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TRACIAL JOINT SPECTRAL MEASURES IN FINITE VON NEUMANN ALGEBRAS AND HANNER INEQUALITIES IN NONCOMMUTATIVE L_p

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Abstract. We construct a positive tracial joint spectral measure for every bounded self-adjoint pair in a finite von Neumann algebra equipped with a faithful normal tracial state. For pencils invertible in every real direction, we obtain a density from Brown's logarithmic potential formula. A 2×2 enlargement and a norm limit extend positivity to arbitrary pencils. Consequently, every real two-dimensional subspace of a Haagerup L_p-space embeds real-linearly and isometrically into a commutative L_p-space for $0 < p < \infty$. This gives the Hanner inequalities for $1 \le p < \infty$. Further consequences include commuting matrix models for shuffle traces with a fixed even number of occurrences and a sign theorem for higher-order spectral shift functions of bounded tracial Schatten perturbations in semifinite von Neumann algebras.

The time available precluded an exhaustive literature search. Readers who find substantial overlap with their own work are invited to contact the author at 3565586261@qq.com.

1. Introduction

For a bounded self-adjoint pair A, B, the tracial laws of $xA + yB$ are indexed by directions $(x, y) \in \mathbb{R}^2$. Heinävaara showed that, for matrices, a radial average of these laws is represented by the linear projections of a single positive measure on the punctured plane [10]. This paper establishes the representation in arbitrary finite tracial von Neumann algebras.

Let $(\mathcal{M}, \tau)$ be a von Neumann algebra with a faithful normal tracial state. For bounded self-adjoint $A, B \in \mathcal{M}$, let $K_{A,B}$ be the closed convex hull of 0 and the pairs $(\omega(A), \omega(B))$ as ω ranges over states of $\mathcal{M}$. For a Borel function $f : \mathbb{R} \to \mathbb{C}$ with $f(0) = 0$ and

$$\int_{-L}^{L} \frac{|f(s)|}{|s|}\,\mathrm{d}s < \infty \qquad (0 < L < \infty),$$

write

$$(\mathcal{H}f)(t) = \int_0^1 \frac{1-r}{r} f(rt)\,\mathrm{d}r.$$

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The transform is bounded on compact intervals by the displayed integrability condition.

Theorem 1.1. *There is a unique positive Radon measure $\mu_{A,B}$ on $\mathbb{R}^2 \setminus \{0\}$, carried by $K_{A,B} \setminus \{0\}$, with*

$$\tau\big((\mathcal{H}f)(xA+yB)\big) = \int f(xa+yb)\,\mathrm{d}\mu_{A,B}(a,b), \qquad x,y \in \mathbb{R}. \tag{1.1}$$

The integral is absolutely convergent and $\int (|a|+|b|)\,\mathrm{d}\mu_{A,B} = \tau(|A|+|B|)/2$.

Păunescu, Rădulescu, and Thom proved the positive-pair measure representation as part of their proof of the Bessis–Moussa–Villani conjecture in finite von Neumann algebras [14, Theorem B]. Their argument also handles a pencil with a nonzero semidefinite direction. For instance, if $A \geq c\mathbf{1}$ with $c > 0$ and $B = B^*$, the scalar construction in their proof of Proposition 4.2 takes $J_a = aA^{-1/2}BA^{-1/2}$, with spectral support now allowed on both sides of zero. The resolvents used for the coefficient identity remain positive for sufficiently small real t, so their argument gives positivity of the distribution weighted by a. The support of the original distribution satisfies $|b| \leq (\|B\|/c)a$, by the Fourier support bound proved in Section 4. Dividing by a gives positivity off the origin. The approximation $A + \varepsilon\mathbf{1}$ and linear changes of coordinates then handle a semidefinite direction. This also explains the sign-definite perturbation case of the trace-derivative application below [14, Propositions 3.2 and 4.2 and Lemma 5.1].

The proof for arbitrary (A, B) has two main steps. First, a 2×2 enlargement makes every pencil $xA + yB$ with $x^2 + y^2 = 1$ invertible and preserves the original pair in a norm limit. Second, we adapt Heinävaara's angular Fourier calculation to finite tracial algebras by replacing the eigenvalue sum with Brown's logarithmic potential. The uniform gap controls the angular cutoffs and forces the trace of the sign of the pencil to vanish. The resulting Brown integral in Section 3 is nonnegative. Related results are the planar pinching defect of Dinh, Le, and Vuong and the finite-matrix determinant comparison of Long, who discusses Brown measure in Remark B.6 [10, 5, 13].

The principal application is the real two-plane representation in Theorem 5.1, obtained from Theorem 1.1 and the reduction theorem of Haagerup, Junge, and Xu. In Schatten classes the Hanner problem was formulated by Ball, Carlen, and Lieb and resolved by Heinävaara. Section 5 gives the range of exponents also accessible by the comparison method of Kumar and Meka. Section 6 proves trace-derivative signs and the fixed-even-occurrence shuffle model, including the six-occurrence case posed by Dinh. Section 7 uses finite corners to transfer the derivative signs to higher-order spectral shift functions for semifinite traces [1, 10, 7, 12, 4].

2. The Fourier distribution and an invertible-pencil reduction

Put

$$g(s)=\int_0^1 \frac{1-r}{r}\big(e^{irs}-1\big)\,\mathrm{d}r, \qquad G(u)=\tau\big(g(u_1A+u_2B)\big) \quad (u\in\mathbb{R}^2).$$

With $\widehat{\phi}(u)=\int e^{-iu\cdot z}\phi(z)\,\mathrm{d}z$, define the tempered distribution $D_{A,B}$ by

$$D_{A,B}(\phi)=\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} G(u)\widehat{\phi}(u)\,\mathrm{d}u.$$

All Fourier integrals use Lebesgue measure. For real s, $|g(s)|\le |s|/2$, by integrating $|e^{irs}-1|\le r|s|$. Thus the integral defining $D_{A,B}$ converges for Schwartz tests. For a norm-convergent sequence of pairs the G's converge pointwise, by continuous functional calculus, and satisfy a common bound $C|u|$. Dominated convergence gives continuity of $D_{A,B}$ in the operator norms of A,B.

