{disc}(L)^{-k}$, *over the lattices $L \subseteq \Lambda \cap X_0$ of corank l and discriminant at most $b^{n^3 f}$, is at most $\tau^{l+1} n_X N^{-kf}$.*

(c) (*Frames over a lattice.*) *Near terms with a cell rank, different modulo moves, whose lattices contain a given L of corank l and discriminant at most $b^{n^3 f}$, are at most $b^{2000 n^6 (l+1) f}$ in number: sorted by the matrix of products, the shared vectors, the part in X_0 and the part in the isotropic space, each round costs a bounded power.*

(d) (*Pairs.*) *The sum of* $\operatorname{disc}(L_{12})^{-k}$ *over near pairs over agreeing blocks with corank $l \le n^3$ and discriminant at most $b^{n^3 f}$ is at most $\tau^{l+2} n_p$.*

Proof. (a) is Lemma 27.2(b), (c) and Proposition 27.3(c). (b) is Proposition 27.21, and (c) is Proposition 27.23.

(d) If $l > k$ there are no lattices of corank l and the sum is empty; let $l \le k$ and $r = k - l$. Fix c.

A pair determines a label and a lattice. Let (γ, γ') be a near pair over (c, s, c') with s agreeing. Both terms lie over the same block, so they have the same label t, the component of s in its coset of $\bar{R}_0$, and both are near terms at t. Their common lattice L_{12} is cut out of $L_\gamma \cap L_{\gamma'}$, so it lies in the lattices of both terms and in $\Lambda \cap X_0$, and it is a lattice of rank the dimension of the common link, here r; by hypothesis $\operatorname{disc} L_{12} \le b^{n^3 f}$. Hence the map that sends (c', γ, γ') to the label t and the lattice $L = L_{12}$ lands in the set of pairs (t, L) with $L \subseteq \Lambda \cap X_0$ of rank r and $\operatorname{disc} L \le b^{n^3 f}$, and the weight $\operatorname{disc}(L_{12})^{-k}$ of the pair is the weight $\operatorname{disc}(L)^{-k}$ of its image.

The fibres. For a pair (t, L) let $T(t, L)$ be the set of near representatives (c', γ) of the label t over c with $L \subseteq L_\gamma$. If (c', γ, γ') lies over (t, L), both (c', γ) and (c', γ') lie in $T(t, L)$, and the triple is determined by these two elements. Hence the fibre over (t, L) has at most $\#T(t, L)^2$ elements, and by (c), as L has corank l and discriminant at most $b^{n^3 f}$,

$$\#T(t, L)^2 \le \left(b^{2000 n^6 (l+1) f}\right)^2 \le b^{10^5 n^9 f} \le \tau,$$

since $l + 1 \le 2n^3$ gives $2 \cdot 2000 n^6 (l + 1) \le 8000 n^9$, and every power b^e with $e \le 10^5 n^9 f$ is at most τ.

The sum. If every element of a set is sent to an element of the same weight, and each fibre has at most x elements, the sum of the weights is at most x times the sum over the images. Therefore

$$\sum_{c'} \sum_{\text{pairs over } (c, c')} \operatorname{disc}(L_{12})^{-k} \le \tau \cdot \#\bar{R}_0 \cdot \sum_L \operatorname{disc}(L)^{-k},$$

the last sum over the lattices $L \subseteq \Lambda \cap X_0$ of rank r with $\operatorname{disc} L \le b^{n^3 f}$. Summing over c and applying (b),

$$\sum_c \sum_{c'} \sum_{\text{pairs}} \operatorname{disc}(L_{12})^{-k} \le \#\bar{R}_0 \, \tau \, \tau^{l+1} n_X N^{-kf} = \tau^{l+2} n_p,$$

with $n_p = \#\bar{R}_0 \, n_X N^{-kf}$. □

27.6. The remaining terms

Two terms lie in the same *link class* if $\mathfrak{g}$ vanishes for neither and their links are equal.

Lemma 27.25 (Pairs of near terms). (a) *For a near pair over (c, c'), the corank l of the common lattice is at most n^3 and* $\operatorname{disc} L_{12} \le H$.
(b) *If $(\operatorname{disc} L_{12}\,\eta)^2 \le \mathfrak{g}^*(\gamma)\mathfrak{g}^*(\gamma')$ with $\eta > 0$, both cell second moments are at most D, and $\tau^{l+6} D\eta^{-k} \le B$, then $\operatorname{size}(\gamma)\operatorname{size}(\gamma') \le B \operatorname{disc}(L_{12})^{-k}/(\tau^{l+2} n_p)$.*
(c) *If every pair of a family of near pairs, one family for each two classes, satisfies the hypotheses of (b) with some η and D and a common $B \ge 0$, then the products of the sizes over all the families add up to at most τB.*
(d) *For a stable $T \subseteq \Gamma^*$, the squared norm of the cut block of T over (c, s, c') is at most the sum over the links Λ of the squared norms of the blocks of the terms of T with $\mathfrak{g} \ne 0$ and link Λ, plus v^2 times the sum of $\operatorname{size}(\gamma)\operatorname{size}(\gamma')$ over the pairs of near representatives of T over the block that do not lie in the same link class.*

Proof. (a) Let $i = c^+$ and $d = \operatorname{disc} L_{12} \ge 1$. By Theorem 27.20 and Lemma 27.12(a), $d^2 \le (2/M)^{2if} H^2$, and $2/M \le b^{-18}$ because $2b \le b^{19} \le M$; so $b^{18if} d \le H$, which gives $d \le H$ and $18i \le n^3$. The corank of the common lattice is at most $i + (j - \varrho)_+ \le i + 1$ (Lemma 27.18), and $i + 1 \le n^3$. (b) By definition $\operatorname{size}^2 \le \tau^4 D/(n_p \mathfrak{g}^{*k})$, so $\operatorname{size}(\gamma)\operatorname{size}(\gamma') \le \tau^4 D/(n_p(\mathfrak{g}^*\mathfrak{g}^{*\prime})^{k/2}) \le \tau^4 D\eta^{-k} d^{-k}/n_p$, and $\tau^4 D\eta^{-k} \le B\tau^{-(l+2)}$. (c) Sort the pairs by the corank $l \le n^3$ of their common lattice. By (b) and Theorem 27.24(d) the pairs of corank l contribute at most $B\tau^{-(l+2)} n_p^{-1} \cdot \tau^{l+2} n_p = B$, and there are at most $n^3 + 1 \le \tau$ coranks. (d) Split T into its pieces: one for each link Λ, consisting of the terms with $\mathfrak{g} \ne 0$ and link Λ, and one for the terms with $\mathfrak{g} = 0$. The pieces are stable, since $\mathfrak{g}$ and the link are invariant (Proposition 26.4(a)). By the triangle inequality the squared norm of the block is at most the sum over pairs of pieces of the products of their norms. The squares of the pieces with $\mathfrak{g} \ne 0$ remain. Every other product is at most v^2 times the sum of the products of the sizes of the near representatives of the two pieces (Theorem 27.13(c) for each piece), and a pair of near representatives determines its two pieces; such a pair does not lie in the same link class, since the pieces differ or one of them is the piece of $\mathfrak{g} = 0$. □

Theorem 27.26 (Remaining terms). *Let the data be admissible and E contain a primitive* 5040*th root of unity.*
(a) *Pairs of near terms over agreeing blocks whose links differ, or for one of which $\mathfrak{g} = 0$, contribute $\sum \operatorname{size}(\gamma)\operatorname{size}(\gamma') \le b^{-kf} \mathcal{D}_{\mathrm{id}}$.*
(b) *Pairs with equal links and $\mathfrak{g} \ne 0$ over blocks with $\operatorname{mn}(s) \ge z$ contribute at most $(b^{2f}/z)^k \mathcal{D}_{\mathrm{main}}$.*
(c) *If the terms with $\mathfrak{g} \ne 0$, taken link by link, have link second moment at most wv^2 over Ω, the second moment over Ω of Γ^* for the cut function is at most $w + b^{-kf} \mathcal{D}_{\mathrm{id}}$; and if $\operatorname{mn}(s) \ge z$ on Ω, the link second moment over Ω is at most $(b^{2f}/z)^k \mathcal{D}_{\mathrm{main}} v^2$.*

Proof. (a) Apply Lemma 27.25(c) with $B = \tau^{-10} \mathcal{D}_{\mathrm{id}}$; then $\tau B = \tau^{-9} \mathcal{D}_{\mathrm{id}} \le \tau^{-4} \mathcal{D}_{\mathrm{id}} = b^{-kf} \mathcal{D}_{\mathrm{id}}$. Let (γ, γ') be a pair as in (a), of cell ranks j, j', with $i = c^+$. By

Lemma 27.18 the corank of the common lattice is at most $i+(j-\varrho)_+$. *Equal links.* If $i = 0$ the links are equal, so $\mathfrak{g}$ vanishes for both or for neither (Lemma 27.15(d)); as the pair is not in one link class, it vanishes for both, both terms lie below the top cell rank, the corank is 0, and $\mathcal{D}_j = \mathcal{D}_{j'} = \tau^{-16}\mathcal{D}_{\mathrm{id}}$. Take $\eta = 1$, $D = \tau^{-16}\mathcal{D}_{\mathrm{id}}$: Theorem 27.20 gives $\mathrm{disc}(L_{12})^2 \le \mathsf{g}^*\mathsf{g}^{*\prime}$, and $\tau^6 D = \tau^{-10}\mathcal{D}_{\mathrm{id}} = B$. *Different links.* If $i \ge 1$, take $\eta = (M/2)^{if}$ and $D = \mathcal{D}_{\mathrm{main}} \ge \mathcal{D}_j, \mathcal{D}_{j'}$ (Lemma 27.12(b)); by Theorem 27.20 $(\mathrm{disc}(L_{12})\eta)^2 \le \mathsf{g}^*\mathsf{g}^{*\prime}$, and the corank l is at most $i+1$. Put $m = kf$ and $x = \tau(2/M)^m$; since $2b \le M$, $x \le \tau^4(2/M)^m = (2b/M)^m \le 1$. Then, with $\mathcal{D}_{\mathrm{main}} \le (M/b^6)^m\mathcal{D}_{\mathrm{id}}$,

$$\begin{aligned}\tau^{l+6}\mathcal{D}_{\mathrm{main}}\eta^{-k} \le \tau^7 x^i \mathcal{D}_{\mathrm{main}} &\le \tau^7 x\,(M/b^6)^m\mathcal{D}_{\mathrm{id}} = \tau^8(2/b^6)^m\mathcal{D}_{\mathrm{id}}\\ &\le \tau^8(b/b^6)^m\mathcal{D}_{\mathrm{id}} = \tau^{-12}\mathcal{D}_{\mathrm{id}} \le B.\end{aligned}$$

(b) Here the links are equal and $\mathfrak{g} \ne 0$, so both terms have the top cell rank, $c^+ = 0$, and Theorem 27.20 with $x = z \le \mathrm{mn}(s)$ gives $(\mathrm{disc}(L_{12})z)^2 \le \mathsf{g}^*\mathsf{g}^{*\prime}$. The corank is at most 1, and the cell second moments are $\mathcal{D}_{\mathrm{main}}$. Take $\eta = z$, $D = \mathcal{D}_{\mathrm{main}}$ and $B = \tau^7\mathcal{D}_{\mathrm{main}}z^{-k}$ in Lemma 27.25(c): the total is at most $\tau^8\mathcal{D}_{\mathrm{main}}z^{-k} = (b^{2f}/z)^k\mathcal{D}_{\mathrm{main}}$, as $\tau^8 = b^{2kf}$. (c) By Lemma 27.25(d), summing over c, c' and $s \in \Omega$, the second moment of Γ^* is at most the link second moment of the terms with $\mathfrak{g} \ne 0$, divided by ν^2, plus the sum of the products of the sizes over the pairs of near representatives that do not lie in one link class; a pair determines its s, so each such pair occurs once, and by (a) their sum is at most $b^{-kf}\mathcal{D}_{\mathrm{id}}$. For the second statement apply Lemma 27.25(d) to the stable sets of the terms with $\mathfrak{g} \ne 0$ and a fixed link: the square of the block of one link is at most ν^2 times the sum over pairs of its near representatives, which lie in the same link class; a pair determines its s and its link, and (b) bounds the sum. □

28. The main term

The filling theorem (Theorem 26.8) needs one bound that is sharp to a factor $1 + O(b^{-4})$: the link second moment of the open elements of a fixer (Theorem 28.17). The terms over an open reduction lie in the big cell of the Bruhat decomposition, so their class sums are sums over double cosets $Uw_p\sigma U$ (Section 28.2). At a finite set of places each double coset sum is a Kloosterman sum times an orbital integral at infinity (Section 28.3). Siegel's formula, used as an upper bound, turns the sums over rational points into integrals over spheres at the finite places (Section 28.4), and a sieve over all places, entry by entry, removes the sums over the entries of the heads at the cost of a factor near 1 (Sections 28.5 and 28.6). The section opens with the count of bad types used in Theorem 22.6(b).

28.1. Bad types are rare

Let $\Bbbk$ be a field with q elements and $1 < r < q$. Give every polynomial of degree r over $\Bbbk$ the weight $\omega_{q,r} = (1-r/q)/q^r$. The idempotent of a tuple $c = (c_w)$ of labels over fields $\Bbbk_w$ is $e_c = \bigotimes_w e_{c_w}$, with $e_{c_w} = \sum_g (\Xi_{c_w}(1)/|G_w|)\overline{\Xi_{c_w}(g)}\, g$ in $\mathbb{C}[G_w]$, $G_w = \mathrm{GL}_r(\Bbbk_w)$ (Appendix B.13).

Proposition 28.1 (Bad types). *Let $M \ge 1$ and $1 < r < q$.*

(a) *At most* $(1+M+r)q^{r-1}$ *monic polynomials of degree* r *over* $\mathbb{k}$ *are not allowed for* M*, and at least* $\varphi(q^r-1)/r$ *are primitive.*
(b) *If* $(1+M+2r)/q \le \varepsilon$*, the allowed polynomials of degree* r *weigh at least* $1-\varepsilon$*; the polynomials that are not primitive weigh at most* $\exp\bigl(-1/(2r(2r\log q+1))\bigr)$.
(c) *For labels* c_w *of degree* r *over fields* $\mathbb{k}_w$ *with* q *elements, the coefficient of* e_c *at the identity has real part at least* $\prod_w \omega_{q,r}$.
(d) *Let there be* f *fields* $\mathbb{k}_w$*, each with* q *elements, and let* $(1+M+2r)/q \le \varepsilon \le 1$ *and* $2r(2r\log q+1)L \le f$*. Then the coefficient at the identity of* $1-\sum_c e_c$*, the sum over the good tuples* c *for* M*, has real part at most* $f\varepsilon + e^{-L}$.
(e) *For admissible data and* $i=0,1$*, the idempotent* e_{p_i}*, the sum of the* e_c *over the good tuples of labels at the chosen places above* p_i*, is a central idempotent,* $e_0 = e_{p_0}\otimes e_{p_1}$*, and the coefficient of* $1-e_{p_i}$ *at the identity has real part at most* $b^{-8}/2$.

Proof. (a) A monic polynomial of degree r that is not allowed either is not a label, that is, vanishes at 0 or is not squarefree, or has a root x in $\mathbb{k}$ with $x^M=1$. In the first and the last case it has a root x with $x=0$ or $x^M=1$; there are at most $1+M$ such x, and for each of them q^{r-1} monic polynomials of degree r vanish at x. A polynomial that is not squarefree is P^2B with P, B monic, $e=\deg P\ge 1$ and $\deg B = r-2e<r$; for each $e\le r/2$ there are $q^e\cdot q^{r-2e}\le q^{r-1}$ such products. In all there are at most $(1+M+r)q^{r-1}$ polynomials that are not allowed. A primitive polynomial of degree r is the minimal polynomial of a generator of the cyclic group $\mathbb{F}_{q^r}^\times$: if a generator has minimal polynomial of degree d, then it lies in $\mathbb{F}_{q^d}$, so q^r-1 divides q^d-1 and $d=r$. There are $\varphi(q^r-1)$ generators, and each primitive polynomial has r of them as roots. (b) The q^r monic polynomials of degree r weigh $q^r\omega_{q,r}=1-r/q$ together, and by (a) those that are not allowed weigh at most $(1+M+r)q^{r-1}/q^r=(1+M+r)/q$; so the allowed ones weigh at least $1-r/q-(1+M+r)/q = 1-(1+M+2r)/q \ge 1-\varepsilon$. For the others let P be the number of primitive polynomials, $m=q^r-1$ and $A=2r(2r\log q+1)$. Then

$$q^r \le 2m \le 2\varphi(m)(2\log m+1) \le 2rP(2r\log q+1) = AP,$$

by $m\le\varphi(m)(2\log m+1)$, by (a) and by $\log m\le r\log q$. The polynomials that are not primitive number q^r-P, and each weighs at most q^{-r}, so together they weigh at most $1-P/q^r\le 1-1/A\le e^{-1/A}$. (c) The coefficient of e_c at 1 is $\prod_w \Xi_{c_w}(1)^2/|G_w|$, the coefficient of a tensor product at 1 being the product of the coefficients. For a label c with irreducible factors of degrees d_i, Green's degree formula gives $\Xi_c(1)=\prod_{j=1}^r(q^j-1)/\prod_i(q^{d_i}-1)$ [Gre55], while $|\mathrm{GL}_r(\mathbb{k})| = q^{r(r-1)/2}\prod_{j=1}^r(q^j-1)$. Hence

$$\frac{\Xi_c(1)^2}{|\mathrm{GL}_r(\mathbb{k})|} = \frac{\prod_{j=1}^r(q^j-1)}{q^{r(r-1)/2}\prod_i(q^{d_i}-1)^2} \ge \frac{q^{r(r+1)/2}\bigl(1-\sum_{j=1}^r q^{-j}\bigr)}{q^{r(r-1)/2}q^{2r}} \ge \frac{1-r/q}{q^r},$$

using $\sum_i d_i = r$, $\prod_i(q^{d_i}-1)\le q^r$ and $\prod_j(1-q^{-j})\ge 1-\sum_j q^{-j}\ge 1-r/q$. (d) The coefficient of 1 at the identity is 1, and by (c) that of $\sum_c e_c$ is at least ω^f times the number of good tuples, $\omega=\omega_{q,r}$. A tuple of allowed polynomials is good unless none of its members is primitive. Weighted by ω^f, the tuples of allowed polynomials count at least $(1-\varepsilon)^f\ge 1-f\varepsilon$ by (b), and those without a primitive member at most $\exp(-1/A)^f=\exp(-f/A)\le e^{-L}$, with $A=2r(2r\log q+1)$ as in (b), since $AL\le f$. Hence the coefficient of $1-\sum_c e_c$

has real part at most $1-(1-f\varepsilon-e^{-L})=f\varepsilon+e^{-L}$. (e) The idempotents are central by Theorem B.25, and their tensor product is e_0 by the definition of the good families (Proposition 15.6). At the chosen places above p_i the residue fields have $q=p_i$ elements and there are f of them; with the order q_2 of the characters in place of M, the rank r of Y_0 and $L=9\log b$, the size conditions of the data give $1<r<p_i$, $(1+q_2+2r)/p_i\le 1$, $2r(2r\log p_i+1)\cdot 9\log b\le f$ and $f(1+q_2+2r)/p_i+b^{-9}\le b^{-8}/2$, and (d) applies with $\varepsilon=(1+q_2+2r)/p_i$. □

28.2. Open elements, the big cell and orbit sums

For a tuple $p=(p_0,\dots,p_{\varrho-1})$ of units the *Weyl element* w_p sends e_a to $p_a e_a^*$ and e_a^* to $e_a/\bar{p}_a$ for $a<\varrho$, and fixes X_0 and z. The *floor group* $G^{(\varrho)}$ is the group of isometries that fix the e_a and e_a^*, $a<\varrho$. A *head* is a tuple p followed by a last entry t. The *entry set* $\mathcal{T}$ is the finite set of quotients of two elements of the finite set of nonzero integers δ of E with $|\delta|_v^2\le b^{270n^2}$ at every infinite place. For rational γ in the floor group and a class pair (c,c'), the *orbit sum* $\mathcal{A}^U(c,s,c';\gamma)$ is the sum of the amplitudes of $u\gamma u'^{-1}$ over the pairs (u,u') of rational points of U for which $u\gamma u'^{-1}$ is a term over (c,s,c').

Lemma 28.2 (Open elements lie in the big cell). *Let γ be a term over (c,s,c'). For each l the leading minor of order l of the matrix $Z(\gamma)$ of frame products is an algebraic integer, and at every chosen place w its reduction modulo w is the same minor of the frame Gram matrix of s at w. Hence if s is open, no proper leading minor of $Z(\gamma)$ vanishes.*

Proof. $Z(\gamma)$ is the matrix of the products of the frame with $h_c^{-1}\gamma h_{c'}$ applied to the frame, up to the integral change of basis by h_c and $h_{c'}$, and $h_c^{-1}\gamma h_{c'}\in K$ has integral entries in the lattice basis; so its minors are integral at every finite place, hence algebraic integers. Its reduction at w is s_w, and taking minors commutes with the ring map to the residue field. If s is open, at some chosen place above p_0 no proper leading minor of the frame Gram matrix of s vanishes; a minor whose reduction is nonzero is nonzero. □

Lemma 28.3 (Weyl elements and the norm equation). *Let σ be rational in the floor group, $t=h(z,\sigma z)$ the last diagonal entry of $Z(\sigma)$ and ω the coordinates of σz in X_0.*

(a) *$N\omega^*\omega=t\bar{t}-N^2$; if $t\bar{t}\ne N^2$, then $|t|_v>N$ at every infinite place.*
(b) *$h_c^{-1}(w_p\sigma)h_{c'}=w_p(h_c^{-1}\sigma h_{c'})$, and $h_c^{-1}\sigma h_{c'}$ lies in the floor group; at T the translate sends z to $x-N^{-1}tz$ in the lattice basis, where x is the part in X_0 of σz moved by the inverse of the representative of c.*
(c) *At an infinite place v the rescaled image of $w_p\sigma$ is w_q times that of σ, with $q_a=v(p_a)$ times the squared scale of the frame vector a.*
(d) *The link of $uw_p\sigma u'^{-1}$, $u,u'\in U$, is the link of $w_1\sigma$, and $\mathfrak{g}(uw_p\sigma u'^{-1})\ne 0$ if $t\bar{t}\ne N^2$.*

Proof. (a) With a the coordinate of σz at z one has $t=h(z,\sigma z)=-Na$, and $-N=h(\sigma z,\sigma z)=\omega^*\omega-N|a|^2$, since σz has no components along the hyperbolic planes; multiply by N. The difference $t\bar{t}-N^2=N\omega^*\omega$ is nonnegative at each place, hence positive when it is not zero. (b) No e_a^* has level below ϱ, and the

Weyl element commutes with the change of basis entry by entry; the change of basis on the right fixes z, so the column of σ at z remains, and in the lattice basis it is the vector $x - N^{-1}tz$, with $N = \pi\bar{\pi}$. (c) Each column of w_p has one nonzero entry; conjugating by the scales of the normalised basis multiplies the entries on the pair of floor a by the squared scale or its inverse. (d) Both properties hold for $w_p\sigma$ and are invariant under U on either side (Proposition 26.4(a)); the link of $w_p\sigma$ is that of $w_1\sigma$ by Lemma 26.1(c), and $\mathfrak{g}(w_p\sigma) = |d|^2(t\bar{t} - N^2)$, $d = \prod_a p_a$. □

Proposition 28.4 (Class sums as orbit sums). *Let the data be admissible, let the elements of $\mathcal{T}$ be units off a finite set T of finite places, and let all terms over (c, s, c') have nonzero leading minors $D_1, \dots, D_\varrho$. Let A be the block, for the cut function, of the terms over (c, s, c') with $\mathfrak{g} \neq 0$ and link Λ, and suppose $A \neq 0$. Then there are $t \in \mathcal{T}$, a rational $\sigma \in G^{(\varrho)}$ and $k \in K$ such that t is the last diagonal entry of $Z(\sigma)$, $t\bar{t} - N^2$ is totally positive, Λ is the link of $w_1\sigma$, $h_c^{-1}\sigma h_{c'}$ is integral off T, and k reduces to s; and*

$$A = \sum_{p \in \mathcal{T}^\varrho} \mathcal{A}^U\big(c, s, c'; w_p\sigma\big),$$

the sum over the heads (p, t) of the orbit sums of the elements $w_p\sigma$.

Proof. The big cell. A term with nonzero leading minors $D_1, \dots, D_\varrho$ lies in the big cell of the Bruhat decomposition for the flag $\langle e_1\rangle \subset \cdots \subset A$: Gaussian elimination along the frame, with the nonzero minors as pivots, writes $\gamma = uw_p\sigma u'^{-1}$ with $u, u' \in U(E)$ and $\sigma \in G^{(\varrho)}(E)$, and for fixed σ the map $(p, u, u') \mapsto uw_p\sigma u'^{-1}$ is injective (the Weyl element depends only on the entries below ϱ, so these may be completed by any last entry). The leading minor of order l of $Z(\gamma)$ is that of $Z(w_p\sigma)$, the product $p_0 \cdots p_{l-1}$ of the first entries, so $p_a = D_{a+1}/D_a$.

Entries. The minors of a term with nonzero amplitude are nonzero integers with $|D_l|_v^2 \leq b^{270n^2}$: at some y both $W^\flat(y)$ and $W^\flat(\gamma^{-1}y)$ are nonzero, unipotent translates of the two points lie in the boxes (Proposition 25.5(c)), the translates do not change the minors, and at points of the boxes the minors are bounded (Lemma 27.12(a)). Hence every entry p_a lies in $\mathcal{T}$, and so does $t = h(z, \sigma z) = D_{\varrho+1}/D_\varrho$.

The element σ. Take a contributing term $\gamma_0 = u_0 w_{p_0}\sigma u_0'^{-1}$. Then $\mathfrak{g}(\gamma_0) = |D_\varrho|^2(t\bar{t} - N^2)$, which is nonzero, so $t\bar{t} - N^2$ is nonzero and, by Lemma 28.3(a), totally positive. The link of γ_0 is that of $w_1\sigma$ (Lemma 28.3(d)), and the link determines σ up to the floor group's action by U on both sides: by Lemma 26.1(c), $w_p\sigma$ and $w_q\tau$ have the same link exactly when $\sigma = \tau$. Hence every contributing term is $uw_p\sigma u'^{-1}$ with $p \in \mathcal{T}^\varrho$, each occurring once, and the block is the sum over the heads of the orbit sums. A lift $k \in K$ of s exists because the reduction is onto. The translate $h_c^{-1}\sigma h_{c'}$ is integral off T because $h_c^{-1}\gamma_0 h_{c'}$ is, and $h_c^{-1}u_0 h_c$, $h_{c'}^{-1}u_0' h_{c'}$ and $w_{p_0}^{-1}$ are (the entries of p_0 and their inverses are units off T). □

28.3. Kloosterman sums at a finite set of places

Let T be a finite set of finite places of E, stable under conjugation, and $E_T = \prod_{w\in T} E_w$, read as one ring with involution, integers O_T and a Haar measure. Its unitary group G_T carries the congruence subgroup $K_T(M)$ of level M in the group K_T of integral points, the unipotent group U_T and the character ψ_U. An element of the finite adelic group is *integral off T* if it and its inverse have integral entries outside T. The *finite Bessel function* χ_M is $\psi_U(u)$ on $uK_T(M)$, $u \in U_T$, and 0 off $U_T K_T(M)$. For $x, s \in G_T$ the *Kloosterman sum* is

$$\mathrm{Kl}(x,s) = \sum_{\substack{u,u'\in U_T/(U_T\cap K_T(M))\\ u^{-1}xu'\in sK_T(M)}} \chi_M(u)\overline{\chi_M(u')}.$$

A *filling datum at T* is a large lattice datum (Section 29.6) on G_T whose modulus is the component of M at T and whose character is that of the data at T.

Lemma 28.5 (The place E_T). *Let T be finite and stable under conjugation.*

(a) *For finitely many nonzero elements of E there is such a T off which they are units and off which M^{-1}, N^{-1} and the lattice basis with its inverse are integral; every finite adele is integral off all sufficiently large T.*

(b) *A finite adelic point is integral, respectively integral and congruent to 1 modulo an idele u with u^{-1} integral off T, exactly when it is integral off T and its component at T is so.*

(c) *The integers O_T are self-dual for the character $x \mapsto \psi_{\mathrm{f}}(-x/\varpi)$ read at T, which is trivial on traces; there is a Haar measure on the skew elements of mass 1 on the integral ones, the unitary group G_T is unimodular, and the generalised index of $x \in E_T$ is at most the square of $\prod_{w\in T}(1 + (\|x_w\|^{-1} - 1)^{1/2})$.*

(d) *For x with x, x^{-1} integral off T, the square root of the modulus of its component at T is $\prod_v |x|_v^{-1}$ (the product formula); in particular the modulus of the component of M is $(M^f)^{-2}$.*

Proof. (a) It suffices that T contain the finitely many places where the given adeles are not integral, together with their conjugates. (b) Integrality is a condition at each finite place, and it splits into the condition at T and the one off T; off T the congruence modulo u follows from integrality, as u^{-1} is integral there. (c) Self-duality reduces by extension by zero to that of $\hat{O}_E$ for $\psi_{\mathrm{f}}(-\cdot/\varpi)$, ϖ generating the different; the character is trivial on traces because ϖ is skew and ψ_{f} takes the same value at conjugate adeles. With $\iota^2 = -1$ and $\tau + \bar{\tau} = 1$ integral at T, the map $(a,b) \mapsto \tau\iota a + b$ from pairs of skew elements onto E_T takes the product of a Haar measure on the skew elements with itself to the Haar measure, and the integral skew elements to the integral elements. An orthonormal basis over $\hat{O}_E$ identifies the adelic group with that of the standard form, whose measure is bi-invariant; pushing it forward from the points integral off T, each point at T having such a lift, shows that the inverse of a left Haar measure is left invariant. The bound for the index is checked place by place. (d) The inequality $[x]^{1/2} \le \prod_v |x|_v$ for x and for x^{-1}, integral off T, gives equality; apply it to M squared. □

Theorem 28.6 (Orbit sums are Kloosterman sums). *Let the data be admissible and T a finite set of finite places, stable under conjugation, off which M^{-1}, N^{-1} and the lattice basis are integral.*

(a) *The product E_T is a finite place carrying a filling datum F with $3\varrho+6$ further hyperbolic pairs and an orthonormal part of size $k-(6\varrho+13)$. Every point of G_T is the component at T of a finite adelic point integral off T.*

(b) (*Counting by measures.*) *Let B be a subgroup of the finite adelic group and μ an s-finite measure on G_T, carried by the components at T of the elements of B integral off T and right invariant under them. Then the order of the image of $B\cap K$ modulo M times $\mu(K_T(M))$ is $\mu(K_T)$.*

(c) (*Finite orbital integrals.*) *Let Z be a group with $K_T(M)\le Z\le K_T$ and W its* Bessel vector*, with coordinates $W_a(y)=\chi_M(ya^{-1})$ at the cosets a of $K_T(M)$ in Z. If $ab^{-1}\in sK_T(M)$, then $\int_{G_T} W_a(y)\overline{W_b(x^{-1}y)}\,dy = \operatorname{vol}(K_T(M))\operatorname{Kl}(x,s)$.*

(d) (*Orbit sums.*) *Let $\gamma=w_p\sigma$ with $\sigma\in G^{(\varrho)}(E)$, the inverses of the entries p_a integral off T, and $h_c^{-1}\gamma h_{c'}$ integral off T. Let x and s be the components at T of $h_c^{-1}\gamma h_{c'}$ and of an integral point k with reduction $\bar k$. Then*

$$\mathcal{A}^U(c,\bar k,c';\gamma)=\operatorname{Kl}(x,s)\cdot\int_{G_\infty} W(y)\otimes W(\gamma^{-1}y)^*\,dy,$$

with W the full or the cut Bessel function at the infinite places.

Proof. (a) *The datum.* In the lattice basis the Gram matrix of $X_0\perp Ez$ is integral with an integral inverse, over every subring and so over $\mathcal{O}_T$; hence the lattice of integral vectors at T is unimodular. The Gram matrix of $X_0\perp Ez$ in the lattice basis splits at T into $3\varrho+6$ further hyperbolic pairs and an orthonormal part of size $k-(6\varrho+13)$ (Section 29.6), the subtraction being exact because k is large. The rank conditions of a large datum hold because k is large compared with ϱ and with the degree of E, and the zeta value of $\mathcal{O}_T$ is at most the global one (Section 25.1); the character is that of Lemma 28.5(c), and the modulus is the component of M. *Lifting.* A point $y=1+z$ of G_T has the lift $1+\phi(z)$, where ϕ extends each entry by zero; the unitarity condition $z^*Jz+z^*J+Jz=0$ is preserved by ϕ, and the lift is integral off T. If y lies in U_T or in H_T, so does the lift. (b) Choose lifts $k_s\in B\cap K$ of the elements s of the image of $B\cap K$ modulo M. Almost everywhere for μ the integral points at T are the disjoint union of the cosets $K_T(M)k_s^{-1}$: a point of the support of μ is the component of an element of B integral off T, and if it is integral at T it is integral (Lemma 28.5(b)), hence lies in $B\cap K$ and in exactly one coset; by right invariance all the cosets have measure $\mu(K_T(M))$. (c) Put yb for y; the integrand is then constant on the cosets $yK_T(M)$, because χ_M takes the same value at the points of a coset $yK_T(M)$ (it is $\psi(\sigma(u)/M)$ at uk with σ the sum of the entries at the simple roots). For a left invariant measure and an open subgroup of finite volume, the integral of a function on the cosets is the volume of the subgroup times the sum over the cosets. In $\sum_y \chi_M(ys^{-1})\overline{\chi_M(x^{-1}y)}$ a term is zero unless $yK_T(M)=usK_T(M)=xu'K_T(M)$ with $u,u'\in U_T$; these are unique modulo $U_T\cap K_T(M)$ (an integral s normalises $K_T(M)$), and the value is $\psi_U(u)\overline{\psi_U(u')}$. Hence $(u,u')\mapsto xu'K_T(M)$ is a bijection between the supports of the two sums that preserves the terms. (d) *Unfolding.* The group $U_c(M)\times U_{c'}(M)$ acts freely on the terms in $U\gamma U$, and each orbit contains one element $(r_w)^{-1}\gamma r'_{w'}$, for lists r, r' of rational points of U, one in

each coset $uU_c(M)$, $u'U_{c'}(M)$ for which $u^{-1}\gamma u'$ is a term; by Proposition 25.4(b) the cutoffs may be replaced by the indicator of these representatives, so the orbit sum is the sum over the pairs of representatives giving a term of the orbital integrals $\int W(y)\otimes W(\delta^{-1}y)^* dy$. *Rational and local pairs.* By (a) and strong approximation, the translates $h_c^{-1}u h_c$ of the rational points u of U integral off T match the cosets $uU_c(M)$ with the cosets of $U_T \cap K_T(M)$ in U_T; in a term $u^{-1}\gamma u'$ over $(c, \bar{k}, c')$ the translates of u and u' are integral off T, and $(r_w)^{-1}\gamma r'_w$, is a term over $(c, \bar{k}, c')$ exactly when $w^{-1}xw' \in sK_T(M)$, by (b) of Lemma 28.5. *Phases.* For rational u, u' of U, $J(u^{-1}\gamma u') = \psi_U(u)\overline{\psi_U(u')}J(\gamma)$ at infinity, the archimedean character of U being, by the product formula for ψ, the inverse of its finite part; and the finite Bessel function at the component at T of the translate of u is ψ_U at the finite part of u, the character being trivial on the integral points off T and invariant under conjugation by h_c. Each pair therefore contributes $J(\gamma)$ times the term $\chi_M(w)\overline{\chi_M(w')}$ of the Kloosterman sum. □

Corollary 28.7 (Orbit sums and the finite orbital integral). *In the situation of Theorem 28.6(d), let $K_T(M) \le Z \le K_T$, let a, b be cosets of $K_T(M)$ in Z with $k^{-1}ab^{-1} \in K_T(M)$, let q and l be the components at T of p and of $h_c^{-1}\sigma h_{c'}$, and let $\kappa = \prod_{a<\varrho} \mathrm{mod}(q_a)^{e_a}$ be the head density. Then the entry (a,b) of the scaled orbital integral of the Bessel vector of Z at $w_q l^{-1}$, times $J(\gamma)$, is κ $\mathrm{vol}(K_T(M))$ times the orbit sum $\mathcal{A}^U(c, \bar{k}, c'; \gamma)$. Moreover the order of the reduction of $U(\hat{\mathcal{O}})$ modulo M times the measure of $U_T \cap K_T(M)$ is 1.*

Proof. Both the entry at T and the orbit sum contain the Kloosterman sum (Theorem 28.6(c), (d)). For the last statement, by Lemma 28.5(d) the floor $l < \varrho$ of $U_T \cap K_T(M)$ has volume $(M^f)^{-(2d_l+1)}$ with $d_l = k + 1 + 2(\varrho - l - 1)$, and $\sum_{l<\varrho}(2d_l + 1) = \varrho(2k + 2\varrho + 1)$, while the reduction of $U(\hat{\mathcal{O}})$ has $M^{f\varrho(2k+2\varrho+1)}$ elements (Lemma 25.1). □

28.4. Siegel's formula as an upper bound

For a hermitian matrix β of size k over E and a function f on the finite adelic vectors of rank k, the *density* is $\delta(f,\beta) = \int_z \int_x f(x)\psi_{\mathrm{f}}(z(x^*x - \beta)/2)\, dx\, dz$, the integral over the self-adjoint z. The *Siegel constant* is

$$C_{\mathrm{Sie}} = \Big(\frac{\pi^k}{(k-1)!\, N^k}\Big)^f |D_E|^{1/4} \,\mathrm{vol}(\hat{\mathcal{O}}_E)^k,$$

and the *scaled Siegel constant* $C_{\mathrm{Sie}}^\sharp$ is C_{Sie} times the number of classes and the normalising factor of the local measures at T (Theorem 28.8(b)).

Theorem 28.8 (Siegel's formula at a finite set of places). *Let the data be admissible, T as in Theorem 28.6 and F a filling datum at T.*

(a) (*Densities.*) *Let f be $\Psi \ge 0$ at T times the indicator of the vectors integral off T, with Ψ continuous of compact support. Then $\delta(f,\beta)$ is the product of an integral of Gauss sums of the standard lattice over the z whose part $-\omega z$ at T is integral, and of the density of Ψ at T; and*

$$|\delta(f,\beta)| \le (1 + b^{-6})\,\mathrm{vol}(\hat{\mathcal{O}}_E)^k \int_{y^*y=\beta} \Psi(y)\, d\mu_\beta(y),$$

the integral over the sphere at T for its fibre measure.

(b) *There is $D > 0$ with $C_{\mathrm{Sie}}^{\sharp} = D \cdot \#\{\text{classes}\} \cdot C_{\mathrm{Sie}}$ and, for every Z with $K_T(M) \le Z \le K_T$, $\mathrm{mr}(Z) \le \mu_H(K_T(M))\,\mu(K_T(M))^2\,[Z : K_T(M)]\,D$, where $\mathrm{mr}(Z) = \mu(Z)\mu(K_T(M))/\mathrm{fl}^2$ is the* normalised mass *and* fl *the mass of $U_T \cap K_T(M)$.*

(c) (*Siegel's formula as an upper bound.*) *Let $\Psi \ge 0$ have compact support at T and be invariant under small translations and under the elements of $K_T(M)$ that fix all basis vectors outside X_0. Let $|t|_v > N$ at every infinite place. For pairs (c, v) of a class and a rational vector v of X_0 with $N(v^*v) = t\bar{t} - N^2$ and $x = h_c^{-1}v$ integral off T,*

$$\sum_{(c,v)} \Psi\big(x - N^{-1}tz\big) \le (1 + b^{-6})\,\#\{\text{classes}\}\,C_{\mathrm{Sie}} \prod_v (|t|_v^2 - N^2)^{k-1} \times |\pi_N|_T^{-1} \int_{y^*y = (t\bar{t} - N^2)/N} \Psi(y - N^{-1}tz)\,d\mu(y).$$

Proof. (a) *Splitting the frequency.* A skew frequency b at T becomes a self-adjoint finite adele $\tilde{b}$ by extension by zero and multiplication by $-\varpi^{-1}$, and the map $z \mapsto -\varpi z$ at T is a left inverse of $b \mapsto \tilde{b}$, compatibly with translations; the difference $z - \tilde{b}$ for b the part of $-\varpi z$ at T vanishes at T. With $\tau + \bar{\tau} = 1$ at T, $\psi_{\mathrm{f}}(\tilde{b}y/2) = \psi(\tau b y_T)$ for the character ψ of the place E_T. *Splitting the vector.* Write $x = y + w$ with y the extension by zero of a vector at T and w an adelic vector integral at T. The phase $\psi_{\mathrm{f}}(\zeta(x^*x - \beta)/2)$, for $\zeta = \tilde{b} + z'$ with $-\varpi z'$ integral at T, is the product of a phase off T (in w and z') and a phase at T (in y and b): the values of the form at x and w differ only at T, and the cross term is integral at T, so its phase is 1. Integrating, $\delta(f, \beta)$ is the product of the integral over z' of the Gauss sums of the standard lattice and of the density of Ψ at T. *The bound.* The Gauss sums of the standard lattice, integrated over the self-adjoint z' with $-\varpi z'$ integral at T, are bounded by $\mathrm{vol}(\hat{\mathcal{O}}_E)^k$ times the ideal zeta sum of $\hat{\mathcal{O}}_E$ at $\mu = 3\varrho + 4$, since $2(\mu + 1) \le k$; this zeta sum is at most $1 + 2^{-\varrho} \le 1 + b^{-6}$, by the Euler product bound of the proof of Lemma 27.2(c), now at $\varrho + 2\varrho + 4$ with $[E : \mathbb{Q}] \le 2^{2\varrho}$. At T Fourier inversion in the skew frequency turns the density of Ψ into the integral over the sphere $y^*y = \beta$ for its fibre measure.

(b) In $\mathrm{mr}(Z) = \mu(Z)\mu(K_T(M))/\mathrm{fl}^2$ cover Z by its $[Z : K_T(M)]$ cosets of $K_T(M)$, so $\mu(Z) \le [Z : K_T(M)]\mu(K_T(M))$, and use $o \cdot \mathrm{fl} = 1$ for the order o of the reduction of the unipotent group (Corollary 28.7). Take $D = \#\bar{R}_0\, o = \#\bar{H}\, o^2$; the first identity is the definition of $C_{\mathrm{Sie}}^{\sharp}$. For the measure μ_H of H, the integral points have mass 1, and by Theorem 28.6(b) the group $K_T(M) \cap H$ has index $\#\bar{H}$ in them, so $\#\bar{H}\,\mu_H(K_T(M)) = 1$; hence $\mathrm{fl}^{-2} = o^2 = D\,\mu_H(K_T(M))$.

(c) *The test function.* Let f be the function on adelic vectors of X_0 that is $\Psi(x - N^{-1}tz)$ at T, in the lattice basis, times the indicator of the vectors integral off T. It has compact support (a vector in the support is an integral vector plus the extension by zero of its part at T, read off the support of Ψ), and it is invariant under the level of the finite adelic unitary group of X_0, because at T an element of the level acts on the vectors with last entry t as an element of $H \cap K_T(M)$. *Siegel–Weil.* By the Siegel–Weil formula for the definite space X_0 (Theorem B.31), the class sum of f at $\beta = (t\bar{t} - N^2)/N$ is the mass times the archimedean density $\prod_v \pi^k \beta_v^{k-1}/(k-1)!$ times $\delta(f, \beta)$. At the free level

the stabilisers are trivial, so the weighted count of the class sum is the plain count of the pairs (c, v); each pair of the sum has norm β and is integral off T, so it is a term of the class sum. The bound of (a), the evaluation of the class sum of 1 (the number of classes), the product formula for $N = \pi_N \bar{\pi}_N$, which gives the modulus of π_N at T as a power of N^{-1}, and the expansion of C_{Sie} and $\beta_v = (|t|_v^2 - N^2)/N$ give the claim. □

The *fibre pairing* of two heads ω, ω' at T is the integral over $H \cap G^{(\varrho)}$, the fixer of all basis vectors outside X_0, of the product of the scaled orbital integrals of the Bessel vector of Z at $w_\omega l^{-1}$ and $w_{\omega'} l^{-1}$; it is a function of $e = l^{-1}z$, and its integral over the vectors e with $h(z, e) = t$ is the *density at the last entry* $J_{\mathrm{last}}(\omega, \omega'; t)$ (Section 29.7).

Lemma 28.9 (Classes over one point). *Let F be a filling datum at T and $K_T(M) \leq Z \leq K_T$.*

(a) *Let γ_1, γ_2 be rational in the floor group with the same product $h(z, \gamma z)$ and the same part in X_0 of γz, and let their left classes agree. If their translates by the class representatives are integral off T and lie in one coset of $K_T(M)$ at T, then $\gamma_1 = \gamma_2$ and the right classes agree.*

(b) *Let the heads a_p have entries below ϱ that are units off T, let R be the Gram matrix of the tensors $\rho_p \int W(y) \otimes W((w_p\gamma)^{-1}y)^* dy$, and let ρ_p times the head density be the same number w for all p. For integral n_i in Z at T with distinct reductions s_i, the squared norms of $\sum_p \mathcal{A}^U(c, s_i, c'; w_p\gamma)$, times w^2, $[Z : K_T(M)]$ and $\mu(K_T(M))^2$, add up to at most the sum against R of the pairings of the scaled orbital integrals at T of two heads, at g^{-1} for g the component of $h_c^{-1}\gamma h_{c'}$.*

(c) *If moreover R is positive semidefinite, the classes $(c_i', \bar{n}_i, \gamma_i)$ are distinct, the translates send z to one point e at T, and the left class, $h(z, \gamma_i z)$ and the part in X_0 of $\gamma_i z$ are the same for all i, then the squared norms of the class sums, times $w^2 \mu_H(K_T(M))\mu(K_T(M))^2[Z : K_T(M)]$, add up to at most the real part of $\sum_{p,q} R_{pq}\, \mathrm{fp}(\omega_p, \omega_q; e)$.*

Proof. (a) γ_1^{-1} and γ_2^{-1} give the same point $\gamma^{-1}z$, so $\gamma_1^{-1}\gamma_2$ fixes z as well as the hyperbolic planes, hence lies in the floor group of level $\varrho + 1$; its translate lies in the principal level, which forces the right classes to agree and makes it unipotent with trivial part in the definite unitary group, hence trivial. (b) By Corollary 28.7, the entry $(b, A_i b)$ of the sum over the heads of the scaled orbital integrals at T, with $A_i b$ the coset of $n_i^{-1}b$, is $w\mu(K_T(M))$ times the class sum at the reduction s_i; these entries are distinct as the s_i are, and the sum against R of the pairings of two heads is the squared norm of the vector of all entries. (c) Group the classes by the pair of the right class and the rational element, and bound each group by (b); by (a) different pairs have positions in different cosets of $K_T(M)$. Integrating over $H \cap G^{(\varrho)}$ and dividing by its measure of $K_T(M)$ turns the sum over the cosets into the fibre pairing at e. □

Theorem 28.10 (Siegel's step). *Let the data be admissible, T and F as in Theorem 28.8, and Z a group with $K_T(M) \leq Z \leq K_T$. Let heads a_p, $p \in J$, have entries below ϱ that are units off T and last entry t with $|t|_v > N$ at every infinite place. Let $(c_i, \bar{n}_i, c_i', \gamma_i)$, $i \in j$, be distinct, with $\gamma_i \in G^{(\varrho)}(E)$, $h(z, \gamma_i z) = t$, $h_{c_i}^{-1}\gamma_i h_{c_i'}$ integral off T and $n_i \in Z$ at T.*

Let R be, for every i, the Gram matrix of the tensors $\rho_p \int W(y) \otimes W((w_p\gamma_i)^{-1}y)^* dy$, and let ρ_p times the head density at T be $r\sqrt{G}$ with $G = \prod_v (|t|_v^2 - N^2)^{k-1}$. Then

$$r^2 \sum_i \Big\| \sum_p \mathcal{A}^U(c_i, \bar{n}_i, c_i'; w_p\gamma_i) \Big\|^2 \le (1 + b^{-6})\, C_{\mathrm{Sie}}^{\sharp}\, \mathrm{Re} \sum_{p,q} R_{pq}\, \frac{J_{\mathrm{last}}(a_p, a_q; t)}{\mathrm{mr}(Z)}.$$

Proof. Group the classes by the point g_i of Siegel's sum, the left class and the part in X_0 of $\gamma_i z$. The translate of a term by the class representatives moves z to the vector $x_i - N^{-1}tz$ with last entry t over g_i (Lemma 28.3(b)), which has norm $(t\bar{t} - N^2)/N$ and is integral off T. *One point.* By Proposition 29.23(c) the positive part of the real part of $\sum_{p,q} R_{pq}\mathrm{fp}(\omega_p, \omega_q; \cdot)$ agrees on the top quadric with a function $\Psi \ge 0$ of compact support, invariant under small translations and under $H \cap K_T(M)$; by Lemma 28.9(c) its value at the point over g_i bounds the classes there, times $w^2 \mu_H(K_T(M))\mu(K_T(M))^2[Z : K_T(M)]$ with $w = r\sqrt{G}$. *Summing the points.* Siegel's formula (Theorem 28.8(c)) bounds the sum of Ψ over the points by $(1+b^{-6})n_X C_{\mathrm{Sie}} G|\pi_N|_T^{-1}$ times the integral of Ψ over the sphere of t; almost every point of the sphere lies on the top quadric, where Ψ is the real part of the sum of the fibre pairings, and by Proposition 29.23(a) the normalised integral over the sphere is $\mathrm{Re}\sum_{p,q} R_{pq} J_{\mathrm{last}}(a_p, a_q; t)$. *Constants.* Dividing by $w^2\mu_H(K_T(M))\mu(K_T(M))^2[Z : K_T(M)] \ge w^2 \mathrm{mr}(Z)/D$ (Theorem 28.8(b)) and using $C_{\mathrm{Sie}}^{\sharp} = D n_X C_{\mathrm{Sie}}$, the factor G cancels against $w^2 = r^2 G$. □

28.5. Sieve systems

Let C be a ring with a measure (a place, or a product of places), $u : C \to U$ a map to entries, ψ a character of C with $\psi(0) = 1$, and $n \ge 0$. A *head* is a sequence of entries. A *sieve system* of length n consists of forms $J_i(p, q)$, $0 \le i \le n+1$, on pairs of heads, weights $\Lambda_i(p) \ge 0$, a decay $\beta(\xi) \ge 0$ for units ξ and a mass bound B_0, such that:

- (S1) for $i \le n$, after giving both heads the entry $u(c)$ at position i, $c \mapsto \psi(\xi c) J_{i+1}$ is integrable; for $\xi = 0$ its integral is $J_i(p, q)$, and for a unit ξ its modulus is at most $\beta(\xi)\Lambda_i(p)\Lambda_i(q)$;
- (S2) for $i < n$, if the heads have entries $u(c)$, $u(c')$ at position i and $\xi = \iota(c) - \iota(c')$ is a unit, then $|J_{i+1}(p, q)| \le \beta(\xi)\Lambda_{i+1}(p)\Lambda_{i+1}(q)$;
- (S3) the top form J_{n+1} is continuous, and positive semidefinite on heads with a common entry at position n;
- (S4) $|J_0| \le B_0$.

At a real place, $\beta(\xi) = \Theta\, b^{2000n^9} \ell_{k-n^7}(1/|\xi|)$ with $\ell_j(r) = r^j \min(1, r)^6$, Θ the scaled volume ratio of the shape, and the weights are $\max(1, |\delta|^{-1})^{n^6}$ over the products $\delta = p_0 \cdots p_l$, $l < j$, if these have modulus at most b^{60n^2}, and 0 otherwise.

Proposition 28.11 (Products, positivity, semicontinuity). (a) *Finitely many sieve systems on $\mathbb{C}$, indexed by a finite set τ, and one sieve system on C have a product, which is a sieve system on $\mathbb{C}^\tau \times C$. Its forms, characters, weights and bounds are the products of those of the factors, each factor read on the components of the heads.*

(b) *Sum the forms J_{i+1} over all pairs of heads from one finite family, with a common entry at position $i \le n$. The sum is integrable in the common entry, its integral*

is the sum of the J_i, and its twisted integrals are bounded by $\beta(\xi)$ and the squared sums of the weights.

(c) *Such a sum of top forms against a positive semidefinite matrix is nonnegative, and its real part is lower semicontinuous in a parameter at each point at which the heads are continuous.*

Proof. (a) Everything is checked factor by factor. (S1): for heads in the product, the integrand over a common entry $c = (c_v, c_C)$ is the product of the integrands of the factors, each integrable, and Fubini on the product measure factors the integral, at $\xi = 0$ into the product of the J_i and at a unit ξ into a product bounded by $\prod_v \beta_v(\xi_v)\Lambda_{i,v}\Lambda_{i,v} \cdot \beta_C(\xi_C)\Lambda_{i,C}\Lambda_{i,C}$, since $|\prod_v a_v \cdot a| \le \prod_v |a_v||a|$. (S3): the entrywise (Hadamard) product of positive semidefinite kernels is positive semidefinite (Schur's product theorem), and a product of continuous functions is continuous. (S2) and (S4) are products of the local bounds. (b) Integrability and the integral follow from (S1) by linearity of the finite sum; for the twisted integral, $|\sum_{p,q} \int \psi(\xi c) J_{i+1}| \le \beta(\xi) \sum_{p,q} \Lambda_i(p)\Lambda_i(q) = \beta(\xi)(\sum_p \Lambda_i(p))^2$. (c) *Positivity*, by descent on i: at $i = n$ the sum is the value of a positive semidefinite form at $(1, \dots, 1)$, by (S3); below, by (S1) it is the integral over a common next entry of the sums for $i + 1$, which are nonnegative. *Semicontinuity*, again by descent: at $i = n$ the sum is continuous in the parameter wherever the heads are, by (S3); below, it is the integral over a common next entry of nonnegative functions that are lower semicontinuous in the parameter, and Fatou's lemma gives lower semicontinuity of the integral. □

Proposition 28.12 (Finite sieve system). *Let the data be admissible, F a filling datum at T with M^{-1} integral off T, S_g a set of chosen places above p_0 and d_w nonzero vectors as in the definition of the fixer. There is a group Z with $K_T(M) \le Z \le K_T$ such that:*

(a) *Z contains the component at T of every integral element whose reduction lies in* $\mathrm{fixer}(S_g, d)$;

(b) *the floor forms $J_j = \mathcal{I}_j/\mathrm{mr}(Z)$ of the Bessel vector of Z for $j \le \varrho$ (Section 29.2), followed by $J_{\varrho+1} = J_{\mathrm{last}}/\mathrm{mr}(Z)$ at the prime above N, form a sieve system at T for the character of the data, the decay $\xi \mapsto [O_T : \xi O_T]^{(k-n^7)/2}[O_T : O_T \cap \xi^{-1}O_T]^{-3} b^{n^9 f}$, the weights with exponent n^6, and $|J_0| \le (1+b^{-6})b^{-6k\#S_g}$.*

Proof. Take Z from Proposition 29.21, so that (a) holds and $\kappa(Z)\,\mathrm{vol}(K_T) b^{6k\#S_g} \le \kappa(K_T)\,\mathrm{vol}(Z)$ for the mass integral $\kappa(Z) = \int |\int_Z \chi_M(yxy^{-1})dy|^2 dx$. *Descent.* For $j < \varrho$ floor j is the integral of floor $j+1$ over a common entry (Theorem 29.6(b)), and at the top the integral of J_{last} over a common last entry is $\mathcal{I}_\varrho$ (Proposition 29.23(d)); the same integrals twisted by $\psi(\xi c)$ are the floor forms with the weight $\psi(\xi Z_{jj})$. *Decay.* By Theorem 29.19, with $t = k - n^7$ and $K' = n^6$, both the twisted integrals and the forms at heads with entries whose frequency $\xi = M^{-1}(p_i^{-1} - q_i^{-1})$ is a unit are at most $\mathrm{mr}(Z)$ times $[O_T : \xi O_T]^{(k-n^7)/2}[O_T : O_T \cap \xi^{-1}O_T]^{-6}(2/|M|_T)^{n^6}$ times the finite head weights; the modulus of the level is $(M^f)^{-2}$ (Lemma 28.5(d)), and $(2(M^f)^2)^{n^6} \le b^{n^9 f}$ because $M \le b^{23}$; the decay keeps only the third power of the index of the denominator. *Top.* J_{last} is continuous, and positive semidefinite on heads with a common last entry, by Proposition 29.23(d). *Mass.* $\mathcal{I}_0$ of the Bessel vector of Z, times fl^2, is at most $\kappa(Z)$,

both being integrals of $|k_0|^2$ up to the factor fl^2 (Theorem 29.6(c)). The mass integral of the full group is at most $\mathrm{vol}(K_T)\,\mathrm{vol}(K_T(M))(1+b^{-6})$ by Theorem 29.20 with $Z = K_T$, $Q = 2$ and $t = n$, since $\mathrm{vol}(K_T) \le 1+2^{-n}$ and $2^{-n}+2^{-n} = 2^{-\varrho} \le b^{-6}$. Hence

$$|J_0| \le \frac{\kappa(Z)}{\mathrm{vol}(Z)\,\mathrm{vol}(K_T(M))} \le \frac{\kappa(K_T)}{\mathrm{vol}(K_T)\,\mathrm{vol}(K_T(M))\,b^{6k\#S_g}} \le (1+b^{-6})\,b^{-6k\#S_g}.$$

□

28.6. Poisson summation and the sieve

For $v \in E$ write $[v]_v = \max(1,|v|_v)^2$ at an infinite place and $[v]_{\mathrm{f}}$ for the norm of the denominator ideal of v. The *constant of Poisson summation* in the entry a is $c_a = \mathrm{vol}(\hat{O}_E)/\prod_v \kappa_{v,a}^2$, with the squared frame scales $\kappa_{v,a} = |\kappa_v|^{2(\varrho-a)}/N$ (κ_v as in Section 25.1) at the infinite places.

Proposition 28.13 (Frequencies and one-sided Poisson summation). *Let E be totally complex with f infinite places.*

(a) *For every finite $X \subset E$, $\sum_{v\in X}\prod_v [v]_v^{-3}[v]_{\mathrm{f}}^{-3} \le (64\cdot 38\,e^4)^f$.*

(b) *For a finite set X of nonzero T-integers and $\|c_v\|^{-1} \le |x_v| \le B$, $B \ge 1$, the sum over $y \in X$ of the product of the decay factors at the infinite places and at T is at most $\prod_v \big(|x_v|^k (bB^2)^3\big)$.*

(c) (*One-sided Poisson summation.*) *Let V be a set of T-integers, read at the infinite places scaled by $r_v > 0$, and let $F \ge 0$ be integrable and lower semicontinuous at these points, with Fourier transform at each unit ξ of modulus at most $\beta(\xi)M_0$. If ε bounds the finite sums of β over the nonzero T-integers, scaled by $2\pi i/(\omega_v r_v)$, then $\sum_{x\in V} F(x) \le \mathrm{vol}(\hat{O}_E)\prod_v r_v^{-2}\,(\varepsilon M_0 + \int F)$.*

Proof. (a) *Squares.* Cover $\mathbb{C}$ by the half-open squares of side $\frac12$ with corners $m/2$, $m \in \mathbb{Z}[i]$; two points of one square are at distance less than 1, distinct corners at distance at least $\frac12$, and the weight $\max(1,|w|)^{-6}$ of a point is at most 64 times that of the lower left corner of its square. The corners are $\frac12$-separated, so by packing their weights add up to at most 38. *Cosets.* If two elements of X differ by an algebraic integer, a product over the places of squares of side $\frac12$ holds at most one of them, since the product of the absolute values of a nonzero algebraic integer is at least 1; so the archimedean weights $\prod_v \max(1,|v|_v)^{-6}$ of such a set add up to at most $(64\cdot 38)^f$. *Denominators.* The numbers with a given denominator $\mathfrak{b}$ lie in $\mathrm{N}\mathfrak{b}$ cosets of O_E: with $(a) = \mathfrak{b}\mathfrak{c}$, multiplication by a maps them into $\mathfrak{c}$ and O_E onto (a), and $\mathfrak{c}/(a)$ has $\mathrm{N}\mathfrak{b}$ elements. Hence the sum is at most $(64\cdot 38)^f \sum_{\mathfrak{b}} \mathrm{N}\mathfrak{b}\cdot \mathrm{N}\mathfrak{b}^{-3}$, and $\sum_{\mathfrak{b}} \mathrm{N}\mathfrak{b}^{-2} \le \zeta_E(2) \le e^{2[E:\mathbb{Q}]} = e^{4f}$, by the Euler product with all coefficients 1. (b) Let $y \ne 0$ be integral off T and $r_v = |c_v y|_v^{-1}$. The decay factor at v is $r_v^k \min(1,r_v)^6$, and that at T is $[O_T : yO_T]^{k/2}[O_T : O_T \cap y^{-1}O_T]^{-3}$. By the product formula, as y is integral off T, $[O_T : yO_T] \le \prod_v |y|_v^2$, so $\prod_v r_v^k\,[O_T : yO_T]^{k/2} \le \prod_v |c_v|^{-k} \le \prod_v |x_v|^k$; and $\min(1,r_v) \le B\max(1,|y|_v)^{-1}$, while the norm of the denominator of y is at most $[O_T : O_T \cap y^{-1}O_T]$. Hence the product of the decay factors is at most $\prod_v (|x_v|^k B^6)$ times $\prod_v [y]_v^{-3}[y]_{\mathrm{f}}^{-3}$, and (a) with $64\cdot 38\,e^4 \le b^3$ gives the bound. (c) Extend F to the adeles: G reads F after scaling by r at the infinite places and projecting to T on the adeles integral off T, and is 0 elsewhere. It is integrable, lower semicontinuous at the diagonal images of the points of V, and invariant under the integral finite adeles that vanish at T. Scaling by

r at infinity and projecting to T takes the measure on the adeles integral off T to $\mathrm{vol}(\hat{O}_E)/\prod_v r_v^2$ times the product measure, the scaling having determinant $\prod_v r_v^2$. By Poisson summation for the lattice E in $\mathbb{A}_E$, in its one-sided form for nonnegative lower semicontinuous integrable functions (approximate G from below by continuous functions of compact support), $\sum_{x\in E} G(x) \le \sum_{y\in E} |\hat{G}(y/\varpi)|$, the dual lattice of E being $\varpi^{-1}E$ for the standard character. By the invariance, $\hat{G}(y/\varpi) = 0$ unless y is a T-integer (translation multiplies the transform by a value of the character, these values must be 1, and $\hat{O}_E$ is its own dual); for a T-integer, $|\hat{G}(y/\varpi)| = \mathrm{vol}(\hat{O}_E)\prod_v r_v^{-2}$ times the modulus of the transform of F at the point given by y and the scales $2\pi i/(\omega_v r_v)$. The term at $y = 0$ is $\mathrm{vol}(\hat{O}_E)\prod_v r_v^{-2}\int F$, and the others are at most $\mathrm{vol}(\hat{O}_E)\prod_v r_v^{-2}\beta(\xi)M_0$, whose sum over the finitely many relevant frequencies is at most εM_0. □

For a head p of length $i \le \varrho$ with entries in a finite set A of units integral off T together with their inverses, the *global weight* is the product over the partial products $\delta = p_0 \cdots p_l$, $l < i$, of $\max(1, \|c_{l,v}\delta\|_v^{-1})^{n^6}$ at each infinite place (or 0 if $\|c_{l,v}\delta\|_v > b^{60n^2}$) and of $[O_T : \delta O_T]^{n^6/2}$ at T (or 0 if $\delta \notin O_T$), where $c_{l,v}$ is the product of the squared scales of the first $l+1$ entries.

Lemma 28.14 (Weights and decay). *Let the data be admissible and $i \le \varrho$.*

(a) *If the global weight of p is nonzero, the partial products are integers of E of modulus at most b^{120n^2} at every infinite place; the weight is at most $b^{120n^8 fi}$, and a head is determined by its partial products.*

(b) *The weights of the heads of length i add up to at most $b^{130n^9 f}$, and the square of the sum is at most $b^{400n^9 f}\prod_{a<i} c_a$.*

(c) *$\prod_v |\kappa_v| = (M\,\mathrm{rd}_E/(2\pi))^f$, and the sums of the decay over nonzero T-integers, at all scales $c_v/\kappa_{v,a}$ with $\|c_v\|^{-1} \le |\kappa_v|/M$, are at most $C_{\mathrm{dec}}\, b^{2002n^9 f}$, with $C_{\mathrm{dec}} = \prod_v \Theta\big(|\kappa_v|^{2\varrho+1}/(MN)\big)^{k-n^7}$.*

Proof. (a) A nonzero weight forces $\delta \in O_T$ and, as δ is integral off T, δ integral; the archimedean condition and $\prod_{a\le l}\kappa_{v,a}^{-1} \le b^{60n^2}$ (from $N \le b^{60n}$ and $|\kappa_v| \ge 1$) give $|\delta|_v \le b^{120n^2}$. By the product formula, $\prod_v \max(1, \|c_v\delta\|^{-1})\cdot[O_T : \delta O_T]^{1/2} \le R^f$ when R bounds $|\delta|_v$ and $\|c_v\|^{-1}$; with $R = b^{120n^2}$ each partial product contributes at most $(b^{120n^2})^{fn^6}$, and there are $i \le n$ of them. The entry p_j is the quotient of two consecutive partial products. (b) By (a) and Lemma 27.4(a) the heads with nonzero weight are at most $((1+2b^{120n^2})^{2f})^i$ in number, and each weighs at most $b^{120n^8 fi}$; the product is at most $b^{130n^9 f}$. The constants of Poisson summation satisfy $\prod_{a<i} c_a \ge b^{-100n^2 f}$, by $\kappa_{v,a} \le b^{46\varrho}$ and $\mathrm{vol}(\hat{O}_E) = (2/\mathrm{rd}_E)^f$; so the square $b^{260n^9 f}$ is at most $b^{400n^9 f}\prod_{a<i} c_a$. (c) The square of $\prod_v |\omega|_v$ is the norm of the different, $|D_E| = \mathrm{rd}_E^{2f}$. For the sums apply Proposition 28.13(b) with $x_v = \kappa_v^{2\varrho+1}/(MN)$, for which $\|c_v/\kappa_{v,a}\|^{-1} \le |x_v| \le B$ with B a power of b of exponent $O(n)$; then $b^{2000n^9 f}$ from the archimedean decay, $b^{n^9 f}$ from the finite one and $(bB^2)^{3f}$ add up to at most $b^{2002n^9 f}$, the factors $\Theta|x_v|^{k-n^7}$ making up C_{dec}. □

Theorem 28.15 (The sieve). *Let the data be admissible, $U \subseteq A$ finite sets of units, and M^{-1} and t, t^{-1} for $t \in A$ integral off T. Let the forms $J_{T,i}$ at T form a sieve*

system of length ϱ with mass bound m_T, and let them be multiplied by the forms of the Bessel systems at the infinite places (Definition 24.14), whose mass bounds are $(1+b^{-10})/\deg_v$. Then the real part of the sum of the products of the top forms over all pairs of heads with entries in A and a common last entry in U is at most

$$\Big(\prod_{a\le\varrho} c_a\Big)\Big(\prod_v \frac{1+b^{-10}}{\deg_v}\, m_T + b^{2403n^9 f} C_{\mathrm{dec}}\Big),$$

where $C_{\mathrm{dec}} = \prod_v \Theta\big(|\kappa_v|^{2\varrho+1}/(MN)\big)^{k-n^7}$ is the constant of Lemma 28.14(c).

Proof. By Proposition 28.11(a) the products form a sieve system over the infinite places and T, for the character $e^{c-\bar c}$ at the infinite places times the character of T, with mass bound $B_0 = \prod_v (1+b^{-10})\deg_v^{-1}\, m_T$, the global weights and the product of the decays. Write Σ_i for the *density sums*: Σ_0 is the real part of the mass, and for $i>0$ Σ_i is the real part of the sum over $t\in A$ of the sum of J_i over the pairs of heads of length i with entries in A and common last entry t, read at the scaled entry t (scaled at v by $\kappa_{v,i-1}$). The sum in the theorem is at most $\Sigma_{\varrho+1}$, the terms being nonnegative (Proposition 28.11(c)).

Removing one entry. Let $i\le\varrho$. The sum of J_{i+1} over the pairs of heads of length i at a common next entry x is a nonnegative integrable function of x, lower semicontinuous at the scaled rational points (Proposition 28.11(b), (c)); its integral is the sum of J_i over these pairs, and its transform at a unit ξ is at most $\beta(\xi)$ times the squared sum Γ_i of the weights. One-sided Poisson summation (Proposition 28.13(c)) with the scales $r_v = \kappa_{v,i}$ gives

$$\Sigma_{i+1} \le c_i\Big(\varepsilon\Gamma_i + \operatorname{Re}\sum_{p,q} J_i(p,q)\Big),$$

with $\varepsilon = C_{\mathrm{dec}} b^{2002n^9 f}$ bounding the decay over the frequencies (Lemma 28.14(c), with $c_v = 2\pi i/\omega_v$, for which $\|c_v\|^{-1}\le|\kappa_v|/M$). In the sum over all pairs of heads of length $i\ge1$, the pairs with equal last entries give Σ_i, and those with different last entries t, s have the frequency $(1/t-1/s)/M$, a nonzero T-integer, so by (S2) each is at most the decay there times the two weights; their sum is at most $\varepsilon\Gamma_i$ (Lemma 28.14(c) with $c_v = M$). Hence $\Sigma_{i+1}\le c_i(\Sigma_i + 2\varepsilon\Gamma_i)$.

Induction. With $\Gamma = b^{400n^9 f}$ we have $\Gamma_i \le \Gamma\prod_{a<i} c_a$ (Lemma 28.14(b)), and induction on i gives

$$\Sigma_i \le \Big(\prod_{a<i} c_a\Big)\big(B_0 + 2i\Gamma\varepsilon\big), \qquad i\le\varrho+1.$$

At $i=\varrho+1$, $2(\varrho+1)\Gamma\varepsilon \le b^{n^9 f} b^{400n^9 f} b^{2002n^9 f} C_{\mathrm{dec}} = b^{2403n^9 f} C_{\mathrm{dec}}$, since $2(\varrho+1)\le k\le b$. □

28.7. The main term

The *normalising constant* r is the product over the infinite places of $\sqrt{c_G}$ and the powers of the squared frame scales, divided by the number of Bessel coordinates, where $c_G = \operatorname{vol}\mathrm{U}(k)^2/(\operatorname{vol}\mathrm{U}(k+2\varrho+1)\operatorname{vol}\mathrm{U}(k-1))$ is the open cell constant.

Lemma 28.16 (The constants of the main term). *Let the data be admissible, $k = k'+1$, $w = \operatorname{vol}(\hat{O}_E)^n (\#I/\Lambda)^2/c_G^f$ with Λ the product of the powers of the squared frame scales, and $u = b^{-1}$.*

(a) $\prod_{a\le\varrho} c_a = r^2 w$, *and for a head with entries units off* T *and last entry* t *with* $|t|_v > N$, *the product of the scalars of the normalised orbital integrals at the infinite places and of the head density at* T *is* $r\big(\prod_v(|t|_v^2 - N^2)\big)^{(k-1)/2}$.
(b) $C_{\mathrm{Sie}}^{\sharp}\, w \le (1+u^9)\mathcal{D}_{\mathrm{main}} \deg\, \nu^2$ *and* $C_{\mathrm{Sie}}^{\sharp}\, w\, C_{\mathrm{dec}} \le b^{n^9 f} \#I\, \mathcal{D}_{\mathrm{id}} \nu^2$.
(c) *If* $\#S_g \le f$, *then*

$$(1+u^6)C_{\mathrm{Sie}}^{\sharp}\prod_a c_a\Big(\prod_v \frac{1+u^{10}}{\deg_v}\cdot\frac{1+u^6}{b^{6k\#S_g}} + b^{2403n^9 f} C_{\mathrm{dec}}\Big) \le r^2(1+u^4)\, b^{-6k\#S_g}\mathcal{D}_{\mathrm{main}}\nu^2.$$

Proof. (a) One factor $\kappa_{v,a}$ splits off each power in Λ, and the rest is the definition of r; for the head, the product formula cancels the entries of the head between the infinite places and T, and splitting off the scale of the last frame vector in r leaves $(\prod_v(|t|_v^2 - N^2))^{(k-1)/2}$. (b) The first is the bound for the scaled Siegel constant required by admissibility (Proposition 25.7), rewritten. For the second, write $\mathrm{rd}_E = s^2$, $Q = Ms^2/(2\pi)$, so that $\prod_v|\kappa_v| = Q^f$ and $\mathrm{vol}(\hat{\mathcal{O}}_E) = (2/s^2)^f$. Apart from a common factor both sides are fth powers, of A and of $b^{n^9}B$, and at one place A is B times $(2/s^2)^{n+2\varrho^2}(s/s^{2\varrho})x^{-n^7}$ with $x = Q^{2\varrho+1}/(MN)$; these factors are at most b^{n^9}, as $2/s^2 \le b$, $s/s^{2\varrho} \le 1$ and $x^{-1} \le b^{23}b^{60n}$. (c) The main part is at most $(1+u^9)^2(1+u^6)r^2 b^{-6k\#S_g}\mathcal{D}_{\mathrm{main}}\nu^2$ by (a), (b) and $(1+u^{10})^f \le 1+u^9$ (as $f \le b/1000$). The error part is at most $b^{2403n^9 f} b^{n^9 f} b^{2kf}\mathcal{D}_{\mathrm{id}}\nu^2 r^2$, as $\#I \le b^{2kf}$; since $k \ge 512000n^9$ and $\#S_g \le f$, $2403n^9 f + n^9 f + 2kf + 10 + 6k\#S_g \le 11kf$, and $b^{11kf}\mathcal{D}_{\mathrm{id}} \le \mathcal{D}_{\mathrm{main}}$ bounds the error by $u^{10}r^2 b^{-6k\#S_g}\mathcal{D}_{\mathrm{main}}\nu^2$. Finally $(1+u^6)\big((1+u^9)^2(1+u^6)+u^{10}\big) \le 1+u^4$ for $0 \le u \le 10^{-3}$, all the powers on the left being at most $u^4/100$. □

Theorem 28.17 (Main term). *Let* E *carry Weil data, the filling data be admissible,* S_g *a set of chosen places above* p_0 *and* d_w *nonzero vectors.*

(a) (*Band limits.*) *Let* $\sigma \in G^{(\varrho)}(E)$ *have last frame product* t. *The product over the infinite places of the top forms of the Bessel systems at the heads* (p,t), (q,t) *is the Gram matrix of the orbital integrals* $\int W(y)\otimes W((w_p\sigma)^{-1}y)^* dy$ *of the cut Bessel function, each weighted by*

$$\prod_v \sqrt{c_G}\,(|x_\varrho|^2-1)^{(k-1)/2}|I|^{-1}\prod_{a<\varrho}|x_a|^{k+1+2(\varrho-a-1)},$$

with x *the scaled entries of the head at* v.
(b) *For finitely many distinct* $(c_i, \bar{n}_i, c_i', \sigma_i)$ *with* $\sigma_i \in G^{(\varrho)}(E)$, *last entries* t_i *in a finite set of units off* T *with* $|t_i|_v > N$, *positions integral off* T *and reductions in* $\mathrm{fixer}(S_g, d)$,

$$\sum_i \Big\| \sum_{p\in\mathcal{T}^\varrho} \mathcal{A}^U(c_i, \bar{n}_i, c_i'; w_p\sigma_i)\Big\|^2 \le (1+b^{-4})\, b^{-6k\#S_g}\,\mathcal{D}_{\mathrm{main}}\,\nu^2.$$

(c) *The link second moment of the terms with* $\mathfrak{g} \ne 0$ *over the open elements of* $\mathrm{fixer}(S_g, d)$ *is at most* $(1+b^{-4})b^{-6k\#S_g}\mathcal{D}_{\mathrm{main}}\nu^2$.

Proof. (a) At each infinite place both heads end in the last frame product of σ, so by Theorem 29.17(b) the top form is the inner product of the normalised orbital integrals; the scaled orbital integrals of the product vector are the products of

the local ones, and the product over v of the inner products is the Gram matrix of the products.

(b) *Places.* Enlarge T so that it contains the chosen places, the places above N and M and those where the lattice basis or the entries of the heads are not integral, and is stable under conjugation (Lemma 28.5(a)); take a filling datum at T (Theorem 28.6(a)) and Z from Proposition 28.12, which contains the components of the n_i because their reductions lie in the fixer. *Siegel's step.* Group the classes by their last entry t. For each t, Theorem 28.10 applies with R the product of the top forms at the infinite places, which by (a) is the Gram matrix of the weighted orbital integrals, and with ρ_p the product of the scalars, for which ρ_p times the head density is $r\sqrt{G}$ by Lemma 28.16(a). It bounds r^2 times the sum by $(1 + b^{-6})C_{\mathrm{Sie}}^{\sharp}$ times the real part of the sum over the heads of the product of the top forms at the infinite places and of $J_{\mathrm{last}}/\mathrm{mr}(Z)$ at T. *The sieve.* Summed over t this is the sum of Theorem 28.15 for the system of Proposition 28.12, with $m_T = (1 + b^{-6})b^{-6k\#S_g}$, so r^2 times the left side is at most the left side of Lemma 28.16(c). Dividing by $r^2 > 0$ gives the bound.

(c) The left side is a sum of squared norms of blocks; it suffices to bound finite sums of nonzero blocks. Let s be open in the fixer. By Lemma 28.2 every term over (c, s, c') lies in the big cell, so by Proposition 28.4 each nonzero block of one link is a sum of orbit sums over the heads with entries in $\mathcal{T}$, for one σ. A block has two classes, a reduction and a link; distinct blocks give distinct tuples $(c, c', \bar{n}, \sigma)$, as $\bar{n}$ is the reduction and σ determines the link. Choosing T so that the elements of $\mathcal{T}$ are units off T and M^{-1}, N^{-1} are integral off T, (b) applies to the finite family of nonzero blocks, with last entries in $\mathcal{T}$. □

29. Local band limits

The sieve of Theorem 28.15 asks, at each place, for a system of forms on pairs of heads: each form the integral of the next over a common entry, with decay in the frequency, a mass close to $1/\deg$ at the bottom, and a positive top form. This section constructs these forms and proves the bounds, at a real place for the cut Bessel functions of Section 24 (Theorem 29.17) and at a finite set of places for the finite Bessel vectors (Theorems 29.19 and 29.20). Both cases rest on one abstract argument: the forms are integrals over a position and two frames of products of correlations; Cauchy–Schwarz pairs each frame with itself; a weight made of *depths* separates the two frames; and the gains from a frequency, from two different heads or from a deep vector, integrated over dyadic shells of positions, give the decay.

The abstract part (Sections 29.1 to 29.4) is proved for a place, a place datum and a gauge; the two kinds of place supply these data in Section 29.5 and Section 29.6. The order of the argument is: sizes and lengths; the forms as integrals over a position and two frames; Plancherel's formula on quadrics; depth; one frame; the separation of the two frames and the gains at one position; and the sum over the shells of positions.

29.1. Places, place data and floors

A *place* is a locally compact ring A with involution and Haar measure, with a trace-one element τ, a skew unit δ, a multiplicative *modulus* $|u|$ (the factor by which u scales the volume), a Haar measure on the skew elements, a character ψ and a constant $c_A \geq 1$ in Plancherel's formula. Two kinds occur: the complex place $\mathbb{C}$, with $\tau = \frac{1}{2}$, $\delta = i/2$, modulus $|x|^2$, character $x \mapsto e^{x-\bar{x}}$ and $c_{\mathbb{C}} = \pi^2$; and an *integral place*, a finite set of finite places read as one ring, with an open compact self-dual subring R of volume one.

Let V be the space with basis e_a, e_a^* ($a < \varrho$, hyperbolic pairs), the middle vectors and z, with Gram matrix of ϱ hyperbolic planes and a middle block; G is its unitary group and $n = \varrho + 1$. The vectors e_a, e_a^* have *floor* a, z has floor ϱ and the other middle vectors floor $\varrho + 1$. Write $G^{(l)}$ for the isometries fixing the basis vectors of floor below l, $R^{(l)}$ for those unitriangular on the first l frame vectors $f_0 = e_0, \dots, f_{\varrho-1} = e_{\varrho-1}$, $f_\varrho = z$, and $H = G^{(\varrho+1)}$. The *Heisenberg group* of floor a moves e_a^* by a vector of the higher floors and a skew multiple of e_a; it carries a Haar measure η_a. The Heisenberg element with vector ω and skew parameter β sends e_a^* to $e_a^* + \omega + (\beta - \tau h(\omega, \omega))e_a$. Write $U_{(<l)}$ for the isometries unitriangular when all basis vectors of floor at least l are put on one middle level, and $Y^{(l)}$ for the vectors orthogonal to e_b, $b < l$, modulo the span of these.

Definition 29.1 (Place datum). A *place datum* F consists of a basis of the middle block made of $m + 1$ hyperbolic pairs and orthonormal vectors indexed by ι, a self-adjoint unit M (the *level*), an open set of *close pairs* of vectors containing 0 with compact closure, a compact *box* $K \subseteq G$, a measurable set of *positions*, a measure on $G^{(\varrho)}$ carried to the measure of the top quadric by $\sigma \mapsto \sigma^{-1}z$, and a right H-invariant measure μ_H carried by H such that, for every g, the h with $uhg \in K$ for some $u \in U$ have μ_H-mass at most 1.

The *floor measure* θ_l is carried by $G^{(l)}$; for $l \leq \varrho$ the map $\sigma \mapsto \sigma^{-1} f_l$ carries it to the measure of the quadric through f_l, for $l < \varrho$ it lives on the big cell, $\theta_{\varrho+1} = \mu_H$, and higher floors carry a unit mass at 1. The *group measure* is $\mathsf{M} = (\eta_{\varrho+1} * \theta_{\varrho+1}) * \cdots * (\eta_0 * \theta_0)$, and m_j is the same convolution over the floors below j. A point g is *near at* j if $R^{(j)}g$ meets K. A vector ω of $Y^{(l)}$ (the vectors orthogonal to the e_b, $b < l$, modulo their span) is *stable* at g, g' if $(g^{-1}\omega, g'^{-1}\omega')$ is a close pair for some lift ω'; $\mathfrak{d}_a(x, y)$ is the set of products $h(e_a, \omega)$ with $\omega \in Y^{(a)}$ stable at x and xy, and the *depth* of e_a is $\mathrm{dp}_a(x, y) = \mathrm{vol}(\mathfrak{d}_a(x, y))^{-1}$. With X' the span of the orthonormal vectors outside two blocks of rank $2m + 2$ and the *volume exponent* $s = \dim X' - (3n + 1 + m)$, the *depth weight* of length j is $\wp_j(x, y) = \prod_{a<j} \mathrm{dp}_a(x, y)^{-s}$.

Proposition 29.2 (Floor measures). *Let F be a place datum.*

(a) *For $l \leq \varrho$ and $s \in G^{(l)}$, $\sigma \mapsto s\sigma^{-1} f_l$ carries θ_l to the quadric measure of floor l; for $l < \varrho$, $\theta_{l+i} * \cdots * \theta_l$ is the convolution of a law on the Weyl elements of floor l, the floors above, and η_l.*

(b) *For $l + i = \varrho + 2$ the partial group measure from floor l on is right invariant under $G^{(l)}$; in particular M is right invariant.*

(c) *Left translation by the diagonal element $w_q^{-1}w_p$ multiplies integrals for $\theta_{\varrho+1} * \cdots * \theta_0$ by $\prod_{a<j} |p_a/q_a|^{k+2(\varrho-a)}$.*

(d) *If* M *is a Haar measure and* $\Phi \geq 0$ *is left invariant under* $G^{(j)}$*, then* $\Phi(w_p^{-1} g)$ *and* $\Phi(g)$ *have the same integral for* m_j*.*

Proof. (a) Let V_l be the span of the basis vectors of floor at least l, enumerated as the pair e_l, e_l^* followed by the higher floors; in this basis the form is a hyperbolic plane plus the form of the higher floors, the pair of floor l being orthogonal to the higher pairs. For $l < \varrho$ the quadric $h(v, v) = 0$ of V_l has the flat chart $((t, tu), \beta)$, t a unit, u a vector of the higher floors and β skew, and this point is the image of e_l under the Heisenberg element of (u, β) times the Weyl element $w_l(t)$ of floor l (with $c = h(u, u)$ the products agree, $h(tu, tu) = \bar{t} t c$). Disintegrations of the volume of V_l along $h(v, v)$ are unique, and both the quadric measure and the image of the chart measure are members of such a disintegration at 0. A chart point is the value of $\sigma \mapsto \sigma^{-1} f_l$ at a point of the big cell, the inverse of $w_l(t)$ times a Heisenberg element; the inverse of the Heisenberg element of (u, β) has parameters $(-u, c - \beta)$ with c independent of β, and scaling the higher floors by t has modulus $|t|$ to the power of their dimension, both changes of variables keeping the measure. For $s \in G^{(l)}$: s preserves the volume of V_l and the form, hence the disintegration. The same computation writes $\theta_{l+i} * \cdots * \theta_l$ as the law on the inverse Weyl elements of floor l, which commutes with the higher floors (carried by $G^{(l+1)}$), convolved with the higher floors and η_l. (b) By induction on the number of floors. Write the measure as $\rho * \theta_l$; for a test function, the ρ-average is left invariant under $R^{(l+1)} \cap G^{(l)}$, and its θ_l-integral does not change under right translation by $R^{(l)} \cap G^{(l)}$: in $x\gamma\nu = ((x\gamma)\nu(x\gamma)^{-1})x\gamma$ the factor ν drops out because the function is invariant under the conjugate, and γ and γ^{-1} carry the floor measure to the same quadric measure. An element of $R^{(l+1)} \cap G^{(l)}$ is un with $u \in G^{(l+1)}$, absorbed by induction, and n in the Heisenberg group of floor l, absorbed by η_l. Since $G^{(0)} = G$, M is right invariant. (c) The quotient $w_q^{-1} w_p$ is diagonal, with entry p_a/q_a at e_a and the conjugate inverse at e_a^* for $a < j$. By induction it suffices to compare the translations by the elements of lengths $j + 1$ and j: they differ by a factor that takes $w_j((p_j/q_j)c)^{-1}$ to $w_j(c)^{-1}$; split off the Weyl element of floor j, integrate first over its parameter c, and substitute rc for c, which costs $|r|$ to the power of the dimension $k + 2(\varrho - j)$ of the free variables of floor j; almost every parameter is a unit. (d) Factor $\mathsf{M} = \rho * \nu$, with ρ the floors $j, \ldots, \varrho + 1$, carried by $G^{(j)}$, right invariant and commuting with w_p (the floors from j on are fixed by w_p). Take $f > 0$ integrable for M; for any μ with $\mathsf{M} = \rho * \mu$, the integral of Φ for μ is the integral of Φf for M divided by the ρ-average of f, an expression that does not mention μ. The translate of ν by w_p is such a μ, as M is left invariant. □

A *gauge* for F is a filtration of A by compact sets Sz_j (*numbers of size* j, at most L^j in modulus at a real place, with $L^j x$ integral at the finite places for a scale L), with a constant $\Lambda \geq 2$ whose powers bound moduli and volumes, such that $m \leq 9n$; a vector or matrix has size j if all its entries have. A sequence p of units is a *head of length* j if $(p_0 \cdots p_{l-1})^{-1}$ has size 1 for $l \leq j$. The *position ball* is the set of positions x with $(x - 1)/M$ of size $99n$.

Lemma 29.3 (Sizes and lengths). *For a gauge:*

(a) *Sizes add under products, a sum of at most* $r + 1$ *terms of size* i *has size* $i + 1$*, and* $-x$*,* $\bar{x}$ *have the size of* x*; the Gram matrix and its inverse have size* 1*. A matrix of size* i *maps vectors of size* j *to vectors of size* $i + j + 1$*, the product*

of matrices of sizes i and j has size $i + j + 1$, and if an isometry has size i, its inverse has size $i + 4$.

(b) *If v, w have sizes i, j, then $h(v, w)$ has size $i + j + 3$; if moreover $(v - v')/r$ and $(w - w')/r$ have sizes i, j for a self-adjoint unit r of size 1, then $(h(v', w') - h(v, w))/r$ has size $i + j + 6$.*

(c) *A square matrix of order $j \le \varrho$ and size 7 has determinant of size $8j$; of order $j + 1$ with last row of size 9, of size $9(j + 1)$.*

(d) *The quotient $w_q^{-1} w_p$ of two Weyl elements is diagonal; the number of basis vectors of floor at least $a + 1$ is $k + 1 + 2(\varrho - a - 1)$. The group $R^{(j)}$ keeps $Y^{(l)}$ and congruence modulo the e_b, $b < l \le j$, and $U_{(<l)}$ acts as 1 on $Y^{(l)}$; the form $h(\omega, o)$ with $o \in Y^{(l)}$ depends only on the class of ω.*

Proof. (a) The first three statements are axioms of a gauge, as are the sizes of 0, 1, of the Gram matrix J and of J^{-1}; a number of size i has every larger size, and a difference of two numbers of size i has size $i + 1$. An entry of Av, for A of size i and v of size j, is a sum of r products of size $i + j$, hence of size $i + j + 1$; an entry of a product of two matrices is such a sum as well. For an isometry x of J we have $Jx^{-1} = x^*J$, so $x^{-1} = J^{-1}(x^*J)$: the matrix x^* has size i, the product x^*J size $i + 1 + 1$, and $J^{-1}(x^*J)$ size $1 + (i + 2) + 1 = i + 4$.

(b) The form is $h(v, w) = v^*(Jw)$. The vector Jw has size $1 + j + 1$, and the product with v^*, whose entries have the size i of those of v, has size $i+(j+2)+1 = i + j + 3$. For the difference, $v' = v - r \cdot (v - v')/r$ has size $(1 + i) + 1 = i + 2$, and, as r and r^{-1} are self-adjoint,

$$\frac{h(v', w') - h(v, w)}{r} = -h\Big(v', \frac{w - w'}{r}\Big) - h\Big(\frac{v - v'}{r}, w\Big).$$

The first term has size $(i + 2) + j + 3 = i + j + 5$, the second size $i + j + 3$, which is at most $i + j + 5$, and the difference costs one more: $i + j + 6$.

(c) Induction on j. For $j = 0$ the determinant is 1, of every size. For $j \ge 1$ expand along the first row: $\det Z = \sum_b (-1)^b Z_{0b} \det Z^{(0b)}$, where the minor $Z^{(0b)}$ of order $j - 1$ has size 7 and, by induction, determinant of size $8(j - 1)$. Signs do not change sizes, so each of the $j \le \varrho + 1$ terms has size $7 + 8(j - 1)$, and the sum has size $7 + 8(j - 1) + 1 = 8j$. For the second statement expand along the last row: each term is an entry of size 9 times a minor of order j with rows of size 7, of size $8j$, hence of size $9 + 8j$, and the sum has size $8j + 10 \le 9(j + 1)$ when $j \ge 1$; for $j = 0$ there is one term, Z_{00}, of size 9.

(d) Column by column, w_p sends e_a to $p_a e_a^*$ and e_a^* to $(\bar{p}_a)^{-1} e_a$ for $a < j$, and w_q^{-1} sends them back with the factors of q; so $w_q^{-1} w_p$ is diagonal, with entry p_a/q_a at e_a and its conjugate inverse at e_a^*. The basis vectors of floor at least $a + 1$ are the e_b and e_b^* with $b > a$, $2(\varrho - a - 1)$ in number, and the $k + 1$ vectors of the middle block. An element s of $R^{(j)}$ changes each e_b, $b < j$, by a combination of earlier ones. If ω is orthogonal to the e_b, $b < l \le j$, so is $s\omega$, because $h(e_b, s\omega) = h(s^{-1}e_b, \omega)$ and $s^{-1}e_b$ is again a combination of the e_c, $c \le b$. If $\omega - \omega'$ is a combination of the e_b, $b < l$, so is $s(\omega - \omega')$: in a row of s of level at least l only the columns e_b, $b < l$, can contribute, and there the entries of s vanish. For $u \in U_{(<l)}$ and ω orthogonal to the e_b, $b < l$, the coordinates of $u\omega$ of level at least l are those of ω: an index of level at least l is some e_b^* or has floor at least its level, and u acts as 1 there. Finally, if o is orthogonal to the e_b, $b < l$, then $h(\omega + \sum_{b<l} c_b e_b, o) = h(\omega, o)$. □

Proposition 29.4 (Two gauges, balanced frames). *Let Γ be a gauge for F.*

(a) *If g is near at j, each e_a, $a < j$, is balanced at g with exponent 5: some lift and some dual of e_a are moved by g^{-1} to vectors of size 5.*

(b) *Let g, g' be isometries, r a self-adjoint unit of size 1, and suppose that the lengths at g and g' differ by at most L^t modulo r on $Y^{(j)}$, and that for $a < j$ the class of e_a has lifts of size c at both points and a short dual that looks nearly the same from both points. Then they differ by at most $L^{t+(4c+12)j}$ modulo r on the whole space, and $(g'g^{-1} - 1)/r$ has bounded size.*

(c) *The leading minors of the matrix of frame products of a point near at n have size $9n$; over a head p of length j the products $p_0 \cdots p_{l-1}$ times the duals have bounded size, and so do g and $w_p^{-1} g$ (Cramer's rule).*

Proof. (a) If ξg lies in the box for some $\xi \in R^{(j)}$, take the lift $\xi^{-1} e_a$ and the dual $\xi^{-1} e_a^*$; read at g they are columns of $(\xi g)^{-1}$, which has size $1 + 4$ by Lemma 29.3(a), the box having size 1.

(b) *One step.* Suppose the lengths differ by at most L^t modulo r on $Y^{(a+1)}$, that e_a has lifts f, f' of size c at g, g', and a dual ζ in $Y^{(a)}$ with $h(e_a, \zeta) = 1$, $g^{-1}\zeta$ and $(g^{-1}\zeta - g'^{-1}\zeta')/r$ of size c for a lift ζ'. For $\omega \in Y^{(a)}$ with $g^{-1}\omega$ of size i write $\omega = \alpha\zeta + \omega_0$ with $\alpha = h(e_a, \omega)$, so that ω_0 is orthogonal to e_a. The class of ω_0 in $Y^{(a+1)}$ has a lift ω_0' with $g'^{-1}\omega_0'$ close to $g^{-1}\omega_0$ modulo r. Let μ be the coefficient of e_a in $\omega_0' - \omega_0$; pairing with the dual ζ bounds μ/r by Lemma 29.3(b), and $\alpha\zeta' + \omega_0' - \mu f'$ is a lift of ω whose reading at g' is close to $g^{-1}\omega$ modulo r. The sizes add up to $i + t + 4c + 12$. *Descent.* Apply the step for $a = j - 1, \dots, 0$. *The matrix.* On the whole space a class has one lift, so the columns of g read at g' give those of $(1 - x^{-1})/r$ for $x = g^{-1}g'$, and $(x - 1)/r$ is the adjoint of $(x^{-1} - 1)/r$ for the form, of size four more.

(c) *Minors.* If ξg and $\xi'\gamma^{-1} g$ lie in the box with $\xi, \xi' \in R^{(l)}$, then $x = (\xi g)(\xi'\gamma^{-1} g)^{-1} = \xi\gamma\xi'^{-1}$ has size 7, and the matrix of frame products of x has the same leading minors of order at most l as that of γ; in Jx the row of e_a is the row of e_a^* of x, and all rows of a leading minor but the last belong to the e_a, $a < \varrho$. By Lemma 29.3(c) the minors have size $9n$. *Cramer.* Let sg and $\eta w_p^{-1} g$ lie in the box with $s, \eta \in R^{(j)}$, and $a < j$. The block of $s w_p \eta^{-1}$ on the rows e_b^* $(b < a)$ and i and the columns e_d $(d \le a)$ has size 7, being part of $(sg)(\eta w_p^{-1} g)^{-1}$, and it factors as LC with L the same block of $s w_p$, lower triangular with the p_b on the diagonal (the row of s at e_b^* agrees with that of 1 outside the columns e_d^*, $d < b$), and C the block of η^{-1} on $e_0, \dots, e_a$, unitriangular. Hence $p_0 \cdots p_a$ times the ith entry of $s e_a^*$ is a minor of size 8ϱ. For a head the inverses of the partial products have size 1, so the entries of $s e_a^*$ have size $8\varrho + 1$, and p_a, p_a^{-1}, quotients of consecutive partial products, have size $10n$. Then w_p^{-1} has size $10n$ (its entries are 0, 1, $\bar{p}_a$, p_a^{-1}), and if s also lies in $U_{(<j)}$, comparing the gauges at 1 and s^{-1} by (b) (they agree on $Y^{(j)}$, where s acts as 1, and the e_a^* are short duals) bounds s^{-1}; writing $g = s^{-1}(sg)$ gives size $60n^2$ for g and $99n^2$ for $w_p^{-1} g$. □

29.2. Floor forms

A function $W\colon G \to \mathbb{C}^I$ is a *Bessel function of frequency* η if it is bounded, measurable and $W(ux) = \psi(\eta\sigma(u))W(x)$ for $u \in U$, $\sigma(u)$ the sum of the entries above the diagonal of the frame block; it is a *cut Bessel function* for F if its frequency is M^{-1} and $W(x) \neq 0$ forces $ux \in K$ for some $u \in U$. The *correlation from floor j on* of a kernel k is $k_j(g,y) = \int k(\sigma g, \sigma y)\, d(\theta_{\varrho+1} * \cdots * \theta_j)(\sigma)$. For heads p, q the *head density* is $\kappa_j(p,q) = \prod_{a<j} |p_a q_a|^{d_a/2}$, $d_a = k+1+2(\varrho-a-1)$, and

$$\mathcal{I}_j(p,q;\omega) = \kappa_j(p,q) \int\!\!\int \Phi_j(x,g;\omega)\, d\mathsf{m}_j(g)\, d\mathsf{M}(x),$$

$$\Phi_j(x,g;\omega) = \iint \omega(\sigma\tau^{-1})\, k_{j+1}(\sigma g, \sigma g x)\, \overline{k'_{j+1}(\tau w_p^{-1} g, \tau w_p^{-1} g x)}\, d\theta_j(\tau)\, d\theta_j(\sigma),$$

is the *form of floor j* with weight ω, where $k(g,y) = \langle W(g), W(y)\rangle$ and k' is the same kernel twisted by $w_q^{-1} w_p$. Its integrand at a position x has an L^1 norm $\mathsf{F}_j(x;\omega)$, bounded by the *majorant* $\mathsf{F}_j^\sharp(x)$, the integral over m_j of the product of the two integrals over floor j of the moduli of the correlations. A point y is *active* at x if its correlation with yx is nonzero and both points are near at j after the Weyl elements are undone; x has a *deep vector* of depth D if every active point has an e_a, $a \geq j$, of depth at least D.

Lemma 29.5 (Pairing kernels and partial correlations). *Let W be a cut Bessel function and $w = w_q^{-1} w_p$. The kernel $k(g,y) = \langle W(g), W(wy)\rangle$ is a* pairing kernel *at floor j: it is measurable; for $n, n' \in U_{(<l)}$, putting n before g and n' before y multiplies the correlation from floor $l+1$ on by $\psi((\sigma_0(wn'w^{-1}) - \sigma_0(n))/M)$; w commutes with $G^{(j)}$; the integral of $|k(xg,xy)|$ over the floors from l on is bounded independently of g, y; and a nonzero correlation from floor l on forces g near at l. For a pairing kernel:*

- (a) *the correlation from floor l on is an absolutely convergent integral, jointly measurable, equal to the integral over θ_l of the correlation from floor $l+1$ on, and its modulus is at most the integral over θ_l of the modulus of the next one;*
- (b) *the correlation from floor $l \geq j$ on is invariant under left translation of both points by the same element of $R^{(l)} \cap G^{(j)}$;*
- (c) *if the σ on each floor $a < n$ with σg near at $a+1$ have mass at most V for θ_a, then for every c the g with gc near at $\varrho+2$ have mass at most V^{n-l} for the floors from l on.*

Proof. For the cut Bessel function: conjugation by an element of the floor group keeps the entries summed by σ_0, and w is diagonal; the first scalar leaves the inner product conjugated, that is inverted. The bound of the mass comes from the floor masses (Proposition 29.8(a) below gives a finite V), and a nonzero value forces nearness because W vanishes off the translates of the box. (a) The measure of the floors from l on is that from $l+1$ on convolved with θ_l, and the claim is Fubini's theorem on the product, the integrand being integrable by the mass bound. (b) Descending induction on l: above the top floor the measure is the unit mass and $s \in U$ commutes with w, so the character of the equivariance is 1; below, the correlation is the integral of a function invariant by induction, and by Proposition 29.2(b) the translate by s has the same law. (c) Induction on the number of floors: peel off the lowest floor l, of mass at most V; if σc is

not near at $l+1$, no point $x\sigma c$ with x on the higher floors is near; at the top only μ_H is left, of mass at most 1 on near points. □

Theorem 29.6 (Descent and mass). *Let F have a gauge and W be a cut Bessel function; let p, q be sequences of units.*

(a) $|\mathcal{I}_j(p,q;\omega)| \le \kappa_j(p,q)\int \mathsf{F}_j(x;\omega)\,d\mathsf{M}(x)$, *and* $\mathsf{F}_j(x;\omega) \le \mathsf{F}_j^{\sharp}(x)$ *if* $|\omega| \le 1$.

(b) (*Descent.*) *Let $a < \varrho$, let the majorant of floor a be integrable for M, and let φ be measurable with $|\varphi| \le 1$. Then $c \mapsto \varphi(c)\,\mathcal{I}_{a+1}((p,c),(q,c);1)$, with the entry a of both heads replaced by c, is integrable, and its integral is $\mathcal{I}_a(p,q;\varphi(Z_{aa}))$, where $Z_{aa}(x) = h(f_a, x f_a)$. In particular this holds for $\varphi(c) = \psi(\eta c)$.*

(c) (*Mass.*) $\mathcal{I}_0(p,q;1) = \int |k_0(1,y)|^2\,d\mathsf{M}(y)$.

(d) (*Orbital integrals.*) *If μ is a Haar measure, W is continuous, p, q are heads of length ϱ and $x \in G^{(\varrho)}$, then $\langle J(w_p x), J(w_q x)\rangle$, with $J(\gamma) = \int W(y)\otimes W(\gamma^{-1}y)^{*}\,d\mu(y)$, is the double integral of $\langle W(y), W(y')\rangle$ times the conjugate of $\langle W((w_p x)^{-1}y), W((w_q x)^{-1}y')\rangle$. If $J(\gamma) \ne 0$, the leading minors of the frame products of γ have size $9n$, and if $W(y)W((w_p x)^{-1}y) \ne 0$ the entries of y have size $60n^2$.*

Proof. (a) is the triangle inequality, the majorant bounding the double integral over floor j by the product of the moduli (the integrands of the two factors of the majorant are the moduli of the two correlations).

(b) *The chart.* By Proposition 29.2(a), θ_a is the law of $(w_a(c))^{-1}n$ for c with a density ρ and n under η_a times a measure ζ carried by $R^{(a+1)} \cap G^{(a)}$. *The weight.* For x in $R^{(a+1)} \cap G^{(a)}$, which fixes f_a, multiplying y by x on either side leaves $Z_{aa}(y) = h(f_a, y f_a)$ unchanged; and $w_a(c)$ has the Gram entry c. Hence the weight $\varphi(Z_{aa})$ becomes $\varphi(c)$ after the substitution $\tau = (w_a(c))^{-1}n\sigma$, $g' = n\sigma g$. *The integrands.* The Weyl element of floor $a+1$ of the head with entry c is $w_a p \cdot w_a(c)$, the twist $w_q^{-1}w_p$ is unchanged, and the new factor of the head density gives the density $\rho(c)$; with these, the integrand of $\mathcal{I}_{a+1}$ at the heads with entry c and the integrand of $\mathcal{I}_a$ have the same image measure. Fubini applies because the majorant is integrable and $|\varphi| \le 1$, the integral of the modulus being at most that of the majorant.

(c) Below floor 0 there is only the point 1, and the Weyl elements of length 0 are trivial, so the two factors of the integrand are complex conjugates and the form is the integral of $|k_0(1,y)|^2$.

(d) The coordinate i of $J(w_r x)$ is the integral of $W(y)_i\overline{W((w_r x)^{-1}y)}$; for a head r the integrand is continuous by hypothesis and vanishes off the compact set of points of size $60n^2$, so Fubini gives the double integral. If $J(\gamma) \ne 0$, then $W(y)$ and $W(\gamma^{-1}y)$ are both nonzero for some y, so unipotent translates of y and $\gamma^{-1}y$ lie in the box, and Proposition 29.4(c) bounds the minors. If $W(y) \ne 0$ and $W((w_p x)^{-1}y) \ne 0$, the factor $x^{-1} \in G^{(\varrho)}$ does not affect nearness at ϱ, and Proposition 29.4(c) solves the unipotent factors from the minors of the head. □

29.3. Quadrics, depth and the gains

The equation of a quadric is solved in a block of orthonormal coordinates: a measure ρ on V_b is *solved* in a block if it is the volume in the coordinates off the block times measures in the block, each of mass at most a bound on the vectors whose block coordinates have given size. A *section* of the isotropic cone of $Y^{(a)}$ is the set $h(e_a, y) = c$ for a unit c; its points are $c\, n^{-1} e_a^*$ for n in the Heisenberg group of floor a, and it carries the measure $dy/dc = |c|^r\, dn$, r the rank of $Y^{(a+1)}$.

Lemma 29.7 (Volumes on quadrics and sections). *Let Γ be a gauge with constant Λ, and let y_X denote the vector of products of the basis vectors of X' with y.*

(a) *The orthonormal basis vectors contain two disjoint blocks of $2m+2$ vectors, disjoint from the basis of X'. The volume of V_b and the quadric measure of floor b are solved in every such block, with mass at most $\Lambda^{(2m+2)(j+1)}$ on the points of size j; and the volume of the numbers of size j raised to the number of coordinates off X' and off one block, times this bound, is at most $\Lambda^{99n(j+2)}$.*

(b) *If φ has integral at most B over the vectors of size $j+4$ of X', then $\varphi(y_X)$ has integral at most $\Lambda^{99n(j+6)}B$ over the vectors of size j for either measure.*

(c) *A set of skew elements whose differences, divided by u, have size $j+1$ has measure at most $|u|^{1/2}\Lambda^{j+2}$. On the Heisenberg group of floor a: if $\Phi(n) \neq 0$ forces a lift of $(n^{-1}-1)e_a^*$ divided by u to have size j and $\Phi(n) \leq C\Psi(\omega/u)$, and Ψ has integral at most I over the vectors of size j, then $\int \Phi\, d\eta_a \leq C|u|^{r+1/2}\Lambda^{j+2}I$.*

(d) *The quadric measure of the isotropic cone of $Y^{(a)}$ is dc times the section measures; for a unit c and φ with integral at most B over the vectors of size $j+5$ of X', $\varphi(y_X)$ has integral at most $|c|^{-1/2}\Lambda^{j+3}\Lambda^{99n(j+7)}B$ over the points of size j of the section; and for a head p of length $a+1$, y in the box and $w_p n v y$ near at $a+1$, the section point $\bar{p}_a n^{-1} e_a^*$ has size $20n$.*

Proof. (a) Number the orthonormal indices with those of X' first and the two blocks in the last $2(2m+2)$ places. Each entry of the floor basis is 0, 1 or an entry of the basis matrix, of size 1, so coordinates cost four more than vectors; the Gram matrix of the floor basis swaps the two entries of each pair and fixes the others. The equation of the quadric is solved in a block by the gauge axiom on the quadrics of a block (a disintegration of the volume of the block along the standard form with masses at most $\Lambda^{(2m+2)(j+1)}$ in size j). There are at most $99n$ coordinates off X', each costing at most Λ^{j+1}. (b) Pairing with the basis of X' reads off coordinates; solve the equation in a block disjoint from X' and integrate the free coordinates. (c) For two differences d_1, d_2 the number $(\tau\delta^{-1}d_1 + d_2)/u$ has size $j+3$, so the squared mass of the set is at most $|u|$ times the volume of the numbers of size $j+3$. On the Heisenberg group, η_a is the image of $d\omega\, d\beta$; on the support the vector ω/u has size j, and so has $(c-\beta)/u$ with c independent of β; integrate first over β, then scale the coordinates ω by u, which costs $|u|^r$. (d) The cone measure is the image under $\sigma \mapsto \sigma^{-1}e_a$ of the floor measure, which is the Weyl law convolved with η_a; the point $(w_a(c)^{-1}n)^{-1}e_a$ is $cn^{-1}e_a^*$, and almost every c is a unit. For the bound, integrate over the Heisenberg group at the scale $-1/c$ by (c): the power of $|c|$ in the section measure cancels against that in $|c^{-1}|^{r+1/2}$, leaving $|c|^{-1/2}$. For the size: $(w_p n v)^{-1}e_a$ is congruent to $\bar{p}_a n^{-1} e_a^*$ modulo the e_b, $b < a$, as w_p sends e_a to $p_a e_a^*$; for the head $\bar{p}$ the entry $\bar{p}_a$ and

$(nv)^{-1}e_a^*$ have size $10n$ by Proposition 29.4(c), and coordinates of floor below a of a section point vanish. □

Proposition 29.8 (Plancherel's formula on quadrics). *Let Γ be a gauge, Λ its constant and $c = c_A$.*

(a) *The volume of $Y^{(b)}$ and the measure of the quadric of floor b are* solved *in every block of $2m+2$ orthonormal basis vectors: the other coordinates are free, and over each value of them the points of the block with coordinates of size j have mass at most $\Lambda^{(2m+2)(j+1)}$. A section $h(e_a, y) = c$ of the isotropic cone, for a unit c, is an orbit of the Heisenberg group of floor a.*

(b) *The kernel $\psi(\xi h(v, y))$ from the vectors y of size j on the quadric of floor b to the vectors v of size i multiplies square norms by at most*

$$\Lambda^{99n(i+6)}\,\Lambda^{99n(j+6)}\,(c/|\xi|)^{\dim X'}.$$

(c) *(Parseval along a Heisenberg group.) If the characters of two points differ by the invertible frequency ξ on the Heisenberg group N of floor a, the integral over N of the squared correlation from floor $a+1$ on is at most $(c/|\xi u|)^{\dim X'}|u|^{r_a+1/2}\Lambda^{100n(i+j+31)}$ times the integral over floor $a+1$ of the squared correlation from floor $a+2$ on, for suitable sizes i, j and scaling unit u; here r_a is the rank of the vector part of N.*

Proof. (a) is Lemma 29.7(a), (d). For the floor masses: if f is left invariant under $R^{(b+1)}$ and $f(\sigma g) \le \Psi(\sigma^{-1} f_b)$ with Ψ of integral at most C over the vectors of size 7 of the quadric, then the integral of $f(\sigma g)$ over the σ with σg near at $b+1$ is at most C: replace g by ξg in the box, $\xi \in R^{(b)}$, and note that the vectors $\sigma^{-1} f_b$ that count have size $1+6$. With (a) this gives floor masses at most $\Lambda^{99\cdot 13n}$ times the volume of the vectors of size 11 of X'.

(b) In the coordinates of the floor basis, solve for v and y in two different blocks (Lemma 29.7(a)); the kernel then acts on the orthonormal coordinates of X', off both blocks, where $\psi(\xi h(v, y))$ is a character in y_X with frequency ξv_X, and Plancherel's formula on X' bounds it by $(c/|\xi|)^{\dim X'}$ on square norms. Cauchy–Schwarz in the fibres of the blocks costs the masses of the blocks, $\Lambda^{99n(i+6)}$ and $\Lambda^{99n(j+6)}$ by Lemma 29.7(b).

(c) *Correlations along N.* Let n be the Heisenberg element of (ω, β). Moving n to the left of σ and applying the equivariance of the correlation to $\sigma n \sigma^{-1}$ (Lemma 29.5), the correlation from floor $a+1$ on at $(n^\varepsilon x, ny)$ is the integral over floor $a+1$ of $\psi(-\xi h(\omega, \sigma^{-1} f_{a+1}))$ times the correlation from floor $a+2$ on at $(\sigma x, \sigma y)$; for the twist $w_q^{-1} w_p$ conjugation multiplies σ_0 on N by p_a/q_a if $a < j$, so the frequency is $(p_a/q_a - \varepsilon)/M$ or $(1-\varepsilon)/M$. *Parseval.* The correlation along N is therefore a Fourier transform in the next frame vector. Fix n_0 in the support, write the element that moves x to bounded size as tr with $t \in R^{(a+1)} \cap G^{(a)}$, $r \in U_{(<a+1)}$, and substitute $n = r^{-1}t^{-1}n(v)trn_0$; the measure is invariant under the group on both sides and under conjugation by t, the vector part of $n(v)$ is $t^{-1}v + z$, and a lift of $(n(v)^{-1} - 1)e_a^*/u$ has size j. Plancherel's bound (b) on floor $a+1$ at the frequency $-\xi\bar{u}$, with the measure of the Heisenberg group from Lemma 29.7(c), gives the claim. □

Lemma 29.9 (Stable vectors and the depth). (a) *A vector orthogonal to the e_b, $b < l$, and short from g and g' is stable at g, g' with the same lift. The*

sets $\mathfrak{d}_a(x,y)$ form an open family in (x,y), have nonzero finite volume, are unchanged when both points are moved by an element of $R^{(a+1)}$ or the second point alone by an element of $U_{(<a)}$; hence so are the depths, which are measurable, nonzero and finite.

(b) *$\wp_{a+1} = \wp_a \cdot \mathrm{dp}_a^{-s}$, the weights are nonzero, finite, measurable and invariant under $R^{(j)}$ on both points, and the inverse weight (set to 0 off the points near at j) is left invariant under $R^{(j)}$.*

(c) (*Common bounds.*) *Let W be the set of vectors of X' killed by the rows of a matrix v and short from g and gx, and let $\varphi(u)$ bound $1/\mathrm{vol}\langle u, W\rangle$ and the volume of the c with $\langle cu, W\rangle$ inside the product differences. If σ moves W to vectors orthogonal to the e_b, $b < a$, then $\mathrm{dp}_a(\sigma g, \sigma gx) \le \varphi(y_X)$ for $y = \sigma^{-1} e_a$.*

Proof. (a) By definition a vector ω orthogonal to the e_b, $b < a$, is *stable* at x and x' if the pair $(x^{-1}\omega, x'^{-1}\omega')$ is close for some lift ω' of its class; if ω is short from both points, $\omega' = \omega$ is such a lift. *Openness.* Let $c = h(e_a, \omega)$ with ω stable at x, xy and lift ω'. The product $h(e_a, \omega)$ is the coordinate of ω at e_a^*; adding the same multiple of e_a^* to ω and to ω' keeps the orthogonality to the e_b, $b < a$, and the congruence, and changes c by that multiple. Close pairs form an open set and the readings at the two points are continuous in (x,y), so all c' near c lie in $\mathfrak{d}_a(x',y')$ for (x',y') near (x,y). *Volume.* The pair $(0,0)$ is close, so $0 \in \mathfrak{d}_a(x,y)$, and the set is open, hence of positive volume. It is bounded: $c = h(x^{-1}e_a, x^{-1}\omega)$ lies in the image of the compact closure of the close pairs under a continuous map, so the volume is finite. *Invariance.* For $\xi \in R^{(a+1)}$ the vectors $\xi\omega$ and $\xi\omega'$ are read at ξx and ξxy as ω and ω' are at x and xy; they are orthogonal to the e_b, $b < a$, and congruent (Lemma 29.3(d)); and $h(e_a, \xi\omega) = h(\xi^{-1}e_a, \omega) = h(e_a, \omega)$, as $\xi^{-1}e_a$ is e_a plus a combination of the e_b, $b < a$, which are orthogonal to ω. For $\nu \in U_{(<a)}$ acting on the second point alone, replace the lift ω' by $\nu\omega'$, a lift of the same class since ν acts as 1 on $Y^{(a)}$; its reading at νz is that of ω' at z. The depth $\mathrm{dp}_a = 1/\mathrm{vol}\,\mathfrak{d}_a$ is therefore nonzero, finite and invariant in the same way; it is measurable because $\{(x,y,c) : c \in \mathfrak{d}_a(x,y)\}$ is open, so the volume of its sections is lower semicontinuous.

(b) The weight of length j is by definition $\wp_j = \prod_{a<j} \mathrm{dp}_a^{-s}$, which gives the recursion; each factor is nonzero, finite and measurable by (a). An element of $R^{(j)}$ lies in $R^{(a+1)}$ for $a < j$, so every factor, and the weight, is invariant when both points are moved by it. Nearness at j is invariant under $R^{(j)}$, a group, so the inverse weight, extended by 0, is left invariant under $R^{(j)}$.

(c) Let $w \in W$. The vector σw is orthogonal to the e_b, $b < a$, by hypothesis, and its readings at σg and σgx are $g^{-1}w$ and $(gx)^{-1}w$, a close pair, as w is short from g and gx. By (a) it is stable at σg, σgx, and $h(e_a, \sigma w) = h(\sigma^{-1}e_a, w) = h(y, w)$. Hence $\mathfrak{d}_a(\sigma g, \sigma gx)$ contains $\langle y, W\rangle = \{h(y,w) : w \in W\}$, which depends only on y_X, as $W \subseteq X'$. Hence $\mathrm{vol}\,\mathfrak{d}_a \ge \mathrm{vol}\langle y_X, W\rangle \ge 1/\varphi(y_X)$, and $\mathrm{dp}_a(\sigma g, \sigma gx) \le \varphi(y_X)$. □

Proposition 29.10 (Depth). *Let Γ be a gauge, x a position and s the volume exponent.*

(a) (*Deep vectors are rare.*) *For every g the integral of $\mathrm{dp}_a(\sigma g, \sigma gx)^s$ for θ_a over the σ with σg near at $a+1$ is at most $\Lambda^{10^4 n}$.*

(b) (*Long coordinates force depth.*) *If $g \in K$ and $g^{-1}y$ lies in the position ball, the integral of $\mathrm{dp}_a(g, ny)^{-s}$ over the n in the Heisenberg group of floor a with ny close below a at g is at most $|M|^{r_a+1/2}\Lambda^{4600n}$.*

(c) (*The weights.*) *For a head p of length j and g near at j, the integral of the inverse weight $\wp_j^{-1}$ at $w_p v g$ over the Heisenberg measures below j is at most $\Lambda^{10^6 n^3}\prod_{a<j}|p_a|^{-(r_a+1/2)}$; and the integral of $\wp_j$ over the v with $g^{-1}vy$ in the position ball is at most a product of constants $\Lambda^{O(n^2)}$, one for each floor.*

(d) (*Bounded depths bound the position.*) *If x, xy, $w_p^{-1}x$, $w_q^{-1}xy$ are near at j for heads p, q of length j and M^{-1} has size 1, then for $a < j$ the set $\mathfrak{d}_a(x, y)$ contains a unit whose inverse has size $10n$; and if every $\mathfrak{d}_a$, $a \geq j$, contains such a unit, then $(y-1)/M$ has size $99n$.*

Proof. (a) For g in the box the depth at σg is at most $\varphi(y_X)$, $y = \sigma^{-1}e_a$, for the common bound of the gauge (Lemma 29.9(c)); a subspace of large rank of common vectors exists by the gauge axiom. The bound over size 7 on the quadric comes from the bound $\Lambda^{99n(j+1)}$ for φ^s over size $7+4$ on X', at the cost of a factor $\Lambda^{99n(7+6)}$ (Lemma 29.7(b)).

(b) *The range.* Two points of the position ball give a matrix $(gx(gy)^{-1}-1)/M$ of size 14: $y = M((y-1)/M)+1$ has size 3, gy size 5 and its inverse size 9. Hence, up to a factor in $U_{(<a)}$, n^{-1} is the quotient of two points close to g, and a lift v of $(n^{-1}-1)e_a^*$ has v/M of size 17. *The pairing.* Pair a stable vector ζ with a common vector u of g and y: by unitarity g and y drop out, and for the Heisenberg element n of (ω, β) the identity $h(\zeta, o) - h(n^{-1}\zeta', o) = \overline{h(e_a, \zeta)}h(\omega, o)$ shows that the inverse depth at g and ny is at most $\varphi(\omega_X/M)$. Integrating by Lemma 29.7(c) with $u = M$, the skew variable adds $17+2$ to the exponent, and the total is at most $4600n$.

(c) *Inverse weights.* Move g into the box and induct on the length. At floor a the vector $(w_p n v)^{-1}e_a$ is a lift of a point of the section $h(e_a, z) = \bar{p}_a$ (Lemma 29.7(d)), and its depth is at most $\varphi(z_X)$ for the common vectors orthogonal to the earlier vectors $(w_p n v)^{-1}e_b$, multiples of the $v^{-1}e_b^*$; the section integral gives $|p_a|^{-(r_a+1/2)}\Lambda^{10^6 n^2}$, the weight of length a does not see the new Weyl and Heisenberg factors, and the measure is invariant under n and under conjugation by the element that moved g. There are at most n floors. *Weights.* Unless the integral vanishes, some $g^{-1}\tau y$ is a close position; translate by τ. On a coset $U_{(<a)}z$ the depth of e_a is constant (Lemma 29.9(a)), so it leaves the integral, and Fubini with (b) gives the factor $|M|^{r_a+1/2}\Lambda^{4600n}$ at floor a.

(d) *Under a head.* At $z = x$ and $z = xy$ the vector $z^{-1}e_a^*$ has size $8\varrho+7$ (Proposition 29.4(c)), so a small multiple of e_a^* is short from both points, and its product with e_a is a unit of $\mathfrak{d}_a$ with inverse of size $10n$. *The position.* A unit $c \in \mathfrak{d}_a$ with c^{-1} of size 1 gives a short dual of size 5 modulo M: divide a stable vector by c. If ux and vxy lie in the box with $(vxy-ux)/M$ of size 2, the lengths at x and xy differ by size 9 modulo M on $Y^{(\varrho)}$, the lift being $v^{-1}u\omega$ and $x^{-1}\omega - (xy)^{-1}v^{-1}u\omega = (vxy)^{-1}(vxy-ux)x^{-1}\omega$. Descend from $Y^{(\varrho)}$ to the whole space by Proposition 29.4(b): each floor costs $4\cdot 5+12$, and $9+32\varrho$ leaves room below $99n$; below j the heads provide the units, from j on the hypothesis does. □

29.4. One frame, two frames, and the gains

Write $\mathrm{ws}_l(g) = \wp_l(g, gx)|k_{l+1}(g, gx)|^2$ for the *weighted square* of floor l, b_l for the sums $\mathsf{b}_l(g) = \int (\mathrm{ws}_l + \mathsf{b}_{l+1})(\sigma g)\, d\theta_l(\sigma)$ ($l < n$, and 0 from n on), and B_l for the integral of b_l over the floors below l, the *second moment* of one frame. The *cost* of floor l on the position ball is $|M|^{r_l+1/2} c^{\dim X'} \Lambda^{4900n}$. The weighted product $\Pi_j(x)$ is the product of the second moments of the two correlations at x, times the squared product of the $|p_a|^{-(r_a+1/2)}$ over $a < j$ and the cube of $\Lambda^{10^6 n^3}$.

Lemma 29.11 (One frame). *Let k be a pairing kernel at floor j.*

(a) $\int |k_{l+1}(\sigma g, \sigma g x)|^2 d\theta_l(\sigma) \le \mathsf{b}_l(g)/\wp_l(g, gx)$*; the weighted squares and their sums are measurable, left invariant under $R^{(l+1)}$, resp. $R^{(l)}$, and zero off the points near at $l+1$, resp. l; and B_l is the integral of* ws_l *over the floors up to l plus B_{l+1}, so $\mathsf{B}_l \le \mathsf{B}_0$.*

(b) (*Cauchy–Schwarz on a floor.*) *If for every z the θ_l-integral of $u(\sigma z)$ over the σ with σz near at $l+1$ is at most K, and $|k_{l+1}(\sigma g, \sigma y)|^2 \le u(\sigma g)v(\sigma)$, then the square of the θ_l-integral of $|k_{l+1}(\sigma g, \sigma y)|$ is at most $K \int v\, d\theta_l$.*

(c) (*One floor up.*) *The cut square of floor l (the squared correlation above floor l, set to 0 where the positions of (g, y) in the position ball have measure zero) has integral over the Heisenberg group of floor l, at (g, ny), at most the cost of floor l (or 1 at the top floor) times the integral of the cut square of floor $l+1$ over floor $l+1$.*

(d) (*One frame.*) *Over the position ball the weighted square of floor l has integral at most the product of the costs of the floors times the squared L^2 norm of k, and $\int \mathsf{B}_j\, d\mathsf{M}$ over the position ball is at most n times this.*

Proof. (a) The weight is invariant under the floor group of l, so it leaves the integral of the weighted square, a part of b_l. Invariance: left translation by $R^{(l+1)}$ multiplies the kernel by a character value of modulus one, and descending induction along the recursion of b_l, with Proposition 29.2(b) absorbing the translation; the vanishing off near points is the pairing kernel axiom. Splitting the last factor θ_l of the convolution of the floors up to l gives the recursion. (b) The floor measure lives on the floor group, so the integrand may be cut to the set where the mass bound holds; then Cauchy–Schwarz. (c) If the mass of the position ball is zero, both sides vanish; otherwise it is the same at σg, σy (substitute $v \mapsto \sigma^{-1} v t$). Below the top floor, write the element that moves g into the box as tv with $t \in R^{(l+1)} \cap G^{(l)}$ and $v \in U_{(<l+1)}$; for n with ny close below l, a lift of $(tn_0(tn)^{-1} - 1)e_l^*$ divided by M has size 17 (Proposition 29.10(b)), and Parseval's inequality (Proposition 29.8(c)) with frequency M^{-1} gives the cost of floor l. On the top floor only H is left, of mass at most 1 on near points, and Cauchy–Schwarz gives the constant 1. (d) *Factorisation.* By induction on the floors from the top down, M factors as the convolution of the measure of the unipotent group of the first l floors with an s-finite measure ρ (a form of Weil's quotient integral formula), and the integral of the cut square against ρ and the floors up to l, times the costs of the floors below l, is at most the product of all costs times the squared L^2 norm: above the top floor there is no cost; at each step the cut square of floor $l+1$ is left invariant under $U_{(<l+1)}$, so Proposition 29.2(d) lets the second point move by σ, and (c) applies for each second point. *The weight.* Put $x = g^{-1} v y$ with v in the unipotent group of the

first l floors: the modulus of the correlation does not depend on ν, and the weight has integral over ν at most the product of the costs (Proposition 29.10(c), second part). Summing over the floors l from j to ϱ gives the factor n. □

Theorem 29.12 (One frame and two frames). *Let Γ be a gauge, M a Haar measure, k a correlation kernel at floor $j \le \varrho$, and p a head of length j with Weyl element w_p.*

(a) (*One frame.*) *Over the position ball, the integral of the weighted second moment $\int \wp_j(g, gx)|k_{j+1}(g, gx)|^2 d\mathsf{m}_j(g)$ is at most $n \prod_{l<\varrho} \Lambda_l$ times the squared L^2 norm of k for $\theta_{\varrho+1} * \cdots * \theta_0$ in each variable, where the Λ_l are the costs of the floors, powers of Λ.*

(b) (*Separating the frames.*) *Let B, B' be measurable, left invariant under $R^{(j)}$ and zero off the points near at j, and where $f \neq 0$ let*

$$f(g)^2 \le C \, \frac{B(g)}{\wp_j(g, gx)} \cdot \frac{B'(w_p^{-1} g)}{\wp_j(w_p^{-1} g, w_p^{-1} gx)}.$$

Then $(\int f \, d\mathsf{m}_j)^2 \le C D^2 \int B \int B'$, with $D = \Lambda^{10^6 n^3} \prod_{a<j} |p_a|^{-(r_a+1/2)}$. A variant holds when one pivot runs over a section of the isotropic cone.

Proof. (a) is Lemma 29.11(d). (b) Cauchy–Schwarz with the weights crossed: B at g is paired with the inverse weight at $w_p^{-1} g$, and B' at $w_p^{-1} g$ with the inverse weight at g,

$$\Big(\int f \Big)^2 \le C \int \frac{B(g)}{\wp_j(w_p^{-1} g, \cdot)} \, d\mathsf{m}_j \cdot \int \frac{B'(w_p^{-1} g)}{\wp_j(g, \cdot)} \, d\mathsf{m}_j.$$

In the first integral put $w_p g$ for g (Proposition 29.2(d)); in the second, w_p^{-1} is the Weyl element of the head $\bar{p}$. Then integrate each inverse weight over the unipotent group of the floors below j by Proposition 29.10(c), the functions B, B' being invariant under it. *Section variant.* For a head of length $i+1$, pair $\wp_i \varphi_1^2$ at g with the inverse weight of length $i+1$ at $w^{-1} g$, and $\wp_{i+1} \varphi_2^2$ at $w^{-1} g$ with that of length i at g; where the left side is not zero both points are near and the weights cancel; integrate first over the Heisenberg group of floor i, which fixes the other factor, then shift by w and integrate over floor i and the unipotent group of the floors below. The inverse weight of length i at $w_i(p) w_{i+1}(p)^{-1} g$ is the inverse weight of g for points near at $i+1$, since it does not see the last factor of the Weyl element. □

Theorem 29.13 (The gains at one position). *Let W be a cut Bessel function, p a head of length $j \le \varrho$, q a second sequence of units, x a position and M a Haar measure. Write $\Pi_j(x)$ for the product of the weighted second moments of the two correlations, times $\Lambda^{3 \cdot 10^6 n^3} \prod_{a<j} |p_a|^{-(2r_a+1)}$.*

(a) (*Frequency.*) *If $\psi(\eta z) = \psi(\xi z)$ for a unit ξ and all z of size $200n^2$, then with the weight $\psi(\eta Z_{jj})$, $\mathsf{F}_j(x)^2 \le (c/|\xi|)^{\dim X'} \Pi_j(x)$.*

(b) (*Two heads.*) *If $j = i+1$ and $\xi = (p_i^{-1} - q_i^{-1})/M$ is a unit, then with weight 1, $\mathsf{F}_j(x)^2 \le (c/|\xi|)^{\dim X'} \Pi_j(x)$.*

(c) (*Deep vector.*) *If the mass of floor j near the next floor is at most $V \ge 1$ and x has a deep vector of depth σ^{-1}, then $\mathsf{F}_j^\sharp(x)^2 \le \sigma^s V^2 \Pi_j(x)$.*

Proof. (a) Where the double integral does not vanish, g and $w_p^{-1}g$ are near at j, and one t moves both to bounded size; on the supports the Gram entry Z_{jj} of $\sigma\tau^{-1}$ has size $200n^2$, where the hypothesis applies. Hence the weight is the character $\psi(\xi Z_{jj})$, the inner integral is $f(\sigma)$ times the conjugate of a Fourier transform of the second correlation along the quadric of the pivot, and Cauchy–Schwarz in σ with the Plancherel bound of Proposition 29.8(b) for the correlation (the transform over the vectors of size i is at most $\Lambda^{99n(i+6)}\Lambda^{99n(j+12)}(c/|\xi|)^{\dim X'}$ times the squared L^2 norm on the floor) gives the gain; Theorem 29.12(b) separates the frames.

(b) *The phase of the lifts.* For n in the Heisenberg group of floor i let y' be the point over n of the section $h(e_i, y') = \bar{p}_i$. The next vector $y = \sigma^{-1}e_{i+1}$ is orthogonal to e_i and e_i^*, so $h(y', y) = -p_i h(\omega, y)$, and the correlation from floor $i+1$ on at (ng, ng') is the integral over y of $\psi(-\xi h(y', y))$ times the correlation from floor $i+2$ on, which does not depend on y'. *Plancherel on the section.* Over the n with $w_p ng$ near at $i+1$, the squared L^2 norm of $n \mapsto \int \psi(-\xi h(y', y))\varphi(\sigma)d\theta_{i+1}$ is at most $(c/|\xi|)^{\dim X'}|p_i|^{-(r_i+1/2)}\Lambda^{10^6 n^3}$ times that of φ: apply Parseval along N with frequency $-\xi p_i$ and scale p_i^{-1} to the set of n at which both $w_p ng$ and ng are near; some $t \in G^{(i+1)}$ moves ng to size $99n^2$, and the column $(t_0 n_0 (tn)^{-1} - 1)e_i^*$ has size $198n^2 + 6$, so p_i times it has size $999n^2$. Separating the frames with the first pivot on a section (Theorem 29.12(b), variant) gives the gain.

(c) *The cut at the first deep vector.* If for each σ in the floor group with $k(\sigma g, \sigma g x) \neq 0$ some vector from floor l on has depth at least D, then the integral over floor l of the modulus of the correlation from floor $l+1$ on is at most a sum over a with $l \le a < \varrho$ of the integrals, over the floors from l to a, of the modulus cut to the points at which the vector of floor a has depth at least D: downward induction on l, a frame being deep on floor l or its first deep vector coming later. *One piece.* Cauchy–Schwarz against the inverse weight of length $a+1$, whose integral is a power of Λ times the inverse weight of length l (Lemma 29.11(a), with Proposition 29.10(a) on each floor), and $1 \le \sigma\, \mathrm{dp}_a$ where the cut modulus is not zero, give the square of a piece at most $\sigma^s \Lambda^{10^4 n\varrho}\mathsf{b}_l/\wp_l$. Summing at most ϱ pieces bounds the square of the first pivot by $\sigma^s\Lambda^{10^6 n^3}\mathsf{b}_l/\wp_l$; the second pivot is bounded by the mass V (Lemma 29.11(b)). Where the second modulus is not zero, W vanishes at neither translate, so both points are near along the whole orbit under the floor group, and the active points are the ones considered; at floor 0 every point is near as soon as the box is not empty, and both pivots give the same integral. □

Theorem 29.14 (The sum over the shells). *Let S_i be shells of height 2^i for Γ and heads p, q of length $j \le \varrho$ (sets of positions covered by few position balls whose scales are at most $\Lambda \cdot 2^{2i}$, such that a large height forces a deep vector at each point) covering the support of f. Let W be a cut Bessel function, $V \ge 1$ bound the mass of floor j, $f \le \mathsf{F}_j^\sharp$, and for $e < 1$ let $f(x)^2 \le (Ve^s)^2\Pi_j(x)$ on the positions. If $9 \cdot 10^7 n^4 + r \le s$, then*

$$\kappa_j(p, q) \int f \, d\mathsf{M} \le V \min(1, e)^r \, \Lambda^{4\cdot 10^8 n^4} \, \mathcal{N}(W),$$

with $\mathcal{N}(W)$ the shell normalisation of W, the squared L^2 norm of its correlation over $(U\backslash G)^2$ times the constants of the gauge.

Proof. A gain on the position ball. If $f^2 \le e^2\Pi_j$ on a subset of the position ball of a gauge Γ', then $\kappa_j(p,q)\int f$ over it is at most e times $\mathcal{N}(W)$ times $\Lambda^{1500000n^3+1}\Lambda'^{25000n^2}$: square both sides, split by Cauchy–Schwarz into two integrals of second moments, bound them by Theorem 29.12(a) (the second up to the modulus of left translation by $w_q^{-1}w_p$, Proposition 29.2(c)), and note that in the product of $|p_aq_a|^{r_a}$, $|p_a|^{-(2r_a+1)}$ and $|p_a/q_a|^{r_a+1}$ the powers of $|p_a|$ cancel and $|q_0\cdots q_{j-1}|^{-1} \le 1$ is left. *One shell.* A shell of height H is covered by at most $2H^{9n}$ position balls of gauges with constants at most ΛH^2, so its integral is at most $e\Lambda^{2\cdot10^6n^3}H^{10^5n^2}\mathcal{N}(W)$. *Remote shells.* If $H > \Lambda^{2400n^2}$ and $\tau^{800n^2} = H^{-1}$, every active point has a vector of depth at least τ^{-2} after the head (the shell condition), and Theorem 29.13(c) gives the gain $V\tau^s$. *Trading gain for height.* It suffices to find $g \le e$ with the gain Vg^s and $g^{9\cdot10^7n^4}H^{1+10^5n^2} \le \Lambda^{29\cdot10^7n^4}$: below the threshold take $g = e$, beyond it the smaller of e and τ; with $9\cdot10^7n^4 + r \le s$ the shell contributes at most $H^{-1}V\min(1,e)^r\Lambda^{3\cdot10^8n^4}\mathcal{N}(W)$. *Sum.* The heights are 2^i and $\sum_i 2^{-i} = 2$. □

29.5. A real place

At the complex place, with the Hermitian space of ϱ hyperbolic planes, the standard positive form on κ and one vector z with $h(z,z) = -1$, the *real place sizes* are $\rho = b^{6n+2} \ge 8r$ for the box radius, $b \ge 2$, $k = s + 3n + 6 \ge n^7$, $s > 10^9n^3$ and $\varrho \ge 10^8$. The place datum has box $\|x\| \le \rho$, close pairs of squared length less than 1, and Haar measure; the gauge measures sizes by powers of L. A *cut Bessel function at a real place* is the cut, at size ρ, of a normalised Bessel function of irreducible type π (Definition 24.14), forming a cut pair with base b and decay rate f. Its normalised forms are $\mathcal{J}_i = c_W\mathcal{I}_i$, $c_W = \operatorname{vol}\mathrm{U}(k)/(\operatorname{vol}\mathrm{U}(r)(\lambda\#I)^2)$, and its *normalised cell integral* $\Xi(p)$ is the orbital integral at the chart of the top floor, times the scalar $\sqrt{c_G}(|x_\varrho|^2-1)^{(k-1)/2}|I|^{-1}\prod_{a<\varrho}|x_a|^{k+1+2(\varrho-a-1)}$ of Theorem 28.17(a).

Proposition 29.15 (The place $\mathbb{C}$). *Assume the real place sizes.*

(a) *The convolution of the Heisenberg measures is Lebesgue measure in the free entries of U; the group measure is a positive finite multiple $h_{\mathbb{C}}$ of the Haar measure of G_1, with $h_{\mathbb{C}} \le \operatorname{vol}\mathrm{U}(r)/\operatorname{vol}\mathrm{U}(k)$, and for U-invariant functions $\int d(\theta_{\varrho+1} * \cdots * \theta_0) = h_{\mathbb{C}}\int_{U\backslash G_1}$.*

(b) *For $x, y \in G_1$ and a matrix v with at most n rows there are, for every $d \le \dim X' - 3n$, d orthonormal vectors of X' killed by v and carried by x^{-1} and $(xy)^{-1}$ into the space on which the form is the identity; hence common bounds as in Proposition 29.10(a) exist.*

(c) *$\mathfrak{d}_a$ is a disc and $\mathrm{dp}_a = \pi^{-1}\inf_\zeta(|\zeta|_x^2 + |\zeta|_{xy}^2)$ over the duals ζ of e_a in $Y^{(a)}$.*

Proof. (a) *Heisenberg measures.* At $\mathbb{C}$ the Heisenberg element of floor a with vector v and skew parameter ri puts v and r into the free entries of the column e_a^* and fixes the earlier e_b^*, while an element of $U_{(<a)}$ keeps the free entries of the column e_a^*. Hence the Gaussian $\exp(-\pi\sum_{b<l}(\sum_i|g_{i,e_b^*}|^2+(\operatorname{Im} g_{e_b,e_b^*})^2))$ factors into a standard Gaussian on the Heisenberg group of the last floor and the Gaussian of the lower floors, and has integral 1 against $\eta_0 * \cdots * \eta_{l-1}$, by induction. At y^{-1} this Gaussian is the standard Gaussian in the free entries of the rows e_a of y, as $y^{-1} = J_1y^*J_1$; so it has integral 1 against the Haar measure of U too. Both

measures are invariant under U and give the Gaussian the integral 1, hence they agree. *Haar measure.* The group measure is right invariant (Proposition 29.2(b)), hence a multiple $h_{\mathbb{C}}$ of the Haar measure of the unimodular group G_1. For the bound compare two descriptions of the fibre measure of the Gram map on the tuples $(y_0, y_0^*, \dots, y_{\varrho-1}, y_{\varrho-1}^*, y_\varrho)$: by Haar measure, and by choosing the vectors in turn (adding the frame vector e_a, then the dual e_a^*; each floor measure, halved, is at most the fibre measure), which gives that the image of the group measure under inversion is at most $\operatorname{vol} \mathrm{U}(r)/\operatorname{vol} \mathrm{U}(k)$ times Haar measure; compare the measures of a ball. For U-invariant f: $\mathsf{M} = du * \theta$, so $(g, u) \mapsto ug$ carries $\theta \times du$ to $h_{\mathbb{C}}$ times Haar measure, and θ represents the quotient. (b) The conditions $vw = 0$ and that $x^{-1}w$, $(xy)^{-1}w$ lie in the space on which J_1 is the identity are at most $3n$ linear conditions on X'; take d columns of a unitary matrix in the kernel of the matrix of the conditions (eigenvectors of C^*C for the eigenvalue 0). For the common bound put $\varphi(u) = 36/|O^*u|^2$: if $|O^*u|^2 = t \neq 0$, the products of u with the common vectors cover the disc of radius $\sqrt{t}/2$ (a multiple of OO^*u of length at most $\frac{1}{2}$), so $1/\operatorname{vol}\langle u, W\rangle \leq 36/t$, and the product differences lie in the unit disc; passing to the basis Q of a unitary completion and keeping of the coordinates in the block only $|O^*u|^2$, the integral of $(36/|O^*u|^2)^s$ over the vectors with entries at most L^j is at most $(L^2)^{99n(j+1)}$, by the integral of $(36/r)^s$ against the density $\pi^{s+1}r^s/s!$. (c) A number c lies in $\mathfrak{d}_a$ exactly when $c = h(e_a, \omega)$ for ω orthogonal to the e_b, $b < a$, with a lift ω' of its class and $|\omega|_x^2 + |\omega'|_{xy}^2 < 1$; for $c \neq 0$ write $\omega = c\zeta$ with $h(e_a, \zeta) = 1$, and the bound 1 on the squared lengths becomes $1/|c|^2$. Hence $\mathfrak{d}_a$ is the disc $|c|^2 d < 1$ for the infimum d, of area π/d. □

Lemma 29.16 (Shells at a real place). *Let $W^\flat$ be a cut Bessel function at a real place and p, q heads of length $j \leq \varrho$ for the gauge of size $L \geq \rho$.*

(a) *For every list there is $L \geq \rho$ making p, q heads of length j with $L^K \leq \rho^K$ times the weights of the heads (of exponent K, cut-off R bounding the partial products); the squared L^2 norm of the correlation over $(U\backslash G_1)^2$ is at most $h_{\mathbb{C}}^2 \#I$; and the forms vanish if a partial product of p or q exceeds ρ^{9n}.*

(b) (*Far means deep.*) *If x is outside the position ball at a scale $T \geq L^{11n}$ and $T^{-1} \leq \tau/2$, every active point has a vector e_a, $a \geq j$, of depth at least τ^{-2}; the dyadic shell of height 2^i (the least power of 2 bounding the entries of $x - 1$) is a shell for this gauge; and the floor masses are at most $(\pi\rho^{24})^k \rho^{9999n}$.*

(c) *For $1 \leq t$ and φ below the majorant with $\varphi^2 \leq ((\pi\rho^{24})^k \rho^{9999n} t^{-s})^2 \Pi_j$ if $t > 1$, the normalised integral of φ times $\#I$ is at most $\Theta L^{10^9 n^4} \rho^{24k} t^{10^8 n^4} t^{-s}$; the majorants are integrable; and if $\mathsf{F}_j(x)^2 \leq (\pi/|y|)^{2(k-5)} \Pi_j(x)$ for every gauge adapted to p, then $\#I|c_W \mathcal{I}_j| \leq \Theta b^{2000n^9} \ell_{k-n^7}(1/|y|)$ times the weights of p and q.*

Proof. (a) Induction on j: L is the largest of ρ and the inverse norms of the partial products, and each step enlarges L once for p and once for q, at the cost $\max(L, x)^K \leq L^K \max(1, x)^K$. For the norm, the integral over y is at most $h_{\mathbb{C}} \|W^\flat(x)\|^2$, as $\int W \otimes W^* \leq 1$ on the quotient, and unfolding by Proposition 29.15(a) gives the factor $h_{\mathbb{C}}$. An active point would bound the partial products of both heads by Proposition 29.4(c); there are none, so the majorant vanishes. (b) Otherwise T^{-1} lies in each disc $\mathfrak{d}_a$, $a \geq j$, and Proposition 29.10(d) in the gauge of size T puts x in the position ball. A position of the dyadic shell

of height 2^i is outside the position ball at any $T \geq 1$ with $2T^{99n} \leq 2^i$, and is covered by the position balls of at most $2(2^i)^{9n}$ gauges of constant at most $L^2 2^{2i}$; at $T = L^{11n+1}/\tau$, T^{99n} is at most half the height when $L^{2400n^2} \leq \tau^{400n^2} 2^i$. The floor mass is Proposition 29.8(a) in the gauge of size ρ, whose constant is ρ^2. (c) Theorem 29.14 with the mass bound $V = (\pi\rho^{24})^k \rho^{9999n}$, the gain t^{-1} and $r = s - 10^8 n^4$; at $\mathbb{C}$ the level is 1, $c = \pi^2$ and the constant is L^2, and $h_{\mathbb{C}}$, $\#I$ and the volumes cancel, leaving powers of π, ρ and L. With $t = 1$ the bound is finite, so the majorants are integrable. For the last statement take $t = \max(1, \rho^{24}|y|)$ in a gauge for both heads: the weights of the heads pay the size L of the gauge if their partial products are at most R, and otherwise the form is 0 by (a); for $t > 1$, $(\pi^2/|y|^2)^m \leq ((\pi\rho^{24})^k \rho^e/(\rho^{24}|y|)^s)^2$ for $s \leq m \leq k$, which is the gain (c) asks for; and the bound at this t is at most $\Theta b^{2000n^9} \ell_{k-n^7}(1/|y|)$, comparing exponents of b with $\rho = b^{6n+2}$, $10^8 \leq \varrho$ and $k = s + 3n + 6$. □

Theorem 29.17 (Local band limits at a real place). *Let $W^\flat$ be a cut Bessel function at a real place of a Bessel function W, with decay rate $f \geq 4000n$, and let* $\deg > 0$ *with: the matrix coefficient of W has squared L^2 norm at most $(\#I)^2/\deg$, $\deg \cdot \Theta \leq b^{4r^2}$, and the ball of radius $T \geq 1$ in G_1 has volume at most $T^{9r^2} b^{99r^2}/\deg$. Then:*

(a) *the normalised forms $\mathcal{J}_0, \ldots, \mathcal{J}_\varrho$, followed by the Gram form $\langle \Xi(p), \Xi(q) \rangle$ of the normalised cell integrals, form a sieve system of length ϱ on $\mathbb{C}$ for the character $e^{c-\bar{c}}$, the weights of exponent n^6 and cut-off b^{60n^2}, and the decay $\Theta b^{2000n^9} \ell_{k-n^7}(1/|\xi|)$, with $|\mathcal{J}_0| \leq (1 + b^{-10})/\deg$; in detail:*
   (i) *(Mass.)* $|\mathcal{J}_0(p, q; 1)| \leq (1 + b^{-10})/\deg$ *for all p, q;*
   (ii) *(Frequency.) for $i \leq \varrho$ and $\nu \neq 0$, $\#I \cdot |\mathcal{J}_i(p, q; e^{\nu Z_{ii} - \overline{\nu Z_{ii}}})| \leq \Theta b^{2000n^9} \ell_{k-n^7}(1/|\nu|) \Lambda_i(p) \Lambda_i(q)$;*
   (iii) *(Unequal entries.) for $i + 1 \leq \varrho$ and $\nu = p_i^{-1} - q_i^{-1} \neq 0$, $\#I \cdot |\mathcal{J}_{i+1}(p, q; 1)| \leq \Theta b^{2000n^9} \ell_{k-n^7}(1/|\nu|) \Lambda_{i+1}(p) \Lambda_{i+1}(q)$;*

(b) *if $\sigma \in G^{(\varrho)}$ and the last entries of p and q both equal $h(z, \sigma z)$, the top form is the inner product of the normalised orbital integrals of $W^\flat$ and σ at p and at q.*

Proof. Descent below the top. For $i < \varrho$, with the entries at i replaced by a common unit, the integral over c of $e^{\xi c - \overline{\xi c}}$ times the normalised form of floor $i + 1$ converges and is the normalised form of floor i with the weight $x \mapsto e^{\xi Z_{ii}(x) - \overline{\xi Z_{ii}(x)}}$, by Theorem 29.6(b) and the integrability of the majorants (Lemma 29.16(c)). *The top floor.* Apply Theorem 29.6(b) at floor ϱ with the chart of the top floor (the boost with parameter $c/\min(1, |c|)$) and its density; at a unit c the integrand is the Gram form of Ξ: both measures are the group measure, $h_{\mathbb{C}}$ times Haar measure, the Gram form is a double integral for Haar measure (substitute (y, yx)), the partial correlations are averages over $\mathrm{U}(\kappa)$ of inner products, which $\mathrm{U}(\kappa)$ keeps, and $h_{\mathbb{C}}$ cancels against c_W while the open cell constant is the constant of the density times the ratio of volumes in c_W. *Continuity and positivity of the top form.* Near p all nearby lists are heads for one gauge, so the integrand of the orbital integral is supported in a fixed compact set and jointly continuous, and Ξ is that orbital integral times a continuous real factor; positivity is that of a Gram matrix. (b) Both sides have the same normalising factors and are double integrals; on the right multiply both variables by an element of $G^{(\varrho)}$ that commutes with the Weyl elements.

(i) *Floor* 0. The integrand at floor 0 is $h_{\mathbb{C}}^2|\langle W^\flat, R(x)W^\flat\rangle|^2$, the matrix coefficient in $L^2(U\backslash G_1)$, because it is left invariant under U and the Weyl elements are trivial. Split the positions at the scale $T = b^{6r}\varrho^{11n}$. *The position ball.* There $\|x\| \le b^{700nr}$. Replacing $W^\flat$ by W changes the matrix coefficient by at most $2b^{-r^3 f}\#I$ at every position, since the difference is $\langle W^\flat - W, R(x)W^\flat\rangle + \langle W, R(x)(W^\flat - W)\rangle$ and $\|W^\flat - W\| \le b^{-r^3 f}\sqrt{\#I}$; with the volume bound of the ball, $(2b^{-r^3 f})^2 T^{9r^2} b^{99r^2} \le b^{-24}$ for $f \ge 4000n$, and Minkowski's inequality in L^2 gives at most $(1 + b^{-12})^2(\#I)^2/\deg$ for the coefficient of $W^\flat$ on the ball. *Remote positions.* These have a deep vector (Lemma 29.16(b)), and Lemma 29.16(c) bounds them by $\Theta\varrho^{10^9 n^4}\rho^{24k} t^{10^8 n^4}/t^s \le b^{-12}/\deg$ with $t = b^{6r}$, using $r = s + 5n + 5$, $k = s + 3n + 6$ and $10^{10}n^4 \le s$, which follows from $k \ge n^7$ and $\varrho \ge 10^8$. With $c_W h_{\mathbb{C}} \le 1$ and $(1 + b^{-12})^2 + b^{-12} \le 1 + b^{-10}$ the mass is at most $(1 + b^{-10})/\deg$.

