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THE EXACT JOINT SPECTRAL ENVELOPE OF THE SECOND-LARGEST ADJACENCY AND LAPLACIAN EIGENVALUES

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Abstract. For finite simple connected graphs on at least two vertices, write λ_2 and μ_2 for the second-largest adjacency and ordinary Laplacian eigenvalues. We determine $F(t) = \inf\{\mu_2(G) : \lambda_2(G) \geq t\}$ for all $0 \leq t \leq 2$ and classify every minimizer. A double-star staircase accumulates at 1. Above $\sqrt{3}$ there are twenty transition levels followed by an infinite sequence of six-level blocks beginning at the smallest possible index 4. The proof combines a sharp maximum-degree reduction, explicit Schur-complement comparisons valid on algebraic intervals, and finite certificates based on exact characteristic-polynomial root counts. We also obtain the optimal coefficient $8/\sqrt{3}$ in a quadratic lower bound and the exact gap for nonstars in Cazenave's inequality.

1. Introduction

The second-largest adjacency and Laplacian eigenvalues describe different features of a graph. A pointwise inequality between the two values gives a constraint on the feasible pairs. Computing the lower boundary under an adjacency threshold also asks which graphs first attain each Laplacian value, which graphs share a value, and how the boundary behaves near a spectral limit.

For connected simple graphs, Cazenave obtained the linear bound $\mu_2 \geq \lambda_2 + 1$ and posed the quadratic bound $\mu_2^2 \geq 4\lambda_2 + 1$ [2, Theorem 1 and Conjecture 6(ii)]. Kramer proved the quadratic inequality, characterized equality, and identified a first nontrivial jump beyond $\lambda_2(P_4)$ [5, Theorem 1 and Corollary 4]. The integer neighbor counts in Cazenave's proof give the ceiling form used below.

Descriptions based on a single spectrum provide pertinent structural information. Li, Guo and Shiu classified the connected graphs with $\mu_2 \leq 3$ [6]. Wu, Yu and Shu treated a larger threshold [9]. Chang and Lin classified the trees with $\mu_2 \leq 2 + \sqrt{3}$ and identified the trees attaining equality [3, Theorem 1.1 and Corollary 3.2]. We use a separate generation argument for the strict, bounded-degree subfamily and its supergraphs since the adjacency

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threshold must also be enforced. Grounded Laplacians are standard principal submatrices [7]. The coupled and uncoupled modes of such matrices are both needed for the present comparison. The limit at 2 also has a connection to the use of small graph extensions in spectral limit point problems [4, 8]; here both coordinates are second-largest eigenvalues and the order comes from joint minimization.

Our main result describes the entire lower boundary up to adjacency threshold 2. The two infinite ranges have different structure. Below 1 a single increasing double-star parameter indexes the steps. Near 2 the same six graph constructions recur at successive values of a triangle-multiplicity parameter; periodicity refers to these constructions, with strictly increasing spectral endpoints. The finite transition and the equality analysis show where this recurrence begins. The spectral inequalities in Section 9 are consequences of this boundary and the signed-neighbor bound.

The proof uses exact finite computations where graph enumeration or univariate sign certification is required. The unbounded quantifiers are handled by a spanning-tree reduction, a sharp rooted structure reduction, and a finite terminating set of comparisons that applies to all multiplicities.

2. Notation and main results

Throughout, a graph is finite, simple, undirected and connected, with $n \geq 2$, unless a subgraph or a matrix block is explicitly under discussion. The eigenvalues of $A(G)$ and $L(G) = D(G) - A(G)$ are ordered decreasingly with multiplicity. We abbreviate

$$a(X) = \lambda_2(X), \qquad b(X) = \mu_2(X),$$

$$\beta = \frac{3+\sqrt{5}}{2}, \qquad C = 2+\sqrt{3}, \qquad h_5 = \frac{5+\sqrt{5}}{2}.$$

For $t \in [0,2]$ set $F(t) = \inf\{b(G) : a(G) \geq t\}$. A level represented by X, after a representative Y, means $F(t) = b(X)$ for $a(Y) < t \leq a(X)$. All assertions of uniqueness refer to graph isomorphism.

Let $S(r,s)$ be the double star whose adjacent centers have r and s private leaves. Write $B(r,r)$ for two disjoint K_r's joined by one bridge. A rooted branch consists of a connected graph B and a nonempty marked subset of its vertices. Form a graph by joining each marked vertex to a common hub outside the branches. Use P_j for an end-marked path on j vertices, T for a path on two vertices marked at both ends, and B for a triangle with one mark. The names $\mathsf{C}_4, \mathsf{C}_5$ denote both-end-marked paths on three and four vertices, respectively. Their subscripts are their cycle lengths after attaching the hub. Let G_k, $k \geq 0$, have one B and $k+1$ branches of type T. Thus $G_0 = B(3,3)$. Addition of a named branch always refers to the same hub. Set $T_j = G_{j-1} + 2\mathsf{P}_1$ for $j \geq 1$.

Theorem 2.1 (Exact envelope and all minimizers)**.** *The infimum defining $F(t)$ is attained for every $t \in [0,2]$. Its values and minimizers are as follows.*

Threshold	*Value and minimizers*
0	1*, attained by all stars of order at least three*
$0 < t < 1$	$b(S(r,1))$*, uniquely, for* $r = \lceil t^2(2-t^2)/(1-t^2) \rceil$
1	β*, a common hub with at least two two-edge arms and any leaves*
$1 < t \le \sqrt{3}$	3*, with the seven types and cutoffs of Section 3*
$\sqrt{3} < t \le a(G_4)$	*the twenty levels of Section 6*
$a(G_4) < t < 2$	*the six-level blocks of Section 7*
2	C*, with the branch family of Section 8*

In the finite transition a representative is unique except at a level ending at $a(G_j)$*, where* T_j *also attains the value up to* $a(T_j)$*. For each* $k \ge 4$*, the block on* $(a(G_k), a(G_{k+1})]$ *has representatives in order*

$$G_{k-1} + \mathsf{C}_5, \quad G_k + \mathsf{P}_1, \quad G_k + \mathsf{P}_2, \quad G_k + \mathsf{P}_3, \quad G_k + \mathsf{P}_4, \quad G_{k+1}.$$

The first five are unique. In the last level T_{k+1} *also attains the value up to* $a(T_{k+1})$*. The smallest starting index for this recurring order is* 4*.*

The proofs below also yield the sharp structural threshold $\Delta \ge 4$ when $a(G) > \sqrt{3}$ and $b(G) < C$, and the best coefficient in $\mu_2^2 \ge c\lambda_2 + 1$ for all graphs. We first record the matrix facts common to the arguments.

For a real symmetric matrix M, let $s_i(M)$ denote its ith largest eigenvalue. We use Cauchy interlacing for principal submatrices and Courant–Fischer for Rayleigh quotients [1]. In particular, $s_2(M-v) \le s_2(M) \le s_1(M-v)$ whenever the entries exist. Adding an edge uv to a graph adds $(e_u - e_v)(e_u - e_v)^{\mathsf{T}}$ to its Laplacian. Padding a subgraph by isolates and applying the min–max principle proves Laplacian eigenvalue monotonicity under subgraph inclusion at any index of the smaller graph. For adjacency matrices we use the corresponding monotonicity only for the largest eigenvalue, by a nonnegative Perron vector.