Lemma 2.1. *To prove positivity of $D_{A,B}$ on $\mathbb{R}^2\setminus\{0\}$ for every finite tracial pair, it suffices to prove it for pairs for which $xA+yB$ is invertible for every $x^2+y^2=1$.*

Proof. For $h>0$ define a self-adjoint pair in $M_2(\mathcal{M})$ by

$$A_h=\begin{pmatrix} A & h\mathbf{1}\\ h\mathbf{1} & 0\end{pmatrix}, \qquad B_h=\begin{pmatrix} B & -ih\mathbf{1}\\ ih\mathbf{1} & 0\end{pmatrix}.$$

For $x^2+y^2=1$ put $C=xA+yB$ and $U=h(x-iy)\mathbf{1}$. Block multiplication gives

$$\begin{pmatrix} C & U\\ U^* & 0\end{pmatrix}^{-1}=\begin{pmatrix} 0 & (U^*)^{-1}\\ U^{-1} & -U^{-1}C(U^*)^{-1}\end{pmatrix}.$$

In particular the inverse norm is at most $h^{-1}+h^{-2}(\|A\|+\|B\|)$, uniformly on the unit circle. Apply the assumed positivity to this pair with $\tau\otimes\mathrm{tr}_2$, where tr_2 is the normalized matrix trace. As $h\downarrow 0$ the pair converges in norm to $(A\oplus 0, B\oplus 0)$. Its Fourier distribution is $\frac{1}{2}D_{A,B}$ because $g(0)=0$. Evaluation on each nonnegative smooth test supported away from zero commutes with the limit. □

3. The density for invertible pencils

For a bounded operator X in a finite tracial von Neumann algebra, put $\ell(X)=\tau(\log|X|)$, allowing the value $-\infty$. We use normalized traces throughout this section. Multiplicativity of the Fuglede–Kadison determinant gives $\ell(XS)=\ell(X)+\ell(S)$ when S is invertible, including when $\ell(X)=-\infty$. The Brown measure ν_T of a bounded T is a probability measure carried by $\mathrm{sp}(T)$, and

$$\ell(T-\zeta\mathbf{1})=\int_{\mathbb{C}}\log|w-\zeta|\,\mathrm{d}\nu_T(w) \qquad (\zeta\in\mathbb{C}). \tag{3.1}$$

For self-adjoint T, ν_T is its tracial spectral distribution [6, 2] [8, Proposition 2.5 and Theorem 2.7].

3.1. **The scalar Fourier identity.**

Lemma 3.1. *For Φ in the Schwartz class on $\mathbb{R}$ and $\lambda \in \mathbb{R}$,*

$$\int_{\mathbb{R}} |r| g(\lambda r) \widehat{\Phi}(r)\, \mathrm{d}r = 2 \int_{\mathbb{R}} \frac{\Phi(s) - \Phi(0) - s\Phi'(0)}{s^2} \log|1 - \lambda/s|\, \mathrm{d}s. \tag{3.2}$$

The integral on the right is absolutely convergent.

Proof. This is Heinävaara's scalar Fourier identity in the present normalization [10, arXiv version, Lemma 3.3]. For $r \neq 0$, differentiating $rg(\lambda r)$ twice gives

$$i\lambda \int_0^1 \big(2(1-v) + (1-v)v\,\partial_v\big) e^{iv\lambda r}\, \mathrm{d}v = i\lambda \int_0^1 e^{iv\lambda r}\, \mathrm{d}v = \frac{e^{i\lambda r} - 1}{r}.$$

The middle equality uses integration by parts in v, with zero endpoint terms. Hence the second derivative of $|r|g(\lambda r)$ is $(e^{i\lambda r} - 1)/|r|$ off zero. The first derivative tends to zero from both sides, so this is also the distributional second derivative. For $\Phi(s) = s^2\Psi(s)$, transfer the two derivatives onto $|r|g(\lambda r)$ in the Fourier pairing and use the identity

$$\int_{\mathbb{R}} e^{-\varepsilon|r|} \frac{1 - e^{i\lambda r}}{|r|} e^{-irs}\, \mathrm{d}r = \log\left(\frac{(s-\lambda)^2 + \varepsilon^2}{s^2 + \varepsilon^2}\right).$$

The latter follows by integrating the difference of the two cosines on the positive half-line. Its right side converges in distributions to $2\log|1 - \lambda/s|$ by local integrability of the logarithms and bounded tails on Schwartz tests.

The difference between the two sides is a linear functional of $\Phi(0)$ and $\Phi'(0)$, since it vanishes on $s^2\Psi(s)$ for every Schwartz Ψ. Test it on $\Phi_\varepsilon(s) = (\alpha + \beta s)e^{-\varepsilon^2 s^2/2}$. Its two values at zero are fixed at α, β. With $r = \varepsilon v$, the absolute Fourier pairing is bounded by a constant times

$$\int_{\mathbb{R}} |v|\, |g(\lambda\varepsilon v)| \big(\varepsilon|\alpha| + |\beta v|\big) e^{-v^2/2}\, \mathrm{d}v.$$

This tends to zero by $|g(s)| \leq |s|/2$. In the right side of (3.2), the numerator becomes $(\alpha+\beta s)(e^{-\varepsilon^2 s^2/2} - 1)$. It is bounded by $C|s|^2$ on compact s-intervals for $0 < \varepsilon < 1$. At infinity the full integrand is $O(|s|^{-2})$, uniformly in ε. Dominated convergence gives zero, so the coefficients of both values at zero vanish. □

3.2. **Symmetric cutoffs in the angular integral.** For the rest of this section suppose $v_1A + v_2B$ is invertible for every real unit vector $v = (v_1, v_2)$. For $z = (a, b) \neq 0$, write

$$\rho = |z|, \qquad C_z = \frac{aA + bB}{\rho^2}, \qquad F_z = bA - aB, \qquad T_z = (\mathbf{1} - C_z)F_z^{-1}.$$

The change of variables used below is the one for matrix pencils in [10, Section 3]. In the uniformly invertible case, its two counterterms can be combined into a symmetric potential integral.

