LOCAL VOLUME MINIMALITY OF MEISSNER'S BODIES

Abstract. Let $M \subset \mathbb{R}^3$ be either of the two Meissner bodies of constant width one. We prove that there exist $c, \varepsilon > 0$ such that

$$\operatorname{vol}(K) - \operatorname{vol}(M) \geq c\,\delta(K, M)^{3/2}$$

for every body K of constant width one with $\delta(K, M) < \varepsilon$, where δ is the Hausdorff distance modulo rigid motions. Thus each Meissner body is a strict local minimizer of volume in the full class of bodies of constant width. The exponent $3/2$ is sharp. The proof uses a normalization by six support values, one-dimensional inequalities on the sphere, and an estimate of the gradient term in the quadratic formula for volume. Sharpness is obtained by changing the retained arc in one short pair of dual edges.

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1. Introduction and main results

A convex body in $\mathbb{R}^3$ is a compact convex set with nonempty interior. It has constant width w if its orthogonal projection onto every line has length w. The Blaschke–Lebesgue problem asks for the smallest volume of a body of a prescribed constant width and for a characterization of the minimizers. In the plane, the Reuleaux triangle is the unique minimizer, up to congruence; see Blaschke [5] and Lebesgue [13]. For background on bodies of constant width, we refer to [14].

In three dimensions, the conjectured minimizers are the two bodies constructed by Meissner and Schilling [15]. Start with a regular tetrahedron of edge length one and intersect the four unit balls centered at its vertices. The resulting Reuleaux tetrahedron has three pairs of opposite circular edges. Retain one edge from each pair and intersect further with the unit balls centered at all points of the three retained edges. This gives a body of constant width one. Up to congruence, the retained edges either meet at one vertex or form a triangle; see Figure 1. We call the resulting bodies *Meissner's bodies*, or *Meissner tetrahedra*. Both have volume

$$V_M = \pi \left(\frac{2}{3} - \frac{\sqrt{3}}{4} \arccos \frac{1}{3} \right) = 0.419860045965\ldots. \tag{1}$$

Their global minimality was conjectured by Bonnesen and Fenchel [8]. For the history of the construction, see Kawohl and Weber [12]; a detailed treatment of the construction and (1) is given by Hynd [10].

Several necessary conditions and restricted minimization results are known. Bayen, Lachand-Robert and Oudet [4] gave an analytic parametrization using the median surface and obtained necessary optimality conditions. Anciaux and Guilfoyle [1] proved that the smooth parts of the boundary of a local volume minimizer have their smaller principal curvature equal to the reciprocal of the width. Meissner's bodies satisfy this condition, as does the tetrahedrally symmetric body U_3

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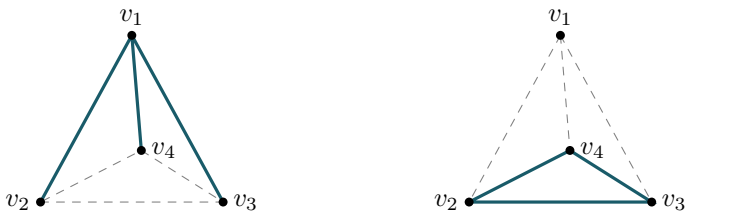


FIGURE 1. The two choices of retained edges, represented by their chords in the generating tetrahedron. The corresponding circular edges are retained in the Meissner body; the other three edges are replaced by spindle surfaces.

constructed by Arman, Bondarenko, Prymak and Radchenko [3]. We use the quadratic support-function formula for volume from [1].

There is substantial numerical evidence for the Meissner conjecture. Oudet [18] studied optimization under a width constraint. Antunes and Bogosel [2] used spherical harmonic approximations of support functions and obtained both Meissner bodies from numerical minimization. These computations motivate a comparison which is uniform over all sufficiently small perturbations.

Meissner polyhedra give another approach to the problem. Montejano and Roldán-Pensado [16] introduced a construction generalizing the Meissner tetrahedra. Hynd [10] proved their density in the class of bodies of constant width. Bogosel [6] expressed their volume in terms of the lengths of dual edges and proved minimality of Meissner tetrahedra among Meissner pyramids. More recently, Bogosel [7] showed that deleting a dangling point (one having exactly two diametric neighbors) from an extremal generating set cannot increase the least surface area among its Meissner completions. For fixed width, surface area and volume are related by Blaschke's formula, so the same comparison holds for volume.

A related result is the extremality of Meissner's bodies under Minkowski addition, proved by Moreno and Schneider [17]. Hynd [11] proved that every Meissner polyhedron is extreme. Here extremality means that the body cannot be written as a nontrivial Minkowski convex combination of other bodies of the same constant width, apart from translations. The local volume comparison below concerns an entire Hausdorff neighborhood in the full class.

For convex bodies K, L, let $d_H(K, L)$ denote the Hausdorff distance, and put

$$\delta(K, L) = \inf_{Q \in O(3),\, a \in \mathbb{R}^3} d_H(K, QL + a). \tag{2}$$

Here and throughout, all bodies have constant width one, unless another width is specified.

Theorem 1.1. *Let M be either Meissner body. There exist $c, \varepsilon > 0$ such that every body $K \subset \mathbb{R}^3$ of constant width one with $\delta(K, M) < \varepsilon$ satisfies*

$$\mathrm{vol}(K) - V_M \geq c\, \delta(K, M)^{3/2}. \tag{3}$$

In particular, M is a strict local volume minimizer modulo rigid motions.

The competing body may have an arbitrary boundary and need not contain any of the four vertices of the generating tetrahedron. The constants in Theorem 1.1 are not evaluated here. For a body K of constant width $w > 0$, scaling gives $\mathrm{vol}(K) - w^3 V_M \geq c w^{3/2} \delta(K, wM)^{3/2}$ when $\delta(K, wM) < \varepsilon w$.

Theorem 1.2. *For either Meissner body M, there is a family of bodies K_t of constant width one, $0 < t < t_0$, such that*

$$\delta(K_t, M) \asymp t^2, \qquad \mathrm{vol}(K_t) - V_M \asymp t^3. \tag{4}$$

Consequently, the exponent $3/2$ in (3) cannot be replaced by any smaller exponent.

The notation $a \asymp b$ means that $cb \le a \le Cb$ for positive constants independent of t. More generally, c, C will denote positive constants depending only on the fixed Meissner body and its fixed coordinate regions. Their values may change at different occurrences.

