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TOPOLOGY OF AMPLITUHEDRA

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Abstract. We prove that every amplituhedron with $m = 2$ is a closed ball, that its source fibers are closed balls, and that its open face strata are open balls. We also prove the closed-ball theorem when m is even and $n-k-m = 3$. For $m = 4$, intersections with projective linear spaces contained in the target Grassmannian are convex polytopes, and the boundary is homotopy equivalent to a sphere. The closed images for $m = 3, 4$ are contractible. We obtain local linear Plücker estimates and uniform Lipschitz lifting and center estimates over each fixed nonnegative lower plane.

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1. Introduction

Let $\mathrm{Gr}(k,n)$ be the real Grassmannian of k-dimensional subspaces of $\mathbb{R}^n$. Its totally nonnegative part $\mathrm{Gr}_{\geq 0}(k,n)$ consists of the planes whose ordered Plücker coordinates have a common nonnegative orientation; $\mathrm{Gr}_{>0}(k,n)$ is defined by strict positivity. For $k,m \geq 1$ and $n \geq k+m$, a positive external matrix is a real $n \times (k+m)$ matrix Z with all ordered maximal minors positive. Writing C for a row frame of a source plane, set

$$\pi_Z(C) = \mathrm{row}(CZ), \qquad \mathcal{A}_{n,k,m}(Z) = \pi_Z\big(\mathrm{Gr}_{\geq 0}(k,n)\big) \subset \mathrm{Gr}(k,k+m).$$

The rank of CZ is k for every nonnegative source (Lemma 2.1). The image is therefore compact. We call $q = n-k-m = \dim \ker Z^{\mathsf{T}}$ the corank of the external data. Interior and boundary are taken in the target real Grassmannian.

Arkani-Hamed and Trnka introduced the amplituhedron in their study of scattering amplitudes in planar supersymmetric Yang–Mills theory [1]. The construction includes cyclic polytopes when $k=1$ and the totally nonnegative Grassmannian when $n=k+m$. Lam placed these images in the broader setting of Grassmann polytopes, obtained by projecting closed positroid cells [17]. The closed-ball conjecture asks whether $\mathcal{A}_{n,k,m}(Z)$ is homeomorphic to a closed km-ball for every positive Z.

The topology of the source provides an earlier model for this question. Postnikov described its positroid cells by planar networks and gave positive coordinates on each cell [23]. Rietsch and Williams proved that closures of cells in totally nonnegative flag varieties are contractible and that their boundaries have the homotopy type of spheres [25]. Galashin, Karp, and Lam proved that the totally nonnegative Grassmannian is a closed ball [10]. Their regularity theorem subsequently established the regular CW structure for totally nonnegative flag varieties; in particular, every closed positroid cell is a ball [11]. These results describe the source cells and their attaching maps. The amplituhedron projection identifies source points, so its topology also involves the geometry of the projection fibers.

Several families of amplituhedra admit explicit topological models. Karp and Williams identified the $m=1$ amplituhedron with the bounded complex of a cyclic hyperplane arrangement and deduced the closed-ball property [13, Theorem 6.16 and Corollary 6.18]. Blagojević, Galashin, Palić, and Ziegler proved contractibility of corank-one Grassmann polytopes and of the fibers of

the projection restricted to closed positroid cells. They also proved the closed-ball theorem for positive external data of corank one and even m [4, Theorems 1.3, 1.7, and 1.8].

A second approach uses flows that strictly improve positivity. Galashin, Karp, and Lam constructed such a flow for the cyclically symmetric amplituhedron with even m [10, Theorem 1.2]. Positive pairing places the whole compact image in one bounded Grassmannian chart, where the induced flow is a contraction. Bloch and Karp developed this method through gradient flows on adjoint orbits. They proved the closed-ball theorem for twisted Vandermonde data and, in particular, for every positive external matrix with $q \leq 2$ [5, Theorem 1.7]. Their Remark 7.19 identifies corank three as a further possible range for a tridiagonal flow construction.

For $m = 2$, triangulations and sign descriptions give additional information. Bao and He constructed a triangulation by a Britto–Cachazo–Feng–Witten (BCFW) recursion [2]. Parisi, Sherman-Bennett, and Williams related the amplituhedron to the hypersimplex through sign patterns, cluster coordinates, and tilings [22]. Łukowski studied its boundaries recursively, gave a graphical enumeration, and conjectured the Eulerian structure of the face poset [19]. Lam defined the face stratification through the embedding

$$Y \longmapsto Z(Y^{\perp}) \in \mathrm{Gr}(2, n)$$

and the rank-two positroid strata. He determined the nonempty strata, their closure relations, and their rank-generating function [18]. Here $Y^{\perp}$ is the Euclidean orthogonal complement of the row plane Y, regarded as a column subspace. Lam's Conjecture 24 asks for open-ball strata, closed-ball stratum closures, a regular CW structure, the closed-ball property of the whole image, and positive geometry. The same conjecture is numbered 4.7 in the published article. Koefler, Pavlov, and Sinn proved the positive-geometry assertion for generic positive external data [15, Theorem B].

For $m = 4$, Even-Zohar, Lakrec, and Tessler proved the BCFW triangulation conjecture. They identified the topological boundary by physical four-bracket hyperplanes and proved that the interior is an open $4k$-ball for every positive external matrix [9, Corollary 8.8 and Theorem 9.2]. Their continuous source section over the interior will also be used below. Convexity in Plücker space gives another approach: Mazzucchelli and Pratt proved an extendable-convexity theorem for $k = m = 2$ and formulated the corresponding question in other parameters [21, Theorem 6.12 and Question 6.2].

We prove the following results.

Theorem 1.1. *For every $1 \leq k \leq n-2$ and every positive $n \times (k+2)$ matrix Z, $\mathcal{A}_{n,k,2}(Z)$ is homeomorphic to a closed $2k$-ball. Every fiber of $\pi_Z : \mathrm{Gr}_{\geq 0}(k, n) \to \mathcal{A}_{n,k,2}(Z)$ is a closed ball. Every nonempty open Lam stratum is an open ball of dimension*

$$2k - 2|L| - \sum_j (|B_j| - 1),$$

where L is its twistor loop set and the B_j are its nontrivial cyclic parallel blocks after the loops are removed. The family over $\mathrm{Gr}_{>0}(k+2, n)$ is locally topologically trivial, preserving these strata.