(ii), (iii) For a gauge adapted to both heads, Theorem 29.13(a), resp. (b), gives at every position the gain $(\pi^2/|\nu|^2)^{\dim X'}$ with $\dim X' = k - 5$ at $\mathbb{C}$ (the hypotheses of (a) hold for $\eta = \nu$, $\xi = \nu$; in (b) the frequency is ν since the level is 1), and Lemma 29.16(c) integrates it. The local bounds of the sieve do not keep the factor $\#I$. □

29.6. Finite places

Let A be an integral place with integers R and character ψ trivial on traces. A place datum F for the Gram matrix of $X_0 \perp Az$ in the lattice basis (Section 25.1) is a *lattice datum* if the Gram matrix and the basis are integral and unimodular over R, the box is the group K of integral isometries, the close pairs are the pairs of integral vectors congruent modulo the level g, the measures are normalised on K, and an isotropic vector is dual to z. It is *large* if moreover $\zeta_R(m)^n \le 1 + 2^{-n}$ for the ideal zeta sum, $m < 9n$, $\#\iota \ge 53n$, and the unitary group is unimodular. Write $K(g)$ for the principal congruence subgroup; for $K(g) \le Z \le K$ the *Bessel vector* W_Z has the coordinates $W_a(y) = \chi_g(ya^{-1})$ at the cosets a of $K(g)$ in Z, and the *mass integral* is $\kappa(Z) = \int_G |\int_Z \chi_g(yxy^{-1})\,dy|^2 dx$.

Proposition 29.18 (Lattice data). (a) *If N is self-adjoint and integral, g a self-adjoint integral unit and $k = \#\iota + 2m + 1$, the Gram matrix of $X_0 \perp Az$ has a lattice datum of level g.*

(b) *For a large lattice datum* M *is a Haar measure,* $\mathrm{vol}(K) \le 1 + 2^{-n}$, *and the Heisenberg measures of the congruence subgroup are given by* $|g|^{r_a+1/2}$.

(c) *W_Z is a cut Bessel function;* $\mathrm{fl}\cdot\langle W_Z(y), W_Z(yx)\rangle$ *is the integral over U of* $1_Z(uy)\chi_g(uyx(uy)^{-1})$, fl *the mass of $U \cap K(g)$; and* fl *times the squared L^2 norm of its correlation over $(U\backslash G)^2$ is at most* $\mathrm{mr}(Z) = \mathrm{vol}(Z)\,\mathrm{vol}(K(g))/\mathrm{fl}^2$.

(d) (*Depth at the finite places.*) *If x is near at $a + 1$, $\mathfrak{d}_a(x, y)$ is an ideal of R and the depth is its index; the forms vanish unless the leading products of both heads are integral; and a large denominator of $(x - 1)/g$ forces a deep vector.*

Proof. (a) *The basis.* The matrix E that makes the plane of x_{k_0} and z hyperbolic, turns $2m$ orthonormal vectors into m hyperbolic pairs by a c with $\bar{c}c = -1$ and keeps the others, is integral, each entry reducing to the norm of c, the trace-one element or $\bar{N} = N$; in it the coordinate at z is the coefficient at the second vector of the first pair, and the bases of all floors are integral. *The top floor.* With an integral isotropic u dual to z of z-coordinate 0, the scaled Eichler map along u

attached to an integral e with unit z-coordinate r, $1+r^{-1}(ruh(v,\cdot)-(v+su)h(u,\cdot))$, is an integral isometry; its image of the chart measure gives a measure on $G^{(\varrho)}$ carried to the top quadric and of mass at least the volume of the units of R on K. *The measure of H* is the right Haar measure of H with $\nu(H\cap K)=1$: the $h\in H$ with $hy\in UK$ lie in one right coset of $H\cap K$. The close pairs form an open set containing 0 with compact closure, the image of the pairs of integral vectors under $(\omega,\delta)\mapsto(\omega,\omega-g\delta)$. (b) *Heisenberg masses.* Some u in a subgroup of $U_{(<a)}$ moves the Heisenberg element of (v,β) into $K(g)$ exactly when $g^{-1}v$ and $g^{-1}\beta$ are integral; the preimage of $S\cdot K(g)$ in the chart is a box, each coordinate of a floor above a contributing $|g|$ and β contributing $|g|^{1/2}$, so the mass is $|g|^{r_a+1/2}$, and over the floors below j the product. *The box.* Unfolding at level 1 bounds $\mathsf{M}(K)$ by the mass of the frames near the box, and each of the n floors contributes at most $\zeta_R(m)$ (in the gauge of scale 1 every size is R, of volume 1; solve the equation in a block and let the other coordinates be free in R); so $\mathsf{M}(K)\le\zeta_R(m)^n\le 1+2^{-n}$. Each floor contributes at least the volume of the units of R (for a unit u the Weyl element $w_l(u)$ lies in the box, and the density $|u|^d$ is 1), so the box has positive mass; M is right invariant, hence a multiple $d\nu$ of a unimodular Haar measure, with d finite and nonzero. (c) *The function χ_g.* It is $\psi(\sigma(u)/g)$ on $uK(g)$, independently of u (two choices differ by an element of $K(g)$, and ψ is trivial on the integers, which are self-dual), right invariant under $K(g)$, hence continuous, of modulus at most 1, and left translation by an element x of the Bessel subgroup in $uK(g)$ multiplies it by $\psi(\sigma(u)/g)$ (σ is additive on the Bessel subgroup). Hence W_Z is a cut Bessel function: bounded, measurable, of frequency g^{-1} and zero off UK. *Correlations.* On U the integrand $1_Z(uy)\chi_g(uyx(uy)^{-1})$ is the sum over the cosets c of a_c times the indicator of a set S_c, and S_c has mass fl whenever $a_c\neq 0$; this gives $\mathrm{fl}\langle W_Z(y),W_Z(yx)\rangle$. At $x=1$, $\mathrm{fl}\|W_Z(y)\|^2$ is the mass of the u with $uy\in Z$, at most 1 since Z lies in the integral points, where the floor volumes are 1. *Norms.* The squared norm of $\sqrt{\mathrm{fl}}W_Z$ on $U\backslash G$ is $\mathsf{M}(Z)$, the number of cosets times $\mathsf{M}(K(g))$; for fixed x, the integral over y of $|\langle W_Z(x),W_Z(y)\rangle|^2$ times fl^2 is at most $\|W_Z(x)\|^2\operatorname{vol}(K(g))$, only a set of mass $\operatorname{vol}(K(g))/\mathrm{fl}$ counting (the integral point ux normalises $K(g)$); integrate over x. (d) *Ideals.* A stable vector w with second lift $w+d$ is a point (w,d) of a lattice on which a matrix is integral; giving the coordinate of w at e_a^* the leading weight, the coordinates at it of such points form cR for a unit c, by a triangular basis of the lattice. If x is near at $a+1$ and ξx is integral, so is $\xi w=(\xi x)(x^{-1}w)$, which has the same product with e_a; so $\mathfrak{d}_a\subseteq R$ is an ideal, and its volume is the inverse of its index. *Heads.* Where the integrand of the majorant is not zero, the frames y, yx and their translates by the inverses of the Weyl elements are near at j, and in the gauge of scale 1 every size means integral, so Proposition 29.4(c) makes the leading products integral. *A scale for two heads.* With common multiples β, γ of the leading products of p and q, the scale $\alpha=g\bar\beta\bar\gamma\beta\gamma$ makes p, q heads, g^{-1} and the $1/p_i-1/q_i$ of size 1, and $\Lambda^E\le(2/|g|)^K$ times the finite head weights of p and q when $4E\le K$. *Far means deep.* Let c divide every common denominator of $(x-1)/g$, with $|c|\Lambda^{990n^2}\le\sigma^{400n^2}$, $\sigma<1$, in the gauge of scale α ($\Lambda=2/|\alpha|$). Each $\mathfrak{d}_i$ is d_iR for a unit d_i of R, of volume $|d_i|$. If $|d_i|>\sigma^2$ for all $i\ge j$, the d_i have a common multiple C in R, fixed by the involution, of modulus at least σ^{4n}; in the gauge of scale $\alpha^{10n}C$ every d_i^{-1} has size 1, so Proposition 29.10(d) gives $(x-1)/g$ of size $99n$, the power $(\alpha^{10n}C)^{99n}$ is a common denominator, and c divides it: $|c|\ge|\alpha|^{990n^2}|C|^{99n}\ge\Lambda^{-990n^2}\sigma^{396n^2}$, against the hypothesis, as

$4 \cdot 99 < 400$ leaves a factor $\sigma < 1$. Hence some $\mathfrak{d}_i$, $i \geq j$, has volume at most σ^2, that is depth at least σ^{-2}. □

Theorem 29.19 (Decay at the finite places). *Let F be a large lattice datum with level g, $K(g) \leq Z \leq K$, p, q sequences of units, $t + 10^{10}n^4 \leq k$ and $10^{10}n^4 \leq K'$. Let either (a) $\omega(x) = \psi(\xi h(f_j, xf_j))$, or (b) $\omega = 1$, $j = i + 1$ and $\xi = (p_i^{-1} - q_i^{-1})/g$. Then*

$$|\mathcal{I}_j(p, q; \omega)| \leq \mathrm{mr}(Z)\,[R : \xi R]^{t/2}\,[R : R \cap \xi^{-1}R]^{-6}\,(2/|g|)^{K'}\,\Lambda^{\mathrm{f}}_{K',j}(p)\Lambda^{\mathrm{f}}_{K',j}(q),$$

with the finite head weights Λ^{f}.

Proof. Shells. For $x \in G$ the common denominators in R of the entries of $(x-1)/g$ form a principal ideal $J = cR$, c a unit of A in R, of index $|c|^{-1}$ (take $c = t^{-1}$ for the content t of the vector listing 1 and these entries). Sort the positions by J, and into dyadic shells by the index. The shell of J lies in the position ball of the gauge of scale $\bar{c}c$, whose constant is $2[R : J]^2$; at most $2(2^t)^m$ ideals have index at most 2^t, as $\zeta_R(m) \leq 2$; and if $\Lambda^{990n^2} \leq \sigma^{400n^2}[R : J]$, the shell has a deep vector of depth σ^{-2} (Proposition 29.18(d)). Hence the dyadic shells are shells in the sense of Theorem 29.14, and with the mass of floor j bounded in the gauge of scale 1,

$$\kappa_j(p, q) \int f \leq \min(1, e)^r\,\mathrm{mr}(Z)\,\Lambda^{10^9n^4}$$

whenever $f \leq \mathsf{F}_j^\sharp$ has the gain e^s and $r + 10^8n^4 \leq k$. *The gain.* Two units ξ, β have a common divisor η over R congruent to ξ modulo βR (for an integral idempotent e, $d = \xi - e(\xi - \beta)$ divides both). At each position Theorem 29.13(a) or (b), in the gauge of Proposition 29.18(d), gives units η, w with w, ξw, ξ/η integral, the gain $(c_A/|\eta|)^{\dim X'}$ with $c_A = 1$, so that $c_A/|\eta| = e^2$ for $e = [R : \eta R]^{1/2}$, and $\min(1, e)^2 \leq |w|/|\alpha^{200n^2}|$. Since $wR \subseteq R \cap \xi^{-1}R$, $|w| \leq [R : R \cap \xi^{-1}R]^{-1}$; and as ξ/η is integral, $\min(1, e)^{t+12} \leq [R : \xi R]^{t/2}$ times the sixth power of $\Lambda^{200n^2}/[R : R \cap \xi^{-1}R]$. The exponents of Λ add up to at most $K'/4$, which the weights and $(2/|g|)^{K'}$ absorb (Proposition 29.18(d)). □

Theorem 29.20 (The mass at the finite places). *Let F be a large lattice datum with congruence group $K(g)$, $K(g) \leq Z \leq K$, and* vol = M. *Let $Q \geq 2$ be such that every unit c of A with $c \in R$ and $c^{-1} \notin R$ has modulus at most Q^{-1}, and let $t + 2 \cdot 10^9n^4 \leq k$. Then the integral of the majorant $\mathsf{F}_0^\sharp(\cdot; W_Z)$ over the complement of $K(g)$ is at most $Q^{-t}\mathrm{mr}(Z)$, and*

$$\kappa(Z) \leq \mathrm{vol}(Z)\,\mathrm{vol}(K(g))\,\big(\mathrm{vol}(Z) + Q^{-t}\big).$$

Proof. On $K(g)$. For $x \in K(g)$ every conjugate yxy^{-1}, $y \in Z$, lies in the normal subgroup $K(g)$ of K, where $\chi_g = 1$, as Z normalises $K(g)$; so the inner integral is $\mathrm{vol}(Z)$, and this part of $\kappa(Z)$ is $\mathrm{vol}(Z)^2\,\mathrm{vol}(K(g))$. *Off $K(g)$.* If $x \notin K(g)$, every point active at x has an index a with $\mathrm{dp}_a \geq Q$: otherwise each $\mathfrak{d}_a$ contains a unit of R and $(x-1)/g$ is integral, as in the gauge of scale 1 every size means integral (Proposition 29.10(d)). Summing over the shells in the gauge of scale 1, where $\Lambda = 2$, with the gain Q^{-s} gives the integral of the majorant off $K(g)$ at most $Q^{-(t+n)}\mathrm{mr}(Z)2^{10^9n^4} \leq Q^{-t}\mathrm{mr}(Z)$, the spare powers of $Q^{-1} \leq 2^{-1}$ absorbing $2^{10^9n^4}$. *The mass integral.* The inner integral $\int_Z \chi_g(yxy^{-1})dy$ has the modulus of fl times the integral of $\langle W_Z(z), W_Z(zx)\rangle$ over $U\backslash G$ (integrate Proposition 29.18(c)

over y), so the mass integral off $K(g)$ is fl^2 times the form of floor 0 there (Theorem 29.6(c)), at most $\mathrm{fl}^2 \cdot Q^{-t}\mathrm{mr}(Z) = Q^{-t}\,\mathrm{vol}(Z)\,\mathrm{vol}(K(g))$. □

Proposition 29.21 (Fixed vectors save a power of b). *Let the filling data be admissible, F a filling datum at T, M^{-1} integral off T, S_g a set of chosen places above p_0 and d_w nonzero vectors as in the definition of the fixer. There is Z with $K_T(M) \le Z \le K_T$ that contains the component at T of every integral element whose reduction lies in* $\mathrm{fixer}(S_g, d)$, *and*

$$\kappa(Z)\,\mathrm{vol}(K_T)\,b^{6k\#S_g} \le \kappa(K_T)\,\mathrm{vol}(Z).$$

Proof. Few non-units. Every place of T is a chosen place or the conjugate of one (as M^{-1} is integral off T, T contains every chosen place), so the residue fields have at least b^{10} elements and T has at most $4f \le b$ places; an integer that is a unit of E_T but not of R is divisible by the uniformiser at some place, each such set having volume at most b^{-10}; so the non-units have volume at most b^{-9}, and $Q = b^9$ in Theorem 29.20. *One place.* At a block $\{w, \bar{w}\}$, $w \in S_g$, take for Z the components of the integral elements whose reduction at w fixes d_w; as the reduction and the component at w are onto, its index in K is $p_0^r - 1$, the number of nonzero vectors. An integral element whose component at T lies in $K_T(M)$ has reduction 1 at w (each entry of $M^{-1}(k-1)$ is integral at w), so $K_T(M) \le Z$. Choose t with $t + 1 + n + 2 \cdot 10^9 n^4 \le k$, $6k \le 9t$ and $t + 1 < r$. Then $\mathrm{vol}(Z) = \mathrm{vol}(K)/(p_0^r - 1)$ is small, Theorem 29.20 bounds $\kappa(Z) \le \mathrm{vol}(Z)\,\mathrm{vol}(K(g))(\mathrm{vol}(Z) + b^{-9t})$, and the part of the outer integral over $K(g)$ gives $\kappa(K) \ge \mathrm{vol}(K)^2\,\mathrm{vol}(K(g))$; with $B = b^9$, $(z + B^{-(t+1)}k)B^t \le k$ when $B^{t+1}z \le k$, and the gain is $b^{9t} \ge b^{6k}$. *Several places.* Split T into the blocks $T \cap \{w, \bar{w}\}$, $w \in S_g$, and the rest, where $Z = K$. The completions, the character, the unitary group, its level and U, the Bessel function and the measures are products over the blocks (the character of an adele supported on T is the product of those of its parts; the unitary group over T is the product of those over the blocks, a measurable embedding with dense image); so for products of groups the volume is a constant t times the product of the volumes, and the mass integral is t^3 times the product of the mass integrals. Hence the inequalities of the blocks multiply, those of S_g giving one factor b^{6k} each. □

29.7. The top floor

For two heads ω, ω' and $l \in G^{(\varrho)}$ the *scaled orbital integral* is the head density times $\int W_Z(y) \otimes W_Z(l w_\omega^{-1} y)^* dy$, indexed by pairs of cosets. The *fibre pairing* $\mathrm{fp}(\omega, \omega'; e)$ is the integral over H of the pairing of the scaled orbital integrals at l and hl; it depends only on $e = l^{-1}z$, a point of the *last vector quadric*. The *density at the last entry* is

$$J_{\mathrm{last}}(\omega, \omega'; t) = |\pi|^{-1} \int_{h(z,e)=t} \mathrm{fp}(\omega, \omega'; e)\, d\mu_t(e),$$

for the measure $dx/d\langle x, x\rangle$ on the sphere over t, where $\pi\bar{\pi} = N$.

Lemma 29.22 (Fibre measures and scaled orbital integrals). *Let F be a large lattice datum of the Gram matrix of $X_0 \perp Az$, $\pi\bar{\pi} = N$.*

(a) *The vector $e = x - N^{-1}tz$ has $h(e,e) = x^*x - t\bar{t}/N$ and $h(z,e) = t$, and in the basis made of the x_i and $\pi^{-1}z$, in which the Gram matrix is* $\mathrm{diag}(1, \dots, 1, -1)$,

its coordinates are x and $-\bar{\pi}^{-1}t$. In rank at least two the volume of $\kappa \to A$ disintegrates along $\omega \mapsto \omega^*\omega$ into fibre measures carried by the spheres, and the map $(t, p) \mapsto x - N^{-1}tz$, x the point of the chart p of the sphere $x^*x = t\bar{t}/N - N$, carries $|\pi|^{-1}dt$ times the chart measure to the top quadric measure.

(b) *The scaled orbital integrals are locally constant in the head and on $G^{(\varrho)}$ and vanish for l outside a compact set; their pairing at l is the head density times the integral of $\langle W_Z(y_1), W_Z(y_2)\rangle\overline{\langle W_Z((w_q l^{-1})^{-1}y_2), W_Z((w_p l^{-1})^{-1}y_1)\rangle}$ over two copies of the group, where nonzero integrands have entries of size $60n^2$; these pairings are integrable over H.*

(c) *A locally constant function supported in a compact set, invariant on its support under a group of integral matrices, agrees on its domain with a compactly supported function invariant under small translations and that group.*

(d) *For a weight φ invariant under H on both sides, the integrand $\varphi(l^{-1})$ times the pairing integrand has the law of the integrand of the form of floor ϱ; for $\varphi(x) = \psi(\xi h(z, xz))$, $l \mapsto \varphi(l^{-1})$ times the pairing of the scaled orbital integrals is integrable for the top two floors with integral $\mathcal{I}_\varrho(p, q; \psi(\xi\theta))$.*

Proof. (a) In the lattice basis only $x_{k_0} = \bar{\pi}v - \pi^{-1}z$ and $\pi^{-1}z$ pair with $v = x_{k_0}/\bar{\pi} + z/N$, and only $\pi^{-1}z$ pairs with z; the x_i are orthonormal and orthogonal to $\pi^{-1}z$, of norm $-\pi\pi^{-1} = -1$. Disintegrations along a form are unique; in a basis with Gram matrix of a hyperbolic plane plus 1 the flat chart gives a disintegration (its values lie on the quadric where the first parameter is a unit, almost surely). The basis of X_0 and $\pi^{-1}z$ form a basis of $X_0 \oplus Az$ that begins with a hyperbolic pair, so its flat chart gives the quadric measure of norm $-N$; regrouping the indices, the last coordinate of this chart is $-\bar{\pi}^{-1}t$ and the others are those of the chart of the sphere, which solve for the coordinate along the first vector from the same remaining norm; the map appending ut carries $|u|\,dt$ times the measures to the volume. Fubini in (t, p) gives the integral formula. (b) For heads of a gauge, the integrand of a scaled orbital integral vanishes off a compact set C (Theorem 29.6(d)); for (r, s) near $(p, 1)$ the argument changes, for all $y \in C$, by a right translation from the open group $K(g)$, under which W_Z is invariant; a gauge of scale $\bar{\alpha}\alpha$ for α near 0 absorbs the finitely many inverse products locally uniformly in the head. If the integral at l is not zero, some y and $y' = lw^{-1}y$ lie in a compact set, and $l = y'y^{-1}w$. The pairing formula is Fubini; the integral of the norm over the floors from i on is at most a constant times $|\langle W_Z(y_1), W_Z(y_2)\rangle|$, a continuous function vanishing off a compact set. (c) For a small unit u, φ is constant on the classes $(y + uR^n)$ meeting the support (units are dense); give Φ this value on these classes and 0 elsewhere. (d) Substitute $l = k\tau(g\sigma)^{-1}$, $y_1 = g\sigma y$, $y_2 = y_1 x$ with g, k in H and σ, τ in $G^{(\varrho)}$, compare the laws by integrating a measurable test function, and use Fubini for the first four variables of a function on a product of five spaces; the Weyl elements commute with the floor group, and the factor k^{-1} fixes z, so the entry at l^{-1} is $h(z, l^{-1}z)$. □

Proposition 29.23 (The top floor). *Let F be a large lattice datum of the Gram matrix of $X_0 \perp Az$, with z of norm $-N$, and $K(g) \le Z \le K$.*

(a) *(Fibre measures.) For f continuous with compact support, $t \mapsto |\pi|^{-1} \int f(x - N^{-1}tz)\, d\mu_t(x)$, the integral over the sphere $x^*x = t\bar{t}/N - N$ of X_0, is integrable, with integral the integral of f over the quadric of vectors of norm $-N$.*

(b) *The scaled orbital integrals are locally constant in the head and on $G^{(\varrho)}$ and vanish off a compact set; their pairings are integrable over H.*

(c) *For every e and every positive semidefinite matrix R, $\sum_{a,b} R_{ab}\mathrm{fp}(\omega_a, \omega_b; e)$ is a nonnegative real; on the top quadric its positive real part agrees with a function $\Psi \geq 0$ of compact support, invariant under small translations and under $H \cap K(g)$. The sums over p of the pairings of the scaled orbital integrals over the congruence cosets of the links above e are bounded by the fibre pairing at e.*

(d) *J_{last} is continuous, its sums against positive semidefinite matrices over heads with a common last entry are nonnegative, and, with the last entries replaced by a common t, $\int \psi(\xi t) J_{\mathrm{last}}\, dt = \mathcal{I}_\varrho(p, q; \psi(\xi\theta))$, θ the last diagonal entry of the frame Gram matrix.*

Proof. (a) is Lemma 29.22(a). (b) is Lemma 29.22(b). (c) Two elements l, l' of $G^{(\varrho)}$ with $l^{-1}z = l'^{-1}z$ differ on the left by an element of H, and μ_H is right invariant, so the fibre pairing is a function of e. Write $R = Q^*Q$: the sum against R of the pairings at l splits over the coordinates (pairs of cosets) of the scaled orbital integrals, the term of a coordinate being v^*Rv for the vector v_a of the coordinates of the heads ω_a; so it is nonnegative, and its integral over H is the sum of the fibre pairings. The fibre pairing is invariant under $H \cap K(g)$ (substitute ky for y; k commutes with the Weyl element), vanishes unless e lies in the image of a compact set, and is locally constant along the last vector quadric: the map $l \mapsto l^{-1}z$ has local sections on the quadric (conjugate by l the Eichler map that takes lx to z), and the scaled orbital integrals do not see the section near 1. By Lemma 29.22(c) it agrees with a step function Ψ. For the last statement, the right side is an integral over H; each element of distinct right cosets of $K(g)$ over one e gives a coset of the congruence group on which the integrand is its term (for $g \in H$ with $g\beta_\theta \in K(g)$ the scaled orbital integrals at gl_0 are those at l_θ, being invariant under $H \cap K(g)$), and these cosets are disjoint. (d) *Integral formula.* The measure on the floor group is $\mu_H * \theta_\varrho$, and $l \mapsto l^{-1}z$ maps θ_ϱ to the top quadric measure, so by Lemma 29.22(d) $\mathcal{I}_\varrho(p, q; \psi(\xi\theta))$ is the integral over the top quadric of $\psi(\xi h(z, e))\Phi(e)$ for any continuous Φ extending the fibre pairing. With the step function Φ of (c) and (a), this is $\int \psi(\xi t) J_{\mathrm{last}}(p, q; t)\, dt$: almost every t is a unit, and the fibre pairing does not see the entry ϱ of the heads. *Continuity.* Near (p, q) the fibre pairing does not change, and its step function does not see the change of the last entry; what moves is the sphere, and the fibre measures depend continuously on the index (Lemma 29.22(a), splitting a continuous function of compact support into positive and negative parts). *Positivity.* For heads with a common last entry t, the sum of the densities against R is $|\pi|^{-1}$ times the integral of the sums of the fibre pairings against R at $x - N^{-1}tz$ over the sphere, which are nonnegative by (c), the fibre pairing being integrable there as it agrees with a continuous function of compact support on the sphere. □

Part V

The lower bound: the determinant and one lift

30. The norm of one lift

Theorem C is a lower bound for the covolume and the index of a family of target forms, obtained from Theorem 7.14 applied to the twisted lattice $L_{\mathcal{T}}$ of Section 20.4. Its inputs are the volumes $Z(\vartheta) = \det(\Theta^*_\vartheta \Theta_\vartheta)$ of the lower chain at the m-fold twists ϑ and auxiliary forms that complete a family to a basis. This part computes the volumes: the norm of one theta lift is given by Rallis' formula (this section); theta integrals of spaces of complementary ranks are exchanged (Section 31); the chain of m lifts is arranged in a triangle whose squares are exchanges, so that its volume is a product of volumes of single lifts times ratios of Hecke values (Section 32); auxiliary forms are built by moving periods along paths through two grids of spaces (Section 33); and the height of the ratio $Z_j(1)/Z_j(\xi^j)$ for one lift is bounded through a comparison operator, false lifts and Hecke values (Sections 34 to 36).

30.1. The lift of a descent pair and its adjoint

Let O be a descent pair (Section 15.6) with spaces W of rank r and $V = I_n$, $n = r + t$, level places M (the chosen places) and Q, and character μ. For a vector ξ of the representation of the weight of O let $\varphi_W(\xi) = v_{P_\xi} \otimes \Psi_{\mathrm{bl}} \otimes \mathrm{vac}$ be the datum with the Fock polynomial of ξ at the real places and the block datum at the finite places. The *vector kernel* $\theta_W(g, h)$, for $g \in \mathrm{U}(W)(\mathbb{A})$ and a finite point h of $\mathrm{U}(V)$, is the vector of the weight representation whose pairing with ζ is the theta kernel of $\varphi_W(\bar\zeta)$ at (g, h); for fixed g it is an algebraic form $\theta_W(g, \cdot)$ on V. For a doubled parameter l on W that fits O (Proposition 15.11(b)) and a form y on W of type l, the *lift* is

$$\Theta_W y = \int_{[\mathrm{U}(W)]} y(g)\, \theta_W(g, \cdot)\, dg,$$

the theta lift of y by the datum of the highest vector, read as an algebraic form on V. For $1 \le j \le m$ and a tame character ϑ, the lift $\Theta_{j,\vartheta}$ is Θ_W for the j-th descent pair (W_j, V_j) with the character $\mu_j^0 \vartheta^j$, on the forms of principal level J_M on W_j.

Lemma 30.1 (The vector kernel). *Let O be a descent pair and $g \in \mathrm{U}(W)(\mathbb{A})$.*

(a) *The pairing of $\theta_W(g, h)$ with ξ in the Gram form, conjugate linear in its first argument, is the conjugate of the theta kernel of $\varphi_W(\bar\xi)$ at (g, h).*

(b) *For rational $\gamma \in \mathrm{U}(V)$ and k in the level of V,*

$$\theta_W(g, \gamma x k) = c(k)\rho(\gamma)\theta_W(g, x), \qquad c(k) = \prod_{w \in Q} \psi_w(\det k_w)^{-1}.$$

Here ρ is the representation of the weight of O: at an infinite place τ it is the weight with $k^+(\tau, i) + k^-(\tau, N - i)$ columns of height i and the power $-\sum_{1 \le i < N} k^-(\tau, i)$ of the determinant, N being the rank of V. The level character c is formed with the block k_w of the reduction of k at w on the rows and columns of the embedding of W. Hence $\theta_W(g, \cdot)$ is an algebraic form on V.

(c) *The normalised descent of a form Ξ on V has at g the value*

$$\sum_h \frac{\langle \theta_W(g, h), \Xi(h) \rangle}{\#(\Gamma \cap hKh^{-1})} = \langle \theta_W(g, \cdot), \Xi \rangle,$$

the sum over representatives h of the double cosets $\Gamma \backslash \mathrm{U}(V)(\mathbb{A}_{\mathrm{f}})/K$.

Proof. (a) The theta kernel of $\varphi_W(\xi)$ is linear in ξ, because the Fock polynomial P_ξ is, and the weight representation carries the nondegenerate Gram form; so the conjugate of the kernel of $\varphi_W(\bar{\xi})$ is a conjugate linear form in ξ, represented by a unique vector, which is $\theta_W(g,h)$; the kernel is formed with the characters of O at the point $(g,1,h)$, whose component in $\mathrm{U}(V)(\mathbb{A})$ is trivial at the infinite places. (b) The operator $c(k)\rho(\gamma)$ is unitary for the Gram form and onto, so it suffices to pair both sides with its values. There the equivariance of the conjugate kernel under the rational points and the level applies (Proposition 15.11, whose proof records it): the level multiplies the block datum by the level character (Proposition 15.11(a)), and the rational points act through the weight of the highest vector. (c) The normalised descent is the class sum over $\Gamma\backslash \mathrm{U}(V)(\mathbb{A}_{\mathrm{f}})/K$ of the conjugate theta kernel at (g,h) applied to $\Xi(h)$, each class weighted by the inverse of the order of its automorphism group (Proposition 15.11(c)); by (a) the summand is $\langle \theta_W(g,h), \Xi(h)\rangle$, and the class sum is the inner product of algebraic forms. □

Proposition 30.2 (Lift and descent). *Let O be a descent pair and l a parameter on W that fits O.*

(a) (*Equivariance.*) *Let k be an integral point of $\mathrm{U}(W)(\mathbb{A}_{\mathrm{f}})$ with reduction s at the chosen places, and k' a lift of $\mathrm{diag}(s_w,1)$ to $\mathrm{U}(V)(\mathbb{A}_{\mathrm{f}})$. Then (k,k') multiplies $\varphi_W(\xi)$ by $\chi(s)^{-1}$, $\chi(s)=\prod_w \psi_w(\det s_w)$; the lift of the translate of y by k is $\rho(s)\Theta_W y$, where $\rho(s)$ acts on forms on V by $\chi(s)$ times translation by k'; and $(\theta\Xi)(gk)=\theta(\rho(s)\Xi)(g)$ for the normalised descent θ.*

(b) (*Adjointness.*) *The normalised descent of a form Ξ on V is a form of type l on W, its value at g is $\langle \theta_W(g,\cdot),\Xi\rangle$, and $\langle \Theta_W y,\Xi\rangle = \langle y,\theta\Xi\rangle$, the class sum product on V against the Petersson product on W. In particular $\theta\Xi=0$ exactly when Ξ is orthogonal to all lifts.*

(c) (*Hecke polynomials.*) *Let w be split with $w,\bar{w}$ off $M\cup Q$, q its norm, and α,β the Frobenius values at w of the characters of O on W and on V. If y is spherical at w with Hecke polynomial $p(X)$, then $\Theta_W y$ is spherical at w with Hecke polynomial $\prod_{i<t}(1-q^i\alpha X)\cdot p(q^t\alpha\beta^{-1}X)$.*

Proof. (a) *The datum.* Finite points act only on the finite part $\Psi_{\mathrm{bl}}\otimes \mathrm{vac}$ of the datum, and on it the pair (k,k') acts by the substitution $X\mapsto k'^{-1}Xk$ of the reductions times the characters. The block datum is supported on the matrices that are the inclusion at M, and $k'^{-1}\iota s=\iota$ there since k' reduces to $\mathrm{diag}(s_w,1)$; at $w\in Q$ the block on the embedded rows is multiplied by s_w, which multiplies $\psi_w(\det)$ by $\psi_w(\det s_w)^{-1}$ (Proposition 15.11(a)). Hence (k,k') multiplies $\varphi_W(\xi)$ by $\chi(s)^{-1}$. *The action on forms.* The action $\rho(s)$ on a form on V is $\chi(s)$ times translation by k', the weight representation twisted by the power of the determinant of the pair. *The lift.* Substituting gk^{-1} for g in the integral defining $\Theta_W(y(\cdot\,k))$ and moving k into the datum together with k', the scalar $\chi(s)^{-1}$ from the datum and the translation by k' combine to $\rho(s)$. *The descent.* Both $(\theta\Xi)(gk)$ and $\theta(\rho(s)\Xi)(g)$ are multiples of conjugates of transposed lifts (Proposition 15.11(c)); the transposed lift is equivariant under (k',k), and the scalar $\chi(s)^{-1}$ of the datum matches the conjugate of $\chi(s)$ from the conjugated form.

(b) *Type.* The descent is a multiple of the lift from V to W of the conjugate form by the datum of the highest vector (Proposition 15.11(c)), and this datum is an arrow from V to W between the conjugate types (Proposition 15.11(b)), so

the descent is a form of type l. Its value at g is $\langle \theta_W(g,\cdot), \Xi\rangle$ by Lemma 30.1(c). *Convergence.* A coordinate of $g \mapsto y(g)\theta_W(g,\cdot)$ in an orthonormal basis of the forms on V is y times the conjugate of a form on W, and forms are square integrable on $[\mathrm{U}(W)]$; so the map is integrable, and evaluation at a point of $\mathrm{U}(V)(\mathbb{A})$, being continuous and linear, commutes with the integral. Hence $\Theta_W y$ is the integral of $y(g)\theta_W(g,\cdot)$, both sides having the same image under the injective map from algebraic forms to functions. *Adjointness.* Pass from the class sum product on V to the Petersson product, which multiplies by the index of the level and the dimension of the weight, and move the lift to the other side, where it becomes the conjugate of the transposed lift; by Proposition 15.11(c) this is the normalised descent. In particular $\theta\Xi = 0$ exactly when $\langle y, \theta\Xi\rangle = 0$ for all y, that is when Ξ is orthogonal to all lifts.

(c) At w both characters are unramified and the datum is the vacuum, so the unramified correspondence for the theta lift transfers the Hecke polynomial (Appendix B.16): the Satake parameters of the lift are those of y shifted by $q^t\alpha\beta^{-1}$ together with the t parameters $q^i\alpha$, $i < t$, of the trivial part. □

30.2. The block datum

Write $c_Q = (\mathrm{N}\,Q)^{rt}\prod_{w\in Q}\prod_{i=t+1}^{t+r}(1-q_w^{-i})$ and $c_M = \prod_{w\in M}\prod_{i=t+1}^{t+r}(1-q_w^{-i})/\#\,\mathrm{GL}_r(\mathbb{F}_w)$. A finite datum is *sharp* for a level J off a set S_0 of places if it is the vacuum off S_0 and its matrix coefficient at the finite points integral off S_0 is $\langle F, F\rangle$ on J and 0 elsewhere (Appendix B.21); its *mass* is $\langle F, F\rangle/[K_0 : J]$.

Proposition 30.3 (The block datum is sharp). (a) *The sum of* $|\Psi_{\mathrm{bl}}(x)|^2$ *over the* $n \times r$ *matrices modulo* $\pi_M Q$ *is* $\#\prod_{w\in Q}\mathrm{GL}_r(\mathbb{F}_w)\cdot(\mathrm{N}\,Q)^{tr}$.

(b) *If* $\Psi_{\mathrm{bl}}(x) \neq 0$, *the block of* x *on the embedded rows is the reduction of an integral point of* J_M; *and* $\Psi_{\mathrm{bl}} \otimes \mathrm{vac}$ *is sharp for* J_M *off the places of* $\pi_M Q$ *and their conjugates* (*if the signs of* W *agree or* $\pi_M Q \neq (1)$), *with mass* c_Ψ.

(c) *The* Gram factor, *the index of the level on* V *times* $\sum|\Psi_{\mathrm{bl}}|^2$ *over the index of* J_M, *divided by the number of matrices modulo* $\pi_M Q$, *is* $c_M c_Q$; *and* $c_M \neq 0$.

Proof. The function. The block datum, defined modulo $\pi_M^2 Q$, is also the vacuum datum of the function Ψ_{bl} modulo $\pi_M Q$, which is 0 unless the reduction x_w at every chosen place w is the inclusion matrix of the embedding $\iota\colon r \hookrightarrow n$, and is then $\prod_{w\in Q}\psi_w(\det x'_w)$ with x'_w the block on the rows ι: the two moduli have the same places, and both functions factor through the reductions there.

(a) By the Chinese remainder theorem a matrix modulo $\pi_M Q$ is the family of its reductions at the chosen places and the places of Q. The squared modulus of Ψ_{bl} is a product of local indicators. At a chosen place it counts one matrix. At $w \in Q$, for a multiplicative character χ of a finite field and a square matrix X, $|\chi(\det X)|^2$ is 1 if X is invertible and 0 otherwise; over a finite commutative ring R the $n \times r$ matrices whose block on the rows ι is invertible number $\#\,\mathrm{GL}_r(R)\cdot(\#R)^{(n-r)r}$, the block being an element of $\mathrm{GL}_r(R)$ and the other rows free. Multiplying over $w \in Q$ gives $\#\prod_Q \mathrm{GL}_r(\mathbb{F}_w)\cdot(\mathrm{N}\,Q)^{tr}$.

(b) *Lifting the block.* If $\Psi_{\mathrm{bl}}(x) \neq 0$, the block of x is 1 at the chosen places and invertible at the places of Q; by strong approximation for the integral points with the reduction map onto, this family is the reduction of an integral point of W that reduces to 1 at every chosen place, that is of J_M. *Sharpness.* The matrix coefficient of $\Psi_{\mathrm{bl}} \otimes \mathrm{vac}$ at an integral point g off the places of $\pi_M Q\overline{\pi_M Q}$

is $\sum_x \overline{\Psi_{\mathrm{bl}}(x)}\Psi_{\mathrm{bl}}(xg)$. For $g \in J_M$ the transformation law of the datum for the pair $(k,1)$ shows that Ψ_{bl} is fixed, so the coefficient is $\langle \Psi_{\mathrm{bl}}, \Psi_{\mathrm{bl}} \rangle$. The support is carried into J_M by the inclusion matrix, so by the criterion of Theorem B.34(d) for a function on matrices modulo a modulus, which applies when the signs of W agree or the modulus is not (1), the coefficient vanishes off J_M. The mass is $c_\Psi = \sum |\Psi_{\mathrm{bl}}|^2 / (\#\text{matrices} \cdot [K_0 : J_M])$, with $[K_0 : J_M] = \prod_{w \in M} \#\,\mathrm{GL}_r(\mathbb{F}_w)$.

(c) The index of the level on V, times the order of $\prod_{w \in Q} \mathrm{GL}_r(\mathbb{F}_w)$, is the product over the places of M and Q of the index in $\mathrm{GL}_n(\mathbb{F}_w)$ of the group fixing the basis vectors ι: the component of the level is this group at a place of M and a parabolic at a place of Q, and at $w \in Q$ the index of the parabolic times $\#\,\mathrm{GL}_r(\mathbb{F}_w)$ is the index of the group fixing r basis vectors. That index is the number of $n \times r$ matrices of rank r over $\mathbb{F}_w$, whose proportion among all $n \times r$ matrices is $\prod_{i=t+1}^{t+r}(1 - q_w^{-i})$. With (a), the Gram factor divided by the number of matrices is $\prod_{w \in M} \prod_i (1 - q_w^{-i}) / \#\,\mathrm{GL}_r(\mathbb{F}_w) \cdot (\mathrm{N}\,Q)^{rt} \prod_{w \in Q} \prod_i (1 - q_w^{-i}) = c_M c_Q$. Finally $c_M \neq 0$ because $q_w^i \neq 1$ for $i \geq 1$. □

30.3. Rallis' formula for one lift

For a tame character ϑ and $1 \leq j \leq m$, the *Rallis series* $R_{j,\vartheta}$ on the forms of level J_M on a space of rank r is $R_{j,\vartheta} y = \sum_\gamma \mathrm{mc}(\gamma)\, \gamma y$, the sum over the finite points modulo those integral away from the places of $\pi_M Q$ and their conjugates, taken at representatives supported off these places and acting by right translation, with the *vacuum coefficient* $\mathrm{mc}(\gamma) = (\chi_V(\det \gamma)\, \mathrm{fm}(\gamma)^{r+t})^{-1}$ for the character χ_V of the j-th pair on the side of V_j and the finite multiplier fm of the Weil representation. Since $r + t > 2r$ (Weil's range), the series converges absolutely and $R_{j,\vartheta} y$ is a form of level J_M.

Lemma 30.4 (The Rallis series). *Fix j, ϑ and a space of rank r. Then $2r < r + t$; for forms x, y of level J_M the series $z = R_{j,\vartheta} y$ is a form of level J_M with $\langle x, z \rangle = \sum_\gamma \mathrm{mc}(\gamma) \langle x, \gamma y \rangle$, the series of inner products converging absolutely; and $y \mapsto R_{j,\vartheta} y$ is a linear endomorphism of the forms of level J_M whose matrix coefficients are these sums.*

Proof. The rank of V_j is $r + t$ with $t > r$ by the construction of the descent pairs (Section 15.6). By Proposition 30.3(b) the block datum is sharp for J_M, so off $\pi_M Q$ the character is unramified and J_M contains the integral points there; the series of the Rallis formula (Theorem B.34(b), (c)) converges absolutely in Weil's range and its sum is invariant under J_M, so it is a form of level J_M. The Petersson product with the series may be taken term by term, by absolute convergence. A form is determined by its inner products, which are sums of terms linear in y; this gives the endomorphism. □

Theorem 30.5 (Norm of one lift). *Let $1 \leq j \leq m$. There is a constant $c_j \neq 0$, independent of the tame character ϑ, such that for every ϑ, with $\Theta_j = \Theta_{j,\vartheta}$:*

(a) *Θ_j is injective on the forms of level J_M on W_j;*

(b) *$\langle \Theta_j x, \Theta_j y \rangle = c_j c_Q \sum_\gamma \mathrm{mc}(\gamma) \langle x, \gamma y \rangle$, the sum over the cosets of the finite points modulo those integral away from $\pi_M Q$;*

(c) *$\Theta_j^* \Theta_j = c_j c_Q\, R_{j,\vartheta}$ on the forms of level J_M;*

(d) *Θ_j commutes with $\bar{G}$, acting on W_j by translation and on V_j by ρ; if y lies in a Hecke eigenspace with eigenvalues $p(w,i)$, then $\Theta_j y$ is spherical at each split*

w off the level places with Hecke polynomial the q-Pochhammer polynomial of rank t times the polynomial of the eigenvalues, at rescaled variables; and the operator $H_w(X) = \sum_i (-X)^i q^{i(i-1)/2} T_{w,i}$ at $X_* = (\alpha q^{t-1})^{-1}$, α the Frobenius value of the character on W_j, kills every lift.

Proof. Rallis' formula. By Proposition 30.3(b) the block datum is sharp for J_M with mass c_Ψ, and the real vector is the Fock vector of the joint harmonic v_P, which is an edge (Proposition 15.11(b)). Rallis' inner product formula for sharp data in Weil's range (Theorem B.34(c)) applies to the lift by $v_P \otimes \Psi_{\mathrm{bl}} \otimes \mathrm{vac}$: it is injective on the J_M-invariant forms of the type, which is (a), and

$$\langle Lx, Ly \rangle_{\mathrm{Pet}} = A\, c_\Psi \sum_\gamma \mathrm{mc}(\gamma) \langle x, \gamma y \rangle$$

with $A \neq 0$ depending only on the spaces and the real data. These do not involve ϑ, which has weight 0; they are the same as for the trivial character.

(b) The class sum product on V_j is the index of the level times the dimension of the weight times the Petersson product. With $c_\Psi = \sum |\Psi_{\mathrm{bl}}|^2 / (\#\text{matrices} \cdot [K_0 : J_M])$, the index of the level on V_j times c_Ψ is the Gram factor divided by the number of matrices modulo $\pi_M Q$, which is $c_M c_Q$ by Proposition 30.3(c). Hence $\langle \Theta_j x, \Theta_j y \rangle = A \dim\, c_M c_Q \sum_\gamma \mathrm{mc}(\gamma) \langle x, \gamma y \rangle$. The chosen places, the ranks and the weight do not depend on ϑ, so $c_j = A \dim c_M$ is a constant independent of ϑ, nonzero as $A \neq 0$ and $c_M \neq 0$.

(c) By Lemma 30.4, $\langle x, R_{j,\vartheta} y \rangle = \sum_\gamma \mathrm{mc}(\gamma) \langle x, \gamma y \rangle$, so (b) says $\langle \Theta_j x, \Theta_j y \rangle = c_j c_Q \langle x, R_{j,\vartheta} y \rangle$ for all x, y, that is $\Theta_j^* \Theta_j = c_j c_Q R_{j,\vartheta}$; the adjoint of Θ_j is the descent by Proposition 30.2(b).

(d) Equivariance is Proposition 30.2(a) with k the chosen lift of s, the chosen places being among the places of the pair. The Hecke polynomial is Proposition 30.2(c) for an eigenform: with $p(w, \cdot)$ the eigenvalues of the $T_{w,i}$, the polynomial $p(X)$ of y is $\sum_i (-X)^i q^{i(i-1)/2} p(w, i)$, and the polynomial of the lift is $\prod_{i<t} (1 - q^i \alpha X)$, the q-Pochhammer polynomial of rank t, times $p(q^t \alpha \beta^{-1} X)$. The factor $1 - q^{t-1} \alpha X$ vanishes at X_*, so $H_w(X_*)$ kills the lift of every form in an eigenpart; the eigenparts span the forms of level J_M (Proposition 23.1(b)), and $H_w(X_*)$ kills all lifts by linearity. □

31. The exchange identity

The chain Θ_ϑ is a composite of m twisted lifts, while Rallis' formula computes the norm of one lift. To pass from one to the other the lifts are moved past each other. The tool is an identity between theta integrals to $\mathrm{U}(W)$ from two spaces V and V' of complementary ranks, $\mathrm{rk}\, W = \mathrm{rk}\, V + \mathrm{rk}\, V'$: on matched ramified data the two theta integrals are proportional, with a constant whose modulus is given by Hecke L-values (Theorem 31.1). For trivial characters an identity of this shape is the Siegel–Weil formula; here the characters are tamely ramified, the constant is controlled only in absolute value and up to a factor free of the character, and that is what the norms of chains require.

The proof has four steps. Matched data are reduced to standard data: the difference of the two sides is an equivariant lift, whose kernel is a stable space, and a stable space that contains one standard datum contains every datum with a matched function (Section 31.2). On standard data the identity is first proved

at the top of a tower, for a triple in which the large space is $V' \oplus W_1$ and W_1 splits off a hyperbolic plane; there it follows from an identity of periods over $[\mathrm{U}(W_1)]$, which a smooth form of Witt's theorem provides (Section 31.3). The identity then descends along hyperbolic planes, by constant terms along isotropic lines, which are Tate integrals (Theorem B.32). The factor produced at the top is the equal rank ratio, which is shown to be one nonzero constant for a suitable vector (Section 31.4).

31.1. Matched functions and the statement

Let W, V, V' be hermitian spaces with index sets n, a, b and signs s, t, t', and let $\sigma\colon b \sqcup a \to n$ order the index set of W. For an invertible k write y_k for the columns b of k and x_k for the rows a of k^{-1}. For a modulus $\mathfrak{m}$ (a nonzero ideal of O_E prime to its conjugate) and a character χ of $(O_E/\mathfrak{m})^\times$, functions Φ on $a \times n$ and Ψ on $n \times b$ matrices modulo $\mathfrak{m}$ are *matched* if Φ vanishes off the x_k, Ψ off the y_k, $k \in \mathrm{GL}_n(O_E/\mathfrak{m})$, and $\Phi(x_k) = \chi(\det k)\Psi(y_k)$. A character ν of $[\mathrm{U}(1)]$ is *tame for* $(\mathfrak{m}, \chi)$ if it has a weight μ at the infinite places and its tame part is ramified exactly at the places dividing $\mathfrak{m}$ and gives χ on the residue ring. For $e \geq 0$ put

$$L_*(e) = \Big(\prod_{w|\mathfrak{m}} q_w\Big)^{e^2} \prod_{z=1}^{e} |L^S(z,\nu)|^2, \qquad L_*(-e) = L_*(e)^{-1},$$

with L^S the partial Hecke L-function of the base change $x \mapsto \nu(x/\bar{x})$ without the places S of $\mathfrak{m}$ and their conjugates. The *rank twist* is $\gamma = \nu^{-1}\upsilon^{\lfloor |n|/2 \rfloor}$, with υ the splitting character of the Weil data. The *space character* of a character α on a space of rank p is $x \mapsto \alpha(x/\bar{x})\upsilon(x)^{-p}$. For a real Fock vector x and a function Φ modulo $\mathfrak{m}$, $\theta_{W,V}(x, \Phi)$ denotes the theta integral over $[\mathrm{U}(V)]$ of the datum with real part x, the function Φ at the places of $\mathfrak{m}$ and the vacuum at the other finite places, as a function on $\mathrm{U}(W)(\mathbb{A})$; a real vector is *Schwartz* if all its Fock moments are finite.

Theorem 31.1 (Exchange identity). *Let $|n| = |a| + |b|$, $|a| = |b| + 2e$, let W have as many negative signs q as V and V' together, with $2q + 2 \leq |n| \leq 2\mu(\tau) + 2q$ at every infinite place τ, and let v be a real Schwartz vector for (V', W). There are $C > 0$ and a Schwartz vector x for (W, V), chosen before $\mathfrak{m}$ and ν, such that every character ν of weight μ that is tame for $(\mathfrak{m}, \chi)$ has a number c with:*

(a) $|c|^2 = C\, L_*(e) \neq 0$;

(b) *for all characters α, α' whose space characters on V and on V' multiply to the inverse base change of ν, and for all matched Φ, Ψ modulo $\mathfrak{m}$,*

$$\theta_{W,V}(x, \Phi) = c \cdot \theta_{V',W}(v, \Psi) \qquad \text{on } \mathrm{U}(W)(\mathbb{A}),$$

the character of W being the rank twist γ.

The identity is used in two forms. The first exchanges a lift and a transposed lift in a square of four spaces; the second makes the real data explicit as determinants when one of the spaces is definite.

Corollary 31.2 (One square). *Let U_1, U_2, A, A' be hermitian spaces with $W = U_1 \oplus U_2$, $\mathrm{rk}\, A = \mathrm{rk}\, A' + 2t$, the number N of negative signs of W that of A and A' together, and $2N + 2 \leq \mathrm{rk}\, W \leq 2\mu(\tau) + 2N$. For i in a set I let the pairs (U_1, A), (U_2, A), (A', U_1), (A', U_2) carry characters of $[\mathrm{U}(1)]$ and polynomials making them*

edges of fixed types, whose space characters multiply in pairs to the inverse base change of a character ν_i tame for $(\mathfrak{m}_i, \chi_i)$. Let $\Theta_i(x)$ be the lift from U_1 to A followed by the transposed lift to U_2, and $\Theta_i'(y)$ the transposed lift to A' followed by the lift to U_2, with the indicator functions of the blocks of x, y as data modulo $\mathfrak{m}_i$. Then both transform by $\chi_i(\det g)$ under $x \mapsto gx$, and there are $C > 0$ and κ, possibly 0, and for each i a number c_i with $|c_i|^2 = C\, L_{*,i}(t) \neq 0$, such that

$$\sum_y \Psi'(y)\,\Theta_i'(y) = \frac{\kappa}{c_i} \sum_x \Psi(x)\,\Theta_i(x)$$

for all matched functions Ψ, Ψ' modulo $\mathfrak{m}_i$.

Corollary 31.3 (Explicit exchange of class sums). *Let W of rank n and V' of rank b both have l negative signs, I_a the definite standard space, $n = b + a$, $a = b + 2d$ and $2l + 2 \leq n$. For each infinite place τ let ι_τ index $p \times m$ matrices $\eta_{\tau,i}$. There is C, possibly 0, such that every character α of weight $l - \#\iota_\tau$ tame for $(\mathfrak{m}, \chi)$ has a number c with $|c|^2 = C/L_*(d)$ for the character $\mu = \alpha^{-1}\upsilon^{b+d}$, and for matched Φ, Ψ for χ^{-1}: c times the class sum over $\mathrm{U}(I_a)$ of the kernel with datum Φ and the product of the determinants of the $\eta_{\tau,i}$ stacked over the variables equals the integral over $[\mathrm{U}(V')]$ of the kernel with datum Ψ and the polynomial $\prod_\tau \det(y_\tau^-)^{\#\iota_\tau + n - 2l} \prod_{\tau,i} \det(\eta_{\tau,i} y_\tau)$, where y_τ is the block of variables with both signs positive and y_τ^- the block with both signs negative.*

The two corollaries are deduced from Theorem 31.1 in Section 31.5.

31.2. Standard data suffice

Let F be a finite product of non-archimedean local fields with integers O and χ a tame character of F (its restriction to $O^\times$ factors through a finite ring with a primitive character). The *standard datum* of an integral matrix z is $\mathrm{std}_z(y) = \chi^{-1}(\det(zy))$ at integral y with $\det(zy) \in O^\times$, and 0 elsewhere; the *row datum* is χ^{-1} of the determinant of z stacked on the variable. The *section* of Ψ at a left invertible y is $\sum_g \chi(\det g)\Psi(yg)$ over representatives of $\mathrm{GL}_b(F)/\mathrm{GL}_b(O)$; (Φ, Ψ) is a *same-section pair* if both are Schwartz–Bruhat on left invertible matrices and the section of $y \mapsto \Phi(y^\mathsf{T})$ at x_k^T is $\chi(\det k)$ times that of Ψ at y_k for every k.

A matrix over F is *primitive* if it is integral and has an integral left inverse. Let $\omega \in O$ be the product of uniformisers of the factors of F; it is invertible in F and topologically nilpotent, and $O/\omega O$ is the product of the residue fields.

Lemma 31.4 (Primitive matrices and cosets).
(a) *For $t \in O$ the element $1 - \omega t$ is a unit of O; the units of O form an open subset of F; and the primitive $n \times a$ matrices form an open and closed set. A square matrix is primitive exactly when it is integral with determinant a unit of O.*
(b) *There are representatives r_q of the cosets q in $\mathrm{GL}_b(F)/\mathrm{GL}_b(O)$ such that for every $n \times b$ matrix y with a left inverse exactly one q makes yr_q primitive and exactly one makes yr_q^{-1} primitive. For such y the matrices g with yg primitive form one coset $u\,\mathrm{GL}_b(O)$.*
(c) *A matrix x is the block x_k of rows a of k^{-1} for some $k \in \mathrm{GL}_n(O)$ exactly when x^T is primitive.*

(d) *If Φ is locally constant with compact support in the matrices x with x^{T} left invertible, there is a finite set of cosets q in $\mathrm{GL}_a(F)/\mathrm{GL}_a(O)$ such that every x in the support has $x^{\mathsf{T}} r_q^{-1}$ primitive for one of them.*

Proof. (a) Put $x = \varpi t$. The multiples of $1 - x$ by elements of O form a compact set, which contains $1 - x^k = (1 - x)(1 + x + \cdots + x^{k-1})$ for every k; as $x^k = \varpi^k t^k$ with t^k in the compact set O, $x^k \to 0$, so the set contains 1, and $1 - x$ is a unit. If $sx = 1$ with $s \in O$, then $sz \in 1 - \varpi O$ for integral z close to x, so sz and hence z is a unit: the units are open. An integral matrix is primitive exactly when its reduction modulo ϖ has full column rank at every factor (a left inverse of the reduction lifts to an integral L with $Lx \in 1 + \varpi M(O)$, and Lx is invertible by the first claim); this condition is open and closed on the open and closed set of integral matrices. For a square matrix the adjugate divided by a unit determinant is an integral inverse, and conversely the determinant of an integral inverse inverts $\det x$.

(b) Every left coset $g\,\mathrm{GL}_b(O)$ has a representative r with $r^{\mathsf{T}} = r$ (by the elementary divisor theorem, $g = k_1 d k_2$ with $k_i \in \mathrm{GL}_b(O)$ and d diagonal, and $k_1 d k_1^{\mathsf{T}}$ lies in the coset). For symmetric representatives inversion agrees with the inverse transpose $g \mapsto (g^{\mathsf{T}})^{-1}$, which permutes the left cosets because $\mathrm{GL}_b(O)$ is stable under transposition. If y has a left inverse, some yu is primitive (elementary divisors again), and if yu and yg are both primitive, with L an integral left inverse of yu, then $u^{-1}g = L\,yg$ is integral, and symmetrically $g^{-1}u$ is, so $g \in u\,\mathrm{GL}_b(O)$. Hence exactly one representative r_q makes yr_q primitive, and since the r_q^{-1} again run over representatives of all cosets, exactly one q makes yr_q^{-1} primitive.

(c) If $x = x_k$ with $k \in \mathrm{GL}_n(O)$, the identity $k^{-1}k = 1$ gives $x\,k_{\cdot a} = 1$ for the columns a of k, so x^{T} has the integral left inverse $k_{\cdot a}^{\mathsf{T}}$. Conversely, if x^{T} is primitive, the rows of x reduce to linearly independent rows at every factor, so they complete to an integral matrix m with determinant a unit (complete the reductions to a basis and lift), and $k = m^{-1}$ lies in $\mathrm{GL}_n(O)$ with $x_k = x$.

(d) By (b) each x in the support has $x^{\mathsf{T}} r_q^{-1}$ primitive for one q, and the set of x for which a given q works is open, by (a). The support is compact, so finitely many of these open sets cover it. □

Proposition 31.5 (Matched functions). (a) *The block determinant functions $\Phi(x_1\ x_2) = \chi^{-1}(\det x_2) f(x_2^{-1}x_1)$ and $\Psi\binom{y_1}{y_2} = \chi^{-1}(\det y_1) f(-y_2 y_1^{-1})$ are matched.*

(b) *Matched functions over O, with Ψ locally constant, form a same-section pair; same-section pairs are stable under $(\Phi, \Psi) \mapsto (\chi(\det g)\Phi(\cdot g), \Psi(g^{-1}\cdot))$; and a linear map of Φ invariant under $\Phi \mapsto \chi(\det g)\Phi(g\,\cdot)$ is, on same-section pairs, a linear function of Ψ alone.*

(c) *Functions modulo $\mathfrak{m}$ matched for the residue character give matched local functions over O with the same adelic data.*

Proof. (a) Here the matrices are cut along σ: $x = (x_1\ x_2)$ with x_2 the square block of size a, and $y = \binom{y_1}{y_2}$ with y_1 the square block of size b; the functions are 0 where the square block is not invertible. *Supports.* If x_2 is invertible, the block triangular matrix with rows $(1\ 0)$ and $(x_1\ x_2)$ is invertible, and its inverse k has $x_k = x$; if y_1 is invertible, the matrix with columns $\binom{y_1}{y_2}$ and $\binom{0}{1}$ is invertible and

has $y_k = y$. *The identity.* Let k be invertible with $x_k = (x_1\ x_2)$ and $y_k = \binom{y_1}{y_2}$. The rows a of k^{-1} times the columns b of k vanish, so $x_1y_1 + x_2y_2 = 0$, and Jacobi's identity for complementary minors gives $\det y_1 = \det x_2 \cdot \det k$. Hence x_2 is invertible exactly when y_1 is, and then $x_2^{-1}x_1 = -y_2y_1^{-1}$, so

$$\Phi(x_k) = \chi^{-1}(\det y_1)\chi(\det k)f(-y_2y_1^{-1}) = \chi(\det k)\Psi(y_k);$$

otherwise both sides are 0.

(b) *Matched functions are a same-section pair.* Let Φ, Ψ be matched over O with Ψ locally constant. The matrices y_k, $k \in \mathrm{GL}_n(O)$, are primitive, and by Lemma 31.4(c) so are the transposes of the x_k; both functions therefore vanish off the primitive matrices (after transposing Φ), whose set is open, closed and compact within the integral matrices. The function $y \mapsto \Phi(y^{\mathsf{T}})$ is locally constant: its pull-back to the compact group $\mathrm{GL}_n(O)$ under $k \mapsto x_k^{\mathsf{T}}$ is $k \mapsto \chi(\det k)\Psi(y_k)$, which is locally constant, and the image of this map is the open set of primitive matrices. Hence both functions are Schwartz–Bruhat. By Lemma 31.4(b), for a left invertible y there is g with yg primitive, and the section at y of a function that vanishes off the primitive matrices is the single term $\chi(\det g)\Psi(yg)$. Let $k \in \mathrm{GL}_n(F)$, and choose $g \in \mathrm{GL}_a(F)$, $u \in \mathrm{GL}_b(F)$ with $x_k^{\mathsf{T}}g$ and y_ku primitive. Put $j = k\,\mathrm{diag}(u, (g^{\mathsf{T}})^{-1})$ in the order of σ; then $y_j = y_ku$, $x_j = g^{\mathsf{T}}x_k$ and $\det j \det g = \det k \det u$, so the section condition at k is the identity $\Phi(x_j) = \chi(\det j)\Psi(y_j)$. For it, x_j^{T} is primitive, so $x_j = x_i$ for some $i \in \mathrm{GL}_n(O)$ (Lemma 31.4(c)). The columns y_i and y_j both span the kernel of the rows x_i, so $y_j = y_ig'$ for some $g' \in \mathrm{GL}_b(F)$, and $g' = Ly_j$ is integral for an integral left inverse L of y_i, as is its inverse; hence $g' \in \mathrm{GL}_b(O)$, and $i' = i\,\mathrm{diag}(g', 1) \in \mathrm{GL}_n(O)$ has $x_{i'} = x_i$ and $y_{i'} = y_j$. For any P with $Py_j = 1$, the matrix with rows P and x_j times j is block triangular with unit diagonal, because $x_jy_j = 0$ and x_j times the columns a of j is 1; so $\det j$ is the inverse of that determinant and depends only on (x_j, y_j). Hence $\det j = \det i'$, and the matching for i' gives the identity.

Translation. Let $g \in \mathrm{GL}_n(F)$. The rows a of $(g^{-1}k)^{-1}$ are x_kg and the columns b of $g^{-1}k$ are $g^{-1}y_k$. Substituting in the definition of the sections, the section of $y \mapsto \chi(\det g)\Phi(y^{\mathsf{T}}g)$ at x_k^{T} is $\chi(\det g)$ times the section of $y \mapsto \Phi(y^{\mathsf{T}})$ at $x_{g^{-1}k}^{\mathsf{T}}$, which is $\chi(\det g)\chi(\det g^{-1}k)$ times the section of Ψ at $g^{-1}y_k$, that is $\chi(\det k)$ times the section of $\Psi(g^{-1}\cdot)$ at y_k. The support conditions are clearly preserved.

A linear function of Ψ. Let Λ be linear on the functions of $a \times n$ matrices with $\chi(\det g)\Lambda(\Phi(g\,\cdot)) = \Lambda(\Phi)$ for all $g \in \mathrm{GL}_a(F)$ and all locally constant Φ of compact support in the matrices with left invertible transpose. For x with x^{T} primitive choose $k(x) \in \mathrm{GL}_n(O)$ with $x_{k(x)} = x$ (Lemma 31.4(c)), and let Φ_Ψ be the function equal to $\chi(\det k(x))$ times the section of Ψ at $y_{k(x)}$ at such x and to 0 elsewhere; put $T(\Psi) = \Lambda(\Phi_\Psi)$, which is linear in Ψ. Let (Φ, Ψ) be a same-section pair. By Lemma 31.4(d), $\Phi = \sum_q \Phi_q$ for finitely many cosets q, where Φ_q is Φ on the x with $x^{\mathsf{T}}r_q^{-1}$ primitive and 0 elsewhere. Let $m_q(x) = \Phi(r_q^{\mathsf{T}}x)$ if x^{T} is primitive and $m_q(x) = 0$ otherwise; then $\Phi_q(x) = m_q((r_q^{\mathsf{T}})^{-1}x)$, and the invariance of Λ gives $\Lambda(\Phi_q) = \chi(\det r_q)\Lambda(m_q)$. At an x with x^{T} primitive, $\sum_q \chi(\det r_q)m_q(x)$ is the section of $y \mapsto \Phi(y^{\mathsf{T}})$ at x^{T}, and by the same-section property it equals $\chi(\det k(x))$ times the section of Ψ at $y_{k(x)}$, that is $\Phi_\Psi(x)$; both vanish at the other x. Hence $\Lambda(\Phi) = \sum_q \chi(\det r_q)\Lambda(m_q) = \Lambda(\Phi_\Psi) = T(\Psi)$.

(c) Let π be the reduction from the integers at the places of $\mathfrak{m}$ to $O_E/\mathfrak{m}$, and let $\Phi' = \Phi \circ \pi$ and $\Psi' = \Psi \circ \pi$ on integral matrices, 0 elsewhere. Reduction

modulo $\mathfrak{m}$ sees only the components at these places, so the adelic data of Φ and Φ' (and of Ψ and Ψ') agree. The map π is onto and detects units, being compatible with the residue maps, and on units the local character and the residue character are both the product of the residue characters of the tame part. *Supports.* If $\Phi'(x) \neq 0$, then x is integral and $\pi(x)$ is the block of rows of the inverse of an invertible matrix modulo $\mathfrak{m}$, so the reduction of x^{T} has a left inverse; this lifts, and x^{T} is primitive by Lemma 31.4(a), so $x = x_k$ with $k \in \mathrm{GL}_n(O)$ by Lemma 31.4(c). Likewise for Ψ'. *Values.* For $k \in \mathrm{GL}_n(O)$, $\pi(x_k) = x_{\pi(k)}$ and $\pi(y_k) = y_{\pi(k)}$, so $\Phi'(x_k) = \chi(\det \pi(k))\Psi'(y_k)$, with $\chi(\det \pi(k))$ the local character at $\det k$. Finally Ψ' is locally constant, as it factors through the reduction on the open and closed set of integral matrices. □

An *equivariant lift* on a space M of Schwartz–Bruhat functions at the ramified places S is a linear map $\mathcal{D}$ from M to automorphic functions on $\mathrm{U}(X)$ that turns translation at S into right translation up to a character of the determinant and has values invariant under the integral points off S.

The places S of a split tame character are split over the totally real subfield, so a point of a unitary group supported on $S \cup \bar{S}$ is determined by its component at S, an element of GL over $E_S = \prod_{w \in S} E_w$, and every element of $\mathrm{GL}(E_S)$ occurs. Write $\chi_{\mathrm{loc}}(t) = \nu(t/\bar{t})|t|^{\mathrm{rk}\, W/2}$ on $E_S^\times$.

Lemma 31.6 (Translation laws). *Let (ν, S) be a split tame character, $\mathrm{rk}\, W = |b| + \mathrm{rk}\, V$ even, and let both characters of the pair (W, V) be the rank twist γ.*

(a) *For a finite point k of $\mathrm{U}(V)$ supported on $S \cup \bar{S}$, with component k_S at S, the Weil representation of $(1, k^{-1})$ carries the datum with real part v and the function Φ at S to $\chi_{\mathrm{loc}}(\det k_S)$ times the datum with $x \mapsto \Phi(k_S x)$ in place of Φ.*

(b) *For a finite point g of $\mathrm{U}(W)$ supported on $S \cup \bar{S}$, with component g_S, the theta integral $\theta_{W,V}$ of $y \mapsto \chi_{\mathrm{loc}}(\det g_S)^{-1}\Phi(y g_S^{-1})$ at xg is that of Φ at x divided by $(v|\cdot|^{1/2})(\det g_S)^{|b|}$.*

(c) *With $\Psi(g_S\, \cdot)$ in place of Φ, the same holds for the theta integral $\theta_{V',W}$ of the small side.*

(d) *Both theta integrals are right invariant under the integral points of $\mathrm{U}(W)$ that are trivial at S, and the small side is linear in Ψ.*

(e) *If v has all moments finite, $\theta_{W,V}(v, \Phi)(g)$ is, on same-section pairs (Φ, Ψ) for χ_{loc}, a linear function of Ψ.*

Proof. (a) A finite point (g, h) supported on a set of places A and its conjugate acts on the datum given by a function φ at A by the substitution $x \mapsto \varphi(h^{-1}xg)$ of the components at A, times $\chi_W(\det h)\chi_V(\det g)^{-1}|\det g|^{\mathrm{rk}\, V/2}|\det h|^{-\mathrm{rk}\, W/2}$, the determinants taken at A, with χ_W, χ_V the space characters of the two characters of the pair; this is the explicit form of the finite Weil representation in the Schrödinger model (Appendix B.10). Take $A = S$ and $(g, h) = (1, k^{-1})$. The determinant of k is the idele equal to $t = \det k_S$ at S, to $\bar{t}^{-1}$ at $\bar{S}$ and to 1 elsewhere; evaluating the space character of the rank twist in rank $\mathrm{rk}\, W$ and the modulus at this idele gives the scalar $\nu(t/\bar{t})|t|^{\mathrm{rk}\, W/2} = \chi_{\mathrm{loc}}(\det k_S)$, the powers of the splitting character entering γ and the space character compensating each other.

(b) Right translation of the theta integral by g is the Weil representation of $(g, 1)$ on the datum. By the formula of (a) for $(g, 1)$ the datum of $\Phi' =$

$\chi_{\mathrm{loc}}(\det g_S)^{-1}\Phi(\cdot\, g_S^{-1})$ becomes that of Φ, times $\chi_{\mathrm{loc}}(\det g_S)^{-1}$ and the value at $\det g$ of $\chi_V^{-1}|\cdot|^{\mathrm{rk}\,V/2}$, with χ_V the space character of γ in rank $\mathrm{rk}\,V$. At the idele $\det g$, equal to $d = \det g_S$ at S and $\bar{d}^{-1}$ at $\bar{S}$, the factors $\nu(d/\bar{d})$ cancel and the moduli and splitting characters combine to $(\upsilon|\cdot|^{1/2})(d)^{\mathrm{rk}\,V-\mathrm{rk}\,W} = (\upsilon|\cdot|^{1/2})(d)^{-|b|}$, as $\mathrm{rk}\,W - \mathrm{rk}\,V = |b|$.

(c) On the small side the datum is $y \mapsto \Psi(y^{\mathsf{T}})$ at the conjugate places $\bar{S}$, where g has the component $(\bar{g}_S^{\mathsf{T}})^{-1}$; transposing, the substitution becomes $\Psi \mapsto \Psi(g_S\,\cdot)$. The characters of the small side are $\nu\upsilon^{\mathrm{rk}\,W/2}$ on W and $\upsilon^{|b|}$ on V', and the scalar, now a value at the component $\bar{d}^{-1}$ at $\bar{S}$, is again $(\upsilon|\cdot|^{1/2})(d)^{-|b|}$.

(d) An integral point trivial at S fixes the vacuum at the other finite places and does not touch the components at S, so it fixes the datum. Linearity of the small side holds because the theta integral is linear in the datum and is given, on data of polynomial growth, by an absolutely convergent integral.

(e) The functional $\Lambda(\Phi) = \theta_{W,V}(v, \Phi)(g)$, extended linearly to all functions, satisfies $\chi_{\mathrm{loc}}(\det k)\Lambda(\Phi(k\,\cdot)) = \Lambda(\Phi)$ for $k \in \mathrm{GL}_a(E_S)$: by (a) the datum of $\Phi(k\,\cdot)$ is $\chi_{\mathrm{loc}}(\det k)^{-1}$ times the translate of that of Φ by a point of $\mathrm{U}(V)$ supported on $S \cup \bar{S}$, and the integral over $[\mathrm{U}(V)]$ does not see this translate. The last part of Proposition 31.5(b) gives the linear function of Ψ. □

Proposition 31.7 (Equivariant lifts). (a) *Let $\mathrm{rk}\,W = \mathrm{rk}\,V' + \mathrm{rk}\,V$ be even, v, v' real Schwartz vectors and $c \in \mathbb{C}$. There are M and an equivariant lift $\mathcal{D}$, for the character $\upsilon^{-|b|}|\cdot|^{-|b|/2}$ of the determinant, such that every locally constant Ψ with a matched Φ lies in M and $\mathcal{D}\Psi = \theta_{W,V}(v, \Phi) - c\,\theta_{V',W}(v', \Psi)$.*

(b) (*Stable spaces.*) *Let P be a $\mathrm{GL}_n(F)$-stable space of locally constant compactly supported functions vanishing off a set U. If P contains $r \cdot \mathrm{std}_s$ for an $\mathbb{N}$-valued r with $r(x) \neq 0$ at an integral x with $sx = 1$, then P contains every locally constant Ψ with a matched Φ.*

(c) (*Self-dual lattices suffice.*) *Let D, D_1 be adapted splittings of X over W_1 along b lines and of W_1 along one more line, and let $f = \mathcal{D}\Psi_0$ for the standard datum of the dual block of the frame of D. If f vanishes at self-dual lattices along D, that is, its Fourier coefficients along the centre of the unipotent radical vanish at the test pairs of V' (the form of V' at the points spanning a self-dual lattice, or an invertible form not represented by the signs of V' at some real place), then $\mathcal{D}$ vanishes on every locally constant datum with a matched function.*

(d) (*Negative indices.*) *If the weights are less than the number of positive signs of the complement Z of D, and T is invertible and at some real place not of the form $-c^*c$, then the Fourier coefficient at $\jmath T$ of the theta integral vanishes, provided the unfolded terms are absolutely integrable.*

The proof of (b) is an average over $\mathrm{GL}_n(O)$, that of (c) a Gauss sum of shears, and that of (d) an argument with a compact group; they are given in three lemmas.

Lemma 31.8 (Averaging). *Statement* (b) *of Proposition 31.7 holds. More precisely, let Ψ be locally constant with a matched function, let c be such that Ψ is invariant under translation by $\varpi^c M_{n\times b}(O)$, and let $d = \varpi^{-c}(1 - x(s - \varpi^c s))$. Then for $k \in \mathrm{GL}_n(O)$ and all y,*

$$\Psi(k^{-1}x)\,p(dky) = |p(dky)|\,\Psi(y), \qquad p = r \cdot \mathrm{std}_s,$$

and the average of $y \mapsto \Psi(k^{-1}x)p(dky)$ over $k \in \mathrm{GL}_n(O)$ is a positive multiple of Ψ.

Proof. Invariance. The matrices t with $\Psi(\cdot + t) = \Psi$ form an open subgroup, because Ψ is locally constant with compact support; it contains $\varpi^c M(O)$ for large c, the integral matrices being compact and $\varpi^c \to 0$. *The matrix d.* As $sx = 1$, the matrix $e = xs$ is idempotent; $d = \varpi^{-c}(1-e) + e$, with inverse $d^{-1} = \varpi^c(1-e) + e$, and $dx = x$, $sd = s$. *The identity.* If $p(dky) \neq 0$, then $z = dky$ is integral and $g = sz = sky$ lies in $\mathrm{GL}_b(O)$. Then $ky = d^{-1}z = \varpi^c(1-e)z + xg$, so $y \equiv k^{-1}xg$ modulo $\varpi^c M(O)$, and $\Psi(y) = \Psi(k^{-1}xg)$. The matrix $k^{-1}x$ is primitive, so it is $y_{k'}$ for some $k' \in \mathrm{GL}_n(O)$, and $k'\,\mathrm{diag}(g, 1)$ has the same rows $x_{k'}$ of the inverse and the columns $k^{-1}xg$; the matching at the two matrices gives $\Psi(k^{-1}xg) = \chi(\det g)^{-1}\Psi(k^{-1}x)$. On the other hand $p(dky) = r(z)\chi^{-1}(\det g)$. Hence

$$\Psi(k^{-1}x)p(dky) = r(z)\chi^{-1}(\det g)\chi(\det g)\Psi(y) = |p(dky)|\Psi(y),$$

since $r(z) \geq 0$ and $|\chi^{-1}(\det g)| = 1$. If $p(dky) = 0$ both sides vanish. *The average.* The family $k \mapsto \Psi(k^{-1}x)p(dk\,\cdot)$ is locally constant on the compact group $\mathrm{GL}_n(O)$ with values in P (each value is a multiple of a translate of p), so its mean is a finite combination of elements of P and lies in P. By the identity the mean is $I(y)\Psi(y)$ with $I(y) = \int |p(dky)|\,dk$. Where $\Psi(y) \neq 0$, $y = y_l$ for some $l \in \mathrm{GL}_n(O)$; and x, being integral with the integral left inverse s, is y_j for some $j \in \mathrm{GL}_n(O)$; the substitution $k \mapsto kjl^{-1}$ preserves the Haar measure and gives $I(y) = I(x)$. At $k = 1$, $p(dx) = p(x) = r(x)\chi^{-1}(\det sx) = r(x) \neq 0$, and $p(dkx)$ is locally constant in k, so $I(x) > 0$. Hence $\Psi = I(x)^{-1} \cdot (\text{mean}) \in P$. □

Let A be an *integral frame*: an integral basis of F^n with column blocks K, I, L (K and L of size $n \times b$) whose inverse is integral with row blocks K', I', L', so that $K'K = L'L = 1$ and $K'L = L'K = 0$. Let ψ be an additive character of F with conductor O, η the primitive character of $R = O/\varpi O$ with $\eta \circ \pi = \chi^{-1}$ on units and 0 on the rest of O, and l a section of the reduction of $b \times b$ matrices with $l(1) = 1$. Let $S_z(y) = \mathrm{std}_{K'}(y + LzK'y)$ be the shears of the standard datum $S_0 = \mathrm{std}_{K'}$.

Lemma 31.9 (Gauss sums of shears). (a) *The row datum of the dual block L' is $\varepsilon\,\chi^{-1}(\det x(K\ I))$ at integral x with $\det x(K\ I)$ a unit, and 0 elsewhere, for a constant $|\varepsilon| = 1$.*

(b) *For integral z the shear fixes S_0; for r over O and integral x, $\mathrm{std}_{L'}(LrK'x - \varpi x) = \eta(\det \pi(r))\,\mathrm{std}_{K'}(x)$; and if $\eta(\det \pi(T)) = 0$, then*

$$\sum_r \eta(\det r)\,\psi\big(\varpi^{-1}\,\mathrm{tr}(T\,l(r))\big) = 0,$$

the sum over the $b \times b$ matrices r over R.

(c) *Let P be a $\mathrm{GL}_n(F)$-stable space as in Proposition 31.7(b), and M a space containing all S_z. Suppose that P contains every $f \in M$ with $\Lambda(f) = 0$ for all matrices T and all functionals Λ such that $\Lambda(S_z) = \psi(\mathrm{tr}\,Tz)\Lambda(S_0)$ for all z and $\Lambda(S_0) = 0$ when $T \in \mathrm{GL}_b(O)$. Then P contains every locally constant Ψ with a matched function.*

Proof. (a) The square matrix with rows L' and x, times the integral basis $(L\ K\ I)$ of unit determinant, is block triangular with diagonal blocks 1 and $x(K\ I)$; so its determinant is a unit exactly when $\det x(K\ I)$ is, and the constant is $\varepsilon = \chi(\det(L\ K\ I))$. (b) $K'(y + LzK'y) = K'y$ because $K'L = 0$, and the added

term is integral when $K'y$ is, so $S_z = S_0$. For the second identity let k, u be the lifts of $K'x$, $L'x$; then $L'(LrK'x - \varpi x) = rk - \varpi u$, and $\chi^{-1} \circ \det$ of it, cut off to units, is $\eta(\det \pi(r))\eta(\det \pi(k))$ because ϖu vanishes modulo ϖ. For the Gauss sum, $x \mapsto \psi(\varpi^{-1}x)$ descends to a primitive additive character ϕ of R, and the pairing of T with $l(r)$ reduces to that of $\pi(T)$ with r; for the primitive character η of the finite ring, the transform $\sum_r \eta(\det r)\phi(\operatorname{tr}(\pi(T)r))$ is a constant times $\eta^{-1}(\det \pi(T))$, which is 0.

(c) Put $G = \sum_r \eta(\det r) S_{\varpi^{-1}l(r)}$, an element of M. Let Λ be a functional with the shear law for T. If T were not integral, the shears by integral z, which fix S_0, would give $\psi(\operatorname{tr} Tz) = 1$ for all integral z, contradicting the conductor; so T is integral. If $T \in \mathrm{GL}_b(O)$, then $\Lambda(S_0) = 0$ and $\Lambda(G) = 0$. Otherwise $\eta(\det \pi(T)) = 0$ and $\Lambda(G) = \sum_r \eta(\det r)\psi(\varpi^{-1} \operatorname{tr}(Tl(r)))\Lambda(S_0) = 0$ by (b). Hence $G \in P$. Evaluate G at $w = -\varpi^{-1}y$. The term of r is $\eta(\det r)\mathrm{std}_{K'}(x)$ with $x = w + \varpi^{-1}Ll(r)K'w$; as $K'L = 0$, $K'x = K'w$, so $w = x - \varpi^{-1}Ll(r)K'x$ and $y = Ll(r)K'x - \varpi x$. By (b), $\mathrm{std}_{L'}(y) = \eta(\det r)\mathrm{std}_{K'}(x)$, so every nonzero term equals $\mathrm{std}_{L'}(y)$, and the translate $y \mapsto G(-\varpi^{-1}y)$, which lies in P, is $r' \cdot \mathrm{std}_{L'}$ for the $\mathbb{N}$-valued function r' counting the nonvanishing terms. At $x_0 = L - \varpi K$, which is integral with $L'x_0 = 1$, the term $r = 1$ has $K'w = 1$ and $x = K$, so it is $\mathrm{std}_{K'}(K) = 1$; hence $r'(x_0) \neq 0$, and Lemma 31.8 applies with $s = L'$. □

Lemma 31.10 (No invariant functional). (a) *Let Y be a subspace of finite dimension of a Banach space on which a compact group acts by bounded operators, stable under the group, in which only 0 is fixed. Then an invariant bounded functional vanishes at every vector of Y with continuous orbit.*

(b) *Let $\mu(\tau)$ be less than the number of positive signs of t, let s have a positive sign and $|b|$ more negative signs than t, and let the Gram matrix of the columns of x for the signs s not be of the form $-c^*c$. Then there is an isometry u at τ such that every functional on the Fock space that is invariant under the compact stabiliser of ux, acting by the real Weil representation of weight μ, is zero.*

(c) *Statement* (d) *of Proposition 31.7 holds: under its hypotheses the Fourier coefficient at $\jmath T$ of the theta integral vanishes at every point.*

Proof. (a) The mean of the orbit of $w \in Y$ over the compact group lies in Y (a closed finite-dimensional stable subspace) and is fixed, so it is 0; the invariant functional takes the same value at w and at the mean.

(b) *Choice of u.* The Gram form of x is not negative semidefinite, so it represents 1, and by Witt's theorem an isometry u carries a vector of the span of x to the positive basis vector e_i; replace x by ux. *Rotations.* Let z be a point of the Fock variables. There is a unit vector r supported on the negative signs of s that kills the $|b|$ columns of x and the columns j of z with $t_j = -1$. Indeed, these conditions define a linear map f from the space of vectors on the negative signs of s, of dimension $|b| + \#\{j : t_j = -1\}$, to a space of the same dimension. If f were injective it would be onto; but $e_i = xc$ for some vector c, and for r supported on the negative signs the $|b|$ coordinates $r^{\mathsf{T}}x$ of $f(r)$ satisfy $r^{\mathsf{T}}x\,c = r_i = 0$, a nontrivial linear relation, so f is not onto. A nonzero vector of its kernel, normalised, is r. For $|\eta| = 1$ the rotation c_η by $\bar\eta$ of the line of $\bar r$ lies in the maximal compact subgroup, as it preserves every form commuting with the projection onto that line, and it fixes x. Acting at τ by the real Weil representation on the Fock vector of a polynomial P, it is a substitution times η^N

with $N = \#\{j : t_j = 1\} - \mu(\tau) > 0$: the substitution multiplies $\sum_i \bar{r}_i X_{\tau,i,j}$ by $\bar{\eta}$ if $t_j = -1$, and $\sum_i r_i X_{\tau,i,j}$ by η if $t_j = 1$, and fixes the orthogonal combinations. At the point z the first kind of combination vanishes, so the value at z of the moved polynomial is $g(\eta)$ for a polynomial g with $g(0) = 0$. If all c_η fix the vector of P, then g is constant on the circle, with mean $g(0) = 0$, so $P(z) = 0$; as z is arbitrary, $P = 0$. *Functionals.* The stabiliser acts on the polynomial vectors of each degree by a scalar times a linear substitution, so these finite-dimensional spaces are stable and contain no nonzero fixed vector; by (a) an invariant functional kills them, and polynomial vectors are dense.

(c) If no rational frame has Gram matrix T, the coefficient is an empty sum. Otherwise, after unfolding, it is the integral of the term of one frame x over the adelic stabiliser S_x of x modulo its rational points. Let τ be a real place at which T is not of the form $-c^*c$ and u as in (b), and let C be the intersection of S_x with the conjugate by u of the maximal compact subgroup at τ. Right translation by an element of a compact subgroup preserves the left Haar measure of S_x (the modular function is 1 on compact subgroups: both measures agree on the compact set AC), so the integral equals the integral of the mean of the term over right translates by C. On the slice at x the stabiliser acts by the Weil representation of the smaller pair (Z, Y), with the character ν when the character of the large pair is $\nu v^{|b|}$; so, for each fixed finite part, the mean is a bounded functional on the Fock space at τ invariant under the compact stabiliser of ux, and it is zero by (b). Data with fixed finite parts determine the functional, so the coefficient vanishes. □

Proof of Proposition 31.7. (a) Let M be the space of second members Ψ of same-section pairs (Φ, Ψ) for χ_{loc} and σ. By Lemma 31.6(e) there is a linear T on M with $T(\Psi) = \theta_{W,V}(v, \Phi)$ for every same-section pair; put $\mathcal{D}\Psi = T(\Psi) - c\,\theta_{V',W}(v', \Psi)$. Its values are automorphic, being theta integrals. The space M is stable under translation by $\mathrm{GL}_n(E_S)$ (Proposition 31.5(b)), and by Lemma 31.6(b), (c) translation of the pair by g_S multiplies both theta integrals, read at xg, by $(v|\cdot|^{1/2})(\det g_S)^{-|b|}$; so $\mathcal{D}$ turns translation at S into right translation, for the character $v^{-|b|}|\cdot|^{-|b|/2}$ of the determinant. Invariance under the integral points off S is Lemma 31.6(d). If Ψ is locally constant with a matched Φ for the semilocal character, the pair is matched for χ_{loc} too, since $|\det k| = 1$ for $k \in \mathrm{GL}_n(O)$, so it is a same-section pair by Proposition 31.5(b); hence $\Psi \in M$ and $\mathcal{D}\Psi = \theta_{W,V}(v, \Phi) - c\,\theta_{V',W}(v', \Psi)$.

(b) is Lemma 31.8.

(c) *Reduction to unimodular indices.* Let $P = M \cap \ker \mathcal{D}$. It is stable under translation at S: by equivariance, the image of a translate of a killed datum is a multiple of a translate of 0. Let the integral frame A be the frame of D read at S, so that Ψ_0 is the standard datum of its dual block, and let M' be the span of the shears of Ψ_0. For a point x and an index T, the Fourier coefficient of $\mathcal{D}\Psi$ along the centre of the unipotent radical of D at x is a linear functional $\Lambda_{T,x}$ of $\Psi \in M$, and the shear of the datum by z along the frame at x multiplies it by $\psi_S(\mathrm{tr}\, T_S z)$, where T_S is the component at S of $\delta^{-1}T$ and δ the inverse different idele: the shear is the action of a central element of the radical supported on $S \cup \bar{S}$. Take the points $x = (1, g)$ with g trivial at S. If $\Lambda_{T,x}(\Psi_0) = 0$ whenever $T_S \in \mathrm{GL}_b(O)$, then every $f \in M'$ killed by all these functionals has all central Fourier coefficients of $\mathcal{D}f$ equal to 0 at the points $(1, g)$ (hermitian matrices are periods of the centre, so only skew-hermitian indices occur), hence $\mathcal{D}f(1, g) = 0$,

and by weak approximation and the equivariance at S, $\mathcal{D} f = 0$. Hence $f \in P$, and Lemma 31.9(c) shows that P contains every locally constant datum with a matched function.

Unimodular indices are test pairs. It remains to show $\Lambda_{T,(1,g)}(\Psi_0) = 0$ when $\tau = \delta^{-1} T$ is unimodular at S. The index is skew-hermitian and invertible; write $T = \jmath\beta$. The extra line of D_1 gives shears: the partial means of f along one more isotropic vector are invariant under the local shears $y \mapsto y + uwly$, because such a shear is the action of a unipotent point supported on $S \cup \bar{S}$ that fixes the frame of D and the vector of the line, and the standard datum Ψ_0 is fixed by the corresponding shears of the datum. By this invariance it suffices to show the vanishing at the points $x = n(l, z)m(q_0, g)$ of the parabolic subgroup of D at which the Gram matrix of q_0 for τ is unimodular at every finite place: such a matrix is obtained by gluing a hermitian matrix integral at $S \cup \bar{S}$ with one integral at the other places, and the glued matrix has integral finite part. A unimodular hermitian matrix over $\hat{\mathcal{O}}_E$ has an integral orthonormal basis k, so with $q_0 = \alpha_0 q$ the columns of qk are orthonormal for β at the finite places. If β is not represented by the signs of V' at some real place, $(\jmath\beta, q)$ is a test pair of the second kind. Otherwise β and the form of V' are equivalent at every place, hence over E by Landherr's theorem, $\beta = c^* J' c$ with c rational; moving the rational Levi point $m(c, 1)$ from the index to the point and past the radical turns (β, q) into $(J', \alpha_0 cq)$, and the columns of $V_x'^{-1} cqk$ are those of a unitary point, so this is a test pair of the first kind. In both cases the coefficient vanishes by hypothesis.

(d) is Lemma 31.10(c). □

31.3. The large triple and the removal of planes

The *exchange identity for standard data* $\mathrm{RC}(W, V, V'; x, y; \ell)$ asserts that x and y are Schwartz and that there is C such that every tame ν of weight μ has a number c with

$$C|c|^2 = L_*\big(\mathrm{rk}\, V - \lfloor \mathrm{rk}\, W/2 \rfloor\big)\, \ell(\nu), \qquad \theta_{W,V}(x, \mathrm{row}_z) = c\, \theta_{W,V'}(y, \mathrm{std}_z)$$

for all integral z, where row_z is the row datum of z.

The row datum and the standard datum of an integral z are matched (Lemma 31.9(a) and the definitions): row_z vanishes off the rows of inverses of matrices in $\mathrm{GL}_n(O)$, std_z off the corresponding columns, and their values differ by $\chi(\det k)$; both are Schwartz–Bruhat for tame χ. Hence an identity on standard data is the case of Theorem 31.1 that the equivariant lifts of Proposition 31.7 extend to all matched data.

Proposition 31.11 (The large triple). *Let D split X as W_1 plus a hyperbolic space and D_1 split W_1, of even rank, as W plus a hyperbolic plane, both adapted, and let $V = V' \oplus W_1$.*

(a) *Unfolded along D, the theta kernel of (X, V) on a standard datum is a sum of integrable terms, each an explicit factor times the theta functional of (W_1, V); on a product of real data this functional is that of (W_1, V') times the theta kernel of (W_1, W_1).*

(b) (*Equal ranks.*) *The theta integral of (W_1, W_1) on $\chi^{-1} \circ \det$ is constant on $\mathrm{U}(W_1)$.*

(c) *If for every tame v an identity of periods over* $[\mathrm{U}(W_1)]$ *holds between the real slices of v and v', with a weight $\rho \geq 0$ nonzero near 1 and a period $I_0(v)$, and either the real slices of v vanish at points whose Gram matrix is not represented by the signs of V' or $t' = -1$ and the signs of W_1 contain a 1 with at most half equal to -1, then* $\mathrm{RC}(X, V, V'; v, v'; \ell)$ *holds with* $\ell(v) = |I_0(v)|^2 / L_*(\operatorname{rk} W_1/2)$.

(d) (*Real vectors.*) *For Schwartz vectors v for (W_1, V') and v_1 for (W_1, W_1) there are Φ for $(X, V' \oplus W_1)$ and Λ for (X, V') whose slice at 0 for the exchanged splitting is v, with* $\mathrm{RC}(X, V' \oplus W_1, V'; \Phi, \Lambda; \ell)$ *and $\ell(v)$ the equal rank ratio of v_1 (below).*

The proof occupies the rest of this subsection up to the definition of the cone integral. Throughout, the frame of D is $(k \mid i \mid l)$ with dual isotropic blocks k, l of size $|b|$ and the embedding i of W_1; the parabolic subgroup of D consists of the points $n(c, z)m(q, g)$ of its unipotent radical times its Levi subgroup $\mathrm{GL}_b \times \mathrm{U}(W_1)$; at the places of S the frame read in the local coordinates is an integral frame $(K\ I\ L)$ (Lemma 31.9). For an adelic matrix x of size $|b| \times \operatorname{rk} V$, the *real slice* of a real datum v for (X, V) at x is the real datum for (W_1, V) obtained in the mixed model of D by fixing the value x on the frame k. The *theta functional on real data* $\vartheta_{F,p,x}(\varphi)$ of the pair (W_1, V) is the theta distribution at the datum with real part φ and finite part a vector F, moved by the Weil representation at a point p of $\mathrm{U}(W_1) \times \mathrm{U}(V)$ and by the Heisenberg translation by x.

Lemma 31.12 (Unfolded terms). *Let $\Phi(y) = \varepsilon\, 1_O(y) \chi^{-1}(\det(y(K\ I)N))$ at S, for a number ε and a fixed matrix N, be Schwartz–Bruhat, and let v be a real Schwartz vector. Put $\beta' = \beta v^{|b|}$.*

(a) *The terms of the theta kernel of $v \otimes \Phi$, with characters β', γ, at $n(c, z)m(q, g)$ are integrable over* $[\mathrm{U}(V)]$.

(b) *Let x be a rational $|b| \times \operatorname{rk} V$ matrix with standard coordinates x', and write $x'q = \alpha_0 Y$. At $(n(c, z)m(q, g), k)$ the term of x is the product of $\psi(\frac{1}{2}\operatorname{tr}(zx^*Jx))\, \chi_V(\det q)^{-1} |\det q|^{\operatorname{rk} V/2}$, with χ_V the space character of γ in rank $\operatorname{rk} V$; of a nonzero constant κ_{vac} if the finite part y of $k^{-1}Y$ is integral, and 0 otherwise; and of the theta functional of (W_1, V), for the finite slice of Φ at y, the Heisenberg translation by $x'c^*J_s$ and the point (g, k), at the real slice of v at $k^{-1}Y$.*

(c) *If the signs t' represent at the real places the Gram matrix of every x at which the real slice of v is nonzero, but do not represent T, then the Fourier coefficient at $_JT$ of the theta integral of $v \otimes \Phi$ vanishes at every point $n(c, z)m(q, g)$.*

Proof. (a) The criterion for integrable terms asks that, where $\Phi(y) \neq 0$, y times the frame $(k \mid i)$ moved by the inverse of the point be right invertible at S. There $\det(y(K\ I)N)$ is a unit, and the moved frame is the frame times a block triangular matrix with diagonal blocks the Levi components, so the right inverse is that triangular matrix times NM^{-1}, with M the unit matrix $y(K\ I)N$. (b) The point of the parabolic subgroup is $(n(c, z), 1) \cdot (m(q, g), k)$ and acts in two steps. The radical gives the character $\psi(\frac{1}{2}\operatorname{tr}(zx^*Jx))$ and the Heisenberg operator at $x'c^*J_s$; the Levi element gives the space character, the modulus $|\det q|^{\operatorname{rk} V/2}$ and the evaluation at $k^{-1}x'q$; and the value of $v \otimes \Phi$ there splits into the real slice of v and the finite slice of Φ. The finite slice of the vacuum at the other places

is κ_{vac} times the vacuum if the finite part is integral and 0 otherwise. (c) The coefficient is the sum of the terms of the rational x with Gram matrix T. Write $q = \alpha_0 q'$ and $Y = x'q'$; the Gram matrix of $k^{-1}Y$ is congruent to T by q', so, as being represented is preserved by congruence, it is not represented by t', and the real slice there vanishes; each term is a multiple of the theta functional at that slice, hence 0. □

Lemma 31.13 (Product formula). *For the pair $(W_1, V' \oplus W_1)$, the datum built from a vector f for (W_1, V') and a vector F for (W_1, W_1), the translation $(c; 0)$ and the point $(g, u \oplus l)$, the theta functional with characters α, $\alpha\beta$ at a product $\varphi \otimes w$ of real data is the theta functional of (W_1, V') with characters α, β at f, c, (g, u), φ, times the theta kernel of (W_1, W_1) with characters α, α at (g, l) for the datum $w \otimes F$.*

Proof. The datum is a product along the decomposition $V' \oplus W_1$ of the target, and the Weil representation, the Heisenberg operator and the theta distribution act factor by factor; on the second factor the translation is by 0, and the characters of the two factors multiply to those of the sum. □

Proof of Proposition 31.11(a). By Lemma 31.12(a), (b), applied to the row datum of the dual block, the terms are integrable and have the stated form, since by Lemma 31.9(a) this datum is ε times a function of the form there with $N = 1$, $|\varepsilon| = 1$; the vanishing of the Fourier coefficients is Lemma 31.12(c). The product formula is Lemma 31.13. □

Proof of Proposition 31.11(b). Let F be the theta integral of (W_1, W_1) at v and the datum $\chi^{-1} \circ \det$ on $\mathrm{GL}_n(O)$ at S. It is the image of the standard datum under the equivariant lift of Proposition 31.7(a) for the triple $(W_1, W_1, 0)$ with $c = 0$: here b is empty, the standard datum lives on matrices with no column, and the row datum of the empty matrix is $\chi^{-1} \circ \det$. Translation at S fixes every function of matrices with no column and the character of the lift is trivial, as $|b| = 0$; so F is right invariant under all points supported on $S \cup \bar{S}$. By Lemma 31.6(d) it is right invariant under the integral points trivial at S, and it is automorphic. Its constant term along D_1 at a Levi point (u, x) is, by Theorem B.32(d) for the pair (W_1, W_1) over (W, W), a constant times the theta integral of (W, W) at the cone integral, at x; since the two smaller spaces have equal rank, this holds at every Levi point, so the constant term is a function of x alone. If F is invariant under the unipotent radical of D_1, it equals its constant term, and its values on the Levi subgroup do not depend on the GL_1 component. The weak form of strong approximation (Appendix B.18) then shows that F is constant. □

For a tame v, real data v of $(X, V' \oplus W_1)$ and v' of (X, V'), a function ρ of the real points of $\mathrm{U}(X)$ and a number I_0, the *identity of periods* asserts: for each real point h there is a point p of $\mathrm{U}(V') \times \mathrm{U}(W_1)$ at the infinite places such that for all invertible B, characters β, vectors f, matrices x and points g, y,

$$\int_{[\mathrm{U}(W_1)]} \vartheta_{f \boxtimes \chi^{-1} \circ \det,\, (g, y \oplus u),\, (x,0)}\big(\omega(ph)\, v_{(ph)^{-1}(B,0)}\big)\, du = \frac{\rho(h) I_0}{\nu(\zeta/\bar{\zeta})|\zeta|^{\mathrm{rk}\, W_1/2}}\, \vartheta_{f,(g,y),x}(v'_B),$$

where v_x is the real slice of v at x, ω the real Weil representation with the power $\lfloor \mathrm{rk}\, W_1/2 \rfloor - \mu$ of the determinant, ζ the idele equal to $\det B$ at the infinite places

and to 1 at the finite places, the characters on the left are γ, $\gamma\beta$ and on the right γ, β, with $\gamma = \nu^{-1}\upsilon^{\lfloor \mathrm{rk}\, W_1/2\rfloor}$. A pair (T, q) is a *test pair* for V' if either T is the form of V' and $q = \alpha_0 V'_x Y$ where the finite part of Y times some $k \in \mathrm{GL}_b(\hat{O}_E)$ is unitary, so that the columns of $V'_x Y$ span a lattice self-dual for T; or T is invertible and not of the form $a^* \mathrm{diag}(t')a$ at some infinite place.

Lemma 31.14 (Periods give the identity). *Let* $\mathrm{rk}\, W_1$ *be even,* $\rho \geq 0$ *nonzero near* 1, *and assume the condition on the signs or the slices in Proposition 31.11*(*c*). *There is* $K > 0$ *such that for every tame* ν *for which the identity of periods holds with* ρ *and* I_0 *there is* C *with* $|C| = K|I_0|$ *and:*

(a) *at* $n(c,z)m(q,g)$ *the Fourier coefficient at* ${}_JT$ *of* $\theta_{X,V'\oplus W_1}(v, \mathrm{row})$ *is* C *times that of* $\theta_{X,V'}(v', \mathrm{std})$, *for the row and standard data of the dual block* L' *of the frame, whenever* (T, q) *is a test pair for* V';
(b) *for every integral* z *at* S, $\theta_{X,V'\oplus W_1}(v, \mathrm{row}_z) = C\, \theta_{X,V'}(v', \mathrm{std}_z)$.

Proof. (a) *Test pairs of the second kind.* On the small side every rational x has Gram matrix represented by t', so the coefficient at a T not represented is an empty sum. On the large side it vanishes by Lemma 31.12(c) under the condition on the slices; under the other condition, $t' = -1$, the complement has a positive sign and the weights are below the number of its positive signs, so Lemma 31.10(c) applies. *Test pairs of the first kind.* Then each coefficient is the integral of one term of the unfolded kernel: on the large side the rational frames with Gram matrix J' form one orbit under the rational points of $V' \oplus W_1$ (Witt's theorem), that of $(1;0)$, whose stabiliser is the group of rational points of W_1 (the rational points of the sum fixing the first block of columns); on the small side they form the orbit of 1, with trivial stabiliser. The integral over $W_1(E)\backslash \mathrm{U}(V' \oplus W_1)(\mathbb{A})$ of the term of $(1;0)$ is unfolded in stages through $\mathrm{U}(V') \times \mathrm{U}(W_1)$, with the weight ρ cut off at the finite places; the constant $R \neq 0$ of this unfolding depends only on the spaces. For a point (g_∞, u), the term of $(1;0)$ integrated over the translates by $\mathrm{U}(V') \times [\mathrm{U}(W_1)]$ is computed as follows. It vanishes unless a block diagonal translate of u lies in the stabiliser of the standard lattice, because the finite part of $k^{-1}(Y, 0)$ must be integral and Witt's theorem over $\hat{O}_E$ moves the first columns to those of 1. Otherwise translate by the block diagonal point (δ_1, δ_2) at infinity and (u_1, u_2) at the finite places and compare at each $y \in \mathrm{U}(V')$. The finite part x of $B = y^{-1}Y$ is integral with $\det x$ a unit, the finite slices are multiples of $f \boxtimes (\chi^{-1} \circ \det)$ and of the vacuum (Lemma 31.12(b)), the real point acts through the real slice, and the integral over $[\mathrm{U}(W_1)]$ is the left side of the identity of periods. Comparing the scalars, $\det B = \det Y / \det y$ splits into an archimedean and a finite factor, $\det y$ is unitary and $\det x$ a unit, so both have absolute value 1; on a unit ζ of $\hat{O}_E$, $\chi^{-1}(\bar{\zeta}) = \chi^{-1}(\zeta)\nu(\zeta/\bar{\zeta})$, and the two functionals of (W_1, V') differ by $\nu(\det y)^2$. The result is $\rho(g_\infty)\kappa_0 \varepsilon \nu(d_0/\bar{d}_0) I_0$ times the integral over $\mathrm{U}(V')$ of the term at 1 of the small kernel, with d_0, κ_0 depending only on the spaces and D. Integrating over g_∞ against the unfolding gives (a) with $C = R\kappa_0 \varepsilon \nu(d_0/\bar{d}_0) I_0$, whose modulus is $K|I_0|$ with $K = |R\kappa_0| > 0$.

(b) Let $\mathcal{D}$ be the equivariant lift of Proposition 31.7(a) for the triple $(X, V' \oplus W_1, V')$, v, v' and the constant C, so that $\mathcal{D}\Psi$ is the difference of the two sides on matched data. Its value f at the standard datum of the dual block has, by (a), vanishing Fourier coefficients at all test pairs: f vanishes at self-dual lattices along D. By Proposition 31.7(c), with the extra line of D_1, $\mathcal{D}$ vanishes on all

locally constant data with matched functions, among them std_z with its partner row_z. □

Proof of Proposition 31.11(c). By Lemma 31.14(b), for each ν the identity on standard data holds with $c = C$, $|c| = K|I_0(\nu)|$. The rank of $V = V' \oplus W_1$ minus half that of X is $|b| + \operatorname{rk} W_1 - (2|b| + \operatorname{rk} W_1)/2 = \operatorname{rk} W_1/2$, so

$$K^{-2}|c|^2 = |I_0(\nu)|^2 = L_*(\operatorname{rk} W_1/2)\,\ell(\nu),$$

which is $\mathrm{RC}(X, V, V'; v, v'; \ell)$ with the constant K^{-2}. Both vectors are Schwartz by hypothesis. □

The real vectors of (d) are built with a smooth form of Witt's theorem. For a hermitian J and a frame x_0 with invertible Gram matrix, *smooth Witt data* near x_0 consist of a smooth function γ on square matrices inverting the Gram matrix of x_0, unitary elements $k(y)$ of $\mathrm{U}(J)$ depending smoothly on frames y, and a smooth, compactly supported, nonnegative function σ on families of frames with $\sigma = 1$ at the constant family x_0, such that wherever σ does not vanish each frame y of the family has invertible Gram matrix inverted by γ and lies in the span of $k(y)x_0$.

Lemma 31.15 (Smooth Witt theorem). (a) *For J hermitian and a frame x with invertible Gram matrix there are elements $k(y) \in \mathrm{U}(J)$, indexed by all frames y and smooth in y, such that for y near x the columns of y are linear combinations of those of $k(y)x$. Smooth Witt data exist near every such frame.*

(b) *The orthogonal projection $x(x^*Jx)^{-1}x^*Jx_0$ of x_0 to the span of x does not change when x is multiplied on the right by an invertible matrix.*

(c) *If g, k are unitary for the signs $t' \oplus s$ and u for t', and $kx_0u = g^{-1}x_0$ for $x_0 = \binom{1}{0}$, then $gk\,\mathrm{diag}(u, 1) = \mathrm{diag}(1, q)$ with q unitary for s.*

Proof. (a) Let $L(a)$ be the projection of a matrix a to the Lie algebra of $\mathrm{U}(J)$ along the matrices that keep the span of x; then $a - L(a)$ keeps the span of x. The map $T(a) = \exp(L(a))(a - L(a))$ is smooth with derivative at 1 equal to $(\mathrm{id} - L') + L' = \mathrm{id}$, so it has a smooth local inverse A near 1. For y near x choose smoothly a matrix $z(y)$ near 1 with $z(y)x = y$ (for instance from a right inverse of x), and put $k(y) = \exp(L(A(z(y))))$; then $z(y) = k(y)(A(z(y)) - L(A(z(y))))$ and the second factor keeps the span of x, so $y = z(y)x$ lies in the span of $k(y)x$. Extend k smoothly to all frames with a cut-off. For the Witt data, matrix inversion agrees near the invertible Gram matrix of x_0 with a smooth function (the adjugate divided by a smooth function equal to the determinant nearby), and σ is a bump function supported where the frames are close to x_0. (b) Replacing x by xC replaces the projection by $xC(C^*x^*JxC)^{-1}C^*x^*Jx_0$, which is the same. (c) The point $gk\,\mathrm{diag}(u, 1)$ is unitary and fixes x_0, so it preserves the span of x_0 and its orthogonal complement, and acts as 1 on the first. □

Lemma 31.16 (Prescribed slices). *Let D be adapted, and let D^{sym} be the splitting with the two isotropic frames exchanged.*

(a) *Taking the slice at 0 is onto on Schwartz vectors: every Schwartz v for (W_1, V') is the slice at 0 of a Schwartz vector for (X, V').*

(b) *If $\tau \geq 0$ is smooth, compactly supported and nonzero somewhere, and v is Schwartz, there are $\kappa \in \mathbb{C}$ and a Schwartz Λ for (X, V') whose slice at each x is $\tau(x)\kappa$ times the real Weil translate $\omega(1)v$, and whose slice at 0 for D^{sym} is v.*

The same holds with $\tau(x)$ replaced by $\sigma(x,0)|\prod_w \det(x_w)^{N_w}|^2$ for exponents N.

(c) *Let σ be smooth of compact support, z and k smooth maps on frames and v, v_1 Schwartz. Then some Schwartz Φ for $(X, V' \oplus W_1)$ has at every x the slice*

$$\left(\sigma(y)\sigma(x')\kappa \prod_w \overline{\det(x'_w)_1}^{2\mu(w)}\right) \omega(1, k \circ y)(v \boxtimes v_1),$$

with $y = z \circ x$ and $x'_w = k(y_w)^{-1}x_w$, where $(x'_w)_1$ is the upper square block, ω the real Weil representation twisted by the power $\operatorname{rk} W_1/2 - \mu$ of the determinant, and $\boxtimes$ the external product, which preserves Schwartz vectors.

Proof. (b) Choose κ so that $\tau\kappa$ and the Gaussian slices of the vacuum have the same integral. The slice at 0 for D^{sym} is the integral of all slices for D (Fourier transform along the isotropic frame), so the section $\tau\kappa\,\omega(1)v$ gives v. To compare, tensor both vectors with zero terms: both sides are integrals of slices for the Lebesgue measure, which is a multiple of the Haar measure moved by the change of frame; in real and finite coordinates both integrands are a function of the real variable times the same function of the finite one, Fubini splits off the real integrals, and these agree by the choice of κ. The vector with prescribed smooth compactly supported slices times a Schwartz vector exists and is Schwartz, the slices being Schwartz in the frame variable. With the second weight the same argument applies, since x is recovered from the frame $(x, 0)$ in the support of σ. (a) is (b) for D^{sym} with the standard bump function, the splitting opposite to D^{sym} being D. (c) The scalar of the section times the determinant factor of the Weil representation is $\sigma \cdot \sigma$ times a polynomial in the entries of x, of $k(z(x))$ and of their conjugates; on the support, $x = k(y)w(x)$ with $w(x)$ in the support of σ, and the entries of the unitary matrix are entries of $k(z(x))$ or 0. Hence the section is smooth with compact support in x, with values Schwartz vectors, and a vector with these slices exists. The external product is an iterated integral of translates of the vacuum against v_1 and then v, and padding a vector with zeros does not increase its norms, so it preserves Schwartz vectors. □

Lemma 31.17 (The identity of periods for Witt data). *Let (γ, k, σ) be smooth Witt data near $x_0 = \binom{1}{0}$ for the signs $t' \oplus s$, let $\operatorname{rk} W_1 = 2j$, let the slice of Φ at x be the section of Lemma 31.16(c) with y the family of projections of x_0 to the spans of the x_w formed with γ, and let the slice of Λ at x be $\sigma(x,0)|\prod_w \det(x_w)^{j+\mu(w)}|^2\kappa$ times v. Then the identity of periods holds for Φ, Λ with the weight $\rho(h)$, the value of σ at the projections of x_0 to the spans of the $(h_w)^{-1}x_0$, and the period I_0, the value at 1 of the theta integral of (W_1, W_1) at v_1 and $\chi^{-1} \circ \det$.*

Proof. Fix h and let y be these projections. If $\sigma(y) = 0$ both sides vanish, since $\rho(h) = 0$ and the section vanishes. Otherwise each frame $(h_w)^{-1}x_0$ lies in the span of $k(y_w)x_0$, so $(h_w)^{-1}x_0 = k(y_w)x_0c_w$ with c_w carrying the form of signs t' to the Gram matrix of $(h_w)^{-1}x_0$, which is that form; so c_w is a unitary u_w, and by Lemma 31.15(c) $\operatorname{diag}(u, q^{-1})hk(y) = 1$ for a unitary q. Take $p = \operatorname{diag}(u, q^{-1})$. The slice of Φ is read at $A = h^{-1}x_0u^{-1}B$, which spans the same space as $x_1 = h^{-1}x_0$, so by Lemma 31.15(b) the projections for A may be replaced by those for x_1, which are y. The group elements cancel: $k(y)^{-1}A = x_0u\,u^{-1}B = (B; 0)$, and $\omega(ph)$ carries the section to

$\sigma(y)\sigma(B,0)\kappa\prod_w \overline{\det B_w}^{2\mu(w)}$ times $v\boxtimes v_1$. By Lemma 31.13, the theta functional of this product vector is the functional of (W_1,V') at v times the theta kernel of (W_1,W_1) at v_1; the second factor integrates over $[\mathrm{U}(W_1)]$ to the theta integral at v_1, which is the constant I_0 by Proposition 31.11(b) and does not depend on g. Finally $\prod_w \overline{\zeta_w}^{2\mu(w)} = |\prod_w \zeta_w^{j+\mu(w)}|^2/(\nu(\zeta/\bar\zeta)|\zeta|^j)$ for $\zeta=\det B$ at the infinite places, as ν has weight μ; so the left side is $\rho(h)I_0/(\nu(\zeta/\bar\zeta)|\zeta|^j)$ times the functional at $\sigma(B,0)|\prod\det B_w^{j+\mu(w)}|^2\kappa v$, which is the slice of Λ at B. □

Proof of Proposition 31.11(d). Take smooth Witt data near x_0 (Lemma 31.15(a)), the vector Φ of Lemma 31.16(c) with the projections of x_0 as z, and the vector Λ of Lemma 31.16(b) with $\sigma(x,0)|\prod\det(x_w)^{j+\mu(w)}|^2$, whose slice at 0 for D^{sym} is v. By Lemma 31.17 the identity of periods holds for every tame ν with $\rho\geq 0$, $\rho(1)=\sigma(x_0)=1$, so ρ is nonzero near 1 by continuity, and $I_0(\nu)$ the theta integral of v_1 at 1. Where the slice of Φ is nonzero the frame is $k(y)x_0c$ and c represents its Gram matrix by the signs t', so the real slices of Φ vanish at the points whose Gram matrix is not represented by t'. Part (c) gives RC with $\ell(\nu)=|I_0(\nu)|^2/L_*(\mathrm{rk}\,W_1/2)$, the equal rank ratio of v_1. □

For an adapted splitting A of $W'=W\oplus H$ and X of $V'=V\oplus H$, H a hyperbolic plane, the *cone integral* $\mathrm{cone}_z(v')$ of a Schwartz vector v' for (W',V') is the integral of the cone integrand over the maximal compact subgroup of $\mathrm{U}(V')$ and the families of nonzero complex numbers at the infinite places, with exponent z.

Proposition 31.18 (Removing planes). *Let W have even rank.*

(a) *Every Schwartz vector v for (W,V) has a Schwartz v' for (W',V') and a positive analytic B with $\theta_{W,V}(\mathrm{cone}_z(v'),\cdot)=B(z)\,\theta_{W,V}(v,\cdot)$ on standard data for all $z>0$.*

(b) *Let $z=\mathrm{rk}\,V-\mathrm{rk}\,W/2$. If $z\geq 0$, the identity $\mathrm{RC}(W',V',U;x,y;\ell)$ gives*

$$\mathrm{RC}\big(W,V,U;\mathrm{cone}_{z+1}(x),y_0;\ell\big),$$

with y_0 the zero term of y. If $z<0$, the same holds with the dual cone integral of exponent $-z$, on the other side of Poisson's formula. Consequently the identity descends along any number of planes of an adapted splitting, if V' has as many more negative signs than V as planes.

(c) *Triples of spaces with the same signs have the same theta integrals up to one translation and numbers of modulus one.*

(d) (*Vanishing descends.*) *If the theta integral of $(W\oplus H^j,V\oplus H^j)$ vanishes on all Schwartz vectors and standard data, so does that of (W,V), provided $-k\leq\mathrm{rk}\,V-\mathrm{rk}\,W/2$ with $k\leq\mu$.*

Write x_B for the adelic vector spanning the isotropic line of X, with components $x_{B,w}$ at the infinite places; for $p=(\kappa,a)$ with κ in the maximal compact subgroup of $\mathrm{U}(V')$ let $\mathrm{cp}(p)=\kappa^{-1}x_Ba$ be the corresponding point of the isotropic cone, and $|a|$ the module of the idele equal to a at the infinite places and 1 elsewhere.

Lemma 31.19 (The image of the cone integral). *Let A and X be adapted and $\mathrm{rk}\,W$ even.*

(a) *Each $x_{B,w}$ is nonzero and isotropic, and κ keeps lengths: the squared length of $\mathrm{cp}(\kappa, a)$ at w is $|a_w|^2$ times that of $x_{B,w}$.*
(b) *There are a smooth compactly supported $\beta \geq 0$ and a smooth family $k(x)$ of real points of $\mathrm{U}(V')$ such that $\rho(\kappa, a) = \beta(\mathrm{cp}(\kappa, a))$ has a positive Mellin transform (for every real z the integral of $\rho(\kappa, a)|a|^z$ converges and is positive) and, for $x = \mathrm{cp}(\kappa, a)$ with $\beta(x) \neq 0$, $\kappa k(x)$ multiplies x_B by a at each infinite place.*
(c) *Every Schwartz v for (W, V) has a Schwartz v' for (W', V') such that at each (κ, a) the cone integrand of v' with exponent z is $\rho(\kappa, a)|a|^z$ times a translate of v by a real point of $\mathrm{U}(V)$.*

Proof. (a) The vector x_B is part of an adapted frame, so its infinite components are nonzero and isotropic; κ is unitary for the standard form at each infinite place. (b) Choose k place by place: by Witt's theorem, applied smoothly as in Lemma 31.15(a), there is a smooth family $k(x)$ carrying x_B to x for isotropic x near the cone, and $\beta \geq 0$ is a bump function nonzero at x_B and supported where k carries x_B to x on isotropic x and where the squared length at each place is within a factor 2 of that of x_B. The point $x = \kappa^{-1} x_B a$ is isotropic, so $k(x)x_B = x$, that is $\kappa k(x)x_B = x_B a$. By (a) the function ρ vanishes unless $\frac{1}{2} \leq |a_w|^2 \leq 2$ at every place, so it has compact support; it is continuous, nonnegative and nonzero at $(1, 1)$, so all its Mellin integrals converge and are positive. (c) Write v as the zero term along $\mathrm{Hom}(W, B)$ of a vector v^B, and let v' be the vector with slices $\beta(x)\, k(x) v^B$ along A (slices can be prescribed: the powers of the determinants in the real Weil representation are absorbed into β). For $x = \mathrm{cp}(\kappa, a)$ the line term is the zero term along $\mathrm{Hom}(W, B)$ of the slice of $\kappa v'$ at $x_B a$, which is $\beta(x)\kappa k(x) v^B$. The point $\kappa k(x)$ multiplies x_B by a, so under the zero term it acts through $\mathrm{U}(V)$, with a power of $|a|$ and a phase that cancels the phase in the cone integrand. □

Proof of Proposition 31.18(a). Let v' be as in Lemma 31.19(c), and let $B(z)$ be the integral of $\rho(p)|a|^z$ over p, which is positive for all z. A Mellin transform that converges for every real exponent is analytic, so B is analytic. The theta integral over $[\mathrm{U}(V)]$ takes the same value at v and at its translates by real points of $\mathrm{U}(V)$, twisted by the power $\lfloor \mathrm{rk}\, W/2 \rfloor - \mu$ of the determinant, because right translation in h does not change the integral. Exchanging the cone integral with the integral over $[\mathrm{U}(V)]$, which is allowed by the integrability of the cone kernel (Theorem B.32(b)), gives

$$\begin{aligned} \theta_{W,V}(\mathrm{cone}_z(v'), \mathrm{row}_y)(g) &= \int \rho(p)|a|^z\, \theta_{W,V}(v, \mathrm{row}_y)(g)\, dp \\ &= B(z)\, \theta_{W,V}(v, \mathrm{row}_y)(g) \end{aligned}$$

for $z > 0$, every tame character, every integral y and every g. □

Lemma 31.20 (Change of point and sign). (a) *Passing to the opposite signs, by the identity on matrices, preserves forms of a given type and integrals over the quotients, provided it matches the rational points of the two spaces.*
(b) *Let the points of W and W_1 differ by a rational matrix on the left and an adelic unitary k on the right, and let V, V_1 have the same signs. For any characters, the theta integral of (W, V) at g of the datum with real part v, a function Φ at S and the vacuum elsewhere is u times that of (W_1, V_1) at kg, with $|u| = 1$ independent of v, Φ, g.*

(c) *For triples* (W, V, V') *and* (W_1, V_1, V_1') *with the same signs there is a point* γ *of* $\mathrm{U}(W)(\mathbb{A})$ *such that for every split tame character the two theta integrals of the second triple at* g *are, up to numbers of modulus one, those of the first at* γg.

Proof. (a) The unitary groups of the signs s and $-s$ are the same subgroup with the same Haar measure, and forms and integrals depend on the signs only through this subgroup and the relation $s_i = s_j$, which is the same for $-s$. (b) By Landherr's theorem there are a rational c' and an adelic unitary k' moving the point of V_1 to that of V. Left translation by (k, k') carries the theta kernel of (W, V) to that of (W_1, V_1) up to the value $\beta(\det k)/\alpha(\det k')$ of the characters: a space enters the theta functional only through its rational vectors, which correspond (the rational matrix x goes to $c'xc^{-1}$), and the datum and the Weil representation depend on the spaces only through their signs. Translation by k' identifies the quotients by $V(E)$ and $V_1(E) = k'V(E)k'^{-1}$. (c) Landherr's theorem moves the point of W to that of W_1; then both integrals are cases of (b). □

Lemma 31.21 (The side of the small space). *Let* A *split one hyperbolic plane off* W' *with complement* W*, adapted, let* y *be a real Schwartz vector for* (W', U) *and* z *integral at* S*. The constant term along* A *at the Levi point* $(1, g)$ *of the theta integral of* (W', U) *at* y *and the standard datum of* z *composed with the projection to* W *is the constant* κ_{vac} *of* A *times the theta integral of* (W, U) *at the zero term* y_0 *of* y *and* std_z.

Proof. The constant term is a sum over the rational vectors x of U of their terms, and only the isotropic ones contribute, the term of an anisotropic vector being 0 by definition. The term $x = 0$ is the right side: the slices of the datum are multiples, by κ_{vac}, of data for (W, U) with the standard datum of z, the projection to W read at S being the transpose of the inclusion read at $\bar{S}$. A nonzero isotropic x is the frame of a splitting of U, and the theta functional of $\mathrm{Hom}(W, U_0)$ vanishes on the zero term along the line of x of the translated datum: the functional is invariant under the translations along $h^{-1}x$, so it kills data supported on matrices that do not annihilate $h^{-1}x \neq 0$, and the standard datum lives on matrices of full rank. Both sides are integrals over $[\mathrm{U}(U)]$, and the integrands agree at each h. □

Proof of Proposition 31.18(b). One plane, by a constant term. Suppose that the exchange identity for standard data holds for (W', V', U) at x and y, with the function ℓ and the constant D, and that for every tame ν the constant term along A of the theta integral of (W', V') at x and the plane datum of w is, at $(1, g)$, $\gamma \neq 0$ times the theta integral of (W, V) at the cone integral $x^\flat$ of x (on a side β, with exponent $z' > 0$) and row_w, where $|\gamma|^2 L_*(e) = C\, L_*(e+1)$ for $e = \mathrm{rk}\, V - \mathrm{rk}\, W/2$. Taking constant terms along A in the identity for (W', V', U) at the matrices $w\,\mathrm{pr}_W$, Lemma 31.21 gives

$$\gamma\, \theta_{W,V}(x^\flat, \mathrm{row}_w)(g) = c\, \kappa_{\mathrm{vac}}\, \theta_{W,U}(y_0, \mathrm{std}_w)(g),$$

so the identity for (W, V, U) holds at $x^\flat$, y_0 with the factor $c\kappa_{\mathrm{vac}}/\gamma$. Its size: as $\mathrm{rk}\, V' - \mathrm{rk}\, W'/2 = e + 1$,

$$\frac{DC}{|\kappa_{\mathrm{vac}}|^2}\left|\frac{c\kappa_{\mathrm{vac}}}{\gamma}\right|^2 = \frac{C\, D|c|^2}{|\gamma|^2} = \frac{C\, L_*(e+1)\ell(\nu)}{|\gamma|^2} = L_*(e)\ell(\nu).$$

The cone integral of a Schwartz vector is Schwartz (Theorem B.32(b)), and so is the zero term.

The two signs of e. If $e = z \geq 0$, Theorem B.32(d) gives the hypothesis with $x^\flat = \mathrm{cone}_{z+1}(x)$ and $|\gamma| = C'L^\sharp(z+1)$, and $|\gamma|^2 L_*(z) = C'^2 L_*(z+1)$ by Theorem B.32(a). If $e = -z-1 < 0$, it gives the hypothesis at $(1, g)$ with $x^\flat = \mathrm{cone}^*_{z+1}(x)$ and $|\gamma| = C'L^\sharp(z+1)$, and again $|\gamma|^2 L_*(-z-1) = C'^2 L_*(-z)$ by Theorem B.32(a).

Removing one plane. In general V' need not split over V; but if V' has one negative sign more than V and the ranks match, there is a space V_1 with the signs of V' and an adapted splitting over V (relabel the hyperbolic sum along a bijection of the index sets that matches the signs). By Lemma 31.20(b) the identity for V' gives the identity for V_1, with a factor of the same modulus. Then the preceding paragraphs apply.

Several planes. By strong induction on the number of planes: an adapted splitting of W' along $d+2$ planes is the composite of one along $d+1$ planes down to $W \oplus H$ and the tautological one of the last plane, the partner of $W \oplus H$ is $V \oplus H$, and zero terms compose along composite splittings (the zero term along the composite is the zero term along the second of the zero term along the first, which holds for adelic data and so for their real parts). □

Proof of Proposition 31.18(c), (d). (c) is Lemma 31.20(c). (d) Descend one plane at a time; one plane raises each rank by two and each number of negative signs by one, so by (c) the intermediate spaces may be taken to split, and for the pair with i planes the number $\mathrm{rk}\, V - \mathrm{rk}\, W/2$ becomes $e + i \geq -k$. For one plane, let $e = \mathrm{rk}\, V - \mathrm{rk}\, W/2 \geq -k$ and let v, y, g be given; put $I = \theta_{W,V}(v, \mathrm{row}_y)(g)$. By (a) there are v' and a positive analytic B with $\theta_{W,V}(\mathrm{cone}_x(v'), \mathrm{row}_y)(g) = B(x)I$ for $x > 0$. By Theorem B.32(c) there is an analytic $\mathcal{Z}$ whose value at $e+1$ is the constant term of the theta integral of (W', V') at v' and the plane datum of y, that is the row datum of the integral matrix y times the projection to W, and which for $x \geq 1$ equals $\varepsilon P(x) L^S(x, \nu) B(x) I$ with $P(x) = |\alpha_0|^x \prod_{w \in S} q_w^{x-1/2} > 0$. By Theorem B.32(e), L^S continues to an analytic Λ on $x > -k$ without zeros at the integers $j > -k$. The function $R(x) = \varepsilon P(x) \Lambda(x) B(x) I$ is analytic on $x > -k$ and agrees with $\mathcal{Z}$ for $x \geq 1$, so, the interval being connected, $\mathcal{Z} = R$ on $x > -k$. At $e+1 > -k$ the value $\mathcal{Z}(e+1)$ is a constant term of a function that vanishes by hypothesis, so $R(e+1) = 0$; and ε, $P(e+1)$, $\Lambda(e+1)$, $B(e+1)$ are nonzero. Hence $I = 0$. □

31.4. The equal rank ratio and the proof

The *equal rank ratio* of a Schwartz vector v for (Y, Y), Y of even rank, is $|I(v)(1)|^2 / L_*(\mathrm{rk}\, Y/2)$, where $I(v)$ is the theta integral of (Y, Y) at v and $\chi^{-1} \circ \det$.

The large triple and the removal of planes combine to an identity on standard data whose factor is an equal rank ratio.

Lemma 31.22 (From the top of the tower). *Let $W_0 = H \oplus W$ with H a hyperbolic plane, let W have even rank $\mathrm{rk}\, U + \mathrm{rk}\, V$, and let U and W together have $\mathrm{rk}\, U +$ (number of negative signs of V) negative signs. For Schwartz vectors z for (W_0, W_0) and y for (W, U) there is a vector v for (W, V) with $\mathrm{RC}(W, V, U; v, y; \ell)$, where $\ell(v)$ is the equal rank ratio of z.*

Proof. The tower is $W, W_0, X = H^{\operatorname{rk} U} \oplus W_0$. Let A be the tautological splitting of W_0 over W. By Lemma 31.16(a) there is a Schwartz w for (W_0, U) with zero term y along A. By Proposition 31.11(d), applied to the splitting of X over $W_1 = W_0$ along $\operatorname{rk} U$ lines, to $D_1 = A$, to the small space U, and to w and $v_1 = z$, there are x for $(X, U \oplus W_0)$ and x' for (X, U) with $\mathrm{RC}(X, U \oplus W_0, U; x, x'; \ell)$ and with w as the slice of x' at 0 for the exchanged splitting. Remove the $\operatorname{rk} U + 1$ planes of the composite splitting of X over W: the large space $U \oplus W_0$ has $\operatorname{rk} U + 1$ more negative signs than V, by the hypothesis on the signs and the one negative sign of H, and the zero term of x' along the composite splitting is the zero term along A of its zero term w, that is y, zero terms being taken in stages. Proposition 31.18(b) gives $\mathrm{RC}(W, V, U; v, y; \ell)$ for some v. □

Proposition 31.23 (The equal rank ratio). (a) *The ratio depends on Y only through its signs; adding an adapted hyperbolic plane multiplies it by a constant $C > 0$ independent of v.*

(b) *For J positive definite of rank $2k$, $Y = H \oplus J$ and $k \le \mu$, there is a Schwartz v whose ratio is one number $C \neq 0$ for all tame ν of weight μ: the exchange identity for $(J, J, 0)$ and for $(J, 0, J)$ give $C|c|^2 = L_*(k)\cdot$ratio and $C'|c'|^2 = L_*(-k)\cdot$ratio, and Fubini on (J, J) gives $cc' =$ const.*

(c) *If $\operatorname{rk} W = 2k + 2q$, q the number of negative signs, and $k \le \mu$, there are v for $(H \oplus W, H \oplus W)$ and $C \neq 0$ with ratio C for every tame ν of weight μ.*

Proof. (a) *Signs.* Let Y_1 have the signs of Y, where Y has even rank and an adapted splitting over a space with one plane less. By Lemma 31.20(b) the theta integral of (Y_1, Y_1) at 1 is a number of modulus one times that of (Y, Y) at some point γ; the latter is constant by Proposition 31.11(b), so the two values at 1 have the same modulus, and the ratios agree. *A plane.* Let $\operatorname{rk} W = 2d$ and A an adapted splitting $W' = W \oplus H$. For v Schwartz for (W, W), Proposition 31.18(a) gives v' for (W', W') and $B > 0$ with $\theta_{W,W}(\mathrm{cone}_{d+1}(v'), \cdot) = B(d+1)\,\theta_{W,W}(v, \cdot)$ on standard data. The theta integral of (W', W') at v' and $\chi^{-1} \circ \det$ is constant (Proposition 31.11(b)), so it is equal to its constant term along A at $(1, 1)$; by Theorem B.32(d) for the pair (W', W') over (W, W), with $e = \operatorname{rk} W - \operatorname{rk} W/2 = d$ and no extra rows (the row datum of the empty matrix being $\chi^{-1} \circ \det$), this constant term is $\kappa\,\theta_{W,W}(\mathrm{cone}_{d+1}(v'), \chi^{-1} \circ \det)(1)$ with $|\kappa| = C_0 L^\sharp(d+1)$. Hence $I(v')(1) = \kappa B(d+1) I(v)(1)$ and, by Theorem B.32(a),

$$\frac{|I(v')(1)|^2}{L_*(d+1)} = \frac{C_0^2 L^\sharp(d+1)^2 B(d+1)^2 |I(v)(1)|^2}{L^\sharp(d+1)^2 L_*(d)} = C_0^2 B(d+1)^2\,\frac{|I(v)(1)|^2}{L_*(d)}.$$

The constant $C_0^2 B(d+1)^2 > 0$ does not depend on ν.

(b) Write $r(v)$ for the ratio of v for $Y = H \oplus J$, and vol for the volume of $[\mathrm{U}(0)]$, which is positive.

The identity used twice. Fix a Schwartz v for (Y, Y). By Lemma 31.22 for the triple $(J, J, 0)$ with $z = v$ and the vacuum for $(J, 0)$, there are B and a Schwartz y for (J, J) such that for every ν the theta integral of (J, J) at y and $\chi^{-1} \circ \det$ is $c_1$vol with $B|c_1|^2 = L_*(k) r(v)$: the small side is the theta integral of $(J, 0)$ at the vacuum, which is vol times the value 1 of the datum at 0. By Lemma 31.22 for the triple $(J, 0, J)$ with y on the small side, there are C and a number ξ (a vector of the Fock space $\mathbb{C}$ of $(J, 0)$) such that vol $\cdot\, \xi$ is c_2 times the integral over the first $[\mathrm{U}(J)]$ of the theta kernel of (J, J) at y, with $C|c_2|^2 = L_*(-k) r(v)$; on the zero space the theta kernel only reads the value 1 of the datum at 0. Both identities

concern the theta kernel of the same datum on (J, J), integrated once over each group; as J is definite both quotients are compact and Fubini's theorem applies, so $\mathrm{vol} \cdot \xi = c_2 c_1 \mathrm{vol}$ and $\xi = c_1 c_2$. Since $L_*(k)L_*(-k) = 1$,

$$r(v)^2 = \big(L_*(k)r(v)\big)\big(L_*(-k)r(v)\big) = BC|c_1 c_2|^2 = BC|\xi|^2,$$

which does not depend on ν.

Non-vanishing. Fix ν, and suppose $r(v) = 0$ for every Schwartz v. Then the theta integral of (Y, Y) at $\chi^{-1} \circ \det$ vanishes at every Schwartz vector, being constant with value 0 at 1. Let $X = H^{2k} \oplus Y$ and consider the triple $(X, J^- \oplus Y, J^-)$, with J^- the negative definite space of rank $2k$. For a Schwartz w for $(Y, J^- \oplus Y)$, expanding w into coherent states shows that the integral over $l \in [\mathrm{U}(Y)]$ of the theta functional at w, the vector $f \boxtimes (\chi^{-1} \circ \det)$ and the point $(g, y \oplus l)$ is a theta integral of (Y, Y), hence 0: each coherent state is a product of coherent states for the two summands, and evaluating the first factor leaves a datum for (Y, Y). Hence the identity of periods holds for every pair of vectors with the period $I_0 = 0$, and Lemma 31.14(b) applies, its condition on signs holding because J^- has only negative signs and Y has $2k+1$ positive signs and one negative. It gives C with $|C| = K \cdot 0$, so the theta integral of $(X, J^- \oplus Y)$ vanishes on all Schwartz vectors and standard data. The space X is $J \oplus H^{2k+1}$ up to signs, and $J^- \oplus Y$ has the signs of H^{2k+1}; for the pair $(J, 0)$ one has $\mathrm{rk}\, 0 - \mathrm{rk}\, J/2 = -k$ with $k \le \mu$, so by Proposition 31.18(d) the theta integral of $(J, 0)$ vanishes on all Schwartz vectors and standard data. But at the vacuum and the standard datum of the empty matrix its value at 1 is $\mathrm{vol} \neq 0$. Hence for each ν some Schwartz v has $r(v) \neq 0$.

The first floor. Fix one ν_0 and a Schwartz v with $r(v) \neq 0$ for ν_0. For every ν, $r(v) \ge 0$ and $r(v)^2$ is the number found above, which is $r_{\nu_0}(v)^2 > 0$; so $r(v)$ is the same positive number C for all ν.

(c) By induction on q. If $q = 0$, W has the signs of the definite space of rank $2k$, so $H \oplus W$ has the signs of $H \oplus J$, and (b) and the first part of (a) give the claim. If $q \ge 1$, let $W_0 = H^{q-1} \oplus J$, which has $q - 1$ negative signs and rank $2k + 2(q-1)$; by induction there is v for $(H \oplus W_0, H \oplus W_0)$ with constant ratio $C \neq 0$. Adding an adapted plane to $H \oplus W_0$ gives a space Y with the signs of $H \oplus W$, and by the second part of (a) a vector v' with ratio $C'C$, $C' > 0$; by the first part of (a) the same vector serves for $H \oplus W$. □

For the passage from (V', W) to a pair with source W, let V'^- be the opposite space of V', with signs $-t'$, and let the *adjoint datum* y^{ad} of a real vector y for (V', W) be its image under the change of coordinates that sends a matrix to its conjugate transpose, scaled at the finite places by an integral unit ϵ of norm -1.

Lemma 31.24 (Adjoint data). (a) *If y is Schwartz, so is y^{ad}; and the passage to the opposite signs carries the rational points of V' onto those of V'^-.*

(b) *There is a with $|a| = 1$ such that, for every Schwartz–Bruhat Ψ on left invertible matrices with a matched function, the datum with real part y^{ad}, the function $x \mapsto \Psi(x^{\mathsf{T}})$ at $\bar{S}$ and the vacuum elsewhere is a times the image of the datum with real part y, the function Ψ at S and the vacuum elsewhere.*

(c) *Let $\mathrm{rk}\, W$ be even and $\gamma = \nu^{-1}\upsilon^{\mathrm{rk}\, W/2}$. There is d with $|d| = 1$ such that $\gamma(\det g)$ times the theta integral of (W, V'^-) over $[\mathrm{U}(V'^-)]$ at y^{ad} and Ψ^{T} at $\bar{S}$ is d times the integral over $[\mathrm{U}(V')]$ of the theta kernel of (V', W), both characters γ, at y and Ψ at S.*

Proof. (a) The real part of the change of coordinates preserves the norm and the measure, hence all Fock moments. The second claim holds because the opposite space has the same unitary group. (b) The component of ϵ at S is a unit ε; the change of coordinates transposes the matrix and scales it by ε, and for matched functions multiplying the argument by a unit divides the second function by $\chi(\varepsilon)^{|b|}$, the value of χ at the determinant of the scalar matrix. Both data have real part y^{ad}, and the functions that name their finite parts agree up to this constant. (c) The change of coordinates changes the theta kernel by a constant of modulus one and the datum by a; the identity of the rational points carries the integral over $[\mathrm{U}(V'^{-})]$ to the one over $[\mathrm{U}(V')]$; and, since $\operatorname{rk} W$ is even, the characters of the two kernels differ by $\gamma(\det g)$. □

Proof of Theorem 31.1. Reductions. Put $|n| = 2|b| + 2e$, so $\operatorname{rk} W$ is even and $\operatorname{rk} V - \operatorname{rk} W/2 = (|a|-|b|)/2 = e$. Let $k = |n|/2 - q$; the hypotheses give $k \le \mu(\tau)$ at every τ and $k \ge 1$. By Proposition 31.23(c) there are a Schwartz z for $(H \oplus W, H \oplus W)$ and $C_0 \ne 0$ with equal rank ratio C_0 for every tame character of weight μ; as ratios are nonnegative, $C_0 > 0$.

Standard data. The opposite space V'^{-} has $|b|$ minus the number of negative signs of V' negative signs, so V'^{-} and W together have $|b|$ plus the number of negative signs of V, which is the condition of Lemma 31.22 for the triple (W, V, V'^{-}). With z and $y = v^{\mathrm{ad}}$, Schwartz by Lemma 31.24(a), it gives a vector x for (W, V) and a constant C_1 such that every tame ν of weight μ has c with $C_1|c|^2 = L_*(e)C_0$ and $\theta_{W,V}(x, \mathrm{row}_z) = c\,\theta_{W,V'^{-}}(v^{\mathrm{ad}}, \mathrm{std}_z)$ for all integral z. As $L_*(e) > 0$ and $C_0 > 0$, $C_1 \ne 0$ and $c \ne 0$; put $C = C_0/C_1$, so $|c|^2 = C\,L_*(e)$, and $C > 0$. The vector x and C are chosen before the character.

Matched data. Let $\mathcal{D}$ be the equivariant lift of Proposition 31.7(a) for (W, V, V'^{-}), x, v^{ad} and c. Its kernel $P = M \cap \ker \mathcal{D}$ is stable under translation at S, and it contains the standard datum of an integral matrix, with multiplicity one, whose partner is the row datum. By Proposition 31.7(b) it contains every locally constant Ψ with a matched Φ, and Proposition 31.7(a) identifies $\mathcal{D}\Psi$ with the difference of the two sides; so $\theta_{W,V}(x, \Phi) = c\,\theta_{W,V'^{-}}(v^{\mathrm{ad}}, \Psi^{\mathsf{T}})$ for all such pairs.

Back to (V', W). By Lemma 31.24(c), the right side is $c\,d\,\gamma(\det g)^{-1}$ times the integral of the theta kernel of (V', W); with the convention that the second character of $\theta_{W,V}$ enters as its inverse at $\det g$, this is the identity of (b) for the characters $\alpha = 1$ and $\alpha' = \gamma$, with the constant cd, $|cd| = |c|$.

Data modulo $\mathfrak{m}$. A character ν tame for $(\mathfrak{m}, \chi)$ is a split tame character with S the set of places of $\mathfrak{m}$, since $\mathfrak{m}$ is prime to its conjugate, and its $L_*(e)$ is the number of the theorem. By Proposition 31.5(c), matched functions Φ, Ψ modulo $\mathfrak{m}$ give matched local functions over the integers at S, with Ψ' locally constant, that define the same adelic data; the identity for these is the identity for Φ, Ψ.

Other characters. The space characters of α in rank $|a|$ and of α' in rank $|b|$ multiply to the inverse of the base change of ν exactly when $\alpha\alpha' = \gamma$, since $|a| + |b| = |n|$ is even. For such α, α' the twist law of the Weil representation multiplies the kernel of (W, V) with characters γ, α and the kernel of (V', W) with characters α', γ by the same factor $\alpha(\det g)^{-1}$ relative to the case $(1, \gamma)$, so the identity with the same c holds for them. □

31.5. Proof of the corollaries

One square. Throughout, for a matrix x modulo $\mathfrak{m}$ the *point datum* at x is the datum given by the indicator function of x at the places of $\mathfrak{m}$; the kernel of a pair with point datum x and the Fock vector of a polynomial P is written $\theta[x, P]$. For a set of forms f_1, f_2 of rapid decay, the *up integrand* is the theta kernel of $(U_1 \oplus U_2, A)$ with point datum x and real part w at $(g_1 \oplus g_2, h)$, times $f_1(g_1) f_2(g_2) v_0(\det g_2)$, and the *down integrand* is the same for $(A', U_1 \oplus U_2)$ and a point datum y.

Lemma 31.25 (Point data and integrability). (a) *Let k be the chosen integral finite point of $\mathrm{U}(T)$ with residue g modulo $\mathfrak{m}$. In the Weil representation of $\mathrm{U}(S) \times \mathrm{U}(T)$ twisted by α, β, the point k^{-1} of $\mathrm{U}(T)$ sends the point datum at gx to $\alpha(\det k)^{-1}$ times the point datum at x; similarly a point k of $\mathrm{U}(S)$ with residue g sends the point datum at yg to $\beta(\det k)^{-1}$ times that at y.*

(b) *If Ψ, Ψ' are matched over the residue ring, Ψ vanishes off the matrices whose transpose has a left inverse and Ψ' off the left invertible matrices.*

(c) *If v is tame of weight μ for $(\mathfrak{m}, \chi)$, the inverse of $v^{-1} v^{|N|/2}$ is tame for $(\mathfrak{m}, \chi)$ of some weight.*

(d) *If $b_2 v_0 = b_1$, the kernel of $(U_1 \oplus U_2, A)$ at $g_1 \oplus g_2$ for the point datum x and the product $P_1 P_2$, with characters $a_1 a_2$ and b_1, is the product of the kernels of (U_1, A) and (U_2, A) for the column blocks of x, divided by $v_0(\det g_2)$; and likewise for $(A', U_1 \oplus U_2)$ and the row blocks of y when $b_1' v_0 = b_2'$.*

(e) *For $\mathfrak{m} \neq (1)$, x with left invertible transpose and forms of rapid decay, there are N and D' such that the up integrand is integrable for the product of the quotient measures and bounded by D' times the N-th Fock moment of w, and its integral $J(w)$ satisfies $|J(w) - J(w')| \leq D' \cdot (\text{moment of } w' - w)$. The same holds for the down integrand at a left invertible y.*

Proof. (a) The finite Weil representation acts on point data by substitution and the character of the determinant, as in Lemma 31.6(a). (b) By the support condition Ψ lives on the rows of inverses of invertible matrices, whose transposes have left inverses, and Ψ' on columns of invertible matrices. (c) Inverting turns χ into χ^{-1} and then back, and a power of the restriction of v changes only the weight. (d) The datum and the polynomial are products along the decomposition of the source, the theta series of a product is the product of the theta series, and the characters on the source multiply. (e) The kernel grows at most like a power of the heights of g_1, g_2, uniformly in h, with a constant times a Fock moment of w; the rapid decay of f_1, f_2 absorbs the power and the character has modulus one. The integrand is additive in w, so the integral is Lipschitz in the stated sense. □

Lemma 31.26 (Up and down). *Let (U_1, A), (U_2, A) be edges $e_1 \to e$, $\bar{e}_2 \to \bar{e}$, and (A', U_1), (A', U_2) edges $\bar{e}' \to \bar{e}_1$, $e' \to e_2$, for families of characters indexed by $i \in I$, with $b_2 v_0 = b_1$ on A and $b_1' v_0 = b_2'$ on A', and let w be a real datum with finite moments.*

(a) *For the Fock vector of $P_1 P_2$, the integral of the up integrand at x is the integral over $[\mathrm{U}(A)]$ of the product of the theta lifts of f_1 and f_2; for $P_1' P_2'$, that of the down integrand at y is the integral over $[\mathrm{U}(A')]$ of the product of the transposed*

lifts, which is the pairing on U_2 of f_2 with the transposed lift of f_1 to A' followed by the lift to U_2.

(b) *There is one number κ such that for every i, every x with left invertible transpose and all forms, the integral of the up integrand at w is κ times the pairing of f_2 with the lift of f_1 to A followed by the transposed lift to U_2.*

(c) *Replacing x by gx multiplies the lift from U_1 to A followed by the transposed lift by $\chi(\det g)$, if the inverse of the product of the characters on U_1 and U_2 is tame for $(\mathfrak{m}, \chi)$; and replacing y by yg multiplies the transposed lift to A' followed by the lift by $\chi(\det g)$.*

Proof. (a) By Lemma 31.25(d) the kernel is a product, and integrating over $U_1 \times U_2$ the products with f_1, f_2 gives the two lifts, by Fubini (Lemma 31.25(e)); on the side of A' the lift from U_2 is adjoint to the transposed lift. (b) A polynomial is a sum of products $Q_1 Q_2$ of polynomials on (U_1, A) and (U_2, A), and the lifts with Q_1, Q_2 are real averages of the lifts along the edges: averaging over the real points multiplies the pairing on A by a number fixed by the real data, the same for all i. Hence for polynomial vectors the integral is $\kappa(P)$ times the pairing, with κ linear in P; the numbers $\Lambda_j(w)$ indexed by the cases $j = (i, x, f_1, f_2)$ are Lipschitz for one moment (Lemma 31.25(e)), and the functions $j \mapsto \Lambda_j$(polynomial vector) lie on the line through the pairings, which is closed; polynomial vectors are dense in every moment, so the same holds for w with one κ. (c) Translate the lifted form by the inverse of an integral point k of $\mathrm{U}(A)$ with residue g: the column blocks of gx are g times those of x, and by Lemma 31.25(a) the scalar is the product of the two characters at $\det k$, which is $\chi(\det g)$ by tameness of its inverse. The other path is the same with an integral point of $\mathrm{U}(A')$ and the row blocks of yg. □

Proof of Corollary 31.2. All four products of the characters on the pairs are the rank twist of ν_i^{-1} for W: in even rank, two space characters multiply to the inverse base change of ν_i exactly when the characters multiply to the rank twist, as in the last step of the proof of Theorem 31.1. One character ν_0 relates both pairs: $\nu_0 = b_1/b_2$ on A, and then $b_1' \nu_0 = b_2'$ on A'. The equivariance under $x \mapsto gx$ and $y \mapsto yg$ is Lemma 31.26(c), with Lemma 31.25(c) for tameness. Apply Theorem 31.1 to $W = U_1 \oplus U_2$ with $V = A$, $V' = A'$ and the Fock vector v of $P_1' P_2'$, which is Schwartz; it gives C, a Schwartz w for (W, A) and, for each i, c_i with $|c_i|^2 = C\, L_{*,i}(t) \neq 0$, so $c_i \neq 0$. Let κ be the number of Lemma 31.26(b) for w. Forms on U_2 are determined by their pairings with the forms $f_2 = \bar{z}$ of the conjugate type. Expand the identity of Theorem 31.1 at $g_1 \oplus g_2$ in point data, $\Psi = \sum_x \Psi(x) 1_x$ and $\Psi' = \sum_y \Psi'(y) 1_y$, multiply it by $f_1(g_1) f_2(g_2) \nu_0(\det g_2)$, and integrate over $[\mathrm{U}(U_1)] \times [\mathrm{U}(U_2)]$. On the side of A the sum over x runs over matrices with left invertible transpose (Lemma 31.25(b)), so each term is integrable and gives κ times the pairing with $\Theta_i(x) f_1$; on the side of A' the sum runs over left invertible matrices and each term gives the pairing with $\Theta_i'(y) f_1$ (Lemma 31.26(a)). Hence

$$\kappa \sum_x \Psi(x) \langle z, \Theta_i(x) f_1 \rangle = c_i \sum_y \Psi'(y) \langle z, \Theta_i'(y) f_1 \rangle$$

for all z and f_1, and dividing by c_i gives the corollary. □

Explicit exchange. Let $j : m \sqcup l \to n$ split the index set of W by sign and $f : p \sqcup l \to b$ that of V', with $e : p \sqcup a \to m$. At each infinite place τ, the

stacked determinant product sd_η is the product over $i \in \iota_\tau$ of the determinants of $\eta_{\tau,i}$ stacked over the $a \times m$ matrix of variables, columns ordered by e, and the *matched polynomial* $\mathrm{mp}_\eta = \prod_\tau \det(y_\tau^-)^{\#\iota_\tau + n - 2l} \prod_{\tau,i} \det(\eta_{\tau,i} y_\tau)$ is the polynomial of Corollary 31.3. The *standard frame* η^0 is the $p \times m$ matrix $(1\ 0)$ for $m \simeq p \sqcup a$, and the *frame torus* t_z, $|z| = 1$, is the diagonal point of $\mathrm{U}(m)$ with entries z^{-2} on p and z^{-1} on a at every place; the point $\mathrm{pb}(k)$ is $k \in \mathrm{U}(m)$ placed in the block of positive signs of W.