Outline of the proof. Write h_K and h_M for the support functions restricted to the unit sphere. A small rigid motion makes their values agree in the twelve directed edge directions of the generating tetrahedron. Only six conditions are independent. In this placement, $g = h_M - h_K$ is nonnegative on four fixed spherical triangles and nonpositive on their antipodes.

The remaining six spherical squares occur in three antipodal pairs. On one square from each pair, the support function of M has a particularly simple description: in a middle band it is supported on a retained circular edge, and in the two outer bands it is supported on unit spherical faces. Along meridians, the inequalities for g therefore change from $g'' + g \le 0$ to $g'' + g \ge 0$.

The volume difference is an exact quadratic expression. We write its linear part as $L(g)$, put $N = \frac{1}{2}\|g\|_{L^1(\mathbb{S}^2)}$, and define a nonnegative quantity $J(g)$ measuring the deviation from sine interpolation across the middle bands. The two principal estimates are

$$L(g) \ge J(g), \qquad L(g) \ge cN, \qquad \int_{\mathcal{H}} |\nabla g|^2 \, d\sigma \le J(g) + C\sqrt{\|g\|_\infty}\, N.$$

Here $\mathcal{H}$ contains one region from each antipodal pair. The last estimate absorbs the negative gradient term, whose coefficient in the volume formula is $1/2$. This yields a positive lower bound by N when g is small.

Concavity along the meridians and the bound $\|\nabla g\|_\infty \le C\sqrt{\|g\|_\infty}$ then give $N \ge c\|g\|_\infty^{3/2}$. Finally, we switch the choice of retained edge in one short dual pair. The resulting support change has order t^2, while the volume excess has order t^3. The normalization argument also shows that allowing rigid motions does not change the order of this distance.

2. Support functions and normalization

Let

$$H_K(x) = \max_{y \in K} x \cdot y, \qquad h_K = H_K|_{\mathbb{S}^2}.$$

We use ∇, ∇^2 and Δ for the gradient, covariant Hessian and Laplacian on $\mathbb{S}^2$, and σ for its area measure. The norm of a symmetric matrix is its operator norm. The constant width condition is

$$h_K(u) + h_K(-u) = 1, \qquad u \in \mathbb{S}^2, \tag{5}$$

and

$$d_H(K, L) = \|h_K - h_L\|_\infty. \tag{6}$$

We recall the support-function facts needed in the proof.

Lemma 2.1. *For a body of constant width one, $h = h_K$ is of class $C^{1,1}$ and its radius matrix*

$$R_h = \nabla^2 h + hI$$

satisfies $0 \le R_h \le I$ almost everywhere. After identifying antipodal tangent spaces, $R_h(u) + R_h(-u) = I$. Moreover,

$$\mathrm{vol}(K) = \frac{\pi}{6} + \frac{1}{2}\int_{\mathbb{S}^2} \left((h - \tfrac{1}{2})^2 - \frac{1}{2}|\nabla h|^2 \right) d\sigma. \tag{7}$$

If g is the difference of two such support functions and $\eta = \|g\|_\infty \le 1$, then

$$\|\nabla^2 g\|_\infty \le 1 + \eta \le 2, \qquad \|\nabla g\|_\infty \le \sqrt{8\eta}. \tag{8}$$

Proof. The regularity is classical; see Howard [9]. One way to see it directly is to use the homogeneous identity $H_K(x) + H_K(-x) = |x|$. Both terms on the left are convex. Their distributional Hessians are positive semidefinite matrix measures whose sum is the smooth Hessian of $|x|$ away from the origin. Hence H_K has locally bounded second derivatives there. Restriction to $\mathbb{S}^2$ gives $h \in C^{1,1}$ and $0 \le R_h \le I$. Differentiating (5) gives the antipodal identity.

Formula (7) is the quadratic volume formula of Anciaux and Guilfoyle [1, Theorem 2], specialized to width one. It applies to $C^{1,1}$ support functions. Alternatively, one can first use the smooth formula and then average over small rotations. The averages preserve constant width and converge in C^1, so both sides of (7) converge.

The radius matrices of the two bodies lie between 0 and I; their difference therefore has operator norm at most one. This proves the Hessian bound. Along a unit speed great circle, Taylor's inequality gives $|g(s) - g(0) - sg'(0)| \le s^2$. Taking $s = g'(0)/2$ and using $|g(s) - g(0)| \le 2\eta$ yields $|g'(0)|^2 \le 8\eta$. □

Fix the generating regular tetrahedron with vertices $v_1, \dots, v_4$, centered at the origin, and write

$$u_{ij} = v_i - v_j \quad (i \ne j).$$

These twelve vectors are unit vectors. Since M contains the tetrahedron and has diameter one, its support point in direction u_{ij} is v_i. In particular,

$$h_M(u_{ij}) = v_i \cdot u_{ij} = \frac{1}{2}. \tag{9}$$

Lemma 2.2. *There exist $C, \varepsilon_0 > 0$ with the following property. If K has constant width one and $d_H(K, M) < \varepsilon_0$, then a rigid motion sends K to a body $\widetilde{K}$ such that*

$$h_{\widetilde{K}}(u_{ij}) = \frac{1}{2} \quad (i \ne j), \qquad d_H(\widetilde{K}, M) \le C d_H(K, M). \tag{10}$$

The required rotation and translation tend to the identity as $d_H(K, M) \to 0$.

There is also a neighborhood of the identity in the group of rigid motions on which

$$\|Q - I\| + |a| \le C \max_{i<j} |h_{QM+a}(u_{ij}) - \tfrac{1}{2}|. \tag{11}$$

Idea of the proof. The six independent support values provide coordinates for rigid motions near M. An infinitesimal motion preserving these values would have zero velocity at all four vertices, and is therefore zero. The uniform Hessian bound turns Hausdorff closeness into C^1 closeness of support functions, which permits the same normalization for every nearby body.

Proof. For $i < j$, consider

$$\Phi_K(Q, t)_{ij} = h_K(Qu_{ij}) - t \cdot Qu_{ij} - \frac{1}{2}, \qquad (Q, t) \in SO(3) \times \mathbb{R}^3.$$

This is the support-value map for $Q^T(K - t)$. At $K = M$ and $(Q, t) = (I, 0)$, its derivative in a direction (Ω, a), with Ω skew-symmetric, is

$$D\Phi_M(I, 0)(\Omega, a)_{ij} = -(a + \Omega v_i) \cdot u_{ij}. \tag{12}$$

Indeed, the derivative of the homogeneous support function at u_{ij} is v_i.

