THE ZALCMAN CONJECTURE FOR ALL SUFFICIENTLY LARGE INDICES

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Abstract. We prove the Zalcman conjecture for all sufficiently large integers n. More precisely, there exists an integer $N \geq 3$ such that every normalized univalent function $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ in the unit disk satisfies

$$|a_n^2 - a_{2n-1}| \leq (n-1)^2, \qquad n \geq N,$$

with equality if and only if f is a rotation of the Koebe function.

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1. Introduction

Let S be the class of injective holomorphic functions on $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ normalized by

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j.$$

We set $a_1 = 1$, write $\mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$, and denote the Koebe function by $k(z) = z/(1-z)^2$. For any function g holomorphic near zero, the notation $a_j(g)$ denotes the coefficient of z^j in its Taylor expansion. A rotation of $f \in S$ means the normalized function $z \mapsto \overline{\lambda} f(\lambda z)$, where $\lambda \in \mathbb{T}$.

Sharp coefficient estimates have played a central role in the theory of univalent functions. The Bieberbach conjecture, proved by de Branges [2], asserts that $|a_j| \leq j$, with equality at any index $j \geq 2$ only for a rotation of k. Zalcman's conjecture concerns the cancellation between a_n^2 and a_{2n-1}. It circulated orally before appearing in the work of Brown and Tsao [1]; a later account is [5, Problem 6.127]. We state the conjecture with its usual equality assertion for $n \geq 3$.

Conjecture 1 (The Zalcman conjecture)**.** *For every $f \in S$ and every integer $n \geq 3$,*

$$|a_n^2 - a_{2n-1}| \leq (n-1)^2.$$

Equality holds if and only if f is a rotation of k.

The inequality at $n = 2$ follows from the area theorem, but its equality class is larger. For example, $f(z) = z/(1-z^2)$ also gives equality. Zalcman's conjecture implies the Bieberbach conjecture; see [1]. Brown and Tsao proved the inequality for starlike and typically real functions, and Ma established further results for close-to-convex functions for $n \geq 4$ and formulated a two-index generalization [12, 13].[1] Krushkal proved the conjecture, including its equality characterization, for $n = 3$ [7, 8]. He also proved the sharp inequalities for $n = 4, 5, 6$ [8, Theorem 1.1].

There is also an asymptotic result for the full class S. Efraimidis and Vukotić [4, Theorem 4.1] proved, in particular,

$$\lim_{n\to\infty} \sup_{f\in S} \frac{|a_n^2 - a_{2n-1}|}{(n-1)^2} = 1.$$

This limit does not give the exact inequality at any prescribed large index. Their Corollary 4.2 also gives eventual strict inequalities for each fixed function that is not a Koebe rotation, with a threshold depending on the function [4, Corollary 4.2].

[1] A function in S is starlike if its image is star-shaped with respect to zero. A normalized analytic function is typically real if $\operatorname{Im} f(z) \operatorname{Im} z > 0$ for nonreal $z \in \mathbb{D}$. A function $f \in S$ is close-to-convex if $\operatorname{Re}(e^{i\theta} f'/g') > 0$ for some $\theta \in \mathbb{R}$ and some $g \in S$ whose image is convex.

Full proofs have been claimed in the recent papers [10, 9]. Appendix B examines the general extremal statement in [9] and gives a counterexample under the interpretation of rotational homogeneity indicated by its Zalcman example. It also examines two auxiliary assertions in [10]: a sharpness claim in its covering lemma and a parameter count for a Schwarzian image. The proof below does not rely on either paper. Its main quantitative ingredient is Theorem 1.2, which gives the exact coefficient bound with a threshold uniform over S.

Theorem 1.1. *There is an absolute integer $N \geq 3$ such that, for every $f \in S$ and every integer $n \geq N$,*

$$|a_n^2 - a_{2n-1}| \leq (n-1)^2.$$

Equality holds if and only if $f(z) = \overline{\lambda} k(\lambda z)$ for some $\lambda \in \mathbb{T}$.

Thus Conjecture 1 holds at every sufficiently large index, with a threshold independent of the function.

The estimate used to prove Theorem 1.1 compares two coefficient deficits while retaining their relative phase.

Theorem 1.2. *There are an absolute constant $C_0 > 0$ and an integer $N_0 \geq 4$ such that, whenever $f \in S$, $n \geq N_0$ is an integer, and $D = n - |a_n| \leq 3$, the numbers $m = 2n-1$ and $\omega = a_n/|a_n|$ satisfy*

$$m - \operatorname{Re}(\overline{\omega}^2 a_m) \leq C_0 D \sqrt{n}\,(\log n)^2. \tag{1.1}$$

Here $D \geq 0$ by the Bieberbach inequality, and $a_n \neq 0$ under the stated hypotheses. The quantity on the left of (1.1) is unchanged by normalized rotations of f. The estimate is uniform even when D tends to zero with n.

We briefly describe the proof. Write $C(z) = \sqrt{f(z)/z}$, with $C(0) = 1$. The positive representation underlying the proofs of de Branges and Weinstein [2, 15, 6] gives comparisons between Milin's logarithmic-coefficient sums at different indices. Retaining the nonnegative remainders in the Lebedev–Milin argument then shows that, after a rotation, the first n coefficients of C are close in squared Euclidean norm to a constant of modulus one. A finite Fourier argument selects the rotation, and the normalization $C(0) = 1$ controls the remaining phase.

Two radial estimates extend this information to indices of size $4n$. The first concerns $f(r)$; the second concerns $f(r) - f(-r)$ and is needed to preserve odd indices. Their proofs use normalized disk automorphisms and two estimates involving only a_2, a_3, a_4. Positivity of the coefficient deficits allows the tails of the resulting power series to be absorbed. Reflections of the coefficient vector then control a_{2n-1}, giving (1.1).

The classical ingredients are the coefficient and growth theorems, Löwner's parametric method [11], the Lebedev–Milin inequalities, and the positive Legendre representation. The additional work lies in the comparisons between indices, the quantitative control of the first coefficients, and the radial estimates that preserve the required phase. We retain the details of these steps, since Milin's nonnegativity alone does not give the comparisons used here.

The main dependencies are shown in Figure 1.

Organization. Section 2 establishes the comparisons between Milin sums. Section 3 proves the coefficient stability estimate. Section 4 treats radial propagation, using the low-degree bounds proved in Appendix A. Section 5 proves Theorems 1.2 and 1.1. Appendix B records the observations concerning the recent claims mentioned above.

Notation. For nonnegative quantities, $X \lesssim Y$ means $X \leq cY$ for an absolute constant $c > 0$. We write $X \gtrsim Y$ when $Y \lesssim X$. For complex quantities, $O(Y)$ denotes a quantity of modulus at most an absolute constant times Y; a subscript indicates permitted dependence. The notation $o(1)$ refers to $n \to \infty$, uniformly under the stated hypotheses. All logarithms of positive real

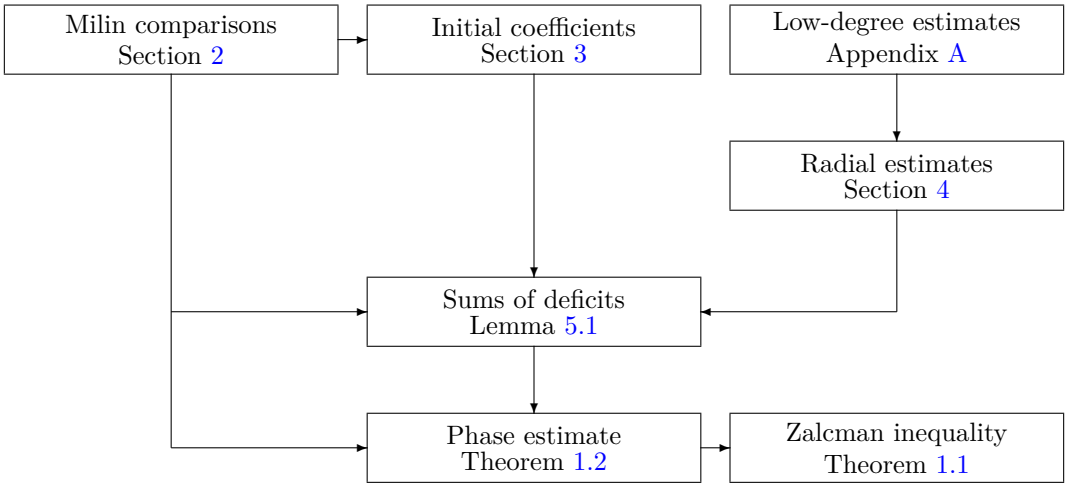

FIGURE 1. An arrow indicates that the argument at its endpoint uses the preceding result.

numbers are natural logarithms. Every threshold described as "sufficiently large" is absolute and, in particular, independent of f and of $D \in [0,3]$. Indices in real intervals are always integers.