Lemma 2.2 (Rayleigh equality)**.** *Suppose* H *is a subgraph of* G *on the same vertex set after padding by isolates. Suppose* W *is two-dimensional and* $v^{\mathsf{T}} L(H) v = b\|v\|^2$ *on* W*. If* $\mu_2(G) = b$*, there is* $0 \ne v \in W$ *with* $L(G)v = bv$ *and* $v_u = v_w$ *for every* $uw \in E(G) \setminus E(H)$*.*

Proof. Choose $v \in W$ orthogonal to a top eigenvector of $L(G)$. Its Rayleigh quotient for $L(G)$ is at most b and at least its quotient for $L(H)$, which is b. Equality on the orthogonal complement yields the eigenvector equation. The difference of the two quadratic forms is a sum of edge squares. □

For a branch with indicator column c, the block in the Laplacian after deletion of the hub is $D_B = L(B) + \operatorname{diag}(c)$. The coupling column to the hub is $-c$; the adjacency block instead has coupling c. Section 5 derives the scalar Schur complements and locates the relevant roots.

3. The lower spectral range

The spectral pairs of K_2 and of a star of order at least three are $(-1, 0)$ and $(0, 1)$, respectively. A connected nonstar contains P_4 or a triangle. Indeed a tree with no P_4 has diameter at most two, and a cycle is a triangle or contains P_4. Since $b(P_4) = 2$ and $b(K_3) = 3$, this proves $F(0) = 1$ with exactly the asserted stars.

Proposition 3.1 (Double-star levels)**.** *For $0 < t < 1$, set*

$$r = \left\lceil \frac{t^2(2 - t^2)}{1 - t^2} \right\rceil.$$

Then $F(t) = b(S(r, 1))$, attained uniquely by $S(r, 1)$. The endpoints are $a_r = a(S(r, 1))$, with $a_0 = 0$, and the rth interval is $(a_{r-1}, a_r]$.

Proof. If $b(G) < \beta$, then K_3 and P_5 are forbidden because $b(K_3) = 3$, $b(P_5) = \beta$. The only possible cycle is C_4. A chord makes a triangle and an external attachment gives P_5, so a graph containing a cycle has to be C_4, whose second adjacency value is zero. A tree with no P_5 is a star or a double star. A double star with at least two leaves on both ends contains $S(2, 2)$, with $b(S(2, 2)) = 3$. Thus with $a(G) > 0$ only the double stars $S(r, 1)$ remain.

Compress the r leaves on one center; the differences of those leaves give zero in the adjacency matrix and 1 in the Laplacian. The remaining quotient matrices, in the order of first leaves, first center, second center, and last leaf, are

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ r & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ -r & r+1 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

Consequently

$$a_r^2 = \frac{r + 2 - \sqrt{r^2 + 4}}{2}, \qquad b_r = b(S(r, 1)),$$
$$p_r(x) = x^3 - (r + 5)x^2 + (3r + 7)x - r - 3,$$

where b_r is the middle root of p_r. The smaller root of $u^2 - (r + 2)u + r$ is positive, is strictly increasing to 1 as $r \to \infty$, and is at least t^2 exactly when $r(1 - t^2) \geq t^2(2 - t^2)$. For example, its strict increase follows from the expression $2r/(r + 2 + \sqrt{r^2 + 4})$. Also $p_r(0) < 0$, $p_r(1) = r$, $p_r(2) = r - 1$, and $p_r(\beta) = -1$. The two smaller roots lie in $(0, 1)$ and $[2, \beta)$, with $b_1 = 2$. Since $p_{r+1}(x) - p_r(x) = -x^2 + 3x - 1 > 0$ on $[2, \beta)$, the middle roots strictly

increase. Dividing their equation by r gives $b_r \to \beta$. The least eligible r therefore gives the unique minimum. □

Proposition 3.2 (The threshold 1)**.** *The value $F(1)$ is β. Its minimizers consist of at least two two-edge arms and any number of leaves at one hub.*

Proof. Proposition 3.1 excludes $b < \beta$ when $a \geq 1$. Suppose $b = \beta$ and $a \geq 1$. If P_5 were absent, the forbidden-subgraph argument in that proposition would still apply and give $a < 1$. Take a P_5 with center u. After deleting u, its two identical grounded arms have top eigenvalue β, with a nonzero coupling at u. Their top eigenvectors, padded by zeros, span a two-dimensional restriction at β. Lemma 2.2 gives an eigenvector with opposite coefficients on the two arms by the equation at u. On one grounded arm the two entries have opposite signs and different absolute values, by the matrix $\left(\begin{smallmatrix} 2 & -1 \\ -1 & 1 \end{smallmatrix}\right)$. The four nonzero coordinates are therefore distinct. No new edge can touch them, except edges already in the initial P_5. Every other branch attaches through u.

If another grounded block has an eigenvalue above β, restrict $L - \beta I$ to that positive mode, one zero mode from an original arm, and e_u. Eliminate the positive mode by a congruence. The remaining 2 by 2 block has a zero diagonal entry for the arm and a nonzero off-diagonal coupling, so its determinant is negative. There would be two positive eigenvalues, a contradiction. Thus every other grounded block has radius at most β. An internal vertex of degree at least two gives $K_{1,2}$ as a subgraph, of Laplacian radius 3. A both-marked two-vertex path also has grounded radius 3. The only other options are single leaves and single-marked two-vertex paths. Conversely, deleting the hub of any such graph gives at least two adjacency blocks with top value 1 and two grounded blocks with top value β, and none above the respective values. Interlacing gives the two second eigenvalues exactly. □

Lemma 3.3 (Signed-neighbor bound)**.** *If $a(G) > 0$, then $b(G) \geq \lceil a(G) \rceil + 1$.*

Proof. A second adjacency eigenvector y has both signs, by orthogonality to the positive Perron vector. At a maximal positive entry, each same-sign neighbor contributes at most that entry and other neighbors contribute nonpositively. Thus there are at least $\lceil a(G) \rceil$ same-sign neighbors. Apply the same argument to $-y$. The graph contains two vertex-disjoint stars of this many leaves. Their top Laplacian value is repeated in their disjoint union. □

Proposition 3.4 (The six-vertex plateau)**.** *For $1 < t \leq \sqrt{3}$, $F(t) = 3$. The seven graph types in Table 1 are the minimizers, each on the interval $(1, a(G)]$ for its own adjacency endpoint.*

Proof. Lemma 3.3 gives the lower bound and G_0 attains it even at $\sqrt{3}$. For equality with $a > 1$, use the two disjoint $K_{1,2}$'s provided by that lemma. The two top Laplacian vectors of these stars are proportional to $(2, -1, -1)$

on each star. Lemma 2.2 shows that new edges cannot leave the support of the resulting vector. By connectedness its support is the whole graph and hence both full stars. There are exactly six vertices. The exact certificate enumerates every same-order supergraph of the two stars, retaining connected graphs with $a > 1$, $b = 3$, and removes repetitions by testing all vertex permutations. It gives Table 1. The last row is G_0, with $a = \sqrt{3}$; the other rows have smaller endpoints. □

Edges added	Adjacency characteristic polynomial
$24, 25$	$x^6 - 6x^4 + 6x^2$
23	$x^6 - 5x^4 + 5x^2$
25	$x^6 - 5x^4 + 6x^2 - 1$
$15, 25, 45$	$x^6 - 7x^4 - 2x^3 + 8x^2 + 4x$
$12, 23$	$x^6 - 6x^4 - 2x^3 + 7x^2 + 4x$
$25, 45$	$x^6 - 6x^4 - 2x^3 + 8x^2 + 4x - 1$
$12, 25, 45$	$x^6 - 7x^4 - 4x^3 + 11x^2 + 12x + 3$

TABLE 1. The seven plateau types. Each graph also contains the four edges $01, 02, 34, 35$. Its endpoint is the second-largest root of the polynomial, with multiplicity.

4. Structural reduction

For P_n the ordinary Laplacian eigenvalues are $2 - 2\cos(j\pi/n)$, $0 \le j < n$. In particular $b(P_{12}) = C$, so $b(G) < C$ forbids P_{12} as a subgraph.