Suppose (12) vanishes for $i < j$. The same equations then hold for the reversed pairs, since $\Omega(v_i - v_j) \cdot (v_i - v_j) = 0$. For each i, the vector $a + \Omega v_i$ is orthogonal to the three independent vectors u_{ij}, $j \ne i$. Thus $a + \Omega v_i = 0$ for every i. Subtracting these equations gives $\Omega = 0$, and then $a = 0$. The derivative is invertible.

Put $\zeta = d_H(K, M)$. Lemma 2.1 implies $h_K \to h_M$ in C^1 as $\zeta \to 0$. In fixed local coordinates for the group of rigid motions, the maps Φ_K are therefore C^1 close to Φ_M. Let $A = D\Phi_M(I, 0)$ in these coordinates. On a sufficiently small fixed coordinate ball, the map $z \mapsto z - A^{-1}\Phi_K(z)$ has Lipschitz constant at most $1/2$ when ζ is small. Its value at zero has norm at most $C\zeta$. It maps

a ball of radius $C'\zeta$ into itself, for a fixed sufficiently large C'. The contraction theorem gives a zero with coordinate norm $O(\zeta)$. This proves (10) for $i<j$; the other values follow from constant width. The estimate for the Hausdorff distance follows from the same bound on the motion.

The map $(Q,a)\mapsto (h_{QM+a}(u_{ij}))_{i<j}$ has the same invertibility property. Its local inverse is Lipschitz, which gives (11). □

For the rest of the local comparison, K is in the normalized placement (10), and

$$g=h_M-h_K,\qquad \eta=\|g\|_\infty. \tag{13}$$

In particular, g is odd and $g(u_{ij})=0$.

3. The spherical regions

3.1. **Triangles and squares.** For each i, let U_i be the spherical triangle generated by u_{ij}, $j\neq i$. On U_i the support point of M is v_i; thus

$$h_M(u)=v_i\cdot u\quad (u\in U_i),\qquad h_M(u)=1+v_i\cdot u\quad (u\in -U_i). \tag{14}$$

Subadditivity of H_K and the normalized values at the three generators imply

$$g\geq 0 \text{ on } U_i,\qquad g\leq 0 \text{ on } -U_i. \tag{15}$$

Indeed, $H_K(\sum_{j\neq i}\lambda_j u_{ij})\leq \sum_{j\neq i}\lambda_j h_K(u_{ij})$ for $\lambda_j\geq 0$, and the expression on the right is $v_i\cdot\sum_{j\neq i}\lambda_j u_{ij}$.

The twelve u_{ij} are the vertices of a cuboctahedron. Radial projection of its faces partitions $\mathbb{S}^2$ into the eight triangles $\pm U_i$ and six spherical squares. Opposite squares correspond to opposite edges of the tetrahedron. From each pair of squares, choose the one associated with the edge retained in M, and denote the collection of three chosen squares by $\mathcal{D}$. Up to sets of area zero,

$$\mathcal{H}=\bigcup_{i=1}^{4}U_i\ \cup\ \bigcup_{D\in\mathcal{D}}D$$

contains exactly one of $u,-u$. We put

$$N=\int_{\mathcal{H}}|g|\,d\sigma=\frac{1}{2}\|g\|_{L^1(\mathbb{S}^2)}. \tag{16}$$

3.2. **Coordinates on a chosen square.** We use separate Euclidean coordinates for each $D\in\mathcal{D}$. Write the four vertices as

$$p_\pm=(0,0,\pm\tfrac12),\qquad q_\pm=(b,\pm\tfrac12,0),\qquad b=\frac{1}{\sqrt2}, \tag{17}$$

so that the retained arc joins q_- to q_+. Changing the origin adds the same linear function to h_M and h_K and leaves g and all radius matrices unchanged. Put

$$\alpha=\arctan b,\qquad \beta=\frac{\pi}{6},\qquad r=\cos\beta=\frac{\sqrt3}{2}. \tag{18}$$

The retained arc is

$$c(s)=r(\cos s,\sin s,0),\qquad -\alpha\leq s\leq\alpha. \tag{19}$$

In longitude and latitude coordinates

$$u(\theta,\phi)=(\cos\phi\cos\theta,\cos\phi\sin\theta,\sin\phi),$$

the square is

$$D=\{|\theta|\leq\alpha,\ |\phi|\leq A(\theta)\},\qquad A(\theta)=\arctan(b\cos\theta). \tag{20}$$

This follows from $|u_y/u_x|\leq b$ and $|u_z/u_x|\leq b$. Notice that $\beta\leq A(\theta)\leq\alpha$. Let

$$D_0=\{|\theta|\leq\alpha,\ |\phi|\leq\beta\},\qquad D_1=D\setminus D_0.$$

These regions are shown in Figure 2.

Lemma 3.1. *On D, in the coordinates* (17)*,*

$$h_M(u(\theta,\phi)) = \begin{cases} r\cos\phi, & |\phi| \le \beta, \\ 1 - \frac{1}{2}|\sin\phi|, & \beta \le |\phi| \le A(\theta). \end{cases} \tag{21}$$

The principal radii on D_0 are 0 *in the meridional direction and $r/\cos\phi$ in the transverse direction. On D_1, both principal radii are one. For a fixed θ, the continuation $A(\theta) \le \phi \le \pi/2$ lies in $-U_{p_-}$, and the continuation $-\pi/2 \le \phi \le -A(\theta)$ lies in $-U_{p_+}$.*

Proof. The body M is contained in the lens $B(p_+,1) \cap B(p_-,1)$ and contains the arc (19). For $|\phi| \le \beta$, the lens is supported at $c(\theta)$, which gives the first formula.