For the source fibers, a symmetric cyclic three-term operator preserves the two-plane $Z(Y^{\perp})$. A zero adjacent minor forces a common nonnegative adjacent line in every source lift. A shear and a positive quotient remove that line. Repeated deletion reduces the full fiber to a section with a strict contracting flow. For the whole image, we prove transversality at every image point: the conormal support of a rank-two positroid stratum lies in every source lift, and positive projection is injective on that lift. Explicit source coordinates and a Whitney calculation then give a proper stratified family. The isotopy lemma identifies its fibers with a spectral closed-ball model. A global product over the positive block-ratio parameters gives the open-stratum assertion.

Theorem 1.2. *Suppose $m \geq 2$ is even and $n = k + m + 3$. For every positive $n \times (k + m)$ matrix Z, the pair*

$$(\mathcal{A}_{n,k,m}(Z), \operatorname{int} \mathcal{A}_{n,k,m}(Z))$$

is homeomorphic to the closed km-ball and its interior. The interior is exactly the image of $\mathrm{Gr}_{>0}(k, n)$.

A positive three-plane admits an affine convex-polygon representation. Three-term barycentric identities give a cyclic tridiagonal operator, and explicit positive diagonal weights make it symmetric. Its spectrum consists of 1, two zeros, and strictly negative eigenvalues. Alternating duality and reciprocal source/external rescaling give a positivity-improving operator preserving the external plane. The induced flow is a strict contraction in one chart containing the whole image. The construction also gives local topological triviality as the positive external data vary.

Theorem 1.3. *Let Z be positive of size $n \times (k + 4)$. If a projective linear space L is contained in* $\mathrm{Gr}(k, k + 4)$ *under the Plücker embedding, then every nonempty intersection $L \cap \mathcal{A}_{n,k,4}(Z)$ is a compact convex polytope in a common positive affine Plücker chart.*

On a section meeting the interior, the physical boundary equations of [9] give the defining halfspaces. An inverse heat flow in the external data places the original closed image strictly inside a nearby expanded image. The section polytopes have uniformly bounded numbers of vertices, and their Hausdorff limit gives the theorem also for sections contained in the boundary.

If a boundary point and an interior point share a $(k - 1)$-plane, the segment in their common pencil increases every active physical form strictly. The implicit-function theorem then gives a local Lipschitz epigraph, so the amplituhedron germ at that boundary point is bi-Lipschitz equivalent to a half-space (Theorem 5.6). Complete pencils over the open domain with an interior completion form a local bundle of closed four-balls (Proposition 5.7).

A compact subset of a manifold is called cellular if it is the intersection of nested embedded closed balls, each contained in the interior of its predecessor.

Theorem 1.4. *For every positive external matrix, the closed amplituhedra with $m = 3$ or $m = 4$ are contractible. For $m = 4$, the image is cellular in its ambient real Grassmannian, and*

$$\partial \mathcal{A}_{n,k,4}(Z) \simeq S^{4k-1}.$$

For $m = 4$, strict external expansion and the open-ball interior supply such neighborhoods. A semialgebraic neighborhood retraction gives a contraction of the image. A second retraction and van Kampen's theorem prove simple connectivity of the boundary; the relative homology sequence then gives the stated homotopy equivalence. For $m = 3$, positive extension of the external plane gives a projection from an $m = 4$ amplituhedron whose fibers are the polytopes of Theorem 1.3. The cell-like mapping theorem transfers contractibility.

We also prove a local estimate on the nonnegative Grassmannian. Normalize its Plücker coordinates by $\sum_I p_I = 1$.

Theorem 1.5. *Let $C_0 \in \mathrm{Gr}_{\geq 0}(k, n)$, let $G = \sum_I b_I p_I$ with $b_I \geq 0$, and let $N = \sum_I c_I p_I$ with $c_I \in \mathbb{R}$. If $N \geq 0$ on the germ of $\{G = 0\}$ at C_0, then there are a neighborhood V of C_0 in* $\mathrm{Gr}_{\geq 0}(k, n)$ *and $K < \infty$ such that*

$$\max\{0, -N(C)\} \leq KG(C) \qquad (C \in V).$$

If F is a nonnegative linear Plücker form and $\{G = 0\} \subset \{F = 0\}$ locally, then $F \leq KG$ locally.

The planar bipartite matching formula of [23, 24] has squarefree monomials. Linear programming duality gives a compact cover by coordinate cube faces on which the parametrizing polynomials have no common zero. Multiaffine interpolation proves the estimate on each face,

and compactness of the entire fiber allows it to descend to the Grassmannian. An anchored decomposition gives the corresponding two-sided bound for smooth functions vanishing on the full zero germ (Theorem 6.8).

Two further constructions concern source lifts. Let $\mathcal{R}_C$ be the span of the vectors in $\ker C$ supported on at most k coordinates, and put $s = \dim \mathcal{R}_C$. For even m, cyclic interval support sums give

$$\dim(\mathcal{R}_C \cap \operatorname{col} Z) = \max\{0, s - q\}.$$

The condition $s \le q$ consequently gives a strictly positive lift with the same numerical image; for odd m, the bound $s \le q - 1$ is sufficient (Theorem 2.9).

For each fixed nonnegative lower plane, ordered simultaneous extension rays give a Hausdorff–Lipschitz bound for all one-row extension fibers, with a constant depending only on Z (Theorem 7.1). This uniformity follows from a positive-projection singular-value bound. Boundary outputs have unique lifts, and the fiber diameter is bounded by the distance to the output boundary. Weighted logarithmic centers give Lipschitz sections on the entire closed output cone (Theorem 7.6). The proof uses the classical weighted least-squares basis formula [3], whose Cauchy–Binet derivation is included.

