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Exact square and near-square Zarankiewicz numbers for $K_{3,3}$ through order 23

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Abstract

We determine the maximum number of ones in an $n \times n$ binary matrix avoiding an all-ones 3×3 submatrix for $17 \leq n \leq 23$. The maxima are $138, 150, 164, 180, 193, 210, 225$. We also determine the intervening $n \times (n+1)$ maxima for $17 \leq n \leq 22$. The extremal 22×22 matrix is unique up to row and column permutations. For the upper bounds we mark a line, enumerate the induced families of sets, and test which families can carry the remaining entries. At 22×23, local bounds on the degrees incident with a column give two incompatible counts of the same incidences at weight 217. A matrix with 216 ones attains the bound. At the next order we obtain $240 \leq z(24, 24; 3, 3) \leq 243$.

AI transparency statement. The author directed an AI-assisted investigation. AI systems developed proofs and verification programs and drafted this article. The upper bounds are computer-assisted. The scope of the checks is described in Section 7.

1 Introduction

Let $z(m, n)$ be the largest number of ones in an $m \times n$ binary matrix in which no three distinct rows have ones in the same three distinct columns. Equivalently, it is the maximum number of edges in a $K_{3,3}$-free bipartite graph with parts of sizes m and n. Thus $z(m, n) = z(n, m)$. We use the maximum-ones convention throughout. The least number of ones forcing a forbidden submatrix is $z(m, n) + 1$.

Theorem 1. *The square and rectangular maxima are as follows.*

n	17	18	19	20	21	22	23
$z(n, n)$	138	150	164	180	193	210	225
$z(n-1, n)$	132	143	156	171	185	201	216

The extremal 22×22 matrix is unique up to independent permutations of its rows and columns.

The first rectangular entry was determined by Wei [5]. Four of the seven square upper bounds follow immediately from the preceding rectangle by averaging the weights after a row deletion. One auxiliary value, $z(19, 21) = 177$, has two uses: an extra eight-one line would be needed to exceed 185 on 20×21, and two such parallel lines would be needed to exceed 193 on 21×21. In both cases removing the lines would leave 178 on 19×21. In contrast, once the 22×23 bound is known, the upper bound of 225 on the next square follows by a single averaging argument.

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Earlier bounds. Table 4 of Collins, Riasanovsky, Wallace and Radziszowski [1] gives upper bounds of 141, 148 and 156 on 17×17, 17×18 and 18×18, respectively. Padhi [2] had already obtained the lower bound of 138 on 17×17. Afrasyab [3] gave $132 \le z(16,17) \le 133$. Wei [5] established $z(16,17) = 132$ and $z(16,18) = 137$. Independent marked-line proofs for Wei's two values are included in the accompanying repository [8]. The 137-one witness there extends Hou's 132-one matrix on 15×18 [4]. The provenance of the first entries is summarized below.

Size	Earlier bound or exact value	Result used here
16×17	132 exactly (Wei)	independent verification
16×18	137 exactly (Wei)	input for 17×18
17×17	138 (Padhi) to 141 (Collins et al.)	upper bound 138
17×18	upper bound 148 (Collins et al.)	exact value 143
18×18	upper bound 156 (Collins et al.)	exact value 150

A consequence is the upper bound $b(2,2,3) \le 17$: two $K_{2,2}$-free colour classes on 17×17 have at most 74 cells each [1], and $74 + 74 + 138 < 289$. Together with the known lower bound this recovers the equality proved by Rowshan and Gholami [6].

2 Marking a line

A *line* is a row or a column. Its degree is its number of ones. The weight of a matrix is the sum of its row degrees. Erasing a one preserves avoidance, so an exclusion at one weight excludes every larger weight on the same dimensions. If a matrix of weight w has m rows, then each one survives $m - 1$ of its m single-row deletions. Consequently

$$(m-1)w \le m\, z(m-1,n), \qquad d_r \ge w - z(m-1,n) \tag{1}$$

for each row degree d_r. The transpose gives the column versions.