Consider the upper band and put $s(u) = 1 + p_- \cdot u$. At $\phi = \beta$, $s = h_M$ and their meridional derivatives agree. At $\phi = A(\theta)$ the normal belongs to the triangle generated by $q_\pm - p_-$ and $p_+ - p_-$, namely $-U_{p_-}$. Hence $s = h_M$ there by (14). The function $f = s - h_M$ satisfies $f_{\phi\phi} + f \ge 0$, because the meridional radius of h_M is at most one. Since $f(\beta) = f_\phi(\beta) = 0$,

$$f(A) = \int_\beta^A \sin(A-t)(f_{\phi\phi} + f)(t)\, dt.$$

The left side is zero and the integrand is nonnegative. Thus $f = 0$ throughout the band. The lower band is treated in the same way. The descriptions of the continued meridians follow from the three generating rays of the corresponding triangles. Finally, differentiating (21) gives the radius matrices. □

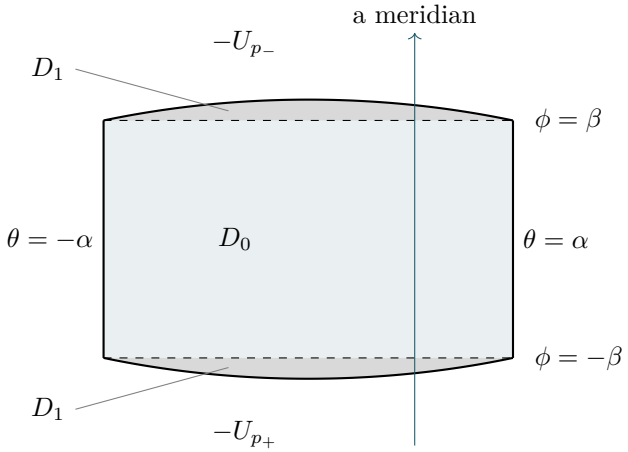


FIGURE 2. A chosen spherical square in (θ,ϕ) coordinates. The middle band has one zero principal radius. The two outer bands have both principal radii equal to one. Each meridian continues into the adjacent spherical face triangles.

3.3. **The volume difference.** Put

$$w = 1 - \operatorname{tr} R_{h_M}.$$

This function is odd almost everywhere. By Lemma 3.1, on a chosen square it has the values

$$w = \begin{cases} 1 - r/\cos\phi, & D_0, \\ -1, & D_1. \end{cases} \tag{22}$$

On U_i it equals one. Expanding (7) and integrating by parts on the whole sphere gives

$$\operatorname{vol}(K) - V_M = L(g) + \int_{\mathcal{H}} \left(g^2 - \frac{1}{2}|\nabla g|^2 \right) d\sigma, \tag{23}$$

where

$$L(g) = \sum_{i=1}^{4} \int_{U_i} g \, d\sigma + \sum_{D \in \mathcal{D}} \int_D w g \, d\sigma. \tag{24}$$

Indeed, the linear term on the whole sphere is $\frac{1}{2}\int_{\mathbb{S}^2}(1 - \operatorname{tr} R_{h_M}) g \, d\sigma$. The integrand is even, as are the quadratic terms. The integration by parts is valid for $C^{1,1}$ functions.

All integrals in the coordinates of (20) use $d\sigma = \cos\phi \, d\phi \, d\theta$. The spherical metric is $d\phi^2 + \cos^2\phi \, d\theta^2$.

4. A lower bound for the linear term

On each middle band D_0, let $B(\theta, \phi)$ solve

$$B_{\phi\phi} + B = 0, \qquad B(\theta, \pm\beta) = g(\theta, \pm\beta),$$

and put $P = g - B$ there. The meridional radius of h_K lies between zero and one. Lemma 3.1 therefore gives, on every continued meridian,

$$g_{\phi\phi} + g \le 0 \quad (|\phi| < \beta), \qquad g_{\phi\phi} + g \ge 0 \quad (\beta < |\phi| < \pi/2). \tag{25}$$

These inequalities can be interpreted in distributions on each meridian. They also follow by restricting the homogeneous convexity inequalities to its plane.

The function $P/\cos\phi$ is concave as a function of $\tan\phi$, and it vanishes at the two endpoints. In particular, $P \ge 0$. Define

$$J(g) = \sum_{D \in \mathcal{D}} \int_{D_0} \left(1 - \frac{r}{\cos\phi}\right) P \, d\sigma. \tag{26}$$

The weight is nonnegative, so $J(g) \ge 0$.

Lemma 4.1. *There is $c_0 > 0$ such that every normalized competitor satisfies*

$$L(g) \ge J(g), \qquad L(g) \ge c_0 N. \tag{27}$$

No smallness assumption on η is needed in this lemma.

Idea of the proof. We integrate the meridional inequalities against a function q which is negative on the middle band and positive just outside it. On the square, $q'' + q$ is exactly the density of the linear term. The positive parts of q are continued into the adjacent face triangles, using only one quarter of their available contribution. Each triangle is used at most three times. A small perturbation of the forcing then gives a uniform lower bound by $\int |g|$.

Proof. Fix θ and write $A = A(\theta) \in [\beta, \alpha]$. Primes in this proof denote differentiation in ϕ. On $[-\beta, \beta]$, let q be the even solution of

$$q'' + q = \cos\phi - r, \qquad q(-\beta) = q(\beta) = 0.$$

The Dirichlet Green function of $d^2/d\phi^2 + 1$ is negative on an interval of length less than π. Since the forcing is positive in the interior, $q < 0$ there. Explicitly,

$$q(\phi) = \frac{1}{2}\phi \sin\phi - r + \left(1 - \frac{1}{2}\beta\tan\beta\right)\cos\phi, \qquad k := q'(\beta) = \frac{\beta}{\sqrt{3}} - \frac{1}{4} > 0. \tag{28}$$

Continue q and q' to the right using $q'' + q = -\cos\phi$ on $[\beta, A]$ and $q'' + q = -\cos\phi/4$ on $[A, \pi/2]$. For $0 < s \le A - \beta$, the sine-kernel formula gives

$$\begin{aligned} q(\beta + s) &= k \sin s - \int_0^s \sin(s - v)\cos(\beta + v) \, dv \\ &\ge k \sin s - r(1 - \cos s) > 0. \end{aligned} \tag{29}$$

For the last inequality, use

$$k > \frac{1}{20}, \qquad r \tan \frac{\alpha - \beta}{2} < \frac{3 - \sqrt{6}}{4(\sqrt{6}+1)} < \frac{1}{25}.$$

The final inequality is equivalent to $71 < 29\sqrt{6}$. At the other endpoint, the forcing is everywhere at most $-\cos\phi/4$, and hence

$$\begin{aligned} q(\pi/2) &\le kr - \frac{1}{4}\int_{\beta}^{\pi/2} \cos^2 v \, dv \\ &= \frac{\pi}{24} - \frac{3\sqrt{3}}{32} < 0. \end{aligned} \tag{30}$$