Formalization. Theorem 1.1 has been unconditionally formalized in Lean 4 using Mathlib. The formalization is available in the Palomar Registry `PALOMAR-2026-11-27-000055` and on GitHub.

Statement on AI usage. The authors acknowledge the use of AI tools for exploratory discussions and language refinement during the preparation of this work. ChatGPT assisted in developing certain technical arguments in the proof of the main theorem. OpenAI's Codex assisted with the Lean 4 formalization of the main theorem.

2. Comparison of Milin functionals

Since $f(z)/z$ is analytic and has no zeros in $\mathbb{D}$, the branches normalized by $C(0) = 1$ and $\log C(0) = 0$ define analytic functions

$$C(z) = \sqrt{f(z)/z} = \sum_{j=0}^{\infty} c_j z^j, \qquad \log C(z) = \sum_{k=1}^{\infty} \frac{t_k}{k} z^k, \qquad c_0 = 1.$$

Put $A_0 = 0$ and, for integers $j \geq 1$, define

$$A_j = \sum_{\ell=0}^{j-1} |c_\ell|^2, \qquad T_j = \sum_{k=1}^{j-1} \frac{j-k}{k}(1 - |t_k|^2).$$

Empty sums are zero. The inequalities $T_j \geq 0$ are the Milin inequalities proved by de Branges [2]. We need comparisons between different indices as well. These follow from a positive representation of the same functionals.

Lemma 2.1. *For every $f \in S$, the numbers T_j are nonnegative. If $n \geq 4$ and $m = 2n-1$, then*

$$\sum_{\substack{1 \leq j \leq n \\ j \equiv n \pmod 2}} T_j \lesssim n \log n \, T_n, \qquad T_m \lesssim \sqrt{n}\, T_j \quad \text{for odd } j \in [3n, 4n].$$

Proof. Let P_ℓ be the Legendre polynomial normalized by $P_\ell(1) = 1$, and let U_M be the Chebyshev polynomial of the second kind, so that $U_M(\cos\theta) = \sin((M+1)\theta)/\sin\theta$. Write $U_M = \sum_{\ell=0}^{M} B_{M\ell} P_\ell$.

By parity, $B_{M\ell} = 0$ unless $\ell \equiv M \pmod 2$. On the remaining indices,

$$\frac{2\ell+1}{M+1} \le B_{M\ell} \le \frac{2\ell+1}{M+1-\ell}, \qquad B_{M\ell} \lesssim \sqrt{\ell+1}. \tag{2.1}$$

Indeed, the generating function

$$\sum_{\ell=0}^{\infty} P_\ell(\cos\theta) w^\ell = (1-we^{i\theta})^{-1/2}(1-we^{-i\theta})^{-1/2}$$

shows that $P_\ell(\cos\theta) = \sum_v p_{\ell v} e^{iv\theta}$, where $p_{\ell v} \ge 0$, $\sum_v p_{\ell v} = 1$, and the support is contained in $|v| \le \ell$, $v \equiv \ell \pmod 2$. Orthogonality gives

$$B_{M\ell} = \frac{2\ell+1}{2}\int_0^\pi \sin((M+1)\theta) P_\ell(\cos\theta)\, d\theta = (2\ell+1)\sum_v p_{\ell v} \frac{M+1}{(M+1)^2 - v^2}.$$

This proves the first two bounds in (2.1). For fixed v, the function $a/(a^2-v^2)$ decreases on $a > |v|$. Thus $B_{M\ell} \le B_{\ell\ell}$. Comparison of leading coefficients gives $B_{\ell\ell} = 4^\ell / \binom{2\ell}{\ell}$. For $\ell \ge 1$ this is at most $2\sqrt{\ell}$: the assertion is an equality at $\ell = 1$, and induction follows from $(2\ell+2)/(2\ell+1) \le \sqrt{(\ell+1)/\ell}$. The case $\ell = 0$ is immediate.

We next establish the representation

$$T_{M+1} = \sum_{\ell=0}^{M} B_{M\ell}\rho_\ell, \qquad \rho_\ell \ge 0, \tag{2.2}$$

where ρ_ℓ is independent of M. We use the following consequence of de Branges' weighted theorem [2, Theorem 2, p. 146]. Suppose that $w_1, \dots, w_L$ are C^1 functions on $[0,\infty)$, $w_1 \ge \cdots \ge w_L \ge w_{L+1} = 0$, w_1 is not identically zero, and

$$w_k' \le 0, \qquad w_k - w_{k+1} = -\frac{w_k'}{k} - \frac{w_{k+1}'}{k+1} \quad (1 \le k \le L).$$

Then

$$\sum_{k=1}^{L} \frac{w_k(0)}{k}(1-|t_k|^2) \ge 0. \tag{2.3}$$

This is the cited theorem after replacing its parameter t by e^u. The normalization follows from $\log(f(z)/z) = 2\sum_{k\ge1} t_k z^k/k$.

For $0 \le q \le 1$, define $b_{\ell k}(q)$ by

$$P_\ell(1-q+q\cos\theta) = \sum_{k=-\ell}^{\ell} b_{\ell k}(q) e^{ik\theta}.$$

Set $b_{\ell k} = 0$ for $|k| > \ell$. The Legendre addition formula gives $b_{\ell k}(q) \ge 0$ and $b_{\ell,-k}(q) = b_{\ell k}(q)$; see [6, Corollary 6, Eq. (36)]. For $\ell \ge 1$ and $1 \le k \le \ell$, put

$$w_{\ell k}(u) = b_{\ell k}(e^{-u}) + 2\sum_{h=k+1}^{\ell} b_{\ell h}(e^{-u}), \qquad w_{\ell,\ell+1} = 0.$$

The Laurent polynomial $L(u,\zeta) = P_\ell(1 - e^{-u} + e^{-u}(\zeta+\zeta^{-1})/2)$ satisfies $(1+\zeta)\partial_u L = (1-\zeta)\zeta\partial_\zeta L$. Comparison of coefficients and summation give $w_{\ell k}' = -k b_{\ell k}(e^{-u})$. Also $w_{\ell k} - w_{\ell,k+1} = b_{\ell k} + b_{\ell,k+1}$, where the coefficients on the right are evaluated at e^{-u}. Moreover, $b_{\ell\ell}(e^{-u}) = 4^{-\ell}\binom{2\ell}{\ell} e^{-\ell u} > 0$, so $w_{\ell 1}$ is not identically zero. Thus these weights satisfy all the hypotheses of (2.3), and the numbers

$$\rho_0 = 0, \qquad \rho_\ell = \sum_{k=1}^{\ell} \frac{w_{\ell k}(0)}{k}(1-|t_k|^2) \quad (\ell \ge 1)$$

are nonnegative.

Finally, $U_M(\cos\theta)=\sum_{r=0}^{M} e^{i(M-2r)\theta}$. Combining this identity with $U_M=\sum_{\ell=0}^{M} B_{M\ell}P_\ell$ and the definition of the weights gives $\sum_{\ell=k}^{M} B_{M\ell}w_{\ell k}(0)=M+1-k$ for $1\le k\le M$. Multiplying by $(1-|t_k|^2)/k$ and summing over k proves (2.2).

