Signed prime Lambert sums and independence in arithmetic progressions

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Abstract

Assign a sign $\varepsilon_p \in \{-1, 1\}$ to each prime p. We prove that $\sum_p \varepsilon_p/(2^p - 1)$ is irrational for every choice of signs. Tao and Teräväinen proved the all-positive case, answering a question of Erdős. We also show that for each fixed modulus $d \geq 2$, the sums of $1/(2^p - 1)$ over primes in the reduced residue classes, together with 1, are linearly independent over $\mathbb{Q}$. As d varies, the class sums span an infinite-dimensional space over $\mathbb{Q}$, and refinement to a common modulus determines every rational relation among them and 1.

1 Signed sums and residue classes

Let $\omega(n)$ count the distinct prime factors of n. Expanding each summand as a geometric series gives

$$\sum_{p \text{ prime}} \frac{1}{2^p - 1} = \sum_{n \geq 1} \frac{\omega(n)}{2^n}.$$

Erdős asked whether the number on the right is irrational [1, p. 102]. Pratt proved this under a quantitative prime tuples conjecture [4], and Tao and Teräväinen proved it unconditionally [5, Theorem 1.3]. Separate the odd primes into their two residue classes modulo four. Their two contributions have irrational sum, since the term at $p = 2$ is $1/3$. What happens if we subtract the 3 mod 4 sum from the 1 mod 4 sum?

The difference is irrational. Give a minus sign to the primes $p \equiv 3 \bmod 4$ and a plus sign to the rest. The signed sum in Theorem 1(i) then equals the difference plus $1/3$.

For $d \geq 2$ write $G_d = (\mathbb{Z}/d\mathbb{Z})^\times$. For $a \in G_d$ let C_a be the primes in the class a mod d, and set

$$P_a(d) = \sum_{p \in C_a} \frac{1}{2^p - 1}.$$

Theorem 1.

(i) *For every assignment of signs $\varepsilon_p \in \{-1, 1\}$ to the primes, $\sum_p \varepsilon_p/(2^p - 1)$ is irrational.*

(ii) *For each $d \geq 2$, the numbers 1 and $P_a(d)$, $a \in G_d$, are linearly independent over $\mathbb{Q}$.*

Both parts follow from a criterion for bounded integer weights u_p. The sum $\sum_p u_p/(2^p - 1)$ is irrational if the primes with $u_p \neq 0$ contribute at least a fixed positive fraction of $\sum_{p \leq x} 1/p$ for all sufficiently large x. For signs, every prime contributes. For a nonzero vector of class weights, the support contains a whole reduced class, whose share tends to $1/\varphi(d)$.

Tao and Teräväinen suggest that their method might extend to other sets resembling the primes [5, discussion after Theorem 1.3]. Each reduced class gives such a set. At modulus 2^j there are 2^{j-1} independent class sums, so the span across all moduli is infinite-dimensional. Passing to a common modulus determines which combinations across moduli are rational (Corollary 3).

Luca and Tachiya [2] proved the analogous independence statement for Lambert series $\sum_{n\ge1} a_n/(2^n-1)$ attached to independent periodic rational sequences on all positive integers. For the sums here, the coefficient of n also contains the indicator of primality. It vanishes on arbitrarily long intervals of composite integers and takes a nonzero value infinitely often in every class of nonzero weight, so the coefficient sequence is aperiodic even after deleting a finite prefix.

Idea of the proof. Tao and Teräväinen combine the binary tails of their series at the vertices of a cube of primes. Under the rationality assumption, the expected distance from the resulting sum to the nearest integer tends to zero. Its complex exponential therefore has mean tending to 1. We first use their cube with a fixed vector of class weights. In the actual sum, one integer sampled from an arithmetic progression determines the residues at all the retained primes. In the model, we draw a uniform residue modulo each of those primes independently. Each prime contributes to the variance in proportion to the *square* of its weight, so opposite signs both contribute to the fluctuations. The angles contributed by individual primes tend uniformly to zero, so the quadratic estimate for the cosine and independence bound the modulus of the mean phase by $\exp(-cV)$, where the total variance V tends to infinity.

The issue in comparing the two means is the contribution of the largest prime divisors, whose products can exceed the length of the sampling interval. We count these divisors separately in each residue class. Each count takes only finitely many values, so it can be recovered exactly by interpolating bounded multiplicative functions. This allows us to apply Tao and Teräväinen's two-point correlation estimate to all the classes on the same sampling interval.