Lemma 3.2. *For $\phi \in C_c^\infty(\mathbb{R}^2 \setminus \{0\})$, put $M_\varepsilon(z) = (\varepsilon|z|)^{-1}$. Then*

(3.3)

$$D_{A,B}(\phi) = \frac{1}{2\pi^2} \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^2} \phi(z) \int_{-M_\varepsilon(z)}^{M_\varepsilon(z)} \big(\ell(\mathbf{1} - C_z - tF_z) - \ell(\mathbf{1} - tF_z)\big)\, \mathrm{d}t\, \mathrm{d}z.$$

At each finite cutoff the two potential integrals converge absolutely.

Proof. Write $v_\theta = (\cos\theta, \sin\theta)$, $u_\theta = (-\sin\theta, \cos\theta)$ and $C(\theta) = v_{\theta,1}A + v_{\theta,2}B$. Compactness of the circle and continuity of inversion give a uniform bound on $C(\theta)^{-1}$. The spectral projection of $C(\theta)$ on $(0, \infty)$ varies continuously in norm. Two projections at norm distance less than one are unitarily equivalent, so its trace is constant. Since $C(\theta + \pi) = -C(\theta)$, this constant is $1/2$. Consequently $\tau(\operatorname{sgn} F_z) = 0$ at every $z \neq 0$.

Set $\Phi_\theta(s) = \int_{\mathbb{R}} \phi(sv_\theta + yu_\theta)\, \mathrm{d}y$. In signed polar coordinates $u = rv_\theta$, $0 \leq \theta < \pi$, $\mathrm{d}u = |r| \mathrm{d}r\, \mathrm{d}\theta$ and $\widehat{\phi}(rv_\theta) = \widehat{\Phi_\theta}(r)$. Applying (3.2) against the tracial spectral distribution of $C(\theta)$ gives

$$D_{A,B}(\phi) = \frac{1}{2\pi^2} \int_0^\pi \int_{\mathbb{R}} \frac{\Phi_\theta(s) - \Phi_\theta(0) - s\Phi_\theta'(0)}{s^2} \ell(\mathbf{1} - C(\theta)/s)\, \mathrm{d}s\, \mathrm{d}\theta. \tag{3.4}$$

The absolute integrals before taking the trace are uniform over spectral values λ in a fixed compact set. In fact on bounded s-intervals the divided numerator is bounded and the logarithms $\log|s|$ and $\log|s - \lambda|$ have uniformly bounded integrals of their absolute values. The integral over $|s| \leq \varepsilon$ is $O(\varepsilon(1 + |\log\varepsilon|))$. At infinity the absolute integrand is $O(|s|^{-2})$. These facts justify the spectral integration and the cutoff used next.

Fix ε smaller than the distance from $\operatorname{supp}\phi$ to the origin. At the cutoff $|s| > \varepsilon$, the three terms of (3.4) are separately absolutely integrable. To transform the term with $\Phi_\theta(s)$, use $z = sv_\theta + yu_\theta$ and $t = y/(s\rho^2)$. At fixed z, $s' = y$, $y' = -s$ and $s^2 + y^2 = \rho^2$, so $\mathrm{d}t/\mathrm{d}\theta = -1/s^2$. Over one period modulo π, the ratio y/s runs once over the real line. The identities $aA + bB = sC(\theta) + y(-\sin(\theta)A + \cos(\theta)B)$ and $F_z = yC(\theta) - s(-\sin(\theta)A + \cos(\theta)B)$ give $C(\theta)/s = C_z + tF_z$. The cutoff becomes $|t| < M_\varepsilon'(z)$, where

$$M_\varepsilon(z) = \frac{1}{\varepsilon\rho}, \qquad M_\varepsilon'(z) = M_\varepsilon(z)\sqrt{1 - \varepsilon^2/\rho^2}.$$

In the terms with $\Phi_\theta(0)$ and $\Phi_\theta'(0)$, use $z = yu_\theta$ and $t = 1/(sy)$. Here $F_z = yC(\theta)$. The Jacobians are $\mathrm{d}z = |y|\mathrm{d}y\, \mathrm{d}\theta$ and $|\mathrm{d}s|/s^2 = |y||\mathrm{d}t|$ at fixed y. The cutoff becomes $|t| < M_\varepsilon(z)$. Since $(b, -a) = yv_\theta$ and $s/y = 1/(t\rho^2)$, the derivative with prefactor s becomes $(b\partial_a - a\partial_b)\phi(z)/(t\rho^2)$. Thus the

cutoff integral in (3.4), before multiplication by $1/(2\pi^2)$, is

$$\begin{aligned}&\int \phi(z) \int_{-M'_\varepsilon}^{M'_\varepsilon} \ell(\mathbf{1} - C_z - tF_z)\,\mathrm{d}t\,\mathrm{d}z \\ &- \int \phi(z) \int_{-M_\varepsilon}^{M_\varepsilon} \ell(\mathbf{1} - tF_z)\,\mathrm{d}t\,\mathrm{d}z \\ &- \int \frac{(b\partial_a - a\partial_b)\phi(z)}{\rho^2} \int_{-M_\varepsilon}^{M_\varepsilon} \frac{\ell(\mathbf{1} - tF_z)}{t}\,\mathrm{d}t\,\mathrm{d}z.\end{aligned}$$

Here all three z-integrals have compact support away from the origin.

On this support F_z, F_z^{-1}, C_z are uniformly bounded, and $M_\varepsilon - M'_\varepsilon = O(\varepsilon)$. Factor $\mathbf{1} - C_z - tF_z = (-t)(\mathbf{1} - T_z/t)F_z$. For $|t| > 2\sup_z \|T_z\|$, both $\mathbf{1} - T_z/t$ and its inverse are uniformly bounded. Its trace logarithm is therefore bounded. Multiplicativity now bounds the change in the first term upon replacing M'_ε by M_ε by $O(\varepsilon(1 + |\log \varepsilon|))$.