It remains to compare the coefficients in this representation. For $\ell\le n-1$ of the same parity as $n-1$,

$$\sum_{\substack{\ell+1\le j\le n\\ j\equiv n \pmod 2}} B_{j-1,\ell}\lesssim (2\ell+1)\log n\lesssim n\log n\, B_{n-1,\ell}.$$

This proves the first comparison. For the second, let $j\in[3n,4n]$ be odd, and consider even indices ℓ. If $0\le\ell\le n-1$, the first two bounds in (2.1) give $B_{m-1,\ell}\lesssim B_{j-1,\ell}$. If $n-1<\ell\le m-1$, they give $B_{j-1,\ell}\gtrsim 1$, while the last bound gives $B_{m-1,\ell}\lesssim\sqrt{n}$. Multiplication by ρ_ℓ and summation finish the proof. □

3. Stability of the initial coefficients

We retain the nonnegative error terms in the classical Lebedev–Milin argument; see [3, §5.1, pp. 144–146] and [14, Chapter II, §2, Lemma 2.2, pp. 33–35]. For $j\ge 1$, define

$$s_j=\sum_{k=1}^{j}|t_k|^2,\qquad u_j=\frac{s_j-j}{j(j+1)},\qquad R_j=s_j-\frac{j^2|c_j|^2}{A_j},\qquad L_j=\log\frac{j}{A_j}-\frac{T_j}{j}.$$

The identity $jc_j=\sum_{\ell=0}^{j-1}t_{j-\ell}c_\ell$ and Cauchy–Schwarz give $R_j\ge 0$. Direct subtraction gives

$$L_{j+1}-L_j=u_j-\log(1+u_j)-\log\left(1-\frac{R_j}{j^2+s_j}\right)\ge 0.$$

Both logarithms are well defined: $u_j>-1$ and $0\le R_j\le s_j<j^2+s_j$. Indeed, the increment of $-T_j/j$ is u_j, and $A_{j+1}/A_j=1+(s_j-R_j)/j^2$. Since $L_1=0$ and $T_j\ge 0$, we obtain $A_j\le j$. The convolution identity $a_j=\sum_{\ell=0}^{j-1}c_\ell c_{j-1-\ell}$ also gives $|a_j|\le A_j$.

Fix $0\le D=n-|a_n|\le 3$. For sufficiently large n,

$$L_n\lesssim\frac{D}{n},\qquad T_n\lesssim D,\qquad \sum_{j=1}^{n-1}\frac{R_j}{j^2}\lesssim\frac{D}{n}.\tag{3.1}$$

The first two estimates follow from $L_n+T_n/n=\log(n/A_n)\le\log(n/(n-D))$. For the third, each $u_j-\log(1+u_j)$ is at most L_n, so $u_j\le 1$ for n sufficiently large. Thus $j^2+s_j\le 4j^2$ for $j<n$. Sum the preceding increment identity and use $-\log(1-x)\ge x$.

Lemma 3.1. *Suppose that $0\le D=n-|a_n|\le 3$ and that n is sufficiently large. After a rotation of f, with the coefficients relabelled, there is $\alpha\in\mathbb{T}$ such that*

$$\sum_{j=0}^{n-1}|c_j-\alpha|^2\lesssim D,\qquad \omega=\alpha^2+O(D/n),\qquad |\alpha-1|^2\lesssim\frac{D(\log n)^2}{\sqrt{n}},\quad \omega=\frac{a_n}{|a_n|}.$$

Proof. The coefficient a_n gives the reflection identity

$$\sum_{\ell=0}^{n-1}|c_\ell-\omega\overline{c_{n-1-\ell}}|^2=2(A_n-|a_n|)\le 2D.$$

For $1\le j\le n$, put $x=\sqrt{A_j}$ and $y=\sqrt{A_n-A_{n-j}}$. Then $x\le\sqrt{j}$, $y^2\ge j-D$, and $|x-y|\le\sqrt{2D}$. With $r_+=\max\{r,0\}$, it follows that

$$\sqrt{j}-x\le(\sqrt{j}-y)_+ +|x-y|\le(1+\sqrt{2})\sqrt{D},$$

Consequently,

$$0 \le j - A_j \lesssim \sqrt{jD}, \qquad A_j \gtrsim j, \qquad \frac{(j-A_j)^2}{A_j} \lesssim D. \tag{3.2}$$

For the middle assertion, use the first bound for large j, and use $A_j \ge 1$ for the remaining bounded range of indices.

The least-squares identity

$$\sum_{\ell=0}^{j-1} \left| t_{j-\ell} - \frac{j}{A_j} c_j \overline{c_\ell} \right|^2 = R_j$$

and orthogonality imply

$$\mathcal{E} := \sum_{0 \le \ell < j < n} |c_j \overline{c_\ell} - t_{j-\ell}|^2 = \sum_{j=1}^{n-1} \left(R_j + \frac{(j-A_j)^2}{A_j} |c_j|^2 \right) \lesssim nD.$$

Here we used (3.1), (3.2), and $A_n \le n$.

For $1 \le h \le \lfloor n/2 \rfloor$, form the vectors $x = (c_j)_{0 \le j < n-h}$ and $y = (c_{j+h})_{0 \le j < n-h}$, and put $e_h = \min_{|v|=1} \|y - vx\|_2^2$. Here $\|\cdot\|_2$ and $\langle \cdot, \cdot \rangle$ denote the Euclidean norm and the Hermitian inner product, respectively. The sum of the squared moduli of the off-diagonal entries of $xx^* - yy^*$ is at most $8\mathcal{E}$, since $t_{j-\ell}$ is invariant under simultaneous translation of the indices. The corresponding diagonal sum is at most $4 \max_{\ell < n} |c_\ell|^2 e_h$. If $s = \|x\|_2^2$, $t = \|y\|_2^2$, and $r = |\langle x, y \rangle|$, then

$$\|xx^* - yy^*\|_{\mathrm{HS}}^2 = \frac{(s-t)^2}{2} + \frac{s+t+2r}{2} e_h \ge \min(s,t) e_h.$$

Here x^* denotes the conjugate transpose of x, and $\|\cdot\|_{\mathrm{HS}}$ is the square root of the sum of the squared moduli of the matrix entries. Both s and t are bounded below by a positive constant times n, whereas

$$|c_\ell|^2 = A_{\ell+1} - A_\ell \le \ell + 1 - A_\ell \lesssim \sqrt{n} \qquad (0 \le \ell < n).$$

Absorption of the diagonal term therefore gives $e_h \lesssim D$.

Let $H = \lfloor n/2 \rfloor$, choose a minimizing phase v_H, and choose $\tau \in \mathbb{T}$ with $\tau^H = v_H$. Set $b_j = \overline{\tau}^j c_j$ for $0 \le j < H$. The cyclic shift by $h = 0$ is the identity. For $1 \le h < H$, choose a phase v_h attaining e_h. The nonwrapping entries of the cyclic shift are controlled by e_h; on the wrapping entries, insert c_{j+h} and use the estimate for e_H. Since $\tau^H = v_H$, the common phase on both parts is $v_h \overline{\tau}^h$, and the total squared error is at most $2(e_h + e_H) \lesssim D$. In particular, if

$$B = \sum_{j=0}^{H-1} |b_j|^2, \qquad Q_h = \sum_{j=0}^{H-1} b_{j+h} \overline{b_j},$$

where indices on b are read modulo H, then $B \gtrsim n$ and $|Q_h| \ge B - O(D)$. Define the discrete Fourier coefficients by $\widehat{b}_\nu = H^{-1/2} \sum_{j=0}^{H-1} b_j e^{-2\pi i j \nu / H}$ and put $p_\nu = |\widehat{b}_\nu|^2 / B$ for $0 \le \nu < H$. Parseval's identity gives

$$\sum_{\nu=0}^{H-1} p_\nu^2 = \frac{1}{HB^2} \sum_{h=0}^{H-1} |Q_h|^2 \ge 1 - O(D/n).$$

Since $\sum_\nu p_\nu = 1$, one of the p_ν is at least $1 - O(D/n)$. Undoing the factor $\overline{\tau}^j$, we obtain $\beta \in \mathbb{C}$ and $\lambda \in \mathbb{T}$, with $\lambda^H = v_H$, such that $\sum_{j=0}^{H-1} |c_j - \beta \lambda^j|^2 \lesssim D$. The estimate for e_H extends this bound to $0 \le j < n$. When $n = 2H + 1$, the last entry is obtained by applying the same estimate to $c_H - v_H c_0$ and $c_{2H} - v_H c_H$.