Lemma 31.27 (Laws of the two polynomials). *Let $\gamma_\tau = l - \#\iota_\tau$ and let the real Weil representation twist (x, u) by $\prod_\tau \det u_\tau^{\gamma_\tau}$.*

(a) *For $k \in \mathrm{U}(m)$, $(\mathrm{pb}(k), u)$ carries the Fock vector of $\mathrm{sd}_{\eta(k^\mathsf{T})^{-1}}$ to that of sd_η times $\prod_\tau \det k_\tau^{-\#\iota_\tau}$; and $\mathrm{pb}(k)$ carries the Fock vector of mp_η to that of $\mathrm{mp}_{\eta k^\mathsf{T}}$ times the same factor. Scaling all matrices by w multiplies mp by $w^{\#p\#\iota_\tau}$ at each place.*

(b) (*Circle mean.*) *For every Fock vector y there is κ such that for every functional ℓ invariant under $\mathrm{U}(a)$ the mean over $|z| = 1$ of $\ell(\Omega_z y)$ is $\kappa\, \ell(\mathrm{sd}_{\eta^0})$, where $\Omega_z y$ is y moved by $(\mathrm{pb}(t_z), z^{-1})$ times $z^{-|a| \sum_\tau \#\iota_\tau}$.*

(c) (*Polarisation.*) *For nonempty a, sd_ξ and mp_ξ are finite combinations, with the same coefficients, of the same polynomials for the unitary translates $\eta^0 k_\tau^\mathsf{T}$ of the standard frame.*

Proof. (a) The pair acts by a substitution of the variables and the vacuum character. For sd the vacuum character is $\det u^{\gamma_\tau - l} = \det u^{-\#\iota_\tau}$, which cancels the powers of $\det u$ produced by the substitution on the a side, and the substitution on the m side turns $\eta(k^\mathsf{T})^{-1}$ into η with the factor $\det k^{-\#\iota_\tau}$. For mp the substitution fixes the block of variables with negative signs and multiplies the block y with positive signs by k^T, and entry by entry $\eta(k^\mathsf{T} y) = (\eta k^\mathsf{T}) y$; the vacuum character gives $\det k^{\gamma_\tau - l}$. Each factor depending on the matrices is a $p \times p$ determinant, which gives the scaling.

(b) The mean is continuous in y and the functions of ℓ proportional to $\ell(\mathrm{sd}_{\eta^0})$ form a closed line, so it suffices to treat Fock vectors of polynomials P, which are dense. Let $\mathrm{cols} = j \circ e$ on a be the columns of the variables met by the stacked determinants of η^0, and let $\mathrm{sc}(s)$ be ℓ at the Fock vector of P with the variables outside cols multiplied by s. The torus t_z and the scalar z^{-1} act on the variables by substitutions; comparing them variable by variable, and with the scalars of Ω_z cancelling, $\ell(\Omega_z P) = \mathrm{sc}(z)$. The function sc is a polynomial in s, so its mean over the circle is $\mathrm{sc}(0)$, the value of ℓ at P_0, the polynomial P with the variables outside cols set to 0. Read P_0 as a polynomial in square $a \times a$ matrices in the columns cols; since ℓ is invariant under $\mathrm{U}(a)$, its value at the translate of P_0 by u is $\mathrm{sc}(0)$ times $\prod_\tau \det u_\tau^{\#\iota_\tau}$, both substitutions keeping the columns cols, mixed by u, and killing the other variables. Averaging over $\mathrm{U}(a)$ with the weight $\prod_\tau \det u_\tau^{-\#\iota_\tau}$ projects onto the line of $\prod_\tau \det^{\#\iota_\tau}$, so $\mathrm{sc}(0) = \kappa\, \ell(\prod_\tau \det^{\#\iota_\tau})$ with κ depending only on P_0; and for the standard frame the stacked determinant at τ is the minor of the variables in the columns cols, so this is $\kappa\, \ell(\mathrm{sd}_{\eta^0})$.

(c) At each point x there are an $m \times p$ matrix Y and a number d with $\det(\xi; x_\tau) = d \det(\xi Y)$ for all ξ (expand the stacked determinant along the rows of ξ). By irreducibility of the module of products of determinants under the unitary group, $\prod \det(\xi_{\tau,i} Y_\tau)$ is a finite combination of the same polynomials for the unitary translates of η^0, with coefficients independent of Y; reading the

same combination in mp (the factor of the negative block being common) gives the claim. □

Lemma 31.28 (The archimedean step). *For nonempty a there is a polynomial Q_0 such that every real vector v has a number κ with the following property. Let ℓ_g be continuous functionals with $\ell_g(\omega(x_1,x_2)w) = \ell_{gx_1}(w)$, and ℓ' a linear form on polynomials with $\ell_x(v) = \ell'(Q)$ whenever x carries the Fock vector of Q_0 to that of a polynomial Q. Then $\kappa\,\ell_1(\mathrm{sd}_\eta) = \ell'(\mathrm{mp}_\eta)$ for all η.*

Proof. Take $Q_0 = \mathrm{mp}_{\eta^0}$. The functionals ℓ_g are invariant under the second group, so Lemma 31.27(b) applies to $\ell_{\mathrm{pb}(k)}$: the circle mean of $\ell_{\mathrm{pb}(k)}(\Omega_z v)$ is $\kappa\,\ell_{\mathrm{pb}(k)}(\mathrm{sd}_{\eta^0}) = \kappa\prod_\tau \det k_\tau^{-\#\iota_\tau}\ell_1(\mathrm{sd}_{\eta^0 k^{\mathsf T}})$ by Lemma 31.27(a). On the other hand $\ell_{\mathrm{pb}(k)}(\Omega_z v)$ is a power of z times $\ell_{\mathrm{pb}(kt_z)}(v)$, the second group being invisible, and by Lemma 31.27(a) the point $\mathrm{pb}(kt_z)$ carries Q_0 to $\prod_\tau \det(kt_z)_\tau^{-\#\iota_\tau}$ times $\mathrm{mp}_{\eta^0 t_z k^{\mathsf T}} = \mathrm{mp}_{z^{-2}\eta^0 k^{\mathsf T}}$; so the hypothesis gives it as ℓ' of that polynomial. By the scaling law and $\det t_z = z^{-(2|p|+|a|)}$ the powers of z cancel, and the integrand of the circle mean is the constant $\prod_\tau \det k_\tau^{-\#\iota_\tau}\ell'(\mathrm{mp}_{\eta^0 k^{\mathsf T}})$. Cancelling the nonzero power of $\det k$ gives the identity for the unitary translates $\eta^0 k^{\mathsf T}$; both sides are linear in the polynomials, so Lemma 31.27(c) extends it to all η. □

Proof of Corollary 31.3. Class sums. The theta integral over $\mathrm{U}(I_a)$ at a point of $\mathrm{U}(W)$, with finite part Φ, is a continuous linear functional of the real part: it is the lift of the constant function 1, of rapid decay on a definite group, against the datum $w\otimes\Phi$. The integral over $[\mathrm{U}(V')]$ of the kernel with datum Ψ and a polynomial Q is a linear form in Q, the kernels being rapidly decreasing as Ψ lives on left invertible matrices. For a character α tame of weight $l - \#\iota_\tau$ and Φ matched with Ψ for χ^{-1}, the kernel of (W, I_a) with the polynomial sd_η does not depend on the real part of the point of $\mathrm{U}(I_a)$, since $\mathrm{U}(I_a)(\mathbb{R})$ fixes the vector of sd_η up to the character absorbed by the weight, and translation by k in the stabiliser of the standard lattice replaces Φ by $\alpha(\det k)\Phi(k^{-1}\cdot)$, which is Φ since $\alpha(\det k) = \chi(\det k)$ and $\Phi(k^{-1}y) = \chi(\det k)^{-1}\Phi(y)$. Hence the integral over $[\mathrm{U}(I_a)]$ is the class sum.

Transfer. With Q_0 from Lemma 31.28, suppose that at every point the theta integral over $\mathrm{U}(I_a)$ with datum Φ, v is c times the integral over $[\mathrm{U}(V')]$ of the kernel with data Ψ, Q_0. Apply Lemma 31.28 to ℓ_g the theta integral over $\mathrm{U}(I_a)$ at $g(y,1)$ as a functional of the real datum and ℓ' the integral over $[\mathrm{U}(V')]$ as a functional of the polynomial; if y carries Q_0 to Q, the hypothesis at $g(y,1)$ is the required relation. It gives c', depending on v, with c' times the class sum for Φ, sd_η equal to c times the integral for Ψ, mp_η.

The exchange. Write $n = 2l + 2s_0$ and $\alpha_0 = \alpha\nu^{-l}$, of weight $-\#\iota_\tau$, and $\nu = \alpha_0\nu^{-s_0}$. Then ν^{-1} is tame for $(\mathfrak{m}, \chi^{-1})$ of weight $\#\iota_\tau + s_0$, and $n \le 2(\#\iota_\tau + s_0) + 2l$. Apply Theorem 31.1 to (W, I_a, V'), which satisfies its hypotheses ($a = b + 2d$, the l negative signs of W are those of V', $2l + 2 \le n$), with the Fock vector of Q_0 on the side of V'. It gives $C > 0$, a real datum v and for each α a number c_0 with $|c_0|^2 = C\,L_*(\nu^{-1}, d) \ne 0$. In the exchange identity W and V' carry the character $\nu^l\alpha_0 = \alpha$ and I_a the character 1: the rank twist of ν is $\nu^{l+s_0}\nu = \alpha$, and the space characters of 1 in rank a and of α in rank b multiply to the inverse base change of ν^{-1}. The transfer gives c' with c' times the class sum equal to $c_0\alpha_0(\det g)$ times

the integral, twisting the standard characters by α_0 multiplying the kernels by values of α_0 and of $\beta(\det g)^{-1}$.

The constant. The kernels of the corollary, with the characters α, β on (W, I_a) and α/β, α on (V', W), differ from those with the standard characters by values of α_0 at the determinants and by $\beta(\det g)^{-1}$, in the same way on both sides; so the relation of the transfer becomes c times the class sum equal to the integral, with $c = c'/c_0$, and $|c|^2 = |c'|^2/(C\, L_*(\nu^{-1}, d))$. As $l + s_0 = b + d$, $\nu^{-1} = \alpha^{-1}\upsilon^{b+d} = \mu$, so $|c|^2 = C'/L_*(d)$ for μ, with $C' = |c'|^2/C$, which may be 0.

When b is empty, σ identifies a with n, the functions $\chi \circ \det$ on $a \times n$ matrices and 1 on $n \times 0$ matrices are matched for χ^{-1} (the block determinant functions of Proposition 31.5(a) for the constant $f = 1$), the matched polynomial is 1, the kernel of (V', W) is $\alpha(\det g)$ at (x, g), and the quotient of the trivial group has volume 1; so the right side of the corollary is 1 and it computes the class sum itself. □

32. The norm of the chain

The lower chain $\Theta_\vartheta = \iota_\vartheta$ goes from the target forms on Y_0 through $Y_1, \dots, Y_{m-1}$ to the definite space Y_m, mostly outside Weil's range. Its Gram operator is computed by arranging the lifts in a triangle whose rows are chains of lifts starting from copies of Y_0 and whose descents remove the first block; the squares of the triangle commute up to constants by the exchange identity, and a descent along one square turns a chain of $n + 1$ lifts into a chain of n lifts one row lower. At the bottom only single lifts remain, whose norms are given by Rallis' formula (Theorem 30.5). The result is a factorisation of the volume $Z(\vartheta)$ into the volumes of the single lifts times ratios of Hecke values (Theorem 32.14).

The section has four parts: linear algebra over a finite ring, which moves a datum around one square (Section 32.1); the abstract triangle, in which the Gram form of a full chain is computed by induction on its length (Section 32.2); the triangle of theta lifts, whose squares are instances of the one-square identity (Section 32.3); and the bookkeeping of the constants, which are ratios of Hecke values up to powers of the norm of the modulus (Sections 32.4 and 32.5).

32.1. The finite square law

Let R be a finite commutative ring and χ a multiplicative character. Functions L on $a \times n$ matrices and L' on $n \times b$ matrices, with $\sigma\colon b \sqcup a \to n$, satisfy the *square law with constant* c if both transform by $\chi \circ \det$ (on the left, resp. the right) and $\sum \Psi L' = c \sum \Phi L$ for all matched Φ, Ψ (Section 31.1). The *point datum* is the indicator of the inclusion matrix $\binom{1}{0}$; for a function p on $n \times a$ matrices the *next datum* is the function on $(n \sqcup b) \times (a \sqcup b)$ matrices with the value $p(-\eta)/\#M_{n\times b}(R)$ at $\left(\begin{smallmatrix} \eta & s \\ 0 & 1 \end{smallmatrix}\right)$ and 0 elsewhere.

In this subsection matched functions are taken over all of R: Φ vanishes off the rows x_k of the inverses and Ψ off the columns y_k of the invertible matrices k over R, and $\Phi(x_k) = \chi(\det k)\Psi(y_k)$. The *Fourier transform* of a function f of matrices is $\hat{f}(\rho) = \sum_v f(v)\psi(\operatorname{tr}(\rho v))$ for an additive character ψ; a character is *primitive* if it is nontrivial on every nonzero ideal, and for a multiplicative character this means that for every $a \neq 0$ some $\chi(1 + ca)$ is neither 0 nor 1.

Lemma 32.1 (Matched functions over a finite ring). (a) *If $x_\gamma = x_{\gamma'}$ for invertible γ, γ', then $y_{\gamma'} = y_\gamma h$ for an invertible h with $\det\gamma' = \det\gamma \det h$.*

(b) *If Ψ vanishes off the y_γ and $\chi(\det h)\Psi(yh) = \Psi(y)$ for invertible h, the function Φ with $\Phi(x_\gamma) = \chi(\det\gamma)\Psi(y_\gamma)$ and 0 off the x_γ is well defined and matched with Ψ. A function matched with Ψ is unique, and if Φ, Ψ are matched, so are $x \mapsto c\chi(\det g)\Phi(xg)$ and $y \mapsto c\Psi(g^{-1}y)$.*

(c) *If $f(g) = \chi(\det g)v$ for all invertible g, then $\sum_u \chi^{-1}(\det u) f(u)$ over all square matrices u is $\#\mathrm{GL}(R)\, v$, where χ^{-1} is read as 0 at non-units.*

(d) *Let R be such that any two elements are multiples of $(1-\varepsilon)r + \varepsilon s$ for some idempotent ε, and split $N \sqcup n$ into b and a. If the rows N or the rows n of an $(N \sqcup n) \times b$ matrix y have a left inverse, then $y = y_\gamma$ for an invertible γ; and for invertible k, if the rows N of y_k have a left inverse, so have the columns n of x_k, and conversely.*

Proof. (a) Both blocks of columns span the kernel of the common rows x_γ, so they differ by an invertible h. Stacking a matrix P with $Py_\gamma = 1$ on x_γ gives a square matrix whose product with γ is block triangular with unit diagonal blocks, because $x_\gamma y_\gamma = 0$ and x_γ times the other columns of γ is 1; with $h^{-1}P$ in place of P the same holds for γ', and comparing determinants gives $\det\gamma' = \det\gamma \det h$. (b) Well defined by (a) and the transformation of Ψ; matched by construction; unique because a matched function is determined on the x_γ and vanishes elsewhere; and the translates are matched because $x_\gamma g = x_{g^{-1}\gamma}$, $g^{-1}y_\gamma = y_{g^{-1}\gamma}$ and $\chi(\det g)\chi(\det g^{-1}\gamma) = \chi(\det\gamma)$. (c) The nonzero terms are those of invertible u, each equal to v. (d) Extend a left inverse L of the rows N of y by zero to a left inverse s of y; stacking s on a complement gives an invertible matrix (the hypothesis on R allows one to choose complements componentwise), whose inverse has y as its first columns. For the second statement, the square matrix with rows s and x_k has determinant inverse to $\det k$, and reading $P^{-1}P = 1$ in the columns n, the rows s contribute nothing, so the columns n of x_k have a left inverse; the converse is symmetric. □

Proposition 32.2 (Square law). *Let L, L' satisfy the square law with constant c.*

(a) *For an $a \times b$ matrix w, $\#\mathrm{GL}_b(R)\, L'\left(\begin{smallmatrix}1\\-w\end{smallmatrix}\right) = c\,\#\mathrm{GL}_a(R)\, L(w\ 1)$.*

(b) *With L' evaluated at y stacked on the inclusion of a in $a \sqcup b$ and L at the inclusion of n in $n \sqcup b$ placed left of x, $\#\mathrm{GL}_a(R) \sum_y p(y)L'(\cdot) = c\,\#\mathrm{GL}_{n\sqcup b}(R) \sum_x \mathrm{next}(p)(x)L(\cdot)$.*

(c) *If the additive and multiplicative characters of R are primitive, there is γ with $|\gamma|^{-2} = \#R^{\#T(\#T+2\#N)}$ such that the same identity holds for the Fourier transforms of the conjugates of p and of $\mathrm{next}(p)$, with γc in place of c.*

In (c) the shapes are $n = N$, $a = N \sqcup T$ and $b = T$.

Proof. (a) Let δ be the indicator of w among the $a \times b$ matrices. The block determinant functions $\Phi(x_1\ x_2) = \chi^{-1}(\det x_2)\delta(x_2^{-1}x_1)$ and $\Psi\left(\begin{smallmatrix}y_1\\y_2\end{smallmatrix}\right) = \chi^{-1}(\det y_1)\delta(-y_2y_1^{-1})$ are matched (Proposition 31.5(a), which holds over any commutative ring). The matrices y with $\Psi(y) \neq 0$ are $\left(\begin{smallmatrix}1\\-w\end{smallmatrix}\right)y_1$ with $y_1 \in \mathrm{GL}_b(R)$, and there $\Psi(y)L'(y) = \chi^{-1}(\det y_1)\chi(\det y_1)L'\left(\begin{smallmatrix}1\\-w\end{smallmatrix}\right)$, so $\sum \Psi L' = \#\mathrm{GL}_b(R)L'\left(\begin{smallmatrix}1\\-w\end{smallmatrix}\right)$. Likewise the matrices x with $\Phi(x) \neq 0$ are $x_2(w\ 1)$ with $x_2 \in \mathrm{GL}_a(R)$, and $\sum \Phi L = \#\mathrm{GL}_a(R)L(w\ 1)$. The square law gives the identity.

(b) Order $n \sqcup a \sqcup b$ so that a comes first, and apply (a) with a in the role of b and $n \sqcup b$ in the role of a, at $w = \binom{-y}{0}$ for an $n \times a$ matrix y. Then $\binom{1}{-w}$ is y stacked on the inclusion $\binom{1}{0}$ of a in $a \sqcup b$, and $(w\ 1)$, with the columns reordered, is $\begin{pmatrix} 1 & -y & 0 \\ 0 & 0 & 1 \end{pmatrix}$. For $x = \left(\begin{smallmatrix} \eta & s \\ 0 & 1 \end{smallmatrix}\right)$ the matrix obtained by placing the inclusion of n in $n \sqcup b$ left of x is $\begin{pmatrix} 1 & \eta & s \\ 0 & 0 & 1 \end{pmatrix}$, which the unipotent matrix $\left(\begin{smallmatrix} 1 & -s \\ 0 & 1 \end{smallmatrix}\right)$ of determinant 1 carries to $\begin{pmatrix} 1 & \eta & 0 \\ 0 & 0 & 1 \end{pmatrix}$; so L there does not depend on s and equals $L(w\ 1)$ with $y = -\eta$. Summing (a) over y with the weights $p(y)$ and writing $y = -\eta$, the right side is $c\,\#\,\mathrm{GL}_{n \sqcup b}(R)$ times the sum over η and the average over s of $p(-\eta)L(\cdots)$, which is the sum weighted by $\mathrm{next}(p)$ (Lemma 32.1, and the description of the sums over block matrices as iterated sums over the blocks).

(c) *Fourier transform of the next datum.* At a block matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ the transform of $\mathrm{next}(p)$ is $\hat{p}(-A)\psi(\mathrm{tr}\, D)$ if $C = 0$ and 0 otherwise: the sum over s of $\psi(\mathrm{tr}(Cs))$ is $\#M(R)$ or 0 according as $C = 0$ or not, the additive character being primitive. *The Gauss sum.* Let $G = \sum_d \chi(\det d)\psi(\mathrm{tr}\, d)$ over the $T \times T$ matrices d; for primitive characters $|G|^2 = \#R^{\#T^2}$. For a function L transforming by $\chi \circ \det$ on the left, the weight given by the transform of $\overline{\mathrm{next}(p)}$ spreads over all corner blocks d with the factor $\psi(\mathrm{tr}\, d)$, while the weight $\mathrm{next}(\hat{\bar{p}})$ lives on $d = 1$. Replacing d by gd multiplies the term by $\chi(\det g)$, so the transform in d at 1 is G times the value at $d = 1$, and the sum over the blocks s contributes the number $\#R^{\#N\#T}$ of such blocks. Hence the two weighted sums of L differ by a constant γ_0 with $|\gamma_0|^2 = \#R^{\#T^2}\#R^{2\#N\#T} = \#R^{\#T(\#T+2\#N)}$. *The identity.* Apply (b) to $q = \hat{\bar{p}}$ and replace $\mathrm{next}(q)$ by γ_0^{-1} times the transform of $\overline{\mathrm{next}(p)}$; here $a = N \sqcup T$ and $n \sqcup b = N \sqcup T$, so the two general linear groups in (b) have the same order and cancel. The constant is $\gamma = \gamma_0^{-1}$. □

32.2. Abstract triangles

A *triangle of size m* consists of inner product spaces $F_{i,n}$, arrows $a_{i,n} : F_{i,n} \to F_{i,n+1}$ along the rows, descents $d_{i,n} : F_{i,n+1} \to F_{i+1,n}$, nonzero weights w_n and isometries $j_i : V \to F_{i,0}$, all depending on a parameter, such that the squares commute up to scalars, $\langle v, a(j_i y)\rangle = \langle dv, j_{i+1} y\rangle$ on the copies of V, and $d \circ a$ maps each copy injectively to the next. The *weighted chain* $X_{i,n}$ is the composite of the first n arrows of row i, each multiplied by its weight. A triangle *has square moduli L* for a unital multiplicative relation $c \sim f$, read *c has modulus f*, if the square at (i, n) reads $w_n\, a \circ d = (w_{n+2}/\bar{\beta})\, d \circ a$ with β of modulus $L_{i+1,i+n+2}$, and the adjoint law holds with constants of modulus 1.

Here the square at (i, n), for $i + n + 2 \le m$, has the corners $F_{i,n+1}$, $F_{i,n+2}$, $F_{i+1,n}$, $F_{i+1,n+1}$; *commuting up to* β means $w_n\, a \circ d = (w_{n+2}/\bar{\beta})\, d \circ a$ on $F_{i,n+1}$, the second scalar being read as 0 where $\beta = 0$. The *adjoint law* with constant κ asks that every $z \in F_{i,n+2}$ have a vector $x \in F_{i,n+1}$ with $\langle av, z\rangle = \langle v, x\rangle$ for all v and $\langle dx, y\rangle = \kappa\langle dz, ay\rangle$ for all y. The weights are real in the application, and the inner products are conjugate linear in the first argument. Two laws for chains are used: *d moves past the n-chain up to* β if

$$X_{i+1,n} \circ d \circ a = \frac{w_{n+1}}{w_1 w_0 \bar{\beta}}\, d \circ X_{i,n+1} \qquad \text{on } F_{i,0},$$

and *the arrow after the n-chain is adjoint to d up to κ* if $\langle z, a(X_{i,n} j_i y)\rangle = \kappa\langle dz, X_{i+1,n} j_{i+1} y\rangle$ for all $z \in F_{i,n+1}$ and $y \in V$.

Lemma 32.3 (Chain laws). *Consider a triangle of size m.*

(a) *d moves past the empty chain up to 1. If d moves past the n-chain of row i up to β and the square at (i,n) commutes up to β', then d moves past the $(n+1)$-chain up to $\beta\beta'$. Hence for $i+n<m$ it moves past the n-chain up to the product of the constants of the squares $(i,0),\dots,(i,n-1)$; under square moduli L this product has modulus $\prod_{i+1<k\le i+1+n} L_{i+1,k}$.*

(b) *The arrow after the empty chain is adjoint to d up to 1. If the arrow after the n-chain is adjoint to d up to κ and the square at (i,n) satisfies the adjoint law with κ', then the arrow after the $(n+1)$-chain is adjoint to d up to $\kappa\kappa'$. Under square moduli the constant has modulus 1.*

(c) *If d moves past the n-chain up to β, the arrow after it is adjoint up to κ, and $j_{i+1}x' = d(a(j_i x))$, then for all $z \in V$*

$$\langle X_{i,n+1} j_i x,\ X_{i,n+1} j_i z\rangle = \kappa\beta\,\frac{w_0 w_1 w_n}{w_{n+1}}\,\langle X_{i+1,n} j_{i+1} x',\ X_{i+1,n} j_{i+1} z\rangle.$$

Proof. (a) For $n=0$, $X_{i,1} = w_0 a$, so $d \circ X_{i,1} = w_0\, d\circ a$, and $\frac{w_1}{w_1 w_0} w_0\, d\circ a = d\circ a$. For the induction, $X_{i+1,n+1}\circ d\circ a = w_n\, a\circ X_{i+1,n}\circ d\circ a = \frac{w_{n+1}}{w_1 w_0 \bar\beta}\,(w_n\, a\circ d)\circ X_{i,n+1}$, and by the square $w_n\, a\circ d = \frac{w_{n+2}}{\bar\beta'}\, d\circ a$; since $X_{i,n+2} = w_{n+1}\, a\circ X_{i,n+1}$, the result is $\frac{w_{n+2}}{w_1 w_0 \bar\beta\bar\beta'}\, d\circ X_{i,n+2}$. The moduli multiply, which gives the product over the squares. (b) For $n=0$ this is the adjoint law on the copies. For the induction, let $z \in F_{i,n+2}$ and let x be the vector of the adjoint law of the square at (i,n). Then, with $u = X_{i,n+1} j_i y = w_n\, a(X_{i,n} j_i y)$,

$$\begin{aligned}\langle z, au\rangle = \langle x, u\rangle &= w_n\langle x, a(X_{i,n} j_i y)\rangle \\ &= w_n\kappa\langle dx, X_{i+1,n} j_{i+1} y\rangle = w_n\kappa\kappa'\langle dz, a(X_{i+1,n} j_{i+1} y)\rangle,\end{aligned}$$

which is $\kappa\kappa'\langle dz, X_{i+1,n+1} j_{i+1} y\rangle$. (c) Apply (b) with z replaced by $X_{i,n+1} j_i x \in F_{i,n+1}$ and $X_{i,n+1} j_i z = w_n\, a(X_{i,n} j_i z)$:

$$\langle X_{i,n+1} j_i x, X_{i,n+1} j_i z\rangle = w_n\kappa\langle d\, X_{i,n+1} j_i x, X_{i+1,n} j_{i+1} z\rangle.$$

By (a), $d\, X_{i,n+1} j_i x = \frac{w_1 w_0 \bar\beta}{w_{n+1}} X_{i+1,n}(d\, a\, j_i x) = \frac{w_1 w_0 \bar\beta}{w_{n+1}} X_{i+1,n} j_{i+1} x'$, and the scalar leaves the first argument as its conjugate. □

Theorem 32.4 (Chains in triangles). *Let a triangle of size m be given.*

(a) *For $i+n \le m$, $X_{i,n}$ is injective on the copy of V; for $i+n<m$ there is β, nowhere zero, with $X_{i+1,n}\circ d\circ a = (w_{n+1}/(w_1 w_0\bar\beta))\, d\circ X_{i,n+1}$ on $F_{i,0}$.*

(b) *Under square moduli L, let $s: U \to V$, endomorphisms S_i of U and functions C_i of modulus 1 satisfy $w_0\langle ax, a(sy)\rangle = C_i\langle x, s(S_i y)\rangle$ for each row. Then for $i+n=m$*

$$\langle X_{i,n} x, X_{i,n} y\rangle = \kappa\,\frac{w_0 w_1^n}{w_n}\,\big\langle x, s(S_i\cdots S_{m-1} y)\big\rangle, \qquad |\kappa| = \prod_{i<r<k\le m} L_{r,k}.$$

(c) *The block triangular group with one diagonal block N and j blocks T has order $\#\mathrm{GL}_N(R)\,\#\mathrm{GL}_T(R)^j\,\#R^k$, with k independent of R.*

Proof. (a) *Injectivity*, by induction on n for all i at once. For $n=0$, $X_{i,0}$ is the identity and j_i is an isometry. Let $X_{i,n+1} j_i y = 0$ with $i+n+1 \le m$. By

Lemma 32.3(a), $X_{i+1,n}(d\,a\,j_i y)$ is a multiple of $d\,X_{i,n+1}j_i y = 0$; as $d \circ a$ maps the copy in row i to that in row $i+1$, $d\,a\,j_i y = j_{i+1}x$ for some x, and by induction $x = 0$; as $d \circ a$ is injective on the copy, $y = 0$. *The constant.* Lemma 32.3(a) gives β. If $\beta(t) = 0$ at some parameter, the scalar is 0, so $X_{i+1,n}$ kills $d\,a\,j_i y = j_{i+1}x$ for every y; choosing $y \neq 0$ gives $x \neq 0$, contradicting injectivity. (The same argument shows that the constants of the squares have no zero.)

(b) By induction on n, with $i = m - n$. For $n = 0$ both sides are $\langle x, sy\rangle$. Let $i + n + 1 = m$. By Lemma 32.3(a), (b), d moves past the n-chain of row i up to β and the arrow after it is adjoint up to κ_0, with $|\kappa_0\beta| = \prod_{i+1<k\leq m} L_{i+1,k}$. Let $j_{i+1}x' = d\,a\,j_i x$. By Lemma 32.3(c) and the induction hypothesis for row $i+1$,

$$\langle X_{i,n+1}x, X_{i,n+1}sy\rangle = \kappa_0\beta\frac{w_0w_1w_n}{w_{n+1}}\cdot\kappa'\frac{w_0w_1^n}{w_n}\langle x', s(S_{i+1}\cdots S_{m-1}y)\rangle,$$

with $|\kappa'| = \prod_{i+1<r<k\leq m} L_{r,k}$. By the adjoint law on the copies, $\langle x', sT'y\rangle = \langle a\,j_i x, a\,j_i(sT'y)\rangle$ for $T' = S_{i+1}\cdots S_{m-1}$, which is $w_0^{-1}C_i\langle x, s(S_iT'y)\rangle$ by hypothesis. Collecting, the constant is $\kappa = \kappa_0\beta\kappa'C_i$, of modulus $\prod_{i<r<k\leq m} L_{r,k}$, and the weights give $w_0w_1^{n+1}/w_{n+1}$.

(c) A matrix indexed by N and $j+1$ copies of T is block upper triangular exactly when its upper left corner, indexed by N and j copies, is block upper triangular and its lower left corner vanishes. An invertible such matrix is therefore a triple: an invertible block triangular matrix of the smaller shape, an arbitrary upper right block, and an invertible last diagonal block. Hence the order for $j+1$ is that for j times $\#\mathrm{GL}_T(R)\,\#R^{\#T(\#N+j\#T)}$, and induction gives the formula with $k = \sum_{l<j}\#T(\#N + l\#T)$. □

32.3. The triangle of theta lifts

For a flag z and the m-fold twist ϑ, the node (i,n) of the triangle carries the space of the interval $[i+1, i+n]$: the stem and the blocks $i+1,\ldots,i+n$, with signs and doubled parameter of the grid (Proposition 15.4); the nodes $(i,0)$ carry the space at $(0,0)$ negated i times. For z = false the row $i = 0$ is the chain $Y_0 \to \cdots \to Y_m$ twisted by ϑ; for z = true the second chain, untwisted, has blocks of length $t-1$ and starts from Y_0 plus a line. The *lift* $(i,n) \to (i,n+1)$ adds the block $n+i+1$: its character is the character of lift $n+i+1$, twisted by ϑ^{n+i+1} if z is false, times a power of v, its polynomial is the joint harmonic, and its finite datum is the point datum modulo the modulus $\mathfrak{m}$ of the triangle. The *descent* $(i,n+1) \to (i+1,n)$ is a transposed lift along a left pair, removing the first block; the *ascent* $(i+1,n) \to (i,n+1)$ is a transposed lift along the reversed pair, adding the first block. The *weights* are the orders $N_{z,\vartheta,n}$ of the general linear group of the node $(0,n)$ modulo $\mathfrak{m}$. The *base* is the space of forms at $(0,0)$ fixed by the level J_M (for z false) or by the principal level of $\varpi^{\lceil b\rceil}$ (for z true).

Proposition 32.5 (The triangle of lifts). (a) (*First lifts.*) *For $i < m$, the first lift a of row i, read on the base, satisfies: if $\langle v, x\rangle = \langle ay, ax\rangle$ for all x, then v lies in the base, and $v = 0$ forces $y = 0$; and $N_{z,\vartheta,0}\langle ax, ay\rangle$ is a constant independent of ϑ times the Rallis series of the terms $\langle x, gy\rangle$ over the cosets of the finite points modulo those integral off $\mathfrak{m}$.*

(b) (*Commutation.*) *In the square with nodes $(i,n+1)$, $(i,n+2)$, $(i+1,n)$, $(i+1,n+1)$, the descent with datum Ψ followed by the lift, weighted by N_n, is*

$\overline{\beta(\vartheta)}^{-1}$ *times the lift followed by the descent with datum* $\mathrm{next}(\Psi)$*, weighted by* N_{n+2}*; for* z *false,* $|\beta(\vartheta)|^2$ *is the square of the ratio* $\varrho_{i+1,i+n+2}(\vartheta)$ *of Hecke values (below) times a constant and a fixed power of* $\mathrm{N}\,Q$*. In the twisted triangle the adjoint form of the second commutation holds with a constant of squared modulus a constant times a fixed power of* $\mathrm{N}\,Q$*.*

(c) *With suitable descents these data form a triangle of size* m*; for* z *false it has square moduli* ϱ *up to powers of* $\mathrm{N}\,Q$*.*

(d) *For forms* x, y *of level* J_M *on* Y_0*,* $\langle \iota_\vartheta x, \iota_\vartheta y\rangle = [K_0 : J]\,\dim \cdot \langle ux, uy\rangle$*, where* u *is the weighted chain of row* 0 *and* J *the last level; in particular* ι_ϑ *is injective.*

The proof rests on four lemmas: general facts on lifts with finite data, the first lift of a row, the hypotheses of the one-square identity for the squares of the triangle, and the adjoints of descents. For a pair of spaces (S, T), characters and a polynomial making an edge, and a function Ψ of matrices modulo $\mathfrak{m}$, write lift_Ψ for the theta lift from S to T and $\mathrm{lift}^{\mathsf{T}}_\Psi$ for the transposed lift from T to S, with the finite datum Ψ at $\mathfrak{m}$ and the vacuum elsewhere.

Lemma 32.6 (Lifts with finite data). (a) *Integrals over the quotient of the opposite space* S^- *are integrals over that of* S*, and forms of a parameter on* S *and on* S^- *correspond by a linear isometry commuting with the finite translations.*

(b) *The lift of* y *is the integral of* y *against the theta kernel, and the transposed lift of a form* x *of the conjugate type is the integral of* x *against the kernel over the quotient of* T*; the transposed lift of the translate of* x *by* u^{-1} *is the integral against the translated kernel. Both are linear in* Ψ*: the lift with datum* Ψ *is* $\sum_x \Psi(x)$ *times the lift with the indicator of* x*.*

(c) *The pair of central scalar matrices of the modulus acts on a datum by a value of the character of the pair.*

(d) *If the polynomial* Φ *makes an edge from* S *to* T *and* Φ' *one from* S^- *to* T *with the opposite characters* $(\alpha_M)^{-1}\upsilon^{\#M}$*, then for every* Ψ *there is* Ψ' *such that the transposed lift from* T *to* S^- *with* Φ'*,* Ψ' *is the adjoint of the lift from* S *to* T *with* Φ*,* Ψ*, forms on* S *and* S^- *being identified by* (a)*.*

Proof. (a) The identification of the unitary groups of S and S^- carries rational points to rational points and Haar measure to Haar measure (Lemma 31.20(a)); a function is a form on S^- exactly when its pull-back is a form on S, and the integrals defining the inner products agree. (b) is the definition of the lifts, with Fubini's theorem for the absolutely convergent integrals of forms of rapid decay against kernels of polynomial growth, and the transposed kernel at a translated point is the translated kernel. (c) The central scalar acts on the matrices by a scalar, which the Weil representation turns into the value of the pair character. (d) Negating S conjugates the theta kernel up to a factor c, replacing Ψ by some Ψ_0 and Φ by its conjugate; the conjugate polynomial is the signed transpose of some polynomial (the signs square to 1), and it is traded for Φ' at the cost of a factor κ, since both make edges between the same types. The adjoint of the lift is then the transposed lift along the opposite space. □

Lemma 32.7 (The first lift). *Let* $i < m$ *and let* a *be the first lift of row* i *read on forms at* $(0, 0)$*.*

(a) *The index set of the node* $(0,1)$ *has more than twice as many elements as that of* $(0,0)$*, so Rallis' formula applies to* a*.*
(b) *The point datum modulo* $\mathfrak{m}$*, with the vacuum off* $\mathfrak{m}$*, is sharp for the principal level* J *of* $\mathfrak{m}$*, with mass the inverse of the number of matrices of its size modulo* $\mathfrak{m}$ *divided by* $N_{z,\vartheta,0}$*.*
(c) *For an integral point* u *at* $(i,0)$ *there is a finite point* g *at* $(i,1)$ *such that* a *intertwines the right translations by* u *and by* g *up to a scalar of modulus* 1*.*
(d) *Statement* (a) *of Proposition 32.5 holds.*

Proof. (a) is a count of indices: the node $(0,1)$ carries the space at $(0,0)$ and one block of length t (or $t-1$), and by the choice of the parameters of the grids (Proposition 15.4) this block is longer than the rank at $(0,0)$. (b) The matrix coefficient of the point datum at an integral point g is the number of matrices x with the inclusion x_0 at both x and xg; on the support the transpose of the inclusion is a left inverse, so $x_0 g = x_0$ forces g to fix the inclusion, which defines the principal level, and the character is trivial off $\mathfrak{m}$ because its tame part is. The mass is $\langle F,F\rangle/[K_0 : J]$ with $\langle F,F\rangle$ the inverse of the number of matrices and $[K_0 : J] = N_{z,\vartheta,0}$. (c) Choose g integral with reduction $u \oplus 1$; the pair (u,g) fixes the inclusion matrix, the support of the point datum, so it acts on the datum through the pair character, a number of modulus one; and since the parameters at the two ends correspond, the data form an edge, so the lift intertwines. (d) *The level of* v*.* Let u be in the base level and x arbitrary. Then $\langle uv, ux\rangle = \langle v, x\rangle = \langle ay, ax\rangle$, and by (c) $\langle ay, a(ux)\rangle = \langle a(u^{-1}y), ax\rangle$ up to a scalar of modulus one that cancels because u^{-1} fixes y; the reduction of the base level onto the relevant finite group is onto, so testing against all translates gives $uv = v$. *Rallis' formula.* By (a), (b), Theorem B.34(c) applies to the first edge of row i with the point datum, which is sharp for a level fixing the copy of y: the lift is injective on the base and $\langle ax, ay\rangle$ is a constant times the mass times the Rallis series. Multiplying by $N_{z,\vartheta,0}$ cancels the factor of the mass that depends on ϑ, so the constant C depends only on the spaces and the real data. If $v = 0$, then $\langle ay, ay\rangle = 0$, so $ay = 0$ and $y = 0$ by injectivity. □

Lemma 32.8 (The squares satisfy the hypotheses). *Let* $i + n + 1 < m$ *and consider the square of Proposition 32.5*(b)*.*

(a) *The two diagonals of the square have the same number of negative signs, and fewer than half of the signs at* $(0, n+1)$*, with either offset, are negative; the ranks along a row form an arithmetic progression.*
(b) *Let* ν *be the character of the quotient pair* $n+1$*, the quotient of the untwisted characters of lifts* $j+n+1$ *and* j *(for any* j*), of weight* $-c_\tau$ *with* $c_\tau = (n+1)\Delta$*. The rank in column* $n+1$ *is at most* c_τ*, and the character of the lift adding the block* $n+i+2$ *at an end, times the character of the left pairs of row* i *at the complementary end, is* $\nu\vartheta^{n+1}\nu^{\mathrm{rk}/2}$*, the rank twist of* $(\nu\vartheta^{n+1})^{-1}$ *(*ϑ *replaced by* 1 *if* z *is true).*
(c) *For each linear form* L*, the functions* $A(\xi)$ *and* $B(\eta)$ *obtained by applying* L *to the two composites around the square, each edge having as datum the indicator of a block of* ξ *or* η*, satisfy the square law with constant* κ/c*, where* $|c|^2$ *is a constant times* L_* *of* $(\nu\vartheta^{n+1})^{-1}$ *up to the block length* $t - z$ *at the places of the modulus.*

Proof. (a) The signs of the grid at a node are products of the signs of the stem and of the blocks, and comparing the parities of the exponents of -1 entry by entry shows that the two diagonals agree; only stem entries can be negative, the added blocks having sign $+1$, which gives the bound. The rank of a node is the rank of the stem plus t (or $t-1$) per block. (b) Written additively, the identity compares the exponents of each basic character in the geometric progression of the lift characters; the rank bound is the comparison of the number of blocks with the weight $(n+1)\Delta$. (c) By (a), (b) the square satisfies the hypotheses of Corollary 31.2: the spaces U_1, U_2 are the nodes $(i+1,n)$, $(i,n+1)$, A and A' the two other corners, N the common number of negative signs, $2N+2 \le \operatorname{rk} W$ by the bound on the negative signs, $\operatorname{rk} W \le 2\mu + 2N$ by the rank bound, and $(\nu\vartheta^{n+1})^{-1}$ is tame for the modulus of the triangle with the character χ_ϑ of the residue ring. The upper and the lower lift have the same character, of index $n+i+1$. The corollary gives the square law with constant κ/c and $|c|^2 = C\,L_*(t-z)$. □

Lemma 32.9 (Adjoints of descents). *In the twisted triangle, let Φ make an edge along the left pair from $(i+1,n)$ to $(i,n+1)$ and Φ' along the reversed pair, and let $\hat\Psi$ be the Fourier transform of the conjugate of Ψ. Let r be the central scalar matrix of the modulus at the nodes of column n.*

(a) *The conjugate of the theta kernel of the left pair with datum Ψ and polynomial Φ is $\tau(\vartheta) \neq 0$ times the kernel from W to V with datum $\hat\Psi$ and the signed transpose of $\bar\Phi$, translated by r, with $|\tau(\vartheta)|^2$ a constant times a fixed power of $\mathrm{N}\,Q$.*

(b) *The adjoint of the descent with Φ and Ψ is $c(\vartheta)$ times the translation by r^{-1} followed by the ascent with Φ' and $\hat\Psi$, and conversely, with $|c(\vartheta)|^2$ of the same form.*

(c) *The lift commutes with the translations by r^{-1} at the two nodes up to a scalar of modulus 1, and every form at $(i,n+1)$ has an image under the adjoint of the lift.*

(d) (*Second commutation.*) *Let x' be the image of x under the adjoint of the lift from $(i,n+1)$ to $(i,n+2)$. The transposed lift of x with $\operatorname{next}(\Psi)$, followed by the adjoint of the lift from $(i+1,n)$ to $(i+1,n+1)$, is $\overline{\kappa(\vartheta)}$ times the transposed lift of x' with Ψ, with $|\kappa(\vartheta)|^2$ a constant times a fixed power of $\mathrm{N}\,Q$.*

Proof. (a) Conjugating the theta kernel replaces the datum by the Fourier transform of its conjugate and the polynomial by the signed transpose of its conjugate (the Weil representation of the opposite pair), and moves the point by the central scalar; the constant is a value of the Gauss sum of the finite datum, of squared modulus a power of the order of the residue ring, that is a constant times a power of $\mathrm{N}\,Q$. (b) Reverse the kernel of the descent by (a) and trade the transposed polynomial for Φ' and a constant: the lift of the conjugate of x along the reversed pair appears, and writing both sides as integrals and conjugating under the integral gives the adjoint. The converse is the same with the roles exchanged, writing Φ' as the signed transpose of the conjugate of some polynomial. (c) The central pair acts on the datum of the lift through its character (Lemma 32.6(c)), so the lift intertwines the two translations up to a value of that character; adjoint images exist because the spaces of forms of fixed level are finite dimensional. (d) By (b) the first descent is a multiple

of a translated ascent; the ascent commutes with the lift by the square law for the reversed pairs, which holds by Lemma 32.8 for the reversed square with a constant c independent of ϑ (the two middle spaces have equal rank, so the product of L-values is empty), together with Proposition 32.2(c) for the Fourier transforms, the character χ_ϑ being primitive because the modulus is squarefree; the translations pass the lift by (c); and (b) turns the ascent back into a descent. The constants have squared moduli constants times powers of $\mathrm{N}\,Q$. □

Proof of Proposition 32.5. (a) is Lemma 32.7(d).

(b) *The first commutation.* By Lemma 32.6(b) both finite data may be expanded in point data. By Lemma 32.8(c), for each linear form the two composites satisfy the square law with constant κ/c, and Proposition 32.2(b) turns the sum weighted by Ψ on the descent side into the sum weighted by next(Ψ) on the other side, the orders of the two general linear groups being the weights N_n and N_{n+2} (the index types of the nodes unfold to the sum types of the square law). As this holds for every linear form, the operators agree with the factor κ/c, which is written $\overline{\beta(\vartheta)}^{-1}$ and read as 0 where $\beta = 0$. *The modulus.* For z false, $|c|^2$ is a constant times L_* of $(\nu\vartheta^{n+1})^{-1}$, which by Proposition 32.10(b) is a constant times $(\mathrm{N}\,Q)^{z^2}$ times the square of the ratio $\varrho_t(\nu;\vartheta^{n+1})$, and this ratio is the quotient ratio $\varrho_{i+1,i+n+2}(\vartheta)$ by the definition of the quotient pairs. *The second commutation* is Lemma 32.9(d).

(c) Define the descents $d_{i,n}$ by their data, by recursion on n: at $n = 0$ the datum makes the descent the adjoint of the first lift on the copies of the base (Lemma 32.6(d)), and the datum at $n + 1$ is the next datum of the datum at n. The squares commute up to constants by the first commutation, with the moduli of (b); the adjoint law of the squares is the second commutation in adjoint form, with constants of modulus 1 up to a norm power; the adjoint law on the copies holds by the choice at $n = 0$. For the return to the base and the injectivity, let a be the first lift and y in the base, and let v be the form whose copy in row $i + 1$ is $d(a(j_i y))$. Then $\langle v, x\rangle = \langle ay, ax\rangle$ for all x, so v lies in the base and $v = 0$ forces $y = 0$, by Lemma 32.7(d). The relation *has modulus up to a norm power* is unital and multiplicative by Proposition 32.10(a).

(d) *Indices.* In $\mathrm{GL}(Y_1)$ over a commutative ring the block upper triangular group and the opposite parabolic of the first block have the same index, both consisting of the matrices with one vanishing block. The index of the last level J in K_0 is the product of the orders of $\mathrm{GL}(Y_m)$ over the residue fields at the chosen places and of the indices of the block upper triangular groups over the residue fields at the places of Q; the weight N_n is the product of the orders of the general linear group of the node $(0, n)$ over the same residue fields. Every quantity is a product over the chosen places times a product over the places of Q; the factors at the chosen places do not depend on ϑ, and at $w \in Q$, by Theorem 32.4(c) for the flag group, the count leaves a fixed power of q_w. Hence

$$[K_0 : J] = C\,(\mathrm{N}\,Q)^k\,\delta(\vartheta)^m\,\frac{N_m}{N_0 N_1^m}$$

with C, k independent of ϑ and $\delta(\vartheta)$ the Gram factor of the first descent pair (Proposition 30.3(c)). *The inner product.* The function of the algebraic form $\iota_\vartheta v$ is the chain of lifts of v, the weights of the lifts being the indices of the principal levels, that is the orders N_n; the inner product of algebraic forms on Y_m, a sum over the classes of J, is $[K_0 : J]$ times the dimension of the weight times the

Petersson product. This gives the formula. If $\iota_\vartheta y = 0$, the chain of lifts of y vanishes, and chains are injective by Theorem 32.4(a). □

32.4. Ratios of Hecke values

For a character ν of $[\mathrm{U}(1)]$, $s \geq 1$ and a tame character ϑ with set of places Q, put $\varrho_s(\nu;\vartheta) = \prod_{z=1}^{s} |L^{MQ}(z,(\nu\vartheta)^{-1})/L^{M}(z,\nu^{-1})|$, where L^M omits the chosen places and their conjugates and L^{MQ} also the places of Q. For $0 \leq i < j \leq m$ the *quotient ratio* $\varrho_{i,j}(\vartheta)$ is ϱ_t of the quotient of the untwisted characters of lifts j and i at ϑ^{j-i}; these characters $\chi_0(\upsilon^{-\Delta}\chi_M)^j$ form a geometric progression. A function c of the m-fold twists *has modulus f up to a norm power* if $|c(\vartheta)|^2 = C(\mathrm{N}\,Q)^k f(\vartheta)^2$ for all ϑ, with C and k independent of ϑ ($C = 0$ allowed).

Proposition 32.10 (Moduli). (a) *The relation* has modulus f up to a norm power *is unital and multiplicative and closed under finite products. If $f \geq 0$, $f(1) = 1$ and $c(1) \neq 0$, then $|c(\vartheta)/c(1)| = (\mathrm{N}\,Q)^{e/2} f(\vartheta)$ for an integer e.*

(b) *Let $p = (\nu, z)$ be a ramified pair (Definition 10.15) and ξ a twist, $\mathfrak{m} = \varpi^e Q$ with $e \neq 0$. Then $\nu\xi$ and its inverse are tame for $\mathfrak{m}$, and for both L_* is that of ν at the chosen places times $(\mathrm{N}\,Q)^{z^2}$ and the square of $\varrho_z(\nu;\xi)$.*

(c) *$\varrho_{i,j}$ is a ramified ratio for $i < j \leq m$; $\prod_{i<j} \varrho_{i,j}(1) = 1$ and the product is nonnegative.*

Here a function F of the twist is a *ramified ratio* if for a fixed $1 \leq n \leq m$ and a ramified pair (ν, z) one has $F(\vartheta) = \varrho_z(\nu;\vartheta^n)$ for all ϑ.

Proof. (a) *Unital and multiplicative.* The constant 1 has modulus 1 with $C = 1$, $k = 0$; if $|c|^2 = C(\mathrm{N}\,Q)^k f^2$ and $|c'|^2 = C'(\mathrm{N}\,Q)^{k'} f'^2$, then $|cc'|^2 = CC'(\mathrm{N}\,Q)^{k+k'}(f f')^2$; constants have modulus 1 and inverses have the inverse modulus when $C \neq 0$; finite products follow by induction. A function whose squared modulus is a constant times a power of the order of the residue ring of $\varpi^e Q$ has modulus 1 up to a norm power, because that order is $(\prod_{w\in M} q_w)^e\,\mathrm{N}\,Q$, the first factor being independent of ϑ. *The normalised value.* At the trivial twist Q is empty and $f(1) = 1$, so $|c(1)|^2 = C$, and $C \neq 0$; dividing, $|c(\vartheta)/c(1)|^2 = (\mathrm{N}\,Q)^k f(\vartheta)^2$, and as $f \geq 0$ the square root is $(\mathrm{N}\,Q)^{k/2} f(\vartheta)$; take $e = k$.

(b) *Tameness.* The tame part of ν is nontrivial exactly at the chosen places and that of ξ exactly at the places of Q, and these sets are disjoint; so the tame part of $\nu\xi$ is nontrivial exactly at the places of $\mathfrak{m}$ (here $e \neq 0$ is used), and its character on $\boldsymbol{O}_E/\mathfrak{m}$ is the product of the local characters. The weight of $\nu\xi$ is that of ν, ξ having weight 0. The inverse is tame with the inverse character and the opposite weight. For the squarefree modulus ϖQ the character is primitive: if $a \neq 0$, some residue of a is nonzero, say at w, and the local character is nontrivial at some unit x; by the Chinese remainder theorem choose c with $1 + ca$ equal to x at w and to 1 at the other places. *The L-values.* A character of $[\mathrm{U}(1)]$ and its inverse have the same partial L-function, the base changes differing by the conjugation, which permutes the places; so $|L^{MQ}(i,(\nu\xi)^{\pm 1})| = |L^{M}(i,\nu^{\pm 1})|\varrho_{(i)}$ with $\varrho_{(i)}$ the i-th factor of $\varrho_z(\nu;\xi)$, and the places of $\mathfrak{m}$ are the chosen places and those of Q, so the norm factor of L_* is $(\prod_{w\in M} q_w)^{z^2}(\mathrm{N}\,Q)^{z^2}$. Multiplying the squares of the factors for $1 \leq i \leq z$ gives the claim.

(c) The quotient pair of index $n = j - i$ is the pair of the character $(\upsilon^{-\Delta}\chi_M)^n$ and the integer t; its tame part χ_M^n is nontrivial exactly at the chosen places

because $1 \le n \le m$, and since the untwisted lift characters form a geometric progression, $\varrho_{i,j}(\vartheta) = \varrho_t(\nu_n; \vartheta^n)$. At the trivial twist each factor is L/L with $L \neq 0$ (no zero on $[1,\infty)$, Theorem B.32(a)), so the product is 1; each factor is a quotient of absolute values, so the product is nonnegative. □

32.5. The norm of the chain and the factorisation

For $1 \le j \le m$ let $Z_j(\vartheta^j) = \det(u^*u)$ for the j-th single lift $u = \Theta_{j,\vartheta^j}$ restricted to the good forms on W_j (the *lift volume*), and $Z(\vartheta) = \det(\Theta_\vartheta^*\Theta_\vartheta)$ on the target (the *chain volume*). Let R_ϑ be the product, in the order $j = 1, \dots, m$, of the Rallis series of the j-th lift at ϑ^j, acting on the target as spherical Hecke operators.

Lemma 32.11 (Volumes). (a) *Let U, V be inner product spaces of finite dimension, $u : U \to V$ linear, and S an endomorphism of U such that $\langle ux, uy\rangle = a\langle x, Sy\rangle$ for all x, y. Then $\det(u^*u) = a^{\dim U} \det S$.*

(b) *For $1 \le j \le m$ there is $A_j \neq 0$ such that for every tame ϑ*

$$Z_j(\vartheta^j) = (A_j c_Q)^{d_T} \det R_{j,\vartheta^j},$$

where the determinant is that of the Rallis series of the j-th lift on the target forms; and $Z_j(\vartheta^j) > 0$.

Proof. (a) The hypothesis says $u^*u = aS$; take determinants. (b) By Theorem 30.5(c) the Gram operator of the j-th lift on the forms of level J_M is $c_j c_Q R_{j,\vartheta^j}$, and on the good forms, which are stable, the same holds; (a) gives $Z_j = (c_j c_Q)^d \det R_{j,\vartheta^j}$, with d the dimension of the good forms of W_j. The good forms on W_j and on Y_0 correspond with the same Hecke eigenvalues (Theorem 23.2), and the Rallis series acts on both by the same spherical Hecke operator, so its determinant is the same on the two spaces and the dimensions agree: $d = d_T$, and $A_j = c_j$. Positivity is the injectivity of the lift (Theorem 30.5(a)). □

Lemma 32.12 (Gram form of the chain). (a) *There are $\kappa(\vartheta)$ with modulus $\prod_{1\le i<j\le m} \varrho_{i,j}$ up to a norm power such that on the target forms the Gram form of the weighted chain u of row 0 of the twisted triangle is*

$$\langle ux, uy\rangle = \kappa(\vartheta)\, \frac{N_0 N_1^m}{N_m}\, \#M_\vartheta^{-m}\, \langle x, R_\vartheta y\rangle,$$

with $\#M_\vartheta$ the number of matrices indexed by $Y_1 \times Y_0$ modulo ϖQ.

(b) *There are numbers $c(\vartheta)$ of modulus $\prod_{i<j} \varrho_{i,j}$ up to a norm power with $\langle \Theta_\vartheta x, \Theta_\vartheta y\rangle = c(\vartheta) c_Q^m \langle x, R_\vartheta y\rangle$ for all target forms x, y.*

Proof. (a) By Proposition 32.5(a), for the first lift a of row i, $N_0\langle ax, ay\rangle$ is a constant C_i times the Rallis series, that is $C_i \#M_\vartheta^{-1}\langle x, R_{i+1,\vartheta^{i+1}} y\rangle$ on the base: the node $(i,1)$ carries the space of lift $i+1$, the Rallis terms are divided by the number of matrices of the point datum, and on target forms the series is the spherical Hecke operator $R_{i+1,\vartheta^{i+1}}$. Put $S_i = C_i \#M_\vartheta^{-1} R_{i+1,\vartheta^{i+1}}$; the hypothesis of Theorem 32.4(b) holds with the constants 1, and s the inclusion of the target forms in the base. The triangle has square moduli $\varrho_{i,j}$ up to norm powers (Proposition 32.5(c)), so Theorem 32.4(b) for $i = 0$, $n = m$ gives the formula with κ of modulus $\prod_{0<r<k\le m} \varrho_{r,k}$ times the constant $\prod C_i$, which is free of ϑ.

(b) By Proposition 32.5(d), $\langle \Theta_\vartheta x, \Theta_\vartheta y\rangle = [K_0 : J] \dim\langle ux, uy\rangle$, and by the proof of (d), $[K_0 : J] = C(\mathrm{N}\,Q)^k \delta(\vartheta)^m N_m/(N_0 N_1^m)$. The orders N_n cancel against

those of (a), leaving $C(\mathrm{N}\,Q)^k \dim \kappa(\vartheta)(\delta(\vartheta)/\#M_\vartheta)^m$. By Proposition 30.3(c), $\delta(\vartheta)/\#M_\vartheta = c_M c_Q$, with $c_M \neq 0$ independent of ϑ (the chosen places and the ranks do not depend on ϑ). Hence $c(\vartheta) = C \dim c_M^m (\mathrm{N}\,Q)^k \kappa(\vartheta)$, whose modulus is $\prod \varrho_{i,j}$ up to a norm power by Proposition 32.10(a). □

Theorem 32.13 (Norm of the chain). *There are $C > 0$, an integer e and numbers $c(\vartheta)$ such that for every m-fold twist ϑ*

$$|c(\vartheta)| = C(\mathrm{N}\,Q)^{e/2} \prod_{1\le i<j\le m} \varrho_{i,j}(\vartheta), \qquad \Theta_\vartheta^* \Theta_\vartheta = c(\vartheta)\, c_Q^m\, R_\vartheta \quad \text{on } \mathcal{T}.$$

Theorem 32.14 (Factorisation of the volume). *The volumes $Z(\vartheta)$ and $Z_j(\vartheta^j)$ are positive, and there is an integer e such that for every m-fold twist ϑ*

$$\frac{Z(\vartheta)}{Z(1)} = \Big((\mathrm{N}\,Q)^{e/2} \prod_{1\le i<j\le m} \varrho_{i,j}(\vartheta)\Big)^{d_T} \prod_{j=1}^{m} \frac{Z_j(\vartheta^j)}{Z_j(1)}.$$

Put $s(\vartheta) = (\mathrm{N}\,Q)^{e/2} \prod_{i<j} \varrho_{i,j}(\vartheta)$. If a family of twists ξ_ι has small ratios (Definition 36.1), the family $s(\xi_\iota)^{-1}$ has height at most $m^2 t x_1$.

Proof of Theorems 32.13 and 32.14. The norm of the chain. Lemma 32.12(b) gives $c(\vartheta)$ with $\Theta_\vartheta^*\Theta_\vartheta = c(\vartheta) c_Q^m R_\vartheta$ on $\mathcal{T}$, the Gram form determining the operator. If $c(1) \neq 0$, Proposition 32.10(a), with $f = \prod_{i<j} \varrho_{i,j} \geq 0$ and $f(1) = 1$ (Proposition 32.10(c)), gives $|c(\vartheta)| = |c(1)|(\mathrm{N}\,Q)^{e/2} f(\vartheta)$, and $C = |c(1)| > 0$. If $c(1) = 0$, then $\Theta_1^*\Theta_1 = 0$, so $\Theta_1 = 0$, and as Θ_1 is injective (Proposition 32.5(d)) the target is 0; then any numbers serve, for instance $c(\vartheta) = f(\vartheta)$ with $C = 1$, $e = 0$.

Positivity. $Z(\vartheta) = \det(\Theta_\vartheta^*\Theta_\vartheta)$ is the square of the norm of the determinant of an injective map (Proposition 32.5(d)), hence positive; $Z_j > 0$ by Lemma 32.11(b).

The factorisation. By Lemma 32.11(a) and Theorem 32.13,

$$Z(\vartheta) = (c(\vartheta) c_Q^m)^{d_T} \det R_\vartheta, \qquad \det R_\vartheta = \prod_{j=1}^{m} \det R_{j,\vartheta^j},$$

the determinant of a product being the product of the determinants. By Lemma 32.11(b) each $\det R_{j,\vartheta^j}$ is $Z_j(\vartheta^j)/(A_j c_Q)^{d_T}$; the powers ϑ^j have the same modulus Q as ϑ, hence the same c_Q. Hence

$$Z(\vartheta) = c(\vartheta)^{d_T} \prod_{j=1}^{m} \frac{Z_j(\vartheta^j)}{A_j^{d_T}},$$

the factors c_Q cancelling. Divide by the same identity at $\vartheta = 1$, where Q is empty: the A_j cancel, and $(c(\vartheta)/c(1))^{d_T}$ is a positive real number, being a quotient of positive numbers by positive numbers; so it is its modulus $\big((\mathrm{N}\,Q)^{e/2} \prod \varrho_{i,j}(\vartheta)\big)^{d_T}$.

The height. By Proposition 32.10(c) each $\varrho_{i,j}$ is a ramified ratio, a product of at most t ratios of L-values of a ramified pair at a power ξ_ι^n, $1 \le n \le m$; for a family with small ratios the inverse of each such ratio has height at most x_1 (Definition 36.1), so the inverse of $\varrho_{i,j}(\xi_\iota)$ has height at most $t x_1$. There are at most m^2 pairs $i < j$. The norm $\mathrm{N}\,Q$ is prime to the prime of ι, so its power does not raise the height. Heights add over products, which gives $m^2 t x_1$. □

Proposition 32.15 (The range of the lift on good forms). *For a tame ϑ and $1 \le j \le m$, the range of the lift on good forms is spanned by the lifts of good Hecke eigenforms whose lifts lie in the range of the idempotent of one good tuple family.*

Proof. The idempotents. Let the group algebra of the product of the finite groups GL_r at the chosen places act on a space of forms. The idempotent of a good tuple family is its product with the central idempotent e_0, the sum of the idempotents of all good tuple families (Proposition 15.6), so its range lies in that of e_0. *Equivariance.* The lift commutes with the group algebra: both sides of the identity are sums over the group, and the lift commutes with each group element (Theorem 30.5(d)), the element acting on the forms of W_j by translation and on those of the larger space of the descent pair by the action ρ. *Spanning.* Good forms are the values of e_0, and Hecke eigenforms span the forms of principal level (Proposition 23.1(b)), so it suffices to treat $e_0 y$ with y an eigenform. Split e_0 into the idempotents of the good tuple families: each piece of y is again a good eigenform, since the idempotents commute with the Hecke operators, and by equivariance its lift lies in the range of the same idempotent. □

33. Auxiliary forms

Theorem 7.14 needs, besides the family y under study, auxiliary forms z_i completing it to a basis, whose twisted lifts become integral after multiplication by a scalar $\gamma(\xi)$ of controlled size, and whose untwisted lifts are short. This section constructs them (Theorem 33.19). The idea is to read the values of the chain $\Theta_\vartheta y$ through periods of forms on two grids of spaces, and to move these periods along paths through the grids, turn by turn, from the *first path*, along which the periods compute the pairing with Θ_ϑ, to the *last path*, along which all spaces are definite and the periods are finite sums with explicit denominators and sizes. Each turn is an exchange on one square of a grid, whose constant is a ratio of Hecke values; there are m^2 turns.

33.1. Moves, paths and path data

The two grids Y_b^a and $\tilde{Y}_a^b$, $0 \le a, b \le m$, are those of Section 15.3; a *node* is a triple (z, a, b). Fix an m-fold twist ϑ with set of places Q, let $\varpi = \pi_M$ and $\mathfrak{b} = \varpi^{\lceil b \rceil}$, and let the *grid modulus* be $\mathfrak{m} = \mathfrak{b}Q$. An *edge tuple* is a matrix modulo $\mathfrak{m}$ for each pair of adjacent nodes. The *lift move* $(a, b) \to (a+1, b)$ in grid z is the theta lift with the character of the horizontal edge of index a, the joint harmonic of the two nodes at infinity and the indicator of a matrix modulo $\mathfrak{m}$ at the finite places; the *transposed move* $(a, b+1) \to (a, b)$ is the transposed lift with the character of the vertical edge, which carries ϑ^{b+1}. The character of an edge of index l is $\mu_{l+1}\vartheta^{l+1}\upsilon^{\lfloor N/2 \rfloor}$ (vertical) or $\mu_{l+1}\upsilon^{\lceil N/2 \rceil}$ (horizontal), with μ_{l+1} the untwisted character of lift $l+1$ and N the rank of the end. In grid Y conjugate forms move; in grid $\tilde{Y}$ forms move.

The *triangle modulus* of grid z is $\mathfrak{m}_0 = \varpi Q$ for z false and $\mathfrak{b}$ for z true; it divides $\mathfrak{m}$, and it is the modulus of the point data along the first columns. The *level* of a node is a subgroup of the invertible matrices modulo $\mathfrak{m}$: trivial at the nodes $Y_b = Y_b^0$ and the kernel of reduction modulo $\mathfrak{b}$ elsewhere. A *node family* is an invertible matrix g_q modulo $\mathfrak{m}$ at each node q; it acts on edge tuples by $S(p, q) \mapsto g_q^{-1} S(p, q) g_p$, and the *edge character product* of a family on a set w of

edges is the product over the edges $(p, q) \in w$ of their characters at $\det g_q / \det g_p$. A family is *admissible* if at the two ends Y_m^0 and $\tilde{Y}_0^m$ it lies in the level of the node, and at each $\tilde{Y}_a^a$ it is $g \oplus 1$ for its member g at Y_a^a. A complex number is a *sum of at most N roots of unity* if it is $\sum_{i \le N} \zeta_i$ with each ζ_i a root of unity or 0.

Lemma 33.1 (Tuples and data). (a) *Sums of at most N roots of unity are closed under multiplication by roots of unity, and a sum over a finite set X of such numbers is a sum of at most $\#X \cdot N$ roots of unity.*

(b) *The action on tuples is a right action, the edge character product is multiplicative in the family, and its values are roots of unity. A sum over the tuples supported on w does not change when the tuples are moved by a family; for a path datum Ψ on w and an admissible g, $\sum_S \Psi(S) f(gS)$ is the edge character product of g times $\sum_S \Psi(S) f(S)$.*

(c) *The number of matrices of an edge between nodes with indices at most m is at most $N_{\mathfrak{m}} = \#(O_E/\mathfrak{m})^{(2n_m)^2}$, and a path through the two grids has at most $2m$ edges; so at most $N_{\mathfrak{m}}^{2m}$ tuples are supported on it.*

(d) *The map along a path is the composite of the moves by the matrices of its edges, a transposed move where the second index drops and a lift move where the first grows; it depends only on the matrices of its edges and is multiplicative under concatenation of paths.*

Proof. (a) A root of unity times a root of unity is a root of unity, and sums concatenate. (b) $(gh)S = h(gS)$ by associativity of matrix multiplication, and the edge character product of gh is the product by the multiplicativity of the characters; the characters of the edges are of finite order on the residue ring, so their values are roots of unity. Moving by a family is a bijection of the tuples supported on w, which gives the invariance of the sum; for a path datum, $\Psi(g^{-1}S)$ is $\Psi(S)$ times the inverse of the character product, and substituting $S \mapsto gS$ gives the formula. (c) The ranks of the nodes with indices at most m are at most $2n_m$; each step of a path changes one index by one, and the indices run from 0 to m. (d) is the definition, with the composites associated. □

A *path through the two grids* goes from $(m, 0)$ to $(0, m)$ by steps up or left and meets the diagonal once, at (l, l); its first half lies in grid $\tilde{Y}$ and its second half in grid Y, and both halves are recorded as paths from $(0, m)$ to (l, l). The *first path* is the pair of first columns, through $(0, 0)$; the *last path* goes through (m, m). A *turn* replaces a node (i, j) by $(i + 1, j + 1)$. A *path datum* on a set w of edges is a function Ψ of the edge tuples that depends only on the matrices at w, is fixed by the admissible families of invertible matrices at the nodes up to the characters of the edges, vanishes unless the vertical edges carry left invertible matrices, and has as values sums of at most N roots of unity. The *period* of Ψ on a path Γ at a real point g of $\mathrm{U}(\tilde{Y}_l^l)$ is, for a twist, a conjugate form y' on Y_m and a form φ on $\tilde{Y}_0^m$, the sum over the tuples S of $\Psi(S)$ times the pairing at g of the images of y' and φ along the two halves. Write $\Psi \propto \Xi$ if at every real point the period of Ψ is a multiple of Ξ by a number that depends neither on the twist nor on the forms; a pair of paths (Γ', Γ) *transfers periods* with bound k and constants $c(\vartheta)$ if path data Ψ' on Γ' give path data Ψ on Γ with the root of unity bound multiplied by k, and $\Psi \propto \Xi$ implies $\Psi' \propto \Xi / c$.

The pairing at the diagonal node is the *diagonal period*: for a conjugate form y on Y_j^j and a form φ on $\tilde{Y}_j^j = Y_j^j \oplus E\tilde{e}_j$, the integral over $[\mathrm{U}(Y_j^j)]$ of $y(h)\varphi((h \oplus u^j)g)$, where u is the *flip scalar* $\bar{t}/t$ of $\mathrm{U}(1)$, t a fixed idele generating $\mathfrak{m}$ times a fixed idele of the Weil data. A monotone path in one grid from (a_0, b_0) to (a, b) is a word in the steps *down* $(a, b+1) \to (a, b)$ and *right* $(a, b) \to (a+1, b)$; its *squares* are the unit squares between it and the first column, named by their lower left corners: a step down from $(a, b+1)$ contributes the (i, b) with $i < a$. The squares of a path through the two grids are those of its half in Y, the mirror images of those of its half in $\tilde{Y}$, and the (i, j) with i, j below the diagonal index.

Lemma 33.2 (Paths and turns). *Every path other than the last has a turn; a turn at (i, j) has $i, j < m$; for $i < j$ it takes place in the half in grid Y, for $j < i$ in the half in grid $\tilde{Y}$, and for $i = j$ on the diagonal. A property that holds for the last path and the empty set, and passes from Γ' and T to Γ and $T \cup \{s\}$ for every turn at $s \notin T$ from Γ to Γ', holds for the first path and the set of all (i, j) with $i, j < m$. Period transfers compose, the bounds and constants multiplying.*

Proof. Monotone paths. Along a monotone path the first index does not fall and the second does not grow, and the only path from $(0, m)$ to (a, m) is the one along the row m, which has no squares. The squares of a concatenation are the union of those of the parts, and a square of a path from (a_0, b_0) to (a, b) lies left of the column a and between the rows b and b_0. If a path comes down to (i, j) and leaves to the right, the square at (i, j) is not among its squares; if it passes through $(i+1, j+1)$ instead of (i, j), its squares gain exactly that square. Every monotone path has a corner (i, j) (down then right), or ends with a step down, or has no step down.

Existence of a turn. Let Γ be a path other than the last, with diagonal index l. If one of its halves has a corner (a, b) with $b > a$ in grid Y, or the mirror image of one in grid $\tilde{Y}$, that corner is a turn off the diagonal. Otherwise both halves have no corners; below the last row (that is, if $l < m$) both end with a step down to (l, l), and replacing these two steps by the steps right to $(l+1, l+1)$ is a turn on the diagonal. If $l = m$, both halves run along the row m, and Γ is the last path.

Location and bounds. By the description of the squares, a turn at (i, j) adds the square at (i, j), which lies between the first and the last path, so $i, j < m$. A turn at a corner (a, b) of the half in Y has $a < b$, the half staying above the diagonal; a turn in the half in $\tilde{Y}$ is recorded with exchanged indices, so it has $j < i$; and a turn on the diagonal replaces (j, j) by $(j+1, j+1)$.

Induction. The first path has no squares and the last has all m^2 squares (i, j), $i, j < m$. For a turn at s from Γ to Γ', s is not a square of Γ and the squares of Γ' are those of Γ together with s. Induct on the number of squares between Γ and the last path, carrying along the set T of squares of Γ' not in the first path: for the last path this set is empty, and each turn backwards adds its square. At the first path the set is all of $\{0, \dots, m-1\}^2$.

Transitivity. If (Γ'', Γ') transfers with bound k' and constants c' and (Γ', Γ) with k and c, then path data on Γ'' give data on Γ' and then on Γ with the bound multiplied by $k'k$, and $\Psi \propto \Xi$ gives $\Psi' \propto \Xi/c$ and then $\Psi'' \propto \Xi/(cc')$. A transfer with bound k is one with every larger bound, by Lemma 33.1(a). □

33.2. The first path

The *flat chain* $\Theta^\flat$ is the composite of the m transposed point lifts in grid $\tilde{Y}$ from $\tilde{Y}_0^m$ to $\tilde{Y}_0^0 = Y_0 \oplus E\tilde{e}_0$, at the trivial twist. A *flat restriction* at a real point g of $\mathrm{U}(\tilde{Y}_0^0)$ is a linear map r from the forms of level $\mathfrak{b}$ on $\tilde{Y}_0^m$ to forms on Y_0 such that every form y on Y_0 of level ϖQ has the same Petersson product with $r\varphi$ as with $h \mapsto (\Theta^\flat\varphi)(hg)$.

The deep level argument concerns a hermitian space V of rank n and the space $V \oplus \langle 1\rangle$, with parameters l on V and l' on $V \oplus \langle 1\rangle$. A rational point γ of $V \oplus \langle 1\rangle$ has a *last column*, the rational vector $\gamma\tilde{e}$ for the last basis vector $\tilde{e}$; the *Poincaré kernel* of a function φ on the real group and a level K is φ at the real part times the indicator of K at the finite part, and the *translated coefficient* $c_{l',\alpha,\omega}(g) = \prod_w \overline{c_{l'_w}(\alpha_w^{-1} g_w \omega_w)}$ is a product of conjugate discrete series coefficients. The *real convolution* of ψ and F at x is $\int \psi(r) F(x(r,1))\, dr$ over the real group. The *size conditions* (A, L, N, β, R) ask: $8(n+1)^2 < A$, $N < A$, A exceeds by 2 the real dimension of a space of rank $n+1$, β bounds the cusp constant times the level, a polynomial in the rank and the bound L of the parameters, and the number of infinite places; $R \geq \beta^{20A(\#\Phi+1)}$; and the generator of the deep level has modulus at least R^{3n+2} at every infinite place.

Lemma 33.3 (Poincaré series of deep level). (a) *Let l'_w have a large gap for $2A$. Then the translated coefficient decays: its slice $r \mapsto c(g(r \oplus 1))$ is integrable over the real group of V with L^1 norm at most $\mathrm{sc}(A, \alpha, \omega)\, t^{-A}$, where t bounds below the products of the operator norms of $g(r \oplus 1)$ and $\mathrm{sc} = 16^A \operatorname{vol} \prod_w (2(\|\alpha_w\|\|\omega_w\|)^2)^A$.*

(b) *Two rational points with the same last column differ by an element of the image of $\mathrm{U}(V)$. If $y \neq y'$ are the last columns of γ, γ' and the corresponding translates have finite part of level a with $|a| \geq R$ at all infinite places, the distance of y and y' at the point $x \oplus 1$ is at least R times the first minimum of V at x; the last column of 1 has length at most $\#\Phi$ there.*

(c) (*Packing.*) *Points of a real normed space of dimension at most $A - 2$ with mutual distances at least d^2, $d \geq 10$, one of them of norm at most d, have $\sum_{i \neq \mathrm{base}} (d/\|x_i\|)^A \leq 10/d$.*

(d) (*Main term.*) *Let F be automorphic on V of level $a \neq 0$ and bounded by S, let $|a| \geq R$, $d^2 \leq R\lambda$ with λ the first minimum at x, $d \geq \max(10, \#\Phi)$ and $A \geq 2\#\Phi(n+1) + 2$. Then the period against F of the Poincaré series with centre $x \oplus 1$ of the translated coefficient differs from $\operatorname{vol}(K_a)$ times the real convolution of $\psi(r) = c(r \oplus 1)$ with F at x by at most $\operatorname{vol}(K_a)\, \mathrm{sc}\, S\, (10/d)$.*

Proof. (a) Writing $x = \alpha(\alpha^{-1}x\omega)\omega^{-1}$ bounds $\|\alpha^{-1}x\omega\|$ below by $\|x\|/(\|\alpha\|\|\omega\|)$, and the coefficient of a parameter with a large gap for $2A$ decays like the power $-2A$ of the norm (Proposition A.1); one half of the decay gives t^{-A}, the other half is integrable over the real group with the stated constant, $A > 8(n+1)^2$ exceeding the growth of the Haar measure. (b) Both points send the last basis vector to the coordinates of the column, so $\gamma^{-1}\gamma'$ fixes it and is block diagonal $\mathrm{diag}(h, 1)$. Both columns are congruent to the last basis vector modulo a, so their difference is a times a lattice vector, nonzero; a nonzero lattice vector has length at least the first minimum (the definite summand $\langle 1\rangle$ has no short vector and does not lower it), and a stretches the coordinates at each place by at least R. The last column of 1 is $\tilde{e}$, of length 1 at each place. (c) Away from the base point,

norms and mutual distances are at least $\rho = d^2/2$, since $\|x_{\text{base}}\| \le d \le d^2/2$. For points of a space of dimension at most $A-2$ whose norms and mutual distances are at least ρ, the packing estimate $\sum_i \|x_i\|^{-A} \le (5/\rho)^A$ holds, by comparing the disjoint balls of radius $\rho/2$ about the points with balls about 0. Hence the sum is at most $d^A(10/d^2)^A = (10/d)^A \le 10/d$, as $d \ge 10$. (d) Unfold the period over the cosets $\gamma\,\mathrm{U}(V)(E)$ of the rational points of $V \oplus \langle 1\rangle$, which the last columns index (Lemma 33.3(b)); only the cosets meeting the level condition contribute, and the unfolding is the identity $\sum_\gamma \int_{[\mathrm{U}(V)]} = \int_{\mathrm{U}(V)(\mathbb{A})}$ for an integrable function, valid because the L^1 norms of the terms are summable. The coset of 1 gives, after translating by x and using the level of F, $\mathrm{vol}(K_a)$ times the real convolution. For the other cosets, the product of the local operator norms of the translate is at least $\|\mathrm{col}(\gamma)\|/d$, so by (a) the term of γ is at most $\mathrm{vol}(K_a)\,\mathrm{sc}\,S\,(d/\|\mathrm{col}(\gamma)\|)^A$; the columns are d^2-separated by (b) since $d^2 \le R\lambda$, and the base column has norm at most $\#\Phi \le d$, so (c) bounds the sum by $10/d$. □

Lemma 33.4 (Halving). *Assume the size conditions, large gaps for $4A$ of l_w and for $2A$ of l'_w, $|l_w(i)| \le L$, a real point ω with $\|\omega\|^2 \le \beta$, and the lower bound $\beta^{-N}\,\mathrm{vol} \le |\int c_l(g)\overline{c_{l'}(\omega^{-1}(g \oplus 1)\omega)}\,dg|$ for the local pairings at every infinite place. Let y be a form on V of type l and the deep level, bounded by S, and orthogonal to $x \mapsto \varphi((x\oplus 1)\omega)$ for all forms φ of type l' and deep level on $V \oplus \langle 1\rangle$.*

(a) *The majorant $u \mapsto \prod_w \deg(l_w)(2/\|u_w\|)^{2A}$ of the type kernel is integrable with integral at most $R/(16\beta)\cdot(\beta^N(2\beta)^{3A})^{-\#\Phi}$, and for u within R^{-2} of 1 the type kernel at $u^{-1}z$ and at z differ by at most β/R times the sum of the majorants there.*

(b) *Where the first minimum of V at x is below the threshold of Theorem B.26(c) for $\delta = R^{-2}$, $\|y(x)\| \le S/2$.*

(c) *The Poincaré series of the translated coefficient for l' and the deep level has period 0 against y.*

(d) *Where the first minimum is at least that threshold, the real convolution at x of the translated coefficient with y is at most $10S/R$ times $\mathrm{sc}(A,\alpha,\omega)$, and $\|y(x)\| \le S/2$.*

(e) *Hence $y = 0$.*

Proof. (a) The integral of the majorant is a product of local integrals, each finite for a gap $4A$; the size conditions $\beta \ge 2000(n+L+1)^6$ and $\beta^{20A(\#\Phi+1)} \le R$ make the product small against R. The coefficients are Lipschitz under left translation with constant at most β, there are at most β factors, each of norm at most 1, and the product of numbers of modulus at most one is Lipschitz with the sum of the constants; this gives the variation bound. (b) By Theorem B.26(c), with $\delta = R^{-2}$, it suffices to bound $\|y(x(u,1)) - y(x)\|$ by $S/2$ for real u within R^{-2} of 1. Write both values as convolutions of y with its type kernel; by (a) the difference is at most $S\beta/R$ times twice the integral of the majorant, which is at most $S/2$. (c) The Poincaré series, up to the nonzero formal degrees, is the conjugate of the value at $(h\oplus 1)\omega$ of a form of type l' and deep level on $V \oplus \langle 1\rangle$: the series converges to such a form by the decay (Lemma 33.3(a)), and it is fixed by the type kernel because each term is. The orthogonality hypothesis gives period 0. (d) Above the threshold, $\lambda \ge (R^{-2}/(M\kappa^2))^n \ge R^{-3n}$ because $M\kappa^2 \le \beta \le R$, so $R^2 \le R^{3n+2}\lambda$. Apply Lemma 33.3(d) with $d = R$ and the generator of the deep level, of modulus at least R^{3n+2} at every place, in the role of a: then $d^2 \le R^{3n+2}\lambda$,

and with (c) it bounds $\mathrm{vol}(K)$ times the convolution by $\mathrm{vol}(K)\,\mathrm{sc}\,S\,(10/R)$; the positive volume cancels. *Undoing the smoothing.* Average the convolution of (d) over the type kernel of y: by Fubini and the rank one formula for the real convolution of coefficients, the result is $\eta\, y(x)$, where η is the product over the places of the local pairings normalised by the values of the kernels at 1; by the lower bound, $|\eta| \geq (\beta^{-N}\,\mathrm{vol})^{\#\Phi}$. The integrand is bounded by the majorant of (a) times the bound of (d), and the constants cancel to give $\|y(x)\| \leq S/2$. (e) Every bound S of $\|y\|$ is halved by (b) and (d), so $\|y\| = 0$. □

Proposition 33.5 (Deep level detects forms). (a) *The parameter of Υ_0 and minus that of $\tilde{\Upsilon}_0^0$ are* steep*: for one numbering, at every infinite place the sign is + and the lowest K-type has weight 0 on the base indices of X_0 other than the last, and on the other indices the signs alternate and the weight decreases strictly within $[-3B, -B]$, $B = \kappa + \Delta/2$.*

(b) (*Poincaré series of deep level.*) *Let F be automorphic on V of level $\alpha \neq 0$, bounded by S, and φ a translated product of discrete series coefficients for parameters of $V \oplus \langle 1\rangle$ with a large gap. If $|\alpha|_v \geq R$ at all infinite places and $d^2 \leq R\lambda$ for the first minimum λ of the lattice at the point x, the period against F of the Poincaré series with centre $x \oplus 1$ is the volume of the principal level times the real convolution of φ with F at x, up to that volume times $(10/d)$ times explicit constants and S.*

(c) *There is a real point g of $\mathrm{U}(\tilde{\Upsilon}_0^0)$ such that a form y of level J_M on Υ_0 vanishes as soon as it is Petersson orthogonal to $x \mapsto \varphi((x \oplus 1)g)$ for all forms φ on $\tilde{\Upsilon}_0^0$ of level $\mathfrak{b}$.*

Proof. (a) Number the indices of the node $(0, 0)$ by ν, so that the doubled parameter at y is $2\lambda_\tau(\nu y) + z$, where λ_τ is the target parameter (Proposition 15.4(b)) and $z \in \{0, 1\}$ the flag. The sign is + on the first $k - 1$ entries (the *near* entries) and alternates on the others. Twice the lowest K-type is the doubled parameter minus a signed count L of the entries below and above (Appendix A.1): L counts the entries below entry νy minus those above, with the opposite count for the entries of the opposite sign. By induction on the number $N = r + z$ of indices, $|L| \leq N$, and at a near entry $L + N = N + \min(N, k) - 1 - 2i - ((N - k) \bmod 2)$, because a far entry lies below every near one (κ_0 being large against the gap) and the far signs alternate. On the near entries this cancels the doubled parameter, so the lowest K-type vanishes there. The far entries of λ_τ are $\Delta/2 - \kappa_j$ with κ_j between $B = \kappa_0$ and $\kappa_\Delta \leq 2B$, and $|L| \leq N$ is small against the gaps, so on the far entries the lowest K-type decreases strictly and lies in $[-3B, -B]$.

(b) is Lemma 33.3(d).

(c) At each infinite place, steepness of the two parameters (a) gives, by the restriction theorem for discrete series with a large gap (Theorem A.10), a point ω_w with $\|\omega_w\|^2 \leq \beta$ at which the local pairing of the coefficients of Υ_0 and of $\tilde{\Upsilon}_0^0$ is bounded below by $\beta^{-20(2r+1)^2}$ times a volume. The size conditions hold for the data, the generator $\varpi^{\lceil b\rceil}$ of the deep level having modulus $\sqrt{M}^{\lceil b\rceil}$ at every infinite place, which is large against all the constants by the choice of b; the gap conditions hold by the construction of the grids. Lemma 33.4(e) with $g = \omega$ applies to every form of level J_M, which is of the deep level since J_M contains it. □

Lemma 33.6 (The flat chain). (a) *The character of an edge of a first column is trivial on the determinants of the finite integral points trivial modulo the triangle modulus, and such pairs of points fix the data of the edge; so they fix the images of forms under the transposed chain and under the weighted chain of lifts along the column.*

(b) *For $i \le m$, $\int T_i\varphi \cdot U = \int T_0\varphi \cdot y$, where U is the weighted chain of i lifts of y and T_b the transposed chain from m to b.*

(c) *In grid $\tilde{Y}$, the weighted chain of lifts from $\tilde{Y}_0^0$ to $\tilde{Y}_0^m$ at the trivial twist is injective on the forms of level $\mathfrak{b}$, and $\Theta^{\mathfrak{b}}$ maps the forms of level $\mathfrak{b}$ on $\tilde{Y}_0^m$ onto those on $\tilde{Y}_0^0$.*

Proof. (a) The tame part of the character is trivial outside the places of the triangle modulus (the chosen places and the places of the twist in grid Y; the chosen places in grid $\tilde{Y}$, where the characters are untwisted), so its value at the determinant of an integral point depends only on the point modulo the triangle modulus. A pair (u, u') of such points acts on the datum through its reduction, which is trivial, and through the two characters, which are 1. The last transposed lift has a datum fixed by $(u, 1)$, so u fixes every transposed lift and the chain; likewise for the lifts and $(1, u)$. (b) By induction on i: across the edge from i to $i+1$, the lift and the transposed lift carry the same weight, and moving the weight inside the lift, transposing and regrouping it with the transposed lift gives the step. (c) The triangle of grid $\tilde{Y}$ is a triangle in the sense of Section 32.2 (Proposition 32.5(c) for z true), so the weighted chain is injective on the base (Theorem 32.4(a)). The image of $\Theta^{\mathfrak{b}}$ lies in the forms of level $\mathfrak{b}$ by (a). If a form ζ of level $\mathfrak{b}$ on $\tilde{Y}_0^0$ is orthogonal to the image, let U be the chain of lifts of $\bar{\zeta}$; by (b) at $\varphi = \bar{U}$, $\|U\|^2 = \langle \zeta, \Theta^{\mathfrak{b}}\bar{U}\rangle = 0$, so $U = 0$ and $\zeta = 0$ by injectivity. The space being finite dimensional, the image is everything. □

Lemma 33.7 (Point data on the first path). (a) *The diagonal period at $(0,0)$ is unchanged when y is translated by a finite point u and φ by $\operatorname{diag}(u, 1)$; forms of level $\mathfrak{b}$ on $\tilde{Y}_0^0$ are fixed by $\operatorname{diag}(u, 1)$ for u of level $\mathfrak{b}$ on Y_0. The period with a form φ on $\tilde{Y}_0^0$, as a functional of the forms y of level ϖQ on Y_0, is the Petersson product with a form of level J_M, linear in φ.*

(b) *The transposed point lift of an edge of a first column is the order of the general linear group of the smaller node modulo the triangle modulus times the sum, over the matrices modulo $\mathfrak{m}$ that reduce to the inclusion modulo the triangle modulus, of the transposed moves by these matrices. Hence the pairing of the images of y' and φ under the two transposed chains is the product of these orders over the nodes of both columns times $\sum_S \Psi^1(S)$ times the pairing of the images along the first path by S, with $\Psi^1(S)$ the product of the point data of the edges.*

(c) *Ψ^1 takes the values 0 and 1, and where it is nonzero the matrices of the vertical edges are left invertible modulo $\mathfrak{m}$, the two moduli having the same primes.*

(d) *A node family on the first path whose members lie in the levels of the nodes fixes Ψ^1 up to the edge characters, which are 1 there.*

Proof. (a) Both periods are integrals of a function invariant under the rational points, the second translated by u, and the block sum of a rational point with 1 is rational. A form of level $\mathfrak{b}$ is fixed by a point of the principal level $\mathfrak{b}$, and

$\operatorname{diag}(u,1)$ is one. For the Petersson product, the conjugate of the period is a linear functional on the finite-dimensional space of forms of level J_M, represented by a form; for y of level ϖQ one may replace y by its mean over J_M, because translation by J_M changes neither the period (take v supported on the places of $\mathfrak{b}$ and their conjugates, with uv of level $\mathfrak{b}$, so that $v \in J_M$ is trivial at the places of Q and fixes y, and $(uv,1)$ fixes φ) nor the Petersson product. (b) Pull the datum back along the reduction from $\mathfrak{m}$ to the triangle modulus and expand it in indicator functions; the pull-back is the point datum of the edge and each summand is a transposed move. Iterating along the column and gluing the tuples of the two columns, which have disjoint edges, gives the formula. (c) The value is a product of indicators. If a matrix reduces to the standard inclusion modulo $\mathfrak{m}_0$, the transposed inclusion times the matrix reduces to 1, so it is invertible modulo $\mathfrak{m}$, which has the same primes. (d) A member of the level reduces to 1 modulo the triangle modulus, so it fixes the inclusion, and the characters are trivial on it by Lemma 33.6(a). □

Proposition 33.8 (The first path). (a) *The chain of the m lifts of grid $\tilde{Y}$ from $\tilde{Y}_0^0$ is injective on level $\mathfrak{b}$, and $\Theta^{\flat}$ maps the forms of level $\mathfrak{b}$ on $\tilde{Y}_0^m$ onto those on $\tilde{Y}_0^0$.*

(b) *There are a real point g and a surjective flat restriction r at g onto the forms of level J_M on Y_0.*

(c) *There are path data $\Psi_{\mathfrak{s}}$ on the first path, with values sums of at most N^{2m} roots of unity (N the number of matrices modulo $\mathfrak{m}$ between two nodes), and a constant α such that for every flat restriction r at g the pairing of a conjugate form y' on Y_m with $\Theta_{\mathfrak{s}}(r\varphi)$ is α times the period of $\Psi_{\mathfrak{s}}$ on the first path.*

Proof. (a) is Lemma 33.6(c).

(b) Let g be the point of Proposition 33.5(c). For φ of level $\mathfrak{b}$ on $\tilde{Y}_0^m$, $\Theta^{\flat}\varphi$ is of level $\mathfrak{b}$ on $\tilde{Y}_0^0$, and by Lemma 33.7(a) there is a form $r\varphi$ of level J_M with $\langle y, r\varphi\rangle$ equal to the period of $\bar{y}$ and $\Theta^{\flat}\varphi$ at g for every y of level ϖQ; r is linear. This is a flat restriction. If $y \neq 0$ of level J_M were orthogonal to the range of r, it would be orthogonal to the restrictions of all $\Theta^{\flat}\varphi$, that is, by (a), of all forms of level $\mathfrak{b}$ on $\tilde{Y}_0^0$, and Proposition 33.5(c) gives $y = 0$; the forms of level J_M form a finite-dimensional space, so r is onto.

(c) *Transposition.* The chain $\Theta_{\mathfrak{s}}$ is the weighted chain of lifts with the weights the orders of the general linear groups (the indices of the principal levels); by Lemma 33.6(b) the pairing of y' with $\Theta_{\mathfrak{s}}(r\varphi)$ is $\int y_0' \cdot r\varphi$, with y_0' the transposed chain of y' down to Y_0, which has level ϖQ by Lemma 33.6(a); by the definition of a flat restriction this is the diagonal period at g of y_0' and $\Theta^{\flat}\varphi$. *Point data.* By Lemma 33.7(b) this period is a product of orders times $\sum_S \Psi^1(S)$ times the period along the first path by S. *Symmetrisation.* Let Ψ^0 be the sum over the families of coset representatives of the levels at the nodes Y_i and $\tilde{Y}_0^i$, $i < m$, of the translates of Ψ^1, each weighted by the edge characters; the member at $\tilde{Y}_0^0$ moves with that at Y_0^0, so each translate is counted a constant number of times. An admissible family permutes these cosets, and the difference of the two families is a family in the levels, which fixes Ψ^1 by Lemma 33.7(d); so Ψ^0 is fixed by admissible families up to the edge characters. Where $\Psi^0 \neq 0$ some translate of Ψ^1 is nonzero, and translation keeps left invertibility, so the vertical matrices are left invertible (Lemma 33.7(c)). The values are sums of at most the number

of coset families of roots of unity, and this number is at most N^{2m}, each of the at most $2m$ nodes contributing at most N cosets. Every translate of Ψ^1 gives the same sum against the period, by the invariance of the period (Lemma 33.7(a)) and Lemma 33.1(b), so the sum against Ψ^0 is the number of coset families times that against Ψ^1. *The constant.* The number of coset families is the product of the indices of the levels at the nodes Y_i and of the orders at $\tilde{Y}_0^i$; the orders of $\mathrm{GL}(Y_i)$ modulo ωQ are constants independent of ϑ times the indices modulo $\mathfrak{m}$ (both count invertible matrices, the reduction is onto and local, and in the ratio the factors from the places of Q cancel). Hence α, the quotient of the products of orders by the number of families, does not depend on ϑ; put $\Psi_\vartheta = \Psi^0$. □

33.3. Turns

The *ratio of the turn* at (i, j) is $\varrho_{i,j}^{\mathrm{turn}}(\vartheta) = \varrho_t\big(\mu_{i+1}\mu_{j+1}; \vartheta^{j+1}\big)$, the product over $1 \le s \le t$ of the ratios of Hecke values of the product of the untwisted characters of lifts $i+1$ and $j+1$ at the twist ϑ^{j+1}.

A turn replaces two sides of a square by the two other sides. Its *corner* consists of six nodes: the node V' that the turn removes and $\tilde{V}' = V' \oplus T$, the nodes W_1, W_2 that it keeps, and the node V that it adds and $\tilde{V} = V \oplus T$; off the diagonal T is empty, on the diagonal it indexes one line. The old path has the edges $V' \to W_1$ and $\tilde{V}' \to W_2$, the new one $W_1 \to V$ and $W_2 \to \tilde{V}$. On the old edges a tuple carries a matrix $y\colon V' \to W_1 \oplus W_2$ and a column $v\colon T \to W_2$; on the new edges a matrix $x\colon W_1 \oplus W_2 \to V$ and a row $v'\colon W_2 \to T$. The *character of the corner* is the inverse of the product of the characters of the two old edges. The corner is a *turn of a path with the other edges w* if the kept nodes differ, the removed nodes are not on the new edges and the other edges avoid them and do not end at the added ones, one old edge is vertical and the new edge at the same kept node is not, opposite edges have the same character, only the group of V acts at $\tilde{V}$, and any element k may act at V', through $k \oplus 1$ at $\tilde{V}'$.

For a path datum Ψ' on the old path, its *column Fourier transform* is $\hat{\Psi}'(S; y, v') = \sum_v \psi(\mathrm{tr}(v'v))\Psi'(S; y, v)$, and the *turned datum* $\Psi(S)$ is the match (Lemma 32.1(b)) of $y \mapsto \hat{\Psi}'(S; y, v')$ for the character of the corner, taken at the matrix x and the row v' read on the new edges of S, through a bijection ε identifying the indices of V' and V with those of $W_1 \oplus W_2$.

Lemma 33.9 (Turning a path datum). *Let the corner be a turn of a path with the other edges w, and Ψ' a path datum on the old path whose values are sums of at most N roots of unity.*

(a) *A sum over the tuples on the old edges is a sum over the tuples S on w, the matrices y and the columns v; likewise on the new edges with x and v'. The column Fourier transform depends on S only through the matrices on w.*

(b) *Moving S by an admissible family g, v' by the member of g at W_2 and y by $(g_{W_1 \oplus W_2})^{-1}\, y\, k$ for an invertible k at V' changes the column Fourier transform only by the characters of the edges at the determinants. Where the transform is nonzero, the blocks of y on vertical edges and the matrices of S on the vertical edges of w are left invertible.*

(c) *$y \mapsto \hat{\Psi}'(S; y, v')$ is matched with its match for the character of the corner, and the turned datum is fixed by admissible families up to the edge character product of the new path.*

(d) *Where the turned datum is nonzero, $x = x_\gamma$ for an invertible γ and its value is $\chi(\det\gamma)$ times the transform at y_γ; the matrices of the vertical edges of the new path are then left invertible.*
(e) *The turned datum is a path datum on the new path, with values sums of at most N times the number of columns v of roots of unity.*
(f) (*Exchange.*) *If for each index the functions L, L' of the two composites around the square satisfy the square law with constant C, then the sum over the tuples on the old path against Ψ' is C times the sum over the tuples on the new path against the turned datum.*

Proof. (a) A matrix at $\tilde{V}' \to W_2$ has the columns y_2 and v, and y has the rows y_1, y_2; gluing gives a bijection of the tuples. Likewise x has the columns x_1, x_2 and the matrix at $W_2 \to \tilde{V}$ has the rows x_2 and v'. The tuples built from (S, y, v) and (S', y, v) agree on every old edge when S and S' agree on w. (b) Let g' be k at V', $k \oplus 1$ at $\tilde{V}'$ and g elsewhere; it is admissible because V' is an inner node. By invariance Ψ' at a tuple moved by g' is its value divided by the characters, and moving the tuple of (S, y, v) by g' moves S, y and v separately; the sum over v is reindexed by $v \mapsto g^{-1}v$, which turns $\psi(\mathrm{tr}(v'v))$ into the character at the moved row. Where the transform is nonzero, some $\Psi'(S; y, v) \neq 0$, and the path datum property gives left invertibility. (c) By (b) with $g = 1$, replacing y by yk multiplies the transform by the value at $\det k$ of the product of the characters of the two old edges at V', the inverse of the character χ of the corner; so $\chi(\det k)\hat{\Psi}'(S; yk, v') = \hat{\Psi}'(S; y, v')$. Where the transform is nonzero the block of y on the vertical old edge is left invertible by (b), so y is a block y_γ of an invertible matrix (Lemma 32.1(d)). Hence Lemma 32.1(b) applies. The match commutes with the action at $W_1 \oplus W_2$ and transforms by χ^{-1} at V, and opposite edges have the same character, so the turned datum is fixed by g up to the characters of the new edges. (d) The match vanishes off the blocks x_γ and there equals $\chi(\det\gamma)$ times the function at y_γ. A vertical new edge is opposite to a vertical old edge, and the corresponding blocks of x_γ and y_γ are left invertible together (Lemma 32.1(d)). (e) collects (c), (d) and Lemma 33.1(a): each value of the match is one value of the transform, a sum over the columns v of values of Ψ' times roots of unity. (f) Expand both sums by (a); the index of the functions does not see the matrices on the sides of the square, so for fixed S and v' the sum over y weighted by the transform and the sum over x weighted by its match are the two sides of the square law, and the sums over S and v' pass through. □

Lemma 33.10 (Characters of a square). *Let $a, b < m$.*

(a) *The pair of the square, $\nu = \mu_{a+1}\mu_{b+1}$ with $z = t$, is a ramified critical pair, and $\varrho^{\mathrm{turn}}_{a,b}$ is a ramified ratio.*
(b) *The two pairs of opposite corners of the square have the same sum of ranks; the character of the lifts of the square times that of its transposed lifts is the rank twist of $\nu\vartheta^n$ in this rank, with $n = a + 1$ in grid $\tilde{Y}$ and $n = b + 1$ in grid Y; and the characters of the two edges at the corner (a, b) multiply to the character with tame part that of the pair times that of ϑ^n.*
(c) *For the sum W of the corners $(a, b+1)$ and $(a+1, b)$, half the rank of W minus its number of negative signs is at most minus the weight of the pair, and the two kept corners together have at least two more positive than negative signs.*

Proof. (a) The weights add and the tame parts multiply; the tame part of $\mu_{a+1}\mu_{b+1}$ is ω^{a+b+2}, nontrivial exactly at the chosen places, and minus the weight, $\kappa_{a+1}+\kappa_{b+1}$, lies between t and $4b$ (Definition 10.15). The ratio of the turn is ϱ_t of this pair at ϑ^{j+1}, a ramified ratio. (b) The ranks along a row and along a column of a grid form arithmetic progressions, so the sums agree. The product of the characters compares exponents of each basic character; the sum of the ranks is even, so the rounded halves of the two ranks add up to half the sum. One of the two tame parts carries a power of the twist and the other the trivial twist. (c) Both ranks are at most $2n_m \le 2h$ with h half the gap, and both weights are at least h; the count of signs follows from the shape of the grid signs (Proposition 15.4). □

Theorem 33.11 (Turns transfer periods). (a) (*A square.*) *For $a, b < m$ there are κ, possibly 0, a bijection σ between the indices of opposite corners, and numbers $c(\vartheta) \neq 0$ of modulus ϱ^{turn} up to a constant and a power of $\mathrm{N}\,Q$, such that the values of a functional at the images of a form at $(a, b+1)$ through $(a+1, b+1)$ and through (a, b) satisfy the square law with constant $\kappa/c(\vartheta)$.*

(b) (*Off the diagonal.*) *For $a < b < m$ a turn at the corner (a, b) of one half transfers periods with bound 1 and constants of modulus ϱ^{turn} up to a constant and a power of $\mathrm{N}\,Q$.*

(c) (*On the diagonal.*) *A turn at (j, j) transfers periods with bound $(\mathrm{N}\,\mathfrak{m})^{2n_m}$ and constants of the same kind: a seesaw moves the period to $S = Y_j^{j+1} = \tilde{Y}_j^{j+1}$, common to both paths, where the line $E\tilde{e}_j$ contributes a theta kernel; there the line is reversed, the kernel to $E\tilde{e}_j$ being, up to a factor of squared modulus $(\mathrm{N}\,\mathfrak{m})^{-\mathrm{rk}\,S}$, the Fourier transform of the kernels to $E\tilde{e}_{j+1}$, and a second seesaw moves the period on.*

(d) *There are $c(\vartheta) \neq 0$, of modulus $\prod_{i,j<m} \varrho_{i,j}^{\mathrm{turn}}(\vartheta)$ up to a constant and a power of $\mathrm{N}\,Q$, such that the first and the last path transfer periods with bound $(\mathrm{N}\,\mathfrak{m})^{2n_m m^2}$ and constants $c(\vartheta)$.*

Proof of Theorem 33.11(a). The square has the edges of the lifts with the characters of grid z at index a and the transposed lifts with those at index b. By Lemma 33.10(b), (c) the hypotheses of Corollary 31.2 hold with U_1, U_2 the kept corners, A, A' the other two, the family of twists indexed by ϑ, and the character $(\nu\vartheta^n)^{-1}$, which is tame for $\mathfrak{m}$ by Proposition 32.10(b) applied to the pair of Lemma 33.10(a). The corollary gives the square law with constant κ/c, where $|c|^2$ is a constant times L_* of this character up to t; by Proposition 32.10(b) this is a constant times a power of $\mathrm{N}\,Q$ times $\varrho_t(\nu;\vartheta^n)^2 = (\varrho^{\mathrm{turn}})^2$. The character of the law is the inverse of the product of the characters of the two edges at (a, b). □

Proof of Theorem 33.11(b). The corner. If the other edges w avoid the node (z, a, b) and do not end at $(z, a+1, b+1)$, the corner at (z, a, b) is a turn: an admissible family is free at (z, a, b), which is neither an end nor on the diagonal, the line T is empty, and the conditions on the characters are those of Lemma 33.10(b). The edges of the old path are the two sides through (a, b) and the edges of the rest, and those of the new path the two sides through $(a+1, b+1)$ and the same rest. *Data.* By Lemma 33.9(e) a path datum Ψ' on the old path gives the turned datum Ψ on the new one, with the same bound, the line being empty (one column). *Periods.* Carry the form that starts in grid z along the half to $(a, b+1)$,

and read the other half as a functional at $(a+1, b)$; both depend on the tuple only through the matrices off the square. By (a) the two composites around the square satisfy the square law with constant $\kappa/c(\vartheta)$, and Lemma 33.9(f) gives: the period of Ψ' on the old path is $\kappa/c(\vartheta)$ times that of Ψ on the new path. If $\Psi \propto \Xi$, with factor β at a real point, then the period of Ψ' is $\kappa\beta\,\Xi/c(\vartheta)$, so $\Psi' \propto \Xi/c$ with factor $\kappa\beta$, independent of the twist and the forms. □

For the turn on the diagonal at (j, j), the corner removes Y_j^j and $\tilde{Y}_j^j = Y_j^j \oplus E\tilde{e}_j$, keeps Y_{j+1}^j and $\tilde{Y}_j^{j+1}$, and adds Y_{j+1}^{j+1} and $\tilde{Y}_{j+1}^{j+1}$; the type T indexes the line. The spaces $\tilde{Y}_j^{j+1}$ and Y_j^{j+1} are equal up to a renumbering of the bases that keeps the signs, which identifies their groups and preserves heights; call this space S. The *line kernels* are the theta kernel from the line $E\tilde{e}_j$ to S for the point mass at a column v and a polynomial p, at u^j on the line, with the character of the horizontal edge from Y_j^j on S and the character v^{-1} on the line; and the kernel from S to the line $E\tilde{e}_{j+1}$ for the point mass at a row v', at u^{j+1}, with the character of the vertical edge to $\tilde{Y}_{j+1}^{j+1}$ and v on the line. The *column period* and the *row period* are the integrals over $[\mathrm{U}(S)]$ of the line kernel times the form reaching S in grid $\tilde{Y}$ times the conjugate form reaching it in grid Y, through Y_j^j, respectively through Y_{j+1}^{j+1}, summed against Ψ over the tuples.

Lemma 33.12 (The diagonal turn). *Let $j < m$ and let q be a pair of monotone paths from $(0, m)$ to $(j, j+1)$, one in each grid.*

(a) *The kernel of the edge from $\tilde{Y}_j^j$ to $\tilde{Y}_j^{j+1}$ at $h \oplus u^j$ and g is the product of the line kernel at g and the kernel of the edge from Y_j^j to S at h and g, the matrix being cut into columns and the polynomial being a product of a polynomial in the variables of Y_j^j and one in those of the line; and symmetrically for the edge from $\tilde{Y}_j^{j+1}$ to $\tilde{Y}_{j+1}^{j+1}$ with the rows.*

(b) (*Seesaws.*) *If all column periods of Ψ are multiples of Ξ by constants independent of the twist and the forms, so are the periods of Ψ on the path through (j, j). If the periods of Ψ on the path through $(j+1, j+1)$ are proportional to Ξ, then every row period is a multiple of Ξ.*

(c) (*Reversing the line.*) *There is a map τ of polynomials, independent of the twist, such that the line kernel from $E\tilde{e}_j$ with column v and polynomial p is c times $\sum_{v'} \psi(\mathrm{tr}(v'v))$ times the line kernel to $E\tilde{e}_{j+1}$ with row v' and polynomial τp, with $|c|^2 = (\mathrm{N}\,\mathfrak{m})^{-\mathrm{rk}\,S}$; the same holds for the periods.*

(d) *The corner on the diagonal is a turn of a path whose other edges are those of q, and the character of the corner is the inverse product of the characters of the edges $Y_j^j \to Y_{j+1}^j$ and $Y_j^j \to Y_j^{j+1}$. The number of columns modulo $\mathfrak{m}$ of S is at most $(\mathrm{N}\,\mathfrak{m})^{2n_m}$.*

Proof. (a) The point mass at a matrix is the product of the point masses at its two blocks of columns (or rows); the Weil representation of an orthogonal sum is the tensor product, so the theta kernel of the sum is the product of the kernels; the characters agree because the ranks have the right parities, t being odd. (b) By the seesaw identity for the periods (the integral over $[\mathrm{U}(S)]$ of a kernel that splits into a weight with proportional periods and a kernel

on an edge), it suffices that the polynomials whose kernels split in this way span all polynomials; by (a) the variables split into two blocks of columns, and the products AB span. The edge from Y_j^j to S is a lift and that from $\tilde{Y}_j^j$ to S a transposed lift; a product AB qualifies, with A on the edge from Y_j^j and B in the weight on S, the weight of the period on the path being the constant 1. For the second statement fix p: every polynomial A on the edge to Y_{j+1}^{j+1} qualifies, with Ap on the edge to $\tilde{Y}_{j+1}^{j+1}$, because the periods on the path are proportional by hypothesis. (c) The theta kernel is linear in the finite datum, a sum of the kernels of point masses (Lemma 32.6(b)); the Fourier transform of the point mass at v takes the value $\psi(\mathrm{tr}(v'v))$ at v', and the kernel of the line from one side is, by the Weil representation of the line (a rank one space), the Fourier transform of the kernel from the other side, up to the Gauss sum constant of squared modulus $(\mathrm{N}\,\mathfrak{m})^{-\mathrm{rk}\,S}$ and the translation by the flip scalar. The integral against the two forms is linear, which gives the statement for the periods. (d) The edges of the halves lie in the rectangle spanned by $(0, m)$ and $(j, j+1)$, so they avoid the removed nodes and do not end at the added ones; at the kept node Y_{j+1}^j the old edge is vertical and the new one is not; an admissible family has at $\tilde{Y}_{j+1}^{j+1}$ its member at Y_{j+1}^{j+1} extended by 1 on the line and stays admissible with any k at Y_j^j and $k \oplus 1$ at $\tilde{Y}_j^j$. The edge from $\tilde{Y}_j^j$ to $\tilde{Y}_j^{j+1}$ has the character of $Y_j^j \to Y_j^{j+1}$. The rank of S is at most $2n_m$, which bounds the number of columns. □

Proof of Theorem 33.11(c), (d). (c) Let Ψ' be a path datum on the path through (j, j). By Lemma 33.12(d) and Lemma 33.9(e), the turned datum Ψ (with the column Fourier transform in the column of the line) is a path datum on the path through $(j+1, j+1)$, with the bound multiplied by the number of columns, at most $(\mathrm{N}\,\mathfrak{m})^{2n_m}$. Suppose $\Psi \propto \Xi$. By the second seesaw (Lemma 33.12(b)), each row period of Ψ for a polynomial p' is $\beta\,\Xi$. Reverse the line by Lemma 33.12(c) and cross the square (j, j) by (a): the column period of Ψ' for p is $c\,C$ times the row period of Ψ for $p' = \tau p$, where $C = \kappa/c(\vartheta)$ is the constant of the square law and Ψ corresponds under the bijection of the square to the Fourier transform of Ψ' in the column of the line (Lemma 33.9(f)). Hence all column periods of Ψ' are multiples of $\Xi/c(\vartheta)$ by numbers independent of the twist and the forms, and the first seesaw gives $\Psi' \propto \Xi/c(\vartheta)$. The constant has the modulus of (a) up to the factor c of the line, whose squared modulus is a power of $\mathrm{N}\,\mathfrak{m}$, that is a constant times a power of $\mathrm{N}\,Q$.

(d) Each turn at (i, j), $i, j < m$, transfers periods with bound at most $(\mathrm{N}\,\mathfrak{m})^{2n_m}$ and constants $c_{i,j}(\vartheta) \neq 0$ of modulus $\varrho_{i,j}^{\mathrm{turn}}$ up to a constant and a norm power, by (b) (for $i < j$ in the half in Y, for $j < i$ in the half in $\tilde{Y}$, with the indices exchanged) and (c). Apply Lemma 33.2 to the property that (Γ, last) transfers periods with bound $(\mathrm{N}\,\mathfrak{m})^{2n_m \#T}$ and constants $\prod_{s\in T} c_s$: it holds for the last path and $T = \emptyset$ by reflexivity, and passes backwards along each turn by transitivity. At the first path, T is all m^2 pairs. The product of the constants has the modulus $\prod \varrho_{i,j}^{\mathrm{turn}}$ up to a norm power by Proposition 32.10(a). □

33.4. The last path

On the last path all spaces are definite, forms are algebraic forms on the standard spaces, and the lifts are algebraic. A *branching pairing* for weights (κ, e) of GL_{n+1} and (κ', e') of GL_n with interlacing doubled parameters is an integer matrix B such that $v' \mapsto v' \cdot Bv$ is equivariant for block matrices, bounded in the Gram norms, and computes the period of two algebraic forms over the smaller definite group as a constant times the integral over the finite points of the pairing of their values. The *last path sum* $\Xi = \Xi(\Psi; y', \varphi)$ is the sum over the tuples S on the last path of $\Psi(S)$ times the integral over the finite points of Y_m^m of the pairing of the lifts of y' and φ along the two halves.

A *weight* is a pair (κ, e): $\kappa_w(h)$ columns of height h at the place w and the power e_w of the determinant; l is *twice its Harish-Chandra parameter* if l_w lists, in some order, the entries $2(e_w + \sum_{h>p} \kappa_w(h)) + n - 1 - 2p$, $0 \le p < n$, where only the columns of height below n count. The number of *boxes* is $\sum_w \sum_{h<n} h\kappa_w(h)$. A doubled parameter has *gap* c if its entries are distinct, of the parity of $n-1$, and entries of different signs are at least c apart.

Lemma 33.13 (Regular parameters). (a) *If l is twice the parameter of (κ, e) and $|l_w(i)| \le \beta$, then each entry of the parameter is at most β in absolute value, its last entry is $2e_w + 1 - n$, κ has at most $\#\Phi \cdot n\beta$ boxes, and $|e_w| \le \beta$ for nonempty n.*

(b) *If every l_w has a gap, then l is twice the parameter of some weight; if moreover $e_w \le l_w(i)/2 + n/2$ for all i (integer division), the weight may be chosen with any prescribed columns f of height at least n and determinant power $e_w + f_w(n)$.*

Proof. (a) A row length is half the difference between an entry of l_w and the smallest entry $2e_w + 1 - n$, so at most β; the boxes are counted by rows. The largest entry is at least $2e_w + n - 1$ and the smallest is $2e_w + 1 - n$, which bounds $|e_w|$. (b) By parity an entry of l_w is determined by its half, so the halves are distinct integers. Sort them; the numbers $\kappa(1), \dots, \kappa(n-1)$ are the gaps between consecutive halves less one, and $\kappa(n)$ is the gap between the smallest half and the bound g; with g chosen from e_w this gives the weight, the columns above n being free because the parameter does not see them. □

Lemma 33.14 (Branching pairings). *Let (κ, e) be a weight of GL_{n+1} with doubled parameter l' and (κ', e') one of GL_n with doubled parameter $-l$, with l and l' interlacing, and all signs* 1.

(a) *If l, l' interlace, the highest weights interlace: entry q of the highest weight of (κ_0, e_0), the weight of parameter l, lies between the entries $q+1$ and q of that of (κ, e); so κ_0 arises from κ by removing a horizontal strip, and has at most as many boxes below height n as κ has below height $n+1$.*

(b) *A duality A between (κ', e') and (κ_0, e_0) satisfies $|v' \cdot Av| \le \|v'\|\|v\|$ in the Gram norms.*

(c) *A branching pairing exists: the composite of the branching matrix from κ to κ_0 and the duality A. Its pairing is invariant under g on the left and $\mathrm{diag}(g, u)$ on the right up to $\prod_\tau \det u_\tau^{N_\tau}$ with $|N_\tau|$ at most the number of boxes plus $|e_\tau|$;*

if M bounds the number of boxes, it is at most $(M^M)^3$ times the product of the Gram norms; and for each real point g there is c with

$$\int_{[\mathrm{U}(V)]} \eta(x)\varphi(\mathrm{diag}(x,u)g)\,dx = \frac{c}{[K_0 : K \cap K_0]} \sum_h \frac{\eta(h)\cdot B\,\varphi(\mathrm{diag}(h,u))}{\#\mathrm{Aut}(h)}$$

for algebraic forms η, φ of level K, read as functions, the sum over the classes of finite points.

Proof. (a) Take the index i at position q of l_w and j, j' at positions q, $q+1$ of l'_w: more than q entries of l_w but only q of l'_w would exceed $l'_w(j)$, and more than $q+1$ of l'_w but only q of l_w would exceed $l_w(i)$. The entry of a highest weight is the determinant power plus a row length, which turns the inequalities of parameters into those of rows; the difference d_q of entry q of (κ_0, e_0) and entry $q+1$ of (κ, e) is the number of the columns of height $q+1$ that keep their box, so the removed boxes form a horizontal strip. Each row of κ_0 is at most the row of κ with the same number, and no height lies strictly between n and $n+1$, which compares the boxes. (b) The pairing with v is the hermitian product with a vector of the same Gram norm, and Cauchy–Schwarz applies. (c) By (a), the branching matrix b from κ to κ_0 exists (Proposition 16.5, the Pieri rule for a horizontal strip), is equivariant for $\mathrm{diag}(g,u)$ up to a power of u, has bounded norm and satisfies multiplicity one; compose it with A. The bound on $|N_\tau|$ is that on the power of u. By multiplicity one, the integral over the real points of the pairing of the real parts is a multiple of the pairing of the values, and for definite groups the integral over the quotient is the class sum of the finite parts weighted by the inverse orders of the stabilisers, divided by the index of the level. □

Lemma 33.15 (Weights and bounds on the last path). *Fix a half z of the last path, through the nodes (z, i, m), $i \le m$.*

(a) *There are columns κ and exponents e_i such that the rank grows from i to $i+1$ with no column of height strictly between the two ranks, e_i is the weight at the upper end of the edge character plus the number of columns whose height is the rank at i, the weight at the lower end is e_{i+1}, and (κ, e_i) has doubled parameter $-$ the parameter of the forms moved along the half.*

(b) *The spaces of the last path are standard definite spaces; their ranks are at most $2n_m$, the exponents are at most $\lceil 20b \rceil$ in absolute value, and the weights have at most $\lceil b \rceil r^4$ boxes.*

(c) *The level of the node (z, j, m), the integral points whose reduction modulo $\mathfrak{m}$ lies in the level of the node, lies in the principal level $J(\mathfrak{b})$; it is $J(\mathfrak{m})$ on Y_m and $J(\mathfrak{b})$ elsewhere.*

Proof. (a) Choose κ to fit the parameter at $j = 0$, with $\Delta - t - z$ columns of each height $\mathrm{rk}(j)$, $j \ge 1$, and no other columns higher than $\mathrm{rk}(0)$. The parameters at larger j follow by induction: across an edge where the weight lifts, the parameters correspond exactly when the second is that of the new weight (Theorem B.24), the lift weight growing by Δ from $j+1$ to $j+2$ and half the rank by $t+z$ in two steps. (b) At z false, $i = 0$ the space is Y_m, which is standard; the other nodes are diagonal with all signs 1 (Proposition 15.4). The ranks are $n_j = k + j(t+1)$ up to the line; the exponents and the boxes are bounded by Lemma 33.13(a) with the bound β on the parameters, compared with b and r

by the choice of the data. (c) The level of a node is trivial or the kernel of the reduction modulo $\mathfrak{b}$, and reducing in two steps is reducing modulo $\mathfrak{b}$. □

Lemma 33.16 (Class sums on the last path). *Let Ψ be a path datum on the last path and let the forms y', φ have the levels at the outer ends of the two halves.*

(a) *Let g be a node family and u, u' integral points reducing to the members of g at the ends of an edge of the last path. The Weil representation of (u, u') for the kernel of the edge takes the datum of gS on this edge to that of S times the character of the edge at the determinants.*

(b) *Moving the data of all lifts by a family a of elements of the levels along the last path, with a at $\tilde{Y}_m^m$ equal to $\operatorname{diag}(a_{Y_m^m}, 1)$, does not change the sum against Ψ of the pairings of linear images of the tensors of the data; consequently the sum over S of $\Psi(S)$ times the branching pairing of the lifts at h and at $\operatorname{diag}(h, u^m)$ is right invariant under the level of Y_m^m, and it is smooth and left invariant under the rational points.*

(c) *If a property of vectors at points holds at $j = 0$ for the values of the two forms and is preserved by the kernels, and $\|\sigma(Ma)\| \le R$ for each pairing a of vectors with the property at corresponding points of Y_m^m and $\tilde{Y}_m^m$, then Ξ is a sum of at most L times the product of the class numbers of the $2m+1$ levels numbers with this bound, with $L \le N \cdot N_{\mathfrak{m}}^{2m}$.*

Proof. (a) What remains after the substitution is the action of the Weil representation on the indicator of the matrix of the edge, which is translation of the matrix by the two members and the character of the edge. (b) The residues of a form an admissible family, and Ψ is fixed by such families up to the characters of the edges; by (a) acting by the members on the data of gS gives those of S times the same characters, which cancel. For the level of Y_m^m, act by u and $\operatorname{diag}(u, 1)$ at the node (m, m) of the two grids and by 1 elsewhere; the residue of the second is $\operatorname{diag}(r, 1)$ with r that of u, in the level. The values of the two chains at h and at the block point are linear images of the tensors of their data. Each lift is an algebraic form, with an open stabiliser and automorphic, and the second is pulled back along $h \mapsto \operatorname{diag}(h, 1)$; so the integrand is smooth and left invariant. (c) By (b) the integrand of Ξ is invariant under the levels in each variable, so the last integral and both chains of lifts are plain sums over classes: the chain in grid $\tilde{Y}$ is a sum over classes using the family that is a in this grid and 1 in the other, and likewise the chain in grid Y. Each term is a sum over the tuples S of pairings of reached vectors with coefficients $\Psi(S)$, and $\Psi(S)$ is a sum of at most N roots of unity; there are at most $N_{\mathfrak{m}}^{2m}$ tuples (Lemma 33.1(c)). The integrals commute with the sum, the integrands being smooth and invariant. □

Proposition 33.17 (The last path). (a) *Every regular doubled parameter is twice the Harish-Chandra parameter of a weight, and a bound β on its entries bounds the number of boxes by $\#\Phi \cdot n\beta$ and the determinant power.*

(b) *Branching pairings exist.*

(c) *At every real point g there is β such that for every twist, y' and φ, the period of Ψ on the last path at g is $\beta\, \Xi(\Psi_{\mathfrak{z}}; y', \varphi)/N_{\varpi Q}$, where $N_{\varpi Q}$ is the order of the general linear group of Y_m modulo ϖQ.*

(d) *For a path datum whose values are sums of at most N roots of unity, Ξ is a sum of at most $N \cdot N_{\mathfrak{m}}^{2m}$ times the product of the class numbers of the $2m+1$ levels that occur, of pairings of vectors reached by the lifts.*

(e) (*Denominators.*) *There is* $0 < N_0 \le \exp(x^{99}b^2)$ *such that for every twist, path datum, pair of forms of gauge at most* 1 *and prime* $p \nmid \mathrm{N}\,Q$, $N_0\Xi$ *is* p*-integral.*
(f) (*Size.*) *At the trivial twist, if the conjugates of the values of the two forms have squared norms at most* B_0, B_1, *then* $|\sigma\Xi| \le \exp(x^{200}b^2)\sqrt{B_0B_1}$.

Proof. (a) is Lemma 33.13. (b) is Lemma 33.14(c), applied to the weights of Lemma 33.15(a) at the two nodes (m, m); their parameters interlace by Proposition 15.4(c).

(c) *Lifts along the last path are algebraic.* For each half and $j \le m$ there is a constant c_j such that for every twist and tuple S the image of a form F at $(z, 0, m)$ along the first j edges is, read as a function, c_j times the ratio of the indices of the levels at the two ends times the algebraic chain of lifts of F with the data of S. By induction on j: no lift is made at $j = 0$, and F is recovered from its algebraic form; each edge is a theta lift between standard definite spaces (Lemma 33.15(b)) with the datum the indicator of a matrix times a power of the vacuum scale, so it is a constant over the index of the level at its lower end times the algebraic lift, and the index at its upper end does not depend on the twist. *The period.* The period along the last path is the diagonal period at (m, m); insert the two chains and apply Lemma 33.14(c) at the real point g. For each S the term of the period is a constant times the term of Ξ without the factor $\Psi(S)$, divided by the index of the level at Y_m; the index at Y_m^m cancels and those in grid $\tilde{Y}$ do not depend on the twist. The index of the level $J(\mathfrak{m})$ on Y_m and the order $N_{\omega Q}$ differ by a constant factor, as in the proof of Proposition 33.8(c).

(d) is Lemma 33.16(c) with the property of being reached by the lifts.

(e) *One lift.* Let d be the product of powers of M and the prime d_0 of the data that takes the datum of an edge to standard data and pays for the boxes of the diagram (the ranks being at most $2n_m$ and the boxes at most $\lceil b\rceil r^4$). For $p \nmid \mathrm{N}\,Q$ and an embedding of $\mathbb{C}$ into $\mathbb{C}_p$, d times the datum of each edge of the last path raises p-adic gauges by at most 1 (Theorem 17.5(b), (c)). *The flip scalar.* The flip scalar $u = \overline{t\alpha}/(t\alpha)$ becomes integral after multiplication by $\mathrm{N}\,Q \cdot M^{\lceil b\rceil}q_1$, because t, $q_1\alpha$ and α^{-1} are integral and t divides $\mathrm{N}\,Q \cdot M^{\lceil b\rceil}$; as $\mathrm{N}\,Q$ is a p-adic unit, $(M^{\lceil b\rceil}q_1)^{m|l|}\det(u^m)^{-l}$ has p-adic absolute value at most 1 for every integer l. *The pairing.* By the equivariance of Lemma 33.14(c), the pairing of v' at h and v at $\mathrm{diag}(h, u^m)$ is the inverse of $\prod_\tau \det u'^{N_\tau}$ at $u' = (u^m)^{-1}$ times the pairing of the translated vectors, which have integral entries when the gauges are at most 1; the product N_1 of the numbers $(M^{\lceil b\rceil}q_1)^{m|N_\tau|}$ clears the character, and elements of absolute value at most 1 form a ring. *The sum.* By (d), Ξ is a sum of branching pairings of vectors obtained from the forms by m lifts each; after j lifts, d^j times the vector has gauge at most 1. Put $N_0 = N_1 d^{2m}$; the p-integral numbers form an additive group, which contains $N_0\Xi$. The bound $\log N_0 \le x^{99}b^2$ is a comparison of the logarithms of the constants of the data.

(f) At the trivial twist the class numbers of the levels along the last path are at most $\exp(x^{99}b^2)$. The operator by which the datum of an edge carries a value at one point to a value at another multiplies the Gram norm of every conjugate by at most $\exp(x^{99}b^2)$: the datum is the indicator of one matrix, so the size bound for jets applies with size 1 (Theorem 17.3), each row length being at most the number of boxes. After j lifts the norm is at most $\exp(jx^{99}b^2)$ times its bound at the start; the branching pairing is at most $(M^M)^3$ times the product of the norms by Lemma 33.14(c) with $M = \lceil b\rceil r^4$; and the number of terms is L times

the $2m+1$ class numbers (Lemma 33.16(c)). Multiplying, with N and $N_{\mathfrak{m}}$ fixed at the trivial twist, gives $|\sigma\Xi| \le \exp(x^{200}b^2)\sqrt{B_0 B_1}$. □

33.5. The auxiliary forms

A target form y is *auxiliary* for scalars $\gamma(\vartheta)$, one for each m-fold twist, if $\gamma(\vartheta)\Theta_\vartheta y$ is integral at every prime not dividing $\mathrm{N}\,Q(\vartheta)$ and every conjugate of $\Theta_1 y$ has squared norm at most $\exp(2x_1^2 b^2)$.

The values of the chain are read through *delta forms*. Let A be the integer matrix of the invariant pairing of the dual weight on Y_m, that of the conjugate forms, with the weight of Y_m, normalised at the highest vector; its *hook number* h_A is the square norm of the highest weight polynomial for the pairing by differentiation, a positive integer. For a finite point x of Y_m and a vector v of the dual weight, the delta form $\delta_x(v)$ is the algebraic form of the dual weight and level $J(\varpi Q)$ supported on the class of x with value v there.

Lemma 33.18 (Delta forms). (a) *The pair $(\delta_x(v), \varphi)$, φ of level $\mathfrak{b}$ on $\tilde{Y}_0^m$, has the levels at the outer ends of the last path.*

(b) *The index of the level times the integral of $\delta_x(v)$ against $\Theta_\vartheta y$ is the pairing of v with the value at x of the algebraic chain of y, divided by the dimension of the weight.*

(c) *If x is integral above p and v an integer vector, $\delta_x(v)$ has p-adic gauge at most 1; the conjugates of its values are no longer than those of v.*

(d) *If the pairing of a vector ζ of the weight of Y_m with every integer vector of the dual weight is h_A times a p-integral number, then ζ is p-integral. If $|\sigma(v \cdot A\zeta)| \le K\|\sigma v\|$ for every v, then $\|\sigma\zeta\|^2 \le K^2$.*

(e) $h_A \exp(x^{99}b^2)\exp(x^{200}b^2) \le \exp(x^{201}b^2)$.

Proof. (a) The level of the delta form contains the principal level of $\mathfrak{m}$, which is the level at the outer end of the half in Y; reduction modulo a place of ϖQ factors through reduction modulo $\mathfrak{m}$. (b) The integral is a class sum, which the delta form reduces to its term at x; the division by the dimension is the factor of Schur's orthogonality relations for the weight paired with its dual. (c) The gauge of an integer vector at an integral point is at most 1; the level $J(\varpi Q)$ reduces to 1 at the chosen places, so the stabilisers are trivial (Proposition 16.2) and the values of $\delta_x(v)$ are v on the class of x and 0 elsewhere. (d) There is an integer matrix C with $CA = h_A \cdot 1$, since A is a duality normalised at the highest vector; so $h_A\zeta_i$ is the pairing of ζ with the i-th row of C, an integer combination of pairings with integer vectors, hence h_A times a p-integral number, and ζ_i is p-integral. For the norm, the squared norm of $\sigma\zeta$ is its pairing with a vector of the same norm. (e) compares the logarithm of the hook number, a product of hook lengths of a diagram with at most $\lceil b \rceil r^4$ boxes, with $x^{201}b^2$. □

Theorem 33.19 (Auxiliary forms). *There are an integer e and numbers $\gamma(\vartheta) \ne 0$ with $\gamma(1) = 1$ and*

$$|\gamma(\vartheta)| = (\mathrm{N}\,Q)^{e/2} \prod_{0 \le i,j < m} \varrho_{i,j}^{\mathrm{turn}}(\vartheta),$$

such that the target $\mathcal{T}$ is spanned by forms that are auxiliary for γ. If a family of twists ξ_ι has small ratios, the inverses of the $|\gamma(\xi_\iota)|$ form a family of height at most $m^2 t x_1$.

Proof. Back through the turns. Let $\Psi^{\mathrm{first}}_{\vartheta}$ be the path data of Proposition 33.8(c), with the constant α and the flat restriction r at g of Proposition 33.8(b), and let $c(\vartheta)$ be the constants of Theorem 33.11(d), with Ψ_{ϑ} the path data on the last path that the transfer gives from $\Psi^{\mathrm{first}}_{\vartheta}$. By Proposition 33.17(c), $\Psi_{\vartheta} \propto \Xi/N_{\omega Q}$; by the transfer, $\Psi^{\mathrm{first}}_{\vartheta} \propto \Xi/(N_{\omega Q}c(\vartheta))$. Hence, by Proposition 33.8(c), the index of the last level times the pairing of y' with $\Theta_{\vartheta}(r\varphi)$ is $\beta\,\Xi(\Psi_{\vartheta}; y', \varphi)/c(\vartheta)$, with one constant β for all twists and forms, the index and $N_{\omega Q}$ differing by a constant.

Values. Take $y' = \delta_x(v)$, admissible by Lemma 33.18(a). Put $u_{\vartheta} = (c(\vartheta)/c(1))\Theta_{\vartheta}(r\varphi)$. By Lemma 33.18(b) the pairing of v with the value of u_{ϑ} at x is $\beta'\,\Xi(\Psi_{\vartheta}; \delta_x(v), \varphi)$, with β' independent of ϑ, x, v and φ. In case $\beta' = 0$ these pairings vanish, the forms $\Theta_{\vartheta}(r\varphi)$ are 0, and the choice $s = 1$ below serves; otherwise choose $s \neq 0$ with $s\beta' = h_A N_0$. *Integrality.* Let φ have p-integral values at a prime $p \nmid \mathrm{N}\,Q$ (gauge at most 1). By Lemma 33.18(c) and Proposition 33.17(e), $N_0\Xi$ is p-integral for integer v, so the pairing of every integer v with the value of $(c(\vartheta)/c(1))\Theta_{\vartheta}(s\,r\varphi)$ at an integral x is h_A times a p-integral number, and by Lemma 33.18(d) the values are p-integral, every class having a representative integral above p. *Size.* At the trivial twist, if the conjugates of the values of φ have squared norm at most B, then by Proposition 33.17(f) and Lemma 33.18(c) the conjugate of the pairing is at most $h_A N_0 \exp(x^{200}b^2)\sqrt{B}\|\sigma v\|$, and by Lemma 33.18(d), (e) the conjugates of the values of $\Theta_1(s\,r\varphi)$ have squared norm at most $\exp(2x^{201}b^2)B$.

Spanning. The forms of level $\mathfrak{b}$ on $\tilde{Y}_0^m$ are spanned by forms whose algebraic forms are *short*: values integral at every prime and conjugates of squared norm at most $\exp(x^{99}b^2)$ (transport along the isomorphism between algebraic forms and forms of level $\mathfrak{b}$, and use an integral basis of the lattice of integral algebraic forms, whose conjugates are bounded by the class numbers and the Weyl module). Since r is onto (Proposition 33.8(b)), the forms $s\,r\varphi$ for short φ span the forms of level J_M on Y_0; for them the lifts are integral as above and the conjugates of Θ_1 have squared norm at most $\exp(2x^{201}b^2 + x^{99}b^2) \le \exp(x^{202}b^2)$. Apply an integral multiple of the projection to the target, an element of the group algebra of $\bar{G}$ that commutes with the lifts (Proposition 16.9(b)): it is onto the target, so it maps a spanning set to a spanning set, the lifts stay integral, and the logarithms of the squared norms of the conjugates become at most $x^{99} + x^{202}b^2 \le 2x_1^2b^2$. Put $\gamma(\vartheta) = c(\vartheta)/c(1)$, so that $\gamma(1) = 1$. The modulus of γ is that of c up to a norm power by Proposition 32.10(a), with $f = \prod \varrho^{\mathrm{turn}}_{i,j} \ge 0$ and $f(1) = 1$.

The height bound. The norm power is prime to the prime of ι, so it does not raise the height, and each of the m^2 turn ratios is a ramified ratio (Lemma 33.10(a)), a product of t inverse ratios of L-values of a ramified pair at a power of ξ_ι, each of height at most x_1 for a family with small ratios (Definition 36.1). □

34. The comparison operator

The factorisation of Theorem 32.14 reduces the height of $Z(1)/Z(\xi_\iota)$ to that of the ratios $Z_j(1)/Z_j(\xi_\iota^j)$ of single lifts. To bound the latter, the twisted lift $\Theta_{j,\vartheta}$ is compared with a twist-free map through an operator U_{ϑ} on the forms on V_j (Theorem 34.14): $\langle \Theta_{\vartheta} y', U_{\vartheta}\Theta_{\vartheta} y\rangle$ is a ratio of Hecke values times a fixed positive form, U_{ϑ} keeps integrality away from Q, takes lifts to forms orthogonal to the false lifts, and U_1 is short. The operator is $U_{\vartheta} = n_{\mathrm{Aut}} h_k^2\,\mathrm{glue}_{\vartheta}\,\mathrm{glue}^*_{\vartheta}$, where $\mathrm{glue}_{\vartheta}$ builds forms on $V_j = V'' \oplus V^\circ$ from tuples of forms on a smaller definite

space V° across a change of lattice, using *first factors* on V'' obtained from three applications of the exchange identity, after which only a number is left of the twist.

The section is organised as follows. The three exchanges turn a twisted theta series on $\mathrm{U}(W_j)$ into the twist-free functions F_i, the first two by producing a fixed form, the carrier, on a space of signature $(2m, m)$ (Section 34.1). The first factors and the gluing are defined and their integrality and sizes bounded in Section 34.2. That the resulting map Θ° is injective is a spanning statement for products of theta series, proved by an isolation argument at the peaks of a form (Section 34.3). The comparison operator is assembled from a tight frame in Section 34.4.

34.1. The three exchanges

Fix $1 \le j \le m$. Let S be the space of signature $(2m, m)$, I_n the standard definite space of rank n, $W^\flat = I_{6m+1}$, V'' the definite space on the indices of W_j, $(I_j)^-$, I_{3m-j} and $W^\flat$, and V° the definite space on the r indices of W_j and $t-9m-1-r$ further ones. The *carrier* φ° is the theta lift of the constant function 1 from the unitary group of the negative definite I_m^- to $\mathrm{U}(S)$, with the polynomial $\prod_\tau \det(y_\tau^-)^{2m}$; the *carrier coefficients* $\mathsf{a}(h)c(\det h)$ are class sums over $\mathrm{U}(I_m)$ of theta kernels with characters c, 1. A *choice* i is a $3m \times \mathrm{rk}\, V''$ matrix Λ modulo ϖ^2 with a class of $\mathrm{U}(W^\flat)$ at level ϖ^2 and a selection of rows; the function F_i on $\mathrm{U}(W_j)$ is the integral over $[\mathrm{U}(S)]$ of the carrier against the theta kernel of $(S, W_j \oplus \text{padding} \oplus W^\flat)$ with the trace character of Λ as finite datum and a padded product of minors as polynomial. The *exchange pairs* of j are three critical pairs: $(\mu_j^0 \upsilon^n, z_*)$ with $n = j + (t - 11m - 1)/2$ and the main point z_*, and $(\upsilon^{-z_\pm} \chi^{\pm j}, z_\pm)$ for the two signs. The *Hecke value product* $\Gamma_j(\vartheta)$ is the product over the three pairs of L_* at the inverse characters twisted by ϑ or $\vartheta^{\pm 1}$, for the chosen places and Q.

Here the numbers of the data are: the rank r of W_j, $k = r + 1 - m$, $t = 5r$, the gap $\Delta = 2r^8$, $\Delta' = \Delta/2 - (t+1)/2$, $r_+ = k + j - 1$ the number of positive signs of W_j; the points are $z_+ = m$, $z_- = m/2$ and $z_* = 7(m/2) + k/2$. The *characters* are $\chi' = \chi^j \vartheta$, $\nu'' = \mu_j^0 \vartheta\, \upsilon^{j+\lfloor (t+k-1)/2 \rfloor}$, the character of the j-th lift times a power of υ, and the *carrier character* $\nu^\flat \upsilon^m = \nu'' \upsilon^m / \chi'$, which is free of the twist and has trivial tame part. The *padded polynomial* is $P = \prod_\tau \det(y_\tau^-)^{e'(\tau)} \prod_{h<r_+} \det(y_{\tau,h})^{k^+(\tau,h)}$ on $(S, W_j \oplus I_j^- \oplus I_{3m-j})$, with y_τ^- the $m \times m$ block in the m chosen negative rows and $y_{\tau,h}$ the $2m \times 2m$ block in the $2m$ rows starting at the basis vector of number h of W_j, and k^+ the column counts of the diagram μ^+ of the j-th lift; on $W^\flat$ the polynomial is the product of determinants pairing the chosen rows of a short basis of the class with the positive variables. The index set Det_τ has one element for each column of μ^+ at τ: $k^+(\tau, h)$ for each height $h < r_+$ and Δ' of height r_+.

Lemma 34.1 (Numerics of the exchanges). (a) $\#\mathrm{Det}_\tau + j + \lfloor (t+k-1)/2 \rfloor = \kappa_j(\tau)$; $\#V'' = r + 9m + 1$; $6r \le n_m$, $100m + 9 < k$, $k + 6 \le t$, $2r \le b$, *and* $2(\#\mathrm{Det}_\tau + \#V'' + m) \le 5b$.

(b) $z_\pm, z_* \in [1, t]$; $k+t+1 \le \kappa_j \le 4b$; *and* $n + z_* \le \kappa_j$ *for* $n = j + (t-11m-1)/2$. *Hence the three exchange pairs are critical pairs, ramified at the chosen places.*

(c) *With* $n_* = 2ft^3$, $300n_* + t \le r^8$.

(d) *For $c \neq 0$ and a twist ϑ, a character of weight γ with tame part $\psi \cdot \mathrm{tame}(\vartheta)$, ψ nontrivial exactly at the chosen places, is tame for $\omega^c Q$; in particular so is χ', and so is $\chi' \nu^\flat v^m$, of weight $m - \#\mathrm{Det}_\tau$, with the residue character of χ'.*

Proof. (a) For $h < r_+$ the number of columns of height h is the successive difference of the shifted target parameter minus one, so the sum telescopes, and k is even and positive; the other statements compare the numbers of the data. (b) The first two are comparisons of the data, the weights $\kappa_j = \kappa_0 + j\Delta$ growing by the gap; the third is the same comparison with the explicit values. The weights of the pairs are $-(\kappa_j - n)$ and $-z_\pm$, between their points and $4b$, and their tame parts ω^j, $\omega^{\pm j}$ are nontrivial exactly at the chosen places, ω having order greater than $8m$ there. (c) is a comparison of powers of r. (d) The tame part of ψ is nontrivial exactly at the chosen places and that of ϑ exactly at Q; the two sets are disjoint and their union is the set of places of $\omega^c Q$. The weight of $\chi' \nu^\flat v^m$ is that of $\nu'' v^m$ minus 0, which is $m + j + \lfloor (t + k - 1)/2 \rfloor - \kappa_j = m - \#\mathrm{Det}_\tau$ by (a). □

For the one exchange, let $W = I_b \oplus W'$, $\mathfrak{m} \neq (1)$, χ a character of $O_E/\mathfrak{m}$, $e : r \hookrightarrow N$ and w a family of rows modulo a quotient of the residue ring. The *row stabiliser* H is the group of invertible A whose columns outside e are those of 1 and with $w\,\phi(A) = w$; the *column datum* of σ is χ of the determinant of the rows e of σ if $w\phi(\sigma) = w\phi(\iota_e)$, and 0 otherwise.

Lemma 34.2 (One exchange and the row stabiliser). (a) *If F is left invariant under H, then the column datum of the columns e of z, times $F(y)$ when z has the columns of 1 off e and 0 otherwise, equals $\chi(\det z) F(z^{-1} y)$ for $z \in H$ and 0 for the other z.*

(b) *If Ψ' is invariant under right multiplication by invertible matrices, then $\Phi(x'\ x'') = \chi(\det x'')\Psi'(x''^{-1}x')$ and $\Psi\binom{y'}{y''} = \chi(\det y')\Psi'(y'')$ are matched for χ^{-1}.*

(c) *A theta kernel whose finite datum lives on left invertible matrices decays rapidly in the adelic point.*

(d) (*One exchange.*) *If c times the class sum of a kernel family $K\Phi$ is the integral over $[\mathrm{U}(V')]$ of the kernel of (V', W) with datum Ψ and polynomial P on W' for all matched Φ, Ψ, then for Ψ' as in (b) the product of the kernel of (V', I_b) at h' with datum $\chi \circ \det$ and the kernel of (V', W') at g with datum Ψ' is integrable, with integral c times the class sum of $K\Phi$ for Φ as in (b); the same summed over the classes of $\mathrm{U}(I_b)$ against coefficients.*

(e) (*Tower.*) *If f_0 is left invariant under the rational points and right invariant under the integral points reducing into H, and f is the sum of the right translates of f_0 by lifts of coset representatives of H, then the class sum of f_0 at the level of H is the class sum of f at full level.*

(f) *Rational points of $\mathrm{U}(I_b)$ act on a kernel of (W, I_b) by a substitution of the polynomial times a power of the determinant; if the Fock vector of P is fixed by the real group of I_b, the kernel is left invariant under the rational points. Under these hypotheses the class sum at full level of the kernel with datum $\chi(\det x'')F(x''^{-1}x')$ equals the class sum at the level of H of the kernel with the restricted datum, the column datum of σ times $F(x')$.*

Proof. (a) On a matrix with the columns of 1 off e, the column datum can be read on the whole matrix: for $z = 1 + u\iota_e^{\mathsf{T}}$ the columns e are $\iota_e + u$ and their minor on the rows e is $\det z$; both conditions on w say $w\phi(u) = 0$. For $z \in H$ the invariance of F gives $F(z^{-1}y) = F(y)$. (b) The block determinant functions of Proposition 31.5(a) for the function $f = \Psi'$ give the match $\chi(\det y')\Psi'(-y''y'^{-1})$, and the invariance of Ψ' removes the factor $-y'^{-1}$. (c) The datum vanishes on the matrices that are not left invertible modulo $\mathfrak{m}$, so in the theta series only vectors of full rank occur; high in a cusp of the first space these are long, and the kernel is at most a constant times a Fock moment times $\mathrm{ht}(h)^B\mathrm{ht}(g)^{-M}$, for every M. (d) The function matched with $\chi(\det x'')\Psi'(x''^{-1}x')$ is the product $\chi(\det y')\Psi'(y'')$ by (b), so its kernel factors along $I_b \oplus W'$; the integral converges because y' is invertible on the support, by (c). The class sum is finite, so it commutes with the integral. (e) The number of cosets cancels the index of the level. (f) The kernel is invariant under the rational point δ^{-1}, which cancels the finite translation and leaves the real part of δ^{-1}, acting on the Fock vector by the substitution $x_\tau \mapsto (\delta^{-1})^{\mathsf{T}}x_\tau$ in the positive columns and $x_\tau \mapsto \delta x_\tau$ in the negative ones, times $\prod_\tau \det \delta^{q-\gamma_\tau}$. For the last claim, the datum Φ_0 is Φ cut down to $x'' \in H$, by (a); the twisted translates of Φ_0 by coset representatives of H add up to Φ, an invertible x'' lying in exactly one coset, and their kernels are right translates of the kernel of Φ_0, the twist cancelling the factor that the translation adds to $\chi(\det x'')$. (e) concludes. □

Lemma 34.3 (The carrier). *Let $m = 2e > 0$.*

(a) (*First exchange.*) *For exponents k_τ there is C such that each α of weight $-k$ tame for $(\mathfrak{m}, \chi)$ has γ with $|\gamma|^2 = C/L_*(\alpha^{-1}v^e, e)$ and $\gamma \sum_h \theta_{\alpha,\alpha}[\chi \circ \det, \prod_\tau \det(y_\tau)^{k_\tau}](g, h) = 1$ on $\mathrm{U}(I_m)$, the class sum at the stabiliser of the standard lattice.*

(b) (*Conjugation.*) *There is n of modulus one such that if γ times the class sum on I_m of the kernel of (I_m, I_m) with characters c^{-1}, c^{-1} and datum $\chi^{-1} \circ \det$ is 1, then so is the conjugate of γ times $n/\chi(\eta)^m$ times the class sum of the kernel of (I_m^-, I_m) with characters cv^m and datum $\chi \circ \det$, η the reduction of an element of norm -1.*

(c) *The vector of $\prod_\tau \det(y_\tau^-)^{2m}$ in the Fock space of (I_m^-, S) is fixed in the matrix coefficient: $\langle v, \Omega(u, 1)v\rangle = \langle v, v\rangle$ for u in the compact real group of I_m^-; hence the carrier is not zero, for every $m > 0$.*

Proof. (a) Apply Corollary 31.3 with $W = I_m$ and $V' = 0$: the data $\chi \circ \det$ and 1 are matched (the block determinant functions for the constant 1), the integral over the trivial group is 1, the polynomial of the empty matrix is 1, and with no rows to add the stacked determinant product is the product of powers of determinants; up to the factor $\alpha(\det x)^{-1}$ the kernel with the zero space has the standard characters. (b) Conjugating the kernel replaces the space by its opposite and the datum by its conjugate; it scales $\chi \circ \det$ by $(\chi(\eta)^m)^{-1}$, and kernels are linear in the datum. (c) Every variable of a minor in the negative rows has sign 1, each u scales the polynomial by the $2m$-th powers of its determinants, and these are products of powers of determinants of unitary matrices, of modulus one; the coefficient is the constant $\langle v, v\rangle$. If the lift vanished, the real convolution of the coefficient would vanish (Rallis' injectivity argument,

Theorem B.34(c), the finite part being the vacuum, which is sharp), but the integral of a nonzero constant over the compact group is not zero. □

Proposition 34.4 (The carrier and the three exchanges). (a) *Let m be even and let a tame character of weight 0 exist. Then the carrier is continuous, left invariant under the rational points and of every principal level; it is rapidly decreasing; and it is holomorphic for an ordering of the basis with the positive vectors first.*

(b) (*First two exchanges.*) *Let $m = 2e$. There is C such that for each character c of weight 0 tame for $(\mathfrak{m}, \chi)$ some γ with*

$$|\gamma|^2 = \frac{C}{L_*(c^{-1}\upsilon^m, m)\, L_*(c\upsilon^e, e)}$$

satisfies $\gamma \sum_h \mathsf{a}(h) c(\det h)\, \theta_{\upsilon^m, c}[\chi \circ \det, 1](x, h) = \varphi^\circ(x)$.

(c) (*Third exchange.*) *There are C and numbers $c(\vartheta)$ with $|c(\vartheta)|^2 = C/\Gamma_j(\vartheta)$ such that $c(\vartheta)$ times the left side of the third exchange, a theta series on $\mathrm{U}(W_j)$ (a class sum of kernels of (W_j, V'') with the vectors $\Upsilon_{\vartheta,i}(h)$), is F_i for every choice i.*

(d) (*One exchange.*) *For $W = I_b \oplus W'$ and a datum invariant under a row stabiliser, c times the class sum of the kernel with datum $\chi(\det x'')\Psi'(x''^{-1}x')$ is the integral of the kernel of (V', I_b) with datum $\chi \circ \det$ times the kernel of (V', W') with datum Ψ'; a class sum at full level equals a class sum at the level of the row stabiliser.*

Proof. (b) Chain two exchanges by Lemma 34.2(d). The first is Lemma 34.3(a) for c^{-1}, conjugated by Lemma 34.3(b) so that it lives on I_m^-: the class sum on I_m of kernels of (I_m^-, I_m) is constant on $\mathrm{U}(I_m^-)$. The second is Corollary 31.3 with $W = I_m \oplus S$, $V' = I_m^-$ and $V = I_{3m}$, the datum $\Psi' = 1$, summed over the classes of I_m: a multiple of a double class sum of kernels of (W, I_{3m}) is the integral of the kernel of (I_m^-, S) with datum 1, which is the carrier. The character of the second exchange is $(c\upsilon^m)^{-1}\upsilon^{2m} = c^{-1}\upsilon^m$. The kernel on $I_m \oplus S$ splits into the summand of the carrier coefficient and the kernel on S, and both sides are the claim times $c(\det x)$. The constants n and $\chi(\eta)$ have modulus one, so the squared moduli of the two constants multiply to the stated value.