2 Integer tails and the prime cube

Fix $d \ge 2$. Clearing denominators in a nontrivial rational relation in Theorem 1(ii) would give a nonzero vector of integers $(w_a)_{a\in G_d}$ with $S = \sum_a w_a P_a(d) \in \mathbb{Z}$. Put $w(p) = w_a$ on C_a, $w(p) = 0$ for $p \mid d$, and $J = \max_a |w_a|$. With $A(m) = \sum_{p\mid m} w(p)$ we have

$$S = \sum_{m\ge1} 2^{-m} A(m), \qquad S \in \mathbb{Z} \implies \sum_{h\ge1} 2^{-h} A(n+h) \in \mathbb{Z} \quad (n \in \mathbb{Z}_{\ge0}). \tag{1}$$

The geometric expansion converges absolutely since $|A(m)| \le J\omega(m)$. For the implication, multiply by 2^n and subtract the integer contributed by $m \le n$. All constants in what follows may depend on d and the fixed weights.

Following Tao and Teräväinen [5, §5, Lemma 5.2], we take an alternating sum over a cube of primes to cancel the first K terms of several binary tails at once. For the parameters K, H, P chosen below, their lemma supplies integers $p_0, v_1, \dots, v_K$ with $|v_j| \le P$ for which the 2^K numbers $p_\epsilon = p_0 + \sum_{j=1}^K \epsilon_j v_j$, $\epsilon \in \{0,1\}^K$, are distinct primes in $[P/2, P]$. Set $b_\epsilon = \sum_{j=1}^K j\epsilon_j v_j$. The kth tail term has shift

$$r_{\epsilon,k} = kp_\epsilon - b_\epsilon = kp_0 + \sum_{j=1}^K (k-j)\epsilon_j v_j.$$

For $1 \le k \le K$ the coefficient at $j = k$ vanishes, so flipping ϵ_k leaves this shift fixed. The same lemma ensures that the later shifts $s_{\epsilon,h} = r_{\epsilon,K+h}$, $1 \le h \le H$, are pairwise distinct.

Choose a large X and put $L = \log\log X$, $t = \log L$, with natural logarithms throughout. Take

$$K = \lfloor 2\log t/\log 2 \rfloor, \quad H = \lfloor t^2 \rfloor, \qquad M = 2\lfloor L \rfloor, \quad H_+ = \lfloor L^2 \rfloor,$$
$$P = \exp(t^3), \qquad\qquad R = X^{1/L^2}, \quad Y = X^{1/100}.$$

Here $\log\log Y - \log\log R = 2t + O(1)$ and $2^K \asymp t^2$. Thus $(\log\log Y - \log\log R)/2^K \to 0$, so averaging over the vertices suppresses the primes between R and Y. At the same time $2^K/L \to 0$, so the

primes below R can still supply a divergent variance of order $L/2^K$. Also $R^{2M} = X^{o(1)}$, which permits comparison of moments through degree M.

Let Q be the product of the distinct primes dividing any nonzero difference of the shifts $s_{\epsilon,h}$ with $1 \le h \le H$ or belonging to the cube $\{p_\epsilon\}$. Impose

$$n \equiv b_\epsilon \bmod p_\epsilon, \qquad n \equiv 0 \bmod p \quad (p \mid Q,\ p \notin \{p_\epsilon\}). \tag{2}$$

The Chinese remainder theorem gives a single class. There are $O((2^K H)^2)$ near-shift differences and $|s_{\epsilon,h}| \ll (K+h)KP$, so the logarithm of the product defining Q is $O(t^{12})$. Thus Q and the absolute values of all the near shifts are $(\log X)^{o(1)}$.

Write $I_x = (x, 2x] \cap \mathbb{Z}$. For now take any $N \in [X^{3/5}, X^{9/10}]$ and draw n uniformly from the class (2) in I_N. Section 3 will choose one such scale for the largest prime factors. For each integer $b \ge 1$ coprime to Q and each b-periodic function f,

$$\mathbb{E} f(n) = \frac{1}{b} \sum_{j \bmod b} f(j) + O(bQ\|f\|_\infty / N). \tag{3}$$

Indeed, one block of b consecutive progression points covers every residue modulo b, and the progression has $N/Q + O(1)$ points.