Replace $f(z)$ by $\lambda f(\overline{\lambda}z)$. This multiplies c_j by $\overline{\lambda}^j$ and leaves $\overline{\omega}^2 a_{2n-1}$ unchanged. In these coordinates take the best constant $\beta = n^{-1}\sum_{j=0}^{n-1} c_j$. Orthogonality to constants gives

$$1 - O(D/n) \le |\beta|^2 \le 1, \qquad \sum_{j=0}^{n-1} |c_j - \beta|^2 \lesssim D.$$

The terms linear in $c_j - \beta$ cancel in the convolution for a_n, so $a_n = n\beta^2 + O(D)$. Thus $\alpha = \beta/|\beta|$ satisfies the first two assertions of the lemma.

To compare α with $c_0 = 1$, put $J = \lfloor \sqrt{n} \rfloor$. For each $1 \le k \le J$, retain the term $\ell = j - k$ in the least-squares identity and sum over $J \le j < 2J$. Equation (3.1) gives $\sum_{J \le j < 2J} R_j \lesssim D$. Moreover,

$$\sum_{J \le j < 2J} \left| \frac{j}{A_j} c_j \overline{c_{j-k}} - 1 \right|^2 \lesssim D.$$

Indeed, all c_ℓ with $\ell < n$ and all ratios j/A_j are bounded, the squared error from the constant α is $O(D)$, and (3.2) gives $\sum_{J \le j < 2J} |j/A_j - 1|^2 \lesssim D$. The triangle inequality in Euclidean space now gives $|t_k - 1|^2 \lesssim D/J$ for $1 \le k \le J$.

The coefficients of C through degree J agree with those of

$$\frac{1}{1-z} \exp\left(\sum_{k=1}^{J} \frac{t_k - 1}{k} z^k \right).$$

The sum of the absolute values of the coefficients in the exponent is $O(\sqrt{D/J}\log n) = o(1)$, uniformly for $D \le 3$. Expansion of the exponential therefore gives $\max_{j \le J} |c_j - 1| \lesssim \sqrt{D/J}\log n$. Averaging $|\alpha - 1|^2 \le 2|c_j - \alpha|^2 + 2|c_j - 1|^2$ over $0 \le j \le J$ proves the last assertion. □

4. Propagation of radial estimates

For $0 < t < 1$ and $\gamma \in \mathbb{T}$, define

$$h_\gamma(t) = 1 - \operatorname{Re}\left(\overline{\gamma} \frac{f(t)}{k(t)} \right), \qquad j_\gamma(t) = 1 - \operatorname{Re}\left(\overline{\gamma} \frac{f(t) - f(-t)}{k(t) - k(-t)} \right).$$

Both quantities are nonnegative by the Bieberbach inequalities. We also use the same formula to define $h_\gamma(-t)$; this latter quantity is not required to be nonnegative.

Lemma 4.1. *There are absolute constants $K, B > 0$ and $r_0 \in (0,1)$ such that, for every $f \in S$, $\gamma \in \mathbb{T}$, and $r_0 \le r \le s < 1$,*

$$\Phi(s) \le K \left(\frac{1-r}{1-s} \right)^B \Phi(r), \qquad \Phi = h_\gamma \text{ or } j_\gamma.$$

Proof. For $x \in \mathbb{R}$, set $F(x) = f(\tanh x)$ and $U(x) = |F'(x)|$. Since a univalent holomorphic function has no critical point, $U(x) > 0$. For $t = \tanh x$, the normalized transform

$$g_x(z) = \frac{f((t+z)/(1+tz)) - f(t)}{(1-t^2)f'(t)}$$

belongs to S. Write $q(x) = a_2(g_x)$, $a(x) = a_3(g_x)$, and $d(x) = 3 - \operatorname{Re} a(x)$. Expansion of $F(x+v) = F(x) + F'(x) g_x(\tanh v)$ gives

$$\frac{F''}{F'} = 2q, \qquad \frac{F'''}{F'} = 6a - 2, \qquad q' = 3a - 2q^2 - 1, \qquad a' = 4a_4(g_x) - 2q - 2qa. \tag{4.1}$$

We will also use $k(\tanh x) = (e^{4x} - 1)/4$.

We first treat h_γ. Put $e(x) = 2 - \operatorname{Re} q(x)$ and $P(x) = e^{-4x}\overline{\gamma}F'(x)$. The area theorem gives $|q^2 - a| \le 1$, while $|q| \le 2$ gives $(\operatorname{Im} q)^2 \le 4e$. The equation for q' in (4.1) implies

$$e' \le 4 - \operatorname{Re}(q^2) = 4e - e^2 + (\operatorname{Im} q)^2 \le 8e.$$

Moreover,

$$\frac{P'}{P} = 2(q-2), \qquad |P(x)| = \exp\left(-2\int_0^x e(u)\,du\right) \quad (x \ge 0).$$

For $x \ge 1$, integration of $e' \le 8e$ backwards over $[x-1, x]$ gives $\int_{x-1}^x e(u)\,du \gtrsim e(x)$. Since $0 \le e \le 4$, it follows that $e(x) \lesssim 1 - |P(x)|^2$. Set $v(x) = 1 - \operatorname{Re} P(x) \ge 0$. The inequalities $e \lesssim v$, $|\operatorname{Im} q|^2 \le 4e$, and $|\operatorname{Im} P|^2 \le 2v$ imply $v' \lesssim v$ for $x \ge 1$. The unnormalized deficit is

$$H_\gamma(x) := k(\tanh x) - \operatorname{Re}(\overline{\gamma}F(x)) = \int_0^x e^{4u}v(u)\,du.$$

For $x \ge 2$, the integral over $[x-1, x]$ is bounded below by a positive constant times $e^{4x}v(x)$. Thus $H_\gamma'(x) \lesssim H_\gamma(x)$.

For j_γ, Lemma A.1 gives $(\operatorname{Im} q)^2 \le d$ and $|\operatorname{Re}(a_4(g_x) - 2q)| \lesssim d$. Rewrite the last equation in (4.1) as $a' = 4(a_4(g_x) - 2q) - 2q(a-3)$. Since $|\operatorname{Im} a|^2 \le 6d$ and $|q| \le 2$, taking real parts shows that $|d'| \lesssim d$. Hence $e^{-C|x-y|}d(y) \le d(x) \le e^{C|x-y|}d(y)$ for an absolute C; this also holds when d vanishes. Choose a continuous argument ψ of F' on $\mathbb{R}$. The identities (4.1) imply

$$U'' = (16 - W)U, \qquad W = 6d - 4(\operatorname{Im} q)^2, \qquad 2d \le W \le 6d, \quad |\psi'| \le 2\sqrt{d}, \quad |(\log U)'| \le 4. \tag{4.2}$$

Let $K_0(v) = \frac{\cosh(4v)-1}{16}$ and $p(u) = \frac{1}{2}|e^{i\psi(u)} - \gamma|^2$. Solving the scalar equation in (4.2) with $U(0) = 1$ and integrating over $[-X, X]$ gives

$$\begin{aligned} J_\gamma(X) &:= k(\tanh X) - k(-\tanh X) - \operatorname{Re}(\overline{\gamma}(F(X) - F(-X))) \\ &= \int_{-X}^X K_0(X - |u|)W(u)U(u)\,du + \int_{-X}^X U(u)p(u)\,du. \end{aligned} \tag{4.3}$$

Indeed, variation of constants on each side of zero expresses U as its homogeneous solution minus the integral of WU against $\sinh(4v)/4$. The term containing $U'(0)$ is odd and disappears after integration. The remaining homogeneous term has integral $\sinh(4X)/2 = k(\tanh X) - k(-\tanh X)$, which proves (4.3). Both integrals on its right are nonnegative.