Proposition 4.1 (Local finiteness)**.** *Let $c < C$. The graphs with $a(G) > \sqrt{3}$ and $b(G) \le c$ have diameter at most 10 and maximum degree at most $D = \lfloor c+2/(C-c) \rfloor$. If such graphs exist, their order is at most $1+D\sum_{i=0}^{9}(D-1)^i$.*

Proof. The forbidden path gives the diameter. Choose a maximum-degree vertex u. Interlacing for $A(G - u)$ gives a component J with adjacency spectral radius greater than $\sqrt{3}$. We construct a unit vector w supported on J with $w^{\mathsf{T}}L(G)w \ge C$ and both its positive-coordinate sum and its negative-coordinate sum of absolute values at most $\sqrt{2}$.

If J contains $K_{1,3}, C_4$, or P_6, take the top Laplacian mode of that subgraph and pad by zeros. The same-sign sums squared are $3/4$ for the star and 1 for the four-cycle. For P_6 the largest mode has opposite coordinates under reversal, with three positive and three negative entries. Each sign has squared norm $1/2$, giving a bound $3/2$. Its eigenvalue is C, and the star and four-cycle each have top eigenvalue 4. Without these subgraphs J is a path on at most five vertices or a triangle or five-cycle. Paths of this order have adjacency spectral radius at most $\sqrt{3}$, by $2\cos(\pi/(n+1))$. In a cycle choose a vertex joined to u, and label it 0. In a triangle use the normalized mode proportional to $(1 + \sqrt{3}, -1, -1)$ of its single-mark grounded block at C. In

the five-cycle take $w = (2, -1, 1, 1, -1)/\sqrt{8}$ in cyclic order. The one-mark quadratic form is $15/4$, and the required squared sums are 2 and $1/2$.

On $\operatorname{span}(e_u, w)$ the diagonal entries of $L(G) - cI$ are $d(u) - c$ and at least $C - c$. The off-diagonal square is at most 2, since the entry is a sum over the neighbors of u of the coordinates of w, up to sign. If $d(u) > c + 2/(C - c)$, this restriction is positive definite, contradicting $b(G) \le c$. The order bound follows by counting vertices in successive layers about one vertex. □

Lemma 4.2. *If $b(G) < C$ and $d(u) \ge 5$, then $\Delta(G - u) \le 2$.*

Proof. If $G - u$ contains a three-leaf star, let W be its four vertices and v its center. If u has at most two neighbors in W, there are three outside W and hence two disjoint three-leaf stars. If u has at least three neighbors in W and is adjacent to v, two common leaves and uv give $K_4 - e$. Otherwise u is joined to all three leaves and at least two outside vertices, giving $K_{2,3}$ with two additional leaves at one degree-three vertex. The first two forbidden graphs have $b = 4$. The third has $b > C$ by the integer-matrix certificate in Section 10. □

Lemma 4.3 (Grounded branch census)**.** *Among paths on $1, \dots, 11$ vertices and cycles on $3, \dots, 11$ vertices, with a nonempty marked set, there are 2450 rooted isomorphism types. Their grounded maximum eigenvalues are below, equal to, or above C for $8, 3, 2439$ types, respectively. The eight subcritical branches are*

$$\mathsf{P}_1, \mathsf{P}_2, \mathsf{P}_3, \mathsf{P}_4, \mathsf{P}_5, \mathsf{T}, \mathsf{C}_4, \mathsf{C}_5.$$

The critical branches are the center-marked three-vertex path, the both-end-marked five-vertex path, and B. Among these eleven only B has adjacency radius greater than $\sqrt{3}$.

Proof. The automorphisms of a path are identity and reversal, and those of a cycle are its dihedral group. Thus enumeration of the nonempty bit masks modulo these explicit actions gives precisely the possible rooted types, including the one-vertex graph. For every orbit the program evaluates the root counts of $L(B) + \operatorname{diag}(c)$ at C and of $A(B)$ at $\sqrt{3}$ as specified in Section 10. A zero at C is counted with its exact multiplicity, which certifies the three critical cases. □

Proposition 4.4 (Bounded maximum degree)**.** *If $\Delta(G) \le 4$ and $b(G) < C$, then $n \le 21$ and $a(G) < 4657/2500$. If $\Delta(G) \le 2$ under the same spectral condition, then $a(G) \le \sqrt{3}$.*

Proof. These restrictions hold for every spanning tree except the asserted adjacency bound, which is tested on supergraphs. Build trees from P_2 by leaf additions subject to $\Delta \le 4$ and the strict Laplacian threshold. Removing a leaf preserves both restrictions. Every eligible tree has a predecessor; any isomorphism to an enumerated predecessor carries the attachment vertex with it. For isomorphism rejection, recursively encode the unordered child subtrees at the tree center. With two centers use the unordered pair of

codes of the halves across the center edge. Equality of codes is equivalent to isomorphism by recursion. There are 198 surviving trees, one of order 21 and none of order 22. Any larger eligible tree would have a 22-vertex predecessor.

For each tree representative choose a fixed order of its nonedges. The search adds edges in increasing index order and prunes as soon as a degree exceeds 4 or $b \geq C$. Every prefix of an eligible supergraph satisfies both restrictions, by degree and Laplacian monotonicity. Thus every admissible graph appears on at least one tree representative, while appearances across representatives can repeat. Exact counts on all 2459 surviving labeled states certify $a < 4657/2500$. If $\Delta \leq 2$, the graph is a path or cycle of order at most 11. Their cosine adjacency spectra yield $a \leq \sqrt{3}$. □

Theorem 4.5 (Sharp joint structure)**.** *If $a(G) > \sqrt{3}$, $b(G) < C$, and $\Delta(G) \geq 4$, then deleting any maximum-degree vertex gives one branch* B *and a multiset of the eight subcritical branches. The bound* 4 *is optimal.*

Proof. First suppose $d(u) = \Delta(G) \geq 5$. Lemma 4.2 and the forbidden P_{12} make every component eligible for Lemma 4.3. Interlacing for $A(G - u)$ gives an active component, with adjacency spectral radius greater than $\sqrt{3}$. Its grounded block reaches or exceeds C. There is at most one such block, because two give $b(G) \geq C$ by interlacing. Other blocks must therefore be subcritical.

For a supercritical active component the principal matrix on u and the component has the form

$$M_B(d) = \begin{pmatrix} d & -c^{\mathsf{T}} \\ -c & D_B \end{pmatrix}.$$

Increasing d adds a positive semidefinite diagonal term. For $s = 8, 7, 6, 5$ test $M_B(\max(s, \mathbf{1}^{\mathsf{T}} c))$ against the rational threshold $37321/10000 > C$. The count of failed exclusions for all supercritical branches and for active branches is shown below. A successful exclusion means the second largest eigenvalue is strictly above the rational threshold.

s	8	7	6	5
all failures	0	1	4	13
active failures	0	0	0	6

The six active residual branches at $d = 5$, with all multisets of subcritical branches of the remaining attachment degree, give 307 rooted combinations. Direct exact spectral comparison excludes all 307. The critical active branch is B.

For maximum degree 4, use the complete finite population in Proposition 4.4. Of the 242 labeled states satisfying $a > \sqrt{3}$, 191 have maximum degree 4. For every one, the deletion of every maximum-degree vertex gives the asserted components. This check examines the components, attachment bits, and exact grounded eigenvalue counts as in Lemma 4.3. For sharpness,

take a triangle on $0, 2, 3$ and append the path $0, 1, 4, 5, 6, 7, 8$. The tested inequalities are $a > \sqrt{3}$, $b < C$. The unique maximum-degree vertex 0 has degree 3, and deletion of it leaves a path on two vertices and a path on six. The certificate records the edge list and both integer characteristic polynomials. □

5. Rooted branches and exact spectral responses

For a branch B with indicator c, set

$$r_B(x) = c^{\mathsf{T}}(xI - A_B)^{-1}c,$$

$$\ell_B(x) = c^{\mathsf{T}}(xI - D_B)^{-1}c, \qquad \psi_B(x) = \frac{\mathbf{1}^{\mathsf{T}}c + \ell_B(x)}{x}.$$

For a fixed hub graph, the two scalar Schur complements are

$$S_A(x) = x - \sum_B r_B(x),$$

$$S_L(x) = x - d(u) - \sum_B \ell_B(x) = x\left(1 - \sum_B \psi_B(x)\right). \tag{5.1}$$

The sums include all branches with their multiplicities, and the degree is $d(u) = \sum_B \mathbf{1}^{\mathsf{T}}c$.