For the third term, write its inner integral using the spectral law of F_z. The scalar integral

$$\int_{-M}^{M} \frac{\log|1 - tq|}{t}\,\mathrm{d}t = \int_0^M \frac{\log|1 - tq| - \log|1 + tq|}{t}\,\mathrm{d}t$$

tends to a fixed constant times $\operatorname{sgn} q$ when $q \neq 0$. Indeed, rescaling the last integral by $|q|$ gives

$$(\operatorname{sgn} q) \int_0^{M|q|} \log\left|\frac{1 - v}{1 + v}\right| \frac{\mathrm{d}v}{v}.$$

The integrand is bounded near zero, has an integrable logarithm at 1, and is $O(v^{-2})$ at infinity. At finite M, the same bounds and the uniform gap justify the spectral integration. The limit is uniform for q in the spectra of the F_z on the support, since $|q|$ is bounded below and $\inf_z M_\varepsilon(z) \to \infty$. Its trace is zero by $\tau(\operatorname{sgn} F_z) = 0$. This removes the third term and proves (3.3). □

3.3. **Evaluation of the logarithmic potentials.**

Lemma 3.3. *With the uniform invertibility assumption of this section, $D_{A,B}$ has density*

$$z \longmapsto \frac{1}{2\pi} \int_{\mathbb{C}} |\operatorname{Im} w|\,\mathrm{d}\nu_{T_z}(w) \tag{3.5}$$

on the punctured plane, with respect to $\mathrm{d}a\,\mathrm{d}b$.

Proof. Write $R_z = F_z^{-1}$. The two operators in (3.3) factor as $(T_z - t\mathbf{1})F_z$ and $(R_z - t\mathbf{1})F_z$. The common $\ell(F_z)$ contributions cancel at each finite cutoff. The inner integral is thus

$$\int_{-M_\varepsilon}^{M_\varepsilon} \big(\ell(T_z - t\mathbf{1}) - \ell(R_z - t\mathbf{1})\big)\,\mathrm{d}t.$$

Each (z,t)-dependent trace logarithm is measurable, as the decreasing limit of the continuous regularizations $\frac{1}{2}\tau\log(X^*X+\delta\mathbf{1})$ with $X=T_z-t\mathbf{1}$ or $X=R_z-t\mathbf{1}$. On compact z-sets away from zero, the Brown measures in question are supported in a common disk, by the operator norm bound. For each z, Fubini's theorem applies to (3.1) on bounded t-intervals, since $\int_{-M}^{M}|\log|t-w||\,\mathrm{d}t$ is bounded uniformly for w in that disk. In particular the integrals in t are well defined even when the log determinant is $-\infty$ at individual t. Subtract $\log|t|$ from each potential and use

$$\lim_{M\to\infty}\int_{-M}^{M}(\log|t-w|-\log|t|)\,\mathrm{d}t=\pi|\operatorname{Im}w|.$$

The convergence is uniform for w in a bounded complex set. To verify this, write $w=x+iy$. For $M>2|x|$ the integral of $\log|t-x|-\log|t|$ on $[-M,M]$ is

$$(M-x)\log(M-x)+(M+x)\log(M+x)-2M\log M,$$

which is $O(x^2/M)$ by Taylor's formula for $r\log r$. The remaining integrand is $\frac{1}{2}\log(1+y^2/(t-x)^2)$. Its integral on the whole line is $\pi|y|$: express it as $\int_0^{|y|}v/((t-x)^2+v^2)\,\mathrm{d}v$ and use Tonelli's theorem. The omitted tails have integral $O(y^2/M)$ for $M>2|x|$.

The limit of the ν_{R_z} integral is zero because R_z is self-adjoint. The uniform bounds just proved apply with $M=M_\varepsilon(z)$ and give uniform convergence on the support of ϕ. They also show that the pointwise limit, the Brown integral in (3.5), is measurable as a limit of integrals of trace logarithms. Now (3.3) gives (3.5). □

4. Support and identification of the measure

Proof of Theorem 1.1. Lemmas 2.1 and 3.3 imply that $D_{A,B}$ is positive on the punctured plane. A positive distribution is represented by a positive Radon measure, which we denote by $\mu_{A,B}$ there.

To see the support bound, write $K=K_{A,B}$ and $h_K(w)=\sup_{z\in K}w\cdot z$. For $u,v\in\mathbb{R}^2$ put $S=(u_1+iv_1)A+(u_2+iv_2)B$. The self-adjoint part of iS is $-v_1A-v_2B\le h_K(-v)\mathbf{1}$, by the definition using states. Differentiating $\|e^{irS}\xi\|^2$ gives $\|e^{irS}\|\le e^{rh_K(-v)}$ for $r\ge0$. Integration of the derivative of the exponential then yields

$$\|e^{irS}-\mathbf{1}\|\le r\|S\|e^{rh_K(-v)}.$$

Here $h_K(-v)\ge0$ since $0\in K$. Thus G extends to an entire function with $|G(u+iv)|\le\frac{1}{2}\|S\|e^{h_K(-v)}$. The Paley–Wiener–Schwartz theorem, with the plus sign convention for the transform $D(e^{i(u+iv)\cdot z})$, gives $\operatorname{supp}D_{A,B}\subseteq K$. In particular G is the Fourier–Laplace transform of this compactly supported distribution [11, Theorem 7.3.1].

For $L(a,b)=xa+yb$ put $X=xA+yB$. The push-forward $L_*D_{A,B}$ is compactly supported and has transform $s\mapsto G(sx,sy)$. Another compactly

supported distribution with that transform is

$$\psi \longmapsto \int_0^1 \frac{1-r}{r}\tau\big(\psi(rX)-\psi(0)\mathbf{1}\big)\,\mathrm{d}r. \tag{4.1}$$

The estimate $\|\psi(rX)-\psi(0)\mathbf{1}\| \le r\|X\|\|\psi'\|_\infty$ gives distributional continuity. The transform is obtained by testing on $\psi(s)=e^{i\xi s}$ and agrees with $G(\xi x,\xi y)$ by the definition of g. Uniqueness of the Fourier transform proves (4.1) for $L_*D_{A,B}$.