Suppose now that $S \in \mathbb{Z}$. The integer tail from (1) at $(n - b_\epsilon)/p_\epsilon$ is defined since $n > b_\epsilon$. Multiplication by a prime adds $w(p)$ to $A(m)$ exactly when $p \nmid m$. Use this identity at $m = (n - b_\epsilon)/p_\epsilon + k$, for which $p_\epsilon m = n + r_{\epsilon,k}$. We obtain

$$\sum_{k \ge 1} 2^{-k} A(n + r_{\epsilon,k}) \equiv -w(p_\epsilon) \sum_{k \ge 1} 2^{-k} \mathbf{1}_{p_\epsilon^2 \mid n + r_{\epsilon,k}} \pmod 1. \tag{4}$$

Here $w(p_\epsilon) \sum_{k \ge 1} 2^{-k}$ is an integer. Since p_ϵ divides Q exactly once, the square-divisibility condition selects one residue modulo p_ϵ in the index of the progression. Its probability is $1/p_\epsilon + O(Q/N)$, uniformly in k.

Write $\sigma_\epsilon = (-1)^{|\epsilon|}$. For $1 \le k \le K$, vertices differing only in ϵ_k have the same kth shift and opposite signs, so their kth terms on the left of (4) cancel. For a two-dimensional cube $r_{\epsilon,1} = p_0 - \epsilon_2 v_2$ and $r_{\epsilon,2} = 2p_0 + \epsilon_1 v_1$, so the two cancellations take place in different coordinates. The sum of the expected absolute errors on the right of (4) is $O_J(2^K(P^{-1} + Q/N)) = o(1)$. Keeping the next H terms gives a random variable T with one contribution for each prime,

$$X_p(n) = w(p) \sum_{h=1}^{H} \sum_{\epsilon} \sigma_\epsilon \, 2^{-K-h} \mathbf{1}_{p \mid n + s_{\epsilon,h}}, \qquad T(n) = \sum_p X_p(n). \tag{5}$$

The omitted tail has mean absolute value $o(1)$. Indeed, for $H < h \le H_+$ each $n + s_{\epsilon,h}$ is positive and at most $3X$. At most 101 prime divisors exceed Y, and at most $O(\log Q)$ primes divide Q. For any remaining prime $p \le Y$, counting one residue class along (2) gives divisibility probability $1/p + O(Q/N)$. Thus the expected number of prime factors is $O(1 + \log Q + L + YQ/N)$. For $h > H_+$, the deterministic bound $\omega(n + s_{\epsilon,h}) \ll \log X + \log(h+1)$ suffices. Summing 2^{-K-h} over the 2^K vertices and then over h bounds the tail by

$$O_J\Big(2^{-H}(1 + t^{12} + L) + 2^{-H_+}(\log X + \log(H_+ + 1))\Big) = o(1).$$

Consequently, with $e(y) = \exp(2\pi i y)$, (4) and the Lipschitz bound for e imply

$$\mathbb{E} e(T) = 1 + o(1). \tag{6}$$

3 Removing the large prime factors

The terms X_p with $p \mid Q$ are constant on (2) and together supply a fixed phase. For $p \nmid Q$ the shifts in (5) occupy distinct residue classes modulo p. At any residue j mod p at most one summand in $X_p(j)$ is nonzero. For a uniform U_p mod p this gives

$$|X_p(j)| \le J\, 2^{-K-1},$$
$$\mathbb{E}X_p(U_p) = 0, \qquad \mathbb{E}X_p(U_p)^2 = \frac{w(p)^2(1-4^{-H})}{3\cdot 2^K p}. \tag{7}$$

Here the signs sum to zero for each h. On squaring, the 2^K terms for that h have the same magnitude, and $\sum_{h=1}^{H} 4^{-h} = (1-4^{-H})/3$.

First consider $R < p \le Y$. For large X, every prime dividing Q is below R. For distinct p, p' in this range the uniform mean of $X_pX_{p'}$ modulo pp' is zero by the Chinese remainder theorem. Summing the periodic-sampling errors from (3) for all pairs and for the diagonal in (7) gives

$$\mathbb{E}\left|\sum_{R<p\le Y} X_p(n)\right|^2 \ll_J 2^{-K} \sum_{R<p\le Y} \frac{1}{p} + \frac{QY^4}{N} \ll_J \frac{1+t}{2^K} + X^{-0.56+o(1)} = o(1).$$

The first error estimate uses at most Y^2 ordered pairs of period at most Y^2. Mertens' theorem bounds the reciprocal sum.