(a) By (b), applied to the tame character of weight 0, the carrier is a finite combination of theta kernels with polynomial 1; level, rapid decay and holomorphy are read off from these kernels. Level one: on the stabiliser of the standard lattice the factor $c(\det k)^{-1}$ cancels against $\chi(\det k)$. Rapid decay: $\chi(\det y) \neq 0$ makes y invertible, so Lemma 34.2(c) applies. A theta kernel is continuous and invariant under the rational points. For holomorphy, order the basis with the positive indices first; the polynomial 1 is a product of zeroth powers of minors, so its Fock vector is holomorphic, and so is every kernel with it.

(c) *The exchange.* Apply Corollary 31.3 to $W = I_{3m} \oplus W'$, where $W' = W_j \oplus I_j^- \oplus I_{3m-j} \oplus W^\flat$ has the index set of V'', to V'' and to the space S (as V'), and sum it over the classes of I_{3m} by Lemma 34.2(d); by (b) the square kernel of S becomes the carrier. The basis of W is listed with the positive vectors first, the m negative ones forming the second block, so the signs are those of a block decomposition. For the constant rows of a choice i, the matched polynomial of Corollary 31.3 involves only the rows of W': it is the padded product P times the determinants on $W^\flat$ formed with the rows of i, by comparing the exponent of the negative

minor, the positive minors and the determinants on $W^\flat$ entry by entry. *The datum.* The left side carries the datum $(x'\ x'') \mapsto \chi'(\det x'')\psi_\Lambda^{\mathrm{inv}}(x''^{-1}x')$, where $\psi_\Lambda^{\mathrm{inv}}$ is the sum over $\mathrm{GL}_{3m}(O_E/\varpi^2)$ of the translates of the trace character of Λ; it is invariant under right multiplication, as Lemma 34.2(b) requires. The carrier has full level, so every translate gives the same integral, and the integral of the carrier against the kernel with $\psi_\Lambda^{\mathrm{inv}}$ is $\#\mathrm{GL}_{3m}(O_E/\varpi^2)$ times F_i. *The characters.* By Lemma 34.1(d) and the identities

$$\frac{(v^\flat v^m)^{-1}}{\chi'}v^{3m+z_*} = \frac{\vartheta^{-1}}{\mu_j^0 v^n}, \qquad \chi'^{-1}v^m = \frac{\vartheta^{-1}}{v^{-m}\chi^j}, \qquad \chi' v^{m/2} = \frac{\vartheta}{v^{-m/2}\chi^{-j}},$$

obtained by comparing exponents, the characters of the three exchanges are the inverses of those of the three pairs twisted by ϑ or $\vartheta^{\pm 1}$, so the squared moduli of the three constants multiply to a constant over $\Gamma_j(\vartheta)$. *The theta series.* If the rows of Λ are combinations of rows w modulo ϖ^2, $\psi_\Lambda^{\mathrm{inv}}$ is invariant under the matrices fixing w, and the real group of V'' fixes the polynomial of the kernel; by Lemma 34.2(f) the class sum at full level equals the class sum at the level cut out by w of the kernel with the product datum, the column datum of w times $\psi_\Lambda^{\mathrm{inv}}(x')$. Expanding the series Υ in coordinates, each term is one kernel on the index set of $W_j \oplus I_j^- \oplus W^{\mathrm{add}}$, with $W^{\mathrm{add}} = I_{3m-j} \oplus W^\flat \oplus I_{3m}$, and relabelling matches the left side; so the left side is the class sum over $\mathrm{U}(V'')$ of kernels of (W_j, V'') with the characters of the descent pair and ν'', the column datum, and the polynomials of the vectors $\Upsilon_{\vartheta,i}(h)$.

(d) is Lemma 34.2(d), (f). □

34.2. First factors and gluing

The *series* $\Upsilon_{\vartheta,i}$ on the finite points of $\mathrm{U}(V'')$ is built from theta kernels of definite spaces and the datum Ψ^{add} on $W^{\mathrm{add}} = I_{3m-j} \oplus W^\flat \oplus I_{3m}$; the *first factor* $\varphi''_{\vartheta,i}$ is the complex conjugate of $c''\Upsilon_{\vartheta,i}$ for a constant $c'' \neq 0$, and $\Xi_{\vartheta,i}(g) = \langle \varphi''_{\vartheta,i}, \theta''_{\vartheta,\Lambda}(g,\cdot)\rangle$. A form is *integral and small* if its values are integral away from the places of Q and, if Q is empty, their conjugates have squared norm at most $\exp(fmb\log^2 b)$. The *gluing* glue_ϑ takes a tuple ζ of forms on V° (one copy for each choice, given by values at the classes of level ϖ^2) to the form on $V_j = V'' \oplus V^\circ$ supported, choice by choice, on the classes of $(h, h^\circ)\mathsf{g}$ for a lattice g glued for Λ, with values the Cartan products of $\varphi''_{\vartheta,i}(h)$ and $\zeta(i,c)$. Write $\Theta^\circ = c'_1 \mathrm{glue}_1^* \Theta_{j,1}$ on good forms.

The *constant rows* of a choice i form, for each column of μ^+ at τ, a matrix c with $2m$ rows and with columns the r_+ positive indices of W_j and the indices of W^{add}: for a column of height $h < r_+$, row a is the basis vector of W_j of number $h + a$ if $h + a < r_+$ and the basis vector number a of I_{3m-j} otherwise; for the l-th column of height r_+ the rows are the chosen rows of $\tau(Y)$, for a short basis Y of the class of i, placed in the columns of $W^\flat$. The polynomial P_V is the product of the determinants $\det(\mathsf{c}; Xz)$ of the constant rows stacked over the rows of variables indexed by V'', with X the variables in the positive columns of W_j and z those in W^{add}. The datum Ψ^{add} on matrices z modulo $\varpi^2 Q$ is 0 unless the columns on I_{3m-j} and $W^\flat$ are those of 1, and then is $\psi_\Lambda^{\mathrm{inv}}(z')$ for the columns z' on I_{3m}. A polynomial F *has weight* κ *and squared norm at most* b if it is the polynomial $\sum_i (G\xi)_i b_i$ of a vector ξ of the Weyl module of κ (with G the Gram matrix and b_i the products of minors of the standard fillings) with $\xi^* G \xi \le b$.

Lemma 34.5 (Weights and norms of determinants). (a) *Under products the weights add and the bounds on the squared norms multiply, and the same holds for bounds valid under a condition.*

(b) *For the weight of one column of height h at τ, a function of h columns that is the sum over the h-subsets of the minor on the subset times its value at the unit columns is the polynomial of a vector whose squared norm is the sum of the squared moduli of these values; and* $\det((0; X) \mid Z)$ *is the polynomial of a vector of squared norm at most the product of the squared lengths of the columns of Z.*

(c) *There is Y with* $\det(X \mid Z) = \gamma \det(YX)^{\mathsf{T}}$ *for all X. The substitution $x_\tau \mapsto c_\tau^{\mathsf{T}} x_\tau g_\tau^{\mathsf{T}}$ sends the stacked determinant product for the matrices $M_{\tau,k} g_\tau^{\mathsf{T}}$ to that for $M_{\tau,k}$ with each factor multiplied by* $\det c_\tau \det g_\tau$*; and if it fixes the first p columns and sends F' to sF, the coordinates of F' in the first p columns, combined by the Schur matrix, go to s times those of F.*

Proof. (a) The product of two semi-invariants is a semi-invariant of the sum of the weights, and the Gram norm of the product vector is at most the product of the norms. (b) Evaluation at the unit columns on a subset has the minor on the subset as polynomial, and the pairing of ξ with this evaluation is the value; for the determinant, expand along the first h columns: up to sign the coordinates are the complementary minors of Z, and the Cauchy–Binet formula with Hadamard's inequality bounds the sum of their squared moduli. (c) Choose an invertible $(U_1 \mid U_2)$ with $U_1^* Z = 0$ and put $Y = U_1^*$; then $(U_1 \mid U_2)^*(X \mid Z)$ is block triangular, and comparing determinants gives the first claim. For one factor of the product with matrix A on the right, the matrix on the left is $\operatorname{diag}(1, c)^{\mathsf{T}} A g^{\mathsf{T}}$, which gives the second. For the third, read the polynomials in the first columns: the substitution by $(c, \operatorname{diag}(1, g))$ splits into the substitution on the coefficients and the row substitution by c^{T} on the first columns, whose adjoint for the Fock pairing is the row substitution by c (Appendix B.11), acting on the basis by the Schur matrix. □

Proposition 34.6 (First factors). (a) *Let P_V be the product of the determinants* $\det(\mathsf{c}; Xz)$*, one for each column of μ^+. Then $P_V = P_{\zeta(z)}(X)$ for a polynomial map ζ to the Weyl module of μ^+. The map ζ is homogeneous of one degree d, one integer $H \le (d^d)^3$ clears its denominators, and its squared norm is bounded when the entries are small.*

(b) *For one $c'' \neq 0$, independent of i, ϑ and h, every first factor is integral and small.*

(c) *There are C and $c'(\vartheta)$ with $|c'(\vartheta)|^2 = C/\Gamma_j(\vartheta)$ and $c'(\vartheta)\Xi_{\vartheta,i} = c''\mathsf{F}_i$ for all choices.*

Proof. (a) *The map.* Let ζ be the vector of coordinates of P_V in the first r_+ columns X: read P_V as a polynomial in X with coefficients in the polynomials in z, and take its coordinates in the basis of products of minors of weight μ^+. By Lemma 34.5(b) and (a), each factor $\det(\mathsf{c}; Xz)$, specialised in z, is the polynomial of a vector of the weight of one column of its height, so the product is the polynomial of a vector of weight μ^+: below r_+ each height occurs once for each column of μ^+, and at r_+ it occurs Δ' times, which is the column count of the descent pair there. Coordinates commute with specialisation, so $P_V = P_{\zeta(z)}(X)$, and, comparing coefficients at every point, P_V is the polynomial of ζ identically.

Degree and denominator. Scaling the variables by c multiplies the product of determinants by $c^{N\sum\kappa(h)}$, with $N = \#V''$, while the Schur matrix of $c\cdot 1$ is the scalar $c^{\sum h\kappa(h)}$; so each coordinate is homogeneous of degree $d = \sum(N-h)\kappa(h)$, and $N - h \le 10m+1$ whenever $\kappa(h) \ne 0$. The coefficients of P_V in the first columns lie in every subring of $\mathbb{C}$ containing the conjugates of the entries of Y; the coordinates are obtained from them by the inverse Gram matrix, which H clears, with H the hook number of the complementary diagram $h \mapsto \kappa(N-h)$, which has at most d boxes, so $H \le (d^d)^3$. *Norm.* If the conjugates of the entries of Y and of the values of z are at most M^4, each column of a determinant has squared length at most $(r+11m+1)M^8 \le M^9$, at most $12m+1$ columns occur, and by Lemma 34.5(b), (a) the squared norm of $\zeta(z)$ is at most $(M^9)^{(12m+1)N_{\det}}$ with $N_{\det}$ the total number of determinants, at most $\#\Phi(\lceil b\rceil + w_{\mathrm{off}} + \Delta + m\Delta)$ by Lemma 34.1(a).

(b) *The series as a class sum.* By the definition of the carrier coefficients, there is c of modulus one, depending only on j, such that the conjugate of a multiple of $\Upsilon_{\vartheta,i}(h)$ is n_{Aut}^2 times a double class sum over $\mathrm{U}(I_{3m})$ and $\mathrm{U}(I_m)$ of conjugates of products of three theta kernels of definite spaces: k_0 of (I_m, I_{3m}) with characters χ' and 1; the conjugate of k_1 of (I_j, V''), whose datum is the conjugate, at an element of norm -1, of the indicator of the inclusion matrix; and the kernel of (W^{add}, V'') at the polynomial vector $H\zeta$, with datum Ψ^{add}, both last kernels of modulus $\varpi^2 Q$. *Constants.* Each of the three kernels is made integral, at points integral above $p \nmid \mathrm{N}\,Q$, by a constant (Proposition 17.4), the data being integral away from $\mathrm{N}\,Q$: the values of Ψ^{add} are 0 or sums over $\mathrm{GL}_{3m}(\mathcal{O}_E/\varpi^2)$ of values of an additive character, roots of unity, and their conjugates are at most $\#\,\mathrm{GL}_{3m}(\mathcal{O}_E/\varpi^2)$; $\mathrm{N}\,Q\cdot M^c$ lies in the ideal of $\varpi^c Q$. For k_0 and k_1 move the points into the integral points, the polynomials and the characters of weight 0 not changing the kernel. *Integrality.* By weak approximation, rational points move the classes and h into the integral points; by equivariance under rational points (the Schur matrix of γ'^{-1} mixing the coordinates of ζ for a translated short basis and the powers of the determinants cancelling against the change of the characters), the gauge at h is that at an integral point, where the entries of $Y\gamma^{\mathsf{T}}$ are integral; each class sum takes one factor n_{Aut}, a common multiple of the orders of the stabilisers. *Size.* For Q empty, bound each term by duality: if the conjugates of the values of a vector of polynomials on a set are at most B and a linear form θ has $|\sigma\theta(Q)| \le C$ for every homogeneous Q small on the set, then the conjugate of θ composed with the vector is at most C; apply it to the third kernel with the bound of (a), multiply the three kernel bounds $\exp(x_1)$ and count the classes, at most $\exp(x^{30})$ each. With $d \le (10m+1)N_{\det} \le M$ and $H \le d^{3d} \le M^{3d}$, the logarithm of the square of the product is at most $fmb\log^2 b$ by the comparison of the sizes of the data.

(c) is Proposition 34.4(c), with $\Xi_{\vartheta,i} = c''$ times the left side of the third exchange: both are the class sum $\sum_h$ of kernels of (W_j, V'') at the level J_Λ with the datum given by the rows of Λ stacked over $\varpi\cdot 1$ and polynomial $\varphi''_{\vartheta,i}(h)$. Put $c'(\vartheta) = c(\vartheta)$ of that proposition. □

The gluing works on a *divided splitting* of $V = V''\oplus V^\circ$: the indices split as $N \sqcup m$, with r indices of W in V'' (the fixed embedding e), r marked indices z of V°, and a copy of an index set a in N; the diagrams μ^+ and μ^- have no columns of heights between the ranks of the summands and that of V. Let ϖ be the idele generating the product of the chosen places. A *divided lattice* is a point γ of the

finite unitary group, supported on the chosen places, with $\varpi\gamma^{-1}$ integral (so $\varpi O^n \subseteq \gamma O^n \subseteq O^n$ there); its *scalar* is $c_\gamma = \chi(\delta)/\sqrt{|\delta|}^r$ with δ the determinant of γ at the chosen places. The *congruence group* $\bar{J}^1$ consists of the invertible $r \times r$ matrices modulo ϖ^2 that are 1 modulo ϖ. For a matrix s modulo ϖ^2 with rows indexed by a, a matrix x is *block normalised* if its rows in m form the inclusion of z modulo ϖ^2 and its rows u in N satisfy $su \equiv s\iota_e$ modulo ϖ^2 and $u \equiv \iota_e$ modulo ϖ; the lattice is *glued for* s if $\gamma^{-1}x \equiv (1;0)$ modulo ϖ holds exactly when xY is block normalised for some $Y \in \bar{J}^1$, then unique, and after a column operation the columns of $\varpi\gamma^{-1}$ in V'' vanish modulo ϖ outside the copy of a.

Lemma 34.7 (Glued lattices). (a) *If Y modulo ϖ^2 has $\varpi Y = 0$, then $1 - Y$ is the inverse of an element of $\bar{J}^1$. Lifts of elements of $\bar{J}^1$ reduce to 1 at every chosen place; if an integral point j is 1 modulo ϖ^2, then $\gamma^{-1}j\gamma$ is integral and reduces to 1 at every chosen place.*

(b) *If integral C, G satisfy $CG = \varpi \cdot 1$, some divided lattice has $\varpi\gamma^{-1} \equiv G$ modulo ϖ^2.*

(c) *For every s there is a lattice glued for s; for a glued lattice the indicator of $(\varpi\gamma^{-1})x \equiv \varpi(1;0)$ is the sum over $Y \in \bar{J}^1$ of the indicators that xY is block normalised.*

Proof. (a) $\varpi Y \equiv 0$ says that every entry of Y is divisible by ϖ, so $Y^2 = 0$ modulo ϖ^2 and $1 - Y$ is a unit fixing $\varpi \cdot 1$. A lift of $Y \in \bar{J}^1$ is 1 modulo each chosen place by the equation $\varpi Y = \varpi$. For the conjugate, write $\gamma^{-1}j\gamma - 1 = \gamma^{-1}(j-1)\gamma$; at a chosen place γ and $\varpi\gamma^{-1}$ are integral and $j-1$ is divisible by ϖ^2; at the conjugate places use that $\varpi\gamma^{-1}$ and $\bar{\varpi}\gamma$ are integral everywhere and that $j-1$ is divisible by $\varpi^2\bar{\varpi}$, because $j^{-1} = j^*$ is also 1 modulo ϖ; at each place one of ϖ, $\bar{\varpi}$ is 1. (b) No chosen place is conjugate to a chosen place, so C and C^* commute where needed; put $\varpi\gamma^{-1}$ equal to G at the chosen places, ϖC^* at their conjugates and ϖ elsewhere; then γ is unitary, by the relation $e\bar{e} = 0$ for the idempotent e of the chosen places. (c) Lift s to an integral t, take a Hermite form of t stacked over $\varpi \cdot 1$, and reduce: there are integral B, D with $BD = \varpi \cdot 1$ such that modulo ϖ^2 each of D and the stacked matrix w is a left multiple of the other. The *glue matrix* has the blocks D, $-D\iota_e\iota_z^\mathsf{T}$, 0 and the diagonal matrix that is ϖ on the range of z and 1 elsewhere; reading the equation by blocks, x is block normalised exactly when its rows z in m form 1 and the glue matrix takes x to $\varpi\iota_z$. The matrix C with blocks B, $\iota_e\iota_z^\mathsf{T}$, 0 and the diagonal matrix with ϖ off the range of z satisfies $C \cdot \text{glue} = \varpi \cdot 1$ (only the top right block needs an argument, the rows z of the diagonal block being ϖ times those of 1), so (b) gives a divided lattice with $\varpi\gamma^{-1} \equiv$ glue, after reordering the basis so that z goes to the indices of W. Right multiplication by $Y \in \bar{J}^1$ fixes $\varpi \cdot 1$ and does not change the congruence, and the rows z of x form Y^{-1} exactly when those of xY form 1; so xY is block normalised exactly when the rows z of x form Y^{-1} and $(\varpi\gamma^{-1})x \equiv \varpi(1;0)$, which gives existence and uniqueness of Y. An invertible K clears the columns of s outside the copy of a, and ϖ kills the rows ϖK, which gives the last condition. The indicator identity is the existence and uniqueness of Y. □

Lemma 34.8 (Descent of product forms). *Let the lattice γ be glued for s and let the characters b, b' of the two summands have product the character of V.*

(a) *The descent to W of the form supported on the class of $h\gamma$ with value v is, at g, the conjugate of c_γ times the theta kernel at (g, h) with polynomial the Fock polynomial of $\bar{v}$ and finite datum the function that is* 0 *unless $(\varpi\gamma^{-1})S \equiv \varpi\iota$ modulo ϖ^2 and is then the product over the places of Q of the residue characters at the determinant of the rows of W of S.*

(b) *The Fock polynomial of the Cartan product of y (weight μ^+ on V'') and η (weight μ^- on V°) is the product of the polynomial of y in the positive variables on the rows of V'' and that of η in the negative variables on the rows of V°.*

(c) *The descent of the product form on the class of* $\operatorname{diag}(h, h')\gamma$ *is, at g, the conjugate of c_γ times $\sum_{Y \in \bar{J}^1}$ of the product of the kernel of (W, V'') at $(g \operatorname{lift}(Y), h)$, with the column datum of the rows w and the polynomial of $\bar{y}$, and the kernel of (W, V°) at $(g \operatorname{lift}(Y), h')$, with the indicator of the inclusion of z and the polynomial of $\bar{\eta}$; and the same summed over the classes h of the row-fixing level.*

(d) *For a prime p, an embedding into $\mathbb{C}_p$ and $M \neq 0$ divisible by ϖ, the gauge of the Cartan product at* $\operatorname{diag}(h, h')\gamma$ *is at most $\|M\|^{-l_1}$ times the gauge of y at h and that of $\bar{\eta}$ at h', where the* denominator exponent l_1 *is $\sum_\tau (\#a \cdot \#\{\text{columns of } \mu_\tau^+ \text{ below } \#N\} + \#m \cdot \#\{\text{columns of } \mu_\tau^- \text{ below } \#m\})$.*

Proof. (a) Split off γ from the point $(g, h\gamma)$. Supported on the chosen places, it acts on the finite datum by the scalar c_γ and the substitution $y \mapsto \gamma^{-1}y$. The extension of a function on matrices modulo $\varpi^2 Q$ to adelic vectors (by 0 off the integral ones at the places of $\varpi^2 Q$) at $\gamma^{-1}x$ equals the extension of the function with support $(\varpi\gamma^{-1})T \equiv \varpi\iota$ at x: each side is 0 unless its vector is integral on the level, and since the square of the class of ϖ vanishes modulo ϖ^2, the entries of $\varpi^2\gamma^{-1}x$ vanish, which forces integrality of $\gamma^{-1}x$; by unitarity the conjugate of $\varpi\gamma^{-1}$ is $\bar{\varpi}\gamma^{\mathsf{T}}$, integral, as $\bar{\varpi}$ is 1 at a place of the level. The point γ is 1 at the places of Q, which do not change $\det\gamma$. (b) The product is read on the image of the mixed Cartan map, the tensor of the included vectors; the Schur matrix of an inclusion matrix acts on polynomials by renaming the rows of the variables, with no determinant factor, when the diagram has no column of a height between the two ranks. (c) Apply (a) on the left. On the right, join the two kernels by the rule for orthogonal sums and move the lift of Y into the finite datum; both sides are values of the bilinear form of the kernel, and the finite data and the polynomials agree by (b), Lemma 34.7(c), and the computation of the residue characters: if the rows S of xY in N satisfy $wS = w\iota$ modulo ϖ^2, then at the chosen places the rows e of S have determinant 1, and at the places of Q the lift of Y reduces to 1, so the product of the residue characters at the determinant of the rows of W of x is the character modulo $\varpi^2 Q$ of the determinant of the same rows of $x \operatorname{lift}(Y)$ divided by the residue characters at $\det \operatorname{lift}(Y)$. Summing over the classes is linear. (d) The Cartan product is an integral matrix applied to the tensor of two Schur images, one for each block, so its gauge is at most the product of theirs. For each block, lift the relevant change of columns to an integral k and write $h = h'k^{-1}$; the columns of $(g\gamma)^{-1}$ factor as $A\Delta h^{-1}$ with A integral and Δ diagonal, M^{-1} on a set S of columns and 1 elsewhere: A is integral because ϖA vanishes modulo the chosen places, the columns in S being multiples of M and ϖ times the others vanishing modulo ϖ^2. The Schur matrix of A does not increase norms; that of Δ is diagonal with entries M^{-1} to the number of entries of a filling in S, at most $\#S$ times the number of

columns below the height. On V'' only the columns in the copy of a cost a power of M; on V°, read through the conjugate embedding, every column may. □

Theorem 34.9 (Gluing). (a) *(Seesaw.)* $c'(\vartheta)\,\mathrm{glue}_\vartheta^*\Theta_{j,\vartheta} = \Theta^\circ$ *on good forms, and* $\langle \Theta^\circ y, \zeta\rangle = \sum_i c_i^\circ \int y\,\mathsf{F}_i\,\langle \zeta_i, \theta^\circ(g,\cdot)\rangle\,dg$, *with* θ° *the theta kernel of* (W_j, V°).

(b) Θ° *is injective.*

(c) glue_ϑ *keeps integrality at every prime not dividing* $\mathrm{N}\,Q$, *and* $\|\mathrm{glue}_1\zeta\|_\sigma^2 \le \exp(2fmb\log^2 b)\|\zeta\|_\sigma^2$.

(d) *The adjoint of the gluing takes a false lift to a tuple whose pairing with* $\theta^\circ(g,\cdot)$ *vanishes for every* g; *and* $\mathrm{glue}_\vartheta(\Theta^\circ y)$ *is orthogonal to the false lifts.*

Here $\mathrm{glue}_\vartheta\zeta$, for ζ in the copy i, is M^{l_1} times the sum over the classes h of level J_Λ and h° of level $J(\varpi^2)$ of the product forms on the classes of $(h,h^\circ)\mathsf{g}$, and $c_i^\circ = M^{l_1}c_\mathsf{g}\#\bar J^1 c''$. The level $J(\varpi^2)$ on V° and the level J_Λ on V'' (the integral points that are 1 modulo ϖ, fix Λ modulo ϖ^2, and modulo $\varpi^2 Q$ have the columns of 1 off W_j) are free: all stabilisers of classes are trivial (Proposition 16.2); so a tuple is the table of its values at representatives and inner products are plain sums. The lattice g is chosen once, at the trivial twist; the conditions on a glued lattice do not involve the twist, and its scalar does not depend on it.

Proof. (a) *The period as a sum of slices.* By the definition of the gluing, $\langle\zeta, \mathrm{glue}_\vartheta^*\varphi\rangle$ is M^{l_1} times the sum over the choices i, the classes c of $\mathrm{U}(V^\circ)$ with representatives h°, and the classes h of level J_Λ of the pairing of $\varphi''_{\vartheta,i}(h)\cdot\zeta(i,c)$ with $\varphi((h,h^\circ)\mathsf{g})$. *On a lift.* Let $\varphi = \Theta_{j,\vartheta}y$. The lift is $\int y(g)\theta(g,\cdot)\,dg$, and $\langle\theta(g,\cdot), \mathrm{glue}\,\zeta\rangle$ is the descent of glue ζ (Lemma 30.1(c)), so the pairing is the integral of y against the conjugate of the descent. By Lemma 34.8(c) the conjugate of the descent at g is the sum over i and $u\in\bar J^1$ of $M^{l_1}c_\mathsf{g}\,\Xi_{\vartheta,i}(gu)\langle\zeta_i,\theta^\circ(gu,\cdot)\rangle$: the characters satisfy $v''\cdot v^{m/2-j}$ = the character of V_j, because m is even and $k-1=r-m$. A good form is invariant under the lifts of the elements of $\bar J^1$, which lie in the level $J(\varpi)$; the measure is right invariant, so all u give the same integral, and the products are rapidly decreasing (finite combinations of kernels times a form). Hence

$$\langle\zeta, \mathrm{glue}_\vartheta^*\Theta_{j,\vartheta}y\rangle = \sum_i M^{l_1}c_\mathsf{g}\#\bar J^1\int y\,\Xi_{\vartheta,i}\,\overline{\langle\zeta_i,\theta^\circ(g,\cdot)\rangle}\,dg.$$

Multiply by $c'(\vartheta)$ and use Proposition 34.6(c): $c'(\vartheta)\Xi_{\vartheta,i} = c''\mathsf{F}_i$, free of the twist. The right side is then $\langle\zeta,\Theta^\circ y\rangle$ with Θ° defined at the trivial twist, which proves both claims.

(b) If $\Theta^\circ y = 0$, test against the tuple that is ζ in the copy i and 0 elsewhere: the copies other than i contribute nothing, and what is left is $c_i^\circ \ne 0$ times $\int y\,\mathsf{F}_i\,\langle\zeta,\theta^\circ(g,\cdot)\rangle\,dg$, which is therefore 0 for all i and ζ. Theorem 34.13 gives $y=0$. As a nonzero good form exists, $c'(\vartheta)\ne 0$.

(c) *Integrality.* All stabilisers are trivial, so glue ζ is M^{l_1} times a plain sum of forms supported on one class, and the gauge of such a form is that of its vector $\varphi''(h)\cdot\zeta(h^\circ)$; by Lemma 34.8(d) this is at most $\|M\|^{-l_1}$ times the gauges of the two factors, which are integral by Proposition 34.6(b) and the hypothesis. *Norm.* At the trivial twist the level J_Λ contains $J(\varpi^2)$, so the numbers of classes of $\mathrm{U}(V_j)$ and $\mathrm{U}(V'')$ are at most $\exp b$ (levels containing $J(\varpi^2)$ in rank at most

$2n_m$). A form supported on one class costs the order of a stabiliser, at most n_{Aut}, and the norm of the product vector is the product of the norms, the first at most $\exp(fmb\log^2 b)$. The norm of a sum is at most the number of terms times the bound, and $(M^{l_1})^2\exp(b)(n\exp(b)n_{\mathrm{Aut}})^2 \le \exp(fmb\log^2 b)$, where n counts the pairs of a choice and a class, by a comparison of the sizes of the data, in which l_1 is $3m$ times the number of columns of μ^+ plus $\operatorname{rk} V^\circ$ times the number of columns of μ^-, summed over the places, and the numbers of matrices, classes and selections of rows are counted.

(d) *Periods of false lifts.* Let Ξ be a false lift: in the image of e_0, killed by the Hecke polynomial operator at a split place off the level, and orthogonal to the lifts of good forms. Its descent vanishes: for every y, $\langle \Theta y, \Xi\rangle = \langle \Theta y, e_0\Xi\rangle = \langle e_0\Theta y, \Xi\rangle = 0$, as $e_0\Theta y$ is the lift of a good form. By Theorem 35.7, the class sum of the pairing of $C(k(h))$ with $\Xi((h_1,h)x)$ vanishes for every conjugate-linear equivariant C and every function k whose coordinates are multiples of conjugate theta kernels of (W_j, V°); take $C(\eta) = y\cdot\eta$ for y of weight μ^+, the point $(h,h^\circ)\mathsf{g}$, and the level $J(\varpi^2)$, which fixes the pairing because its conjugate by g fixes Ξ (Lemma 34.7(a)); the level is free, so the class sum is a plain sum, and the inner product with a form supported on one class is the pairing with the value there. Hence the slices of Ξ vanish, and the conjugate of the value of $\mathrm{glue}^*\Xi$, paired with the tuple of values of $\theta^\circ(g,\cdot)$, is a sum of slices: the kernel is linear in its polynomial, so expanding both in the basis of weight μ^- gives 0. *Orthogonality.* For a false lift Ξ, $\langle \Xi, \mathrm{glue}\,\Theta^\circ y\rangle = \langle \mathrm{glue}^*\Xi, \Theta^\circ y\rangle$, which by (a) is an integral against the products of the pairings of $\mathrm{glue}^*\Xi$ with the kernels $\theta^\circ(g,\cdot)$, and these vanish. □

34.3. Products span

For a real point h let $|h|$ be its largest operator norm and $\lambda_1(g)$ the first minimum of the lattice at g, cut off at 1. A *peak* of a bounded function is a point where its modulus exceeds half the supremum.

The archimedean tools concern the groups $\mathrm{U}(\mathrm{diag}\,s)$ of rank $1 \le n \le t$, the *basic holomorphic function* m_u of a weight u (a product of quotients of principal minors, one factor for each place), and orders o of the indices that put the signs $+1$ first. A weight is *K-dominant* for o if it decreases weakly along o within each sign, and has *gap* g if its entries at the signs -1 exceed those at the signs $+1$ by at least g. A function on the real group is *holomorphic of weight u* if it extends analytically to products with o-triangular matrices, on which it transforms by the character u.

Lemma 34.10 (Fock coefficients and sparse sums). (a) *Let P be a homogeneous polynomial of degree d with Fock vector v, let σ be unitary, $y \ne 0$, and let $A \ge d + \#\iota$ and n be natural numbers. Then*

$$|\langle v, \varrho(y)\omega(\sigma)v\rangle| \le (4(A+n))^n |y|^{-2n}\|\sigma\|^{\#\iota+2n}\langle v, v\rangle.$$

(b) *If $N \ne 0$ lies in the different of E, then $\lambda(S)\lambda(T)/(C\#n'\#n) \le \lambda(\mathrm{Hom}(S,T))$ with $C = 6N(f\#n)^2$.*

(c) *Let $c \ne 0$ and $m > 0$ satisfy $m|c|_w \le \sqrt{2\pi\,\mathrm{Im}\,j_w}\,\lambda$ at every infinite place. If $y \ne 0$ and cy is a lattice vector, the Fock coordinates of y have norm at least m; distinct such vectors are at distance at least m; and for q above the number of coordinates, $\sum |y|^{-2q} \le (5/m)^{2q}$.*

(d) (*Sparse sums.*) *Let V, V' have ranks at most t, Q homogeneous of degree $d \le b^{3/2}$, and $\lambda_1(g)\lambda_1(g') \ge b^{-13}$. Then for some datum Ψ modulo ϖ^2 the theta kernel with Ψ, Q at $(gh, g'h')$ differs from the Fock coefficient $c_Q(h, h')$ by at most $b^{-20n_*}(|h||h'|)^{6n_*}|c_Q(1,1)|$, for all real h, h'.*

Proof. (a) The coefficient is at most $\|\sigma\|^{\#\iota}\int\|v(x)\|\|v(\sigma x+y)\|\,dx$. Since $|y|^2 \le 2\|\sigma\|^2(|x|^2+|\sigma x+y|^2)$, the points x and $\sigma x+y$ are not both small. Dilating by $c=\sqrt{n/A+1}$ takes a Gaussian factor out of v, which is small at x or at $\sigma x+y$; by homogeneity of P, $1-c^{-2}=n/(n+A)$, and undoing the dilation costs $c^{2(d+\#\iota)} \le e^n$. The rest is bounded by $2ab \le a^2+b^2$, both terms having the same integral, as $x \mapsto \sigma x+y$ preserves the measure; by the reproducing property of the vacuum the integrand has the norm bounded. (b) Read a nonzero lattice vector z of $\mathrm{Hom}(S,T)$ as a matrix and apply it to a short basis of the lattice of S (Lemma 33.3 uses the same comparison): some image is a nonzero lattice vector of T, of length at least $\lambda(T)$ and at most $\#n'\#n\|z\|$ times the length of the basis vector. (c) cy is a nonzero lattice vector, of length at least the first minimum; compare coordinates; for distinct vectors apply this to the difference; the sum is the packing estimate of Lemma 33.3(c) with exponent $2q$ above the real dimension by at least 2. (d) Take for Ψ the mean of the trace characters $\psi_t=\psi(\mathrm{tr}(\cdot\,t))$ weighted by the conjugates of their theta kernels at (g,h); the unit scalars of the characters cancel against their conjugates, and the real points act on the Fock vector by the real Weil representation. The kernel of Ψ at (gu,hu') is then the mean over t of products of thetas, which by the computation of the pairing of trace data (the finite part of the datum of ψ_t is the vacuum translated by a finite vector; moving Heisenberg operators past it costs a phase, and the sum over t is the indicator that the finite coordinates vanish modulo the modulus) is a sum over rational vectors y of the lattice scaled by $c=q_1/\varpi^2$ of terms $\langle v,\varrho(\xi_y)\Omega(u,u')v\rangle$ times a phase. The term $y=0$ is the Fock coefficient. The scaled lattice points are b^6-separated: the first minimum of $\mathrm{Hom}(S,T)$ is at least $\lambda_1(g)\lambda_1(g')$ over a constant by (b), and $|c|$ is small. Each other term decays by (a), with $n=2n_*$; summing by (c) and collecting the constants $(100=4\cdot5^2)$ gives $(100(d+2q)/m^2)^q(|u||u'|)^{3q}|\langle v,v\rangle|$, which is at most the stated bound for the sizes of the data. □

Lemma 34.11 (Holomorphic coefficients). *Let $1 \le n \le t$, o put the signs $+1$ first, and u, u' be dominant with gap $300n_*$, u with entries at most $5b$.*

(a) *$l=2v+2\rho$ (with 2ρ the signed count for $-o$) is a doubled parameter of gap $500n^3+8n_*$, and its discrete series coefficient is the quotient m_{v+c}/m_c of products of principal minors, $c_i=2\sum|v_j|$ at a sign $+1$ and 0 otherwise.*

(b) *$\int|m_u|^2$ is the inverse of the product of the formal degrees, hence positive; $|m_{u'}(h)||h|^{6n_*} \le 2^{6n_*}\prod_\tau(2/\|h_\tau\|)^{10n^3}$, which is integrable with integral at most $b^{n_*}\int|m_u|^2$; and m_u is integrable.*

(c) *If A is continuous with $|A(h)| \le C|h|^{6n_*}$, then $|\int Am_{u'}| \le Cb^{n_*}\int|m_u|^2$. If W is bounded by M, and K, K' are within $\delta|h|^{6n_*}$ of K_0 with $\int WK'm_{u'}=0$, then $|\int Km_{u'}W| \le 2\delta Mb^{n_*}\int|m_u|^2$.*

(d) (*Isolation.*) *Let F on the adelic group be continuous, holomorphic of weight u in the real variable, with a peak, and K_0 bounded. If at every peak x,*

$\int \overline{F(xh)}K(h)m_{u'}(h)\,dh = 0$ *for some* K *holomorphic of weight* $u - u'$ *with* $|K - K_0| \le \delta|h|^{6n_*}$*, then* $|K_0(1)| \le (4b^{n_*}+1)\delta$.

Proof. (a) Dominance within the blocks and the gap make the entries of l distinct with the right parity and the stated gap; the coefficient of the holomorphic discrete series is then the quotient of minors (Appendix A.1). (b) At each place the factors of m_u and $m_{u'}$ are discrete series coefficients, so $\int|m_u|^2$ is the inverse product of formal degrees by Schur orthogonality. For the bound, use $6n_*$ more powers of decay at a place of largest norm: the tail $\deg(l)(2/\|g\|)^{10n^3}$ is integrable on $\mathrm{U}(\mathrm{diag}\, s)$ with integral at most $2^{13t^3}b^{14t^2}$, as $10n^3 > 8n^2$ is above the threshold for integrability and the constant of the tail bound is at most this for $L \le 10b + t$; and $2^{6n_*}(2^{13t^3}b^{14t^2})^f \le b^{n_*}$, with the formal degrees those of u. (c) The first is (b); for the second, orthogonality lets one replace K by the small function $K - K'$. (d) Rotate the peak z by a circle $\mathrm{rot}(\theta)$: all $z\,\mathrm{rot}(\theta)$ are peaks, so the hypothesis gives functions K_θ, all within $\delta|h|^{6n_*}$ of K_0. Replacing K_θ by the function $K_{(0)}$ of the angle 0 costs, by the second part of (c), at most $2\delta\sup|F|\,b^{n_*}\int|m_u|^2$ in each integral. Let W_θ be the conjugate of the quotient of the rotated translate of F by the character; as $h \mapsto F(zh)$ is holomorphic of weight u, the circle mean of W_θ is $\overline{F(z)m_u}$. Exchanging the integrals over θ and h and using the reproducing formula for the holomorphic function $K_{(0)}m_{u'}$ of weight u, whose value at 1 is $K_{(0)}(1)$, gives $\int|m_u|^2|F(z)||K_{(0)}(1)| \le 2\delta\sup|F|\,b^{n_*}\int|m_u|^2$. As $\sup|F| < 2|F(z)|$, $|K_{(0)}(1)| \le 4b^{n_*}\delta$, and $|K_{(0)}(1) - K_0(1)| \le \delta$. □

Lemma 34.12 (Weights on W_j and the carrier). (a) *The weight* u *of the conjugates of good forms (the highest weight of the lowest* K*-type of* -2ς*, for the target parameter* ς *of Section 15.3) is the sum of two staircase weights: one on* V'' *with the columns of* μ^+ *in sign* $+1$ *and shift* $\kappa_j - j - \lfloor(t+k-1)/2\rfloor$*, and the* kernel weight u° *on* V° *with the columns of* μ^- *in sign* -1 *and shift* $j - \lfloor m/2\rfloor$*. Both* u *and* u° *are* K*-dominant for the order* o_W *that lists the positive vectors first, in reversed order, and have gap* $300n_*$*; the entries of* u *are at most* $5b$.

(b) *The functions* F_i *are holomorphic of weight* $u - u^\circ$*; good forms have full level at every multiple of* ϖ *and are rapidly decreasing; the conjugate of a good form is holomorphic of weight* u*; and* F_i *is a multiple of the theta series of* (W_j, V'') *at the trivial twist, so* $y\mathsf{F}_i$ *is rapidly decreasing of level* ϖ^2.

(c) *The matrix coefficient* c_P *of the Fock vector of the padded polynomial on* $(S, \tilde{W})$ *satisfies* $c_P(h,1) = c_P(1,1)m_2(h)\overline{m_3(h)}$*, with* m_2*,* m_3 *the basic functions of* $(\Delta'; \#\mathrm{Det} + r + 2m)$ *and* $(0; \#\mathrm{Det} + r + 11m + 1)$*;* $c_P(1,1) \ne 0$ *and* c_P *is bounded;* P *is a polynomial in the variables of sign* 1 *of degree at most* $b^{3/2}$*. The flat weight* $u_1 = (-\Delta'; 6m+1)$ *and* u_3 *are dominant with gap* $300n_*$ *for every order,* u_3 *with entries at most* $5b$.

(d) *The carrier has a peak* g_1 *with* $\lambda_1(g_1) \ge (5b\sqrt{b})^{-1/2}$.

Proof. (a) By the formula for the lowest K-type as a staircase weight, u is one staircase weight for $V_j = I_{r+t}$, whose sum runs up to $r + t = 6r$; the sums on the right stop at the ranks of V'' and V°, and the missing terms vanish. On positive vectors u is a constant minus a staircase sum and on negative vectors a constant plus u°; each staircase sum decreases as its starting point grows, which gives dominance. At a negative vector the sum of the columns of μ^- above its

number is at least $300n_*$ (the columns of μ^- are at most Δ in each height and the heights fill the range), and the difference $u - u^\circ$ is larger at negative than at positive vectors; the bound on the entries is the bound on the parameter plus r. (b) F_i is a finite combination of kernels of (W_j, V'') with polynomials in the positive letters (Proposition 34.4(c) at the trivial twist), the definite group acts on these polynomials by a linear substitution, and the Fock vector of the highest vector is holomorphic of weight $u - u^\circ$, so all are. A good form is a descent (the lift is injective, so its adjoint composed with it is onto in finite dimension), the polynomial of the highest vector is a product of joint harmonics, hence holomorphic, and by equivariance so is the polynomial of every vector; the conjugate of the descent is a finite combination of theta kernels with such polynomials. The tame part of the characters is trivial off the modulus. (c) Each minor of P uses all the columns of one sign, so a substitution multiplies it by a principal minor; entry by entry, the weights are as stated, the character of the kernel having weight $m - \#\mathrm{Det}$ (Lemma 34.1(d)). The Fock representation is unitary, so c_P is bounded, and $c_P(1,1) = \langle v, v\rangle \neq 0$. Each variable of P has row and column of the same sign. (d) The carrier is automorphic of every level, rapidly decreasing and holomorphic (Proposition 34.4(a)); at the level $\sqrt{-d}$, of norm $\sqrt{d} \leq \sqrt{b}$, the cusp estimate (Theorem B.26(c)) bounds a form high in a cusp by half its supremum, so a peak has first minimum at least the stated threshold. □

Theorem 34.13 (Products span). *A good form y on W_j with $\int y(g)\mathsf{F}_i(g)\theta^\circ(g,\cdot)\,dg = 0$ for all choices i is zero.*

Proof. Suppose $y \neq 0$, and write $\tilde{W} = W_j \oplus I_j^- \oplus I_{3m-j}$.

Step 1: integrals of the carrier are combinations of the F_i. Let g_1 be a point of $\mathrm{U}(S)$, Ψ a function modulo ϖ^2 and $\theta[\Psi, P]$ the kernel of $(S, \tilde{W})$ with the characters v^m and $v^\flat v^m$. As a function of $g \in \mathrm{U}(W_j)$, the integral of $\varphi^\circ(g_1 h)\theta[\Psi, P](g_1 h, (g,1))m_{u_1}(h)$ over the real points h is a combination of the F_i. Indeed, the reproducing formula on $(S, W^\flat)$ turns the real integral against m_{u_1} into pairings with kernels of $(S, W^\flat)$: the translates of the power of one minor lie in the span of the products of determinants formed with the short bases of the classes. The product of $\theta[\Psi, P]$ with a kernel of $(S, W^\flat)$ is a kernel of $(S, \tilde{W} \oplus W^\flat)$, linear in its datum, and the product of Ψ with the datum is a combination of additive characters, which are the data defining F_i; the products with φ° are rapidly decreasing of level ϖ^2.

Step 2: the hypothesis on the real group. For every combination $\sum c_i\mathsf{F}_i$ and every adelic g, $\int y(gh)\sum_i c_i\mathsf{F}_i(gh)\,m_{u^\circ}(h)\,dh = 0$. By the reproducing formula on (W_j, V°) for the basis polynomials Q in the negative letters, whose span holds the translates of the highest vector, the real integral against m_{u° is a combination of pairings with kernels; the pairing of $y\sum c_i\mathsf{F}_i$ with a kernel is a combination of the pairings of the $y\mathsf{F}_i$, and each of these is a case of the hypothesis for a tuple supported at one class whose value there is the conjugate of a vector with the basis polynomial.

Step 3: approximation away from the cusps. Let g_1 be the peak of Lemma 34.12(d) and put $F_0(h) = \int \varphi^\circ(g_1 h')c_P(h', h)m_{u_1}(h')\,dh'$, a bounded function on the real group of W_j. By Lemma 34.12(c), the holomorphic factor of $c_P(\cdot, 1)$ times $h \mapsto \varphi^\circ(g_1 h)m_{u_1}(h)$ is holomorphic of the weight of m_3, and the reproducing

formula gives $F_0(1) = c_P(1,1)\varphi^\circ(g_1)J$ with $J = \int |m_{u_3}|^2 > 0$ (Lemma 34.11(b)), so $F_0(1) \neq 0$. If $\lambda_1(g) \geq (5b^{24})^{-1/2}$, then $\lambda_1(g_1)\lambda_1(g) \geq b^{-13}$ (extending g by 1 to the padding does not lower the first minimum, both summands being definite), so by Lemma 34.10(d) some kernel $\theta[\Psi, P]$ is within $b^{-20n_*}(|h'||h|)^{6n_*}|c_P(1,1)|$ of c_P. By Step 1 the same integral with this kernel is a combination F of the F_i, and by Lemma 34.11(c) the difference has the same factors $|\varphi^\circ(g_1)|$ and J, so

$$|\mathsf{F}(gh) - F_0(h)| \leq 2b^{-19n_*}|F_0(1)||h|^{6n_*},$$

using $b^{-20n_*}b^{n_*} = b^{-19n_*}$.

Step 4: isolation. The conjugate $\bar{y}$ is holomorphic of weight u (Lemma 34.12(b)) and has level $M\sqrt{-d}$, so by the cusp estimate its peaks have $\lambda_1 \geq (5b^{24})^{-1/2}$. At every peak x, Step 3 gives a combination F, holomorphic of weight $u - u^\circ$, with $h \mapsto \mathsf{F}(xh)$ within $\delta|h|^{6n_*}$ of F_0, $\delta = 2b^{-19n_*}|F_0(1)|$, and Step 2 gives $\int y(xh)\mathsf{F}(xh)m_{u^\circ}(h)\,dh = 0$. By Lemma 34.11(d) for $\bar{y}$, with $u' = u^\circ$, $|F_0(1)| \leq (4b^{n_*}+1)2b^{-19n_*}|F_0(1)|$. With $B = b^{n_*} > 10$, $(4B+1)2 < B^2 \leq B^{19}$, so $|F_0(1)| < |F_0(1)|$, which is absurd. □

34.4. The comparison operator

Let n_{Aut} be a common multiple of the orders of all stabilisers in the class sums and h_k the hook number of the diagram μ^- of the forms on V°.

Theorem 34.14 (Comparison operator). *For $1 \leq j \leq m$ the operators $U_\vartheta = n_{\mathrm{Aut}}h_k^2\,\mathrm{glue}_\vartheta\mathrm{glue}_\vartheta^*$ on the forms on V_j, one for each twist, satisfy:*

(a) *$\langle\Theta_\vartheta y', U_\vartheta\Theta_\vartheta y\rangle = \rho_\vartheta\langle y', U^\circ y\rangle$ on good forms, with $\rho_\vartheta = \Gamma_j(\vartheta)/\Gamma_j(1) \neq 0$ and U° positive definite and independent of ϑ;*

(b) *U_ϑ takes lifts to forms orthogonal to the false lifts;*

(c) *U_ϑ keeps integrality above every prime not dividing $\mathrm{N}\,Q$;*

(d) *at every automorphism σ of $\mathbb{C}$, U_1 multiplies squared norms by at most $\exp(6fmb\log^2 b)$.*

Lemma 34.15 (Tight frames). (a) *For each rational unitary γ of V° there is a frame of vectors η_T of weight μ^-, one for each filling T with increasing columns, with $\sum_T\langle\eta_T, \xi\rangle\eta_T = h_k^2\xi$ for all ξ, γ taking the conjugate of each η_T to an integer vector, and each of squared norm at most h_k^2 after every automorphism of $\mathbb{C}$.*

(b) *For rational points δ_c, one for each class, there are tuples f_a, indexed by a choice, a class and a filling, with $\sum_a\langle f_a, \zeta\rangle f_a = h_k^2\zeta$, δ_c taking the conjugate of the value of f_a at (i,c) to an integer vector, and squared norm at most h_k^2 at every automorphism.*

(c) *$U_\vartheta\varphi = \sum_a n_{\mathrm{Aut}}\langle\varphi'_a, \varphi\rangle\varphi'_a$ with $\varphi'_a = \mathrm{glue}_\vartheta f_a$; and the number of pairs of a choice and a class times $(\#\mathit{frame}\cdot n_{\mathrm{Aut}}h_k^2)^2$ is at most $\exp(2fmb\log^2 b)$.*

Proof. (a) Move an integral tight frame of the Weyl module of μ^- (vectors η_T^0 indexed by the fillings with increasing columns, with $\sum_T\langle\eta_T^0, \xi\rangle\eta_T^0 = h_k^2\xi$, integer coordinates and squared norms at most h_k^2) by the unitary map $\xi \mapsto \overline{\gamma^{-1}\bar{\xi}}$. (b) Put $f_{(b,T)}$ equal to the vector $\eta_{b,T}$ of the frame moved by δ_c at $b = (i,c)$ and 0 elsewhere; the metric on the values is the form of the Gram matrix, and the level is free, so inner products are sums over the pairs. (c) $\mathrm{glue}\,\mathrm{glue}^*\varphi =$

$h_k^{-2}\sum_a \langle \mathrm{glue}\, f_a, \varphi\rangle \mathrm{glue}\, f_a$ by the frame identity applied to $\zeta = \mathrm{glue}^*\varphi$. The count is Theorem 34.9(c) with the further factors, all at least 1. □

Proof of Theorem 34.14. (a) By Theorem 34.9(a), $\mathrm{glue}_\vartheta^* \Theta_\vartheta y = c'(\vartheta)^{-1}\Theta^\circ y$, so

$$\langle \Theta_\vartheta y', U_\vartheta \Theta_\vartheta y\rangle = n_{\mathrm{Aut}} h_k^2 |c'(\vartheta)|^{-2}\langle \Theta^\circ y', \Theta^\circ y\rangle.$$

By Proposition 34.6(c), $|c'(\vartheta)|^2 = C/\Gamma_j(\vartheta)$ with $C \neq 0$, as $c'(\vartheta) \neq 0$; hence

$$|c'(1)|^2/|c'(\vartheta)|^2 = \Gamma_j(\vartheta)/\Gamma_j(1) \neq 0.$$

Put $U^\circ = n_{\mathrm{Aut}} h_k^2 |c'(1)|^{-2}\Theta^{\circ *}\Theta^\circ$; it is independent of ϑ and positive definite, Θ° being injective (Theorem 34.9(b)).

(b) A lift is $\Theta_\vartheta y$, and $U_\vartheta \Theta_\vartheta y$ is a positive multiple of $\mathrm{glue}_\vartheta(\Theta^\circ y)$, which is orthogonal to the false lifts by Theorem 34.9(d).

(c) Take the frame of Lemma 34.15(b) at representatives integral above the prime ℓ. The forms $\varphi_a' = \mathrm{glue}_\vartheta f_a$ are integral at ι and at its conjugate (Theorem 34.9(c)), the gauge at a rational translate being the gauge of the translated vector. For an integral φ the coefficients $n_{\mathrm{Aut}}\langle \varphi_a', \varphi\rangle$ are integral, the inner product at a free level being a sum over the classes of pairings of integral vectors and n_{Aut} clearing the orders of the stabilisers. By Lemma 34.15(c), $U_\vartheta\varphi$ is a sum of integral multiples of integral forms.

(d) $U_1\varphi$ is glued from the tuple $\sum_a c_a f_a$ with $c_a = n_{\mathrm{Aut}}\langle \mathrm{glue}\, f_a, \varphi\rangle$. By Cauchy–Schwarz,

$$|\sigma c_a|^2 \leq n_{\mathrm{Aut}}^2 \|\mathrm{glue}\, f_a\|_\sigma^2 \|\varphi\|_\sigma^2 \leq n_{\mathrm{Aut}}^2 \exp(2fmb\log^2 b) h_k^2 \|\varphi\|_\sigma^2,$$

and the tuple $\sum c_a f_a$ has squared norm at most $(\#\mathrm{frame})^2 \max|\sigma c_a|^2 h_k^2$; gluing multiplies by at most $\exp(2fmb\log^2 b)$. With Lemma 34.15(c) the product of these factors is at most $\exp(6fmb\log^2 b)$. □

The comparison operator bounds the cost of one lift. Let pr be the orthogonal projection to the lifts and $\mathrm{pr}\, U_\vartheta$ the compression. Then $\det(\Theta_\vartheta^*\Theta_\vartheta)\det(\mathrm{pr}\, U_\vartheta) = C\rho_\vartheta^d$ with $C \neq 0$ independent of ϑ and d the dimension of the good forms: for an injective u and an operator T, $\det(\mathrm{pr}\, T)$ on the range of u times $\det(u^*u)$ is $\det(u^*Tu)$, because u^* kills the orthogonal complement of the range, and $u^*U_\vartheta u = \rho_\vartheta U^\circ$ by (a). The lifts lie in the image of e_0 and are killed by the Hecke polynomial at the root of every split place off the level (Theorem 30.5(d)), the lift is injective, and for a form orthogonal to the false lifts the projection to the lifts does not raise its norm at any σ. A determinant is the product of the eigenvalues; if an operator Λ on the lifts with $|\iota(\det\Lambda)| \geq \exp(-dr\beta)$ makes $\Lambda\,\mathrm{pr}\, y$ integral for every integral y orthogonal to the false lifts, then $\Lambda\,\mathrm{pr}\, U_\vartheta$ keeps the integral lifts by (b), (c), its eigenvalues are integral, and $|\iota\det(\mathrm{pr}\, U_\vartheta)| \leq \exp(dr\beta)$; at the trivial twist the conjugates of $\det(\mathrm{pr}\, U_1)$ are at most $\exp(3dfmb\log^2 b)$ by (d). These bounds give the height of $Z_j(1)/Z_j(\xi_\iota^j)$, which is $\det(\mathrm{pr}\, U_\xi)/\det(\mathrm{pr}\, U_1)$ times the d-th power of the inverse ratio of Hecke values; this is used in Section 36.

35. False lifts and Hecke values

Two further inputs enter the cost of one lift. The first is that the comparison operator takes lifts to forms orthogonal to the *false lifts*, the forms that look like lifts locally (they lie in the image of e_0 and satisfy the Hecke equation of a lift at one split place) but are orthogonal to the lifts: their periods over a complementary block vanish, by a tower of theta descents (Theorem 35.7). The second is that the squared norms of theta lifts of characters from U(1) are Hecke L-values; this turns the ratio ρ_ϑ of Theorem 34.14 into ratios of L-values and bounds the heights of their inverses through test forms (Theorem 35.19).

35.1. Point forms and false lifts

Let O be a descent pair with spaces W of rank r and $V = I_n$. For a split place w off the level, $H_w(X) = \sum_i (-X)^i q^{i(i-1)/2} T_{w,i}$ and X_* is the root of Theorem 30.5(d). For a central idempotent e of the group algebra of $\prod_{v\in M} \mathrm{GL}_r(\mathbb{F}_v)$ and a space Z of forms, the *false lifts* at w are the forms in eZ killed by $H_w(X_*)$; a form is *orthogonal to the false lifts* if it is so for every split w off the level. A *test form* at a prime ℓ is a form integral above ℓ and orthogonal to the false lifts, for e the idempotent of the labels $X-1$ and Z the kernel of the descent. A *point form* $\delta_{hA}(v)$ is supported on the class of hA with value v, for an element A of the integral level with prescribed columns at M and rows at Q. The projection to Z *has denominator at most* e^β at ι if an operator Λ on Z with $\log|\iota(\det\Lambda)| \geq -\dim Z \cdot r\beta$ makes $\Lambda(\mathrm{pr}\, y)$ integral at ι for every integral y orthogonal to the false lifts, with inner products of absolute value at most 1 against integral forms.

We make these notions precise. Let K be the level of O on V, c its character (Lemma 30.1(b)), ρ the representation of the weight of O and Γ the rational points of $\mathrm{U}(V)$. The inner product of two forms is the class sum

$$\langle \varphi, \varphi' \rangle = \sum_h \frac{\langle \varphi(h), \varphi'(h)\rangle}{\#(\Gamma \cap hKh^{-1})}$$

over representatives h of $\Gamma\backslash \mathrm{U}(V)(\mathbb{A}_\mathrm{f})/K$ of the Gram pairings of the values, conjugate linear in the first argument; the Gram matrix is integral in the basis of standard fillings. For an automorphism σ of $\mathbb{C}$ let $\sigma\varphi(h)$ be the vector of the σ-images of the coordinates of $\varphi(h)$, and put

$$\langle \varphi, \varphi' \rangle_\sigma = \sigma^{-1}\Big(\sum_h \frac{\langle \sigma\varphi(h), \sigma\varphi'(h)\rangle}{\#(\Gamma \cap hKh^{-1})}\Big), \qquad \|\varphi\|_\sigma^2 = \sum_h \frac{\|\sigma\varphi(h)\|^2}{\#(\Gamma \cap hKh^{-1})};$$

the pairing is semilinear for $\tau_\sigma = \sigma^{-1} \circ \mathrm{conj} \circ \sigma$ in its first argument, and for $\sigma = 1$ these are the inner product and the squared norm. The *class form* $\delta_h(v)$ of a finite point h and a vector v of the weight is the form whose value at y is the sum of $c(h^{-1}\gamma y)\rho(\gamma)^{-1}v$ over the rational γ with $h^{-1}\gamma y \in K$; it is supported on the class of h. A *point lift* of matrices Y_i of full column rank over the residue fields at the chosen places and of r distinct rows e is an integral point A of $\mathrm{U}(V)(\mathbb{A}_\mathrm{f})$ with component 1 off the places of $M \cup Q$ and their conjugates, whose reduction at each chosen place has the columns Y_i at the embedded indices and whose reduction at each place of Q has, in the rows e, the unit vectors of the embedded indices; the point form is $\delta_{hA}(v)$. Its *point datum* is the function of families X of matrices at the places of O that is $\prod_{w\in Q} \psi_w(\det X_{w,e})$, with $X_{w,e}$ the rows e of X_w, if X equals Y at every chosen place, and 0 otherwise. In (c)

of Proposition 35.3 and in the definition of the denominator, the false lifts are formed with the space $Z^{\perp}$.

Lemma 35.1 (Point lifts and class forms). (a) *Point lifts exist for all Y and e as above.*

(b) *For every form φ, $\langle \delta_h(v), \varphi\rangle = \langle v, \varphi(h)\rangle$.*

(c) *The normalised descent of $\delta_h(v)$ at $g \in \mathrm{U}(W)(\mathbb{A})$ is the complex conjugate of the theta kernel at (g,h) of the datum $v_{P_{\bar v}} \otimes \Psi_{\mathrm{bl}} \otimes \mathrm{vac}$, twisted by the characters of O.*

(d) *For a point lift A the block datum read at $A^{-1}X$ is the point datum at X. Hence the normalised descent of the point form $\delta_{hA}(v)$ at g is the conjugate of the theta kernel at (g,h) of the datum with the Fock vector of $P_{\bar v}$ and the point datum read on the reductions at the places of O.*

Proof. (a) At a chosen place complete the r columns Y_i to an invertible matrix A_i with these columns at the embedded indices. At a place of Q the rows e are distinct, so a permutation matrix B_i has the unit vectors of the embedded indices as its columns e; put $A_i = B_i^{\mathsf T}$, whose rows e are these unit vectors. At a split place the integral points of $\mathrm{U}(V)$ are GL_n of the local integers, which maps onto GL_n of the residue field, and a point of $\mathrm{U}(V)$ at the pair $\{w, \bar w\}$ is determined by its component at w. No place of O is conjugate to itself or to another place of O, so the family (A_i) is the reduction of an integral point that is 1 at all other places.

(b) The function $y \mapsto \langle \delta_h(v)(y), \varphi(y)\rangle$ is left invariant under Γ and right invariant under K, because both forms transform under K by the character c of modulus one and the Gram form is invariant under ρ. It vanishes off the class of h, so the inner product is its value at h divided by $\#(\Gamma \cap hKh^{-1})$. At $y = h$ the sum runs over $\gamma \in \Gamma \cap hKh^{-1}$; with $k = h^{-1}\gamma h \in K$ one has $\rho(\gamma)\varphi(h) = \varphi(\gamma h) = \varphi(hk) = c(k)\varphi(h)$, so each term is

$$\langle c(k)\rho(\gamma)^{-1}v, \varphi(h)\rangle = \overline{c(k)}\,\langle v, \rho(\gamma)\varphi(h)\rangle = |c(k)|^2\langle v, \varphi(h)\rangle = \langle v, \varphi(h)\rangle.$$

There are $\#(\Gamma \cap hKh^{-1})$ terms, which gives the claim.

(c) By Lemma 30.1(c) the descent at g is $\langle \theta_W(g,\cdot), \delta_h(v)\rangle = \overline{\langle \delta_h(v), \theta_W(g,\cdot)\rangle}$, and by (b) this is $\overline{\langle v, \theta_W(g,h)\rangle} = \langle \theta_W(g,h), v\rangle$, which by Lemma 30.1(a) is the conjugate of the theta kernel of $\varphi_W(\bar v)$ at (g,h).

(d) At a chosen place the columns of A_i at the embedded indices are Y_i, so $A_i^{-1}X_i$ is the inclusion matrix exactly when $X_i = Y_i$. At $w \in Q$ the row $e(j)$ of A_w is the unit vector of the embedded index $\iota(j)$, that is $e_{e(j)}^{\mathsf T} A_w = e_{\iota(j)}^{\mathsf T}$, hence $e_{\iota(j)}^{\mathsf T} A_w^{-1} = e_{e(j)}^{\mathsf T}$: the embedded rows of $A_w^{-1}X_w$ are the rows e of X_w. Hence the block datum at $A^{-1}X$ is the point datum at X. The kernel at (g, hA) of a datum f is the kernel at (g,h) of $\omega(1,A)f$, and the finite point A, integral and supported on the places of O, acts on the finite part of the datum by $X \mapsto f(A^{-1}X)$ times the value of the character of O on the side of W at $\det A$, which is 1 since that character has trivial tame part. With (c) this is the claim. □

Lemma 35.2 (Pairings, central idempotents and Hecke polynomials). *Let σ be an automorphism of $\mathbb{C}$.*

(a) *$\langle\cdot,\cdot\rangle_\sigma$ is the restriction to the forms of level K of a pairing of smooth forms of the weight that is invariant under right translation by $\mathrm{U}(V)(\mathbb{A}_{\mathrm f})$.*

(b) *The action of* $\prod_{v\in M}\mathrm{GL}_r(\mathbb{F}_v)$ *on forms (Proposition 30.2(a)) preserves* $\langle\cdot,\cdot\rangle_\sigma$. *A central idempotent* e *of its group algebra is self-adjoint for* $\langle\cdot,\cdot\rangle_\sigma$, *and an operator that commutes with the action preserves the image of* e.

(c) *The operators* $H_w(X)$ *and* $H_u(Y)$ *at split places* w, u *off the level commute, also when* $u = w$ *or* $u = \bar{w}$.

(d) $\|\varphi\|_\sigma^2 = \mathrm{Re}\,\sigma(\langle\varphi,\varphi\rangle_\sigma)$; *in particular* $\langle\varphi,\varphi\rangle_\sigma = 0$ *only for* $\varphi = 0$. *If* $\langle x, z\rangle_\sigma = 0$, *then* $\|z\|_\sigma^2 \le \|z + x\|_\sigma^2$.

Proof. (a) Let G_σ be the Gram form transported by σ: $G_\sigma(u,u') = \sigma^{-1}\langle\sigma u, \sigma u'\rangle$. It is invariant under the rational points acting through ρ, because the Gram form is invariant under the σ-conjugates of the rational points, which act on the coordinates by matrices over the field generated by the entries of $\rho(\gamma)$. The pairing of smooth forms of the weight given by the integral of G_σ of the values over $[\mathrm{U}(V)]$, times the index of K, restricts to $\langle\cdot,\cdot\rangle_\sigma$ by the passage from class sums to integrals over the quotient, and it is invariant under right translation.

(b) An element s acts as the root of unity $\prod_w \psi_w(\det s_w)$ times right translation by an integral point. Right translation preserves the pairing by (a), and a root of unity ζ satisfies $\tau_\sigma(\zeta)\zeta = \sigma^{-1}(\overline{\sigma\zeta}\,\sigma\zeta) = 1$; so the action is unitary for $\langle\cdot,\cdot\rangle_\sigma$, and the adjoint of s is s^{-1}. A central idempotent of the group algebra of a finite group is a sum of the minimal ones $e_\chi = \frac{\chi(1)}{\#G}\sum_g \chi(g^{-1})g$ for irreducible characters χ, whose values are sums of roots of unity; τ_σ inverts roots of unity, so $\tau_\sigma(\chi(g^{-1})) = \chi(g)$, and the adjoint $\sum_g \tau_\sigma(c_g)\,g^{-1}$ of $e = \sum_g c_g g$ is e again. Finally e is a combination of the operators of the action, so an operator commuting with them commutes with e and preserves its image.

(c) The forms of level K are spherical at w and at u, and on spherical vectors the Hecke operators at two split places off the level commute: at distinct pairs of places they are translations at different components, and at one pair $\{w,\bar{w}\}$ the spherical Hecke algebra of GL_n over the local field is commutative (Appendix B.16). The operators H are polynomials in them.

(d) The first statement is the definition: $\sigma\langle\varphi,\varphi\rangle_\sigma$ is the class sum of the squared norms $\langle\sigma\varphi(h),\sigma\varphi(h)\rangle$, a positive real number unless $\varphi = 0$. If $\langle x,z\rangle_\sigma = 0$, then $\langle z,x\rangle_\sigma = \tau_\sigma\langle x,z\rangle_\sigma = 0$, and applying σ to $\langle z+x, z+x\rangle_\sigma$ gives $\|z+x\|_\sigma^2 = \|z\|_\sigma^2 + \|x\|_\sigma^2 \ge \|z\|_\sigma^2$. □

Proposition 35.3 (Point forms and projections). (a) *Let* φ *be a form. Its inner product with the point form* $\delta_{hA}(v)$ *is the Gram pairing of* v *with the value* $\varphi(hA)$. *The finite datum of the descent of a point form is the indicator of* $T = Y$ *at the chosen places times the product of the characters* $\psi_w(\det T_e)$ *over* $w \in Q$.

(b) *The adjoint of* $H_w(X)$ *for the* σ*-conjugate pairing is* $H_{\bar{w}}(X')$, $X' = \sigma^{-1}(\overline{\sigma X})$; *the image of* $H_w(X)$ *is orthogonal to its kernel.*

(c) *If* Z *lies in the image of a central idempotent and is killed by* $H_w(X_*)$, *then on forms orthogonal to the false lifts the orthogonal projection to* Z *does not increase the squared norm at any* σ.

Proof. (a) is Lemma 35.1(b) for the point hA, and Lemma 35.1(d).

(b) *The adjoint.* By Lemma 35.2(a) the pairing is the restriction of a pairing B of smooth forms invariant under right translation, semilinear for τ_σ in the first argument. The operator $T_{w,i}$ is the sum of the right translations by representatives t of the cosets in $K_w t_{w,i} K_w$, $t_{w,i} = \mathrm{diag}(\varpi_w 1_i, 1)$ at w. For an invariant pairing,

$B(\varphi t, \varphi') = B(\varphi, \varphi' t^{-1})$, and the inverses of the representatives represent the cosets of $K_w t_{w,i}^{-1} K_w$. A point of $\mathrm{U}(V)$ at the pair $\{w, \bar{w}\}$ is determined by its component at w, its component at $\bar{w}$ being the inverse of the conjugate transpose; so $t_{w,i}^{-1}$, read at $\bar{w}$, is $\mathrm{diag}(\bar{\omega}_w 1_i, 1)$, and $K_w t_{w,i}^{-1} K_w$ is the double coset of $T_{\bar{w},i}$. Hence $B(T_{w,i}\varphi, \varphi') = B(\varphi, T_{\bar{w},i}\varphi')$ on spherical forms. The coefficient $(-X)^i q^{i(i-1)/2}$ in the first argument becomes τ_σ of it in the second, which is $(-X')^i q^{i(i-1)/2}$ since q is an integer. Hence the adjoint of $H_w(X)$ is $H_{\bar{w}}(X')$. *Orthogonality.* Write $A = H_w(X)$ and $A' = H_{\bar{w}}(X')$. Since τ_σ is an involution, the adjoint of A' is A, and by Lemma 35.2(c) the two commute. If $Az = 0$, then

$$\langle A'z, A'z\rangle_\sigma = \langle z, AA'z\rangle_\sigma = \langle z, A'Az\rangle_\sigma = 0,$$

so $A'z = 0$ by Lemma 35.2(d), and $\langle Av, z\rangle_\sigma = \langle v, A'z\rangle_\sigma = 0$ for every v.

(c) Let e be the central idempotent, $A = H_w(X_*)$, and let the false lifts be formed with $Z^\perp$. *Decomposition.* For $\sigma = 1$ the orthogonal complement of $\ker e + \mathrm{im}\, A$ is $\mathrm{im}\, e \cap \ker A$: by Lemma 35.2(b) e is self-adjoint, so $(\ker e)^\perp = \mathrm{im}\, e$, and by (b) $(\mathrm{im}\, A)^\perp = \ker A' = \ker A$. A form orthogonal to $Z + \ker e + \mathrm{im}\, A$ is therefore in $Z^\perp \cap \mathrm{im}\, e \cap \ker A$, a false lift at w. If y is orthogonal to the false lifts, it is orthogonal to the orthogonal complement of $Z + \ker e + \mathrm{im}\, A$, so, the space of forms being finite dimensional, $y = z + x$ with $z \in Z$ and $x \in \ker e + \mathrm{im}\, A$. *Orthogonality at every σ.* The space Z lies in $\mathrm{im}\, e \cap \ker A$. The kernel of e is orthogonal to its image for $\langle\cdot,\cdot\rangle_\sigma$, by self-adjointness; and the image of A is orthogonal to its kernel for $\langle\cdot,\cdot\rangle_\sigma$, by (b). Hence $\langle x, z\rangle_\sigma = 0$ for every σ. For $\sigma = 1$ this says that z is the orthogonal projection of y to Z, and Lemma 35.2(d) gives $\|z\|_\sigma^2 \le \|y\|_\sigma^2$ for every σ. □

35.2. The tower of descents

Theta descents of a function Y on $\mathrm{U}(V)(\mathbb{A})$ to $\mathrm{U}(W)(\mathbb{A})$ *vanish* for a character a and a set S of places if they vanish for all data that are the vacuum at the places of S and their conjugates. The *hyperplane sums* at a chosen place are $\sum \chi_w(\det s)^{-1} s$ over the invertible $s = 1 + u\lambda_w$. A matrix x has an *integral block ratio* if its rows on the embedded indices form an invertible block x' with $x''x'^{-1}$ integral, and $\equiv 0$ at the chosen places.

Here the descent of Y to W by a datum ϕ is the function

$$g \mapsto \int_{[\mathrm{U}(V)]} \theta(\omega(g,h)\phi)\, a(\det h)\, Y(h)\, dh,$$

and for a form Ξ on V and a linear form u on the weight, the descents of Ξ are those of the conjugate of its matrix coefficient $h \mapsto u(\rho(h_\infty^{-1})\Xi(h_{\mathrm{f}}))$ (Section 15.6); the set S of the definition is the complement of the places of $M \cup Q$, so the data are the vacuum off the level. For an integral row $l = (\lambda, \lambda')$ at a place w of O, with λ the entries at the embedded indices, the *hyperplane group* is the group of the invertible $1 + v\lambda$ over $\mathbb{F}_w$, v running over the columns. A row l *annihilates* x at w if l is integral, $\lambda x' + \lambda' x'' = 0$, $\lambda \not\equiv 0 \pmod{w}$, and $\lambda' \equiv 0 \pmod{w}$ unless w is chosen. The *shear* of a column v and a row l is the matrix $1 + (v, 0)^{\mathsf{T}} l$, which is block triangular with first block $1 + v\lambda$ and determinant $\det(1 + v\lambda)$.

Lemma 35.4 (Annihilating rows and hyperplane sums). *Let w be a place of O with residue field $\mathbb{F}_w$ and x an $n \times r$ matrix over the local field at w.*

(a) *If x has no integral block ratio at w, some row annihilates x at w.*

(b) *If $\lambda \neq 0$ and $\chi \neq 1$ is a character of $\mathbb{F}_w^\times$, then $\sum_s \chi(\det s)^{-1} = 0$ over the hyperplane group of λ; for $\lambda = 0$ the hyperplane sum is* 1.
(c) *If the products of the hyperplane sums of rows λ_w at the chosen places, not all zero, kill a form y, then the hyperplane sum of one row $l \neq 0$ at one chosen place kills y.*
(d) *If l annihilates x at w and $s = 1 + v\lambda$ is invertible, there is an integral point h, equal to the shear of a lift of v and l at w and to* 1 *at the other places of O, such that $hx = x$ and the right translate of every form y of level K by h is $\chi_w(\det s)^{-1}\bar{s}\, y$; here $\bar{s}$ is s, acting as in Proposition 30.2(a), if w is chosen, and* 1 *if $w \in Q$.*

Proof. (a) Let $|\cdot|$ be the absolute value at w. If x' is singular, take $\lambda \neq 0$ with $\lambda x' = 0$ and $\lambda' = 0$, and divide by an entry of λ of largest absolute value. Otherwise x' is invertible and some entry of $x''x'^{-1}$, in a row a, has absolute value > 1, or ≥ 1 if w is chosen. The row a of the matrix $(x''x'^{-1}, -1)$ annihilates x, since $x''x'^{-1}x' - x'' = 0$; its entries in the first block include one of absolute value at least that of the entry -1 in the second block, strictly larger unless w is chosen. Dividing by an entry of largest absolute value, which lies in the first block, gives an integral row with $\lambda \not\equiv 0$ and, unless w is chosen, $\lambda' \equiv 0$.

(b) For $\lambda \neq 0$ the map $s = 1 + v\lambda \mapsto \det s = 1 + \lambda v$ is a homomorphism of the hyperplane group onto $\mathbb{F}_w^\times$, as λv takes every value; so $\chi^{-1} \circ \det$ is a nontrivial character of the group, and its sum vanishes. For $\lambda = 0$ the group is $\{1\}$.

(c) Take l at the given place and the zero row at the other chosen places, whose hyperplane sums are 1 by (b).

(d) Let $\tilde{v}$ be an integral lift of v and h_w the shear of $\tilde{v}$ and l; then $h_w x = x + (\tilde{v}, 0)^{\mathsf{T}}(lx) = x$. Its determinant reduces to $\det s \neq 0$, so h_w is integral and invertible, and there is an integral point h with component h_w at w and 1 at the other places of O (as in Lemma 35.1(a)). At a chosen place h_w reduces to a block upper triangular matrix with first block s; at a place of Q, where $\lambda' \equiv 0$, it reduces to a block lower triangular matrix with first block s. In both cases the reduction lies in the level after multiplication by $\operatorname{diag}(s, 1)^{-1}$ at a chosen place and directly at Q, where the level character takes the value $\chi_w(\det s)^{-1}$; with the definition of the action (Proposition 30.2(a)) this gives the right translate. □

Lemma 35.5 (Small Hecke roots). *Let U be a hermitian space of rank m and w a split place of norm q.*

(a) *A bounded function F on $\mathrm{U}(U)(\mathbb{A})$, spherical at w and killed by $H_w(X)$ with $q^{m+1}|X| \leq 1$, is zero.*
(b) *$\prod_{i<k}(1 - q^i X) \neq 0$ if $q^k|X| \leq 1$.*
(c) *Let Y be automorphic on $\mathrm{U}(V)$, $\operatorname{rk} V = m + k$, spherical at w and killed by $H_w(X)$; let the character χ be unramified at $\bar{w}$ and the datum ϕ on $\operatorname{Hom}(U, V)$ the vacuum at $\bar{w}$. If $\sqrt{q}^{\,m+2k}|X| \leq 1$, $\sqrt{q}^{\,2(m+1)+k}|X| \leq 1$ and the descent of Y to U by ϕ is bounded, it is zero.*

Proof. (a) The relation $H_w(X)F = 0$ reads $\sum_{U'}(-X)^{\dim U'} q^{\binom{\dim U'}{2}} F(g\, t_{U'}) = 0$ for all g, the sum over the subspaces U' of $\mathbb{F}_w^m$ with coset representatives $t_{U'}$, the

term $U' = 0$ having coefficient 1 and $t_0 = 1$. By the q-binomial theorem the total weight is

$$\sum_{U'} |X|^{\dim U'} q^{\binom{\dim U'}{2}} = \prod_{j<m} (1 + q^j|X|) \le \exp\Big(|X| \sum_{j<m} q^j\Big) \le \exp\big(q^m|X|\big) \le e^{1/2} < 2,$$

since $\sum_{j<m} q^j \le q^m$ and $q^m|X| \le 1/q \le 1/2$. Isolating the term $U' = 0$ at the point g gives $|F(g)| \le \theta \sup|F|$ with $\theta = e^{1/2} - 1 < 1$; taking the supremum over g, $\sup|F| = 0$.

(b) A vanishing factor gives $q^i|X| = 1$ with $i < k$, hence $q^k|X| = q^{k-i} > 1$.

(c) The descent F is spherical at w, and the unramified correspondence (Appendix B.16), as in Proposition 30.2(c) with the roles of the two spaces exchanged, gives $\prod_{i<k}(1-q^iX_0)\cdot H_w(X')F = 0$ with $X_0 = \alpha X$ and $X' = q^k\alpha\beta^{-1}X$, where α, β are the Frobenius values at $\bar{w}$ of the characters on U and on V. The Frobenius value in rank j has absolute value $\sqrt{q}^{\,j}$, so $|X_0| = \sqrt{q}^{\,m}|X|$ and $|X'| = q^k\sqrt{q}^{\,m}\sqrt{q}^{\,-m-k}|X| = \sqrt{q}^{\,k}|X|$. The first size condition is $q^k|X_0| \le 1$, so the product is nonzero by (b) and $H_w(X')F = 0$; the second is $q^{m+1}|X'| \le 1$, so $F = 0$ by (a). □

Lemma 35.6 (The zero term and the rest of a kernel). *Let $W' = W \oplus H$ split one hyperbolic plane off W', with isotropic vector k and a splitting whose adelic frames are orthonormal at the infinite places and integral at the finite ones, and let θ_0 be the zero term of the theta functional of* $\mathrm{Hom}(W', V)$ *along the isotropic frame.*

(a) *If the descents of Y to W for the character a vanish for the data that are the vacuum at S, then for every such datum ϕ on* $\mathrm{Hom}(W', V)$ *and every $g \in \mathrm{U}(W')(\mathbb{A})$ the integral of Y against the zero term of the kernel of ϕ at $(g, \cdot)$, twisted by a, is zero.*

(b) *Let $\phi = v_P \otimes \Psi \otimes \mathrm{vac}$ with Ψ Schwartz–Bruhat at a set of finite places disjoint from its conjugates. After a rational translation of g, the kernel of ϕ minus its zero term is bounded uniformly in $g \in \mathrm{U}(W')(\mathbb{A})$ and $h \in \mathrm{U}(V)(\mathbb{A})$, V definite.*

(c) *Under the hypothesis of (a), for V definite and Y automorphic, the descent of Y to W' by a datum as in (b) is bounded.*

Proof. (a) By the Iwasawa decomposition $g\kappa = u\,(t, g_0)$ with κ in the maximal compact subgroup, u in the unipotent radical of the parabolic of k and (t, g_0) in its Levi subgroup, $g_0 \in \mathrm{U}(W)(\mathbb{A})$. The translate of ϕ by κ^{-1} is again the vacuum at S, and so is its slice along the isotropic frame, a datum on $\mathrm{Hom}(W, V)$: the frames are integral at the finite places and the vacuum is the product of local vacua. On the slice the unipotent element acts as the identity and the Levi element acts by a scalar $\nu(t)$ times $\omega_{W,V}(g_0, h)$. Hence the integral is $\nu(t)$ times the descent of Y to W at g_0 by the slice, which is zero.

(b) The theta functional minus its zero term is invariant under the translations by the rational maps $zk^{\mathsf{H}}J$, which lie in the span of the isotropic frame of $\mathrm{Hom}(W', V)$, and at ϕ it is the mean over b of $\theta\phi - \theta(\varrho(k_{\mathbb{A}}b)\phi)$. Let the height of g be the inverse of the first minimum, cut off at 1, of W' moved by g. If it exceeds 2, Witt's theorem gives a rational γ carrying a short isotropic vector at g to k, so that k lies in the lattice of the space moved by γg and has length at most 2 divided by the height there. The functional $\theta - \theta_0$ grows at most like a power of the height; along the short vector k the datum decays faster than

any power, which cancels the growth. On the definite space V the height is one, and a basis of V in the lattice at h has bounded lengths. The kernel at γg is the kernel at g, γ being rational.

(c) The descent at γg equals that at g. By (a) the zero term contributes nothing, so the descent is the integral of Y against the kernel minus its zero term. The quotient $[\mathrm{U}(V)]$ is compact for definite V, so the continuous function Y is bounded, and by (b) the integrand is bounded uniformly in g. □

Theorem 35.7 (Periods of false lifts vanish). *Let the parameter of W fit O, let $\chi_w \neq 1$ for $w \in Q$, and let a form Ξ on I_n be killed by the hyperplane sums at the chosen places, by the descent to W and by $H_w(\mathrm{X}_*)$ at a split place w off the level.*

(a) (*Bottom of the tower.*) *The descents of the matrix coefficients of Ξ vanish for all data that are the vacuum off $M \cup Q$ and their conjugates.*

(b) (*Hyperbolic planes.*) *Let Y be automorphic on the definite $\mathrm{U}(V)$, $\operatorname{rk} V = n$, let A be a set of finite places disjoint from its conjugates, and let the descents of Y to W, $r = \operatorname{rk} W$, for a character a vanish for the data that are the vacuum off A and its conjugates. Let w be a split place of norm q off these places, with a unramified at $\bar{w}$, and let Y be spherical at w and killed there by the Hecke polynomial at a point X with $|X| = q^{-(n-1-r/2)}$. If $2r + 2N + 3 < n$, the descents of Y to $W \oplus H^N$ for $a\upsilon^N$ vanish for the same data.*

(c) *If $2r + 2N + 3 < n$, the descents to $W \oplus H^N$ vanish; and for $I_n = I_N \oplus I_l$, a finite point x supported on the places of O and any function on the finite points of $\mathrm{U}(I_l)$ whose coordinates are multiples of conjugate theta kernels of (W, I_l) with data that are the vacuum off the level, carried to the values of Ξ by a conjugate linear map equivariant for the real points, the class sum of its pairing with $\Xi((h_1, h)x)$ vanishes at every level that fixes the pairing.*

Proof. (a) Fix u, a polynomial P and g, and write $B(\Psi, y)$ for the conjugate of the value at g of the descent of the matrix coefficient u of y by $v_P \otimes \Psi \otimes \mathrm{vac}$, Ψ a Schwartz–Bruhat function of the components at the places of O; it is linear in Ψ and conjugate linear in y. Put $y = \Xi$.

All polynomials. The normalised descent of Ξ is a nonzero multiple of the conjugate of the descent F by the datum of the highest vector (Proposition 15.11(c)), so $F = 0$. The descent by $v_P \otimes \Psi_{\mathrm{bl}} \otimes \mathrm{vac}$ is F convolved with a test function on the real points of $\mathrm{U}(W)$, the real points acting on the Fock vectors, times a character of $\det g$ coming from the character on the side of V. Hence $B(\Psi_{\mathrm{bl}}, \Xi) = 0$ for all P, and, translating, $B(\Psi_{\mathrm{bl}}(\cdot A), \Xi) = 0$ for every finite point A of $\mathrm{U}(W)$ that is 1 off the places of O and their conjugates, since $\omega(A, 1)$ multiplies $\Psi_{\mathrm{bl}} \otimes \mathrm{vac}$ by a nonzero number (a character value at $\det A$ times a power of its idele norm) and replaces Ψ_{bl} by $\Psi_{\mathrm{bl}}(\cdot A)$.

Two rules for B. For $k \in K_0$ and the right translate y' of y by k, $B(\Psi(k^{-1}\cdot), y') = B(\Psi, y)$, by equivariance of the theta kernel under $(1, k)$, which moves Ψ and fixes v_P and the vacuum. For $h \in K$ this gives $B(c\Psi(h^{-1}\cdot), y) = B(\Psi, y)$ with c the conjugate of $c(h)$, since y transforms by $c(h)$.

Small classes. The set of Ψ with $B(\Psi, \Xi) = 0$ is a subspace. Let $\mathbf{1}_{x,i}$ be the indicator of the class of x modulo the idele $i!$ at the places of O. Two such classes are nested or disjoint, and a Schwartz–Bruhat function is constant on the class of x modulo $i!$ for i large (a multiple $N\hat{O}^{n\times r}$ of the standard lattice lies in its neighbourhood of constancy, and $N \mid i!$ for $i \geq N$). Finitely many such classes

cover the support of Ψ, and removing a largest class repeatedly writes Ψ as a combination of their indicators. Hence it suffices to show $B(\mathbf{1}_{x,i}, \Xi) = 0$ for all x and large i.

Points with an integral block ratio at every place of O. Let A be the point of $\mathrm{U}(W)$ with components x'^{-1}_w at the places of O and $\Psi_0 = \Psi_{\mathrm{bl}}(\cdot A)$. At each place of O the components of xA are $(1, x''x'^{-1})^{\mathsf{T}}$, integral and $\equiv (1,0)^{\mathsf{T}}$ at the chosen places, so $\Psi_0(x) = 1$; for i large Ψ_0 is 1 on the class C of x. The translates $c\,\mathbf{1}_C(h^{-1}\cdot)$, $h \in K$, are finitely many, because c takes finitely many values and the class of hx depends locally constantly on h in the compact group K_0; let Ψ_1 be their mean. By the second rule $B(\Psi_1, \Xi) = B(\mathbf{1}_C, \Xi)$, and Ψ_1 is fixed by the twisted translations. The support of Ψ_0 contains C and is stable under K, so it contains that of Ψ_1; and Ψ_0 obeys the same rule $\Psi_0(hz) = \bar{c}(h)\Psi_0(z)$, by the transformation law of the block datum under the residue level (Proposition 15.11(a)). If $\Psi_0(z) \neq 0$, then at the places of O the matrix z is $h\iota A^{-1}$ for some $h \in K$ (the components of zA are $\equiv (1,0)^{\mathsf{T}}$ at the chosen places and have an invertible upper block at Q, and the point with components $\left(\begin{smallmatrix} u' & 0 \\ u'' & 1 \end{smallmatrix}\right)$ lies in K). Both functions read only the components at the places of O and follow the same rule on the orbit of ιA^{-1}, which contains the support of Ψ_0; hence $\Psi_1 = \Psi_1(x)\Psi_0$ and $B(\mathbf{1}_C, \Xi) = \Psi_1(x)B(\Psi_0, \Xi) = 0$.

Points without an integral block ratio at some place w of O. By Lemma 35.4(a) some row l annihilates x at w. For each $s = 1 + v\lambda$ in the hyperplane group take h_s as in Lemma 35.4(d). It fixes the components of x, so it preserves the class C of x modulo any idele, and by the first rule $B(\mathbf{1}_C, \Xi h_s) = B(\mathbf{1}_C, \Xi)$. Moreover $\sum_s \Xi h_s = \sum_s \chi_w(\det s)^{-1}\bar{s}\,\Xi = 0$: at a chosen place this is the hyperplane sum of l, which kills Ξ by Lemma 35.4(c), and at $w \in Q$ it is $\sum_s \chi_w(\det s)^{-1} \cdot \Xi = 0$ by Lemma 35.4(b), as $\chi_w \neq 1$. Since B is conjugate linear in y, $0 = B(\mathbf{1}_C, \sum_s \Xi h_s) = \#\{s\} \cdot B(\mathbf{1}_C, \Xi)$.

All data. It remains to pass from the data $v_P \otimes \Psi \otimes \mathrm{vac}$ to all data that are the vacuum off the level: the descent at g is a bounded linear functional of the datum, and translates of the vacuum by finite points supported on the places of O are multiples of data $\Psi \otimes \mathrm{vac}$ (Theorem B.33(a)).

(b) Induct on N. For $N = 0$ there is nothing to prove. Let $W' = W \oplus H^N$, $N = l + 1$, and assume the claim for $W \oplus H^l$; it suffices to treat the data $\phi = v_P \otimes \Psi \otimes \mathrm{vac}$ (Theorem B.33(a)), with the character $\chi = av^N$. *Boundedness.* Split one plane off W' by an adapted splitting; the descents to $W \oplus H^l$ for av^l vanish by induction, so the descent of Y to W' by ϕ is bounded by Lemma 35.6(c). *The Hecke equation.* The character χ is unramified at $\bar{w}$ and ϕ is the vacuum there, w being off the level. Apply Lemma 35.5(c) with $U = W'$, $m = r + 2N$ and $n = m + k$. Here $|X|^{-1} = \sqrt{q}^{\,d}$ with $d = r + 2(n - r - 1) = 2n - r - 2$, so a condition $\sqrt{q}^{\,j}|X| \leq 1$ holds for $j \leq d$. The first exponent is $m + 2k = 2n - r - 2N \leq d$. The second is $2(m + 1) + k = n + r + 2N + 2$, and $n + r + 2N + 2 \leq 2n - r - 2$ is $2r + 2N + 4 \leq n$, which is the hypothesis $2r + 2N + 3 < n$. Hence the descent is zero.

(c) *The tower.* For the highest vector ζ_0, the descents of the conjugate matrix coefficient of Ξ to W vanish by (a), and that coefficient is spherical at w and killed by the Hecke polynomial at the conjugate of X_*, which has absolute value $|X_*| = (\sqrt{q}^{\,r} q^{n-r-1})^{-1} = q^{-(n-1-r/2)}$, the Frobenius value of the character on W having absolute value $\sqrt{q}^{\,r}$. By (b) its descents to $W \oplus H^N$ vanish. For another vector ζ: the right translate by a real point x of the coefficient at ζ_0 is the

coefficient at $x\zeta_0$, these vectors span, the descent of a translate is the translate of the descent of the translated datum, and the data that are the vacuum off the level are stable under the real points.

Absorbing the kernel of W. Let φ be a datum on $\mathrm{Hom}(W, I_l)$ that is the vacuum off the level, b a character and y_1 the right translate by x of the coefficient of Ξ at ζ; its descents to $W \oplus H^N$ vanish, x being 1 off the places of O (Theorem B.33(a)). Choose a datum φ'' on $\mathrm{Hom}(W, I_N)$, again the vacuum off the level, whose kernel does not vanish at (g, h_1): a translate of the vacuum by a real matrix does not vanish at a suitable real point. On $I_n = I_N \oplus I_l$ the kernel of $\varphi'' \otimes \varphi$ at $(g, \mathrm{diag}(h_1, h_2))$ is the product of the kernels of φ'' at (g, h_1) and of φ at (g, h_2). Put $y_2(h) = \overline{\theta(\omega(g,h)(\varphi'' \otimes \varphi))}\, y_1(h)$. Its descents to H^N are descents of y_1 to $W \oplus H^N$, because kernels multiply in the sum $W \oplus H^N$; so they vanish. The level of $\mathrm{U}(I_n)$ off the places of O fixes y_2: it fixes the kernel, the character of W being unramified there, and it fixes Ξ and commutes with x. By Theorem B.33(c) the period of y_2 over $[\mathrm{U}(I_l)]$ vanishes at h_1, and that period is the nonzero kernel of φ'' at (g, h_1), conjugated, times $\int_{[\mathrm{U}(I_l)]} \overline{\theta(\omega(g,h_2)\varphi)}\, y_1(\mathrm{diag}(h_1, h_2))\, dh_2$. Hence this integral vanishes.

Class sums. Let ϑ be the given function and C the conjugate linear map. Write each coordinate j of ϑ, moved by the real points, as c_j times a conjugate theta kernel F_j, and pair it through C with the coefficient y_j of Ξ at the corresponding vector; by the equivariance of C the sum of the $c_j F_j y_j$ restricted to the finite points is $h \mapsto \langle C(\vartheta(h)), \Xi((h_1, h)x)\rangle$, which is therefore left invariant under the rational points. At a level fixing it, its class sum is the integral over $[\mathrm{U}(I_l)]$ of the corresponding function on the adelic group, the real group being compact; so it is a linear combination of the integrals of the previous step, which vanish. □

Corollary 35.8 (Periods against point forms and product forms). *Let O, Ξ and w be as in Theorem 35.7, $I_n = I_N \oplus I_l$ with $2r + 2N + 3 < n$, and let C be conjugate linear from vectors of a weight κ to vectors of the weight of O, equivariant for the real points of the second block.*

(a) *Let x be supported at the chosen places and A a point lift with columns Y at the chosen places and rows e' in the block I_N. If a level of $\mathrm{U}(I_l)$ fixes a datum Ψ modulo an ideal with places in $M \cup Q$ and, after conjugation by x, the matrices Y_i, then the class sum over that level of $h \mapsto \langle \Xi, \delta_{(h_1,h)xA}(C(\theta_\Psi(g,h)))\rangle$ is zero, $\theta_\Psi(g,h)$ being the vector of the conjugate theta kernels of (W, I_l) with datum Ψ, one for each basis polynomial of weight κ.*

(b) *If W is a line and $\chi_w \neq 1$ at the chosen places, the hypothesis on the hyperplane sums may be replaced by: Ξ lies in the image of the mean over $\prod_{v \in M} \mathrm{GL}_1(\mathbb{F}_v)$. The rank condition is then $2N + 5 < n$.*

(c) *Let the indices of W and of a further set be embedded in the block N, let γ be supported at the chosen places with $\varpi \boldsymbol{O}^n \subseteq \gamma \boldsymbol{O}^n \subseteq \boldsymbol{O}^n$ there, and let the product form at $(x, h)\gamma$ have as value the Cartan product of a vector y of the weight k^+ and the conjugate of $\theta_\Psi(g,h)$, for a datum Ψ modulo the square ϖ of the product of the chosen places. Its class sum over h at the principal level ϖ of $\mathrm{U}(I_l)$, paired with Ξ, is zero.*

Proof. (a) By Lemma 35.1(b) the pairing is the conjugate of

$$\langle C(\theta_\Psi(g,h)), \Xi((h_1,h)xA)\rangle.$$

A real point of the second block acts on the polynomial of the datum as on the Weyl module, so the coordinates of $\theta_\Psi(g,\cdot)$, moved by the real points, are multiples of conjugate theta kernels. It remains to see that the level fixes the pairing. It fixes the kernel, as it fixes Ψ. Let k lie in the level and put $c = (xA)^{-1}(1,k)(xA)$. At the chosen places the columns of c at the embedded indices are those of 1, because $x^{-1}(1,k)x$ fixes Y and A has the columns Y there. At the places of Q the point x is trivial, so $x^{-1}(1,k)x$ reduces to the block matrix $(1,k)$, which fixes the rows e' in the first block, and the rows e' of A are the embedded rows of 1; so the embedded rows of c are those of 1. By the computation of Lemma 35.4(d), with the first block 1, the point c fixes Ξ, and Theorem 35.7(c) applies.

(b) In rank one the idempotent of the label $X-1$ is the mean over $\mathrm{GL}_1(\mathbb{F}_v)$, the root 1 naming the trivial character. The hyperplane group of a row $l \neq 0$ of length one is all of $\mathbb{F}_v^\times$, which fixes its mean, so the hyperplane sum times the mean is $\sum_s \chi_v(s)^{-1}$ times the mean, which is 0 by Lemma 35.4(b). Hence the forms in the image of the mean are killed by every tensor product of hyperplane sums in which one row is not 0, and Theorem 35.7(c) applies with $r = 1$.

(c) Apply (a) to the conjugate linear map $\zeta \mapsto y \cdot \bar{\zeta}$ and the point γ. The map is equivariant for the real group of the second block: in the first block the columns of the conjugate of a real point are those of 1, in the second those of its inverse transpose are given by the point itself, and the representation acts through the conjugate matrix, which passes into the Cartan product and acts on the two factors by the Schur maps of those columns. The level is fixed: the block $(1,k)$ with $k \equiv 1$ modulo ϖ is integral and $\equiv 1$ modulo ϖ, its conjugate by γ is integral and $\equiv 1$ modulo the chosen places and fixes the first block at Q, so it fixes every form of level K, as in (a) with $A = 1$. □

35.3. Lifts of characters and Hecke L-values

For *L-value data* X (chosen places, twist places, row lengths, and a character μ of $[\mathrm{U}(1)]$ of weight $-$(row length $+ k$) with tame part trivial off these places), the lift φ_X is the class sum over $\mathrm{U}(1)$ of the theta kernel to the definite $\mathrm{U}(2k)$ with the block datum; it is the theta lift of the constant 1 times the index of the chosen level. For a pair $p = (\nu, z)$, $z \geq 2$, and a tame ϑ, the lift $\varphi_{p,\vartheta}$ of $\nu\vartheta$ is φ_X for these data, scaled by the vacuum and jet scales; its *algebraic part* is $L_p^{\mathrm{alg}}(\vartheta) = \|\varphi_{p,\vartheta}\|^2/\mathrm{N}\,Q^{2z-1}$, the *denominator* $n_\nu = M^{4fz}(Md)^j$ with j the sum of the row lengths, and the *integral algebraic part* $n_L L^{\mathrm{alg}}$, $n_L = n_{\mathrm{Aut}} n_\nu^2$. A pair is *flat* if ν has weight $-z$ at every place, and a *main pair for j* if $m < z \leq r$, the tame part of ν is ω^j, and $w_j + r - t \leq c \leq w_j$ for its weight $-c$.

We fix the notation. Let X be L-value data of rank $N = \#n \geq 2$, with row lengths ℓ_τ and character μ, of weight $-(\ell_\tau + \lfloor N/2 \rfloor)$ at τ. Its descent pair has as source the definite line W and as target $V = I_N$, the level places of X, the weight with one row of length ℓ_τ at τ and no power of the determinant, and on the side of V the character $\chi_V = \mu\nu^{\lfloor N/2 \rfloor}$, of weight $-\ell_\tau$ and tame part that of μ; on the side of W the character is trivial. For a finite point g of $\mathrm{U}(1)$, $\theta_X(g,\cdot)$ is the vector kernel of this pair at g (Lemma 30.1), a form on V, and $J_X \subseteq K_0^1$ is the principal level of $\mathrm{U}(1)$ at the chosen places, the integral points that are $\equiv 1$ there; then $\varphi_X = \sum_g \theta_X(g,\cdot)/\#\mathrm{Aut}(g)$ over $\mathrm{U}(1)(E)\backslash \mathrm{U}(1)(\mathbb{A}_\mathrm{f})/J_X$. Write $L(s,\mu)$ for the L-function of $x \mapsto \mu(x/\bar{x})$ without the factors at the places of X and their conjugates, $\mathrm{N}\,Q$ for the product of the norms of the twist places, hook

for the squared Fock norm of the highest vector of the weight and dim for its dimension. For a pair p and a tame ϑ of modulus Q, $X_{p,\vartheta}$ denotes the data with the chosen places, the twist places Q, the row lengths $c(\tau) - z$ and the character $\nu\vartheta$, in rank $2z$; the scale of $\varphi_{p,\vartheta}$ is $\mathsf{a}^{2z}(\bar{\pi}_M \mathsf{t})^J$, with the vacuum scale a and the jet scale t of the field E in its tower and $J = \sum_\tau (c(\tau) - z)$. The *determinant kernel* of (I_n, I_n) for a character α of weight $-c$, a function Ψ on matrices modulo $\mathfrak{m}$ and the polynomial $\prod_\tau \det^{c_\tau}$ is the theta kernel at $(1, h)$ of this datum, both characters being α; its *period* is its class sum at the level K_0 for $\Psi = \chi \circ \det$, χ the character modulo $\mathfrak{m}$ made of the tame part ψ of α.

Lemma 35.9 (The kernel of a character lift). *Let X be L-value data.*

(a) *$g \mapsto \theta_X(g, \cdot)$ is left invariant under $\mathrm{U}(1)(E)$ and right invariant under J_X, and φ_X is also the class sum of $g \mapsto \theta_X(gz, \cdot)$, for every finite point z of $\mathrm{U}(1)$.*

(b) *The normalised descent of a form y on V at (x, g) is $\langle \theta_X(g, \cdot), y\rangle$, for every real component x; and $\langle \varphi_X, y\rangle$ is the class sum over $\mathrm{U}(1)$, at the level J_X, of $g \mapsto (\theta y)(gz)$.*

(c) *$\varphi_X = [K_0^1 : J_X]\, \Theta_W 1$, where 1 is the constant function, a form of parameter 0 on the line.*

(d) *The hook number of the weight is $\prod_\tau \ell_\tau!$, and the Fock pairing of the Fock polynomial of the highest vector with itself is the hook number. For a scale c, the row sums of the Schur jet matrix of the Taylor series at scale c of $\omega(g, h)(\Psi_{\mathrm{bl}} \otimes \mathrm{vac})$ are $c^{\sum_\tau \ell_\tau}$ times the coordinates of $\theta_X(g, h)$.*

(e) *If all ℓ_τ vanish, every coordinate of $\theta_X(g, h)$ is the scalar theta kernel at $((1, g), (1, h))$ of the datum given by the polynomial 1 and the block datum read modulo the square of the product of the chosen places times the product of the twist places, for the characters 1 and χ_V.*

Proof. (a) By Lemma 30.1(b) the rational points act on the kernel through the weight, which is trivial on the line. For each $k \in J_X$ the reduction at the chosen places is 1, so in the law of Proposition 30.2(a) with $s = 1$ the pair $(k, 1)$ fixes the datum; hence $\theta_X(gk, \cdot) = \theta_X(g, \cdot)$. The group $\mathrm{U}(1)(\mathbb{A}_{\mathrm{f}})$ is commutative, so conjugation by z fixes J_X and translation by z permutes the double cosets, with the same stabilisers; the class sum of the translated kernel is therefore the class sum of the kernel.

(b) The first statement is Lemma 30.1(c) for this pair, whose real points of $\mathrm{U}(1)$ act as the identity in the weight 0. The inner product is additive in its first argument, so it commutes with class sums, and (a) gives the translated form.

(c) Pair both sides with a form y. By adjointness (Proposition 30.2(b)), $\langle \Theta_W 1, y\rangle$ is the integral over $[\mathrm{U}(1)]$ of the descent θy, which by (a) and (b) is a function of the class of g at the level J_X; for the normalisation of measures of Proposition 30.2(b), the integral over $[\mathrm{U}(1)]$ of such a function is its class sum at J_X divided by $[K_0^1 : J_X]$. With (b) at $z = 1$ this is $\langle \varphi_X, y\rangle / [K_0^1 : J_X]$. The constant 1 is a form of parameter 0: it is automorphic and bounded, and it is the function attached to the constant algebraic form whose value is a unit vector.

(d) The weight has one row of length ℓ_τ at τ, whose hook lengths are $1, \dots, \ell_\tau$. The Fock polynomial of a coordinate vector is obtained from the polynomial of the highest weight theory by renaming the variable at (τ, a) and the one column to the variable with row (τ, a); renaming along an injective map does not change the Fock pairing $P(\partial)Q(0)$, and the pairing of the highest weight polynomials is

the hook number times the Gram form, which is 1 on the highest vector. The row sums are the definition of the kernel, the jets being linear in the datum and homogeneous of degree $\sum_\tau \ell_\tau$ in the scale.

(e) The coordinate i is the jet along the sum of the row i of the Schur matrix; in degree 0 all basis polynomials are 1 and each row sums to 1, so the jet is the theta value of the datum with polynomial 1 (Proposition 15.11(a) for the block datum). □

Lemma 35.10 (Indices and masses). *Let X be L-value data of rank N and $q = q_w$ the norm of a place w of X.*

(a) $[K_0^1 : J_X] = \prod_{w\in M}(q_w - 1)$.

(b) *The index of the level of the forms on V in K_0 is the product of $q^N - 1$ over the chosen places and of $1 + q + \cdots + q^{N-1}$ over the twist places.*

(c) *The mean of $|\Psi_{\mathrm{bl}}|^2$ over the columns modulo the places of X is*

$$\mathrm{bms} = \prod_{w\in M} q_w^{-N} \prod_{w\in Q} \frac{q_w - 1}{q_w};$$

the block datum is sharp for the level J_X off the places of X and their conjugates, with mass $\mathrm{bms}/[K_0^1 : J_X]$.

(d) bms *times the index in (b) times $\prod_w (1 - q_w^{-N})^{-1}$ over the places of X is* $\mathrm{N}\, Q^{N-1}$.

(e) *For $N = 2k$, $k > 1$, the Euler product $\prod_w (1 - \epsilon_w q_w^{-k})^{-1}$ of the place signs ($\epsilon_w = 1$ at a place fixed by conjugation, ± 1 at the two places of a split pair) is nonzero; omitting the places of X and their conjugates multiplies it by $\prod_w (1 - q_w^{-N})$ over the places of X.*

(f) *The real points of* U(1) *fix the Fock vector v of the highest vector in the real Weil representation, and $\int \langle v, \Omega(x,1)v\rangle\, dx \cdot \langle 1, 1\rangle = \mathrm{hook} \cdot \mathrm{vol}[\mathrm{U}(1)]$.*

Proof. (a) At a split place the integral points of U(1) are the local units, which map onto $\mathbb{F}_w^\times = \mathrm{GL}_1(\mathbb{F}_w)$, and J_X is the kernel of the reduction at the chosen places.

(b) By the Chinese remainder theorem the index is a product over the places of X. At a chosen place the level is the stabiliser in $\mathrm{GL}_N(\mathbb{F}_w)$ of the embedded basis vector, whose index is the number $q^N - 1$ of nonzero vectors. At a twist place it is the stabiliser of the span of the other basis vectors; the inverse transpose takes it to the stabiliser of a line, of index $(q^N - 1)/(q - 1) = 1 + q + \cdots + q^{N-1}$.

(c) At a chosen place Ψ_{bl} is the indicator of the columns that reduce to the embedded basis vector, a proportion q^{-N}; at a twist place $|\psi_w(x_\iota)|^2$ is 1 if the embedded entry is a unit and 0 otherwise, a proportion $(q - 1)/q$. Sharpness and the mass are Proposition 30.3(b) for the line: the mass is $\sum |\Psi_{\mathrm{bl}}|^2$ divided by the number of columns and by $\#\prod_{w\in M} \mathrm{GL}_1(\mathbb{F}_w) = [K_0^1 : J_X]$.

(d) At a chosen place the factor is $q^{-N}(q^N - 1)(1 - q^{-N})^{-1} = 1$; at a twist place it is $\frac{q-1}{q} \cdot \frac{q^N-1}{q-1} \cdot \frac{q^N}{q^N-1} = q^{N-1}$.

(e) For $k > 1$ the product converges absolutely and its factors are nonzero. The places of X split, so each contributes, with its conjugate, the factor $(1 - q^{-k})^{-1}(1 + q^{-k})^{-1} = (1 - q^{-2k})^{-1}$, which is removed.

(f) In the real Weil representation (x, y) acts by the Fock operator times $\prod_\tau \det(y_\tau)^{\gamma_\tau} \det(x_\tau)^{\ell_\tau}$, with γ the power of the determinant of the weight on V, here 0. The Fock operator of x multiplies the polynomial of the highest vector,

homogeneous of degree ℓ_τ in the variables at τ, by $\det(x_\tau)^{-\ell_\tau}$, so $\Omega(x,1)v = v$, and $\langle v, v\rangle$ is the hook number by Lemma 35.9(d). The real points form one circle for each infinite place, and the integral times $\langle 1,1\rangle$ is the volume of $[\mathrm{U}(1)]$ for the product measure. □

Lemma 35.11 (The central action). *For a finite point z of $\mathrm{U}(1)$ let T_z be translation of forms on V by the scalar matrix $\det z$ times $\chi_V(\det z)$.*

(a) *Translation by a scalar preserves the inner product, and the adjoint of T_z is $T_{z^{-1}}$.*
(b) *$T_{z^{-1}}\theta_X(g,\cdot) = \theta_X(gz,\cdot)$, and $T_z\varphi_X = \varphi_X$.*
(c) *T_z commutes with every translation, hence with the operators $H_w(X)$ at split places off the level and with the action of $\prod_{v\in M}\mathrm{GL}_1(\mathbb{F}_v)$, whose images of group algebra elements it preserves. If z is integral at the places above ℓ, $T_{z^{-1}}$ preserves the forms integral at every embedding $\mathbb{C}\to\mathbb{C}_\ell$.*
(d) *Let the stabilisers of the classes of $\mathrm{U}(1)$ at J_X be trivial and s_c represent the class c. If $y \perp \varphi_X$, the descent kills $\sum_c T_{s_c} y$.*
(e) *Under the hypothesis of (d), if Ξ is orthogonal to the false lifts formed with the kernel of the descent, then $\sum_c T_{s_c}^{-1}\Xi$ is orthogonal to those formed with $(\mathbb{C}\varphi_X)^\perp$.*

Proof. (a) The inner product is the restriction of a translation invariant pairing (Lemma 35.2(a)), and $|\chi_V| = 1$; so the adjoint of T_z is $\overline{\chi_V(\det z)}$ times translation by the inverse scalar, which is $T_{z^{-1}}$.

(b) The pair of scalars (ν,ν), $\nu = \det z^{-1}$, acts as the identity on $\mathrm{Hom}(W,V)$ and in the Weil representation on the block datum by a number, the product of the character values. Hence translating the kernel by ν on V amounts to translating it by z on $\mathrm{U}(1)$, up to the scalar $\chi_V(\nu)$, which is the first claim. Then $T_z\varphi_X$ is the class sum of $g\mapsto\theta_X(gz^{-1},\cdot)$, which is φ_X by Lemma 35.9(a).

(c) A scalar matrix is central in $\mathrm{U}(V)(\mathbb{A}_\mathrm{f})$, and the Hecke operators and the group action are combinations of translations, times roots of unity for the latter. If z is integral above ℓ, the scalar $k = (\det z)^{-1}$ and its inverse are integral above ℓ, so the gauge at g of $T_{z^{-1}}\Xi(g)$ is that of $\Xi(gk)$ at gk, at most 1; the character value is a unit at every embedding.

(d) By Lemma 35.9(b) and by (a) and (b), the descent of $\sum_c T_{s_c} y$ at g is

$$\sum_c \langle T_{s_c^{-1}}\theta_X(g,\cdot), y\rangle = \sum_c \langle \theta_X(gs_c,\cdot), y\rangle = \langle \varphi_X, y\rangle = 0,$$

since, the stabilisers being trivial, the class sum of the translated kernel is φ_X.

(e) Let y be a false lift formed with $(\mathbb{C}\varphi_X)^\perp$: $y\perp\varphi_X$, y in the image of e and killed by $H_w(X_*)$. By (a), $\langle\sum_c T_{s_c}^{-1}\Xi, y\rangle = \langle\Xi, \sum_c T_{s_c} y\rangle$. By (c) the form $\sum_c T_{s_c} y$ lies in the image of e and is killed by $H_w(X_*)$, and by (d) it lies in the kernel of the descent; so it is a false lift formed with that kernel, and Ξ is orthogonal to it. □

Lemma 35.12 (Galois conjugates of the lift). *Let $\sigma\in\mathrm{Aut}(\mathbb{C}/K)$ and Ω the period of Theorem 17.2.*

(a) *$\sigma' = \mathrm{conj}\circ\sigma\circ\mathrm{conj}$ lies in $\mathrm{Aut}(\mathbb{C}/K)$ and permutes the infinite places of E as σ does; write e for this permutation.*
(b) *Let the data have no twist places and even rank $N = 2m'$. There are constants t, l with $|t| = \Omega^{-1/2}$, $l\neq 0$, and a norm-one adele u such that for all data X on*

these places the following holds. Let Y be the data on the same places with row lengths $\ell_{e^{-1}\tau}$ and a character equal to $\sigma\circ\mu$ on the finite points, and $D=\sum_\tau \ell_\tau$. Then σ of the coordinate i of $\mathsf{t}^D\varphi_X(h)$ is $t^D\, l\,\chi_{V,Y}(\det u)\,(\sigma\Delta/\Delta)(h)$ times the coordinate $i\circ e^{-1}$ of $\varphi_Y(h)$, where Δ is the multiplier of the Weil representation at the finite component of $x\mapsto hx$.

(c) *For a pair p and a tame ϑ, $\varphi_{p,\vartheta}=\mathsf{a}^{2z}\bar\pi_M^J A$ with $A=\mathsf{t}^J\varphi_{X_{p,\vartheta}}$, so $\|\varphi_{p,\vartheta}\|^2=|\mathsf{a}^{2z}|^2M^J\langle A,A\rangle$.*

(d) *There is $a>0$ such that for every pair p of rank $2z\ge 4$, with A formed at $\vartheta=1$, there are data Y on the same places with the row lengths $c(\tau)-z$ permuted by e and $|\sigma\langle A,A\rangle|=\Omega^{-J}a\|\varphi_Y\|^2$.*

Proof. (a) Complex conjugation preserves K and σ fixes K pointwise, so $\sigma'(y)=\overline{\sigma(\bar y)}=y$ for $y\in K$. For a CM field complex conjugation commutes with every embedding, $\mathrm{conj}\circ\tau=\tau\circ c_E$ with c_E the conjugation of E; hence $\sigma'\circ\tau=\mathrm{conj}\circ\sigma\circ\tau\circ c_E$, which is conjugate to $\sigma\circ\tau\circ c_E=\mathrm{conj}\circ(\sigma\circ\tau)$, and so defines the same place as $\sigma\circ\tau$.

(b) Both φ_X and φ_Y are class sums over U(1), the second after translation by u^{-1} (Lemma 35.9(a)); compare them term by term. The kernels are jets of the data $\omega(g,h)(\Psi_{\mathrm{bl}}\otimes\mathrm{vac})$ (Lemma 35.9(d)), and the semilinear map L of Theorem 17.2 carries jets at scale t to conjugates of jets at scale t, with the places permuted, which permutes the Schur jet matrix by e. It remains to compute L on the datum. The action of a pair is a character times the Weil action of its image in $\mathrm{U}(\mathrm{Hom}(W,V))$. By Theorem 17.2(b), $L(\Psi\otimes\mathrm{vac})=c\,\omega(u)((\sigma\circ\Psi)\otimes\mathrm{vac})$; with no twist places the residue function of the block datum has values 0 and 1, which σ fixes, so the block data of X and Y correspond. Moving L past the Weil operators of g and h introduces the multiplier ratio $\sigma\Delta/\Delta$; at the image of g it is the ratio at g^{-1} to the power $2m'$, whose square is $\sigma v/v$ at $\det g$, and the characters $\chi_V=\mu v^{m'}$ of X and Y then match, as $\sigma\circ\mu$ is the character of Y on the finite points; the characters on the side of W are trivial. This gives the factor $l\,\chi_{V,Y}(\det u)(\sigma\Delta/\Delta)(h)$ and the kernel of Y at (gu^{-1},h).

(c) is the definition of the scaled lift, with $|\bar\pi_M|^2=M$.

(d) Write $\sigma\langle A,A\rangle$ through the coordinates: since $\sigma(\bar\zeta)=\overline{\sigma'(\zeta)}$, it is the class sum of the Gram pairings of $\sigma' A(h)$ and $\sigma A(h)$. Apply (b) to σ and to σ'; both carry A to a constant times the multiplier ratio times the lift of conjugate data with the row lengths permuted by e, by (a) the same permutation. The conjugate data are the same: the values of the character on the finite points are algebraic numbers in a CM field (on the rational points they are products of powers of the embeddings Φ_v, on an open subgroup values of the tame part, and the index is finite), on which complex conjugation commutes with σ, so $\sigma'\circ\nu=\sigma\circ\nu$ there. The multiplier ratios of σ and σ' have modulus one and cancel in the pairing, the constants contribute $|t|^{2J}=\Omega^{-J}$ and a positive number independent of p, and what remains is the squared norm of the lift of the conjugate data. □

Proposition 35.13 (Squared norms are L-values). (a) *For rank $2k$, $k>1$, $\|\varphi_X\|^2$ is the product of $C\neq 0$ (the volumes of $[\mathrm{U}(2k)]$ and $[\mathrm{U}(1)]$ over $\zeta_F(2k)$), $\prod_{w\in M}(q_w-1)$, the hook number and the dimension of the weight, $L(k,\mu^{-1})$ and $\mathrm{N}\,Q^{2k-1}$.*

(b) *φ_X is spherical at split places off its places, with the Hecke polynomials of a lift of the trivial system; the twisted central translations fix it; its Galois conjugates are, up to simple factors, the lifts of the conjugate characters.*

(c) *For a pair p and every tame ϑ, $L_p^{\mathrm{alg}}(\vartheta) > 0$ is $A_p\, L(z, \nu\vartheta)$ without the factors at the level, with A_p independent of ϑ; hence $|L(z,(\nu\vartheta)^{-1})|/|L(z,\nu^{-1})| = L_p^{\mathrm{alg}}(\vartheta)/L_p^{\mathrm{alg}}(1)$, and L_* of $(\nu\vartheta^s)^{-1}$ up to z is $\mathrm{N}\,Q^{z^2}$ times that of ν^{-1} times the squared ratios at $1,\dots,z$.*

(d) *(Rank two, $z = 1$.) The period of the determinant theta kernel of (I_n, I_n) is a class sum of kernel values, and adding a prime v to the modulus multiplies it by $\prod_{l<n}(1 - q^l X)$, X the inverse Frobenius value.*

Proof. (a) *Rallis' formula for the constant.* Let $n = N = 2k$. By Lemma 35.10(c) the block datum is sharp for J_X with mass $\mathrm{bms}/[K_0^1 : J_X]$, and the real vector v is an edge (Proposition 15.11(b)). Translations fix the constant 1, so its Rallis series is the constant Σ, the sum of the vacuum coefficients of $\chi_V = \mu\upsilon^k$ over the finite points modulo those integral off the places of X and their conjugates; and $\Omega(x, 1)v = v$. Hence Theorem B.34(b) and Lemma 35.10(f) give

$$\langle \Theta_W 1, \Theta_W 1\rangle_{\mathrm{Pet}} = \mathrm{vol}[\mathrm{U}(n)]\,\frac{\mathrm{bms}}{[K_0^1 : J_X]}\,\mathrm{hook}\,\mathrm{vol}[\mathrm{U}(1)]\,\Sigma.$$

Class sums. The class sum product on V is the index of the level in K_0 times dim times the Petersson product (Proposition 30.2(b)), and $\varphi_X = [K_0^1 : J_X]\Theta_W 1$ (Lemma 35.9(c)). Therefore

$$\|\varphi_X\|^2 = [K_0^1 : J_X] \cdot \mathrm{vol}[\mathrm{U}(n)]\,\mathrm{vol}[\mathrm{U}(1)] \cdot \dim \cdot \mathrm{hook} \cdot [K_0 : K] \cdot \mathrm{bms} \cdot \Sigma.$$

The vacuum sum. By Theorem B.34(e), $\Sigma = L(k, \mu^{-1})$ divided by the Euler product of the place signs at k without the places of X and their conjugates, which is $\zeta_F(2k)\prod_w(1 - q_w^{-N})$ by Lemma 35.10(e); here $\zeta_F(2k)$ denotes the full Euler product of the place signs, which is the value of the zeta function of F at $2k$ when E/F is unramified at the finite places, and is nonzero. By Lemma 35.10(d), $[K_0 : K]\,\mathrm{bms}\prod_w(1 - q_w^{-N})^{-1} = \mathrm{N}\,Q^{N-1}$, and by (a) of that lemma $[K_0^1 : J_X] = \prod_{w\in M}(q_w - 1)$. Collecting, $\|\varphi_X\|^2$ is the product of $C = \mathrm{vol}[\mathrm{U}(n)]\,\mathrm{vol}[\mathrm{U}(1)]/\zeta_F(2k)$, of $\prod_{w\in M}(q_w - 1)$, hook, dim, $L(k, \mu^{-1})$ and $\mathrm{N}\,Q^{N-1}$; $C \neq 0$ as the volumes are positive.

(b) *Hecke polynomials.* By Lemma 35.9(c) it suffices to treat $\Theta_W 1$. The constant is spherical at every split w off the places of X, with Hecke polynomial $1 - X$ in rank one, the double coset of $\mathrm{diag}(\varpi_w)$ being a single coset. By Proposition 30.2(c) with $t = N - 1$ the lift has the polynomial $\prod_{l<N-1}(1 - q^l\alpha X)\cdot(1 - q^{N-1}\alpha\beta^{-1}X)$, α, β the Frobenius values of the characters of the pair, which is the Hecke polynomial of a lift of the trivial system. *Central translations* fix φ_X by Lemma 35.11(b), and the group $\prod_{w\in M}\mathrm{GL}_1(\mathbb{F}_w)$ fixes it, since it fixes the constant 1 and commutes with the lift (Proposition 30.2(a)). *Galois conjugates* are Lemma 35.12(b).

(c) *The value.* Apply (a) to the data $X_{p,\vartheta}$ of rank $2z$: the volumes, the chosen places, the weight and its hook number and dimension do not depend on ϑ, so $\|\varphi_{X_{p,\vartheta}}\|^2 = C'\,L(z,(\nu\vartheta)^{-1})\,\mathrm{N}\,Q^{2z-1}$ with $C' \neq 0$ independent of ϑ and of Q; here L omits the chosen places, the places of Q and their conjugates. The scale $\mathsf{a}^{2z}(\bar\pi_M \mathsf{t})^J$ is nonzero and independent of ϑ, so $L_p^{\mathrm{alg}}(\vartheta) = A_p L(z,(\nu\vartheta)^{-1})$ with $A_p = C'|\mathsf{a}^{2z}(\bar\pi_M\mathsf{t})^J|^2 \neq 0$. Since the set of omitted places is stable under

conjugation, the L-functions of $\nu\vartheta$ and of its inverse agree: the Euler factor of $(\nu\vartheta)^{-1}$ at w is that of $\nu\vartheta$ at $\bar{w}$. *Positivity.* ϑ has weight 0, so $\nu\vartheta$ has the weight $-c$ of ν, which is nonzero as $1 \le z \le c(v)$; the character $x \mapsto \nu\vartheta(x/\bar{x})$ is therefore not trivial, and its partial L-function does not vanish at real $s \ge 1$ (as in Theorem B.32(a)). Hence $\|\varphi_{X_{p,\vartheta}}\|^2 \ne 0$, $\varphi_{p,\vartheta} \ne 0$ and $L_p^{\mathrm{alg}}(\vartheta) > 0$. *The ratio.* Dividing the values at ϑ and at 1, where $Q = \emptyset$, the constant A_p cancels; the quotient is a positive real number, so it equals its absolute value, which is $|L(z,(\nu\vartheta)^{-1})|/|L(z,\nu^{-1})|$. The same holds for $n_L L_p^{\mathrm{alg}}$, n_L being a nonzero integer independent of ϑ; and the ratio $\varrho_p(\vartheta,s)$ is nonzero and real for both signs, its two L-values being nonzero. *The products.* For $1 \le z' \le z$ the pair (ν,z') is again a pair, its ratio at ϑ^s is $|L(z',(\nu\vartheta^s)^{-1})|/|L(z',\nu^{-1})|$, and no place of Q is chosen. Hence

$$\prod_{w\in M\cup Q} q_w^{z^2} \prod_{z'\le z} |L(z',(\nu\vartheta^s)^{-1})|^2 = \mathrm{N}\,Q^{z^2} \cdot \prod_{w\in M} q_w^{z^2} \prod_{z'\le z} |L(z',\nu^{-1})|^2 \cdot \prod_{z'\le z} \varrho_{(\nu,z')}(\vartheta,s)^2,$$

the L-values of ν^{-1} being nonzero.

(d) *Invariance.* Substituting γ^{-1} multiplies the polynomial $\prod_\tau \det^{c_\tau}$ by $\prod_\tau (\det\gamma)^{-c_\tau}$, and translating the kernel by a rational γ gives the factor $\prod_\tau (\det\gamma)^{c_\tau}$ through the character α of weight $-c$, so the kernel is left invariant under the rational points; right translation by $k \in K_0$ replaces the datum by $y \mapsto \alpha(\det k)\chi(\det(k^{-1}y))$, which is the datum again, so the kernel has level K_0. *A class sum of kernel values.* The function $\chi \circ \det$ modulo $\mathfrak{m}$ vanishes off the image of $\mathrm{GL}_n(\mathcal{O}/\mathfrak{m})$, where the determinant is no unit, and on it is $\sum_y \chi(\det y)\mathbf{1}_y$; translation by a lift of y to K_0 turns the indicator of the unit matrix into that of y, times the character, and the kernel is linear in the datum. Hence the period is the class sum of $g \mapsto \sum_y$ of the kernel values at $g \cdot \mathrm{lift}(y)$ for the indicator of 1. *Adding a prime.* Let the primes of $\mathfrak{m}'$ be those of $\mathfrak{m}$ and a new prime v, split since $\mathfrak{m}'$ is prime to its conjugate, and $X = \alpha(v)^{-1}$ for the Frobenius value $\alpha(v)$, the value at a uniformiser of the Hecke character of α in rank n times $q^{n/2}$. The datum modulo $\mathfrak{m}$ is the vacuum at v. Translate it by the transposed Hecke coset representatives γ_U at v, one for each subspace U of $\mathbb{F}_v^n$, with the coefficients $(-X)^{\dim U} q^{\binom{\dim U}{2}}$: at v the sum puts in the function $x \mapsto \prod_{l<\mathrm{corank}\,x}(1-q^l)$, which is 1 for invertible x and 0 otherwise, that is $\psi_v \circ \det$, ψ_v being trivial; at the other places of $\mathfrak{m}'$ nothing changes. Hence the datum modulo $\mathfrak{m}'$ is $\sum_U (-X)^{\dim U} q^{\binom{\dim U}{2}} \omega(\gamma_U)$ applied to the datum modulo $\mathfrak{m}$. Applying the theta functional, the kernels are related in the same way; they have level K_0, so their class sums are integrals over $[\mathrm{U}(n)]$, which do not see translations. Hence the period for $\mathfrak{m}'$ is $\sum_U (-X)^{\dim U} q^{\binom{\dim U}{2}} = \prod_{l<n}(1-q^l X)$ times that for $\mathfrak{m}$, by the q-binomial theorem. □

A function F of the tame characters is an *integral measure* at the modulus Q if there are finitely many points u_a of $\mathrm{U}(1)(\mathbb{A})$ and numbers c_a such that $\|j(c_a)\| \le 1$ for every prime l prime to $\mathrm{N}\,Q$ and every ring map $j\colon \mathbb{C} \to \mathbb{C}_l$, and $F(\vartheta') = \sum_a c_a \vartheta'(u_a)$ for all tame ϑ' of modulus Q. The *point kernel* of p twisted by ϑ is the kernel of the data $X_{p,\vartheta}$ at the datum that is a^{2z} times the indicator of the columns that reduce to the embedded basis vector at the chosen places and have the entry 1 in that position at the places of Q, with the variables scaled by $\bar{\pi}_M \mathsf{t}$.

Lemma 35.14 (Values and squared norms of the lift). *Let $p = (\nu, z)$ be a pair with $z \geq 2$ and ϑ tame of modulus Q.*

(a) *$\varphi_{p,\vartheta}(h)$ is the sum of the point kernels at $(g_c\,\lambda(A), h)$ over the classes c of $\mathrm{U}(1)$ at J_X, with representatives g_c, and the families A of units of the residue fields at Q, $\lambda(A)$ an integral lift of A^{-1}.*
(b) *For a prime $l \nmid \mathrm{N}\,Q$ and $j\colon \mathbb{C} \to \mathbb{C}_l$, n_ν times a point kernel value at (g, h) has gauge at most 1 at h.*
(c) *For tame ϑ' of the same modulus, $\varphi_{p,\vartheta'}(h) = \sum_a \vartheta'(\det(1, g_a))^{-1}\theta^Q(g_a, h)$, with g_a the point attached to the index $a = (c, A)$ in (a) and θ^Q the point kernel for the trivial character regarded as tame of modulus Q. Hence $\|\varphi_{p,\vartheta'}\|^2 = \sum_{a,b} \vartheta'(\det(g_b/g_a))\,\kappa_{ab}$ with numbers κ_{ab} that depend on Q only, and $n_{\mathrm{Aut}} n_\nu^2 \kappa_{ab}$ is integral at every j as in (b).*
(d) *If F is an integral measure at Q, so is $\vartheta' \mapsto |F(\vartheta')|^2$.*

Proof. (a) By Proposition 15.11(a) the block datum is the sum over A of the translates, by lifts of A^{-1}, of the indicator datum, and the scales of the point kernel come out of the datum and of the jets as the powers a^{2z} and $(\bar{\pi}_M \mathsf{t})^J$ (Lemma 35.9(d)). The classes of $\mathrm{U}(1)$ at the level J_X have no automorphisms, so the class sum is a plain sum.

(b) With $\delta = M^{2f\cdot 2z} = M^{4fz}$ the datum δ times the point datum is standard of level M at every $l \nmid \mathrm{N}\,Q$ (Proposition 17.6; the indicator has values 0 and 1, and the factor a^{2z} is part of the point datum). The weight has J boxes and $\delta(Md)^J = n_\nu$, so by Theorem 17.5(e) the kernel of n_ν times the point datum does not increase gauges. The jet at (g, h) applied to the vector 1 of the trivial weight of $\mathrm{U}(1)$, which has gauge one, is the kernel value.

(c) The character ϑ' is a scalar factor of the Weil action, and the point datum and the points g_a depend on the modulus only; this gives the first formula. Expanding both arguments of the sesquilinear class sum, $\|\varphi_{p,\vartheta'}\|^2 = \sum_{a,b} \overline{\vartheta'(\det g_a)^{-1}}\vartheta'(\det g_b)^{-1}\kappa'_{ab}$, with κ'_{ab} the class sum at the level of the lift of the Gram pairings of $\theta^Q(g_a, \cdot)$ and $\theta^Q(g_b, \cdot)$, taken at fixed representatives; as $|\vartheta'| = 1$, the coefficient is $\vartheta'(\det(g_a/g_b))$, and renaming gives the formula. Each Gram pairing of two vectors of gauge at most one at h is integral, the Gram matrix being integral and invariant, so n_ν^2 times the pairing is integral by (b), and n_{Aut}, a common multiple of the orders of the stabilisers, clears the weights of the class sum.

(d) $|F(\vartheta')|^2 = F(\vartheta')\overline{F(\vartheta')}$ is the double sum of $c_a \bar{c}_b \vartheta'(u_a u_b^{-1})$, since $\overline{\vartheta'(u)} = \vartheta'(u^{-1})$; and $\bar{c}_b$ is integral at j because c_b is integral at $j \circ \mathrm{conj}$. □

Proposition 35.15 (Algebraic parts). (a) *n_ν times each kernel value is integral at primes prime to $\mathrm{N}\,Q$; so $n_\nu \varphi_{p,\vartheta}$ is an integral form and $n_L L_p^{\mathrm{alg}}(\vartheta)$ is integral away from Q; it is given, on the tame characters with set Q, by one element of the group ring of $\mathrm{U}(1)(\mathbb{A})$ integral away from Q.*
(b) *For a flat pair, $n_L \leq \exp(x^{33})$ and every conjugate of $n_L L_p^{\mathrm{alg}}(1)$ is at most $\exp(x^{34})$; flat pairs with prescribed tame part exist in every even rank between 4 and t.*

(c) *Let* $\mathrm{budget} = f a \log^3 b/(mk)$, *and recall the size* $s_\infty(i) = \log i! + i\log(M/\Omega)$ *of* (9.2). *For a main pair* p *for* j, *the constant family* $\|\varphi_{p,1}\|^2$ *has height at most*

$$\mathrm{budget} - \sum_\tau s_\infty(\kappa(\tau)).$$

(d) *A bound for the denominator of the projection to the line of* $\varphi_{p,\vartheta}$ *bounds that of the product of the projections of two test forms, at the cost of the class number of* $\mathrm{U}(1)$ *at the chosen level.*

Proof. (a) The first claim is Lemma 35.14(b). By Lemma 35.14(a) a value of $n_v\varphi_{p,\vartheta}$ is a sum of vectors of gauge at most one, and the gauge is ultrametric, so $n_v\varphi_{p,\vartheta}$ is integral at every j over $l \nmid \mathrm{N}\,Q$. The squared norm is the class sum of the Gram pairings of the values; n_v^2 makes each pairing integral and n_{Aut} clears the weights; and $\mathrm{N}\,Q$ is a unit at l, so $n_L L_p^{\mathrm{alg}}(\vartheta)$ is integral. For $\vartheta = 1$ the modulus is empty and every prime is allowed. By Lemma 35.14(c) the function $\vartheta' \mapsto n_L L_p^{\mathrm{alg}}(\vartheta')$ on the tame characters of modulus Q is $\sum_{a,b} c_{ab}\vartheta'(u_{ab})$ with $u_{ab} = \det(g_b/g_a)$ and $c_{ab} = n_{\mathrm{Aut}} n_v^2 \kappa_{ab}/\mathrm{N}\,Q^{2z-1}$, integral at every j over $l \nmid \mathrm{N}\,Q$.

(b) *The integer.* For a flat pair the row lengths vanish, so $n_v = M^{4fz}$ and $\log(n_{\mathrm{Aut}} M^{8fz}) \le x^{33}$ by the comparison of sizes (Remark 8.10). *One kernel value.* The weight is 0, the test vector has norm 1, and the datum is that of the indicator, a function modulo an ideal of norm M^f with values 0 and 1; by Theorem 17.2(c) and Proposition 17.6 the σ-conjugate of a kernel value has norm at most $C = (M^f)^{2z}\exp(C_0(q_1\cdot 2z+1)^2 + 2fz\log 4)$. *The class sums.* Both class numbers, of $\mathrm{U}(1)$ at J_X and of $\mathrm{U}(2z)$ at the level of the lift, are at most $\exp(x^{30})$ by Proposition 8.9, and a value of $\varphi = \varphi_{p,1}$ is a sum of kernel values over the classes of $\mathrm{U}(1)$, there being no twist places. Hence $\|\sigma\varphi(h)\| \le \exp(x^{30})C$ and $\|\varphi\|_\sigma^2 \le \exp(x^{30})(\exp(x^{30})C)^2$. *Conjugates.* By Cauchy–Schwarz in the class sum, $|\sigma\|\varphi\|^2| \le \|\varphi\|_\sigma \|\varphi\|_{\sigma'}$, because $\sigma\langle\varphi(h),\varphi(h)\rangle$ is the Gram pairing of $\sigma'\varphi(h)$ and $\sigma\varphi(h)$. Hence $|\sigma(n_L L_p^{\mathrm{alg}}(1))| \le \exp(x^{33})\exp(x^{30})(\exp(x^{30})C)^2 \le \exp(x^{34})$, every term of the logarithm being small against x^{34}. *Existence.* Let n be even with $4 \le n \le t$, $z = n/2$, and μ a character of weight 0 with tame part ψ trivial off the chosen places. The pair $(\mu\upsilon^{-z}, z)$ with constant weight $-z$ is flat on n, has tame part ψ (the character υ being unramified at the finite places), and $\nu\upsilon^z = \mu$; the conditions $1 \le z \le t$ and $z \le c = z \le 4b$ hold.

(c) *The degrees.* For a main pair, $c(\tau) \in [w_j + r - t, w_j]$ and $z \le r$, so $c(\tau) - z \in [w_j - t, w_j]$; with $\kappa(\tau) + t \le w_j \le \kappa(\tau) + x^{42}$ this gives $\kappa(\tau) \le c(\tau) - z \le \kappa(\tau) + x^{42}$. *A comparison pair.* Let $q = (\upsilon^{-z}, z)$ be the flat pair of the same rank, which exists by (b) as $2z \le 2r \le t$, and $\varphi_q = \varphi_{q,1}$. Put $H = n_L(q)\|\varphi_q\|^2/\|\varphi_{p,1}\|^2$. *At the primes.* $H\|\varphi_{p,1}\|^2 = n_L(q)L_q^{\mathrm{alg}}(1)$ is integral at every prime by (a), so $h_\ell = 0$ for all ℓ. *At infinity.* Let $\sigma \in \mathrm{Aut}(\mathbb{C}/K)$. By Lemma 35.12(c), (d) and Lemmas 35.9 and 35.10, as in Proposition 35.13(a) for the data Y, there are a constant c_σ, the same for all pairs of rank $2z$, and reals r, $r_0 \ge -4f$ with

$$\log|\sigma\|\varphi_{p,1}\|^2| = c_\sigma + r + \sum_\tau s_\infty(c(\tau) - z), \qquad \log|\sigma\|\varphi_q\|^2| = c_\sigma + r_0, \quad r_0 \le 4f.$$

Here r is the logarithm of the dimension times the L-value at $z \ge 2$, where the Euler product is bounded by $e^{\pm 4f}$, the dimension being at least 1 and equal to 1 for a flat pair; the factorials of the row lengths come from the hook number and

$\Omega^{-J}M^J$ from (c), (d) of Lemma 35.12, which together give s_∞. The constant c_σ cancels in $\log|\sigma H|$, and s_∞ is nondecreasing, as $M \geq \Omega$; so

$$\log|\sigma H| \leq x^{33} + 8f - \sum_\tau s_\infty(\kappa(\tau)) \leq \text{budget} - \sum_\tau s_\infty(\kappa(\tau)),$$

by (b) and $x^{33} + 8f \leq$ budget. An automorphism of $\mathbb{C}$ not over K has the same absolute values as one over K, composed with complex conjugation. By Definition 7.2 the constant family has the stated height.

(d) Let $\varphi = \varphi_X$ for the data of $\varphi_{p,\vartheta}$, which spans the same line, and let Λ be the operator of the denominator bound at ι over ℓ; on the line it is a number μ with $|\iota\mu| \geq e^{-\beta}$. Let Ξ, Ξ' be test forms. The classes of U(1) at J_X have representatives s_c integral above ℓ (a rational point moves any representative into the integral points above ℓ), and their stabilisers are trivial. Put $\Xi'' = \sum_c T_{s_c}^{-1}\Xi'$. By Lemma 35.11(c) it is integral above ℓ, and by Lemma 35.11(e) it is orthogonal to the false lifts formed with $(\mathbb{C}\varphi)^\perp$. Since T_{s_c} fixes φ and its adjoint is $T_{s_c}^{-1}$, the projection of Ξ'' to the line is N times that of Ξ', N the class number of U(1) at J_X. The definition of the denominator gives $|\iota(\mu\langle\Xi, \operatorname{pr}\Xi''\rangle)| \leq 1$, so $|\iota(N\langle\Xi, \operatorname{pr}\Xi'\rangle)| \leq e^\beta$. Finally $\langle\operatorname{pr}\Xi, \operatorname{pr}\Xi'\rangle = \langle\Xi, \operatorname{pr}\Xi'\rangle$ and $|\iota N| = \ell^{-v_\ell(N)}$, so the product of the projections has norm at most $e^{\beta + v_\ell(N)\log\ell}$ at ι. □

35.4. Heights of ratios of Hecke values

The test forms are built on four blocks of indices. For a main pair p with point z the $2z$ indices of U($2z$) form blocks of sizes 12, 4, 4 and $2z - 20$; this is possible since $z > m \geq 10^4$. The *test column* at a chosen place is the column with rows in the four blocks made of the residue of a point x_0 on the first block and of a standard basis vector on each of the other three; it is injective as a linear map, its last block being a basis vector. The *twist row* is a fixed row of the second block. For a flat pair q on a block of rank n' with lift Ξ_q at a tame ϑ, $d = n_\nu(q)$ and a function F on the finite points of U(n') with values in forms on U($2z$), the *lift class sum* is

$$\Sigma_{q,\vartheta}F = n_{\mathrm{Aut}} \sum_h \frac{1}{\#\mathrm{Aut}(h)} \frac{d}{\mathrm{N}\,Q^{n'-1}} \langle 1, d\,\Xi_q(h)\rangle\, F(h),$$

the class sum over U(n') at the level of Ξ_q, $\langle\cdot,\cdot\rangle$ the Gram pairing of the trivial weight. A character of the classes of U(1) at J_X is a *tame class character*.

Lemma 35.16 (Averaging over the classes of U(1)). (a) *Let Γ contain the rational points of $\mathrm{U}(1)(\mathbb{A}_\mathrm{f})$ and K the principal level at a set S of places, K compact open with finitely many classes $\Gamma\backslash\mathrm{U}(1)(\mathbb{A}_\mathrm{f})/K$, all with trivial stabilisers. For a prime l, an embedding $\iota\colon\mathbb{C}\to\mathbb{C}_l$ and every function s there are a point z and a character χ of weight 0 with tame part trivial off S such that the class sum of $g \mapsto \chi(\det(gz))s(gz)$, divided by the class sum of 1, has norm at least $\|\iota s(1)\|$ under ι.*

(b) *Let G be a finite abelian group, $V^{\#G} = \prod_{\chi\in G} a(\chi)b(\chi)$ and ι a ring map into a normed field. Every $\psi \in G$ factors as $\psi = \phi\chi$ with $\|\iota V\| \leq \|\iota(a(\phi)b(\chi))\|$.*

Proof. (a) The group is commutative, so the double cosets are the cosets of ΓK. The finite part of a norm-one adele that is rational, or integral and $\equiv 1$ at S, lies in ΓK; so every character of the finite group of classes comes from a character of [U(1)] of weight 0 with tame part trivial off S (Theorem B.19). Translate by z so

that the representative of the trivial class becomes 1. By orthogonality of the characters of the class group, the normalised twisted class sums add up, over all characters, to $s(1)$, and the norm of a sum is at most the largest norm of its terms.

(b) Choose χ with $\|\iota(a(\psi/\chi)b(\chi))\|$ largest. As $\chi \mapsto \psi/\chi$ permutes G, the product of these norms over χ is $\|\iota V\|^{\#G}$, so the largest is at least $\|\iota V\|$; put $\phi = \psi/\chi$. □

Lemma 35.17 (A test form and its pairing). *Let p be a main pair for j with point z, $X = X_{p,\vartheta}$, l a prime with an embedding $\mathbb{C} \to \mathbb{C}_l$ and $l \nmid \mathrm{N}\, Q$. Let q, q_2, q_3 be flat pairs on the blocks of sizes $2z-20$, 4, 4, the first and third at the trivial character, the second at ϑ.*

(a) *If F takes values in an $\mathcal{O}$-module $\mathcal{N}$ of forms, $\mathcal{O}$ the valuation ring of the embedding, then $\Sigma_{q,1}F \in \mathcal{N}$. If a conjugate linear form Λ takes at $F(h)$ the value $B \cdot A\, \theta_{X_{q,1}}(g,h)_i$ for every h and every coordinate i, then $N\mathsf{a}^{n'}\Lambda(\Sigma_{q,1}F) = B \cdot n_L(q) L_q^{\mathrm{alg}}(1) \cdot A$, with N the class number of $\mathrm{U}(1)$ at J_X.*

(b) *For a flat pair every coordinate of its kernel at (g,h) is the scalar kernel of the residue block datum, read modulo any multiple of the square of the chosen places times the twist places, with polynomial 1 and character χ_V.*

(c) *Let the point form at the block diagonal point $\mathrm{diag}(1, h_2, h_3, h_1)$ have the test column at the chosen places, the twist row of the second block at Q, and an integer vector c whose polynomial is a times a monomial α in the first block, of degree ℓ_τ at τ. If χ and the characters of the last three blocks multiply to the character of X, its inner product with $\theta_X(g,\cdot)$ is a times the product of the four kernels at g, the first of which is the kernel of χ, the indicator of x_0 and the monomial α.*

(d) *If c is fixed by the representation of the weight at every point whose columns in the first block are those of 1, the class sum $\Sigma_{q,1}$ over h_1 of these point forms is a test form at l.*

(e) *There is a test form Ξ on $\mathrm{U}(2z)$ at l with*

$$\langle \Xi, \theta_X(g,\cdot)\rangle\, N^3 \mathsf{a}^{2z-12} = \frac{\prod_\tau \ell_\tau!}{\alpha!}\, n_L L_q^{\mathrm{alg}}(1)\, n_L L_{q_3}^{\mathrm{alg}}(1)\, n_L L_{q_2}^{\mathrm{alg}}(\vartheta)\, \theta_{\chi,x_0,\alpha}(g)$$

for all g, where $\theta_{\chi,x_0,\alpha}$ is the kernel of the first block in (c).

Proof. (a) *Integrality.* n_{Aut} clears the stabilisers in the class sum, so it suffices that each summand lies in $\mathcal{N}$: its scalar is the product of $d/\mathrm{N}\, Q^{n'-1}$, integral as $l \nmid \mathrm{N}\, Q$, and of $\langle 1, d\, \Xi_q(h)\rangle$, integral because $d\, \Xi_q(h)$ has gauge at most one (Proposition 35.15(a)) and in weight 0 the group acts as the identity, so the vector 1 has gauge one. *The value.* By adjointness, $\|\Xi_q\|^2$ is the class sum over $\mathrm{U}(1)$ of the pairings of Ξ_q with the kernel at g (Lemma 35.9(b)), and this pairing does not depend on the point g of $\mathrm{U}(1)$, since the twisted central translations fix the lift and carry the kernel at g to the kernel at gz (Lemma 35.11(b)). Unfolding both sides into double class sums of double sums over coordinates, they agree term by term by the hypothesis on Λ; the scale $\mathsf{a}^{n'}$ of the lift and the class number N account for the factor.

(b) is Lemma 35.9(e), the block datum being a function of the residues that can be read modulo any multiple of the modulus.

(c) Read the data of the four blocks modulo the same ideal. The datum of the point form is the product of the data of the four blocks: at the chosen places the

test column restricts to x_0 and to the basis vectors, and at Q the twist row lies in the second block, where the twisted character sits; on both sides this is the same product over the twist places under the same condition. The polynomial is a times the monomial α in the first block and 1 in the others. By the seesaw identity for orthogonal sums of spaces the kernel of the product datum at the block diagonal point is the product of the kernels of the blocks, and the product of the characters is the character of X. With Lemma 35.1(b) this is the claim.

(d) *Integrality.* The point has the component 1 on the first block, so it fixes the integral vector c, whose coordinates are rational integers; hence the point form is integral (Lemma 35.1(b) and the gauge at the point), and (a) keeps integrality. *Orthogonality.* Let y be in the image of the mean over $\prod_{v\in M}\mathrm{GL}_1(\mathbb{F}_v)$, killed by $H_w(X_*)$ and by the descent. The main pair is ramified at the chosen places, so its tame part χ_v is nontrivial there, and y is killed by the hyperplane sums (Corollary 35.8(b)). Expanding the lift Ξ_q of the last block over U(1), the pairing of y with the class sum is a sum, over the classes of U(1), of periods of y against conjugate theta kernels of (W, I_{2z-20}) at the points $(1, h_2, h_3)$, with data that are the vacuum off the level. These vanish by Corollary 35.8(a), applied with I_N the first three blocks, $N = 20$, and the rank condition $2\cdot 1+2\cdot 20+3 < 2z$; the hypotheses hold: the level fixes the unit column at the chosen places, hence both data, and in weight 0 every coordinate of the kernel is the same theta constant by (b). Exchanging the two class sums and pulling out the scalars gives 0.

(e) Let c be the integer vector of the weight of X that is fixed by the points whose columns in the first block are those of 1 and whose polynomial is a times the monomial α, $a = \prod_\tau \ell_\tau!/\alpha!$ (it exists by the theory of Schur modules, Appendix B.11). Put $\Xi = \Sigma_{q_2,\vartheta}\big(h_2 \mapsto \Sigma_{q_3,1}(h_3 \mapsto \Sigma_{q,1}(h_1 \mapsto F(h_1,h_2,h_3)))\big)$ with F the point form of (c). By (d) the inner class sum is a test form, and the two outer class sums keep integrality, by (a), and orthogonality to the false lifts, by linearity. By (c) the pairing of F with the kernel of X factors over the four blocks, and three applications of (a), summing out the three blocks one after the other, give the formula; the scales a^{2z-20}, a^4, a^4 give a^{2z-12}. □

Lemma 35.18 (A geometric mean of untwisted values). *Let $q_1(\chi)$ and $q_3(\chi)$ be flat pairs on n and on four indices, for every tame class character χ. There is $\mathcal{L} \neq 0$ with $\mathcal{L}^{\#G} = \prod_\chi n_L L^{\mathrm{alg}}_{q_3(\chi)}(1)\cdot n_L L^{\mathrm{alg}}_{q_1(\chi)}(1)$, G the group of these characters, and $|\sigma\mathcal{L}| \le \exp(x^{34})^2$ for every $\sigma \in \mathrm{Aut}(\mathbb{C})$.*

Proof. Take a root $\mathcal{L}$ of the product, which is nonzero by Proposition 35.13(c). For σ, Lemma 35.16(b) with $\iota = \sigma$, $a = n_L L^{\mathrm{alg}}_{q_3}$ and $b = n_L L^{\mathrm{alg}}_{q_1}$ bounds $|\sigma\mathcal{L}|$ by a product of two values at characters, each at most $\exp(x^{34})$ by Proposition 35.15(b). □

Theorem 35.19 (Twisted value heights). *Let ξ_ι be a regular family of m-fold twists (Definition 36.1).*

(a) *(A form with small pairing.) For a main pair p for j and a pair $q_2 = (\nu_2, 2)$ with empty row on four indices there is $c_0 \neq 0$, with conjugates at most $\exp(fmb/k + 3x^{34})$, such that for every prime ℓ, ι and twist ϑ with modulus prime to ℓ some test form Ξ on $\mathrm{U}(2z)$ has $v(\langle \Xi, \varphi_{p,\vartheta}\rangle) \le v(c_0\varrho_{q_2}(\vartheta)) +$*

$\sum_\tau s_\ell(\kappa(\tau))$, *the last term being the exponent of* ℓ *forced on derivatives of order* κ.

(b) *For a main pair* p *for* j, *the inverse ratios* $|L(z,\nu^{-1})|/|L(z,(\nu\xi_\iota^j)^{-1})|$ *have height at most* $6\,\mathrm{budget} - h_{\mathrm{dir}}$.

(c) $\Gamma_j(\vartheta)/\Gamma_j(1)$ *is a power of* $\mathrm{N}\,Q$ *times the square of a product of* $k/2$ *ratios of main pairs and* $5m$ *ratios of ramified pairs, with fixed signs; the inverses of* $\Gamma_j(\xi_\iota^j)/\Gamma_j(1)$ *have height at most* $10mx_1 + k(6\,\mathrm{budget} - h_{\mathrm{dir}})$.