Lemma 2. *Mark t columns in the support of one row. Restrict each other row to these columns and regard the result as a block on t points. Two blocks intersect in at most two points. The load of point j, its number of occurrences in the blocks, is $c_j - 1$, where c_j is its column degree. If two rows share u columns, the sum of degrees of those columns is at most $2u + 2(m-2)$ in a matrix with m rows.*

Proof. Three common points of two blocks give three common entries of their rows and the marked row. For the last assertion the two rows contribute $2u$ entries on the shared columns and each other row contributes at most two. □

To enumerate the blocks, begin with all possible families of the largest allowed size, often five. Adjoin smaller blocks in descending order. A block of size three or more cannot repeat. A block of size at most two can repeat, and the empty block accounts for a row missing the mark. Permuting the points or the blocks gives an equivalent projection. We canonicalize the bipartite incidence graph with points and blocks in separate vertex classes using nauty [7]. Repeated blocks have distinct vertices. In particular the enumeration must include the empty family of five-blocks whenever the load bound permits it.

For a mark comprising the entire support of a row, the missing columns avoid that row. On the remaining rows generate column supports S of the permitted sizes. For each pair of fixed columns A, B require $|S \cap A \cap B| \le 2$. Upon choosing a column impose the same test on all newly formed pairs. This checks exactly the triples involving a new column. If the mark uses only part of the

row, a missing column through that row can use only blocks with pairwise intersection at most one. The completion tests which exclude this possibility explicitly check how many such blocks can be chosen.

Choosing a mark. Give column j a weight e_j and set $W_r = \sum_{j\sim r} e_j$, where $j \sim r$ denotes a one. The identity

$$\sum_r W_r = \sum_j c_j e_j \tag{2}$$

can force a row with high incident weight even when ordinary averaging only gives its degree. With $e_j = c_j - b$ and nonnegative weights, a degree-t row of incident weight W_r gives a projection of total load $(b-1)t + W_r$. Thus each extra unit forced by (2) raises the threshold in the block enumeration.

3 The maxima through order 21

Here and below the accompanying finite enumerations are part of the proof. Appendix A identifies their inputs and entry points. We describe the covering arguments since an empty list of completions gives an upper bound only if every hypothetical matrix has a mark on the list.

The first square. At weight 139 on 17×17, deletion and the bounds of [1, Table 4] at heights 16, 15, 14 and 13 give minimum degree six and total deficit below eight at most three on each side. Indeed there cannot be two sixes, one six and two sevens, or four sevens on the same side. Six common columns of two rows would need at least $48 - 3 = 45$ entries, exceeding the capacity $12 + 2 \cdot 15 = 42$. Marking ten entries of a long line now gives blocks of size at most five and total load at least 67. For the five-blocks the $\{\pm 1\}$ incidence vectors have squared norm ten and mutual inner products at most -2. The squared norm of their sum bounds their number by six. Enumerating the compatible fours and the remaining blocks first excludes degrees above ten, then leaves 519 relabelling keys for a mark of degree ten. None extends.

For this argument the inputs at heights $16, 15, 14, 13$ are, in order, $133, 126, 118, 110$ as upper bounds on seventeen columns. The six-row case also uses the published uniqueness of the 128-one 16×16 matrix. A column of degree five in a 133-one core would have to extend that square. Testing its $\binom{16}{5}$ possible supports in each orientation excludes such an extension.

The remaining lines have degrees six through nine. If a six exists, transpose as needed and delete a six-row. Otherwise delete a seven-row if one exists. The respective 16×17 cores have 133 and 132 ones, so each has a nine-row. Mark a nine-column if available and otherwise a nine-row. With minimum degree eight on both sides of the original matrix, mark a nine-column in the full square. These cases exhaust the possible degrees. Completing the nine-point projections, including restoration of the deleted row when present, excludes 139. The repository gives two witnesses for 138, distinguished even up to transposition by their maximum line degrees, ten and nine.