We claim that $J_\gamma'(X) \lesssim J_\gamma(X)$ for $X \ge 2$. Differentiate (4.3). On $X - |u| \ge 1/2$, $K_0'(X - |u|) \lesssim K_0(X - |u|)$. On either remaining half-unit interval, the inequalities $|d'| \lesssim d$ and $|(\log U)'| \le 4$ give $W(u)U(u) \lesssim d(\varepsilon X)U(\varepsilon X)$, where $\varepsilon = 1$ or -1 is the sign of the endpoint. The first integral over $X - 1 \le \varepsilon u \le X - 1/2$ controls this last product from below. Thus the derivative of the first integral is at most a constant times $J_\gamma(X)$.

The second integral supplies a point u_0 in the same inner interval such that $U(\varepsilon X)p(u_0) \lesssim J_\gamma(X)$. By (4.2) and the comparability of d on that interval, $|\psi(\varepsilon X) - \psi(u_0)|^2 \lesssim d(\varepsilon X)$. It follows that $U(\varepsilon X)p(\varepsilon X) \lesssim J_\gamma(X)$. These are exactly the boundary terms in the derivative of the second integral, proving the claim.

Integrate the inequalities for H_γ and J_γ, and divide by their increasing positive Koebe normalizations. If $x = \operatorname{arctanh} r$ and $y = \operatorname{arctanh} s$, then $e^{2(y-x)} = \frac{(1+s)(1-r)}{(1-s)(1+r)} \le 2\frac{1-r}{1-s}$. Taking $r_0 = \tanh 2$ and enlarging the absolute constants proves the lemma. □

5. The coefficient estimate and the Zalcman inequality

5.1. **Sums of coefficient deficits.** Work in the rotation supplied by Lemma 3.1, and put

$$\gamma = \alpha^2, \qquad b_\ell = \overline{\alpha}c_\ell, \qquad \delta_j = j - \operatorname{Re}(\overline{\gamma}a_j) \quad (j \ge 1).$$

The Bieberbach inequalities give $\delta_j \geq 0$. The convolution formula for a_j gives

$$\delta_j = j - A_j + \frac{1}{2}\sum_{\ell=0}^{j-1} |b_\ell - \overline{b_{j-1-\ell}}|^2 \leq j - A_j + 2\sum_{\ell=0}^{j-1} |b_\ell - 1|^2. \tag{5.1}$$

Since $1 - x \leq -\log x$ for $x > 0$, $j - A_j \leq T_j + jL_j$. For $j \leq n$, use this inequality, (3.1), and Lemmas 2.1 and 3.1 in (5.1). For the sum over both parities, use instead (3.2). This gives

$$\sum_{\substack{1\leq j\leq n\\ j\equiv n \pmod 2}} \delta_j \lesssim nD\log n, \qquad \sum_{j=1}^{n} \delta_j \lesssim n^{3/2}\sqrt{D}. \tag{5.2}$$

Lemma 5.1. *Let I_n be the set of positive odd integers if n is odd, and the set of all positive integers if n is even. For $0 \leq D \leq 3$ and all sufficiently large n,*

$$\sum_{\substack{1\leq j\leq 4n\\ j\in I_n}} \delta_j \lesssim nD\log n.$$

Proof. Write $h = h_\gamma$ and take $s = e^{-1/n}$, $r = e^{-R/n}$, where the absolute constant $R > 2$ will be fixed below. For $0 < t < 1$,

$$h(t) = (1-t)^2 \sum_{j=1}^{\infty} \delta_j t^{j-1}.$$

For any subseries with the same prefactor, its terms with $j \leq n$ at r are bounded by R^2/n^2 times their unweighted sum. Its tail with $j > n$ is at most $R^2 e^{-(R-1)}$ times the corresponding subseries at s, because $(1-r)/(1-s) \leq R$ and $(r/s)^{j-1} \leq e^{-(R-1)}$ for $j > n$. Fix R so large that $2KR^{B+2}e^{-(R-1)} < 1/4$, where K, B are from Lemma 4.1. For sufficiently large n, both radii belong to the range of that lemma.

Suppose first that n is odd, and let

$$O(t) = (1-t)^2 \sum_{\substack{j\geq 1\\ j \text{ odd}}} \delta_j t^{j-1}.$$

The identity $j_\gamma(t) = \frac{2k(t)}{k(t)-k(-t)}\, O(t)$ has a prefactor between 1 and 2. Lemma 4.1 therefore gives $O(s) \leq 2KR^B O(r)$. The first estimate in (5.2), followed by absorption of the tail, yields $O(s) \lesssim D\log n/n$.

Now suppose that n is even. Applying the same argument to the full series and using the second estimate in (5.2) first gives the preliminary bound $h(s) \lesssim \sqrt{D/n}$. Let

$$V(t) = (1-t)^2 \sum_{\substack{j\geq 1\\ j \text{ even}}} \delta_j t^{j-1}.$$

The even and odd parts satisfy

$$h(r) = 2V(r) + \left(\frac{1-r}{1+r}\right)^2 h(-r). \tag{5.3}$$

To control the last term, use the analytic logarithm

$$G(z) = \log((1-z)C(z)) = \sum_{k=1}^{\infty} \frac{t_k - 1}{k} z^k, \qquad G(0) = 0.$$

Summing the definition of T_j gives

$$\sum_{k=1}^{\infty} \frac{1-|t_k|^2}{k} s^k = (1-s)^2 \sum_{j=1}^{\infty} T_j s^{j-1} \geq 0.$$

The interchange is absolutely convergent: apply Cauchy's coefficient estimate to $\log C$ on a circle of radius strictly greater than $\sqrt{s}$. It follows that

$$\sum_{k=1}^{\infty} \frac{|t_k - 1|^2}{k} s^k \le -2 \operatorname{Re} G(s) = \log \frac{k(s)}{|f(s)|} \lesssim h(s).$$

For the last step, use $|f(s)|/k(s) \ge 1 - h(s)$ and the preliminary bound, which gives $h(s) = o(1)$ uniformly for $D \le 3$. Cauchy–Schwarz now yields

$$|G(-r)|^2 \lesssim h(s) \sum_{k=1}^{\infty} \frac{(r^2/s)^k}{k} \lesssim h(s) \log n = o(1).$$

The lower growth theorem [3, Theorem 2.6] gives $|w| \ge 1$ for $w = \overline{\gamma} f(-r)/k(-r) = \overline{\gamma} e^{2G(-r)}$. Hence, by Lemma 3.1,

$$h(-r) = 1 - \operatorname{Re} w \le \frac{1}{2}|1 - w|^2 \lesssim h(s) \log n + \frac{D(\log n)^2}{\sqrt{n}}.$$

In (5.3), the terms of $2V(r)$ with $j \le n$ are $O_R(D \log n/n)$ by (5.2); the remaining terms are bounded by $2R^2 e^{-(R-1)} h(s)$. Also $(1-r)/(1+r) \le R/n$. Using Lemma 4.1, we obtain

$$h(s) \le C_R \frac{D \log n}{n} + \frac{1}{4} h(s) + C_R \frac{\log n}{n^2} h(s) + C_R \frac{D(\log n)^2}{n^{5/2}}.$$

Here C_R depends only on the fixed constant R. For large n, the two terms containing $h(s)$ are absorbed, and the last term is bounded by a constant times $D \log n/n$. Thus $h(s) \lesssim D \log n/n$.

