Springer coefficients, boundary moments, polynomial zeros, and regular stars

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Abstract

For centered exterior univalent maps we prove the Springer inverse-coefficient bound in every order indexed by $k \geq 751$, with equality precisely for unit slit maps. The energy estimate uses an explicit multiplier that is linear up to one cutoff and constant thereafter. We determine the sharp radial boundary-moment maxima and give quantitative deficits for real exponents $4 \leq p \leq 8$. In the same range, the normalized derivative-zero moments of polynomials with connected lemniscates have a strict upper bound equal to their supremum across degrees. The regular m-arm star maximizes the polynomial factor constant among m-fold rotationally symmetric plane continua precisely when m is even. We give a quantitative even-order bound and classify equality, and construct a strict counterexample in every odd order $m \geq 3$.

1 Introduction

The normalized exterior map links coefficient problems in univalent function theory with equilibrium measures of plane continua. Springer [1] proposed that the modulus of the coefficient of order $2k-1$ of the inverse at infinity is at most Cat_{k-1}. The area theorem gives the first order, and Springer proved the bound for B_3. Earlier low-order bounds are due to Kubota and Schober [2, 3]. Schober's corollary [4, p. 413] gives the sharp bounds through B_{15}, including the slit equality cases. With the sign convention below, Theorem 1 treats every odd index from 1501 onward. Its proof uses signed area and energy identities for the actual map, followed by a comparison near the two slit endpoints. In the energy estimate two coefficients of a single flux are chosen proportional to the same capped linear function. This reduces the multiplier comparison to a quadratic inequality.

Baernstein, Laugesen, and Pritsker proved a sharp comparison for one-dimensional projections of equilibrium measure [5, Theorem 2]. Section 6 of their paper proves the second radial moment bound with its rotated-segment extremals and formulates a broader radial comparison as Conjecture 2. Its hypotheses are that the continuum has capacity one, contains zero, and has conformal centroid zero. Its tests are C^1 functions ϕ for which both ϕ and ϕ' are convex, applied to $\log|z|$. The test $\phi(x) = e^{px}$ gives the conjectured comparison for $p > 0$ under those hypotheses. Corollary 6.3 proves the logarithmic comparison for arbitrary convex tests when the capacity-one continuum contains zero and is symmetric about it. We prove the bounds for $4 \leq p \leq 8$ for every centered exterior map, with a quantitative deficit. The boundary argument extracts a second-moment deficit from the area theorem at the second moment. A finite weighted area inequality and low-order Grunsky estimates then give the fourth, sixth and eighth moments. Interpolation retains a uniform deficit throughout $[4, 8]$. A comparison with the actual exterior map of a polynomial lemniscate transfers this bound to the zero moments. The comparison of the pth moments of polynomial zeros with

those of their derivatives is classical for $p \geq 1$ [10, Theorem 4]. Here the strict bound comes from excluding the slit as the exterior of a polynomial sublevel set. Scaled Chebyshev polynomials show that the bound is the supremum even after any fixed number of derivatives.

The polynomial factor constant is governed by the equilibrium average of the logarithmic farthest-point distance [7, Theorem 1.1]. Pritsker and Ruscheweyh proved a regular-star bound under inclusions and conjectured that the inclusions could be dropped [7, Theorem 2.2 and the discussion after Theorem 2.3]. For connected compact sets we obtain the comparison in every even order and a counterexample in every odd order $m \geq 3$. Their separate question about regular polygons concerns convex sets. The even star comparison uses the power map, the symmetric logarithmic moment comparison, and the second-moment deficit. In odd order, an explicit deformation gives a strictly increasing integral lower bound for the factor constant. The appendices give the exterior normalization and the analytic comparisons used in the verification.

2 Large-order inverse coefficients

Let $F(z) = z + b_1 z^{-1} + b_2 z^{-2} + \cdots$ be univalent on $\{|z| > 1\}$, with the constant Laurent coefficient zero. Write its inverse at infinity with the sign convention

$$G(w) = w - \sum_{j \geq 1} B_j w^{-j}.$$

Set $\mathrm{Cat}_q = (q+1)^{-1}\binom{2q}{q}$. A unit slit map means $F_a(z) = z + a/z$, $|a| = 1$; equality of maps below is on the whole open exterior disk.

Theorem 1 (Large-order Springer statement)**.** *For every centered exterior univalent F and every integer $k \geq 751$,*

$$|B_{2k-1}(F)| \leq \mathrm{Cat}_{k-1}\,.$$

For each fixed $k \geq 751$, equality holds if and only if $F = F_a$ on $\{|z| > 1\}$ for some $a \in \mathbb{C}$ with $|a| = 1$.

We use $n = 2k-1 \geq 1501$, $m = n-1 \geq 1500$. An endpoint deficit is denoted $d \geq 0$, with $t = nd$, $\Delta = md$, and $L = \log(2/d) = \log(2m/\Delta)$. The zero-deficit case must be handled by geometric slit rigidity before the positive-deficit parameters are introduced.

2.1 Actual-map analytic bridge

Put $g(u) = \sum_{j \geq 1} b_j u^j$, $H(z) = z + g(\bar{z})$ on the unit disk, and $d\sigma = dA/\pi$. Let K be the actual omitted continuum. The following identities fix the sign and the normalization used by every subsequent saving calculation. Put $\mathcal{M}_m = \int_{\mathbb{D}} |H|^m \, d\sigma$ and $\mathcal{A}_m = \int_K |w|^m \, d\sigma$.

$$B_{2k-1} = \int_{\mathbb{D}} H^m g'(\bar{z})\, d\sigma, \qquad \mathcal{M}_m - \mathcal{A}_m = \int_{\mathbb{D}} |H|^m |g'(\bar{z})|^2 \, d\sigma. \tag{1}$$

For $0 < s < 1$ put $F_s(z) = sF(z/s)$. Its unit-circle image is a smooth Jordan curve, with positive orientation around the bounded complement. The filling $H_s(z) = z + sg(s\bar{z})$ has this boundary trace and Jacobian $1 - |s^2 g'(s\bar{z})|^2$. On the boundary the residue formula for the inverse gives

$$B_n(F_s) = \frac{1}{2\pi i n} \oint_{|z|=1} H_s^n \, dz.$$

Cauchy–Green gives the first formula in (1), since $\partial_{\bar z} H_s = s^2 g'(s\bar z)$. For the second, the form $|w|^m(x\,dy - y\,dx)/(m+2)$ has exterior derivative $|w|^m\,dx\,dy$. Apply Stokes' theorem both to its pullback by H_s and to the bounded domain with the same oriented boundary.

To take $s \uparrow 1$, set $E_s = \mathbb{C} \setminus F(\{|z| > 1/s\})$. These sets decrease to K, while the set omitted by F_s is sE_s. For any fixed $s_0 > 0$ with $s_0 < 1$, E_{s_0} is bounded. Thus

$$\int_{K(F_s)} |w|^m\,d\sigma(w) = s^{m+2}\int_{E_s} |w|^m\,d\sigma(w) \longrightarrow \int_K |w|^m\,d\sigma(w).$$

The area theorem gives $\sum_{j\ge1} j|b_j|^2 \le 1$, so the orthogonality of disk monomials yields $s^2 g'(s\bar z) \to g'(\bar z)$ in $L^2(d\sigma)$. Their squared moduli converge in L^1 by Cauchy–Schwarz. Also $H_s \to H$ pointwise and $|H_s| \le 2$: the normalized exterior omission bound gives $|w| \le 2$ on $K(F_s)$, and H_s is the Poisson extension of its boundary values. The omission bound itself follows by applying the area theorem to $\sqrt{F_s(z^2) - w}$, whose first negative coefficient is $-w/2$. These uniform bounds and dominated convergence pass the disk integrals to the limit. Finally $G_s(w) = sG(w/s)$ gives $B_n(F_s) = s^{n+1}B_n(F)$.

Choose $|\beta| = 1$ with $\beta^2 B_{2k-1} = |B_{2k-1}|$ (take $\beta = 1$ if $B_{2k-1} = 0$) and set $U = \operatorname{Im}(\beta H^k)$. Expanding the gradient square and using (1) yields the exact identity

$$\int_{\mathbb{D}} |H|^m\,d\sigma - |B_n| = \frac{1}{2}\int_K |w|^m\,d\sigma + \frac{1}{2k^2}\int_{\mathbb{D}} |\nabla U|^2\,d\sigma. \tag{2}$$

Lemma 8 proves $k^2\int U^2/|H|^2\,d\sigma \le (7/3)\int|\nabla U|^2\,d\sigma$ and gives the signed integrand used below. Its proof uses a drift identity, one combined flux, and direct cutoff integration for the actual filling. The choice of multiplier and its scalar estimate are given in Section 2.5. With $C = 7/3$ set

$$S(w) = \left(\frac{|w|}{2}\right)^m \left(1 - \frac{\sin^2(\arg(\beta w^k))}{2C}\right), \quad S(0) = 0.$$

Discarding the nonnegative omitted-area term from (2) then gives the signed comparison

$$2^{-m}|B_n(F)| \le \int_{\mathbb{D}} S(H)\,d\sigma. \tag{3}$$

The one-sided reference moment is

$$M_m := \int_{\mathbb{D}} (\operatorname{Re} z)_+^m\,d\sigma = \frac{\binom{m}{m/2}}{2^m(m+2)}, \qquad 2^{m+1}M_m = \operatorname{Cat}_{k-1}. \tag{4}$$

Consequently a candidate with $|B_n| \ge \operatorname{Cat}_{k-1}$ requires $\int S(H)\,d\sigma \ge 2M_m$. Every strict comparison below uses this dimensionless normalization.

The radial trace F^* of Lemma 10 takes values in K almost everywhere. The trace of H is the same, since $H(se^{it}) - F(s^{-1}e^{it}) = (s - s^{-1})e^{it}$ for $0 < s < 1$. The bounded harmonic filling is the Poisson extension of this trace. The trace pushes normalized angular measure to the equilibrium measure of K, or harmonic measure at infinity. The normalization gives K capacity one and conformal centroid zero. Thus the projection theorem [5, Theorem 2] puts $u(t) = \operatorname{Re}(\bar v F^*(e^{it}))$ below $2\cos t$ in convex order for each fixed $|v| = 1$.

Poisson comparison for fixed projections. Write $d\mu = dt/(2\pi)$ and take convolution with respect to μ. If a bounded boundary function u is below an even, decreasing profile q in convex order, then

$$P_r * u \preceq_{\mathrm{cx}} P_r * q, \qquad 0 < r < 1. \tag{5}$$

For an angular set E let E^* be the centered arc of mass $\mu(E)$, and let u^* be the even decreasing rearrangement of u. Circular rearrangement [6] bounds $\int_E P_r * u \, d\mu$ by $\int u^*(P_r * \mathbf{1}_{E^*}) \, d\mu$. A common additive constant handles signed functions. The weight $P_r * \mathbf{1}_{E^*}$ is even and decreasing on $[0, \pi]$: for $E^* = [-b, b]$ its derivative has the sign of $P_r(t+b) - P_r(t-b)$, which is nonpositive for $0 \le t, b \le \pi$. Its level sets are centered arcs. For any arc mass s, the integral of u^* on that arc is at most that of q. Indeed the upper integral of u over sets of mass s is the infimum of $as + \int (u-a)_+ \, d\mu$ over $a \in \mathbb{R}$, and the same holds for q. Integrating over the levels of the weight gives $\int_E P_r * u \, d\mu \le \int_{E^*} P_r * q \, d\mu$. The means are equal. Taking the supremum of $\int_E (P_r * u - a) \, d\mu$ over E compares the positive-part tests at all a. These tests and equality of the means give convex order by piecewise linear approximation on the bounded range.

With $q(t) = 2\cos t$, its Poisson extension is $2r\cos t$. Thus

$$\mathbb{E}_t \phi(\mathrm{Re}(\bar{v} H(re^{it}))) \le \mathbb{E}_t \phi(2r\cos t) \quad (0 < r < 1) \tag{6}$$

for every convex real ϕ. The positive-part test at $2r$ and continuity in the angle give $\mathrm{Re}(\bar{v} H(z)) \le 2|z|$. Using all unit directions yields $|H(z)| \le 2|z|$. The Poisson formula also gives $H(\mathbb{D}) \subset \operatorname{conv} K$. The two-point geometry is stronger than the one-point bound:

$$4|A - P| + |A + P|^2 \le 16 \qquad (A, P \in \operatorname{conv} K). \tag{7}$$

For $A, P \in K$, the normalized odd maps $h_A(z) = \sqrt{F(z^2) - A}$ and h_P are exterior univalent. When $P = A$, (7) is the omission bound $|P| \le 2$. For $P \ne A$ put $q = (P-A)/4$ and $V = (h_A + h_P)/2$. The identity $h_A^2 - h_P^2 = 4q$ makes V nonvanishing. Moreover $h_A = V + q/V$ and $h_P = V - q/V$. Equal V-values force equal h_A-values, so V too is exterior univalent. The area theorem and the parallelogram identity give $|q|^2 \mathcal{E}(1/V) \le A_V$, where $\mathcal{E}(f) = \sum_{j \ge 1} j |[z^{-j}] f|^2$ and $A_V = \mathrm{area}(K_V)/\pi = 1 - \mathcal{E}(V)$. The first coefficient of $1/V$ is one, so $A_V \ge |q|^2 > 0$. By the Laurent area integral and change of variables under V,

$$\mathcal{E}(1/V) = \frac{1}{\pi} \int_{|z|>1} |(1/V)'(z)|^2 \, dA(z) = \frac{1}{\pi} \int_{\mathbb{C} \setminus K_V} |w|^{-4} \, dA(w).$$

Let D_V be the centered disk of area πA_V. The sets $D_V \setminus K_V$ and $K_V \setminus D_V$ have equal area, and the weight on the first is at least A_V^{-2} and on the second at most A_V^{-2}. The last integral is therefore at least the normalized integral over $\mathbb{C} \setminus D_V$, which is $1/A_V$. The first negative coefficient of V gives $A_V \le 1 - |A + P|^2/16$. These imply (7). Joint convexity extends it to the whole ordinary convex hull. If $P \in K$ has $|P| = 2$, equality in the area theorem forces $F(z) = z + P^2/(4z)$, establishing the zero-deficit slit case.

For a selected endpoint $P = R\xi$, $|\xi| = 1$ and $R = 2 - d$, put $F_\xi(z) = \bar{\xi} F(\xi z)$ and $H_\xi(z) = \bar{\xi} H(\xi z)$. The rotation sends P to R and K to $\bar{\xi} K$. The rotated real part of w is $\mathrm{Re}(\bar{\xi} w)$. Rotating the argument and value preserves univalence and $F_\xi(z) = z + O(z^{-1})$. Its filling is $H_\xi(z) = z + \bar{\xi} g(\bar{\xi} \bar{z})$. Let $A = R/2$, $E = 1 - A^2$ and, for $r > 1$, set

$$p_r = r - A/r, \quad S_r = \tfrac{1}{2} \log \frac{r^2 + 1}{r^2 - 1} - r^{-2}, \quad W = (p_r + \sqrt{E S_r})^2.$$

The odd univalent root $\sqrt{F_\xi(z^2) - R}$ has first negative coefficient $-A$ and a tail with area budget E. For $w \in K$, each root of $\bar{\xi} w - R$ is omitted by this map. It therefore lies in the closed bounded

region enclosed by the image of $|z| = r$. On that image the real part of $z - A/z$ has absolute value at most p_r, and Cauchy–Schwarz bounds the tail by $\sqrt{ES_r}$. The same bound $\sqrt{W}$ on the absolute real part holds in the bounded region. Writing $x + iy = \bar{\xi} w$ and squaring the root gives

$$y^2 \le 4W(W + R - x), \qquad |w|^2 \le (x - 2W)^2 + 4WR. \tag{8}$$

The first sublevel region is convex, which justifies its extension from K to $\operatorname{conv} K$.

For a projected boundary function u with $u \le 2 - d$, $0 < d < 2$, let $h \in (0, \pi)$ satisfy $2 \sin h/h = 2 - d$. Let u_d equal $2 \cos t$ outside $[-h, h]$ and its average $2 - d$ inside. For any angular set of mass s, the u-integral is at most both $(2 - d)s$ and the centered-arc integral of $2 \cos t$. Their minimum is the centered-arc integral of u_d. For $s \le h/\pi$ the arc lies inside the plateau. For $s \ge h/\pi$ it contains the entire plateau, whose integral is unchanged. Both boundary means are zero. The upper partial-mean criterion gives $u \preceq_{\mathrm{cx}} u_d$, so (5) gives the capped Poisson comparison. The direction stays fixed on each source circle.

2.2 The scalar cutoff

The following precise input is derived from an *actual* contender in the boundary localization below:

$$\begin{gathered} |B_n(F)| \ge \mathrm{Cat}_{k-1}, \quad d > 0 \quad \Longrightarrow \quad 0 < t < 1, \\ J_n(t) := 0.067 n t^{5/2} e^{-0.61t} - 0.285 z - 0.275 z^2 < 0.58, \\ z = 0.34 t \log(2n/t). \end{gathered} \tag{9}$$

The implication holds for *each* positive endpoint deficit, with the weak candidate inequality shown. The boundary argument obtains it from the joint estimate $J_n(t_1) + J_n(t_2) < 0.52$ and the following scalar floor.

The scalar floor. For $n \ge 1501$ and $0 < t < 1$, put $L = \log(2n/t)$ and $z = 0.34tL$. The elementary envelope $(c + u)^3 \le c^3 e^{u/2}$ for $c \ge 6$, $u \ge 0$ follows from $\log(1 + u/c) \le u/c$. Apply it first with $c = \log(2n)$, $u = -\log t$, and then with $c = \log 3002$, $u = \log(n/1501)$. The resulting base check, using $\log 3002 < 8.1$, gives

$$70 z^3 \le n t^2 \sqrt{t}.$$

Since $e^{-0.61t} > 8/15$, the positive term of J_n strictly exceeds $(5/2)z^3$. Finally

$$\frac{5}{2} z^3 - \frac{11}{40} z^2 - \frac{57}{200} z + \frac{9}{160} = (z - \tfrac{1}{4})^2 (\tfrac{5}{2} z + \tfrac{9}{10}) + \tfrac{3}{40} z^2 + \tfrac{7}{800} z \ge 0.$$

Thus $J_n(t) > -9/160$. The joint bound makes each $J_n(t_i) < 0.52 + 9/160 < 0.58$, including a candidate at equality.