For a real symmetric block matrix M and a coupling column c, the resolvent response is $\sum_\theta \|\Pi_\theta c\|^2/(x-\theta)$, with Π_θ orthogonal projection onto the θ-eigenspace. Discard zero weights and collect repeated poles. The full matrix decomposes into the hub coupled to one vector for each distinct pole, and the orthogonal eigenspaces of the grounded block sum. Each open gap between coupled poles and each exterior interval has exactly one coupled root. Indeed the relevant Schur complement has derivative $1 + \sum_\theta \|\Pi_\theta c\|^2/(x-\theta)^2 > 0$ and opposite signs at the two ends of the interval. An uncoupled mode stays in the full spectrum. For repeated poles the multiplicity lost by collecting weights also supplies uncoupled eigenvectors.

Tables 2 and 3 give all responses needed for the high branch family. They follow from a determinant identity. For $K = xI - M$,

$$\det K - \det(K - cc^{\mathsf{T}}) = (\det K)c^{\mathsf{T}}K^{-1}c.$$

Dividing the block characteristic polynomial by the monic reduced denominator gives the factors from modes orthogonal to the coupling. The extra two critical branches, which are used only at $t = 2$, are included in Table 4. Their grounded maximum eigenvalue equals C.

Lemma 5.1 (Root comparison)**.** *Consider a hub graph formed from one* B *and subcritical branches. In the interval $(\sqrt{3}, 2)$, the sign of $S_A(x)$ is negative, zero, or positive according as $a(G) > x$, $a(G) = x$, or $a(G) < x$. For $x \in (181/50, C)$ above every grounded eigenvalue of its subcritical branches, the same rule holds for $S_L(x)$ and $b(G)$. In particular, if $a(G) > \sqrt{3}$, the second adjacency eigenvalue is the unique coupled root in $(\sqrt{3}, 2)$.*

Proof. The adjacency block sum has exactly one eigenvalue greater than $x \in (\sqrt{3}, 2)$ and no zero eigenvalue after shifting by x. It comes from B at 2. Under the stated conditions the same holds for the Laplacian block sum, with its single eigenvalue at C from B. A symmetric congruence splits $xI - M$ into that shifted block sum and the scalar complement. Its negative count is 1 plus the scalar negative count, and its nullity is the scalar nullity. This proves the signs. The top adjacency pole 2 is coupled, so the top full eigenvalue is strictly bigger than 2, the next coupled root is strictly below it, and no uncoupled mode is at 2. Thus $a(G) < 2$. Using the analogous facts at C also gives $b(G) < C$ for this entire branch family. □

The C_5 branch has a reversal-antisymmetric Laplacian factor $x^2 - 5x + 5$. Its larger root is h_5, which lies above the coupled root for one of the graphs in the finite transition. For P_5, the largest grounded eigenvalue ρ_5 belongs to the reduced denominator and lies in $(3.68, 3.683)$. All other subcritical grounded eigenvalues are below $181/50 = 3.62$. If P_5 occurs, the second coupled root for L lies strictly between ρ_5 and C. Every uncoupled mode, including duplicates at ρ_5, lies at or below ρ_5. In this case $b(G) > \rho_5$.

Proposition 5.2. *The characteristic polynomials of G_k are*

$$\begin{aligned} p_{A,k}(x) &= (x-1)^k (x+1)^{k+2} Q_k(x), \\ Q_k(x) &= x^4 - 2x^3 - (2k+4)x^2 + (2k+6)x + 4k + 3, \\ p_{L,k}(x) &= x(x-3)^{k+2}(x-1)^k R_k(x), \\ R_k(x) &= x^3 - (2k+8)x^2 + (8k+17)x - 2k - 6. \end{aligned}$$

The two second eigenvalues increase strictly to 2 and C, respectively, and $a(G_0) = \sqrt{3},\ b(G_0) = 3$.

Proof. For the coupled part, split vertices into the hub, the bridge vertex at the other end, the latter's two other triangle vertices, and all non-hub vertices of the hub triangles. Compression to constant coordinates on these four cells yields the displayed factors Q_k and xR_k; the zero-sum directions within the cells give the other factors. Equivalently one may multiply the Schur complement in (5.1) by all grounded block determinants. The values for $k = 0$ follow by factoring the two polynomials. Now $Q_k(\sqrt{3}) > 0$ for $k > 0$, while $Q_k(2) = -1$. Also $R_k(3) = 4k > 0$ for $k > 0$, and $R_k(C) = -2$. Thus for $k \geq 1$ the second eigenvalues lie strictly between the base values and the respective top poles, by the coupled-root rule. Adding a T decreases each complement strictly at the old second root. Its responses are positive in those intervals. This gives strict increase. At any fixed point sufficiently close to the top pole from below, adding more copies of T makes the complement tend to $-\infty$; hence the limits are the top poles. □

6. The finite transition

Fix $q = 3661371/10^6$. Full-matrix root counts give $b(G_4) < q < C$. Every graph with $a(G) > \sqrt{3}$ and $b(G) \le q$ is in the rooted family of Theorem 4.5 or in the bounded-degree population of Proposition 4.4. We independently include all graphs from that population that meet the two spectral thresholds.

In the rooted family, P_5 is excluded by interlacing: its grounded top eigenvalue exceeds q, so along with the grounded eigenvalue C in B it would give $b(G) > q$. All other subcritical blocks are strictly below q. By Lemma 5.1, $b(G) \le q$ is equivalent to the rational budget

$$\sum_{B \neq \mathsf{B}} \psi_B(q) \;\le\; 1 - \psi_{\mathsf{B}}(q). \tag{1}$$

Every cost in the sum is positive. Enumerating the multiplicity of each of the seven allowable branch types from zero up to the integer part of the remaining budget divided by its cost gives all nonnegative solutions without repetition. There are 795. An exact check shows that all 795 have $b(G) \le b(G_4)$, 773 of them have $a(G) > \sqrt{3}$, and 570 have $a(G) > a(G_1)$. This distinction between the budget population and joint-threshold populations is used below.

Theorem 6.1 (Twenty transition levels)**.** *On $(\sqrt{3}, a(G_4)]$ the ordered representatives are the four rows below, read from left to right and top to bottom.*

Levels	*Representatives in order*
1–4	$G_0 + \mathsf{P}_1,\ G_0 + \mathsf{P}_2,\ G_0 + \mathsf{P}_3,\ G_1$
5–8	$G_1 + \mathsf{P}_1,\ G_1 + \mathsf{P}_2,\ G_1 + \mathsf{P}_3,\ G_2$
9–14	$G_1 + \mathsf{C}_5,\ G_2 + \mathsf{P}_1,\ G_2 + \mathsf{P}_2,\ G_2 + \mathsf{P}_3,\ G_1 + \mathsf{C}_5 + \mathsf{P}_2,\ G_3$
15–20	$G_2 + \mathsf{C}_5,\ G_3 + \mathsf{P}_1,\ G_3 + \mathsf{P}_2,\ G_3 + \mathsf{P}_3,\ G_2 + \mathsf{C}_5 + \mathsf{P}_2,\ G_4$

The adjacency and Laplacian second eigenvalues each increase strictly along the displayed list. At the level represented by G_j for $1 \le j \le 4$, T_j also attains the same value for t up to its adjacency endpoint $a(T_j)$, strictly between the level's two endpoints. These are the only extra minimizers. The first three representatives are also denoted H, J, K in order. No graph with $b(G) \le b(G_4)$ has $a(G) > a(G_4)$.