For $p > Y$, the product of two primes can exceed the sample length. We instead count their contributions at each shifted integer. Put $\Lambda = (\log X)^{1/4}$ and define

$$D_a(m) = \#\{p \mid m : p > Y,\ p \in C_a\}.$$

For $1 \le m \le 3X$ the product of 102 primes above Y would exceed $3X$ for all sufficiently large X. Consequently $D_a(m) \le 101$. Tao and Teräväinen [5, §5.14] split the very large primes into shorter ranges and estimate how often two prime factors fall in the same range. Here the bounded range of D_a lets us recover the class count exactly from multiplicative functions to which a two-point estimate applies.

Lemma 2. *There is an absolute $c > 0$ such that for every sufficiently large X (depending on d) there is a real $N \in [X^{3/5}, X^{9/10}]$ with the following property. Let n be uniform on the integers of I_N in any one residue class modulo $B \le \Lambda^c$. For $a, b \in G_d$ and distinct integer shifts $|r|, |s| \le \Lambda^c$,*

$$\left|\mathbb{E}\big[(D_a(n+r)-\mu_a)(D_b(n+s)-\mu_b)\big]\right| \ll_d \Lambda^{-c}, \qquad \mu_a = \frac{1}{|I_N|}\sum_{m\in I_N} D_a(m).$$

The same N works for all these choices.

Proof. For each a and each of 102 distinct fixed numbers $z_j \in (0,1]$, let $g_{a,j}(m) = z_j^{D_a(m)}$. Since D_a adds on coprime arguments, the $g_{a,j}$ are real, 1-bounded multiplicative functions. There are real constants c_j, independent of X, such that

$$k = \sum_j c_j z_j^k \qquad (0 \le k \le 101). \tag{8}$$

Indeed, interpolate the polynomial z^k at the 102 nodes and differentiate at $z = 1$. Thus (8) recovers D_a exactly on the whole range of integers that occurs in the lemma.

Tao and Teräväinen adapted Pilatte's decoupling estimate [3, Theorem 2.4 and Remark 2.6] to obtain their correlation theorem [5, Theorem 3.1]. Its equidistributed case requires a bound on all

arithmetic progressions. For $g = g_{a,j}$ write $\delta_g(x) = |I_x|^{-1} \sum_{m \in I_x} g(m)$. Uniformly for $X^{0.4} \le x \le X$, any positive modulus v, and any class b mod v, we need

$$\sum_{\substack{m \in I_x \\ m \equiv b \bmod v}} g(m) = \frac{x}{v} \delta_g(x) + O(x/\Lambda). \tag{9}$$

For $v > \Lambda$ this follows by counting integers. For $v \le \Lambda$, expand $g(m)$ over the squarefree divisors $\ell \mid m$ supported on primes in C_a above Y. The coefficient of ℓ is $(z_j - 1)^{\omega(\ell)}$, of absolute value at most one. Since $(\ell, v) = 1$, counting multiples of ℓ in the progression and comparing with $1/v$ times the unrestricted count costs $O(1)$ per divisor.

There are $O(x/\log X)$ possible divisors $\ell \le 2x$. For $\ell > 1$ write $\ell = up$, where p is its largest prime. For each eligible $u \le 2x/Y$, the bound $\pi(y) \ll y/\log y$ gives $O(x/(u \log Y))$ choices of p. Since u is squarefree and all its prime factors exceed Y, the sum of $1/u$ is at most $\prod_{Y < p \le 2X}(1 + 1/p) = O(1)$ by Mertens' theorem and $Y = X^{1/100}$. As $\log Y = (\log X)/100$, the accumulated counting error gives (9). Replacing $|I_x|$ by x costs $O(1)$. The other condition of [5, Theorem 3.1(i)] asks for $g(p) = 1$ when $\exp((\log X)^{1/11}) \le p \le \exp((\log X)^{1/10})$. That interval lies below Y.

Apply the theorem with parameter Λ to every ordered pair (g, g') of the $102\varphi(d)$ test functions and to every $(g, 1)$. The functions may depend on X through Y, but their number is fixed. For each pair the exceptional set in $[\sqrt{X}, X]$ has logarithmic measure $O(\Lambda^{-c} \log X)$. Their union still has measure $o(\log X)$, whereas $[X^{3/5}, X^{9/10}]$ has logarithmic measure $3 \log X/10$. Choose N there outside the union. The theorem bounds $\mathbb{E}[(g(n + r) - \delta_g(N))g'(n + s)]$ by $O(\Lambda^{-c})$, uniformly in the stated shifts and progression. Its normalization B/N differs from the reciprocal progression count by a relative $O(B/N)$. Subtract the estimate with second function 1, multiplied by $\delta_{g'}(N)$. Applying (8) to both centered factors proves the lemma. □