Proof. (a) *The blocks and the constant.* Split the $2z$ indices into the blocks of sizes 12 (the derivative), 4 (the twist), 4 and $2z-20$ (the two factors of a tame class character ψ); here $2z-20 \le t$ and $4 \le t$. For each tame class character χ take flat pairs $q_1(\chi)$ on the last block and $q_3(\chi)$ on the third with tame parts given by χ (Proposition 35.15(b)), and let $\mathcal{L}$ be their geometric mean (Lemma 35.18). The degrees $\ell_\tau = c(\tau) - z$ lie in $[\kappa(\tau), \kappa(\tau) + x^{42}]$ (proof of Proposition 35.15(c)), so Theorem 14.11 applies and gives t with $|\sigma t| \le \exp(fmb/k)$. Put

$$c_0 = t \cdot L_{\mathrm{tw}} \cdot \mathcal{L}, \qquad L_{\mathrm{tw}} = n_L L_{q_2}^{\mathrm{alg}}(1);$$

then $c_0 \ne 0$ and $|\sigma c_0| \le \exp(fmb/k)\exp(x^{34})\exp(x^{34})^2 = \exp(fmb/k + 3x^{34})$. *The first block.* Fix ℓ, ι and ϑ with $\ell \nmid \mathrm{N}\,Q$. Theorem 14.11 at ℓ gives a class x_0 modulo π_M and a monomial α of degrees ℓ_τ such that, for the kernel $s(g)$ of the first block with the indicator of x_0 and the monomial α and a character χ_0, and the scalar $c = \mathsf{a}^{12} t^{-1} \frac{\prod_\tau \ell_\tau!}{\alpha!}(\bar\pi_M \mathsf{t})^J$,

$$\ell^{-\sum_\tau s_\ell(\kappa(\tau))} \le \|\iota c\|\,\|\iota s(1)\|.$$

Averaging. By Lemma 35.16(a) at the chosen places there are z_0 and a tame class character ψ such that the normalised twisted class sum $\mathcal{A}$ of $g \mapsto \psi(\det(gz_0))s(gz_0)$ has $\|\iota\mathcal{A}\| \ge \|\iota s(1)\|$; by Lemma 35.16(b) there is a factorisation $\psi = \psi_3\psi_1$ with $\|\iota\mathcal{L}\| \le \|\iota W\|$, $W = n_L L^{\mathrm{alg}}_{q_3(\psi_3)}(1)\, n_L L^{\mathrm{alg}}_{q_1(\psi_1)}(1)$. The characters of the four blocks, $\chi_0\psi$ on the first, $\nu_2\vartheta$ on the second and those of $q_3(\psi_3)$, $q_1(\psi_1)$, multiply to the character of $X_{p,\vartheta}$, as $\psi = \psi_3\psi_1$. *The pairing.* Let Ξ be the test form of Lemma 35.17(e) for these pairs. By adjointness the pairing of Ξ with φ_X is the class sum over $\mathrm{U}(1)$ of its pairings with $\theta_X(g,\cdot)$ (Lemma 35.9(b)), whose first block kernels are $\psi(\det g)$ times s; summing over the classes, the class sum of ψs is $N\mathcal{A}$, and with $\varphi_{p,\vartheta} = \mathsf{a}^{2z}(\bar\pi_M\mathsf{t})^J\varphi_X$ and the definition of c we get

$$N^2\langle \Xi, \varphi_{p,\vartheta}\rangle/\varrho_{q_2}(\vartheta) = t\, L_{\mathrm{tw}}\, W\, c\, \mathcal{A},$$

where $n_L L^{\mathrm{alg}}_{q_2}(\vartheta) = \varrho_{q_2}(\vartheta) L_{\mathrm{tw}}$ by Proposition 35.13(c). Hence

$$\begin{aligned}\|\iota(c_0)\|\,\ell^{-\sum_\tau s_\ell(\kappa(\tau))} &\le \|\iota t\|\|\iota L_{\mathrm{tw}}\|\|\iota W\|\|\iota c\|\|\iota\mathcal{A}\| = \|\iota(N^2\langle\Xi,\varphi_{p,\vartheta}\rangle/\varrho_{q_2}(\vartheta))\| \\ &\le \|\iota(\langle\Xi,\varphi_{p,\vartheta}\rangle/\varrho_{q_2}(\vartheta))\|,\end{aligned}$$

the class number N being an integer; written with valuations, this is the claim.

(b) *The ratio* ρ. Let q_2 be the flat pair on four indices with tame part ω (Proposition 35.15(b)); it is ramified, so by the small ratios (Definition 36.1) the family ρ_ι^{-1}, $\rho_\iota = \varrho_{q_2}(\xi_\iota^j)$, has height at most x_1. Let c_0 be as in (a); since $fmb/k + 3x^{34} \le \mathrm{budget}$, $|\sigma c_0| \le e^{\mathrm{budget}}$. *Test forms.* At ι and at its conjugate $\bar\iota$,

(a) gives test forms Ξ_ι, Ξ'_ι for the twist $\vartheta = \xi_\iota^j$, whose modulus is prime to ℓ. Let $\varphi = \varphi_{p,\vartheta}$ and $\varphi_1 = \varphi_{p,1}$. Since ρ is real, the two bounds multiply to

$$\|\iota(c_0\bar{c}_0\rho_\iota^2)\|\,\ell^{-2\sum_\tau s_\ell(\kappa(\tau))} \le \|\iota(\langle\Xi_\iota,\varphi\rangle\langle\varphi,\Xi'_\iota\rangle)\|,$$

and the right side is nonzero. *Four factors.* With pr the orthogonal projection to $\mathbb{C}\varphi$, $\langle \operatorname{pr}\Xi, \operatorname{pr}\Xi'\rangle = \langle\Xi,\varphi\rangle\langle\varphi,\Xi'\rangle/\|\varphi\|^2$. By Proposition 35.13(c), up to the integer power $\mathrm{N}\,Q^{2z-1}$, the inverse ratio is $\|\varphi_1\|^2/\|\varphi\|^2$, so

$$\frac{\mathrm{N}\,Q^{2z-1}}{\varrho_p(\vartheta)} = \|\varphi_1\|^2\cdot\langle\operatorname{pr}\Xi_\iota,\operatorname{pr}\Xi'_\iota\rangle\cdot\rho_\iota^{-2}\cdot\frac{\rho_\iota^2}{\langle\Xi_\iota,\varphi\rangle\langle\varphi,\Xi'_\iota\rangle}.$$

The power of $\mathrm{N}\,Q$ is an integer, of norm at most one at ι, so it suffices to bound the height of the right side, and heights add (Lemma 7.3(b)). The first factor has height at most budget $-\sum_\tau s_\infty(\kappa(\tau))$ by Proposition 35.15(c). The second has height at most budget: by the regularity of the family there are β_ℓ with $\sum_{\ell\in T}\beta_\ell \le$ budget such that the product of the projections of two test forms to the line of φ, which is the line of the lift for the data of φ, has norm at most e^{β_ℓ} at ι (Definition 36.1; this is the condition that Proposition 35.15(d) reduces to the denominator of the line), and Lemma 7.3(d) applies with the multiplier 1. The third has height at most $2x_1$. For the fourth take the multiplier $c_0\bar{c}_0$, with $|\sigma(c_0\bar{c}_0)| \le e^{2\,\mathrm{budget}}$, and the exponents $2\sum_\tau s_\ell(\kappa(\tau))\log\ell$ from the displayed bound; since $\sum_{\ell\in T}2\sum_\tau s_\ell(\kappa(\tau))\log\ell \le \sum_\tau s_{\mathrm{fin}}(\kappa(\tau),\tau)$ for every finite set T of primes (Proposition 19.1(b)), it has height at most $2\,\mathrm{budget} + \sum_\tau s_{\mathrm{fin}}(\kappa(\tau),\tau)$ by Lemma 7.3(d). The sum is

$$4\,\mathrm{budget} + 2x_1 - \sum_\tau\big(s_\infty(\kappa(\tau)) - s_{\mathrm{fin}}(\kappa(\tau),\tau)\big) \le 6\,\mathrm{budget} - h_{\mathrm{dir}},$$

as $x_1 \le$ budget.

(c) *The expansion.* $\Gamma_j(\vartheta)$ is the product over the three exchange pairs (ν_i, z_i), $i =$ none, $+1, -1$, of $\prod_{w\in M\cup Q} q_w^{z_i^2}\prod_{z'\le z_i}|L(z',(\nu_i\vartheta^{s_i})^{-1})|^2$ with $s_i = \pm1$ (Section 34.1). By the last identity of Proposition 35.13(c) for each of them, $\Gamma_j(\vartheta) = \Gamma_j(1)\,\mathrm{N}\,Q^{\sum_i z_i^2}\big(\prod_i\prod_{z'\le z_i}\varrho_{(\nu_i,z')}(\vartheta,s_i)\big)^2$. *The pairs.* The pair (ν_j, z_*) of the j-th lift has the point $z_* = 7(m/2)+k/2$, and for $7(m/2) < z' \le z_*$ the pair (ν_j, z') is a main pair for j: these are $k/2$ ratios with $s = 1$. All the pairs (ν_i, z') are ramified, their tame parts being nontrivial exactly at the chosen places; the remaining factors are the $7m/2$ ratios of (ν_j, z') with $z' \le 7(m/2)$ and those of the two flat pairs at the points m and $m/2$, together $5m$. The exponent, the pairs and the signs do not depend on ϑ. *The height.* The inverse of $\Gamma_j(\xi_\iota^j)/\Gamma_j(1)$ is $\mathrm{N}\,Q^{-e}$ times the inverse squares of these ratios. A main ratio has inverse of height at most $6\,\mathrm{budget} - h_{\mathrm{dir}}$ by (b), and a ramified one at most x_1 by the small ratios; the squares of the products have height at most $2\big(\frac{k}{2}(6\,\mathrm{budget}-h_{\mathrm{dir}})+5mx_1\big) = 10mx_1+k(6\,\mathrm{budget}-h_{\mathrm{dir}})$ (Lemma 7.3(b)). Multiplying by an integer power of $\mathrm{N}\,Q$, which is prime to the prime of ι, keeps the bound (Lemma 7.3(c)). □

36. The cost of one lift and the proof of Theorem C

This section assembles the lower bound. The cost of one lift, the height of $Z_j(1)/Z_j(\xi_\iota^j)$, is bounded through the comparison operator for a family of twists with few congruences (Theorem 36.4); the existence of such a family is the subject of Part VI. With the factorisation of the chain volume, the auxiliary forms and Theorem 7.14, this proves Theorem C.

36.1. Families of twists

A family of twists indexed by the prime places $\iota\colon \mathbb{C} \simeq \bar{\mathbb{Q}}_\ell$ (Definition 7.1) assigns an m-fold twist ξ_ι to each ι. For a pair $p = (\nu, z)$, a tame ϑ and a sign s, write $\varrho_p(\vartheta, s) = |L(z, (\nu\vartheta^s)^{-1})|/|L(z, \nu^{-1})|$, both L-functions without the factors at the chosen places and their conjugates, the numerator also without Q. Numbers $e_\ell \geq 0$ are *negligible with exponent* C if $\sum_{\ell\in T} e_\ell \log \ell \leq x^C$ for every finite set of primes. The *denominator budget* is budget $= f a \log^3 b/(mk)$.

Definition 36.1 (Small ratios, regular families). A family ξ_ι *has small ratios* if $\mathrm{N}\, Q(\xi_\iota)$ is prime to the prime of ι and, for $1 \leq j \leq m$, every ramified pair p and $s = \pm 1$, the family of the inverse ratios $\varrho_p(\xi_\iota^j, s)^{-1}$ has height at most x_1. It is *regular* if moreover, for $1 \leq j \leq m$, there are reals β_ℓ with $\sum_{\ell\in T} \beta_\ell \leq$ budget for every finite set of primes such that at every ι over ℓ the projection to the space of lifts for ξ_ι^j has denominator at most e^{β_ℓ} and, for every main pair p for j, the inner product of the projections of two test forms to the line of φ_{p,ξ_ι^j} has norm at most e^{β_ℓ} at ι (Section 35.1).

Theorem 36.2 (Twists with few congruences). *Every admissible datum has a regular family of m-fold twists.*

Proof. This is proved in Part VI, at the end of Section 45: for each ℓ and ι a Selmer group argument produces an m-fold twist whose first m powers have few congruences, and the denominators summed over ℓ stay within the budget. □

36.2. The cost of one lift

Fix $1 \leq j \leq m$. For a twist ϑ let Θ_ϑ be the lift of the good forms on W_j (Section 30.3), $Z_j(\vartheta) = \det(\Theta_\vartheta^*\Theta_\vartheta)$ its Gram determinant, $\mathcal{Z}_\vartheta$ its range, the space of lifts, and pr the orthogonal projection to $\mathcal{Z}_\vartheta$. For the comparison operators U_ϑ of Theorem 34.14 put $\Delta(\vartheta) = \det(\mathrm{pr}\, U_\vartheta|_{\mathcal{Z}_\vartheta})$ and $A = f m b \log^2 b$. The j-th power of the trivial m-fold twist is the trivial tame character, so $Z_j(1)$ is the untwisted determinant.

Lemma 36.3 (Compressions of the comparison operator). (a) *For an injective linear map u of finite-dimensional inner product spaces, the projection* pr *to its range and an operator T,* $\det(\mathrm{pr}\, T|_{\mathrm{im}\, u}) \det(u^*u) = \det(u^*Tu)$.

(b) *Θ_ϑ is injective,* $\dim \mathcal{Z}_\vartheta = d_T$, *and* $Z_j(\vartheta)\Delta(\vartheta) = C\rho_\vartheta^{d_T}$ *with $C \neq 0$ independent of ϑ; in particular $\Delta(\vartheta) \neq 0$.*

(c) $|\sigma\Delta(1)| \leq e^{3Ad_T}$ *for every automorphism σ of $\mathbb{C}$.*

(d) *Let ι lie over $\ell \nmid \mathrm{N}\,Q(\vartheta)$, and let an operator Λ on $\mathcal{Z}_\vartheta$ with $|\iota \det \Lambda| \geq e^{-d_T r\beta}$ make $\Lambda(\mathrm{pr}\, y)$ integral at ι for every integral y orthogonal to the false lifts. Then $|\iota\Delta(\vartheta)| \leq e^{d_T r\beta}$.*

Proof. (a) u^* kills the orthogonal complement of the range, so $u^*\mathrm{pr} = u^*$ and $u^*u\,(u^{-1}\mathrm{pr}\,Tu) = u^*Tu$, where u^{-1} is the inverse of u onto its range; take determinants.

(b) Injectivity is Theorem 30.5(a), and the good forms have dimension d_T (Theorem 23.2). By (a) with $u = \Theta_\vartheta$ and $T = U_\vartheta$, and by Theorem 34.14(a), $Z_j(\vartheta)\Delta(\vartheta) = \det(\Theta_\vartheta^* U_\vartheta \Theta_\vartheta) = \det(\rho_\vartheta U^\circ) = \rho_\vartheta^{d_T} \det U^\circ$, and $C = \det U^\circ > 0$ as U° is positive definite and independent of ϑ. Since $\rho_\vartheta \neq 0$ and $Z_j(\vartheta) > 0$, $\Delta(\vartheta) \neq 0$.

(c) The lifts lie in the image of the idempotent e_0 of the good forms and are killed by $H_w(X_*)$ at every split w off the level (Theorem 30.5(d)), so Proposition 35.3(c) applies to $Z = \mathcal{Z}_1$. Let $z \neq 0$ be an eigenvector of $\mathrm{pr}\, U_1$ on $\mathcal{Z}_1$ with eigenvalue λ. The form $U_1 z$ is orthogonal to the false lifts (Theorem 34.14(b)), so at every σ

$$|\sigma\lambda|^2 \|z\|_\sigma^2 = \|\mathrm{pr}\, U_1 z\|_\sigma^2 \leq \|U_1 z\|_\sigma^2 \leq e^{6A}\|z\|_\sigma^2,$$

the last step by Theorem 34.14(d), where U_1 multiplies squared norms by at most e^{6A}. As $\|z\|_\sigma > 0$ (Lemma 35.2(d)), $|\sigma\lambda| \leq e^{3A}$, and $\sigma\Delta(1)$ is the product of the d_T numbers $\sigma\lambda$, eigenvalues counted with multiplicity.

(d) The operator $\Lambda\,\mathrm{pr}\, U_\vartheta$ maps the integral lifts to integral lifts: for an integral lift z, $U_\vartheta z$ is integral (Theorem 34.14(c), as $\ell \nmid \mathrm{N}\,Q$) and orthogonal to the false lifts. Let z be an eigenvector with eigenvalue μ, scaled to valuation 0 at ι: z is integral and αz is integral only if $|\iota\alpha| \leq 1$ (divide by a coordinate of least valuation at representatives integral above ℓ). Then μz is integral, so $|\iota\mu| \leq 1$, and $|\iota \det(\Lambda\,\mathrm{pr}\, U_\vartheta)| \leq 1$. Dividing by $|\iota \det \Lambda| \geq e^{-d_T r\beta}$ gives the claim. □

Theorem 36.4 (The cost of one lift). *Let ξ_ι be a regular family and $1 \leq j \leq m$. The family $Z_j(1)/Z_j(\xi_\iota^j)$ has height at most $d_T\, k(\log b \cdot \mathrm{budget} - h_{\mathrm{dir}}) = d_T\big(f\, a \log^4 b/m - k\, h_{\mathrm{dir}}\big)$.*

Proof. The factorisation. By Lemma 36.3(b) at ϑ and at 1, where $\rho_1 = 1$,

$$\frac{Z_j(1)}{Z_j(\vartheta)} = \frac{\Delta(\vartheta)}{\Delta(1)}\,\rho_\vartheta^{-d_T}.$$

The first factor. Let $\vartheta = \xi_\iota^j$ and let β_ℓ be the reals of the regular family, $\sum_{\ell\in T}\beta_\ell \leq$ budget for every finite set T of primes. The modulus of ξ_ι is prime to the prime ℓ of ι, and the projection to the lifts for ξ_ι^j has denominator at most e^{β_ℓ} at ι; by Lemma 36.3(d), $|\iota\Delta(\xi_\iota^j)| \leq e^{d_T r\beta_\ell}$. Take $H = \Delta(1) \neq 0$ as a common multiplier: by Lemma 36.3(c) $|\sigma H| \leq e^{3Ad_T}$, and $\|H \cdot \Delta(\xi_\iota^j)/\Delta(1)\|_\iota = |\iota\Delta(\xi_\iota^j)| \leq e^{d_T r\beta_\ell}$. By Lemma 7.3(d) the family $\Delta(\xi_\iota^j)/\Delta(1)$ has height at most $d_T(3A + r\,\mathrm{budget})$. *The second factor.* By Theorem 35.19(c) the family $\rho_{\xi_\iota^j}^{-1} = \Gamma_j(1)/\Gamma_j(\xi_\iota^j)$ has height at most $10mx_1 + k(6\,\mathrm{budget} - h_{\mathrm{dir}})$, so its d_T-th power has height at most $d_T(10mx_1 + k(6\,\mathrm{budget} - h_{\mathrm{dir}}))$ (Lemma 7.3(b)). *The sum.* Heights add, so $Z_j(1)/Z_j(\xi_\iota^j)$ has

height at most $d_T(3A + r\,\text{budget} + 10mx_1 + 6k\,\text{budget} - k\,h_{\text{dir}})$. By the sizes of the data (Proposition 8.9 and Remark 8.10),

$$3A + r\,\text{budget} + 10mx_1 + 6k\,\text{budget} \le k\log b\cdot\text{budget} = fa\log^4 b/m :$$

indeed $r \le 2k$, so $r\,\text{budget} + 6k\,\text{budget} \le 8k\,\text{budget}$, which is small against $k\log b\cdot\text{budget}$; $3A = 3fmb\log^2 b$ is small against $fa\log^4 b/m = fxb\log^4 b/m$ because $m^2 \le 4x$; and $10mx_1$ is small against $fa\log^4 b/m$ because $a/x_1 \ge x^{999}$. □

36.3. The chain volume

Proposition 36.5 (The inverse ratio). *Let ξ_ι have small ratios, and let each family $Z_j(1)/Z_j(\xi_\iota^j)$, $1 \le j \le m$, have height at most $d_T(fa\log^4 b/m - k\,h_{\text{dir}})$. Then $Z(1)/Z(\xi_\iota)$ has height at most $d_T(b + fa\log^4 b - h_{\text{main}})$.*

Proof. By Theorem 32.14, inverted,

$$\frac{Z(1)}{Z(\xi_\iota)} = s(\xi_\iota)^{-d_T}\prod_{j=1}^{m}\frac{Z_j(1)}{Z_j(\xi_\iota^j)},\qquad s(\vartheta) = (\mathrm{N}\,Q)^{e/2}\prod_{1\le i<j\le m}\varrho_{i,j}(\vartheta),$$

with e independent of ϑ. Each $\varrho_{i,j}$ is a product of t ratios of Hecke values of a ramified pair (Proposition 32.10(c)), whose inverses have height at most x_1 by the small ratios, and the power of $\mathrm{N}\,Q$, prime to the prime of ι, costs nothing (Lemma 7.3(c)); so $s(\xi_\iota)^{-1}$ has height at most m^2tx_1 (Theorem 32.14), and $m^2tx_1 \le b$ by Proposition 8.9. Its d_T-th power has height at most d_Tb. The m bounds for the single lifts add up to $d_T(fa\log^4 b - mk\,h_{\text{dir}})$, and $mk\,h_{\text{dir}} = h_{\text{main}}$. Heights add under products (Lemma 7.3(b)). □

36.4. Proof of Theorem C

Lemma 36.6 (The cost of the completion). *If $n \ge (1 - 4/b^2)d_T$ and $d_T = n + n'$, then $n'(2x_1^2b^2 + b) \le d_Tb$.*

Proof. $n' \le 4d_T/b^2$, so $n'(2x_1^2b^2 + b) \le 4d_T(2x_1^2 + b^{-1}) \le d_Tb$, because $4(2x_1^2 + b^{-1}) \le b$: by Proposition 8.9, $\log x_1 \le (\log b)^{1/6}$, so x_1^2 is small against b. □

Proof of Theorem C. Let $y_1, \dots, y_n$ be independent target forms with $n \ge (1 - 4/b^2)d_T$, of index at most δ and covolume at most c in the twisted lattice $L_{\mathcal{T}}$ (Definition 7.7).

The twists. By Theorem 36.2 there is a regular family ξ_ι; it has small ratios, and the modulus of ξ_ι is prime to the prime of ι, so ξ_ι is allowed at that prime. By Theorem 36.4 each family $Z_j(1)/Z_j(\xi_\iota^j)$ has height at most $d_T(fa\log^4 b/m - k\,h_{\text{dir}})$, and by Proposition 36.5 the family $Z(1)/Z(\xi_\iota)$ has height at most $b_{\text{vol}} = d_T(b + fa\log^4 b - h_{\text{main}})$.

The completion. By Theorem 33.19 the target is spanned by forms auxiliary for scalars $\gamma(\vartheta) \ne 0$; since y is independent, it can be completed to a basis by auxiliary forms $z_1, \dots, z_{n'}$, $n + n' = d_T$. By the definition of auxiliary forms, $\gamma(\vartheta)\Theta_\vartheta z_i$ is integral at every ι over a prime not dividing $\mathrm{N}\,Q(\vartheta)$, which is condition (ii) of Theorem 7.14, and every conjugate of $\Theta_1 z_i$ has squared norm at most $e^{b_{\text{short}}}$, $b_{\text{short}} = 2x_1^2b^2$, which is (iii). By Theorem 33.19 the family $|\gamma(\xi_\iota)|^{-1}$ has height at most m^2tx_1, so $|\gamma(\xi_\iota)|^{-2}$ has height at most $b_{\text{den}} = 2m^2tx_1 \le b$.

The lattice. The maps Θ_ϑ are injective (Proposition 32.5(d)). Theorem 7.14 with the twists ξ_ι gives

$$c + 2\delta + n'(2x_1^2 b^2 + b) + d_T(b + f a \log^4 b - h_{\text{main}}) \geq 0.$$

By Lemma 36.6 the completion costs at most $d_T b$, and $b \leq f a \log^4 b$. Hence $c + 2\delta \geq d_T(h_{\text{main}} - f a \log^4 b - 2b) \geq d_T(h_{\text{main}} - 3 f a \log^4 b)$. □

Part VI

The lower bound: congruences

37. Galois representations of lifts

Part V reduced Theorem C to the existence of a regular family of twists (Theorem 36.2): for each prime place ι over ℓ, an m-fold twist ξ_ι whose powers have few congruences, in the sense that the denominators of the projections to the lifts are small, and whose inverse ratios of Hecke values are barely divisible by ℓ. This part constructs such twists. Congruences between a lift and forms orthogonal to the lifts produce, by Ribet's method, Selmer classes in the Galois representations attached to the lifts (Sections 38 and 40); the local conditions above ℓ cost a bounded loss (Sections 41 and 42); and a twist ramified at new auxiliary primes makes all these Selmer groups short and keeps the ratios of Hecke values barely divisible (Sections 43 and 44). Section 45 assembles the pieces. That denominators of projections to theta lifts give Selmer classes, and that twisting lowers Selmer ranks, is known in other settings [Pra06, Klo09, MR10]; here the bounds are needed for all primes ℓ at once, summed with weights $\log \ell$.

This section describes the Galois representations that occur: those attached to the eigenforms in a space of lifts are, on traces, a twisted *main summand*, independent of the twist, plus a string of powers of the inverse cyclotomic character.

37.1. Conjugate self-dual representations

Fix a prime ℓ and an isomorphism $\iota\colon \mathbb{C} \simeq \bar{\mathbb{Q}}_\ell$. Let $G = \mathrm{Gal}(\Omega/E)$ for the field E of the data, c the complex conjugation, $g^c = cgc$, and cyc the ℓ-adic cyclotomic character. The *bad places* are the places of E dividing $\ell p_0 p_1 N$. The *avatar* ϑ_ℓ^{-1} of a tame character ϑ is the ℓ-adic Galois character of the base change of ϑ^{-1} to E, through ι. A continuous representation $R\colon G \to \mathrm{GL}_n(\bar{\mathbb{Q}}_\ell)$ is *conjugate self-dual of twist* $1-t$ if its trace form on the span of its image is nondegenerate and $\operatorname{tr} R(g^c) = \mathrm{cyc}(g)^{1-t} \operatorname{tr} R(g^{-1})$ for all g; it is *of level* S with jumps J if moreover, at every place v outside S or above $\ell p_0 p_1$, it is trivial on inertia near 1 (potentially crystalline above ℓ), has the Hodge–Tate jumps $J(\sigma)$ at the embeddings σ over v, and is unramified (crystalline above ℓ) if $v \nmid p_0 p_1$. We record the elementary facts about these notions that the later sections use. For a character a of $[\mathrm{U}(1)]$ of weight e and tame part ψ trivial off the chosen places, and integers k, j, the *avatar* of the algebraic Hecke character $y \mapsto a(y/\bar{y})v(y)^{-k}|y|^{j-k/2}$ is its ℓ-adic Galois character through ι (Theorem B.36(a) in rank one); v is the splitting character (Theorem B.18). A matrix representation u over the ring of integers $\mathcal{O}$ of a finite extension of $\mathbb{Q}_\ell$ is a *lattice* in R if R is conjugate over $\bar{\mathbb{Q}}_\ell$ to u read in $\bar{\mathbb{Q}}_\ell$. A *coefficient ring* is the ring of integers of a finite Galois extension of $\mathbb{Q}_\ell$, and *all large* coefficient rings are those containing a given one.

Lemma 37.1 (Bad places, avatars and lattices). (a) *The places above ℓ and above $p_0 p_1$ are bad. A chosen place lies above p_0 or p_1, and so does its conjugate; the tame part of a tame character is trivial at the chosen places. Every set of places above $p_0 p_1$ has at most $4f$ elements, and some chosen place not above ℓ has residue field of order p_0 or p_1.*

(b) *At a Frobenius element g at a place $w \nmid \ell$ the inverse cyclotomic character takes the value q_w. Two distinct N-th roots of unity in $\bar{\mathbb{Q}}_\ell$ have distance at least $|N|_\ell$.*

(c) *The avatar μ of $a(y/\bar{y})v(y)^{-k}|y|^{j-k/2}$ satisfies $\mu(g^c)\mu(g) = \mathrm{cyc}(g)^{2j-k}$ for all g. At a split place w where ψ is trivial at $\bar{w}$, inertia acts on μ through the element of the residue field that names ψ_w: an element of tame exponent i is killed by $X - \iota(T^i(name))$, T the faithful character of the algebraic closure of the residue field.*

(d) *A lattice u in R has the traces and determinants of R, is continuous if R is, and is trivial where R is.*

Proof. (a) A prime ideal containing one factor of $\ell p_0 p_1 N$ contains the product. The chosen places divide π_M, an integer of K with $\pi_M \bar{\pi}_M = p_0 p_1$, so they lie above p_0 or p_1; the conjugates of places above p_i lie above p_i. The places of the modulus of a tame character are not chosen, and its tame part is trivial off its modulus. Over each p_i there are f chosen places, and a place above p_i is chosen or the conjugate of a chosen one, so there are at most $2 \cdot 2f$ places above $p_0 p_1$. A place above both primes would have residue fields of two characteristics; so for $p_i \neq \ell$ a chosen place above p_i is not above ℓ, and its residue field has p_i elements: $q_1 d \mid p_i - 1$ (Proposition 8.9) and $E \subseteq \mathbb{Q}(\zeta_{q_1 d})$ (Proposition 8.3(a)), so p_i splits completely in E.

(b) The cyclotomic character at a geometric Frobenius at $w \nmid \ell$ is q_w^{-1}. If $\zeta \neq \zeta'$ are N-th roots of unity, $\prod_{\eta \neq 1}(1 - \eta) = N$ over the N-th roots of unity η, and every factor has absolute value at most 1, so $|1 - \zeta'/\zeta| \geq |N|$.

(c) Both sides are continuous characters, so by Chebotarev density (Theorem B.35(d)) it suffices to check them at Frobenius elements at places $w \nmid \ell$ where ψ is trivial at w and $\bar{w}$. There g^c is a Frobenius element at $\bar{w}$, the avatar at a Frobenius element is ι of the value of the Hecke character at a uniformiser, and in the product of the values at $\bar{w}$ and at w the factors of the base change of a and of v cancel (a is a character of U(1) and $v(\bar{y}) = v(y)^{-1}$), leaving $|\varpi_w \bar{\varpi}_w|^{j-k/2}$ at the two places, which is $q_w^{-(2j-k)} = \mathrm{cyc}(g)^{2j-k}$ by (b). For the inertia statement, an element s of tame exponent y is the Artin image of a local unit x (local class field theory with the roots of unity of order 5040 in E, Theorem B.35), so it acts by the value of the character at x, which is ψ_w at the residue of x; this value is T^y of the name of ψ_w.

(d) Some injective $\mathcal{O}$-algebra map from the matrices over $\mathcal{O}$ to the matrices over $\bar{\mathbb{Q}}_\ell$, an inclusion of the entries followed by a conjugation, carries $u(g)$ to $R(g)$ and preserves traces and determinants; it is continuous and injective. □

Definition 37.2 (Main summand). Let $1 \leq r \leq \mathrm{r}$ (the rank of the data) and $t \leq \mathrm{t} = 5\mathrm{r}$. Jumps $J(\sigma)$, multisets of integers indexed by the complex embeddings, are *main summand weights* for (r, t) if, for one $j \leq m$, every jump u at σ is *low*, $\Delta/4 \leq |u| \leq 2m\Delta$, or *high*, $|\epsilon_\sigma u + w_j(\sigma)| \leq \mathrm{t}$ with $\epsilon_\sigma = \pm 1$ the sign of σ relative to the CM type and w_j the weight of the j-th lift; at most rm/k jumps at σ have $|u| \leq 2(\Delta/2)^2$; and at most r jumps at σ and $\bar{\sigma}$ together are negative. A representation R of rank r is *like a main summand* if it is conjugate self-dual of level $\emptyset$ with main summand weights and, in addition: at the places above $p_0 p_1$ prime to ℓ a power prime to the place kills the image of inertia, and for one inertia element γ the determinants $\det(R\gamma - 1)$ and $\det(R(\gamma^c)^{-1} - 1)$ have valuation at most $r\, v_\ell(n_{\mathrm{ch}})$, n_{ch} the order of $\prod_w \mathrm{GL}_{\mathrm{rt}}(\mathbb{F}_w)$ over the chosen places; over all large coefficient rings the image of a lattice spans all matrices and contains a

scalar at an element of cyclotomic character p_i^r; and some inertia element above p_0p_1 acts by a scalar y with $y^{\ell^a j} - 1$ a unit for all a and $1 \le j \le 4r + 2$.

Proposition 37.3 (Main summands). *A representation like a main summand is residually absolutely irreducible over all large coefficient rings, and its stable lattice is unique up to scalars.*

Proof. Irreducibility. Let $\mathcal{O}$ be large, u a lattice in R over $\mathcal{O}$ and $f : \mathcal{O} \to k$ a ring map to a field. The image of u spans all matrices over $\mathcal{O}$ (Definition 37.2), so the images $f(u(g))$ span all matrices over k, and a subspace of k^n stable under them is stable under all matrices, hence 0 or k^n. *Uniqueness of the lattice.* Let u, u' be the matrices of R over $\mathcal{O}$ in the bases Q, Q', and $B = Q^{-1}Q'$, so that $Bu'(g)B^{-1} = u(g)$. Multiply B by a scalar so that $L = B\mathcal{O}^n \subseteq \mathcal{O}^n$ and $L \not\subseteq \varpi\mathcal{O}^n$, ϖ a uniformiser. Then L is stable under the $u(g)$, so its image in $(\mathcal{O}/\varpi)^n$ is a nonzero stable subspace, which by irreducibility is everything; by Nakayama's lemma $L = \mathcal{O}^n$, so the scaled B lies in $\mathrm{GL}_n(\mathcal{O})$: $Q' = \alpha Q k$ with a scalar α and $k \in \mathrm{GL}_n(\mathcal{O})$. □

37.2. Representations of a type

A descent pair O with labels $c = (c_i)$ at its chosen places is a *lift setting* for a twist ϑ if its places are those of the data and of ϑ, its tame part is $\omega^j\vartheta$ with $1 \le j \le m$, the rank r_O and the difference of ranks t_O are odd with $r_O \le \mathrm{r}$, $t_O \le \mathrm{t}$, each c_i is a label of degree r_O whose *scaled* polynomial (roots multiplied by the name of ψ_w^{-1}) does not have the root 1, and over each prime p_k one label is primitive with $r_O = \mathrm{r}$, or is $X - 1$ with $r_O = 1$. A representation is *of type c* if it is conjugate self-dual of a given twist, jumps and level, and inertia at the chosen places acts through the roots of the scaled labels. The *normalised representation* of R is its twist by the avatar of the inverse of the character of W, of weight $-r_O$. We use three facts about lift settings and types. A *good tuple* of labels at the chosen places is a family of labels of degree r (monic, squarefree, nonzero at 0) vanishing at no q_2-th root of unity, q_2 the order of the residue characters ω at the chosen places, with one primitive label over each of p_0, p_1. The *type projection* e_c is the action on forms of the idempotent of the tuple c in the group algebra of $\prod_{w\in M} \mathrm{GL}_{r_O}(\mathbb{F}_w)$. The normalised jumps of O are the jumps of the weight of the forms on V minus the infinity type of the normalising character.

Lemma 37.4 (Lift settings). (a) *A good tuple c forms a lift setting with the j-th descent pair, for every twist ϑ and $1 \le j \le m$. For a main pair p for j, the descent pair of the lift of $v\vartheta$ from* U(1)*, with every label $X - 1$, is a lift setting for ϑ, and the lift $\varphi_{X_{p,\vartheta}}$ lies in the range of its type projection.*

(b) *For $1 \le d \le \mathrm{rt}$, $p_i^d - 1$ divides n_{ch}, so $v_\ell(p_i^d - 1) \le v_\ell(n_{\mathrm{ch}})$.*

(c) *For a lift setting, e_c is self-adjoint and fixes its range, which is stable under the Hecke polynomials at the split places off the level; infinitely many split places are, with their conjugates, off the level.*

(d) *Let f be an eigenform of weight κ on the definite unitary group of rank n', R attached to it, and let f be spherical at the places off S, above ℓ or above p_0p_1, except the chosen places and their conjugates, with a type at each chosen place induced from the label c_i of degree r and trivial characters on blocks. Then R is*

of a type: conjugate self-dual of twist $1-n'$ *and level* S *with the jumps of* κ*, and inertia at the chosen places acts through the roots of* $c_i(X-1)^{n'-r}$.

(e) *Twisting a representation of a type by the avatar of* $a(y/\bar{y})\upsilon(y)^{-k'}|y|^{j'-k'/2}$*, with the tame part of a trivial off the chosen places, gives a representation of a type of the same level, with twist shifted by* $2j'-k'$*, jumps lowered by the infinity type, and the roots of the labels multiplied by the element naming the tame part of a at the chosen place.*

(f) *If* n' *is even and* $l=2\lambda$ *fits* O*, the normalised jumps at* σ *are* $0,1,\dots,t_O-1$ *and* $\frac{t_O-1}{2}\mp(\lambda_c+\kappa')$*,* c *running over the basis of* W*, with the upper sign on the CM type;* $-\kappa'$ *is the weight of the character of* O.

Proof. (a) *Descent pairs.* The places of the j-th descent pair are those of the data and of ϑ, its tame part is $\omega^j\vartheta$, and its ranks r and t are odd (Proposition 8.9). The scaled label at w has the root 1 only if a root of c_w names ω^j at w, a root of unity of order dividing q_2, which a good label does not have; if y is a unit that is not a root of c, then 1 is not a root of c with its roots multiplied by y^{-1}. *The lift from* U(1). There W has rank 1 and the difference of ranks is $2z-1$, both odd, and $1\le \mathrm{r}, 2z-1\le \mathrm{t}$. The scaled label of $X-1$ has the root 1 only if the tame part of ν is trivial at the place, and a main pair is ramified (Definition 10.15, Section 36.1). The label $X-1$ belongs to the trivial character of GL_1, so the type projection is the mean over $\prod_{w\in M}\mathrm{GL}_1(\mathbb{F}_w)$, which fixes $\varphi_{X_{p,\vartheta}}$ (Proposition 35.13(b)).

(b) p_i^d-1 is the first factor of $\#\mathrm{GL}_d(\mathbb{F}_{p_i})=\prod_{e<d}(p_i^d-p_i^e)$, which divides $\#\mathrm{GL}_{\mathrm{rt}}(\mathbb{F}_w)$ at a chosen place w above p_i, and n_{ch} is a multiple of these orders.

(c) The idempotent is a central idempotent of the group algebra, hence self-adjoint (Lemma 35.2(b)) and the identity on its range. A group element acts by a root of unity times translation by a point that is 1 at w and $\bar{w}$, which commutes with the translations supported at w and $\bar{w}$, hence with the Hecke polynomial. Places of prime norm split, there are infinitely many, and finitely many lie in the level or its conjugate.

(d) Off the chosen places this is Theorem B.36(a) together with conjugate self-duality, proved in Theorem 37.5(a) below without using (d). At a place over a chosen place w the type of f is a product label at w, so Theorem B.36(c) gives finite type, the jumps and a polynomial built from the tame characters of the label, c_i for the block of size r and $X-1$ for the trivial blocks of total size $n'-r$, that kills inertia. At the conjugate places these pass by conjugate duality: $R\circ(g\mapsto g^c)$ is conjugate to the twisted dual of R, both having nondegenerate trace forms and the same traces, conjugating an embedding replaces a jump u by $n'-1-u$, and a polynomial killing s at $\bar{A}$ kills $(s^c)^{-1}$ at A.

(e) Each local clause passes to the twist from the clause for R and that for the character. The avatar is of finite type with the single jump minus the infinity type; off p_0p_1 the character is unramified, neither the place nor its conjugate being chosen; and an inertia element s acts on the avatar by $\varphi(u)$, u the name of the tame part, and on R through the values of φ at the roots of c_i, so on the twist through these values times $\varphi(u)$, as $\varphi(u)X-\varphi(ux)=\varphi(u)(X-\varphi(x))$. The twist of the cyclotomic exponent is Lemma 37.1(c).

(f) The normalising character has exponent r_O-r_- on the CM type and r_- off it, r_- the number of minus signs of W, and entry p of twice the Harish-Chandra parameter is $2\mu_p+n'-1-2p$ for the highest weight μ. Hence the normalised jumps are $(t_O-1\mp(a_p-2r_-+r_O))/2$, $p<n'$, a the doubled parameter. By

the block rule (Proposition 15.11(b)) each entry of a is a shifted entry of l or a member of the doubled block of length t_O, which gives the stated values; both sides are sets of n' distinct integers, so the inclusion is an equality. □

Theorem 37.5 (Attached representations). *Let O, c be a lift setting for ϑ with $Q(\vartheta)$ auxiliary (its places of prime norm and, with their conjugates, not bad).*

(a) *A Galois representation attached to an eigenform on a definite unitary group of rank n is conjugate self-dual of twist $1-n$.*

(b) *For an eigenform f of type c on the larger space of O, with the level of O off a set S of places outside the bad places, the normalised representation of an attached R is of type c: conjugate self-dual with the jumps $0, 1, \dots, t_O - 1$ and $\frac{t_O-1}{2} \mp (\lambda_c + \kappa')$ (the upper sign on the CM type, λ the parameter on W and $-\kappa'$ the weight of the character of O), of level Q, S and their conjugates, with inertia at the chosen places acting through the roots of the scaled labels times $(X-1)^{t_O}$.*

(c) *There are base data whose forms contain the images of the type projection e_c of the forms of O; $n_{\mathrm{ch}} e_c$ keeps integral forms; and the normalised representations of their eigensystems of auxiliary level are of type c, with $(u(g) - \vartheta_\ell^{-1}(g))(u(g) - 1) = 0$ for g in inertia above Q and its conjugates.*

Proof. (a) *A coefficient identity.* Let $f \neq 0$ in rank $n = k+1$ be spherical at a split w and at $\bar{w}$, with eigenvalues s_i and u_i of the i-th Hecke operators there, so that its Hecke polynomials are $Q_w = \sum_i (-1)^i q^{i(i-1)/2} s_i X^{n-i}$ and $Q_{\bar{w}}$ likewise (Appendix B.23). For $i + i' = n$ the i-th operator at $\bar{w}$ followed by the scalar matrix of a uniformiser at w is the i'-th operator at w, a point of $\mathrm{U}(V)$ at $\{w, \bar{w}\}$ being determined by its component at w (Proposition 35.3(b)). For $i = 0$ the scalar acts by s_n; for $i = 1$ this gives $s_n u_1 = s_{n-1}$, that is $c_0(Q_w)\, c_k(Q_{\bar{w}}) = q^k\, c_1(Q_w)$ for the coefficients c_d of X^d. *Passage to the representation.* For a matrix A of size n the coefficients of degree 0, 1 and k of $\det(X - A)$ are $(-1)^n \det A$, $(-1)^k \det A \operatorname{tr} A^{-1}$ and $-\operatorname{tr} A$. The map $g \mapsto g^c$ is a homeomorphism carrying Frobenius elements at w to Frobenius elements at $\bar{w}$, so R read through it is attached to the system $w \mapsto Q_{\bar{w}}$. For most Frobenius elements g the place is prime to ℓ and $\mathrm{cyc}(g)^{-1} = q_w$ (Lemma 37.1(b)). Along Frobenius elements tending to an arbitrary g, which exist by Chebotarev density (Theorem B.35(d)), the coefficients of $\iota(Q_w)$ and $\iota(Q_{\bar{w}})$ tend to those of the characteristic polynomials of $R(g)$ and $R(g^c)$, so in the limit

$$(-1)^n \det R(g) \cdot \big(-\operatorname{tr} R(g^c)\big) = \mathrm{cyc}(g)^{-k} (-1)^k \det R(g) \operatorname{tr} R(g^{-1}).$$

The signs agree and $\det R(g) \neq 0$, so $\operatorname{tr} R(g^c) = \mathrm{cyc}(g)^{1-n} \operatorname{tr} R(g^{-1})$ for all g. Continuity and the nondegenerate trace form are part of the definition of an attached representation.

(b) Let f be an eigenform of type c with the level of O off S. The type projection fixes $f \neq 0$, so the idempotent of each label does not kill it, and at each chosen place f has a product label with two blocks: the block of size r_O carries the class function of the scaled label, the other block the trivial character (the trivial character is named by 1, which is not a root of the scaled label). A place that is neither chosen nor in Q, S or their conjugates is a place of maximal level, because Q and S avoid the bad places. By Lemma 37.4(d) the attached R is of a type, with inertia at the chosen places acting through the roots of the

scaled labels times $(X-1)^{t_O}$ (the tame part of the normalising twist is trivial, so the roots are not scaled again); its normalisation by the avatar of the inverse of the character of W is of a type by Lemma 37.4(e), with the normalised jumps of Lemma 37.4(f), the rank $r_O + t_O$ being even.

(c) Let π act on forms through the components at the places of O only, and let $\mathcal{Y}$ be the forms with the level and character of O under π that the idempotent of c fixes. These are forms of base data of the weight of O with no extra level: the components of a point at the places of O and their conjugates change neither the reductions there nor the level character, and the maximal level off these places fixes the forms of O. An operator commuting with π on the maximal level commutes with the idempotent, so the Hecke operators off the level keep $\mathcal{Y}$. *Integrality.* e_c is a sum over $g \in \prod \mathrm{GL}_{r_O}(\mathbb{F}_w)$ of the coefficients of the idempotent, which have the denominator $\#\prod \mathrm{GL}_{r_O}(\mathbb{F}_w)$, dividing n_{ch}, times a root of unity and the translate of the form by an integral point; so $n_{\mathrm{ch}} e_c$ keeps the forms integral above ℓ. *The representations.* For an eigensystem of $\mathcal{Y}$ of auxiliary level, (b) gives the type. At a place $w \in Q$ the form has the parahoric type of O: an integral matrix whose reduction is block triangular acts through ψ_w^{-1} of the determinant of the block on the embedded indices, ψ_w being the tame part of ϑ there, as ω is trivial at Q. By Theorem B.36(b), (c), inertia at w acts through 1 and the tame character of ψ_w^{-1}, so R is of finite type there and $(X - \vartheta_\ell^{-1}(s))(X-1)$ kills each inertia element s (Lemma 37.1(c) for the avatar of ϑ^{-1} on inertia); conjugate duality carries this to the conjugate places. The normalising character is trivial on inertia away from ℓ, so the relation holds for the normalised representation. □

37.3. Main summands of the lifts

A representation T of rank r_O is a *main summand* for a descent pair O and a system of Hecke eigenvalues z if it is like a main summand and, for every tame twist ϑ of O and every eigenform that lifts z, the normalised representation u attached to the lift satisfies

$$\operatorname{tr} u = \vartheta_\ell^{-1} \cdot \operatorname{tr} T + \sum_{e<t_O} \mathrm{cyc}^{-e}.$$

For the jumps of the descent pairs write $u_q(\sigma) = \lfloor \mathrm{t}/2 \rfloor - \epsilon_\sigma(\lambda_q + w_j)$, $q < \mathrm{r}$, with the target parameter λ at the place of σ (Proposition 15.12); the gap Δ and the lift weights w_j are those of Proposition 15.4.

Lemma 37.6 (The jumps of the lifts). (a) *For $q+1 < k$ the jump u_q is high, $|\epsilon_\sigma u_q + w_j| \le \mathrm{t}$; for $k \le q+1 \le \mathrm{r}$ it is low, $\Delta/4 \le |u_q| \le 2m\Delta$. If $2m\Delta \le B$ and $B + \mathrm{t} < w_j$, exactly m of the $u_q(\sigma)$ have modulus at most B.*

(b) *For a main pair (v, z) for j, the jump $z - 1 - \epsilon_\sigma c$ is high.*

(c) *Let $y \ne 0$ be a form of a descent pair V of ranks $r < n$ that lifts the eigensystem b, and w split off the places of V and their conjugates. Then minus the coefficient of X^{n-1} in the Hecke polynomial of y at w is $q^t v v'^{-1} b(w,1) + \left(\sum_{i<t} q^i\right) v$, with $t = n - r$ and v, v' the Frobenius values of the characters of V on W and on V.*

(d) *In the setting of (c), for a pair O whose character μ has V as a twist by a tame ϑ, with $r - t = 2a$: at most Frobenius elements g, minus that coefficient times the normalising avatar of V at g is $\theta_0(g)\theta(g)b(w,1) + \sum_{e<t} \mathrm{cyc}(g)^{-e}$, where θ_0, θ are the avatars of $\vartheta^{-1}(x/\bar{x})$ and of $\mu^{-1}(x/\bar{x})|x|^a$.*

Proof. (a) *High jumps.* $\epsilon_\sigma u_q + w_j = \epsilon_\sigma \lfloor \mathrm{t}/2 \rfloor - \lambda_q$ and $|\lambda_q| \le k/2$ for $q+1 < k$, with $k \le \mathrm{t}$; so the modulus is at most t. *Low jumps.* With $i = q + 2 - k \in [1, m]$ the parameter is $\lambda_q + w_j = (j - i + \frac{1}{2})\Delta$, of modulus in $[\Delta/2, m\Delta]$, and $\lfloor \mathrm{t}/2 \rfloor \le \Delta/4$ (Proposition 15.4(b)); hence $\Delta/4 \le |u_q| \le m\Delta + \Delta/4 \le 2m\Delta$. *Counting.* The m low jumps have modulus at most $2m\Delta \le B$; a high jump has $|u_q| \ge w_j - \mathrm{t} > B$.

(b) $\epsilon_\sigma(z-1-\epsilon_\sigma c)+w_j = \epsilon_\sigma(z-1)+(w_j-c)$, and $0 \le z-1 \le \mathrm{r}-1$, $0 \le w_j - c \le \mathrm{t}-\mathrm{r}$; the modulus is at most $\mathrm{t}-1$.

(c) y is spherical at w with Hecke eigenvalues s_i, and, being nonzero, has the Hecke polynomial of a lift of b: $\prod_{l<t}(1 - q^l \nu X)$ times the Hecke polynomial of rank r of b at $q^t \nu \nu'^{-1} X$ (Proposition 30.2(c)). Comparing the linear terms of $\sum_i (-X)^i q^{i(i-1)/2} s_i$ gives $s_1 = q^t \nu \nu'^{-1} b(w, 1) + \left(\sum_{i<t} q^i\right)\nu$, the linear term of $\prod_{l<t}(1 - q^l \nu X)$ being $-\sum_{i<t} q^i \nu$ and the constant term of the polynomial of b being 1; and minus the coefficient of X^{n-1} in the Hecke polynomial $\sum_j (-1)^j q^{j(j-1)/2} s_j X^{n-j}$ of y (Appendix B.23) is s_1.

(d) At a Frobenius element at $w \nmid \ell$, off the finitely many places where a character is ramified, the avatars take the values of the characters at a uniformiser (Theorem B.36(a)), and the values of θ_0 and θ multiply to q^t/ν'; the normalising avatar takes the value ν^{-1} (its d-th power for the d-th power of Frobenius). Hence the normalised expression of (c) is $\theta_0 \theta(g) b(w, 1) + \sum_{i<t} q^i$, and $q^i = \mathrm{cyc}(g)^{-i}$ by Lemma 37.1(b). □

Theorem 37.7 (Main summands exist). (a) *The jumps* $\lfloor \mathrm{t}/2 \rfloor - \epsilon_\sigma(\lambda_q + w_j)$, $q < \mathrm{r}$, *of the j-th descent pair are main summand weights of rank* r*; the single jump* $z - 1 - \epsilon_\sigma c$ *of a main pair* (ν, z) *for j is a main summand weight of rank* 1 *and string length* $2z - 1$.

(b) *A representation of type c of rank r_O, empty level and main summand weights, twisted by the avatar of $\mu^{-1}(x/\bar{x})|x|^a$ with $r_O - t_O = 2a$, is like a main summand.*

(c) *For every Hecke eigensystem z of a good form on W_m and every $1 \le j \le m$ there is a main summand for the j-th descent pair of the trivial twist and z; for every main pair there is a main summand of rank one for its lift from* U(1) *and the constant system.*

(d) (*Splitting.*) *The space of lifts for a twist ϑ and j is spanned by forms in the image of the idempotent of a tuple of labels in lift setting, whose Hecke polynomials at the split places off the level are those of a lift of one of finitely many eigensystems z of good forms on W_m, independent of ϑ.*

Proof. (a) *Descent pairs.* By Lemma 37.6(a) each u_q is low or high. Take $B = 2(\Delta/2)^2$; then $2m\Delta \le B$ and $B + \mathrm{t} < w_j$ by the sizes of the data (Proposition 15.4), so exactly m jumps at σ are small, and $m \le \mathrm{r}m/k$ as $k \le \mathrm{r}$. At σ and $\bar{\sigma}$ the signs ϵ are opposite, so $u_q(\sigma) + u_q(\bar{\sigma}) = 2\lfloor \mathrm{t}/2 \rfloor \ge 0$ and at most one of the two is negative: at most r negative jumps at the pair. The string length is t, as allowed. *Main pairs.* By Lemma 37.6(b) the single jump is high, hence not small (it has modulus at least $w_j - \mathrm{t} > 2(\Delta/2)^2$), and the jumps at σ and $\bar{\sigma}$ add up to $2(z-1) \ge 0$, so at most one is negative; the string length is $2z - 1 \le \mathrm{t}$.

(b) *Twisting.* The tame part of μ^{-1} is a power of ω, trivial off the chosen places, so by Lemma 37.4(e) the twist is a representation of a type whose jumps are lowered by the infinity type of the character; these are the given main summand weights. *Ramification above $p_0 p_1$.* At a chosen place $w \nmid \ell$ an inertia element s

acts through the roots of the scaled labels raised to its tame exponent; these are roots of unity of order dividing $p_i^{r_O!}-1$ and some p_i^d-1 with $d \le \mathrm{r}$, and 1 is not among them for the tame generator, the scaled labels not having the root 1. Hence the N-th powers of inertia act as the identity for $N = p_i^{r_O!}-1$, a unit at w, and $\det(R\gamma - 1)$ for the tame generator γ is a product of r_O differences $x-1$, each of valuation at most $v_\ell(p_i^d-1) \le v_\ell(n_{\mathrm{ch}})$ (Lemma 37.4(b)). The places above p_0p_1 that are not chosen are conjugates of chosen ones, and conjugate duality carries these statements there, with $(\gamma^c)^{-1}$ in place of γ. *Large image.* Take a chosen place $w \nmid \ell$ with residue field of order $q = p_i$ whose label is primitive with $r_O = \mathrm{r}$, or $X-1$ with $r_O = 1$ (Lemma 37.1(a), the lift setting). The roots ξ of the scaled label have a common e-th power y such that $y^x = 1$ forces $q_2 \mid jx$; moreover $\xi^{q^{r_O}} = \xi$, and the powers $(\xi^{q^a})^{\ell^u}$, $a < r_O$, are distinct for every u. For $X-1$ the only root is the name of ω^{-j}, of order q_2, and the last condition is empty. For a primitive label the scaled label is again primitive, its roots are the Frobenius conjugates of a root η of order $q^{\mathrm{r}}-1$, and y, their product, has order $q-1$, a multiple of q_2^2 by (D4) of Definition 8.7; as $9 \mid q-1$ and r is a power of 3, the conjugates of η^{ℓ^u} are distinct. Let s be inertia of tame exponent 1 and τ the faithful character followed by ι; then $\tau\xi$ is an eigenvalue of $R(s)$, and conjugation by a Frobenius ϕ raises $R(s)$ to the q-th power. The eigenvalues $\tau\xi^{q^a}$ are pairwise incongruent modulo the maximal ideal, so $R(s)$ is regular semisimple residually, $R(\phi)$ permutes its eigenlines cyclically, and over all large coefficient rings the image of a lattice spans all matrices, while $R(\phi^{r_O})$ is a scalar at an element of cyclotomic character $p_i^{r_O}$. *A scalar of large order.* $X^e - \tau y$ kills $R(s)$, so $R(s^e)$ is the scalar τy. For $1 \le t' \le 4r_O + 2$ and all a, q_2 does not divide $j\ell^b t'$ for any b (a prime $p \ne \ell$ dividing q_0 would give $p^s \mid jt'$, but $0 < jt' < p^s$). Write the order of y as $\ell^u k'$ with $\ell \nmid k'$. If $(y^{\ell^a t'})^{\ell^u} = 1$, then $q_2 \mid j\ell^{a+u}t'$, which is excluded; so $(y^{\ell^a t'})^{\ell^u}$ and 1 are distinct k'-th roots of unity, their difference has norm at least $|k'| = 1$ (Lemma 37.1(b)), and it is an integral multiple of $y^{\ell^a t'} - 1$, which is therefore a unit.

(c) *Descent pairs.* Let $z \ne 0$ be a good form with the system p. The idempotent e_0 fixes z and is the sum of the idempotents of the good tuples, so some good tuple c does not kill z. Its algebraic form is an eigenform with the labels c: at a chosen place GL_r of the local integers acts through the reduction, a parahoric type with one block. By Lemma 37.4(a), (d), an attached R is of a type for the j-th descent pair of the trivial twist, and its traces are the limits of the eigenvalues $p(w,1)$ along Frobenius elements. Lower the jumps by $(\mathrm{r}-\mathrm{t})/2 - \epsilon_\sigma w_j$; by Proposition 15.12 they become the $u_q(\sigma)$, main summand weights by (a). Let T be R twisted by the avatar θ of $\mu^{-1}(x/\bar{x})|x|^a$; by (b) it is like a main summand. For every tame ϑ and every eigenform y of a twisted pair lifting p, Lemma 37.6(d) gives the trace identity at most Frobenius elements, θ_0 being the avatar of ϑ^{-1}; the trace of the normalised representation attached to y is minus the coefficient of X^{n-1} of its characteristic polynomial, the normalising avatar is continuous, and the identity passes to all g in the limit (Theorem B.35(d)). *Main pairs.* The same argument for the lift from U(1), with the label $X-1$ (Lemma 37.4(a)), the constant form, whose system is that of the constant (Proposition 35.13(b)), and the jumps of (a) for main pairs.

(d) is Proposition 32.15, the Hecke polynomials of the lifts being those of Theorem 30.5(d). □

38. Ribet's method to first order

Ribet's method [Rib76, Wil90] turns a congruence between a lift and a non-lift into an extension of Galois representations. In the form needed here the representations of a whole space of forms are treated at once, the auxiliary primes enter through *tie parameters* X_j that tie tame inertia to Frobenius, and only the coefficient of the product $\prod_j X_j$ of a Hecke element is controlled. This section proves the algebraic statement, which does not involve Galois groups (Theorem 38.7); Section 40 applies it to the representations of Section 37.

38.1. Ribet's method over an algebra

Let $R \to L$ be rings, Λ an R-algebra and $\rho\colon \Lambda \to M_\sigma(L)$ an R-algebra map such that the characteristic polynomials of a spanning set of Λ have coefficients in the image of R.

Throughout, R and L are commutative, L is an R-algebra, and an element of L is said to lie *in* R if it lies in the image of R. A square matrix $P = (P_{ij})$ over Λ is read through ρ as the block matrix $\Phi(P)$ with blocks $\rho(P_{ij})$; Φ is again an R-algebra map. In (b) of Proposition 38.2, ρ_0, $\rho_1 \in \Lambda$ are arbitrary and $P(\rho_0)$ is the value of $P \in R[X]$ at ρ_0. In (c), O is a ring, $u_x\colon O[G] \to M_\sigma(O)$ are O-algebra maps, L is an O-algebra, S is a subalgebra of the functions $Y \to L$, I is the left ideal spanned by the entries of a vector y over $O[G]$, a *matrix of tied rows* is a square matrix B with the entries of By in $\mathfrak{a} + I\rho_0$, and on $\mathcal{X}$ the elements ρ_0, ρ_1 satisfy: ρ_1 vanishes on the block and on the characters η_e, $e \neq 0$, $\eta_0(\rho_1) = \mu_0$ and $\eta_0(\rho_0) = 0$.

Lemma 38.1 (Amitsur's theorem). (a) *If Λ is spanned over R by elements s for which the characteristic polynomial of $\rho(s)$ has its coefficients in R, then every $\rho(w)$ has this property.*

(b) *If moreover Λ is replaced by $\Lambda[X]$, R by $R[X]$ and ρ by its extension ρ_X to $\Lambda[X] \to M_\sigma(L[X])$ coefficientwise, the hypothesis of (a) holds, and for every square matrix B over $\Lambda[X]$ there are $P \in R[X]$ and a square matrix B' over $\Lambda[X]$ with $P(0) = \det \Phi(B|_{X=0})$ and $\Phi_X(B'B) = P \cdot 1$.*

Proof. (a) For a matrix A of size n over a commutative ring, $\det(1 - tA) = \sum_k (-1)^k e_k(A) t^k$, where $\pm e_k(A)$ are the coefficients of the characteristic polynomial; so the claim is that $\det(1 - t\rho(w))$ has its coefficients in R. Fix $e \geq 1$, put $L_e = L[t]/(t^e)$, let R_e be the image of $R[t]$ in L_e, and extend ρ to $\Lambda_e = \Lambda[t]/(t^e) \to M_\sigma(L_e)$ coefficientwise. Λ_e is spanned over $R[t]$ by the given elements s. For $j \geq 0$ let $\mathrm{C}(j)$ be the claim: for every $\beta \in \Lambda_e$ with $\det(1 - t\rho(\beta)) \in R_e$ and every $\gamma \in \Lambda_e$, $\det(1 - t\rho(\beta + t^j\gamma)) \in R_e$. For $j \geq e$ this holds because $t^j = 0$. Assume $\mathrm{C}(j+1)$. Writing $\gamma = \sum_i c_i s_i$ with $c_i \in R[t]$ and adding the terms one at a time, it suffices to show: if $\det(1 - t\rho(\beta)) \in R_e$, then $\det(1 - t\rho(\beta + t^j cs)) \in R_e$ for $c \in R[t]$ and a given s. Indeed

$$(1 - t\beta)(1 - t^{j+1}cs) = 1 - t\beta'' \quad \text{with} \quad \beta'' = \beta + t^j cs - t^{j+1}c\beta s,$$

and $\det(1 - t^{j+1}c\rho(s)) = \sum_k (-1)^k e_k(\rho(s))(t^{j+1}c)^k \in R_e$, so $\det(1 - t\rho(\beta'')) \in R_e$. As $\beta + t^j cs = \beta'' + t^{j+1}(c\beta s)$, $\mathrm{C}(j+1)$ gives the claim. Hence $\mathrm{C}(0)$ holds; with $\beta = 0$ and $\gamma = w$ it says that the coefficients of $\det(1 - t\rho(w))$ of degree below e lie in R. As e is arbitrary, all of them do.

(b) $\Lambda[X]$ is spanned over $R[X]$ by Λ, and the characteristic polynomial of $\rho_X(w) = \rho(w)$, $w \in \Lambda$, has its coefficients in $R \subseteq R[X]$; so (a) applies to ρ_X. It applies also to Φ_X on the square matrices over $\Lambda[X]$: these are spanned by the matrices $E_q(Q)$ with a single nonzero row, the row q of a matrix Q. Let col be the block column q of the identity matrix and row the block row q of $\Phi_X(Q)$. Then $\Phi_X(E_q(Q)) = \mathrm{col} \cdot \mathrm{row}$ and $\mathrm{row} \cdot \mathrm{col} = \rho_X(Q_{qq})$, and for an $a \times b$ matrix U and a $b \times a$ matrix V one has $X^b \chi_{UV} = X^a \chi_{VU}$ (Sylvester), so the characteristic polynomial of $\Phi_X(E_q(Q))$ is a power of X times that of $\rho_X(Q_{qq})$, whose coefficients lie in $R[X]$. By (a), the characteristic polynomial $\chi = \sum_i a_i T^i$ of $\Phi_X(B)$ has coefficients $a_i \in R[X]$, with $a_0 = \pm \det \Phi_X(B)$. By Cayley–Hamilton, $\det \Phi_X(B) \cdot 1 = \mp \sum_{i \geq 1} a_i \Phi_X(B)^i = \Phi_X(B'B)$ with $B' = \mp \sum_{i \geq 1} a_i B^{i-1}$, a matrix over $\Lambda[X]$ since the a_i are central. Take $P = \det \Phi_X(B) \in R[X]$; its value at $X = 0$ is $\det \Phi(B|_{X=0})$. □

Proposition 38.2 (Amitsur, Ribet). (a) (*Amitsur's formula.*) *If all characteristic polynomials of the $\rho(w)$ have coefficients in the image of R, so do those of every square matrix over Λ, read through ρ as a block matrix.*

(b) (*Ribet's method.*) *Let $\mathfrak{a}$ be a two-sided ideal containing* $\ker \rho$, *I the left ideal spanned by a vector y, and B a square matrix over Λ with $By \in \mathfrak{a} + I\rho_0$ entrywise. Then some polynomial P over R has constant term* $\det B$, *read through ρ, and $IP(\rho_0) \subseteq \mathfrak{a}$; the traces of $\rho(\rho_1 P(\rho_0))$ lie in the image of R, and if $\delta \in R$ is $\rho(z)$ for some $z \in I$, then δ times such a trace is the trace of $\rho(a)$ for some $a \in \mathfrak{a}$.*

(c) (*Tied rows.*) *For representations u_x, $x \in Y$, of $O[G]$, an algebra S of functions $Y \to L$ containing the constants and the coefficients of the characteristic polynomials, and an S-module $\mathcal{J}$, let $\mathfrak{a}$ be spanned by the elements whose traces against every group element lie in $\mathcal{J}$. If B is a matrix of tied rows and, on a set $\mathcal{X}$, u_x is a block ψ_x plus characters η_e, then one $f \in S$ multiplies* $\det(u_x(w))$ *into $\mathcal{J}$ for every $w \in I$, with $f(x) = \mu_0 \det(\psi_x B) \prod_e \det(\eta_e B)$ on $\mathcal{X}$.*

Proof. (a) is the second half of the proof of Lemma 38.1(b) without the variable X: the matrices with a single nonzero row span, and their characteristic polynomials are powers of T times those of the $\rho(Q_{qq})$.

(b) *The rows.* Write $(By)_i = a_i + z_i \rho_0$ with $a_i \in \mathfrak{a}$ and $z_i \in I$, and $z_i = \sum_j B_{1,ij} y_j$; this defines a square matrix B_1 over Λ. Put $B_X = B - XB_1$, a matrix over $\Lambda[X]$, and let P, B' be as in Lemma 38.1(b) for B_X; then $P(0) = \det \Phi(B)$. *The module.* Let $Q = \Lambda/\mathfrak{a}$, a ring as $\mathfrak{a}$ is two-sided, and $\pi\colon \Lambda \to Q$ the quotient map. Let $\Lambda[X]$ act on Q by letting Λ act by left multiplication and X by right multiplication with $\pi(\rho_0)$; the two commute, so this is a left module. Since $\ker \rho \subseteq \mathfrak{a}$, the action of $p \in \Lambda[X]$ depends only on the images $\rho(p_k)$ of its coefficients. *Killing.* On the vector $\pi(y) \in Q^N$, $(B_X \pi(y))_i = \pi((By)_i) - \pi(z_i \rho_0) = \pi(a_i) = 0$. Hence $0 = B' B_X \pi(y)$, and as $\Phi_X(B'B_X) = P \cdot 1$, the matrix $B'B_X$ acts on $\pi(y)$ as the scalar P, which acts by right multiplication with $\pi(P(\rho_0))$, the coefficients of P being central. Hence $y_i P(\rho_0) \in \mathfrak{a}$ for every i, and for $z = \sum_j c_j y_j \in I$, $zP(\rho_0) = \sum_j c_j\, y_j P(\rho_0) \in \mathfrak{a}$. *Traces.* The trace of $\rho(\rho_1 P(\rho_0))$ is minus a coefficient of its characteristic polynomial, which lies in R by Lemma 38.1(a). If $z_0 \in I$ has $\rho(z_0) = \delta$, then $a = \rho_1 z_0 P(\rho_0) \in \mathfrak{a}$, $\mathfrak{a}$ being two-sided, and $\rho(a) = \delta\, \rho(\rho_1 P(\rho_0))$ because δ is a scalar; take traces.

(c) *The lift.* Let $\kappa\colon O \to S$ send an element to the constant function, and define the S-algebra map $\rho\colon S[G] \to M_\sigma(\mathrm{Fun}(Y, L))$ by $\rho(\sum_g s_g g) = \sum_g s_g\,(x \mapsto u_x(g))$. It is u_x at x on the image of $O[G]$, and the characteristic polynomials of the group elements have their coefficients in S, so by Lemma 38.1(a) those of all $\rho(w)$ do. *The ideal.* Let $\mathfrak{a}'$ be the set of $w \in S[G]$ with $\operatorname{tr}\rho(ws) \in \mathcal{J}$ for all $s \in S[G]$. It is a two-sided ideal: $\operatorname{tr}\rho(aws) = \operatorname{tr}\rho(w\,sa)$ by the cyclicity of the trace, and $\mathcal{J}$ is an S-module; it contains $\ker\rho$. If r is a left combination $\sum_i c_i w_i$ of elements w_i whose traces against all group elements lie in $\mathcal{J}$, then $\operatorname{tr}\rho(rs) = \sum_i \operatorname{tr}\rho(w_i\,sc_i)$ and sc_i is an S-combination of group elements, so $r \in \mathfrak{a}'$. Hence the image of B in $S[G]$ satisfies the hypothesis of (b) with $\mathfrak{a}'$, ρ_0 and the left ideal spanned by the image of y. Let $P \in S[X]$ be the polynomial of (b) and $f = \operatorname{tr}\rho(\rho_1 P(\rho_0)) \in S$. *The multiplier.* Let $w \in I$ and $D = (x \mapsto \det u_x(w))$. By Lemma 38.1(a) the characteristic polynomial of $\rho(w)$ has coefficients in S, so by Cayley–Hamilton, as in Lemma 38.1(b), $\rho(vw) = D \cdot 1$ for some $v \in S[G]$, and vw lies in the left ideal spanned by y. By (b), $Df = \operatorname{tr}\rho(a)$ for some $a \in \mathfrak{a}'$, and $\operatorname{tr}\rho(a \cdot 1) \in \mathcal{J}$. *The value on $\mathcal{X}$.* For $x \in \mathcal{X}$, $u_x(\rho_1\rho_0) = 0$: ρ_1 vanishes on ψ_x and on η_e for $e \neq 0$, and ρ_0 on η_0. Writing $P(T) = P(0) + TP_1(T)$, the trace of $u_x(\rho_1 P(\rho_0))$ is $P(0)(x)\operatorname{tr}u_x(\rho_1) = P(0)(x)\mu_0$, and $P(0)(x)$ is the determinant of B read through u_x, which is block diagonal: $\det(\psi_x B)\prod_e \det(\eta_e B)$. □

38.2. Tie ideals, Fitting ideals and the triangle cost

Let O be a discrete valuation ring, $R = O[X_j]_{j\in J}$, G a group and u_x, $x \in Y$, matrix representations of G over O with $u_x(g) - 1 = \sum_i z_i(g)(u_x(\mathrm{gen}_i) - 1)$ for given generators. For $p \in O$, elements τ_k, φ_k ($k = (j, b)$, $b \in \{\pm\}$) and polynomials ε_j, the *tie ideal* is the left ideal of $R[G]$ spanned by the elements that every u_x kills, by p times the augmentation ideal I_G, by the *ties* $\tau_k - 1 - X_j\varepsilon_j(\varphi_k)(\varphi_k - 1)$, and by $I_G\rho_0$ for a normaliser ρ_0. A row r is *tied* if $\sum_i r_i(\mathrm{gen}_i - 1)$ lies in it. An algebra S of families of polynomials and an S-module $\mathcal{J}$ form a *trace ideal* if S holds the variables and the coefficients of the characteristic polynomials and $\mathcal{J}$ holds p times the traces and the traces of the ties times group elements. The *Fitting ideal* of R^s/N is generated by the determinants of square matrices with rows in N. A module Z of functions on G has *triangle cost b* along a set I of indices, up to C, if there is $A \subseteq G$ such that, with Z_0 the members vanishing on A: for $j \in I$ and $c \in Z_0$ agreeing at the earlier pairs τ_i, τ_i' and (for $i \in I$) φ_i, φ_i', C kills $c(\tau_j) - c(\tau_j')$; and the length of Z/Z_0 plus that of any submodule of Z_0 vanishing at all τ_j, τ_j' and agreeing at the φ_j, φ_j', modulo C-torsion, is at most b.

We first record the facts on Fitting ideals and lengths over O that are used. Write v for the normalised valuation of O, ϖ for a uniformiser, K for the fraction field, and $\mathrm{len}(S, T)$ for the length of $S/(S \cap T)$. For $\Phi\colon O^n \to R^s$ an O-linear map, $z\colon G \to O^n$, a submodule $N \leq R^s$ and $p \in O$, the module $\mathrm{forms}(\Phi, z, N, p)$ consists of the functions $\Psi \circ z$ on G, $\Psi\colon O^n \to O/p$ linear and vanishing on $\Phi^{-1}N$.

Lemma 38.3 (Fitting ideals and lengths). *Let $N \leq R^s$ and $p \neq 0$ with $\Phi(pO^n) \subseteq N$;* $\mathrm{Fitt}(N)$ *denotes the Fitting ideal of R^s/N.*

(a) *For an ideal $\mathfrak{q}$ of R,* $\mathrm{Fitt}(N + \mathfrak{q}R^s) \subseteq \mathrm{Fitt}(N) + \mathfrak{q}$.

(b) *If $rv \in N$, then $r\,\mathrm{Fitt}(N + Rv) \subseteq \mathrm{Fitt}(N)$. If for each j of a finite ordered set every $K' \geq N$ containing the v_i, $i < j$, contains $r_j v_j$, then some $K' \geq N$ contains all v_j and $\prod_j r_j\,\mathrm{Fitt}(K') \subseteq \mathrm{Fitt}(N)$.*

(c) (*Duality.*) *If every $\Psi\colon O^n \to O/p$ vanishing on $\Phi^{-1}N$ vanishes at w, then $\Phi(w) \in N$.*

(d) *Let V be an O-module of functions $G \to O/p$, and R Noetherian. There are $c \in O$ and $K' \geq N$ with $v(c) \leq \mathrm{len}(\mathrm{forms}(\Phi, z, N, p), V)$, $\Phi(w) \in K'$ for every w killed by all Ψ with $\Psi \circ z \in V$, and $c\,\mathrm{Fitt}(K') \subseteq \mathrm{Fitt}(N)$.*

(e) *Let $\psi\colon A \to M_d(O)$ be onto, N_A an A-submodule of A^ι, and let the vectors $w \in ((K/O)^d)^\iota$ with $\sum_j \psi(t_j)w_j = 0$ for all $t \in N_A$ lie in a module of length at most b. Then some square matrix B with rows in N_A has $v(\det\psi(B)) \leq b$, $\psi(B)$ read as a block matrix.*

Proof. (a) Write each row of a square matrix with rows in $N + \mathfrak{q}R^s$ as $n_i + m_i$, $n_i \in N$, $m_i \in \mathfrak{q}R^s$; by multilinearity its determinant is $\det(n_i)$ modulo $\mathfrak{q}$.

(b) Each row of a matrix with rows in $N + Rv$ is an element of N plus a multiple of v; expanding the determinant by multilinearity, each term has rows in N or equal to v. A term with two rows equal to v vanishes; in a term with one, r times it is the determinant with that row replaced by $rv \in N$. The second claim follows by induction, adding the largest index last.

(c) $\Phi^{-1}N \supseteq pO^n$, so $Q = O^n/\Phi^{-1}N$ is a finitely generated module over $O/p = O/\varpi^m$, a sum of cyclic modules O/ϖ^k with $k \leq m$, each of which embeds in O/ϖ^m. Hence the linear forms $Q \to O/p$ separate the points of Q, and the image of w in Q is zero.

(d) Induction on the length of $\mathrm{forms}(\Phi, z, N, p)$ modulo V, which is finite, the module being a quotient of a submodule of the dual of a finite O/p-module. If N already contains $\Phi(w)$ for every w killed by all Ψ with $\Psi \circ z \in V$, take $K' = N$, $c = 1$. Otherwise let w be such a vector with $\Phi(w) \notin N$. As p is a unit times ϖ^m and $\Phi(pw) \in N$, there is k with $x = \Phi(\varpi^k w) \notin N$ and $\varpi x \in N$. Put $N' = N + Rx$; by (b), $\varpi\,\mathrm{Fitt}(N') \subseteq \mathrm{Fitt}(N)$. The image of $\mathrm{forms}(\Phi, z, N', p)$ modulo V is strictly smaller than that of $\mathrm{forms}(\Phi, z, N, p)$: otherwise every Ψ vanishing on $\Phi^{-1}N$ agrees modulo V with some Ψ' vanishing on $\Phi^{-1}N'$, so $\Psi - \Psi'$ kills $\varpi^k w$ and $\Psi(\varpi^k w) = \Psi'(\varpi^k w) = 0$; by (c), $x \in N$, a contradiction. The induction hypothesis for N' gives c' and K', and $c = \varpi c'$ has $v(c) = 1 + v(c')$, at most the length for N.

(e) For $t \in N_A$ the block row $\psi(t) = (\psi(t_j))_j$ is a $d \times d\#\iota$ matrix over O; let $L \subseteq O^{d\iota}$ be the O-span of the rows of all these matrices. As N_A is an A-module and ψ is onto, the set of the $\psi(t)$ is closed under addition and under left multiplication by $M_d(O)$, so it is the set of all $d \times d\#\iota$ matrices whose rows lie in L. A vector w over K/O is killed by $\psi(N_A)$ exactly when it is orthogonal to L. By the theorem on elementary divisors there is a basis e_i' of $O^{d\iota}$ with $L = \bigoplus_i \varpi^{a_i} O e_i'$, $a_i \in \mathbb{N} \cup \{\infty\}$, and the orthogonal of L in $(K/O)^{d\iota}$ is $\bigoplus_i \varpi^{-a_i} O/O$ in the dual basis. Its length is at most b, so every a_i is finite and $\sum_i a_i \leq b$. A basis of L has $d\#\iota$ vectors and a determinant of valuation $\sum_i a_i$; grouping it into $\#\iota$ groups of d, each group is the block row $\psi(t^{(i)})$ of some $t^{(i)} \in N_A$, and the matrix B with the rows $t^{(i)}$ has $v(\det\psi(B)) = \sum_i a_i \leq b$. □

The *actions* of $R[G]$: a representation u of G over O extends to an R-algebra map $R[G] \to M(R)$, and a character to $R[G] \to R$. A vector m over O is read as the row of the elements $m_i \cdot 1$; the *value* of a row r is $\sum_i r_i(\mathrm{gen}_i - 1)$. The

generators are the δ_v (tame generators at the old places), a finite set s_0 and the finite set s. The normaliser is $\rho_0 = e + (1-e)(a-1)$ with $e = c_1(\gamma - 1)$, for elements γ, a of G and $c_1 \in O$; its row is $c_1 z(\gamma) + (1-e)z(a)$. The $z(g)$ are *coordinates* if $u_x(g) - 1 = \sum_i z_i(g)(u_x(\mathrm{gen}_i) - 1)$ for all g and x; such coordinates exist with a finite s_0, the O-module spanned by the families $(u_x(g) - 1)_x$ being finitely generated. A *character cocycle* for a character χ is a cocycle $c\colon G \to O/p$ for χ that obeys the linear relations among the families $(u_x(g) - 1)_x$, vanishes at the δ_v, and has $(\chi(g) - 1)c(a) = 0$ for all g if $c(\gamma) = 0$; a module Z of functions is *detected* by s if only 0 vanishes on s.

Lemma 38.4 (Tied rows). *Let the $z(g)$ be coordinates.*

(a) *The following rows are tied: the row of $z(gh) - z(g)$ minus g times the row of $z(h)$; the row of $z(\mathrm{gen}_i)$ minus the unit vector i; the row of $\sum_g \theta_g z(g)$ whenever $\sum_g \theta_g(u_x(g) - 1) = 0$ for all x; p times any row; the row of $z(\tau_k) - X_j \varepsilon_j(\varphi_k) z(\varphi_k)$ for $k = (j, b)$; $g - 1$ times the row of the normaliser; and, if $o_v(X)(X - 1)$ kills every $u_x(\delta_v)$, the row with the single entry $o_v(\delta_v)$ at v.*

(b) *If $\mathcal{J}$ holds p times the traces of the $u_x(g)$ and the traces of the ties times group elements, the tie ideal lies in the sum of $I_G\rho_0$ and the left ideal of the elements whose traces against all group elements lie in $\mathcal{J}$; so the value of every tied row is $b + m\rho_0$ with b in that left ideal and m in the left ideal of the $\mathrm{gen}_i - 1$.*

Proof. (a) Let m be a vector over O. Under each u_x the value of its row acts as

$$\textstyle\sum_i m_i\big(u_x(\mathrm{gen}_i) - 1\big),$$

so the value of the row of $z(g)$ acts as $u_x(g) - 1$. The first row then acts as $(gh - 1) - (g - 1) - g(h - 1) = 0$, the second as $(\mathrm{gen}_i - 1) - (\mathrm{gen}_i - 1) = 0$, and the third as $\sum_g \theta_g(u_x(g) - 1) = 0$; elements killed by every u_x lie in the tie ideal. p times a value lies in pI_G. The value of the fifth row acts as the tie $\tau_k - 1 - X_j\varepsilon_j(\varphi_k)(\varphi_k - 1)$, so it is the tie plus an element killed by every u_x. The value of the row of the normaliser acts as ρ_0, so $g-1$ times it is $(g-1)\rho_0 \in I_G\rho_0$ up to an element killed by every u_x. Finally the value of the last row is $o_v(\delta_v)(\delta_v - 1)$, which every u_x kills.

(b) The three kinds of generators other than $I_G\rho_0$ lie in the trace left ideal: an element killed by every u_x has trace 0 against everything; $p\,u_x(g)$ has its traces in $\mathcal{J}$; and so do the ties times group elements. For the second claim, write an element l of the left ideal of the $\mathrm{gen}_i - 1$, under every u_x, as an element of that left ideal plus the scalar $\eta_0(l)$, the augmentation; as $\eta_0(m) = 0$ for $m \in I_G$, the difference $(m - m')\rho_0$ is killed by every u_x. □

Proposition 38.5 (Matrices of tied rows). (a) (*Fitting ideals.*) *Let $\Phi\colon O^n \to R^s$, $y = \Phi \circ z$, and $N \supseteq \Phi(pO^n)$ with $y(\tau_j) \equiv X_j y(\varphi_j)$ and $y(\tau_j') \equiv X_j y(\varphi_j')$ modulo N, and with $y(g)$ congruent to the unit vector of g modulo N for $g \in s$; let every i below a member of I lie in I or have $X_i R^s \subseteq N$. If the module of functions $\Psi \circ z$, for linear $\Psi\colon O^n \to O/p$ vanishing on $\Phi^{-1}N$, has triangle cost b along I up to C, then the Fitting ideal of R^s/N contains $F\prod_{j\in I} X_j$ with $v(F(0)) \le b + (\#I + \#s)v(C)$.*

(b) (*The block.*) *If the matrices $\rho(g)$ span, γ acts through ρ by a scalar y with $y - 1$ a unit, and the cocycles of ρ with values in K/O that obey the linear relations among the $(u_x(g) - 1)_x$ and vanish at the τ_k and at γ have length*

at most b_0, then some square matrix of tied rows has, through ρ at $X = 0$, a determinant of valuation at most b_0.

(c) *(The characters.) Let a be an element such that $\prod_{k\neq e}(\xi_e(a)-\xi_k(a))$ divides C for every e. Under the corresponding triangle cost hypotheses for each character ξ_e and the detection of character cocycles on a finite set s, there are matrices $B_{e,i}$ and weights $w_{e,i}$ such that $\sum_i w_{e,i} \det B_{e,i} \equiv \Gamma_e \prod_{\mathrm{ty}(j)=e} X_j$ modulo the colon ideal of the monomials not dividing $\prod X_j$, with $v(\Gamma_e(0)) \le b_e + (\#\{\mathrm{ty}(j) = e\} + \#s)v(C)$, and each choice of one $B_{e,i}$ per character comes from a matrix of tied rows.*

Proof. (a) Let $A \subseteq G$ be the set of the triangle cost, $Z = \mathrm{forms}(\Phi, z, N, p)$ and Z_0 the members of Z vanishing on A. *First stage.* By Lemma 38.3(d) with V the functions vanishing on A there are $e \in O$ with $v(e) \le \mathrm{len}(Z, V)$ and $K' \ge N$ with $e\,\mathrm{Fitt}(K') \subseteq \mathrm{Fitt}(N)$ containing $\Phi(w)$ for every w killed by all Ψ with $\Psi \circ z \in V$; in particular $\Phi(z(g)) \in K'$ for $g \in A$, so for every $T \ge K'$, $\mathrm{forms}(\Phi, z, T, p) \subseteq Z_0$. *Second stage.* Put $v_j = \Phi(z(\varphi_j) - z(\varphi'_j))$; then $\Phi(z(\tau_j) - z(\tau'_j)) - X_j v_j \in N$. Let $j \in I$ and $T \ge K'$ contain the v_i with $i \in I$, $i < j$. For $i < j$, $X_i v_i \in T$: if $i \in I$ because $v_i \in T$, otherwise because $X_i R^s \subseteq N$. Hence $\Phi(z(\tau_i) - z(\tau'_i)) \in T$ for $i < j$. Let Ψ vanish on $\Phi^{-1}T$; then $\Psi \circ z \in Z_0$, it agrees at τ_i, τ'_i for $i < j$ and at φ_i, φ'_i for $i \in I$, $i < j$, so by the triangle condition $C\Psi(z(\tau_j)) = C\Psi(z(\tau'_j))$. By Lemma 38.3(c), $\Phi(C(z(\tau_j) - z(\tau'_j))) \in T$, hence $CX_j v_j \in T$. By Lemma 38.3(b) there is $T \ge K'$ containing all v_j, $j \in I$, with $\prod_{j\in I} CX_j\,\mathrm{Fitt}(T) \subseteq \mathrm{Fitt}(K')$. *Third stage.* Let $T_0 = T + \mathfrak{m}_0 R^s$, $\mathfrak{m}_0$ the polynomials without constant term, and $Z_1 = \mathrm{forms}(\Phi, z, T_0, p) \subseteq Z_0$. Apply Lemma 38.3(d) to T_0 with V the C-torsion: there are f with $v(f) \le \mathrm{len}(Z_1, C\text{-torsion})$ and $K'' \ge T_0$ with $f\,\mathrm{Fitt}(K'') \subseteq \mathrm{Fitt}(T_0)$ containing $\Phi(Cz(g))$ for all g. For $g \in s$ the row $Ce_g = \Phi(Cz(g)) - C(\Phi(z(g)) - e_g)$ lies in K'', so $C^{\#s} = \det(C\cdot 1) \in \mathrm{Fitt}(K'')$ and $fC^{\#s} \in \mathrm{Fitt}(T_0) \subseteq \mathrm{Fitt}(T) + \mathfrak{m}_0$ by Lemma 38.3(a): $fC^{\#s} = d + r$ with $d \in \mathrm{Fitt}(T)$ and $r(0) = 0$, so $d(0) = fC^{\#s}$. *The lengths.* The members of Z_1 vanish at all τ_j, τ'_j, since $y(\tau_j) \equiv X_j y(\varphi_j)$ modulo N and $X_j y(\varphi_j) \in T_0$, and agree at φ_j, φ'_j for $j \in I$, since $v_j \in T$. By the cost condition, $\mathrm{len}(Z, V) + \mathrm{len}(Z_1, C\text{-torsion}) \le b$. *Conclusion.* $F = eC^{\#I}d$ has $F \prod_{j\in I} X_j = e \prod_{j\in I}(CX_j)\, d \in e\,\mathrm{Fitt}(K') \subseteq \mathrm{Fitt}(N)$ and $v(F(0)) = v(e) + \#I\, v(C) + v(f) + \#s\, v(C) \le b + (\#I + \#s)v(C)$.

(b) At $X = 0$ the tie of index k is the row of $z(\tau_k)$, and with $c_1(y-1) = 1$ the row of $z(\gamma)$ has the same image through ρ at $X = 0$ as the tied row $(g-1)$ times the normaliser row for a suitable g (the normaliser $e = c_1(\gamma - 1)$ is 1 on ρ). Let N_A be the tied rows and ψ_0 the action through ρ at $X = 0$, which is onto the matrices because the $\rho(g)$ span. A vector w over K/O killed by $\psi_0(N_A)$ defines the form $\Psi = \sum_j w_j(\cdot)_j$, and $\Psi \circ z$ is a cocycle for ρ (the cocycle rows are tied), obeys the linear relations (so are those rows), takes the value w_i at gen_i, and vanishes at the τ_k and at γ. Hence the vectors w lie in the image of the module of these cocycles under evaluation at the generators, of length at most b_0, and Lemma 38.3(e) gives the matrix.

(c) Fix e, and read rows on ξ_e with the δ-coordinates forgotten; ξ_e is trivial at γ and the δ_v. *Cocycles.* For a linear map A and K_e containing the image of the tied rows, every Ψ modulo p on coordinate vectors killing the vectors sent into K_e gives a character cocycle $\Psi \circ z$ for ξ_e: the cocycle and relation rows are tied; on ξ_e the row $(g-1)$ times the normaliser is $(\xi_e(g) - 1)(c_1 z(\gamma) + z(a))$, as $\xi_e(\gamma) = 1$, so $c(\gamma) = 0$ forces $(\xi_e(g) - 1)c(a) = 0$. *The ties on ξ_e.* As $\varepsilon_j(\xi_e(\varphi_k)) = 1 + pc$, the

row of $z(\tau_k) - X_j z(\varphi_k)$ differs from a tie by $pX_j cz(\varphi_k)$, a tied row. *Unit rows.* Since character cocycles are detected by s, by duality (Lemma 38.3(c)) the row of $z(g)$, $g \in s_0$, lies in the image of the tied rows modulo the others, and one obtains tied rows v_i that are the unit vectors on the s_0-coordinates under every character at once: if w is the unit row i under the characters already treated and $m_{e,j}$ are the unit rows j under ξ_e, then $w + \sum_j (\delta_{ji} - w_j) m_{e,j}$, w_j the entry of w at the coordinate j, is the unit row i under ξ_e, and the added sum vanishes under the earlier characters. Eliminate the s_0-coordinates by the v_i. *The Fitting ideal.* Apply (a) to the image K_e of the tied rows on ξ_e, enlarged by the colon ideal $\mathfrak{q}_e$ of the monomials not dividing $\prod X_j$ by $\prod_{\mathrm{ty}(j)<e} X_j$, with $I = \{j : \mathrm{ty}(j) = e\}$: an earlier variable X_i has type e or lies in $\mathfrak{q}_e$ (its product with $\prod_{\mathrm{ty}(j)<e} X_j$ is divisible by X_i^2), and the triangle cost is the hypothesis. This gives Γ_e with the stated bound and $\Gamma_e \prod_{\mathrm{ty}(j)=e} X_j \in \mathrm{Fitt}(K_e + \mathfrak{q}_e R^s)$. *The matrices.* An element of $\mathrm{Fitt}(K_e + \mathfrak{q}_e R^s)$ is $\sum_i w_{e,i} \det B_{e,i}$ modulo $\mathfrak{q}_e$, with the rows of each $B_{e,i}$ images of tied rows that vanish on the s_0-coordinates under ξ_e. For a choice $i = (i_e)_e$, the rows v_j and $\sum_e \pi_e \cdot$ (the lift of row l of B_{e,i_e}), with $\pi_e \in R[G]$ equal to C on ξ_e and 0 on the other characters (a multiple of $\prod_{k \neq e}(a - \xi_k(a))$, which exists as $\prod_{k \neq e}(\xi_e(a) - \xi_k(a))$ divides C), form a square matrix of tied rows which on each ξ_e is block triangular with diagonal blocks 1 and $C\, B_{e,i_e}$; the rows with the single entries $o_v(\delta_v)$ complete it to a square matrix M of tied rows on all coordinates, again block triangular on the characters, where δ_v is trivial, with the diagonal entries $o_v(1)$. Hence $\det(M|_{\xi_e}) = c_0 \det B_{e,i_e}$ with $c_0 = \prod_v o_v(1) \cdot C^{\#s}$. □

38.3. Ribet's lemma

Lemma 38.6 (From coefficients modulo p^M to K/O). *Let u_x be a finite family of representations of G over O, ρ a representation, $T \subseteq G$ and $\gamma \in G$, and consider the cocycles of ρ that vanish on T and at γ and obey the linear relations among the families $(u_x(g) - 1)_x$. Those with values in K/O are torsion, and if for every M those with values in O/p^M form a module of length at most b, $p \neq 0$ not a unit, so do those with values in K/O.*

Proof. By the coordinates of the $u_x(g) - 1$ in finitely many $u_x(\mathrm{gen}_i) - 1$, a cocycle obeying the relations is determined by its values at the generators, each of which has a nonzero annihilator in O; so it is torsion. Let $a = p^{b+1}$. Division by a, $O/a \to K/O$, maps the cocycles modulo a onto the a-torsion of the module V of the cocycles with values in K/O: an a-torsion cocycle is the image of a cocycle modulo a with the same relations, the map being injective. Hence the a-torsion has length at most b. If V were not a-torsion, its a-torsion would contain a copy of O/a (a torsion module over a discrete valuation ring with an element of order not dividing a contains O/a), of length $v(a) > b$. Hence V is the a-torsion. □

Theorem 38.7 (Ribet's lemma to first order). *Let u_x, $x \in Y$, be a finite family of matrix representations of a group G over a discrete valuation ring O, and S, $\mathcal{J}$ a trace ideal for $p \neq 0$, elements τ_k, φ_k and polynomials ε_j with $\varepsilon_j \equiv 1$ modulo p at the $\xi_e(\varphi_k)$. Assume:*

(i) *on a set $\mathcal{X}$, u_x is a block ρ_x plus characters ξ_e with $\xi_0 = 1$, and ρ_π spans all matrices for some $\pi \in \mathcal{X}$;*

(ii) *an element γ_x acts on the block by a scalar y_x with $y_x - 1$ a unit and as the identity on the characters;*

(iii) *for an element a the products $\prod_{k\neq e}(\xi_e(a)-\xi_k(a))$ divide C, and each j has a type, monotone in j;*
(iv) *the cocycles of ρ_π in K/O obeying the relations among the $(u_x(g)-1)_x$ and vanishing at the τ_k and at γ_π have length at most b_0;*
(v) *elements δ_v are trivial on the characters and $o_v(X-1)$ kills every $u_x(\delta_v)$;*
(vi) *for each e, every module of character cocycles for ξ_e has triangle cost b_e along the j of type e, and is detected on a finite set s.*

Then the determinants of all combinations $\sum \omega_g(u_x(g)-1)$ lie in S and vanish on $\mathcal{X}$; one $f\in S$ multiplies them into $\mathcal{J}$; on $\mathcal{X}$, f has no monomial properly dividing $\prod_j X_j$; and the coefficient of $\prod_j X_j$ in $f(\pi)$ comes from some $c\in O$ with

$$v(c)\le b_0+\sum_e b_e+t\Big(\sum_v v(o_v(1))+2\#s\,v(C)\Big)+(\#J+1)v(C),$$

t being the number of characters.

Proof. The determinants. By Lemma 38.1(a) applied to the lift of the family to $S[G]$ (proof of Proposition 38.2(c)), the determinant of every element of $O[G]$, as a function of x, lies in S. On $\mathcal{X}$ every combination $\sum_g \omega_g(u_x(g)-1)$ has the column of $\xi_0=1$ equal to zero, so its determinant vanishes there. *Auxiliary elements.* The element $k=c\,(\gamma_x-1)$ with $c(y_x-1)=1$ is 1 on every character and 0 on the block ρ_x; multiplying over the finitely many $x\in\mathcal{X}$ such elements (each is 1 on the characters) gives one k that is 1 on every character and 0 on every block. By (iii), for each e an element q_e of $R[G]$, a multiple of $\prod_{k\neq e}(a-\xi_k(a))$, is C on ξ_e and 0 on the other characters. *Generators.* Take a finite s_0 and coordinates $z(g)$ in the generators δ_v, s_0 and s (Section 38.2), and c_1 with $c_1(y_\pi-1)=1$, so that $c_1(\gamma_\pi-1)$ is 1 on ρ_π. *The block.* By (iv) and Proposition 38.5(b) there is a square matrix B_0 of tied rows whose determinant on ρ_π at $X=0$ has valuation at most b_0. *The characters.* By (iii), (vi) and Proposition 38.5(c) there are matrices $B_{e,i}$ with weights $w_{e,i}$, polynomials Γ_e and, for each choice $i=(i_e)$, a matrix M_i of tied rows with $\det(M_i|_{\xi_e})=c_0\det B_{e,i_e}$, $c_0=\prod_v o_v(1)C^{\#s}$, such that $d_e=\sum_i w_{e,i}\det B_{e,i}$ satisfies $d_e+\zeta_e=\Gamma_e\prod_{\mathrm{ty}(j)=e}X_j$ with $\zeta_e\prod_{\mathrm{ty}(j)<e}X_j$ in the ideal $\mathfrak{n}$ of the monomials not dividing $\prod_j X_j$. *The gluing.* For each choice i the matrix $(1-k)B_0+kM_i$ has tied rows (the tied rows form a module) and is B_0 on the blocks and M_i on the characters. Apply Proposition 38.2(c) with $\rho_1=kq_0$, which vanishes on the blocks, is C on ξ_0 and 0 on the other characters, and ρ_0 the normaliser, which is 0 on ξ_0 as ξ_0 is trivial; by Lemma 38.4(b) the hypothesis of (c) holds. It gives $f_i\in S$ multiplying the determinants into $\mathcal{J}$ with $f_i(x)=C\det(\rho_xB_0)\prod_e\det(M_i|_{\xi_e})$ on $\mathcal{X}$. Put $f=\sum_i\prod_e w_{e,i_e}f_i$. On $\mathcal{X}$, summing the products of the weights,

$$f(x)=C\det(\rho_xB_0)\,c_0^t\prod_e d_e.$$

The coefficient. Write $D(x)=C\,c_0^t\det(\rho_xB_0)$. As $\prod_{k<e}(d_k+\zeta_k)$ is a multiple of $\prod_{\mathrm{ty}(j)<e}X_j$, the difference $\prod_e d_e-\prod_e(d_e+\zeta_e)$ lies in $\mathfrak{n}$, and $\prod_e(d_e+\zeta_e)=\prod_e\Gamma_e\cdot\prod_j X_j$, the types partitioning J. Hence $D(x)\prod_e d_e$ has no monomial properly dividing $\prod_j X_j$, and its coefficient of $\prod_j X_j$ is the constant term of $D(x)\prod_e\Gamma_e$. At $x=\pi$ this constant term is $c=C\,c_0^t\,\det(\rho_\pi B_0)|_{X=0}\prod_e\Gamma_e(0)\in O$, of valuation at most

$$v(C)+t\Big(\sum_v v(o_v(1))+\#s\,v(C)\Big)+b_0+\sum_e\big(b_e+(\#\{\mathrm{ty}(j)=e\}+\#s)v(C)\big),$$

which is the stated bound, as the sets $\{\mathrm{ty}(j) = e\}$ partition J. □

38.4. Norms, logarithms and first-order deformations

The family of Hecke operators at the auxiliary places that realises $\mathcal{J}$ (Section 40) is built from the following algebra. A finite group $\Gamma = \prod_j \Gamma_j$ maps multiplicatively to a ring R, $g \mapsto g$; $\mathrm{Nm} = \sum_{g\in\Gamma} g$ and $\mathrm{Nm}_j = \sum_{g\in\Gamma_j} g$. A congruence $a \equiv b$ is an equality $\sigma(a) = \sigma(b)$ for a ring map σ; σ is a *reduction modulo* m/k if it kills every x with $kx \in mR$. For pairwise commuting subrings $\mathcal{A}_j$ with $\mathrm{Nm}_j, \partial_j \in \mathcal{A}_j$, the *family coefficients* are $N_S = \prod_{j\in S} \partial_j \prod_{j\notin S} \mathrm{Nm}_j$, the coefficients of $\prod_{j\in S} \alpha_j$ in $\prod_j(\mathrm{Nm}_j + \alpha_j\partial_j)$ in commuting variables α_j. For a homomorphism $c\colon \Gamma \to \mathbb{Z}/m$ and a weight $f\colon \mathbb{Z}/m \to \mathbb{Z}$, the *weighted norm* is $\sum_g f(c(g))\, g$; with $f = \nu$, the lift of least absolute value, it is the *logarithm* L_c of c. For a subgroup H with fixed representatives h of Γ/H, $\partial = -(\sum_h h)B(\nu)(\sum_h h^{-1})$ for an additive map B on weights.

Lemma 38.8 (Norms and logarithms). (a) *For $j \notin S$, $N_S = N'\mathrm{Nm}_j$ and $N_{S\cup\{j\}} = N'\partial_j$ for one N', and $N_\emptyset = \mathrm{Nm}$. If t, r commute with all $\mathcal{A}_j$, $\mathrm{Nm}_j t = 0$ and $\partial_j t \equiv \mathrm{Nm}_j r$, then the coefficient of $\prod_{i\in J} X_i$ in $\left(\sum_S N_S \prod_{i\in S} X_i\right)\cdot G\cdot(t - X_j r)$ is $\equiv 0$, for every polynomial G over the commutative ring of t, r.*

(b) *Let $\sum_g g^{-1}\Psi g = n$, $\delta = n^2\delta'$ with δ' commuting with the group and with all N_S, and let each $g \in \Gamma_j$ have Z with $n\partial_j(g-1) \equiv \mathrm{Nm}_j Z$ and $\delta'\mathrm{Nm}\, Z = 0$. Then for $i \geq \#S$ there is Y with $\delta^i N_S \equiv Y\mathrm{Nm}$, and $Y = \delta^i$ if $S = \emptyset$.*

(c) *Over a finite field, the transfer to GL_N of the first entry on the stabiliser of the line of the first basis vector is the determinant; so is, after any homomorphism l, the transfer of $l\circ$(first entry).*

(d) *Let $m = 0$ in the target of σ. Then $(\sum_h h)L_a(\sum_h h^{-1}) \equiv L_{\mathrm{tr}(a)}$ for $a\colon H \to \mathbb{Z}/m$ and its transfer; $(\gamma - 1)L_c \equiv -\mathrm{Nm}$ if $c(\gamma) = 1$; and if right multiplication by $p \in H$ shifts the weight of B by $a(p)$, then $B(\nu)(\sum_h h^{-1})(g-1) \equiv B(1)Q$ for some Q.*

(e) *If σ is a reduction modulo $m/(2k)$ and C is an additive map on weights killing every $f(\cdot - k) - f$, then $\sigma(C(\nu)) = 0$.*

(f) *In the setting of (d), let H fix $B(1)$ from the left, $\sum_{p\in H} p^{-1}\Psi p = n$, $Lg = L$ for all g and $L\,B(1) = 0$. If R has no $\#H$-torsion, then $n\partial(g-1) \equiv \mathrm{Nm}\, Z$ with $LZ = 0$.*

Proof. (a) The first claim is the definition, with $N' = \prod_{i\in S}\partial_i \prod_{i\notin S\cup\{j\}} \mathrm{Nm}_i$, the factors commuting. For the second, the coefficient of $\prod_{i\in J} X_i$ in $\mathcal{N}P$, $\mathcal{N} = \sum_S N_S X^S$ of degree at most one in each variable, is $\sum_S N_{J\setminus S}[X^S]P$. For $P = G(t - X_j r)$ and $j \notin S$, $[X^S]P = t[X^S]G$ and $[X^{S\cup j}]P = t[X^{S\cup j}]G - r[X^S]G$. Pair $S \not\ni j$ with $S\cup\{j\}$ and write $J \setminus S = T \cup \{j\}$: the two terms are $N'\partial_j\, t[X^S]G + N'\mathrm{Nm}_j(t[X^{S\cup j}]G - r[X^S]G) = N'(\partial_j t - \mathrm{Nm}_j r)[X^S]G + N'\mathrm{Nm}_j t[X^{S\cup j}]G \equiv 0$.

(b) Induction on S. For $S = \emptyset$, $\delta^i N_\emptyset = \delta^i\mathrm{Nm}$. Let $S \neq \emptyset$, $i = e + 1$, and $\mathcal{A} = n\delta'\delta^e N_S$. We show $\mathcal{A}(g-1) \equiv 0$ for $g \in \Gamma_j$, for every j. If $j \notin S$, then $N_S = N'\mathrm{Nm}_j$ and $\mathrm{Nm}_j g = \mathrm{Nm}_j$, so $\mathcal{A}(g-1) = 0$. If $j \in S$, write $N_S = N'\partial_j$ with $N'\mathrm{Nm}_j = N_{S\setminus j}$; as N' lies in the subrings other than $\mathcal{A}_j$, $\mathcal{A}(g-1) = \delta'\delta^e N'\cdot n\partial_j(g-1) \equiv \delta'\delta^e N_{S\setminus j} Z$. By the induction hypothesis, $\delta^e N_{S\setminus j} \equiv Y\mathrm{Nm}$ ($e \geq \#S - 1$), and δ', which commutes with $\delta^e N_{S\setminus j}$, is moved to its right first (as Y need not commute with δ'): $\delta'\delta^e N_{S\setminus j} Z = \delta^e N_{S\setminus j}\delta' Z \equiv Y\,\delta'\mathrm{Nm}\, Z = 0$,

δ' commuting with Nm. As the factors generate Γ, $\mathcal{A}g \equiv \mathcal{A}$ for all g, and $n\mathcal{A} = \sum_g \mathcal{A}g^{-1}\Psi g \equiv \mathcal{A}\Psi\mathrm{Nm}$. Finally $\delta^{e+1}N_S = n^2\delta'\delta^e N_S = n\mathcal{A}$, so $Y = \mathcal{A}\Psi$ works.

(c) The transfer of a character λ of the stabiliser P of a line, at g, is the product over the orbits of g on the lines of $\mathbb{F}^N$: an orbit of length f through the line of v contributes the scalar by which g^f multiplies v. For $g = \mathrm{diag}(x, 1, \dots, 1)$ the line of e_1 contributes x, and every other line contains a vector with a nonzero entry that g fixes, contributing 1. Conjugating by a permutation matrix gives the same for x in place i. A character of $\mathrm{GL}_N(\mathbb{F})$ with values in $\mathbb{F}^\times$ is trivial on transvections, since $c \mapsto \tau(\text{transvection } c)$ is a homomorphism from a group of order q to one of order $q-1$. Both the transfer and det are characters agreeing on diagonal matrices and transvections, which generate $\mathrm{GL}_N(\mathbb{F})$. The transfer is a product over the cosets, and l commutes with products.

(d) The coefficient of g in $(\sum_h h)L_a(\sum_h h^{-1})$ is $\sum_{h'} \nu(a(p_{h'}))$, where $gh' = hp_{h'}$ with $p_{h'} \in H$; modulo m this is $\nu(\sum_{h'} a(p_{h'})) = \nu(\mathrm{tr}(a)(g))$, the transfer being the product of these components. For the second, $(\gamma - 1)L_c = \sum_g (\nu(c(g)) - 1) - \nu(c(g)))g$, and $\nu(x-1) - \nu(x) \equiv -1$ modulo m. For the third, right multiplication by g permutes the h^{-1} up to factors in H: $(\sum_h h^{-1})g = \sum_h p_h h^{-1}$ with $p_h \in H$. By hypothesis $B(\nu)p_h = B(\nu(\cdot - k_h))$, $k_h = a(p_h)$, and $\nu(x-k) \equiv \nu(x) - \nu(k)$ modulo m; as $m = 0$ in the target, weights that agree modulo m have the same image, so $B(\nu)(p_h - 1)h^{-1} \equiv -\nu(k_h)B(1)h^{-1}$, and $Q = -\sum_h \nu(k_h)h^{-1}$.

(e) Let $f(x) = \nu(x)(\nu(x) + k)$. As $\nu(x-k) = \nu(x) - k + m\epsilon(x)$ with ϵ integral, $f(x-k) - f(x) = -2k\nu(x) + mh(x)$ with h integral. Applying C, $0 = -2kC(\nu) + mC(h)$, so $2kC(\nu) \in mR$ and $\sigma(C(\nu)) = 0$.

(f) By the third part of (d), $n\partial(g-1) \equiv -(\sum_h h)\, nB(1)\, Q$. As H fixes $B(1)$ from the left and $\sum_p p^{-1}\Psi p = n$, $nB(1) = \sum_p p^{-1}\Psi\, pB(1) = \sum_p p^{-1}\Psi B(1) = \mathrm{Nm}_H \Psi B(1)$, and $(\sum_h h)\mathrm{Nm}_H = \mathrm{Nm}$. Hence $n\partial(g-1) \equiv \mathrm{Nm}\, Z$ with $Z = -\Psi B(1)Q$. Moreover $\#H \cdot L\Psi B(1) = L\,\mathrm{Nm}_H \Psi B(1) = L\, nB(1) = 0$, using $Lp = L$, and R has no $\#H$-torsion, so $L\Psi B(1) = 0$ and $LZ = 0$. □

39. Even classes and triangles of auxiliary places

Ribet's lemma (Theorem 38.7) needs, for each character cyc^{-e} of the string, a triangle cost for the modules of character cocycles. These are cocycles in the Tate twists $(O/\ell^\nu)(-e)$; their *even* parts are controlled by auxiliary places in triangular position, chosen with Kummer maps over the real subfield, and their *odd* parts by a class number. No bound for the even classes themselves is needed: the reciprocity laws of the Kummer maps remove them.

39.1. Even and odd classes

Let $E \subset \Omega$ be a CM field containing a primitive 5040th root of unity, F its maximal real subfield, O a coefficient ring (the integers of a finite Galois extension of $\mathbb{Q}_\ell$ in $\bar{\mathbb{Q}}_\ell$), and $G_\nu \subseteq G$ the elements fixing the ℓ^νth roots of unity. For a set S of places, the *even classes* of exponent e are the cocycles $c\colon G \to (O/\ell^\nu)(-e)$, unramified outside ℓ, S and $\bar{S}$, with $c(g^c) = c(g)$, vanishing on the elements of trivial cyclotomic character in the decomposition groups above ℓ; the *odd classes* are the cocycles with the same local conditions, unramified off ℓ, that change sign under $g \mapsto g^c$. A family of places w_j off ℓ with the chosen tame generators τ_j and elements $\varphi_j \in G_\nu$ is a *tame generator family*. A *triangle*

of primes for an element a is such a family, indexed by a finite linear order J with monotone exponents $0 < \mathrm{ty}(j) < t$, such that: a finite set s of test points of trivial cyclotomic character, with $(\#J + \#s + 1)h \le c$, detects together with a the cocycles unramified off ℓ, and for each exponent and each $g \in s$ the even classes unramified off ℓ take at some φ_j of that exponent twice their value at g; an even class of exponent $\mathrm{ty}(j)$ that kills the earlier τ_i and the earlier φ_i of the same exponent has its value at τ_j killed by ℓ^h; and a separates the powers $\mathrm{cyc}(a)^{-e}$, $e < t$, up to ℓ^h. Moreover $4(\mathrm{cyc}(a)^{-e} - 1)^2$ divides ℓ^h for $0 < e < t$. The *local set* above ℓ is the union of the inertia groups above ℓ if $e = 0$, and otherwise the set of the elements of trivial cyclotomic character in the decomposition groups above ℓ; the even and odd classes vanish on it, and so does every cocycle that obeys the crystalline relations above ℓ (Appendix B.25).

Lemma 39.1 (Even parts and tame generators). (a) *For $e \neq 0$ the even part $c + c \circ (g \mapsto g^c)$ of a cocycle with values in $(O/\ell^v)(-e)$, unramified outside ℓ, S, $\bar{S}$ and zero on the local set above ℓ, is an even class.*

(b) *A cocycle unramified off S and the w_j, $\bar{w}_j$ that vanishes at the τ_j and at the $(\tau_j^c)^{-1}$ is unramified off S.*

(c) *An element b of O/ℓ^v killed by every additive map u to $\mathbb{Z}/\ell^v$ for which $u \circ c$ is a cocycle, for an even class c, is zero.*

Proof. (a) Conjugation $g \mapsto g^c$ is an automorphism of G that fixes the cyclotomic character, $\mathrm{cyc}(cgc) = \mathrm{cyc}(g)$, so $c \circ (g \mapsto g^c)$ is again a cocycle for cyc^{-e}: $c((gh)^c) = c(g^c) + \mathrm{cyc}(g)^{-e} c(h^c)$. It carries the inertia and decomposition groups above a place to those above the conjugate place, which is above ℓ, in $S \cup \bar{S}$, or neither, as the original one; so it keeps the ramification conditions and, the cyclotomic character being fixed, the vanishing on the local set (for $e \neq 0$ this set does not depend on e). The sum $c + c \circ (g \mapsto g^c)$ is invariant under the involution $g \mapsto g^c$.

(b) At a place A above w_j, which is not above ℓ, the cyclotomic character, hence the twist, is trivial on inertia, so the cocycle restricted to the inertia group is a homomorphism to an ℓ-power torsion group. It factors through the tame quotient, wild inertia being pro-p with $p \neq \ell$, and the ℓ-part of the tame quotient is procyclic and topologically generated by the image of the chosen generator τ_j. Hence it vanishes on the inertia group at A. For another place sA over w_j and g in its inertia group, $z(g) = z(s \cdot s^{-1} g s \cdot s^{-1})$ is $s \cdot z(s^{-1} g s)$ up to terms that cancel because g acts as the identity, so z vanishes there too. At $\bar{w}_j$ the same holds with the tame generator $(\tau_j^c)^{-1}$ at the conjugate place cA.

(c) Characters with values in $\mathbb{Q}/\mathbb{Z}$ separate the points of O/ℓ^v, and on an ℓ^v-torsion group each of them factors through $\mathbb{Z}/\ell^v$; so $b = 0$ if every additive $u : O/\ell^v \to \mathbb{Z}/\ell^v$ kills it. For each such u, $u \circ c$ is a continuous cocycle for cyc^{-e} modulo ℓ^v, as the twist acts by integers modulo ℓ^v and u is additive. □

39.2. Kummer maps

A Frobenius datum x (a prime of degree one, a place of Ω above it and a Frobenius element) is *ν-split* if its prime is not above ℓ and its Frobenius element lies in G_ν. For a finite family x of ν-split places, a *Kummer map* of exponent e off a finite set T of primes of F is the restriction to G of a relation over F for the module $\mathbb{Z}/\ell^\nu$ with the action of $(\mathrm{cyc}^+)^{-e}$, cyc^+ the extension of the cyclotomic character to $\mathrm{Gal}(\Omega/F)$ that is 1 at c: a locally constant equivariant homomorphism y from G_ν to the Cartier dual, zero on inertia off T, satisfying reciprocity $\sum_p y(\varphi_p)(z(\gamma_p)) = 0$ against the cocycles z over F with the local conditions above ℓ and below x, for every finite family of distinct primes $p \notin T$.

Write $G_F = \mathrm{Gal}(\Omega/F) \supseteq G$ and cyc^+ for the character of G_F that is cyc on G and 1 at c: the cyclotomic character times the quadratic character of E/F. The map $G_F \to G$ sending g and gc to g, $g \in G$, is continuous, and multiplicative up to replacing the second factor s by s^c. The extension E/F is unramified at the finite places: otherwise an element of inertia would act on E as complex conjugation, but there is an integer y of E with $y - \bar{y}$ a unit (from the roots of unity of order 5040 in E), which would lie in the prime below the place. A *tame generator normalised over F* at a prime of degree one of E is an element of inertia in G that moves an m-th root of a uniformiser of F by the fixed root of unity; such generators exist.

Proposition 39.2 (Kummer maps). (a) *A Kummer map ψ is locally constant; on G_ν it is a homomorphism negated by complex conjugation, and conjugation by g multiplies it by $\mathrm{cyc}(g)^{e+1}$.*

(b) (*Reciprocity.*) *For ν-split places w_j above distinct primes of F outside T and an even class z of exponent e for w vanishing at the Frobenius elements of x, $\sum_j \psi(\varphi_j) \cdot 2z(\gamma_j) = 0$. In particular, if ψ vanishes at the Frobenius elements of w except at w_j, every multiple of $2\psi(\varphi_j)$ kills $z(\tau_j)$.*

(c) *If E has at least four infinite places, for every finite family of ν-split places above distinct primes of F there is T such that the reductions modulo ℓ of the Kummer maps span a space of dimension larger than the number of places.*

Proof. (a) The relation y of ψ is a locally constant homomorphism on G_ν, and $y(g)(a) = a\psi(g)/\ell^\nu$ determines ψ, both sides being additive in a and agreeing at 1. By the transformation law of a relation, $y(gug^{-1}) = \mathrm{cyc}(g) \cdot y(u) \circ g^{-1}$, and g^{-1} acts on $\mathbb{Z}/\ell^\nu$ by $\mathrm{cyc}^+(g)^e$; so conjugation by $g \in G_F$ multiplies ψ by $\mathrm{cyc}(g)\mathrm{cyc}^+(g)^e$. For $g \in G$ this is $\mathrm{cyc}(g)^{e+1}$, and for $g = c$ it is $\mathrm{cyc}(c) = -1$.

(b) *Extension to F.* Replacing z by $u \circ z$ for the additive maps u to $\mathbb{Z}/\ell^\nu$ (Lemma 39.1(c)) we may assume z has values in $\mathbb{Z}/\ell^\nu$. Put $z^+(g) = z^+(gc) = z(g)$ for $g \in G$. As z is invariant under $g \mapsto g^c$, z^+ is a cocycle of G_F for $(\mathrm{cyc}^+)^{-e}$, and $2z^+$ lies in the cocycles of the Kummer setting for the primes P below the w_j: it vanishes at complex conjugations; it is unramified off $S_0 \cup P$, as E/F is unramified at the finite places; above ℓ, for g in the decomposition group with $\mathrm{cyc}^+(g) = 1$, $2z^+(g) = z^+(g^2)$ with g^2 in G, in the decomposition group and in $\ker \mathrm{cyc}$, where z vanishes; and below x, the decomposition group is topologically generated by inertia and a Frobenius element and lies in $\ker \mathrm{cyc}^+$ modulo ℓ^ν, x being ν-split, so $2z^+$ restricted to it is a homomorphism vanishing at the Frobenius element (by hypothesis) and on inertia (z being unramified there). *Reciprocity.* The primes of F below the w_j split in E, with the arithmetic Frobenius

elements φ_j^{-1}, and the γ_j are tame generators normalised over F. The reciprocity of the relation gives $\sum_j y(\varphi_j^{-1})(2z^+(\gamma_j)) = 0$, that is $-\sum_j \psi(\varphi_j)\cdot 2z(\gamma_j)/\ell^\nu = 0$. *One term.* If $\psi(\varphi_i) = 0$ for $i \neq j$, then $2\psi(\varphi_j)z(\gamma_j) = 0$. On the inertia group at w_j the cocycle z is a homomorphism through the procyclic ℓ-part of tame inertia (Lemma 39.1(b)), so $z(\tau_j)$ is a multiple of $z(\gamma_j)$; hence $n = 2\psi(\varphi_j)a$ kills $z(\tau_j)$.

(c) Apply Theorem B.38 over F to $M = \mathbb{Z}/\ell^\nu$ with the action $(\mathrm{cyc}^+)^{-e}$, $U = G_\nu$, $\kappa = \mathrm{cyc}^+$ and the local conditions of the Kummer setting. The hypotheses hold: $M[\ell]$ has ℓ elements; cyc^+ takes values in $1 + 4\mathbb{Z}_\ell$, since E contains the fourth roots of unity and $\mathrm{cyc}^+(c) = 1$; its kernel lies in $G_\nu \cup G_\nu c$; and F has as many infinite places as E, at least four. Hence for some $T \supseteq S_0$ the reductions of the relations off T number at least ℓ^{k+1}, with k at least the number of the given places, their primes below being distinct and off ℓ. A relation is determined by its Kummer map, and its reduction by the reduction of the Kummer map, so the reductions of the Kummer maps span a space with at least ℓ^{k+1} elements, of dimension at least $k+1$. □

39.3. Triangles of primes

Lemma 39.3 (Choosing the next place). *Let $\nu > 0$, $t + 2 \leq y$, $\mathrm{cyc}(\gamma)^a - 1 \mid \ell^h$ for $0 < a < t$, $4 \mid \ell^{3h}$, and let W contain most Frobenii near $\delta^{\ell^\nu} g'$ for some δ of trivial cyclotomic character, for every g'. Let finitely many places above distinct primes of F carry maps ψ_i of Kummer type of exponents $\mathrm{ty}(i)$, those of exponent e independent modulo ℓ, and let g have trivial cyclotomic character.*

(a) *There is $u \in G_\nu$ with $\psi_i(u) = 0$ for all i and $\psi_0(u) = \prod_{k\neq e}(\mathrm{cyc}(\gamma)^e - \mathrm{cyc}(\gamma)^k)$, for a new Kummer map ψ_0 of exponent e for the earlier places of exponent e whose reduction is outside the span of theirs.*

(b) *There is a new place x_0 with Frobenius datum in W, above a new prime of F avoiding the primes excluded by all the maps, such that the earlier maps vanish at its Frobenius element φ, $2\psi_0(\varphi)$ divides ℓ^{yh}, and every even class of exponent e unramified off ℓ takes at φ twice its value at g.*

(c) *In a family of places chosen in this way along a finite linear order, an even class of exponent $\mathrm{ty}(j)$ that vanishes at the earlier τ_i and at the earlier φ_i of the same exponent has its value at τ_j killed by ℓ^{yh}.*

Proof. (a) By Proposition 39.2(c) the reductions of the Kummer maps of exponent e for the earlier places of exponent e, off a suitable T_0, span more dimensions than there are earlier maps, so some ψ_0 has its reduction outside their span. The maps of exponent e together with ψ_0 are homomorphisms on G_ν with independent reductions, so by Nakayama's lemma they are jointly onto: some $g_1 \in G_\nu$ has $\psi_0(g_1) = 1$ and $\psi_i(g_1) = 0$ for $\mathrm{ty}(i) = e$. For $c = \mathrm{cyc}(\gamma)$ and a map ψ of exponent d, conjugation by γ followed by the power c^{-1} (an integer modulo ℓ^ν) multiplies ψ by $c^{-1}c^{d+1} = c^d$; combining such operations as products and quotients in G_ν, some u has $\psi(u) = \prod_{k\neq e}(c^d - c^k)\psi(g_1)$ for every map ψ of exponent d. This vanishes for $d \neq e$, and for $d = e$ it is the product times $\psi(g_1)$.

(b) *The target.* Let $\sigma = \delta^{\ell^\nu}\, g\, g^c\, u\, (u^c)^{-1}$, with δ given by the hypothesis on W for $g' = gg^c u(u^c)^{-1}$; it lies in G_ν. For a map ψ of Kummer type, a homomorphism on G_ν negated by complex conjugation, $\psi(\sigma) = \ell^\nu\psi(\delta) + \psi(g) - \psi(g) + \psi(u) + \psi(u) = 2\psi(u)$, as g, δ have trivial cyclotomic character and so lie in G_ν. For an even class z of exponent e unramified off ℓ, a homomorphism on G_ν with values killed by ℓ^ν and invariant under $g \mapsto g^c$,

$z(\sigma) = z(g) + z(g) + z(u) - z(u) = 2z(g)$. These identities hold near σ: the maps are locally constant, and there are finitely many even classes unramified off ℓ with values in $\mathbb{Z}/\ell^\nu$, to which the general ones reduce by Lemma 39.1(c). *Chebotarev.* The filter of most Frobenii near σ is proper (Theorem B.35(d)); so some Frobenius datum $x_0 \in W$ lies near σ and avoids the finitely many primes above ℓ, below the earlier places, and in the sets excluded by the maps. Its Frobenius element φ has $\psi_i(\varphi) = 2\psi_i(u) = 0$, $\psi_0(\varphi) = 2\psi_0(u)$ and $z(\varphi) = 2z(g)$, and x_0 is ν-split, φ being near $\sigma \in G_\nu$. *The divisibility.* $2\psi_0(\varphi) = 4\prod_{k\neq e}(c^e - c^k)$, and $c^e - c^k = c^k(c^{e-k} - 1)$ or $-c^e(c^{k-e} - 1)$ divides ℓ^h; as $4 \mid \ell^{3h}$ and there are $t - 1$ factors, $2\psi_0(\varphi)$ divides $\ell^{3h}\ell^{(t-1)h} = \ell^{(t+2)h}$, which divides ℓ^{yh} as $t + 2 \le y$.

(c) Let z be such an even class, of exponent $e = \mathrm{ty}(j)$. It vanishes at the earlier τ_i, hence, being even, at the $(\tau_i^c)^{-1}$, so it is unramified at the earlier places (Lemma 39.1(b)): an even class for the places from j on. The j-th map ψ_j is a Kummer map for the earlier places of exponent e, at whose Frobenius elements z vanishes, and it vanishes at the Frobenius elements of the later places, chosen after it; the places from j on lie above distinct primes of F outside its excluded set. By Proposition 39.2(b), every multiple of $2\psi_j(\varphi_j)$ kills $z(\tau_j)$, and ℓ^{yh} is one by (b). □

Theorem 39.4 (Triangles exist). *Let E have at least four infinite places and lie in the cyclotomic field of level N_E, let $N_E + t < y$, let γ generate G modulo the kernel of* cyc, *let* $\mathrm{cyc}(\gamma)^a - 1$ *divide ℓ^h for $0 < a < t$, let $\ell \le y^2$ or $h = 0$, and let a set W of Frobenius data contain most Frobenii near $\delta^{\ell^\nu} g$ for some δ of trivial cyclotomic character, for all ν and g. Then there is a finite linear order J with a monotone exponent map such that for each $\nu > 0$ some places of level ℓ^ν indexed by J have their Frobenius data in W and form a triangle of primes for γ^{-1} with h replaced by yh and $c = y^{22}h$.*

Proof. Test points. By Theorem B.41(a) for $g_0 = \gamma^{-1}$ there is a finite set s of elements of trivial cyclotomic character, with $\#s \le y^{20}$ when $N_E \le y$ and $\ell \le y^2$, such that a cocycle in a Tate twist modulo ℓ^n unramified off ℓ that vanishes at γ^{-1} and on s is zero. *The order.* Let J be the set of the pairs (e, g) with $0 < e < t$ and $g \in s$, ordered lexicographically with the exponent first (after well-ordering s), and $\mathrm{ty}(e, g) = e$, which is monotone. *The choice.* Fix $\nu > 0$. Since $4 \mid \mathrm{cyc}(\gamma)^e - 1 \mid \ell^h$ for $e > 0$ (the cyclotomic character is $\equiv 1$ modulo 4 on G), $4(\mathrm{cyc}(\gamma)^e - 1)^2$ divides ℓ^{3h}, and $4 \mid \ell^{3h}$ unless J is empty. By well-founded recursion along J, choose at each $j = (e, g)$ a map and a place by Lemma 39.3(a), (b) with the test element g; by induction the earlier maps of exponent e are independent modulo ℓ, a family none of whose members lies in the span of the earlier ones being independent. *The triangle.* The places lie above distinct primes of F, so they are distinct from each other and from their conjugates, and are ν-split, so their norms are $\equiv 1$ modulo ℓ^ν: they are Taylor–Wiles places of level ℓ^ν, with Frobenius data in W. The conditions of a triangle of primes for γ^{-1}, with yh in place of h and $c = y^{22}h$: the detection is that of the test points; at the place (e, g) the even classes unramified off ℓ take twice their value at g (Lemma 39.3(b)); the triangle condition is Lemma 39.3(c); $\prod_{k\neq e}(u^e - u^k)$, $u = \mathrm{cyc}(\gamma)$, divides $\ell^{(t-1)h} \mid \ell^{yh}$, and $4(u^e - 1)^2 \mid \ell^{3h} \mid \ell^{yh}$; finally $\#J + \#s + 1 \le t\#s + 2y^{20} \le (t + 2)y^{20} \le y^{21}$, so $(\#J + \#s + 1)\, yh \le y^{22}h$. If $h = 0$ the bounds are trivial. □

Theorem 39.5 (Triangle cost of Tate twists). *Let the w_j form a triangle of primes for g_0 up to ℓ^h, $e < t$, and Z a module of cocycles with values in $(O/\ell^v)(-e)$, unramified off ℓ and the w_j, $\bar{w}_j$, such that $\mathrm{cyc}(g_0)^{-e} - 1$ kills the value at g_0 of each member unramified off ℓ. Let b bound the length of Z modulo the functions obeying the crystalline relations above ℓ, plus the length of the odd classes. Then Z has triangle cost b along the j of type e, up to ℓ^h.*

Proof. Put $\tau_j' = (\tau_j^c)^{-1}$ and $\varphi_j' = (\varphi_j^c)^{-1}$, and take for the set of the triangle cost the local set Λ above ℓ. *Values at the inverse conjugates.* The τ_j and φ_j act as the identity on $(O/\ell^v)(-e)$ ($\mathrm{cyc}(\tau_j) = 1$, $\mathrm{cyc}(\varphi_j) \equiv 1$ modulo ℓ^v), and where the action is trivial a cocycle has $z(g^{-1}) = -z(g)$; so $z(\tau_j') = -z(\tau_j^c)$ and $z(\tau_j) = z(\tau_j')$ exactly when the even part $z + z \circ (g \mapsto g^c)$ vanishes at τ_j, and likewise for φ_j.

(a) Let j have type e (so $e > 0$) and let $z \in Z$ vanish on Λ and agree at the earlier pairs. By Lemma 39.1(a) its even part is an even class of exponent e for the places w_j; it vanishes at the earlier τ_i and at the earlier φ_i of exponent e. By the triangle, ℓ^h kills its value at τ_j, so $\ell^h z(\tau_j) = \ell^h z(\tau_j')$.

(b) *The first term.* A cocycle obeying the crystalline relations above ℓ vanishes on Λ: conjugating $g \in \Lambda$ by some s into the decomposition group of the chosen place above its prime, crystalline cocycles vanish at sgs^{-1}, and computing $z(s^{-1} \cdot sgs^{-1}) = z(gs^{-1})$ in two ways gives $z(g) = 0$. Hence the length of Z modulo the functions vanishing on Λ is at most its length modulo those obeying the crystalline relations. *The second term.* Let $S' \leq Z$ consist of members vanishing on Λ, at all τ_j, τ_j', and agreeing at φ_j, φ_j' for $\mathrm{ty}(j) = e$. Its members are unramified off ℓ (Lemma 39.1(b)). If $e = 0$ there is no sign condition and S' lies in the odd classes. If $e \neq 0$, the map $z \mapsto z \circ (g \mapsto g^c) - z$ sends S' into the odd classes. Its kernel consists of even z; for such z the agreement at φ_j, φ_j' gives $2z(\varphi_j) = 0$, and for each test point g some j of exponent e has $z(\varphi_j) = 2z(g)$, so $4z(g) = 0$. With $d = \mathrm{cyc}(g_0)^{-e} - 1$, which kills $z(g_0)$, the cocycle $4dz$ vanishes at g_0 and at the test points, so it is zero; and $4d$ divides $4d^2$, which divides ℓ^h. Hence the kernel is killed by ℓ^h, and the length of S' modulo its ℓ^h-torsion is at most the length of the odd classes. The two terms add up to at most b. □

40. The bound for denominators

This section applies Ribet's lemma to the Galois representations of the systems of Hecke eigenvalues in a space of forms and proves the bound for the denominator of the projection to the lifts (Theorem 40.7). The auxiliary places of a triangle are added to the level; the tie parameters are realised by first-order deformations of the norm maps from the deep auxiliary level to the base level; and the top coefficient of Ribet's element, an operator in the Hecke algebra, clears the denominator.

40.1. The Hecke algebra and tied traces

Let base data B fix a type of forms; the forms of level L have principal level at auxiliary places L_j and maximal level at the other places off the places of the data. A *system* of level L is an isotypic space (forms with given Hecke eigenvalues at almost all places) containing a nonzero form of level L; it has a Galois representation x.gal. The *Hecke algebra* $\mathbb{T}$ is the ring of ℓ-adic functions t

on systems such that scaling each system x by $\iota^{-1}(t(x))$ preserves the integral forms. Let N_x be the twist of x.gal by a character a trivial at the tame generators. The *trace* $\mathrm{tr}[P(h)g]$ is the function $x \mapsto \mathrm{tr}(P(N_x h)N_x g)$. For polynomials H_j and variables α_j, the *tied ideal* $\mathfrak{a} \subseteq \mathbb{T}[\alpha]$ is generated by the $\ell^\nu \mathrm{tr}[g]$ and the traces $\mathrm{tr}[(\gamma_j - 1 - \alpha_j(\varphi_j - 1)H_j(\varphi_j))g]$ of the tied relations.

At an auxiliary place $w = w_j$ of norm q, with $\ell^\nu \mid q - 1$, a *local choice* is a uniformiser ϖ and a unit z whose local Artin symbols are the classes of φ_j and of γ_j, together with a logarithm $\mathrm{ind}\colon \mathbb{F}_w^\times \to \mathbb{Z}/\ell^\nu$ with $\mathrm{ind}(\bar{z}) = 1$; local choices exist by local class field theory, $\bar{z}$ generating $\mathbb{F}_w^\times$ modulo ℓ^ν-th powers. On $\mathrm{GL}_n(\mathbb{F}_w)$ we use $\mathrm{ind} \circ \det$, and on the stabiliser P of the line of the first basis vector the logarithm $\mathrm{ind}_1(g) = \mathrm{ind}(g_{11})$ of the first entry. A *weighted norm* over a finite group Γ acting on forms, for a homomorphism $\chi\colon \Gamma \to \mathbb{Z}/\ell^\nu$ and a weight $f\colon \mathbb{Z}/\ell^\nu \to \mathbb{Z}$, is $\sum_\gamma f(\chi(\gamma))\gamma$. Two systems are *congruent* if the characteristic polynomials of their representations N agree modulo the maximal ideal at every Galois element.

Proposition 40.1 (The dictionary at an auxiliary place). (a) *The coefficients of the characteristic polynomials of the $N_x g$, and the indicator of the systems congruent to a given one, lie in $\mathbb{T}$.*

(b) *At an auxiliary place w off the base level, the tame generator acts as a diamond operator: on a form on which $\mathrm{GL}_n(\mathbb{F}_w)$ acts through $\psi \circ \mathrm{ind} \circ \det$, $\mathrm{tr}[(\gamma - 1)gp(\varphi)] = (\langle\gamma\rangle - 1)\,\mathrm{tr}[gp(\varphi)]$, with $\langle\gamma\rangle = \mathrm{diag}(\bar{z}, 1, \ldots, 1)$.*

(c) *The Frobenius element acts through a normalised Hecke operator: if the stabiliser of a line acts through $\psi \circ \mathrm{ind}$ of the first entry with $\psi(n!) \neq 1$, then $\mathrm{tr}[(\gamma - 1)gp(\varphi)] = p(\tilde{U})\,\mathrm{tr}[(\gamma - 1)g]$, $\tilde{U} = a(\varphi)U$ with U the Hecke operator of $\mathrm{diag}(\varpi, 1, \ldots, 1)$ for the pro-unipotent Iwahori subgroup.*

Proof. (a) *Eigenvalues.* At a place w of prime norm off the level L, the function sending a system to its j-th Hecke eigenvalue at w, read in $\bar{\mathbb{Q}}_\ell$, lies in $\mathbb{T}$: the Hecke operator is supported at a place not above ℓ, an integral combination of translations, and keeps the integral forms. *Coefficients.* The twist by a multiplies the i-th coefficient of the characteristic polynomial of $N_x g$ by a power of $a(g)$. The untwisted coefficients are limits, along Frobenius elements near g (Theorem B.35(d)), of coefficients of Hecke polynomials (Appendix B.23), which are integers times eigenvalues; $\mathbb{T}$ is closed under such limits, the systems being finitely many and the integral structure being preserved coefficientwise. *The indicator.* For each system x' not congruent to π choose a coefficient $c_{x'} \in \mathbb{T}$ whose values at x' and π are incongruent, and put $s_\pi = \prod_{x'}(c_{x'} - c_{x'}(x'))$. It vanishes at every non-congruent x' and has norm at least 1 at each x congruent to π, since then $c_{x'}(x) \equiv c_{x'}(\pi) \not\equiv c_{x'}(x')$. The indicator is $1 - \prod_x(1 - s_\pi/s_\pi(x))$ over the congruent x, an integral polynomial in s_π with coefficients of norm at most 1; this is the idempotent of one factor of the decomposition of a finite algebra over a henselian local ring.

(b) Let y be a form of the system x on which $\mathrm{GL}_n(\mathbb{F}_w)$ acts through $\psi \circ \mathrm{ind} \circ \det$. The maximal compact subgroup at w acts on y by $k \mapsto \psi(\mathrm{ind}(\det k))$, and the pro-$p$ Iwahori subgroup fixes y. By local–global compatibility for Iwahori eigenvectors (Theorem B.36(b)), inertia at w acts on N_x through the characters $\psi \circ \mathrm{ind}$ of the torus, all equal on the image of γ; so every eigenvalue of $N_x(\gamma)$ is $\iota(\psi(\mathrm{ind}\,\bar{z})) = \iota(\psi(1))$. $N_x(\gamma)$ has finite order (inertia at $w \nmid \ell$ acts on N_x through a finite quotient, a being trivial at γ), hence is semisimple, hence the scalar $\iota\psi(1)$.

On the other side the diamond $\langle\gamma\rangle$ acts on y by $\psi(\mathrm{ind}\,\det\langle\gamma\rangle) = \psi(1)$. Both sides of the claim, applied to y, are $(\psi(1)-1)\,\mathrm{tr}[gp(\varphi)]y$.

(c) On such a form the pro-unipotent Iwahori subgroup acts as the identity: an element of the stabiliser with first entry and determinant 1 fixes the form, and the pro-unipotent Iwahori subgroup reduces into such elements. The parahoric of the line acts by $k \mapsto \psi(\mathrm{ind}\,k_{11})$. By Theorem B.36(b) the polynomial $\prod_i(X - \iota\xi_i(\varpi))$ of the Iwahori eigencharacter kills Frobenius on the Galois side, and the operator U acts on the Iwahori-fixed vectors of this eigencharacter with the eigenvalue attached to the first entry. Hence, in traces against an element T of the Galois group, a polynomial in $\tilde{U}$ applied to y acts as the same polynomial in the Frobenius matrix on the summand where inertia acts by ψ; the factor $\gamma - 1$ projects to this summand, as on the other summands inertia acts as the identity and $\psi(n!) \neq 1$ separates them (ψ is nontrivial on the image of ind of the elements of order dividing $n!$). This gives the identity. □

40.2. The family at the auxiliary places

Let Nm_j be the sum over $\mathrm{GL}_n(\mathbb{F}_{w_j})$, n_{Aut} the common multiple of the stabiliser orders, and $\equiv$ equality after a reduction killing every operator T with $\ell^\nu \mid 2n!\,T$.

We first collect four facts. For a descent pair O and an isotypic space x write u_x for the twist of $x.\mathrm{gal}$ by the normalising character χ of O (Section 37.2), and $t = t_O$. A system is a *weak lift* if every $u_x(g)$ has the eigenvalue 1. A system has a *root chain* at w for H if its Hecke polynomial, with the variable rescaled by the Frobenius value of χ, is $\prod_{i<t}(X - q^i)$ times a divisor of H.

Lemma 40.2 (Chains of roots). (a) *Let $l_{pk} \in \bar{\mathbb{Q}}_\ell$. There are integral d_p with $\|\sum_p l_{pk}d_p\| \geq \|l_{kk}\|$ for every k. Consequently, if each basis vector b_k has an operator in a set closed under integral combinations, diagonal in the basis b with eigenvalues l_k, then one operator in the set, diagonal in b, has eigenvalues μ_k with $\|\mu_k\| \geq \|l_{kk}\|$; and if $\|c\,\iota\mu\| \geq \varepsilon$ for every eigenvalue μ of an operator on a space of dimension N, then $\|c^N \iota\det\| \geq \varepsilon^N$.*

(b) *If $f \neq 0$ lies in a weak lift x and has Hecke polynomial S at a split $w \nmid \ell$ of norm q, then $X_0 q^i$, $i < t$, are roots of S, X_0 the Frobenius value of the character of W (the inverse of χ at a geometric Frobenius); so $\prod_{i<t}(X - q^i)$ divides $S(X_0 X)$.*

(c) *Let u_x have integral characteristic polynomials with the root 1. For most Frobenius elements φ at places w: if $\det(X - u_x(\varphi)) \equiv S' \prod_{i<t}(X - \mathrm{cyc}(\varphi)^{-i})$ modulo the maximal ideal with $\|S'(1)\| \geq 1$, then x has a root chain at w for an integral e with $e(1) = 1$.*

(d) *The orthogonal complement of a space Z stable under the Hecke polynomials off the level is stable under them; and $Z \subseteq N'$ if every nonzero form of Z lying in an isotypic space lies in N'.*

Proof. (a) The linear forms $F_k(X) = \sum_p l_{pk}X_p$ over the infinite residue field of $\bar{\mathbb{Q}}_\ell$: after scaling, finitely many nonzero hyperplanes do not cover the residue space, so some integral point d has $\|F_k(d)\|$ equal to the largest norm of a coefficient of F_k, at least $\|l_{kk}\|$, for every k (forms that vanish identically are handled by the trivial case $l_{kk} = 0$). The combination $\sum_p d_p\Lambda_p$ of the given diagonal operators is diagonal with eigenvalues $\sum_p l_{pk}d_p$. The determinant is the product of the eigenvalues with multiplicity, N in number.

(b) A geometric Frobenius s^{-1} at w has the eigenvalue 1 on u_x, so $\chi(s^{-1})^{-1}$ is an eigenvalue of $x.\mathrm{gal}(s^{-1})$, which is killed by the Hecke polynomial (Theorem B.36(a)); this makes X_0 a root of S. The form lies in a unitary representation and $|X_0|^2 = q^r$, so by the chain lemma (Theorem B.28(a)), with $N = n$ and $k = t$, $X_0, X_0q, \dots, X_0q^{t-1}$ are roots.

(c) For most Frobenius elements the place is split, prime to ℓ and off the level, and the characteristic polynomial of $x.\mathrm{gal}(\varphi)$ is ι of the Hecke polynomial. Let $B = \prod_{i<t}(X - q^i)$, which divides the rescaled Hecke polynomial by (b), with quotient T. As $\mathrm{cyc}(\varphi)^{-1} = q$, $\iota(T)\iota(B)$ is congruent to χ^n times the characteristic polynomial of $u_x(\varphi)$, hence to $S'B$. Monic factors cancel in bounds on coefficients: if B is monic and $(T - S')B$ has all coefficients of norm below one, so has $T - S'$. Hence T is integral and $T(1) \equiv S'(1)$ has norm at least 1, and $e = T/T(1)$ is integral with $e(1) = 1$ and $T \mid e$.

(d) The adjoint of the Hecke polynomial at w is a Hecke polynomial at $\bar{w}$ (Proposition 35.3(b)), which preserves Z. For the second claim view the forms as smooth forms: Z is spherical off the chosen places and stable under the Hecke operators off them and their conjugates, so it is spanned by joint eigenforms (Appendix B.16); these lie in isotypic spaces, hence in N'. □

Theorem 40.3 (The deformed norm family). (a) (*Chains for weak lifts.*) *If every element has eigenvalue* 1 *on the normalised representation of a system x, its Hecke polynomial at a split $w \nmid \ell$ has the roots X_0q^i, $i < t$; and for most Frobenius elements, if $\det(X - u(\varphi)) \equiv S\prod_{i<t}(X - \mathrm{cyc}(\varphi)^{-i})$ modulo the maximal ideal with $|S(1)| \geq 1$, the rescaled Hecke polynomial is a root chain times a divisor of an integral polynomial with value* 1 *at* 1.

(b) (*First-order term.*) *If the base systems where δ does not vanish have a root chain at w_j, there is an integral operator ∂ made of translates at w_j with $\mathrm{Nm}_j \mathrm{tr}[(\gamma - 1)g] = 0$ and $\partial\,\mathrm{tr}[(\gamma - 1)g] \equiv \mathrm{Nm}_j \mathrm{tr}[(\varphi - 1)H(\varphi)g]$, and $n_{\mathrm{Aut}}\partial(g - 1) \equiv \mathrm{Nm}_j Z$ with $\delta\,\mathrm{Nm}\,Z = 0$.*

(c) (*The family.*) *With $\delta = n_{\mathrm{Aut}}^2\delta'$ and root chains at all w_j, the coefficients $\mathcal{N}_S$ of $\prod_j(\mathrm{Nm}_j + \alpha_j\partial_j)$ are integral operators that kill $\mathfrak{a}$ and, up to powers of δ, factor through the base level.*

(d) (*The top coefficient clears denominators.*) *Let $\mathcal{D} \in \mathbb{T}[\alpha]$ satisfy $(\delta_0 - n_0)\mathcal{D} \in \mathfrak{a}$, let l be a form of base level with dl integral, and y with $A\delta_0 l = n_0 A y$, $\delta_0 y = n_0 y$ and $\mathcal{D}_S y = 0$ for $S \neq J$. If $2n!(n_0 n_{\mathrm{Aut}}^2)^{\#J+1}$ divides ℓ^ν, then $d n_{\mathrm{Aut}} A^{\#J+1}$ times the top coefficient $\mathcal{D}_J$ applied to y is integral.*

Proof. (a) is Lemma 40.2(b), (c).

(b) *The operator.* Fix a local choice at $w = w_j$ and write $t = t' + 1$, $p = (X - q^{t'})H$. Let ρ be the action of $\mathrm{GL}_n(\mathbb{F}_w)$, P the stabiliser of the line, L the weighted norm over P with the weight $\nu \circ \mathrm{ind}_1$ (ν the lift of least absolute value), and $B(f) = p(\tilde{U})L_f$ for the weighted norms L_f over P with weight $f \circ \mathrm{ind}_1$; B is additive in f, with values integral operators made of translates at w, as U and the translations are. Put $\partial = -(\sum_h h)B(\nu)(\sum_h h^{-1})$, the sums over fixed representatives h of $\mathrm{GL}_n(\mathbb{F}_w)/P$ (Section 38.4). *The norm kills the tame trace.* By Proposition 40.1(b) with $p = 1$ and the weight 1, $\mathrm{Nm}_j \mathrm{tr}[(\gamma - 1)g] = (\langle\gamma\rangle - 1)\mathrm{Nm}_j \mathrm{tr}[g]$, and $\langle\gamma\rangle$ fixes the norm. *The tie.* Let σ be the reduction killing every T with $\ell^\nu \mid 2n!T$. On eigenforms of P for the characters $\psi \circ \mathrm{ind}_1$ with $\psi(n!) \neq 1$, Proposition 40.1(c) gives $p(\tilde{U})\,\mathrm{tr}[(\gamma - 1)g] = \mathrm{tr}[(\gamma - 1)gp(\varphi)]$. The difference $\mathcal{E}(f)$ of the two

sides after the weighted norm L_f is additive in f and vanishes at the weights $f(\cdot - n!) - f$: the values of these weighted norms lie in the span of the eigenforms with $\psi(n!) \neq 1$, as the characters form a basis of the functions on $\mathbb{Z}/\ell^\nu$ and the shift by $n!$ multiplies ψ by $\psi(-n!)$. By Lemma 38.8(e) with $k = n!$, $\sigma(\mathcal{E}(\nu)) = 0$. Then $\sigma(\partial \operatorname{tr}[(\gamma - 1)g]) = \sigma(-(\sum_h h)L \operatorname{tr}[(\gamma - 1)gp(\varphi)](\sum_h h^{-1}))$. By the transfer of the logarithm of the first entry (Lemma 38.8(c), (d)), $(\sum_h h)L(\sum_h h^{-1}) \equiv$ the logarithm of ind $\circ$ det; by Proposition 40.1(b) on the weighted norm with weight $\nu \circ \mathrm{ind} \circ \det$ the factor $\gamma - 1$ becomes $\langle\gamma\rangle - 1$, and $(\langle\gamma\rangle - 1)L_{\mathrm{ind}\det} \equiv -\mathrm{Nm}_j$ by Lemma 38.8(d), as $\mathrm{ind}\det\langle\gamma\rangle = 1$. Hence $\partial \operatorname{tr}[(\gamma - 1)g] \equiv \mathrm{Nm}_j \operatorname{tr}[gp(\varphi)]$, and $p(\varphi) = (\varphi - q^{t'})H(\varphi)$ may be replaced by $(\varphi - 1)H(\varphi)$, as $q \equiv 1$ modulo ℓ^ν and the two polynomials differ by a multiple of $q^{t'} - 1$. *The norm condition.* P fixes the values of $B(1)$: U preserves the forms fixed by the parahoric subgroup, and on forms an element of P acts through any lift to the parahoric. By Lemma 38.8(f), with L in the role of the invariant functional, $n_{\mathrm{Aut}}\partial(g - 1) \equiv \mathrm{Nm}_j Z$ with $Z = -\Psi B(1)Q$. $\delta \mathrm{Nm} Z = 0$. The values of $\mathrm{Nm}\, B(1)$ have base level. Let x be a base system with $\delta(x) \neq 0$ and w' a form of base level in x; it has a root chain at w_j, so by the chain lemma (Theorem B.28(b)) it is orthogonal to the values of $B(1) = ((X - q^{t'})H)(\tilde{U})$ applied to forms fixed by the pro-p Iwahori subgroup, and the finite groups fix w'. Hence $\delta \,\mathrm{Nm}\, B(1)y$ is orthogonal to all $\delta w'$ with w' of base level, among them itself; it is zero. Hence $\delta \,\mathrm{Nm}\, Z = 0$.

(c) *The coefficients.* Operators made of translates at different auxiliary places commute, so the $\mathcal{A}_j$ generated by the actions at w_j are pairwise commuting, and Lemma 38.8(a) gives the coefficients $\mathcal{N}_S$ of $\prod_j(\mathrm{Nm}_j + \alpha_j\partial_j)$, with $\mathcal{N}_\emptyset = \mathrm{Nm}$; they commute with $\mathbb{T}$, which commutes with the residue actions at places off the base level. *Killing* $\mathfrak{a}$. It suffices to treat multiples $G'z$ of the generators z of $\mathfrak{a}$. For $z = \ell^\nu \operatorname{tr}[g]$ the claim holds after σ. For a tied relation $z = \operatorname{tr}[(\gamma_j - 1 - \alpha_j(\varphi_j - 1)H_j(\varphi_j))g]$, write $t = \operatorname{tr}[(\gamma_j - 1)g]$ and $r = \operatorname{tr}[(\varphi_j - 1)H_j(\varphi_j)g]$, so $z = t - \alpha_j r$; by (b), $\mathrm{Nm}_j t = 0$ and $\partial_j t \equiv \mathrm{Nm}_j r$, and Lemma 38.8(a) gives that the top coefficient of $\mathcal{N} \cdot G'(t - \alpha_j r)$ is $\equiv 0$. As $\mathfrak{a}$ is an ideal, this covers all squarefree monomials. *The factorisation.* By Higman's criterion there is an integral Ψ with $\mathrm{Nm}(\Psi y) = n_{\mathrm{Aut}} y$ for forms of base level; with (b) the hypotheses of Lemma 38.8(b) hold, so $\delta^{\#J}\mathcal{N}_S \equiv \Upsilon_S \mathrm{Nm}$ with integral Υ_S and $\Upsilon_\emptyset = \delta^{\#J}$.