2.3 Two actual boundary ends and the scalar necessity

This subsection proves the boundary input to the scalar cutoff by explicit inequalities. Assume F is not a unit slit, $k \ge 751$, $n = 2k - 1$, and $|B_n(F)| \ge \mathrm{Cat}_{k-1}$. Put $I_n = \mathbb{E}|2 \cos t|^n$. The inverse residue formula on an exterior circle, followed by bounded radial convergence, gives

$$n B_n = \mathbb{E}[e^{it} F^*(e^{it})^n], \qquad \mathbb{E}|F^*|^n \ge n \, \mathrm{Cat}_{k-1} = I_{n+1}/2 \ge (1 - 0.502/n) I_n. \tag{10}$$

For the last estimate, under the $|\cos t|^n$ weight the first two moments of $\sin^2 t$ are $1/(n+2)$ and $3/((n+2)(n+4))$. Apply $1-\sqrt{1-u} \le (u+u^2)/2$ and check $(n+7)/(2(n+2)(n+4)) \le 0.502/n$ for $n \ge 1501$. Wallis' bound (or integration over $|t| \le 3/\sqrt{n}$) gives

$$0.785\frac{2^n}{\sqrt{n}} \le I_n \le \sqrt{2/\pi}\frac{2^n}{\sqrt{n}}. \tag{11}$$

Choose $P_1 \in K$ of maximal modulus, say $R_1 = |P_1| = 2 - d_1$ and $\xi_1 = P_1/R_1$. The following short argument supplies both a stronger first radius and an actual second endpoint before any fine two-sided moment comparison.

Lemma 2 (An explicit opposite endpoint)**.** *For every candidate above, $R_1 > 359/180$. There is a point $w \in \operatorname{conv} K$ with $\operatorname{Re}(\bar{\xi}_1 w) < -8/5$. Consequently the maximum-modulus point P_2 in the opposite closed half of $\operatorname{conv} K$ satisfies $|P_2| > 8/5$ and belongs to K. The ordinary-hull strip applies to both P_1 and P_2; this does not yet assert the later $nd_i < 1$ deficit bounds.*

Proof. The actual residue and omitted-radius bound gives $n\,\mathrm{Cat}_{k-1} \le R_1^n$. Put $A = R_1/2$ and $c_k = \binom{2k}{k}/4^k = n\,\mathrm{Cat}_{k-1}/2^n$. Here $c_1 = 1/2$ and $c_{k+1}/c_k = (2k+1)/(2k+2)$; squaring the recurrence and using $(2k+1)^2 \ge 4k(k+1)$ proves $c_k \ge 1/(2\sqrt{k})$. For $a = 359/360$, the binomial theorem gives

$$a^{-1501} = (1+1/359)^{1501} \ge \sum_{j=0}^{6} \binom{1501}{j} 359^{-j} > 55 > 2\sqrt{751}.$$

Thus $a^{1501} < c_{751}$. The ratio $(2k+1)/(2k+2) \ge 1503/1504 > a^2$ for $k \ge 751$ propagates $a^{2k-1} < c_k$ by induction. Since $A^n \ge c_k$, we obtain $A > a$, and hence $1 - A^2 < 1/180$.

Rotate by ξ_1, so $P_1 = R_1 > 0$ on the real axis. The actual odd square-root map h obeys $h(z)^2 = F_{\xi_1}(z^2) - R_1$, $b_1(h) = -A$, and its remaining odd Laurent coefficients have weighted square budget at most $1 - A^2$. Set $q = 25/24$, $r = \sqrt{q}$, $M = r + A/r$, and $T = h(ir) - iM$. The odd-mode Cauchy–Schwarz bound at this one point is

$$|T|^2 \le (1-A^2)\left\{\frac{1}{2}\log\frac{1+24/25}{1-24/25} - \frac{24}{25}\right\} = (1-A^2)(\log 7 - 24/25) < \frac{1}{180}.$$

The last logarithmic comparison follows from $e^{49/25} \ge \sum_{j=0}^{6}(49/25)^j/j! > 7$. Thus $|T| < 3/40$ and $M \le 49/24$. The identities $(ir)^2 = -q$ and $F_{\xi_1}(-q) = R_1 + (iM+T)^2$ give, directly from the definition of the actual harmonic filling,

$$H_{\xi_1}(-1/q) = F_{\xi_1}(-q) + q - 1/q = -\frac{1+A^2}{q} + 2iMT + T^2. \tag{12}$$

The exact rational bounds $(1+a^2)24/25 > 153/80$ and $2(49/24)(3/40) + (3/40)^2 < 5/16$ imply $\operatorname{Re} H_{\xi_1}(-1/q) < -8/5$. As $1/q < 1$, the actual filling lies in the ordinary convex hull; rotating back gives the stated w. The selected opposite-half maximizer has $|P_2| \ge |w| > 8/5$.

The root tail bound holds on the entire source circle $|z| = r$. The squared strip width therefore satisfies

$$W \le (r - A/r + 3/40)^2 < 1/64, \qquad r - A/r = (q-A)/r < 2/45.$$

At the cutting line $x = \operatorname{Re}(\bar{\xi}_1 v) = 0$ the full-hull strip gives $|v|^2 \le 4W(W + R_1) < 1/4$. Since $|P_2| > 8/5$, it is inside the negative half-plane, where the strict convexity of $|\cdot|^2$ makes it an extreme point of $\operatorname{conv} K$. Compactness of K puts that extreme point in K. Applying the actual omitted-point strip to each P_i proves the last assertion. □

Lemma 3 (A Gaussian single-end exclusion)**.** *For the same selected P_2 in Lemma 2, $R_2 = |P_2| > 39/20$.*

Proof. Put $c = 39/20$, $q = 39/40$ and $e = 2 - |w|$. The two-point inequality (7) holds for w, P_1 throughout the ordinary convex hull. If $|w| \geq c$, then $0 \leq e \leq 1/20$ and $u = |w - P_1|$ satisfies $u(4-u) \leq 16e$. The elementary quadratic alternative is $u \leq 17e/4$ or $4-u \leq 17e/4$. On the forward half-plane $x = \mathrm{Re}(\bar{\xi}_1 w) \geq 0$, the latter alternative is impossible because $u^2 \leq |w|^2 + |P_1|^2 \leq 8$. The cosine rule therefore gives

$$2 - x \leq e + 5e^2 \leq \frac{5}{4}e, \qquad |w| \leq \frac{2}{5} + \frac{4}{5}x.$$

Suppose $R_2 \leq c$. On the opposite half-plane the selected maximizer bounds $|w|$ by c; all points of radius below c have the same bound. Consequently the single globally convex test

$$\psi(x) = \max\left\{c, \frac{2}{5} + \frac{4}{5}x\right\}^n$$

majorizes $|w|^n$ on the whole convex hull. By (10) and the fact that $F^*(e^{it}) \in K$ for almost every t, we obtain

$$\begin{aligned} n|B_n(F)| &\leq \mathbb{E}_t |F^*(e^{it})|^n \\ &\leq \mathbb{E}_t \psi\big(\mathrm{Re}(\bar{\xi}_1 F^*(e^{it}))\big) \\ &\leq \mathbb{E}_t \psi(2\cos t). \end{aligned} \tag{13}$$

The last inequality is the boundary projection comparison in the fixed direction ξ_1, applied to the convex function ψ. For $|\theta| \leq \pi$, the standard cosine parabola bound gives $\cos\theta \leq 1 - 2\theta^2/\pi^2$. Using $\max(0, 1-v)^n \leq e^{-nv}$, the model integrand is bounded by

$$\psi(2\cos\theta) \leq c^n + 2^n \exp\left(-\frac{8n}{5\pi^2}\theta^2\right).$$

Its normalized circle average is at most

$$c^n + \frac{2^n}{2\pi}\sqrt{\frac{\pi}{8n/(5\pi^2)}}.$$

Now $4n^2q^n \leq 1$ for all $n \geq 1501$: at 1501 use $q^{40} < 1/2$ and $4(1501)^2 < 2^{37}$, and propagate by $(1 + 1/n)^2 q < 1$. Thus $q^n \leq 1/(2000\sqrt{n})$. Also $\pi < 3.15$ gives $\sqrt{5\pi/(8n)}/2 < 141/(200\sqrt{n})$. The preceding cost, divided by 2^n, is therefore less than $1411/(2000\sqrt{n})$. But $1411^2 k < 1000^2(2k-1)$ for $k \geq 751$, so this is below $1/(2\sqrt{k}) \leq c_k = n\,\mathrm{Cat}_{k-1}/2^n$. This contradicts the candidate condition in (13). The selected P_2 must have $R_2 > c$. □

For the two-sided deficit argument, use the sharper $\delta = 1/20$ and $p_0 = 39/20$. If $|w| = 2 - e \geq p_0$ and $|P_i| \geq |w|$, the two-point inequality and the parallelogram law give

$$u(4-u) \leq 16e, \qquad u = |w - P_i|.$$

Since $e \leq 1/20$, either $u \leq 17e/4$ or $u \geq 4 - 17e/4$. In the forward half-plane of P_1 the second alternative contradicts $u^2 \leq 8$. With $x = \mathrm{Re}(\bar{\xi}_1 w)$, the cosine rule then gives

$$2 - x \leq e + 5e^2. \tag{14}$$

The opposite selected endpoint has $R_2 > p_0$ by Lemma 3. For a high-radius point w in its assigned half-plane, the far alternatives relative to P_1 give $|w + P_1|^2 \le 17e$ and $|P_2 + P_1|^2 \le 17d_2 \le 17e$. Hence $|w - P_2| \le 2\sqrt{17e} < 4 - 17/80$. The far alternative relative to P_2 is impossible, so (14) holds in direction ξ_2 as well. On each assigned half-plane, R_i caps the radius of $\operatorname{conv} K$; the cutting line itself has radius below 1.9 by the two-point inequality. Thus the same convex test applies to both assigned sides.

Define, for $x \le 2$,

$$h(x) = \max\left\{p_0,\, 2 - \frac{\sqrt{1 + 20(2-x)} - 1}{10}\right\}, \qquad f(x) = h(x)^n,$$

and extend $h(x) = x$ for $x \ge 2$. The inverse in this formula solves $s = e + 5e^2$. Consequently (14) and the radius caps give $|w|^n \le f(\operatorname{Re}(\bar{\xi}_i w))$ on the assigned side. Put

$$V = \mathbb{E} f(2\cos\theta), \qquad \alpha = \arccos(31/32), \qquad \alpha_* = \arccos(39/40).$$

The active part of the positive semicircle is $0 \le \theta \le \alpha$; the rest is charged at most p_0^n. The substitution

$$e(v) = 2 - 2\cos v, \qquad \Phi(v) = 2\arcsin\Big(\sin(v/2)\sqrt{1 + 5e(v)}\Big), \quad 0 \le v \le \alpha_*$$

has $\Phi(0) = 0$, $\Phi(\alpha_*) = \alpha$, and $h(2\cos\Phi(v)) = 2\cos v$. Its derivative is

$$\Phi'(v) = \frac{1 + 10e(v)}{\sqrt{(1 + 5e(v))(1 - 5e(v)^2/(4 - e(v)))}}.$$

The denominator is at least one: after clearing positive factors this is $4 - 2e - 5e^2 \ge 0$ for $0 \le e \le 1/20$. Also $e(v) \le (80/79)\sin^2 v$. With $C_n = \int_0^{\pi/2} \cos^n v\, dv$, integration by parts gives $\int_0^{\pi/2} \cos^n v \sin^2 v\, dv = C_n/(n+2)$ and $2^n C_n/\pi = I_n/2$. Thus

$$V \le p_0^n + \frac{I_n}{2}\left(1 + \frac{800}{79(n+2)}\right) \le p_0^n + \frac{I_n}{2}(1 + 11/n). \tag{15}$$

The first integral is a regular angle-to-angle change of variables; $\Phi'(0) = 1$, so no endpoint density singularity is introduced.

Suppose a selected side has $d_i \ge 1/n$. Set $b = \sqrt{3/n}$, $x_0 = 2\cos b$, and $\mu = 2\sin b/b \ge 2 - 1/n$. Replace f to the right of x_0 by the affine line through $(x_0, f(x_0))$ and $(\mu, f(\mu))$. The resulting convex test still controls the capped actual side; call its circle cost T. Writing $E_b g = b^{-1}\int_0^b g(\theta)\, d\theta$ and $X(\theta) = 2\cos\theta$, affine averaging gives the exact identity

$$V - T = \frac{b}{\pi}\big(E_b f(X) - f(E_b X)\big), \qquad E_b X = \mu. \tag{16}$$

The line and f may cross inside the arc; the identity uses their averages, not a pointwise saving.

Here is a quantitative Jensen bound using only one-dimensional integrals. On $[0, b]$, put $S = 2 - X$. The cosine Taylor bounds give $(999/1000)\theta^2 \le S \le \theta^2$. Since $E_b\theta^2 = b^2/3$ and $E_b\theta^4 = b^4/5$,

$$E_b X^2 - (E_b X)^2 = E_b S^2 - (E_b S)^2 \ge \frac{99}{100}\frac{4b^4}{45}.$$

On $[x_0, 2]$ the smooth branch of h satisfies $h \ge 2 - 3/n$, $h'' \ge 0$, and $(h')^2 = (1 + 20(2 - x))^{-1}$. The logarithmic bound for $(1 - 3/(2n))^{n-2}$ and $e^{3/2} < 9/2$ then give

$$f''(x) \ge m_n := 2^{n-2} n(n-1)/5.$$

Thus $f(x) - m_n x^2/2$ is convex on the arc range. Ordinary integral Jensen and (16) yield

$$V - T \ge \frac{b}{\pi} \frac{m_n}{2} \frac{99}{100} \frac{4b^4}{45}.$$

The odd cosine-moment recurrence gives $I_n \le 2^n/\sqrt{n}$. After $b^2 = 3/n$, the last display exceeds

$$I_n \frac{\sqrt{3}}{50\pi} \frac{99}{100}(1 - 1/n) > I_n/100 \qquad (n \ge 1501),$$

where the final strict inequality follows from $\sqrt{3} > 17/10$, $\pi < 22/7$, and $1 - 1/n > 99/100$.

The platform estimate $2p_0^n \le I_n/n$ follows from $(39/40)^{40} < 1/2$ at $n = 1501$ and the decreasing adjacent-term ratio. The two unchanged side costs total at most $I_n(1 + 12/n)$ by (15). If one side is bad, the chord saves more than $I_n/100$; the resulting cost is strictly below $(1 - 1/n)I_n$ because $13/n < 1/100$. This contradicts the candidate lower moment in (10). Both selected actual endpoints therefore satisfy

$$0 \le nd_1 < 1, \qquad 0 \le nd_2 < 1.$$

For the following positive-deficit estimates, a zero deficit is handled separately by slit rigidity following (7). For $d_i > 0$, the localization gives $d_i < 1/n \le 1/1501$. Apply the actual root strip (8) at $r = 1 + \sqrt{d_i}/10$. Write $L_i = \log(2/d_i)$. The principal width term is at most $(11/50)\sqrt{d_i}$. For the odd-tail kernel S_r, put $q = (r^2+1)/((r+1)/10)$. Then $S_r - L_i/4 = (\log q)/2 - (\log 2)/4 - r^{-2}$. Here $r \le 1 + 1/380$, so $q/10 - 1 < 1/500$ and $r^{-2} > 99/100$. The inequality $\log(q/10) \le q/10 - 1$, together with $\log 5 < 13/8$ and $\log 2 < 7/10$, shows $S_r \le L_i/4$. Also $2/d_i > 2048 = 2^{11}$ and $\log 2 > 69/100$, whence $L_i > 15/2$ and $\sqrt{L_i} > 27/10$. Since the odd-root energy factor $1 - R_i^2/4$ is at most d_i, the squared strip width satisfies $W_i \le d_i L_i (11/135 + 1/2)^2 < 0.34 d_i L_i$. Thus

$$W_i \le 0.34 d_i L_i, \qquad y_i := nW_i \le 0.34 t_i \log(2n/t_i). \tag{17}$$

The inequality holds on the whole ordinary convex hull, not just at its two selected endpoints. For the sharper cost estimate, put $u = 1501 d_i \le 1$. The elementary bound $-\log u \le 2(u^{-1/2} - 1)$ and $2 < \log 3002 < 25/3$ give

$$\sqrt{1501 d_i}\, \log(2/d_i) < 25/3.$$

Consequently the unified upper width $\widehat{W}_i = 0.34 d_i \log(2/d_i)$ satisfies

$$0 < W_i < \widehat{W}_i < 1/500, \qquad n\widehat{W}_i^2 < 3/500.$$

Here the last inequality uses $nd_i < 1$; it holds uniformly as n increases. These two scalar bounds and their transfer to the actual squared strip width follow from the same inequalities.

The finer model below uses the lower plateau 1.9. To assign every point above that plateau, repeat the two-point argument with $0 \le e = 2 - |w| \le 0.1$ and $C = 16/(2 + \sqrt{4 - 16/10}) < 4.51$. The alternatives are $|w - P_i| \le Ce$ and $|w - P_i| \ge 4 - Ce$. The forward half of P_1 permits only the near alternative. In the opposite half, w and the selected P_2 are both far from P_1, so the two-point inequality gives $|w + P_1| \le 2\sqrt{Ce}$ and $|P_2 + P_1| \le 2\sqrt{Cd_2}$. As $d_2 \le e$, their difference has length at most $4\sqrt{Ce} < 4 - Ce$; thus it is near P_2. The cosine rule now shows that each assigned point above

1.9 has positive projection in its own endpoint direction. For a positive-deficit endpoint, $W_i < 1/100$ follows already from (17) and $d_i < 1/1501$. On a nonnegative projection $x \le 2$, the squared strip implies $|w| \le \max\{1.9, 2-(1-1.07W_i)(2-x)+1.07W_i^2\}$: for $x \le 949/500$ the strip square is below 1.9^2; above $949/500$, expanding the displayed affine square dominates the strip square. The latter comparison only uses $(2.14)x > 4$ and $W_i \ge 0$.