Choose the scale N supplied by Lemma 2 for the rest of the proof. The bounds for Q and the shifts in Section 2 place the actual progression within its range. Fix a retained h and a class C_a. The sign sum annihilates μ_a, so write

$$V_{a,h} = 2^{-K} \sum_{\epsilon} \sigma_\epsilon (D_a(n + s_{\epsilon,h}) - \mu_a).$$

On squaring there are 2^K diagonal terms, bounded by an absolute constant each. For distinct vertices the shifts are different, so Lemma 2 bounds every off-diagonal term. Thus $\|V_{a,h}\|_2^2 \ll 2^{-K} + \Lambda^{-c} = o(1)$ uniformly in h and a. The weighted sum over all these primes is exactly $\sum_{a \in G_d} w_a \sum_{h=1}^{H} 2^{-h} V_{a,h}$. Since d is fixed, the triangle inequality in L^2 and $\sum 2^{-h} \le 1$ show that its L^2 norm tends to zero.

It remains to examine $\mathcal{P} = \{p \le R : p \nmid Q\}$ and $T_1 = \sum_{p \in \mathcal{P}} X_p(n)$. Taking moduli eliminates the fixed phase from $p \mid Q$. The discarded terms have mean absolute value $o(1)$, so (6) gives

$$|\mathbb{E} e(T_1)| = 1 + o(1). \tag{10}$$

4 Independent residues and the variance

Take independent U_p, uniform modulo p, for $p \in \mathcal{P}$. Write $Z_p = X_p(U_p)$, $T_* = \sum Z_p$, and $V = \sum \mathrm{Var}(Z_p)$. Equation (7) is exact in this model. Taking logarithms in Williams's Mertens theorem gives the reciprocal-prime sum in each reduced class [6, Theorem 1]. Omitting the primes dividing Q costs at most $J^2 \sum_{p \le Q} 1/p \ll_J 1 + \log\log(3Q)$. Hence

$$\sum_{p \in \mathcal{P}} \frac{w(p)^2}{p} = \left(\frac{1}{\varphi(d)} \sum_{a \in G_d} w_a^2 \right) \log\log R + O_{d,J}(1 + \log\log(3Q)). \tag{11}$$

The coefficient of $\log\log R$ is positive for every nonzero vector of class weights. Since $\log\log R = L - 2t$ and $\log\log(3Q) = O(\log t)$, we obtain

$$V \sim \frac{\sum_{a \in G_d} w_a^2}{3\varphi(d)} \frac{L}{2^K} \longrightarrow \infty, \qquad V = o(M).$$

The divergence will force cancellation in the independent model. The upper bound $V = o(M)$ will allow us to transfer that cancellation using only M moments.

Remark (Bounded integer weights). For bounded integer weights u_p, the sum $\sum_p u_p/(2^p - 1)$ is irrational whenever $\sum_{p \le x} u_p^2/p \ge \eta \log\log x$ for some $\eta > 0$ and all sufficiently large x. On the support, $1 \le u_p^2 \le (\sup_q |u_q|)^2$, so Mertens' theorem identifies this condition with the one in the introduction. In Theorem 1(i), $u_p^2 = 1$ at every prime.

If the Lambert sum had denominator b, the weights $w(p) = bu_p$ would give the integer tails in (1). The divisor count for (9) allows any subset of the primes above Y, so the proof of Lemma 2 gives one sampling interval for the finitely many nonempty level sets $\{p : u_p = j\}$. Omitting $p \mid Q$ costs $O_{b,u}(1 + \log\log(3Q)) = o(L)$. The lower bound at R and Mertens' upper bound now give $V \asymp_{b,u} L/2^K$ by (7). The cube estimates use only integrality and boundedness of the weights, and the argument below applies since $V \to \infty$ and $V = o(M)$.