There is therefore a first zero $\gamma \in (A, \pi/2)$ after β. Reflect the construction to the left and stop at $-\gamma$. On the two exterior intervals $q \ge 0$, while $g'' + g \ge 0$ by (25). At the stopping points $g \le 0$ by Lemma 3.1 and (15). Also $q'(\gamma) \le 0$ and $q'(-\gamma) \ge 0$. Integration by parts gives

$$\begin{aligned} \int_{-\gamma}^{\gamma} (q'' + q) g \, d\phi &= \int_{-\gamma}^{\gamma} q(g'' + g) \, d\phi + [q'g - qg']_{-\gamma}^{\gamma} \\ &\ge \int_{-\beta}^{\beta} q(g'' + g) \, d\phi = \int_{-\beta}^{\beta} (\cos\phi - r) P \, d\phi. \end{aligned} \tag{31}$$

For the last equality, replace g by P and integrate by parts again, using the zero endpoint values of P and q.

The part of the left side over $[-A, A]$ is $\int_{-A}^{A} wg \cos\phi \, d\phi$. The remaining parts are one quarter of the integral of $-g$ over subsets of the two adjacent face triangles. Integrate in θ and sum over the three chosen squares. Each such parametrization is one-to-one on its triangle portion, apart from a pole of area zero. A fixed triangle can be used at most three times, since there are three chosen squares. After antipodal identification, the total amount used is at most $\frac{3}{4}\sum_i \int_{U_i} g \, d\sigma$. Consequently,

$$L(g) \ge J(g) + \frac{1}{4}\sum_{i=1}^{4} \int_{U_i} g \, d\sigma. \tag{32}$$

This proves the first inequality.

For the second inequality, choose a small number $\epsilon > 0$ and replace the forcing on the whole square by

$$q_\epsilon'' + q_\epsilon = q'' + q - \epsilon \, \operatorname{sgn}(g) \cos\phi. \tag{33}$$

The forcing in the continued triangle portions remains $-\cos\phi/4$. First solve the perturbed Dirichlet problem on $[-\beta, \beta]$, and then continue separately from each endpoint. The Green kernel and its first derivative are bounded on this fixed interval. The initial-value sine kernel is bounded on the subsequent intervals. It follows that the change in q is at most $C\epsilon$ in C^1, uniformly in A and in the measurable function $\operatorname{sgn}(g)$.

We record why the required signs persist uniformly. On the middle interval, q is strictly negative away from the endpoints, and its endpoint derivatives are $-k, k$. The zero endpoint values and the C^1 bound preserve this sign for sufficiently small ϵ. On the first exterior interval, (29) is at least

$$\left(k - r \tan \frac{\alpha - \beta}{2}\right) \sin s.$$

The change there is $O(\epsilon s)$, because the value at $s = 0$ is still zero. Thus positivity persists up to A. Finally, the strictly negative uniform bound in (30) persists at $\pi/2$. On each side we can stop at the first exterior zero, which lies beyond the square.

Apply (31) with q_ϵ and these stopping points. Its middle integral is nonnegative by (25) and $q_\epsilon < 0$. The change of the forcing on the square contributes $-\epsilon \int_D |g|\, d\sigma$. The same triangle counting therefore gives

$$L(g) \geq \epsilon \sum_{D \in \mathcal{D}} \int_D |g|\, d\sigma + \frac{1}{4} \sum_{i=1}^{4} \int_{U_i} g\, d\sigma.$$

Fixing such an ϵ proves the second inequality in (27), with $c_0 = \min(\epsilon, 1/4)$. □

5. Control of the gradient term

The negative term in (23) has to be estimated uniformly over all nearby bodies. The following lemma supplies the needed coefficient.

Lemma 5.1. *If K is normalized and $\eta = \|g\|_\infty \leq 1$, then*

$$\int_{\mathcal{H}} |\nabla g|^2\, d\sigma \leq J(g) + C\sqrt{\eta}\, N. \tag{34}$$

Idea of the proof. On a vertex triangle, a suitable planar version of g is nonnegative and concave. Its maximum and total variation are therefore controlled by its integral. Concavity along meridians gives a similar estimate for g_ϕ on the squares. The transverse derivative requires a more precise argument. On the outer bands, $g_{\theta\theta}$ is bounded below by $-C\sqrt{\eta}$. On the middle band the additional negative term is exactly $-w \cos^2 \phi$. Subtracting the sine interpolant B and integrating by parts makes this term appear as $J(g)$, with coefficient one.

5.1. **The triangular regions.** Put

$$H = \sqrt{8\eta}, \qquad m = \max_{\bigcup_i U_i} g. \tag{35}$$

By Lemma 2.1, $\|\nabla g\|_\infty \leq H$. In particular, all first coordinate derivatives on the fixed squares are bounded by CH.

Fix one of the triangles and let n be its central unit normal. In gnomonic coordinates write

$$u = \frac{n + x}{\sqrt{1 + |x|^2}}, \quad x \perp n, \qquad f(x) = \sqrt{1 + |x|^2}\, g\left(\frac{n + x}{\sqrt{1 + |x|^2}}\right).$$

The spherical triangle is represented by a fixed bounded planar triangle T. By (14), $f(x) = v_i \cdot (n + x) - H_K(n + x)$ is concave and nonnegative. If x_* maximizes f, then $f \geq \frac{1}{2} \max_T f$ on $(T + x_*)/2$. The coordinate factors and the area Jacobian are bounded above and below on T. It follows that

$$m \leq C \sum_i \int_{U_i} g\, d\sigma \leq CN. \tag{36}$$

On every line parallel to either coordinate axis, the total variation of a nonnegative concave function is at most twice its maximum. Integrating along these lines gives $\int_T |\nabla f| \leq C \max_T f$. Returning to the sphere and summing over the four triangles yields

$$\sum_i \int_{U_i} |\nabla g|\, d\sigma \leq Cm, \qquad \sum_i \int_{U_i} |\nabla g|^2\, d\sigma \leq CHm. \tag{37}$$

5.2. **The meridional derivative.** Fix a meridian of a chosen square and put

$$t = \tan\phi, \qquad F(t) = \frac{g(\theta,\phi)}{\cos\phi}, \qquad \tau = \tan\beta. \tag{38}$$

A direct differentiation gives

$$g_{\phi\phi} + g = \frac{F''(t)}{\cos^3\phi}. \tag{39}$$

Thus F is concave on $[-\tau,\tau]$ and convex on each of the exterior half-lines. The values of g at the poles are zero, because these poles are directed edge normals $p_+ - p_-$ and $p_- - p_+$. The Lipschitz property of g therefore makes F bounded as $t \to \pm\infty$. A bounded convex function on a right half-line is nonincreasing; on a left half-line it is nondecreasing. Consequently, the maximum of F on the square slice is attained in $[-\tau,\tau]$.