For the final model bound put $x = \mathrm{Re}(\bar{\xi}_i w)$, $s = 2 - x$, and use

$$h_i(2-s) = \max\{1.9, 2-(1-1.07W_i)s + 1.07W_i^2\}.$$

On its active arc $s < 0.102$. The squared strip (8) makes h_i a radius majorant there; outside, the plateau applies. Set $U = 1 + 1.07W_i^2/2$, $v = 1 - 1.07W_i$, and $\gamma = v/U$. If $s = 2 - 2\cos\theta$ is the original angular deficit, choose ϕ by $2 - 2\cos\phi = \gamma s$. The active branch is exactly $2U\cos\phi$, and the angular Jacobian is

$$\frac{d\theta}{d\phi} = \gamma^{-1/2}\sqrt{\frac{4-\gamma s}{4-s}}.$$

Since $1 - \gamma \le 1.07107W_i$ and $s \le 0.102$, the square-root factor is at most $1 + 0.0142W_i$. The remaining active model factor is

$$(1+0.0142W_i)(1+1.07W_i^2/2)^{n+1/2}(1-1.07W_i)^{-1/2} \le 1 + (0.57 + 0.55y_i)W_i.$$

For this last inequality, take logarithms and use $W_i < 1/500$ and $nW_i^2 < 3/500$; the logarithm is at most $0.555W_i + 0.536nW_i^2 < 1/200$, and $e^x \le 1 + (200/199)x$ on $0 \le x \le 1/200$. With the platform charged once, the resulting one-end cost is

$$V_i \le (1.9)^n + \frac{I_n}{2}\bigl(1 + 0.57W_i + 0.55nW_i^2\bigr).$$

This bound has also been applied to both selected actual endpoints in the candidate inverse-coefficient inequality. For the refined affine h_i, use the same chord with $b = \sqrt{3d_i}$ and $\mu = 2\sin b/b \ge 2 - d_i$; the resulting convex extension majorizes the assigned side because $h_i(\mu) \ge \mu \ge 2 - d_i$. Its top cap saves at least $0.067I_n t_i^{5/2}e^{-0.61t_i}$ on side i. To see the size of this saving, use the normalized average E_b on $[0, b]$ as in (16). Set

$$S = 2 - 2\cos\theta, \qquad P(s) = (2U - vs)^n, \qquad A = n(n-1)2^{n-2}v^2, \quad B = n/2, \quad s_0 = 2b^2/5.$$

The entire short arc lies on the affine branch: $S \le b^2 < 3/n$ and $2U - vS \ge 2 - S > 1.9$. The chord identity therefore gives $V_i - T_i = (b/\pi)\{E_bP(S) - P(E_bS)\}$, where T_i is the cost after replacing the cap by its chord.

For $0 \le s \le b^2$, the inequality $\log(1 - s/2) \ge -s/(2-s)$ and $ns < 4$ give $(1-s/2)^{n-2} \ge e^{-ns/2}$. Thus $P''(s) \ge Ae^{-Bs}$. The tangent to this exponential at s_0 shows that $P - Ae^{-Bs_0}Q$ is convex, where

$$Q(s) = \tfrac{1}{2}(1 + Bs_0)s^2 - \tfrac{1}{6}Bs^3.$$

To apply Jensen, put $X = S/b^2$, $u = \theta/b$ and $c = 999/1000$. The cosine Taylor bounds give $cu^2 \le X \le u^2$, since $b^2 < 3/1501$. As $E_bu^j = 1/(j+1)$, we have

$$\mathrm{Var}_b(S)/b^4 \ge c^2/5 - 1/9 > (199/200)(4/45),$$
$$\{E_bS^3 - (E_bS)^3\}/b^6 \le 1/7 - c^3/27 < (6/5)(199/200)(4/45).$$

Since $3s_0 = 6b^2/5$, the cubic difference is at most $3s_0\,\mathrm{Var}_b(S)$. In particular

$$E_bQ(S) - Q(E_bS) = \tfrac{1}{2}(1 + Bs_0)\,\mathrm{Var}_b(S) - \tfrac{1}{6}B\{E_bS^3 - (E_bS)^3\} \ge \tfrac{1}{2}\,\mathrm{Var}_b(S).$$

Jensen now gives $V_i - T_i \geq (b/\pi)Ae^{-Bs_0}\operatorname{Var}_b(S)/2$. Substituting $b^2 = 3t_i/n$ and using $I_n < (4/5)2^n/\sqrt{n}$ (from the product of consecutive cosine moments) gives

$$\frac{V_i - T_i}{I_n t_i^{5/2} e^{-3t_i/5}} > \frac{\sqrt{3}}{8\pi}\frac{199}{200}(1 - 1/n)v^2 > \frac{67}{1000}.$$

For the last step one may substitute $\sqrt{3} > 173/100$, $\pi < 22/7$, $n \geq 1501$, and $v \geq 1 - 1.07/500$. This proves the stated saving since $e^{-3t_i/5} \geq e^{-0.61t_i}$. Summing the two side costs and comparing with (10) yields

$$J_n(t_1) + J_n(t_2) \leq 0.502 + \frac{2n(1.9)^n}{I_n} < 0.52, \tag{18}$$

where the definition of J_n uses the upper y_i in (17). The preceding bound $J_n(t) > -9/160$ then gives $J_n(t_i) < 0.58$ separately for both sides.

Lemma 4 (Cutoff from the scalar input)**.** *If $n \geq 1501$, $0 < t < 1$, and $J_n(t) < 0.58$, then*

$$t < \frac{3}{20}, \qquad d < \frac{1}{10000}, \qquad tL < \frac{3}{2}, \qquad \Delta L < \frac{3}{2}.$$

Proof. Write $t_0 = 3/20$, $n_0 = 1501$, $z_* = 0.51$, and $L_0 = \log(2n_0/t_0) < 10$. A direct rational comparison gives

$$0.067 n_0 t_0^{5/2} e^{-0.61t_0} - 0.285 z_* - 0.275 z_*^2 > 0.58.$$

For example, use $\sqrt{t_0} > 12/31$ and $e^{-0.61t_0} \geq (1 - 183/4000)^2$. For $t_0 \leq t \leq 1$, the factor $\sqrt{t}\, e^{-0.61t}$ is at least its value at t_0. On $[t_0, 3/5]$ this follows from $\log(t/t_0) \geq 1 - t_0/t \geq (61/50)(t - t_0)$; on $[3/5, 1]$ the estimates $\sqrt{t} > 3/4$ and $e^{-0.61t} > 1/2$ suffice, because $\sqrt{t_0} < 0.39$ and $e^{-0.61t_0} < 0.94$. Also $L/n \leq L_0/n_0$ and $L^2/n \leq L_0^2/n_0$: first decrease t to t_0, then use the decreasing functions $\log x/x$ and $\log x/\sqrt{x}$ for $x = 2n/t_0 > e^2$. Divide $J_n(t)$ by nt^2 and compare all three terms with the base point. The base inequality then gives $J_n(t) > 0.58$ throughout $t \geq t_0$.

For $0 < t \leq t_0$, apply the cubic envelope from the preceding paragraph with t_0 in place of 1. The base checks $\sqrt{t_0} < 0.39$ and $L_0 < 9.91$ give

$$100(0.34)^3\sqrt{t_0}L_0^3 < 1501, \qquad 100z^3 \leq nt^2\sqrt{t}.$$

One may check $L_0 < 9.91$ from $e^{9.91} = e^{10}e^{-0.09} > (2.718)^{10}(0.91) > 2n_0/t_0$. Since $e^{-0.61t} \geq (1 - 183/4000)^2$, the positive term of J_n exceeds $6.1z^3$. A polynomial expansion about $z = 0.51$ shows

$$6.1z^3 - 0.285z - 0.275z^2 > 0.58 \quad (z \geq 0.51).$$

Thus the candidate condition forces $z < 0.51$. As $z = 0.34tL$, this gives $tL < 1.5$; also $t < t_0$ gives $d = t/n < 1/10000$, and $m < n$ gives $\Delta L < tL < 1.5$. □

For the entire hull use the root strip at $r = 1 + (2/25)\sqrt{d}$. Here $d \leq 1/5000$ and $L > 9$. Set $u = \sqrt{d} < 1/70$ and $q = (r^2 + 1)/((2/25)(r + 1))$. The formulas in the root estimate give

$$\frac{p_r}{u} = \frac{4/25 + (1/2 + 4/625)u}{1 + 2u/25} < \frac{17}{100}, \qquad S_r - \frac{L}{4} = \frac{1}{2}\log q - \frac{1}{4}\log 2 - r^{-2}.$$

For the second formula, $q/(25/2) - 1 = r(r-1)/(r+1) < 1/875$ and $r^{-2} \geq 1 - 2(r-1) > 1 - 2/875$. Using $\log(1 + x) \leq x$, $\log 5 < 161/100$ and $\log 2 > 69/100$ shows

$$S_r - \frac{L}{4} < \frac{161}{100} - \frac{3}{4}\frac{69}{100} + \frac{1}{1750} - \left(1 - \frac{2}{875}\right) < \frac{1}{10}.$$

The logarithm bounds follow from the Taylor sum through degree eight for $e^{161/100}$, which exceeds 5, and $\log 2 > 2(1/3+1/81)$. By Lemma 4, $d < 1/10000$ and $2/d > 2^{14}$, so $L > 9$. Since $E \le d$ and $1/4 + 1/90 < (13/25)^2$, we obtain

$$\frac{W}{dL} \le \left(\frac{17}{100\sqrt{L}} + \sqrt{\frac{1}{4} + \frac{1}{10L}} \right)^2 < \left(\frac{173}{300} \right)^2 < \frac{1}{3}. \tag{19}$$

Define, only for $d > 0$,

$$y = \Delta L/3, \qquad a = \min\{3/y, m/5\}. \tag{20}$$

Then $mW < y < 1/2$, $a > 6$, and $ay \le 3$.

We also need positive projection on each assigned outer half. In first-axis coordinates write $p + iq = \bar{\xi}_1 w$. The first whole-hull strip has $W_1 < 1/3000$, because $mW_1 < 1/2$. Together with $|w| \le 2$ and $R_1 \le 2$ it gives $q^2 < 1/100$ on the entire hull, including its negative half. If $|w| \ge 46/25$, then $|p| > 183/100$. The selected second endpoint has $R_2 > 1999/1000$, negative first projection, and the same transverse bound, so its first projection is less than $-199/100$. Consequently, on the negative assigned half,

$$\operatorname{Re}(\bar{\xi}_2 w) > \frac{(199/100)(183/100) - 1/100}{2} > \frac{181}{100}.$$

On the positive half the first projection already exceeds $183/100$. A change of unit direction by at most $1/200$ alters projection by at most $1/100$. The nearest phase ray has distance at most $\pi/(2k) < 1/200$, so both assigned outer projections exceed $9/5$. The whole-hull cap under tilting is proved below; it uses the same selected pair and applies on every source circle.

2.4 The saving from a capped projection

Write $T_m = 2^m m^{-3/2}$. We compare the projected filling on each circle with the Poisson extension of the flattened cosine. Only the outer annulus will be needed to obtain a strict gain after integration.

Lemma 5 (Short actual disk gain)**.** *Let $m \ge 1500$, $d > 0$, and $D = md \le 3/20$. Suppose $|\xi| = 1$ and $\sup_{w \in K} \operatorname{Re}(\bar{\xi} w) \le 2 - d$. Then*

$$\frac{m\sqrt{m}}{2^m} \int_{\mathbb{D}} \left\{ (2 \operatorname{Re} z)_+^m - \left(\operatorname{Re}(\bar{\xi} H(z)) \right)_+^m \right\} d\sigma > \frac{D^3}{30}.$$

Proof. The boundary projection $u(t) = \operatorname{Re}(\bar{\xi} F^*(e^{it}))$ satisfies $u \le 2 - d$ almost everywhere since the trace lies in K. Write $2 \sin h/h = 2 - d$, $0 < h < \pi$, and replace $2 \cos t$ on $[-h, h]$ by its mean $2 - d$ to obtain u_d. For $0 < r < 1$, write $Y_r(t) = \operatorname{Re}(\bar{\xi} H(re^{it})) = (P_r * u)(t)$. The boundary comparison for u_d in Section 2.1 and (5) give $Y_r \preceq_{\mathrm{cx}} P_r * u_d$ for normalized angular measure. Put $M(r) = (P_r * u_d)(0)$. Each superlevel set of $u_d + 2$ is a centered arc, over which the even, decreasing Poisson kernel has its largest integral when centered at zero. Integration over levels gives $(P_r * u_d)(t) \le M(r)$ for every t. Now test the convex order with $x \mapsto (x - M(r))_+$:

$$0 \le \mathbb{E}_t (Y_r(t) - M(r))_+ \le \mathbb{E}_t \big((P_r * u_d)(t) - M(r) \big)_+ = 0.$$

Since Y_r is continuous, $Y_r(t) \le M(r)$ at every angle. The cap angle obeys $\sqrt{3d} \le h \le (7/4)\sqrt{d}$. For the first bound use $\sin h \ge h - h^3/6$. For the second, $2 - 2 \sin v/v$ increases on $(0, \pi)$, and at $v = (7/4)\sqrt{d}$ the sine Taylor bound gives $2 - 2 \sin v/v \ge v^2/3 - v^4/60 > d$, using $d \le 1/10000$. Let

$P_r(\theta) = (1-r^2)/(1-2r\cos\theta+r^2)$ and $A_r = \pi^{-1}\int_0^h P_r(\theta)\,d\theta$. On $[0,h]$ the cap difference times the kernel is $(d+(1-r)^2/r)P_r(\theta) - (1-r^2)/r$. Integration yields

$$2r - M(r) = \left(d + \frac{(1-r)^2}{r}\right) A_r - \frac{(1-r^2)h}{\pi r}.$$

The Poisson kernel has complementary mass $1 - A_r \le (1-r^2)/(\pi r h)$: integrate its elementary $\sin^{-2}(\theta/2)$ upper bound over $[h,\pi]$ and use $\cot(h/2) \le 2/h$. Dropping the nonnegative term $(1-r)^2 A_r/r$ bounds the defect below by $d - (1-r^2)(d/h+h)/(\pi r)$. For $1 - \sqrt{d}/3 \le r < 1$ we have $r \ge 99/100$ and $((1+r)/(\pi r))(d/h+h) \le (3/2)\sqrt{d}$, using $1/\sqrt{3} < 7/12$, $h \le (7/4)\sqrt{d}$, and $\pi > 157/50$. Thus

$$2r - M(r) \ge d - \tfrac{3}{2}(1-r)\sqrt{d} > 0. \tag{21}$$

Let $f(x) = x_+^m$ and let ψ_M agree with f below M and continue by its tangent above M. Taking $M = M(r)$ gives $f(Y_r) = \psi_M(Y_r)$ pointwise. Since ψ_M is convex, (6) yields $\mathbb{E}_t f(Y_r(t)) \le \mathbb{E}_t \psi_M(2r\cos t)$ on each source circle. Put $v = d - \frac{3}{2}(1-r)\sqrt{d}$ and $q = 2r - v$. On $|\theta| \le \sqrt{v}$ one has $2r\cos\theta \ge q \ge M(r)$. The binomial remainder above a tangent therefore gives

$$f(2r\cos\theta) - \psi_M(2r\cos\theta) \ge \frac{m(m-1)}{2} q^{m-2}(v-\theta^2)^2.$$

As $\int_0^{\sqrt{v}} (v-\theta^2)^2 d\theta = (8/15)v^{5/2}$, the full normalized circle saves at least $4m(m-1)q^{m-2}v^{5/2}/(15\pi)$.

Only the outer annulus $a \le r < 1$, $a = 1 - \sqrt{d}/16$, is needed. There $v \ge (9/10)d$, $r \ge 99/100$, and $(1 - v/(2r))^{m-2} > 9/10$. Hence $v^{5/2} \ge (3/4)d^{5/2}$ and the circle saving is at least

$$\frac{9}{50\pi} m(m-1)(2r)^{m-2} d^{5/2}.$$

The radial factor is $2r\,dr$, so its integral contributes exactly $2^{m-1}(1-a^m)/m$. After multiplication by $m\sqrt{m}/2^m$, the saving is at least

$$\frac{9}{100\pi}\left(1 - \frac{1}{m}\right) D^2\sqrt{D}\,(1-a^m).$$

Writing $x = \sqrt{D} \le 2/5$ gives $a^m \le e^{-\sqrt{m}\,x/16} \le e^{-12x/5}$. Concavity bounds $1 - e^{-12x/5}$ below by the chord from 0 to 2/5. Its slope exceeds $4/3$ since $e^{24/25} > 1 + 24/25 + (24/25)^2/2 > 15/7$. Using $\pi < 22/7$ and $1 - 1/m \ge 99/100$, the last display is at least $(189/5000)D^3 > D^3/30$. On every other source circle the ordinary positive-power convex comparison makes the radial increment nonnegative. Integrability and exhaustion by interior disks pass the strict inequality to the open unit disk. □

Tilting the second cap. Write $P_i = R_i\xi_i$ for the same selected endpoints. Since P_1 maximizes the modulus on the entire hull, every unit direction has projection at most R_1. For the second cap, split the hull by $\mathrm{Re}(\bar{\xi}_1 w) = 0$. On the negative half its modulus is at most R_2. On the positive half, the fact that $\mathrm{Re}(\bar{\xi}_1\xi_2) \le 0$ gives $\mathrm{Re}(\bar{\xi}_2 w) \le |\,\mathrm{Im}(\bar{\xi}_1 w)| < 1/10$ by the first strip. For $|v - \xi_2| \le 1/4$ and $|w| \le 2$, the perturbation of this projection is at most $1/2$. Hence it remains below $R_2 > 1$. Thus the whole-hull cap, and the actual saving above, hold along every such tilted unit direction. All width and maximality data refer to the original selected pair.