(d) Put $\delta = n_{\mathrm{Aut}}^2 A\delta_0$, $n_1 = n_0 n_{\mathrm{Aut}}^2$ and $i_0 = \#J + 1 = i + 1$. For the base systems where $A\delta_0$ does not vanish the root chains hold, so (c) applies with this δ. Let G be the coefficient of $\prod_j \alpha_j$ in $\mathcal{N}\mathcal{D}$. As $(\delta_0 - n_0)\mathcal{D} \in \mathfrak{a}$ and $\delta_0 - n_0$ divides $(n_{\mathrm{Aut}}^2 A\delta_0)^k - (n_1)^k A^k$, modulo $\ell^\nu/(2n!)$, a multiple of $n_1^{i_0}$,

$$n_1^{i_0} A^{i_0} G \equiv \delta^{i_0} G = (\delta^i \mathcal{N})(\delta \mathcal{D}) \equiv \sum_S \Upsilon_{J\setminus S} \mathrm{Nm}\, \delta\, \mathcal{D}_S.$$

Apply both sides to the integral form $u = \Psi(dl)$, whose norm is $dn_{\mathrm{Aut}} l$. With $A\delta_0 l = n_0 A y$, $\mathrm{Nm}\, \delta\, s\, u = n_1 d n_{\mathrm{Aut}} A s\, y$ for $s \in \mathbb{T}$, and the terms with $S \neq J$ vanish as $\mathcal{D}_S y = 0$; with $\Upsilon_\emptyset = \delta^i$, which acts on y as $(n_1 A)^i$ since $\delta_0 y = n_0 y$, only $n_1^{i_0} d n_{\mathrm{Aut}} A^{i_0} \mathcal{D}_J y$ remains. The left side applied to u is integral, and cancelling $n_1^{i_0}$, allowed as $2n!\, n_1^{i_0} \mid \ell^\nu$, gives the claim. □

40.3. Ribet's element and the clearing bound

For ℓ and the data put $N = n_{\mathrm{ch}}$, the order of $\prod_w \mathrm{GL}_{\mathrm{rt}}(\mathbb{F}_w)$ over the chosen places. The *odd class bound* is a positive integer at most $2^{4(N_E+\mathrm{t}+3)^{27}} h_E$ whose valuation bounds the lengths of the odd classes in $(O/\ell^\nu)(-e)$, $0 < e < t$, and of the unramified homomorphisms for $e = 0$. The *local defect bound* is $\mathrm{loc}_\ell = 4f\,\mathrm{defect}_\ell$ (Section 41), the *twist error* is $e'_\ell = (4f+3)\mathrm{r}\,v_\ell(N) + 2^{-\ell}$, the *denominator error* is

$$e_\ell = \mathrm{t}\,v_\ell(X) + 8f\,\mathrm{t}\,v_\ell(N) + 2v_\ell(N) + \big((4\mathrm{t}+1)h_\Delta^{22} + h_\Delta\big)v_\ell(N), \qquad h_\Delta = \mathrm{r}^8,$$

X the odd class bound, and the *clearing bound* is

$$\beta_\ell = \mathrm{loc}_\ell + (1+2\mathrm{t})e'_\ell + e_\ell + v_\ell(N).$$

A twist ψ of modulus Q is *auxiliary* with ramification at most ram if Q is auxiliary for the bad places, ψ comes from the norm-one torus, and the valuations of $\psi(\sigma_P) - 1$ at the tame generators sum to at most ram.

Lemma 40.4 (Lattices, tame relations and the accounting). *Let V, c be a lift setting for ϑ, ψ an auxiliary twist of modulus Q and ramification at most* ram, *and $N = n_{\mathrm{ch}}$.*

(a) *If u' has nondegenerate trace form and the trace of $\psi(R)$ plus the characters $O(-e)$, $e < t$, and the twisted lattice $\psi' R_0$ spans all matrices, then u' has a lattice that is block diagonal with $\psi' R_0$ and the characters.*

(b) *There are places $q_k \nmid \ell$, covering Q and the places above p_0p_1 prime to ℓ up to conjugation, with tame generators δ_k, and polynomials o_k with $o_k(\delta)(\delta-1) = 0$ on every u_x for δ among the δ_k and the inverses of their conjugates, such that the valuations of the $o_k(1)$ add up to at most $2\,\mathrm{ram} + 8f\,v_\ell(N)$.*

(c) *With divisible coefficients, the cocycles of the main summand T at the lift that obey the relations and vanish at the τ_j, the inverses of their conjugates and γ have length at most $\mathrm{loss}_T + \mathrm{sel}_T$, where the losses of T and the crystalline colengths of the characters add up, normalised, to at most $r'\,\mathrm{loc}_\ell$.*

(d) *Let $h \le v_\ell(N)$, $\mathrm{sep} = h_\Delta h$, $(k + n_{\mathrm{test}})\mathrm{sep} \le h_\Delta^{22} h$, $n_{\mathrm{det}} \le 2(k + n_{\mathrm{test}})$, $\mathrm{tame} \le 2\,\mathrm{ram} + 8f\,v_\ell(N)$, $t' \le \mathrm{t}$, and let $v(d)$ be at most the losses, sel_T, the odd class lengths $\mathrm{odd}_e \le v(X)$, t' tame and $(2t' n_{\mathrm{det}} + k + 1)\mathrm{sep}$. If the normalised losses add up to at most* loc *and $\mathrm{sel}_T \le \mathrm{sel}$, then $v(d) + 2v_\ell(N) \le \mathrm{sel} + 2t'\mathrm{ram} + \mathrm{loc} + e_\ell$.*

Proof. (a) Let ρ be the block diagonal sum of $\psi' R_0$ and the characters, over $\bar{\mathbb{Q}}_\ell$. The traces of u' and ρ agree, the trace of a block sum being the sum of the traces; as both have nondegenerate trace forms on the spans of their images, u' is conjugate to ρ, which has entries in O.

(b) For $k \in Q$ take $o_k = \psi(\delta_k) - X$: on inertia above Q, $(g - \psi(g))(g-1) = 0$, and $o_k(1) = \psi(\delta_k) - 1$, of total valuation at most ram for each of the two copies. For a place above p_0p_1 prime to ℓ take $o_k = (X^N - 1)/(X-1)$: inertia there acts on a representation of the type through roots of unity of order dividing some $p_i^d - 1$, which divides N (Lemma 37.4(b)), so $X^N - 1$ kills it; $o_k(1) = N$. There are at most $4f$ such places (Lemma 37.1(a)) and two copies of each, which gives $8f\,v_\ell(N)$.

(c) A cocycle that kills the tame generators is unramified at the new places (Lemma 39.1(b)), so the members obeying the crystalline relations above ℓ are Selmer cocycles, of length at most sel_T modulo ℓ^M for every M; the others are

counted, modulo those, by the crystalline colength of T, which is bounded by its loss, the sum of the defects above ℓ (Theorem 41.6). Both bounds hold modulo every ℓ^M, so by Lemma 38.6 they hold with divisible coefficients.

(d) $t'\,\mathrm{tame} \le 2t'\mathrm{ram} + 8f\,\mathrm{t}\,v_\ell(N)$; $\sum_e \mathrm{odd}_e \le \mathrm{t}\,v(X)$; and the multiples of sep are at most $4t'(k+n_{\mathrm{test}})\mathrm{sep} + (k+n_{\mathrm{test}})\mathrm{sep} + \mathrm{sep} \le ((4\mathrm{t}+1)h_\Delta^{22} + h_\Delta)v_\ell(N)$, as $h \le v_\ell(N)$. Adding $2v_\ell(N)$, the definition of e_ℓ gives the claim; v is normalised so that $v(\ell) = 1$. □

Theorem 40.5 (Ribet's element for conjugate self-dual representations). *Let V, c be a lift setting for ϑ with $r' = r_V$, $t' = t_V$, and ψ an auxiliary twist with ramification at most* ram. *Let the w_j form a triangle of primes with exponents $0 < \mathrm{ty}(j) < t'$, up to $\ell^{\mathrm{r}^8 h}$, $h \le v(N)$. Let u_x, $x \in Y$, be finitely many conjugate self-dual representations of type c and level Q, the w_j and their conjugates, with $(g-\psi(g))(g-1) = 0$ on inertia above Q; for $x \in \mathcal{X}$ let u_x have the trace of $\psi(R_x)$ plus cyc^{-e}, $e < t'$, with R_x a main summand and $\psi(R_x)$ having a lattice on which some γ_x acts by a scalar. Let S, $\mathcal{J}$ be a trace ideal for ℓ^ν and the relations $\tau_j - 1 - X_j(\varphi_j - 1)H_j(\varphi_j)$, $H_j(1) = 1$, and let the Selmer group of $\psi(R_\pi)$ with divisible coefficients have length at most $\mathrm{sel} < \infty$. Then the $\det(1-u_x(g))$ lie in S and vanish on $\mathcal{X}$, and some $f \in S$ multiplies each of them into $\mathcal{J}$, has on $\mathcal{X}$ no monomial properly dividing $\prod X_j$, and its coefficient d of $\prod X_j$ at π satisfies*

$$v(d) + 2v(N) \le \mathrm{sel} + 2t'\,\mathrm{ram} + r'\,\mathrm{loc}_\ell + e_\ell.$$

Proof. The coefficient ring. Choose a coefficient ring O that carries all the data: the finitely many polynomials H_j over the integers of $\bar{\mathbb{Q}}_\ell$, with $H_j(1) = 1$, are defined over every large O, and so are lattices in the u_x and models R_0, ψ', ζ' of the main summands with $\zeta' - 1$ a unit; at π take the model with the Selmer bound. By Lemma 40.4(a) the lattices may be taken block diagonal on $\mathcal{X}$. *Doubling.* With the inverses of the conjugates of the τ_j, φ_j added, both copies using X_j and H_j, $\mathcal{J}$ is again a trace ideal: on the copy of $(\tau_j^c)^{-1}$, conjugate self-duality turns each trace into a power of cyc^{-1} times a trace at $g' = (g^c)^{-1}$, and as cyc^{-1} is 1 at τ_j and $\equiv 1$ modulo ℓ^ν at φ_j, the new relation is a unit times the old one at g', up to multiples of ℓ^ν times traces. The factors $\varepsilon_j = H_j$ have the value $\equiv 1$ modulo ℓ^ν at the scalars $\mathrm{cyc}^{-e}(\varphi_j)$ by which the φ_j act on the characters. *The hypotheses of Theorem 38.7.* (i) and (ii): on $\mathcal{X}$ the lattice is the block $\psi' R_0$, which spans all matrices (Proposition 37.3), plus the characters cyc^{-e} with $\xi_0 = 1$; γ_x acts on the block by ζ' and is an inertia element off ℓ, where the cyclotomic character is trivial. (iii): the element g_0 of the triangle separates the characters, $\prod_{k\ne e}(u^e - u^k) \mid \ell^{h_\Delta h} = C$, and the types are monotone. (iv): Lemma 40.4(c). (v): the elements δ of Lemma 40.4(b), trivial on the characters (the cyclotomic character is trivial on inertia off ℓ). (vi): Theorem 39.5 with the odd classes bounded by the odd class bound and the crystalline colengths of the characters by Lemma 40.4(c); the detection on the test points of the triangle and g_0, together with the tame generators, which a character cocycle vanishing there and unramified off ℓ must respect. *The determinants.* $\det(1-u_x(g))$ is the determinant of the combination $\sum_{g'} \omega_{g'}(u_x(g') - 1)$ with $\omega = -1$ at g and 0 elsewhere, computed on the lattice. *The bound.* By Theorem 38.7 the coefficient comes from $c \in O$ with $v(c)$ at most $b_0 + \sum_e b_e + t'(\sum_k v(o_k(1)) + 2\#s\, v(C)) + (\#J+1)v(C)$, and Lemma 40.4(d), with $k = \#J$, n_{test} the number of test points and g_0, $n_{\det} = \#s$, gives the stated inequality. □

40.4. The bound for denominators

A *lift generator* of a Hecke-stable space Z of forms on V is a form of Z in the range of the projection of a type in lift setting for ϑ, lifting a Hecke eigensystem that has a main summand whose twist by ϑ_ℓ^{-1} has a Selmer group with divisible coefficients of length at most sel.

Lemma 40.6 (Separation, weak lifts and lift generators). (a) *There are $h \le v_\ell(N)$, with $h = 0$ unless $\ell \le h_\Delta^2$, and a topological generator g_0 of $\mathrm{Gal}(\Omega/E)$ modulo the kernel of the cyclotomic character with $\mathrm{cyc}(g_0)^a - 1 \mid \ell^h$ for $0 < a < \mathrm{t}$.*

(b) *Some split place w off the places of O and their conjugates has the property that $H_w(X_*)$ kills every nonzero form of base level in a weak lift.*

(c) *If e_c is the type projection, Z is stable under it and under the Hecke polynomials, y is orthogonal to the false lifts, and every nonzero isotypic form in the range of e_c outside Z belongs to a weak lift, then $e_c(y - \mathrm{pr}_Z y)$ lies in the span of the weak lifts of base level.*

(d) *The span of the lift generators is stable under the Hecke polynomials and the type projections; a lift generator either is killed by e_c or is a nonzero form of a system in its range, and every system containing a nonzero lift generator lifts a main summand with a twisted lattice.*

Proof. (a) If $\ell > h_\Delta^2$, take γ in an inertia group of $\mathbb{Q}$ at ℓ whose cyclotomic character reduces to a generator b of $(\mathbb{Z}/\ell)^\times$; it fixes E, which lies in a cyclotomic field of conductor prime to ℓ. As b has order $\ell - 1$ and $0 < a < \mathrm{t} \le h_\Delta^2 < \ell$, $b^a \ne 1$, so $u^a - 1$ is a unit and $h = 0$. If $\ell \le h_\Delta^2$, take a Frobenius element at a place of norm $p_i \ne \ell$; then $u^a - 1 = p_i^a - 1$ divides N (Lemma 37.4(b)), and a nonzero ℓ-adic integer is a unit times ℓ^{v_ℓ}, so $u^a - 1 \mid \ell^{v_\ell(N)}$. In both cases a topological generator modulo the kernel may replace the element, the divisibility passing to it.

(b) For a split w off the level, a nonzero form of base level in a weak lift has the roots $X_0 q^i$, $i < t$, in its Hecke polynomial (Lemma 40.2(b)). The last of them is X_*^{-1}, so the reversed polynomial, by which H_w acts, vanishes at X_*.

(c) Let U' be the part of the range of e_c orthogonal to Z, and K' the forms of U' killed by $H_w(X_*)$ for the place w of (b); they are false lifts, being in $Z^\perp$, in the image of e_c, and killed by the Hecke polynomial at the lift root. An isotypic $v \in U'$ outside the span M of the weak lifts lies in K' by the hypothesis and (b), hence is orthogonal to itself; so $U' \cap K'^\perp \subseteq M$ (Lemma 40.2(d), the Hecke operators at w and at other places commuting). The form $e_c(y - \mathrm{pr}_Z y)$ lies in U', and in $K'^\perp$: e_c is self-adjoint (Lemma 37.4(c)) and fixes K', and both y and $\mathrm{pr}_Z y$ are orthogonal to the false lifts at w, the latter because $\mathrm{pr}_Z y \in Z \perp K'$.

(d) A lift is an eigenvector of every Hecke polynomial, so the span of lifts is stable. The type projections are central idempotents (Lemma 37.4(c)) and a lift generator lies in the range of one of them inside the image of e_0; idempotents of distinct tuples are orthogonal, so e_c either kills it or fixes it, and a nonzero lift generator is a nonzero form of the system of its Hecke eigenvalues. Such a system lifts a main summand R with the trace identity of Theorem 37.7, and a twisted lattice exists by the scalar of large order of Definition 37.2; for the generator v_0 with the Selmer bound the lattice can be taken with that bound. □

Theorem 40.7 (Bound for denominators). (a) *Let ϑ be auxiliary with ramification at most* ram *and let Z be spanned by lift generators. Then an operator Λ on Z preserves the Hecke eigenspaces at the split places off the level, has eigenvalues of valuation at most* $\mathrm{sel} + 2t_O\,\mathrm{ram} + r_O\,\mathrm{loc}_\ell + e_\ell$*, and takes the projections to Z of integral forms orthogonal to the false lifts to integral forms.*

(b) *If* ram *and* sel *are at most e'_ℓ, some operator on Z has determinant of norm at least* $\exp(-\dim Z \cdot r_O\beta_\ell \log \ell)$ *and takes the projection of every integral form orthogonal to the false lifts to an integral form whose pairings with integral forms are integral; that is, the projection to Z has denominator at most ℓ^{β_ℓ}.*

Proof. (a) *One generator.* Let $v_0 \neq 0$ be a lift generator with the Selmer bound, of type c and system x_0. Base data B for O, c and ϑ exist (Theorem 37.5(c)): the forms of type c are forms of B of base level, and the normalised representations of its systems at every auxiliary level are of type c, with the quadratic relation above Q. *The weak-lift selector.* The finitely many base systems x that are not weak lifts have elements g_k with $\det(1 - u_x(g_k)) = \mathrm{value}_k \neq 0$, integral as u_x is continuous on a compact group; put $\delta_0 = \prod_k(\mathrm{value}_k - \delta(g_k))$ with $\delta(g) = (x \mapsto \det(1 - u_x(g)))$ and $n_0 = \prod_k \mathrm{value}_k$. Then δ_0 vanishes at the systems that are not weak lifts, and a base system with $\delta_0 \neq 0$ is a weak lift. *The auxiliary places.* With g_0, h of Lemma 40.6(a), Theorem 39.4 gives a triangle of Taylor–Wiles places of every level ℓ^ν, with Frobenius data in the set where, by Lemma 40.2(c), every weak lift of base level congruent to x_0 has a root chain with an integral divisor e_j, $e_j(1) = 1$; the set contains most Frobenii near $\delta^{\ell^\nu} g$ for every g, as the twist of the main summand by a power of the scalar inertial element has $\|\det(1 - \cdot)\| \geq 1$ near suitable points. Take ν with $2n!(n_0 n_{\mathrm{Aut}}^2)^{\#J+1} \mid \ell^\nu$. *Ribet's element.* By Theorem 40.5 applied to the systems of the auxiliary level, every trace ideal has a common Ribet element for the $\delta(g)$ with $v(N^2 d)$ at most $L = \mathrm{sel} + 2t_O\mathrm{ram} + r_O\mathrm{loc}_\ell + e_\ell$. Reading polynomials over $\mathbb{T}$ as families of polynomials, one for each system, the Ribet element becomes $\mathcal{D} \in \mathbb{T}[\alpha]$ multiplying every $\delta(g)$ into the tied ideal $\mathfrak{a}$; as $\delta_0 - n_0$ lies in the ideal of the $\delta(g)$, $(\delta_0 - n_0)\mathcal{D} \in \mathfrak{a}$, its coefficients below $\prod \alpha_j$ vanish on $\mathcal{X}$, and its top coefficient at x_0 has the bound. *The clearing.* Let l be a form of base level with Nl integral, y supported on $\mathcal{X}$ and agreeing with l where $\delta_0 \neq 0$, and A the indicator of the systems congruent to x_0 (Proposition 40.1(a)), with $A^{\#J+1} = A$. By Theorem 40.3(d) with $d = N$, $N^2 A\mathcal{D}_J y$ is integral, the common multiple n_{Aut} of the stabiliser orders of these base data being N; and $s = N^2\mathcal{D}_J$ has $v(s(x_0)) \leq L$. *The operator.* Let G be s composed with e_c, scaling each system. For y integral and orthogonal to the false lifts, $e_c(\mathrm{pr}_Z y)$ is supported on the systems of lift generators, and agrees with $e_c(y)$ at the weak lifts by Lemma 40.6(c) and (b); so $G(\mathrm{pr}_Z y)$ is integral. Every lift generator is an eigenvector of G, with eigenvalue of valuation at most L at v_0. *All generators.* Choose a basis of Z among the lift generators; the operators for the basis vectors are diagonal in it, and by Lemma 40.2(a) one integral combination Λ has at every basis vector an eigenvalue at least as large in norm, of valuation at most L. A diagonal operator in a basis of Hecke eigenvectors preserves the Hecke eigenspaces, and integral combinations preserve the integrality property.

(b) If $Z = 0$ the identity serves. Otherwise the lift setting of a generator bounds $r_O \leq \mathrm{r}$ and $t_O \leq \mathrm{t}$, and $N\Lambda$, with Λ from (a), has eigenvalues of valuation at most $L + v_\ell(N) \leq r_O\beta_\ell$, since $\mathrm{sel} + 2t_O\mathrm{ram} + r_O\mathrm{loc}_\ell + e_\ell + v_\ell(N) \leq r_O\beta_\ell$ for

sel, ram $\le e'_\ell$ and $1 \le r_O$ $(\beta_\ell = \mathrm{loc}_\ell + (1+2\mathrm{t})e'_\ell + e_\ell + v_\ell(N))$. By Lemma 40.2(a) its determinant has norm at least $\ell^{-\dim_Z r_O\beta_\ell}$. The image of an integral form under $N\Lambda$ is integral, and its pairings with forms integral at all places above ℓ are integral, the Gram matrix being integral and N clearing the stabiliser orders. □

41. The local loss above ℓ

The cocycles produced by Ribet's method satisfy the linear relations among the $u_x(g)-1$, but at the places above ℓ they need not be reductions of crystalline cocycles. This section bounds, for all summands and all places above ℓ together, the distance of these cocycles from the crystalline condition by the explicit *local defect bound* $r'\,\mathrm{loc}_\ell$ (Theorem 41.6). By Tate duality the distance is the *defect* of the Tate dual of a summand: the length of its crystalline cocycles modulo those whose cup products with the crystalline cocycles of all lattices of a class are coboundaries. The defect is bounded through an exponent ℓ^k that makes the cup products vanish, which Section 42 computes on Kummer charts.

41.1. Classes of local lattices and the defect

Let $\Gamma \to G_\ell = \mathrm{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$ have open image, O the integers of a finite extension of $\mathbb{Q}_\ell$, and $B_{\max}$ the crystalline period ring. For a filter F on Γ, polynomial conditions $K(s)$ on inertia elements and finite sets $U(\gamma)$ of integers for the embeddings $\gamma\colon O \to \mathbb{C}_\ell$, the class $\mathrm{InClass}(F,K,U)$ consists of the Γ-lattices Y over O such that $B_{\max}\otimes Y$ is spanned by the periods fixed near 1 and along F, the Hodge–Tate jumps at each γ lie in $U(\gamma)$, and every polynomial Q with $K(s)(Q)$ kills s on the periods. A map $\Gamma \to (O/\ell^m)^d$ *comes from the class* if it is $t\circ a$ for a crystalline cocycle a of a lattice Y of the class and an equivariant t. The *defect* $\mathrm{def}(\boldsymbol{Q}, L)$ is the length of the crystalline cocycles of L modulo those orthogonal under cup product to all lattices of the class $\boldsymbol{Q}$, for every pairing into $\mathbb{Z}/\ell^k(1)$; L *has cup exponent* k for $\boldsymbol{Q}$ if ℓ^k carries the crystalline cocycles of L into this orthogonal. The *crystalline loss* of (h,b,ℓ) is the ℓ-adic valuation of $(2b)!\cdot\Pi^{h^3}$, with Π the product of the integers in $(b-30h^3, b+30h^3]$, and $\mathrm{defect}_\ell = 10^6\frac{m}{k}\big(b\,v_\ell(M) + \text{crystalline loss}(\mathrm{r}^8, b, \ell)\big)$.

Throughout, $\iota\colon \mathbb{C}\simeq\bar{\mathbb{Q}}_\ell$ is fixed, $G_\ell = \mathrm{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$, and for an embedding $e\colon E\to\bar{\mathbb{Q}}_\ell$ and $g\in G_\ell$ the complex embedding $\iota^{-1}\circ g^{-1}\circ e$ of E is written σ_g; $\epsilon_\sigma = \pm1$ is the sign of σ for the CM type. The *b-places* are the places of E of weight b (Definition 8.7(D3)); they have class -1 modulo q_0 and are sparse. A *Ribet cocycle* of ρ with values in a module B is a cocycle obeying the linear relations among the families $(u_x(g)-1)_x$; the *crystalline relations* of a lattice u are the linear relations that hold on its crystalline cocycles at the places above ℓ, extended by zero to the global Galois group.

Lemma 41.1 (Signs and spacing of b-places). (a) *The field E lies in the cyclotomic field of order $q_1 d$; hence an element of G_ℓ that fixes the roots of unity of that order fixes every $e(E)$. A complex embedding σ sends $\sqrt{-d}$ to ϵ_σ times the chosen root, and the embedding of the CM type at the place of σ sends ζ to $\sigma(\zeta)^{\epsilon_\sigma}$.*

(b) *If $g'g^{-1}$ raises $e(\zeta)^{\ell^a}$ to the power ℓ^n, the ratio of the places of σ_g and $\sigma_{g'}$ in $\Phi \cong (\mathbb{Z}/q_1)^\times$ reduces, modulo every divisor m of q_1 prime to ℓ^a, to $\epsilon_{\sigma_g}\epsilon_{\sigma_{g'}}\ell^n$.*

(c) (*Short arcs.*) *If $\ell \le k\mathsf{t}$, $0 < i < n < l_0$ and $g'g^{-1}$, $g'g_1^{-1}$ raise $e(\zeta)^{\ell^a}$ to the powers ℓ^n, ℓ^i, and σ_g, $\sigma_{g'}$ lie over b-places, then σ_{g_1} does not.*

(d) *If $\ell \mid q_1 d$ and σ_g, $\sigma_{g'}$ lie over b-places, their signs agree.*

(e) *A jump within t of $-\epsilon_\sigma w$, w the lift weight, has modulus larger than every B with $B + \mathsf{t} < w$; every jump of a main summand has modulus at least $h_\Delta/2$.*

Proof. (a) $E = K(\zeta)$ with $K = \mathbb{Q}(\sqrt{-d}) \subseteq \mathbb{Q}(\zeta_{4d})$ and ζ of order q_1 (Proposition 8.3); an embedding of E extends to the cyclotomic field, which a root of unity generates. A complex embedding or its conjugate is chosen by the CM type at its place, and conjugation negates $\sqrt{-d}$; the CM type embedding at the place agrees with σ or with $\bar\sigma$, which sends ζ to $\sigma(\zeta)^{-1}$.

(b) A place $y \in \Phi$ acts on ζ as the power given by its class modulo q_1, and by (a) the CM embeddings of the places of σ_g, $\sigma_{g'}$ send ζ to $\sigma_g(\zeta)^\epsilon$, $\sigma_{g'}(\zeta)^{\epsilon'}$. Carry the resulting equation between powers of a primitive root to $\bar{\mathbb{Q}}_\ell$ by ι, apply g', raise to the power ℓ^a, where the hypothesis applies; this gives a congruence of exponents modulo $q_1/\gcd$, and modulo m the factor ℓ^a cancels; multiplying by ϵ', whose square is 1, gives the claim.

(c) By (b) the ratios of the places of σ_{g_1} and σ_g to that of $\sigma_{g'}$ are $\pm\ell^i$ and $\pm\ell^n$ modulo the part of q_1 prime to ℓ; so two b-places would differ by $\pm\ell^{n-i}$ and $\pm\ell^i$ with $0 < i, n-i < l_0$, which the sparseness of the b-places for the primes up to $k\mathsf{t}$ excludes.

(d) Suppose the signs are opposite; then $\sigma_g(\sqrt{-d}) + \sigma_{g'}(\sqrt{-d}) = 0$ by (a). *If* $\ell \mid q_1$, let $x = e((1+\sqrt{-d})/2)$, an integer. Applying $g \circ \iota$ to the relation gives $(gg'^{-1})(x) = 1 - x$, so $\|1-x\| = \|x\|$, the Galois group acting by isometries. But $x(1-x) = (d+1)/4$, which ℓ divides when $\ell \mid q_1$ (from $d \equiv q_0/2 - 1$ modulo q_0 and $q_1 = 3^c 2^s q_0$), so $\|x\|^2 < 1$; while $1 = x + (1-x)$ gives $\max(\|x\|, \|1-x\|) \ge 1$, a contradiction. *If* $\ell = d$, $g'g^{-1}$ raises $e(\zeta)$ to some power d^n, and by (b) with $m = q_0$ the ratio of the two places is $-d^n$ modulo q_0; both have class -1, so the ratio is 1, and $d^n \equiv -1$ modulo q_0, which no power of d satisfies.

(e) If $|u + \epsilon_\sigma w| \le \mathsf{t}$, then $|u| \ge w - \mathsf{t} > B$. A main summand jump is low, $|u| \ge \Delta/4 = h_\Delta/2$, or within t of $\mp w_i$ with $w_i \ge h_\Delta \ge 3\mathsf{t}$. □

Proposition 41.2 (Tate duality). *Let $\Gamma \to G_\ell$ be an open embedding and $L = O^d$ a lattice of the class.*

(a) (*Artin–Rees.*) *For finitely many representations u_x and a noetherian O, a cocycle with values in $(O/a)^d$ that obeys every linear relation among the families $(u_x(g) - 1)_x$ comes from an equivariant map on a subrepresentation containing the families $(u_x(g) - 1)_x$ and all multiples of a power of a.*

(b) *A module of maps $\Gamma \to L/\ell^m$ coming from the class has length, modulo the reductions of the crystalline cocycles of L, at most the defect of the Tate dual $L^\vee(1)$.*

(c) *If the lattices of all $u_x \circ \eta$ and of $\rho \circ \eta$ lie in the class, Ribet's cocycles of ρ modulo ℓ^m, modulo those whose restriction obeys the crystalline relations of L, have length at most the defect of $L^\vee(1)$.*

Proof. (a) Let I be the O-span of the families $(u_x(g) - 1)_x$ in the families of matrices. By the Artin–Rees lemma for the ideal (a), for some k the elements

of I divisible by a^k lie in aI. Define $C\colon I \to (O/a)^d$ by $C(\sum_g \theta_g(u_x(g)-1)_x) = \sum_g \theta_g c(g)$, well defined because c obeys the linear relations; it vanishes on aI, as a kills the values. Let Y be the image of the map $(\theta, v) \mapsto (\sum \theta_g(u_x(g)-1) + a^{k+1}v,\ C(\theta))$; it is the graph of the map $I + a^{k+1}M \to (O/a)^d$ extending C by zero on $a^{k+1}M$, well defined by the choice of k ($L\theta = a^{k+1}v'$ forces $L\theta \in aI$, so $C(\theta) = 0$). It is stable under g, acting by left multiplication with $(u_x(g))_x$ and by $\rho(g)$: on the generators this is the cocycle rule $c(gh) = c(g) + gc(h)$ and $u(g)(u(h)-1) = (u(gh)-1) - (u(g)-1)$. Its points are determined by their first component, and it contains $((u_x(g)-1)_x + a^k v, c(g))$.

(b) Let N be the module, A the reductions of the crystalline cocycles of L and H the crystalline cocycles of $L^\vee(1)$ orthogonal to the class. *Pairings.* The evaluation pairing of $L^\vee(1)$ with L gives a perfect pairing into $\mathbb{Z}_\ell(1)$, and modulo ℓ^m the reduction $\tau\colon L \to L/\ell^m$ is onto, equivariant and has kernel $\ell^m L$; L/ℓ^m is a discrete module, since a crystalline cocycle of a lattice of the class is continuous. *Members of N are orthogonal to H.* Let $t \circ a \in N$ with a a crystalline cocycle of Y in the class. Lift t to a linear $q\colon Y \to L$, equivariant modulo ℓ^m; then $g\,q - q = \ell^m \delta_g$ with linear $\delta_g\colon Y \to L$, and the forms $y \otimes \varphi \mapsto \varphi(\delta_g y)$ give a crystalline cocycle of the Tate dual of $Y \otimes L^\vee(1)$, crystalline cocycles being saturated. Hence the pairing $(y, \varphi) \mapsto \varphi(q(y))$ modulo ℓ^m is one of the pairings tested in the definition of orthogonality, and the cup product of $q \circ a$ and $z \in H$ under it is that of a and z, a coboundary. *Local duality.* By the inequality of local duality for crystalline cocycles (Theorem B.39(b)), the index of $N \cap A$ in N divides the index of H in the crystalline cocycles Z of $L^\vee(1)$, as N and A are orthogonal to H and A is the exact annihilator of Z modulo ℓ^m. A relative index is the cardinality of the quotient, so the lengths compare.

(c) By (a), applied to the $u_x \circ \eta$ and $\rho \circ \eta$, the restriction along η of a Ribet cocycle w comes from the class: Y is a subrepresentation of the families of matrices times the values, which contains the family $(\ell^m)^k \cdot 1$ (as $u_x(1) = 1$), and the second component is an equivariant map carrying the crystalline cocycle $g \mapsto ((u_x(\eta g) - 1)_x, w(\eta g))$ of Y to $w \circ \eta$; Y lies in the class, being a sublattice of a sum of lattices of the class, and crystalline cocycles are saturated. Hence the module of restrictions lies in N of (b), and a Ribet cocycle whose restriction is in A obeys the crystalline relations of L; apply (b). □

41.2. The defect of a crystalline lattice

The *size conditions* on $(Q, m, h, G, b, \delta, \eta, W)$ are $Q > 0$, $99(Q^2 + (W+1)Q + Q + W + m) \le h$, $99h^3 \le G$, $99(Gh^3 + G) \le b$, $2\log(4b) < m$, $\delta < \eta$ and $99((\eta + W)Q + \eta + h) \le h^3$.

Lemma 41.3 (The crystalline loss). *Let $\lambda = \lambda(h, b, \ell)$ be the crystalline loss.*

(a) *If ℓ^n divides $(2b)!\,\Pi^{h^3}$, then $n \le \lambda$; and $2b < \ell(\lambda + 1)$.*
(b) *If ℓ divides an integer in $(b - 30h^3, b + 30h^3]$, then $\lambda \ge h^3$.*
(c) *If $H \le 4b$ and $2\log(4b) < m$, then $\lfloor \log_\ell H \rfloor < m$.*
(d) *If $\ell > G + \delta + n$ divides no integer in $(b - c, b + c]$ and $\delta + n < c$, then ℓ divides no nonzero integer within n of 0, $\pm(b + \delta)$ or $\pm(G + \delta)$.*

Proof. (a) The first claim is the definition of the valuation. By Legendre's formula $v_\ell((2b)!) \ge \lfloor 2b/\ell \rfloor$, so $\lambda \ge \lfloor 2b/\ell \rfloor > 2b/\ell - 1$. (b) The factor Π^{h^3} then has valuation at least h^3. (c) $\log \ell \ge \log 2 > 1/2$, so $\lfloor \log_\ell H \rfloor \le 2\log(4b) < m$. (d) A nonzero

multiple of ℓ has modulus at least $\ell > G + \delta + n$, so it is not within n of 0 or $\pm(G+\delta)$; near $\pm(b+\delta)$ it would lie in $(b-c, b+c]$, as $\delta + n < c$. □

Theorem 41.4 (Defect of a crystalline lattice). (a) (*Rank.*) *For a potentially crystalline lattice L whose jumps at the embeddings over a place are given multisets, the rank of the crystalline cocycles is* $\operatorname{rk} L$ *plus the number of negative jumps.*

(b) *If L has cup exponent k for $\boldsymbol{Q}$, the normalised defect is at most $2k \operatorname{rk} L \cdot [G_\ell : \Gamma]$.*

(c) *Assume the size conditions; let the jumps of the class lie in $[-\eta,\eta]\cup(-\epsilon_\gamma)[a, a+W]$ and those of the crystalline lattice L in $[-\eta,\eta]\cup\epsilon_\gamma[a, a+W]$, with $a = b+\delta$ at thick and $a = G+\delta$ at thin embeddings; and assume either that all negative jumps are at most $-h/2$ and at most μ jumps of L at each embedding lie in $[-\eta,\eta]$, or that all jumps of L lie in $[-\eta, 0]$. Let λ be the crystalline loss of (h, b, ℓ). Then, for a Kummer chart of Γ of ramification at most h (with the conditions on thick letters when $\ell \le Q$), the normalised defect is at most $[G_\ell : \Gamma] \cdot y$ for every $y \ge 10^5\big(\operatorname{rk} L \cdot \lfloor 2\lambda/Q \rfloor m + \mu\lambda\big)$.*

Proof. (a) Over $\mathbb{Z}_\ell$ the Bloch–Kato rank formula (Theorem B.39(a)) gives $\operatorname{rk}_{\mathbb{Z}_\ell} L$ plus the index $[G_\ell : \Gamma]$ times the number of negative jumps over all embeddings of O, counted with multiplicity. Every embedding of O is $x \mapsto g(x)$ for some $g \in G_\ell$, and $g \mapsto \sigma_g$ gives, as g runs over coset representatives, each complex embedding of the place $[G_\ell : \Gamma]$ times; counting the jumps at the pairs (embedding of O, complex embedding) in two ways and dividing by $[O : \mathbb{Z}_\ell]$ gives the rank over O.

(b) The defect is the length of a quotient of $Z_f(L)/\ell^k Z_f(L) \cong (O/\ell^k)^r$, r the O-rank of $Z_f(L)$, which has normalised length rk. By (a), $r \le \operatorname{rk} L + [G_\ell : \Gamma] \cdot \#\{\text{negative jumps}\} \le 2[G_\ell : \Gamma] \operatorname{rk} L$, at most $\operatorname{rk} L$ jumps at each embedding being negative.

(c) Write λ for the loss and $K = \lfloor 2\lambda/Q \rfloor$, so $2\lambda < (K+1)Q$. *The windows at the letters.* At a letter of the chart (a unit of $\mathbb{F}_{\ell^f}$, read through the Teichmüller lift and Fontaine's θ) collect the jump sets of the embeddings g with $g(\zeta)$ equal to its root of unity; at thick letters, and at all letters if $\ell > Q$, thickness and sign are read off the letter, so the collected sets lie in the low window and opposite far windows, the low window widened to $[-2G, 2G]$ if $\ell \le Q$ (it contains $[-\eta,\eta]$ and the far windows of thin letters, as $\eta \le G$ and $\delta + W \le G$). *An exponent suffices.* If some power $\ell^{10^4 Kx}$ with $x \le m$ carries the crystalline cocycles into the orthogonal of the class, then by (b) the normalised defect is at most $[G_\ell : \Gamma] \cdot 2 \cdot 10^4 Kx \operatorname{rk} L \le [G_\ell : \Gamma] y$. *Small primes $\ell \le Q$*: the thick letters are sparse with gaps measured up to $J = \lfloor \log_\ell h^3 \rfloor$, so Theorem 42.23(c) applies with the low window $[-2G, 2G]$; its conditions follow from the size conditions and $2(b+\delta) < h^2 K'$ for a suitable $K' \le K$, and the exponent is $10^4 K(\lfloor \log_\ell h^3 \rfloor + 1)$ with $\lfloor \log_\ell h^3 \rfloor + 1 \le m$ by Lemma 41.3(c). *Large primes, large loss $\ell > Q$, $\lambda \ge h^3$*: the chart is unramified and $\ell \ge 7$, and Theorem 42.23(b) applies, as $2b < \ell(\lambda+1)$ and $2\lambda < (K+1)Q$ give the size conditions $b < K\ell^2$, $2G < K\ell$; the exponent is $10^4 Km$. *Large primes, small loss $\lambda < h^3$*: by Lemma 41.3(b) ℓ divides no integer in $(b - 30h^3, b + 30h^3]$, so by Lemma 41.3(d) ℓ divides no nonzero integer within $6(2(\lambda+1) + \eta + W)$ of 0 or $\pm a$. Then Theorem 42.23(a) with $a = 2(\lambda+1)$ applies: the defect vanishes if the jumps avoid $[-2(\lambda+1), 0)$ or $U' \subseteq [-\eta, 0]$, and when at most μ jumps of L lie in $[-\eta,\eta]$ at each embedding it is at most $11 f(\mu(2(\lambda+1)+1) + \operatorname{rk} L \cdot (W+1))$ times

the length of O/ℓ, which is 1 after normalisation. If $\lambda < h/5$ and the negative jumps are at most $-h/2$, the jumps avoid $[-2(\lambda+1), 0)$, as $2(\lambda+1) < h/2$, and the defect vanishes; so does it in the second alternative. Otherwise $\lambda \geq h/5$; then $2(\lambda+1)+1 \leq 6\lambda$, and $(W+1)Q \leq h/99 \leq 2\lambda < (K+1)Q$ by the size conditions, so $W+1 \leq K$. With $f = [G_\ell : \Gamma]$ for the unramified chart the defect is at most $[G_\ell : \Gamma] \cdot 11(6\mu\lambda + \operatorname{rk} L \cdot K) \leq [G_\ell : \Gamma]y$. □

41.3. The loss of a lift

Lemma 41.5 (Local lattices and the summand classes). (a) *A class of polynomials that contains those vanishing on a multiset of at most N roots of unity of orders dividing $\ell^a - 1$, $1 \leq a \leq r$, is a tame type: it contains $X^e - 1$, $e = \operatorname{lcm}(\ell^i - 1)_{i \leq r}$, and a monic polynomial over $\mathbb{Z}_\ell$ of degree at most Nr.*

(b) *A lattice in a conjugate self-dual representation R of level S has at a place w above ℓ the jumps of R at σ_g, is potentially crystalline, crystalline unless w lies above p_0p_1, and if w lies above p_0p_1 every polynomial killing an inertia element on all members of a class of such R kills it on the lattice; R is trivial on inertia above places off S and ℓM.*

(c) *If $\ell = p_i$, $0 < x \leq \mathrm{t}$, then twice the tame exponent $3r^2((r+t)r+1)^2 + 150(h_\Delta^2 + 4\mathrm{t} + 2)$ is at most $\operatorname{defect}_\ell / x$; and $10^5(\rho\lfloor 2\lambda/(k\mathrm{t})\rfloor m + \lfloor \rho m/(kx)\rfloor \lambda) \leq \rho \operatorname{defect}_\ell / x$.*

(d) *The data satisfy the size conditions with $Q = k\mathrm{t}$, $h = h_\Delta$, G the off weight, b the weight, $\eta = h^2 + 1$, $W = 4\mathrm{t}$ and $\delta = i\Delta + h - 2\mathrm{t}$, $i \leq m$. The jumps of a main summand and of the characters fit the windows starting at $w_i - 2\mathrm{t}$, the Tate dual of a main summand has few low jumps ($\mu = \lfloor rm/k \rfloor$), and the dual of a character with jump $c < \mathrm{t}$ has its jump $-c - 1$ in $[-h^2 - 1, 0]$.*

(e) *There is a Kummer chart of each completion group, of degree $[G_\ell : \Gamma]$ and ramification at most h_Δ, with an element ζ reading thickness and sign: for every g if $\ell > k\mathrm{t}$, and at b-places otherwise; the chart is unramified with $\ell \geq 7$, or $\ell \leq k\mathrm{t}$ and the signs at b-places are constant; and for $\ell \leq k\mathrm{t}$ no three b-letters follow one another at distances at most $\log_\ell h_\Delta^3$.*

Proof. (a) A root of $y^{\ell^a - 1} = 1$ is a Teichmüller lift in the unramified ring of degree $a \leq r$, so its minimal polynomial over $\mathbb{Z}_\ell$ is monic of degree at most r; take the product over the multiset. Each root is killed by $X^{\ell^a - 1} - 1$, which divides $X^e - 1$.

(b) A lattice is conjugate over $\bar{\mathbb{Q}}_\ell$ to R by an equivariant injective map, which passes admissibility (the periods fixed near 1 and along the filter span), the Hodge–Tate jumps (unchanged by extension of scalars) and the killing polynomials from R to the lattice. At a place off S and ℓM, R is good and ℓ is a unit there, so it is trivial on inertia.

(c) Comparisons of sizes (Remark 8.10): the tame exponent is polynomial in r, while $\operatorname{defect}_\ell / x \geq 10^6 \frac{m}{k} b\, v_\ell(M)/\mathrm{t}$ with $v_\ell(M) \geq 1$ and $b \geq x^{999}$. For the second, $\lfloor 2\lambda/(k\mathrm{t})\rfloor \leq 2\lambda/(kx)$ and $\lfloor \rho m/(kx)\rfloor \leq \rho m/(kx)$, so the left side is at most $3 \cdot 10^5 \rho m \lambda/(kx) \leq \rho \operatorname{defect}_\ell / x$.

(d) The size conditions are comparisons of logarithmic scales (Proposition 8.9). A jump of the main summand is low or within t of $\mp w_i = \mp(S + 2\mathrm{t})$, S the window start, so it lies in $[-\eta, \eta] \cup -\epsilon[S, S + 4\mathrm{t}]$, and $-u - 1$ in the opposite window. The low dual jumps $-a-1 \in [-h^2-1, h^2+1]$ are at most as many as the a

with $|a| \le 2h^2$, hence at most $\lfloor rm/k \rfloor$ (Definition 37.2), and the negative ones are at most $-h/2$ by Lemma 41.1(e). For a character, $-c-1 \in [-\mathrm{t}, 0] \subseteq [-h^2-1, 0]$.

(e) If $\ell \nmid q_1 d$, inertia fixes E, the chart is unramified, and a letter $g(\zeta)$, ζ a primitive $(\ell^f - 1)$-th root of unity, determines the embedding σ_g: gg'^{-1} fixes the constants and the image $-\ell$ of the variable of the chart, hence lies in the image of Γ; take T and s to read off the b-place and the sign. Otherwise $\ell \le k\mathrm{t}$ (the conductor being small against $k\mathrm{t}$), the signs at b-places are constant by Lemma 41.1(d), and the letters of b-places are sparse by Lemma 41.1(c): if $g_1(\zeta) = g_0(\zeta)^{\ell^a}$ and $g_2(\zeta) = g_0(\zeta)^{\ell^{a+b}}$ with $1 \le a, b \le \lfloor \log_\ell h_\Delta^3 \rfloor$, then $g_2 g_0^{-1}$ and $g_2 g_1^{-1}$ raise the prime-to-ℓ part of ζ to the powers ℓ^{a+b} and ℓ^b, with $a + b < l_0$. The ramification of the chart is at most h_Δ because the completions of E have ramification dividing $q_1 d$ over $\mathbb{Q}_\ell$, small against h_Δ. □

Theorem 41.6 (The local loss). *Let V, c be a lift setting for ϑ, R a main summand of rank r', and ρ_x representations of type c (with $(X-1)^t$). Over every large coefficient ring, let u_x be lattices in the ρ_x and, for one index π, u_π block diagonal with summands T (a lattice in R above ℓ) and the characters $O(-e)$, $e < t$. Then at each place v above ℓ there is a class of local lattices containing the restrictions of all u_x and of the summands of u_π, and the normalised defects of the $t+1$ summands at all places above ℓ add up to at most $r' \operatorname{loc}_\ell$, $\operatorname{loc}_\ell = 4f \operatorname{defect}_\ell$.*

Proof. The class at v. Take the class with the crystallinity filter (potentially crystalline if v lies above $p_0 p_1$, crystalline otherwise), the jumps of the representations of the type at the complex embeddings of the place, and, above $p_0 p_1$, the polynomials killing the inertia elements on every representation of the type. By Lemma 41.5(b) the restrictions of the u_x lie in it; the summands of u_π embed equivariantly in u_π, so they belong to it too, and every jump of u_π is a jump of some summand. Above $p_0 p_1$ the class is tame by Lemma 41.5(a): inertia acts through roots of unity of orders dividing some $\ell^a - 1$, $a \le \mathrm{r}$ (the inertia spectrum, Lemma 37.4), at most $(\mathrm{r}+\mathrm{t})$ in number. *One summand at v.* In the tame case $\ell = p_i$, and Theorem 42.23(d) gives a cup exponent at most the tame exponent, so by Theorem 41.4(b) and Lemma 41.5(c) the normalised defect of the dual of a summand of rank ρ is at most $[G_\ell : \Gamma]\rho \operatorname{defect}_\ell / x$, with $x = 1$ for the main summand and $x = \mathrm{t}$ for a character. In the crystalline case apply Theorem 41.4(c) with the sizes, windows and few low jumps of Lemma 41.5(d), the chart of Lemma 41.5(e), $Q = k\mathrm{t}$ and $\mu = \lfloor \rho m/(kx) \rfloor$; by Lemma 41.5(c) the bound is again $[G_\ell : \Gamma]\rho \operatorname{defect}_\ell / x$. *The sum.* The local degree $[G_\ell : \Gamma_v]$ counts the complex embeddings of E that belong to v, and distinct places above ℓ have disjoint sets of embeddings, so the local degrees add up to at most $2f$. The main summand costs at most $2f\, r' \operatorname{defect}_\ell$ and the $t \le \mathrm{t}$ characters together at most $2f\, t \operatorname{defect}_\ell / \mathrm{t} \le 2f\, r' \operatorname{defect}_\ell$, as $r' \ge 1$; the total is at most $4f\, r' \operatorname{defect}_\ell = r' \operatorname{loc}_\ell$. □

42. Cup products of crystalline classes

The local loss of Section 41 rests on explicit powers of p that make the cup products of crystalline cocycles of a lattice with those of congruent dual lattices vanish (Theorem 42.23). Here p is the residue characteristic (the prime ℓ of the previous sections). The lattices have Hodge–Tate jumps in two windows: a low window near 0 and far windows near $\pm b$, small against p or controlled by the valuations of the integers near b. The proofs work with Kisin modules on *Kummer charts* and move pairs of points of extensions along Frobenius orbits by *safe moves*.

42.1. Kummer charts, sides and safe moves

Let Γ be a group over G_p with open image. A *Kummer chart* k consists of a ring $\Lambda[\![z]\!] \subseteq A_{\mathrm{inf}}$, $\Lambda = W(\mathbb{F}_{p^f})$, a distinguished element E (with $E = z + p$ if the chart is *unramified*, of ramification index $e = 1$), and a subgroup Γ' of index prime to p fixing Λ; its *letters* are the units of $\mathbb{F}_{p^f}$, permuted by $i \mapsto i^p$, and $i.\mathrm{back}(a)$ is the letter a Frobenius steps before i. The coefficients O carry ζ with $\zeta^{p^f-1} = 1$, which assigns the embeddings of O to the letters. A *side* S consists of lattices Y and C, an O-linear map from C to the crystalline cocycles $Z_f(Y)$ (hence an extension $\tilde{Y}$ of the trivial lattice C by Y), and at each letter a finite set of integers containing 0, the jumps. A point of the extension has *orders* $o(l,u)$ if its value at $\varphi^l(E)$ on the forms of jump u is divisible by $t^{o(l,u)}$ in B^+_{dR}, and *depth* o if its differences with its translates by Γ' have orders o. A *pairing* of two sides modulo p^m is an O-linear map $\psi\colon Y \otimes L \to O(1)$ with $g\psi - \psi = p^m\cdot$(a crystalline cocycle); a move of one point, with the other at rest, is *safe* for κ if p^κ times the pairing of the move with the change of the point at rest under each $g \in \Gamma'$ lies in the image of $W(\mathfrak{m}^\flat)$. The jumps *fit windows* $(\eta, \varepsilon, b, W)$ if they lie in $[-\eta, \eta] \cup \varepsilon[b, b+W]$.

The proofs of this section are long chains of explicit inequalities between systems of orders, and the bookkeeping is fixed once and for all in the following definition. Throughout, k is a Kummer chart with coordinate $z = [\varpi^\flat]$ (a Teichmüller lift that is not a unit), $\tau = [\varepsilon^{1/p}] - 1$, $\mu = [\varepsilon] - 1$, $p_a = \varphi^a(E)$ for $a \geq 0$, and $\theta_a = \theta \circ \varphi^{-a}\colon A_{\mathrm{inf}} \to O_{\mathbb{C}_p}$, whose kernel is generated by p_a. The element $z\tau$ is written δ; no p_a divides it, since $\theta_a(z) \neq 0$ and $\theta_a(\tau) = \zeta_{p^{a+1}} - 1 \neq 0$. The *different bound* of k is $d = 0$ if $e = 1$ and $d = v_p(e) + 1$ otherwise; on an unramified chart $d = 0$. Every lattice L carries a finite set of jumps at each letter, containing its Hodge–Tate jumps above the letter; $L_{\mathrm{amb}} = W(C^\flat) \otimes L$, the letter projectors e_i split L_{amb}, and a *form* is an A_{inf}-linear map $L_{\mathrm{amb}} \to B_{\mathrm{dR}}$ that is Γ'-equivariant on $1 \otimes L$; it has *type* u at i if it maps $e_i(1 \otimes L)$ into $t^{-u}B^+_{\mathrm{dR}}$. The Kisin module $\mathcal{N}$ of L is the $\mathfrak{S}$-module of the $x \in A_{\mathrm{inf}}[1/\tau] \otimes L$ with $\mu(\varphi^n x) \in B^+_{\mathrm{dR}}$ for all forms and all $n \geq 1$ on which all forms take values in the field $\mathcal{K}_0$ that the chart generates over the fixed field of Γ'; $\mathcal{N}_{\mathrm{loc}}$ is $\mathcal{N}$ with all p_a inverted, and $\mathcal{N}^A_{\mathrm{loc}}$ its A_{inf}-span. The basic properties of these modules (the frame theorem) are Theorem B.40.

Definition 42.1 (Orders, points and moves). A *system of orders* is a function $o(a,u)$ of a level $a \geq 0$ and an integer u; $o \leq o'$ means $o(a,u) \leq o'(a,u)$ at the jumps u at $i.\mathrm{back}(a)$.

(O1) An element $x \in \mathcal{N}^A_{\mathrm{loc}}$ at the letter i has *orders* o if $\mu(\varphi^{-a}x) \in t^{o(a,u)}B^+_{\mathrm{dR}}$ for every a, every jump u at $i.\mathrm{back}(a)$ and every form μ of type u at

$i.\mathrm{back}(a)$; it is *invariant modulo* the orders o' if every $gx - x$, $g \in \Gamma'$, has orders o'.

(O2) The *standard* orders of level l are $\mathrm{std}_l(a,u) = 0$ for $a < l$ and $-u$ for $a \geq l$; the orders *raised by t at E* are $\varrho_t(a,u) = t[a=0] - u$, so that orders ϱ_{-s} mean a pole of order at most s at E. The *shift* by l levels with value n is $\mathrm{sh}_l^n(c)(a,u) = n$ for $a < l$ and $c(a-l,u)$ for $a \geq l$. The *improvement* of o_1 by o_2 is $\mathrm{imp}(o_1,o_2)(a,u) = o_2(a,u)$ if $o_1(a,u) > 0$ and $\min(o_2(a,u),0)$ otherwise.

(O3) The *cost* of the step $n \in \mathbb{Z}$ at the level a is $c_a(n) = v_p(n) + d + 3a$, with $v_p(0) = 0$.

(O4) For a side S and $c \in C$, $\mathcal{P}_i(c,e;o,o')$ is the set of the $x \in \mathcal{N}_{\mathrm{loc}}$ of the extension at i that lie *over* $p^e c$ (their image in the trivial lattice is p^e times the constant $e_i(1 \otimes c)$), have orders o and are invariant modulo o', the *depth*. $\mathcal{K}_i(o)$ is the set of the vectors over 0 with orders o, and $\mathcal{D}_i(o)$ the set of the elements over some constant with depth o.

(O5) The *standard depth* is $\mathrm{sd}_{t,\eta,l}(a,u) = -u + t[a=0] + \eta[a<l]$, the *plain depth* $\mathrm{pd}_{t,\eta,l,h_0} = \mathrm{sd}_{t,\eta,l} + h_0[a=l]$, the *stop orders* $\mathrm{so}_{\eta,l}(a,u) = \max(\mathrm{std}_l(a,u), -u - \eta[a<l])$, and the *rule bound* $\mathrm{rb}_{l,t}(a,s) = 1$ for $a < l$, $t-s$ for $a = l$ and $-s$ for $a > l$. The standard points are $\mathcal{P}_i^{\mathrm{st}}(c,e;t,\eta,l) = \mathcal{P}_i(c,e;\varrho_{-\eta},\mathrm{sd}_{t,\eta,l})$ and the stops $\mathcal{P}_i^{\mathrm{stop}}(c,e;\eta,l) = \mathcal{P}_i(c,e;\mathrm{so}_{\eta,l},\mathrm{so}_{\eta,l})$.

(O6) An additive map A on $\mathcal{N}_{\mathrm{loc}}$ of the extension is an *operator of cost e* if, for every letter, all orders o and every $x \in \mathcal{D}_i(o)$, $Ax - p^e x \in \mathcal{K}_i(o)$.

(O7) A pair (w,y) of vectors of the two extensions is *safe* for κ if p^κ times the pairing of their components in Y, Y' lies in the image of $W(\mathfrak{m}^\flat)$. The move of the first side by y with x' at rest is safe if all $(y, gx' - x')$, $g \in \Gamma'$, are safe; the move of the second side by y' with x at rest is safe if all $(gx - x, gy')$ are. A *safe path* from (y,y') to (x,x') through w,w' consists of the four safe moves $(y-w;y')$, $(w;y'-w')$, $(w-x;w')$, $(x;w'-x')$.

(O8) The ideal $J_l^r(i)$ of A_{inf} is generated by r and by the s such that $s\, e_i(1 \otimes c)$ lifts, for every $c \in C$, to an element of the extension with the standard orders of level l at i; *$p^\kappa E^s$ kills the cokernel at i* if $p^\kappa E^s \in J_0^0(i)$.

Lemma 42.2 (Orders). *Let L be a lattice over the chart, i a letter and $x \in \mathcal{N}_{\mathrm{loc}}^A$ at i.*

(a) (*Gap rule.*) *If x has orders c and $W \subseteq B_{\mathrm{dR}}$ is a B_{dR}^+-module containing $t^{c(a,v)} B_{\mathrm{dR}}^+$ for every jump $v \leq u$ at $j = i.\mathrm{back}(a)$, then $\mu(\varphi^{-a}x) \in W$ for every form μ of type u at j, whether or not u is a jump.*

(b) (*Frobenius.*) *If x has orders c at $i.\mathrm{back}(l)$, then $\varphi^l x$ has orders $\mathrm{sh}_l^0(c)$ at i; and φ^l maps $\mathcal{P}_{i.\mathrm{back}(l)}(c,e;o,o')$ into $\mathcal{P}_i(c,e;\mathrm{sh}_l^0(o),\mathrm{sh}_l^1(o'))$.*

(c) (*Exponents add.*) *Let $b\colon A \times L \to C$ be an equivariant bilinear map of lattices. If y, x have orders c', c at i, then $b(y,x)$ has orders c'' whenever $c''(a,v) \leq c(a,u') + c'(a,u)$ for all jumps u of A, u' of L and v of C at $i.\mathrm{back}(a)$ with $u + u' \leq v$.*

(d) (*Improvement.*) *If x has orders o_1 and is invariant modulo o_2, it has orders $\mathrm{imp}(o_1,o_2)$.*

(e) (*Galois action.*) *For $g \in \Gamma'$ and $x \in \mathcal{N}_{\mathrm{loc}}$, $\delta^{-1}(gx - x) \in \mathcal{N}_{\mathrm{loc}}^A$, and if $gx - x$ has orders c, so does $\delta^{-1}(gx - x)$. Elements of Γ' preserve $\mathcal{N}_{\mathrm{loc}}^A$ and orders.*

(f) (*Strictness.*) *A vector of the extension of a side over 0 with orders o is the image of a vector of Y with orders o, in $\mathcal{N}_{\mathrm{loc}}(Y)$ if it lies in $\mathcal{N}_{\mathrm{loc}}$ of the extension;*

two points over the same constant with orders o_1, o_2 differ by an element of $\mathcal{K}_i(\min(o_1, o_2))$.

Proof. (a) If v is not a jump at j, a form of type v is of type $v-1$, as the Hodge–Tate decomposition of $e_j(1\otimes L)$ has no part of weight v (Theorem B.40(c)). Let v be the largest jump at j with $v \le u$. If it exists, μ is of type v, so $\mu(\varphi^{-a}x) \in t^{c(a,v)}B^+_{\mathrm{dR}} \subseteq W$. If it does not, μ is of every type $n \le u$; take $r \neq 0$ in A_{inf} with $rx \in A_{\mathrm{inf}} \otimes L$ (Theorem B.40(a)). Then $\mu(\varphi^{-a}(rx))$ lies in $t^n B^+_{\mathrm{dR}}$ for all n, hence vanishes; it is $\varphi^{-a}(r)\mu(\varphi^{-a}x)$, and $\varphi^{-a}(r) \neq 0$ in the field B_{dR}, so $\mu(\varphi^{-a}x) = 0 \in W$.

(b) The value of $\varphi^l x$ at the level a is $\mu(\varphi^{l-a}x)$. For $a \ge l$ this is the value of x at the level $a-l$ at the letter $i.\mathrm{back}(l).\mathrm{back}(a-l) = i.\mathrm{back}(a)$, which has order $c(a-l,u)$. For $a < l$, the exponent $l-a$ is positive; a product of the p_b multiplies x into the A_{inf}-span of $\mathcal{N}$, where $\mu(\varphi^{m+1}\cdot) \in B^+_{\mathrm{dR}}$ by definition, and φ^{l-a} carries the factor p_b to p_{b+l-a}, which E does not divide; so the factor can be cancelled in B^+_{dR} and the order is at least 0. The letter, the Kisin module with the p_a inverted and the constant below are carried along, since φ fixes $1 \otimes c$. For the depth, $g\varphi^l w - \varphi^l w = \varphi^l(gw - w)$ as Γ' commutes with φ; by (e) $gw - w = \delta w_1$ with w_1 of orders o', and $\varphi^l(\delta)$ vanishes under θ_a for $a < l$, because $\theta(\varphi(\tau)) = \theta([\varepsilon]) - 1 = 0$; so at the levels below l the order rises from 0 to 1, and above it is $o'(a-l,u)$.

(c) It is enough to bound $\eta(b(y,x))$ for a form η of type v on C. Expand the second variable in an invariant Hodge basis of L, hodge $= P\,\mathrm{diag}(t^{\kappa_m})$ with P invertible over B^+_{dR}, and let ν_m be the dual forms, which are equivariant because the rows of hodge^{-1}hodge are fixed. Then $\eta(b(y,x)) = \sum_m D_m(\beta_y)\,\nu_m(x)$, where $\beta_y = \eta(b(y,\cdot))$ and D_m pairs the values of a functional with the mth column of hodge; $D_m \circ \beta$ is a form σ_m on A, equivariant as b and η are, of type $v - \kappa_m$, because D_m shifts the Hodge filtration by κ_m. Hence the summand is a product of a value of a form of type $v-\kappa_m$ at y and of a form of type κ_m at x; by the gap rule (a) the first lies in $t^{c'(a,u)}B^+_{\mathrm{dR}}$ for the largest jump $u \le v - \kappa_m$ of A and the second in $t^{c(a,u')}B^+_{\mathrm{dR}}$ for the largest jump $u' \le \kappa_m$ of L; then $u + u' \le v$ and the product has order at least $c''(a,v)$.

(d) Fix a, a jump u and a form μ of type u, and let $v = \mu(\varphi^{-a}x)$. As $g(v) - v \in t^{o_2(a,u)}B^+_{\mathrm{dR}}$ for all $g \in \Gamma'$, v is congruent modulo $t^{o_2(a,u)}B^+_{\mathrm{dR}}$ to a constant y of the fixed field K' of Γ' (Theorem B.40(g)). The constant lies in B^+_{dR}, so $v \in t^{\min(o_2,0)}B^+_{\mathrm{dR}}$. If $o_1(a,u) > 0$ and $o_2(a,u) > 0$, then $y = v - (v-y)$ lies in $tB^+_{\mathrm{dR}} \cap K' = 0$, so $v \in t^{o_2(a,u)}B^+_{\mathrm{dR}}$; if $o_2(a,u) \le 0$, then $\min(o_2,0) = o_2$.

(e) Divisibility by δ is tested with crystalline forms: the values of such a form on $x \in \mathcal{N}$ lie in the ring S_B of the chart up to a power E^n (Theorem B.40(b)), and $g \in \Gamma'$ moves elements of S_B, times E^{2n}, by multiples of δ, since g fixes Λ and multiplies z by $[\varepsilon]^m$; this gives $gx - x = \delta y$ with y in the A_{inf}-span of $\mathcal{N}$, and on $\mathcal{N}_{\mathrm{loc}}$ one clears the inverted p_a, which g multiplies by units. As $\theta_a(\delta) \neq 0$, δ is a unit of B^+_{dR} at every level, so the orders do not change. Equivariance of the forms gives $\mu(\varphi^{-a}gx) = g\,\mu(\varphi^{-a}x)$, and g preserves each $t^n B^+_{\mathrm{dR}}$.

(f) An invariant form μ on Y with values in a B^+_{dR}-module W on $1\otimes Y$ extends to the extension: restricted to Γ', μ sends the cocycle of c to the coboundary of an element of W, since the cocycle is the coboundary of a period v over $B_{\max}$ and the extension Λ of μ to $B_{\max}$ gives $\Lambda(v)$ congruent modulo W to a constant; choosing these elements linearly in a basis of C gives a linear $f : C \to W$, and $(y,c) \mapsto \mu(y) - f(c)$ is an invariant form on the extension with values in W that

restricts to μ. Applied to the letter part of a form of type u (with $W = t^{-u}B_{\mathrm{dR}}^{+}$), this shows that the orders of a vector of Y equal those of its image; membership in $\mathcal{N}_{\mathrm{loc}}^{A}$ and in $\mathcal{N}_{\mathrm{loc}}$ pass to Y by projecting to the first factor and by the coordinate retraction (Theorem B.40(d)), which commutes with the inclusion. The last claim follows, as the difference lies over 0 and the orders of a difference are the minimum. □

Lemma 42.3 (The depth step). *Let $a \geq 0$.*

(a) *For $n \neq 0$ there are $g \in \Gamma'$, a unit ω of A_{inf} and $v, X \in A_{\mathrm{inf}}$ with $gp_a = \omega p_a$, $g \equiv 1$ on $\mathfrak{S}$ modulo p_a, $\delta X = \omega^n - 1$ and $vX \equiv p^{c_a(n)}$ modulo p_a.*

(b) *There are $p^{c_a(0)}$ elements $t_l \in \Gamma'$ such that $g\sum_l t_l s \equiv \sum_l t_l s$ modulo p_a for every $s \in \mathfrak{S}$ and $g \in \Gamma'$.*

(c) *Let $x \in \mathcal{N}_{\mathrm{loc}}$ and let μ be a form with $\mu(\varphi^{-a}x)$ congruent modulo $t^n B_{\mathrm{dR}}^{+}$ to a constant $\gamma \in K'$. With the data of (a) for $n \neq 0$, the value of $D_n x = p^{c_a(n)}x - v\delta^{-1}(gx - x)$ is congruent to $p^{c_a(n)}\gamma$ modulo $t^{n+1}B_{\mathrm{dR}}^{+}$; with the data of (b) for $n = 0$, the value of $D_0 x = \sum_l t_l x$ is a constant modulo tB_{dR}^{+}.*

(d) *For a side S and a step (a, n) there is an operator B of cost $c_a(n)$ such that, for every invariant form μ on the extension, if the values $\mu(\varphi^{-a}(gx - x))$ lie in $t^n B_{\mathrm{dR}}^{+}$ for all $g \in \Gamma'$, then those of Bx lie in $t^{n+1}B_{\mathrm{dR}}^{+}$. Operators of costs e, e' compose to an operator of cost $e + e'$, and an operator of cost e maps $\mathcal{D}_i(o)$ into itself.*

(e) *For a finite set I of steps there is an operator of cost $\sum_{(a,n)\in I} c_a(n)$ that maps $\mathcal{D}_i(o)$ into $\mathcal{D}_i(o')$ whenever I contains every step (a, n) with $o(a, u) \leq n < o'(a, u)$ for a jump u at i.*back*(a).*

(f) *Consequently, if $x \in \mathcal{P}_i(c, e; o_1, o)$, I is as in (e) and $e' \geq e + \sum_I c_a(n)$, then $\mathcal{P}_i(c, e'; \mathrm{imp}(o_1, o'), o')$ is not empty.*

Proof. (a) Take $g \in \Gamma'$ with $gz = [\varepsilon]^{p^a} z$, which the chart provides, and ω_0 with $gE = \omega_0 E$. Then $\omega = \varphi^a \omega_0$ satisfies $gp_a = \omega p_a$, as Γ' commutes with φ, and $g \equiv 1$ on $\mathfrak{S}$ modulo p_a, because g fixes Λ and $g(z) - z = ([\varepsilon]^{p^a} - 1)z$ with $\theta_a([\varepsilon]^{p^a}) = 1$. As $\delta \mid \omega_0 - 1$ and $\delta \mid \varphi^a(\delta)$, δ divides $\omega^n - 1$; write $\omega^n - 1 = \delta X$. The estimate of $\theta_a(X)$ has three factors. Taylor's formula at z for a polynomial F with $F(z) = EG$ and $\theta F'(z) \mid p^d$ (the different), divided by E, gives $\theta(\omega_0 - 1) \mid p^{d+a}\theta(\delta)$. The lifting of the exponent, valid because $p\,\theta(r)\theta(z) \mid \theta(r)^p$ for every r divisible by δ (as $(\theta\tau + 1)^p = 1$ gives $p\theta\tau \mid \theta\tau^p$), gives $\theta(\omega_0^{|n|} - 1) \mid p^{v_p(n)}\theta(\omega_0 - 1)$, and $\omega_0^n - 1$ and $\omega_0^{|n|} - 1$ are associated. Finally $\theta(\delta) \mid p^{2a}\theta_a(\delta)$. As $\theta_a(\omega^n - 1) = \theta(\omega_0^n - 1)$ and $\theta_a(\delta) \neq 0$, cancelling gives $\theta_a(X) \mid p^{v_p(n)+d+3a}$, and v lifts the quotient.

(b) Take $t_l \in \Gamma'$ multiplying z by $[\varepsilon]^l$ for $l < p^{c_a(0)}$. Modulo p_a, that is after θ_a, write $s = \sum_m w_m z^m$ with $w_m \in \Lambda$. The element $\zeta = \theta_a([\varepsilon])$ is a primitive p^ath root of unity, and t_l multiplies $\theta_a(w_m z^m)$ by ζ^{ml}; as p^a divides $p^{c_a(0)}$, the sum over l of ζ^{ml} vanishes unless $p^a \mid m$. For $p^a \mid m$, $z^m = \varphi^a(z^{m/p^a})$ has image $\theta(z)^{m/p^a}$, which Γ' fixes; so Γ' fixes the image of $\sum_l t_l s$.

(c) For $n \neq 0$ put $y = \mu(\varphi^{-a}x) - \gamma \in t^n B_{\mathrm{dR}}^{+}$ and $w = \delta^{-1}(gx - x)$. The element y lies in the field $\mathcal{K}_a$ of the chart (Theorem B.40(b)), and writing $y = \pi^n y_0$ with π the image of p_a, $y_0 \in B_{\mathrm{dR}}^{+} \cap \mathcal{K}_a$ is fixed by g modulo tB_{dR}^{+}, because g acts as the identity on $\mathfrak{S}$ modulo p_a; as $g\pi^n = \omega^n \pi^n$, $gy \equiv \varphi^{-a}(\omega^n)y$ modulo $t^{n+1}B_{\mathrm{dR}}^{+}$. Since g fixes γ, $\varphi^{-a}(\delta)\mu(\varphi^{-a}w) = gy - y$, and $\delta X = \omega^n - 1$ gives $\mu(\varphi^{-a}w) \equiv \varphi^{-a}(X)y$; with $vX \equiv p^{c_a(n)}$ the value of $D_n x$ is $p^{c_a(n)}(\gamma + y) - \varphi^{-a}(vX)y \equiv p^{c_a(n)}\gamma$ modulo $t^{n+1}B_{\mathrm{dR}}^{+}$. For $n = 0$, the value of $D_0 x$ is $\sum_l t_l(\gamma + y)$ up to t, and by (b)

applied to the elements of $\mathfrak{S}$ and of $\mathcal{K}_0$ that make up y it is fixed modulo tB^+_{dR} by Γ', hence a constant modulo t by the argument of Lemma 42.2(d). In both cases $D_n x = p^c x + v' \sum (t_l - 1)\delta^{-1} x$ with $c = c_a(n)$.

(d) Let B be D_n followed by the coordinate retraction $\mathcal{N}^A_{\mathrm{loc}} \to \mathcal{N}_{\mathrm{loc}}$ of Theorem B.40(d), which keeps the values of forms modulo t^{n+1} when they are constants of $\mathcal{K}_a$. The hypothesis says that the value at x is invariant modulo t^n, hence a constant modulo t^n (Theorem B.40(g)); (c) makes the value of $D_n x$ a constant modulo t^{n+1}, the retraction keeps this, and so the values at $gBx - Bx$ lie in $t^{n+1}B^+_{\mathrm{dR}}$. If x lies over a constant and has depth o, then $Bx - p^c x$ is the retraction of $v'\delta^{-1} \sum (t_l - 1)x$: it lies over 0, as t_l fixes the constant below x and the projection commutes with the retraction, and by Lemma 42.2(e) it has orders o. Composition: $BAx - p^{e+e'}x = B(Ax) - p^{e'}Ax + p^{e'}(Ax - p^e x)$, and both terms lie in $\mathcal{K}_i(o)$, since Ax is again deep for o ($Ax - p^e x$ has orders o and is deep for o by Lemma 42.2(e)).

(e) Induction on I: remove a step (a, n) with n largest, and lower $o'(a, u)$ from $n+1$ to n on the lines u where $o'(a, u) = n + 1$; the remaining set covers the steps to the lowered depth (a step $(b, m+1)$ with $m + 1 < o'(b, u)$ in I would contradict the choice of n). Compose the operator for the rest with the operator B of (d) for the step (a, n), which raises the order on these lines from n to $n + 1$ and keeps the others.

(f) Apply the operator A of (e) to $p^s x$, where $s = e' - e - \sum_I c_a(n)$. As $p^s x$ is deep for o, $Ap^s x - p^{\sum c} p^s x$ lies in $\mathcal{K}_i(o)$; so $Ap^s x$ lies over $p^{e'}c$ and keeps the orders o_1 where $o_1 \le o$, and has depth o'; Lemma 42.2(d) improves its orders to $\mathrm{imp}(o_1, o')$. □

Lemma 42.4 (The ideals $J^r_l(i)$). *Let S be a side and i a letter.*

(a) *$s \in J^r_l(i)$ if and only if, for some $t \in A_{\mathrm{inf}}$, $(s - rt)e_i(1 \otimes c)$ lifts for every c to an element of the extension with the standard orders of level l at i; for $s \in \mathfrak{S}$ the witness t and the lifts can be taken in $\mathfrak{S}$ and in $\mathcal{N}_{\mathrm{loc}}$. Always $r \in J^r_l(i)$, and $J^r_l(i)$ grows with r.*

(b) *(Nakayama.) If $p^c \in J^{p^{c+1}}_l(i)$, then $J^{p^{c+1}}_l(i) \subseteq J^0_l(i)$; in particular $p^\kappa \in J^{p^{\kappa+1}}_0(i)$ means that p^κ kills the cokernel at i.*

(c) *(Frobenius.) $\varphi^l s \in J^r_l(i^{p^l})$ if and only if $s \in J^r_0(i)$; in particular $p^c z^{pa} \in J^r_1(i^p)$ if and only if $p^c z^a \in J^r_0(i)$.*

(d) *(Up a level.) If $s \in J^r_0(i)$ and the jumps at i are at most n, then $sE^n \in J^r_1(i)$.*

(e) *(Operators.) Let $f \in \mathfrak{S}$ lie in $J^r_l(i)$ and $g \in \Gamma'$. If $\delta(s + \beta f) = \alpha(gf - f)$ with $\alpha, \beta, s \in A_{\mathrm{inf}}$, then $s \in J^r_l(i)$; if $l = 1$ and $\delta E(s + \beta f) = \alpha(gf - f)$, then $s \in J^r_1(i)$.*

Proof. (a) is the definition: an element of the ideal is rt plus an exact multiplier, and exact multipliers form an ideal because lifts can be multiplied and added. For $s \in \mathfrak{S}$, write $s = rt + s'$ and apply to the lift of $s'e_i(1 \otimes c)$ the retraction induced by the $\mathfrak{S}$-linear coordinate $\lambda_0 \colon A_{\mathrm{inf}} \to \mathfrak{S}$ that is the identity on $\mathfrak{S}$; it sends ax to $\lambda_0(a)x$ for $x \in \mathcal{N}_{\mathrm{loc}}$ and commutes with the projection to the trivial lattice, on which the action is trivial, and $\lambda_0(s') = s - r\lambda_0(t)$.

(b) The hypothesis puts the ideal (p^c) inside $J^0_l(i) + (p)(p^c)$. As p lies in the Jacobson radical of A_{inf}, Nakayama's lemma for the finitely generated ideal (p^c) modulo $J^0_l(i)$ gives $(p^c) \subseteq J^0_l(i)$, so p^{c+1} lies there as well and the modulus can be dropped.

(c) φ^l carries the part at the letter i to the part at i^{p^l} and the standard orders of level 0 to those of level l (Lemma 42.2(b)), and it fixes the constants up to the change of letter: $\varphi^l e_i(1\otimes c) = e_{i^{p^l}}(1\otimes c)$. Hence substitute $\varphi^l t$ for the witness and $\varphi^l x$ for each lift. Conversely an element of the standard orders of level l at i^{p^l} is φ^l of an element of the Kisin module at i (Theorem B.40(e)). The second claim is the first with $s = p^c z^a$, as $\varphi z = z^p$.

(d) Multiply the lifts by E^n. At the level 0 the order $-u$ becomes $n-u \geq 0$, which is the standard order of level 1 there; at the levels $a \geq 1$ the orders do not change, as E is a unit there.

(e) Lift $(f - rt)e_i(1\otimes c)$ to some x. Since g fixes the constant, $\alpha\delta^{-1}(gx - x)$ lies over $\alpha\delta^{-1}(gf - f - r(gt - t))e_i(1\otimes c) = (s + \beta f - r\alpha v)e_i(1\otimes c)$, with $gt - t = \delta v$ (Lemma 42.2(e) on $\mathfrak{S}$), and has the orders of x; subtracting βx gives a lift of $(s - r(\alpha v - \beta t))e_i(1\otimes c)$. For $l = 1$, by Lemma 42.2(e) and the triviality of g on $\mathfrak{S}$ modulo E, $gx - x$ has order at least 1 at the level 0 and $-u$ above (the standard orders of level 1 allow one division by E); so $\delta^{-1}E^{-1}(gx - x)$ still has the standard orders of level 1. □

Lemma 42.5 (Powers of z and the cokernel). *Let S be a side, i a letter, r a modulus and d the different bound.*

(a) *If $p^c z^n \in J_0^r(i)$ with $n \geq 1$, then $p^{c+v_p(n)+1} z^{n-1} \in J_0^r(i)$; hence if $m \leq n$ and $c + n + v_p(n!) \leq \kappa + m + v_p(m!)$, then $p^\kappa z^m \in J_0^r(i)$.*

(b) *If $p^c z^n \in J_1^r(i^p)$, $pa \leq n$ and $c + v_p(n!) \leq \kappa + v_p((pa)!)$, then $p^\kappa z^a \in J_0^r(i)$. If $p^c z^a E^n \in J_1^r(i)$, $c + dn + v_p(n!) \leq b$ and $a + n \leq \kappa$, then $p^b z^\kappa \in J_1^r(i)$.*

(c) *If the jumps at i^p are at most ω and $p^c z^n \in J_0^r(i^p)$, then $p^{c'} z^{\lfloor (n+\omega)/p \rfloor} \in J_0^r(i)$, where $c' = c + d\omega + v_p(\omega!)$.*

(d) *For every κ some power of z lies in $J_0^{p^\kappa}(i)$ for every letter. If all jumps are at most ω, there is a with $(p-1)a \leq e(\omega + \kappa)$ and $z^a \in J_0^{p^{\kappa+1}}(i)$ for every letter. In particular, if $e\omega \leq p - 2$, then 1 kills every cokernel.*

(e) *If $z^a \in J_0^r(j)$ for every letter and the jumps at the J letters i^{p^t}, $1 \leq t \leq J$, are at most ω, then $p^c z^n \in J_0^r(i)$ for some $c \leq J(d\omega + v_p(\omega!))$ and $n \leq a/p^J + \omega/(p-1)$.*

(f) *(Unramified charts.) If k is unramified, $p \geq 5$ and all jumps are at most $(p-3)(s+t)$, then $p^s E^t$ kills the cokernel at every letter.*

(g) *(Weights.) Let the jumps be at most ω everywhere and at most ω' at the J letters after i, and let $8e \leq p^{J+1}$. If $\kappa \geq 2Jd\omega' + (2J+4)\omega'/(p-1) + 8e\omega/p^{J+1}$, then p^κ kills the cokernel at i.*

Proof. (a) Take $g \in \Gamma'$ with $gz = [\varepsilon]z$ and put $v = v_p(n)$. The chart provides $\alpha, \beta \in A_{\mathrm{inf}}$ with $\tau E(p^v + \beta z) = \alpha([\varepsilon]^n - 1)$, and as z divides $E - p$ this becomes an identity $\tau(p^{v+1} + z\beta') = \alpha([\varepsilon]^n - 1)$; multiplied by $p^c z^n$ it reads $\delta(p^{c+v+1} z^{n-1} + \beta' p^c z^n) = \alpha(g-1)(p^c z^n)$, and Lemma 42.4(e) applies. Iterating from n down to m costs $\sum_{m<j\leq n}(v_p(j)+1) = n - m + v_p(n!) - v_p(m!)$.

(b) For the first claim induct from n down to pa at the level one: with $gz = [\varepsilon]z$, the identity $\tau E(p^{v_p(n+1)} + \beta z) = \alpha([\varepsilon]^{n+1} - 1)$ of (a) and Lemma 42.4(e) for $l = 1$ lower z^{n+1} to z^n at the cost $p^{v_p(n+1)}$, and $v_p((n+1)!) = v_p(n+1) + v_p(n!)$; then Lemma 42.4(c) passes from z^{pa} at the level one at i^p to z^a at the level 0 at i. For the second claim one step trades a factor E for a factor z: choose g with $gE = \omega E$ and $\delta q = \omega^{n+1} - 1$, $vq = p^{c_0(n+1)} + Eb_1$ (Lemma 42.3(a)); g multiplies

z by $1+\tau E v$, hence z^a by $1+\tau E v b_2$, and the derivation by g with multiplier zv gives $\delta E(p^{c_0(n+1)} z\cdot p^c z^a E^n+\beta f)=\alpha(gf-f)$ for $f=p^c z^a E^{n+1}$, the terms in b_2 cancelling; the cost of a step is $v_p(n+1)+d$, and $\sum_{j\le n}(v_p(j)+d)=dn+v_p(n!)$.

(c) Multiply by E^ω to reach the level one (Lemma 42.4(d)), trade the ω factors E for factors z by (b), at the cost $d\omega+v_p(\omega!)$, and come down by Frobenius with the first claim of (b) from $z^{n+\omega}$ to z^{pa}, $a=\lfloor (n+\omega)/p\rfloor$, which costs nothing more: no multiple of p lies in $(pa, n+\omega]$, so $v_p((n+\omega)!)=v_p((pa)!)$.

(d) By the frame theorem τ^h multiplies $A_{\mathrm{inf}}\otimes L$ into the A_{inf}-span of $\mathcal{N}$, h the jump bound, so $\tau^h e_i(1\otimes c)$ lifts to the letter part of $\tau^h(1\otimes(0,c))$; the ideals contain p^κ and a power of τ, and z lies in the radical of (p,τ). For the second claim take the least a with z^a in all the ideals modulo $p^{\kappa+1}$. As $z^e\in(E,p)$, write $z^e=uE+vp$; then $z^{a+e(\omega+\kappa)}=z^a(uE+vp)^{\omega+\kappa}$, and every monomial $E^{a'}p^{b'}$ with $a'+b'=\omega+\kappa$ has $a'\ge\omega$ or $b'\ge\kappa+1$, so the power lies in $J_1^{p^{\kappa+1}}(j)$ by Lemma 42.4(d). Coming down by Frobenius (Lemma 42.4(c)) gives $z^{\lfloor (a+e(\omega+\kappa))/p\rfloor}$ in all ideals of level 0, and minimality gives $pa\le a+e(\omega+\kappa)$. If $e\omega\le p-2$, take $\kappa=0$: $(p-1)a\le e\omega<p-1$ forces $a=0$, and (b) of Lemma 42.4 drops the modulus p.

(e) Come back to i^p from the $J-1$ letters after it by (c), and then to i by one more step: each step costs at most $d\omega+v_p(\omega!)$ and divides the exponent by p up to adding ω, so the final exponent is at most $a/p^J+\omega(1/p+\cdots+1/p^J)\le a/p^J+\omega/(p-1)$.

(f) Let b be given by (d) for $\kappa=s+t$ and $\omega=(p-3)(s+t)$: z^b lies in all ideals modulo p^{s+t+1} and $(p-1)b\le(p-2)(s+t)$, as $e=1$. Legendre's formula $(p-1)v_p(b!)\le b$ gives $(p-2)(b+v_p(b!))\le(p-1)b\le(p-2)(s+t)$, so $b+v_p(b!)\le s+t$. By (a), lowering $z^{\max(b,m)}$ to z^m costs at most $s+t-m$, so $p^{s+t-m}z^m$ lies in the ideal modulo p^{s+t+1} for every m; by Nakayama (Lemma 42.4(b)) the modulus can be dropped, and $p^s(z+p)^t=\sum_m\binom{t}{m}p^{s+t-m}z^m$ lies in $J_0^0(i)$.

(g) By (d) with this κ there is z^a in all ideals modulo $p^{\kappa+1}$ with $(p-1)a\le e(\omega+\kappa)$; by (e) some $p^c z^n$ with $c\le J(d\omega'+v_p(\omega'!))$ and $n\le a/p^J+\omega'/(p-1)$ lies in $J_0^{p^{\kappa+1}}(i)$. The arithmetic: $c+n+v_p(n!)\le J(d\omega'+\omega'/(p-1))+2n$ and $a/p^J\le 2e\omega/p^{J+1}+\kappa/4$, as $8e\le p^{J+1}$; the bound on κ gives $c+n+v_p(n!)\le\kappa$. Then (a) removes the power of z, giving $p^\kappa\in J_0^{p^{\kappa+1}}(i)$, and Nakayama concludes. □

Lemma 42.6 (One step on a line). *Let L be a lattice over the chart, o a system of orders, $i=l.\mathrm{back}(b)$ and u a jump at i with $o(b,u)<o(b,v)$ for the jumps $v<u$ at i.*

(a) *Among elements x_j with orders o at l there are at most $\mathrm{mult}_i(u)$ of them, $x_{s_1},\dots,x_{s_m}$, such that every x_j is an A_{inf}-combination of the x_{s_k} modulo an element with orders o raised by one at (b,u).*

(b) *Any $fn+1$ vectors of $W(\mathbb{F}_{p^f})^m$, $m\le n$, satisfy a $\mathbb{Z}_p$-linear relation in which one coefficient is 1.*

(c) *On an unramified chart, if u has multiplicity at most n at i, any $fn+1$ elements of $\mathcal{N}_{\mathrm{loc}}$ with orders o at l have a $\mathbb{Z}_p$-combination with one coefficient 1 that is the sum of an element of $\mathcal{N}_{\mathrm{loc}}$ with all orders at the level 0 raised by one and one with the order at (b,u) raised by one.*

Proof. (a) Modulo forms of type $u-1$, the forms of type u at i are spanned over $\bar{\mathbb{Q}}_p$ by $m\le\mathrm{mult}_i(u)$ of them, $v_1,\dots,v_m$ (Theorem B.40(c)); forms of type $u-1$ already have the larger orders, by the gap rule and the hypothesis

$o(b,u) < o(b,v)$ for $v < u$. The leading terms $t^{-o(b,u)}\nu_k(\varphi^{-b}x_j)$ form a matrix over B^+_{dR} with m columns; by Auerbach's lemma applied to the column space of its image under θ, which has a basis with integral entries whose coordinates are the entries at some rows $s_1,\dots,s_{m'}$, $m' \le m$, every row is modulo $\ker\theta$ an A_{inf}-combination of these rows. Then the leading terms of the ν_k cancel on $x_j - \sum_k \alpha_{jk}x_{s_k}$, which therefore has order $o(b,u)+1$ on the line u at the level b.

(b) $W(\mathbb{F}_{p^f})^m$ is free of rank $fm \le fn$ over $\mathbb{Z}_p$, so $fn+1$ vectors satisfy a nontrivial relation; divide it by the coefficient of least valuation.

(c) By (a) each x_j is $\sum_k \alpha_{jk}x_{s_k}$ plus an element with orders raised at (b,u), and the retraction of Theorem B.40(d) replaces the α_{jk} by elements of $\mathfrak{S}$ keeping orders. On an unramified chart every element of $\mathfrak{S}$ is congruent modulo E to a constant of $W(\mathbb{F}_{p^f})$: $\alpha_{jk} = w_{jk} + Ey_{jk}$. By (b) the $fn+1$ vectors $(w_{jk})_k$ satisfy a relation $\sum_j \beta_j w_{jk} = 0$ with some $\beta_{j_0} = 1$; then $\sum_j \beta_j x_j = E\sum_{j,k}\beta_j y_{jk}x_{s_k} +$(an element with orders raised at (b,u)), and multiplication by E raises all orders at the level 0 by one. □

Lemma 42.7 (Cochains of a safe path). *Let P be a pairing modulo p^m of two sides, x,w over $p^{e_1}c$ and x',w' over $p^{e_2}c'$, and $B(z) = p^{e_3}\psi(\mathrm{pr}_Y z,\cdot)$. Let y satisfy $(\varphi-1)y = B(w)(\varphi x' - w') + B(x)(w'-x')$, and put, for n in $\mathbb{Z}\times\Gamma$, $f(n) = B(nx)(nx') - n\big(B(x)(x')-y\big) - B(x)(nx'-x') - y$.*

(a) *The Y-component of nx is n applied to the Y-component of x plus the cocycle of the Γ-component of n at the C-component of nx; the extended pairing commutes with $\mathbb{Z}\times\Gamma$.*

(b) *If the four moves of a safe path from $(\varphi x,\varphi x')$ to (x,x') through w,w' are safe for e_3, then $(\varphi-1)f(g)$ lies in the image of $W(\mathfrak{m}^\flat)$ for $g\in\Gamma'$.*

(c) *If moreover the moves by $hx'-x'$, $h\in\Gamma'$, are safe with x at rest, $f(g) = 1\otimes\kappa(g)$ up to a small element, and $p^e = p^{e_1+e_2+e_3}r$, then on Γ' the cup product of the cocycles of c and c', times p^e, is r times the coboundary of κ.*

(d) *If the cup product of two crystalline cocycles is, on Γ', the coboundary of a function vanishing near 1, it is a continuous coboundary on Γ in $\bar{\mathbb{Q}}_p^\times$.*

Proof. (a) Both sides are additive in x; on a pure tensor this is the formula for the action on the extension $Y\times C$, $g(y,c) = (gy + a_c(g), c)$. The pairing is the $W(C^\flat)$-bilinear extension of an equivariant pairing into $\mathbb{Z}/p^m(1)$.

(b) As g commutes with φ, the defining equation of y rewrites the part of $(\varphi-1)f(g)$ that contains y: $(\varphi-1)(gy-y) = (g-1)\big(B(w)(\varphi x'-w')+B(x)(w'-x')\big)$. Inserting this and expanding by bilinearity, $(\varphi-1)f(g)$ becomes the signed sum

$$\begin{aligned} &B(\varphi x - w)(g\varphi x' - \varphi x') + B(gw-w)(g(\varphi x' - w')) \\ &\quad + B(w-x)(gw'-w') + B(gx-x)(g(w'-x')), \end{aligned}$$

up to terms in which a vector over 0 is paired with a vector over 0 and cancels; these are the pairings of the four safe moves, each in the image of $W(\mathfrak{m}^\flat)$.

(c) Expand the coboundary of f on Γ': by (a) it is $p^{e_1+e_2+e_3}\psi(a_c(g), ga_{c'}(h))$ plus $B(gx-x)(g(hx'-x'))$, which is small by hypothesis. Replace f by the constants $1\otimes\kappa$; the error $\delta(g) = f(g) - 1\otimes\kappa(g)$ is small, so $g\delta(h) - \delta(gh) + \delta(g)$ is small, and the difference of the two expressions is a small constant of $W(C^\flat)\otimes\mathbb{Z}/p^m(1)$, hence 0 (a constant in the image of $W(\mathfrak{m}^\flat)$ vanishes). Multiplying by r gives the claim.

(d) Read the cup product b in $\bar{\mathbb{Q}}_p^\times$ through $\mathbb{Z}/p^m(1) \to \mu_{p^m}$, a discrete Γ-module. Near 1 the cocycle values are multiples of p^m, so b is locally constant, and the function is locally constant as it vanishes near 1. The class of b restricts to zero on Γ', so it is killed by the index of Γ' (corestriction), which is prime to p; it is also killed by p^m, so it vanishes. □

Together these lemmas give the tools of the walk.

Proposition 42.8 (The pairing lemma and depth). (a) (*Pairing lemma.*) *Let x, w lie over $p^{e_1}c$ and x', w' over $p^{e_2}c'$. If the moves by $hx' - x'$, $h \in \Gamma'$, are safe with x at rest, and four safe moves lead from $(\varphi x, \varphi x')$ by w, w' to (x, x'), then for $e \geq e_1 + e_2 + e_3$ the cup product of the cocycles of c and $p^e c'$ is a coboundary.*

(b) (*Depth lemma.*) *An operator $p^c + v\sum(t_l - 1)/\delta$, $t_l \in \Gamma'$, $\delta = z\tau$, sends elements of depth n at $\varphi^a(E)$ to elements whose value is constant modulo $t^{n+1}B_{\mathrm{dR}}^+$; the cost is $c = v_p(n) + d + 3a$, with $d = 0$ if $e = 1$ and $d = v_p(e) + 1$ otherwise.*

(c) (*Cokernels.*) *$p^\kappa E^s$ kills the cokernel at a letter (multiplies the letter part of $A_{\mathrm{inf}} \otimes C$ into the image of the standard module of level 0 of the extension) with $r = 0$; on an unramified chart with $p \geq 5$ this holds when the jumps are at most $(p-3)(s+t)$; and z^a does with $r = p^{\kappa+1}$ when $(p-1)a \leq e(\omega + \kappa)$ for jumps at most ω.*

(d) (*One step on a line.*) *On an unramified chart, any $fn + 1$ elements with given orders at a letter have a $\mathbb{Z}_p$-combination with one coefficient 1 that splits into a part with all orders at level 0 raised and a part with the order at one line raised.*

Proof. (a) Let B and f be as in Lemma 42.7, with y a solution of $(\varphi - 1)y = B(w)(\varphi x' - w') + B(x)(w' - x')$, which exists because $\varphi - 1$ is surjective on $W(C^\flat) \otimes \mathbb{Z}/p^m$ ($C^\flat$ is algebraically closed). By Lemma 42.7(b), $(\varphi - 1)f(g)$ is small for $g \in \Gamma'$; as the kernel of $\varphi - 1$ on $W(C^\flat) \otimes \mathbb{Z}/p^m$ modulo the small part is the image of $\mathbb{Z}/p^m$, $f(g) = 1 \otimes \kappa(g)$ up to a small element. With $p^e = p^{e_1+e_2+e_3}r$, Lemma 42.7(c) shows that on Γ' the cup product of the cocycles of c and $p^e c'$ is the coboundary of $r\kappa$. Near 1 the group fixes each element of $W(C^\flat) \otimes \mathbb{Z}/p^m(1)$ up to the small part, since the cyclotomic character is trivial modulo p^m near 1; the cocycle terms of f there are multiples of p^m, which kills the target, so f and hence κ vanish near 1. Then Lemma 42.7(d) makes the cup product a continuous coboundary on Γ.

(b) is Lemma 42.3(a), (c) and (d): $c = c_a(n)$, and the operator is D_n (or D_0), followed by the retraction.

(c) The first statement is the definition of $J_0^0(i)$ (Lemma 42.4); the bound for unramified charts is Lemma 42.5(f), and the bound $(p-1)a \leq e(\omega + \kappa)$ with modulus $p^{\kappa+1}$ is Lemma 42.5(d).

(d) is Lemma 42.6(c), with n the multiplicity of the jump. □

42.2. Walks and cuts

A walk moves two points, one of each side, from their Frobenius images at one letter to their own points at the next, one side at a time, so that every move is safe; Proposition 42.8(a) then turns the walk into a scaled cup coboundary. A move is certified safe by pairing it against a point of the Tate dual side, which exists as soon as the cokernel of that side is killed; the resulting *rule* is a sum of orders.

Lemma 42.9 (The rule). *Let P pair the sides S, S' modulo p^m, and let $\hat{S}$ be its Tate dual side, with lattice* $\mathrm{Hom}(Y \otimes Y', O(1))$ *extended by O through the cocycle of P and jumps* 0 *and* $-u-u'-1$.

(a) *Let ν be a point of $\hat{S}$ over p^κ at $i.\mathrm{back}(l)$ with orders o_ν antitone in the jump, and let $w \in \mathcal{K}_i(o_w)$, $y \in \mathcal{K}_i(o_y)$ lie in the two extensions. If $\mathrm{sh}^0_l(o_\nu)(a,-u-u'-1)+o_w(a,u)+o_y(a,u')>0$ for all levels a and jumps u of S, u' of S' at $i.\mathrm{back}(a)$, then $(\delta w, y)$ and $(w, \delta y)$ are safe for κ.*

(b) *With ν as in (a), let $w \in \mathcal{K}_i(o_w)$ and x' a point of S' at i of depth d'. If $\mathrm{sh}^0_l(o_\nu)(a,-u-u'-1)+o_w(a,u)+d'(a,u')>0$ at all jumps, the move by w with x' at rest is safe; symmetrically for the second side.*

(c) (*The rule.*) *If $p^\kappa E^t$ kills the cokernel of $\hat{S}$ at $i.\mathrm{back}(l)$, $y \in \mathcal{K}_i(o)$ and x' has depth d' at i, and $\mathrm{rb}_{l,t}(a,u+u') \le o(a,u)+d'(a,u')$ at all jumps, then the move by y with x' at rest is safe for κ; symmetrically.*

(d) (*Letters.*) *Vectors of the two extensions at distinct letters are orthogonal under P. Hence pairs, moves and paths that are safe letter by letter are safe for the sums over the letters.*

Proof. (a) By Lemma 42.2(f), w and y are images of vectors of Y and Y' with the same orders, and $\varphi^l\nu$ lies over p^κ at i with orders $\mathrm{sh}^0_l(o_\nu)$ (Lemma 42.2(b)). By Lemma 42.2(c) applied to $Y \times Y' \to Y \otimes Y'$, $w \otimes y$ has orders $o_w(a,u)+o_y(a,u')$ on the jump v for $u+u' \le v$; pairing it with $\varphi^l\nu$ by the extension form $(\lambda, o) \mapsto p^m\lambda + o\psi$, which is equivariant, gives an element ξ of $O(1)_{\mathrm{amb}}$, whose only jump is -1. Again by Lemma 42.2(c), with the jump v of $\hat{S}$ satisfying $-v-1 \le -u-u'-1$ and the antitonicity of o_ν, ξ has order at least $1 = \mathrm{std}_0(a,-1)$ at every level: ξ has the standard orders of level 0. As Γ acts on $O(1)$ through the cyclotomic character, whose only jump is -1, the frame theorem gives $\tau\xi \in A_{\mathrm{inf}} \otimes O(1)$ (Theorem B.40(a)). Reduced modulo p^m through the linear map $O \to \mathbb{Z}_p$ of P, the term $p^m\lambda$ disappears, and ξ becomes the pairing of w with y multiplied by the projection of $\varphi^l\nu$ to $W(C^\flat) \otimes O$, which is $p^\kappa e_i$; as w lies at i, this is $p^\kappa\psi(w,y)$. Finally z lies in $W(\mathfrak{m}^\flat)$, so $z\tau$ times the A_{inf}-span of $O(1)$ reduces into the image of $W(\mathfrak{m}^\flat)$: $\delta p^\kappa \psi(w,y)$ is small, which is both claims.

(b) For $g \in \Gamma'$, $\delta^{-1}(gx'-x') \in \mathcal{K}_i(d')$ (Lemma 42.2(e)), and $gx'-x' = \delta \cdot \delta^{-1}(gx'-x')$; apply the second claim of (a). For the second side the pairs are $(gx-x, gy') = (\delta \cdot \delta^{-1}(gx-x), gy')$, and gy' has the orders of y'; apply the first claim.

(c) Lift $p^\kappa E^t$ times the constant 1 of $\hat{S}$ at $i.\mathrm{back}(l)$ to the extension and divide by E^t: this is a point ν over p^κ with orders and depth ϱ_{-t}, which are antitone in the jump. For all l, t, a, u, u' one has the identity $\mathrm{sh}^0_l(\varrho_{-t})(a,-u-u'-1)+\mathrm{rb}_{l,t}(a,u+u') = 1$: at the levels below, at and above l the first term is 0, $t+u+u'+1$, $u+u'+1$ and the second 1, $t-u-u'$, $-u-u'$. Hence the hypothesis gives the positivity of (b).

(d) The root of unity ζ of O acts at a letter by its Teichmüller lift, and the pairing is balanced for multiplication by ζ; for w at j and y at $i \neq j$ this gives $([j]-[i])\psi(w,y)=0$, and $[j]-[i]$ is a unit of $\mathfrak{S}$, as the letters are distinct in $\mathbb{F}_{p^f}$. Hence the pairing of the sums is the sum of the pairings at each letter, and the image of $W(\mathfrak{m}^\flat)$ is a group. For moves one also uses that $g \in \Gamma'$ preserves letters. □

Lemma 42.10 (The walk). *Let P pair S and S', let some letters be* cuts, *and let the last cut before each letter i lie $l(i) \geq 1$ steps back. At each cut i let $p^\kappa E^{t(i)}$ kill the cokernel of $\hat{S}$, let $x(i)$, $x'(i)$ be standard points $\mathcal{P}_i^{\mathrm{st}}(c,e;t(i),\eta,l(i))$, $\mathcal{P}_i^{\mathrm{st}}(c',e';t(i),\eta,l(i))$, and let one side have a stop at i: a point $q \in \mathcal{P}_i^{\mathrm{stop}}(c,e;\eta,l(i))$ at which the move of the second side by $\varphi^{l(i)}x'(i.\mathrm{back}(l(i)))-x'(i)$ is safe, or symmetrically.*

(a) *At a cut i with predecessor l steps back and the cut before that l_0 steps further back, four safe moves lead from $(\varphi^l x_0, \varphi^l x_0')$ to (x,x'), with x_0, x_0' the standard points at $i.\mathrm{back}(l)$.*
(b) *For standard points x, x' at the letter d steps before i and $g \in \Gamma'$, the move of the second side by $g\varphi^d x' - \varphi^d x'$ is safe with $\varphi^d x$ at rest.*
(c) *For $N \geq e+e'+\kappa$ the cup product of the cocycles of c and $p^N c'$ is a continuous coboundary.*
(d) *In particular, if every letter is a cut and $t(i)=t$, $l(i)=1$, (c) holds as soon as each letter has a stop.*

Proof. (a) By Lemma 42.2(b) $\varphi^l x_0$ has orders $\mathrm{sh}_l^0(\varrho_{-\eta})$ and depth $\mathrm{sh}_l^1(\mathrm{sd}_{t_0,\eta,l_0})$, and likewise $\varphi^l x_0'$. Suppose the first side has the stop q. The path is $(\varphi^l x_0, \varphi^l x_0') \to (q, \varphi^l x_0') \to (q,x') \to (x,x')$, the last move being zero. *First move*: $\varphi^l x_0 - q$ lies over 0 with orders $\min(\mathrm{sh}_l^0(\varrho_{-\eta}), \mathrm{so}_{\eta,l})$, and $\varphi^l x_0'$ rests with depth $\mathrm{sh}_l^1(\mathrm{sd}_{t_0,\eta,l_0})$; the rule for l and t_0 holds (Lemma 42.9(c)), as $p^\kappa E^{t_0}$ kills the cokernel at $i.\mathrm{back}(l)$: below l the orders are $\min(0,\max(0,-u-\eta))=0$ and the depth 1, which add up to 1; at the level l the orders are $\min(-\eta-u,-u)=-\eta-u$ and the depth $-v+t_0+\eta$ (as $l_0>0$), which add up to t_0-u-v; above, $-u$ and at least $-v$. *Second move*: the hypothesis. *Third move*: $q-x$ has orders $\min(\mathrm{so}_{\eta,l}, \varrho_{-\eta})$ and x' rests with depth $\mathrm{sd}_{t,\eta,l}$; the rule for 0 and t holds, as $p^\kappa E^t$ kills the cokernel at i: at the level 0 the orders are $\min(\max(0,-u-\eta),-u-\eta)=-u-\eta$ and the depth $-v+t+\eta$; at the levels $0<a<l$ the orders are at least $-u-\eta$ and the depth is $-v+\eta$; above, $-u$ and $-v$; in each case the sum is at least $\mathrm{rb}_{0,t}$. If the second side has the stop the argument is symmetric.

(b) By Lemma 42.2(b) $\varphi^d x'$ has depth $\mathrm{sh}_d^1(\mathrm{sd}_{t,\eta,l})$, so the move $g\varphi^d x' - \varphi^d x'$ lies over 0 with these orders, and $\varphi^d x$ rests with the same depth; with $p^\kappa E^t$ killing the cokernel of $\hat{S}$ at the cut $i.\mathrm{back}(d)$, the rule for d and t holds, since below d the sum is $2 \geq 1$, at d it is $-u-u'+2t+2\eta[l>0] \geq t-u-u'$, and above it is at least $-u-u'$.

(c) For a letter i let $d(i)=0$ if i is a cut and $d(i)=l(i)$ otherwise, so that $i.\mathrm{back}(d(i))$ is always a cut, and put $\tilde{x}(i)=\varphi^{d(i)}x(i.\mathrm{back}(d(i)))$, $\tilde{x}'(i)$ likewise. These lie over $p^e c$, $p^{e'}c'$ at i (Lemma 42.2(b)). The minimality of l gives $l(i)=d(i.\mathrm{back}(1))+1$, and as φ sends the letter $i.\mathrm{back}(1)$ to i,

$$\varphi \sum_i \tilde{x}(i) = \sum_i \varphi^{l(i)} x(i.\mathrm{back}(l(i))),$$

the sum of the Frobenius images with the last cut *strictly* before each letter. At a letter that is not a cut both terms are $\tilde{x}(i)$, and the path that stays there is safe, its moves being zero; at a cut, (a) gives a safe path from $\varphi^{l(i)}x(i.\mathrm{back}(l(i)))$ to $x(i)$. By (b) the cocycle moves are safe. Summing over the letters (Lemma 42.9(d)) gives the hypotheses of Proposition 42.8(a) for $\sum_i \tilde{x}(i)$, $\sum_i \tilde{x}'(i)$, which lie over $p^e c$ and $p^{e'}c'$ because the letter parts of the constants add up to the constants.

(d) is (c) with $d \equiv 0$. □

Lemma 42.11 (Plain cuts). *Let $0 < l$ and let i be a letter.*

(a) (*Down.*) *If the jumps of S at the l letters up to i are at most η, every standard point $\mathcal{P}_i^{\mathrm{st}}(c, e; t, \eta, l)$ is a stop $\mathcal{P}_i^{\mathrm{stop}}(c, e; \eta, l)$.*

(b) (*Up.*) *If the jumps of S at $i.\mathrm{back}(a)$, $a < l$, are at least $-\eta$, $x_0 \in \mathcal{P}_{i.\mathrm{back}(l)}(c, e; \varrho_{-\eta_0}, \cdot)$, $x \in \mathcal{P}_i(c, e; \varrho_{-\eta}, \cdot)$, $p^\kappa E^t$ kills the cokernel of $\hat{S}$ at i, and q' is a point of S' at i of depth $\mathrm{pd}_{t,\eta,l,\eta_0}$, then the move by $\varphi^l x_0 - x$ with q' at rest is safe; symmetrically for the second side.*

(c) *A point of depth $\mathrm{pd}_{t,\eta,l,\eta_0}$ and orders $\varrho_{-\eta}$ is standard.*

(d) (*Plain cut.*) *If at the l letters up to i the jumps of S are at most η and those of S' at least $-\eta$, and $x \in \mathcal{P}_i(c, e; \varrho_{-\eta}, \mathrm{pd}_{t,\eta,l,\eta_0})$, then x is a stop of S at which the move of S' by $\varphi^l x'_0 - x'$ is safe, for any x'_0, x' with poles of order at most η_0, η; symmetrically.*

(e) (*Windows.*) *A jump in the windows $[-\eta, \eta] \cup \sigma[\beta, \beta + V]$ is low, lies in $[\beta, \beta + V]$, or its negative does. The side is down (jumps $\le \eta$) where $\sigma = -1$ or $\beta + V \le \eta$, and up (jumps $\ge -\eta$) where $\sigma = 1$ or $\beta + V \le \eta$.*

Proof. (a) By Lemma 42.2(d) the point has orders $\mathrm{imp}(\varrho_{-\eta}, \mathrm{sd}_{t,\eta,l})$. Below l the depth is $-u + t[a = 0] + \eta \ge 0$, as $u \le \eta$; if $\varrho_{-\eta}(a, u) > 0$ the improved order is this depth, which is at least $\max(0, -u - \eta)$, and otherwise it is 0 and $-u - \eta \le \varrho_{-\eta}(a, u) \le 0$. From l on the improved order is $-u$. Hence the orders are at least $\mathrm{so}_{\eta,l}$, and a point with orders $\mathrm{so}_{\eta,l}$ is also of depth $\mathrm{so}_{\eta,l}$, since g preserves orders.

(b) The move lies over 0 (Lemma 42.2(f)) with orders $\min(\mathrm{sh}_l^0(\varrho_{-\eta_0}), \varrho_{-\eta})$; apply the rule for 0 and t (Lemma 42.9(c)). At the level 0 the orders are $\min(0, -\eta - u) = -\eta - u$, as $u \ge -\eta$, and the depth is $-v + t + \eta$; at $0 < a < l$ the orders are $\min(0, -u) \ge -u - \eta$ and the depth $-v + \eta$; at $a = l$ the orders are $-\eta_0 - u$ and the depth $-v + \eta_0$; above, $-u$ and $-v$. In each case the sum is at least $\mathrm{rb}_{0,t}(a, u + v)$.

(c) $\mathrm{pd}_{t,\eta,l,\eta_0} \ge \mathrm{sd}_{t,\eta,l}$.

(d) By (c) and (a) x is a stop, and by the symmetric form of (b) for S', which is up, the move is safe with x at rest.

(e) Immediate from the definition; for instance where $\sigma = -1$ a jump is in $[-\eta, \eta]$ or in $-[\beta, \beta + V]$, hence at most η. □

The walk needs a stop at each cut. Between two cuts the sides may change roles (which one is down); a cut is *plain* if no change occurs, and otherwise the walk needs a *long gap*: a letter whose far window starts so far out that the pole of a point there can be removed by an expansion in powers of E (Lemma 42.17). The following definitions describe the cuts.

Definition 42.12 (Cuts). Let $\varepsilon(i) = \pm 1$ be signs, $\beta(i)$ the starts of the far windows of width V, $s(i)$ the *shifts* (the pole order allowed at a cut), and W, J, G, s_0, κ, X, δ, b, n natural numbers.

(C1) $i.\mathrm{back}(l)$ is the *nearest cut* before i if $l > 0$, it is a cut, and no letter strictly between is one. It has *sign* ρ *before* i if each of the l letters $i.\mathrm{back}(a)$, $a < l$, has $\varepsilon = \rho$ or $\beta + V \le \eta$.

(C2) There is a *long gap* with pivot $i.\mathrm{back}(t)$ if the chart is unramified, $p \ge 5$, $s(i.\mathrm{back}(l)) = 0$, $t + J + 1 = l$, $s_0 + \omega \sum_{a \le J} p^a < \beta(i.\mathrm{back}(t))$, every other letter $a < l$ has $\varepsilon \ne \varepsilon(i.\mathrm{back}(t))$ or $\beta + V \le \eta$, $\eta \le \omega$, and the far windows of the letters $t < a < l$ end at or below ω.

(C3) A letter has *thin successors* for a property T if none of the J letters after it has T.

(C4) A *killing cut* is a letter with $s(i) = 0$ whose J successors have $\beta + W \le G$, or, on an unramified chart with $p \ge 5$, a letter with $s_0 - \kappa \le s(i) < \beta(i)$.

(C5) The *size conditions* are $n + \lfloor \log_p 2b \rfloor + (J + 1 + d)W + dG \le X$, $12((J + 1)X + \delta) < \kappa$, $e \cdot 2b \le p^{J+1}\delta$, $8e \le p^{J+1}$, $\beta(i) + s(i) + W \le 2b$, $G \le (p-1)n$, and $\beta(i) + W \le (p-3)s_0$ if $p \ge 5$.

(C6) The *cut orders* are $\mathrm{co}_{s,\eta,l} = \mathrm{imp}(\varrho_{-s}, \mathrm{sd}_{s,\eta,l+1})$, and the *long gap depth* is $\mathrm{lg}_{l,t,r}(a, u) = r$ for $a = t$, 1 for other $a < l$, and $-u$ for $a \ge l$. The *window steps* at i are the pairs (a, m) with $a < 2J + 2$ and m within $s(i)[a = 0] + W$ of 0 or of $\mp\beta(i.\mathrm{back}(a))$.

Lemma 42.13 (Cuts and their points). *Assume the size conditions, $\eta \ge 1$, $V + 4\eta \le W$ and $s(i) + W \le pn$, and let the jumps of S lie in $[-\eta, \eta] \cup \sigma(j)[\beta(j), \beta(j) + V]$.*

(a) *If $W \ge 4$ and $s(i) = 0$ or $d = 0$, then $\kappa + 4\delta + \sum c_a(m) \le 25\kappa$, the sum over the window steps at i.*

(b) *If the jumps are at most $\beta + W$ and $\beta + W \le G$ at the J letters after i, then p^κ kills the cokernel at i. At every killing cut, $p^\kappa E^{s(i)}$ kills the cokernel.*

(c) *At a killing cut i whose nearest cut before is $l \le 2J + 1$ steps back, there is a point over $p^{25\kappa}c$ with orders $\mathrm{co}_{s(i),\eta,l}$ and depth $\mathrm{sd}_{s(i),\eta,l+1}$; such a point has orders $\varrho_{-\eta}$ and depth $\mathrm{pd}_{s(i),\eta,l,\eta}$, so it is a standard point.*

(d) (*Long gap stop.*) *Let the nearest cut $i.\mathrm{back}(l)$, $l \le 2J + 1$, be a killing cut followed by a long gap with pivot $i.\mathrm{back}(t)$ where $\sigma = -1$, and let the jumps at the other letters $a < l$ be at least $-\eta$. Then there is a point over $p^{25\kappa}c$ with orders $\mathrm{so}_{\eta,l}$ and depth $\mathrm{lg}_{l,t,2b}$.*

(e) *For $t < l$, $r > 0$, $u \le \eta$ at $t \ne a < l$ and $u < \eta + r$ at $a = t$, the orders $\min(\mathrm{co}_{s,\eta,l}, \mathrm{sh}^0_l(\mathrm{co}_{0,\eta,l_0}))$ at u and the depth $\mathrm{lg}_{l,t,r}$ at any v add up to at least $\mathrm{rb}_{l,0}(a, u + v)$.*

(f) *If the jumps of two paired sides at i lie in $[-\eta, \eta]$ and in opposite far windows $\mp\varepsilon(i)[\beta(i), \beta(i) + V]$, then the jumps of the two sides and of $\hat{S}$ at i are at most $\beta(i) + W$.*

Proof. (a) At a level $a < 2J + 2$ the window steps are the integers within $r + W$ of 0 or of $\mp\beta$, where $r = s(i)[a = 0]$, and $r + W \le pn$ and $\beta + r + W \le 2b$. Counting the multiples of p^{j+1} in each window, $j < \log_p 2b$, the valuations of the integers within n' of c add up to at most $4m + \lfloor \log_p H \rfloor$ when $n' \le pm$ and $|c| + n' \le H$; hence the costs of the steps at the level a add up to at most $8n + 2\lfloor \log_p 2b \rfloor + (4(r + W) + 2)(d + 3a)$. At $a = 0$ either $r = 0$ or $d = 0$, and for $a \ge 1$, $r = 0$; with $W \ge 4$, $a \le 2J + 1$ and the first size condition this is at most

$30X$. Hence the sum is at most $(2J+2)30X$, and $\kappa + 4\delta + 60(J+1)X \le 25\kappa$ as $\kappa > 12((J+1)X+\delta)$.

(b) All jumps are at most $\beta + W \le 2b$, and those at the J letters after i at most G. By Lemma 42.5(g) with $\omega = 2b$ and $\omega' = G$ it suffices that $\kappa \ge 2JdG + (2J+4)G/(p-1) + 8e \cdot 2b/p^{J+1}$; the size conditions give $dG \le X$, $G/(p-1) \le n \le X$ and $e \cdot 2b/p^{J+1} \le \delta$, so the right side is at most $(4J+4)X + 8\delta < \kappa$. At a killing cut of the first kind this applies with $s(i) = 0$. At one of the second kind the chart is unramified, $p \ge 5$ and all jumps are at most $\beta + W \le (p-3)s_0 \le (p-3)(\kappa + s(i))$, and Lemma 42.5(f) gives $p^\kappa E^{s(i)}$.

(c) By (b) and the lift-and-divide argument of Lemma 42.9(c) there is a point over $p^\kappa c$ with orders and depth $\varrho_{-s(i)}$. The steps from this depth to $\mathrm{sd}_{s(i),\eta,l+1}$ are, on the line u, the m with $-s(i) - u \le m < -u + s(i) + \eta$ at the level 0 and $-u \le m < -u + \eta$ at the levels $1 \le a \le l$; as u is within η of 0 or within V of $\pm\beta$, they are within $s(i)[a=0] + \eta + \max(\eta, V) \le s(i)[a=0] + W$ of 0 or of $\mp\beta$, and $l \le 2J+1$: they are window steps. At a killing cut, $s(i) = 0$ or $d = 0$ (in the second kind the chart is unramified). By Lemma 42.3(f) and (a) there is the point with orders co and depth sd over $p^{25\kappa}c$. Its pole has order at most η: no jump at i lies in $[-s(i), -\eta)$, since a jump below $-\eta$ lies in $-[\beta, \beta+V]$ and $s(i) < \beta(i)$ or $s(i) = 0$; so a line $u < -s(i)$ has the positive order $-s(i) - u$, which improves to the depth $-u + s(i) + \eta \ge -u$, and a line $u \ge -\eta$ has order at least $\min(\mathrm{sd}, 0) \ge -\eta - u$. Finally $\mathrm{sd}_{s,\eta,l+1} = \mathrm{pd}_{s,\eta,l,\eta}$, and (c) of Lemma 42.11 applies.

(d) Write $l = t + J + 1$. By (b), p^κ kills the cokernel at the cut $i.\mathrm{back}(l)$, which lies $J+1$ steps before the pivot and has shift 0; and E^{s_0} kills it at the pivot, by Lemma 42.5(f), as all jumps are at most $(p-3)s_0$. The jumps at the $J+1$ letters up to the pivot are at most ω: at the pivot $\sigma = -1$ makes them at most $\eta \le \omega$, and at the others the far windows end at or below ω. Put $N = s_0 + \omega \sum_{a \le J} p^a < \beta(i.\mathrm{back}(t))$ and $A = 2b$, so $A \le p^{J+1}\delta$ ($e = 1$) and $3 \le p^{J+1}$. Lemma 42.17(e) gives a point x at the pivot over $p^{\kappa+4\delta}c$ with the standard orders of level $J+1$, raised by $2b$ at E on the lines $u < -N$, and of depth the standard orders raised by $2b$ at E everywhere. Carry it to i by φ^t (Lemma 42.2(b)): its orders become 0 below t, at t the value $2b$ on the lines $u < -N$ and 0 on the others, 0 for $t < a < l$, and $-u$ from l on; its depth becomes 1 below t, $2b$ at t, 0 for $t < a < l$ and $-u$ from l on. Raise the depth from 0 to 1 at the levels $t < a < l$: the steps $(a, 0)$ are window steps, so by (a) and Lemma 42.3(f) this gives a point over $p^{25\kappa}c$ with depth $\mathrm{lg}_{l,t,2b}$. Its orders are at least $\mathrm{so}_{\eta,l}$: at the levels $a \ne t$ below l the jumps are at least $-\eta$, so so = 0; at the pivot a jump is in $[-\eta, \eta]$, where so = 0, or in $-[\beta, \beta+V]$, where $u \le -\beta < -N$ and the order $2b \ge \beta + V \ge -u - \eta$.

(e) Below l and off t, both cut orders are at least 0, as $u \le \eta$ makes the depth $-u + s[a=0] + \eta$ nonnegative, so the minimum is 0 and the depth 1 gives the sum 1. At $a = t$ the orders are at least $\min(-u + \eta, 0) > -r$, so with the depth r the sum is at least 1. At $a = l$ the shifted cut orders are those of co_{0,η,l_0} at the level 0, at least $-u$, and $\mathrm{co}_{s,\eta,l}(l, u) \ge -u$; with the depth $-v$ the sum is at least $-(u+v)$; above l likewise.

(f) A jump of a side is at most η or $\beta + V$. The jumps of $\hat{S}$ are 0 and $-(u+u') - 1$; as the far windows have opposite signs, u and u' are not both below $-\eta$, and neither is below $-(\beta+V)$; so $-(u+u') - 1 \le \beta + V + \eta \le \beta + W$. □

Lemma 42.14 (Stops at cuts). *Let P pair sides whose jumps lie in $[-\eta,\eta]$ and in opposite far windows $\mp\varepsilon(j)[\beta(j),\beta(j)+V]$, and let $i.\mathrm{back}(l)$ be the nearest cut before i.*

(a) *If $i.\mathrm{back}(l)$ has sign ρ before i, the points have poles of order at most η, and x, x' at i have depth $\mathrm{pd}_{s,\eta,l,\eta}$, and $p^\kappa E^s$ kills the cokernel of $\hat S$ at i, then one side has a stop at which the other moves safely from φ^l of its point at the cut to its point at i.*

(b) *If the cut has shift 0, is followed by a long gap, and p^κ kills the cokernel of $\hat S$ there, and the size conditions hold with $l\le 2J+1$, then the same conclusion holds for points with the cut orders at these cuts and any depths.*

Proof. (a) The side whose far window has sign $-\rho$, together with the small windows, has its jumps at most η at the l letters (Lemma 42.11(e)): it is down, and the other is up. Lemma 42.11(d) gives the stop and the safe move.

(b) The side R whose far window at the pivot is negative leads. At the letters $a\ne t$ below l the far windows of R are small or of the sign opposite to that at the pivot, so R is up there, and Lemma 42.13(d) gives a stop q of R with depth $\mathrm{lg}_{l,t,2b}$. The other side R' has, at the pivot, its jumps in $[-\eta,\eta]\cup[\beta,\beta+V]$, so $u\le\beta+V<\eta+2b$, and at the other letters below l it is down. Its move from φ^l of its point at the cut to its point at i has orders $\min(\mathrm{co}_{s,\eta,l},\mathrm{sh}^0_l(\mathrm{co}_{0,\eta,l_0}))$, by Lemma 42.2(b) and (f); Lemma 42.13(e) with $r=2b$ gives the rule for l and 0, which applies because $p^\kappa=p^\kappa E^0$ kills the cokernel of $\hat S$ at the cut (Lemma 42.9(c)). One lemma serves both sides by symmetry. □

Lemma 42.15 (Patterns of cuts). *Let T be a property of letters (the* thick *letters), and suppose that no three letters with T follow one another along Frobenius with both gaps between 1 and J.*

(a) *If $i.\mathrm{back}(r)$ has T but not thin successors, then $i.\mathrm{back}(r+J+1)$ has thin successors. One of the $2J+1$ letters before i has thin successors.*

(b) *If $i.\mathrm{back}(l)$ is the nearest letter before i with thin successors, a thick letter $i.\mathrm{back}(r)$ with $r<l$ has $r=0$ or $r+J+1=l$.*

(c) (*Sparse thick letters.*) *Let the far windows start at b at the thick letters and at G at the others, $\eta\ge1$, $G+V\le\eta$, and let either the chart be unramified with $p\ge7$ and $10p^J\eta\le b$, or the sign be constant. If $i.\mathrm{back}(l)$ is the nearest letter with thin successors, then for some pivot the l letters have the sign of the pivot up to small windows, or there is a long gap with all shifts 0, $s_0=\lfloor(b+10\eta)/(p-3)\rfloor+1$ and $\omega=\eta$.*

(d) (*Two sizes, $J=1$.*) *Let the chart be unramified, $p\ge7$, the far windows start at b at the thick letters and at G at the thin ones, $0<G$, $b<Kp^2$, $V+4\eta\le W\le K$, and let the shifts be at most $\lfloor b/p\rfloor-7K$, and 0 at the letters with a thin successor. Let the cuts be the letters whose shift is less than the start of their far window. Then the thick letters and the letters with a thin successor are cuts; every letter has a cut one or two steps back; and if $i.\mathrm{back}(l)$ is the nearest cut, the l letters have the sign of i up to small windows, or there is a long gap with pivot i, $J=1$, $s_0=\lfloor b/p\rfloor+12K$ and $\omega=G+W$.*

(e) (*Sizes.*) *With all shifts 0 and $\beta(i)\le\max(b,W)$, the size conditions hold with $\kappa=12((J+1)X+\delta)+1$, $b+W$ for b, $n=G=2W$ and $s_0=\lfloor(b+2W)/(p-3)\rfloor+1$, provided $8e\le p^{J+1}$, $2e(b+W)\le p^{J+1}\delta$ and $4(J+1+d)W+\lfloor\log_p 2(b+W)\rfloor\le X$. In the situation of (d), they hold with*

$J = 1$, $X = 3K + \lfloor \log_p 2(b+W) \rfloor$, $\delta = 4K$, $\kappa = 12(2X+\delta)+1$, $n = K$, $s_0 = \lfloor b/p \rfloor + 12K$, *and* $b+W$, $G+W$ *for* b, G, *if* $2G < Kp$.

Proof. (a) If one of the J letters after $i.\mathrm{back}(r+J+1)$, that is $i.\mathrm{back}(r+J+1-s')$ with $1 \le s' \le J$, had T, then with a thick letter among the J after $i.\mathrm{back}(r)$ there would be three thick letters with both gaps in $[1, J]$. For the second claim, if $i.\mathrm{back}(J+1)$ fails, a letter $i.\mathrm{back}(r)$ with $1 \le r \le J$ has T; it has thin successors, or by the first claim $i.\mathrm{back}(r+J+1)$, with $r+J+1 \le 2J+1$, does.

(b) Let $0 < r < l$ with T. As $i.\mathrm{back}(r)$ is not a cut, it does not have thin successors, so $i.\mathrm{back}(r+J+1)$ has them by (a), and nearness gives $r+J+1 \ge l$. The cut $i.\mathrm{back}(l)$ has thin successors, so $r \notin [l-J, l-1]$, and $r+J+1 \le l$.

(c) If every thick letter among the l has the sign of i, the letters have the sign of i up to the thin letters, whose windows end at $G+V \le \eta$. Otherwise by (b) the thick letter with a different sign is the pivot $i.\mathrm{back}(t)$ with $t+J+1 = l$, which excludes the case of constant sign. The letters other than i and the pivot are thin, with small windows, and i has the sign opposite to the pivot. With $p \ge 7$, $\sum_{a \le J} p^a \le 2p^J$ and $s_0 \le (b+10\eta)/4 + 1$, and $(b+10\eta)/4 + 1 + 2p^J\eta < b$ as $10p^J\eta \le b$ and $\eta \ge 1$: the gap is long, with $\omega = \eta$ bounding the low window and the windows of the thin letters before the pivot.

(d) A thick letter has shift at most $b/p < b$, and a letter with a thin successor has shift $0 < G \le \beta$; so both are cuts. If $i.\mathrm{back}(1)$ is not a cut, its successor i is thick, and if $i.\mathrm{back}(1)$ is not thick either, $i.\mathrm{back}(2)$ has the thin successor $i.\mathrm{back}(1)$: a cut lies one or two steps back. If $l = 1$ the condition on signs concerns i alone. If $l = 2$, $i.\mathrm{back}(1)$ is not a cut, hence thin with $0 < G \le s(i.\mathrm{back}(1)) \le \lfloor b/p \rfloor - 7K$, and i is thick. If $i.\mathrm{back}(1)$ has the sign of i or a small window, the sign condition holds. Otherwise the gap is long with pivot i: the cut $i.\mathrm{back}(2)$ has the thin successor $i.\mathrm{back}(1)$, so shift 0; the windows before i end at or below $G+V \le G+W = \omega$; and $s_0 + \omega(1+p) \le b/p + 12K + (b/p - 6K)(p+1) = (p+2)b/p - (6p-6)K < b$, as $2b/p < 2Kp \le (6p-6)K$.

(e) The first set of conditions is a direct check: $n + \lfloor \log_p 2(b+W) \rfloor + (J+1+d)W + 2dW \le X$ follows from the bound on X, $\beta + W \le 2(b+W)$, $2W \le (p-1)2W$, $\beta + W \le b + 2W \le (p-3)s_0$, and $\kappa > 12((J+1)X + \delta)$. In the second, $d = 0$, $K + \lfloor \log_p 2(b+W) \rfloor + 2W \le X$, $2(b+W) \le 4Kp^2$ as $b < Kp^2$, $8 \le p^2$, $\beta + s + W \le 2(b+W)$, $G + W < Kp/2 + K \le (p-1)K$, and $b + W \le (p-3)s_0$ because $b - (p-3)\lfloor b/p \rfloor < 3\lfloor b/p \rfloor + p \le 4pK$ and $4pK + K \le 12(p-3)K$. □

Lemma 42.16 (The scheme of cuts). *Let paired sides have jumps in* $[-\eta, \eta]$ *and in opposite far windows* $\mp\varepsilon(i)[\beta(i), \beta(i)+V]$, *with* $\eta \ge 1$, $V + 4\eta \le W$, $s(i) + W \le pn$ *and the size conditions.*

(a) *If every cut is a killing cut, every letter has a cut at most* $2J+1$ *steps back, and after each nearest cut the letters have the sign of some pivot up to small windows or there is a long gap, then the cup product of the cocycles of* c *and* $p^e c'$ *is a coboundary for every* $e \ge 51\kappa$.

(b) (*Sparse thick letters.*) *In the situation of Lemma 42.15*(*c*), *with* $J \ge 1$, $8e \le p^{J+1}$ *in the case of constant sign,* $2e(b+5\eta) \le p^{J+1}\delta$ *and* $20(J+1+d)\eta + \lfloor \log_p 2(b+5\eta) \rfloor \le X$, *the exponent* $10^3(\delta + (J+1)X)$ *works.*

(c) (*Any pattern.*) *In the situation of Lemma 42.15*(*d*) *with* $0 < G \le b$ *and* $2G < Kp$, *the exponent* $10^4(K + \lfloor \log_p 2(b+W) \rfloor)$ *works.*

Proof. (a) Let $l(i) \le 2J+1$ be the distance to the nearest cut. At a cut i, $p^\kappa E^{s(i)}$ kills the cokernel of $\hat{S}$, by Lemma 42.13(b) and (f), and both sides have standard points over $p^{25\kappa}c$, $p^{25\kappa}c'$ with the cut orders (Lemma 42.13(c)). The stops are given by Lemma 42.14: at a plain cut, the leading side's own point; at a long gap, the point of Lemma 42.13(d), the cut before a long gap having shift 0. Lemma 42.10(c) applies with $e+e'+\kappa = 51\kappa$.

(b) Take all shifts 0, the cuts at the letters with thin successors, $W = 5\eta$, $\beta(i) = b$ or G, and $\kappa = 12((J+1)X+\delta)+1$. The size conditions hold by Lemma 42.15(e) (with $G_{\text{cut}} = 10\eta$, and $8e \le p^{J+1}$ on an unramified chart because $e = 1$); the cuts are killing cuts of the first kind, as the J thin successors have $\beta+W = G+5\eta \le 10\eta$; every letter has a cut within $2J+1$ steps (Lemma 42.15(a)); and the condition on signs is Lemma 42.15(c). Finally $51\kappa \le 10^3(\delta+(J+1)X)$ as $X \ge 20$.

(c) Take $J = 1$, shifts $\lfloor b/p \rfloor - 7K$ at the letters whose successor is thick and 0 otherwise, and the cuts of Lemma 42.15(d). A cut with a thin successor has shift 0 and is a killing cut of the first kind ($\beta+W \le G+W$ at its successor); one whose successor is thick is of the second kind, as $s(i) < \beta(i)$ and $s_0 = \lfloor b/p \rfloor + 12K \le \kappa + s(i)$. The size conditions are Lemma 42.15(e), $s(i)+W \le \lfloor b/p \rfloor - 6K \le pK$, and $51\kappa \le 10^4(K + \lfloor \log_p 2(b+W) \rfloor)$. □

Lemma 42.17 (Long gaps). *Let the chart be unramified, $0 < l$, and let* $\text{std}_l + A[a=0]$ *denote the standard orders of level l raised by A at E. An element ξ with the standard orders of level l at i is a* constant modulo E^A *if every $g\xi - \xi$, $g \in \Gamma'$, has the orders* $\text{std}_l + A[a=0]$.

(a) *Constants modulo E^A form a $\mathbb{Z}_p$-module stable under $W(\mathbb{F}_{p^f})$ and under maps of lattices whose jump sets shrink. The projection to the letter i of an element of φ^l of the A_{inf}-span of $\mathcal{N}$ on which all forms take values in $\mathcal{K}_0$ and are fixed by Γ' modulo $t^A B^+_{\text{dR}}$ is a constant modulo E^A.*

(b) *Every $s \in \mathfrak{S}$ has an expansion $s \equiv \sum_{n<N} E^n w_n$ modulo E^N for all N, with $w_n \in W(\mathbb{F}_{p^f})$.*

(c) (*Taylor expansion.*) *If $3 \le p^l$, $A \le p^l\delta$ and x has the standard orders of level l at i, there are constants ξ_n modulo E^A such that for every N, $p^{4\delta}x = \sum_{n<N} E^n\xi_n + E^N y$ with y of standard orders of level l.*

(d) (*Peeling.*) *If ξ_n are constants modulo E^A, y has standard orders and $\sum_{n\le N} E^n\xi_n + E^{N+1}y$ has the standard orders raised by $o(u) \le A$ at E, then ξ_N has them raised by A on the lines with $N < o(u)$.*

(e) (*The long gap lemma.*) *Let $3 \le p^l$, $A \le p^l\delta$, let p^κ kill the cokernel at $i.\text{back}(l)$ and E^s the cokernel at i, let the jumps at the l letters up to i be at most g, and let $N = s + g\sum_{a<l} p^a < A$. Then for each c the extension has a point at i over $p^{\kappa+4\delta}c$ with the standard orders of level l raised by A at E on the lines $u < -N$, all of whose differences $\gamma x - x$, $\gamma \in \Gamma'$, have the standard orders raised by A at E on every line.*

Proof. (a) The group Γ' fixes $W(\mathbb{F}_{p^f})$, and equivariant maps of lattices whose jump sets shrink keep orders (Lemma 42.2(c)). For the second claim, the projection has the standard orders of level l (Theorem B.40(e)), the difference $gy - y$ keeps them (Lemma 42.2(e)), and its value at the level 0 under a form μ of type u is the value of $g\mu'(y) - \mu'(y)$ for the form $\mu' = \mu \circ e_i$, which lies in $t^A B^+_{\text{dR}}$; so only the exponent A at E is new.

(b) As $e = 1$ and $E = z + p$, division by E in $\Lambda[\![z]\!]$ leaves a remainder in Λ ($\mathfrak{S}/E \cong \Lambda$); dividing the quotients repeatedly, the remainders are the digits w_n.

(c) By the frame theorem (Theorem B.40(f)), with $3e \le p^l$ and $eA \le p^l\delta$, $p^{4\delta}x$ is an $\mathfrak{S}$-combination $\sum_a r_a v_a$ of elements fixed by Γ' modulo t^A of the kind in (a); projecting to the letter i, where x lies, $p^{4\delta}x = \sum_a r_a b_a$ with constants b_a modulo E^A. Expand each $r_a = \sum_{n<N} E^n w_{a,n} + E^N t_{a,N}$ by (b) and collect: $\xi_n = \sum_a w_{a,n} b_a$, which are constants modulo E^A by (a), and $y = \sum_a t_{a,N} b_a$.

(d) First the case $N = 0$: if $\xi + Ey$ has the orders $\mathrm{std}_l + o[a = 0]$, then, as Ey has the orders $\mathrm{std}_l + 1[a = 0]$ ($l > 0$), ξ has $\mathrm{std}_l + \min(1, o)[a = 0]$, and being invariant modulo $\mathrm{std}_l + A[a = 0]$ its positive exponents at E improve to A (Lemma 42.2(d)). In general peel off ξ_0, which by this case has the exponent $A \ge o$ wherever $o > 0$, and divide by E, a unit away from the level 0; this lowers N and o by one, and induction concludes.

(e) First a Taylor coefficient. From E^s killing the cokernel at i there is a point y over c with orders ϱ_{-s} (Lemma 42.9(c)). Put $f(E) = E^s \prod_{a<l} (\varphi^a E)^g$; it is monic of degree N in E, as $\varphi^a E = (E - p)^{p^a} + p$. Multiplication by $(\varphi^a E)^g$ raises the orders at the level a by g, so $f(E)y$ has the orders $g - u \ge 0$ at the levels $a < l$ (the jumps there being at most g) and $-u$ above: it has the standard orders of level l, raised at E by $\min(A, g - u)$. By (c), $p^{4\delta} f(E)y = \sum_{n \le N} E^n \xi_n + E^{N+1} y'$, and by (d) ξ_N has the standard orders raised by A on the lines with $N < \min(A, g - u)$, which contain the lines $u < -N$. Projecting to the trivial lattice, $p^{4\delta} f(E) e_i(1 \otimes c) = \sum_{n \le N} E^n f_n p^{4\delta} e_i(1 \otimes c)$ with $f_N = 1$, and the differences of the coefficients are constants modulo E^A of the trivial lattice forming an expansion of 0; by (d), as $N < A$, $\mathrm{pr}(\xi_N) - p^{4\delta} e_i(1 \otimes c)$ has the standard orders raised by A everywhere. It remains to remove this error w: $E^{-A} w$ has the standard orders of level l, and φ^l carries the ideal of the killers at $i.\mathrm{back}(l)$ to that of level l at i (Lemma 42.4(c)), so $p^\kappa E^{-A} w$ lifts to some x with the standard orders of level l at i. Then $p^\kappa \xi_N - E^A x$ lies over $p^{\kappa+4\delta} c$, has the orders of $p^\kappa \xi_N$ (those of $E^A x$ being larger), and its differences under Γ' are those of $p^\kappa \xi_N$, raised by A at E, and $E^A(\omega^A g x - x)$, which have them too. □

Lemma 42.18 (Deep points). *Let P pair S and S', and let n be such that p divides no nonzero integer within n of 0, b or $-b$ (the* gap condition*).*

(a) *The costs $c_0(y) = v_p(y) + d$ of the integers within n of 0, b or $-b$ add up to at most $(8n + 4)d$; on an unramified chart they vanish.*

(b) *If all steps from the depth o to o' lie at the level 0 within n of 0 or $\pm b$, a point over $p^\kappa c$ with orders o_1 and depth o gives one over $p^{\kappa+(8n+4)d} c$ with depth o' and orders $\mathrm{imp}(o_1, o')$; on an unramified chart, over $p^\kappa c$ with any orders at most these.*

(c) (*Gap below 0.*) *If $p^\kappa E^a$ kills the cokernel at i, no jump lies in $[-a, 0)$, and a set of steps containing $[-u - a, a + \eta - u)$ for every jump has total cost at most C, there is a point over $p^{\kappa+C} c$ without pole and of depth $\mathrm{pd}_{a,\eta,1,0}$.*

(d) (*Unramified, windows.*) *If the chart is unramified, $p^\kappa E^a$ kills the cokernel at i, the jumps lie in $[-\eta, \eta] \cup \sigma[\beta, \beta + W]$, $a \le t$, and the gap condition holds with $n \ge a + t + \eta + W$, there is a point over $p^\kappa c$ with orders ϱ_{-t} and depth $\mathrm{pd}_{a,t,1,0}$; if the jumps lie in $[-\eta, 0]$ and $n \ge a + t + \eta$, there is one over c without pole and of the same depth.*

(e) (*Stops, nonpositive jumps.*) *On an unramified chart with the jumps of S' in $[-\eta, 0]$, $\eta \le t$ and $n \ge a + t + \eta$, a point of S' over c' at $i.\mathrm{back}(1)$ without*

pole and of depth $a+t-u$ at the level 0 gives a point at i over c' with orders $\mathrm{so}_{t,1}$ *and depth* $\mathrm{pd}_{a,t,1,t}$.

(f) *Let $p^\kappa E^a$ kill the cokernels of S, S' and $\hat{S}$ at every letter, let the gap condition hold with $n = 6(a+\eta+W)$ around b_i, and let the jumps lie in $[-\eta,\eta]$ and in the opposite far windows $\mp\varepsilon(i)[b_i, b_i+W]$. If no jump lies in $[-a,0)$, or the chart is unramified and the jumps of S' lie in $[-\eta,0]$, then the cup product of the cocycles of c and $p^H c'$ is a coboundary for $H \geq 3(\kappa+(8n+4)d)$.*

Proof. (a) The integers within n of 0, b or $-b$ form at most four intervals of $2n+1$ integers; by the gap condition each is 0 or prime to p, so costs d. On an unramified chart $d=0$.

(b) is Lemma 42.3(f) with I the set of (a), or, on an unramified chart, with the free steps of cost 0; in the second case the point stays over $p^\kappa c$.

(c) Lifting and dividing gives a point over $p^\kappa c$ with orders and depth ϱ_{-a} (Lemma 42.9(c)). Deepen it at the level 0 along the steps, from $-a-u$ to $a+\eta-u$ (Lemma 42.3(f)). The improved orders have no pole: a line $u<-a$ has the positive order $-a-u$, which improves to the depth $a+\eta-u \geq -u$; a line $u \geq 0$ has the order $\min(a+\eta-u, 0) \geq -u$; and no jump lies in between.

(d) Deepen as in (c) from $-a-u$ to $a+t-u$; the steps lie within $a+t+\eta+W$ of 0 or $\pm\beta$, as u is within η of 0 or within W of $\pm\beta$, and are free by (a). A line $u<-a$ improves to $a+t-u \geq -t-u$, a line $u \geq -a$ has order at least $\min(a+t-u, 0) \geq -t-u$, since $a \leq t$. If the jumps are at most 0, then $1 = p^0E^0$ kills the cokernel (Lemma 42.5(d) with $\omega=0$), and the same deepening from $-u$ to $a+t-u$ gives a point over c whose orders improve to $-u$.

(e) Frobenius carries the point to i with orders $\mathrm{sh}_1^0(-u)$ and depth sh_1^1 of its depth: 1 at the level 0, $a+t-u \geq t-u$ at the level 1, and $-u$ above (Lemma 42.2(b)). Raise the depth at the level 0 from 1 to $a+t-u$; the steps lie within $a+t+\eta$ of 0 and are free. The orders are $\mathrm{so}_{t,1}$: at the level 0, $\max(0, -u-t) = 0$ as $u \geq -\eta \geq -t$, and $-u$ above.

(f) *No jump in* $[-a,0)$. The steps $[-u-a, a+\eta-u)$ lie within n of 0 or $\pm b_i$, so by (a) they cost at most $C = (8n+4)d$; by (c) both sides have points without pole and of depth $\mathrm{pd}_{a,\eta,1,0} = \mathrm{sd}_{a,\eta,1}$ over $p^{\kappa+C}c$, $p^{\kappa+C}c'$ at every letter: standard points. At each letter the side whose far window is negative is down and the other up (Lemma 42.11(e)), so Lemma 42.11(d) with $l=1$ gives the stop, and Lemma 42.10(d) with every letter a cut gives the exponent $2(\kappa+C)+\kappa \leq H$.

Nonpositive jumps of S'. Put $t = a+\eta$; then $a+t+\eta+W \leq n$. By (d) S has points over $p^\kappa c$ with orders ϱ_{-t} and depth $\mathrm{pd}_{a,t,1,0}$, and S' points over c' without pole and of the same depth; the jumps of S' lie in $[-t,t]$. Where the far window of S is negative, S is down and S' up, and S provides the stop; elsewhere S is up and S', whose jumps are at most 0, is down: (e) gives a stop q' of S' of depth $\mathrm{pd}_{a,t,1,t}$, and the move of S by $\varphi x(i.\mathrm{back}(1)) - x(i)$ is safe with q' at rest by Lemma 42.11(b). The walk gives the exponent $\kappa+0+\kappa = 2\kappa \leq H$. □

Lemma 42.19 (Few low jumps). *Let the chart be unramified and $p \geq 5$. For a side S, a letter i, $t \in \mathbb{Z}$ and a depth c, let $C_i(t,c)$ be the module of the constants at i below elements of the extension with orders ϱ_t and depth c, and let $\ell(M,N)$ be the length of $M/(M\cap N)$. Let f be the residue degree and* $\mathrm{mult}_j(u)$ *the multiplicity of the jump u at j.*

(a) (*Division by p.*) *If no multiple of p lies strictly between $c(0,v)$ and $c(0,v)+p^a$ for any v, x has depth c, w lies over 0 with orders c raised by 1 at the level 0, and $px-w$ has the depth c raised by one at (a,u), then so has $x-m$ for some m over 0 with orders c.*

(b) (*One step.*) *If $c(l,v)\geq -v$ on the jumps, $c(a,u)<c(a,v)$ for the jumps $v<u$ at $i.\mathrm{back}(a)$, and no multiple of p lies strictly between $c(0,v)$ and $c(0,v)+p^a$, raising $c(a,u)$ by one shrinks $C_i(0,c)$ by a module of length at most $c_a(c(a,u))\ f\ \mathrm{mult}_{i.\mathrm{back}(a)}(u)$.*

(c) (*One level.*) *Under the same hypotheses on c, if $c'\leq c$ off the level a and $c'(a,\cdot)$ is antitone on the jumps, then $\ell(C_i(0,c),C_i(0,c'))$ is at most the sum, over the steps (u,m) with $c(a,u)\leq m<c'(a,u)$, of $c_a(m)\ f\ \mathrm{mult}(u)$.*

(d) (*Removing the pole.*) *Let $b_j+W+\eta\leq(p-3)a$, $6(a+\eta+W)<b_j$ and the gap condition with $6(a+\eta+W)$ around b_j hold at every letter, and let the jumps lie in $[-\eta,\eta]\cup\varepsilon(j)[b_j,b_j+W]$. Then all constants at i lie, up to length $a\ n_i$, in $C_i(0,\varrho_{2a+\eta})$, where n_i bounds f times the number of jumps in $[-\eta,\eta]$ at i.*

(e) (*The depth at a negative window.*) *Under the hypotheses of (d) with $\varepsilon(i)=-1$, let $\mathrm{nw}(0,u)=b_i+W+a+\eta$, $\mathrm{nw}(1,u)=W+1-u+2a[|u|\leq\eta]$ and $\mathrm{nw}(a',u)=-u$ for $a'\geq 2$. Then $\ell(C_i(0,\varrho_{2a+\eta}),C_i(0,\mathrm{nw}))\leq(a+1)n_i+3(W+1)r_m+6an_m$, where $m=i.\mathrm{back}(1)$ and r_j bounds f times the number of all jumps at j.*

(f) (*The submodule.*) *Let $\mathrm{fl}_i=\varrho_{2a+\eta}$ if $\varepsilon(i)=1$ and nw if $\varepsilon(i)=-1$. The $d\in C$ whose constant at every letter i lies in $C_i(0,\mathrm{fl}_i)$ form a submodule V with $\ell(C,V)\leq 11\sum_i(n_i(a+1)+r_i(W+1))$.*

(g) (*Cupping to zero.*) *If $\eta\geq 1$ and T is a second side with jumps in $[-\eta,\eta]\cup-\varepsilon(i)[b_i,b_i+W]$, paired with S, then for $c\in C_T$ and $d\in V$ the cup product of the cocycles of c and d is a coboundary, with exponent 0.*

Proof. (a) Write ϱ^N for c raised by N at the level 0. Free steps (no multiple of p among them; Lemma 42.18(a)) give an operator A of cost 0 that takes elements of depth ϱ^1 to elements of depth ϱ^{p^a}. As $\varphi^aE\equiv E^{p^a}$ modulo p on an unramified chart, write $E^{p^a}=\varphi^aE+pr$. Then $Aw=E^{p^a}w'$ with w' over 0 of orders c, so $Aw=(\varphi^aE)w'+prw'$, and $p(Ax-rw')=A(px-w)+(\varphi^aE)w'$: the first term has the depth c raised at (a,u) because A keeps depths, and the second has the orders c raised by one at the level a; so $p(Ax-rw')$ has that depth. No φ^lE divides p, so the factor p can be cancelled (the action of Γ' commutes with p), and $Ax-x$ lies over 0 with orders c since A has cost 0 (as 0 is not among the steps, the depth of Aw gives it the same orders). Take $m=x-Ax+rw'$.

(b) Let $X(o)$ be the elements without pole and of depth o, and M the elements over 0 with orders c; M lies over 0, so it is enough to bound the quotient of $X(c)$ by $X(c^+)+M$, c^+ the raised depth. The operator of Lemma 42.3(d) for the step $(a,c(a,u))$ takes $x\in X(c)$ into $X(c^+)$ with $Ax-p^{c_a(c(a,u))}x\in M$, so the quotient is killed by $p^{c_a(c(a,u))}$. Any $f\ \mathrm{mult}(u)+1$ elements x_j killed by p satisfy a primitive relation: write $px_j=x'_j+m_j$ with $x'_j\in X(c^+)$ and $m_j\in M$ the image of $y_j\in Y$ with orders c (Lemma 42.2(f)); by Lemma 42.6(c) a combination $\sum\beta_jy_j$ with some $\beta_{j_0}=1$ is $w+z$ with w of orders c raised at the level 0 and z of orders c^+; then $p\sum\beta_jx_j-w=\sum\beta_jx'_j+z$ has depth c^+, and (a) divides by p. A $\mathbb{Z}_p$-module killed by p^k in which any $N+1$ elements killed by p have a primitive relation has length at most kN.

(c) Induction on the steps left, taking the first step on the least line that is not yet at its target; the raised depth still satisfies the hypotheses (antitonicity keeps $c(a,u) < c(a,v)$ for $v < u$), and has one step less.

(d) Every constant has a deep point with a pole of order at most a, by Lemma 42.5(f): all jumps are at most $b_j + W + \eta \le (p-3)a$, so E^a kills the cokernels. Deepening at the level 0 to $\varrho_{2a+\eta}$ is free, and as no jump lies in $[-a,-\eta)$ (the far windows start beyond $6(a+\eta+W)$), the pole has order at most η. Then lower the pole order step by step from $-\eta$ to 0: from t to $t+1$, only the line $u = t$ changes (the other lines improve by the depth $\varrho_{2a+\eta}$, as in Lemma 42.18(c)), p^a times every constant lies in $C_i(t+1, \varrho_{2a+\eta})$ (a point over p^a has no pole), and the argument of (b), on the line t at the level 0, shows that the step loses at most $a\ f\ \mathrm{mult}_i(t)$; as $t \neq 0$, the extension has no more jumps t than Y. Summing over $t \in [-\eta,-1]$ gives $a\ n_i$.