In both parity cases, $s^{j-1} \ge e^{-4}$ for $j \le 4n$ and $(1-s)^2 \gtrsim n^{-2}$. The asserted unweighted sum follows. □

5.2. **Completion of the proofs.** We first prove Theorem 1.2.

Proof. Lemma 5.1 permits the choice of indices

$$j_1 \in [3n/2, 7n/4] \cap I_n, \qquad j_2 \in [2n, 9n/4] \cap I_n, \qquad j_3 \in [3n, 4n] \cap (2\mathbb{Z} + 1)$$

such that $\delta_{j_i} \lesssim D \log n$. Indeed, each of these sets contains at least a positive constant times n indices. For $i = 1, 2$, the reflection $\ell \mapsto j_i - 1 - \ell$ pairs the indices $j_i - n \le \ell < j_i$ with $0 \le \ell < n$. By (5.1),

$$\sum_{\ell=j_i-n}^{j_i-1} |b_\ell - 1|^2 \le 4\delta_{j_i} + 2 \sum_{\ell=0}^{n-1} |b_\ell - 1|^2.$$

The initial and reflected blocks cover $0 \le \ell < m = 2n - 1$, so $\sum_{\ell=0}^{m-1} |b_\ell - 1|^2 \lesssim D \log n$.

Since $j_i - A_{j_i} \le \delta_{j_i} \lesssim D \log n$ and $D \le 3$, $j_i \log \frac{j_i}{A_{j_i}} \lesssim D \log n$. Monotonicity of L_j gives $L_m \le L_{j_2} \lesssim D \log n/n$, while $T_{j_3} \lesssim D \log n$. Lemma 2.1 and (5.1) now imply

$$\delta_m \le T_m + m L_m + 2 \sum_{\ell=0}^{m-1} |b_\ell - 1|^2 \lesssim D\sqrt{n} \log n.$$

It remains to replace the phase γ by ω^2. Set $u = \gamma \overline{\omega}^2$. Since $\gamma = \alpha^2$ and $|\omega - \gamma| \lesssim D/n$, Lemma 3.1 gives

$$|1 - u| \le |1 - \gamma| + 2|\omega - \gamma|, \qquad |1 - u|^2 \lesssim \frac{D(\log n)^2}{\sqrt{n}}.$$

For $|z| \le 1$ and $|u| = 1$, expanding $zu = 1 + (z-1) + (u-1) + (z-1)(u-1)$ gives $1 - \operatorname{Re}(zu) \le 2(1 - \operatorname{Re} z) + 2(1 - \operatorname{Re} u)$. Here we used $|1 - z|^2 \le 2(1 - \operatorname{Re} z)$ and $|1 - u|^2 = 2(1 - \operatorname{Re} u)$. Apply this with $z = \overline{\gamma} a_m/m$ and multiply by m. The preceding two bounds give (1.1). Normalized rotations leave its left side unchanged, so the estimate holds for the original function. □

We now prove Theorem 1.1.

Proof. Let $m = 2n-1$ and suppose that $|a_n^2 - a_m| \geq (n-1)^2 = n^2 - m$. Write $A = |a_n|$ and $D = n - A$. Since $|a_m| \leq m$, we have $A^2 \geq n^2 - 2m$. For sufficiently large n, this implies $0 \leq D \leq 3$ and $A > 0$. Put $\omega = a_n/A$ and $b = \overline{\omega}^2 a_m$. Squaring the assumed inequality and using $|b| \leq m$ gives

$$(n^2 - A^2)(n^2 + A^2 - 2m) \leq 2A^2(m - \operatorname{Re} b).$$

The two factors on the left are at least nD and $n^2/2$, respectively, once n is sufficiently large. Since $A \leq n$, $m - \operatorname{Re} b \geq \frac{nD}{4}$. On the other hand, Theorem 1.2 gives $m - \operatorname{Re} b \leq C_0 D\sqrt{n}(\log n)^2$. Choose an absolute N so large that all preceding estimates apply and $C_0(\log n)^2/\sqrt{n} < 1/4$ for $n \geq N$. Then $D = 0$.

The equality case of the Bieberbach inequality implies that f is a Koebe rotation. Conversely, for $f(z) = \overline{\lambda} k(\lambda z)$, $a_n^2 - a_{2n-1} = (n-1)^2\lambda^{2n-2}$, so equality holds. Every other function satisfies the strict inequality. □

Appendix A. Two coefficient estimates

The following estimates are used in the proof of Lemma 4.1. The first is already contained in the classical Löwner proof of the third-coefficient bound; see [3, §3.5, pp. 93–94]. We include the calculation because the same quantities are needed for the second estimate.

Lemma A.1. *There is an absolute constant C such that every $f \in S$ satisfies*

$$(\operatorname{Im} a_2)^2 \leq 3 - \operatorname{Re} a_3, \qquad |\operatorname{Re}(a_4 - 2a_2)| \leq C(3 - \operatorname{Re} a_3). \tag{A.1}$$

Proof. By slit-map density [3, Theorem 3.2, p. 80] and continuity of the first four coefficients, it suffices to consider maps whose images are the plane with a Jordan arc extending to infinity removed. Such a map f is the initial member of a radial slit chain F_u: the domains $F_u(\mathbb{D})$ increase with u, and $F_0 = f$, $F_u(0) = 0$, and $F_u'(0) = e^u$. The required parametric representation is [3, Theorem 3.3, §3.3, and Chapter 3, Exercise 8, p. 117]. This class and both asserted inequalities are invariant under $f(z) \mapsto -f(-z)$, so we may assume that $\operatorname{Re} a_2 \geq 0$.

Write $t = e^{-u}$, $g_t = tF_{-\log t}$, and choose the measurable driving function $\eta : (0,1] \to \mathbb{T}$ so that the reverse-time Löwner equation is

$$t\partial_t g_t(z) = g_t(z) - z\frac{1 - \eta(t)z}{1 + \eta(t)z} g_t'(z).$$

Thus $g_1 = f$ and $g_t \in S$ for $0 < t \leq 1$. All differential equations below hold for almost every t. The equations for the first two coefficients, integrated from $t = 0$ using their uniform boundedness, give

$$a_2 = 2\int_0^1 \eta(t)\,dt, \qquad a_3 = a_2^2 - 2\int_0^1 t\eta(t)^2\,dt.$$

In the rest of the proof, integrals with no displayed limits are over $[0,1]$. Set

$$y(t) = \frac{1 - \operatorname{Re}\eta(t)}{2}, \qquad v(t) = \operatorname{Im}\eta(t),$$
$$Y = \int y, \qquad \nu = \int v, \qquad E = \int ((1-t)y + ty^2).$$

Then $0 \leq y \leq 1$, $v^2 = 4y(1-y)$, and $Y \leq 1/2$. For $d = 3 - \operatorname{Re} a_3$, direct substitution yields

$$d = 16(E - Y^2) + 4\nu^2, \qquad E - Y^2 = \int (y - Y)^2 + \int (1-t)y(1-y). \tag{A.2}$$

Since $\operatorname{Im} a_2 = 2\nu$, the first inequality in (A.1) follows.