The appendix gives a rational corank-three example with all maximal minors, the characteristic polynomial, and the entries of the two additive compounds printed explicitly.

2. Positive projection and contractive flows

2.1. **Conventions and positive projection.** All vector spaces are real and carry their standard Euclidean inner products. We identify row and column realizations of a subspace by transpose, denoted ^{T} or ^{T}. Write $E^{\perp}$ for the orthogonal complement of E, e_i for the standard coordinate vectors, e_i^* for their dual covectors, and E_{ij} for the matrix with its only nonzero entry equal to 1 in position (i, j). The symbols I_r and I denote identity matrices of the indicated or ambient size. For a vector v, $\operatorname{supp}(v) = \{i : v_i \neq 0\}$. The notation $\operatorname{Mat}_{a\times b}(\mathbb{R})$ denotes real $a \times b$ matrices. A matrix is totally nonnegative if all its minors of every size are nonnegative, and totally positive if all are positive. Vector norms are Euclidean and norms of linear maps are induced operator norms. On a matrix space we use the Frobenius inner product $\langle X, Y\rangle = \operatorname{tr}(X^T Y)$ and its associated norm when specified. For a nonempty subset S, $\operatorname{dist}(x, S) = \inf_{s\in S} \|x - s\|$. The open and closed unit balls in $\mathbb{R}^D$ are B^D and $\overline{B}^D$; $S^{D-1} = \partial\overline{B}^D$ for $D \ge 1$.

A k-plane is represented by a full-rank $k \times n$ row matrix C. For an increasing k-subset $I \subset [n] = \{1, \dots, n\}$, let $\Delta_I(C)$ be the determinant of its columns indexed by I. For an $n \times k$ column frame U, the same notation denotes the determinant of its rows indexed by I. The conditions $C \ge 0$ and $C > 0$ mean that all these maximal minors are nonnegative and positive, respectively, after one common orientation choice. The corresponding spaces are $\operatorname{Gr}_{\ge 0}(k, n)$ and $\operatorname{Gr}_{>0}(k, n)$. At rank zero both spaces are a point and the empty determinant is 1. A positive external $n \times d$ matrix has all increasing d-row minors positive. For a row plane Y, its annihilator $\ker Y$ means $Y^{\perp}$ as a column subspace. For a row subspace V, write $VZ = \{vZ : v \in V\}$ for its image. Products of explicitly specified matrices are numerical matrix products; the image of a row frame C is written $\operatorname{row}(CZ)$.

The source positroid of $C \in \operatorname{Gr}_{\ge 0}(k, n)$ is its set of bases

$$M_C = \{I \subset [n] : |I| = k,\ \Delta_I(C) > 0\}.$$

A positroid is any basis set obtained in this way. Its positive cell is $\Pi_{M,>0} = \{C \in \operatorname{Gr}_{\ge 0}(k, n) : M_C = M\}$. For positroids of the same rank, $M \le M'$ means inclusion of their basis sets. The dual matroid has bases $[n] \setminus I$ for $I \in M$. A loop belongs to no basis, and a coloop belongs to every basis. The rank of $S \subset [n]$ is $r_M(S) = \max_{I\in M} |I \cap S|$. A circuit is a minimal set S

with $r_M(S) < |S|$. The closure of S is $\mathrm{cl}_M(S) = \{i : r_M(S \cup \{i\}) = r_M(S)\}$, and a flat is a set equal to its closure. The restriction $M|U$ has rank function $r_{M|U}(S) = r_M(S)$ for $S \subset U$; the contraction M/F has rank function $r_{M/F}(S) = r_M(S \cup F) - r_M(F)$ on $[n] \setminus F$. Components are the equivalence classes generated by pairs in a common circuit; a matroid is connected if it has one component. A direct sum on disjoint ground sets has the sum of their rank functions.

A positive plane admits a complete positive flag [6, Lemma 3.4]: for $E \in \mathrm{Gr}_{>0}(d,n)$ and $0 \le r \le d$ there is $W \in \mathrm{Gr}_{>0}(r,n)$ with $W \subset E$, and for $d < n$ there is a positive $(d+1)$-plane containing E. Alternating orthogonal duality is the identity

$$E \longmapsto A_n E^{\perp}, \qquad A_n = \mathrm{diag}(1,-1,1,-1,\ldots), \tag{2.1}$$

which sends positive and nonnegative planes to positive and nonnegative planes of complementary dimension. Indeed, complementary minors of an orthogonal complement differ by the sign of the permutation listing I and I^c; multiplying the rows indexed by I by their alternating signs makes this sign independent of I. A single orientation choice removes the remaining common sign.

For a covector α, exterior contraction is defined by

$$\iota_\alpha(v_1 \wedge \cdots \wedge v_d) = \sum_{j=1}^{d} (-1)^{j-1} \alpha(v_j) v_1 \wedge \cdots \wedge \widehat{v}_j \wedge \cdots \wedge v_d,$$

where the hat means that the indicated factor is omitted. The positive-subflag assertion has the following finite proof. Let p be a positively oriented decomposable d-vector for E, with smallest and largest coordinates $p_{\min} > 0$ and $p_{\max}$. For $d \ge 2$ put

$$\epsilon = \frac{p_{\min}}{2p_{\max} + p_{\min}}, \qquad \alpha_\epsilon = \sum_{j=1}^{n} (-\epsilon)^{j-1} e_j^*.$$

For a $(d-1)$-set I, let j_0 be its least missing index. In the I-coordinate of $\iota_{\alpha_\epsilon} p$, the term indexed by j_0 is $\epsilon^{j_0-1} p_{I \cup \{j_0\}} > 0$: all $j_0 - 1$ earlier indices lie in I, so the contraction and coefficient signs cancel. The absolute sum of the later terms is at most

$$p_{\max} \frac{\epsilon^{j_0}}{1-\epsilon} = \frac{p_{\min}}{2} \epsilon^{j_0 - 1}.$$

Every coordinate of this contraction is therefore strictly positive. A nonzero contraction of a decomposable d-vector is a decomposable $(d-1)$-vector for a subspace of E: choose a basis of E whose last $d-1$ vectors are in $\ker \alpha_\epsilon$. This proves one-step descent, and iteration gives every lower positive rank. Applying descent to $A_n E^{\perp}$ and then alternating duality gives a positive one-step extension of E. Descent from rank one ends at the zero plane; under duality this is the extension from rank $n-1$ to $\mathbb{R}^n$. Thus the flag facts used below include the endpoint cases.