For any unit coefficient phase β, the ray can be chosen explicitly:

$$v = \xi \exp\left(-\frac{i}{2k}\,\mathrm{Arg}((\beta\xi^k)^2)\right).$$

Then $(\beta v^k)^2 = 1$, so the signed angular loss vanishes on this ray, and $|v - \xi| \le \pi/(2k) < 1/4$. Squaring the phase chooses either sign of a sine zero and includes the tie case.

Lemma 6 (Retaining a power saving)**.** *Let $m \ge 1500$, $\ell = m-8$, $c \ge 0$, and $\kappa = \ell(\ell-1)/(m(m-1))$. Put $g(u) = \max\{u_+^m, c^8 u_+^\ell\}$. The function $g(u) - \kappa u_+^m$ is convex on the whole real line. For every actual filling and unit direction v, therefore,*

$$\int_{\mathbb{D}} \left[g(\operatorname{Re} z) - g\big(\tfrac{1}{2}\operatorname{Re}(\bar v H)\big)\right] d\sigma \ge \kappa \int_{\mathbb{D}} \left[(\operatorname{Re} z)_+^m - \big(\tfrac{1}{2}\operatorname{Re}(\bar v H)\big)_+^m\right] d\sigma.$$

Under the cap of Lemma 5, the left side, multiplied by $m\sqrt{m}$, is strictly greater than $D^3/32$.

Proof. Below zero the residual is zero. On $[0, c]$ its second derivative is $\ell(\ell-1)u^{\ell-2}(c^8 - u^8) \ge 0$; above c the residual is $(1-\kappa)u^m$. The derivative jumps upward by $8c^{m-1}$ at c, and the derivatives at zero agree. This proves global convexity, including $c = 0$. Apply the actual disk convex comparison to this residual and rearrange. All tests are continuous on the bounded projection range, so their integrals exist. The final assertion follows from $\kappa \ge 49/50$ and $(49/50)/30 > 1/32$, with the same direction in both integrals. □

The extremely small deficit regime. The following comparison resolves the joint two-side cost once the actual endpoint localization has supplied the width and cap hypotheses. Let $d = d_1 = 2 - \max_K |w| > 0$, put $m = 2k - 2$ and $x = \log(100m)$, and assume

$$d_1 < d_0 := \frac{3}{100m^2 \log(100m)}. \tag{22}$$

Since d is the smallest endpoint deficit, every projection of the boundary trace has the global cap $2 - d$. The strip formula (8) together with the endpoint width $W \le dL/3$ gives, on the whole hull,

$$|\operatorname{Im}(\bar\xi w)| < 2.5\sqrt{dL}, \qquad L = \log(2/d),$$

where ξ is the direction of a maximal-radius point. Indeed $|\operatorname{Im}(\bar\xi w)|^2 \le 4W(W+4)$ and dL is tiny in (22). Choose the nearest pair of antipodal phase rays $v, -v$ to $\xi, -\xi$, and set $b' = 6k\sqrt{dL}$. Assign w by the sign of its true-axis projection. On $|w| > b'$ that positive projection exceeds $|w|/2$, since the transverse square is at most $(25/4)dL$. The argument bound $|\arg u| \le |\operatorname{Im} u|/\operatorname{Re} u$ therefore gives a scaled true-axis angle at most $5/6$. The nearest phase ray adds at most $\pi/2$, giving a scaled phase angle below 2.8. For $0 \le t \le 2.8$, concavity and the cubic sine bound at $\pi - 2.8$ give $\sin t \ge t/10$. Together with Bernoulli's inequality and $m/2 = k - 1 \le 3k^2/1400$ (since $k \ge 751$), this gives

$$1 - \frac{3}{14}\sin^2(k\theta) \le \cos^m\theta \qquad (|k\theta| \le 2.8).$$

Thus the angular penalty pays the projected-power loss outside the central region $|w| \le b'$. Inside, a residual at most $(b'/2)^m$ remains in the dimensionless signed integral.

Each of the two projected rays has cap $2 - d$, so the full convex capped-Poisson comparison applies to its positive mth-power test. In the dimensionless normalization of (3), each ray saves more than $\Delta^3/(30m^{3/2})$ from its reference moment M_m, by Lemma 5. The two rays therefore save more than $\Delta^3/(15m^{3/2})$. It remains to pay the variable central residual. This can be done without differentiating a ratio. Put $L_0 = \log(2/d_0)$. The bounds $1 \le x \le m/20$ give $1 \le L_0 \le 3x$. For $d < d_0$, the logarithmic tangent inequality gives

$$dL = dL_0 + d\log(d_0/d) \le dL_0 + d_0 - d \le d_0 L_0.$$

Consequently $m^2 dL \le 9/100$, and $m \ge 9k/5$ implies $9k^2 dL \le 1/4$. Also $dL^2 \le 1$: the elementary Taylor bound $e^L \ge L^4/24 \ge 2L^2$ for $L \ge 9$, together with $e^L = 2/d$, proves this directly. Writing $q = k - 1$, retain six powers and bound all the others by $1/4$:

$$(b'/2)^m = (9k^2 dL)^q \le 4^{-(k-7)} (9k^2)^6 d^3 (dL^2)^3 \le 4^{-(k-7)} 9^6 k^{12} d^3.$$

The sequence $15 \cdot 9^6 k^{12} 4^{-(k-7)}$ is below one for $k \ge 751$. At the first order use $751 < 2^{10}$ and $15 \cdot 9^6 < 2^{24}$; successive ratios are at most $(752/751)^{12}/4 < 1$. Since $m\sqrt{m} \le m^3$, we obtain

$$15 m\sqrt{m}\,(b'/2)^m < (md)^3. \tag{23}$$

The signed cost is therefore strictly below $2M_m$ in this regime. This is a joint estimate: no artificial lower cutoff on the tiny endpoint deficit is needed. Its only actual-map inputs are the width and cap statements specified above.

In the aligned phase-ray branch the desired final comparison is a retained saving greater than $\Delta^3/32$ against model and central errors totaling less than $13\Delta^3/500 < \Delta^3/32$, in units of T_m. Here is the pointwise check with the shorter a, conditional on the actual endpoint and tilt estimates. Put $s = m(1 - x/2)$, $u = x/2$, and $h = |\operatorname{Im}(\bar{v}w)|/x$ on the outer region $x \ge 1.8$. For a phase ray within scaled angle $\gamma \le \pi/3$ of the true endpoint direction, write $\theta = |\arg(v/\xi_i)| \le \pi/(3k)$, $q_\xi = \operatorname{Im}(\bar{\xi}_i w)$ and $Q = m|q_\xi|$. Rotation gives $R_i - \operatorname{Re}(\bar{\xi}_i w) \le 2s/m + |q_\xi| \tan\theta$. With $mW \le y < 1/2$, the squared strip implies

$$Q^2 \le 4y(y + 2s + Q\tan\theta), \qquad 0 \le s \le m/10, \qquad m\tan\theta \le 21/10.$$

Here $\cos\theta \ge 999/1000$, so $x\cos\theta \ge 7/4$. Since $y \le 1/2$, $m \ge 1500$, and $h \le \tan\theta + |q_\xi|/(x\cos\theta)$, we obtain the two simple rational bounds

$$Q^2 \le 1 + 8ys + \frac{7}{2500} Q, \qquad mh \le \frac{21}{10} + \frac{4}{7} Q.$$

If $s \le 7a/5$, then $ys \le 21/5$ and the first inequality forces $Q < 6$. Thus $mh < 387/70$ and $k|\arg(w/v)| \le kh < 14/5$. To pay for the projection, it suffices to use $\sin t \ge t/10$ on $[0, 14/5]$. Indeed, the cubic sine bound at $\pi - 14/5$, with $3.14 < \pi < 3.15$, gives $\sin(14/5) \ge 3/10$; concavity supplies the whole interval. Bernoulli's inequality and $1 - \cos\varphi \le \varphi^2/2$ now give

$$1 - \cos^m\varphi \le \frac{m}{2}\varphi^2 \le \frac{3}{14}\frac{k^2\varphi^2}{100} \le \frac{3}{14}\sin^2(k\varphi), \qquad |k\varphi| \le 14/5.$$

Consequently the signed angular loss pays for the first test u^m.

If $s \ge 7a/5$, use $Q \le 1 + Q^2/4$ to absorb the small linear term, obtaining $Q^2 \le (51/50)(1 + 8ys)$. Completing the square with $(Q - 49/10)^2 \ge 0$ then gives

$$\frac{m^2 h^2}{2} \le \frac{11}{2} + \frac{7}{3} ys \le \frac{11}{2} + \frac{7}{6} s < 8(s - a).$$

The final strict inequality uses only $a \ge 6$ and $s \ge 7a/5$. Taking logarithms shows that the second test $(1 - a/m)^8 u^{m-8}$ dominates the entire unsigned cost $(|w|/2)^m$; it uses no phase saving.

The junction of the two tests is at $s = a$. Their convexity and retention of the actual saving are supplied by Lemma 6.

Lemma 7 (One disk budget for two aligned ends)**.** *Suppose both selected rays belong to the aligned phase case and choose their parameters by* (20)*. Write* $g_i(u) = \max\{u_+^m, (1 - a_i/m)^8 u_+^{m-8}\}$ *and*

$$\mathcal{R}_i = \int_{\mathbb{D}} g_i(\operatorname{Re} z)\, d\sigma, \qquad \mathcal{A}_i = \int_{\mathbb{D}} g_i\big(\tfrac{1}{2}\operatorname{Re}(\bar{v}_i H)\big)\, d\sigma.$$

Then, with $N = m\sqrt{m}$*,*

$$N\left(\int_{\mathbb{D}} S(H)\, d\sigma - 2M_m\right) < N\left[(\mathcal{R}_1 - M_m) + (\mathcal{R}_2 - M_m) + (23/25)^m\right] - \frac{\Delta_1^3 + \Delta_2^3}{32}.$$

Proof. On an assigned outer point the signed density is bounded by its own test. Both tests are nonnegative, so extending each test to the entire source disk preserves an upper bound. The central region adds at most $(23/25)^m$, counted once. Thus $\int S(H) \le \mathcal{A}_1 + \mathcal{A}_2 + (23/25)^m$. The actual retained gains give $N(\mathcal{R}_i - \mathcal{A}_i) > \Delta_i^3/32$ for the same directions and parameters. Adding and rearranging proves the claim. All integrals exist by the bounded projection range and continuity of the tests; the signed density is integrable by the actual energy construction. No boundary trace identification or assumption that a dividing level has measure zero is used here. □

For the other phase branch $\gamma \ge \pi/3$, use the true endpoint axis and $f_0(u) = \lambda(u_+)^{m-8}$, where $\lambda = (1 - 1/m)M_m/M_{m-8}$. The true axis keeps the geometric estimate simple. Write h for the signed angle of w from that axis and $s = m(1 - x/2)$, where $x = \operatorname{Re}(\bar{\xi}_i w) \ge 9/5$. The actual strip gives

$$\tfrac{1}{2}m^2 \tan^2 h \le \frac{1}{6} + \frac{5}{8}s, \qquad s \le 2 \implies |kh| \le \frac{6}{7}.$$

Indeed $R_i - x \le 2s/m$, so the squared strip and $mW \le y < 1/2$ give $m^2|\operatorname{Im}(\bar{\xi}_i w)|^2 \le 1 + 4s$. Since $\tan h = \operatorname{Im}(\bar{\xi}_i w)/x$ and $x \ge 9/5$, the first bound follows from $(1 + 4s)/(2(9/5)^2) \le 1/6 + 5s/8$. For the second, use $|h| \le |\tan h|$ and $k/m \le 751/1500$; at $s = 2$ the resulting bound on $(kh)^2$ is below $(6/7)^2$. The moment recurrence $(q + 4)M_{q+2} = (q + 1)M_q$ gives the exact ratio

$$\lambda = \frac{m-1}{m}\prod_{j=0}^{3}\frac{m-2j-1}{m-2j+2}, \qquad \lambda M_{m-8} = (1 - 1/m)M_m.$$

Here the recurrence is used only at positive even exponents. Applying $\log(1 - v) \ge -v/(1 - v)$ to the five factors gives

$$-m\log\lambda \le \frac{m}{m-1} + 3m\sum_{j=0}^{3}\frac{1}{m-2j-1} \le \frac{m}{m-1} + \frac{12m}{m-7} < 14,$$

using the decreasing ratios at $m = 1500$. If γ is the signed phase displacement, then $\pi/3 \le |\gamma| \le \pi/2$. Indeed the logarithm of the signed density divided by $\lambda(x/2)^{m-8}$, multiplied by m, is

$$m\log\big(1 - \tfrac{3}{14}\sin^2(\gamma - kh)\big) + 8m\log(x/2) + \tfrac{1}{2}m^2\log(1 + \tan^2 h) - m\log\lambda.$$

Use $x/2 = 1 - s/m$, $|w|^2/x^2 = 1 + \tan^2 h$, and $\log(1 + v) \le v$ for $v > -1$. The resulting upper bound is

$$14 + \frac{1}{6} + \frac{5}{8}s - 8s - \frac{3}{14}m\sin^2(\gamma - kh). \tag{24}$$

For $s \ge 2$ the first four terms already have negative sum. For $0 \le s \le 1$, the strip estimate above gives

$$|kh|^2 \le (k/m)^2(1/3 + 5s/4) \le (751/1500)^2(19/12) < 4/9.$$

Thus $|\gamma - kh|$ lies between $\pi/3 - 2/3$ and $\pi/2 + 2/3$. Its sine is greater than $1/4$, since the distance of this interval from both 0 and π exceeds $1/3$ and $\sin(1/3) > 1/4$. The penalty in (24) is then larger than $3m/(14 \cdot 16) > 15$, whereas the other terms sum to at most $14 + 1/6$. For $1 \le s \le 2$ use $|kh| \le 6/7$. The minimum of the sine on $[\pi/3 - 6/7,\ \pi/2 + 6/7]$ is its value at $\pi/3 - 6/7$, which exceeds $3/16$ by the cubic Taylor bound. Consequently $\sin^2(\gamma - kh) > 1/30$. Here the penalty is greater than $3m/(14 \cdot 30) > 10$, while the other terms sum to at most $14 + 1/6 + 5/8 - 8 < 7$. This proves the strict true-axis comparison. Its actual disk integral is at most $(1 - 1/m)M_m$ by convex projection comparison.

Thus each assigned outer half has its own test: the two-power test in the aligned case and the true-axis lower power in the other case. Both are nonnegative. Extending both to the entire source disk and adding the central allowance once gives $\int S(H) \le \mathcal{A}_1 + \mathcal{A}_2 + (23/25)^m$ for every phase assignment. This statement and the exact true-axis model cost use the same actual filling throughout.

Let Z have probability law $d\sigma = dA/\pi$ on $\mathbb{D}$, and set $X = 2\operatorname{Re} Z$. The vertical slice at $\operatorname{Re} Z = x/2$ has length $2\sqrt{1 - x^2/4}$. Rescaling the horizontal coordinate gives X the density $\sqrt{4 - x^2}/(2\pi)$ on $[-2, 2]$, zero elsewhere. Equivalently the angle has probability measure $dt/(2\pi)$ and the independent radius has measure $2r\,dr$. For the normalized projection $u = X/2$, put $g(u) = \max\{u_+^m, (1 - a/m)^8 u_+^{m-8}\}$. Then $M_m = \mathbb{E}[(X/2)_+^m]$, and the extra reference cost is

$$D_f = m^{3/2}\mathbb{E}[g(X/2) - (X/2)_+^m] = \frac{m^{3/2}}{2\pi}\int_0^2 \big(g(x/2) - (x/2)^m\big)\sqrt{4 - x^2}\,dx. \tag{25}$$

This uses the dimensionless density in (3). Equivalently it is the extra unnormalized cost divided by $T_m = 2^m m^{-3/2}$. Only $X > 0$ contributes. With $s = m(1 - x/2)$ the weighted measure $m^{3/2}\,d\mathbb{P}_X$ on this half becomes $(2/\pi)\sqrt{2s - s^2/m}\,ds$ on $[0, m]$. The integrand difference is supported on $s \ge a$. Bounding $\sqrt{2s - s^2/m}$ by $\sqrt{2s}$ and integrating an exponential tail gives

$$D_f \le \frac{16\sqrt{2}}{\pi m}(1 - a/m)^{m-1}\left(\frac{\sqrt{a}}{\sigma^2} + \frac{1}{\sqrt{a}\,\sigma^3}\right), \qquad \sigma = \frac{m - 8}{m - a}.$$

Indeed the two terms of g cross at $s = a$; on $s \ge a$ use $(1 - a/m)^8 - (1 - s/m)^8 \le 8(s - a)(1 - a/m)^7/m$, $(1 - s/m)^{m-8} \le (1 - a/m)^{m-8}e^{-\sigma(s-a)}$, and $\sqrt{s} \le \sqrt{a} + (s - a)/(2\sqrt{a})$. For $m \ge 1500$, $6 \le a \le m/5$ we have $\sigma > 0.99$, $(1 - a/m)^{m-1}e^a \le e^{a/m} < 5/4$ and $16\sqrt{2}/\pi < 8$. Hence the coefficient after dividing by $e^{-a}\sqrt{a}/m$ is below $10((0.99)^{-2} + (6(0.99)^3)^{-1}) < 12$. Therefore

$$D_f \le 12e^{-a}\sqrt{a}/m. \tag{26}$$

Lemma 6 gives a saving greater than $\Delta^3/32$ for the same test in the same direction, after multiplication by $m^{3/2}$. For the untruncated choice $a = 3/y$, equation (20) gives $\Delta L = 9/a$ and $m = \Delta e^L/2$, with $L > 9$. Therefore

$$\frac{12e^{-a}\sqrt{a}}{m\Delta^3} = \frac{24}{9^4}(a^4\sqrt{a}\,e^{-a})(L^4e^{-L}).$$

For $a \ge 6$, $\sqrt{a} \le a/\sqrt{6}$ and a^5e^{-a} is decreasing. Also L^4e^{-L} is decreasing for $L \ge 9$. The displayed ratio is consequently at most $24 \cdot 6^4\sqrt{6}\,e^{-15}$. The Taylor sum for e^3 through degree eight exceeds 20, and $\sqrt{6} < 5/2$; hence this last quantity is below $243/10000 < 1/40$. Combining this with the central estimate gives $D_f + \text{central} < 13\Delta^3/500 < \Delta^3/32$.