At $\phi = \pm A(\theta)$, the triangle data give $|g| \le m$. Monotonicity on the exterior intervals, followed by concavity on the middle interval, gives the uniform bound

$$g \ge -Cm \quad \text{on every } D \in \mathcal{D}. \tag{40}$$

Choose a fixed C_1 large enough that $F + C_1 m \ge 0$ on the square slice. On $[-\tau,\tau]$, this is a nonnegative concave function. A comparison with the two line segments joining a point of maximum to the endpoints gives

$$\max_{[-\tau,\tau]}(F + C_1 m) \le C \int_{-\tau}^{\tau} (F + C_1 m)\, dt.$$

On each of the three pieces of the square slice, F is monotone or concave, and its total variation is at most twice its oscillation. Since $\cos\phi$ is bounded away from zero on D, this proves

$$\int_{-A(\theta)}^{A(\theta)} |g_\phi|\, d\phi \le C\left(m + \int_{-\beta}^{\beta} |g|\, d\phi\right). \tag{41}$$

Multiplying by the uniform derivative bound H and integrating in θ, we obtain, using (36),

$$\sum_{D\in\mathcal{D}} \int_D g_\phi^2\, d\sigma \le CHN. \tag{42}$$

5.3. **The transverse derivative.** For the transverse unit vector e_θ, the radius matrix has the component

$$R_g(e_\theta, e_\theta) = \frac{g_{\theta\theta}}{\cos^2\phi} - \tan\phi\, g_\phi + g.$$

Since $R_{h_K} \le I$, Lemma 3.1 implies

$$\frac{g_{\theta\theta}}{\cos^2\phi} \ge \begin{cases} -w - CH, & D_0, \\ -CH, & D_1. \end{cases} \tag{43}$$

Here we used $|g| \le \eta \le H$ and $|g_\phi| \le H$.

Let $b_\pm(\theta) = g(\theta, \pm\beta)$. The outer-band inequality implies

$$b_\pm'' \ge -CH \quad \text{in distributions on } (-\alpha,\alpha). \tag{44}$$

To justify the trace, fix a compact subinterval of $(-\alpha,\alpha)$ and approach $\pm\beta$ by latitudes in D_1 for which (43) holds almost everywhere. Uniform convergence of the restrictions of g permits passage to the limit against nonnegative smooth test functions. Compact subintervals exhaust the required interval.

The interpolant introduced in Section 4 is

$$B(\theta,\phi) = \frac{\sin(\beta-\phi)}{\sin(2\beta)}\, b_-(\theta) + \frac{\sin(\beta+\phi)}{\sin(2\beta)}\, b_+(\theta). \tag{45}$$

The coefficients are bounded and nonnegative on D_0. By (44), (40) and the derivative bound,

$$B_{\theta\theta} \geq -CH, \qquad |B_\theta| \leq CH, \qquad B \geq -Cm, \qquad 0 \leq P = g - B \leq |g| + Cm. \tag{46}$$

Fix $|\phi| < \beta$ and put $I = [-\alpha, \alpha]$. Twice integrating by parts in θ gives the identity

$$\int_I g_\theta^2 \, d\theta = [g g_\theta - B g_\theta + B_\theta g]_{\partial I} - \int_I P g_{\theta\theta} \, d\theta - \int_I g B_{\theta\theta} \, d\theta. \tag{47}$$

The endpoint values of g are bounded by m, since the vertical sides of the square lie on triangle boundaries. The endpoint values of B are bounded by Cm, since its four corner values are triangle data as well. Thus the boundary term in (47) has absolute value at most CHm.

In the last integral, replace g by $g + C_2 m \geq 0$, where C_2 is a fixed constant from (40). The resulting correction is $C_2 m [B_\theta]_{\partial I}$, of absolute value at most CHm. The lower bound for $B_{\theta\theta}$ now gives

$$-\int_I g B_{\theta\theta} \, d\theta \leq CH \left(m + \int_I |g| \, d\theta \right).$$

Use $P \geq 0$, (43) and (46) for the remaining integral. We obtain

$$\int_I g_\theta^2 \, d\theta \leq \cos^2 \phi \int_I wP \, d\theta + CH \left(m + \int_I |g| \, d\theta \right). \tag{48}$$

For an outer latitude, $\beta < |\phi| < \alpha$, put $I_\phi = \{\theta : A(\theta) \geq |\phi|\}$. This is an interval whose endpoints lie on triangle boundaries. One integration by parts, using $g + C_2 m \geq 0$, gives

$$\begin{aligned} \int_{I_\phi} g_\theta^2 \, d\theta &= [(g + C_2 m) g_\theta]_{\partial I_\phi} - \int_{I_\phi} (g + C_2 m) g_{\theta\theta} \, d\theta \\ &\leq CH \left(m + \int_{I_\phi} |g| \, d\theta \right). \end{aligned} \tag{49}$$

All functions on these slices are $C^{1,1}$; the identities and inequalities hold on almost every slice, which suffices for Fubini's theorem.

Conclusion of the proof of Lemma 5.1. Since $|\nabla g|^2 = g_\phi^2 + g_\theta^2 / \cos^2 \phi$, divide (48) and (49) by $\cos^2 \phi$ and integrate against $\cos \phi \, d\phi$. The coefficient of $\int_{D_0} wP \, d\sigma$ is one. All other terms are bounded by CHN using (36) and the fixed coordinate bounds. Adding (37) and (42) proves (34). □

6. Proof of the local estimate

Combining (23) with Lemmas 4.1 and 5.1 gives

$$\begin{aligned} \operatorname{vol}(K) - V_M &\geq L(g) - \frac{1}{2} J(g) - C\sqrt{\eta} \, N \\ &\geq \left(\frac{c_0}{2} - C\sqrt{\eta} \right) N. \end{aligned} \tag{50}$$

Thus the volume difference is bounded below by a positive multiple of N when η is sufficiently small. It remains to compare N with the uniform norm.