Proof. For each qualifying multiset, build the full integer matrices and isolate their second-largest roots. Sort by exact b, grouping equal second Laplacian roots by interval isolation and polynomial gcd. In each group keep every graph whose a is greater than the largest a in the strictly earlier groups. A discarded graph is feasible only where an earlier group supplies a strictly lower objective value. In a retained group the same b is attained up to the largest a, and the other retained graphs have their own adjacency endpoints. This procedure gives the displayed representatives and four additional minimizers. It handles the hidden value $b(G_1 + \mathsf{C}_5) = h_5$ by using the full Laplacian.

For the remaining graphs the bounded-degree enumeration yields 156 labeled candidates with $a > \sqrt{3}$ and $b \le q$ after deduplicating identical labeled edge lists across tree representatives. For each candidate the program finds the first displayed level whose adjacency endpoint is at least the candidate's a. It checks that the candidate's b is at least the value there. If equal, a degree-filtered exact bijection check identifies the candidate with a minimizer already retained at that value. No additional endpoint occurs. Graphs with $b > q$ cannot lower any displayed value. The algorithm and exact polynomial comparisons are specified in Section 10. □

7. The infinite period-six staircase

Write $q_A = 49/25$, $q_L = 181/50$, and $q_* = 367514/100000$. Exact comparisons give $a(G_4) > q_A$, $b(G_4) > q_L$, and $q_L > h_5$. Consider G with $a(G) > a(G_4)$ and $b(G) < C$. Theorems 4.5 and 6.1 and Proposition 4.4 imply that G is in the rooted family and $b(G) > b(G_4)$.

Lemma 7.1 (Dominance comparisons)**.** *In a hub graph with these spectral bounds, each replacement in Table 5 increases a strictly. All decrease b strictly except $2\mathsf{P}_1 \to \mathsf{T}$, which preserves b.*

Proof. On $(q_A, 2)$ the new total r minus the old total r is strictly positive for every row, including all ten pairs in the last row. On (q_L, C) the old total ψ minus the new one is strictly positive for the strict rules other than $\mathsf{P}_5 \to \mathsf{T}$, and is identically zero for $2\mathsf{P}_1 \to \mathsf{T}$. In the P_5 rule the sign is positive on (ρ_5, C). The response table and Appendix A give the precise sign tests. For example,

$$\psi_{\mathsf{C}_4} - 2\psi_{\mathsf{T}} = \frac{-2(x^2 - 4x + 1)}{(x-1)(x^2 - 4x + 2)},$$

$$\psi_{\mathsf{C}_5} + \psi_{\mathsf{P}_3} - 2\psi_{\mathsf{T}} = \frac{-(x-3)(x^2 - 4x + 1)(x^2 - x - 1)}{(x-1)(x^2 - 3x + 1)(x^3 - 5x^2 + 6x - 1)}.$$

In the P_5 rule the reduced numerator has constant positive sign from 3.68 to C. Its denominator divides the product of the characteristic polynomials of D_{P_5} and D_{T}, and is positive above ρ_5.

Apply Lemma 5.1 at the two *old* second eigenvalues. Every new adjacency branch has spectral radius at most $\sqrt{3}$. No new Laplacian branch has a grounded eigenvalue above q_L, so the old b is still a valid comparison point. In particular, any P_5 already present satisfies $b > \rho_5$. Both inertia comparisons therefore give the asserted eigenvalue inequalities. The new a is still above $a(G_4)$. By Theorem 6.1, the new b is above $b(G_4)$ as well, so the same comparison is available on the next step. □

Theorem 7.2 (Period six and its smallest index)**.** *For each integer $k \ge 4$, the representatives of F on $(a(G_k), a(G_{k+1})]$ are exactly those in Theorem 2.1. They are strictly ordered in both second eigenvalues. The asserted order fails at $k = 3$. All minimizers are as in that theorem.*

Proof. Every replacement strictly decreases the number of branches other than T. The process therefore reaches a terminal multiset, which consists solely of T's, or contains one additional P_j for $1 \le j \le 4$ or one C_5. For at most four T's and no extra branch or one P_j, the adjacency value is below q_A. With C_5 the same is true for at most three T's. It suffices to test the largest multiplicity in each of these cases at q_A, since adding a T strictly decreases the adjacency complement there. Thus the terminal graphs in the range of interest are exactly the listed forms, with starting indices as claimed.

For consecutive forms from G_k through G_{k+1}, the responses of the unchanged branches cancel. The new minus old adjacency difference is positive on $(q_A, 2)$ for each of the six rows of Table 6. For the first five rows the new minus old Laplacian difference is positive on (q_L, C). The last row is positive on (q_*, C), and full-matrix root counts give $b(G_4 + \mathsf{P}_4) > q_*$. Since adding copies of T gives a Laplacian supergraph after padding by isolates, $b(G_k + \mathsf{P}_4) \ge b(G_4 + \mathsf{P}_4)$ for every $k \ge 4$. Starting with G_4, Lemma 5.1 thus gives strict ordering of the entire sequence in both coordinates. The rational counts $b(G_3 + \mathsf{P}_4) > 3.662 > b(G_4)$ show that index 3 fails. Every $t < 2$ in the tail is covered, by Proposition 5.2. Replacing any graph with $a \ge t$, $b < C$ terminates at a listed graph with adjacency value at least t and no larger objective. The earliest listed graph with $a \ge t$ gives the minimum. Graphs with $b \ge C$ have strictly larger objective.

For equality, no strict rule can occur along a replacement chain from a minimizing graph, because it gives a feasible graph with strictly lower b. Every minimizer is therefore obtained by inverse replacements $\mathsf{T} \mapsto 2\mathsf{P}_1$ of its terminal representative. The sign tests in Table 7 show that on the first five levels one inverse replacement takes the adjacency value below the preceding endpoint. On the sixth level one such replacement gives

$$a(G_k + \mathsf{P}_4) < a(T_{k+1}) < a(G_{k+1}),$$

while two take it below the preceding endpoint. Further inverse replacements decrease the adjacency response at the tested endpoint, so they cannot recover feasibility. All of these comparisons take place in $(q_A, 2)$. The identity $\psi_{\mathsf{T}} = 2\psi_{\mathsf{P}_1}$ preserves the relevant Laplacian root for T_{k+1}, with no hidden mode at the comparison point. This proves exactly the additional equality range claimed. The argument also excludes coincidences of response values at an individual algebraic point, because any graph outside the weak-replacement possibilities admits a strict improvement along its terminating chain. □

8. The endpoint $t = 2$

Theorem 8.1. *$F(2) = C$. A graph attains this value if and only if it is formed at one hub from at least two copies of B and any multiset of branches among the eleven types in Lemma 4.3. Every such graph has $a(G) = 2$.*

Proof. Suppose $a(G) \geq 2$. Use a second adjacency eigenvector y with both signs. If some maximal positive entry has at least three positive neighbors, choose a $K_{1,3}$ on those four positive vertices. Otherwise every maximal positive entry has exactly two positive neighbors and the eigenvector equation forces equality throughout at these maxima. In this case $a(G) = 2$, all positive neighbors of a maximum are maxima, and no maximum has a negative neighbor. The maximum vertices thus induce a union of cycles. At least one vertex of a cycle is adjacent to a zero vertex, by connectedness. At that vertex choose a $K_{1,3}$ whose leaves are its two cycle neighbors and that zero neighbor. Apply the same construction to $-y$. The two stars are disjoint or share a single zero leaf. In the disjoint case $b(G) \geq 4 > C$. In the common-leaf case use that leaf u as hub. Deleting it leaves two center-marked P_3's, both grounded at C. Interlacing gives $b(G) \geq C$.