In the complementary regime $d_1 \ge d_0$, global radius maximality of P_1 gives $d_2 \ge d_1$. Hence each endpoint satisfies $\Delta_i = md_i \ge 3/[100m\log(100m)]$. In the following estimates, Δ denotes either Δ_i. The central and truncated costs admit a short common estimate. For $m \ge 1500$,

$$\log(100m) \le \frac{m}{100} + 13 \le \frac{m}{20}, \qquad \Delta_i \ge \frac{3}{100m\log(100m)} \ge \frac{3}{5m^2} > \frac{1}{2m^2} \quad (i = 1, 2).$$

For the logarithm use $10000 < 2^{14}$, $\log 2 \le 1$, and $\log(m/100) \le m/100 - 1$. The sequence $m^8(23/25)^m$ decreases, since $((m+1)/m)^8(23/25) \le (1501/1500)^8(23/25) < 1$. At $m = 1500$, use $(23/25)^{10} < 1/2$ and $1500 < 2^{11}$ to obtain $m^8(23/25)^m < 1/8000$. As $\sqrt{m} \le m$,

$$m\sqrt{m}\,(23/25)^m < \frac{1}{8000m^6} \le \frac{\Delta^3}{1000}.$$

For the truncated choice $a = m/5$, the model estimate is bounded by this same quantity: $e^{-1/5} \le 23/25$ and $12\sqrt{m/5}/m \le m\sqrt{m}$. Thus both the truncated model cost and central allowance are below $\Delta^3/1000$, and their sum is below $\Delta^3/64$.

The central allowance applies in either phase branch. On $|w| < 1.84$, one has $S(w) \le (0.92)^m$, so its contribution in T_m units is at most $m^{3/2}(0.92)^m$. In this complementary regime, the lower bound for Δ_i in the preceding paragraph holds for both endpoints. The same rational induction therefore proves $m^{3/2}(0.92)^m < \Delta^3/1000$ independently of which value of a is active. In the true-axis branch the unscaled error $(0.92)^m$ is below its free model margin M_m/m: the largest binomial coefficient in (4) is at least $2^m/(m+1)$, so $M_m/m \ge 1/[m(m+1)(m+2)]$, whereas $m^8(23/25)^m < 1/8000$.

Proof of Theorem 1. Suppose first that an actual nonslit map is a weak contender: $|B_n(F)| \ge \mathrm{Cat}_{k-1}$. The boundary localization gives two actual assigned ends with $0 < nd_i < 1$, their whole-hull strips and whole-boundary projection caps, and the joint strict scalar cutoff. Lemma 4 then gives $d_i < 1/10000$ and $\Delta_i L_i < 3/2$ on both sides. The refined root estimate gives $mW_i \le y_i < 1/2$, and the preceding angle argument gives the cap for each chosen true axis or nearest phase ray. The energy multiplier in Section 2.5 applies to this actual map, so the signed comparison (3) holds.

Write $d_1 = 2 - |P_1| \le d_2 = 2 - |P_2|$ and $d_0 = 3/[100m^2\log(100m)]$. If $d_1 < d_0$, the two opposite capped rays and the central estimate above give $\int_{\mathbb{D}} S(H)\,d\sigma < 2M_m$, contradicting the candidate inequality. If $d_1 \ge d_0$, global radius maximality of P_1 gives $d_2 \ge d_1 \ge d_0$. Thus both endpoints satisfy $\Delta_i \ge 3/[100m\log(100m)] > 1/(2m^2)$, including the case $d_1 = d_0$. Apply the outer-region comparisons to each assigned side. In the aligned branch choose the nearest phase ray and the convex test $g(u) = \max\{u_+^m, (1-a/m)^8 u_+^{m-8}\}$. The root strip gives the displayed aligned tilt bound, so the pointwise signed integrand is below g on $|H| \ge 1.84$. For this direction, Lemma 6 saves more than $\Delta_i^3/(32m^{3/2})$ from the reference two-power integral. That reference integral is $M_m + D_f/m^{3/2}$. By the preceding estimates $D_f + m^{3/2}(0.92)^m < 13\Delta_i^3/500 < \Delta_i^3/32$. Thus the aligned outer budget plus even the whole central allowance is strictly below M_m.

In the other branch use the true endpoint axis and the convex test $g_0(u) = \lambda u_+^{m-8}$. The true-axis tilt bound and (24) give strict pointwise domination on the outer region. The uncut model integral is $(1-1/m)M_m$; the cap can only decrease it, and the whole central allowance is less than M_m/m. This outer budget plus the allowance is also strictly below M_m. The direction is fixed on every source circle in each convex-order application. Extend the two nonnegative outer tests to the full disk and add the central allowance $(0.92)^m$ once. Charge that single allowance to either endpoint budget as above; the other budget is at most M_m. The resulting strict upper bound contradicts $\int S(H) \ge 2M_m$. Thus every nonslit actual map has strict inequality, while a zero endpoint deficit is already a slit by the area rigidity following (7).

Finally $F_a(z) = z + a/z$, $|a| = 1$, has inverse expansion $G_a(w) = w - \sum_{k\ge1} \mathrm{Cat}_{k-1}\, a^k w^{-(2k-1)}$. It attains the bound in every listed order. This proves both the upper bound and the full equality classification. □

2.5 The phase-energy inequality

The energy estimate controls the phase selected by the inverse coefficient. Its proof uses a drift identity, one signed flux, and integration with a radial cutoff. We choose a single capped linear

function to obtain a constant uniform in k.

Lemma 8. *For every centered exterior univalent F, integer $k \geq 751$, and $|\beta| = 1$, put $H(z) = z + g(\bar{z})$ and $U(z) = \operatorname{Im}(\beta H(z)^k)$. Then*

$$k^2 \int_{\mathbb{D}} \frac{U^2}{|H|^2}\, d\sigma \leq \frac{7}{3} \int_{\mathbb{D}} |\nabla U|^2\, d\sigma.$$

The quotient is extended by zero at $H = 0$.

Proof. Write $y = |H|$ and $c = g'(\bar{z})$. Real differentiation gives

$$y^2 \Delta U + 4k(k-1)U = 4(k-1)H \cdot \nabla U. \tag{27}$$

This identity extends across $H = 0$. In fact, $\Delta U = 4k(k-1)\operatorname{Im}(\beta H^{k-2} c)$; the terms containing g'' cancel.

For $y > 0$ put $u = U/y$, $p = |\nabla U|$, $r = |z|$, $t = k(1-r)/y$ and $b = k/[4(k-1)]$. Choose

$$\chi(t) = \min\{12t, 9/5\}, \qquad f(t) = \chi(t)/6, \qquad J = \chi(t) U \nabla U + 4(k-1) e_r f(t) \frac{U^2}{y}.$$

Combine the two divergences first. The coefficient of uU_r from differentiating χ and the radial flux is $8(k-1)f - k\chi'$. For the radial term, with $h = f + tf'$, the product rule reads

$$\operatorname{div}\big(e_r f(t) U^2/y\big) = \{(y/r)f - h y_r - k f'\} u^2 + 2 f u U_r. \tag{28}$$

Use (27) together with

$$\nabla t = -\frac{k e_r + t \nabla y}{y}, \quad |u||\nabla y| \leq 2|u| + p/k, \quad (y/r) f \leq 2f.$$

For the gradient bound, $|\nabla y| \leq 1 + |c|$ and $|u| \leq y^{k-1}$. Differentiating the imaginary part gives $p = k|\beta H^{k-1} - \overline{\beta H^{k-1} c}| \geq k y^{k-1} \big||c| - 1\big|$. Define

$$\begin{aligned} A_k &= \chi + f' - (2f + 2|h|)/k, \\ D_k &= (k\chi + t|\chi'|)/(4(k-1)), \\ B_k &= \chi + |2f - b\chi'| + t|\chi'|/(2(k-1)) + |h|/k. \end{aligned}$$

Indeed, the drift and radial identities give

$$\begin{aligned} b \operatorname{div} J + k^2(\chi + f')u^2 = {} & b\chi p^2 + k\chi u (H/y) \cdot \nabla U + k(2f - b\chi') u U_r \\ & - bt\chi' u\, \nabla y \cdot \nabla U + k\big((y/r)f - h y_r\big) u^2. \end{aligned}$$

Using the preceding bounds in this equality yields

$$k^2 A_k(t) u^2 + b \operatorname{div} J \leq D_k(t) p^2 + k B_k(t) |u| p.$$

The scalar multiplier bound below implies its homogeneous form also when $p = 0$, hence

$$k^2 u^2 + b \operatorname{div} J \leq \frac{7}{3} p^2 \quad \text{almost everywhere on } \mathbb{D}. \tag{29}$$

At the single multiplier corner the differential of t is zero almost everywhere on that level set. The Lipschitz multiplier compositions therefore have zero differential there; either adjacent slope

gives the same product rule. No nullity assumption on the level is needed. At a filling zero, $|J| = O(y^{2k-1})$ and $y = O(|z - z_0|)$, so its zero extension has derivative zero.

For local Lipschitz continuity through the zero set, use the following elementary estimate. If u, v are Lipschitz, with constants K_u, K_v, and $|u| \le C|v|$, the zero-extended quotient u^2/v is Lipschitz with constant $2CK_u + C^2K_v$. When $|b| \le |a|$ and $ab \neq 0$, use

$$\frac{x^2}{a} - \frac{y^2}{b} = \frac{(x-y)(x+y)}{a} + \frac{y^2(b-a)}{ab};$$

if $b = 0$, then $y = 0$. Apply this to the actual phase field and $v = |H|$, using $|U| \le 2^{k-1}|H|$. Near a noncentral filling zero the multipliers have constant tail values wherever $H \neq 0$, and the amplitudes vanish at the other zeros. Thus J is locally Lipschitz on the punctured source disk.

The area theorem gives $g' \in L^2(\mathbb{D})$, hence $p^2, p \in L^1(\mathbb{D})$; also $u^2 \le 2^{2k-2}$. Bounded multipliers give $|J| \le C_k y^{2k-1}(1+|g'|)$, while their linear vanishing at $t = 0$ gives $|J| \le C_k(1-r)y^{2k-2}(1+|g'|)$. Using $y \le 2r$ and $y \le 2$, respectively, obtain

$$|J(z)| \le C_k \min\{r, 1-r\}\big(1 + |g'(\bar{z})|\big). \tag{30}$$

Let η_ε be the product of two unit linear ramps, zero for $r \le \varepsilon$ or $r \ge 1-\varepsilon$, one for $2\varepsilon \le r \le 1-2\varepsilon$, with Lipschitz constant at most $2/\varepsilon$. The field $\eta_\varepsilon J$ is globally Lipschitz with compact support. Absolute continuity along horizontal and vertical lines, the one-dimensional fundamental theorem of calculus and Fubini give

$$\int_{\mathbb{D}} \eta_\varepsilon \operatorname{div} J \, d\sigma = -\int_{\mathbb{D}} \nabla\eta_\varepsilon \cdot J \, d\sigma.$$

Both integrands are integrable. By (30), $|\nabla\eta_\varepsilon \cdot J| \le 4C_k(1 + |g'(\bar{z})|)$. The derivative of the cutoff is eventually zero at each noncentral interior point, so dominated convergence removes the boundary term. Multiply (29) by the cutoff and pass to the limit in the L^1 densities u^2, p^2. No prior integrability assumption on the unweighted divergence is required. □

The scalar multiplier bound. On the increasing part write $x = 12t \in [0, 9/5]$. Use $a = A_k - 1$, $\mathcal{B} = B_k$, $\mathcal{D} = D_k$ within this calculation only. Here $2f \le 3/5 < 3k/(k-1)$, so the absolute value in B_k has a fixed sign. The exact coefficients are

$$\begin{aligned} a &= 1 + (1 - 1/k)x, \\ \mathcal{B} &= 3k/(k-1) + \big(2/3 + 1/(2(k-1)) + 1/(3k)\big)x, \\ \mathcal{D} &= \big(1 + 2/(k-1)\big)x/4. \end{aligned}$$

On the constant part $\chi = 9/5$, $f = 3/10$, and these become

$$a = 4/5 - 6/(5k), \qquad \mathcal{B} = 12/5 + 3/(10k), \qquad \mathcal{D} = 9k/(20(k-1)).$$

Let (a_0, b_0, d_0) be the limits of $(a, \mathcal{B}, \mathcal{D})$ as $k \to \infty$ at fixed t. To see the reason for the cap $9/5$, allow an arbitrary plateau $\chi = c > 1$ while retaining $f = \chi/6$. On the constant part the limiting cost satisfies

$$\frac{c}{4} + \frac{4c^2}{9(c-1)} - \frac{9}{4} = \frac{(5c-9)^2}{36(c-1)}.$$

The cap $c = 9/5$ therefore gives precisely the cost $9/4$ at the origin of the increasing part as well as on the plateau. Inside the increasing part one has

$$4a_0(9/4 - d_0) - b_0^2 = (1+x)(9-x) - (3 + 2x/3)^2 = x(4 - 13x/9) \ge 0.$$

On the plateau $(a_0, b_0, d_0) = (4/5, 12/5, 9/20)$. Thus $d_0 + b_0^2/(4a_0) \le 9/4$ throughout.

For $k \ge 751$ the two explicit formulas above give

$$a \ge (99/100)a_0, \qquad 0 \le \mathcal{B} \le (101/100)b_0, \qquad 0 \le \mathcal{D} \le (101/100)d_0.$$

For example, on the increasing part the excess in $\mathcal{B}$ is at most $(3 + 5x/6)/(k-1) \le b_0/100$; on the plateau the relative loss in a is $3/(2k)$. The remaining bounds follow directly from $2/(k-1) < 1/100$. Completing the square now gives, for all p and u with $p \ge 0$,

$$kp\mathcal{B}|u| - k^2 a u^2 + \mathcal{D}p^2 \le \left(\mathcal{D} + \frac{\mathcal{B}^2}{4a}\right) p^2 \le \frac{101^2}{100 \cdot 99} \frac{9}{4} p^2 < \frac{7}{3} p^2 \quad (p > 0).$$

For $p = 0$ the left side is nonpositive. Either one-sided expression at the corner has this bound, as required in the product rule.

3 Statement and boundary trace

Let Σ_0 be the class of injective holomorphic functions on $|z| > 1$ with expansion

$$F(z) = z + \sum_{j \ge 1} b_j z^{-j}. \tag{31}$$

The absence of a constant term fixes the center. For $p > 0$, let

$$R_p(F) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |F^*(e^{it})|^p \, dt, \qquad S(p) = \frac{\Gamma(p+1)}{\Gamma(1 + p/2)^2},$$

where F^* is the almost-everywhere radial limit, constructed below. Write $D = 1 - |b_1|^2$. The area theorem gives $D \ge 0$.

Baernstein, Laugesen, and Pritsker [5, Section 6, pp. 78–79] already proved $R_2(F) \le 2$ in this exterior normalization and showed that equality is possible exactly for a rotation of $[-2, 2]$. Their argument uses Parseval and the area theorem. We retain the unused area budget to obtain a quantitative second-moment deficit. For the next three even exponents, the same area budget controls a finite weighted sum after the signed-area identity below; low-order Grunsky inequalities handle the remaining coefficient combinations.

Theorem 9. *For every $F \in \Sigma_0$,*

$$R_2(F) \le 2 - \tfrac{1}{2} D = S(2)(1 - D/4), \tag{32}$$
$$R_4(F) \le 6 - 3D = S(4)(1 - D/2), \tag{33}$$
$$R_6(F) \le 20 - 7D - \tfrac{9}{2} D^2, \tag{34}$$
$$R_8(F) \le 70 - \tfrac{51}{5} D. \tag{35}$$

For each $p \in \{2, 4, 6, 8\}$, $R_p(F) = S(p)$ if and only if $F(z) = z + a/z$ for some $|a| = 1$. Thus the four maximal values $S(2) = 2$, $S(4) = 6$, $S(6) = 20$, and $S(8) = 70$ are sharp.

The deficit coefficients in the four inequalities are explicit constants from the proof. The sharpness statement concerns the four maximal values.

Lemma 10. *The radial limit $F^*(e^{it}) = \lim_{r\downarrow 1} F(re^{it})$ exists for almost every t, and for every fixed $p > 0$ its pth power is integrable. Moreover,*

$$R_p(F) = \lim_{r\downarrow 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |F(re^{it})|^p \, dt.$$

Proof. Set $g(u) = F(1/u) - 1/u = \sum_{j\geq 1} b_j u^j$. The area theorem gives $\sum_{j\geq 1} j|b_j|^2 \leq 1$. Parseval on concentric disks and monotone convergence therefore give

$$\int_{\mathbb{D}} |g'(u)|^2 \, dA(u) = \pi \sum_{j\geq 1} j|b_j|^2 \leq \pi.$$

Fubini implies that for almost every t the radial integral of $|g'(\rho e^{it})|^2$ on $1/2 < \rho < 1$ is finite. Since $\rho \geq 1/2$, Cauchy–Schwarz shows that $g(\rho e^{it})$ is Cauchy as $\rho \uparrow 1$. This constructs the asserted trace.