Lemma 2.1. *Let Z be positive of size $n \times d$. For every $C \in \mathrm{Gr}_{\ge 0}(k,n)$ with $k \le d$, the matrix CZ has rank k. The induced map is continuous on the whole nonnegative Grassmannian, and is a smooth submersion on the open ambient locus where CZ has rank k. Its positive-source image is open in* $\mathrm{Gr}(k,d)$.

Proof. Choose a positive k-plane in $\mathrm{col}\, Z$ and a positive frame $U = ZT$ for it, with T of size $d \times k$. Cauchy–Binet gives

$$\det(CU) = \sum_{|I|=k} \Delta_I(C) \Delta_I(U) > 0. \tag{2.2}$$

Every summand is nonnegative, and at least one source minor is nonzero. Hence CZ has rank k. Continuity follows from the Grassmannian construction on this transverse locus.

For a surjective linear map $a : \mathbb{R}^n \to \mathbb{R}^d$, the derivative of its Grassmannian map at a plane V on which it is injective is

$$\mathrm{Hom}(V, \mathbb{R}^n/V) \longrightarrow \mathrm{Hom}(aV, \mathbb{R}^d/aV).$$

It is induced by the isomorphism $a|_V$ and the surjection on quotient spaces, and is therefore surjective. The positive Grassmannian is open in the real Grassmannian, so its image under this submersion is open. □

In the setting of the lemma, all target points lie in the affine Plücker chart

$$\ell(Y) = \det(YT) > 0. \tag{2.3}$$

Normalizing a target Plücker vector by $\ell = 1$ makes the closed image compact in this chart. This follows either from continuity on the compact source or directly from (2.2), after normalizing the source minors to have sum one.

The coherent cyclic shift of a k-plane moves the last coordinate to the first and multiplies it by $(-1)^{k-1}$. Moving that coordinate past the other $k-1$ indices shows that every ordered maximal minor retains its sign. If $d-k$ is even, the coherent cyclic signs for source rank k and external rank d agree.

2.2. **Additive compounds.** For an $n \times n$ matrix M, let $M^{[r]}$ denote its additive rth compound: the operator on $\bigwedge^r \mathbb{R}^n$ defined by

$$M^{[r]}(v_1 \wedge \cdots \wedge v_r) = \sum_{a=1}^{r} v_1 \wedge \cdots \wedge Mv_a \wedge \cdots \wedge v_r.$$

Differentiating both sides at zero, or using uniqueness for the resulting linear differential equation, gives

$$\bigwedge^r e^{tM} = e^{tM^{[r]}}. \tag{2.4}$$

The entries on the left are precisely the ordered r-minors of e^{tM}.

A real square matrix is Metzler if its off-diagonal entries are nonnegative. It is irreducible if the directed graph of its nonzero off-diagonal entries is strongly connected.

Lemma 2.2 (Compound positivity)**.** *Let $1 \le r < n$. Suppose that M has positive entries in both directions between every pair $i, i+1$, and zero entries outside the diagonal, these neighboring positions, and the two corners. Suppose the corner entries are either both zero or both have sign $(-1)^{r-1}$ and positive absolute value. Then $M^{[r]}$ is an irreducible Metzler matrix. Consequently every ordered r-minor of e^{tM} is positive for $t > 0$. If M is symmetric, its eigenvalues $\lambda_1 \ge \cdots \ge \lambda_n$ satisfy $\lambda_r > \lambda_{r+1}$, and its highest r-dimensional spectral space is positive.*

Proof. Use increasing wedges indexed by r-subsets of $[n]$. An off-diagonal compound entry replaces one occupied index by an unoccupied neighbor. An ordinary neighboring move requires no permutation. A move across the corner permutes the moved index past $r-1$ other indices, and its sign is canceled by the assumed corner sign. All such entries are positive. The graph is connected even using only the ordinary path edges: successively slide occupied indices to the left until the set is $\{1, \ldots, r\}$.

Choose c large enough that $M^{[r]} + cI$ is entrywise nonnegative. For any two vertices, a path between them gives a positive entry in some power of this nonnegative matrix. The exponential series is therefore entrywise positive for positive time. Equation (2.4) proves the minor assertion.

When M is symmetric, the eigenvalues of $M^{[r]}$ are the sums $\lambda_{i_1} + \cdots + \lambda_{i_r}$ over $1 \le i_1 < \cdots < i_r \le n$, counted with multiplicity. Perron–Frobenius applied to the positive exponential shows

that the largest such sum is simple and has a strictly positive eigenvector. If $\lambda_r = \lambda_{r+1}$, there would be two linearly independent highest wedges, a contradiction. The wedge of an oriented orthonormal basis of the highest r-space is its Perron vector and has all coordinates positive. □

For $r = n$, the only compound minor is $\det(e^{tM}) = e^{t \operatorname{tr} M} > 0$. In particular, for the ordinary path matrix

$$H_n = \sum_{i=1}^{n-1} (E_{i,i+1} + E_{i+1,i}), \tag{2.5}$$

for $t > 0$, e^{tH_n} has all minors positive. Right multiplication by it sends every nonnegative k-plane to a positive k-plane. Letting $t \downarrow 0$ proves the density of $\mathrm{Gr}_{>0}(k,n)$ in $\mathrm{Gr}_{\geq 0}(k,n)$. The latter is compact, since it is a closed subset of the real Grassmannian.

2.3. **The flow-ball lemma.** The following is the diagonal linear version of the contractive-flow lemma of [10, Lemma 2.3].