Lemma 6.1. *For every normalized competitor with $\eta \leq 1$,*

$$N \geq c\eta^{3/2}. \tag{51}$$

Idea of the proof. If a fixed proportion of the maximum occurs on a triangle, concavity there already gives $N \geq c\eta$. Otherwise, the positive maximum occurs on a square. Meridian concavity spreads it across a fixed meridional interval, while the gradient bound spreads it across a transverse interval of length comparable to $\sqrt{\eta}$. Their product gives the exponent $3/2$.

Proof. Use the notation m from (35) and choose a small fixed number $a > 0$. If $m \geq a\eta$, then (36) gives $N \geq ca\eta \geq ca\eta^{3/2}$.

Suppose $m < a\eta$. By (40), the negative values of g on the chosen squares have absolute value at most $Ca\eta$, and its values on all triangles have absolute value at most $a\eta$. Choose a so small that both are less than η. Oddness then implies that g attains the value η on one of the chosen squares.

On that meridian, the function F of (38) has its maximum in the middle band. Since $\cos\phi \geq r$ there, some $(\theta_0, \phi_0) \in D_0$ satisfies $g(\theta_0, \phi_0) \geq r\eta$. The gradient estimate in (8) gives an interval $I \subset [-\alpha, \alpha]$ of length at least $c\sqrt{\eta}$, adjacent to θ_0 if necessary, on which

$$g(\theta, \phi_0) \geq \frac{r}{2}\eta.$$

For each $\theta \in I$, $F + C_1 m$ is nonnegative and concave on $[-\tau, \tau]$. The same line-segment comparison used in Section 5 gives

$$\int_{-\beta}^{\beta} |g(\theta, \phi)| \cos\phi \, d\phi \geq c_1\eta - C_3 m.$$

Decrease the fixed a once more so that $C_3 a \leq c_1/2$. Integrating over I proves (51). □

Proof of Theorem 1.1. The infimum in (2) is attained: rotations form a compact set, and translations in a minimizing sequence are bounded. Apply an initial rigid motion so that $d_H(K, M) = \delta(K, M)$, and then use Lemma 2.2. For the resulting support difference,

$$\delta(K, M) \leq \eta \leq C\delta(K, M).$$

Choose ε so small that the coefficient in (50) is at least $c_0/4$. Lemma 6.1 now yields

$$\mathrm{vol}(K) - V_M \geq cN \geq c'\eta^{3/2} \geq c'\delta(K, M)^{3/2}.$$

Volumes and the distance δ are unchanged by the rigid motions. This proves the theorem. □

7. A short arc switch and sharpness

Idea of the construction. Insert two nearby points into the interior of one retained circular edge. The edge is now split into three pieces. Keep the two outer pieces, and replace the middle piece by the other edge in its dual pair. The general Meissner construction guarantees constant width. The volume change can be computed from this single pair. Its first nonzero term is cubic in the separation of the inserted points, whereas the support change is quadratic.

7.1. **The construction.** Keep the coordinates (17) and the arc (19) for any one of the edges retained in M. For $0 < t < \alpha$, put

$$c_- = c(-t), \qquad c_+ = c(t), \qquad X_t = \{p_-, p_+, q_-, q_+, c_-, c_+\}.$$

A diameter-one set of $n \geq 4$ points is called extremal if it has $2n-2$ diametric pairs. The six points of X_t have diameter one and ten diametric pairs. There are the six pairs of tetrahedral vertices and the four pairs joining $c_\pm$ to $p_\pm$. All other distances are chords of the circle (19) with angular separation less than 2α, and hence are less than one. Thus X_t is an extremal six-point set.

We use the following standard construction of Meissner polyhedra [16, 10]. For a finite extremal set X of diameter one, the edges of the ball polyhedron $\bigcap_{x\in X} B(x, 1)$ occur in dual pairs. The four cross distances between the endpoints of two dual edges equal one. Select one edge from each pair. Intersecting the unit balls centered at X and at all points of the selected edges gives a body of constant width one. The selected edges are retained in the resulting body. Degree-two vertices are included in this construction.

For X_t, the original edge is subdivided at c_-, c_+. The inner subarc $c([-t, t])$ is dual to a circular arc with endpoints p_-, p_+. To describe this arc, put

$$\rho_t = \sqrt{1 - r^2 \sin^2 t}. \tag{52}$$

It is the short arc on the negative x side of the circle

$$(x - r\cos t)^2 + z^2 = \rho_t^2, \qquad y = 0. \tag{53}$$

Indeed, it is the intersection of the two unit spheres centered at c_-, c_+, and it joins p_- to p_+. The parametrization

$$d_t(v) = (r\cos t - \rho_t \cos v,\, 0,\, \rho_t \sin v), \qquad |v| \le \arcsin\frac{1}{2\rho_t},$$

shows that the arc lies in all the other required balls. Every subchord between one endpoint and an arc point has length at most the full chord, which is one. For $q_\pm$, the squared distance is

$$|d_t(v) - q_\pm|^2 = \rho_t^2 + (r\cos t - b)^2 + \frac{1}{4} - 2\rho_t(r\cos t - b)\cos v.$$

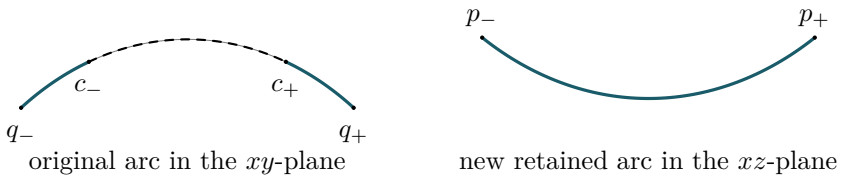
Since $r\cos t > b$ for $t < \alpha$, this expression has its maximum one at the two endpoints.

The two other edges retained in M remain available in the refined ball polyhedron. They lie in M, so they satisfy the new ball constraints. Their interiors lie strictly inside the balls centered at c_- and c_+. To verify strictness, let z be an interior point of one of these edges. Its endpoints are a p vertex and a q vertex, and $z_x > 0$. For completeness, if v is the circular angle measured from its midpoint, then its x coordinate, up to reversing v, is

$$\frac{b}{2}\left(1 + \frac{\sin v}{\sin\alpha}\right), \qquad -\alpha < v < \alpha.$$

Since $z, c(s) \in M$, we have $|z - c(s)| \le 1$ on $[-\alpha, \alpha]$. Equality at an interior parameter $s = \pm t$ would make $z_x \cos s + z_y \sin s$ have a local minimum there. At any stationary point with $|s| < \pi/2$ and $z_x > 0$, its second derivative is $-z_x/\cos s < 0$. This is impossible.