We will also use $E \le C_0 d$ for an absolute constant C_0. Put $\sigma = (\int (y-Y)^2)^{1/2}$ and $Q = \int (1-t)y(1-y)$. If $\sigma^2 \ge Y^2/64$, then $E = Y^2 + \sigma^2 + Q \le 65(\sigma^2 + Q) \le 65d/16$. Otherwise, the function $x \mapsto x(1-x)$ is 1-Lipschitz on $[0,1]$, and Cauchy–Schwarz gives

$$Q \ge \frac{Y(1-Y)}{2} - \frac{\sigma}{\sqrt{3}} \ge \frac{Y}{4} - \frac{Y}{8\sqrt{3}} \ge \frac{Y}{8}.$$

In this case $E \le Y \le d$. Thus $E \le 65d/16$ in either case. Notice also that

$$Y^2 \le E, \qquad \int y^2 \le E, \qquad \int v^2 \le 4Y, \qquad \int (1-t)v^2 = 4Q \le d/4.$$

To handle $a_4 - 2a_2$, let k^{-1} be the local inverse of k at zero and define

$$H(t,w) = t^{-1} g_t(k^{-1}(tw)).$$

These functions are defined for w near zero, and $H(1, k(z)) = f(z)$ for z near zero. Differentiating the reverse-time equation gives

$$\partial_t H = V_t(w)\partial_w H, \qquad V_t(w) = 2(\eta(t)-1)w^2 \frac{1-z}{1+\eta(t)z}, \qquad z = k^{-1}(tw).$$

Since $z = tw - 2t^2w^2 + O(w^3)$, write $V_t(w) = A_t w^2 + B_t w^3 + C_t w^4 + O(w^5)$, where

$$A_t = 2(\eta(t)-1), \qquad B_t = -2t(\eta(t)^2 - 1), \qquad C_t = 2t^2(\eta(t)^2-1)(\eta(t)+2).$$

For $H(t,w) = w + h_2(t)w^2 + h_3(t)w^3 + h_4(t)w^4 + O(w^5)$, comparison of coefficients gives

$$h_2' = A_t, \qquad h_3' = B_t + 2A_t h_2, \qquad h_4' = C_t + 2B_t h_2 + 3A_t h_3, \qquad h_j(0) = 0.$$

The initial values follow from $h_j(t) = O(t^{j-1})$ as $t \downarrow 0$: the first four coefficients of $g_t \in S$ are uniformly bounded. The identity $H(1, k(z)) = f(z)$ gives $a_4 - 2a_2 = 8h_2(1) + 6h_3(1) + h_4(1)$. Integrating the three differential equations therefore gives $a_4 - 2a_2 = I_1 + I_2 + I_3$, where

$$\begin{aligned} I_1 &= \int (8A_t + 6B_t + C_t)\, dt, \\ I_2 &= \int_{0<s<t<1} (12A_tA_s + 3A_tB_s + 2B_tA_s)\, ds\, dt, \\ I_3 &= \left(\int A_t\, dt\right)^3. \end{aligned}$$

For the first integral, direct expansion gives

$$\operatorname{Re}(8A_t + 6B_t + C_t) = -32y(t)\big(1 - t(1-y(t))\big)\big(1 - 2t(1-y(t))\big).$$

It follows that $|\operatorname{Re} I_1| \le 32E$.

For the second integral, use

$$\operatorname{Re} A_t = -4y(t), \quad \operatorname{Im} A_t = 2v(t), \quad \operatorname{Re} B_t = 16ty(t)(1-y(t)), \quad \operatorname{Im} B_t = -4tv(t) + 8ty(t)v(t).$$

The real part of its integrand equals $-(48 - 24s - 16t)v(s)v(t) + \mathcal{R}(s,t)$, with

$$|\mathcal{R}(s,t)| \le C\big(y(s)y(t) + (y(s)+y(t))|v(s)v(t)|\big).$$

Consequently,

$$\int_{s<t} |\mathcal{R}(s,t)|\, ds\, dt \le C\left(Y^2 + \int y|v| \int |v|\right) \le C\left(Y^2 + 4Y\sqrt{E}\right) \le CE.$$

The symmetric extension of $48-24s-16t$ is $8+24(1-s)+24(1-t)-8\min(1-s,1-t)$. Its integral against $v(s)v(t)$ over the square has modulus at most Cd: use $\nu^2 \le d/4$ from (A.2) and

$$\left|\int (1-t)v(t)\,dt\right|^2 \le \frac{1}{2}\int (1-t)v(t)^2\,dt \le d/8,$$

$$\left|\iint \min(1-s,1-t)v(s)v(t)\,ds\,dt\right| \le \left(\int \sqrt{1-t}\,|v(t)|\,dt\right)^2 \le \int (1-t)v(t)^2\,dt \le d/4.$$

Thus $|\operatorname{Re} I_2| \le C(E+d) \le Cd$.

Finally, $\int A_t\,dt = -4Y + 2i\nu$, so

$$|\operatorname{Re} I_3| = |-64Y^3 + 48Y\nu^2| \le 64Y^3 + 48Y\nu^2 \le C(E+d) \le Cd.$$

Combining the estimates proves the second inequality in (A.1). The result for all of S follows by density. □

Appendix B. Remarks on two recent coefficient theorems

B.1. **A general extremal statement.** Theorem 2 of [9] asserts that a weakly rotationally homogeneous polynomial coefficient functional of the stated form has modulus bounded by its value at the Koebe function. There is an ambiguity in the terminology: the displayed transformation rule in [9, §1.1], with its prescribed choice of opposite rotation angles, would require the functional itself to be invariant under normalized rotations. This excludes the Zalcman functional listed there as an example. We use the interpretation indicated by that example, namely invariance of the modulus under normalized rotations.

Under this interpretation, the general bound requires additional restrictions, as the following example shows.

Lemma B.1. *The functional*

$$\mathcal{J}(f) = \left(4a_3(f)^2 - 7a_5(f)\right)^3$$

is not maximized in modulus on S by a Koebe rotation.

Proof. The function $h(z) = z/(1-z^2)$ belongs to S, since

$$h(z) - h(w) = \frac{(z-w)(1+zw)}{(1-z^2)(1-w^2)}$$

does not vanish for distinct $z, w \in \mathbb{D}$. Its coefficients satisfy $a_3(h) = a_5(h) = 1$, whereas $a_3(k) = 3$ and $a_5(k) = 5$. Thus

$$|\mathcal{J}(h)| = 27 > 1 = |\mathcal{J}(k)|.$$

Under the normalized rotation $f(z) \mapsto \overline{\lambda} f(\lambda z)$, the functional is multiplied by λ^{12}, so every Koebe rotation has the same modulus 1. □

The example in Lemma B.1 also satisfies the separation condition in Theorem 1 of [9]. Indeed, on the larger set of pre- and post-rotations $k_{\tau,\theta}(z) = e^{-i\theta}k(e^{i\tau}z)$, with $\tau, \theta \in \mathbb{R}$,

$$|4a_3(k_{\tau,\theta})^2 - 7a_5(k_{\tau,\theta})| = |36e^{i(\tau-\theta)} - 35| \ge 1.$$

Continuity of the coefficient functional and compactness of this rotation set in the topology of locally uniform convergence give a neighborhood disjoint from its zero set. Moreover, all monomials in $\mathcal{J}$ involve only a_3, a_5, and their total algebraic degrees are between 3 and 6. Thus the degree restriction in that theorem does not remove the example. This is why the general extremal statement is not used in the present proof.

B.2. **The sharpness assertion in the covering lemma.** Lemma 11(a) of [10, p. 389] makes a covering assertion for a family of normalized univalent functions whose Schwarzians range over a connected complex submanifold of universal Teichmüller space. If $A > 0$ is the maximum of the modulus of the second coefficient in that family, it asserts that the common covered disk has sharp radius $1/(2A)$, and that a function omits a point on the circle of that radius if and only if its second coefficient has modulus A. The following example satisfies the stated hypotheses with a one-dimensional complex submanifold, but the asserted sharpness and equality characterization both fail.