The next three rectangles and squares. At 144 on 17×18 a degree-nine column can be forced and no column has greater degree. Project onto the nine rows containing the mark or the eight rows missing it, according to the weight on each side. Let S denote the weight on the nine incident rows away from the column. At $S \geq 65$ the inside enumeration gives 3,671,498 projections. Otherwise the eight outside rows have at least 71 ones. Their minimum degree is seven, eight or nine, giving respectively 7,298,758, 12,057,949 and 825 projections. These four cases give 23,029,030 in total. None completes. Appending a five-one column to the second 138-one square gives 143. At 151 on 18×18 every degree is at least eight. Marking eleven points on a largest line of degree at

least eleven requires nine five-blocks among seventeen blocks, with total load at least 77. No such projection has the required loads. For maximum degrees ten and nine there are respectively 978 and 1,821 projections of the required types. In the latter case a nine-column meeting at least three nine-rows is forced. None completes. The 150-one witness has three lines of degree ten and fifteen of degree eight on each side.

At 157 on 18×19 the search marks a row after removing columns of degree seven. The separate cases for zero and one such column and the bounds on the smaller rectangles leave two through nine columns to restore. A compatible single column need not extend to the required family of columns. For example, in a cross-check with three columns removed, 313 of 5,964 completed cores accepted one column and none accepted three. The completed enumeration excludes 157 for each number removed. Two witnesses give 156. One accepts an eight-one row to give 164 on 19×19, attaining the bound $\lfloor 19 \cdot 156/18 \rfloor$.

At 172 on 19×20, all degrees are at least eight, and an eight-row and an eight-column cannot meet: deleting both would leave 157 ones on 18×19. If there is an eight-row, its eight columns have degree at least nine. It gives 114 possible marks, none of which completes. Otherwise the rows have degrees $10, 9, \ldots, 9$. Marking ten rows on any column of degree at least ten gives an empty catalogue. There must then be a nine-column through the ten-row, and its 81 possible projections all fail to complete. Thus $z(19, 20) \leq 171$ and $z(20, 20) \leq 180$ by (1). For the lower bounds take all translates of $\{0, 2, 3, 6, 7, 8, 9, 11, 15\}$ in $\mathbb{Z}/20\mathbb{Z}$ as rows, and delete a row for the rectangle. Direct triple intersection gives the required avoidance.

One exhaustion for three values. At weight 178 on 19×21 let s be the number of seven-columns. Every column has degree at least seven, and deleting three seven-columns would leave 157 on 19×18. Thus $0 \leq s \leq 2$. Remove all s. The remaining column degrees have the form $8 + e_j$ and the core row degrees are at least eight, by the respective 18-row bounds 170, 163 and 156. The sum of the e_j is $10 + s$. For a row of degree d the projection has load $7d + W_r$. The two searches mark a row maximizing $d + W_r$ or $2d + W_r$. Applying (2) and the point-load bounds gives the finite cases below. The counts include different choices of mark for the same core.

s	0	1	2
Marked cases	155,847	174,743	184,800
Extensions to weight 178	0	0	0

The reductions at 186 on 20×21 and 194 on 21×21 force, respectively, one eight-row and two parallel eight-lines (transposing if necessary) whose removal gives exactly this 178-one case. This yields the three upper bounds 177, 185 and 193. For the reverse inequalities, start with the 193-one square and delete one or both of its eight-rows. Details of the two score bounds and the reductions appear in the 178-one exhaustion and eight-line reductions.

4 A 159-case proof at 21×22

Proposition 3. *$z(21, 22) = 201$ and $z(22, 22) = 210$.*

Proof. A column extends the 193-one square to 201. An additional row gives 210. For the upper bounds, (1) gives $z(21, 22) \leq 202$ from $z(21, 21) = 193$, and $z(20, 22) \leq 193$ from $z(20, 21) = 185$. Hence at weight 202 every row and column has degree at least nine. Lemma 2 bounds pair intersections by five on both sides.

If a row has degree at least eleven, mark eleven of its entries. The twenty blocks have total load at least 88 and size at most five. There must be at least eight five-blocks. There are six compatible families of at least eight five-blocks on eleven points up to relabelling. For a family of size m, one needs at least $28 - 2m$ compatible four-blocks, since the total load with q fours is at most $60 + 2m + q$. None of the six families admits them. Thus all rows have degree nine or ten, with thirteen of degree ten.