Lemma 2.3 (A compact set entered strictly by a contraction)**.** *Let*

$$f_t(x_1, \dots, x_D) = (e^{-a_1 t} x_1, \dots, e^{-a_D t} x_D),$$

where every $a_i > 0$. Let $K \subset \mathbb{R}^D$ be compact, let $0 \in \operatorname{int} K$, and suppose

$$f_t(K) \subset \operatorname{int} K \qquad (t > 0).$$

Then $(K, \operatorname{int} K)$ is homeomorphic to $(\overline{B}^D, B^D)$. The conclusion for $D = 0$ means that K is a point.

Proof. Assume $D > 0$. Choose $\varepsilon > 0$ such that the closed ε-ball lies in $\operatorname{int} K$, and write $\Sigma = \{u : \|u\| = \varepsilon\}$. Every nonzero orbit meets Σ once: its norm is continuous, strictly decreases with time, and tends to infinity and zero at the two ends of time. For $u \in \Sigma$, put

$$\tau(u) = \sup\{s \geq 0 : f_{-s}(u) \in K\}.$$

This number is finite by boundedness and positive by the choice of ε. The set of such s is the interval $[0, \tau(u)]$. Indeed, if s is allowed then every smaller nonnegative time is allowed by forward invariance. Closedness gives its endpoint. The strict-entry hypothesis shows that all earlier points on the orbit are interior, so the endpoint is its unique boundary point.

For any $\eta > 0$ smaller than $\tau(u)$, $f_{-\tau(u)+\eta}(u)$ is interior to K, whereas $f_{-\tau(u)-\eta}(u)$ is outside K. Both properties persist for nearby u'. The interval property then puts $\tau(u')$ between $\tau(u) - \eta$ and $\tau(u) + \eta$. Thus τ is continuous, and in particular bounded on Σ.

Identify a point of the punctured unit ball with (r, u) where $0 < r \leq 1$ and $u \in \Sigma$. Define

$$h(r, u) = f_{-\tau(u) - \log r}(u), \qquad h(0) = 0. \tag{2.6}$$

Uniform contraction and boundedness of τ prove continuity at zero. The unique orbit intersection with Σ and with the boundary proves bijectivity. A continuous bijection from a compact space to a Hausdorff space is a homeomorphism. Formula (2.6) sends $r = 1$ exactly to ∂K. □

Proposition 2.4 (Spectral model)**.** *Let S be real symmetric, with eigenvalues $\mu_1 \geq \cdots \geq \mu_n$ counted with multiplicity. Suppose its highest k-dimensional spectral space F is positive, $\mu_k > \mu_{k+1}$, and e^{tS} has positive k-minors for every $t > 0$. Let E be an invariant d-plane containing F, spanned by eigenvectors of S, with $k < d \leq n$. If a positive external matrix Z spans E, then $\mathcal{A}_{n,k,d-k}(Z)$ is a closed $k(d-k)$-ball and its positive-source image is its interior.*

Proof. Choose an orthonormal eigenbasis beginning with a positive oriented frame U_F of F. For every nonnegative C, (2.2) gives $\det(CU_F) > 0$. Thus the whole source lies in the single graph chart $\mathrm{row}(I_k, X)$ relative to $F \oplus F^\perp$. Compactness makes its chart image bounded. The condition that E is spanned by eigenvectors means that, after an invertible target coordinate change, projection to E discards exactly the graph columns outside E.

The row flow $C \mapsto Ce^{tS}$ induces

$$X_{ab} \longmapsto e^{t(\mu_b - \mu_a)} X_{ab}, \qquad a \le k < b,$$

and the same formula on retained target columns. Every exponent is negative. Let K be the full image in this target chart and U the image of $\mathrm{Gr}_{>0}(k,n)$. Cauchy–Binet gives $f_t(K) \subset U$ for $t > 0$. The set U is open by Lemma 2.1, contains the fixed point F, and is dense in K by density of the positive source. If $x \in \operatorname{int} K$, then $f_{-t}(x) \in K$ for sufficiently small $t > 0$; hence $x \in U$. Thus $U = \operatorname{int} K$, and Lemma 2.3 applies. □

The ordinary path matrix (2.5) supplies such a model for every $k < d \le n$: its highest d-plane is positive by Lemma 2.2, and all the required spectral gaps are strict. For $d = n$, the target is the source Grassmannian itself. This proves, in the form used below, both the existence of a positive-data closed-ball model and the closed-ball theorem for $\mathrm{Gr}_{\ge 0}(k,n)$.

2.4. **A criterion for a positive lift.** For a full-rank $k \times n$ source matrix C, define its short-relation space by

$$\mathcal{R}_C = \mathrm{span}\{v \in \ker C : |\operatorname{supp}(v)| \le k\} \subset \mathbb{R}^n.$$

This space depends only on the source plane. The support condition includes all relations witnessing the vanishing of a maximal minor.

Proposition 2.5 (A short-relation criterion)**.** *Let Z be a positive $n \times d$ matrix, where $d = k + m$, and let C represent a point of $\mathrm{Gr}_{\ge 0}(k,n)$. If*

$$\mathcal{R}_C \cap \operatorname{col} Z = 0,$$

then there is a smooth family of full-rank matrices C_t, defined for $0 \le t < \epsilon$, such that

$$C_0 = C, \qquad C_t Z = CZ, \qquad \mathrm{row}(C_t) \in \mathrm{Gr}_{>0}(k,n) \quad (0 < t < \epsilon).$$

Consequently, $\mathrm{row}(CZ)$ is an ambient interior point of the amplituhedron. If Y is a boundary point, then $\mathcal{R}_C \cap \operatorname{col} Z \neq 0$ for every nonnegative lift C of Y.

Proof. Choose a matrix B whose rows form a basis of $\mathcal{R}_C^\perp$, and write $C = U_0 B$. This is possible because every row of C annihilates $\mathcal{R}_C$. We first check that, for all U sufficiently close to U_0, the matrix UB is nonnegative and has exactly the same vanishing maximal minors as C.