Retaining all three subarcs of the original edge and the other two original retained edges gives M itself. Define K_t by changing only the inner choice: retain the arc (53) in place of $c([-t, t])$, and keep the other four choices. It is a Meissner polyhedron of constant width one, containing X_t. Figure 3 shows the change.



original arc in the xy-plane new retained arc in the xz-plane

FIGURE 3. The short arc switch. The two solid outer pieces on the left are retained. The dashed middle piece is replaced by the arc on the right. Every point of the latter is at unit distance from both c_- and c_+. The drawings show the circular arcs in their respective planes, with separate vertical offsets.

7.2. **The volume change.** Let

$$\ell = |c_+ - c_-| = 2r\sin t, \qquad x = 2\arcsin(\ell/2), \qquad y = \frac{\pi}{3}. \tag{54}$$

For angular chord lengths a, b, put

$$\mathcal{F}(a, b) = 2b\cos(b/2)\arcsin\left(\frac{\sin(a/2)}{\cos(b/2)}\right). \tag{55}$$

Here the first argument corresponds to the retained edge. To specify the normalization, that edge has circular radius $\cos(b/2)$, central angle $2\arcsin(\sin(a/2)/\cos(b/2))$, and hence $\mathcal{F}(a, b)$ is its arc

length multiplied by b. Combining the wedge and spindle contributions in Bogosel's formula [6, Section 3] gives

$$\mathrm{vol} = \frac{2\pi}{3} - \frac{1}{2}\sum \mathcal{F}(a,b). \tag{56}$$

In particular, three pairs with $a = b = \pi/3$ give (1).

The two bodies M and K_t differ only in the order of the arguments for the inner pair. Therefore

$$\mathrm{vol}(K_t) - V_M = \frac{1}{2}\big(\mathcal{F}(x,y) - \mathcal{F}(y,x)\big). \tag{57}$$

For fixed $y = \pi/3$, Taylor expansion at $x = 0$ gives

$$\begin{aligned}\mathcal{F}(x,y) &= xy + \frac{y\tan^2(y/2)}{24}x^3 + O(x^5),\\ \mathcal{F}(y,x) &= xy + \frac{2\tan(y/2) - y}{8}x^3 + O(x^5).\end{aligned}$$

The linear terms cancel. Since $x = \ell + O(\ell^3)$ and $\ell = \sqrt{3}\,t + O(t^3)$, we obtain

$$\mathrm{vol}(K_t) - V_M = k\ell^3 + O(\ell^5) = 3\sqrt{3}\,kt^3 + O(t^5), \qquad k = \frac{10\pi/9 - 2\sqrt{3}}{48} > 0. \tag{58}$$

The positivity follows, for example, from $\pi > 9\sqrt{3}/5$.

7.3. **The Hausdorff distance.** The midpoint $(r\cos t - \rho_t, 0, 0)$ of the new retained arc belongs to K_t. Conversely, averaging the two squared-distance inequalities for the balls centered at c_-, c_+ gives

$$(x - r\cos t)^2 + y^2 + z^2 \le \rho_t^2 \quad \text{for } (x,y,z) \in K_t.$$

Thus $\min_{K_t} x = r\cos t - \rho_t$. Constant width implies $h_{K_t}(e_x) = 1 + \min_{K_t} x$, while $h_M(e_x) = r$. Consequently,

$$\begin{aligned}h_M(e_x) - h_{K_t}(e_x) &= r(1-\cos t) - (1-\rho_t)\\ &= \frac{r(1-r)}{2}\,t^2 + O(t^4).\end{aligned} \tag{59}$$

This gives $d_H(K_t, M) \ge ct^2$ for small t.

For the upper bound, let S consist of the four vertices and the three retained arcs generating M, so that $M = \bigcap_{s\in S} B(s,1)$. Every point of S is within

$$\omega_t = r(1-\cos t)$$

of K_t: the generators outside the lost inner arc belong to K_t, and the inner arc is within ω_t of the chord $[c_-, c_+] \subset K_t$. Since K_t has diameter one,

$$K_t \subset \bigcap_{s\in S} B(s, 1+\omega_t).$$

Translate a fixed inball $B(0, r_0) \subset M$ to the origin. For each $s \in S$, $|s| \le 1 - r_0$. Hence, for $z \in K_t$,

$$\left|\frac{z}{1+\omega_t/r_0} - s\right| \le \frac{|z-s| + (\omega_t/r_0)|s|}{1+\omega_t/r_0} \le 1.$$

It follows that $K_t \subset (1+\omega_t/r_0)M$, and so $h_{K_t} - h_M \le C\omega_t$. This support difference is odd. The same estimate therefore holds for its negative, proving

$$d_H(K_t, M) \asymp t^2. \tag{60}$$

We finally check that rigid motions cannot remove this quadratic displacement. Work again with the original centered tetrahedron. Because K_t contains its four vertices and has diameter one,

$$h_{K_t}(u_{ij}) = \frac{1}{2} \quad (i \ne j).$$

Let $Q_t M + a_t$ realize $\delta(K_t, M)$. Since $K_t \to M$, the distance of these motions from the set of symmetries of M tends to zero. Indeed, translations are bounded, and any convergent subsequence has a limiting motion sending M to itself. Composing with a symmetry of M, we can therefore take the motions near the identity. Their six support values differ from $1/2$ by at most $\delta(K_t, M)$. Lemma 2.2 gives

$$\|Q_t - I\| + |a_t| \le C\delta(K_t, M).$$

Together with the triangle inequality for d_H, this implies $d_H(K_t, M) \le C'\delta(K_t, M)$. The reverse comparison $\delta(K_t, M) \le d_H(K_t, M)$ is immediate. Thus

$$\delta(K_t, M) \asymp t^2. \tag{61}$$

Equations (58) and (61) prove Theorem 1.2. If $p < 3/2$, they also give

$$\frac{\mathrm{vol}(K_t) - V_M}{\delta(K_t, M)^p} \asymp t^{3-2p} \longrightarrow 0,$$

which proves the asserted sharpness.

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