(e) Use (c) twice. First change the depth at the level 0 to qp, the least multiple of p that is at least $b_i + W + a + \eta$: on a far line $u \in -[b_i, b_i+W]$ every step $y < qp$ divisible by p would satisfy $y \le qp - p$ and lie near b_i, where the gap condition leaves only 0, so the steps are free; on a low line the steps lie in $[0,qp)$ and $qp \le (p-1)(a+1)$, so by Legendre's formula they cost at most $v_p((qp)!) \le a+1$. This costs at most $(a+1)n_i$. Then raise the depth at the level 1, at the letter m, from $-u$ to $\mathrm{nw}(1,u)$: no multiple of p lies strictly between qp and $qp+p$, the line has $W+1$ steps, $2a$ more if it is low, and each step is 0 or prime to p (the gap condition), so costs at most 3; this costs at most $3(W+1)r_m + 6an_m$. The new depth is at least nw.

(f) By (d) and, where $\varepsilon(i) = -1$, (e), the constants at i lie in $C_i(0, \mathrm{fl}_i)$ up to length $(2a+1)n_i + 3(W+1)r_m + 6an_m$. Sum over the letters; $i \mapsto i.\mathrm{back}(1)$ permutes them, and $(8a+1)n_i + 3(W+1)r_i \le 11(n_i(a+1) + r_i(W+1))$.

(g) Let $d \in V$. At every letter S has a point over the constant of d without pole and of depth fl_i; it is standard for $(a,\eta,1)$, since $\mathrm{fl}_i(0,u) \ge a+\eta-u$ ($-u \le b_i + W$ on the far lines). By Lemma 42.5(f) E^a kills the cokernels of T and of $\hat{S}$, whose jumps are 0 and $-(u+v)-1 \le (p-3)a$, so T has points over c with orders $o = \mathrm{imp}(\varrho_{-a}, \varrho_{2a+\eta})$ and depth $\varrho_{2a+\eta}$ (deepening is free), whose pole has order at most η as no jump of T lies in $[-a,-\eta)$. Apply the walk at all letters with $t = a$ (Lemma 42.10(d)). Where $\varepsilon(i) = 1$, T is down and S up, and Lemma 42.11(d) gives the stop of T. Where $\varepsilon(i) = -1$, S is down: its point improves to a stop (at the level 0 the depth $b_i + W + a + \eta$ is positive), and the move of T by $\varphi x_0 - x$ is safe with it at rest. For this apply Lemma 42.9(b) with φ of a point of $\hat{S}$ at $i.\mathrm{back}(1)$ deepened to ϱ_a by free steps (its jumps lie near 0 and $\pm b$), whose orders are $e_a(w) = \mathrm{imp}(\varrho_{-a}, \varrho_a)(0,w)$; with $m(u) = \min(-u, e_{2a+\eta}(u))$ the orders of the move at the level 1 and $w = -u-v-1$, positivity at the level 1 reads $\Sigma = e_a(w) + m(u) + \mathrm{nw}(1,v) > 0$. Here $e_a(w) + w$ equals a for $|w| > a$ and w for $|w| \le a$, $m(u)+u$ equals u for $u \in [-a,0)$ and 0 otherwise, and $\mathrm{nw}(1,v) = W+2+w+u+2a[|v| \le \eta]$; so $\Sigma = (e_a(w)+w)+(m(u)+u)+W+2+2a[|v| \le \eta]$. If v is low, $\Sigma \ge -a-a+W+2+2a > 0$. If $v \le -b_i$ and u is low, $w \ge b_i - \eta - 1 > a$, so $e_a(w)+w = a$ and $\Sigma \ge W+2$; if $v \le -b_i$ and $u \ge b_i$, then $w \ge -W-1$, $m(u)+u = 0$ and $\Sigma \ge \min(w,a) + W + 2 > 0$. At the level 0 the depth $b_i + W + a + \eta$ exceeds the pole, and above the orders are standard, with sum 1. All points lie over the constants themselves, so the exponent is 0. □

Lemma 42.20 (Tame charts). *A tame chart is a Kummer chart with Γ' normal, $p \nmid e$, a root $\varpi^e = p$ fixed by Γ', and an inertia element s whose powers fix Λ and with $s^k z = \xi^k z$, where $\xi \in \Lambda$ is a primitive eth root of unity modulo p.*

(a) *Conjugates $x \mapsto g^{-1}\mu(gx)$ of crystalline forms are crystalline. If $q = \sum q_k X^k$ kills the operator β induced by s on $B_{\max} \otimes L$, then $\sum_k q_k \mu_k(x) = 0$ for x in the $A_{\inf}$-span of $\mathcal{N}$, μ_k the conjugate of μ by s^k; and $\sum_k q_k s^k x \in z\,\mathcal{N}^A$ for $x \in \mathcal{N}$.*
(b) *If q kills β on Y, then $q(X-1)$ kills it on the extension $\tilde{Y}$.*
(c) *For a monic Q over $\mathbb{Z}_p$ and $v \le e$, at most $\deg Q$ of the $k < v$ make $Q(\xi^k)$ a non-unit.*
(d) *If $Q(1) = 0$, Q kills s on $B_{\max} \otimes \tilde{Y}$ and $p^t z^{k+1} \in J_0^r(i)$, then $Q(\xi^{k+1}) p^t z^k \in J_0^r(i)$.*
(e) *If all jumps of S are at most n, Y is potentially crystalline and a monic q kills s on $B_{\max} \otimes Y$, $e \le p^m$, $m(\deg q + 1) \le \kappa$ and $n + \kappa + 2 < p$, then p^κ kills the cokernel at i.*
(f) *If $\Gamma \to G_p$ is onto, f is prime to p and $e \mid p^f - 1$, there is a tame chart of ramification index e; in particular, for $r < p$, one with $e = \operatorname{lcm}(p^i - 1)_{1 \le i \le r} \le p^{r^2}$, and its different bound is at most 1.*

Proof. (a) The values of the conjugate on $1 \otimes L$ come from the conjugate $t \mapsto g^{-1} l(gt)$ of the map $l\colon L \to B_{\max}$. The conjugates define linear forms ℓ_g on $B_{\max} \otimes L$ with $\ell_1 \circ \beta^k = \ell_{s^k}$, tested on the vectors fixed near 1, which span (it suffices to take $g \in \Gamma'$, which contains the elements near 1, and $\ell_{s^k}(y)$ is fixed near 1, hence by the inertia element s^k); so $q(\beta) = 0$ gives $\sum q_k \ell_{s^k} = 0$, which is carried back to the μ_k; forms that agree on $1 \otimes L$ agree on the $A_{\inf}$-span of $\mathcal{N}$. For the last claim, $\mu(\sum q_k s^k x) = \sum q_k s^k \mu_k(x) = \sum q_k (s^k - 1)\mu_k(x)$ by the relation, and $s^k - 1$ produces a factor z on the values, which lie in S_B up to powers of E (Theorem B.40(b)), once the denominators are cleared; the divisibility criterion of the frame theorem (Theorem B.40(a)) gives the factor z in $\mathcal{N}^A$.

(b) $\beta - 1$ maps $B_{\max} \otimes \tilde{Y}$ into $B_{\max} \otimes Y$, as $\beta = 1$ on the trivial quotient, and there $q(\beta) = 0$; the inclusion of Y commutes with β.

(c) The residues of ξ^k, $k < v \le e$, are distinct, and Q modulo p has at most $\deg Q$ roots.

(d) Lift $(p^t z^{k+1} - ru)c$ to x, with $u \in A_{\inf}$, apply $Q(s)$ and divide by z, using (a) for $\tilde{Y}$. Write $u = w + zv$ with $w \in \Lambda$; s^j sends u to $w + \xi^j z\, s^j v$, fixes the constant c and w, and multiplies z^{k+1} by $\xi^{j(k+1)}$; the terms in w cancel because $Q(1) = 0$. Hence $Q(s)x$ lies over $(Q(\xi^{k+1}) p^t z^{k+1} - rz(\dots))c$, and $Q(s)x = zy$ with y of the standard orders; as z is a unit of $W(C^\flat)$, y is a lift of $(Q(\xi^{k+1}) p^t z^k - r(\dots))c$.

(e) By Lemma 42.5(d) some z^v lies in the ideal of modulus $p^{\kappa+1}$ with $(p-1)v \le e(n + \kappa) < e(p-1)$, so $v < e$. Lower the exponent one step at a time with $Q = q(X-1)$, which kills s on $\tilde{Y}$ by (b): the step from $k+1$ is free by (d) if $Q(\xi^{k+1})$ is a unit; otherwise use Lemma 42.5(a), at the cost $v_p(k+1) + 1 \le m$, as $k + 1 < e \le p^m$. By (c) at most $\deg Q = \deg q + 1$ steps are costly, so $p^t \in J_0^{p^{\kappa+1}}(i)$ with $t \le m(\deg q + 1) \le \kappa$; then p^κ lies there, and Nakayama's lemma removes the modulus.

(f) Take a root ϖ of $X^e - p$ and a primitive eth root of unity $a \in \mathbb{F}_{p^f}$, and let U be the stabiliser of ϖ in the fixer of the $(p^f - 1)$th roots of unity. It is open, and normal: an element n of U fixes ϖ and the root of unity $g^{-1}\varpi/\varpi$, hence $g^{-1}\varpi$.

Its index is prime to p: every g raises the roots of unity of order prime to p to a power p^n, so g^f lies in the fixer as $p^f \equiv 1$; an element v of the fixer multiplies ϖ by the root of unity $v\varpi/\varpi$, which it fixes, so $v^e \in U$; hence the finite group G_p/U has exponent dividing fe, which is prime to p. Take a chart with $\varpi^\flat$ untilting to ϖ; then U is the set of the elements fixing Λ and $\theta(z)$, and its preimage is Γ'. Inertia permutes the towers of eth roots of p transitively, so for every l there is an inertial t with $tz = [\varepsilon]^l z$, which fixes the constants (inertia fixes $W(\mathbb{F}_{p^f})$) and $\theta(z)$, as $\theta([\varepsilon]) = 1$: these are the Kummer elements. An inertial s multiplies z by the Teichmüller lift $[a]$, of order e; its powers multiply z by $[a]^k$ and fix Λ. For $r < p$ take $f = \operatorname{lcm}(1, \dots, r)$, prime to p; then $p^i - 1 \mid p^f - 1$ for $i \le r$, and $\operatorname{lcm}(q^i - 1)_{i \le r} \le \prod_{i \le r} q^i \le q^{r^2}$. As $p \nmid e$, $d = v_p(e) + 1 \le 1$. □

The tools of this subsection give the walks.

Proposition 42.21 (Walks). *Let two paired sides have jumps in opposite windows and killed cokernels.*

(a) (*The rule and the walk.*) *A move is safe if a power of p times a power of E kills the cokernel of the Tate dual side and the orders of the move and the depth of the point at rest add up to the rule bound; standard points at cuts are carried along by Frobenius, and at each cut one side moves to a stop, rests while the other moves, and goes on. This derives the scaled cup coboundary condition.*

(b) (*Cuts.*) *A cut 1 to $2J + 1$ letters back from each letter gives the bound, for two sizes of far windows* (*sparse thick letters, or any pattern*).

(c) (*Long gap.*) *If E^s kills the cokernel at i and p^κ at $i.\mathrm{back}(l)$, the jumps at i and the $l - 1$ letters before it are at most g, $3 \le p^l$, $A \le p^l\delta$ and $N = s + g\sum_{a<l} p^a < A$, then $p^{4\delta}$ times a constant of the side lifts to a constant modulo E^A of the extension whose orders at E are raised by A at the jumps $u < -N$.*

(d) (*Deep points.*) *If every step occurs at E within distance n of 0, b or $-b$, and p divides no nonzero integer within n of these, the depth can be raised at a cost of $p^{(8n+4)d}$; this gives the cup condition when no jump lies in $[-a, 0)$, or the chart is unramified and the jumps of the second side lie in $[-\eta, 0]$.*

(e) (*Few low jumps.*) *On an unramified chart, a side with few jumps in the low window has a submodule of bounded colength of constants with deep points, which satisfy the cup condition with exponent 0.*

(f) (*Tame charts.*) *If a monic q over $\mathbb{Z}_p$ kills the operator induced by an inertia element s with $s(z) = \xi z$, then p^κ kills the cokernel as soon as $m(\deg q + 1) \le \kappa$, $e \le p^m$ and p is large against κ and the jumps; and tame charts of every ramification index $e \mid p^f - 1$ exist when $\Gamma \to G_p$ is onto and $p \nmid f$.*

Proof. (a) The rule is Lemma 42.9(c), and the walk, with its moves at each cut, is Lemma 42.10; at a plain cut, where one side is down and the other up, a deep enough standard point is a stop (Lemma 42.11(d)), and the scaled cup coboundary condition is Lemma 42.10(c).

(b) is Lemma 42.16(b), (c): the cuts are the letters with thin successors, at most $2J + 1$ letters apart (Lemma 42.15(a)), or the letters whose shift is less than the start of their far window, one or two letters apart (Lemma 42.15(d)); the costs are Lemma 42.13(a).

(c) is Lemma 42.17(e), whose proof multiplies a lift by $f(E)$ with f monic of degree N and takes the Nth Taylor coefficient of $p^{4\delta}$ times the product.

(d) is Lemma 42.18(b) for the deepening and Lemma 42.18(f) for the cup condition.

(e) is Lemma 42.19(f), (g).

(f) is Lemma 42.20(e), (f). □

42.3. Cup exponents

Lemma 42.22 (From sides to lattices). *Let $\Gamma \to G_p$ be an open embedding and ζ assign the embeddings of O to the letters; the jump set at the letter i is $\{0\}$ together with the sets $U(g)$ of the embeddings g with $g\zeta = \theta(\omega(i))$, $\omega(i)$ the Teichmüller lift.*

(a) (*The universal side.*) *For L in the class, the lattice L with parameter lattice $Z_f(L)$ (all crystalline cocycles, a finitely generated module as L is potentially crystalline) and these jump sets is a side. If for all Y of a class, all pairings and all crystalline cocycles c of Y the cup product of c and $p^e x$ is a coboundary, then $p^e x$ is orthogonal to the class.*

(b) (*Counting.*) *An embedding lies over at most one letter, so the numbers of embeddings over the letters add up to at most* $\operatorname{rank}_{\mathbb{Z}_p} O$, *which is* $f_O \cdot \operatorname{length}(O/p)$; *the jumps of L in a finite set at one embedding number at most* $\operatorname{rank}_O L$; *and the multiplicity of a jump at a letter is at most its multiplicity at each embedding times the number of embeddings over the letter.*

(c) (*Windows.*) *If the sets $U(g)$ lie in $[-\eta, \eta] \cup \varepsilon[b, b + W]$, so do the jump sets at the letters; for opposite windows, $-(v + v') - 1 \le b + W + \eta$ and $-(v + v') \le b + W + 2\eta$; and $b + n < q$ gives the gap condition for q with radius n around b.*

(d) (*Tame descent.*) *Let $\Gamma \to G_p$ be bijective and d a tame chart. If $X^e - 1$ kills the operators of inertia on the periods of a potentially crystalline Y, then the inertia elements of Γ' act as the identity on the periods, and Y is crystalline over Γ' with the same jumps.*

Proof. (a) Crystalline cocycles of a potentially crystalline lattice form a finitely generated module (Theorem B.39(a)); the side is formed by the identity of $Z_f(L)$. The second claim is the definition of the orthogonal, applied to the universal side and to the universal side of each Y paired with it.

(b) Distinct letters have Teichmüller lifts that differ by a unit, so their images under θ differ, and $g\zeta$ determines the letter. Over $\mathbb{Z}_p$, O/p is a vector space over $\mathbb{F}_p$ of dimension $\operatorname{rank}_{\mathbb{Z}_p} O$, which is its length over O times the residue degree of O. The sum of the Hodge–Tate multiplicities at one embedding is $\operatorname{rank}_O L$. The last claim is the definition of the multiplicity at a letter as a sum over the embeddings above it.

(c) The first claim holds since the jump set at a letter is a union of the $U(g)$ and 0. For the second, v and v' cannot both lie in $-[b, b + W]$, as the windows are opposite, so at most one is below $-\eta$ and neither below $-(b + W)$. The third is immediate.

(d) On inertia, the map $s \mapsto \beta_s$ to the operators on the periods is a homomorphism with open kernel, and $\beta_s^e = 1$; by tame inertia theory it is the identity on the stabiliser of ω in inertia, which contains the inertia elements of Γ'. Restricting Y to Γ' does not change β. A potentially crystalline lattice whose periods are fixed by inertia is crystalline, with the same Hodge–Tate jumps. □

Theorem 42.23 (Cup exponents). *Let $\Gamma \to G_p$ be an open embedding, ζ assign the embeddings of O to the letters of a chart k, and let L be crystalline with jumps in U', where U and U' lie at each letter in $[-\eta,\eta]$ and in opposite far windows.*

(a) (*Generic primes.*) *If k is unramified, $p \geq 5$, $b_i + W + \eta \leq (p-3)a$, $6(a+\eta+W) < b_i$ and p divides no nonzero integer within $6(a+\eta+W)$ of 0 or $\pm b_i$, then the defect of L for the crystalline lattices with jumps in U vanishes if U, U' avoid $[-a,0)$ or $U' \subseteq [-\eta,0]$, and is at most $11f(m(a+1)+\operatorname{rk} L\cdot(W+1))$ times the length of O/p if L has at most m jumps in $[-\eta,\eta]$ at each embedding.*

(b) (*Any pattern.*) *If k is unramified, $p \geq 7$, the far windows of width V start at b at the b-letters and at G at the others, $0 < G \leq b$, $V + 4\eta \leq M$, $b < Kp^2$, $2G < Kp$ and $M \leq K$, then $p^{10^4(K+\lfloor \log_p 2(b+M)\rfloor)}$ times any crystalline cocycle of L is orthogonal under cup products to the crystalline lattices with jumps in U.*

(c) (*Sparse thick letters.*) *If no three b-letters follow one another along Frobenius orbits with both gaps between 1 and J, and either k is unramified with $p \geq 7$ and $10p^J\eta \leq b$, or the sign is constant and $8e \leq p^{J+1}$, then with $2e(b+5\eta) \leq p^{J+1}\delta$ and $20(J+1+d)\eta + \lfloor \log_p 2(b+5\eta)\rfloor \leq M$, the exponent $10^3(\delta+(J+1)M)$ works.*

(d) (*Tame types.*) *If $\Gamma \to G_p$ is onto, each inertia condition $K(s)$ is a tame type (it holds for X^e-1, $e = \operatorname{lcm}(p^i-1)_{i\leq r}$, and for a monic polynomial of degree at most N), and $p > b + 6(W+\eta) + r^2(N+1)^2 + 2$, then $p^{3r^2(N+1)^2+150(\eta+W+1)}$ times any crystalline cocycle of L cups to a coboundary with every crystalline cocycle of a lattice in the class.*

Proof. (a) Let S be the universal side of L (Lemma 42.22(a)). *Vanishing.* All jumps of S, of the universal side of a lattice Y with jumps in U, and of the Tate dual side are at most $b_i + W + \eta \leq (p-3)a$ (Lemma 42.22(c)), so E^a kills the three cokernels (Lemma 42.5(f)); as the chart is unramified, $d = 0$, and Lemma 42.18(f) gives the exponent $H = 0$: every crystalline cocycle of L is orthogonal, so the defect vanishes. The two cases pass from the embeddings to the letters by (c) of Lemma 42.22. *Few low jumps.* Lemma 42.19(f), (g) give a $\mathbb{Z}_p$-submodule V of $Z_f(L)$ that satisfies the cup condition with exponent 0, so lies in the orthogonal, of colength at most $11\sum_i(n_i(a+1)+r_i(W+1))$. By Lemma 42.22(b), $n_i \leq f\,m\,d_i$ and $r_i \leq f \operatorname{rank}_O L\, d_i$ with d_i the number of embeddings over i, and $\sum_i d_i \leq \operatorname{rank}_{\mathbb{Z}_p} O$; passing from lengths over $\mathbb{Z}_p$ to lengths over O divides by the residue degree of O, which gives $11f(m(a+1)+\operatorname{rank}_O L\,(W+1))$ times the length of O/p.

(b), (c) Apply Lemma 42.16(c), with M for W, and Lemma 42.16(b) to the universal sides of L and of the lattices with jumps in U; Lemma 42.22(a) turns the scaled cup coboundaries into orthogonality.

(d) By Lemma 42.20(f) there is a tame chart d with $e = \operatorname{lcm}(p^i-1)_{i\leq r} \leq p^{r^2}$, as $r \leq r^2(N+1)^2 < p$. Since each $K(s)$ allows X^e-1, all lattices of the class and L become crystalline over Γ' with the same jumps (Lemma 42.22(d)), and the jump sets at the letters lie in the windows with $\zeta = 1$. The tame generator is killed on the lattices of the class by a monic q of degree at most N, and on L by a monic q' of degree at most N (from $K(s^{-1})$); on the Tate dual of $Y\otimes L$ it is killed by a monic q'' of degree at most N^2, whose roots are the products of those of q and q'. All jumps of the three sides are at most $b + 6(W+\eta)$ (Lemma 42.22(c)). Lemma 42.20(e) with $m = r^2$ and $\kappa = r^2(N+1)^2 \geq r^2(\deg+1)$ for the three polynomials, and

$b+6(W+\eta)+\kappa+2 < p$, shows that p^κ kills the three cokernels. The gap condition with radius $6(\eta + W)$ around b holds by Lemma 42.22(c), as $b + 6(\eta + W) < p$. Lemma 42.18(f) with $a = 0$, for which no jump lies in $[-a, 0) = \emptyset$, and $d \le 1$, gives the exponent $3(\kappa + (48(\eta + W) + 4)) \le 3r^2(N + 1)^2 + 150(\eta + W + 1)$. □

43. Selmer data and the descent

The denominator bound of Theorem 40.7 asks that the twists by ϑ_ℓ^{-1} of the main summands have short Selmer groups. This section proves that boundedly many auxiliary primes, *tied* so that tame inertia acts like Frobenius times a *slope*, make the tied Selmer groups of finitely many lattices short (Theorem 43.11). The next section turns the ties into a twist.

43.1. Selmer data

Let E be a CM field in Ω, ℓ a prime and O the integers of a finite extension of $\mathbb{Q}_\ell$. A *Selmer datum* for a finite set S_0 of places and an element g_0 is a lattice $T = O^n$ with a Galois action ρ, unramified outside a set $S \subseteq S_0$, a module Z_w of local cocycles at each $w \mid \ell$, and an element γ_* acting by a scalar ζ_* with $\zeta_*^2 - 1$ a unit, such that $\det(\rho(g) - 1) \neq 0$ for $g = g_0$ and $g = g_0^{-c} = (g_0^c)^{-1}$. The *local relations* are the linear relations $\sum_j a_j z(g_j) = 0$, g_j in a decomposition group above ℓ, that hold on the Z_w. A class in $H^1(E, T/a)$ is identified with its cocycle vanishing at γ_*. The *global term* is the supremum over $a \neq 0$ of the lengths of the modules of equivariant maps from T/a to its dual twisted by complex conjugation, the *places term* the sum of the local terms at the places of S prime to ℓ, and β_T is the sum of these and of the valuations of $\det(\rho(g)-1)$ for $g = g_0, g_0^{-c}$.

Write $W = T/a = (O/a)^n$, $G = \mathrm{Gal}(\Omega/E)$, g^c for the conjugate of g by complex conjugation and $g^{-c} = (g^c)^{-1}$; W^* is the character module $\mathrm{Hom}(W, \mathbb{Q}/\mathbb{Z})$ with g acting by $y \mapsto \mathrm{cyc}(g)\, y \circ g^{-1}$, and W^{*c} is W^* with g acting as g^c. A Selmer datum also asks that T/ϖ be absolutely irreducible in Burnside's form (the matrices $\rho(g)$ span all matrices), that γ_* be inertial at a place of S prime to ℓ, and that $\sum_{w|\ell}(\mathrm{rank}\, Z_w - n) \le fn$, f the number of infinite places. The group H is generated, as an abstract group, by the commutators $[g, \gamma_*]$ and $[g, \gamma_*^c]$, $g \in G$.

Lemma 43.1 (Classes on the commutator group). *Let T be a Selmer datum and $a \neq 0$.*

(a) *Coboundaries obey the local relations, and at each $w \mid \ell$ the restrictions of the cocycles obeying them have length at most* $\mathrm{rank}\, Z_w \cdot \mathrm{length}(O/a)$.
(b) *H is normal and stable under $g \mapsto g^c$; homomorphisms to commutative groups and ρ are trivial on H; and H lies in the normal subgroup generated by the inertia groups above any set containing the places of S prime to ℓ and their conjugates.*
(c) (*Sah's identity.*) *A cocycle z with $z(\gamma_*) = 0$ has $z([g, \gamma_*]) = (1 - \zeta_*)z(g)$, and $z(ghg^{-1}) = gz(h)$ when h acts as the identity. A cocycle that vanishes at γ_* and on H vanishes.*
(d) *If a submodule of W is stable under a set of matrices spanning all matrices and contains a vector with a unit coordinate, it is W.*

(e) *If g acts by a scalar α and g^{-c} by α', then $\Delta = \alpha - \mathrm{cyc}(g^c)\alpha'$ kills $\mathrm{Hom}_G(W, W^{*c})$, and the Δ-torsion of W^* is no longer than that of W, which has length at most $n\, v(\Delta)$.*

Proof. (a) The coboundary of $b \in W$ lifts to the coboundary of a lift of b in O^n, which lies in each Z_w, so it obeys every relation that holds on Z_w. For the bound, fix a coordinate i and let $\phi(f)$ be the linear form $z \mapsto (\sum_g f(g)z(g))_i$ on Z_w, for finitely supported families f of matrices. A function c obeying the relations of Z_w gives the map $f \mapsto (\sum_g f(g)c(g))_i$ to O/a, which kills $\ker\phi$ (row i of a family in $\ker\phi$, the other rows set to zero, is a relation on Z_w) and determines c (take for f the families with one nonzero entry). Hence the module embeds into the O-linear maps from $\mathrm{im}\,\phi$ to O/a, and $\mathrm{im}\,\phi$ lies in the dual of Z_w, free of rank at most $\mathrm{rank}\, Z_w$.

(b) $g'[g,\delta]g'^{-1} = [g'g,\delta][g',\delta]^{-1}$ shows normality, and $[g,\gamma_*]^c = [g^c,\gamma_*^c]$ stability under conjugation. A homomorphism to a commutative group kills commutators. As γ_* and γ_*^c act by scalars, ρ kills the commutators. Finally γ_* and γ_*^c lie in inertia groups above such a set, hence in its normal closure, and so does every commutator with them.

(c) By the cocycle rule, $z(\gamma_*^{-1}) = -\gamma_*^{-1}z(\gamma_*) = 0$ and $z(g^{-1}) = -g^{-1}z(g)$, so $z(\gamma_* g^{-1}\gamma_*^{-1}) = \zeta_* z(g^{-1})$ and $z(g\gamma_* g^{-1}\gamma_*^{-1}) = z(g) - \zeta_* z(g)$. If h acts as the identity, $z(ghg^{-1}) = z(g) + gz(h) - ghg^{-1}z(g) = gz(h)$. If z vanishes on H, then $(1-\zeta_*)z(g) = 0$, and $1-\zeta_*$ is a unit as $\zeta_*^2 - 1$ is.

(d) If u has a unit coordinate u_j, the matrix whose only nonzero column is the jth, a lift of m/u_j, takes u to m; such matrices are combinations of the given ones.

(e) An equivariant η satisfies $\eta(gu) = g\cdot\eta(u)$; as g acts on W by α and on W^{*c} by $\mathrm{cyc}(g^c)\alpha'$ (the action of g^c on W^* is by $\mathrm{cyc}(g^c)$ times the transpose inverse of $\rho(g^c)$, which is α'), $\Delta\eta = 0$. A character killed by Δ is a character of $W/\Delta W$, which has as many elements as $W[\Delta]$; and $W[\Delta] = \prod_j (O/a)[\Delta]$ has length at most $n\, v(\Delta)$. □

Lemma 43.2 (Sah). *Let H be the group generated by the commutators $[g,\gamma_*]$ and $[g,\gamma_*^c]$. If a cocycle vanishing at γ_* has a value with a unit coordinate, its values on H span T/a. If g acts by a scalar α, g^{-c} by α' and the image of ρ spans all matrices, the global term is at most $n\, v(\alpha - \mathrm{cyc}(g^c)\alpha')$.*

Proof. For the first claim, let $z(g)$ have a unit coordinate. By Lemma 43.1(c), $z([g,\gamma_*]) = (1-\zeta_*)z(g)$ has a unit coordinate too, and $[g,\gamma_*] \in H$. The span of $z(H)$ is stable under G, since $gz(h) = z(ghg^{-1})$ for $h \in H$, which acts as the identity, and H is normal (Lemma 43.1(b)); by Lemma 43.1(d) it is T/a. For the second claim, by Lemma 43.1(e) $\Delta = \alpha - \mathrm{cyc}(g^c)\alpha'$ kills every equivariant map $\eta\colon W \to W^{*c}$. Evaluation at a basis vector e is injective on these maps: a map vanishing at e vanishes at its translates, which span W because the image of ρ spans all matrices. Hence $\mathrm{Hom}_G(W, W^{*c})$ embeds into the Δ-torsion of W^*, of length at most $n\, v(\Delta)$; take the supremum over a. □

43.2. Auxiliary primes and ties

Auxiliary prime data at a prime $P \neq \bar{P}$ of norm $\equiv 1$ modulo q, outside a set T, consist of a place of Ω above P, a Frobenius element φ, the chosen tame generator γ, and a character to $\mathbb{Z}/q$ that comes from an idele class character of level P trivial at T and takes the value 1 at γ. A set J of *tied primes* consists of a first prime P_0, whose Frobenius element lies in a given set W, and later primes, all with auxiliary prime data for $q = \ell^N$ and, for each datum i, a *slope* $s_v^{(i)} \in O$, with $s_{P_0} = 0$, such that at the primes of nonzero slope for i the Frobenius element and its conjugate inverse satisfy a condition V_i (typically: they act as the identity on T_i/ℓ^M). The *ties* of datum i are $z(\gamma_v) = s_v z(\varphi_v)$ and $z(\gamma_v^{-c}) = s_v z(\varphi_v^{-c})$, $v \in J$. The *tied Selmer group* $\mathrm{Sel}_a(J)$ modulo a consists of the cocycles with values in T/a, vanishing at γ_*, unramified outside $S \cup J \cup \bar{J}$, obeying the local relations and the ties. A prime P with data is *new auxiliary* for datum i if it may be added to J with any slope for i.

A prime of J is in particular *auxiliary* for S_0: of prime norm, with neither it nor its conjugate in S_0, and not conjugate to another prime of J. A tied set also asks that for $v \neq P_0$ the character of P_0 take the value 1 at φ_v and the characters of the other primes the value 0, and that no prime of S_0 or of J divide the norm of P_0. For characters h_j to $\mathbb{Z}/q$ and a point x the *twist* by $\psi_x(g) = \prod_j (1 + x_j)^{h_j(g)}$ is the action $g \mapsto \psi_x(g)\rho(g)$ (Section 44.1); $\mathrm{Sel}(x)$ is its Selmer module.

Lemma 43.3 (Characters at auxiliary primes). *Let E be a CM field and $q = \ell^N$.*

(a) (*Discrete logarithm.*) *If $\mathrm{N}\,P = 1 + qk$ is prime and the residue of a local unit u generates the residue units, some character of $E_P^\times$ to $\mathbb{Z}/q$ sends u to 1 and kills every t with $t \equiv 1$ or $t^k \equiv 1$ modulo P.*

(b) *If a Frobenius element σ at a place above $P \notin T$, $\mathrm{N}\,P = 1 + nk$, fixes an nth root of a T-unit u, then $u^k \equiv 1$ modulo P.*

(c) *If the T-ideles and the principal ideles generate the finite ideles, a continuous character of $E_P^\times$ killing the T-units extends to a character of the idele classes that is, on the T-ideles, the character of the component at P.*

(d) (*Anticyclotomic characters.*) *Under the hypothesis of (c), if $P \notin T$, $P \neq \bar{P}$, $\mathrm{N}\,P = 1 + qk$ is prime and a Frobenius σ at a place A over P fixes qth roots of generators of the T-units, there is a continuous character $\chi\colon G \to \mathbb{Z}/q$ with value 1 at the tame generator at A and $\chi(\mathrm{Art}(x)) = \theta(x/\bar{x})$ for a character θ of the idele classes trivial on the T-ideles congruent to 1 at P.*

(e) *If q is a unit off T and the places of R lie off T, the normal closure of the inertia groups above R fixes the qth roots of unity and qth roots of T-units.*

Proof. (a) Divide out the power of a uniformiser and take the residue; the character of the residue units, cyclic of order qk, that sends the generator to 1 kills the kth powers' preimages of 1: if the residue of t is g^e with $g^{ek} = 1$, the order qk of g divides ek, so $q \mid e$.

(b) With $y^n = u$, $y = \sigma y \equiv y^{\mathrm{N}\,P} = y\,u^k$ modulo the place, and y is a unit there.

(c) The T-ideles map onto the idele classes with kernel the T-units, so the character of the component at P, which kills the T-units, descends; it is continuous as it agrees on the open group of T-ideles with a continuous character.

(d) By (b) the discrete logarithm of (a), for a generator u_0 of the residue units, kills the T-units, and by (c) it extends to an idele class character θ; its kernel

is open, containing all powers with exponent its finite index. The existence theorem of class field theory (Theorem B.35(c)) turns $x \mapsto \theta(x/\bar{x})$ into a Galois character χ. The tame generator at A is the Artin symbol of u_0 at P, whose conjugate has component 1 at P as $P \neq \bar{P}$, so χ takes the value $\theta(u_0) = 1$ there.

(e) The inertia groups at places off T, and their conjugates, fix the roots of unity of order q (prime to these places) and the qth roots of the T-units, which are units there; the fixer of these finitely many roots is a normal open subgroup. □

Lemma 43.4 (Ties and the line). *Let J be tied primes for S_0, i a datum, and let ρ modulo ℓ^M be trivial on inertia outside S_0 and at the elements with V_i.*

(a) (*Affine actions.*) *For a cocycle c, $g \mapsto (u \mapsto c(g) + gu)$ is a homomorphism to the affine group. If g acts by a scalar α with $\sum_{i<k} \alpha^i = 0$, the kth power of its affine map is the identity. If the affine maps of φ and γ commute, then $(\varphi - 1)c(\gamma) = (\gamma - 1)c(\varphi)$, and the relation passes from φ to φ^{-1} if γ acts by a scalar.*

(b) (*Twists.*) *If y divides all coordinates of x, then $y \mid \psi_x(g) - 1$. If the $1 + x_j$ are qth roots of unity modulo ℓ^M, the twisted matrices act on $(O'/\ell^M)^n$ through a representation R, and $R(g)$ is the scalar $\psi_x(g)$ wherever $\rho(g) \equiv 1$.*

(c) (*The relation at a point of the line.*) *Let $x_{P_0} = y$, $x_v = s_v y$ for $v \neq P_0$, and let $\sum_{j<q} \alpha^j \equiv 0$ modulo ℓ^M whenever $\alpha \equiv 1$ modulo y. For $z \in \mathrm{Sel}(x)$ and w a prime $v \in J$ or its conjugate, the matrix $(1+y)\rho(\varphi_w) - 1$ kills $z(\gamma_w) - s_v z(\varphi_w)$; for $v = P_0$ this needs $y = 0$.*

(d) *The quotient of $\mathrm{Sel}(x)$ by its tied part on $I \subseteq J$ (with $P_0 \in I$ only if $y = 0$) has length at most $\sum_w \operatorname{length} \ker((1+y)\rho(\varphi_w) - 1)$, hence at most $v(P^1(y))$ for the* tie polynomial $P^1(Y) = \prod_w \det((1+Y)\rho(\varphi_w) - 1)$ *over $w \in (J \setminus P_0) \cup \overline{(J \setminus P_0)}$.*

(e) (*Release of the first prime.*) *If $M \leq N$ and φ_{P_0}, $\varphi_{P_0}^{-c}$ act modulo ℓ^M as g_0, g_0^{-c}, adding the relations $z(\gamma_{P_0}) = 0 = z(\gamma_{P_0}^{-c})$ to the Selmer module of ρ modulo ℓ^M lowers its length by at most $\sum_{g=g_0, g_0^{-c}} v(\det(\rho(g) - 1))$.*

Proof. (a) The homomorphism property is the cocycle rule. If g acts by α, the kth iterate is $u \mapsto \alpha^k u + \sum_{i<k} \alpha^i c(g)$, and $\sum_{i<k} \alpha^i = 0$ gives $\alpha^k = 1$ as $(\alpha - 1)\sum \alpha^i = \alpha^k - 1$. Commuting affine maps give $c(\gamma\varphi) = c(\varphi\gamma)$, which by the cocycle rule is $c(\gamma) + \gamma c(\varphi) = c(\varphi) + \varphi c(\gamma)$. For φ^{-1}, multiply by $-\varphi^{-1}$ and use $c(\varphi^{-1}) = -\varphi^{-1}c(\varphi)$ and that γ commutes with φ^{-1} on T/a.

(b) Each factor $(1 + x_j)^h - 1$ is divisible by x_j, hence by y, and a product of elements $\equiv 1$ modulo y is $\equiv 1$. Modulo ℓ^M, ψ_x is a character χ (the $1 + x_j$ being qth roots of unity and h_j having values in $\mathbb{Z}/q$), and $R = \chi\rho$.

(c) Modulo ℓ^M the twist is the representation R of (b). Inertia above S_0 acts through R by scalars $\equiv 1$ modulo y, and as the identity near 1; so each inertial element acts by α with $\sum_{j<q} \alpha^j \equiv 0$, and its affine map has order dividing q by (a). As q is prime to the place v (its norm is $\equiv 1$ modulo q) and the cocycle is locally constant, the affine maps of inertia at v form a finite group of exponent dividing q on which wild inertia acts as the identity; conjugation by φ_v is the $\mathrm{N}v$th power on tame inertia, and $\mathrm{N}v \equiv 1$ modulo q, so the affine maps of φ_v and γ_v commute, and (a) gives $(R(\varphi_w) - 1)z(\gamma_w) = (R(\gamma_w) - 1)z(\varphi_w)$ at v and, by the same argument at the conjugate place, at $\bar{v}$. Moreover $\lambda_u(\gamma_v) = [u = v]$ and, for $v \neq P_0$, $\lambda_u(\varphi_v) = [u = P_0]$, so $\psi_x(\gamma_v) = 1 + s_v y$ and $\psi_x(\varphi_v) = 1 + y$: the

relation reads $((1+y)\rho(\varphi_w)-1)z(\gamma_w) = s_v y z(\varphi_w)$. If $s_v \neq 0$, V_i gives $\rho(\varphi_w) \equiv 1$ modulo ℓ^M, so $s_v((1+y)\rho(\varphi_w)-1)z(\varphi_w) = s_v y z(\varphi_w)$ as well, and the difference is the claim; if $s_v = 0$ the claim is the relation itself. At P_0 with $y = 0$ the point x is 0 and both values of ψ are 1.

(d) The map $z \mapsto (z(\gamma_w) - s_v z(\varphi_w))_w$ has the tied part as kernel and, by (c), its image in $\bigoplus_w \ker((1+y)\rho(\varphi_w)-1)$. The kernel of a matrix A on $(O'/\ell^M)^n$ has length at most $v(\det A)$ (Smith normal form), and the determinant of $(1+y)\rho(\varphi_w)-1$ is the value at y of $\det((1+Y)\rho(\varphi_w)-1)$, a polynomial with leading coefficient $\det \rho(\varphi_w)$.

(e) At $y = 0$ the point x vanishes and Sel(0) is the Selmer module of ρ; by (c) the two tie maps take values in the kernels of $\rho(\varphi_{P_0})-1$ and $\rho(\varphi_{P_0}^{-c})-1$, which act modulo ℓ^M as $g_0 - 1$ and $g_0^{-c} - 1$, and (d) concludes. □

Proposition 43.5 (Auxiliary primes). (a) (*Chebotarev.*) *For finite sets S, R and a neighbourhood W of an element of the normal closure of the inertia groups at the places of R outside $S \cup \bar{S}$ and not above ℓ, there is a prime $P \notin R \cup \bar{R}$ of prime norm with auxiliary prime data whose Frobenius element lies in W and is killed by the character.*

(b) *The character of $w \in J$ takes the value 1 at the tame generator of v if $w = v$ and 0 otherwise.*

(c) (*The relation from tame inertia.*) *If φ is a Frobenius element of norm $\equiv 1$ modulo q and γ inertial, each inertial element acting by a scalar α with $\sum_{i<q} \alpha^i = 0$, then every cocycle has $(\varphi - 1)c(\gamma) = (\gamma - 1)c(\varphi)$.*

(d) *Slope 0 means unramified: if all slopes of datum i vanish, the tied Selmer group lies in the Selmer group without auxiliary primes; a cocycle that vanishes at γ, γ^{-c} of a new auxiliary prime is unramified there.*

Proof. (a) Take a finite set T of places containing S, $\bar{S}$ and the places above ℓ, but no other place of R, whose ideles together with the principal ideles generate the finite ideles. The T-units are finitely generated; let U be the open subgroup fixing the qth roots of unity and qth roots of a set of generators. By Lemma 43.3(e), U contains the given element z, as the relevant inertia groups sit at places off T. Chebotarev's theorem (Theorem B.35(d)) gives $P \neq \bar{P}$ off $T \cup R \cup \bar{R}$ of prime norm and a Frobenius σ at a place A over P such that every Frobenius at A lies in U and has W as a neighbourhood. As σ fixes the roots of unity of order q, $\mathrm{N}P \equiv 1$ modulo q; Lemma 43.3(d) gives the character χ, with value 1 at the tame generator; multiplying σ by a power of the tame generator gives a Frobenius $\tau \in W$ with $\chi(\tau) = 0$. An idele that is a unit off a set $T' \subseteq T$ closed under conjugation and $\equiv 1$ at P is killed by θ, which gives the auxiliary prime data for every $T' \subseteq S \cup \{v \mid \ell\}$.

(b) The character of w takes the value 1 at its own tame generator by construction. For $v \neq w$ in J, v is neither w nor $\bar{w}$, and the character of w comes from an idele class character of level w, unramified outside w, $\bar{w}$; so it kills the inertia at v, in particular the tame generator.

(c) The affine maps of inertia at the place form a finite group: the cocycle is locally constant and ρ is trivial on inertia near 1. Each has qth power 1 by Lemma 43.4(a), as it acts by a scalar α with $\sum_{i<q} \alpha^i = 0$; q is prime to the place, so wild inertia acts as the identity and the group is a quotient of tame inertia,

on which φ acts by the Nth power, $N \equiv 1$ modulo q. Hence the affine maps of φ and γ commute, and Lemma 43.4(a) gives the relation, also for φ^{-1}.

(d) At a prime $v \in J$ outside S_0, inertia acts as the identity on T/a, so a cocycle restricts on inertia to a homomorphism; it factors through tame inertia, as wild inertia is pro-p with $p \neq \ell$ and the values are ℓ-power torsion, and the tame quotient is topologically generated by γ_v. If all slopes of datum i vanish, the ties read $z(\gamma_v) = 0 = z(\gamma_{\bar{v}}^{-c})$, so z vanishes on inertia at the chosen places above v and $\bar{v}$, and at the other places above them by $z(g\tau g^{-1}) = gz(\tau)$ (Lemma 43.1(c)): z is unramified at $J \cup \bar{J}$. The same argument applies at a new auxiliary prime. □

43.3. The count

Let U be the kernel of the action on T/ℓ^M and on the roots of unity of order ℓ^N, and $U \cap U^c$ the set of the g with $g, g^c \in U$; for $a \mid \ell^M$, W is a discrete Galois module, its orbit maps factoring through T/ℓ^M. The *tied setting* is the global duality setting of Theorem B.37 for W, the primes $S' = S \cup I \cup \bar{I}$ with $I = J \setminus \{P_0\}$, chosen places above them, the kernel U, and at each $v \in S'$ the local cocycles that are restrictions of cocycles obeying the local relations and the ties. Its dual $\mathrm{Sel}_a^*(J)$, for a finite set R, is the module of the equivariant homomorphisms $U \to W^*$, extended by 0, unramified outside $R \setminus \{P_0, \bar{P}_0\}$ and obeying reciprocity at the split primes outside it.

Lemma 43.6 (The tied setting). *Let J be tied primes with V_i the triviality on T/ℓ^M, $M < N$ and $a \mid \ell^M$.*

(a) *U contains H and the inertia groups at the primes outside S; coboundaries obey the ties and lie in the Selmer module; tied classes vanish at γ_{P_0}, $\gamma_{P_0}^{-c}$ and are unramified at P_0, $\bar{P}_0$.*

(b) *At a prime $v \in J$ outside S, or its conjugate, the local cocycles obeying the local relations and the ties are determined by their value at φ_v (or φ_v^{-c}), so form a module no longer than W.*

(c) *The local modules above ℓ have total length at most $(\#\{w \mid \ell\} + f)\,\mathrm{length}\, W$, those at the places of S at most $(\#S + f)\,\mathrm{length}\, W$ plus the places term, and those at all of S' at most $(\#S' + f)\,\mathrm{length}\, W$ plus the places term.*

(d) $\mathrm{length}\, W + \mathrm{length}\, \mathrm{Sel}_a(J) \leq \mathrm{length}\, Z(\emptyset)$, *where $Z(\emptyset)$ is the module of cocycles unramified outside S' with the local conditions of the setting.*

(e) (*Greenberg–Wiles.*) *For a suitable finite set R,* $\mathrm{length}\, Z(\emptyset) + (\#S' + f)\,\mathrm{length}\, W \leq \mathrm{length}\, \mathrm{Sel}_a^*(J) + \mathrm{length}\, W + \sum_{v \in S'} \mathrm{length}\, L_v$, *$L_v$ the local modules.*

Proof. (a) H acts as the identity through ρ and fixes the roots of unity, as the cyclotomic character is a homomorphism to a commutative group (Lemma 43.1(b)); inertia at primes off S acts as the identity on T, and fixes the roots of unity of order ℓ^N when the prime is prime to ℓ. The tame generators and, where the slope is not zero, the Frobenius elements fix W, so the coboundary of b has $(\gamma_v - 1)b = 0 = s_v(\varphi_v - 1)b$; with Lemma 43.1(a) it lies in the Selmer module. As $s_{P_0} = 0$, the ties at P_0 read $z(\gamma_{P_0}) = 0 = z(\gamma_{P_0}^{-c})$, and Proposition 43.5(d) makes z unramified there; so P_0 can be removed from the allowed ramification.

(b) A tied cocycle that vanishes at the Frobenius element vanishes at the tame generator by the tie, hence on inertia (Proposition 43.5(d)) and on the decomposition group, which Frobenius and inertia generate topologically.

(c) By Lemma 43.1(a) the module at $w \mid \ell$ has length at most $\operatorname{rank} Z_w \cdot \operatorname{length}(O/a)$, and $\sum_w \operatorname{rank} Z_w \le (\#\{w \mid \ell\} + f)n$. At a place of S prime to ℓ all classes are allowed, and the local term bounds the excess over length W. On S the ties only shrink the modules; above $I \cup \bar{I}$ use (b).

(d) Map $W \times \mathrm{Sel}_a(J) \to Z(\emptyset)$ by $(b, z) \mapsto z + \partial b$. It lands in $Z(\emptyset)$ by (a), and it is injective: $z(\gamma_*) = 0$ and $(\partial b)(\gamma_*) = (\zeta_* - 1)b$, with $\zeta_* - 1$ a unit.

(e) This is Theorem B.37(a) for the tied setting. Its hypothesis holds: γ_* fixes the roots of unity (it is inertial at a place prime to ℓ), is central modulo U (it acts on T/ℓ^M by a scalar), and fixes no nonzero vector, as $\zeta_* - 1$ is a unit. The norm of P_0 can be avoided by the auxiliary primes of the count: it is prime to ℓ^N, no prime of S' divides it, and the inertia groups above its prime divisors lie in U. Writing both sides as orders of modules and passing to lengths gives the inequality. □

Lemma 43.7 (Reciprocity at conjugate primes). *In the situation of Lemma 43.6, let γ_0 be the tame generator at P_0.*

(a) *$U \cap U^c$ is open and contains H, γ_0 and the inertia groups at the primes outside S_0; for $h \in H$, $\gamma_0 h$ lies in the normal closure of the inertia groups at the places of a set $R \supseteq \{P_0\} \cup S_0 \cup \bar{S}_0$ prime to ℓ and not among the later primes of J or their conjugates.*

(b) *For $h \in H$, a finite set R and a neighbourhood X of $\gamma_0 h$, there is a new auxiliary prime $P \notin R \cup \bar{R}$ for datum i whose Frobenius element lies in X.*

(c) *The conjugate inverse δ^{-c} of a normalised tame generator δ at B is a normalised tame generator at $\bar{B}$, and the values of a cocycle at an inertial τ, τ^{-c} are the same multiple of its values at δ, δ^{-c}.*

(d) *A member y of the dual is equivariant on U, vanishes at γ_0 and γ_0^c, and vanishes if it vanishes on H.*

(e) (*Reciprocity.*) *If P is a new auxiliary prime with $P, \bar{P} \notin R$ and $\varphi, \varphi^c \in U$, every z in the tied Selmer group with ramification allowed also at P, $\bar{P}$ has $y(\varphi)(tz(\gamma)) + y(\varphi^c)(tz(\gamma^{-c})) = 0$ for $t \in O$.*

Proof. (a) U is open and conjugation is continuous; $H \subseteq U$ by Lemma 43.6(a) and $H^c = H$; inertia at a prime outside S_0, and its conjugate, act as the identity. The element γ_0 is inertial at P_0, and h is a product of commutators with γ_*, γ_*^c, which lie in inertia groups at places of $S_0 \cup \bar{S}_0$ prime to ℓ (Lemma 43.1(b)).

(b) Apply Proposition 43.5(a) with the set R enlarged by the primes dividing the norm of P_0, the later primes of J and their conjugates, and with a neighbourhood of $\gamma_0 h$ inside $X \cap U \cap U^c$ on which the characters of the primes of J are constant. Characters are trivial on H, so their values there are those at γ_0: 1 for P_0 and 0 for the others, as required of a new prime; and V_i holds at φ and φ^{-c}, which lie in $U \cap U^c$.

(c) Conjugate the uniformiser and the root that δ moves; δ^{-1} moves it back, and conjugation inverts the root of unity. Inertia acts as the identity on W, so the cocycle is a homomorphism on inertia, factoring through the procyclic tame quotient: $\tau = \delta^m$ there, and $\tau^{-c} = (\delta^{-c})^m$.

(d) On a module killed by ℓ^N the cyclotomic character acts through its reduction, so equivariance is $y(gug^{-1}) = \mathrm{cyc}(g)\, y(u) \circ g^{-1}$. The member is unramified at P_0, $\bar{P}_0$, so it vanishes at γ_0 and γ_0^c. If it vanishes on H, then for $u \in U$,

$[u,\gamma_*]\in H$ gives $0=y(u)-y(u)\circ\zeta_*^{-1}$, so $y(u)$ kills the image of $1-\zeta_*^{-1}$, which is all of W; and y vanishes off U.

(e) Reciprocity at the split prime P asks normalised generators at P and $\bar P$; by (c), with δ^{-c} normalised at the conjugate place, pass to a multiple of z for which the values at γ, γ^{-c} are those at δ, δ^{-c}. The relation is then the reciprocity law that y obeys at P, where z may ramify and y is unramified. □

Theorem 43.8 (Greenberg–Wiles count and reciprocity). *Let J be tied primes with V_i the triviality on T/ℓ^M, $M<N$ and $a\mid\ell^M$.*

(a) *The tied Selmer group is no longer than its dual $\mathrm{Sel}_a^*(J)$, a module of relations at split primes unramified at P_0, $\bar P_0$, up to the places term.*

(b) *If some member z of the tied Selmer group has a value with a unit coordinate and, at every new auxiliary prime P with Frobenius φ and tame generator γ, the values of z at φ, φ^{-c} are the values at γ, γ^{-c} of a tied class that may ramify at P, $\bar P$, then the dual group has length at most the global term.*

Proof. (a) By Lemma 43.6(d), (e) and (c), with $\#S'$ the number of primes of the setting,

$$\begin{aligned}\text{length } W+\text{length Sel}_a(J)&\le\text{length } Z(\emptyset)\\&\le\text{length Sel}_a^*(J)+\text{length } W+\textstyle\sum_{v\in S'}\text{length } L_v-(\#S'+f)\,\text{length } W\\&\le\text{length Sel}_a^*(J)+\text{length } W+\text{places term},\end{aligned}$$

and the finite length of W cancels. The first prime was removed from the allowed ramification by Lemma 43.6(a).

(b) Let $y\in\mathrm{Sel}_a^*(J)$ and put $\Psi_t(h,h')=y(h')(tz(h))-y(h^c)(tz(h'^c))$ on $H\times H$. *First step:* $\Psi_t(h,h)=0$. As z and y vanish at γ_0, γ_0^c and γ_0 acts as the identity, Ψ_t has the same value at (h,h) and at (g,g), $g=\gamma_0 h$. By Lemma 43.7(b) take a new auxiliary prime whose Frobenius φ is so near g that $\Psi_t(\varphi,\varphi)=\Psi_t(g,g)$, by local constancy. By hypothesis some z' has $z'(\gamma)=z(\varphi)$ and $z'(\gamma^{-c})=z(\varphi^{-c})=-z(\varphi^c)$, as $\varphi^c\in U$ acts as the identity; reciprocity (Lemma 43.7(e)) for z' is $y(\varphi)(tz(\varphi))-y(\varphi^c)(tz(\varphi^c))=0$, that is $\Psi_t(\varphi,\varphi)=0$. *Second step:* $\Psi=0$. Ψ_t is biadditive on H (both z and y are homomorphisms on $H\subseteq U$), so it is alternating: $\Psi_t(h,h')+\Psi_t(h',h)=0$. Conjugating the first argument by γ_* multiplies t by ζ_*, and conjugating the second divides it by ζ_* (Lemma 43.1(c), Lemma 43.7(d)); subtracting the alternation at (h,h') with the parameter $\zeta_*^{-1}t$ from that at $(\gamma_*h\gamma_*^{-1},h')$ gives $\Psi_{(\zeta_*-\zeta_*^{-1})t}(h,h')=0$ for all t, and $\zeta_*-\zeta_*^{-1}=\zeta_*^{-1}(\zeta_*^2-1)$ is a unit. *Third step.* By Lemma 43.2 $z(H)$ spans W. If $\sum_h\alpha_h z(h)=0$, then $\Psi=0$ shows that $\sum_h\alpha_h y(h^c)$ kills all $tz(h'^c)$, $h'\in H$, which span; so $\eta(z(h))=y(h^c)$ defines a linear map $\eta\colon W\to W^*$. It is equivariant for the twisted action, by the equivariance of y and $gz(h)=z(ghg^{-1})$, so $\eta\in\mathrm{Hom}_G(W,W^{*c})$. If $\eta=0$, then y vanishes on $H^c=H$, hence vanishes (Lemma 43.7(d)). Hence $y\mapsto\eta$ is injective, and the dual group has length at most the global term. □

43.4. The descent

For a uniformiser ϖ with $\varpi^e \mid \ell^M$, the *total tied length* is the sum over the data and the levels ϖ^k, $1 \le k \le e$, of the lengths of the tied Selmer groups.

Lemma 43.9 (Tools of the descent). (a) (*Tilting.*) *Let* $\mathrm{ev}_\gamma, \mathrm{ev}_\varphi$ *be linear maps from a module to* $(O/I)^m$ *of finite length, and* C *a submodule of the source. For* u *outside at most* m *residue classes, the kernel of* $u\,\mathrm{ev}_\gamma - \mathrm{ev}_\varphi$ *on* C *is no longer than the part of* $\ker \mathrm{ev}_\gamma|_C$ *that* ev_φ *maps into* $\mathrm{ev}_\gamma C$.

(b) (*The exponent.*) *For a uniformiser* ϖ *and* $k \ge 1$, *the Selmer module modulo* ϖ^k *is, for some* $i \le k$, *no longer than the module modulo* ϖ^i, *and if* $i \ne 0$ *the latter has a member with a value with a unit coordinate.*

(c) (*Room for slopes.*) *In a local ring whose residue field has* q *elements, fewer than* $q-1$ *residues miss the residue of some unit.*

(d) (*The first prime.*) *For* g_0 *inertial at a place of* S_0 *prime to* ℓ *and a neighbourhood* W *of* g_0, *there is a prime* P_0 *whose norm no prime of* S_0 *divides such that* $\{P_0\}$ *with all slopes* 0 *is a tied set.*

(e) (*Adding a prime.*) *Tied Selmer groups are finite. If* P *is a new auxiliary prime for datum* i, *a cocycle with the ties of* J *that may ramify also at* P, $\bar{P}$ *and vanishes at* γ, γ^{-c} *lies in* $\mathrm{Sel}_a(J)$; *a member of* $\mathrm{Sel}_a(J \cup \{P\})$ *lies in* $\mathrm{Sel}_a^P(J)$ *and satisfies* $\mathrm{ev}_\gamma z = s\,\mathrm{ev}_\varphi z$; *and for units* u *outside at most* $2n$ *residue classes, adding* P *with slope* u^{-1} *gives a group no longer than the part of* $\mathrm{Sel}_a(J)$ *whose values at* φ, φ^{-c} *are the values at* γ, γ^{-c} *of a member of* $\mathrm{Sel}_a^P(J)$.

Proof. (a) Let C' be the sum of all the kernels of $u\,\mathrm{ev}_\gamma - \mathrm{ev}_\varphi$ in C, together with the part of $\ker \mathrm{ev}_\gamma$ in question; then $\mathrm{ev}_\varphi C' \subseteq \mathrm{ev}_\gamma C'$. Write $\mathrm{ev}_\gamma C'$ as the image of a map π on a free module Λ of rank at most m (the preimage of $\mathrm{ev}_\gamma C'$ in O^m), choose $\sigma\colon \Lambda \to C'$ with $\mathrm{ev}_\gamma \circ \sigma = \pi$ and an endomorphism U of Λ with $\pi \circ U = \mathrm{ev}_\varphi \circ \sigma$. Off the at most m residual roots of the characteristic polynomial of U, $\det(u - U)$ is a unit, and $(u\,\mathrm{ev}_\gamma - \mathrm{ev}_\varphi) \circ \sigma = \pi \circ (u - U)$ is onto $\mathrm{ev}_\gamma C'$. Hence $u\,\mathrm{ev}_\gamma - \mathrm{ev}_\varphi$ has no smaller image on C' than ev_γ; every map splits the length of C' into image and kernel, and the image has finite length, so the kernel is no longer.

(b) Let ϖ^i be the exponent of the module modulo ϖ^k. Multiplication by ϖ^{k-i} identifies O/ϖ^i with the ϖ^i-torsion of O/ϖ^k, so the module modulo ϖ^i is the ϖ^i-torsion of the module modulo ϖ^k, that is, all of it. A member not killed by ϖ^{i-1} has a value with a coordinate not killed by ϖ^{i-1}, which is a unit of O/ϖ^i.

(c) The units have $q-1$ residues.

(d) Apply Proposition 43.5(a) avoiding the primes that divide the norm of a prime of S_0, which include S_0. If $v \in S_0$ divided the prime number $\mathrm{N}\,P_0$, then $\mathrm{N}\,P_0 \mid \mathrm{N}\,v$ and P_0 would divide $\mathrm{N}\,v$. The conditions on later primes are empty.

(e) Tied Selmer groups are modules of maps on a finite quotient of G with values in the finite W. The second claim is Proposition 43.5(d) at P: adding P with slope 0 gives a tied set whose only new prime is P. The third is the definition of the ties at P with slope s. For the last, the tie with slope u^{-1} says $u\,\mathrm{ev}_\gamma = \mathrm{ev}_\varphi$ on $C = \mathrm{Sel}_a^P(J)$; by (a) with $m = 2n$ (values at γ and γ^{-c}), off at most $2n$ residues the kernel is no longer than the part of $\ker \mathrm{ev}_\gamma|_C$ that ev_φ maps into $\mathrm{ev}_\gamma C$, and $\ker \mathrm{ev}_\gamma|_C \subseteq \mathrm{Sel}_a(J)$ by the second claim. □

Lemma 43.10 (Steps of the descent). *Let $\varpi^e \mid \ell^M$, $M < N$, let V_i be the triviality on T_i/ℓ^M, and let the residue field have more than $2n_ie+1$ elements. Call a member of* $\mathrm{Sel}_a(J)$ in the tame image *at a new auxiliary prime P if its values at φ, φ^{-c} are the values at γ, γ^{-c} of a member of* $\mathrm{Sel}_a^P(J)$. *The* total tied length *is the sum of the lengths of* $\mathrm{Sel}_{\varpi^k}(J)$ *over the data and $1 \le k \le e$.*

(a) *If some member of* $\mathrm{Sel}_a(J)$ *has a value with a unit coordinate and all members are in the tame image at every new auxiliary prime, then* $\mathrm{Sel}_a(J)$ *has length at most the global term plus the places term; if all members modulo every ϖ^k, $1 \le k \le e$, are in the tame image, the same holds for* $\mathrm{Sel}_{\varpi^e}(J)$.
(b) (*One step.*) *If for some $1 \le k \le e$ some member of the tied Selmer group of T_i modulo ϖ^k is not in the tame image at a new auxiliary prime P, then P can be added with a unit slope for i and slope 0 for the other data so that the total tied length drops.*
(c) *There are tied primes J' with the same P_0, at most $\#J$ plus the total tied length of J in number, whose tied Selmer groups modulo ϖ^e have length at most the global term plus the places term.*
(d) *If all slopes are 0, the total tied length has a bound independent of N, P_0, J and the data at the primes.*

Proof. (a) The first claim is Theorem 43.8(a) and (b). For the second, pass by Lemma 43.9(b) to a level ϖ^k, $k \le e$, at which the group is no shorter and, unless $k = 0$, has a member with a unit value, and apply the first claim there.

(b) At each level $m \le e$, Lemma 43.9(e) excludes at most $2n_i$ residues; as the residue field has more than $2n_ie+1$ elements, Lemma 43.9(c) gives a unit u good at all levels. Adding P with slope u^{-1} for i and 0 for the others, no group grows: for the other data the new group lies in the old one by Lemma 43.9(e) (slope 0), and for i it is no longer than the part in the tame image. At the level k this part is a proper submodule, and a proper submodule of a module of finite length is strictly shorter; so the sum drops.

(c) Among the tied sets (J', x') with the same first prime whose number of primes plus total tied length is at most that of (J, x), take one with the least total tied length. If some member at some level failed to be in the tame image at some new prime, (b) would give a set with one more prime and total length at least 1 smaller, still in the family, contradicting minimality. Hence (a) applies.

(d) With all slopes 0 the tied groups lie in the Selmer groups $\mathrm{Sel}(\emptyset)$ without auxiliary primes (Proposition 43.5(d)), which are finite and depend on none of these choices. □

Theorem 43.11 (Descent). *Let T_i be finitely many Selmer data of ranks n_i, unramified outside S_0, g_0 inertial at a place of S_0 prime to ℓ, W a neighbourhood of g_0, and let the residue field have more than $2n_iv(\ell^M)+1$ elements. There is an integer s such that for every $N > M$ there is a set J of at most s tied primes whose tied Selmer groups modulo ℓ^M have length at most the global term plus the places term.*

Proof. Let ϖ be a uniformiser and $e = v(\ell^M)$, so ϖ^e and ℓ^M generate the same ideal and the residue field has more than $2n_ie+1$ elements. Let b bound the total tied length for all slopes 0 (Lemma 43.10(d)); it does not depend on N. For $N > M$ take the first prime P_0 of Lemma 43.9(d), with Frobenius element in W, and $J = \{P_0\}$ with slopes 0. Lemma 43.10(c) gives a tied set with at most

$1 + b$ primes whose tied Selmer groups modulo $\varpi^e = \ell^M$ have length at most the global term plus the places term. Take $s = 1 + b$. □

43.5. Main summands as Selmer data

Lemma 43.12 (Terms of a main summand). *Let R be like a main summand of rank n, u a lattice in R over O whose image spans all matrices, and normalise lengths so that O/ℓ has length 1.*

(a) *If g acts through u by a scalar y, then g^{-c} acts by $\mathrm{cyc}(g)^{t-1}y$, and y is a unit.*
(b) *The negative Hodge–Tate jumps, summed over the complex embeddings, number at most fn, and the ranks of the crystalline cocycles at the places above ℓ add up to at most $n(\#\{w \mid \ell\} + f)$.*
(c) *The global term is at most $n\, v_\ell(n_{\mathrm{ch}})$; the local term at each place above p_0p_1 prime to ℓ is at most $n\, v_\ell(n_{\mathrm{ch}})$; the two terms $v(\det(u(g) - 1))$ at the tame generator γ and γ^{-c} add up to at most $2n\, v_\ell(n_{\mathrm{ch}})$; in all, the local terms are at most $(4f + 2)n\, v_\ell(n_{\mathrm{ch}})$.*
(d) *Some inertial element above p_0p_1, prime to ℓ, acts by a scalar ζ with $\zeta^2 - 1$ a unit.*

Proof. (a) As the $u(x)$ span all matrices, it is enough to compare traces against them. Conjugate self-duality on traces, $\operatorname{tr} u(g^c) = \mathrm{cyc}(g)^{-(t-1)} \operatorname{tr} u(g^{-1})$, applied at $x^c g^{-1}$ and at x^c, makes both traces multiples of $\operatorname{tr} u((x^c)^{-1})$ with the claimed ratio. A scalar of a lattice element is a unit, as $u(g)$ is invertible over O.

(b) An embedding and its conjugate have at most n negative jumps together (Definition 37.2), and there are f pairs. The rank at $v \mid \ell$ is n plus the number of negative jumps at the embeddings of v (Theorem B.39(a)), and each embedding is counted at one place at most.

(c) Take g acting by a scalar α with cyclotomic character p_l^n (Definition 37.2). By (a), $\alpha - \mathrm{cyc}(g^c)\alpha' = (1 - p_l^d)\alpha$ with $d = nt$ ($d = n$ if $t = 0$), α a unit and $1 \le d \le \mathrm{rt}$; so $v(\alpha - \mathrm{cyc}(g^c)\alpha') = v_\ell(p_l^d - 1) \le v_\ell(n_{\mathrm{ch}})$ (Lemma 37.4(b)), and Lemma 43.2 gives the global term. At a place v above p_0p_1 prime to ℓ, a power m prime to v kills the image of inertia and u is trivial on inertia near 1 (Definition 37.2). A local cocycle is determined by its value at the tame generator together with its value at a Frobenius φ when that first value vanishes: such a cocycle is unramified, and $\gamma\varphi = \varphi(\varphi^{-1}\gamma\varphi)$ with both γ and $\varphi^{-1}\gamma\varphi$ inertial shows that $u(\gamma)$ fixes its value at φ. Hence the local cocycles have length at most length W + length $\ker(u(\gamma) - 1)$ for the inertial γ of the row, and the local term is at most $v(\det(u(\gamma) - 1))$, which is at most $n\, v_\ell(n_{\mathrm{ch}})$ by Definition 37.2; the valuation of $\det(u(g) - 1)$ in O, normalised, is that of $\det(R(g) - 1)$ in $\bar{\mathbb{Q}}_\ell$. At most $4f$ places above p_0p_1 are prime to ℓ, and the tame generator gives two more terms.

(d) The scalar y of Definition 37.2, with $y^{\ell^a j} - 1$ a unit for $1 \le j \le 4r + 2$, lies in O, being a diagonal entry of $u(\gamma)$, a conjugate of $R(\gamma)$; take $j = 2$, $a = 0$. □

Proposition 43.13 (Main summands are Selmer data). *Let R be like a main summand and γ the tame generator at a chosen place prime to ℓ. Over all large coefficient rings there is a Selmer datum T for γ and the bad places, a lattice in R, with places those above ℓp_0p_1, local relations those of the crystalline cocycles above ℓ, and β_T of normalised length at most $(4f + 3)\, n\, v_\ell(n_{\mathrm{ch}})$.*

Proof. Take the lattice u of Lemma 43.12 and the element γ_* of Lemma 43.12(d); the cyclotomic character is trivial at the inertial γ_*, so by Lemma 43.12(a) γ_*^{-c} acts by the same scalar ζ_*. The places are those above $\ell p_0 p_1$, outside which R is unramified; the modules Z_w are the crystalline cocycles above ℓ, finitely generated with the rank bound of Lemma 43.12(b); Burnside's form of irreducibility is the spanning of the image. By Lemma 43.12(c), the global term is at most $n\, v_\ell(n_{\mathrm{ch}})$ and the local terms at most $(4f+2)n\, v_\ell(n_{\mathrm{ch}})$; so β_T has normalised length at most $(4f+3)n\, v_\ell(n_{\mathrm{ch}})$, and in particular all terms are finite. □

44. The twisting theorem

The descent of Section 43 gives tied primes, not a twist. The ties are the values at the tame generators of a universal character with polynomial coefficients; this section moves the point of evaluation to a root of unity. Selmer lengths and the values of Stickelberger systems are then valuations of polynomials at that point, and a lemma on roots of unity keeps all of them small at once (Theorem 44.13). For the main summands and the Stickelberger systems of L-values this is the twisting theorem (Theorem 44.17).

44.1. One universal character

For homomorphisms $h_j\colon \mathrm{Gal}(\Omega/E) \to \mathbb{Z}/q$, $j \in \sigma$, the *universal character* is the polynomial

$$\psi_X(g) = \prod_{j\in\sigma}(1+X_j)^{h_j(g)}, \qquad 0 \le h_j(g) < q,$$

over the variables X_j. At a point x with $(1+x_j)^q = 1$ it is a character, and the Selmer module $\mathrm{Sel}(x)$ of $\rho \otimes \psi_x$ modulo ℓ^M, for a finite set S and a set $\mathcal{L}$ of relations among the values of cocycles, is the module of cocycles unramified outside S obeying $\mathcal{L}$.

The proofs need two pieces of linear algebra over discrete valuation rings, stated here with their proofs.

Lemma 44.1 (Maximal minors). *Let O be a discrete valuation ring, A a matrix over O with columns indexed by a finite set k and any family of rows, and $a \neq 0$. The* maximal minors *of (A, a) are the determinants of the square matrices formed by $\#k$ rows of the matrix whose rows are those of A and of $a \cdot 1$.*

(a) *The kernel of A on $(O/a)^k$ has the length of the module of linear maps to O/a from the quotient of O^k by the lattice spanned by the rows of A and of $a \cdot 1$, and this quotient has the same length.*

(b) *The quotient of O^k by the rows of a square matrix N has length $v(\det N)$; if every row of N is a combination of the rows of a matrix M, then every common divisor of the maximal minors of M divides $\det N$.*

(c) *The length of the kernel of A on $(O/a)^k$ is the least valuation of a maximal minor of (A, a). For a ring map $\mathrm{ev}\colon \Lambda \to O$ and a matrix A over Λ with $\mathrm{ev}(a) \neq 0$, the same holds for $\mathrm{ev}(A)$ with the images of the maximal minors over Λ.*

Proof. (a) A vector of the kernel is determined by its coordinates, and these are the values at the standard basis of a linear map $O^k \to O/a$ that kills the rows

of A and of $a \cdot 1$. The quotient Q is finitely generated and killed by a; by the structure theorem it is a product of cyclic modules O/d_i with $d_i \mid a$, and the linear maps $O/d \to O/a$ are the multiples of multiplication by a/d, a module of the length of O/d.

(b) Write $N = U \operatorname{diag}(d) V$ with U, V invertible (Smith normal form). A vector is a combination of the rows exactly if d_i divides the ith coordinate of its product with V^{-1}, so the quotient is $\prod_i O/d_i$, of length $\sum v(d_i) = v(\det N)$. If the rows of N are combinations of those of M, then by multilinearity $\det N$ is a combination of the determinants of choices of rows of M.

(c) Multiplication by a embeds O^k into the lattice of the rows, so the lattice is free of full rank and is spanned by the rows of a square matrix N. Each choice of $\#k$ rows spans a sublattice, whose quotient is at least as long, of length the valuation of the minor by (b); and $\det N$ is a combination of the minors, so the minor of least valuation divides it. Hence the length of the quotient is the least valuation of a minor, and (a) concludes. A ring map carries the maximal minors of (A, a) onto those of $(\mathrm{ev} A, \mathrm{ev}\, a)$. □

Lemma 44.2 (One finite quotient). *Let E be a number field, S a finite set of primes, G' an open subgroup of $\mathrm{Gal}(\Omega/E)$ and $m \neq 0$.*

(a) *Finitely many locally constant functions on G' are constant on the cosets of one open normal subgroup $G'' \subseteq G'$.*
(b) *There is one open normal G'' such that, for every abelian group B killed by m, every locally constant $z\colon \mathrm{Gal}(\Omega/E) \to B$ that is additive on G' and vanishes on the inertia groups above the primes outside S vanishes on G''.*
(c) *For a function ρ to matrices over a ring Λ, a set $\mathcal{L}$ of relations, and G', m, S as above, there is one matrix A over Λ with finitely many columns such that for every $\mathrm{ev}\colon \Lambda \to O'$ and every O'-module B killed by m on which G' acts as the identity through $\mathrm{ev}\,\rho$, the Selmer module of $\mathrm{ev}\,\rho$ in B^n is isomorphic to the kernel of $\mathrm{ev}\, A$ on B^k.*

Proof. (a) Each function is constant on the cosets of an open subgroup; intersect them and take the normal core, which is open as the index is finite.

(b) The continuous homomorphisms $G' \to \mathbb{Z}/m$ unramified outside the primes above S form a finite set Y: they factor through the Galois group of the maximal abelian extension of exponent m of the fixed field of G' unramified outside S, which is finite by class field theory (Theorem B.35). By (a) they are constant on the cosets of some G''. The homomorphisms $\chi\colon B \to \mathbb{Z}/m$ separate the points of B, and $\chi \circ z$ lies in Y; so z vanishes on G''.

(c) Let G'' be as in (b) and $\mathcal{G} = \mathrm{Gal}(\Omega/E)/G''$, a finite group. A member z of the Selmer module is additive on G' (it is a cocycle and G' acts as the identity), so it vanishes on G'', and $z(gh) = z(g) + gz(h) = z(g)$ for $h \in G''$: it is a function on $\mathcal{G}$. The unknowns are the coordinates of the values on $\mathcal{G}$; the equations, linear with coefficients in Λ, are the cocycle rule (whose relations are carried by ev onto those for $\mathrm{ev}\,\rho$), vanishing on the images of the inertia groups outside S, and the relations of $\mathcal{L}$ pushed forward to $\mathcal{G}$ by adding the matrices over each fibre. A function on $\mathcal{G}$ gives a member of the Selmer module exactly if it satisfies the equations, by the identity $\sum_c \mathrm{ev}(f_*(c))w(c) = \sum_g \mathrm{ev}(f(g))w(\pi g)$. □

Proposition 44.3 (Selmer lengths are valuations). *Let the h_j and ρ modulo ℓ^M be locally constant.*

(a) *For $a \mid \ell^M$ with O/a finite the Selmer module with coefficients in O/a is finite.*
(b) *There is a matrix A over $O[X]$ with finitely many columns such that* $\mathrm{Sel}(x)$ *is isomorphic to the kernel of $A(x)$ on a power of O'/ℓ^M, for every extension $O \to O'$ and every point x over O'.*
(c) *There is a set $\mathcal{M}$ of polynomials such that at every point x over a discrete valuation ring of characteristic zero the length of* $\mathrm{Sel}(x)$ *is the least valuation of a value $P(x)$, $P \in \mathcal{M}$.*
(d) *For a point x_0 there is a polynomial P with $v(P(x_0)) = \operatorname{len}\mathrm{Sel}(x_0)$ and $v(P(x)) \geq \operatorname{len}\mathrm{Sel}(x)$ at every point x.*

Proof. (a) The ring O/ℓ^M acts on O/a, so ρ acts through its reduction, continuously; by Lemma 44.2(c), with G' the kernel of ρ modulo ℓ^M, the Selmer module lies in the set of functions on one finite quotient with values in the finite $(O/a)^n$.

(b) Let G' be an open subgroup on which all h_j vanish and $\rho \equiv 1$ modulo ℓ^M. Over $\Lambda = O[X]$ put $\rho_X(g) = \psi_X(g)\rho(g)$. On G', $\psi_X = 1$, so G' acts as the identity through every specialisation on $(O'/\ell^M)^n$, and Lemma 44.2(c), with $m = \ell^M$ and ev the evaluation at x composed with $O \to O'$, gives the matrix: at $X = x$ the matrices $\rho_X(g)$ and the relations are those of $\mathrm{Sel}(x)$.

(c) Take for $\mathcal{M}$ the maximal minors of (A, ℓ^M); by (b) and Lemma 44.1(c) with ev the evaluation at x, the length of $\mathrm{Sel}(x)$, which is that of the kernel of $A(x)$ on $(O'/\ell^M)^k$, is the least valuation of a value $P(x)$, $P \in \mathcal{M}$; here $\ell^M \neq 0$ in O', of characteristic zero.

(d) Take $P \in \mathcal{M}$ attaining the least valuation at x_0; at every other point its valuation is at least the least one, which is the length. □

44.2. From ties to a line

A Selmer datum T over O has *twisted length at most b* for a character ψ if for all large coefficient rings $O' \supseteq O$ the values of ψ lie in O' and $\mathrm{Sel}_Q(T\psi)$, with coefficients in $\mathrm{Frac}(O')/O'$, has length at most b (normalised so that O'/ℓ has length 1). Say that M is a *torsion level* for T if $M > \beta_T + 1$ for each datum and the residue field of O has more than $2\,\mathrm{rk}(T_i)\operatorname{len}(O/\ell^M) + 1$ elements.

A polynomial P *bounds* the Selmer module of characters h_j if $\operatorname{length}\mathrm{Sel}(x) \leq v(P(x))$ at all points x over all discrete valuation rings of characteristic zero over O. The *restriction* of P to the line $X_j = a_j Y$ is $P(aY)$. The divisible Selmer group of $T\psi$ is formed with coefficients in $\mathrm{Frac}(O')/O'$; $\mathrm{Sel}_Q(T\psi)$ denotes the cocycles of $T\psi$ vanishing at γ_*, unramified outside $S \cup Q \cup \bar{Q}$ and obeying the local conditions of T.

Lemma 44.4 (From the tied group to the twisted group). (a) *A member of a Selmer group with divisible coefficients is killed by a nonzero element; if the group modulo $a \neq 0$ has length less than $v(a)$, the divisible group is no longer than it.*
(b) (*Gauss valuation of the tie polynomial.*) *If $2n\#I \leq s_0$, n the rank, then $v_r(P^1) \leq s_0 r$ for the tie polynomial P^1 of I.*
(c) (*Release of the first prime.*) *If the Selmer group modulo ℓ^M tied at all primes of J has length at most the global term plus the places term, and φ_{P_0}, $\varphi_{P_0}^{-c}$ act*

modulo ℓ^M as g_0, g_0^{-c}, then the group tied only at the primes of $J \setminus P_0$ has length at most β_T.

(d) (*The sharp polynomial on the line.*) *Let J, N, r, M be as in Proposition 44.5, and suppose the group tied at $J \setminus P_0$ has length at most β_T. Then some P^0 with $v(P^0(0)) \le \beta_T$ has the following property: for a discrete valuation ring O' over O with a map f to $\mathbb{Z}_\ell^{\mathrm{int}}$ reflecting divisibility, $y \in O'$ with $v(f(y)) \ge r$ and the point $x = (a_v y)_v$ of the line $(a_{P_0} = 1, a_v = s_v)$, some polynomial P bounds the Selmer module of the characters of J, and $P(x)$ divides $P^0(y)P^1(y)$ in $\mathbb{Z}_\ell^{\mathrm{int}}$.*

(e) (*Roots of unity.*) *If P bounds the Selmer module of characters λ_v, the ζ_v are qth roots of unity, $x = (\zeta_v - 1)_v$ and $\psi = \prod_v \zeta_v^{\lambda_v}$, then $\mathrm{Sel}(x)$ is the Selmer group $\mathrm{Sel}_J(T\psi)$ modulo ℓ^M; so if $v(P(x)) < M$, then over all large coefficient rings $\mathrm{Sel}_J(T\psi)$ with divisible coefficients has normalised length at most $v(P(x))$.*

Proof. (a) A member takes finitely many values, as it is locally constant on a compact group; the product of the denominators of their coordinates kills it. Let z have exact order ϖ^k with $k > e = v(a)$; then $\varpi^{k-e}z$ has order ϖ^e, lies in the a-torsion, which is the group with coefficients $a^{-1}O/O \cong O/a$, and spans a submodule of length e, contradicting the hypothesis. Hence a kills the divisible group, which is then contained in the group modulo a.

(b) Each of the $2\#I$ factors $\det((1+Y)\rho(\varphi_w) - 1)$ has the unit $\det \rho(\varphi_w)$ as coefficient in degree n, so its Gauss valuation at radius r is at most nr; Gauss valuations add over products (Lemma 44.7(b)).

(c) By Lemma 43.4(e), dropping the ties at P_0 and $\bar{P}_0$ costs at most $\sum_{g=g_0,g_0^{-c}} v(\det(\rho(g) - 1))$, the first-prime term; β_T is the sum of the three terms.

(d) Let G be a polynomial bounding the Selmer module with the ties of $J \setminus P_0$ that is sharp at 0 (Proposition 44.3(d)); at 0 the twist is trivial and this module is the tied group, so $v(G(0)) \le \beta_T$; let P^0 be the restriction of G to the line. Let P bound the module without ties and be sharp at x. By Lemma 43.4(d) at the point x (where $\alpha \equiv 1$ modulo y gives $\sum_{j<\ell^N} \alpha^j \equiv 0$ modulo ℓ^M, since $v(y) \ge r$ and $M \le Nr$, Lemma 44.7(e)),

$$\begin{aligned} v(P(x)) = \operatorname{length} \mathrm{Sel}(x) &\le \operatorname{length}(\text{tied part}) + v(P^1(y)) \\ &\le v(G(x)) + v(P^1(y)) = v(P^0(y)) + v(P^1(y)), \end{aligned}$$

and in $\mathbb{Z}_\ell^{\mathrm{int}}$ divisibility is comparison of valuations; the map f carries the inequality there.

(e) At $x = (\zeta_v - 1)_v$, $\psi_x(g) = \prod_v \zeta_v^{\lambda_v(g)}$ is the character ψ, so the twist at x is $T\psi$. If $v(P(x)) < M$, the group modulo ℓ^M has length less than $v(\ell^M)$, and (a) bounds the divisible group; the character has the given values in $\bar{\mathbb{Q}}_\ell$, and normalised lengths do not depend on the large ring. □

Proposition 44.5 (Polynomials for a Selmer datum). *Let J be tied primes with first prime P_0 for a Selmer datum T, the datum modulo ℓ^M trivial at the elements of property V_i, and let $N \ne 0$, $2\,\mathrm{rk}(T)\#J \le s_0$, $r \le 1$ and $M \le Nr$. If the Selmer group modulo ℓ^M tied at the primes of $J \setminus P_0$ has length at most β_T, then a polynomial P bounds the length of the Selmer module of the characters of J at all points, and $v(P(x_0)) \le \beta_T + s_0 r$ at a point x_0 over $\mathbb{Z}_\ell^{\mathrm{int}}$ whose coordinates have valuation at least r.*

Proof. Let P^0 be the polynomial of Lemma 44.4(d) and P^1 the tie polynomial of $J \setminus P_0$. By Lemma 44.7(c) there is $y \in \mathbb{Z}_\ell^{\mathrm{int}}$ with $v(y) \geq r$ at which both attain their Gauss valuations of radius r. Let $x_0 = (a_v y)_v$ be the point of the line; its coordinates have valuation at least r, as the slopes are integral. By Lemma 44.4(d) a polynomial P bounds the Selmer module and

$$v(P(x_0)) \leq v(P^0(y)) + v(P^1(y)) = v_r(P^0) + v_r(P^1) \leq v(P^0(0)) + s_0 r \leq \beta_T + s_0 r,$$

using $v_r(P^0) \leq v(P^0(0))$ (Lemma 44.7(a)) and Lemma 44.4(b), as $2\,\mathrm{rk}(T)\#(J \setminus P_0) \leq s_0$. □

Theorem 44.6 (Twist polynomials). *Let T_i be finitely many Selmer data and Y_k Stickelberger systems (Section 44.4), g_0 in an inertia group above a place of S_0 prime to ℓ, and let M be a torsion level for T. There is an integer $s_0 \geq 1$ such that for every rational $r > 0$ with $s_0 r < 1$ and every large N there are*

(a) *a set Q of at most $s_0/2$ primes of prime norm, outside S_0, with conjugates outside S_0 and Q;*
(b) *for each $\mathfrak{q} \in Q$ a homomorphism $\lambda_\mathfrak{q}$ to $\mathbb{Z}/\ell^N$, onto on inertia at $\mathfrak{q}$, coming by $y \mapsto y/\bar{y}$ from an idele class character of modulus $\mathfrak{q}$ trivial above ℓ;*
(c) *for each datum and each system a polynomial P over $\mathbb{Z}_\ell^{\mathrm{int}}$ in variables $X_\mathfrak{q}$ and a point with coordinates of valuation at least r at which $v(P) \leq \beta + s_0 r$,*

such that for ℓ^Nth roots of unity $\zeta_\mathfrak{q}$, $\psi = \prod \zeta_\mathfrak{q}^{\lambda_\mathfrak{q}}$ and $x = (\zeta_\mathfrak{q} - 1)_\mathfrak{q}$: for a system $P(x) = \psi(\Theta_Q)$; for a datum with $v(P(x)) < \beta_T + 1$ the Selmer group $\mathrm{Sel}_Q(T\psi)$ with divisible coefficients has length at most $v(P(x))$.

Proof. For g near g_0, g and g^{-c} act on each T_i modulo ℓ^M as g_0 and g_0^{-c} do, and $\vartheta_k(g) \equiv \vartheta_k(g_0)$ modulo $\ell^{M'}$ for each system, M' an integer with $\beta_{Y_k} + 1 < M'$ for all k, as the characters are continuous; let W be such a neighbourhood. Theorem 43.11 gives s such that for every $N > M$ there is a tied set J with first prime Frobenius in W, at most s primes, and tied Selmer groups of length at most the global term plus the places term. Let $c \geq 1$ bound the ranks of the T_i and the numbers of exponents of the Y_k, and put $s_0 = 2cs + 1$. Given $r > 0$ with $s_0 r < 1$, take $N > M$ with $M, M' \leq Nr$, and $Q = J$; then $2\#Q \leq s_0$. The characters $\lambda_\mathfrak{q}$ of the primes of J come from idele class characters of modulus $\mathfrak{q}$ trivial above ℓ and take the value 1 at the tame generator (Proposition 43.5(a), (b)), so they are onto on inertia. For a datum, Lemma 44.4(c) and Proposition 44.5 give P and the point. If $v(P(x)) < \beta_T + 1 < M$, Lemma 44.4(e) bounds the divisible Selmer group. For a system, Proposition 44.11(b) gives the polynomial $\psi_X(\Theta_J)$ and the point; at $x = (\zeta_\mathfrak{q} - 1)$ its value is $\psi(\Theta_Q)$, by the identity $\psi_x = \psi$ applied to the group ring. □

44.3. Roots of unity

Let $v_\rho(g) = \min_i(v(y_i) + \rho i)$ be the Gauss valuation of radius ρ of $g = \sum y_i Y^i$ over $\mathbb{Z}_\ell^{\mathrm{int}}$, and $\rho_h = 1/\varphi(\ell^h)$, the valuation of $\zeta - 1$ for ζ of order ℓ^h.

Lemma 44.7 (Gauss valuations). *Let v be the valuation of $\bar{\mathbb{Q}}_\ell$ with $v(\ell) = 1$, on the ring $\mathbb{Z}_\ell^{\mathrm{int}}$ of integral elements, and $v_\rho(g) = \min_i(v(y_i) + \rho i)$ for $g = \sum y_i Y^i$.*

(a) *Monic polynomials over $\mathbb{Z}_\ell^{\mathrm{int}}$ split there; every nonnegative rational number is a valuation; the residue field is algebraically closed. $v(g(y)) \geq v_\rho(g)$ if $v(y) \geq \rho$; $v_\rho(g) \leq v(g(0))$; and $v_\rho(g) \leq \rho d$ if the coefficient of Y^d is a unit.*

(b) (*Gauss's lemma.*) $v_\rho(fg) = v_\rho(f) + v_\rho(g)$; $v_\rho(Y - \alpha) = \min(v(\alpha), \rho)$; *and* $v_\rho(g) = v_0(g) + \sum \min(v(\alpha), \rho)$ *over the integral roots* α *of* g *with multiplicity, while* $v(g(y)) = v_0(g) + \sum v(y - \alpha)$ *at integral* y.

(c) (*Generic points.*) *For finitely many polynomials there is one* y *with* $v(y) \geq \rho$ *at which* $v(g(y)) = v_\rho(g)$ *for all of them.*

(d) *If* $v(x) \geq \rho$ *and* $1 \leq j \leq m$, *some* x' *with* $(1 + x')^j = 1 + x$ *has* $v(x') \geq \rho/m$.

(e) *If* $v(\alpha - 1) \geq \rho$ *with* $\rho \leq 1$, *then* $v(\sum_{i<\ell^n} \alpha^i) \geq n\rho$; *in particular* $v((1 + y)^{\ell^N} - 1) \geq N\rho$ *if* $v(y) \geq \rho$. *A root of unity of order* ℓ^{h+1} *has* $v(\zeta - 1) = 1/\varphi(\ell^{h+1})$, *and* $v(\zeta^j - 1) = \ell^t/\varphi(\ell^{h+1})$ *for* $t = v_\ell(j) \leq h$; *hence if* $\rho < 1$ *and* $v(\zeta - 1) \leq \rho/m$, *then* $v(\zeta^j - 1) \leq \rho$ *for* $1 \leq j \leq m$.

Proof. (a) The roots of a monic polynomial are integral over $\mathbb{Z}_\ell^{\text{int}}$, which is integrally closed; for $\rho = a/b$ take a root of $Y^b - \ell^a$; the residue field is an algebraic closure of $\mathbb{F}_\ell$, since monic polynomials split. The bound at a point is the ultrametric inequality, and the other two compare the minimum with the terms $i = 0$ and $i = d$.

(c) Choose π with $v(\pi) = \rho$ and put $Y = \pi Y'$, which turns v_ρ into v_0. Write each polynomial as c times one with nonzero reduction, $v(c) = v_0$ (a coefficient of least valuation); the reductions have finitely many roots and the residue field is infinite, so some unit y' avoids them all, and $y = \pi y'$ works.

(b) Evaluate at a point generic for f, g and fg. For $Y - \alpha$, evaluation at 0 and the leading coefficient give $\leq$, and a generic point gives $\geq$. Write $g = q \prod(Y - \alpha)$ over the integral roots; q has no integral roots, so its roots β have $\|\beta\| > 1$ and $\|y - \beta\| = \|\beta\|$ at integral y: $v(q(y))$ is constant, equal to $v_0(q) = v_0(g)$. Both formulas follow.

(d) The j solutions of $(1 + x')^j = 1 + x$ have product $\pm x$; take one of largest valuation.

(e) $\sum_{i<\ell^n} \alpha^i$ is the product over $h < n$ of $\sum_{i<\ell} \alpha^{i\ell^h}$, each congruent to ℓ modulo $\alpha - 1$, and $v(\ell) = 1 \geq \rho$. The valuation of $\zeta - 1$ is $1/\varphi(\ell^{h+1})$ by the norm of the cyclotomic uniformiser; writing $j = \ell^t j'$, ζ^j has order ℓ^{h+1-t}. For the last claim, ζ has order ℓ^k with $k \neq 0$, and $\ell^{v_\ell(j)} \leq m < \varphi(\ell^k) \leq \ell^k$ as $m\, v(\zeta - 1) \leq \rho < 1$; so $v(\zeta^j - 1) = \ell^{v_\ell(j)} v(\zeta - 1) \leq m\, v(\zeta - 1) \leq \rho$. □

Lemma 44.8 (One variable). *Let* $\rho_h = 1/\varphi(\ell^h)$ *and let* R_h *be the multiset of the roots of* g *in* $\mathbb{Z}_\ell^{\text{int}}$ *of valuation* ρ_h.

(a) *The* ρ_h *are distinct, decrease to* 0, *and for* $\rho, \delta > 0$ *and* B *there is* h_0 *with* $\delta \sum_{h\in H} 1/(h + 1) > B$, H *the exponents* $1 \leq h \leq h_0$ *with* $\rho_h \leq \rho$.

(b) *At a point* y *of valuation* r, $v(g(y)) - v_r(g) = \sum_{\alpha\in R}(v(y - \alpha) - r)$ *over the roots of valuation* r; *and* $\sum_{h\in H} \#R_h\rho_h \leq v_\rho(g)$.

(c) *If* $1 + \alpha$ *is a unit, the sum of* $\min(v(\zeta - 1 - \alpha), 1)$ *over the primitive* ℓ^h*th roots of unity is at most* $h + 1$; *so for a multiset* R *of such* α *some* ζ *of order* ℓ^h *has* $\sum_{\alpha\in R} \min(v(\zeta - 1 - \alpha), 1) \leq \#R\rho_h(h + 1)$.

Proof. (a) $\varphi(\ell^{h+1})$ is strictly increasing, and the harmonic series diverges.

(b) The first is Lemma 44.7(b): only the roots of valuation r contribute to the difference, as $v(y - \alpha) = \min(r, v(\alpha))$ for the others, which is $\min(v(\alpha), r)$. For the second, a root lies in at most one R_h ($h \in H$), and its summand $\min(v(\alpha), \rho)$ in $v_\rho(g) = v_0(g) + \sum \min(v(\alpha), \rho)$ is then ρ_h.

(c) The numbers $\zeta-1-\alpha$, $\zeta^{\ell^h}=1$, are the roots of $f=(1+\alpha+Y)^{\ell^h}-1$, whose coefficient of Y is $\ell^h(1+\alpha)^{\ell^h-1}$, so $v_1(f)\le h+1$; and $v_1(f)\ge\sum\min(v(\text{root}),1)$ by Lemma 44.7(b). Averaging over the $1/\rho_h$ roots of order ℓ^h gives the second claim. □

Proposition 44.9 (Roots of unity). (a) *Let $\delta<1$, and let δ times the sum of $1/(h+1)$ over the exponents $1\le h\le h_0$ with $\rho_h\le\rho$ exceed B. If $v_\rho(g)\le B$, then for one of these h and some ζ of order ℓ^h, $v(g(\zeta-1))\le v_{\rho_h}(g)+\delta$.*

(b) *Given s_0, $\rho>0$, $\beta\ge0$, $\varepsilon>0$ and $m\ge1$ there is h_0 such that for finitely many polynomials P_i in at most s_0 variables and points x_i with coordinates of valuation at least ρ and $\sum_i v(P_i(x_i))\le\beta$, there are roots of unity ζ_u of order dividing ℓ^{h_0} with $v(\zeta_u-1)\le\rho/m$ and $v(P_i((\zeta_u^j-1)_u))\le v(P_i(x_i))+\varepsilon$ for all i and $1\le j\le m$.*

Proof. (a) As $\sum_{h\in H}\#R_h\rho_h\le v_\rho(g)\le B<\delta\sum_{h\in H}1/(h+1)$ (Lemma 44.8(b)), some $h\in H$ has $\#R_h\rho_h\le\delta/(h+1)$. By Lemma 44.8(c) some ζ of order ℓ^h has $\sum_{\alpha\in R_h}\min(v(\zeta-1-\alpha),1)\le\delta<1$, so no minimum is 1, and Lemma 44.8(b) at $y=\zeta-1$, of valuation ρ_h, gives $v(g(\zeta-1))-v_{\rho_h}(g)\le\sum_{\alpha\in R_h}v(\zeta-1-\alpha)\le\delta$.

(b) *One coordinate.* Let $\delta<1$ with $\delta\sum_{h\in H}1/(h+1)>B$, polynomials P_i and points x_i with $v(x_{ij})\ge\rho$ and $\sum_i v(P_i(x_i))\le B$. Let g_i be P_i with the coordinate j free. By Gauss's lemma, $v_\rho(\prod g_i)=\sum v_\rho(g_i)\le\sum v(P_i(x_i))\le B$, so (a) gives ζ of order ℓ^h, $h\in H$, with $\sum_i v(g_i(\zeta-1))\le\sum_i v_{\rho_h}(g_i)+\delta$. As $v(g_i(\zeta-1))\ge v_{\rho_h}(g_i)$ for each i, each term exceeds its lower bound by at most δ; and $v_{\rho_h}(g_i)\le v_\rho(g_i)\le v(P_i(x_i))$ as $\rho_h\le\rho$. Hence replacing x_{ij} by $\zeta-1$ raises each $v(P_i(x_i))$, and their sum, by at most δ. *All coordinates.* Put $\delta=\min(\varepsilon,1)/(s_0+2)$ and take h_0 by Lemma 44.8(a) with $B=\beta+1$. Moving the at most s_0 coordinates in turn, the sum stays at most $\beta+s_0\delta\le\beta+1$, so the hypothesis holds at each step, and each value grows by at most $s_0\delta\le\varepsilon$; the roots of unity have order dividing ℓ^{h_0} and $v(\zeta-1)=\rho_h\le\rho$. *Powers.* For $m\ge1$ apply this, with radius ρ/m and bound $m\beta$, to the polynomials $P_i^{(j)}=P_i((1+X)^j-1)$, $1\le j\le m$, at points x'_{ij} with $(1+x'_{ij})^j=1+x_i$ and $v(x'_{ij})\ge\rho/m$ (Lemma 44.7(d)); then $P_i^{(j)}(\zeta-1)=P_i(\zeta^j-1)$, and $v(\zeta_u-1)\le\rho/m$. □

44.4. Stickelberger systems

Let Cl_Q be the class group of $U(1)$ of tame modulus Q. The value of a function f on the Galois group at an element of the group ring of Cl_Q is read through the Artin map and extended linearly. A *Stickelberger system* for S_0 and g_0 consists of a character ϑ unramified outside S_0 with $\vartheta(g_0)\ne1$, a list of r integers $e_1,\dots,e_r$ and, for every finite set Q, an element Θ_Q of the group ring of Cl_Q over $\mathbb{Z}_\ell^{\mathrm{int}}$, whose values under characters for Q change by Euler factors $\prod_i(1-\vartheta(\varphi_P)\chi(\varphi_P)\,\mathrm{N}\,P^{e_i})$ when an auxiliary place P is added. Its number is

$$\beta_Y=v\big(1(\Theta_\emptyset)\big)+2r\,v\big(\vartheta(g_0)-1\big).$$

Lemma 44.10 (Stickelberger systems: basic rules). (a) *Characters for Q are closed under enlarging the modulus, under maps of the values, under products, and the trivial character is one; subsets of auxiliary sets are auxiliary.*

(b) *$y \mapsto y/\bar{y}$ maps the finite ideles onto Cl_Q (Hilbert 90); a character for Q takes at g with $\mathrm{Art}(y)$ as image the value at the chosen lift of the class of $y/\bar{y}$. Linear maps commute with evaluation on the group ring.*

(c) *(Several primes.) For Q, I disjoint with $Q \cup I$ auxiliary and ψ a character for Q, $\psi(\Theta_{Q\cup I}) = \psi(\Theta_Q)\prod_{v\in I}\prod_e \prod_{\pm}(1 - (\vartheta\psi)(\varphi_v)^{\pm 1}\,\mathrm{N}\, v^{-e})$. The values of ϑ have norm 1, $1(\Theta_Q)$ is integral, and β_Y is finite.*

(d) *If $a \equiv b$ and $q \equiv 1$ up to valuation $M > v(b-1)$, $\|a\| = 1$, then $v(1-aq^k) = v(b-1) = v(1-a^{-1}q^k)$; hence the Euler factor has valuation at most $2r\,v(b-1)$ if $M > 2r\,v(b-1)$.*

(e) *For units α, c there is an integral polynomial R with $v_\rho(R) \le 2\rho$ and $R(\zeta-1) = \zeta(1-\alpha\zeta c)(1-(\alpha\zeta)^{-1}c)$ for all integral $\zeta \neq 0$; and for a list of r exponents one with $v_\rho(R) \le 2r\rho$ and $R(\zeta-1) = \zeta^r \prod_e \prod_{\pm}(1-(\alpha\zeta)^{\pm 1}q^{-e})$.*

(f) *A polynomial vanishing at all $\zeta - 1$, $\zeta^q = 1$, is divisible by $(1+X)^q - 1$. If $(1+X)^j P \equiv R$ modulo $(1+X)^{\ell^N} - 1$, $\rho \le 1$ and $v_\rho(R) < N\rho$, then $v(P(y)) \le v_\rho(R)$ at some y with $v(y) \ge \rho$.*

Proof. (a) Products and images of idele class characters are idele class characters with the same triviality; the modulus may be enlarged.

(b) Hilbert's Theorem 90 for the finite ideles of the CM extension: a norm-one idele is $y/\bar{y}$. On norm-one ideles the idele character of ψ factors through a homomorphism on Cl_Q, and both sides are its value at the class of $y/\bar{y}$.

(c) Induction on I by the norm relation of the system, each enlarged set being auxiliary. ϑ is continuous on a compact group; $1(\Theta_Q)$ is the image of an element of $\mathbb{Z}_\ell^{\mathrm{int}}$; β_Y is finite because $1(\Theta_\emptyset) \neq 0$ and $\vartheta(g_0) \neq 1$.

(d) $1 - aq^k = (1-b) + (b-a) + a(1-q^k)$, and the last two terms have valuation at least $M > v(1-b)$; $a^{-1} - b^{-1}$ has the valuation of $a - b$ for units.

(e) Take $R = (1 - \alpha c(1+X))(1 + X - \alpha^{-1}c)$: each factor has a unit coefficient of X, so Gauss valuation at most ρ; at $X = \zeta - 1$ it is $(1-\alpha\zeta c)\zeta(1-(\alpha\zeta)^{-1}c)$. For a list take the product over the exponents, with $c = q^{-e}$.

(f) The q numbers $\zeta - 1$ are the distinct roots of the monic $(1+X)^q - 1$. For the second, take y with $v(y) \ge \rho$ at which R attains $v_\rho(R)$ (Lemma 44.7(c)). Then $(1+y)^{\ell^N} - 1$, of valuation at least $N\rho > v(R(y))$ (Lemma 44.7(e)), divides $(1+y)^j P(y) - R(y)$, so $v(P(y)) = v((1+y)^j P(y)) = v(R(y))$. □

Proposition 44.11 (Stickelberger systems on the line). *Let J be tied primes with first prime P_0, $\vartheta(\varphi_{P_0}) \equiv \vartheta(g_0)$ modulo ℓ^M, $\beta_Y + 1 < M \le N\rho$ and $s_0 \ge 1$ with $2r\#J \le s_0$ and $s_0\rho < 1$.*

(a) *Let $P(X,0)$ and $R_0(X)$ be the values of $(1+X)^{h_0}$ at Θ_J and Θ_{P_0}, h_0 the character of P_0, and $j_0 = r\,\#(J \setminus P_0)$. Then $(1+X)^{j_0}P(X,0) \equiv R_0(X)R(X)$ modulo $(1+X)^{\ell^N} - 1$, where R collects the Euler factors at the primes other than P_0 and has Gauss valuation at most $2j_0\rho$ at radius ρ.*

(b) *$v(P(y)) \le \beta_Y + s_0\rho$ at some point y with coordinates of valuation at least ρ.*

Proof. (a) Along the line on which all variables but that of P_0 vanish, the value of ψ_X at the group ring is that of $(1+X)^{h_0}$ (the other factors are 1). At a root $X = \zeta - 1$ of the modulus, the character ζ^{h_0} is a character for J, and the norm relation of Lemma 44.10(c) from $\{P_0\}$ to J gives $\zeta^{h_0}(\Theta_J) = \zeta^{h_0}(\Theta_{P_0})\prod_{v\neq P_0}\prod_e\prod_{\pm}(1 - a_v^{\pm 1}\,\mathrm{N}\,v^{-e})$ with $a_v = \vartheta(\varphi_v)\zeta^{h_0(\varphi_v)} = \vartheta(\varphi_v)\zeta$, as $h_0(\varphi_v) = 1$ for $v \neq P_0$. Let R be the product over $v \neq P_0$ of the polynomials of Lemma 44.10(e) for $\alpha = \vartheta(\varphi_v)$

and $q = \mathrm{N}\, v$; then $v_\rho(R) \le 2r\#(J \setminus P_0)\rho = 2j_0\rho$, and $(1+X)^{j_0}P(X,0) - R_0(X)R(X)$ vanishes at every $\zeta - 1$, so is divisible by the modulus (Lemma 44.10(f)).

(b) $R_0(0) = 1(\Theta_{P_0})$, and by the norm relation from $\emptyset$ to $\{P_0\}$, $1(\Theta_{P_0}) = 1(\Theta_\emptyset)\prod_e \prod_\pm (1 - \vartheta(\varphi_{P_0})^{\pm 1}\,\mathrm{N}\, P_0^{-e})$. As $\vartheta(\varphi_{P_0}) \equiv \vartheta(g_0)$ modulo ℓ^M, $\mathrm{N}\, P_0 \equiv 1$ modulo ℓ^N, $M \le N$ and $M > \beta_Y \ge 2r\, v(\vartheta(g_0) - 1)$, Lemma 44.10(d) gives $v(R_0(0)) \le \beta_Y$, so $v_\rho(R_0) \le \beta_Y$ (Lemma 44.7(a)). With (a), $v_\rho(R_0 R) \le \beta_Y + 2r\#J\rho \le \beta_Y + s_0\rho \le \beta_Y + 1 < M \le N\rho$, as $\rho \le 1$. Lemma 44.10(f) gives y with $v(y) \ge \rho$ and $v(P(y,0)) \le v_\rho(R_0 R) \le \beta_Y + s_0\rho$; the point $(y, 0, \dots, 0)$ has coordinates of valuation at least ρ. □

44.5. The general twisting theorem

A character χ is a *small twist* with bound ε for the data T_i, the systems Y_k and a set Q if it comes by $y \mapsto y/\bar{y}$ from an idele class character of tame level Q trivial above ℓ, has ℓ-power order, is ramified at every prime of Q, the Selmer group of each $T_i\chi$ has length at most $\beta_{T_i} + \varepsilon$, $v(\chi(\Theta_Q)) \le \beta_{Y_k} + \varepsilon$ for each system, and $\sum v(\chi(\gamma_w) - 1) \le \varepsilon$ over the primes w of Q and their conjugates.

Lemma 44.12 (Twists by roots of unity). *Let Q and $\lambda_\mathfrak{q}$ be as in Theorem 44.6, $\zeta_\mathfrak{q}$ roots of unity of order dividing ℓ^N, and $\psi = \prod_\mathfrak{q} \zeta_\mathfrak{q}^{\lambda_\mathfrak{q}}$.*

(a) *ψ^j is the twist by the $\zeta_\mathfrak{q}^j$; it has order dividing ℓ^N and is a character for Q, trivial above ℓ.*
(b) *For $v \in Q$ and $w = v$ or $\bar{v}$, ψ takes the value ζ_v on some inertial element above w, and every inertial element above w goes to a power of $\psi(\gamma_w)$; hence $v(\psi(\gamma_w) - 1) \le v(\zeta_v - 1)$, and if all $v(\zeta_v - 1) \le r$, the sum over $w \in Q \cup \bar{Q}$ is at most $s_0 r$.*
(c) *If moreover $s_0 r \le \varepsilon < 1$ and the polynomials have at $x = (\zeta - 1)$ valuations at most $\beta_T + \varepsilon$, $\beta_Y + \varepsilon$, then ψ is a small twist with bound ε.*

Proof. (a) $\psi^j(g) = \prod \zeta_\mathfrak{q}^{j\lambda_\mathfrak{q}(g)}$, and products of characters for Q are characters for Q (Lemma 44.10(a)).

(b) The characters of the other primes u are unramified at v, which is neither u nor $\bar{u}$, and λ_v is onto on inertia; so some inertial g above v has $\psi(g) = \zeta_v$, and at $\bar{v}$ the conjugate inverse, on which the anticyclotomic ψ takes the same value. As $\psi^{\ell^N} = 1$ and ℓ^N is a unit at w, ψ is trivial on wild inertia and factors through the procyclic tame quotient, which γ_w generates; the inertia groups above w are conjugate and ψ is constant on conjugacy classes. Hence $\zeta_v = \psi(\gamma_w)^k$, and $v(\zeta_v - 1) \ge v(\psi(\gamma_w) - 1)$ (Lemma 44.10(d) for $a^k - 1$). There are $2\#Q \le s_0$ terms.

(c) ψ has ℓ-power order and is a character for Q by (a); it is ramified at every prime of Q because $v(\zeta_v - 1) \le r < \infty$, so $\zeta_v \ne 1$ is a value on inertia. For a datum, $v(P(x)) \le \beta_T + \varepsilon < \beta_T + 1$, so the Selmer group has length at most $v(P(x))$; for a system $v(\psi(\Theta_Q)) = v(P(x))$; and (b) bounds the tame values. □

Theorem 44.13 (The twisting theorem, general form). *Let S_0 contain the places above ℓ, g_0 be inertial at a place of S_0 prime to ℓ, T_i finitely many Selmer data and Y_k finitely many Stickelberger systems, M a torsion level for T, and let $m \ge 1$ and $\varepsilon > 0$. There are an auxiliary set Q and a character ψ such that each of $\psi, \psi^2, \dots, \psi^m$ is a small twist with bound ε.*

Proof. Let s_0 be the integer of Theorem 44.6. Choose a rational $r > 0$ with $s_0 r \le \varepsilon/2$ and $s_0 r < 1$, then h_0 by Proposition 44.9(b) for s_0, the radius r, the bound $\sum_i(\beta_{T_i} + s_0 r) + \sum_k(\beta_{Y_k} + s_0 r)$, the error $\varepsilon/2$ and m, and only then $N \ge h_0$ large enough for Theorem 44.6. The twist polynomials come with points at which $v(P) \le \beta + s_0 r$; Proposition 44.9(b) gives roots of unity $\zeta_{\mathfrak{q}}$ of order dividing ℓ^{h_0}, hence ℓ^N, with $v(\zeta_{\mathfrak{q}} - 1) \le r/m$ and $v(P((\zeta^j - 1))) \le \beta + s_0 r + \varepsilon/2 \le \beta + \varepsilon$ for all data, systems and $1 \le j \le m$. By Lemma 44.7(e), $v(\zeta_{\mathfrak{q}}^j - 1) \le r$ for $j \le m$. Lemma 44.12(c) for ψ^j, the twist by the $\zeta_{\mathfrak{q}}^j$, concludes. □

44.6. Stickelberger systems of L-values

A *critical pair* $p = (\nu, z)$ is a character ν of $U(1)$ of weight $-c$ with $1 \le z \le c$; it is *ramified* if the tame part of ν is nontrivial exactly at the chosen places. For a twist ξ with modulus Q the ratio $\varrho_p(\xi, s) = |L(z, (\nu\xi^s)^{-1})|/|L(z, \nu^{-1})|$ is taken without the Euler factors at the chosen places and, in the numerator, at Q.

Lemma 44.14 (Measures and characters). *Let $p = (\nu, z)$ be a ramified critical pair.*

(a) ($z \ge 2$.) *The integral algebraic part is nonzero at the trivial character, is an integral measure at each modulus, is multiplied by $(1 - a\,\mathrm{N}\,v^{-z})(1 - a^{-1}\,\mathrm{N}\,v^{-z})$ when a place v is added, with a the value at a uniformiser of the base change of $\nu\vartheta$, and at a twist equals its value at 1 times the ratio ϱ_p.*

(b) ($z = 1$.) *The determinant period $\Lambda(\vartheta)$ on the rank-two unitary group of $\nu\vartheta v$ (weight $c - 1$, v the splitting character restricted to $\mathrm{U}(1)$) is nonzero, and there is $C \ne 0$ independent of ϑ with $C\|\Lambda(\vartheta)\|^2$ equal to the norm of the modulus times $|L(1, (\nu\vartheta)^{-1})|^2$ without the Euler factors at the modulus and the chosen places; adding a place of norm q multiplies Λ by $(1 - X)(1 - qX)$, X^{-1} the Frobenius value; and some $N \ne 0$ makes $\|n_{\mathrm{ch}} N \Lambda\|^2$ an integral measure.*

(c) *The product of the norms of an auxiliary set avoiding the bad places is prime to ℓ and to the level. An anticyclotomic Galois character of auxiliary modulus I trivial above ℓ is the avatar of the inverse of a tame character of weight 0 and modulus I.*

(d) *The character $\vartheta = \vartheta_{p,\delta}$ attached to the base change of $\nu^{-\delta}$ is unramified outside the bad places; at the tame generator of a chosen place w not above ℓ it is a root of unity $\ne 1$ of order dividing $\mathrm{N}\,w - 1$, so $v(\vartheta(g_0) - 1) \le v(\mathrm{N}\,w - 1)$; and at a Frobenius φ off the bad places and the modulus of τ, $\vartheta(\varphi)\tau_{\mathrm{av}}^{-1}(\varphi) = \iota(y)^\delta$ for the value y of the base change of $\nu\tau^\delta$ at the chosen uniformiser.*

Proof. (a) The integral algebraic part is an integral measure by Proposition 35.15(a). By Proposition 35.13(c) it is nonzero, and the ratio of the L-values at a twist is $L_p^{\mathrm{alg}}(\vartheta)/L_p^{\mathrm{alg}}(1)$. Adding v, the coefficients of the Euler product at $\{v, \bar{v}\}$ are set to zero, which multiplies it by the two inverse Euler factors; the places removed for the enlarged modulus are those for the old one together with v and $\bar{v}$.

(b) $\nu\vartheta v$ has weight $-(c - 1)$ and the product of the tame parts. The exchange identity gives an element γ with $\gamma\Lambda = 1$ and $\|\gamma\|^2 = C/(\text{the } L\text{-value product})$ (Lemma 34.1), and the insertion formula is Proposition 35.13(d) in rank two. The period is a weighted sum of kernel values at points depending only on the modulus (Proposition 35.13(d)); moving each point by a rational element until it is integral, N times a value is integral, and n_{ch} over the order of the

automorphism group of a class clears the weights; so on characters of a given modulus F is $\sum_a c_a \vartheta(u_a)$ with c_a integral away from the modulus.

(c) If the prime $\mathrm{N}\,w$ divided ℓMN, then w would be bad. The character η with $\psi(\mathrm{Art}\,x) = \eta(x/\bar{x})$, restricted to U(1), kills rational points and integral points $\equiv 1$ modulo I, so it factors through the class group of tame level I and comes from a tame character μ; then ψ is the avatar of μ^{-1}.

(d) The residue characters of ν are trivial off the chosen places, so the base change is unramified off the bad places, and ϑ kills inertia there: the class of an inertial element is the local Artin symbol of a unit. At w the tame generator is the Artin symbol of a unit whose residue generates the residue units, and the residue character of ν at w is nontrivial (the pair is ramified), of order dividing $\mathrm{N}\,w - 1$. Distinct roots of unity of order dividing n are at distance at least $\|n\|$, so $v(\vartheta(g_0) - 1) \le v(\mathrm{N}\,w - 1)$. At φ, replace φ by a geometric Frobenius, where the avatars are known, and multiply the two values. □

Proposition 44.15 (L-value systems). (a) (*Twist measures.*) *For a ramified critical pair there is a function F of tame characters, nonzero at the trivial character, that is at each modulus an integral measure, is multiplied by inverse Euler factors with denominators q^l, $l \in I$, when a place of norm q is added, and has $F(\vartheta) = F(1)\,\mathrm{N}\,Q(\vartheta)^m \varrho_p(\vartheta)^{m+1}$. For $z \ge 2$, $I = \{z\}$, $m = 0$ and F is an integral multiple of the algebraic part of the L-value; for $z = 1$, $I = \{0, 1\}$ and $m = 1$.*

(b) (*Galois characters.*) *For I auxiliary for the bad places, every locally constant ψ with $\psi(\mathrm{Art}\,x) = \eta(x/\bar{x})$ for an idele class character η of modulus I trivial above ℓ is the Galois character of a tame character of weight 0 and modulus I.*

(c) (*L-value systems.*) *Let w be a chosen place not above ℓ and g_0 its tame generator. There is a Stickelberger system for the bad places and g_0 with $\beta_Y \le v(1(\Theta_\emptyset)) + 4v(\mathrm{N}\,w - 1)$ and, for every twist ξ of auxiliary modulus Q whose Galois character ψ of ξ^{-1} is a character for Q,*

$$\|\psi(\Theta_Q)\| = \|1(\Theta_\emptyset)\|\,\|\iota_\ell(\varrho_p(\xi, \delta))\|^i, \qquad i = 2 \text{ if } z = 1, \quad i = 1 \text{ otherwise.}$$

Proof. (a) is Lemma 44.14(a) for $z \ge 2$, with $I = \{z\}$ and $m = 0$, and for $z = 1$ the function $F = \|n_{\mathrm{ch}} N\Lambda\|^2$ of Lemma 44.14(b): adding a place multiplies it by $|(1 - X)(1 - qX)|^2$, whose factors, read through the conjugate pair of places, are the inverse Euler factors with denominators q^0 and q^1; and at a twist $F(\vartheta) = F(1)\,\mathrm{N}\,Q(\vartheta)\varrho_p(\vartheta)^2$, the constant C cancelling.

(b) is Lemma 44.14(c).

(c) At an auxiliary modulus I the measure is given by points u_j and coefficients α_j integral at ι_ℓ, the norms of the places of I being prime to ℓ; let Θ_I be the image of $\sum_j \alpha_j [u_j]$ under $u \mapsto u^{-\delta}$ in the group ring of Cl_I. Then $\psi(\Theta_I) = \iota F(\tau^\delta)$ for ψ the avatar of τ^{-1}, by (b) and Lemma 44.10(b); in particular $1(\Theta_\emptyset) = \iota F(1) \ne 0$. Let $\vartheta = \vartheta_{p,\delta}$, unramified off the bad places with $\vartheta(g_0) \ne 1$ (Lemma 44.14(d)), and the exponents those of I. Adding a place v multiplies $\iota F(\tau^\delta)$ by $\prod_e \prod_\pm (1 - a^{\pm 1}\,\mathrm{N}\,v^{-e})$ with $a = \vartheta(\varphi)\psi(\varphi)$, by the Euler factors of (a) and Lemma 44.14(d): this is the norm relation. As at most two exponents occur and $v(\vartheta(g_0) - 1) \le v(\mathrm{N}\,w - 1)$, $\beta_Y \le v(1(\Theta_\emptyset)) + 4v(\mathrm{N}\,w - 1)$. At a twist ξ of auxiliary modulus Q, ξ^δ is again a twist and $\mathrm{N}\,Q$ is an ℓ-adic unit, so $\|\psi(\Theta_Q)\| = \|\iota F(\xi^\delta)\| = \|\iota F(1)\|\|\iota\varrho_p(\xi, \delta)\|^{m+1}$ with $m + 1 = i$. □

44.7. The twisting theorem

Put $e'_\ell = (4f+3)\mathrm{r}\, v_\ell(n_{\mathrm{ch}}) + 2^{-\ell}$, the twist error of Section 40.3.

Lemma 44.16 (Passing to the main summands). (a) *A relation that holds on the crystalline cocycles of u above ℓ, with coefficients mapped to O', holds on those of the twist of $u \otimes O'$ by a character trivial on the decomposition groups above ℓ. If a set of relations contains the images of the class relations of T, the corresponding Selmer module is no longer than $\mathrm{Sel}_Q(T\psi)$.*

(b) *A character for an auxiliary modulus Q is trivial at γ_*; a small twist satisfies the auxiliary twist conditions.*

(c) *If $\mathrm{Sel}_Q(T\psi)$ has length at most b over all large rings, then over all large rings some lattice in ψR on which γ_* acts by ζ_* has crystalline Selmer group for Q and γ_* of length at most b.*

(d) *$4v(p_i - 1) + 2^{-\ell} \le e'_\ell$ for $i = 0, 1$. The avatar of $(\tau^j)^{-1}$ is the jth power of that of τ^{-1}, and if it is nontrivial on inertia above $w \in Q$ the tame part of τ at w is nontrivial.*

(e) *For finitely many representations like main summands there are a coefficient ring and Selmer data T_i for the bad places and the tame generator at a chosen place w prime to ℓ, with $\beta_{T_i} \le c = (4f+3)\mathrm{r}\, v(n_{\mathrm{ch}})$, and $c+2$ is a torsion level for T.*

Proof. (a) O' is finite and torsion-free over O, hence projective, so it suffices to test the relation with each O-linear form $l\colon O' \to O$. As the character is trivial on the decomposition group, l is equivariant there and $l \circ z$ is a crystalline cocycle of u for each crystalline cocycle z of the twist. Both modules have the same places, so only the relations differ, and more relations give a smaller module.

(b) γ_* is inertial at a place of S_0 prime to ℓ, where the character, of modulus Q disjoint from S_0 and $\bar{S}_0$, is unramified, and its inertia elements are Artin symbols of units; the second claim keeps two clauses of the definition.

(c) The lattice is $T\psi$, conjugate to ψR; $\psi(\gamma_*) = 1$ by (b), so γ_* still acts by ζ_*, and (a) compares the Selmer groups.

(d) $p_i - 1$ divides n_{ch} (Lemma 37.4(b)), $4 \le (4f+3)\mathrm{r}$, and $2^{-\ell}$ appears on both sides. The avatar of a power is computed on Artin symbols, which exhaust the abelianised Galois group. An inertial element above w is the local Artin symbol of a unit, and the avatar's value there is the tame part of τ at the residue.

(e) Proposition 43.13 over all large rings, and arbitrarily large rings have a large residue field; the bound c gives $c + 2 > \beta_{T_i} + 1$. □

Theorem 44.17 (The twisting theorem). *Let D be admissible data, $\iota_\ell\colon \mathbb{C} \to \bar{\mathbb{Q}}_\ell$ an isomorphism and R_i finitely many ℓ-adic representations like main summands. There is a character τ of $U(1)(\mathbb{A})$ of weight 0, tamely ramified on a finite set Q, such that each power $\xi = \tau^j$, $1 \le j \le m$, is ramified at every place of Q and satisfies:*

(a) *the Galois character ψ of ξ^{-1} is anticyclotomic and trivial above ℓ; the places of Q have prime norm; Q, $\bar{Q}$ and the bad places are pairwise disjoint; and $\sum_{w \in Q} v(\psi(\gamma_w) - 1) \le e'_\ell$;*

(b) *for each i there are an inertial element γ away from ℓ and a scalar ζ, $\zeta - 1$ a unit, such that over all large coefficient rings γ acts by ζ on a lattice in ψR_i whose crystalline Selmer group for Q and γ, with coefficients in $\mathrm{Frac}(O)/O$, has length at most e'_ℓ;*

(c) *for each ramified critical pair $p = (\nu, z)$ with $z \le \mathrm{t}$, $z \le c \le 4b$, and each sign s, $v(\iota_\ell \varrho_p(\xi, s)) \le e'_\ell$, that is, $\|\iota_\ell \varrho_p(\xi, s)^{-1}\| \le \ell^{e'_\ell}$.*

Proof. Let T_i, $c = (4f + 3)\mathrm{r}\, v(n_{\mathrm{ch}})$ and the chosen place w, prime to ℓ, with tame generator g_0 be those of Lemma 44.16(e). For each ramified pair p and sign δ take the L-value system $Y_{p,\delta}$ of Proposition 44.15(c), with $\beta_Y - v(1(\Theta_\emptyset)) \le 4v(\mathrm{N}\, w - 1)$. There are finitely many ramified pairs (Proposition 10.16). Apply Theorem 44.13 with the given m and $\varepsilon = 2^{-\ell}$, and let ψ be the character. By Lemma 44.14(c) ψ is the avatar of the inverse of a tame character τ, and by Lemma 44.16(d) the avatar of $(\tau^j)^{-1}$ is ψ^j, which is ramified at every place of Q, so the tame part of τ^j is nontrivial there: τ is an m-fold twist. (a) is Lemma 44.16(b) for ψ^j, with $\sum_{w \in Q} v(\psi^j(\gamma_w) - 1) \le \varepsilon \le e'_\ell$. (b) is Lemma 44.16(c) with $b = \beta_{T_i} + \varepsilon \le c + \varepsilon \le e'_\ell$, the scalar ζ_* with $\zeta_*^2 - 1$ a unit. (c) By the norm identity of Proposition 44.15(c) at $\xi = \tau^j$, $\|\psi^j(\Theta_Q)\| = \|1(\Theta_\emptyset)\|\|\iota_\ell \varrho_p(\xi, \delta)\|^i$, with $1(\Theta_\emptyset) \ne 0$ integral and $i \ge 1$; so $v(\psi^j(\Theta_Q)) \le \beta_Y + \varepsilon$ gives $i\, v(\iota_\ell \varrho_p) \le \beta_Y - v(1(\Theta_\emptyset)) + \varepsilon \le 4v(\mathrm{N}\, w - 1) + 2^{-\ell} \le e'_\ell$, as $\mathrm{N}\, w$ is one of p_0, p_1 (Lemma 44.16(d)); and $v(\iota x) \le e'_\ell$ gives $\|\iota x^{-1}\| \le \ell^{e'_\ell}$. □

45. Regular families of twists

This section proves Theorem 36.2. For each prime place ι over ℓ the twisting theorem gives an m-fold twist whose main summands have short Selmer groups and whose ratios of L-values are ℓ-adically large; the bound for denominators of Section 40 turns the Selmer bound into a clearing bound β_ℓ; and the clearing bounds, summed over ℓ with weights $\log \ell$, stay within the denominator budget.

45.1. The eigensystems of the lifts

Proposition 45.1 (Splitting the space of lifts). *Let ϑ be a twist and $1 \le j \le m$.*

(a) *The space of lifts for ϑ and j, the range of the theta lift of the good forms on W_j for the descent pair of ϑ and j, is spanned by forms y such that y lies in the range of the type projection of a tuple c of labels in lift setting, and at the split places off the level the Hecke polynomial of y is that of a lift of a system q of Hecke eigenvalues occurring in a good form on W_m.*

(b) *The systems q in (a) are finitely many and do not depend on ϑ or j.*

(c) *For a main pair p for j and the descent pair of the trivial twist, the trivial ℓ-adic character of rank one is of the type of the labels $X - 1$, and some rank one representation is a main summand for the lift of p.*

Proof. (a) The space of lifts is the range of the lift $L^{\mathrm{good}}_{\vartheta,j}$ of the good forms on W_j, those of principal level at the chosen places in the image of the idempotent e_0, the sum of the idempotents e_c of the good tuples c of labels. The good forms decompose into joint eigenspaces of the Hecke operators (Proposition 23.1), and e_0 is the sum of the e_c; so the range is spanned by the lifts y of good eigenforms on W_j that lie in the image of some e_c, which is contained in that of e_0. The labels c for which some twisted label has the root 1 contribute false lifts or zero (Theorem 30.5), so the remaining c are in lift setting for the descent pair of ϑ and j (Lemma 37.4(a)). The lift intertwines the Hecke operators at the split places off the level with the lift of the Hecke polynomials (Proposition 30.2(c)),

so the Hecke polynomial of y is that of a lift of the eigensystem q of its source. This is Theorem 37.7(d).

(b) The eigensystem q of a nonzero good eigenform on W_j also occurs in a good form on W_m: W_j embeds into W_m compatibly with the Hecke action off the level and with the good idempotent, so the image of the eigenform is a nonzero good form on W_m with the same eigenvalues. The good forms on W_m form a finite-dimensional space, so the systems occurring there are finitely many (Proposition 23.1); they depend neither on ϑ nor on j.

(c) For the descent pair of the lift of p from U(1) to U($2z$), with the trivial twist, take for every chosen place the label $X - 1$. The trivial ℓ-adic representation of rank one is conjugate self-dual with the parameter 1; it satisfies the local conditions of the empty level at all finite places, with the single jump 0 at each embedding; and inertia at a chosen place acts through 1, the only root of $X - 1$. Hence it is of the type of these labels. The main pair is in lift setting for these labels (Section 37.2). After the twist of Theorem 37.7(b), with $r_O = 1$ and the string of length $2z - 1$, the jump 0 becomes $0 - (1 - z - \epsilon_\sigma(-c)) = z - 1 - \epsilon_\sigma c$, the main summand weight of rank one of Theorem 37.7(a); so Theorem 37.7(b), (c) give the rank one main summand. □

45.2. A twist for one prime

Theorem 45.2 (A twist with few congruences at one prime). *For a prime ℓ and $\iota\colon \mathbb{C} \simeq \bar{\mathbb{Q}}_\ell$ there is an m-fold twist τ such that $\mathrm{N}\, Q(\tau)$ is prime to ℓ and, for $\vartheta = \tau^j$, $1 \le j \le m$:*

(a) *for every ramified pair p and sign s, $\|\varrho_p(\vartheta, s)^{-1}\|_\iota \le \ell^{e'_\ell}$;*

(b) *the projection to the space of lifts for ϑ and j has denominator at most ℓ^{β_ℓ} at ι, and for every main pair p for j the inner product of the projections of two test forms to the line of the lift $\varphi_{p,\vartheta}$ has norm at most $\ell^{\beta_\ell} h_\ell$ at ι, where h_ℓ is the ℓ-part of the class number of $U(1)$ at the chosen level.*

Proof. The family of main summands. Let $\mathcal{S}$ be the finite set of systems of Proposition 45.1(b). For each $1 \le j \le m$ and $q \in \mathcal{S}$ let $R_{j,q}$ be a main summand for the j-th descent pair of the trivial twist and q (Theorem 37.7(c)), and for each main pair p for some j let R'_p be the rank one main summand of Proposition 45.1(c). All of them are like main summands, and the main pairs are finitely many (Proposition 10.16). The twisting theorem (Theorem 44.17) for the joined family gives an m-fold twist τ such that every $\vartheta = \tau^j$ has (i) auxiliary modulus, a character ψ of ϑ^{-1} from the norm-one torus, and ramification at most e'_ℓ; (ii) for each member of the family, a lattice in its twist by ψ whose crystalline Selmer group with divisible coefficients has length at most e'_ℓ; (iii) the bound (a).

The modulus. The places of $Q(\tau)$ have prime norm and avoid the bad places, which contain those above ℓ; so $\mathrm{N}\, Q(\tau)$ is prime to ℓ (Lemma 44.14(c)).

The space of lifts. Fix j and $\vartheta = \tau^j$. The descent pairs of ϑ and of the trivial twist differ by the twist of their character of U(1) by ϑ, so a main summand $R_{j,q}$ for the trivial twist, twisted by ψ, has the Selmer bound (ii) for ϑ. By Proposition 45.1(a) the space of lifts Z is spanned by forms y in the image of e_c for labels c in lift setting, which lift a system $q \in \mathcal{S}$; with $R_{j,q}$, each such y is a lift generator in the sense of Section 40. Theorem 40.7(b), with ramification and Selmer length at most e'_ℓ, gives an operator Λ on Z with $\|\iota \det \Lambda\| \ge \ell^{-\dim Z \cdot r\beta_\ell}$, r the rank of W_m, that clears the denominators of the projection to Z on integral

forms orthogonal to the false lifts for e_0; a fortiori the same holds with any bound $\beta' \geq \beta_\ell$.

The line of a main pair. Let p be a main pair for j. The lift $\varphi_{p,\vartheta}$ has the labels $X - 1$ at the chosen places and lifts the constant system 1 (Proposition 45.1(c)), and the lattice of R'_p twisted by ψ has the Selmer bound (ii). Theorem 40.7(b) for the one-dimensional space spanned by $\varphi_{p,\vartheta}$ and the idempotent of the labels $X - 1$ gives an operator with $\|\iota \det\| \geq \ell^{-\beta_\ell}$ that clears the denominators of the projection to this line on integral forms orthogonal to the false lifts. The classes of U(1) at the chosen level have trivial automorphism groups (Proposition 16.2), so the inner product of two integral forms is 1/#classes times an integral sum (Proposition 35.13); with the clearing operator, the inner product of the projections of two test forms to the line has norm at most $\ell^{\beta_\ell} h_\ell$ at ι, where $h_\ell = \ell^{v_\ell(\#\text{classes})}$. □

45.3. Summing over the primes

Lemma 45.3 (Weighted sums of valuations). *Let T be a finite set of primes.*

(a) *If $1 \leq a \leq b$ are integers, $\sum_{\ell \in T} v_\ell(a) \log \ell \leq \log b$.*
(b) $\sum_{\ell \in T} 2^{-\ell} \log \ell \leq 2$.
(c) *The product Π of the integers in $(b - 30h^3, b + 30h^3]$, for $b \geq 0$ and the truncated difference, is at most $(b + 30h^3)^{60h^3}$.*

Proof. (a) $\sum_{\ell \in T} v_\ell(a) \log \ell$ is at most the sum over all primes dividing a, which is $\log a$ by unique factorisation, and $\log a \leq \log b$. (b) $\log \ell \leq \ell$, and $\sum_{n \geq 1} n 2^{-n} = \frac{1/2}{(1-1/2)^2} = 2$. (c) There are at most $60h^3$ factors, each at most $b + 30h^3$. □

Proposition 45.4 (The error terms are negligible). *Let x be the length of the prime sum.*

(a) $\sum_{\ell \in T} \mathrm{loc}_\ell \log \ell \leq 4 \cdot 10^6 e^{180} f\,(m/k)\, b \log b$ *for every finite set T of primes.*
(b) *The odd class bound is at most $2^{4(N_E + \mathrm{t} + 3)^{27}} h_E$, and its ℓ-adic valuations are negligible with exponent* 2000.
(c) *The twist errors e'_ℓ are negligible with exponent* 999, *the denominator errors e_ℓ with exponent* 4000, *and the valuations $v_\ell(h)$ of the class number of $U(1)$ at the chosen level with exponent* 30.
(d) $\sum_{\ell \in T} (\beta_\ell + v_\ell(h)) \log \ell \leq$ budget *for every finite set T of primes.*

Proof. All sums are over a finite set T of primes. (a) By definition $\mathrm{defect}_\ell = 10^6 \frac{m}{k} \big(b\, v_\ell(M) + v_\ell((2b)!\, \Pi^{h^3})\big)$ with $h = \mathrm{r}^8$, and by Lemma 45.3(a), (c),

$$\sum_{\ell \in T} \mathrm{defect}_\ell \log \ell \leq 10^6 \frac{m}{k} \Big(b \log M + \log\big((2b)!\, (b + 30h^3)^{60h^6}\big) \Big);$$

the bracket is at most $e^{180} b \log b$ by Proposition 8.9, and $\mathrm{loc}_\ell = 4f\, \mathrm{defect}_\ell$. (b) The bound for the odd class bound is Theorem B.39(c). By Lemma 45.3(a) the weighted sum of its valuations is at most $\log(2^{4(N_E + \mathrm{t} + 3)^{27}} h_E) = 4(N_E + \mathrm{t} + 3)^{27} \log 2 + \log h_E$, and $\log h_E \leq x^{30}$; this is at most x^{2000} by Proposition 8.9. (c) For the twist errors, by Lemma 45.3(a), (b) the weighted sum is at most $(4f + 3)\mathrm{r} \log n_{\mathrm{ch}} + \sum_\ell 2^{-\ell} \log \ell \leq (4f + 3)\mathrm{r} \log n_{\mathrm{ch}} + 2$. The denominator error is t times the valuation of the odd class bound plus $c\, v_\ell(n_{\mathrm{ch}})$, with $c = 8f\mathrm{t} + 2 + (4\mathrm{t} + 1)h^{22} + h$ (Section 40.4), so its weighted sum is at most $\mathrm{t}\, x^{2000} + c \log n_{\mathrm{ch}}$,

by (b) and Lemma 45.3(a). The class number of U(1) at the chosen level is at most $\exp(x^{30})$, so its weighted sum is at most x^{30}. The three bounds are at most x^{999}, x^{4000} and x^{30} by Proposition 8.9, as $\log n_{\mathrm{ch}} \le x^{32}$ (Proposition 16.2). (d) By definition $\beta_\ell = \mathrm{loc}_\ell + (1+2\mathrm{t})e'_\ell + e_\ell + v_\ell(n_{\mathrm{ch}})$, so by (a), (c) and Lemma 45.3(a) the sum is at most

$$4 \cdot 10^6 e^{180} f \, \frac{m}{k} \, b \log b + (1+2\mathrm{t})x^{999} + x^{4000} + \log n_{\mathrm{ch}} + x^{30},$$

which is at most budget $= f a \log^3 b/(mk)$ by Proposition 8.9. □

45.4. Proof of the existence of regular families

Proof of Theorem 36.2. For each prime place ι over ℓ let ξ_ι be the twist τ of Theorem 45.2. Its modulus is prime to ℓ. For a ramified pair p, a sign s and $1 \le j \le m$ the family $\varrho_p(\xi_\iota^j, s)^{-1}$ has height at most x_1: take $H = 1$, $\alpha = 0$ and $h_\ell = e'_\ell$ in Definition 7.2; condition (ii) is Theorem 45.2(a), and (iii) holds since $\sum_{\ell \in T} e'_\ell \log \ell \le x^{999} \le x_1$ by Proposition 45.4(c) and Proposition 8.9. Hence the family has small ratios. With $\beta'_\ell = (\beta_\ell + v_\ell(h)) \log \ell$ the denominators of Theorem 45.2(b) are at most $e^{\beta'_\ell}$, and $\sum_{\ell \in T} \beta'_\ell \le$ budget by Proposition 45.4(d). Hence the family is regular. □

Appendices

Appendix A. Discrete series with a large gap

This appendix proves the lower bound for periods of discrete series coefficients that enters Theorem 24.4(c): over the group of a *stationary position*, the volume of a large ball is at most a power of the parameters times the modulus of the period (Theorem A.9). The proof is Laplace's method. Near the identity a coefficient with a large gap is an explicit phase times a number close to the interval $[0,1]$ (Theorem A.3); in a stationary position the phases of the two coefficients cancel against the character of the unipotent part (Proposition A.7); so the integrand lies in a cone about the positive axis, up to an error that decays away from the compact subgroups (Proposition A.4). Integration and a covering argument give the bound (Theorem A.6). The restriction from $\mathrm{U}(\varepsilon \oplus 1)$ to $\mathrm{U}(\varepsilon)$ is the special case of one vector (Theorem A.10). Related bounds, for fixed rank, are in [NV21, He17, Sun07]; what is used here is that all losses are powers of the weights.

A.1. Coefficients, gaps and decay

Let $\varepsilon\colon n \to \{\pm 1\}$ be a vector of signs, $\mathrm{U}(\varepsilon)$ the unitary group of $\mathrm{diag}(\varepsilon)$, and $K = \mathrm{U}(\varepsilon) \cap \mathrm{U}(n)$ its maximal compact subgroup, the block diagonal unitary matrices, with Haar measure of mass one. For a matrix g write a_g for its *compact part*, the entries g_{ij} with $\varepsilon_i = \varepsilon_j$, and $y_g = g - a_g$ for its *noncompact part*. A doubled Harish-Chandra parameter is an integer vector $l = 2\lambda$ with distinct entries congruent to $\#n - 1$ modulo 2; it has *gap* A if $|\lambda_i - \lambda_j| \geq A$ whenever $\varepsilon_i \neq \varepsilon_j$, and *large gap* h if it has gap $500(\#n)^3 + h$. Let ρ be the half sum of the roots positive for l, ρ_c that of the compact ones, and $\Lambda = \lambda + \rho - 2\rho_c$ the highest weight of the lowest K-type. For an integer weight w let $m_w(g) = \prod_t \det_{\{w \geq t\}} g$, the product of the principal minors on the sets $\{i : w_i \geq t\}$, those with $t \leq 0$ divided by $\det g$. *Flensted-Jensen's coefficient* [FJ80] is

$$\Xi_l(g,z) = \frac{1}{\|m_\Lambda\|_K^2} \int_K m_\Lambda(gk)\,\overline{m_\Lambda(k)}\, m_{-\lambda-\rho}(k^* z k)^{-1}\, dk, \qquad c_l(g) = \Xi_l(a_g, a_g^{-1} g),$$

the coefficient of the highest weight vector of the lowest K-type, normalised by $c_l(1) = 1$.

Proposition A.1 (Size and decay). *Let l have gap A and $g \in \mathrm{U}(\varepsilon)$.*

(a) $|c_l(g)| \leq 1$; *for every integer* $0 \leq c \leq A$, $|c_l(g)|^2 |\det a_g|^c \leq 1$ *and* $|c_l(g)| \leq (2/\|g\|)^c$ *in the operator norm.*

(b) *If l has large gap h, then* $|c_l(g)|^4 \leq (1 + \|y_g\|^2)^{-h}$ *in the Frobenius norm.*

(c) (*Lipschitz.*) *If l has large gap and entries at most L, then* $|c_l(xg) - c_l(x)| \leq 100(\#n + L + 1)^5 \|g - 1\|$; *in particular* $|c_l(g)| \geq 1/2$ *when* $\|g - 1\| \leq (200\beta^5)^{-1}$ *in the operator norm and* $\beta \geq \#n + L + 1$.

(d) (*The lowest K-type.*) *Under the hypotheses of (c),* $\|m_\Lambda\|_K^2 \geq \big(4(2000(\#n + L + 1)^6)^{2\#n^2}\big)^{-1}$; *hence* $\|m_\Lambda\|_K^2 \geq e^{-5000T}$ *for every* $T \geq \max(1, \#n^2, \#n^2 \log(\#n + L + 1))$.

Proof. (a) is the square-integrability estimate for Flensted-Jensen's function: the integrand is bounded by Hadamard's inequality for the minors, and each minor of $a_g k$ grows like a power of $\det a_g$ along the noncompact directions, which the gap converts into decay [FJ80]. (c) Join 1 to $g = 1 + A$ by the curve $\gamma(t) =$

$1+tA(1-\frac{t-1}{2}A)^{-1}$, which stays in $\mathrm{U}(\varepsilon)$ because $(\gamma u)^*\mathrm{diag}(\varepsilon)(\gamma u)-u^*\mathrm{diag}(\varepsilon)u = t(g^*\mathrm{diag}(\varepsilon)g-\mathrm{diag}(\varepsilon))=0$ for $u=1-\frac{t-1}{2}A$. The derivative of $c_l(x\gamma(t))$ is the value of a Lie algebra element of norm at most $12(\#n+L+1)^5\|A\|$ applied to the highest vector of the lowest K-type, a vector of norm one, and the mean value inequality gives the bound; at $x=1$, $c_l(1)=1$. (d) K has mass one and lies in the unit ball, and on K the coefficient $c_l(k)$ is $m_\Lambda(k)$. By (c), $|c_l|\ge 1/2$ on a ball of radius $(200\beta^5)^{-1}$ about 1, and the covering argument of Lemma A.5 with $D=200\beta^5$, $R=1$, applied to K, gives $1\le(2000(\#n+L+1)^6)^{2\#n^2}\int_K 4|c_l|^2$. Taking logarithms, $\log 2000\le 2000$ and $\log 4\le 4$ give the second bound. (b) The block of $g^*\,\mathrm{diag}(\varepsilon)\,g=\mathrm{diag}(\varepsilon)$ that commutes with $\mathrm{diag}(\varepsilon)$ reads $a_g^*a_g=1+y_g^*y_g$. If s_i are the singular values of y_g, then $|\det a_g|^2=\det(1+y_g^*y_g)=\prod_i(1+s_i^2)\ge 1+\sum_i s_i^2=1+\|y_g\|^2$. By (a) with $c=h\le A$, $|c_l(g)|^4\le|\det a_g|^{-2h}\le(1+\|y_g\|^2)^{-h}$. □

A.2. Principal minors near the identity

For a set S of indices let P be the coordinate projection to S, $\det_S$ the principal minor on S, and $\|[P,Z]\|$ the Frobenius norm of the two blocks of Z that cross S. For an injective $o\colon n\to\mathbb{Z}$ and a matrix Z let u and v be the strictly lower and upper triangular parts, for the order of o, of the compact part of Z, and $\hat g=\exp(-u)\exp(Z)\exp(-v)$.

Lemma A.2 (One minor). (a) *For unitary k, $|\det_S k|^2\le\exp(-\|[P,k]\|^2/2)$.*

(b) *Let Q be an orthogonal projection, $\delta\le 1/20$, and A_1,A_2,A_3 of norm at most δ with $\|[Q,A_i]\|\le\zeta$. Then*

$$\det\bigl(1+Q(e^{A_1}e^{A_2}e^{A_3}-1)\bigr)=\exp\Bigl(\sum_i \mathrm{tr}\,QA_i+\sum_{i\le j}{}'\,\mathrm{tr}\,QA_i(1-Q)A_j+r\Bigr),$$

with $|r|\le 10^4\delta\zeta^2$, where $\sum'$ halves the terms with $i=j$.

(c) *Let o be smaller on S than off S, Z in the Lie algebra of $\mathrm{U}(\varepsilon)$ with $\|Z\|\le\delta\le 1/20$, and $k\in K$. Then $\det_S(k^*\hat g k)=\exp\bigl(\mathrm{tr}\,PZ+\|[P,Z]\|^2/4+r\bigr)$ with $|r|\le 10^5\delta(\|[P,Z]\|^2+\|[P,k]\|^2)$.*

(d) *If l has gap h and $\|Z\|\le 1/20$, then $\Xi_l(1,\exp Z)=\Xi_l(1,\hat g)$ for $o=l$.*

Proof. (a) Put $M=(1-P)kP$ and $Y=(1-P)+PkP$. Then $\det_S k=\det Y$, and since $k^*k=1$, $Y^*Y=(1-P)+Pk^*PkP=1-M^*M$. For unitary k the two crossing blocks have the same norm: $\|(1-P)kP\|^2=\mathrm{tr}\,P-\mathrm{tr}\,Pk^*PkP=\|Pk(1-P)\|^2$, so $\|[P,k]\|^2=2\|M\|^2$. If $s_i\le 1$ are the singular values of M, $|\det Y|^2=\prod_i(1-s_i^2)\le\exp(-\sum_i s_i^2)=\exp(-\|[P,k]\|^2/2)$.

(b) Write $e^{A_1}e^{A_2}e^{A_3}=1+A+R$ with $A=\sum_i A_i+\sum_{i<j}A_iA_j+\frac12\sum_i A_i^2$ and $\|R\|\le\delta^3e^{3\delta}$, and $\log\det(1+QX)=\mathrm{tr}\log(1+QX)$ with the logarithm $\int_0^1(1+sM)^{-1}M\,ds$ for $\|M\|\le 1/2$. Expanding to second order, $\mathrm{tr}\,QA-\frac12\,\mathrm{tr}(QA)^2$ equals the stated quadratic expression up to terms of order three, because $\mathrm{tr}\,QA_iQA_j=\mathrm{tr}\,QA_iA_j-\mathrm{tr}\,QA_i(1-Q)A_j$. The rest r is of order three in the A_i. Scale the crossing parts of the A_i by a complex parameter τ: $r(\tau)$ is holomorphic for $|\tau|\le\delta/\zeta$, even in τ (conjugation by $2Q-1$), and bounded by $C\delta^3$; it vanishes at $\tau=0$ to second order because then the A_i commute with Q and the determinant is exactly $\exp\sum\mathrm{tr}\,QA_i$. Schwarz's lemma at radius δ/ζ gives $|r(1)|\le C\delta^3(\zeta/\delta)^2$; if $\delta\le\zeta$ the third order bound suffices directly. Tracking the series gives $C\le 10^4$.

(c) The minor is the determinant of (b) for $A_1 = -u$, $A_2 = Z$, $A_3 = -v$ and $Q = kPk^*$, with $\zeta = \|[P,Z]\| + 2\delta\|[P,k]\|$. Since u and v are strictly triangular inside the blocks and o is smaller on S, $\operatorname{tr} Qu$ and $\operatorname{tr} Qv$ contribute only through the crossing parts of k, and the quadratic sum differs from $\|[P,Z]\|^2/4$ by at most $18\delta\|[P,k]\|\zeta$ plus $\delta\|[P,k]\|^2$; the elementary inequality $\delta\kappa^2 + 18\delta\kappa\zeta + 10^4\delta\zeta^2 \le 10^5\delta(x^2+\kappa^2)$ for $\zeta \le x+\kappa$ gives the bound. (d) The matrices $\exp Z$ and $\hat g$ are near 1, hence accretive, and block diagonal triangular factors multiply the integrand of Ξ_l by a character of their diagonal, which is 1 because the diagonal of a strictly triangular matrix vanishes. □

A.3. The Gaussian shape

For a weight w let $Q_w(X) = \sum_{i,j} |w_i - w_j| |X_{ij}|^2$.

Theorem A.3 (Gaussian shape of the coefficient). *Let l have large gap A and entries at most L, let $T \ge \max(1, \#n^2, \#n^2 \log(\#n + L + 1))$ and $\delta T \le 10^{-12}$. For $g \in \mathrm{U}(\varepsilon)$ with $\|g - 1\| \le \delta$ (Frobenius norm) there is $\varrho \in [0,1]$ with*

$$\left| \frac{c_l(g)}{\exp(\operatorname{tr} \Lambda \log g)} - \varrho \right| \le \frac{\varrho}{10} + e^{-100T}.$$

One may take $\varrho = \exp(-Q_{\lambda+\rho}(\log g)/4)$.

Proof. Put $Z = \log g = \log(1 + (g-1))$; then Z lies in the Lie algebra of $\mathrm{U}(\varepsilon)$ and $\|Z\| \le 2\delta$. Let $q = Q_{\lambda+\rho}(Z)/4 \ge 0$ and $\varrho = e^{-q} \in [0,1]$; write $\mu = \Lambda$.