If $\Delta_I(C) = 0$ for a k-element set I, the square submatrix C_I has a nonzero kernel vector. Extend this vector by zero outside I to obtain $v \in \mathcal{R}_C$. Since $Bv = 0$, the same nonzero vector is in the kernel of $(UB)_I$ for every U. Thus $\Delta_I(UB) = 0$ identically. Each of the remaining maximal minors is strictly positive at U_0, after fixing the nonnegative orientation of C. There are finitely many such minors, so they remain strictly positive in one neighborhood of U_0. At least one is positive, which also proves full rank in that neighborhood.

Put $G_t = \exp(tH_n)$, with H_n as in (2.5). Lemma 2.2 shows that every ordered minor of G_t is strictly positive for $t > 0$. Set

$$M_t = BG_tZ.$$

The matrix $M_0 = BZ$ has full column rank. Indeed, $BZx = 0$ would put Zx in $\ker B \cap \operatorname{col} Z = \mathcal{R}_C \cap \operatorname{col} Z = 0$, and then $x = 0$ because Z has full column rank. Consequently, for $|t|$ sufficiently small, the smooth matrix

$$L_t = (M_t^\mathsf{T} M_t)^{-1} M_t^\mathsf{T}$$

is defined and satisfies $L_tM_t = I_d$. Define

$$U_t = U_0 + (CZ - U_0M_t)L_t, \qquad C_t = U_tBG_t. \tag{2.7}$$

At $t = 0$ the correction vanishes, and direct multiplication gives

$$C_tZ = U_tM_t = U_0M_t + (CZ - U_0M_t)L_tM_t = CZ.$$

Moreover $U_t \to U_0$, so the first part of the proof makes U_tB nonnegative and of rank k for all sufficiently small t. For every k-element set J, Cauchy–Binet now gives

$$\Delta_J(C_t) = \sum_{|I|=k} \Delta_I(U_tB)\det(G_t)_{I,J} > 0 \qquad (t > 0).$$

All summands are nonnegative, at least one first factor is positive, and every second factor is positive. This proves all the assertions about the family.

The image of any strictly positive source is an ambient interior point by Lemma 2.1. Applying this to C_t proves the interior assertion. Its contrapositive, applied to each source in a boundary fiber, is the final assertion. □

2.5. **Cyclic support sums and automatic transversality.** For $I \subset [n]$, write $\mathbb{R}^I$ for its coordinate subspace and $H_I = H \cap \mathbb{R}^I$. A cyclic interval consists of consecutive coordinates in the circular order; the empty and full sets are allowed.

Lemma 2.6 (Ordinary interval restrictions and absorption)**.** *Let $H \in \mathrm{Gr}_{\geq 0}(h, n)$.*

(i) *If I is an ordinary interval, H_I is nonnegative in its own dimension.*
(ii) *Suppose $J = [a, b]$ and $H_J = \mathbb{R}v$, where $v_a, v_b > 0$ and $v_i \geq 0$ on J. Every internal index $a < i < b$ for which $v_i = 0$ is a zero coordinate of the whole space H.*
(iii) *Under the hypotheses of* (ii)*, if $v_i > 0$ for every $i \in J$, put*

$$T = I_n - \sum_{i=a}^{b-1} \frac{v_{i+1}}{v_i} E_{i,i+1}.$$

Then HT is nonnegative, $vT = v_ae_a$, and T^{-1} is an invertible totally nonnegative matrix. Within the J block,

$$(T^{-1})_{ij} = \frac{v_j}{v_i} \quad (a \leq i \leq j \leq b),$$

and the other entries are those of the identity outside that block and zero between the block and its complement.

Proof. For (i), let $f = \dim H_I$ and choose $h - f$ columns outside I which form a basis of the outside projection. Use a row basis of H whose first f rows span H_I. A maximal minor on these fixed outside columns and any f columns of I factors into the corresponding f-minor of H_I and one fixed nonzero outside determinant. Because I is an interval, the sorting sign is independent of the f selected indices. All these minors therefore have one common nonnegative orientation, proving (i).

For (ii), write H_j for column j of a row frame of H, and choose an $(h-1)$-element column basis K of the outside projection. A linear functional on the column space which produces the row v annihilates these columns. If $v_i = 0$, write $H_i = \sum_{j\in K} \gamma_j H_j$ in any row matrix for H. For $j \in K$ put $K' = K \setminus \{j\}$. All indices in K' lie outside $[a, b]$. Expanding modulo their span gives the exact identity

$$\Delta_{K'\cup\{i,b\}}(H) = -\frac{v_b}{v_a}\Delta_{K'\cup\{a,i\}}(H).$$

The two interval columns occupy the same consecutive positions in the two ordered determinants. Both minors are nonnegative, so both vanish. Since $K \cup \{a\}$ is a basis, this forces $\gamma_j = 0$. This holds for every j, so $H_i = 0$. For $h = 1$ the conclusion follows directly from $H = \mathbb{R}v$.

For a nonnegative plane, its bounded affine permutation is

$$f_H(i) = \min\{j \geq i : H_i \in \operatorname{span}(H_{i+1}, \ldots, H_j)\},$$

where columns are extended periodically for this span calculation. For each cyclic starting index a, let I_a be the greedy column basis: scan $a, a+1, \ldots, a+n-1$, keeping a column exactly when it increases the span of those already kept, and reduce the retained indices modulo n. These bases form the Grassmann necklace. Use the coherent cyclic orientation for their minors. Lam's bridge-removal theorem [17, Proposition 7.10 and Lemma 7.11] states that if

$$i + 1 \leq f_H(i) < f_H(i+1) \leq i + n,$$

then $H(I_n - tE_{i,i+1})$ is nonnegative for

$$t = \frac{\Delta_{I_{i+1}}(H)}{\Delta_{(I_{i+1} \setminus \{i+1\}) \cup \{i\}}(H)} > 0.$$

Suppose the current positive line is supported on $[a, \ell]$, where $\ell > a$. The outside columns span a hyperplane. Every column from $\ell + 1$ through $a - 1$ in cyclic order lies in that hyperplane, whereas column a is outside it. Thus

$$f_H(\ell) = n + a, \qquad \ell \leq f_H(\ell - 1) \leq n + a - 1 < n + a \leq n + \ell - 1.$$

These are exactly the bridge inequalities at $i = \ell - 1$. The greedy basis starting at ℓ consists of ℓ and a basis of the outside columns: they already span the whole column space before the cyclic scan reaches a. Expanding the two necklace minors along the row v shows that their ratio is $v_\ell / v_{\ell-1}$. There is no sorting-sign change between two adjacent columns with all other chosen columns outside the interval. The bridge removes coordinate ℓ from v and preserves nonnegativity. The new outside projection still has rank $h - 1$: it contains the old outside basis and is annihilated by the new nonzero supported row. The same argument can therefore be repeated.