To compare characteristic functions we use that M is even. For any ordered m-tuple of primes in $\mathcal{P}$, its product of variables has period $b = \prod_{p \text{ distinct}} p \le R^m$. Its uniform mean modulo b agrees exactly with its expectation in the model. Repeated occurrences of a prime use the same residue U_p. By (7) the product is bounded by one for all sufficiently large X. There are at most R^m ordered tuples, hence (3) gives, uniformly for $0 \le m \le M$,

$$|\mathbb{E}T_1^m - \mathbb{E}T_*^m| \ll QR^{2M}/N = o(1).$$

Put $\theta_0 = 2\pi \exp(1)$. For either $\theta = \pm\theta_0$, centering and $|Z_p| \le J2^{-K-1}$ imply $\mathbb{E}\exp(\theta Z_p) \le \exp(C\theta^2 \operatorname{Var}(Z_p))$ for large X, by the real Taylor estimate on a bounded interval. Independence and $y^M/M! \le \exp(|y|)$ therefore give

$$\frac{(2\pi)^M}{M!}\mathbb{E}T_*^M \le \exp(-M) \sum_{\theta = \pm\theta_0} \mathbb{E}\exp(\theta T_*) \le 2\exp(-M + O(V)) = o(1).$$

Expand $e(T_1)$ and $e(T_*)$ to degree $M - 1$. The discrepancies of the polynomial terms sum to $O(e^{2\pi}QR^{2M}/N)$. The Taylor remainder for a real argument x is bounded by $(2\pi|x|)^M/M!$. The last display and the same moment comparison at the even degree M bound both expected remainders by $o(1)$. In particular $\mathbb{E}e(T_1) = \mathbb{E}e(T_*) + o(1)$.

Finally let Z_p' be an independent copy of Z_p. Since $|Z_p - Z_p'| \le J2^{-K}$, the elementary inequality $1 - \cos(2\pi x) \gg x^2$ for $|x| \le 1/4$ gives

$$|\mathbb{E}e(Z_p)|^2 = \mathbb{E}\cos(2\pi(Z_p - Z_p')) \le \exp(-c_0 \operatorname{Var}(Z_p))$$

with an absolute $c_0 > 0$. Independence yields $|\mathbb{E}e(T_*)| \le \exp(-c_0 V/2) \to 0$. This contradicts (10) and proves Theorem 1. □

5 Refining the modulus

Theorem 1(ii) also describes all rational combinations of sums with different moduli. For example, $P_1(4) = P_1(8) + P_5(8)$. When the larger modulus has an additional prime factor, a finite correction can appear. The primes congruent to 2 mod 3 consist of 2 itself and the primes congruent to 5 mod 6, so $P_2(3) - P_5(6) = 1/3$.

Corollary 3. *Let $d_1, \dots, d_k \geq 2$, let $a_i \in G_{d_i}$, and let $c_1, \dots, c_k \in \mathbb{Q}$. Set $D = \operatorname{lcm}(d_1, \dots, d_k)$ and define the periodic weight on all integers by*

$$W(n) = \sum_{i=1}^{k} c_i \, \mathbf{1}_{n \equiv a_i \bmod d_i}.$$

Then $\sum_i c_i P_{a_i}(d_i)$ is rational if and only if $W(b) = 0$ for every $b \in G_D$. In that case its value is $\sum_{p \mid D} W(p)/(2^p - 1)$, summed over the distinct prime divisors of D.

Proof. Partition the primes coprime to D by their classes modulo D. This gives the exact identity

$$\sum_i c_i P_{a_i}(d_i) = \sum_{b \in G_D} W(b) P_b(D) + \sum_{p \mid D} \frac{W(p)}{2^p - 1}.$$

The second term is rational. By Theorem 1(ii) at modulus D, the first is rational precisely when all its coefficients vanish. □

Verification

Two AI collaborators independently reviewed the mathematics of the initial November 26, 2026 manuscript, including the results now stated as Theorem 1(ii) and Corollary 3. Both checked the correlation estimate on the chosen progression, the signed second moments and the transfer to independent residues, taking the cited results of Tao and Teräväinen and of Williams as inputs. One also reviewed the exposition and the comparison with earlier work. A separate audit of that manuscript derived the bounded-weight criterion and its consequence for arbitrary signs, Theorem 1(i), and independently reproduced the finite checks below. A subsequent exposition review covered the November 26 revision featuring the arbitrary-sign result.

From the root of the source bundle, `python3 checks/check_identities.py` reproduces the finite arithmetic. It checks (8) on all 102 monomials, mixed moments on a complete system of residues modulo $11 \cdot 13$, and the square-divisibility counts in (4) on a four-vertex cube. With $p_0 = 11$ and $(v_1, v_2) = (2, 6)$, the vertices are $11, 13, 17, 19$. Set $K = H = 2$ and assign weights 1 and -1 to the primes 37 and 43, respectively. Over all $37 \cdot 43$ residue pairs the second moment of their sum is $\frac{5}{64}(\frac{1}{37} + \frac{1}{43}) = \frac{25}{6364}$, as predicted by (7).

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