Write $c_j = 9 + e_j$. Then $\sum e_j = 4$. If one $e_j \geq 2$, that column must meet a ten-row, since it has degree at least eleven and there are only eight nine-rows. Otherwise four columns have degree ten. If every ten-row met at most one of the four, the eight nine-rows would supply at least $40 - 13 = 27$ entries on them. By discrete convexity the number of column triples on these rows would be at least $5\binom{3}{3} + 3\binom{4}{3} = 17$, whereas the available capacity is $2\binom{4}{3} = 8$. Either way a ten-row has $W_r \geq 2$.

Mark that row. Its twenty blocks on ten points have point loads from eight to twelve and total load at least 82. There are 159 canonical projections. The twelve missing columns avoid the marked row. They have degrees from nine to eleven, because at most two excess units remain outside the mark. Testing compatible supports as in Section 2 gives zero completions to weight 202. Finally $z(22, 22) \leq \lfloor 22 \cdot 201/21 \rfloor = 210$. □

Classification at equality. In a 210-one square all degrees are at least nine. Give a column weight $c_j - 9$. A row of degree at least eleven has incident weight at most one: otherwise its eleven heaviest points give one of 338 projections of load at least 90, none of which extends. The same restriction holds on transposing. Thus a degree-d row with $d \geq 11$ requires at least $d - 1$ nine-columns. Two such rows can share at most five of these, and three can share at most two. Distributing their required incidences among the nine-columns and applying the same bounds on the other side leaves seven pairs of degree profiles up to transposition. In each case (2) forces a ten-row of incident weight at least five or a nine-row of weight at least eight.

The ten-row gives 13,205 projections. Exhausting their completions returns one square. To enumerate the nine-row marks, omit a point of least degree, enumerate eight-point projections, then restore the point on blocks meeting pairwise in at most one point. This gives 81 distinct nine-point projections, all without completions. The resulting square agrees up to relabelling with the 210-one witness. In either orientation none of its $\binom{22}{7}$ proposed seven-point column extensions is admissible. The classification and this last test are both used in the next rectangle. See the classification at weight 210 for the seven profiles and the reconstruction.

5 The last rectangle

Figure 1 gives a 225-one square. Its last row has nine ones, so the other rows attain 216. We now exclude a 22×23 matrix of weight 217. A row of degree nine will be called a nine-row, and similarly for other degrees and for columns.

Lemma 4 (Degree restrictions at 217)**.** *Every row has degree at least nine. There are $k = 1$ or 2 eight-columns, and all other columns have degree at least nine. Each eight-column contains at most one nine-row. There are at least seven nine-rows. A nine-row avoiding the eight-columns satisfies $\sum_{j \sim r}(c_j - 9) \leq 6$. A ten-column meets at least three nine-rows, and a column of degree $c \geq 11$ meets at least $c - 5$.*

We give the reductions behind this computational lemma, since using only ordinary deletion bounds at this step would allow too many profiles. A column of degree below seven would leave more than 210 on deletion. Degree seven is impossible by the extension test on the unique square.

For the row bound one needs $z(21,23) \le 208$. At weight 209 the degree and projection tests force four or five eight-columns, say q, and every nine-row avoids them. Write b for the number of nine-rows. On columns of degree at least ten give each entry weight $c-9$. Those column weights sum to $q+2$, each column meets at least three nine-rows, and each nine-row carries weight at most three. Hence $b \ge q+2$. On each eight-column the sum of the incident row degrees minus ten per incidence is at most three. A row of degree $10+s$, $s>0$, needs at least $s+2$ eight-columns. Summing its excesses on these columns gives $3(b-1) \le 3q$: the positive row surplus above ten sums to $b-1$, and no nine-row meets these columns. The two inequalities are incompatible. The computations supplying these local bounds are itemized in the 209-one exclusion. Deleting a row at weight 217 now proves the row bound.

On 22×23 an all-high-degree column side is ruled out by weighted marks. Deleting eight-columns then leaves weights $209, 201, 193, 185$ after one through four deletions. The smaller rectangular bounds and the eight-column count exclude five or more. The four-column test and the three-column test exclude four and three. Deleting one eight-column leaves a 209-one square with at most one eight-row, since deleting an eight-row of that square leaves 201 on 21×22 with minimum row degree nine. This proves the asserted incidence bound on the eight-column.