For $\psi \in C_c^\infty(\mathbb{R}\setminus\{0\})$, a cutoff equal to one near K turns $\psi\circ L$ into a test on the punctured plane. Its pairing with $D_{A,B}$ is therefore its integral against $\mu_{A,B}$. This identifies the restriction of $L_*\mu_{A,B}$ to $\mathbb{R}\setminus\{0\}$ with the radial average of the tracial spectral measure of X, with weight $(1-r)/r$. Both measures are locally finite on the punctured line. For the first, the preimage in K of a compact set away from zero stays away from the planar origin. For the radial average, only r bounded below can contribute on such a set if $X\neq 0$. The identity therefore extends to Borel f with $f(0)=0$. Apply it first to $|f|$ and use $\mathcal{H}|f|(t)\le\int_{-\|X\|}^{\|X\|}|f(s)|/|s|\,\mathrm{d}s$ on the spectrum of X. This gives absolute convergence and (1.1). Taking $f(s)=|s|$ in the two coordinate directions gives the stated first moment.

Finally, taking $f(s)=s^j$ for $j\ge 1$ determines the moment of each $(xa+yb)^j$. Comparing coefficients determines every moment of positive total degree. The finite measure $(a^2+b^2)\mu_{A,B}$ on K, extended by zero at the origin, is therefore determined by polynomial density in $C(K)$. Division by a^2+b^2 on the punctured plane proves uniqueness. □

For an example, take the Pauli matrices $A=\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ and $B=\left(\begin{smallmatrix}0&-i\\i&0\end{smallmatrix}\right)$ with the normalized trace. Here $C_z^2=\rho^{-2}\mathbf{1}$, $F_z^2=\rho^2\mathbf{1}$, and $C_zF_z=-F_zC_z$. Thus $T_z^2=\rho^{-2}(1-\rho^{-2})\mathbf{1}$. Formula (3.5) gives density $(2\pi)^{-1}\rho^{-2}\sqrt{1-\rho^2}$ on $0<\rho<1$ and zero for $\rho>1$. Its two second coordinate moments are both $1/6$, as in (1.1).

5. Real two-dimensional subspaces and Hanner inequalities

Theorem 5.1. *Let $\mathcal{N}$ be any von Neumann algebra and $0<p<\infty$. Every real two-dimensional subspace of the Haagerup space $L_p(\mathcal{N})$ embeds real-linearly and isometrically into a commutative L_p-space. Consequently, for $1\le p\le 2$ and $X,Y\in L_p(\mathcal{N})$,*

$$\|X+Y\|_p^p+\|X-Y\|_p^p \ge (\|X\|_p+\|Y\|_p)^p+\big|\|X\|_p-\|Y\|_p\big|^p. \tag{5.1}$$

For $2\le p<\infty$, the reverse inequality holds.

Proof. For bounded self-adjoint A,B in a finite tracial algebra, put $f(s)=|s|^p$ in (1.1). The scalar transform is $\mathcal{H}f=f/[p(p+1)]$. We can represent the resulting norm on S^1 by pushing $p(p+1)|z|^p\mathrm{d}\mu_{A,B}(z)$ forward under

$z \mapsto z/|z|$. If σ is the resulting finite positive measure, then

$$\|xA + yB\|_p^p = \int_{S^1} |xu_1 + yu_2|^p \, \mathrm{d}\sigma(u).$$

Its mass is at most $c_p(\|A\|_p^p + \|B\|_p^p)$ by the coordinate norm identities and $(a^2 + b^2)^{p/2} \le c_p(|a|^p + |b|^p)$.

For arbitrary bounded A, B, use the self-adjoint dilations

$$A^{\#} = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}, \qquad B^{\#} = \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix}.$$

For real x, y, write $Z = xA + yB$. Then $xA^{\#} + yB^{\#} = Z^{\#}$ and $|Z^{\#}|^p$ has diagonal blocks $|Z^*|^p$ and $|Z|^p$. These blocks have equal trace by polar decomposition. With the normalized matrix trace the p-norm of $Z^{\#}$ is thus $\|Z\|_p$. For an unnormalized finite trace, multiply the sphere measure by the trace of the identity.

In a finite algebra, approximate tracial L_p elements in the p-norm or quasi-norm by bounded elements. For such a sequence of pairs, the norms of both coordinates stay bounded. The mass bound for σ is uniform, so a subsequence of their measures converges weakly on S^1. For $0 < p < 1$, convergence of $\|xA_j + yB_j\|_p^p$ follows from the p-triangle inequality. For $p \ge 1$ use norm convergence. Each function $u \mapsto |xu_1 + yu_2|^p$ is continuous, so a single weak limit gives the representation for all real x, y.

For general $\mathcal{N}$, the reduction theorem of Haagerup, Junge, and Xu gives a common space X_p with isometric embeddings of $L_p(\mathcal{N})$ and of an increasing directed family of finite tracial L_p-spaces with dense union. For each integer j, approximate X, Y within $1/j$ in a common member of the family. The representation in that finite tracial space and the same weak compactness argument give a measure σ representing the pair in $L_p(\mathcal{N})$. Sending $xX + yY$ to the function $u \mapsto xu_1 + yu_2$ defines the isometry on its real span, with linear relations preserved by the norm identity. The classical Hanner inequalities for these functions give (5.1) and its reversal [7, Theorem 3.1 and Remarks 3.2, 3.4] [9]. □

The argument of Kumar and Meka also yields the Hanner inequalities for $p \ge 3$ in arbitrary von Neumann algebras. For finite-spectrum operators in a finite tracial algebra, use the overlap weights $W_{ij} = \tau(P_i Q_j)$ of the two spectral resolutions. These are nonnegative with row sums $\tau(P_i)$ and column sums $\tau(Q_j)$. In their proof of the cubic comparison, an active spectral projection P_i at parameter s satisfies $C(s)P_i = (s\mathbf{1} + Y_s)P_i$, where $C(s) = X_s + Y_s$ and $Y_s = \sum_j y_j(s) Q_j$. The bound replacing their vector estimate is

$$\tau(P_i C(s)_+^2 P_i) \le \tau(P_i C(s)^2 P_i) = \sum_j (s + y_j(s))^2 W_{ij}.$$

Their Klein inequality follows by summing the scalar convexity inequality against spectral overlap weights. The remaining scalar estimates and the extension to $f(x) = |x|^p$ use the same weights, with the total mass $\tau(\mathbf{1})$.