Proposition B.2. *There is a family satisfying the hypotheses of* [10, Lemma 11(a)]*, with* $A = 5/4$*, such that every member maps* $\mathbb{D}$ *onto a domain containing the disk of radius* $1/(\sqrt{5}+1/8) > 1/(2A)$*. The maximum* $|a_2| = A$ *is attained in this family.*

Proof. Write $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, $\mathbb{D}^* = \{z \in \mathbb{C} : |z| > 1\} \cup \{\infty\}$, and $G = \{\tau \in \mathbb{C} : |\tau| < 1/64\}$. Set

$$F_\tau(\zeta) = \zeta + \frac{1}{4\zeta^2} + \frac{\tau}{\zeta}(1 - \zeta^{-3})^2, \qquad \zeta \in \mathbb{D}^*.$$

These functions are univalent and admit quasiconformal extensions. For completeness, an extension to the closed unit disk is

$$W_\tau(z) = z + \frac{1}{4}\overline{z}^{\,2} + \tau(\overline{z} - 2\overline{z}^{\,4} + \overline{z}^{\,7}).$$

It agrees with F_τ on the unit circle. On each of the closed unit disk and its exterior, the perturbation of the identity has Lipschitz constant at most $1/2 + 16|\tau| < 3/4$. For the exterior, this follows from $|\zeta^{-j} - \eta^{-j}| \le j|\zeta - \eta|$ when $|\zeta|, |\eta| \ge 1$. On the disk it follows by differentiating the polynomial in $\overline{z}$. Subdividing a line segment at its intersections with the unit circle shows that the glued perturbation has the same global Lipschitz bound. Thus the glued map is a bi-Lipschitz homeomorphism of the plane (surjectivity follows from the contraction principle), and extends to a homeomorphism of the sphere fixing infinity. On the disk the complex dilatation $\partial_{\overline{z}} W_\tau / \partial_z W_\tau$ has modulus at most $1/2 + 16|\tau| < 3/4$, and outside the disk it vanishes. The extension is consequently quasiconformal. Write $K_\tau = W_\tau(\overline{\mathbb{D}}) = \widehat{\mathbb{C}} \setminus F_\tau(\mathbb{D}^*)$.

Put $A = 5/4$. For $\zeta = e^{i\theta}$ and $s = \sin(3\theta/2)$, we have

$$|F_0(\zeta)|^2 = A^2 - s^2, \qquad A - |F_0(\zeta)| = \frac{s^2}{A + |F_0(\zeta)|} \ge \frac{2}{5}s^2.$$

Since $|F_\tau(\zeta) - F_0(\zeta)| = 4|\tau|s^2$, it follows that $|F_\tau(\zeta)| \le A$, with equality exactly when $\zeta^3 = 1$. At these three points $F_\tau(\zeta) = A\zeta$. The maximum principle applied to the subharmonic function $|W_\tau|^2$ therefore gives $K_\tau \subset \{|z| \le A\}$, and $A \in K_\tau$.

For every $c \in K_\tau$, define

$$f_{\tau,c}(z) = \frac{1}{F_\tau(1/z) - c} = \frac{z}{1 - cz + \tau z^2 + \frac{1}{4}z^3 - 2\tau z^5 + \tau z^8}.$$

The value at zero is defined by continuity. Each $f_{\tau,c}$ is univalent in $\mathbb{D}$, satisfies $f_{\tau,c}(0) = 0$ and $f'_{\tau,c}(0) = 1$, and has second coefficient c. It also admits a quasiconformal extension as a map of the sphere. Let

$$\chi(\tau) = S_{f_{\tau,0}}, \qquad S_f = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2.$$

We use the Schwarzian model of universal Teichmüller space: its points are the Schwarzians of univalent holomorphic functions on $\mathbb{D}$ admitting quasiconformal extensions to $\widehat{\mathbb{C}}$. Thus χ takes its values in this space. It is holomorphic with values in the Banach space of holomorphic functions on $\mathbb{D}$ having finite norm $\|\varphi\| = \sup_{z\in\mathbb{D}}(1 - |z|^2)^2|\varphi(z)|$. Indeed, write $f_{\tau,0}(z) = z/Q_\tau(z)$. On $\overline{\mathbb{D}}$, the displayed denominator satisfies $|Q_\tau| \ge 3/4 - 4|\tau| > 11/16$, and the numerator of $f'_{\tau,0}$ satisfies

$|Q_\tau - zQ'_\tau| \geq 1/2 - 16|\tau| > 1/4$. These bounds give holomorphic dependence of the Schwarzian in this norm, locally uniformly in τ. Furthermore, $\chi(\tau)(0) = -6\tau$. Evaluation at zero is a continuous linear functional on this Banach space. Thus $\chi(G)$ is a holomorphic graph over this coordinate, and is a connected, embedded, one-dimensional complex submanifold.

Two locally univalent functions with the same Schwarzian differ by a Möbius transformation. The normalization at zero therefore shows that all holomorphic univalent solutions of $S_f \in \chi(G)$, $f(0) = 0$, and $f'(0) = 1$ are precisely the functions $f_{\tau,c}$ with $\tau \in G$ and $c \in K_\tau$. Here the restriction $c \in K_\tau$ is exactly the condition that the Möbius transformation introduce no pole in $\mathbb{D}$. Consequently the maximum modulus of the second coefficient is A; it is attained, for example, by $f_{0,A}$.

There is, however, a larger disk covered by every member of this family. To see this directly, let $z, w \in \overline{\mathbb{D}}$ and put $p = |z + w|$, $q = |z - w|$. Since $p^2 + q^2 \leq 4$ and $q \leq 2$,

$$|W_0(z) - W_0(w)| \leq q + \frac{1}{4}pq \leq q + \frac{1}{2}p \leq \sqrt{5}.$$

Hence $\operatorname{diam} K_0 \leq \sqrt{5}$, where $\operatorname{diam} K = \sup_{z,w\in K} |z - w|$. Also $\sup_{|z|\leq 1} |W_\tau(z) - W_0(z)| \leq 4|\tau|$. It follows that

$$\operatorname{diam} K_\tau \leq \sqrt{5} + 8|\tau| < D_0 := \sqrt{5} + \frac{1}{8} < \frac{5}{2} = 2A.$$

For $c \in K_\tau$ and $0 < |w| < 1/D_0$, the point $c + 1/w$ lies outside K_τ, and hence belongs to $F_\tau(\mathbb{D}^*)$. Thus $w \in f_{\tau,c}(\mathbb{D})$; zero also belongs to this image. Every member of the family therefore covers the disk of radius $1/D_0 > 1/(2A)$. In particular, $f_{0,A}$ covers the entire circle of radius $1/(2A)$ despite maximizing the modulus of its second coefficient. This contradicts the asserted sharpness and equality characterization in [10, Lemma 11(a)]. □

This example concerns the auxiliary covering lemma as stated. It does not contradict the Zalcman inequality, nor does it exclude a separate argument for the special families used later in that paper.

B.3. **The Schwarzian image of an affine-extension family.** A separate issue occurs in [10, p. 389, Lemma 12]. The family introduced in [10, p. 377, Section 1.3] is

$$F_{b_0,b_1;t}(z) = z + b_0 t + b_1 t^2 z^{-1}, \qquad |z| > 1,$$

with $|b_0| \leq 2$, $|b_1| \leq 1$, and $|t| \leq 1$. Its image under the Schwarzian derivative is asserted there to be three-dimensional. The following calculation shows that the stated parameter count is incorrect.

Proposition B.3. *For $c = b_1 t^2$ and $|c| < 1$, the Schwarzian derivative $S_F = F'''/F' - \frac{3}{2}(F''/F')^2$ of $F = F_{b_0,b_1;t}$ is*

$$S_F(z) = -\frac{6c}{(z^2 - c)^2}.$$

Consequently, this Schwarzian image depends on one complex parameter.

Proof. We have $F' = 1 - cz^{-2}$, $F'' = 2cz^{-3}$, and $F''' = -6cz^{-4}$. Substitution gives the displayed formula. It is independent of b_0 and depends on b_1, t only through c. Distinct values of c give distinct Schwarzians, since the coefficient of z^{-4} is $-6c$. □

Thus the three-parameter description does not give the three-dimensional Schwarzian image claimed in the cited lemma. In particular, holomorphic dependence on those parameters does not justify the assertion in its proof that this image is open in the ambient Schwarzian space.

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