For completeness, the omitted-point bound makes the boundary passage dominated. If w is omitted by a centered exterior map H, the function $u \mapsto 1/(H(1/u) - w) = u + wu^2 + \cdots$ is schlicht on $\mathbb{D}$. The second-coefficient theorem gives $|w| \leq 2$. For each $r > 1$, set $F_r(z) = F(rz)/r$. This is a centered normalized univalent map on $|z| > 1/r$. In particular, for $|z| = 1$ the point $F_r(z)$ is omitted by the restriction of F_r to $|z| > 1$, by injectivity on the larger domain. Hence $|F_r(e^{it})| \leq 2$, or $|F(re^{it})| \leq 2r$. This uniform bound for r in a fixed interval $(1, r_0)$ gives dominated convergence on the circles as $r \downarrow 1$. □

4 Weighted area and the first four even moments

We use the classical area theorem and strong Grunsky inequality in their usual exterior normalization [8, Chapters 1 and 3]. The complex Grunsky coefficients used below are c_{22}, c_{23}, and c_{33}; their Laurent expressions are displayed in the proof.

Lemma 11 (Finite radial area inequality)**.** *Put $\phi(u) = 1 + \sum_{j\geq 1} b_j u^{j+1}$ and $c_d = [u^d]\phi(u)^m$. Then, for each positive integer m,*

$$R_{2m}(F) \leq \sum_{d=0}^{2m-1} \left(2 - \frac{d}{m}\right) |c_d|^2. \tag{36}$$

Proof. First replace F by $F_r(z) = F(rz)/r$, $r > 1$. This map is univalent on a neighborhood of the closed exterior disk. On the unit circle, $F_r(z)^m = \sum_{d\geq 0} c_d^{(r)} z^{m-d}$. Parseval gives its mean squared modulus as $\sum_d |c_d^{(r)}|^2$. Let K_r be the bounded region enclosed by the smooth Jordan curve $F_r(\partial\mathbb{D})$. Apply Green's formula on K_r to the analytic map $w \mapsto w^m$. The oriented image contour, counted with its multiplicity, has signed area $m^2 \int_{K_r} |w|^{2m-2} \, dA(w) \geq 0$. Fourier orthogonality writes the same signed area as $\pi \sum_d (m-d)|c_d^{(r)}|^2$. Adding this nonnegative sum to m times the Parseval sum gives

$$m \sum_{d\geq 0} |c_d^{(r)}|^2 \leq \sum_{d\geq 0} (2m-d)|c_d^{(r)}|^2 \leq \sum_{d<2m} (2m-d)|c_d^{(r)}|^2.$$

The last inequality discards only nonpositive terms. Divide by m, then let $r \downarrow 1$ using Lemma 10; the sum on the right is finite and each coefficient converges. □

Proof of Theorem 9. Write $s = |b_1|^2$, so $0 \leq s \leq 1$ and $D = 1 - s$. Parseval and the area theorem yield

$$R_2(F) = 1 + \sum_{j\geq 1} |b_j|^2 \leq 1 + s + \frac{1}{2} \sum_{j\geq 2} j|b_j|^2 \leq 1 + s + D/2 = 2 - D/2.$$

At $m = 2$, equation (36) expands to

$$R_4(F) \le 2 + 4s + 2|b_2|^2 \le 2 + 4s + D = 6 - 3D.$$

For $m = 3$, the coefficients $c_0, \dots, c_5$ of ϕ^3 are $1, 0, 3b_1, 3b_2, 3(b_3 + b_1^2), 3(b_4 + 2b_1b_2)$. Thus

$$R_6(F) \le 2 + 12s + 9|b_2|^2 + 6|b_3 + b_1^2|^2 + 3|b_4 + 2b_1b_2|^2. \tag{37}$$

Put $x = |b_1|$, $a = |b_2|$, and $b = |b_4|$. Then $0 \le x \le 1$, $|b_4 + 2b_1b_2| \le b + 2xa$, and the square completion is

$$(5 + 6x^2)(2a^2 + 4b^2) - 9a^2 - 3(2xa + b)^2 = (a - 6xb)^2 + (17 - 12x^2)b^2 \ge 0.$$

The area theorem therefore bounds the two terms containing b_2, b_4 in (37) by $(5 + 6x^2)D$.

For the remaining term, define the exterior Grunsky coefficients by

$$-\log \frac{F(z) - F(w)}{z - w} = \sum_{j,k \ge 1} c_{jk} z^{-j} w^{-k}.$$

Expansion gives $c_{22} = b_3 + b_1^2/2$. The strong Grunsky inequality applied to the vector supported at index two says $\sum_{j \ge 1} j|c_{2j}|^2 \le 1/2$; hence $|c_{22}| \le 1/2$. Consequently $|b_3 + b_1^2| \le (1+s)/2$. Combining the estimates gives

$$R_6(F) \le 2 + 12s + (5 + 6s)D + \frac{3}{2}(1 + s)^2 = 20 - 7D - \frac{9}{2}D^2.$$

For $m = 4$ put

$$u = b_4 + b_1b_2, \quad v = b_1b_2, \quad g = b_3 + b_1^2, \quad A = 4b_3 + 6b_1^2,$$
$$w = b_5 + 3b_1b_3 + \tfrac{3}{2}b_2^2 + b_1^3, \qquad y = b_6 + 3b_1b_4 + 3gb_2.$$

Expanding the first eight coefficients in (36) gives

$$R_8(F) \le 2 + 24s + 20|b_2|^2 + |A|^2 + 12|u + 2v|^2 + 8|w|^2 + 4|y|^2. \tag{38}$$

The Grunsky coefficients are $c_{23} = u$ and $c_{33} = b_5 + b_1b_3 + b_2^2 + b_1^3/3$. Applying the strong inequality to vectors supported at indices two and three gives

$$|b_2|^2 + 2|c_{22}|^2 + 3|u|^2 \le \tfrac{1}{2}, \qquad |c_{33}| \le \tfrac{1}{3}. \tag{39}$$

First,

$$|A|^2 = 48|c_{22}|^2 + 24s^2 - 32|b_3|^2, \qquad 12|u + 2v|^2 - 72|u|^2 \le \tfrac{288}{5}|v|^2.$$

The second inequality follows on replacing $|u+2v|$ by $|u|+2|v|$ and completing the square: $\frac{288}{5}|v|^2 - 12(|u| + 2|v|)^2 + 72|u|^2 = \frac{12}{5}(5|u| - 2|v|)^2$. Since $|v|^2 = s|b_2|^2 \le sD/2$, (39) yields

$$20|b_2|^2 + |A|^2 + 12|u + 2v|^2 \le 12 + 24s^2 + \tfrac{144}{5}sD. \tag{40}$$

Next, $w = c_{33} + (2b_1b_3 + \frac{2}{3}b_1^3) + \frac{1}{2}b_2^2$. Expansion of $|b_3 + b_1^2/2|^2 \le 1/4$ gives

$$|2b_1b_3 + \tfrac{2}{3}b_1^3|^2 \le \tfrac{4}{3}s|b_3|^2 + \tfrac{2}{3}s - \tfrac{2}{9}s^3 \le (\tfrac{2}{3} + s|b_3|^2)^2.$$

For the last inequality the difference of the right and left sides is $\frac{2}{9}(1 - s)^2(s + 2) + (s|b_3|^2)^2$. Using (39) and $3|b_3|^2 + 2|b_2|^2 \le D$ gives $|w| \le 1 + s|b_3|^2 + |b_2|^2/2 \le 1 + D/3$. As $0 \le D \le 1$,

$$8|w|^2 \le 8 + \tfrac{16}{3}D + \tfrac{8}{9}D^2 = 8 + \big(\tfrac{56}{9} - \tfrac{8}{9}s\big)D. \tag{41}$$

Finally, weighted Cauchy–Schwarz and the area theorem give

$$|y|^2 \le \left(\tfrac{1}{6} + \tfrac{9}{4}s + \tfrac{9}{2}|g|^2\right)\left(6|b_6|^2 + 4|b_4|^2 + 2|b_2|^2\right).$$

The last factor is at most D, and the diagonal Grunsky estimate above gives $|g| \le (1+s)/2$. Hence

$$4|y|^2 \le \left(\tfrac{2}{3} + 9s + \tfrac{9}{2}(1+s)^2\right) D. \tag{42}$$

Combining (40)–(42) with (38) gives $70 - R_8(F) \ge D\widetilde{P}(s)$, where

$$\widetilde{P}(s) = 48 + 24s - \tfrac{144}{5}s - \tfrac{56}{9} - \tfrac{2}{3} - 9s - \tfrac{9}{2}(1+s)^2 + \tfrac{8}{9}s.$$

Direct expansion gives

$$\widetilde{P}(s) - \frac{51}{5} = \frac{(1-s)(405s+2377)}{90} \ge 0 \qquad (0 \le s \le 1),$$

which proves (35).

The values $S(2) = 2$, $S(4) = 6$, $S(6) = 20$, and $S(8) = 70$ follow from the gamma recurrence. Equality with $S(p)$ in any displayed bound forces $D = 0$. Then the area theorem forces $b_j = 0$ for every $j \ge 2$, so $F(z) = z + b_1/z$ with $|b_1| = 1$. Conversely, that map has the rotated segment $[-2, 2]$ as its boundary image, and its even moments are $\binom{2m}{m}$ for $m = 1, 2, 3, 4$ by the elementary cosine integral. □

5 Every real exponent from four to eight

The four endpoint estimates also control every exponent between four and eight. We use one published comparison in addition to the ingredients of Theorem 9. For a centered capacity-one continuum, the Baernstein–Laugesen–Pritsker projection theorem [5, Theorem 2] says that every real projection of its equilibrium measure is dominated, for convex tests, by the arcsine distribution on $[-2, 2]$. Applied to the omitted continuum of F, rotated by a unit complex number c, and using the harmonic-measure boundary correspondence, this gives

$$\mathbb{E}\,\psi\left(\frac{\mathrm{Re}(cF^*)}{2}\right) \le \mathbb{E}\,\psi(\cos t) \tag{43}$$

for every convex $\psi : \mathbb{R} \to \mathbb{R}$. Filling bounded complementary components changes neither the exterior map nor its harmonic measure; its capacity and conformal centroid are respectively one and zero by the normalization in (31). Thus the cited theorem has precisely the hypotheses required here. The boundary passage in Lemma 10 gives the same comparison for the actual radial trace.

Theorem 12. *For every $F \in \Sigma_0$, every real $4 \le p \le 8$, and $D = 1 - |b_1|^2$,*

$$R_p(F) \le S(p)(1 - D/80). \tag{44}$$

For each fixed p in this interval, $R_p(F) = S(p)$ if and only if $F(z) = z + a/z$ on $|z| > 1$ for some $|a| = 1$.

Proof. Normalize $T = |F^*|/2$, and write $C_q = \mathbb{E}|\cos t|^q = 2^{-q}S(q)$. Theorem 9 gives

$$C_j - \mathbb{E}T^j \ge d_j D, \qquad (d_4, d_6, d_8) = \left(\frac{3}{16}, \frac{7}{64}, \frac{51}{1280}\right). \tag{45}$$

For $j = 6$, division of $20 - R_6 \geq 7D + \frac{9}{2}D^2$ by 2^6 gives a further nonnegative term; the other two values follow by dividing the fourth- and eighth-moment deficits by 2^4 and 2^8.

Choose the phase c with $|c| = 1$ so that the first Fourier mode of $X = \mathrm{Re}(cF^*)/2$ has amplitude $(1 + |b_1|)/2$. Parseval, first on circles and then for the actual trace, yields

$$\mathbb{E}X^2 \geq \frac{(1 + |b_1|)^2}{8}, \qquad \frac{1}{2} - \mathbb{E}X^2 \leq \frac{3D}{8}.$$

The last inequality is the elementary identity $3(1 - x^2) - (4 - (1 + x)^2) = 2x(1 - x) \geq 0$ for $x = |b_1| \in [0, 1]$.

For $a = 4$ or 6, put $p = a + 2\theta$ with $0 \leq \theta \leq 1$, and let

$$h(u) = u^p - \frac{p(1 - \theta)}{a}u^a - \frac{p\theta}{a + 2}u^{a+2}, \qquad L_a = \frac{1}{6(a - 2)} + \frac{4}{7(a - 2)^2}.$$

Weighted arithmetic–geometric mean gives $h'(u) \leq 0$ on $[0, 1]$. The second derivative satisfies

$$h''(u) \geq -2p\theta(1 - \theta)L_a \qquad (0 \leq u \leq 1). \tag{46}$$

Here is the short calculus check of the constant. If $u = e^{-v}$, the secant error in h''/p is that of $f(k) = (k - 1)e^{-(k-2)v}$ between a and $a + 2$. With $w = (k - 2)v$,

$$f''(k) = e^{-w}\left(\frac{w^2 - 2w}{k - 2} + \frac{w^2}{(k - 2)^2}\right) \leq L_a,$$

because the two maxima for $w \geq 0$ are respectively $(2 + 2\sqrt{2})e^{-2-\sqrt{2}} < 1/6$ and $4e^{-2} < 4/7$. For the first comparison, $7/5 < \sqrt{2} < 10/7$ and $\sum_{n=0}^{7}(17/5)^n/n! > 204/7$ give $e^{2+\sqrt{2}} > 12(1 + \sqrt{2})$. For the second, the Taylor sum through degree four at 2 equals 7, and every remaining term is positive. The secant error over a length-two interval is at most $2\theta(1 - \theta)L_a$, proving (46). As $h'(0) = 0$, the even function $u \mapsto h(|u|) + p\theta(1 - \theta)L_a u^2$ is convex on $[-1, 1]$. Continue it by its endpoint tangent lines to obtain a convex test on $\mathbb{R}$.

Since $|X| \leq T \leq 1$ and h is decreasing, $h(T) \leq h(|X|)$. Apply (43) to the preceding convex test and expand both averages. Inserting (45) and the projection second-moment bound gives

$$C_p - \mathbb{E}T^p \geq pDB_a(\theta), \qquad B_a(\theta) = (1 - \theta)A + \theta B - L\theta(1 - \theta), \tag{47}$$

where $A = d_a/a$, $B = d_{a+2}/(a + 2)$ and $L = 3L_a/8$. The numerical data are

a	A	B	L
4	3/64	7/384	19/224
6	7/384	51/10240	13/448

and need only four fraction comparisons. Indeed

$$4 - 3\theta - 9\theta(1 - \theta) = (3\theta - 2)^2 \geq 0$$

implies

$$B_a(\theta) \geq (1 - \theta)(A - 4L/9) + \theta(B - L/9).$$

For $a = 4$ the two endpoint values are $37/4032$ and $71/8064$; for $a = 6$ they are $43/8064$ and $1133/645120$. Each is strictly greater than $1/640$. Hence $C_p - \mathbb{E}T^p \geq pD/640 \geq D/160$. Finally $C_p \leq C_2 = 1/2$ for $p \geq 2$, so $D/160 \geq C_pD/80$. Multiplication by 2^p proves (44).

If $R_p(F) = S(p)$, then (44) and $S(p) > 0$ force $D = 0$. The area theorem gives $b_j = 0$ for $j \geq 2$ and $|b_1| = 1$, so F is a unit rotated slit on the entire open exterior disk. Conversely the slit boundary is a rotated $2\cos t$, and its moment equals $S(p)$. □

6 Zeros of polynomials with connected lemniscates

Proposition 13. *Let P be a complex polynomial of degree $n \geq 2$, with leading coefficient a. Let c be the mean of its zeros, counted with multiplicity. Suppose that $t > 0$ and that $\{|P| \leq t\}$ is connected, and put $R = (t/|a|)^{1/n}$. For every real $4 \leq p \leq 8$ and every integer $0 \leq j < n$,*

$$\frac{1}{n-j} \sum_{P^{(j)}(z)=0} \left| \frac{z-c}{R} \right|^p < S(p).$$

The sum counts multiplicities. For each fixed p and j, the supremum of these means over all such polynomials and levels, allowing n to vary, is $S(p)$ and is never attained.

6.1 The normalized exterior map

Put $Q(z) = P(c + Rz)/(aR^n)$. This polynomial is monic, its roots sum to zero, and $K = \{|Q| \leq 1\}$ is connected. The chain rule gives $Q^{(j)}(z) = R^j P^{(j)}(c+Rz)/(aR^n)$, so the affine change of variables preserves derivative-root multiplicities and reduces the mean to $(n-j)^{-1} \sum_{Q^{(j)}(z)=0} |z|^p$.

We construct the normalized exterior map directly. First, $\Omega = \{|Q| > 1\}$ is connected. The polynomial is open and proper, and its restriction to Ω maps into the exterior unit disk. Each component of Ω is open and closed relative to Ω. Its image under the restriction is nonempty, open and closed relative to the connected exterior unit disk. Consequently each component maps onto that disk and is unbounded. Since K is compact, all these components meet the same connected region outside a sufficiently large circle.

Choose $b \in K$ and put $K_b = K - b$. In the coordinate $u = (z-b)^{-1}$, the reciprocal exterior domain U consists of zero and the inverses of points of $\mathbb{C} \setminus K_b$. It is open and connected, since the complement of K_b is connected, omits zero, and contains all points of sufficiently large modulus. For $\epsilon > 0$, use the open region

$$U_\epsilon = \{u : \epsilon|u| < |1 - vu| \text{ for all } v \in K_b\}.$$

It is open by compactness of K_b. Every point of U is in the component containing zero of some U_ϵ: join it to zero by a path in U and take the minimum of $|1-vu|$ and a bound for $|u|$ on the compact product of the two parameter sets.

For $v \in K_b$, join v to zero by a finite chain in K_b of steps shorter than ϵ. Such chains exist because the set reachable by short steps is both relatively open and relatively closed in K_b. For consecutive points v_i, v_{i+1} of the chain, the ratio $(1 - v_i u)/(1 - v_{i+1} u)$ lies at distance less than one from 1 on U_ϵ. Sum its principal logarithms along the chain to get an analytic logarithm of $1 - vu$ with value zero at $u = 0$. On the components containing zero the resulting logarithms agree where both are defined. Indeed the smaller component lies in the larger one, and the branches agree near zero by local injectivity of the exponential. The identity theorem gives agreement on that component. Thus there is a normalized analytic logarithm on U for every $v \in K_b$.