Norm approximation, self-adjoint dilation and Haagerup reduction give the general $p \geq 3$ inequality. The dual Hanner inequality gives $1 < p \leq 3/2$, and the $p = 1$ case follows from the triangle inequality. Theorem 5.1 includes $3/2 < p < 3$ and gives an isometric representation for the entire range $0 < p < \infty$ [12, Sections 2.1, 3 and 4.1] [1, 7].

6. Trace derivatives and fixed shuffle occurrences

Theorem 6.1. *Let $m \geq 1$ be an integer, and let A, B be bounded self-adjoint elements of a finite tracial von Neumann algebra. For smooth real f on $\mathbb{R}$ with $f^{(m)} \geq 0$, the mth derivative of $t \mapsto \tau f(A+tB)$ is nonnegative for even m. For odd m, the derivative is nonnegative if $B \geq 0$ and nonpositive if $B \leq 0$.*

Proof. The trace is smooth locally in t. Indeed, on a bounded t-interval one can replace f by a compactly supported smooth function equal to it on a neighborhood of all relevant spectra. In the Fourier integral for this function, the iterated Duhamel formula bounds the jth derivative of $e^{iu(A+tB)}$ in norm by $|u|^j \|B\|^j$. For $j = 1$, cyclicity under the trace gives $\frac{\mathrm{d}}{\mathrm{d}t}\tau f(A + tB) = \tau(f'(A + tB)B)$. This proves the sign for $m = 1$.

Suppose $m \geq 2$. For $s \in \mathbb{R}$, apply (1.1) to $(A - s\mathbf{1}, B)$ and $r \mapsto (r_+)^{m-1}$, where $r_+ = \max\{r, 0\}$. Its $\mathcal{H}$ transform is $1/(m(m-1))$ times that function. Thus

$$\tau\big((A + tB - s\mathbf{1})_+^{m-1}\big) = m(m-1) \int (a + tb)_+^{m-1} \, \mathrm{d}\mu_{A-s\mathbf{1},B}(a, b).$$

For $b \neq 0$ the mth distribution derivative in t of $(a + tb)_+^{m-1}$ is $(m - 1)!b^m|b|^{-1}\delta_{-a/b}$, by the chain rule and scaling of the Dirac measure. It is zero for $b = 0$. On bounded t-intervals the original integrand is at most $C(|a| + |b|)$ on the compact support of the planar measure. The Dirac coefficient is bounded by $C|b|$ there. Hence the interchange with distribution differentiation is justified by the first moment in Theorem 1.1. The coefficient is nonnegative for even m. If $B \geq 0$, the measure is carried by $b \geq 0$, by the support bound involving states. For $B \leq 0$ it is carried by $b \leq 0$. This gives the odd sign assertion for the truncated powers.

To recover f, fix a compact interval of t, an interval containing the relevant spectra in its interior, and a nonnegative $\chi \in C_c^\infty(\mathbb{R})$ equal to one on that spectral interval. The function

$$r \longmapsto \frac{1}{(m-1)!} \int_{\mathbb{R}} (r - s)_+^{m-1} \chi(s) f^{(m)}(s) \, \mathrm{d}s$$

has mth derivative $f^{(m)}$ on the spectral interval. It therefore agrees there with f up to a polynomial of degree at most $m - 1$. The trace of that polynomial contributes zero to the mth derivative in t. To interchange the s integral with distribution differentiation, pair first with a test in t; the truncated-power trace is jointly continuous and bounded on the relevant compact (s, t)-sets. The nonnegative weight $\chi f^{(m)}$ preserves the sign just

proved for each s. The smoothness of the trace in t now gives the pointwise derivative sign. The truncated-power approach is the one used for matrices by Heinävaara [10]. □

6.1. **Fixed occurrences in shuffle words.** For integers $n, m \geq 0$, let $S_{n,m}(A, B)$ be the sum of the words containing n copies of A and m copies of B.

Corollary 6.2. *If A, B are self-adjoint contractions and $m \geq 2$ is even, there is a positive measure λ_m on $[-1, 1]$ of mass $\tau(B^m)$ such that*

$$\binom{n+m}{m}^{-1} \tau(S_{n,m}(A, B)) = \int_{-1}^{1} x^n \, \mathrm{d}\lambda_m(x) \quad (n \geq 0). \tag{6.1}$$

For this fixed m, any finite $N \subset \{0, 1, 2, \ldots\}$ and any $\varepsilon > 0$, there are commuting real diagonal contractions X, Y of some common size d with

$$|\tau(S_{n,m}(A, B)) - \mathrm{tr}_d(S_{n,m}(X, Y))| < \varepsilon \qquad (n \in N),$$

where tr_d is normalized. Both assertions also hold at an odd m for $B \geq 0$.