Suppose equality holds. Each selected star then came from a cycle in its sign-maximal set. Use the top vector on each center-marked P_3, with center entry 1 and both leaf entries $-1/(C-1)$. Their span in the original shared-leaf subgraph has a constant quadratic restriction at C. By Lemma 2.2 the coefficients are opposite and nonzero, from the equation at u. Among the six support vertices, only the two leaves on the same side can have equal coordinates. Those leaves each have a further neighbor in their sign-maximal cycle. It must be the other leaf. Thus the two chosen branches are single-marked triangles, and any other vertex can connect to them only through u. If an additional grounded block had an eigenvalue $> C$, its positive mode, a coupled zero mode of a triangle and e_u would give two positive eigenvalues of $L - CI$ by the three-dimensional congruence from Proposition 3.2.

It remains to identify the additional blocks of radius at most C. An internal $K_{1,3}$ would give radius at least 4 by padding and adding a nonnegative diagonal. Hence the components are paths and cycles. In a component of order at least six choose a P_6 subpath containing a marked vertex. The Laplacian top mode of P_6, at C, has no zero entry. Adding the diagonal mark strictly raises its Rayleigh quotient. Such a component is impossible. The finite census in Lemma 4.3 gives exactly the eleven stated types.

Conversely, for any hub graph in the statement, deleting the hub leaves at least two critical grounded blocks and every grounded eigenvalue is at most C. Interlacing yields $b = C$. The adjacency spectral radii are at most 2, since the blocks are paths or cycles; the two triangle blocks reach 2. Interlacing gives $a = 2$. □

9. Sharp spectral inequalities

Theorem 9.1. *For every graph in the stated domain,*

$$b(G)^2 \;\geq\; \frac{8}{\sqrt{3}}a(G) + 1.$$

The coefficient $8/\sqrt{3}$ is largest possible and equality holds exactly for stars of order at least three and for G_0. For a nonstar,

$$b(G)^2 - 4a(G) - 1 \;\geq\; 5 - 2\sqrt{5},$$

with equality exactly for P_4.

Proof. Put $s = 8/\sqrt{3}$. If $a \leq 0$, separate K_2 and the other stars, which have the spectral pairs already given. Every nonstar has $b \geq 2$. For $0 < a \leq a_1$, use $b \geq 2$ and $a_1 = (\sqrt{5}-1)/2$, which gives $1 + sa_1 < 4$. In the next range $a_1 < a \leq a_2$, use $b \geq b_2 > 9/4$ and $a_2 < 4/5$. For $a_2 < a \leq 1$ use $b \geq b_3 > 121/50$. The two lower bounds on b_r follow by evaluating $p_2(9/4)$ and $p_3(121/50)$ positively in $(2, \beta)$, and $a_2^2 = 2 - \sqrt{2}$. In both ranges these bounds give $b^2 > sa + 1$. For $1 < a \leq \sqrt{3}$, $b \geq 3$, with equality in the quadratic inequality possible only at G_0 by Proposition 3.4. For $\sqrt{3} < a \leq 2$ the first transition level already gives $b > 16/5$, and $(16/5)^2 > 1 + 2s$. For $a > 2$, put $k = \lceil a \rceil \geq 3$; then $b^2 \geq (k+1)^2 > 1 + sk$ by Lemma 3.3. The G_0 equality makes s sharp.

For the nonstar bound, the same intervals give explicit lower bounds for $b^2 - 4a - 1$. At $0 < a \leq a_1$ the gap is at least $3 - 4a_1 = 5 - 2\sqrt{5}$. Equality requires $b = 2$ and $a = a_1$, which forces the first double star P_4. The next two intervals give gaps greater than $(9/4)^2 - 4(4/5) - 1$ and $(121/50)^2 - 5$, respectively. On $1 < a \leq \sqrt{3}$ the lower bound is $8 - 4\sqrt{3}$; on $\sqrt{3} < a \leq 2$ it is strictly larger than $(16/5)^2 - 9$; and for $a > 2$ it is at least $k(k-2) \geq 3$. The $a \leq 0$ nonstar gap is at least 3. All these bounds outside the first positive interval exceed $5 - 2\sqrt{5}$. □

For completeness, the signed-neighbor bound gives strictness of $b \geq a + 1$ at noninteger $a > 0$. If $a = k$ is a positive integer and $b = k + 1$, the two signed stars and Lemma 2.2 force their joint support to be the whole graph. Each sign set has $k + 1$ vertices. Sum $Ay = ky$ over the positive set. Its internal contribution is at most k times the sum of the positive entries, and its external contribution is strictly negative by connectedness. This is impossible. The nonpositive pairs above leave equality in the linear bound only for stars, including K_2.

Proposition 9.2. *Along $B(r, r)$ with $r \geq 2$, the positive linear gap $b - a - 1$ tends to zero.*

Proof. Zero-sum directions on the non-bridge vertices of each clique have adjacency value -1 and Laplacian value r. On the four remaining classes, symmetry and antisymmetry between the cliques give adjacency quotient blocks $\left(\begin{smallmatrix} r-2 & 1 \\ r-1 & \pm 1 \end{smallmatrix}\right)$ and Laplacian quotient blocks $\left(\begin{smallmatrix} 1 & -1 \\ -(r-1) & r \mp 1 \end{smallmatrix}\right)$, with coordinates ordered as non-bridge then bridge in each clique. The resulting second eigenvalues are

$$b = r, \qquad a = \frac{r - 3 + \sqrt{r^2 + 2r - 3}}{2}.$$

In particular $b - a - 1 = 2/(r + 1 + \sqrt{r^2 + 2r - 3}) \to 0$. □

10. Exact computational certification

The computations used in the proof take integer graph matrices, rational shifts, or shifts in a real quadratic field as inputs. They use no floating-point eigensolver. We describe the root-count test and the independent recovery of the finite search populations.

Lemma 10.1 (Shifted sign count)**.** *If a real monic polynomial p has only real roots, the number of roots strictly above q, with multiplicity, equals the number of sign variations of the nonzero coefficients of $p(q+z)$. The smallest nonzero exponent of z is the multiplicity at q.*

Proof. Remove the zero roots of $p(q + z)$. If its degree is m, Descartes' rule applied to this polynomial and its reflection bounds the positive and negative root counts by two variation counts. The sum of the latter counts is at most m. In fact between successive nonzero coefficients the total contribution is at most their difference in exponents. The root counts add to m, so both bounds are exact. □

For each symmetric integer matrix the verifier computes its characteristic polynomial by Berkowitz's division-free algorithm as implemented in SymPy. Horner shifting clears positive denominators. A quadratic-field coefficient $u + v\sqrt{d}$ with opposite summand signs is compared using u^2 and dv^2. Thus $q = 1, 2, \sqrt{3}, 3, \beta, h_5, C$ and all rational cutoffs in the paper are tested without choosing tolerances. For a second eigenvalue, a strict upper bound is equivalent to at most one root at or above the threshold; a strict lower bound is equivalent to at least two roots strictly above it.

To compare two second eigenvalues at unspecified algebraic points, bisect a rational spectral bracket by the same count. Disjoint isolating intervals decide the sign. For overlapping intervals, ensure that each contains just one distinct root of the relevant polynomial, then test the intersection for a root of their exact polynomial gcd. If no root is shared, further bisection separates the two distinct algebraic numbers. When bisection lands on the second root itself, compare that rational point directly.

The root count is also used for the component and forbidden-graph tests in Section 4. The certificate includes the strict degree-three counterexample and the rooted shape counts for each maximum-degree class. The program checks all translated graph labels on the six-vertex plateau using the permutation action of S_6; no isomorphism heuristic is used. A tree is represented by its center code, and the supergraph recursion and the branch budget use explicit monotonicity and ordered extension. The graph-generation proofs in Sections 4 and 6 fix the scope of every finite calculation.