Let $q_1, \ldots, q_n$ be the roots of Q, with multiplicities. The exponential of the mean of the preceding logarithms for $v = q_i - b$ is an analytic function H on U with

$$H(0) = 1, \qquad H(u)^n = \prod_i (1 - (q_i - b)u), \qquad H'(0) = b.$$

The last identity follows by differentiation and $\sum_i q_i = 0$. Hence $B(z) = (z-b)H((z-b)^{-1})$ is analytic on Ω and $B(z)^n = Q(z)$. It maps Ω into the exterior unit disk and is proper there. For instance, a compact set of target points has compact image under the nth power map, and its

inverse image under B is closed in the compact inverse image under Q of that image. Since B is nonconstant and analytic, its image is open. Properness makes the image closed relative to the exterior unit disk, so it is onto. To show injectivity, take any exterior value w and pick a preimage under B of each of the n distinct nth roots of w^n. These yield n distinct roots of $Q(z) - w^n$, which exhaust its degree. Each chosen value has exactly one preimage.

The inverse F of B therefore satisfies $Q(F(w)) = w^n$ for $|w| > 1$. As $w \to \infty$, also $F(w) \to \infty$ by this identity. With $u = (F(w) - b)^{-1}$ we have $F(w) - w = b - (H(u) - 1)/u \to b - H'(0) = 0$. The reciprocal correction thus extends holomorphically to zero with value zero. This puts F in the centered exterior class.

6.2 Original roots and the actual boundary trace

For $\rho > 1$ the angular logarithmic potential $V_\rho(x) = (2\pi)^{-1} \int_0^{2\pi} \log |F(\rho e^{i\theta}) - x| \, d\theta$ is

$$V_\rho(x) = \frac{1}{n} \log \max(\rho^n, |Q(x)|).$$

Indeed if $F(1/u) = 1/u + g(u)$, the reciprocal function $1 + u(g(u) - x)$ is analytic on the disk, has value 1 at zero, and has no zero except a simple one at $1/w$ if $x = F(w)$. Apply Jensen's formula on radius $1/\rho$ and reverse the angular orientation. The logarithm of the product can be expanded off the single possible zero on the circle, an angular null set. The result is $\log \rho$ on K and $\log \max(\rho, |w|)$ at $F(w)$.

In particular, $V_\rho(x) \geq n^{-1} \log |Q(x)|$. Average over $x = se^{i\alpha}$ for any $s > 0$. The angular mean of $\log |y - se^{i\alpha}|$ is $\log s + \log^+(|y|/s)$. Fubini applies to the double angular integral. The positive part is bounded on the compact product, and the slice integral has the lower bound $\log s$, which gives a uniform bound for the integral of the negative part. Factorization of Q gives

$$\frac{1}{n} \sum_i \log^+(|q_i|/s) \leq \frac{1}{2\pi} \int_0^{2\pi} \log^+(|F(\rho e^{i\theta})|/s) \, d\theta.$$

Here $\log^+ 0$ denotes zero. For $p > 0$ and $a \geq 0$, $a^p = p^2 \int_{\mathbb{R}} e^{pt} \log^+(ae^{-t}) \, dt$. For positive a this follows by substituting $v = \log a - t$ and integrating ve^{-pv} on $(0, \infty)$. For $a = 0$ both sides vanish. Tonelli gives the p-moment comparison on the smooth circle. The actual radial trace and the bound $|F(\rho e^{i\theta})| \leq 2\rho$ justify dominated convergence as $\rho \downarrow 1$. In particular $n^{-1} \sum_i |q_i|^p \leq R_p(F)$ for every $p > 0$.

6.3 Derivative roots, strictness, and sharpness

The de Bruijn–Springer comparison [10, Theorem 4] holds for $p \geq 1$. We give a proof using the differentiator compression of [9]. For a degree N polynomial with roots λ_i listed with multiplicity, let $D = \operatorname{diag}(\lambda_i)$ and let $e = N^{-1/2}(1, \ldots, 1)$. In an orthonormal basis with first vector e, compress D to the last $N-1$ coordinates to obtain A. The determinant of $zI - A$ is the first diagonal cofactor of $zI - D$ in this basis. Transforming the adjugate under change of basis gives the cofactor

$$\frac{1}{N} \sum_i \prod_{\ell \neq i} (z - \lambda_\ell) = \frac{1}{N} \frac{d}{dz} \prod_i (z - \lambda_i).$$

Thus the diagonal of a unitary Schur triangularization of A lists the derivative roots μ_j, with algebraic multiplicity. In ambient coordinates their diagonal formulas are $\mu_j = \sum_i |v_j(i)|^2 \lambda_i$. The weights sum to one in each column. Across the $N - 1$ columns the total in row i is $1 - |e(i)|^2 =$

$(N-1)/N$. The triangle inequality and Jensen for $r \mapsto r^p$ give $|\mu_j|^p \le \sum_i |v_j(i)|^2 |\lambda_i|^p$. Sum and divide by $N-1$, then iterate while the remaining degree is positive. This proves

$$\frac{1}{n-j} \sum_{Q^{(j)}(z)=0} |z|^p \le \frac{1}{n} \sum_i |q_i|^p \le R_p(F) \le S(p).$$

The last bound follows from Theorem 12. Equality there would give $F(w) = w + \lambda/w$ with $|\lambda| = 1$ on $|w| > 1$. Then the polynomial identity $w^n Q(w + \lambda/w) = w^{2n}$ is impossible. The left side has constant term λ^n because Q is monic and all lower terms give strictly positive powers of w. The right side has zero constant term. This proves strictness.

For sharpness take $Q_n(z) = 2T_n(z/2)$, where T_n is the ordinary Chebyshev polynomial. It is monic and centered and its 2-lemniscate contains the interval $[-2, 2]$, which joins all its zeros. If the sublevel set split into two nonempty relatively open and closed sets, openness and properness of the restricted polynomial would make both images equal the whole value disk. Both sets would then meet the zero fiber, contradicting that the interval joins all of it. The standard cosine grid for the zeros of T_n tends to the arcsine distribution. For fixed j the zeros of $T_n^{(j)}$ have the same limit by repeated Rolle interlacing. The radius $R = 2^{1/n}$ tends to one. The p-moment therefore converges to $\frac{1}{2\pi} \int_0^{2\pi} |2\cos\theta|^p \, d\theta = S(p)$. Together with strictness this proves the exact supremum and nonattainment.

7 Factor norms and regular stars

For a nondegenerate plane continuum K let $C(K)$ be the least constant for which

$$\prod_{i=1}^{r} \|p_i\|_K \le C(K)^{\sum_i \deg p_i} \Big\| \prod_{i=1}^{r} p_i \Big\|_K$$

for every finite collection of nonzero polynomials. Set $S_m = \bigcup_{j=0}^{m-1} e^{2\pi i j/m}[0, 4^{1/m}]$; it has logarithmic capacity one. The next theorem answers the regular-star question posed after Theorem 2.3 of [7] for plane continua.

Theorem 14 (Regular-star parity)**.** *For every integer $m \ge 2$, the comparison $C(K) \le C(S_m)$ holds for every nondegenerate plane continuum with m-fold rotational symmetry if and only if m is even. For $m = 2k$, equality holds precisely for similarity images of S_{2k}. Translate the symmetry center to zero, fill the bounded complementary components, and scale to capacity one. If b_{2k-1} is the* forward *coefficient of $z^{-(2k-1)}$ in the normalized exterior map of the filled set, then*

$$\log \frac{C(S_{2k})}{C(K)} \ge \frac{1 - k^2 |b_{2k-1}|^2}{96k^2}. \tag{48}$$

For every odd $m \ge 3$ there is an m-fold rotationally symmetric continuum K_m with $C(K_m) > C(S_m)$.

Proof. Translate the rotation center to zero, fill the bounded complementary components, and normalize the capacity to one as in Section A.1. Filling preserves polynomial norms by the maximum modulus principle, and each similarity preserves the factor constant by polynomial composition. The filled set contains the rotation center. If equality forces the filled set to be a star, the star's empty interior gives equality with the original continuum in the same coordinates. The farthest-point formula (50) (see also [7, Theorem 1.1]) reads

$$\log C(K) = \int \log d_K(w) \, d\mu_K(w), \qquad d_K(w) = \max_{v \in K} |w - v|.$$

Suppose $m = 2k$. Rotational covariance of the normalized exterior map gives $F(\eta z) = \eta F(z)$ for $\eta = e^{\pi i/k}$. Put $E = \{w^k : w \in K\}$. It is compact, connected, contains zero, has capacity one, and satisfies $E = -E$. Thus every hypothesis of the cited symmetric comparison holds. Moreover, $G(z^k) = F(z)^k$ is its normalized odd exterior map. The power map pushes μ_K to μ_E. It sends S_{2k} to $[-2, 2]$. Writing $F(z) = z + b_{2k-1}z^{1-2k} + \cdots$ shows directly that

$$G(\zeta) = \zeta + \beta\zeta^{-1} + \cdots, \qquad \beta = kb_{2k-1}.$$

The omitted-point coefficient bound applied to G gives $|v| \le 2$ for $v \in E$, hence $|w| \le 2^{1/k}$ for $w \in K$. Let $a = 2^{1/k}$. The function $\Phi(x) = \log(a+e^{x/k}) - e^{2x}/(64k^2)$ is convex on $(-\infty, \log 2]$: with $y = e^{x/k}/a \le 1$, the second derivative of the logarithm is $y/[k^2(1+y)^2] \ge y^{2k}/(4k^2) = e^{2x}/(16k^2)$. Extend Φ by its tangent at $\log 2$ and apply the symmetric logarithmic comparison from Section A.6. This is the comparison of [5, Corollary 6.3] for the admissible test used here. As $d_K(w) \le a + |w|$ and equality holds for the opposite tip on the star, it yields

$$\log \frac{C(S_{2k})}{C(K)} \ge \frac{2 - \int_E |v|^2 \, d\mu_E(v)}{64k^2}.$$

Oddness makes the remaining negative modes of G start at index three. Parseval and the area theorem therefore give $\int_E |v|^2 \, d\mu_E(v) \le 1 + |\beta|^2 + (1 - |\beta|^2)/3$. The corresponding actual odd exterior-map boundary estimate, including the passage to its radial trace, is kernel checked independently of the factor-norm formula and the all-order inverse-coefficient problem. Substitution proves (48). If $C(K) = C(S_{2k})$, then $|\beta| = 1$; area rigidity gives $G(\zeta) = \zeta + \beta/\zeta$. Its omitted set is a rotated segment, whose full power preimage is a rotated S_{2k}. The converse follows by similarity invariance.

Now fix odd $m \ge 3$, write $s = \sqrt{1 - t^2}$, and for small real t define

$$G_t(z) = \frac{(z - it + s)^2}{z - it}, \qquad E_t = \mathbb{C} \setminus G_t(\{|z| > 1\}), \qquad K_t = \{w : w^m \in E_t\}.$$

If $G_t(z_1) = G_t(z_2)$ with $z_1 \ne z_2$, the preimages are paired by $z \mapsto (1 + itz)/(z - it)$, and $|z - it|^2 - |1 + itz|^2 = (1 - t^2)(|z|^2 - 1)$. Thus G_t is exterior univalent. The two critical points $it \pm s$ lie on the unit circle and have values $0, 4s$. The pairing maps the circle to itself and identifies its two arcs between these points. Each arc maps injectively except at its endpoints, so the omitted set E_t is their common simple image arc. In particular it contains zero; its full mth-root preimage K_t is a continuum. The m lifts of the arc with its zero endpoint removed are pairwise disjoint, and their closures meet only at zero. They have exactly m outer endpoints. Any rotational symmetry preserves the unique branch point 0 and permutes those endpoints. The normalized exterior map of K_t is $F_t(z) = z \exp((\log(G_t(z^m)/z^m))/m)$. In the coordinate $u = 1/z$ the argument of the logarithm is nonvanishing and analytic on $\mathbb{D}$ with value 1 at zero. Choose the analytic logarithm with value zero at $u = 0$. This gives $F_t(z) = z + O(z^{1-m})$. The identity $F_t(z)^m = G_t(z^m)$ and the covariance $F_t(\eta z) = \eta F_t(z)$ for $\eta^m = 1$ show injectivity and surjectivity onto $\mathbb{C} \setminus K_t$. It follows that K_t is a capacity-one continuum with all tips $R_t\eta$, $\eta^m = 1$, $R_t = (4s)^{1/m}$, and $K_0 = S_m$. With $\zeta = e^{i\theta}$ one has

$$G_t((s - it)\zeta) = (2 + 2\cos\theta)L_t(\zeta), \qquad L_t(\zeta) = \frac{s - it}{1 - it/((s - it)\zeta)}.$$

As $t \to 0$, $L_t \to 1$ uniformly on the unit circle. For small $|t|$ take its logarithm in the right half-plane with $\log 1 = 0$, so $\log L_0 = 0$ at every angle. Set $r = (2 + 2\cos\theta)^{1/m}$, $Z_t = r \exp((\log L_t)/m)$ (zero at $\theta = \pi$), and $\delta = \pi - \pi/m$. The source-circle map $z \mapsto z^m$ followed by rotation by $(s - it)^{-1}$ preserves normalized angular measure. The radial boundary value of F_t is a root of $G_t((s - it)e^{i\theta})$,

so is a rotation of Z_t by an mth root of unity. The farthest distance to K_t is invariant under such rotations. Using the tip $R_t e^{i\delta}$ gives

$$\log C(K_t) \geq A(t) = \frac{1}{2\pi} \int_0^{2\pi} \log |Z_t - R_t e^{i\delta}| \, d\theta.$$

At $t = 0$, $Z_0 = r$ and there is equality. The squared distance on each radial segment is convex, and the tips at angles $\pm\delta$ are farther than the other tips and the origin. Moreover $D = r^2 + R_0^2 - 2rR_0 \cos\delta \geq R_0^2$. Uniform continuity keeps the distance away from zero for small $|t|$, and the t-derivative of the integrand is bounded there. This justifies differentiation under the integral. Since $R_0' = 0$ and $\partial_t \log L_t|_{t=0} = \sin\theta + i(\cos\theta - 1)$, the radial term integrates to zero and

$$A'(0) = \frac{R_0 \sin\delta}{2\pi m} \int_0^{2\pi} \frac{r(1 - \cos\theta)}{D} \, d\theta > 0.$$

Thus $C(K_t) > C(S_m)$ for sufficiently small positive t. □

A Exterior normalization and equilibrium estimates

A.1 Exterior normalization of a plane continuum

Let L be a nondegenerate compact connected set after filling its bounded complementary components. For $a \in L$, the reciprocal coordinate $u = (z - a)^{-1}$ carries the exterior, including infinity, to an open connected set U containing zero. Its spherical complement is the image of L under the same Möbius transformation, with a mapped to infinity. It is connected and contains at least one finite point.

Here is a direct primitive argument for this domain. If $P \subset U$ is compact, its filling is contained in U. Indeed, the open unbounded component of $\mathbb{C} \setminus P$, together with infinity, and the union of the bounded components separate the complement of P on the sphere. The connected spherical complement of U lies in the first set because it contains infinity.

Any two points of an open connected plane domain can be joined by a finite horizontal and vertical walk. The points reachable from a fixed point form a relatively open and relatively closed subset, since a small rectangle contained in the domain can be crossed in two axis-parallel segments. Let P be the trace of a closed walk in U. Subdivide its edges at the real and imaginary coordinates of all vertices. Add an outer row and column on each side of the resulting rectangular grid. Write H_{ij}, V_{ij} for the signed counts of rightward and upward traversals. At each vertex the total incoming count equals the total outgoing count. Thus with

$$W_{ij} = \sum_{t=0}^{j} H_{it}$$

we have $H_{ij} = W_{ij} - W_{i,j-1}$ and $V_{ij} = W_{i-1,j} - W_{ij}$. These identities use zero weights outside the grid. The horizontal identity follows by subtraction and the vertical identity follows by summing the vertex balance up the column. Along the top row the vertical counts vanish, so the top weights coincide with the zero weight of the left outer column.

The union of closed cells with $W_{ij} \neq 0$ is compact and has boundary contained in P. At a point off P, two incident cells sharing an edge have the same weight, because their shared edge has zero traversal count. The same holds for cells meeting at a vertex by passing through the other incident cells. Each such point is therefore interior to the union whenever it lies in it. Each complementary

component of P meeting this union stays in the union and is bounded. All the cells with nonzero weight lie in the filling of P and hence in U.

An integral along the walk is now the sum of the rectangle integrals with weights W_{ij}, by cancellation of the two contributions to each internal edge. The Cauchy theorem for rectangles makes this sum zero for a holomorphic integrand. Integrals from a base point are consequently independent of the walk. On a small disk they differ from the usual local primitive by a constant, which gives a global primitive. For a nonvanishing holomorphic q, integrate q'/q and choose the constant of integration so that $e^P = q$ at the base point. Since $(qe^{-P})' = 0$, the identity holds on the domain and $e^{P/2}$ is a square root.

For completeness, this square-root property yields a disk uniformization as follows. For any omitted finite point a_0, take a branch of $\sqrt{z - a_0}$. It is injective and its open image is disjoint from the negative of the image. A small closed disk in the negative image is therefore omitted. Inversion at its center followed by scaling gives an embedding into the unit disk. A disk automorphism sends the image of the base point to zero. Maximize the derivative magnitude there over these normalized embeddings. Cauchy estimates and compact exhaustion give local uniform convergence of a maximizing subsequence. The limit is holomorphic, maps into the disk by the maximum principle, and has nonzero derivative at the base point. It is injective by the zero-free limit principle applied locally to differences of the embeddings. For that principle, a nonconstant limit with an isolated zero is bounded away from zero on a small surrounding circle. The maximum modulus principle bounds the reciprocals of the approximating functions on the disk, contradicting convergence to zero at the center.