Step 1: a mean over K. Since g is within δ of 1 it is accretive, and $c_l(g) = \Xi_l(1,g) = \Xi_l(1,\hat g)$ by Lemma A.2(d) with $o = l$. The weight $-\lambda - \rho$ has the order opposite to that of l, so on each set $\{-\lambda - \rho \ge t\}$ the order $o = l$ is smaller than off it. Multiplying Lemma A.2(c) over these sets, and using that $\sum_t$ of the projections is $\operatorname{diag}(-\lambda - \rho)$ up to a constant and $\sum_t \|[P_t, X]\|^2 = Q_{\lambda+\rho}(X)$ (exactly $|w_i - w_j|$ of the sets separate i from j),

$$m_{-\lambda-\rho}(k^* \hat g k) = \exp\bigl(q - \operatorname{tr} \mu Z - r(k)\bigr), \qquad |r(k)| \le 10^6 \delta \bigl(q + Q_\mu(k) + \#n^2\bigr).$$

Here the trace against $\lambda + \rho$ was replaced by that against μ: the difference $2\rho_c$ has entries at most $\#n$, so the traces differ by at most $\#n^2\|Z\|$; and $Q_{\lambda+\rho}(k) \le Q_\mu(k) + 2\#n^2$ for the same reason, since $\|k\|^2 = \#n$. Substituting in Ξ_l,

$$\frac{c_l(g)}{e^{\operatorname{tr} \mu Z}} - \varrho = \frac{1}{\|m_\mu\|_K^2} \int_K |m_\mu(k)|^2 \, e^{-q} \left(e^{r(k)} - 1\right) dk.$$

Step 2: the integrand. Put $m = |m_\mu(k)|^2$; by Lemma A.2(a), multiplied over the sets $\{\mu \ge t\}$ and with $|\det k| = 1$, $m \le e^{-Q_\mu(k)/2}$. Write $Q = Q_\mu(k)$. If $|r(k)| \le 1/20$ then $|e^r - 1| \le \frac{21}{20}|r| \le \frac{1}{10}$. Otherwise $q + Q + \#n^2 > (2 \cdot 10^7 \delta)^{-1} \ge 5 \cdot 10^4 T$, because $\delta T \le 10^{-12}$; as $\#n^2 \le T$, $q + Q \ge 4 \cdot 10^4 T$, and $|r| \le 10^6 \delta (q + Q + \#n^2) \le 2 \cdot 10^{-6}(q+Q)$. Then $e^{-q} m |e^r - 1| \le 2 e^{-q - Q/2 + |r|} \le 2 e^{-(q+Q)/3} \le e^{-5100T}$. In all cases

$$e^{-q} m |e^{r(k)} - 1| \le \frac{e^{-q}}{10} m + e^{-5100T}.$$

Step 3. Integrating over K (mass one) and dividing by $\|m_\mu\|_K^2 \ge e^{-5000T}$ (Proposition A.1(d)) gives $|c_l(g) e^{-\operatorname{tr} \mu Z} - \varrho| \le \varrho/10 + e^{-100T}$. □

A.4. Laplace data and the cone

Let ε, η be signs on n and m, l and k doubled parameters for $\mathrm{U}(\varepsilon)$ and $\mathrm{U}(\eta)$ with lowest weights $\mu = \Lambda_l$ and $\mu' = \Lambda_k$, and $j\colon S \to \mathrm{U}(\varepsilon)\times\mathrm{U}(\eta)$ a homomorphism of a locally compact group, $j(s) = (g, g')$. The parameter l is *steep* for B and a splitting $n = X \sqcup \{1, \dots, M\}$ if on X the signs are $+1$ and $\mu = 0$, while on the other coordinates the signs alternate $+, -, \dots$ and μ decreases strictly in $[-3B, -B]$. The pair (j, φ) is a *Laplace datum* if l is steep; $\mu' \geq 0$, $\mu' \leq 3B$ unless $\mathrm{U}(\eta)$ is compact, and $\mu' = 0$ on a copy $\tau(X)$ of X of sign $+1$; the *phase* $\varphi(s) = \operatorname{tr}\mu(g-1) - \operatorname{tr}\mu'(g'-1)$ is additive and imaginary and $\operatorname{tr}(g-1) = \operatorname{tr}(g'-1)$; and every unitary u on X occurs, with $g-1$ and $g'-1$ equal to $u-1$ placed along X and $\tau(X)$. The *Laplace integrand* is $\theta(s) = c_l(g)\,\overline{c_k(g')}\,e^{-\varphi(s)}$.

Proposition A.4 (The cone). *Let (j, φ) be a Laplace datum with $B \geq 1$, let both parameters have large gap h and entries at most L, let $\#n + \#m + L \leq \beta$, $N = (\#n + \#m)^2$, $T = N\log\beta \geq 1$, $h \geq 10^{30}T^3$, and write $G(s) = (2/\|g \oplus g'\|)^{9N}$.*

(a) *(Localization.) If $0 < r \leq 1/6$ and $\|y_g\|^2 + \|y_{g'}\|^2 \leq r^2$, then $\theta(s) = \theta(t)$ for some t with $\|g_t - 1\|, \|g'_t - 1\| \leq 4r$.*

(b) *(Far.) If $T\rho = 10^{-13}$ and $\|y_g\|^2 + \|y_{g'}\|^2 \geq \rho^2$, then $|\theta(s)| \leq e^{-100T}$.*

(c) *(Near.) If $\|g-1\|, \|g'-1\| \leq \delta$ with $\delta T \leq 10^{-12}$, then $|\theta(s)| \leq 2\operatorname{Re}\theta(s) + 12e^{-100T}$.*

(d) *(Cone and decay.) For every s, $|\theta(s)| \leq G(s)$ and $|\theta(s)| \leq 2\operatorname{Re}\theta(s) + 6\beta^{-50N}G(s)$.*

(e) *If $\|g \oplus g' - 1\| \leq (200\beta^5)^{-1}$, then $|\theta(s)| \geq 1/4$.*

Proof. Since φ is imaginary, $|\theta(s)| = |c_l(g)||c_k(g')|$.

(a) Weight the row relations of $g^*\operatorname{diag}(\varepsilon)g = \operatorname{diag}(\varepsilon)$ by $-\mu$ and those of g' by μ' and add: because $\operatorname{tr}\mu(g-1) - \operatorname{tr}\mu'(g'-1)$ is imaginary the real parts of the traces cancel, and since $-\mu \geq B$ off X and $\leq 3B$, the rows of $g-1$ off X have squared norm at most $6(\|y_g\|^2 + \|y_{g'}\|^2)$, and likewise for g' off $\tau(X)$. The block a of g on X is then within r of a unitary matrix; write $a = k_1 z_1$ with k_1 unitary and $z_1 \geq 0$, and translate s by an element realising k_1^{-1} on X. This changes neither coefficient (the weights vanish on X, so m_μ and $m_{\mu'}$ are invariant under such left translations), nor the phase (it is additive and vanishes on $\mathrm{U}(X)$), nor the noncompact parts. After translation the block on X is z_1, within r of 1; the rows and columns off X contribute $6r^2$ each and the block r^2, so $\|g_t - 1\| \leq 4r$; the equality of traces bounds $g'_t - 1$ in the same way.

(b) By Proposition A.1(b), $|\theta(s)|^4 \leq (1 + \|y_g\|^2)^{-h}(1 + \|y_{g'}\|^2)^{-h} \leq (1+\rho^2)^{-h}$. Since $\log(1+\rho^2) \geq \rho^2/2$ for $\rho \leq 1$ and $h\rho^2/2 \geq 10^{30}T^3 \cdot 10^{-26}T^{-2}/2 = 5000T$, this is at most e^{-400T}.

(c) The parameters $\#n, \#m$ and L satisfy the hypotheses of Theorem A.3 with this T. Since $\overline{c_k(g')} = c_{-k}(g')$ and $\varphi(s)$ equals the phase evaluated at $\log g$ and $\log g'$ (additivity along the one-parameter groups), $\theta(s) = \kappa_l\kappa_k$ with $\kappa_l = c_l(g)/\exp(\operatorname{tr}\mu\log g)$ and $\kappa_k = c_{-k}(g')/\exp(-\operatorname{tr}\mu'\log g')$. By Theorem A.3 there are $a, c \in [0,1]$ with $|\kappa_l - a| \leq a/10 + t$, $|\kappa_k - c| \leq c/10 + t$, $t = e^{-100T} \leq 1$. Hence $|\theta(s) - ac| \leq 0.21ac + 4t$, and a complex number within $0.21p + 4t$ of a real $p \geq 0$ satisfies $|z| \leq 2\operatorname{Re} z + 12t$.

(d) The decay: let the larger of $\|g\|, \|g'\|$ be $\|g\|$; then $|\theta(s)| \leq |c_l(g)| \leq (2/\|g\|)^c$ for $c \leq h$ by Proposition A.1(a), and $\|g \oplus g'\| = \max(\|g\|, \|g'\|)$; take $c = 18N \leq h$. The cone: if $\|y_g\|^2 + \|y_{g'}\|^2 \geq \rho^2$ with $T\rho = 10^{-13}$, (b) gives $|\theta| \leq e^{-100T}$, and every

z with $|z| \le t'$ satisfies $|z| \le 2\operatorname{Re} z + 3t'$. Otherwise (a) with $r = \rho \le 1/6$ moves s to t with both components within 4ρ of 1, and (c) with $\delta = 4\rho$, $\delta T = 4 \cdot 10^{-13}$, applies. In both cases $|\theta| \le 2\operatorname{Re}\theta + 12e^{-100T} = 2\operatorname{Re}\theta + 12a^2$ with $a = \beta^{-50N}$. The decay with $c = 18N$ gives $|\theta| \le G^2$. If $G \ge 2a$, then $12a^2 \le 6aG$. If $G < 2a$, then $2\operatorname{Re}\theta + 6aG \ge -2|\theta| + 3G^2 \ge G^2 \ge |\theta|$. In both cases $|\theta| \le 2\operatorname{Re}\theta + 6aG$, and $|\theta| \le G^2 \le G$ since $G \le 1$ where $\|g \oplus g'\| \ge 2$, while $|\theta| \le 1 \le G$ elsewhere.

(e) By Proposition A.1(c) each coefficient has modulus at least $1/2$, since $\|g - 1\|, \|g' - 1\| \le \|g \oplus g' - 1\|$ and $\beta \ge \#n + L + 1, \#m + L + 1$. □

A.5. Laplace's method

Lemma A.5 (Covering). *Let $J\colon S \to \mathrm{U}(\zeta)$ be a homomorphism to the unitary group of a sign vector on a set of size d, ν a right invariant measure on S, $D, R \ge 1$ and $3DR^2 \le \kappa$. Then*

$$\nu\{s : \|J(s)y\| \le R\} \le \kappa^{2d^2}\, \nu\{s : \|J(s) - 1\| \le 1/D\}$$

for every y. In particular $\nu\{\|J(s)\| \le \beta + 1\} \le \beta^{16d^2}\nu\{\|J(s) - 1\| \le (200\beta^5)^{-1}\}$ for $\beta \ge 10^{12}$, and, if J is proper and ν finite on compacts, for $c > 8d^2$ the function $(2/\|J(s)y\|)^c$ is integrable with integral at most $16^c\nu\{\|J(s) - 1\| \le 1\}$.

Proof. Among the s with $\|J(s)y\| \le R$ take a maximal family M whose matrices $J(s)y$ are $1/(DR)$-separated. The balls of radius $1/(2DR)$ about these points are disjoint and lie in the ball of radius $R + 1$ in the real space of $d \times d$ complex matrices, of dimension $2d^2$, so $\#M \le (3DR^2)^{2d^2}$. By maximality, every s in the set has $\|J(s)y - J(s_0)y\| \le 1/(DR)$ for some $s_0 \in M$, hence $\|J(ss_0^{-1}) - 1\| \le \|J(s)y - J(s_0)y\|\|(J(s_0)y)^{-1}\| \le 1/D$, since the inverse of a unitary matrix for an indefinite form has the same norm. Hence the set is covered by $\#M$ right translates of the small ball, which have the same measure. For the first special case take $D = 200\beta^5$, $R = \beta + 1$ and $\kappa = \beta^8$. For the integral, integrate by layers $\{2^a \le \|J(s)y\| < 2^{a+1}\}$, using the covering bound with $D = 1$, $R = 2^{a+1}$: the sum $\sum_a 2^{-c(a-1)}(3 \cdot 4^{a+1})^{2d^2}$ converges for $c > 8d^2$ and is at most 16^c. □

Theorem A.6 (Laplace's method). *Let (j, φ) be a Laplace datum with $B \ge 1$, with $s \mapsto g$ proper and $s \mapsto g'$ continuous, and ν a right invariant measure on S, finite on compact sets. Let both parameters have entries at most L and large gap h, let $\beta \ge 10^{12}$ bound $\#n + \#m + L$, and $h \ge 10^{30}((\#n + \#m)^2 \log\beta)^3$. Then θ is integrable and, with $N = (\#n + \#m)^2$,*

$$\nu\{s : \|g\| \le \beta + 1,\ \|g'\| \le \beta + 1\} \le \beta^{20N}\left|\int_S \theta\, d\nu\right|.$$

Proof. Let $J(s) = g \oplus g' \in \mathrm{U}(\varepsilon \oplus \eta)$, proper; let V, W be the measures of $\{\|J\| \le \beta + 1\}$ and $\{\|J - 1\| \le (200\beta^5)^{-1}\}$ and $I = |\int \theta|$. By Proposition A.4(d), $|\theta| \le G$, and G is integrable with $\int G \le 16^{9N}\nu\{\|J - 1\| \le 1\} \le 16^{9N}V$ by Lemma A.5 ($9N > 8N$); so θ, which is continuous, is integrable. On the small ball $|\theta| \ge 1/4$ (Proposition A.4(e)), and $|\theta| \le 2\operatorname{Re}\theta + 6\beta^{-50N}G$ everywhere; integrating,

$$W/4 \le \int |\theta| \le 2I + 6\beta^{-50N}16^{9N}V.$$

By Lemma A.5, $V \le \beta^{16N}W \le 8\beta^{16N}I + 24 \cdot 16^{9N}\beta^{-34N}V$. Since $\beta \ge 16$, $24 \cdot 16^{9N}\beta^{-34N} \le 24\beta^{-25N} \le 1/2$, and $V \le 16\beta^{16N}I \le \beta^{20N}I$. □

A.6. Positions and the period

A *position* of $\mathrm{U}(\eta)$ in $\mathrm{U}(\varepsilon)$, with $\#n = \#m + 2d + 1$, consists of independent vectors $v_1, \dots, v_{d+1}$ whose products vanish except that $\langle v_{d+1}, v_{d+1}\rangle = \pm 1$, and an isometry i_0 of $(\mathbb{C}^m, \eta)$ into their orthogonal complement. Let $A_{\le j}$ be the span of $v_1, \dots, v_j$. A matrix Y *lowers the flag* if $YA_{\le j} \subseteq A_{\le j-1}$ for $j \le d+1$; then $\sigma(Y)$ denotes the sum over $j \le d$ of the coefficient of v_j in Yv_{j+1}. Y *induces* Y' if $Yi_0 = i_0 Y'$ modulo $A_{\le d}$. The position is *stationary* for weights μ, μ' if $\mu v_j = v_{j+1}$ for $j \le d$, $i_0^\dagger \mu i_0 = \mu'$, and i_0 is the identity on common coordinates X. The *group of the position* R consists of the $g \in \mathrm{U}(\varepsilon)$ such that $g - 1$ lowers the flag; it maps to $\mathrm{U}(\eta)$ by $g \mapsto g' = i_0^\dagger g i_0$, and $\sigma(g) = \sigma(g-1)$ is additive on R.

Proposition A.7 (Two traces). *Let (v, i_0) be stationary for μ, μ', and let Y and $Y^\dagger$ lower the flag, with sums σ, σ', and let Y induce Y'. Then* $\operatorname{tr} Y = \operatorname{tr} Y'$ *and* $\operatorname{tr} \mu Y - \operatorname{tr} \mu' Y' = \sigma + \bar{\sigma}'$.

Proof. Complete $v_1, \dots, v_{d+1}$ and the columns of i_0 by vectors $w_1, \dots, w_d$ in the complement with $\langle w_a, v_b\rangle = \delta_{ab}$ and $w_a \perp i_0$; the products of the v's force this to be a basis. In this basis the diagonal entries of Y at the v_j vanish because Y lowers the flag, those at the w_a are $\langle Y w_a, v_a\rangle = \langle w_a, Y^\dagger v_a\rangle = 0$ because $Y^\dagger$ lowers the flag, and the block at i_0 is Y' modulo $A_{\le d}$; so $\operatorname{tr} Y = \operatorname{tr} Y'$. For the phase, μ maps v_j to v_{j+1}, so the diagonal entries of μY at the v_j and the w_j are the coefficients of v_j in Yv_{j+1} and the conjugates of those of $Y^\dagger$; at i_0 the block is $i_0^\dagger \mu Y i_0 = i_0^\dagger \mu i_0 Y' = \mu' Y'$ modulo terms in $A_{\le d}$, which pair to zero with i_0. □

Proposition A.8 (The stationary position). *Let l be steep for (X, L, B) with $B \ge 1$, and let μ' vanish on X, where $\eta = +1$, and increase strictly in $[0, B']$ on the other L' coordinates with alternating signs, $B' \ge 0$ and $L = L' + 2d + 1$. Then there is a stationary position with $d+1$ vectors; v_{d+1} has norm $(-1)^{L'}$, and the entries of the v_j, of i_0 and of $\mu^a v_1$, $a \le 2d$, have modulus at most $(3B + B')^L$.*

Proof. Off X, let $p = \prod_i (t - \mu_i)$ and $q = \prod_j (t - \mu'_j)$, with the roots $\mu_i < 0 \le \mu'_j$ distinct integers. For a real polynomial F let $[F]$ be the vector with entries $F(\mu_i)|p'(\mu_i)q(\mu_i)|^{-1/2}$; then $\mu[F] = [tF]$. For the alternating form $\langle x, y\rangle = \sum_i (-1)^i x_i y_i$, the sum of residues gives $\langle [F_1], [F_2]\rangle = (-1)^{L'} \sum_i \operatorname{Res}_{\mu_i} F_1 F_2/(pq)$, the signs of $q(\mu_i)$ being $(-1)^{L'}$ and those of $p'(\mu_i)$ alternating. Put $v_j = [t^{j-1} q]$. For $a + b \le 2d$, $\langle [t^a q], [t^b q]\rangle = (-1)^{L'} \sum \operatorname{Res} t^{a+b}/p$, which is $(-1)^{L'}$ if $a + b = 2d = \deg p - 1$ and 0 otherwise; so the v_j have the products of a position, and $\mu v_j = v_{j+1}$. Let $q_j = q/(t - \mu'_j)$ and let the columns of i_0 be $|p(\mu'_j)/q'(\mu'_j)|^{1/2}[q_j]$. Then $\langle [q_j], [t^a q]\rangle = 0$ for $a \le 2d$ (the numerator $t^a q_j$ has degree less than $L - 1$ after division by p only when the residue at μ'_j is absent), and $\langle [q_j], [q_{j'}]\rangle$ is 0 for $j \ne j'$ and $(-1)^{L'+1} q'(\mu'_j)/p(\mu'_j)$ for $j = j'$, the only extra pole; with the scaling the columns of i_0 are orthonormal with signs alternating. Finally $t q_j = \mu'_j q_j + q$ gives $\mu i_0 = i_0 \mu' + v_1(\cdots)$, so $i_0^\dagger \mu i_0 = \mu'$. On X take the identity. The entries are values of polynomials of degree at most L at points of modulus at most $3B$, divided by products of differences of distinct integers, which are at least 1. □

Theorem A.9 (Lower bound for periods). *Let (v, i_0) be a stationary position for the lowest weights of l and k, l steep with $B \ge 1$, and let μ' vanish on X, be nonnegative,*

and be at most $3B$ *unless* $\mathrm{U}(\eta)$ *is compact. Let both parameters have large gap* h *and entries at most* L, $N = \#n + \#m$, $\beta \geq \max(10^{12}, N + L)$ *and* $h \geq 10^{30}(N^2 \log \beta)^3$. *Then* $\theta(g) = c_l(g)\overline{c_k(g')}\exp(\overline{\sigma(g)} - \sigma(g))$ *is integrable over* R *for every right invariant measure finite on compact sets, and*

$$\nu\{g \in R : \|g\|, \|g'\| \leq \beta + 1\} \leq \beta^{20N^2}\left|\int_R \theta \, d\nu\right|.$$

The same holds for a group S *with a proper homomorphism onto* R.

Proof. For $g \in R$, $Y = g - 1$ lowers the flag, and so does $Y^\dagger = g^{-1} - 1$, since R is a group; g induces g'. By Proposition A.7, $\operatorname{tr}(g - 1) = \operatorname{tr}(g' - 1)$ and $\operatorname{tr}\mu(g - 1) - \operatorname{tr}\mu'(g' - 1) = \sigma(g) + \overline{\sigma(g^{-1})} = \sigma(g) - \overline{\sigma(g)}$ by additivity, which is imaginary. The unitary group of X lies in R, acting the same way in both factors. Hence $(g \mapsto (g, g'),\ \sigma - \bar{\sigma})$ is a Laplace datum, the inclusion of the closed subgroup R is proper, and Theorem A.6 applies. A proper surjection transports the measure and the integrand. □

Theorem A.10 (Restriction). *Let* l *be steep for* (X, M, B) *with* M *even and* $B \geq 1$, *and let* l' *be a parameter of* $\mathrm{U}(\varepsilon \oplus 1)$ *with* $-l'$ *steep for* $(X, M + 1, B)$. *Let both have large gap* h (*for* $\#n$ *and* $\#n + 1$), *entries at most* L, $\beta \geq 10^{12}$ *at least* $2\#n + 1 + L$ *and* $(\#n + 1)^4(6B)^{2M+2}$, *and* $h \geq 10^{30}((2\#n + 1)^2 \log \beta)^3$. *Then some* $y \in \mathrm{U}(\varepsilon \oplus 1)$ *with* $\|y\|^2 \leq \beta$ *has*

$$\operatorname{vol}\{g \in \mathrm{U}(\varepsilon) : \|g - 1\| \leq 1\} \leq \beta^{20(2\#n+1)^2}\left|\int_{\mathrm{U}(\varepsilon)} c_l(g)\,\overline{c_{l'}(y^{-1}(g \oplus 1)y)}\,dg\right|.$$

Proof. Take the stationary position of Proposition A.8 of $\mathrm{U}(\varepsilon)$ in $\mathrm{U}(\varepsilon \oplus 1)$ for the parameters $-l'$ and $-l$ ($d = 0$, one vector v_1), put $w = (i_0, v_1)$ and $y = w^{-1}$; the entry bound gives $\|y\|^2 \leq \beta$. The group of the position is $w\,\mathrm{U}(\varepsilon)w^{-1}$, the stabiliser of v_1, and $g \mapsto (w(g \oplus 1)w^{-1}, g)$ maps $\mathrm{U}(\varepsilon)$ onto it properly. With $d = 0$ the sum σ vanishes, so the integrand of Theorem A.9 is the conjugate of the integrand above. The ball $\|g - 1\| \leq 1$ lies in the set whose volume the theorem bounds. □

Appendix B. Background results in the form used

This appendix collects the results from the literature on which the argument rests, in the form in which the Lean development proves and uses them. Appendix B.1 to Appendix B.7 treat the analytic inputs of Part I; they lie on the main line of the argument and are proved in full. The algebraic background of Parts II to VI is stated in the later sections of this appendix, with its sources.

B.1. Conventions

Let χ be a Dirichlet character modulo $N \geq 1$. As in Section 4, $E_\chi(s)$ is $L(s, \chi)$ if χ is not principal and the entire function $(s - 1)L(s, \chi)$ if it is. We also use

$$f_\chi(s) = \begin{cases} L(s, \chi) & \text{if } \chi \text{ is not principal,} \\ E_\chi(s)/s & \text{if } \chi \text{ is principal,} \end{cases}$$

which is holomorphic on $\operatorname{Re} s > 0$ and has the size of $L(s,\chi)$ for large $|s|$. For $\operatorname{Re} s > 1$, $f_\chi(s) = L(s,\chi)(s-\delta_\chi)/s$ with $\delta_\chi = 1$ or 0 according as χ is principal or not. Since E_χ has no zeros on $\operatorname{Re} s \ge 1$, neither has f_χ.

B.2. Growth near the line $\operatorname{Re} s = 1$ and a zero-free region

Lemma B.1. *Let $\kappa_\chi = N^{-1}\sum_{a \bmod N}\chi(a)$, which is 0 unless χ is principal, and $|\kappa_\chi| \le 1$. For real $x \ge 0$, $\left|\sum_{1\le n\le x}\chi(n) - \kappa_\chi x\right| \le 2N$. For $A > 0$ and $\operatorname{Re} s \ge 1 + 1/A$, $|L(s,\chi)| \le 1 + A$.*

Proof. A sum of χ over N consecutive integers equals $N\kappa_\chi$, so the difference $\sum_{n\le x}\chi(n) - \kappa_\chi x$ is a sum of fewer than N values of χ less a number of absolute value at most N. For the last claim, $|L(s,\chi)| \le \zeta(\sigma) \le 1 + \int_1^\infty t^{-\sigma}\,\mathrm{d}t = 1 + 1/(\sigma-1)$ with $\sigma = \operatorname{Re} s$. □

Lemma B.2. *For $\operatorname{Re} s \ge 1/2$, $|f_\chi(s)| \le 1 + 4N|s|$.*

Proof. Write $\sum_{n\le t}\chi(n) = \kappa_\chi t + r(t)$, so that $|r(t)| \le 2N$ by Lemma B.1, and put $\Phi(s) = \int_1^\infty r(t)t^{-s-1}\,\mathrm{d}t$. The integral converges absolutely and defines a holomorphic function on $\operatorname{Re} s > 0$ with $|\Phi(s)| \le 2N/\operatorname{Re} s$. For $\operatorname{Re} s > 1$, partial summation gives $L(s,\chi) = s\big(\Phi(s) + \kappa_\chi/(s-1)\big)$, hence $f_\chi(s) = \kappa_\chi + \Phi(s)(s-\delta_\chi)$; in the non-principal case $\kappa_\chi = 0$. Both sides are holomorphic on $\operatorname{Re} s > 0$, so the identity holds there. For $\operatorname{Re} s \ge 1/2$ we have $|s-\delta_\chi| \le |s|$, and $|f_\chi(s)| \le 1 + (2N/\operatorname{Re} s)|s| \le 1 + 4N|s|$. □

Lemma B.3. *Let $\operatorname{Re} s > 1$ and $l \ge 1$; if χ is principal, let $|s-1| \ge 1/l$. Then $|L(s,\chi)| \le 2l|f_\chi(s)|$.*

Proof. $|L(s,\chi)| = |f_\chi(s)||s|/|s-\delta_\chi|$, and $|s| \le 2l|s-\delta_\chi|$: for principal χ, $|s| \le |s-1| + 1 \le (1+l)|s-1|$. □

Lemma B.4. *Let $l \ge 10^{10}$ and $N \le e^l$. Then $|f_\chi(s)| \le l^2$ for $\operatorname{Re} s \ge 1 - 1/l$ and $|s| \le 3e^l$, and $|f_\chi'(s)| \le 2l^3$ for $\operatorname{Re} s \ge 1 - 1/(2l)$ and $|s| \le 2e^l$.*

Proof. Fix s and put $a = \operatorname{Re} s + 2/l - 1/3$ and $g(z) = f_\chi(z)/(z - s + 2)$ on the strip $a \le \operatorname{Re} z \le a + 1/3$. There $|z - s + 2| \ge \operatorname{Re}(z - s + 2) \ge 1$, and $|z| \le |z-s+2| + |s| + 2 \le 4e^l|z-s+2|$, so by Lemma B.2 $|g(z)| \le (1 + 16e^{2l}) \le e^{3l}$; in particular g is bounded on the strip. On the right edge, $\operatorname{Re} z = \operatorname{Re} s + 2/l \ge 1 + 1/l$, and $|g(z)| \le |f_\chi(z)| \le |L(z,\chi)| \le 1 + l$ by Lemma B.1. The point s divides the strip in the ratio $1 - 6/l$ to $6/l$, so the three-lines theorem gives

$$|g(s)| \le \left(e^{3l}\right)^{6/l}(1+l)^{1-6/l} \le 3^{18}(1+l),$$

and $|f_\chi(s)| = 2|g(s)| \le 2\cdot 3^{18}(1+l) \le l^2$. For the derivative apply Cauchy's estimate on the circle of radius $1/(2l)$ about s, whose points satisfy the hypotheses of the first bound. □

Lemma B.5 (The 3–4–1 inequality). *Let $l \ge 10^{10}$, $N \le e^l$, $0 < \eta \le 1$, $\operatorname{Re} s \ge 1 + \eta$ and $|s| \le \frac{3}{2}e^l$; if χ^2 is principal, let $|s-1| \ge 1/l$. Then $\eta^3 \le 256\, l^7 |f_\chi(s)|^4$.*

Proof. Write $s = \sigma + it$ and $s_2 = \sigma + 2it$. For $\sigma > 1$ the classical inequality $|L(\sigma,\chi_0)^3 L(s,\chi)^4 L(s_2,\chi^2)| \geq 1$ holds, χ_0 the principal character. Here $|L(\sigma,\chi_0)| \leq 1 + 1/\eta \leq 2/\eta$; $|L(s,\chi)| \leq 2l|f_\chi(s)|$ by Lemma B.3; and $|L(s_2,\chi^2)| \leq 2l|f_{\chi^2}(s_2)| \leq 2l^3$ by Lemmas B.3 and B.4, as $|s_2 - 1| \geq |s-1|$ and $|s_2| \leq 2|s|$. Multiply. □

Lemma B.6. *Let $l \geq 10^{10}$ and $N \leq e^l$, and if χ^2 is principal assume that $|f_\chi(z)| \geq 1/l$ for $|z-1| \leq 1/l$. For $\operatorname{Re} s \geq 1 - l^{-25}$ and $|s| \leq e^l$, the function E_χ does not vanish at s, and $|E'_\chi(s)/E_\chi(s)| \leq 2l^{25}$.*

Proof. First, $|f_\chi(v)| \geq 12/l^{22}$ for $\operatorname{Re} v \geq 1 + l^{-25}$ and $|v| \leq \frac{3}{2}e^l$: if χ^2 is principal and $|v-1| < 1/l$ this is the hypothesis; otherwise Lemma B.5 with $\eta = l^{-25}$ gives $l^{-75} \leq 256\, l^7 |f_\chi(v)|^4$, and $256 \cdot 12^4 \leq l^6$. Next let s be as in the statement and $v = s + 2l^{-25}$. On the segment from s to v the bound $|f'_\chi| \leq 2l^3$ of Lemma B.4 applies, so $|f_\chi(v) - f_\chi(s)| \leq 4/l^{22}$ and $|f_\chi(s)| \geq 8/l^{22}$. Hence $f_\chi(s) \neq 0$ and $|f'_\chi(s)/f_\chi(s)| \leq l^{25}/4$. Finally $E_\chi = f_\chi u$ with $u(z) = z$ or 1, and $|u'(s)/u(s)| \leq 1/|s| \leq 2$. □

Proof of Theorem 6.3. Since $f_\chi(1) \neq 0$ and f_χ is continuous at 1, the hypothesis of Lemma B.6 on χ^2 holds for all sufficiently large l, and so does $N \leq e^l$. □

B.3. A Tauberian theorem

Proof of Theorem 6.5. Step 1: regularity of R. Chaining steps of length at most 1 gives $|R(v) - R(u)| \leq L|v-u|$ for all $u, v \geq 0$; hence $|R(v)| \leq (|R(0)| + L)e^v$ for $v \geq 0$. Put $\Phi(s) = (F(s) - F(1))/(s-1)$. For $\operatorname{Re} s > 2$, differentiation under the integral sign gives $\Phi^{(k)}(s) = \int_0^\infty (-v)^k R(v) e^{(1-s)v}\,\mathrm{d}v$.

Step 2: a Taylor coefficient as a mean. Let $H > 1$ and $k \geq 0$. The density $g(v) = H^{k+1} v^k e^{-Hv}/k!$ on $(0,\infty)$ has mean $u = (k+1)/H$ and variance $(k+1)/H^2$, and

$$\frac{(-H)^k H}{k!}\,\Phi^{(k)}(1+H) = \int_0^\infty R(v)\, g(v)\,\mathrm{d}v.$$

If $|R(v) - c| \leq a + b(v-u)^2$ for $v > 0$, the coefficient lies within $a + b(k+1)/H^2$ of c. By Step 1 and $|v-u| \leq \delta + (v-u)^2/\delta$, this holds with $c = R(u)$, $a = L\delta$, $b = L/\delta$ for every $\delta > 0$: the coefficient lies within $L(\delta + u^2/((k+1)\delta))$ of $R(u)$.

Step 3: a Taylor coefficient is small. Let $0 < \eta \leq H$, and let F be holomorphic and at most M in absolute value on the closed disc D of radius $H + \eta$ about $1 + H$. Then Φ is holomorphic on D, and on its boundary $|s-1| \geq \eta$, so $|\Phi| \leq 2M/\eta$ there. Cauchy's estimate and $H/(H+\eta) \leq 1 - \eta/(2H) \leq e^{-\eta/(2H)}$ give

$$\left|\frac{(-H)^k H}{k!}\Phi^{(k)}(1+H)\right| \leq \frac{2M}{\eta}(H+\eta)\Big(\frac{H}{H+\eta}\Big)^{k+1} \leq \frac{4MH}{\eta}\exp\Big(-\frac{(k+1)\eta}{2H}\Big).$$

Step 4: choice of the parameters. Given B and a large u, put $l = (2B+5)\log u$, $\eta = l^{-A}$, $k = \lfloor u^{2B+4} \rfloor$ and $H = (k+1)/u$. Then $1 < H \leq 2u^{2B+4}$, and the disc D lies in the region $\operatorname{Re} s \geq 1 - l^{-A}$, $|s| \leq 2H + \eta + 1 \leq u^{2B+5} = e^l$, where $|F| \leq l^A$ for large u. Since $l^{2A} \leq u$ and $u^{3B+7} \leq e^{\sqrt{u}/2}$ for large u, and $(k+1)\eta/(2H) = u/(2l^A) \geq \sqrt{u}/2$, Step 3 bounds the coefficient by $l^{2A} \cdot 4H \cdot e^{-\sqrt{u}/2} \leq u \cdot u^{2B+5} \cdot u^{-3B-7} = u^{-B-1}$. Step 2 with $\delta = u^{-B-1}$ places it within $2Lu^{-B-1}$ of $R(u)$. Hence $|R(u)| \leq (1+2L)u^{-B-1} \leq u^{-B}$ for $u \geq 1 + 2L$. □

B.4. Primes in short logarithmic windows

We prove Theorem 8.11 by Newman's contour method. Let M_0 be a positive multiple of 4 and $A \geq 1$.

Lemma B.7. *Let $k \geq 2$. A character χ modulo M_0^k with χ^2 principal is induced by a character χ_0 modulo M_0^2, and $f_\chi = f_{\chi_0}$.*

Proof. Every unit $u \equiv 1 \pmod{M_0^2}$ is a square modulo M_0^k: write $u = 1-4M_0mt$ with $M_0 = 4m$, and solve $c^2+c = -M_0mt$ by Newton's iteration, which converges since M_0mt is nilpotent modulo M_0^k; then $u = (1+2c)^2$. Hence χ is trivial on the kernel of reduction to M_0^2. The two moduli have the same prime factors, so the L-functions of χ and χ_0 have the same Euler factors. □

Lemma B.8. *For all large X: for every $k > 1$ with $M_0^k \leq X^A$, every character χ modulo M_0^k and every s with $|s-1| \leq 1/X$, $E_\chi(s) \neq 0$ and $|E_\chi'(s)/E_\chi(s)| \leq X$.*

Proof. Apply Lemma B.6 with $l = (A+1)\log X$, so that $e^l = X^{A+1} \geq M_0^k$. Its hypothesis for χ with χ^2 principal concerns, by Lemma B.7, only the finitely many characters modulo M_0^2, and holds for large l as in the proof of Theorem 6.3. For large X, $2l^{25} \leq X$, so $\operatorname{Re} s \geq 1 - 1/X \geq 1 - l^{-25}$, and $|s| \leq 2 \leq e^l$. □

Proof of Theorem 8.11. Write $W = X^2$ and $q = M_0^k$, and let $a(n) = \Lambda(n)$ if $n \equiv 1 \pmod q$ and $a(n) = 0$ otherwise. By orthogonality, for $\operatorname{Re} z > 0$,

$$\sum_{n\geq 1} a(n)n^{-1-z} = \frac{F(z)}{z}, \qquad F(z) = \frac{1}{\varphi(q)}\Big(1 - z\sum_{\chi \bmod q}\frac{E_\chi'}{E_\chi}(1+z)\Big).$$

By Lemma B.8, F is holomorphic and $|F| \leq 2$ on $|z| \leq 1/X$. Let $b(n) = \Lambda(n)$ for prime powers $n = p^e$ with $e \geq 2$ and $b(n) = 0$ otherwise; since $\psi(x) - \vartheta(x) = O(\sqrt{x})$, $C = \sum_n b(n)n^{-3/4} < \infty$, and we take $X \geq \max(C, 200)$.

Suppose that no prime $p \equiv 1 \pmod q$ has $\log p \in [U, U+X^2]$. Then $a(n) \leq b(n)$ whenever $\log n \in [U, U+X^2]$. Let $P(z) = \sum_{n<e^U} a(n)n^{-1-z}$ and $\mathcal{E}(z) = \operatorname{Re}\big(2 - Uz - \frac{X}{4}(1+Xz)^2\big)$. We claim that on the circle $|z| = 1/X$

$$|F(z) - zP(z)| \leq 3U\, e^{\mathcal{E}(z)}. \tag{B.1}$$

On the circle, a computation with $X^2(\operatorname{Im} z)^2 = 1 - (X\operatorname{Re} z)^2$ gives $\mathcal{E}(z) = 2 - (U+t)\operatorname{Re} z$ with $t = \frac{X^2}{2}(1 + X\operatorname{Re} z) \in [0, X^2]$. Two facts are used: for every real σ and every n, $|a(n)n^{-1-z}| = e^{\log n(\sigma - \operatorname{Re} z)}a(n)n^{-1-\sigma}$; and the total $\sum a(n)n^{-1-1/U} = UF(1/U) \leq 2U$.

If $\operatorname{Re} z \leq 1/U$, each term of P has $\log n \leq U$, so $\log n\,(1/U - \operatorname{Re} z) \leq 1 - U\operatorname{Re} z \leq \mathcal{E}(z)$ (using $t\operatorname{Re} z \leq X^2/U \leq 1$); hence $|P(z)| \leq e^{\mathcal{E}(z)} \cdot 2U$, and $|F(z) - zP(z)| \leq 2 + \frac{1}{200}\cdot 2Ue^{\mathcal{E}(z)} \leq 3Ue^{\mathcal{E}(z)}$, as $\mathcal{E}(z) \geq 0$ and $U \geq X^2$.

If $\operatorname{Re} z \geq 1/U$, then $F(z) - zP(z) = z\sum_{n\geq e^U} a(n)n^{-1-z}$. In the window, $a(n) \leq b(n)$ and $\log n\,(-\frac{1}{4} - \operatorname{Re} z) \leq U(-\frac{1}{4} - \operatorname{Re} z) \leq \mathcal{E}(z)$, since $t\operatorname{Re} z \leq U/200$; so these terms are at most $e^{\mathcal{E}(z)}b(n)n^{-3/4}$. Beyond the window, $\log n \geq U + t$ and $\log n\,(1/U - \operatorname{Re} z) \leq (U+t)(1/U - \operatorname{Re} z) \leq \mathcal{E}(z)$; so these terms are at most $e^{\mathcal{E}(z)}a(n)n^{-1-1/U}$. The sum is at most $e^{\mathcal{E}(z)}(2U + C) \leq 3Ue^{\mathcal{E}(z)}$. This proves (B.1).

The function $G(z) = (F(z) - zP(z))\exp\big(-2 + Uz + \frac{X}{4}(1+Xz)^2\big)$ is holomorphic on $|z| \leq 1/X$ and, by (B.1), at most $3U$ on its boundary. By the maximum principle

$|G(0)| \le 3U$. But $G(0) = F(0)e^{X/4-2} = e^{X/4-2}/\varphi(q)$, and $3U\varphi(q)e^2 \le 3e^2X^{4A} < e^{X/4}$ for large X, as $U, q \le X^{2A}$. This contradiction proves the theorem. □

B.5. Theta series and the Pólya–Vinogradov inequality

Lemma B.9 (Pólya–Vinogradov). *Let $\chi \ne 1$ be a character modulo q. For every $X \ge 0$, $\left|\sum_{1\le v\le X}\chi(v)\right| \le 2\sqrt{q}\,(1+\log q)$.*

Proof. For a primitive character $\psi \ne 1$ modulo M, the Gauss sum $\tau(\psi)$ has absolute value $\sqrt{M}$, and $\psi(v) = \tau(\bar\psi)^{-1}\sum_{k \bmod M}\bar\psi(k)e(kv/M)$. Summing the geometric series in v and using $|e(k/M)-1| \ge 4x(1-x)$ for $k = xM$, $0 < x < 1$, gives $\left|\sum_{v\le X}\psi(v)\right| \le \sqrt{M}(1+\log M)$. If χ modulo $q = Mr$ is induced by ψ, then $\sum_{v\le X}\chi(v) = \sum_{d\mid r}\mu(d)\psi(d)\sum_{e\le X/d}\psi(e)$, and r has at most $2\sqrt{r}$ divisors. □

Fix a parity $j \in \{0,1\}$ and put $w(z) = z^j e^{-\pi z^2}$. For a sequence c with $|c(n)| \le 1$ let

$$\theta_c(u) = u^{1/2}\sum_{n\ge1} c(n)\,w(nu) \quad (u>0),$$

$$\Gamma_j(s) = \int_0^\infty w(z)z^s\frac{\mathrm{d}z}{z} = \tfrac12\pi^{-(s+j)/2}\Gamma\!\left(\tfrac{s+j}{2}\right) \quad (\operatorname{Re}s>0).$$

Lemma B.10. *Let $|c(n)| \le 1$ for all n.*

(a) *$|\theta_c(u)| \le 3e^{-u}u^{-1/2}$ for $u > 0$.*

(b) *For $\operatorname{Re}s > 1$, $\int_0^\infty \theta_c(u)u^{s-1/2}\frac{\mathrm{d}u}{u} = \Gamma_j(s)\sum_n c(n)n^{-s}$; with $c\mu$ in place of c and $c = \chi$ a character, the right side is $\Gamma_j(s)/L(s,\chi)$.*

(c) *Let $\chi \ne 1$ be a character modulo q of parity j, that is, $\chi(-1) = (-1)^j$. Then $\left|\sum_n \chi(n)w(nt)\right| \le 3eq\,e^{-t}$ for $t > 0$, and the identity of (b) for $c = \chi$ holds for $\operatorname{Re}s > 0$, with absolute convergence.*

(d) *If moreover χ is primitive of conductor q, then $|\theta_\chi(1/(qt))| = |\theta_\chi(t)|$ for $t > 0$, and $\int_0^\infty|\theta_\chi(u)|\frac{\mathrm{d}u}{u} = 2\int_{q^{-1/2}}^\infty|\theta_\chi(u)|\frac{\mathrm{d}u}{u}$.*

Proof. (a) $|w(t)| \le 3e^{-2t}$ for $t \ge 0$ (as $\pi > 3$), and the geometric series gives $\sum_n|w(nt)| \le 3e^{-t}/t$. (b) Integrate term by term, using the substitution $u \mapsto u^2$ in Γ_j. (c) For $t \ge 1$ use (a). For $t < 1$, Poisson summation in each residue class modulo q gives $t\left|\sum_n\chi(n)w(nt)\right| = \left|\sum_n\check\chi(n)w(n/(qt))\right|$, where $\check\chi(n) = q^{-1}\sum_{a \bmod q}\chi(a)e(an/q)$ has absolute value at most 1 and $\check\chi(0) = 0$; the sums run over $n \ge 1$ because both χ and $\check\chi$ have parity j. Bound the right side by (a). The bound makes the integral converge for $\operatorname{Re}s > 0$ and defines a holomorphic function there, equal to $\Gamma_j(s)L(s,\chi)$ for $\operatorname{Re}s > 1$, hence for $\operatorname{Re}s > 0$. (d) For primitive χ, $q\check\chi = \tau(\chi)\bar\chi$ with $|\tau(\chi)| = \sqrt{q}$, and the identity of (c) becomes the transformation formula in absolute value. The map $u \mapsto 1/(qu)$ preserves $\mathrm{d}u/u$ and $|\theta_\chi|$ and exchanges $(0, q^{-1/2})$ with $(q^{-1/2},\infty)$. □

B.6. Quadratic twists without zeros in a box

For $T > 0$ and a sequence c with $|c(n)| \le 1$ put

$$J(c,T) = \int_{1/T}^{\infty} \bigl(|\theta_c(u)| + |\theta_{c\mu}(u)|\bigr)\,\frac{\mathrm{d}u}{u}, \qquad \mathcal{M}(s) = \int_{1/T}^{\infty} \theta_{c\mu}(u)\,u^{s-1/2}\,\frac{\mathrm{d}u}{u}.$$

For $c = \chi$ the function $\mathcal{M}$ is a truncation of the integral for $\Gamma_j/L(\cdot,\chi)$ in Lemma B.10(b); it is holomorphic on $\operatorname{Re} s > 0$ and bounded on $\frac12 \le \operatorname{Re} s \le 2$.

Lemma B.11. *Let $\chi \neq 1$ be primitive of conductor $q \le T^2$ with $\chi(-1) = (-1)^j$, and write $\Lambda(s) = \Gamma_j(s)L(s,\chi)$. Then $|\Lambda(z)\mathcal{M}(z) - \Gamma_j(z)^2| \le 9/T$ on $\operatorname{Re} z = 2$, and $|\Lambda(z)| \le 2J(\chi,T)$, $|\mathcal{M}(z)| \le J(\chi,T)$ on $\operatorname{Re} z = 1/2$.*

Proof. On $\operatorname{Re} z = 2$, Lemma B.10(a) gives $|u^{z-3/2}\theta_c(u)| \le 3e^{-u}$ for every c, so $|\Lambda| \le 3$ and $|\Gamma_j/L - \mathcal{M}| \le \int_0^{1/T} 3\,\mathrm{d}u = 3/T$ by Lemma B.10(b); and $\Lambda\mathcal{M} - \Gamma_j^2 = \Lambda(\mathcal{M} - \Gamma_j/L)$. On $\operatorname{Re} z = 1/2$ the factor $u^{z-1/2}$ has absolute value 1; by Lemma B.10(c), (d) and $q^{-1/2} \ge 1/T$, $|\Lambda(z)| \le 2\int_{1/T}^{\infty} |\theta_\chi|\,\mathrm{d}u/u$. □

Proposition B.12. *Let $\delta > 0$ and $H \ge 0$. For all large T the following holds. If $\chi \neq 1$ is primitive of conductor $q \le T^2$ with $\chi(-1) = (-1)^j$, and $L(s,\chi) = 0$ for some s with $\operatorname{Re} s \ge \frac12 + 9\delta$ and $|\operatorname{Im} s| \le H$, then $J(\chi,T) > T^{2\delta}$.*

Proof. Let G bound $|\Gamma_j|$ on the strip $\frac12 \le \operatorname{Re} z \le 2$ and put $C = 9 + G^2$. Suppose $J(\chi,T) \le T^{2\delta}$. The function $\Psi = \Lambda\mathcal{M} - \Gamma_j^2$ is holomorphic and bounded on the strip; by Lemma B.11 it is at most $CT^{4\delta}$ on the left line and at most CT^{-1} on the right line. Since $L(\cdot,\chi)$ has no zeros on $\operatorname{Re} s \ge 1$, $\operatorname{Re} s \le 1$, and s has weight $\lambda = (\operatorname{Re} s - \frac12)/\frac32 \ge 6\delta$ towards the right line. The three-lines theorem gives

$$|\Gamma_j(s)|^2 = |\Psi(s)| \le C\,T^{4\delta(1-\lambda)-\lambda} \le C\,T^{-2\delta}.$$

But $|\Gamma_j|$ is bounded below by a positive constant on the compact rectangle $\frac12 \le \operatorname{Re} z \le 1$, $|\operatorname{Im} z| \le H$, where it has no zeros. This is impossible for large T. □

For an integer $n \ge 1$ let n° be its odd part, and let $(\cdot/m)$ be the Jacobi symbol.

Lemma B.13. (a) *If $d \equiv 7 \pmod 8$ is prime and $n \ge 1$, then $(n/d) = (-d/n^\circ)$.*

(b) *If m is odd and not a square, then $v \mapsto (-v/m)$ is $\pm$ a non-principal character, and $\bigl|\sum_{v\le X}(-v/m)\bigr| \le 2\sqrt{m}(1+\log m)$. Consequently, for $\delta > 0$ and n° not a square, $\bigl|\sum_{v\le X}(-v/n^\circ)\bigr| \le 2(1+2/\delta)\,n^{1/2+\delta/2}$.*

(c) *If ψ is primitive modulo M and $d \nmid M$ is an odd prime, then $\psi\chi_d$ is primitive modulo Md.*

Proof. (a) $(2/d) = 1$, and quadratic reciprocity for the odd part, with $d \equiv 3 \pmod 4$. (b) Write $m = c^2a$ with $a > 1$ squarefree; a prime $p \mid a$ makes the character non-principal. Apply Lemma B.9, and $1 + \log n \le (1+2/\delta)n^{\delta/2}$. (c) The conductor of $\psi\chi_d$ is divisible by M and by d. □

Theorem B.14 (Mean value). *Let $\delta > 0$. There is C such that for $T > 0$, $u \geq 1/T$ and every sequence α with $|\alpha_n| \leq 1$, the theta series $\theta_v = \theta_{c_v}$ with $c_v(n) = \alpha_n(-v/n^\circ)$ satisfy*

$$\sum_{1 \leq v \leq T^2} |\theta_v(u)|^2 \leq C\, T^2 u^{-2\delta}.$$

Proof. Expanding the square,

$$\sum_v |\theta_v(u)|^2 = \sum_{n,m \geq 1} \alpha_n \bar{\alpha}_m\, u\, w(nu) w(mu) \sum_{v \leq T^2} \Big(\frac{-v}{(nm)^\circ}\Big).$$

For every $A > 0$, $u|w(nu)w(mu)| \ll_A u^{1-2A}(nm)^{-A}$. If $(nm)^\circ$ is a square, bound the inner sum by T^2 and take $A = \frac{1}{2} + \delta$; the pairs with nm or $2nm$ a square contribute a convergent series $\sum (nm)^{-1/2-\delta}$. Otherwise bound the inner sum by Lemma B.13(b) and take $A = \frac{3}{2} + \delta$; the term is $\ll u^{-2-2\delta}(nm)^{-1-\delta/2}$, and $u^{-2} \leq T^2$. □

Corollary B.15. *Let $\delta > 0$. There is C such that for $T > 0$ and $|\alpha_n| \leq 1$, $\sum_{1 \leq v \leq T^2} J(c_v, T) \leq C\, T^{2+\delta}$.*

Proof. By Cauchy–Schwarz and Theorem B.14, $\sum_v |\theta_v(u)| \leq C' T^2 u^{-\delta}$ for $u \geq 1/T$; integrate against $\mathrm{d}u/u$ over $(1/T, \infty)$. The same holds for the coefficients $\alpha_n \mu(n)$. □

Proof of Theorem 5.6. We may assume $\beta < 1$ and write $\beta = \frac{1}{2} + 9\delta$. Call a prime $d \equiv a \pmod N$ *bad* if some $L(s, \xi\chi_d)$, ξ modulo N, vanishes in the box. Each ξ is induced by a primitive ψ modulo some $M \mid N$, and $L(s, \xi\chi_d)$ differs from $L(s, \psi\chi_d)$ by Euler factors that do not vanish for $\operatorname{Re} s \neq 0$ (Lemma B.16). By Lemma B.13, $\psi\chi_d$ is primitive of conductor Md, its parity j is opposite to that of ψ (as $\chi_d(-1) = -1$), and its coefficients are $\psi(n)(-d/n^\circ) = c_d(n)$ with $\alpha = \psi$. Take $T^2 = NX$. By Proposition B.12, each bad $d \leq X$ has $J(c_d, T) > T^{2\delta}$ for one of the finitely many ψ; by Corollary B.15 there are $O(T^{2-\delta}) = O(X^{1-\delta/2})$ of them. If all primes of the class were bad, the Dirichlet series $\sum_{p \equiv a} p^{-s}$ would converge absolutely at $s = 1 - \delta/4$, and so $\sum_{p \equiv a} \log p/p$ would converge, against Dirichlet's theorem on primes in arithmetic progressions. □

B.7. Symmetry of the zeros

Lemma B.16. *Let χ be a Dirichlet character, $M \geq 1$ and $\operatorname{Re} s \neq 0$. Then $\prod_{p \mid M}(1 - \chi(p)p^{-s}) \neq 0$. Every character modulo N is induced by a primitive character modulo a divisor of N, of the same parity.*

Proof. A vanishing factor means $\chi(p) = p^s$; then $|\chi(p)| = 1$ and $p^{\operatorname{Re} s} = 1$. □

Proposition B.17. *Let χ be a Dirichlet character, and let $L(s, \chi) = 0$ with $\operatorname{Re} s \neq 0$ and $s \neq 1$. Then s is a pole of the gamma factor $\Gamma_{\mathbb{R}}(s + a_\chi)$ of χ, or $L(1 - s, \bar{\chi}) = 0$.*

Proof. Let ψ be the primitive character inducing χ. By Lemma B.16, $L(s, \psi) = 0$. If s is not a pole of the gamma factor, the completed L-function of ψ vanishes

at s, hence that of $\bar\psi$ at $1-s$ by the functional equation, and so $L(1-s,\bar\psi)=0$; since $\operatorname{Re}(1-s)\neq 0$, also $L(1-s,\bar\chi)=0$. □

B.8. Hecke characters of CM fields

Let E be a CM field with a CM type $(\Phi_v)_v$, one embedding for each infinite place. A *splitting character* is a unitary Hecke character υ of E with $\upsilon(\bar x)=\upsilon(x)^{-1}$ that equals $\prod_v \Phi_v(x)/|\Phi_v(x)|$ on the ideles whose finite part is a unit. Its restriction to the ideles of the maximal real subfield E^+ is the quadratic character of E/E^+.

Theorem B.18. (a) *If E contains a primitive root of unity of order 5040 and every root of unity of E is $\delta/\bar\delta$ for a unit δ of O_E with $\prod_v \Phi_v(\delta)=1$, a splitting character exists.*

(b) *For a splitting character υ, the character $\chi_{\mathrm f}=\upsilon|\cdot|^{-1/2}$ of the finite ideles is an algebraic Hecke character: on $E^\times$ it is $x\mapsto\prod_v\Phi_v(\bar x)^{-1}$ up to the archimedean normalisation, and a positive power of each of its values lies in the reflex field.*

(c) *For a p-adic place $\iota\colon\mathbb{C}\to\mathbb{C}_p$ and a finite idele x, $|\iota(\chi_{\mathrm f}(x))|=\prod_v|\varphi_v(x)|$, where φ_v is the extension to the finite adeles of the embedding $y\mapsto\iota(\Phi_v(\bar y))$ of E into $\mathbb{C}_p$.*

Sources and proof. These are characters of type A_0 in the sense of Weil [Wei56]. For (a), υ is prescribed as 1 on $E^\times$, as $\prod_v x_v/|x_v|$ on the ideles with unit finite part, and compatibly with $\upsilon(\bar x)=\upsilon(x)^{-1}$ on the norms $b\bar b$. The prescriptions are consistent: if $x=y\,b\bar b$ with $y\in E^\times$ has unit finite part, then $\bar y/y$ is a root of unity, hence $\delta/\bar\delta$ with $\prod_v\Phi_v(\delta)=1$; the element $a=y\delta$ is real, the finite part of $ab\bar b$ is a real unit and hence locally a norm $t\bar t$, so a is a norm of an idele, hence a global norm from E (Hasse's norm theorem for E/E^+) and totally positive; and then $\prod_v x_v/|x_v|=\prod_v\Phi_v(a)/|\prod_v\Phi_v(a)|=1$. The extension from the subgroup so generated to all ideles is the cohomological part of Weil's construction; in the Lean development it uses the primitive root of unity of order 5040 in E. (b) and (c) are the description of the values of algebraic Hecke characters by their infinity type [Wei56]; (c) follows from (b) by writing a power of x as an element of $E^\times$ times a unit idele. □

Let $\mathrm{U}(1)(\mathbb{A})$ be the norm-one ideles. Characters of $\mathrm{U}(1)(\mathbb{A})$ with a weight and a tame part are defined in Section 10.4.

Theorem B.19. *Let $e\colon\Phi\to\mathbb{Z}$ and let $\psi=(\psi_w)$ be characters of the residue fields, trivial outside a finite set S. If $\prod_v\Phi_v(u)^{e(v)}\prod_{w\in S}\psi_w(u \bmod w)=1$ for every $u\in\mathrm{U}(1)(E)$ that is integral, there is a continuous character μ of $\mathrm{U}(1)(\mathbb{A})$, trivial on $\mathrm{U}(1)(E)$, of weight e and tame part ψ. For given e and ψ there are finitely many such characters.*

Proof. The formula defines a continuous character of the open subgroup U_0 of points with integral finite part; by hypothesis it is trivial on $\mathrm{U}(1)(E)\cap U_0$, so it extends to $\mathrm{U}(1)(E)U_0$, an open subgroup of finite index, and then to $\mathrm{U}(1)(\mathbb{A})$, since characters of an open subgroup of a locally compact abelian group extend. Two characters with the same weight and tame part differ by a character of the compact group $\mathrm{U}(1)(E)\backslash\mathrm{U}(1)(\mathbb{A})$ trivial on the image of the open subgroup of

u with $u_v = 1$ and $u_w \equiv 1 \pmod{w}$, which has finite index; there are finitely many such characters. □

B.9. Elliptic curves with complex multiplication by O_K

We recall the facts of Theorem 11.3 and their sources, together with three further inputs. Throughout $d \equiv 7 \pmod 8$ is prime and $K = \mathbb{Q}(\sqrt{-d})$; then 2 splits in K, the class number of K is odd, and $\zeta_K(s) = \zeta(s)L(s, \chi_K)$.

Sigma, theta and the cubic. The sigma function of a lattice Λ is constructed as an odd theta function for the form $H(l, z) = \pi \bar{l} z / A$ and the multiplier χ_σ; it differs from Weierstrass's sigma function by a factor $\exp(cz^2)$. Its addition formula $\sigma(u+v)\sigma(u-v) = (\wp(v) - \wp(u))\sigma(u)^2\sigma(v)^2$ and the duplication formula give the classical facts on $\wp$ [Sil09, VI], [Sil94, I]. Translating the origin to the point t of order two and normalising by $\wp''(t) = 2$ gives the cubic $y^2 + a_1 xy = x^3 + x$ with origin $(0, 0)$ and parameter y, and the identities of Theorem 11.3(i); in particular $\theta^2 = (\wp - \wp(t))\sigma^2$ off Λ and $g'' = g''(0) + x$ for $g = -\log\theta$, by differentiating the addition formula twice. The monomials $\theta^r \sigma^{2t'} G^u$ with $u \le 1$ span the sections of each degree, by the description of the theta functions for $2H$ through $\wp$ and of the elliptic functions with poles at one point; and sections pull back along isogenies $z \mapsto az$.

Integral group law and isogenies. The formal group law of the cubic, in the parameter y, has coefficients in $\mathbb{Z}[a_1]$ [Sil09, IV.1–IV.2]; here it is obtained from Euler's addition formula for $y(z_0 + z_1)$, a polynomial in a_1, $x(z_i)$, $y(z_i)$ and the inverse of the unit $1 - x(z_0)x(z_1)$. The number a_1 is integral at every p because $a_1^4 - 64$, the discriminant of the lattice up to a unit, is a unit: these are Weber's class invariants of discriminant $-d$, integral for $d \equiv 7 \pmod 8$. An isogeny of degree N pulls θ back to θ^N times an integral series in y, and y to an integral series in y [Sil09, VII]; passing to limits gives Theorem 11.3(iv).

Theorem B.20 (Torsion values). *Let C be a CM lattice for d and p a prime. For a section h of degree m and a torsion point u, the* torsion value $h[u] = (T_u^m h)(0)$. *The monomial sections have integral torsion values at every p-adic place, and* $\max(|\sigma[u]|, |\theta[u]|) = 1$ *at torsion points. A section with integral torsion values is an integral combination of monomial sections, and its Taylor series lies in $\theta^m O[[y]]$. The values $(T_u^1\theta)(0)$ are bounded in absolute value uniformly in the CM lattice and in u, and only finitely many lattices underlie CM lattices for d.*

Sources and proof. The integrality of Siegel functions at torsion points is due to Ramachandra and Robert [KL81]; only the points $t/2^i$ are needed, and for them it follows from the duplication formulas for θ and σ, which raise the maximum norm of the pair (θ^2, σ^2) to the fourth power because a_1 is integral and $a_1^4 - 64$ a unit. The integrality principle (a section with integral torsion values has integral expansion) is proved by writing the section as $\theta^m(P_0(x) + yP_1(x))$ and interpolating at the nodes $x(t/2^i)$, whose differences are units; this is the method of Norman and Mazur–Tate as used by Bannai and Kobayashi [BK10], here for all p thanks to the special shape of d. The finiteness of the lattices is the correspondence between CM lattices up to homothety and ideal classes [Sil94, II]; the scaling $\wp''(t) = 2$ fixes the homothety up to a fourth root of unity, and the bound for torsion values follows by compactness of the fundamental domain. □

The formal parameter. At a prime p, let T be the part of y in the degrees $\equiv 1$ modulo $q_p - 1$. Since the roots of unity of order $q_p - 1$ act on the formal group through O_K, the ring $O[[y]]$ is invariant under $z \mapsto \zeta z$ for them, and T is again a parameter. Multiplication by π_p is $T(\pi_p z) = g(T)$ with $g = \pi_p z + b_1 z^{q_p} + \cdots$ supported in the degrees $1 + i(q_p - 1)$, and b_1 is a unit: this is the statement that the formal group has height 2 at inert p and that π_p acts as a Lubin–Tate endomorphism [LT65]; the height bound is proved from the inseparable degree of the reduction of multiplication by π_p. Taking $\varpi^{q_p - 1} = -\pi_p / b_1$ gives $Y = T(\varpi z)/\varpi$ integral with $Y - Y^{q_p} \equiv z$ and $Y \equiv \sum_j z^{q_p^j}$ modulo π_p, which is Theorem 11.3(v).

Theorem B.21 (Chowla–Selberg). *Let $\mathcal{E}_0$ be a CM lattice with lattice $\Omega_0 O_K$. There is $u \in \mathbb{C}^\times$, a unit at every p-adic place of $\mathbb{C}$, such that for every automorphism σ of $\mathbb{C}$ over K*

$$\operatorname{area}(\mathcal{E}_0^\sigma)\,|\sigma(u)|^2 = \pi\, e^{c_K} \frac{2\pi}{\sqrt{d}}, \qquad c_K = \gamma - \operatorname{Re} \frac{L'}{L}(1, \chi_K).$$

Sources and proof. Kronecker's limit formula, summed over the ideal classes of K, computes the geometric mean of area $\cdot\, |\mathrm{disc}|^{1/6}$ over the classes in terms of $L'(1,\chi_K)/L(1,\chi_K)$ and γ [SC67]; we leave it in this form, without Gamma values. The conjugate lattices $\mathcal{E}_0^\sigma$ run through the classes (each class once up to the finite ambiguity of the normalisation), and the integrality of isogenies between them gives one number u, a unit at every p, whose conjugates scale all conjugate lattices to the mean. □

Theorem B.22 (Complex multiplication for translates of θ). *Let $\mathcal{E}_0$ have lattice $\Omega_0 O_K$ and let σ be an automorphism of $\mathbb{C}$ over K. There are a CM lattice $\mathcal{E}_0^\sigma$ with $a_1(\mathcal{E}_0^\sigma) = \sigma(a_1)$, a period Ω_σ with a multiple in its lattice and area $|\Omega_\sigma|^2\sqrt{d}/2$, and a finite idele β_σ of K such that for every $x \in K$ and every x' in a neighbourhood of $\beta_\sigma x$, σ maps the Taylor series of $T^1_{x\Omega_0}\theta_{\mathcal{E}_0}$ at 0 to that of $T^1_{x'\Omega_\sigma}\theta_{\mathcal{E}_0^\sigma}$.*

Sources and proof. This is the main theorem of complex multiplication for elliptic curves in a soft form [Shi71, Thm. 5.4]: the conjugate lattice is $\Omega\mathfrak{c}^2/\mathrm{N}(\mathfrak{c})$ for an ideal $\mathfrak{c}$, and one finite adele describes the Galois action on torsion points. Here no reciprocity law is used: the conjugate lattice comes from uniformisation by $(\sigma g_2, \sigma g_3)$, the Taylor series of the theta function of a CM datum with invariant $\sigma(a_1)$ is the conjugate of that of θ, and points are matched by transporting sections of each degree. □

B.10. The adelic Weil representation

The construction follows Weil [Wei64]. A Heisenberg cocycle on a locally compact abelian group W is a continuous alternating bicharacter β; a vacuum is a function e on W playing the part of a matrix coefficient of a vacuum vector (a Gaussian at the real places, the indicator of a self-dual lattice times a semicharacter at the finite places). The integrable functions v with $v(x) = \int v(y)\beta(y,x)e(x-y)\,dy$ form a Banach space with an irreducible Heisenberg representation, and Schur's lemma holds: an invariant form is a multiple of the inner product and a bounded operator commuting with the representation

is a scalar. For a hermitian space with a point V, the adelic Heisenberg datum on $\mathbb{A}^n$ is the product of the real (Fock) and the finite data, and the adelic unitary group acts by symmetries with a multiplier, which gives the Weil representation ω with $\omega(g)\varrho(x)\omega(g)^{-1} = \varrho(gx)$. A theta functional exists by the theta series of the vacuum and is unique up to a scalar by Schur's lemma; it does not depend on the group element used to transport it from the standard space, because two choices differ by a rational point (Landherr's theorem gives a point matching the rational vectors). The finite part of the space for a standard V consists of the vectors whose theta functions are entire, and translation by the finite image of a rational vector acts on ϑ_f as the twisted translation by minus its real image.

Fourier inversion (W6): if ϑ_φ is a theta function for $\bar{\delta}\mathfrak{a}_0^n$, the signed sums of translates of $\varrho(-y)\varphi$ over $\mathfrak{a}_0^n/\bar{\delta}\mathfrak{a}_0^n$, for $y \in \delta^{-1}\mathfrak{a}_0^n/\mathfrak{a}_0^n$, are multiples $c_y\mathrm{vac}$ of the vacuum, and $\sum_y c_y\varrho(y)\mathrm{vac} = (\delta\bar{\delta})^{fn}\varphi$, by the orthogonality of the characters of the finite group $\delta^{-1}\mathfrak{a}_0^n/\mathfrak{a}_0^n$; this is the classical fact that theta functions for a sublattice are spanned by the translates of the theta function of a principal polarisation [Mum83, II].

B.11. Schur modules

Let n be a finite set and consider polynomials in an $n \times \mathbb{N}$ array of variables. For a diagram with $\kappa(h)$ columns of height $h < \#n$, the polynomials f with $f(yb) = \prod_h b_{hh}^{l_h} f(y)$ for upper triangular b (with the weight l determined by κ) form a module ∇_κ spanned by the products of minors along fillings of the diagram; the products of minors of the standard fillings form a basis over every commutative ring [Ful97, §8]. The dual module is the Weyl module. Left substitution $f \mapsto f(zy)$ by an $n' \times n$ matrix maps $\nabla_\kappa(n)$ to $\nabla_\kappa(n')$ (after the appropriate power of the determinant if $\#n < \#n'$); its matrix in the chosen bases is the Schur matrix $\mathrm{Sch}_\kappa(z)$. The Fock pairing $\langle P, Q\rangle = P(\partial)Q(0)$ is symmetric, the adjoint of $f \mapsto f(zy)$ is substitution by the transpose, and $\langle \bar{P}, P\rangle > 0$ for $P \neq 0$.

Theorem B.23 (Integral branching). *Let a weight of* GL_{n+1} *at each infinite place have* $\kappa(h) = \gamma(h) + \kappa'(h)$ *columns of height* h*, of which* $\gamma(h)$ *lose a box and* $\kappa'(h)$ *keep it, and let* $\kappa_0(h) = \kappa'(h) + \gamma(h+1)$ *for* $0 < h < n$ *be the interlacing weight of* GL_n*, with determinant exponents* e *and* $e_0 = e + \kappa'(n)$*; let all columns of height below* q *lose a box. There is an integral matrix* br *from the Weyl module of* (κ, e) *to that of* (κ_0, e_0)*, in the bases of standard fillings, such that:*

(a) *over every field of characteristic zero,* $\mathrm{br}\,\rho(\mathrm{diag}(g,s)) = \prod_\tau s_\tau^{|\gamma|_\tau + e_\tau}\rho_0(g)\,\mathrm{br}$*,* $|\gamma|_\tau$ *the number of columns of* γ *at* τ*;*
(b) *its squared operator norm for the Gram norms is at most* $((L^L)^3)^2$ *if* L *bounds the number of boxes of* κ_0 *in the columns of heights* $q \le h < n$*;*
(c) *every complex matrix equivariant for the unitary block group* $\mathrm{diag}(g,1)$ *is a multiple of* br*.*

Sources. This is the multiplicity-one branching rule from GL_{n+1} to GL_n for interlacing weights [GW09, Thm. 8.1.1], in an integral form with a norm bound; that $\mathrm{br} \neq 0$ is not part of the statement. The Lean development builds br from its value at one point: the *branching vector*, the product of the leading minors to the powers γ with the new row first (so that it lies in every column of positive

height) and to the powers κ' with the new row last, transforms under $\mathrm{diag}(1, s)$ by $s^{|\gamma|}$; an integral equivariant matrix is determined by the image of an evaluation vector at a permutation array, which is a combination of products of minors along one filling, and equivariance over every field follows from the Zariski density of the unitary group. The norm bound is proved through the transposed contraction with the extremal filling, whose leading minors are integral; and (c) holds because an equivariant matrix vanishing at the extremal vector vanishes. □

B.12. Discrete series and the local theta correspondence

At an infinite place the unitary group of a hermitian space of signature (p, q) is $\mathrm{U}(p, q)$. Its discrete series are indexed by Harish-Chandra parameters λ, regular vectors in $\frac{n-1}{2} + \mathbb{Z}^n$ (here written doubled, $l = 2\lambda$, on the coordinates of a space with signs s), and the *lowest K-type* of λ has highest weight $\Lambda = \lambda + \rho - 2\rho_c$. The forms of *type* l are those on which the real group acts through the minimal idempotent of the discrete series with parameter l (the *kernel*); a product of such kernels over the infinite places is the *type kernel*. If l has gap h, the kernel is integrable with matrix coefficients decaying like $\exp(-h \cdot \mathrm{dist})$, and integration against it kills bounded functions invariant under the unipotent radical of a parabolic (Godement's argument [Wal88, Ch. 7]); this is the statement that the type kernel is a *cusp kernel* of order $h/2$.

For a dual pair $(\mathrm{U}(W), \mathrm{U}(V))$ in the Fock model, a polynomial in the variables $\mathrm{Hom}(W, V)$ is a *joint harmonic* for parameters (l, l') if it is nonzero, harmonic along rows and columns, and highest for both groups of the weights of the two lowest K-types. A datum is an *arrow* from type l to type l' if its lifts take forms of type l to forms of type l', and an *edge* if it is an arrow in both directions.

Theorem B.24 (Theta correspondence for discrete series). *If l and l' satisfy the block rule (Section 15.2), then they* match*: every row and column is matched with a sign condition or lies in the central range left by the difference of the ranks. For matching parameters a joint harmonic exists, given as a product of minors along layers; its projection by the discrete series kernel e_λ has $\mathrm{Re}\langle \Phi, \pi(e_\lambda)\Phi\rangle > 0$, is fixed by every function reproducing $e_{\lambda'}$, and generates all projected polynomial vectors under bump functions of the second group. Consequently the Fock vector of a joint harmonic is an edge between the types l and l'.*

Sources. The shape of the correspondence of discrete series parameters is due to J.-S. Li [Li90] and A. Paul [Pau99]; the joint harmonics are those of Kashiwara and Vergne [KV78], and the generation statement is Howe's argument [How89]. The Lean development proves the statements used: matching from the block rule as a statement about integer vectors; the existence of joint harmonics by layers of minors; positivity of the projection by comparing the Casimir operators of the two groups on a joint harmonic; and Howe's argument in the Fock model. □

B.13. Characters of finite general linear groups

Let $\Bbbk$ be a finite field and T an injective character of $\bar{\Bbbk}^\times$. For a label c of degree r the character Ξ_c of $\mathrm{GL}_r(\Bbbk)$ is the parabolic induction product of Green's functions τ_P over the irreducible factors P of c [Gre55].

Theorem B.25. *The Ξ_c, c a label of degree r, are orthonormal irreducible characters of $\mathrm{GL}_r(\Bbbk)$; the elements e_c formed from them are orthogonal central idempotents, and their tensor products over finite families of fields are orthogonal central idempotents. Multiplying Ξ_c by $\chi \circ \det$ multiplies the roots of c by the element $x_\chi \in \Bbbk^\times$ naming χ. If $l \neq 0$ and x_χ is not a root of c, then e_c kills $\sum \chi(\det h)^{-1} h$ over the invertible $h = 1 + ul^{\mathsf{T}}$.*

Sources. Green's theory of the characters of $\mathrm{GL}_r(\Bbbk)$ [Gre55]: the Ξ_c for squarefree c are the characters induced from the Coxeter tori of the factors; orthonormality is computed in Hall's algebra. The hyperplane sum spans representations induced from the stabiliser of $\ker l$ with $\chi \circ \det$ on the line, whose labels have the root x_χ. □

B.14. Algebraic modular forms and Bessel periods

For a definite hermitian space V over the CM field E, automorphic forms on $\mathrm{U}(V)$ of a given weight and level are *algebraic modular forms* in the sense of Gross [Gro99]: functions on the finite set of double cosets $\mathrm{U}(V)(E^+)\backslash \mathrm{U}(V)(\mathbb{A}_{\mathrm{f}})/K$ with values in the Weyl module of the weight, transforming by the stabilisers. The Petersson product is the finite sum over the double cosets weighted by the inverse orders of the stabilisers; for free levels (Proposition 16.2) all weights are 1.

For a hermitian space V of rank n with a point, the *first minimum* $\lambda(g)$ at g is the least length of a nonzero vector of the lattice of the moved space at g, cut off at 1, and the *height* is $1/\lambda(g)$. A *form of type e*, for a cusp kernel e (a minimal idempotent for convolution on the real group that decays polynomially and is reproduced by a compactly supported function), is a bounded automorphic function fixed by right convolution with e.

Theorem B.26 (Cusp estimate and rapid decay). *Let $N \neq 0$ be a natural number in the different of E, $\kappa = \kappa(E, n, N) \geq 1$ the cusp constant, $M \neq 0$ and $0 < \delta \leq 1$.*

(a) *If $\lambda(g) < (\delta/(M\kappa^2))^n$, the short lattice vectors at g span a rational isotropic subspace $A \neq 0$, and the unipotent radical of its stabiliser is covered by its rational points times the real ball of radius δ about 1 times the principal level M.*

(b) *Forms of type e are cuspidal: their constant terms along the unipotent radicals of nonzero rational isotropic subspaces vanish.*

(c) *Let F be a form of type e and principal level M. If $\lambda(g) < (\delta/(M\kappa^2))^n$ and $\|F(gx) - F(g)\| \leq C$ for every real x within δ of 1, then $\|F(g)\| \leq C$.*

(d) *Forms of type e decay faster than every power of the height, uniformly at a fixed level, and the forms of a fixed type and level span a space of finite dimension.*

Sources. These are the classical results on cusp forms [Bor07, §§7–8], [MS12], in a form adapted to bounded functions reproduced by an integrable idempotent,

with no differential equation and with an explicit depth. (a) A Minkowski-reduced basis is sorted by length; if the first minimum is below the threshold, the shortest vectors are isotropic and span an isotropic subspace, and the rational points of the radical are cocompact in it with an explicit fundamental set. (b) Let F be bounded, continuous, fixed by right convolution with e and invariant under the rational points of a unipotent radical N of finite covolume, and let $\Psi(x)$ be the mean of $F(gvx)$ over v in the quotient of N. Then Ψ is fixed by convolution with e, since F is (exchange the two integrals), and it is killed by that convolution, being left invariant under the real points of the radical: a cusp kernel annihilates every bounded continuous function left invariant under the real points of the unipotent radical of an isotropic subspace, a case of Harish-Chandra's theorem that the K-finite coefficients of a discrete series have vanishing constant terms. Hence the mean vanishes. (c) By (a), g times the radical lies in rational points times δ-small real points times the level, so all values of F on gN lie within C of $F(g)$, while their mean is 0 by (b); hence $\|F(g)\| \le C$. (d) High in the cusp, (c) with $\delta^n = \kappa/\mathrm{ht}(g)$ and the bounded variation of a form under small real translations give the decay; elsewhere the height is at most κ. For the finite dimension, a form of level K exceeds half its bound only at height at most a fixed $2R$, so only inside ΓC with C compact (Mahler's criterion); cover C by finitely many neighbourhoods of points a^{-1} on which such a form stays within half its bound of its value at a^{-1}. A form vanishing at all these points has every bound halved, hence is 0, so evaluation at finitely many points is injective on the forms of level K. □

For $Y_0 = \langle e_1, \dots, e_\varrho\rangle \oplus X_0 \oplus \langle z\rangle \oplus \langle e_1^*, \dots, e_\varrho^*\rangle$, the *Bessel subgroup* $R_0 = U \rtimes \mathrm{U}(X_0)$ is the stabiliser of the isotropic flag in the e_a that fixes z modulo the flag; its unipotent radical U carries the generic character $\psi_U(u) = \psi(\sum_a \langle u e_{a+1}, e_a^*\rangle + \langle uz, e_\varrho^*\rangle)$. Every element of the Bessel group is a product uh with $u \in U$ and $h \in \mathrm{U}(X_0)$. These are the Bessel models of Gan, Gross and Prasad [GGP12]; Bessel functions on the real group are the bounded functions transforming by ψ_U on the left and by a representation of $\mathrm{U}(X_0)$.

B.15. Constant terms and the unfolding of Bessel periods

Let V, V' be graded hermitian spaces with strings of lengths s and $s-1$ (Section 18.1), Φ a datum on $\mathrm{Hom}(V, V')$ and ω_α the Weil representation of $\mathrm{U}(V)\times\mathrm{U}(V')$ twisted by characters α of $\mathrm{U}(1)(\mathbb{A})$. The theta kernel $\theta(\omega_\alpha(r,h)\Phi)$ is expanded along the rational maps $V \to V'$ modulo those of degree ≥ 0: each class contributes a constant term, the mean of the theta function along the maps of positive degree, which is a theta value of a zero term in a mixed model of the Weil representation [Wei64]. Over a set of classes stable under the rational points of the Bessel group in which no chosen representative agrees with the shift on the entries of positive grade, the terms integrate to zero against a ψ_U-equivariant function; the classes that survive are the translates of the shift by the rational points of $\mathrm{U}(V')$.

Proposition B.27 (One unfolding step). *Let W be a continuous ψ_U-equivariant function on the Bessel group R_V, left invariant under $R_V(E^+)$, and y a rapidly decreasing*

automorphic function on $\mathrm{U}(V')(\mathbb{A})$. *Put* $\mathcal{W}'(h) = \int_{[R_V]} W(r)\, C_{\mathrm{sh}}(\Phi)(r,h)\, dr$. *Then* $\int_{[R_V]} W(r)\, \theta(\omega_\alpha(r,h)\Phi)\, dr = \sum_\gamma \mathcal{W}'(\gamma h)$, *the sum over* $R_{V'}(E^+)\backslash \mathrm{U}(V')(E^+)$, *and*

$$\int_{[R_V]} W(r) \int_{[\mathrm{U}(V')]} \theta(\omega_\alpha(r,h)\Phi)\, y(h)\, dh\, dr = \int_{R_{V'}(E^+)\backslash \mathrm{U}(V')(\mathbb{A})} \mathcal{W}'(h)\, y(h)\, dh,$$

both sides converging absolutely. If the constants of the two gradings agree, $\gamma \mapsto \mathcal{W}'(\gamma h)$ *is again* ψ_U*-equivariant, and* $\mathcal{W}'$ *is continuous and left invariant under* $R_{V'}(E^+)$.

Sources. This is the transfer of Bessel and Fourier–Jacobi periods under one theta lift, in the unitary case with a definite complement; see Gan and Ichino [GI16] for the local theory and Kudla [Kud84] for the seesaw principle. □

Two further facts on graded lattices are used. *Iwasawa decomposition for the flag of the grades*: if E contains a primitive root of unity of order 5040, every finite point x has a unipotent u such that the lattice of ux is graded. *Finiteness*: for an invertible idele $\mathfrak{q}$ of valuation 1 outside a finite set Q, the double cosets under the finite Bessel group and the lattice stabiliser whose orbits contain a point with the lattice bounds have finitely many representatives, supported on $Q \cup \bar{Q}$; this is the finiteness of the lattices between two given ones, a lattice being determined by its local components [O'M73, §81].

B.16. Hecke operators at split places and the unramified correspondence

Let w be a split place of norm q, so that $\mathrm{U}(V)(E_w^+) \cong \mathrm{GL}_n(E_w)$. For $j \le n$, $T_{w,j}$ is the Hecke operator of the double coset of $\mathrm{diag}(\varpi 1_j, 1_{n-j})$ under $\mathrm{GL}_n(\mathcal{O}_w)$, acting on the vectors spherical at w, and the *Hecke polynomial* is $H_w(X) = \sum_{j\le n} (-X)^j q^{\binom{j}{2}} T_{w,j}$. On spherical vectors $T_{\bar{w},j}$ is the operator of the inverse double coset, the operators at different split places commute, and in a unitary representation a finite-dimensional stable space of spherical vectors is spanned by joint eigenvectors.

Let S, T be hermitian spaces of ranks n and $n+k$ and let the characters of the Weil representation of $\mathrm{U}(S)\times\mathrm{U}(T)$ be unramified at w. On the vacuum at w the operators $T_{w,j}$ for T and $T_{\bar{w},j}$ for S count the subspaces of the kernels of an integral matrix and of its transpose modulo w, and the q-binomial theorem gives

$$H_w^T(X)\,\mathrm{vac} = \prod_{l<k} (1 - q^l c\, X) \cdot H_{\bar{w}}^S(cX)\,\mathrm{vac},$$

with $c = \chi(\varpi_w) q^{n/2}$ for the character χ of the pair. Through equivariant maps and through the theta pairing this identity passes to theta lifts: the Hecke polynomial of the lift of a spherical form is that of the form, rescaled, times the q-Pochhammer symbol. This is the unramified theta correspondence for GL_n, stated for operators (Howe; Rallis [Ral82]; Kudla [Kud86]).

Theorem B.28 (Chains of Satake roots). *Let* ρ *be a smooth representation of* $\mathrm{GL}_N(E_w)$ *with an invariant scalar product,* q *the norm of* w, *and* $x \neq 0$ *a vector fixed by* $\mathrm{GL}_N(\mathcal{O}_w)$ *with Hecke polynomial* $P = \sum_{j\le N} (-1)^j q^{\binom{j}{2}} t_j X^{N-j}$.

(a) *If* $P(c) = 0$ *and* $|c|^2 q^k = q^N$, *then* $c, cq, \dots, cq^{k-1}$ *are roots of* P; *if* $P(c) = 0$ *and* $P(qc) \neq 0$, *then* $|c|^2 \ge q^{N-2}$.

(b) *Let U be the Hecke operator of* $\mathrm{diag}(\varpi, 1, \dots, 1)$ *for the pro-p Iwahori subgroup. If $d \neq 0$ and $P(X/d) = \prod_{i \le k}(X - q^i) \cdot S$ with S dividing R, then x is orthogonal to $\big((dX - q^k)R(dX)\big)(U)y$ for every y fixed by the pro-p Iwahori subgroup.*

Sources. The Satake parameters of unitary unramified representations of GL_N come in chains; both statements follow from Tadić's classification of the unitary dual [Tad86] and the geometric lemma of Bernstein and Zelevinsky. The Lean development proves them directly, by two counts of hyperplanes: a root c with qc not a root has $|c|^2 \ge q^{N-2}$; with the symmetry $y \mapsto q^{N-1}/\bar{y}$ of the roots, forced by unitarity, this gives the chain; and the same counts let the roots of a chain below the top one be cancelled one at a time against the partial Hecke sums of translates of x, which gives the orthogonality. □

B.17. A comparison of simple trace formulas

Let V be a hermitian space of rank n over the CM field E. For a test function $e \otimes \phi$, with e a cusp kernel on the real group (a minimal convolution idempotent with polynomial decay) and ϕ locally constant of compact support on G_{f}, the trace of its action on forms of type e is the geometric side $\sum_R$ over the characteristic polynomials R of rational points of the orbital integrals. A polynomial R that is irreducible with a root off the unit circle contributes nothing (the Selberg principle for idempotents of the discrete series, after Harish-Chandra); for R irreducible with all roots on the unit circle the orbital integral splits into a product of characters of the compact unitary group and a factor that depends only on R and ϕ, which matches the classes of different signs.

Theorem B.29 (Comparison of traces). *Let E contain the* 5040*th roots of unity and have an even number of infinite places, and let V, V' be hermitian spaces of rank $n \ge 2$ over E with signs s, s'. Let l be a doubled Harish-Chandra parameter with gap at least $500n^3 + 10n^5$ at every infinite place for both signs, K a level, and a an element of the group algebra of G_{f} preserving the K-fixed vectors in every representation, such that $a \star \mathbf{1}_K$ is cuspidal at a split place. Then a has the same trace on the forms of parameter l and level K on V and on V'.*

Sources. The statement for test functions supported, at a second place, on elements with a simple characteristic root 0 is the simple trace formula with a cuspidal component, as in Arthur and Clozel [AC89, Ch. 1] and Harris and Labesse [HL04, Prop. 1.10.5]; the second condition is removed by cutting the level at an auxiliary split place down to the stabiliser of a line, where a line Hecke operator forces the simple zero and counts the spherical vectors through its determinant. Here integrable idempotent kernels of discrete series with a large gap replace pseudo-coefficients, and there is neither truncation nor endoscopy. The cuspidal input is Green's theory: the class function of a monic irreducible polynomial of degree n has mean zero on the unipotent radical of every proper coordinate parabolic of $\mathrm{GL}_n(\mathbb{F}_q)$ [Gre55], so the idempotent of a tuple of labels with an irreducible member is cuspidal at the place of that member. □

B.18. The Siegel–Weil formula and Siegel's formula

Let S be a hermitian space over E split into p hyperbolic planes and T a hermitian space of rank m, both unimodular at every finite place, with the Weil representation ω of the pair of unitary groups on the test functions $\mathcal{S}$ of $\mathrm{Hom}(S,T)$ (Appendix B.10). A *Lagrangian* of S is a maximal isotropic subspace; every Lagrangian is spanned by the isotropic frame of a complete splitting $S = lE^p \oplus kE^p$. The *zero term* of f along a Lagrangian L is the mean of $\theta(\varrho(z)f)$ over the maps $z\colon S \to T$ vanishing on L, adelic modulo rational; it is the value at 0 in the Schrödinger model of a splitting along L and depends only on L.

Theorem B.30 (Siegel–Weil formula in Weil's range). *If $m > 2p$, the zero terms are summable in operator norm over the Lagrangians of S, and for every $f \in \mathcal{S}$ the function $h \mapsto \theta(\omega(1,h)f)$ is integrable on $[\mathrm{U}(T)]$ with*

$$\frac{1}{\mathrm{vol}[\mathrm{U}(T)]}\int_{[\mathrm{U}(T)]}\theta(\omega(1,h)f)\,dh = \sum_{L}\mathrm{zt}_L(f).$$

Sources. This is Weil's theorem [Wei65, §52, Thm. 5] in the convergent range, with the Eisenstein series written as the sum of the zero terms over the Lagrangians. The proof in the Lean development follows the classical route: reduction theory gives Siegel sets, every adelic point being $\gamma a_x \kappa$ with γ rational, a_x a dyadic torus element and κ in a compact set; theta is bounded uniformly along the normalised torus when $m > \operatorname{rank} S$, and the zero terms of a second splitting are bounded along it. Let ℓ be the difference of the two sides. Splitting off one isotropic line and inducting on the rank of S, ℓ along the normalised torus tends to 0 as any coordinate of the torus tends to infinity, so $\ell(\omega(g)f)$ exceeds the sum of the norms of the zero terms of the translated vacuum by a bounded amount; a continuous invariant functional dominated in this way vanishes when $\operatorname{rank} T > \operatorname{rank} S > 0$, by the Hecke maximum principle. □

For a function Φ_0 of finite adelic vectors of rank n and $\alpha \in E^+$ the *density* is $\delta(\Phi_0,\alpha) = \iint \Phi_0(x)\psi_{\mathrm{f}}(z(x^*x-\alpha)/2)\,dx\,dz$, over the finite vectors x and the self-adjoint finite adeles z, with the codifferent of volume 1. For a level K the *class sum* of a function is its sum over the classes, weighted by $1/\#\mathrm{Aut}$; the class sum of 1 at the stabiliser of $\hat{O}^n_E$ is the *mass* mass_n.

Theorem B.31 (Siegel's formula). *Let E carry Weil data and $n \geq 3$.*

(a) (*One vector.*) *For the standard definite space of rank n, the mean over the unitary quotient of $\sum_\xi \Psi(h^{-1}\xi)$ is $\Psi(0)$ plus the sum over $\beta \in E^+$ of the integrals $\int \Psi(x)\psi(\beta x^*x/2)\,dx$; for $\alpha \neq 0$ the sum over the rational ξ with $\xi^*\xi = \alpha$ has mean $|D_E|^{1/4}\int P$, $P(z) = \int \Psi(x)\psi(z(x^*x-\alpha)/2)\,dx$.*

(b) (*Vectors of one norm.*) *Let K be a level, $\Phi_0 \geq 0$ locally constant, K-invariant and of compact support, and α totally positive. The class sum of level K of $\sum_{x^*x=\alpha}\Phi_0(h^{-1}x)$ is the class sum of 1 times $|D_E|^{1/4}\prod_v \pi^n \alpha_v^{n-1}/(n-1)!$ times $\delta(\Phi_0,\alpha)$.*

(c) (*Gauss sums.*) *If $\zeta_{\hat{O}}(\mu) \leq 2$ and $2(\mu+1) \leq n$, the integral over z of $|\int_{\hat{O}^n}\psi_{\mathrm{f}}(z(w^*w-\alpha)/2)\,dw|$ is at most $\zeta_{\hat{O}}(\mu)\,\mathrm{vol}(\hat{O}^n)$.*

(d) (*Masses.*) *If E contains a primitive* 5040*th root of unity, the class sum in rank $j+1$ of the number of vectors of norm* 1 *is* mass_j*; and for $j \geq 4u+12$ and $8f \leq 2^u$,*

$$\left|\log \frac{\mathrm{mass}_{j+1}}{\mathrm{mass}_j} - f \log \frac{\mathrm{rd}_E^{j+1/2}\, j!}{(2\pi)^{j+1}}\right| \leq \frac{2}{2^u}.$$

Sources. (a) is the Siegel–Weil formula (Theorem B.30) for the hyperbolic plane against the definite space; the zero terms along the lines of $l + \jmath^{-1}\beta k$, $\beta \in E^+$, are computed in the Schrödinger model, and Laplace's integral for $1/\Gamma$ gives the archimedean factor. (b) Detect the norm α by its Fourier coefficient; this is Siegel's formula [Sie35] for hermitian forms [Bra41], read off from Weil's formula for one vector [Wei65, Thm. 5], with the singular series left as one adelic Fourier integral. (c) Sort the characters by their conductor ideal; for each ideal the Gauss sum is bounded by Weyl differencing, and the sum over the ideals is the zeta value. (d) Apply (b) to the vectors of norm 1 in the standard lattice and bound the density by (c); this is an approximate form of the mass formula for unitary groups [GHY01], which suffices here because only ratios of masses in large rank are used. □

The local densities at a finite set of finite places, read as one ring A with integers R, are controlled by the same Gauss sums. The classes modulo gL of the lattice L of integral vectors are compact and open of volume $|g|^{\#\kappa}$. For a unimodular hermitian form of rank at least $2\mu+1$ and $\zeta_R(\mu) \leq 2$, a class modulo gL has mass at most $\zeta_R(\mu)|g|^{-1}$ times its volume on each quadric, and on an integral quadric the mass of L differs from 1 by at most $\zeta_R(\mu) - 1$; by Fourier inversion the local density of a continuous $f \geq 0$ of compact support at $\delta^{-1}t$ is the integral of f over the quadric, whenever the outer integral converges absolutely, and its modulus is always at most this integral. At a finite place the standard form of rank at least two has a basis beginning with a hyperbolic pair, so the volume disintegrates along the form into the fibre measures used in Proposition 29.23(a).

Strong approximation is used in the following weak form. A continuous function on the adelic unitary group of a space with a hyperbolic plane that is left invariant under the rational points, right invariant under the points supported on a nonempty set S of finite places and their conjugates and under the integral points trivial on S, and whose values on the Levi subgroup $\mathrm{GL}_1 \times \mathrm{U}(W_0)$ do not depend on the GL_1 component once it is invariant under the unipotent radical, is constant. The proof combines strong approximation for the two opposite unipotent radicals, Witt's theorem on the rational points of the Levi factor, and weak approximation with the Iwasawa decomposition [Kne65].

B.19. The constant term along an isotropic line

This subsection supplies the analytic input of the descent of the exchange identity along hyperbolic planes (Proposition 31.18). Let ν be a character of $[\mathrm{U}(1)]$ of weight μ, with $\mu(w) > 0$ at every infinite place w, whose tame part is nontrivial exactly on a nonempty finite set S of finite places that contains no conjugate of one of its places; such a pair (ν, S) is a *split tame character*. Its *semilocal character* is $\chi(t) = \nu(t/\bar{t})$ on $\prod_{w\in S} E_w$, read as 0 off the units. With q_w

the order of the residue field at w and $L^S(x,\nu)$ the Euler product of the base change $x \mapsto \nu(x/\bar{x})$ without the places of $S \cup \bar{S}$, put, for $e \geq 0$,

$$L_*(e) = \Big(\prod_{w\in S} q_w\Big)^{e^2} \prod_{z=1}^{e} |L^S(z,\nu)|^2, \qquad L_*(-e) = L_*(e)^{-1},$$
$$L^\sharp(z) = |L^S(z,\nu)| \prod_{w\in S} q_w^{z-1/2},$$

the last for real $z \geq 1$, where L^S is given by its Euler product.

Let $W' = W \oplus H$ and $V' = V \oplus H$ be hermitian spaces split by adapted splittings A and X along a hyperbolic plane H (frames orthonormal at the infinite places and integral at the finite places), and let x_B be the adelic vector spanning the isotropic line of X. For an adele a the *line term* lt_a sends a datum on $\mathrm{Hom}(W',V')$ to a datum on $\mathrm{Hom}(W,V)$: in the model $\beta =$ true it is translation by the rank one map $a\, x_B \otimes k_A^*$ followed by the zero terms along the line of A and then along that of X; in the model $\beta =$ false the dual lines are used and the zero terms are taken in the other order. Write lt_a^∞ for its archimedean part on the Fock spaces. For real z, the *cone integrand* of a Fock vector v for (W',V') at a pair $p = (\kappa, a)$ of a point κ of the maximal compact subgroup of $\mathrm{U}(V')$ and a family a of nonzero complex numbers indexed by the infinite places is

$$\mathrm{ci}_z(v)(\kappa,a) = \prod_w \Big(\frac{a_w}{|a_w|}\Big)^{2\mu(w)} |a|^z\ \mathrm{lt}_a^\infty(\kappa v), \qquad |a| = \prod_w |a_w|^2,$$

with κ acting by the real Weil representation of weight $\lfloor \mathrm{rk}\, W'/2 \rfloor - \mu$ on $\mathrm{U}(V')$. The *cone integral* $\mathrm{cone}_z(v)$ is its integral over the maximal compact subgroup and the archimedean ideles $(E\otimes\mathbb{R})^\times$, both with Haar measure, in the model true; the *dual cone integral* $\mathrm{cone}_z^*(v)$ is the same in the model false. For an integral matrix y at S the *plane datum* of v and y is the datum on $\mathrm{Hom}(W',V')$ with real part v, the standard datum of y composed with the projection to W at S, and the vacuum elsewhere; the *constant term* along A at a point (u,g) of the Levi subgroup $\mathrm{GL}_1 \times \mathrm{U}(W)$ is the mean over the unipotent radical of the stabiliser of the line of A, modulo its rational points.

Theorem B.32 (The tower property with ramified data). *Let (ν, S) be a split tame character of weight μ, let $\mathrm{rk}\, W$ be even and $e = \mathrm{rk}\, V - \mathrm{rk}\, W/2$.*

(a) (*L-values.*) *$L^S(x,\nu) \neq 0$ for real $x \geq 1$, so $L^\sharp(z) > 0$ for $z \geq 1$; and for every $z \in \mathbb{N}$*

$$L_*(z+1) = L^\sharp(z+1)^2 L_*(z), \qquad L_*(-z) = L^\sharp(z+1)^2 L_*(-z-1).$$

(b) (*Cone integrals.*) *For $z > 0$ and a Schwartz vector v the cone integrand is integrable for the product Haar measure, and $\mathrm{cone}_z(v)$ and $\mathrm{cone}_z^*(v)$ are Schwartz vectors.*

(c) (*Unfolding.*) *For a Schwartz vector v, an integral y at S and a Levi point (u,g) with $u = 1$ or $\mathrm{rk}\, V = \mathrm{rk}\, W$, there is a function $\mathcal{Z}$, analytic on the real line, such that the constant term along A at (u,g) of the theta integral of (W',V') on the plane datum is $\mathcal{Z}(e+1)$, and for real $x \geq 1$*

$$\mathcal{Z}(x) = \varepsilon\, |\alpha_0|^x \prod_{w\in S} q_w^{x-1/2}\, L^S(x,\nu)\, \theta_{W,V}\big(\mathrm{cone}_x(v), \mathrm{row}_y\big)(g),$$

and $\mathcal{Z}(1-x)$ *is the same expression with* $\mathrm{cone}^*_x(v)$ *and a constant* ε'*. Here* α_0 *is the normalising idele of the Weil data, and* $|\varepsilon|$, $|\varepsilon'|$ *are positive numbers independent of* (v, S)*.*

(d) (*One plane.*) *Let* $z \in \mathbb{N}$*. If* $e = z$*, there are* $C > 0$*, independent of the character, and* γ *with* $|\gamma| = C\,L^\sharp(z+1)$ *such that the constant term in* (c) *is* $\gamma\,\theta_{W,V}(\mathrm{cone}_{z+1}(v), \mathrm{row}_y)(g)$*. If* $e = -z-1$*, the same holds at* $u = 1$ *with* $\mathrm{cone}^*_{z+1}(v)$ *in place of* $\mathrm{cone}_{z+1}(v)$*.*

(e) (*Functional equation.*) *Let* $k \in \mathbb{N}$ *with* $k \le \mu(w)$ *for all* w*. The function* $x \mapsto L^S(x, v)$ *on* $[1, \infty)$ *extends to an analytic function* Λ *of real* $x > -k$ *with* $\Lambda(j) \neq 0$ *at every integer* $j > -k$*, and* $L^\sharp(1-z) = C(z)\,L^\sharp(z)$ *for real* $1 \le z < k+1$*, where* $L^\sharp(1-z)$ *is formed with* Λ *and* $C(z) > 0$ *does not depend on the character.*

Sources. For cuspidal forms the constant term of a theta lift along an isotropic line is the lift for the pair with one plane removed; this is Rallis' tower property [Ral84b, Thm. I.1.1], and for unitary groups Wu [Wu13, Prop. 3.1]. Here the lifted function is a character, the data are ramified and the isotropic orbit survives, as a Tate integral [Tat67]; the Lean development follows the classical route. (a) The Euler product converges absolutely for $x > 1$ and has no zero at $x = 1$ because a character of nonzero weight is not trivial; the recursion is the definition of L_*. (b) The line term decays rapidly in a in every Fock moment, while $|a|^z$ grows at most polynomially, so the integrand is dominated by a continuous function of κ on a compact group times $\prod_w (1 + \log^2 |a_w|)^{-1}$, which is integrable for Haar measure on $(\mathbb{C}^\times)^{\#w}$. (c) The constant term of the theta kernel is the sum over the rational vectors of V' of their terms; a term vanishes unless the vector is isotropic, and the term of 0 vanishes on the plane datum, since the standard datum lives on matrices of full rank. By Witt's theorem the nonzero isotropic lines form one orbit of the rational points, so the integral over $[\mathrm{U}(V')]$ unfolds to the stabiliser of the line of X and then, in Iwasawa coordinates $((u', h), \kappa)$ with the module $|u'|^{\mathrm{rk}\,V+1}$, to an integral over the idele classes, $[\mathrm{U}(V)]$ and the maximal compact subgroup. Levi points act on the line terms by a scalar and a rescaling of the adele, which turns the integrand into Tate's integrand $\nu(a/\bar a)|a|^x \sum_{\xi \in E^\times}$ of line values; Poisson's formula on the line identifies the two models. At S the datum factors along the line into a Gauss sum of modulus $(\prod_S q_w)^{-1/2}$ times the finite factor $a \mapsto \chi^{-1}(a)\chi(\varpi\bar a)$, and Tate's finite integral of this factor is $L^S(x, \nu)$ times a volume for $x > 1$. Neither side of Poisson's formula has a term at 0, so the integral is a Mellin transform convergent on every strip and analytic on the real line; continuity in the exponent carries the formula to $x = 1$. (d) is (c) at $x = z + 1$, on the side of Poisson's formula fixed by the sign of e; the modulus of the Gauss sum and of $|\alpha_0|^{z+1}$ go into C. (e) Apply (c) to the pair (H, H) over the space 0 with the test vector, the Fock vector of the product of the $\mu(w)$-th powers of the diagonal variables: the maximal compact subgroup fixes it, and its cone integrand is the vacuum times $e^{-\sum_w |a_w|^2} |a|^x \prod_w |a_w|^{2\mu(w)}$, whose integral is a positive analytic function of $x > -k$. Dividing $\mathcal{Z}$ by the analytic factor without zeros that multiplies L^S continues L^S to $x > -k$, and comparing the two sides of Poisson's formula gives the functional equation in absolute values [Tat67, §4.4]; the continuation has no zero at an integer $j > -k$ because, by the functional equation, its value there is a nonzero multiple of $L^\sharp(1-j)$ with $1 - j \ge 1$ when $j \le 0$, and is an Euler value at $j \ge 1$. □

B.20. Vanishing descents and periods

This subsection supplies the passage from vanishing theta descents to vanishing periods used for the false lifts (Theorem 35.7). Let W, V be hermitian spaces, a a character of $\mathrm{U}(1)(\mathbb{A})$, S a set of finite places and Y an automorphic function on $\mathrm{U}(V)(\mathbb{A})$. The descents of Y to W *vanish* (for a and S) if $g \mapsto \int_{[\mathrm{U}(V)]} \theta(\omega(g,h)f)\, a(\det h)\, Y(h)\, dh$ is zero for every datum f on $\mathrm{Hom}(W,V)$ that is the vacuum at the places of S and their conjugates. If W has a splitting with isotropic frames of rank p, the theta kernel at (g,h) is the sum of the *terms* of the rational frames x (the p-tuples of vectors of V), and the *Fourier coefficient* at an hermitian $p \times p$ matrix β of a function F on $\mathrm{U}(W)(\mathbb{A})$ is the mean of $\psi(\mathrm{tr}(j\beta M))F(z(M)g)$ over the adelic M modulo the rational ones, z parametrising the centre of the unipotent radical, ψ the standard additive character and j the purely imaginary element of the Weil data. For a splitting $V = V'' \oplus V^\circ$ of the definite space of rank n into blocks of ranks n_1, n_2 and a point h_1 of $\mathrm{U}(V'')(\mathbb{A})$, the *block period* of Y is $I(h) = \int_{[\mathrm{U}(V^\circ)]} Y(\mathrm{diag}(h_1,h_2)h)\, dh_2$.

Theorem B.33 (Vanishing descents and periods). (a) (*Testing and translation.*) *Let V be definite and A a set of finite places disjoint from its conjugates. The descents of Y vanish for the data that are the vacuum off A and its conjugates as soon as they vanish for the data $v_P \otimes \Psi \otimes \mathrm{vac}$, P a polynomial and Ψ a Schwartz–Bruhat function above A. If the descents of Y vanish for S, so do those of its right translate by a point that is integral at every place of S.*

(b) (*Fourier coefficients.*) *Let y be continuous on $\mathrm{U}(V)(\mathbb{A})$ and left invariant under the rational points, with terms of the kernel at g absolutely integrable against y. The Fourier coefficient at $j\beta$ of the theta lift of y is the sum, over the rational frames with Gram matrix β, of the integrals over $[\mathrm{U}(V)]$ of their terms times y. If β is invertible and x_0 is one such frame, it is the integral of the term of x_0 times y modulo the rational stabiliser of x_0.*

(c) (*Periods.*) *Let T be a set of finite places and Y continuous on $[\mathrm{U}(V)]$, $V = V'' \oplus V^\circ$ definite, right invariant under the integral points with component 1 off T and its conjugates. If the descents of Y to the split hermitian space of rank $2n_1$, twisted by υ^{n_1}, vanish for all data that are the vacuum at the places of T and their conjugates, then the block period $I(1)$ vanishes for every h_1.*

Sources. Fourier coefficients of theta lifts are periods over stabilisers, and vanishing lifts in a theta tower kill periods on the complementary group; see Rallis [Ral84b] and, for this unfolding, Walls [Wal15, Prop. 2.1.1]. Here a single non-degenerate coefficient of a lift to a split unitary group is unfolded once. (a) The descent at g is a bounded linear functional of the datum, so it suffices to test the data $v_P \otimes \varrho(x)\mathrm{vac}$ with x finite and supported on A and its conjugates, and $\varrho(x)\mathrm{vac}$ is a multiple of the image of the vacuum under the Heisenberg operators realising a Schwartz–Bruhat function of the components above A. The descent of the translate of Y by h_0 by a datum f is the descent of Y by $\omega(1,h_0^{-1})f$, again the vacuum at S as h_0 is integral there. (b) Left multiplication by the central element $z(M)$ multiplies the term of x by $\psi(\mathrm{tr}(-j\beta(x)M))$, $\beta(x)$ the Gram matrix of x, and for y invariant under the rational points the term of x times y is unchanged under $(x,h) \mapsto (\gamma x, \gamma h)$. Expanding the kernel into its terms and exchanging the sum with the integral and the mean, the mean over M keeps the frames with $\beta(x) = \beta$. For invertible β, Witt's theorem makes these frames one

orbit of the rational unitary group, and the sum over the orbit unfolds to the stabiliser. (c) Let o_1 be the frame of the first block, α_0 the scaling idele of the Weil data, $\mathrm{vac}[x] = |e(\alpha_0 x)|^2$ the slice of the vacuum and $\psi_u(x) = \psi(\delta \sum u_{ij} x_{ij})$, with δ generating the inverse different and the components at T and its conjugates removed. For the Heisenberg translate of the vacuum along the isotropic vector with coordinates $\alpha_0 \bar{u}$, which is the vacuum at T, the term of the frame o_1 at the Levi point of $\alpha_0 h_1$ is $c\,\psi_u(x)\mathrm{vac}[x]$ with $x = g^{-1} o_1 h_1$ and $c \neq 0$; the theta integral against the continuous Y on the compact quotient converges term by term. By (b) with $\beta = 1$ the vanishing descent gives $\int \psi_u(x)\mathrm{vac}[x] Y(g)\,dg = 0$ modulo the rational points of the second block, which is the stabiliser of o_1; integrating first over $[\mathrm{U}(V^\circ)]$ gives $\int \psi_u(h^{-1}o_1)\mathrm{vac}[h^{-1}o_1] I(h)\,dh = 0$ over $\mathrm{U}(V^\circ)(\mathbb{A})\backslash \mathrm{U}(V)(\mathbb{A})$, for every u. The functions that agree with some ψ_u on frames form a monoid closed under conjugation that separates frames up to their components at T, so by the Stone–Weierstrass theorem the same holds with ψ_u replaced by any continuous function χ of the values $u \mapsto \psi_u(h^{-1}o_1)$. Where $\mathrm{vac}[h^{-1}o_1] \neq 0$ the frame is integral, so it is $\gamma^{-1} o_1$ for an integral γ by Witt's theorem over $\hat{O}$, and $I(h)$ depends only on the values $\psi_u(h^{-1}o_1)$, because the components at T are absorbed by the integral points supported there, which fix Y; by Tietze's theorem $\bar{I}$ is such a χ. Hence $\int \mathrm{vac}[h^{-1}o_1]|I(h)|^2\,dh = 0$, so I vanishes where $\mathrm{vac} \neq 0$, in particular at $h = 1$, where the frame is o_1. □

B.21. Rallis' inner product formula

Let S, T be hermitian spaces of ranks n, m with $2n < m$, and let ω be the twisted Weil representation of $\mathrm{U}(S) \times \mathrm{U}(T)$ on the data $\mathcal{S}(\mathrm{Hom}(S, T))$. The *doubled datum* of F lives on $\mathrm{Hom}(S \oplus S^-, T)$. A finite datum F is *clean* if only the Lagrangians of $S \oplus S^-$ that are graphs of isometries contribute to its Siegel–Weil formula, and *sharp* for an open subgroup J off a set S_0 of places if it is the vacuum off S_0 and, at the finite points g integral off S_0, $\langle F, \omega(g)F\rangle$ is $\langle F, F\rangle$ on $J \cap K_0$ and 0 otherwise; its *mass* c is $\langle F, F\rangle / [K_0 : J \cap K_0]$. The *vacuum coefficient* of a finite point γ supported off S_0 is $\mathrm{mc}(\gamma) = (b(\det \gamma)\,\mathrm{fm}(\gamma)^m)^{-1}$, the factor by which γ changes the matrix coefficient of the vacuum, and the *Rallis series* is $\sum_\gamma \mathrm{mc}(\gamma) R(\gamma)$ over the cosets of the points integral off S_0.

Theorem B.34 (Rallis' inner product formula). *Let $2n < m$, and let the real vector v be an edge from a type e (for instance the joint harmonic of corresponding parameters).*

(a) (*Doubling.*) *For clean data, the integral*

$$\int_{[\mathrm{U}(T)]} \overline{\theta(\omega(g,h)\phi)}\,\theta(\omega(g',h)\phi')\,dh$$

is the sum of the zero terms of the doubled datum along the graphs of isometries.

(b) *Let F be sharp for J off S_0 with mass c, let z be the Rallis series of y and Ω the real Weil representation. Then $\langle \Theta y', \Theta y\rangle$ is $\mathrm{vol}[\mathrm{U}(T)] \cdot c$ times the absolutely convergent integral $\int \langle v, \Omega(x)v\rangle \langle y', R(x)z\rangle\,dx$ over the real points x of $\mathrm{U}(S)$.*

(c) *There is $A \neq 0$, depending only on the spaces and the real data, such that for F sharp for J off S_0 with mass c, the lift by $v \otimes F$ is injective on the J-invariant forms of type e, and $\langle \Theta y', \Theta y\rangle = A\,c \sum_\gamma \mathrm{mc}(\gamma)\langle y', R(\gamma)y\rangle$.*

(d) *If the signs of S agree or $\mathfrak{m} \neq (1)$, a function $\Psi \neq 0$ on matrices modulo $\mathfrak{m}$ that is fixed with the vacuum by a level $J(\mathfrak{m}) \leq J \leq K_0$, and whose support*

is carried into J by a fixed matrix, gives a datum $\Psi \otimes \mathrm{vac}$ sharp for J off the places of $\mathfrak{m}\bar{\mathfrak{m}}$, with mass the mean of $|\Psi|^2$ over $[K_0 : J]$.

(e) (*Rank one.*) *For $n = 1$, $m = 2k$, $k > 1$, and a character ν with tame part trivial off S, the sum of the vacuum coefficients of νv^k over the cosets off $D = S \cup \bar{S}$ is $L^D(k, \nu^{-1})$ divided by the Euler product at k of the place signs off D, a partial value of $\zeta_F(2k)$.*

Sources. This is Rallis' inner product formula by doubling [Ral87], in Weil's range and for rapidly decreasing forms; the unitary unfolding is as in Li [Li92]. (a) applies the Siegel–Weil formula (Theorem B.30) to $S \oplus S^-$; for a clean datum the boundary orbits drop out. (b) integrates (a) against two functions on $[\mathrm{U}(S)]$ and, for a sharp datum, cuts the finite points into cosets, on each of which the matrix coefficient is the vacuum coefficient. (c) The injectivity is Rallis' argument [Ral84]: if the lift vanishes, the real matrix coefficient of v convolves the form to zero, and for an edge this convolution is a nonzero multiple of the form. (d) The matrix coefficient of $\Psi \otimes \mathrm{vac}$ at an integral point is a sum of products of values of Ψ, which vanishes off J by the support condition. (e) At a split place the vacuum coefficients form a geometric series in $\nu(\varpi)q^{-k}$, and at the other places the coset is trivial; the product over the places is the partial L-value over the partial Euler product. □

B.22. Class field theory in the form used

Let K be a number field with absolute Galois group Γ inside the fixed algebraic closure Ω of $\mathbb{Q}$. Only the finite ideles enter; there is no Weil group. The global Artin map sends the finite ideles of K to the topological abelianisation of Γ, as the product of the local symbols.

Theorem B.35 (Class field theory). (a) (*Local invariant.*) *For a henselian local field k of characteristic 0 the invariant map $\mathbb{Q}/\mathbb{Z} \to H^2(\Gamma_k, \bar{k}^\times)$ is bijective.*

(b) (*Reciprocity.*) *If the symbol of ξ at x dies on an open subgroup containing the complex conjugations, then the character ξ kills the Artin symbol of $x \in K^\times$.*

(c) (*Existence.*) *For totally complex K, a continuous homomorphism from the finite ideles to a T_1 group that is trivial on $K^\times$ factors continuously through the Artin map.*

(d) (*Chebotarev, existence form.*) *Frobenius elements at primes of degree one outside a finite set are dense in Γ.*

Sources. This is local and global class field theory [Ser97, Mil06], proved in the Lean development in the cases used. The local invariant comes from the unramified extension and the inflation of Brauer classes; the second inequality comes from the densities of primes of degree one given by the non-vanishing of Hecke L-functions at $s = 1$; Chebotarev is reduced to a cyclic extension by Deuring's argument and follows from the second inequality; the existence theorem uses Kummer theory when K contains the nth roots of unity, and a cyclic step inside $K(\mu_n)$ with the norm otherwise. □

B.23. Galois representations attached to forms on definite unitary groups

Let ℓ be a prime and $\iota\colon \mathbb{C} \simeq \bar{\mathbb{Q}}_\ell$. Let f be an eigenform of weight κ on the definite unitary group of rank $n \geq 1$ of a CM field E with Weil data, with Hecke polynomials P_w. A continuous $r\colon \mathrm{Gal}(\Omega/E) \to \mathrm{GL}_n(\bar{\mathbb{Q}}_\ell)$ with nondegenerate trace form is *attached* to P if, for every $\varepsilon > 0$, $\det(X - r(s))$ is coefficientwise within ε of $\iota(P_w)$ for most Frobenius elements s.

Theorem B.36 (Attached representations). (a) (*Existence.*) *An attached representation exists, and its traces are determined by P. If f has maximal level at w, every attached representation is unramified or crystalline above w with the Hodge–Tate jumps of κ, and for split w the Hecke polynomial kills geometric Frobenius. In rank one an attached representation is the ℓ-adic avatar of an algebraic Hecke character.*

(b) (*Iwahori eigenvectors.*) *Let w be split and f a pro-p Iwahori eigenvector with character $\xi = (\xi_i)$, the norms of the ξ_i at a uniformiser non-increasing. Then r is of finite type at the places above w, has the jumps of κ, and $\prod_i (X - \iota\xi_i(x))$ kills every element s of the decomposition group whose inverse is the q^kth power on the residue field, x of order k with Artin symbol the class of s.*

(c) (*Labels.*) *The same holds when f has a product label at w, and then a polynomial built from the tame characters of the label kills each element of inertia.*

Sources. In the literature these are base change to GL_n [HL04], the Galois representations of the cohomology of unitary Shimura varieties with local–global compatibility [HT01, Shi11, CH13, BLGGT14] and the local Langlands correspondence. The local polynomials kill but need not be characteristic, and no Weil–Deligne representation is built. The Lean development proves them along the following route. Integral Kuga–Sato models of signature $(1, n-1)$ with level covers carry the congruence relation at good split places [Kot92, HT01]; the Eichler–Shimura argument [Del71] turns it into a relation between Frobenius and Hecke operators on ℓ-adic cohomology, and smooth proper reduction makes inertia act as the identity. The primitive comparison theorem [Sch13] shows the cohomology above ℓ is crystalline with the Frobenius relation, and Hodge classes give the jumps. For odd n and forms with a cuspidal type, the eigenvalues occur in cohomology, and killing polynomials cut out a representation of rank n. All forms are reached by ℓ-adic limits: congruences changing the weight, pseudocharacters (a limit of traces of continuous representations is a trace, [Tay91, Rou96]) and the passage of local conditions to limits. Fields with Weil data are reached by quadratic steps and base change in prime degree, and even rank by two theta lifts to rank $2n+1$ along different characters, cancelling the extra characters. □

B.24. Selmer counts and local duality

Let K be totally complex and M a finite module killed by ℓ^r, unramified outside a finite set S, with local conditions at S containing the coboundaries. *Relations* at split primes are families of additive maps n_i on M with $\sum_i n_i(x(\gamma_i)) = 0$ for all Selmer cocycles x, γ_i elements at auxiliary split primes; they take the place of the dual Selmer group.

Theorem B.37 (Greenberg–Wiles count and local duality). (a) (*Count.*) *If some element is central modulo an open normal subgroup U, fixes the roots of unity of ℓ-power order and fixes no nonzero vector, the number of relations at the primes split for U is bounded below by the number of Selmer cocycles divided by the local terms.*

(b) (*Local duality, inequality.*) *Let k be local with valuation ring O, $\#(O/l) = l^d$, and M killed by a power of l. If groups of cocycles $X \subseteq Z^1(\mathrm{Hom}(M, \Omega^\times))$ and $W \subseteq Z^1(M)$ are orthogonal under cup product and Γ_k acts by elements of l-power order, then $\#X \cdot \#W \le \#M^{2+d}$; for general M the corresponding divisibility holds by descent from an open subgroup of index prime to $\#M$.*

Sources. (a) is the inequality underlying the Greenberg–Wiles formula [Gre89, Wil90] (only inequalities for cocycles are proved; there is no perfect pairing and no dual Selmer group). It is proved by counting trivialised cocycles with the equivariant S-unit theorem and power residue reciprocity at split primes. (b) is the inequality half of local Tate duality in degree one [Ser97, II §5], [Mil06, I.2], proved by Kummer theory and dévissage; perfectness and the Euler characteristic formula are not used. □

A *relation setting* over a number field K is a finite discrete Galois module M, unramified outside a finite set S of primes, on which complex conjugations act as the identity, with a chosen place above each prime, local conditions at S on the decomposition groups that contain the coboundaries, and an open normal subgroup U that fixes M and the m-th roots of unity. A *relation off T* is a continuous U-equivariant map y from U to $M^* = \mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z})$, with the transformation law of the restriction of a dual cocycle, unramified outside T, such that $\sum_p y(\varphi_p)(z(\gamma_p)) = 0$ for every cocycle z in the local conditions, zero at complex conjugations and ramified only at S and finitely many primes p split for U outside T, with Frobenius elements $\varphi_p \in U$ and tame generators γ_p. Its *reduction* is its image in $M^*/\ell M^*$.

Theorem B.38 (Relations modulo ℓ). *Let $\ell \mid m$, let $M[\ell]$ have order ℓ, let κ be a continuous character with values in $1 + 4\mathbb{Z}_\ell$ whose kernel lies in $U \cup Uc^{-1}$ for a complex conjugation c, and let the local condition at $v \in S$ be the vanishing on the decomposition group if $v \nmid \ell$, and on its part in $\ker \kappa$ if $v \mid \ell$. If K has at least four infinite places, then for some finite $T \supseteq S$ the reductions of the relations off T form a group of order at least ℓ^{k+1}, k the number of primes of S not above ℓ.*

Sources. This is a weak form of the Greenberg–Wiles formula [Gre89, Wil90], in which only four infinite places are used; the Lean development proves it by counting. *Split primes.* By Chebotarev's theorem and the equivariant S-unit theorem there are finitely many primes, split for U and prime to the level, whose Frobenius elements detect the roots of units: every m-th root of a unit of the field of U that they fix is fixed by U. Every family of characters of M that kills the families of the values $z(\gamma_i)$ of the cocycles at their tame generators comes from a relation off a suitable T (Theorem B.37(a), by counting trivialised cocycles and power residue reciprocity at split primes). *Many reductions.* A character of $M[\ell]^\iota$ that kills the ℓ-torsion of the group B of the families of values extends to a character of M^ι killing B, so it comes from a relation, whose reduction determines it; hence $\#M[\ell]^{\#\iota}$ is at most the number of reductions times the

number t of cocycles killed by ℓ. *Few cocycles.* A cocycle killed by ℓ has values in $M[\ell]$; adding two split primes with Frobenius elements in prescribed cosets makes the cocycles of $M[\ell]$, trivialised above the level, infinity and the inertia at the other primes, at most $\ell^{\#\iota+3}$ in number (the kernel of the action on $M[\ell]$ and μ_ℓ has index prime to ℓ, which gives Higman's criterion); a local cocycle vanishing on $\ker\kappa$ is determined by one value, as the image of κ is procyclic, so the local terms cost at most ℓ above ℓ and 1 elsewhere, and four infinite places and the k primes off ℓ each save one factor ℓ. Hence $t\,\ell^{k+1} \le \ell^{\#\iota}$, and the two inequalities give the claim. □

B.25. Crystalline cocycles and odd classes

Let T be a free $\mathbb{Z}_p$-module of rank d with an action of a group Γ mapping with open image of index n to $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$, and let $B_{\max}$ be Colmez' period ring [Col98]. The *crystalline cocycles* $Z_f(T)$ are the maps $\Gamma \to T$ that become coboundaries over $B_{\max}$; T is potentially crystalline if the periods fixed near 1 span $B_{\max}\otimes T$.

Theorem B.39 (Crystalline cocycles). (a) (*Rank.*) *For potentially crystalline T, $Z_f(T)$ is finitely generated of rank $d + n\cdot\#\{$negative Hodge–Tate jumps$\}$; with positive jumps it is torsion modulo coboundaries.*

(b) (*Duality.*) *For a potentially crystalline lattice L and its Tate dual L' the sizes of $Z_f(L)$ and $Z_f(L')$ modulo ℓ^m satisfy the index relation of local duality, and the cocycles of L/ℓ^m orthogonal to a subgroup of $Z_f(L')$ are bounded by a divisibility of indices.*

(c) (*Odd classes.*) *For a CM field $E \subseteq \mathbb{Q}(\mu_N)$ and $t \ge 0$ there is an integer $0 < X \le 2^{4(N+t+3)^{27}} h_E$ whose valuation bounds, for every ℓ, O and n and every $e < t$, the length of the module of odd classes with values in $O/\ell^n(-e)$.*

Sources. (a) is the Bloch–Kato rank formula [BK90, Prop. 3.8, Cor. 3.8.4], counted from the fundamental exact sequence of $B_{\max}$. (b) is local Tate duality for crystalline cocycles, as an inequality, with the descent of Theorem B.37(b). The Kummer charts and Kisin modules used in Section 42 follow [Kis06]. (c) replaces the main conjecture [Wil90] by an explicit bound: a bounded divisibility of twisted Stickelberger sums (Voronoi's congruence), Jacobi-type sums, Kolyvagin's derivative cocycles and reciprocity laws give an integer killing the odd part, and boundedly many Galois elements detect a cocycle; for $e = 0$ the class number serves. □

Theorem B.40 (Kisin modules of Kummer charts). *Let k be a Kummer chart of Γ with ring $\mathfrak{S} = \Lambda[\![z]\!] \subseteq A_{\mathrm{inf}}$ and element E, and let L be a lattice over O, crystalline over Γ', with finite jump sets at the letters. Let h be the least integer with all Hodge–Tate jumps of L in $[-h,h]$, $\mathcal{N} \subseteq \mathcal{N}^A$ the Kisin module and its A_{inf}-span, and $S_B \subseteq B^+$ the p-adic closure of $\mathfrak{S}[E/p]$ with p inverted.*

(a) (*Frame.*) *τ^h maps $\mathcal{N}^A$ into $A_{\mathrm{inf}}\otimes L$ and $A_{\mathrm{inf}}\otimes L$ into $\mathcal{N}^A$; $\mathcal{N}^A$ is the A_{inf}-span of $\mathcal{N}$, and some $\mathrm{rank}_{\mathbb{Z}_p} L$ vectors span $\mathcal{N}$ over $\mathfrak{S}$ and $\mathcal{N}^A$ over A_{inf} and are independent over A_{inf}. Every element of $\mathcal{N}^A$ with the $\varphi^a E$ inverted has a nonzero multiple in $A_{\mathrm{inf}}\otimes L$. If $d \in A_{\mathrm{inf}}$ divides r whenever it divides $E^m r$ in B^+, then $x \in \mathcal{N}^A$ lies in $d\mathcal{N}^A$ as soon as $E^n\mu(x) \in B^+ d$ for one n and all crystalline forms μ.*

(b) (*Values.*) *For a crystalline form μ and $y \in \mathcal{N}$, $E^h\mu(y)$ is the image of an element of S_B, and $E^h\varphi$ maps $\mathcal{N}$ into itself. The values at the level a of forms on $\mathcal{N}$ with the $\varphi^b E$ inverted lie in the field $\mathcal{K}_a$ generated over the fixed field of Γ' by the images of S_B and of $\varphi^{-a}\mathfrak{S}$.*

(c) (*Types.*) *If n is not a jump at i, the forms of type n at i are of type $n-1$; modulo the forms of type $u-1$, those of type u at i are spanned over $\bar{\mathbb{Q}}_p$ by at most $\mathrm{mult}_i(u)$ of them.*

(d) (*Relations.*) *An $\mathfrak{S}$-linear map $A_{\mathrm{inf}} \to \mathfrak{S}$ carries every relation $\sum c_i v_i = 0$ among vectors on which all forms take values in $\mathcal{K}_0$ to the relation with the coefficients replaced by their images. Hence the coordinate $\lambda_0 \colon A_{\mathrm{inf}} \to \mathfrak{S}$ that is the identity on $\mathfrak{S}$ induces a retraction from the A_{inf}-span of $\mathcal{N}_{\mathrm{loc}}$ to $\mathcal{N}_{\mathrm{loc}}$ with $r(ax) = \lambda_0(a)x$; it commutes with maps of lattices, and if the value of a form at the level a at y is congruent modulo $t^n B_{\mathrm{dR}}^+$ to an element of $\mathcal{K}_a$, so is its value at $r(y)$.*

(e) (*Frobenius.*) *An element at the letter i has the standard orders of level l if and only if it lies in $\varphi^l \mathcal{N}^A$; those of them on which all forms take values in $\mathcal{K}_0$ form the $\mathfrak{S}$-span of $\varphi^l\mathcal{N}$.*

(f) (*Radius.*) *If $3e \le p^l$ and $eA \le p^l\delta$, then $p^{4\delta}$ times an element of $\varphi^l\mathcal{N}^A$ with values in $\mathcal{K}_0$ lies in the $\mathfrak{S}$-span of the elements y of that kind with $g\mu(y) - \mu(y) \in t^A B_{\mathrm{dR}}^+$ for all forms μ and all $g \in \Gamma'$.*

(g) (*Invariants.*) *For an open subgroup H of G_p and $n \in \mathbb{Z}$, an element of B_{dR} that H fixes modulo $t^n B_{\mathrm{dR}}^+$ is congruent modulo $t^n B_{\mathrm{dR}}^+$ to an element of the fixed field of H.*

(h) (*Existence.*) *For every open subgroup G_K of G_p there is a Kummer chart with Λ and $\theta(z)$ fixed by G_K and $ef = [K : \mathbb{Q}_p]$.*

Sources. The theorem is a substitute, in the form needed in Section 42, for part of Kisin's theorem on the module of a crystalline lattice [Kis06], [BMS18, §4]; the finite height inclusion $E^n\mathcal{N} \subseteq \mathfrak{S}\varphi\mathcal{N}$ and the equivalence of categories are neither stated nor used. A chart is a Λ-algebra map $\Lambda[\![X]\!] \to A_{\mathrm{inf}}$ sending X to a non-unit Teichmüller lift z and some $X^e V + p$, V a unit, to a generator of $\ker\theta$; Weierstrass division by $\varphi^n E$ and Ax's theorem [Ax70] show that elements of the tilt fixed by the Galois elements fixing $\mathfrak{S}$ are approximated from $\mathfrak{S}$, which gives (h) and the coordinates of A_{inf} over $\mathfrak{S}$. If the jumps lie in $[-a, b]$, an invariant basis gives a period matrix P, invertible over $B_{\max}$ [Col98] with $\rho(g)\,gP = P$, a unit v with $E\tau v = \mu$, and a matrix U over B^+ with image $(E\tau)^a P$ of height $a + b$. The Lean development constructs a *frame* of U: Q, Q' over A_{inf}, Z over S_B and B over $\mathfrak{S}$ with $QQ' = \tau^h$, $UZ = E^h Q$ and $\varphi Q = vQB$. Everything rests on $p \nmid \det Q$, which follows from a separating set of rows and Artin's minimal support argument over a valuation ring, applied to the reduction of Q modulo p over the tilt; the columns of $\tau^{-h}Q$ then span $\mathcal{N}$, which gives (a), (b), (d) and (e), and a radius lemma for the Frobenius matrix gives (f). (c) is the analogue for one chart of the bounds on Hodge–Tate weights of Wach modules [Wac96], [Ber04, §III], in the form of upper bounds by the multiplicity of each jump. (g) is Tate's computation of the Galois cohomology of $\mathbb{C}_p(n)$ [Tat67b], by dévissage along $t^m B_{\mathrm{dR}}^+ / t^{m+1} B_{\mathrm{dR}}^+ \cong \mathbb{C}_p(m)$. □

Theorem B.41 (Detecting points and odd classes). *Let $E \subseteq \mathbb{Q}(\mu_N)$ be a number field, ℓ a prime and g_0 a generator of* $\mathrm{Gal}(\Omega/E)$ *modulo the kernel of the ℓ-adic cyclotomic character.*

(a) *There is a finite set s of elements of trivial cyclotomic character, with $\#s \le y^{20}$ whenever $N \le y$ and $\ell \le y^2$, such that for every coefficient ring O and all n, e, a locally constant cocycle in $(O/\ell^n)(-e)$ unramified off ℓ that vanishes at g_0 and on s is zero.*

(b) *Let E be CM and $e \ge 1$. There is an integer $0 < X \le 2^{(N+e+3)^6}$ such that $4X^2$ kills the odd classes of exponent e attached to each character of conductor dividing N, and, for $\ell > N + e + 3$, $v_\ell(X) + 1$ elements detect such a class.*

Sources. For large ℓ a bound of this kind follows from the main conjecture of Mazur and Wiles [Wil90]; the Lean development proves only an explicit upper bound at finite level, after Greither's method. (a) It is a cocycle analogue, with a crude bound, of the class field theory bound for the number of generators of the Galois group of the maximal pro-ℓ extension unramified outside ℓ. Put $M = N\ell^2$ and $K = \mathbb{Q}(\mu_M) \supseteq E$. The points are representatives of the cosets of $\mathrm{Gal}(\Omega/K)$ in $\mathrm{Gal}(\Omega/E)$ and points detecting the homomorphisms on $\mathrm{Gal}(\Omega/K)$ unramified off ℓ (generators of a ray class group of K, of number at most a power of M). If $Mz = 0$, then $\mathrm{Gal}(\Omega/K)$ acts as the identity on the values of z, so z is a homomorphism there and the second set detects it; that $Mz = 0$ follows by induction on the power of ℓ killing z, as $\ell \mid M$. Writing each point as an element of the kernel of the cyclotomic character times a limit of powers of g_0 and keeping the first factor gives s, the cocycle being also required to vanish at g_0. (b) Voronoi's congruence gives a bounded divisibility of twisted Stickelberger sums by X; Jacobi-type sums give units with prescribed valuations at a place above a split prime; Kolyvagin's derivative construction turns them into cocycles with prescribed values on inertia and Frobenius; and the reciprocity law, read on each component, shows that $4X^2$ kills the component, an analogue of Stickelberger's theorem, and that for large ℓ the stated number of elements detects a cocycle. For $e = 0$ the class number serves (Theorem B.39(c)). □

Appendix C. The Lean development

Every numbered statement of the paper that belongs to the argument carries an anchor: the list of declarations of the Lean development that state and prove it. The anchors are checked against the rebuilt environment (Appendix E): every anchored name is a declaration of the Lean development, and the anchored declarations, together with the declarations of the same module folders that they use, cover the cone of the main theorem. The table below gives, for each section, the page where it begins, the number of its numbered statements and the number of distinct anchored declarations, and beneath it the module folders in which these declarations lie, in the order of first use, grouped by the root of the module name. The full table, statement by statement, with every declaration and its module, is the file `supplement/lean-anchors.md` of the source repository; it is generated from the same anchors.

§	Section and Lean module folders	Page	Stat.	Decl.
1	Introduction `Main, Statements.`	3	2	9
4	A weighted prime sum and its Laplace transform *New:* `Reduction.Zeros`; *Lib:* `Analytic.EntireLFunction.`	15	9	21
5	Landau's lemma and the one-sided criterion *Lib:* `Analytic.Landau, Analytic.QuadraticTwist`; *New:* `Reduction.Zeros.`	18	8	8
6	The prime number theorem for the remainders *New:* `Reduction.Zeros, Reduction.OneSidedBound`; *Lib:* `Analytic.PrimeNumberTheorem.`	21	6	13
7	Heights and twisted lattices *New:* `Reduction.Lattices.`	23	14	40
8	The parameters and the data *New:* `Reduction.Data, Reduction.PrimeSums, Reduction.Zeros`; *Lib:* `General.NumberTheory, Analytic.Primes.`	28	14	74
9	Prime sums, the three theorems and the main inequality *New:* `Reduction.Data, Reduction.PrimeSums, Reduction.ShortAndEnough, Determinant.Twist, Reduction.MainInequality, Reduction.OneSidedBound, Main`; *Lib:* `Analytic.ZeroSymmetry.`	33	11	38
10	The field E at a prime *New:* `Derivatives.Field`; *Lib:* `ClassField.HeckeCharacter.`	40	16	124
11	The theta function of a CM curve at a non-split prime *Lib:* `General.NumberTheory, CM.FormalGroup, CM.Sigma, CM.Integrality, General.Algebra`; *New:* `Derivatives.FormalGroup, Reduction.Data, Derivatives.Divisibility.`	45	9	108
12	Couplings, the normal form and the count *New:* `Derivatives.Divisibility, Derivatives.FormalGroup`; *Lib:* `CM.Integrality, Weil.Construction, GLn.SchurModule.`	51	13	128
13	The integral vacuum and standard data *Lib:* `Weil.Construction, Adelic.Unitary, General.LinearAlgebra, Weil.MixedModel, Weil.Fock, CM.ChowlaSelberg, General.NumberTheory, Adelic.AutomorphicForms, CM.FormalGroup`; *New:* `Derivatives.Vacuum, Derivatives.Field, UpperBound.Size, Derivatives.Sharpness.`	58	21	229
14	The divisibility is attained *New:* `Derivatives.Sharpness`; *Lib:* `CM.Integrality.`	68	11	177
15	Spaces, weights and forms *New:* `Reduction.Data, UpperBound.Objects, UpperBound.Seesaw`; *Lib:* `DiscreteSeries.Theta, DiscreteSeries.FlenstedJensen, Weil.Lift, LocalGL.Green, General.Groups, Weil.Periods.`	78	12	334
16	Chains of lifts, the composite lift and the Poincaré map *Lib:* `Weil.CompactLift, General.LinearAlgebra, Adelic.Hecke, Shimura.HeckeTower, GLn.Branching, Weil.Periods`; *New:* `UpperBound.Objects, UpperBound.Size, UpperBound.Seesaw.`	88	11	269

§	Section and Lean module folders	Page	Stat.	Decl.
17	Theta kernels between definite spaces *Lib:* Weil.DualPair, Adelic.AutomorphicForms; *New:* UpperBound.Kernels, OneLift.ComparisonOperator, UpperBound.Seesaw.	96	6	102
18	The seesaw identity *Lib:* Weil.Periods, Adelic.AutomorphicForms; *New:* UpperBound.Seesaw, Reduction.Lattices, UpperBound.Divisibility.	102	15	459
19	Divisibility of the composite lift *New:* UpperBound.Size, UpperBound.Divisibility, UpperBound.Objects, Reduction.ShortAndEnough; *Lib:* Adelic.AutomorphicForms, GLn.SchurModule.	121	13	288
20	Size of the composite lift and the proof of Theorem A *New:* UpperBound.Size, Spanning.Unfolding, Reduction.IntegralTargetForms, Reduction.Data, Determinant.Chain, Reduction.ShortAndEnough, Spanning.SecondMoment.	135	7	112
21	Elementary divisors of edge Hecke operators *New:* UpperBound.ElementaryDivisors, Spanning.SecondMoment; *Lib:* General.Groups, General.Algebra, GLn.SchurModule.	141	14	406
22	The second moment and the spanning theorem *New:* Spanning.SecondMoment, Reduction.ShortAndEnough; *Lib:* Adelic.Unitary.	155	6	50
23	Equality of multiplicities *New:* Spanning.Multiplicities; *Lib:* Adelic.TraceFormula.	158	2	58
24	Bessel functions at a real place *Lib:* DiscreteSeries.Bessel, General.Groups, DiscreteSeries.Bruhat; *New:* Spanning.Bessel.	161	15	248
25	Filling data and the unfolding of the second moment *Lib:* General.Groups; *New:* Spanning.Unfolding, Spanning.RemainingTerms, Spanning.SecondMoment.	175	7	184
26	Terms, cells and the filling theorem *New:* Spanning.RemainingTerms, Spanning.Unfolding; *Lib:* General.Groups, General.Measure, General.LinearAlgebra.	184	9	158
27	The remaining terms *New:* Spanning.RemainingTerms; *Lib:* General.NumberTheory, General.LinearAlgebra, General.Measure.	192	26	633
28	The main term *New:* UpperBound.Objects, Spanning.MainTerm, Spanning.SecondMoment, Spanning.Unfolding, Spanning.Frames, Spanning.Bessel; *Lib:* LocalGL.Green, DiscreteSeries.Bruhat, General.Measure.	215	17	266
29	Local band limits *New:* Spanning.Frames.	230	23	747
30	The norm of one lift *New:* Determinant.Norms.	254	5	67
31	The exchange identity *New:* Determinant.Exchange.	258	28	268
32	The norm of the chain *New:* Determinant.Chain.	285	15	183

§	Section and Lean module folders	Page	Stat.	Decl.
33	Auxiliary forms	297	19	390
	New: `Determinant.AuxiliaryForms, Determinant.Twist.`			
34	The comparison operator	315	15	386
	New: `OneLift.ComparisonOperator, OneLift.LiftVolumeHeight.`			
35	False lifts and Hecke values	332	19	326
	New: `OneLift.TestedForms, OneLift.FalseLifts, OneLift.HeckeValues.`			
36	The cost of one lift and the proof of Theorem C	355	6	24
	New: `Determinant.Twist, Congruences.FitTwistsExist, OneLift.LiftVolumeHeight.`			
37	Galois representations of lifts	360	7	124
	New: `Congruences.GaloisRepresentations.`			
38	Ribet's method to first order	368	8	245
	New: `Congruences.FirstOrder;` *Lib:* `General.LinearAlgebra, General.Algebra.`			
39	Even classes and triangles of auxiliary places	377	5	105
	New: `Congruences.FirstOrder.`			
40	The bound for denominators	382	7	145
	New: `Congruences.FirstOrder.`			
41	The local loss above ℓ	391	6	86
	New: `Congruences.LocalLoss.`			
42	Cup products of crystalline classes	397	23	410
	New: `Congruences.Lifting.`			
43	Selmer data and the descent	422	13	177
	New: `Congruences.Selmer.`			
44	The twisting theorem	433	17	247
	New: `Congruences.Selmer;` *Lib:* `General.Algebra, General.NumberTheory.`			
45	Regular families of twists	445	4	19
	New: `Congruences.FitTwistsExist.`			
A	Discrete series with a large gap	450	10	263
	New: `Appendix.DiscreteSeries.`			
B	Background results in the form used	456	41	1533
	Lib: `Analytic.PrimeNumberTheorem, Analytic.Primes, Analytic.PolyaVinogradov, Analytic.DirichletTheta, Analytic.QuadraticTwist, Analytic.ZeroSymmetry, ClassField.Global, Weil.Construction, ClassField.HeckeCharacter,` `CM.Sigma, CM.ChowlaSelberg, CM.Integrality, CM.FiniteLattices, CM.FormalGroup, General.NumberTheory, CM.ConjugateCurve, GLn.SchurModule, GLn.Branching,` `DiscreteSeries.FlenstedJensen, Adelic.AutomorphicForms, DiscreteSeries.CuspKernel, DiscreteSeries.Theta, Weil.Lift, LocalGL.Green, Adelic.Hecke, Weil.Periods, DiscreteSeries.Bruhat,` `General.Groups, DiscreteSeries.Bessel, Weil.Unramified, General.Combinatorics, LocalGL.SphericalChainLemma, Adelic.TraceFormula, SiegelWeil.Formula, SiegelWeil.Mass, SiegelWeil.OneVector,`			

§	Section and Lean module folders	Page	Stat.	Decl.
	`SiegelWeil.Tower`, `SiegelWeil.RallisInner`, `ClassField.Local`, `GaloisRep.Main`, `GaloisRep.LocalGlobal`, `GaloisRep.Pseudocharacter`, `Shimura.IntegralModel`, `Shimura.Eigenspace`, `Shimura.Transfer`, `Shimura.CongruenceRelation`, `PadicComparison.CrystallineFrobenius`, `PadicComparison.HodgeTate`, `PadicComparison.Primitive`, `PadicHodge.Limit`, `ClassField.GreenbergWiles`, `ClassField.LocalDuality`, `PadicHodge.BlochKato`, `Cyclotomic.OddSha`, `PadicHodge.Kisin`, `PadicHodge.RadiusLemma`, `PadicHodge.TateSen`, `Cyclotomic.Kolyvagin`, `Cyclotomic.Stickelberger`, `Cyclotomic.Voronoi`.			

Appendix D. Constants and exponents

The argument fixes a number of explicit constants and exponents: the moduli of the reduction, the sizes of the data, the exponents of the three main theorems, and the losses of Laplace's method and of the congruences. The table lists each of them as it is printed in the text, with the file of the Lean development that fixes it. Each entry occurs, as printed, in the text of the paper and, as a Lean expression, in the named file. Constants that the argument only bounds, such as the constant C_0 of Remark 8.8, and the comparison of sizes of Remark 8.10, are not listed; their role is described where they occur.

Constant or exponent	Lean file
$18480 \mid q_0$	`New/Reduction/Data/FieldsAndPlaces.lean`
$d \equiv q_0/2 - 1 \pmod{q_0}$	`New/Reduction/OneSidedBound.lean`
$d_0 = 9240N - 1$	`New/Reduction/OneSidedBound.lean`
$q_0 = 18480N$	`New/Reduction/OneSidedBound.lean`
$d > 64$	`New/Reduction/Data/FieldsAndPlaces.lean`
non-split primes $p \geq 13$	`New/Reduction/Data/FieldsAndPlaces.lean`
$l_0 = 3^{\lfloor \log_3 s \rfloor + 4}$	`New/Reduction/Data/FieldsAndPlaces.lean`
$27s < l_0 \leq 81s$	`New/Reduction/Data/FieldsAndPlaces.lean`
$q_2 = (q_0 d)^{s+6}$	`New/Reduction/Data/FieldsAndPlaces.lean`
$(\log b)^{100} \leq x$	`New/Reduction/Data/AdmissibleData.lean`
$x^{10^6} \leq x_1$	`New/Reduction/Data/AdmissibleData.lean`
$x_1^{10^6} \leq b$	`New/Reduction/Data/AdmissibleData.lean`
$(\log x_1)^6 \leq \log b$	`New/Reduction/Data/AdmissibleData.lean`
$\log\log b \geq 10^{12} C_0$	`New/Reduction/Data/AdmissibleData.lean`
$m \in [\sqrt{x}, 2\sqrt{x}]$	`New/Reduction/Data/AdmissibleData.lean`
$r \in [x^5, 3x^5]$	`New/Reduction/Data/AdmissibleData.lean`
$2^s = m^{22}$	`New/Reduction/Data/AdmissibleData.lean`
$s \leq 32 \log x$	`New/Reduction/Data/AdmissibleData.lean`

Constant or exponent	Lean file
$t = 5r$	`New/Reduction/Data/AdmissibleData.lean`
$\Delta = 2r^8$	`New/Reduction/Data/AdmissibleData.lean`
$p_0, p_1 \in [b^{10}, b^{11}]$	`New/Reduction/Data/AdmissibleData.lean`
factor $b^{-15/2}$ in N_-	`New/Reduction/Data/AdmissibleData.lean`
$\log N \le \log N_- + \sqrt{\log b}$	`New/Reduction/Data/AdmissibleData.lean`
$m \log b \le \log N \le 30m \log b$	`New/Reduction/Data/LevelAndTwists.lean`
Chebyshev bound $6z$	`New/Reduction/Zeros/PrimeNumberTheorem.lean`
count error $13t$	`New/Reduction/Zeros/PrimeNumberTheorem.lean`
slow variation $26/y$	`New/Reduction/OneSidedBound.lean`
zero-free region l^{-25}	`Lib/Analytic/PrimeNumberTheorem/ZeroFreeRegion.lean`
bound l^{26} for G_q	`New/Reduction/Zeros/PrimeNumberTheorem.lean`
one-sided constant $64\varphi(q_0) + 3$	`New/Reduction/OneSidedBound.lean`
Theorem A: error $f a \log^4 b$	`New/Reduction/ShortAndEnough.lean`
Theorem B: count $5d_T \log(1 + 2b^2)$	`New/Reduction/ShortAndEnough.lean`
Theorem B: share $1 - 4/b^2$	`New/Reduction/ShortAndEnough.lean`
Theorem C: $3 f a \log^4 b$	`New/Determinant/Twist/LongForms.lean`
main inequality $f a \log^5 b$	`New/Reduction/MainInequality.lean`
index $10 \log(1 + 2b^2)$ per dimension	`New/Reduction/MainInequality.lean`
weights within x^{42}	`New/Reduction/Data/LevelAndTwists.lean`
gap $500(\#n)^3 + h$	`Lib/DiscreteSeries/FlenstedJensen/Defs.lean`
Laplace gap $10^{30}(N^2 \log \beta)^3$	`New/Appendix/DiscreteSeries/Periods.lean`
Laplace loss β^{20N}	`New/Appendix/DiscreteSeries/Periods.lean`
small ball $(200\beta^5)^{-1}$	`New/Appendix/DiscreteSeries/Periods.lean`
one minor $10^5\delta$	`New/Appendix/DiscreteSeries/OneMinor.lean`
twist error $(4f + 3)\mathrm{r}\, v_\ell(n_{\mathrm{ch}}) + 2^{-\ell}$	`New/Congruences/GaloisRepresentations/SelmerHypotheses.lean`
local defect $\mathrm{loc}_\ell = 4f\ \mathrm{defect}_\ell$	`New/Congruences/FirstOrder/TraceRelations.lean`
defect $10^6 \frac{m}{k}(b\, v_\ell(M) + \dots)$	`New/Congruences/LocalLoss/Basic.lean`
crystalline loss window $(b - 30h^3, b + 30h^3]$	`New/Congruences/LocalLoss/Basic.lean`
clearing bound $\beta_\ell = \mathrm{loc}_\ell + (1 + 2\mathrm{t})e'_\ell + e_\ell + v_\ell(N)$	`New/Congruences/FirstOrder/TraceRelations.lean`
odd class bound $2^{4(N_E+\mathrm{t}+3)^{27}} h_E$	`New/Congruences/FirstOrder/TraceRelations.lean`
budget $f a \log^3 b/(mk)$	`New/Reduction/Data/AdmissibleData.lean`

Constant or exponent	Lean file
local loss sum $4 \cdot 10^6 e^{180} f(m/k) b \log b$	`New/Congruences/FitTwistsExist.lean`

Appendix E. Record of the formal verification

This appendix records how the Lean development on which the paper rests was rebuilt and checked for the present text. The development is the readable edition `grh` of the archive [Ant26], distributed under the Apache License 2.0; the argument it formalises is due to Alpöge and Claude [Alp26]. Nothing in it was changed.

E.1. Versions and hashes

item	value
Lean	4.35.0, commit `eb6cafb7290e8bd5f2937522d8d804f58f02bafc`
Lake	5.0.0 (Lean 4.35.0)
Mathlib	commit `56a005855ea17e0b0958f73ec801ae140b0a1679`
Lean sources	803 files under `src`
`src/Statements.lean`	SHA-256 `79ef7ca7...f33b47bf`
`src/Main.lean`	SHA-256 `40da180a...1a67a17c6d`
all of `src`	SHA-256 of the sorted list of the SHA-256 sums, `5ef6bb24...c334d2eea`

The full hashes are in the files `ledger/statements.sha256` and `ledger/sources.sha256` of the source repository; the first two agree with the files of the rebuilt archive.

E.2. Statements

The Lean statements of the five theorems behind Theorems 1.1 and 1.2 are in the file `src/Statements.lean`. It imports only Mathlib and states them in terms of Mathlib's `riemannZeta` and `DirichletCharacter.LFunction`. The file `Challenge.lean` repeats the five statements with `sorry` in place of the proofs and imports only `Statements`; it is the input of a comparison tool that checks that the theorems proved in `Main` have exactly these statements and use only the permitted axioms `propext`, `Quot.sound` and `Classical.choice`.

E.3. Build and axioms

On 26 November 2026 the archive was built from these sources with `lake exe cache get` (the Mathlib cache of the pinned commit) and `lake build`, with five concurrent Lean processes. The build completed all 9852 jobs without error in two hours and thirty-seven minutes. The command `lake env lean scripts/PrintAxioms.lean` then printed, for each of the five theorems,

```
depends on axioms: [propext, Classical.choice, Quot.sound].
```

In particular none of the five depends on `sorryAx` or on an axiom introduced by the archive.

E.4. Kernel replay

The kernel of Lean checks every declaration when it is added; a replay checks the stored environment again, independently of the elaborator and of the build. The command

```
lake env leanchecker -fresh Main
```

replays every constant of the environment of `Main`, including Lean's core library, Mathlib and all modules of the archive, into a fresh environment through the kernel. It was run on the rebuilt archive on 26 and 27 November 2026, as a single process; it finished after fifty-three minutes with exit status 0 and reported no error, so every constant of the environment, among them the five theorems and everything they depend on, was accepted by the kernel a second time.

E.5. The anchors of the paper

From the built environment every declaration of the modules of the archive was exported with its module, its type and the declarations it uses. Every name in an anchor (Appendix C) is a declaration of this export. The *cone* of the five theorems, the declarations on which they depend, has 49822 members, of which 9621 lie in the modules under `New`; the anchors of the paper, together with the declarations of the same module folders that they use, cover 98.7 per cent of these. The remaining declarations are auxiliary lemmas of the module folders, used only inside proofs.

Declarations

Attribution. The proof expounded here is due to Levent Alpöge, working with Claude [Alp26]. The Lean development `grh-lean` [Ant26], on which the exposition is based, was produced at Anthropic in collaboration with Claude; it is Copyright 2026 Anthropic, PBC, and is distributed under the Apache License 2.0. Names of its declarations and short extracts of its statements are quoted under that licence. This paper claims no part of the proof.

This paper. A research paper of the GAI-MUA programme of WiseMotif.ai, it contributes the exposition: the reading of the formal proof; the selection and organisation of the argument, with the chain of the proof set out in Section 2.1; the proofs written in mathematical language; the correspondence with the Lean declarations; and the independent rebuild of the development recorded in Appendix E.

Use of AI. WiseMotif.ai is an AI-native company, and its research and development are carried out with AI assistance throughout. The author designed the overall framework of the paper; AI tools assisted with the drafting and with the collation of the literature. The author has reviewed the entire text and is responsible for the whole of it.

Status. The formal proof has been checked by the Lean kernel, again in the rebuild recorded in Appendix E, and, according to the authors of `grh-lean`, by independent proof checkers. This exposition has not been refereed; where it and the Lean development differ, the development is authoritative.

Licence and data. This paper is distributed under the Creative Commons Attribution 4.0 International licence; the quoted Lean material remains under

the Apache License 2.0. The LaTeX source of the paper, the table of Lean anchors and the record of the rebuild are deposited with the paper at Zenodo, doi:10.5281/zenodo.24593828, and at https://github.com/AlienMuse/the-rhythm-of-primes.

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