Three further marked cases are needed. An eight-column with incident row-degree sum at least 84 gives 900 projections. A ten-column with two nine-rows and eight ten-rows gives 23,526. A nine-row avoiding the eight-columns with incident excess exactly seven gives 15,766. None completes. Together with the earlier high-degree marks these tests give the excess bound of six, at least three nine-rows on a ten-column, and at most five rows of degree above nine on any longer column. If there are at most five nine-rows, all positive column excess is on $10+k$ ten-columns. These require $3(10+k)$ incidences with the nine-rows, which can supply at most thirty. For a nine-row touching an eight-column, the latter capacity follows by deleting that column and then the residual eight-row: the resulting 201-one rectangle needs at least three eight-columns, forcing the original nine-row to meet at least $4-k$ nine-columns.

At exactly six nine-rows the incidence count forces all columns to have degree eight, nine or ten. There are three units of positive row surplus above degree ten. If a ten-column meets exactly three nine-rows and a longer row, the mark consists of three nine-rows, six ten-rows and one eleven-row. A second longer row would require one eleven-row and eight further rows of degree at least ten outside this mark. None of its 129,226 projections admits even the corresponding partial placement. For clarity, let m' count nine-rows on a ten-column. With two eight-columns, all these counts equal three. With one eight-column, the sum of $m'-3$ over the eleven ten-columns is at most two. A ten-column with $m'>3$ can hold at most $m'-2$ units of row excess above ten. If at least two long rows exist, all their ten-column incidences must now lie on these exceptional columns. They have room for at most four such incidences, whereas two long rows together need at least five: even counting multiplicities, they can use at most seventeen positions outside the ten-columns. A single long row would have degree thirteen and could meet only ten-columns with $m' \ge 5$. There are none if $k=2$. For $k=1$ there is at most one, forcing the row to use all twelve columns outside the tens. A ten-row avoiding the eight-column needs at least six nine-columns. Two rows could not share those six, by Lemma 2 and their total degrees on those columns. Details of these capacities and the six-row case are in the six-nine-row exclusion. This completes the reductions used in Lemma 4.

Lemma 5 (Column scores)**.** *Give row r weight d_r-9 and define $W_j=\sum_{r\sim j}(d_r-9)$. At weight 217,*

c_j	8	9	≥ 10
W_j *at most*	11	9	7.

Proof. The 900-projection exclusion gives $W_j \le 83-8\cdot9 = 11$ for an eight-column. Any two

columns share at most five rows. Indeed, six shared rows need at least 54 ones. Their two columns and the other 21 columns supply at most $12 + 21 \cdot 2 = 54$. Equality forces every one of the six rows to have degree nine and each remaining column to meet exactly two of them. An eight-column would then meet either six or two nine-rows, contradicting Lemma 4.

Each row has degree at most fifteen. If it meets t columns of degree at least nine, it shares at least $8t$ incidences on those columns with the 21 other rows. No other row shares six such columns, by Lemma 2 and $6 \cdot 9 > 12 + 20 \cdot 2$. Hence $8t \le 105$ and the at most two eight-columns give the bound.

For a nine-column of score at least ten mark all nine points. For a column of degree at least ten and score at least eight mark its ten heaviest incident rows. These retain at least eight units of score, since weights are nonnegative integers. The 22 other columns give blocks of size at most five. The loads lie between eight and fourteen. Catalogue all block multisets with the indicated minimum total. A ten-point mark of score at least nine with at least five positive-weight points was already excluded by the earlier ten-point test.

Points	Minimum load sum	Projections	Remaining	Passing
9	82	60,469	60,469	0
10	88	453,448	452,058	0

There are two tests. For a column of degree greater than ten, a further row on the column needs eight of the outside columns. Their blocks must meet pairwise in at most one marked point. No remaining ten-point mark admits an eight-clique in this compatibility graph. For a full mark of $p = 9$ or 10 points, write M for the total of the marked row degrees, and let h of those rows have degree nine. The $22 - p$ outside rows need a gain of $217 - M - 9(22 - p)$ above their baseline of nine each. At most $22 - p - \max(0, 7 - h)$ of them can supply gain. Enumerate the supports of degrees ten to fifteen on the 22 outside columns. Choose compatible longer rows in decreasing degree order, then ten-rows, testing each new triple. Two equal supports of size ten would share at least eight columns of degree at least nine, already impossible. A completion must attain the required gain within the available number of rows. No projection in either row of the table passes this test. □