For the sign calculations in Section 7, reduce each response difference to a rational function. Count roots of its numerator and denominator in the specified open interval by a Sturm sequence and test one rational sample.

At an endpoint, take the one-sided sign of each Sturm polynomial from the lowest nonzero term in its exact shifted expansion. This permits zeros at an omitted endpoint such as C. For the exceptional P_5 denominator, the grounded characteristic-polynomial divisibility and ρ_5 bound supply the sign immediately above the pole. These interval certificates are uniform in all branch multiplicities.

A second implementation from the research audit used rational symmetric congruence and pivoted inertia on the finite matrices. The supplied verifier instead recomputes the determinant polynomials and all shifted signs. The paper and the verifier give computer-assisted mathematics in the usual sense. The claim does not involve verification by a proof-assistant kernel.

Code and data availability

The verifier, exact reference certificate, execution log, and figure data generator are available in the public companion archive at fixed commit `0429292`. The archive includes instructions and pinned dependencies for regenerating the certificate. The Python code is MIT-licensed. The original data and documentation are CC BY 4.0.

Use of artificial intelligence and contributions

Gemini was used in the main exploratory phase for graph searches, candidate extremizers, conjecture formation, and development of arguments. ChatGPT supported research coordination, counterexample search, and review. Codex independently reconstructed the finite verification, audited and revised mathematical arguments, and prepared this manuscript and its reproduction package. The human contributors directed and organized the project and are the parties responsible for decisions about the work and its submission. The contributors report checking and understanding parts of the work. They do not claim independent human verification of every proof.

Appendix A. Response and sign certificates

For the first table, P_j joins the hub by one edge at an end of the j-vertex path. The remaining marks are as defined in Section 2. The x multiplying ψ in (5.1) is positive on all comparison intervals of Section 7.

For Tables 5–7, the two strict sign tests are root freedom of the reduced numerator and denominator over the indicated interval and positivity at $x = 199/100$ for adjacency or $x = 37/10$ for the Laplacian. The exception at the moving pole is treated in the proof of Lemma 7.1. Thus Table 6, for example, certifies the last comparison for every $k \geq 4$ after the single lower bound on $b(G_4 + \mathsf{P}_4)$.

Branch	$r_B(x)$	$\psi_B(x)$
P_1	$\dfrac{1}{x}$	$\dfrac{1}{x-1}$
P_2	$\dfrac{x}{x^2-1}$	$\dfrac{x-2}{x^2-3x+1}$
P_3	$\dfrac{x^2-1}{x(x^2-2)}$	$\dfrac{(x-3)(x-1)}{x^3-5x^2+6x-1}$
P_4	$\dfrac{x(x^2-2)}{x^4-3x^2+1}$	$\dfrac{(x-2)(x^2-4x+2)}{(x-1)(x^3-6x^2+9x-1)}$
P_5	$\dfrac{x^4-3x^2+1}{x(x^2-1)(x^2-3)}$	$\dfrac{(x^2-5x+5)(x^2-3x+1)}{x^5-9x^4+28x^3-35x^2+15x-1}$
T	$\dfrac{2}{x-1}$	$\dfrac{2}{x-1}$
C_4	$\dfrac{2x}{x^2-2}$	$\dfrac{2(x-3)}{x^2-4x+2}$
C_5	$\dfrac{2(x-1)}{x^2-x-1}$	$\dfrac{2(x-2)}{x^2-3x+1}$
B	$\dfrac{x-1}{(x-2)(x+1)}$	$\dfrac{x-3}{x^2-4x+1}$

TABLE 2. Responses of the eight subcritical branches and the singly marked triangle.

Branch	Adjacency factor	Grounded Laplacian factor
$\mathsf{P}_1, \ldots, \mathsf{P}_5$	1	1
T	$x+1$	$x-3$
C_4	x	$x-2$
C_5	x^2+x-1	x^2-5x+5
B	$x+1$	$x-3$
Center-marked P_3	x	$x-1$
Both-end-marked P_5	x^2-1	$(x-3)(x-1)$

TABLE 3. Uncoupled factors within individual branches. Shared poles can produce further uncoupled modes in the sum.

Branch	$r_B(x)$	$\psi_B(x)$
Center-marked P_3	$\dfrac{x}{x^2-2}$	$\dfrac{x-3}{x^2-4x+1}$
Both-end-marked P_5	$\dfrac{2(x^2-2)}{x(x^2-3)}$	$\dfrac{2(x^2-5x+5)}{(x-2)(x^2-4x+1)}$

TABLE 4. Responses of the two critical paths used in the endpoint classification.

Old	New	$\Delta r > 0$ on	$-\Delta\psi$
P_5	T	$(q_A, 2)$	> 0 on (ρ_5, C)
C_4	$2\mathsf{T}$	$(q_A, 2)$	> 0 on (q_L, C)
$2\mathsf{C}_5$	$2\mathsf{T} + \mathsf{P}_1$	$(q_A, 2)$	> 0 on (q_L, C)
$\mathsf{C}_5 + \mathsf{P}_1$	$\mathsf{T} + \mathsf{P}_2$	$(q_A, 2)$	> 0 on (q_L, C)
$\mathsf{C}_5 + \mathsf{P}_2$	$\mathsf{T} + \mathsf{P}_3$	$(q_A, 2)$	> 0 on (q_L, C)
$\mathsf{C}_5 + \mathsf{P}_3$	$2\mathsf{T}$	$(q_A, 2)$	> 0 on (q_L, C)
$\mathsf{C}_5 + \mathsf{P}_4$	$2\mathsf{T}$	$(q_A, 2)$	> 0 on (q_L, C)
$2\mathsf{P}_1$	T	$(q_A, 2)$	0 identically
$\mathsf{P}_i + \mathsf{P}_j$, $1 \le i \le j \le 4$, $(i,j) \ne (1,1)$	T	$(q_A, 2)$	> 0 on (q_L, C)

TABLE 5. The seventeen dominance rules. Differences use new minus old. The last row comprises nine rules.

Old changed branches	New changed branches	Interval for $\Delta\psi > 0$
T	C_5	(q_L, C)
C_5	$\mathsf{T} + \mathsf{P}_1$	(q_L, C)
P_1	P_2	(q_L, C)
P_2	P_3	(q_L, C)
P_3	P_4	(q_L, C)
P_4	T	(q_*, C)

TABLE 6. Consecutive representatives. For each row $\Delta r > 0$ on $(q_A, 2)$.

Comparison	Positive difference of adjacency responses
Level 1, one inverse replacement	$2r_{\mathsf{T}} - r_{\mathsf{C}_5} - 2r_{\mathsf{P}_1}$
Level 2, one inverse replacement	$r_{\mathsf{C}_5} - 3r_{\mathsf{P}_1}$
Level 3, one inverse replacement	$r_{\mathsf{T}} - r_{\mathsf{P}_1} - r_{\mathsf{P}_2}$
Level 4, one inverse replacement	$r_{\mathsf{T}} + r_{\mathsf{P}_2} - 2r_{\mathsf{P}_1} - r_{\mathsf{P}_3}$
Level 5, one inverse replacement	$r_{\mathsf{T}} + r_{\mathsf{P}_3} - 2r_{\mathsf{P}_1} - r_{\mathsf{P}_4}$
Level 6, two inverse replacements	$r_{\mathsf{T}} + r_{\mathsf{P}_4} - 4r_{\mathsf{P}_1}$
Level 6, one inverse vs. preceding	$2r_{\mathsf{P}_1} - r_{\mathsf{P}_4}$
Level 6, representative vs. one inverse	$r_{\mathsf{T}} - 2r_{\mathsf{P}_1}$

TABLE 7. Equality-range differences on $(q_A, 2)$. In the first six rows the preceding level has the larger response. The last two rows use the indicated larger graph.