Proof. For a polynomial q nonnegative on $[-1, 1]$, choose a polynomial f with $f^{(m)} = q$. The choice of antiderivative does not affect the mth derivative of $\tau f(A + tB)$. Fix $0 < \delta < 1$ and set $A_\delta = (1 - \delta)A$. For $|t| < \delta/2$, the spectra of $A_\delta + tB$ lie in $[-1 + \delta/2, 1 - \delta/2]$. Multiply q by a nonnegative smooth cutoff supported in $(-1, 1)$ and equal to one on this interval. Its mth antiderivative agrees with f up to a lower-degree polynomial there. Theorem 6.1 applies to the modified antiderivative, and we may pass to $\delta \downarrow 0$ in the polynomial trace derivative. This proves that

$$q \longmapsto \frac{\mathrm{d}^m}{\mathrm{d}t^m} \tau f(A + tB)\bigg|_{t=0}$$

is a positive linear functional on the polynomials on $[-1, 1]$. Its value on 1 is τB^m. Applying positivity to $\|q\|_\infty \pm q$, extending by polynomial density, and using the Riesz representation theorem gives λ_m. For $q(x) = x^n$ take $f(x) = n! x^{n+m}/(n + m)!$. In the expansion of its trace along $A + tB$, the coefficient of t^m is $n!/(n + m)!$ times the shuffle trace. This proves (6.1).

For the model, partition $[-1, 1]$ into intervals on which the finitely many functions $\binom{n+m}{m} x^n$, $n \in N$, have small oscillation. Replace the mass in each interval by one atom and round the atomic weights down to multiples of $1/d$ for large d. The resulting measures approximate all the prescribed integrals and have total mass at most $\tau(B^m) \leq 1$. Realize an atom x of weight k/d by k diagonal entry pairs $(x, 1)$, filling the remaining slots by $(0, 0)$. The normalized shuffle traces of this pair equal the rational atomic integrals. In even order the passage from the positive functional to the model is Dinh's criterion [4, Theorem 5.1]. □

Dinh established the even cases $m = 2, 4$ and posed $m = 6$ in terms of the polynomial trace-derivative functional. The corollary proves that case

and every higher fixed even case. The approximating pair is chosen for the stipulated m and list N. The joint shuffle-word problem of Pascoe and Woerdeman requires one pair to approximate tests at different occurrences simultaneously [4, Theorem 5.1 and Conjecture 6.1] [15].

7. Higher-order spectral shift functions

Let $(\mathcal{N}, \mathrm{tr})$ be a von Neumann algebra with a faithful normal semifinite trace. For a self-adjoint H affiliated with $\mathcal{N}$, a bounded self-adjoint $W \in L_m(\mathcal{N}, \mathrm{tr})$, and $f \in C_c^\infty(\mathbb{R})$, write

$$\mathcal{R}_m(f; H, W) = f(H+W) - \sum_{j=0}^{m-1} \frac{1}{j!} \frac{\mathrm{d}^j}{\mathrm{d}t^j} f(H+tW)\Big|_{t=0}.$$

The spectral shift function $\eta_{m,H,W} \in L_1(\mathbb{R})$ represents the trace of this remainder by

$$\mathrm{tr}\big(\mathcal{R}_m(f; H, W)\big) = \int_{\mathbb{R}} f^{(m)}(r) \eta_{m,H,W}(r)\, \mathrm{d}r. \tag{7.1}$$

We use the existence, perturbation continuity, and mass formula of Potapov, Skripka, and Sukochev [16, Theorems 1.1 and 2.1 and equation (2.7)].

Theorem 7.1. *Let H be self-adjoint affiliated with $\mathcal{N}$, and let $V = V^* \in \mathcal{N} \cap L_m(\mathcal{N}, \mathrm{tr})$ for an integer $m \geq 1$. If m is even, then $\eta_{m,H,V} \geq 0$ almost everywhere. If m is odd and $V \geq 0$ (respectively $V \leq 0$), then $\eta_{m,H,V} \geq 0$ (respectively $\eta_{m,H,V} \leq 0$) almost everywhere. In these cases*

$$\|\eta_{m,H,V}\|_1 = \frac{\mathrm{tr}(|V|^m)}{m!}.$$

7.1. Continuity at a perturbation with finite trace support.

Lemma 7.2. *Suppose $H_k \to H$ in the strong resolvent sense, with all operators self-adjoint and affiliated with $\mathcal{N}$. If $W \in \mathcal{N}$ is self-adjoint and has support projection of finite trace, then*

$$\mathcal{R}_m(f; H_k, W) \longrightarrow \mathcal{R}_m(f; H, W) \quad \text{in } L_1(\mathcal{N}, \mathrm{tr})$$

for $f \in C_c^\infty(\mathbb{R})$.

Proof. For a self-adjoint K as above, set $U_K(s) = e^{ius(K+tW)}$, with u, t fixed. Iteration of the bounded-perturbation Duhamel identity gives

$$\frac{\mathrm{d}^m}{\mathrm{d}t^m} e^{iu(K+tW)} = m!(iu)^m \int_{\Delta_m} U_K(s_0) W U_K(s_1) \cdots W U_K(s_m)\, \mathrm{d}s. \tag{7.2}$$

Here Δ_m is parameterized by $s_1, \ldots, s_m \geq 0$ with $\sum_{j=1}^m s_j \leq 1$ and $s_0 = 1 - \sum_{j=1}^m s_j$, and its volume is $1/m!$. One obtains the formula by expanding $e^{iu(K+(t+h)W)}$ at $h = 0$: each Duhamel iteration inserts one factor $iuhW$ and adds one simplex variable. The term of order $m+1$ with the perturbed unitary still present has norm bounded by $|uh|^{m+1}\|W\|^{m+1}/(m+1)!$. The full

iterated series converges in operator norm for every h, giving the derivatives even for unbounded K.

Let p be the support of W. For unitaries $U_k \to U$ strongly with their adjoints, put $Z_k = (U_k - U)W$. Normality of the finite functional $X \mapsto \operatorname{tr}(W^*XW)$ gives $\|Z_k\|_2 \to 0$. For $m > 2$ use $\|Z_k\|_m^m \le \|Z_k\|^{m-2}\|Z_k\|_2^2$. For $m = 1$ use $Z_k = Z_k p$ and $\|Z_k\|_1 \le \operatorname{tr}(p)^{1/2}\|Z_k\|_2$. The same argument applied to adjoints gives $W(U_k - U) \to 0$ in L_m. Consequently products in (7.2) converge in L_1 when the unitaries converge strongly with their adjoints. Indeed, group each factor W with its preceding unitary and group the final W also with the terminal unitary. The resulting m factors converge in L_m, and a telescoping sum with Hölder's inequality gives convergence of the product. The bound for each product is $\|W\|_m^m$.