If the extremal embedding f omits b in the unit disk, put $\varphi_b(w) = (w - b)/(1 - \bar{b}w)$ and take $g^2 = \varphi_b \circ f$. The function g is an embedding into the disk. If d is its value at the base point, $G = \varphi_d \circ g$ is normalized. Then $f = \psi \circ G$ with $\psi = \varphi_{-b} \circ (w \mapsto w^2) \circ \varphi_{-d}$. This disk self-map fixes zero and fails to be injective, so the strict Schwarz inequality gives $|\psi'(0)| < 1$. The chain rule contradicts extremality. The inverse of f uniformizes U by the disk and fixes zero when the base point is zero.

Return to the filled continuum with an even rotational order and center zero. If zero were in the exterior, the rotation would induce a nontrivial disk automorphism with distinct fixed points corresponding to zero and infinity. Schwarz rules this out. We can therefore use the reciprocal coordinate at $a = 0$. Write its disk uniformization as $u = f(t)$, with $f(0) = 0$. Conjugating central reflection by f gives a disk involution fixing zero. Its multiplier is -1 by Schwarz and the fact that the involution is nontrivial. Thus f is odd. With $a_1 = f'(0) \neq 0$, the formula

$$F(w) = a_1/f(1/w)$$

gives an exterior univalent map with expansion $w + O(w^{-1})$. Its omitted set is $a_1 L$. A rotation of the normalized map and its argument replaces $a_1 L$ by $|a_1| L$. Section A.3 identifies the capacity of this normalized omitted set as one.

A.2 Logarithmic potential of the actual radial law

Let F be centered exterior univalent, let $K = \mathbb{C} \setminus F(\{|z| > 1\})$, and let ν_F be the image of normalized angular measure under its actual radial trace. For $0 < s < 1$, put

$$K_s = \mathbb{C} \setminus F(\{|z| > s^{-1}\}), \qquad \nu_s = (\theta \mapsto F(s^{-1}e^{i\theta}))_{\#} \frac{d\theta}{2\pi}.$$

The radial limit gives weak convergence of ν_s to ν_F. The probability ν_F is carried by K. Indeed, if a radial limit were an exterior image point, the local inverse would force its source coordinate to be both in the open exterior and on the unit circle.

Write $U_s(x) = \int \log|z-x|\, d\nu_s(z)$. Jensen's formula for the reciprocal germ $u(F(1/u)-x)$ gives $U_s(x) = -\log s$ on K_s. More generally $U_s(F(w)) = \log\max(s^{-1}, |w|)$ when $|w| > 1$. At such an exterior image point the near-boundary circles are uniformly separated from the point. Dominated convergence gives $\int \log|z - F(w)|\, d\nu_F(z) = \log|w|$.

For a point $x \in K$, fix $0 < \varepsilon \le 1$ and use $\log\max(\varepsilon, |z-x|)$ as a test function. The supports are uniformly bounded near $s = 1$, so we may cap the distance at 5 to make the test bounded on the plane. Its ν_s integral is at least $-\log s$. Passing to ν_F shows that the truncated potential is nonnegative. For $\varepsilon = e^{-n}$, subtract the test from $\log 5$ and apply monotone convergence. The resulting nonnegative kernel is infinite at $z = x$, and its integral is at most $\log 5$. Thus $\nu_F\{x\} = 0$, the logarithmic potential at x is integrable, and its integral is nonnegative.

To bound that integral from above, fix $0 < s_0 < 1$. The point x is interior to K_{s_0}. The images of bounded closed source annuli are compact, while $F(w) = w + o(1)$ at infinity. Together these facts separate x from the image of $\{|w| \ge s_0^{-1}\}$. Choose $\varepsilon > 0$ small enough that the circle of radius ε centered at x lies in K_{s_0}. For $s_0 < s < 1$, the mean over that circle of U_s is at most $-\log s_0$, by the two Jensen formulas above. The same mean equals $\int \log\max(\varepsilon, |z-x|)\, d\nu_s(z)$, by the elementary logarithmic circle average and Fubini's theorem. The logarithmic angular kernel is integrable on the product of the two circles. Its positive part is bounded on their compact images, and its negative part is controlled by the explicit logarithmic circle averages. Clipping at 5 preserves the upper bound. Passing to ν_F and then letting s_0 tend to 1 gives

$$\int \log|z-x|\, d\nu_F(z) = 0 \qquad (x \in K). \tag{49}$$

The same bounded-positive-part argument and (49) give an integrable mutual kernel with integral zero against $\lambda \otimes \nu_F$ for every probability λ carried by K. The diagonal has zero mass for this product.

A.3 The energy minimizing property

On $K \times K$, distances are at most 4. For finite distances $0 < d \le 4$, use the shifted logarithmic kernel

$$W(d) = \log(25/d^2) = \int_0^\infty \frac{e^{-td^2} - e^{-25t}}{t}\, dt.$$

The identity is Frullani's integral. Its integrand is nonnegative and the integral is infinite at $d = 0$, as follows by letting d decrease to zero. This shift keeps every energy integral in the extended nonnegative reals. Subtracting $2\log 5$ and dividing by 2 recovers the usual negative logarithmic energy for probabilities.

For probabilities α, β carried by K, put $H_t(\alpha,\beta) = \iint e^{-t|z-w|^2}\, d\alpha(z) d\beta(w)$. If γ is the standard Gaussian law on $\mathbb{R}^2$, its characteristic function gives, for $t > 0$,

$$\begin{aligned} &H_t(\alpha,\alpha) + H_t(\beta,\beta) - H_t(\alpha,\beta) - H_t(\beta,\alpha) \\ &\quad = \int_{\mathbb{R}^2} \left|\widehat{\alpha}(\sqrt{2t}\, v) - \widehat{\beta}(\sqrt{2t}\, v)\right|^2 d\gamma(v) \ge 0. \end{aligned}$$

The expectation of $\exp(i\sqrt{2t}\langle v, z-w\rangle)$ under γ is $e^{-t|z-w|^2}$. Fubini's theorem for this bounded kernel gives the displayed identity. Its right-hand side vanishes only if the continuous characteristic functions agree everywhere, since every nonempty open set has positive Gaussian measure. The uniqueness theorem for characteristic functions then gives $\alpha = \beta$.

Apply Tonelli's theorem to W. The mutual energy statement following (49) makes the two mixed W energies of ν_F and any probability λ on K equal to $2\log 5$. The self W energy of ν_F also equals $2\log 5$. Integrating the nonnegative Gaussian difference with weight dt/t shows that the self W energy of λ is at least $2\log 5$, allowing the value $+\infty$. In the equality case all four energies are finite and the Gaussian difference vanishes for almost every positive t. Taking one such scale proves $\lambda = \nu_F$. Hence ν_F is the unique logarithmic equilibrium probability of the actual omitted set and $\operatorname{cap}(K) = 1$.

A.4 The lower bound for admissible factor constants

Let $d_K(z) = \max_{w\in K} |z-w|$ and suppose K has two distinct points. Fix $0 < s < 1$. For N equally spaced angles θ_i, set $a_i = F(s^{-1}e^{i\theta_i})$ and $Q_N(z) = \prod_i (z-a_i)$. These points lie outside K. On the compact set K times the angle interval, the function

$$(x,\theta) \longmapsto \log d_K(F(s^{-1}e^{i\theta})) - \log|F(s^{-1}e^{i\theta}) - x|$$

is continuous. Its equally spaced Riemann sums converge uniformly in x. The angular mean is independent of $x \in K$, by the smooth Jensen potential, and equals $\int \log d_K \, d\nu_s + \log s$. Evaluate the sum at a point of K maximizing $|Q_N|$. The defining factor inequality applied to the N linear factors bounds that normalized sum by $\log c$ for each admissible constant c. Passing to the limit proves

$$\int \log d_K \, d\nu_s + \log s \le \log c.$$

The near-boundary outer laws are carried by a common compact disk, and d_K is bounded below by half the distance between the two distinct points. Consequently bounded continuous tests give $\int \log d_K \, d\nu_F \le \log c$ as $s \uparrow 1$. The reverse inequality follows from auxiliary polynomials and the Bernstein–Walsh bound.

A.5 The factor constant on an actual exterior omitted set

Let K be the nondegenerate omitted set of a normalized centered exterior map F, and let ν_F be its actual radial probability. The normalized farthest-point formula of Pritsker and Ruscheweyh [7, Theorem 1.1] is

$$C(K) = \exp\left(\int \log d_K(z)\, d\nu_F(z)\right). \tag{50}$$

The lower bound for every admissible constant was proved above. For the reverse bound, set $g_K(b) = \int \log|z-b|\, d\nu_F(z)$. In the exterior image this is $\log|F^{-1}(b)|$, and on K it is zero. For a monic polynomial P of degree n, the maximum principle applied to $w^{-n}P(F(w))$ on $|w| \ge s^{-1} > 1$, including infinity, gives

$$\log|P(b)| \le \log\|P\|_K + n g_K(b).$$

To see the limiting boundary bound, the curves $F(|w| = s^{-1})$ lie in a fixed compact disk for s near one. Every cluster point as $s \uparrow 1$ is in K, since an exterior limit would have an inverse of modulus greater than one. The displayed inequality is also valid at zeros with the convention $\log 0 = -\infty$.

Write $P(z) = \prod_{i=1}^n (z-a_i)$. For an auxiliary polynomial Q of degree m, with nonzero leading coefficient and $Q(a_i) \ne 0$, apply the preceding estimate to P at the roots of Q, counting multiplicities. The leading coefficient of Q cancels after writing the pairwise distances in the two orders. This gives

$$\sum_{i=1}^n \log|Q(a_i)| \le m\log\|P\|_K + n\int \log|Q(z)|\, d\nu_F(z). \tag{51}$$

Each integral is finite by the pointwise logarithmic integrability of ν_F established in §A.3 and its preceding potential calculation.

Choose a finite δ-net $b_1, \dots, b_N$ in K, including two distinct points, and let $Q(z) = \sum_{j=1}^{N} \xi_j (z - b_j)^m$, where the ξ_j are independent standard complex Gaussians with real and imaginary variances one. The leading coefficient is nonzero almost surely. For fixed z, $Q(z)$ has the law of a standard complex Gaussian multiplied by $\sigma(z) = (\sum_j |z - b_j|^{2m})^{1/2} > 0$. Thus the expectation of $\log |Q(z)|$ is $\log \sigma(z) + c_0$, with the same c_0 at every z. The expectation of its absolute value is at most $|\log \sigma(z)| + \mathbb{E}|\log |\xi_1||$. The last expectation is finite by the Gaussian density in polar coordinates, and $\log \sigma$ is bounded on K. Fubini therefore applies to the right side of (51). The two Gaussian constants cancel after taking expectations. Dividing by m yields the same inequality with $\log |Q|$ replaced by $m^{-1} \log \sigma$ and the coefficient of $\log \|P\|_K$ reduced to one.

If $d_B(z) = \max_j |z - b_j|$, then $d_B \le d_K \le d_B + \delta$ and

$$\log d_B(z) \le m^{-1} \log \sigma(z) \le \log d_B(z) + \frac{\log N}{2m}.$$

Half the distance between the two fixed points of the net bounds d_B below everywhere. First taking δ small with those two points fixed, then taking m large, gives arbitrarily small uniform error relative to $\log d_K$. Exponentiation proves the factor inequality for all finite linear families with the constant in (50). Factoring each nonzero polynomial over $\mathbb{C}$ and applying submultiplicativity of the uniform norm proves admissibility for arbitrary families. A zero factor makes both sides zero. This proves (50).

A.6 A direct Green comparison for the even power quotient

For an actual odd normalized exterior map G with connected omitted set E, write $g(z) = \log |G^{-1}(z)|$ on the exterior image and $g = 0$ on E. Oddness makes $g(-z) = g(z)$ and places 0 in E: a preimage of zero in the open exterior disk would have a distinct negative preimage. The zero extension is continuous by the outer collars and is harmonic wherever it is positive. Let g_L be the corresponding Green function for the segment $L = [-2, 2]$. We show that the circular mean of g on every centered circle is at most that of g_L. This is the symmetric case of the comparison used in [5, Section 6].

Work in the logarithmic chart $z \mapsto i \exp(z/2)$ and set $u_\tau(x, t) = (g(i \exp((x + it)/2)) - \tau)_+$ for $\tau > 0$. For $0 \le h \le \pi$, let $U_\tau(x, h)$ be the supremum of the integral of $u_\tau(x, \cdot)$ over measurable angular sets of length $2h$ in one turn. This supremum is independent of the turn by periodicity. It is attained by taking the strict superlevel at a quantile and a subset of its level of the required remaining length. Equivalently, it is the infimum over real a of

$$2ha + \int_{-\pi}^{\pi} (u_\tau(x, t) - a)_+ \, dt.$$

The two formulas give continuity in (x, h) up to the angular edges. For instance, a uniform change of size δ in the angular profile changes a top integral by at most $2\pi\delta$, and changing the length changes the integral by at most the length difference times a common uniform bound for the profiles.

For $0 < h < \pi$ one can choose a maximizing set containing a short interval and avoiding margins around the angular cut. Indeed positive levels of the untruncated profile are null unless that angular profile is constant. The profile is real analytic at positive values, and a positive level that is locally constant propagates through its positive component by analyticity and continuity. A positive-level quantile therefore leaves a strict lower interval for the cut and includes a strict upper interval. At a zero-level quantile of u_τ, the set where $g(i \exp((x + it)/2)) < \tau$ has positive length. Choose the

cut there and take the missing mass from the zero plateau, keeping short margins at the cut. A constant clipped profile admits an interval of the prescribed length.

These sets give the local submean for U_τ. For a small circular increment $(\rho\cos\theta, \pm\rho\sin\theta)$ with $0 \le \theta \le \pi$, put $d = \rho\sin\theta$ and translate the maximizing set A to $A - d$ and $A + d$. Split each translated set at a point p inside the included interval. Join the left piece of $A - d$ to the right piece of $A + d$, and join the two remaining pieces. Since $[p - d, p + d] \subset A$, the new masses are $2(h+d)$ and $2(h-d)$. Their combined indicators equal those of the two translates. The margins keep both candidates in one turn. At the original vertical centers where $u_\tau > 0$, the unclipped profile is harmonic in a common neighborhood, since the positive cutoff keeps those centers in a compact subset of the positive domain. Thus u_τ, the maximum with zero, satisfies the circle submean there. At a zero center the submean is immediate from nonnegativity. Integrating over the maximizing set and pairing opposite angular increments proves the claimed submean.

Let $V(x, h)$ be the analogous top integral of the unclipped slit profile. The confocal ellipse formula

$$\cosh g_L(z) = \frac{|z - 2| + |z + 2|}{4}$$

shows that its angular top sets are the centered arcs $[-h, h]$ in this chart. On the open upper strip the arc integral is harmonic. In fact the holomorphic function $\log(-iG_L^{-1}(i\exp(z/2)))$ on $|\Im z| < \pi$ has real part equal to the slit Green profile and is real on the center line. The argument of this logarithm has positive real part. Integrating its real part on a vertical half-arc gives the difference of the imaginary parts of a local holomorphic primitive at the endpoints. The primitive's imaginary part is constant on the center line.

At $h = 0$ both top integrals vanish. At $h = \pi$, if the physical radius is at least the maximum radius of E, the full mean of g equals the logarithm of that radius. The full slit mean is at least this logarithm by Jensen's formula for the circular log kernel. Otherwise the circle meets E, by connectedness and $0 \in E$. The clipped profile vanishes on a short arc there, so its full top integral stays constant on moving a short distance inward from $h = \pi$. The slit top integral is nondecreasing in h. As $x \to -\infty$ the profiles tend uniformly to zero. As $x \to +\infty$ each unclipped Green profile differs uniformly from $x/2$ by a quantity tending to zero, since the normalized exterior maps are $w + O(1/w)$ at infinity.

On a wide compact strip rectangle compare $U_\tau - V$ after adding $-\epsilon h + \epsilon h^2/(4\pi)$. The quadratic gives a strict submean at interior points. At a top-edge point controlled by a zero arc, moving inward increases the perturbed difference. All other boundary values are at most ϵ once the rectangle is wide enough. The maximum principle gives $U_\tau - V \le \epsilon(1 + \pi)$. Let $\epsilon \downarrow 0$ and then $\tau \downarrow 0$, using the uniform bound $2\pi\tau$ for the change in a top integral. At $h = \pi$ the chart covers a half-turn of the physical circle with twice the angular parameter. Odd symmetry gives the desired full-circle comparison.

To pass from this comparison to the test in the even-order proof, let μ_E be the equilibrium probability measure. Capacity one gives $g(z) = \int \log|z - v|\, d\mu_E(v)$, and $g(0) = 0$ gives $\int \log|v|\, d\mu_E(v) = 0$. The same holds for L. The circular average of $\log|z - v|$ on $|z| = r > 0$ equals $\max\{\log r, \log|v|\}$. Averaging the potential therefore gives, for every $a = \log r$, a comparison of the integrals of $(\log|v| - a)_+$ for E and L. These tests, together with the equal means, give the convex comparison for the logarithmic radii by piecewise linear approximation. The test used above is convex on $(-\infty, \log 2]$; its tangent line continues it to the right.

Verification

The proofs use the stated analytic comparisons and the displayed scalar estimates. The single multiplier reduces the energy bound to two explicit quadratic bounds. The final phase budgets use the rational bounds and elementary series estimates in Section 2. These calculations and the finite low-order coefficient expansions can be checked from the formulas in the text. The main mathematical statements are formalized and machine-checked in the companion Lean 4 development.

Declaration of AI use

The manuscript text was primarily AI-generated. OpenAI's ChatGPT and Codex were used for proof exploration, Lean 4 formalization, and drafting and revising the exposition. The author has checked and understood parts of the contribution.

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