Proposition 6. *$z(22, 23) = 216$ and $z(23, 23) = 225$.*

Proof. Suppose there are 217 ones. Write a for the number of nine-rows and m for the number of columns of degree at least ten. On the latter columns put $e_j = c_j - 9$, $E = \sum e_j = 10 + k$ and $D = \sum(e_j - 1) = E - m$. The rows above degree nine have degrees $10 + s_r$, where $s_r \ge 0$ and $\sum s_r = a - 3$. Summing the scores of Lemma 5 first by rows and then by columns gives

$$\sum_j W_j = 10(22 - a) + 11 \sum s_r + \sum s_r^2 \ \ge \ 184 + 2a,$$
$$\sum_j W_j \le 11k + 9(23 - k - m) + 7m = 207 + 2k - 2m.$$

Here $s_r^2 \ge s_r$. Integrality gives $a + m \le 11 + k$, or $a \le D + 1$.

Now let T count incidences between the nine-rows and the m columns, each incidence counted with weight e_j. A column of weight one contributes at least three. A column of weight $e \ge 2$ meets at least $e + 4$ nine-rows, and

$$e(e + 4) - 3e - 6(e - 1) = (e - 2)(e - 3) \ge 0.$$

Hence $T \geq 3E + 6D$. A nine-row avoiding the eight-columns contributes at most six. At most k nine-rows meet an eight-column, and each can contribute at most $E \leq 12$. Thus $T \leq 6a + 6k \leq 6D + 6 + 6k$. Together these require $3(10 + k) \leq 6 + 6k$, so $k \geq 8$. This contradicts $k \in \{1, 2\}$. The witness attains 216. Applying (1) gives $z(23, 23) \leq \lfloor 23 \cdot 216/22 \rfloor = 225$, also attained. □

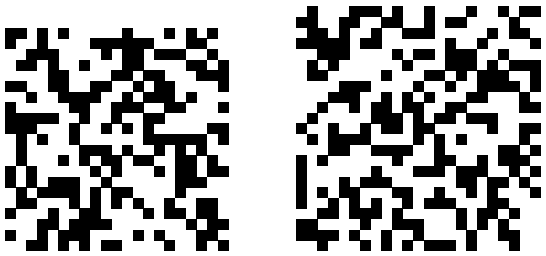

Figure 1: Extremal squares of weights 193 (left) and 225 (right). Removing the two eight-rows of the first square gives 177 on 19×21. Removing the last row of the second gives 216 on 22×23. Each black cell is a one.

6 The next order

There is a short cyclic construction one order further. In $\mathbb{Z}/24\mathbb{Z}$ take the translates of $\{0, 1, 2, 3, 5, 8, 11, 13, 17, 18\}$. Every triple of translates has intersection of size at most two. This gives 240 ones on the square and 230 after deleting any row. Column deletion gives $z(23, 24) \leq \lfloor 24 \cdot 225/23 \rfloor = 234$.

Proposition 7. $z(24, 24) \leq \max\{240, z(23, 24) + 9\} \leq 243$.

Proof. Above weight 240 some row has at least eleven entries. Suppose every column has degree at least ten and mark eleven entries of that row. A six-point block would require 60 incidences on its six columns, above their capacity $12 + 22 \cdot 2 = 56$. Thus the 23 blocks have sizes at most five, and total load at least $11 \cdot 9 = 99$. Write m and q for the number of fives and fours. Counting first relative to four, then to three, gives $m \geq 7$ and $q \geq 30 - 2m$. There are 21 compatible families with at least seven fives on eleven points up to relabelling. For each, enumerate the choices of $30 - 2m$ pairwise compatible fours. Only $m = 7$ contributes, so all 23 blocks are specified. Of the 18 choices on the representative families, thirteen have minimum point load eight and five have minimum seven. None reaches nine. Hence a column of degree at most nine exists. Delete it and use transposition. □

At weight 234 on 23×24 every column has degree at least nine and at least six have degree exactly nine. Each such deletion would leave an extremal 225-one square. In both orientations of the stored square exactly one five-point column extension is admissible, and none of size six is. A 234-one rectangle would therefore require a different 225-one square. Whether another 225-one square admits such an extension remains open in this analysis.