For fixed $t \in [0,1]$, $H_k + tW \to H + tW$ in strong resolvent sense by the Neumann expansion at any ζ with $|\operatorname{Im}\zeta| > \|W\|$. The unitaries in (7.2) and their adjoints therefore converge strongly for every u, s. For fixed K, the products are L_1-continuous in the real parameters u, s, t, by strong continuity of the unitaries and the multiplication argument above. In the t variable continuity follows from the Duhamel norm bound $|us||t - t'|\|W\|$ for the difference of the unitaries. The simplex integrals of the products are therefore Bochner integrals in L_1, and (7.2) has L_1 bound $|u|^m\|W\|_m^m$.

Use Fourier inversion for f and integrate (7.2) with weight $(1-t)^{m-1}/(m-1)!$ on $0 \le t \le 1$ to obtain the Taylor remainder. The operator formula and the L_1 integral agree by testing against bounded operators of finite trace support, which separate the affiliated operators in $\mathcal{N} + L_1(\mathcal{N}, \operatorname{tr})$. The Fourier and Duhamel bounds justify these pairings in both norms. Since $|u|^m|\widehat{f}(u)|$ is integrable, dominated convergence in the simplex, Fourier, and Taylor integrals gives the claimed L_1 convergence. □

7.2. **The finite-corner argument.**

Proof of Theorem 7.1. We may assume $V \neq 0$, since the remainder vanishes for $V = 0$. Let $e_m = 1$ when m is even or $V \ge 0$, and $e_m = -1$ when m is odd and $V \le 0$. To test positivity, take $q \in C_c^\infty(\mathbb{R})$ with $q \ge 0$, and let f be its mth antiderivative that vanishes to the left of $\operatorname{supp} q$. With $\chi \in C_c^\infty(\mathbb{R})$ equal to one on $[-1,1]$, put $f_L(r) = \chi(r/L)f(r)$. For sufficiently large L,

$$\|f_L^{(m)} - q\|_\infty \le C_q/L.$$

Indeed, the term $\chi(r/L)q(r)$ equals $q(r)$ for L large. Each other term has $j \ge 1$ derivatives of the cutoff, of size $O(L^{-j})$ on $L \le |r| \le CL$, and $m - j$ derivatives of f there. On each component of this transition region f is a polynomial of degree at most $m - 1$, so that derivative is $O(L^{j-1})$. The resulting C_q is independent of the operator pair.

For a bounded pair H_0, W in a finite tracial algebra, assume the same sign condition on W as on V and use the same e_m. Normalize the finite trace to apply Theorem 6.1 to $f_L + (C_q/L)r^m/m!$, whose mth derivative is nonnegative. Integrate the signed trace derivative in Taylor's formula

along $H_0 + tW$, then restore the original normalization. The polynomial contribution to the remainder is exactly $(C_q/L)W^m/m!$, with $e_m W^m = |W|^m$. Thus

$$e_m \operatorname{tr}\big(\mathcal{R}_m(f_L; H_0, W)\big) \geq -\frac{C_q}{Lm!}\operatorname{tr}(|W|^m). \tag{7.3}$$

Now put $V_j = V\mathbf{1}_{\{|V|>1/j\}}$. Its support p has trace at most $j^m\|V\|_m^m$. The truncation preserves the sign and converges to V in L_m. For fixed j, approximate H by bounded finite-spectrum functions H_k of H, obtained by truncating at $\pm k$ and rounding with mesh $1/k$. Pointwise convergence on the spectral line gives strong resolvent convergence by the spectral theorem.

Let P_ℓ be the finitely many spectral projections of H_k. Compress to the projection e that is the join of the range projections of $P_\ell p$. Each range projection lies below P_ℓ and has trace at most $\operatorname{tr}(p)$ by polar decomposition. Thus e has finite trace and commutes with H_k. It contains p because $\sum_\ell P_\ell p = p$. Both H_k and V_j commute with e, and the Taylor remainder on its complement is zero since V_j vanishes there. Applying (7.3) in the corner gives the bound for (H_k, V_j) with the original trace. If $V_j = 0$, that bound is immediate.

Keep L, j fixed and pass to $k \to \infty$ by Lemma 7.2. Then take $j \to \infty$ by the perturbation continuity of the trace remainder in L_m and by $\operatorname{tr}(|V_j|^m) \to \operatorname{tr}(|V|^m)$. Formula (7.1) gives

$$e_m \int f_L^{(m)}(r)\eta_{m,H,V}(r)\,\mathrm{d}r \geq -\frac{C_q}{Lm!}\operatorname{tr}(|V|^m).$$

Let $L \to \infty$. Since $\eta_{m,H,V} \in L_1(\mathbb{R})$, the uniform approximation of q shows $\int q\, e_m\eta_{m,H,V} \geq 0$ for every nonnegative compactly supported smooth q. Thus $e_m\eta_{m,H,V} \geq 0$ almost everywhere. The mass identity $\int \eta_{m,H,V} = \operatorname{tr}(V^m)/m!$ gives the norm formula [16, Theorem 2.1 and equation (2.7)]. □

For ordinary Schatten perturbations in $\mathcal{B}(\mathcal{H})$, the sign assertion appears in Pradhan and Skripka's Theorem 4.1 [17]. They also establish sign results for classes of relatively Schatten perturbations of semibounded operators on a Hilbert space [18]. Theorem 7.1 allows any faithful normal semifinite trace and concerns bounded perturbations in the corresponding tracial L_m-ideal. For existence of trace formulas with certain unbounded tracial perturbations, see Chattopadhyay, Coine, Giri, and Pradhan [3].

AI declaration

OpenAI Codex assisted with formulating the results, developing the mathematical arguments, checking the proofs, searching and comparing the literature, and drafting and revising the manuscript. Its mathematical contributions include development of the block enlargement and Brown-potential argument and of the proofs of the applications.

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