Perform these removals for $\ell = b, b-1, \ldots, a+1$. The matrix units in this decreasing order have zero pairwise forward products, so their product is precisely T. Each inverse factor is an ordinary adjacent positive elementary matrix, hence totally nonnegative. Their product is T^{-1}, and multiplication of the factors gives the displayed entries. The singleton interval case has $T = I_n$ and needs no bridge. □

Theorem 2.7 (Cyclic support sums)**.** *Let $H \in \mathrm{Gr}_{\geq 0}(h, n)$ and let*

$$V = \sum_{I \in \mathcal{I}} H_I, \qquad s = \dim V,$$

where $\mathcal{I}$ is a family of cyclic intervals. If $h - s$ is even, then $V \in \mathrm{Gr}_{\geq 0}(s, n)$.

Proof. We induct on s, allowing all values of h and n. The zero space is immediate, as is $s = h$, when $V = H$. We may assume $0 < s < h$. Remove zero summands. No remaining support can become the full coordinate set after deleting zero coordinates of H, since its supported space would then be H and would force $s = h$. Rotate one nonzero interval I_0 to an ordinary interval using the rank-h coherent cyclic shift. Its seam sign $(-1)^{h-1}$ agrees with the rank-s seam sign, so proving nonnegativity in this order suffices for the original order.

By Lemma 2.6(i), $F = H_{I_0}$ is nonnegative. Let $p_1 < \cdots < p_f$ be its lexicographically first column basis and normalize those columns to the identity. The last pivot row v is zero before

$a = p_f$: a column encountered earlier lies in the span of the earlier greedy columns. For $j > a$ its entry is

$$v_j = \frac{\Delta_{\{p_1,\dots,p_{f-1},j\}}(F)}{\Delta_{\{p_1,\dots,p_f\}}(F)} \geq 0.$$

Let b be its last nonzero index and $J = [a, b]$. The other pivots are outside J, so $F_J = \mathbb{R}v$; since $J \subset I_0$, also $H_J = \mathbb{R}v$. Any internal zero of v is a zero coordinate of the whole H by Lemma 2.6(ii). Delete these coordinates. The intervals remain cyclic intervals in the inherited order, and the dimensions h, s do not change. Thus v is positive on J. Obtain T from part (iii).

A surviving interval I cannot be a proper subset of J, because $H_J = \mathbb{R}v$ and every coordinate of v on J is nonzero. Write c for the cyclic right endpoint of I and set

$$I^+ = \begin{cases} I \cup \{c+1\}, & a \leq c < b, \\ I, & \text{otherwise.} \end{cases}$$

Then

$$(\mathbb{R}^I + \mathbb{R}v)T = \mathbb{R}^{I^+} + \mathbb{R}e_a. \tag{2.8}$$

Indeed, a coordinate can leave I under T only across its right endpoint c, and only when $a \leq c < b$. In that case $a \in I$: an interval ending at c and avoiding a would be a proper subinterval of J. Also $c + 1 \notin I$, so $v \notin \mathbb{R}^I$. The left side of (2.8) is contained in $\mathbb{R}^{I^+}$, contains e_a, and has dimension $|I| + 1 = |I^+|$, proving equality. This includes a wrapping interval whose omitted gap is inside J. In the other case T preserves $\mathbb{R}^I$, and the identity follows directly from $vT = v_a e_a$.

Because $v \in H$, we have

$$H \cap (\mathbb{R}^I + \mathbb{R}v) = H_I + \mathbb{R}v.$$

Intersect (2.8) with HT and quotient by $\mathbb{R}e_a$. Intersection commutes with this quotient because both spaces contain e_a. Delete coordinate a and multiply every remaining coordinate before a by -1. This common signed quotient makes $\overline{H}$ nonnegative of rank $h - 1$, and sends VT to

$$\overline{V} = \sum_{I \in \mathcal{I}} (\overline{H} \cap \mathbb{R}^{I^+ \setminus \{a\}}), \qquad \dim \overline{V} = s - 1.$$

Every displayed support is a cyclic interval in the smaller order. The parity difference is unchanged. Induction gives $\overline{V} \geq 0$. Reinserting the coordinate coloop and undoing the prefix signs makes VT nonnegative: for a maximal minor containing a, the prefix sign cancels its Laplace-expansion sign, independently of the rank. All other maximal minors are zero. Now $V = (VT)T^{-1}$ is nonnegative by Cauchy–Binet. Reinsert the deleted zero coordinates and undo the cyclic shift. This completes the induction. □

Lemma 2.8 (Interval generation of short relations)**.** *For $C \in \mathrm{Gr}_{\geq 0}(k, n)$, let C_J denote the columns indexed by J in any row frame of C. Then*

$$\mathcal{R}_C = \sum_{\substack{J \text{ cyclic interval} \\ \operatorname{rank} C_J < k}} (\ker C \cap \mathbb{R}^J).$$

For each cyclic starting index, it suffices to use the longest interval of rank less than k starting there. Thus at most n explicit matrix kernels generate $\mathcal{R}_C$.