Appendix B. Spectral plots and graph constructions

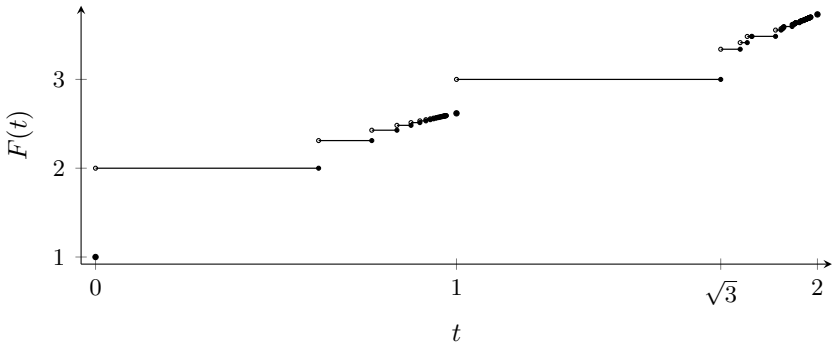


Figure 1. The global envelope. Filled endpoints are included and open endpoints excluded. Only finitely many steps are drawn below 1 and 2; the missing steps accumulate at the marked limits. Heights and coordinates in the plots are for display.

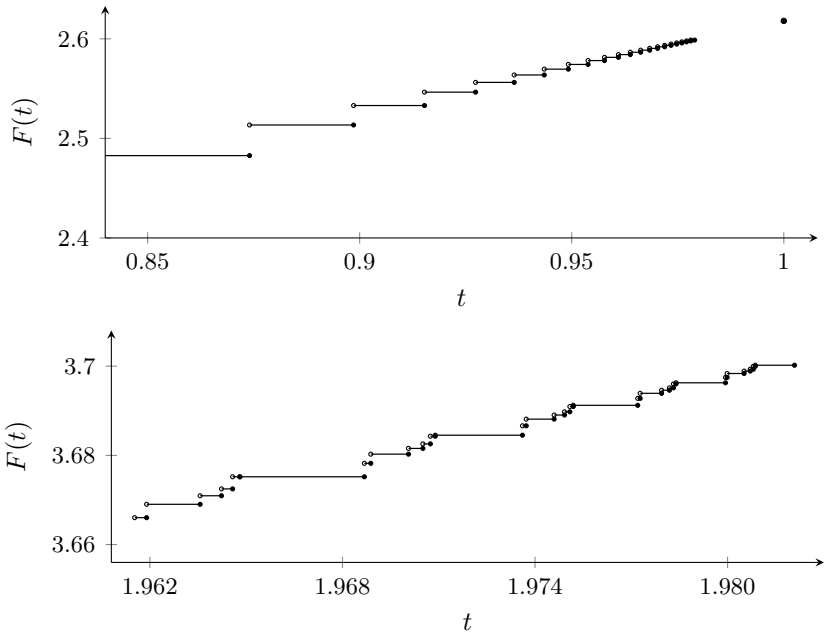


FIGURE 2. Local views of the double-star levels $4 \le r \le 24$ and the high-spectrum blocks $4 \le k \le 8$, with the high panel clipped at the visible right boundary. Both staircases continue beyond the displayed steps.

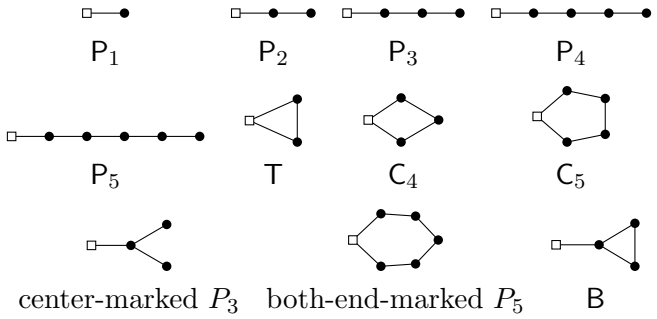


FIGURE 3. The eight subcritical branches (first two rows) and the three critical branches (bottom row). The square is the common external hub and is omitted from the grounded matrix.

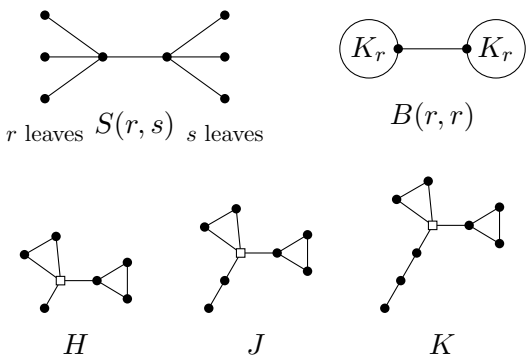


FIGURE 4. General double-star and barbell constructions (top), and the first three high-spectrum minimizers (bottom). The three leaves on each side of the double star illustrate the two free multiplicities.

Stage	Triangle branches at the hub	Extra branch
G_k	$k+1$	none
$G_{k-1}+\mathsf{C}_5$	k	C_5
$G_k+\mathsf{P}_j$ $(j=1,2,3,4)$	$k+1$	P_j
G_{k+1}	$k+2$	none

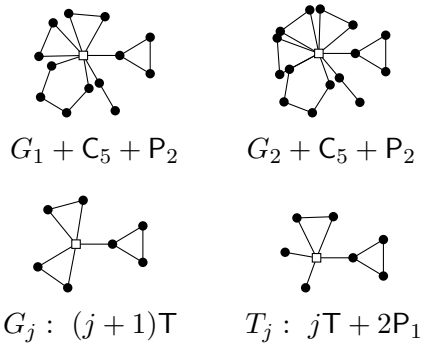
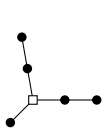
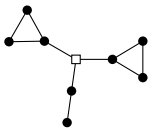


FIGURE 5. One B is common to every periodic-stage construction. The middle row shows the two special transition graphs with all branches. The last row compares the tied constructions schematically; the displayed multiplicities give their exact parameter dependence.



$t = 1 : \geq 2$ two-edge arms, any leaves



$t = 2 : \geq 2$ copies of B

FIGURE 6. Endpoint families. At $t = 2$ any further branches from Figure 3 are allowed. The drawings show examples and the labels specify all multiplicities.

References

[1] Andries E. Brouwer and Willem H. Haemers, *Spectra of graphs*, Universitext, Springer, New York, 2012, DOI.
[2] Tristan Cazenave, *Lemmas and conjectures in spectral graph theory*, 2026, arXiv:2611.03879v1.
[3] Haoran Chang and Hongying Lin, *On the second largest Laplacian eigenvalue of trees*, RAIRO Oper. Res. **59** (2025), no. 5, 3069–3077, DOI.
[4] Alan J. Hoffman, *On limit points of spectral radii of non-negative symmetric integral matrices*, Graph Theory and Applications (Y. Alavi, D. R. Lick, and A. T. White, eds.), Lecture Notes in Math., vol. 303, Springer, 1972, DOI, pp. 165–172.
[5] Alexander M. Kramer, *A proof of a conjecture of Cazenave on graph eigenvalues*, 2026, Hexagon preprint 2611.00232v1.
[6] Jianxi Li, Ji-Ming Guo, and Wai Chee Shiu, *On the second largest Laplacian eigenvalues of graphs*, Linear Algebra Appl. **438** (2013), no. 5, 2438–2446, DOI.
[7] Mohammad Pirani and Shreyas Sundaram, *On the smallest eigenvalue of grounded Laplacian matrices*, IEEE Trans. Automat. Control **61** (2016), no. 2, 509–514, DOI.
[8] James B. Shearer, *On the distribution of the maximum eigenvalues of graphs*, Linear Algebra Appl. **114/115** (1989), 17–20, DOI.
[9] Yarong Wu, Guanglong Yu, and Jinlong Shu, *Graphs with small second largest Laplacian eigenvalue*, European J. Combin. **36** (2014), 190–197, DOI.

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