7 Computational checks

For a lower bound, encode every row as an integer with a bit for each column. An independent witness check consists of summing the bit counts of the rows, then checking $\text{popcount}(r_i \& r_j \& r_k) \leq 2$

for each $i < j < k$. For the preparation of this article all 18 stored witness files were checked in both orientations from a clean export of the revision in [8]. The eight-orbit construction of the 225-one square can also be regenerated with `z23-k33/construct.py`. Clean-export replays for this article also recovered the 171-one and 201-one bounds, the unique 210-one square, and the two final catalogues excluding weight 217. Other enumeration counts are reported from the individual proof records in [8].

For an upper bound there are three separate claims to check: a line or deletion of the asserted kind must exist, the catalogue must contain every possible projection at that mark, and the completion test must retain any valid completion. The numerical counts in the tables refer to the indicated stage of the search. A positive control for the last column-score calculation hides rows in the 216-one witness and recovers a compatible placement. The final two exclusions can be run with

```
cd z22x23-k33
python3 verify_217.py --jobs 4
```

This command checks the two rows of the table in Lemma 5. The directory's proof index records the separate verifications on which the degree restrictions depend. The final proof of $z(21, 23) \leq 208$ uses incidence counts, so the large DRAT archives for the earlier individual four-column case are unnecessary for Theorem 1. Run the Python drivers with assertions enabled (omit `-O` and `-OO` and leave `PYTHONOPTIMIZE` unset). The native replays use Python 3.10 or later, a C++20 compiler and 64-bit nauty. The degree-profile replay for the three-eight-column case also uses OR-Tools, SciPy and NumPy. It certifies 298 exclusions among 302 profile pairs. These are program-checked finite searches. Their correctness depends on the coverage arguments, the enumerators and the completion routines.

A Guide to the finite computations

All paths below are relative to the repository in [8]. The linked notes specify the degree inequalities, branching rules, multiplicities, and counts for their respective computations. Parameters and counts refer to the same revision.

Target or input	Computational step	Replay or proof index
17×17	Six-, seven-, and nine-line cases	`z17-k33/verify_exact.py`
17×18	Both sides of a nine-column	`z17x18-k33/verify_exact.py`
18×18	Marks of size 9, 10, 11	`z18-k33/verify_exact.py`
18×19	Joint seven-column extension	`z18x19-k33/verify_exact.py`
19×20	Marks in both orientations	`z19x20-k33/verify_exact.py`
19×21	Three counts of seven-columns	`z19x21-k33/verify_weight_178.py`
20×21, 21×21	Reductions to 178	`z20x21-k33/weight-186.md`
21×22	Marked ten-row	`z21x22-k33/verify_exact.py`
22×22	Census at 210	`z22-k33/verify_unique.py`
21×23	Exclusion of 209	`z21x23-k33/no-209.md`
22×23	Number of eight-columns	`z22x23-k33/README.md`
	Local degree restrictions	`z22x23-k33/verify_217_loads.py`
	Exclusion of six nine-rows	`z22x23-k33/verify_six_nines.py`
	Final column-score bounds	`z22x23-k33/verify_217.py`

For the small calculation in Section 6 the following pseudocode uses `z17x18-k33/fives.cpp` and the clique iterator in `z17x18-k33/verify.py`, both in the cited revision. Compatibility means

intersection of size at most two. The outer loop visits each family size, including the nonmaximal families.

```
for F in canonical compatible families of at least seven
         5-subsets of an 11-set:
    P = all 4-subsets compatible with every member of F
    for Q in cliques(P, size = 30 - 2*len(F)):
        assert len(F) == 7 and len(Q) == 16
        assert some point lies in fewer than 9 members of F union Q
```

Declaration of AI-generated research and authorship

Jason Davies initiated the investigation, selected the sequence of square and near-square cases, and directed the computational work. AI systems working under his direction developed mathematical reductions, wrote enumeration and checking programs, and produced this manuscript draft. The programs and the scope of their checks are described above so that the calculations can be examined independently.

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