Proof. If $\operatorname{rank} C_J = p < k$, its kernel is generated by circuit relations supported on at most $p + 1 \leq k$ columns. This gives one inclusion. For the converse, a relation supported on at most k columns is a sum of circuit relations supported on subsets of those columns: subtract a suitable multiple of a minimal-support relation to cancel one nonzero coordinate, and induct on the support size. Coordinate loops give singleton intervals, so delete them first.

Let n_0 be the number of remaining columns and M their loopless column positroid. For a circuit S with $|S| \le k$, the flat $F = \mathrm{cl}_M(S)$ is proper, since $r_M(S) = |S| - 1 < k$, and its restriction is connected. The circuit restriction is connected, and adding an element in its closure creates no new component: that element has a circuit with a subset of S. Bonin's characterization [7, Theorem 3.2(3)] states that, for a connected flat with $2 \le |F| \le n_0 - 2$, the contraction M/F is the direct sum of its restrictions to the maximal cyclic gaps U_j disjoint from F. If $|F| = n_0 - 1$, the same assertion has only one summand and is immediate. These cases cover F, since a circuit of a loopless matroid has at least two elements.

Some gap has positive contracted rank $q_j = r_{M/F}(U_j)$, because F is proper. Its complement J is a cyclic interval containing S, and the direct-sum rank formula gives

$$\operatorname{rank} C_J = r_M(F) + \sum_{i \ne j} r_{M/F}(U_i) = k - q_j < k.$$

The original circuit vector therefore belongs to the indicated supported kernel with its original coefficients. Inserting the deleted loops turns intervals back into intervals and does not increase their ranks. This proves the equality.

Every deficient interval starting at a fixed index is contained in the longest such interval. Their supported kernels are correspondingly nested. Taking one maximal interval for each start proves the final assertion. Empty intervals contribute zero. □

Theorem 2.9 (Automatic transversality of the relation space)**.** *Let $C \in \mathrm{Gr}_{\ge 0}(k,n)$, let Z be positive of size $n \times d$, where $d = k + m$, and put $s = \dim \mathcal{R}_C$ and $q = n - d$. If m is even, then*

$$\dim(\mathcal{R}_C \cap \operatorname{col} Z) = \max\{0, s - q\}. \tag{2.9}$$

In particular, $s \le q$ gives the exact same-output strictly positive deformation of Proposition 2.5. For odd m, the sufficient condition $s \le q - 1$ gives the same deformation. Thus, for every m, a sufficient condition is

$$\dim \mathcal{R}_C \le n - k - m - (m \bmod 2).$$

Proof. Use alternating duality (2.1) and set

$$H = A_n(\ker C), \quad h = n - k, \quad V = A_n \mathcal{R}_C, \quad r = n - s, \quad B = \mathcal{R}_C^{\perp}.$$

Then H is nonnegative, and Lemma 2.8 expresses V as a sum of cyclic interval restrictions of H. If $h - s = r - k$ is even, Theorem 2.7 and alternating duality make B a nonnegative r-plane.

If $h - s$ is odd, V is proper in H. Choose an inclusion-minimal cyclic interval I for which $H_I \not\subset V$. Removing one endpoint lowers $\dim H_I$ by at most one and gives a space contained in V, by minimality. Hence $V + H_I$ has dimension $s + 1$; this also holds for a singleton I. Its dimension has the parity of h, so its alternating orthogonal complement is a nonnegative $(r-1)$-plane contained in B.

Every nonnegative p-plane W contains nonnegative subplanes of each rank $0 \le d' \le p$. Indeed, $\dim(W \cap \mathbb{R}^{[1,j]})$ increases from 0 to p in steps of at most one, and each such prefix restriction is nonnegative by Lemma 2.6(i). It follows that B contains a nonnegative d-plane whenever $r \ge d$ and m is even. In the opposite-parity case $r - d$ is then a positive odd integer, so $r - 1 \ge d$. For odd m the same conclusion holds under $r \ge d + 1$. Pairing that subplane with Z and applying Cauchy–Binet shows $\operatorname{rank}(BZ) = d$, where B in this expression denotes any row basis. Equivalently, $\mathcal{R}_C \cap \operatorname{col} Z = 0$. This proves both sufficient conditions and the case $r \ge d$ of (2.9).

Suppose m is even and $r < d$. If $r - k$ is even, B is itself nonnegative, and Lemma 2.1 makes its projection injective. If $r - k$ is odd, V has a cyclic interval subsum of dimension $s - 1$. To construct it, grow each of its generating intervals one endpoint at a time and add the resulting supported spaces to the accumulated sum. Each dimension increase is at most one, all intermediate sums lie in V, and the final sum is V. Thus every intermediate dimension

occurs. The chosen $(s-1)$-dimensional sum has the parity of h, so its alternating orthogonal complement is a nonnegative $(r+1)$-plane containing B. Here $r+1 \le d$. Positive projection is injective on that whole plane, hence also on B. We have proved

$$\operatorname{rank}(BZ) = \min\{r, d\}.$$

The kernel of the pairing map from $\operatorname{col} Z$ to B^* is exactly $\mathcal{R}_C \cap \operatorname{col} Z$. Its nullity is $d - \min\{r, d\} = \max\{0, s-q\}$, proving the formula. The argument includes $d = n$, when projection is invertible. Finally apply Proposition 2.5 to obtain the stated deformation. □

Corollary 2.10. *For even m, every nonnegative lift C of an amplituhedron boundary point satisfies $\dim \mathcal{R}_C > n-k-m$. A target has a strictly positive lift if and only if it has a nonnegative lift satisfying $\dim \mathcal{R}_C \le n-k-m$.*

Proof. The first assertion follows from the interior conclusion of Proposition 2.5. For the second assertion, a strictly positive source has no nonzero relation supported on at most k coordinates, so its relation space is zero. The converse is the preceding theorem. □

Example 2.11 (The odd-m equality case)**.** Take $k = 2$, $n = 4$, $d = 3$, and

$$C = \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{pmatrix}.$$

In lexicographic order the six maximal minors of C are $(1, 1, 0, 1, 1, 1)$, and the four maximal minors of Z are $(2, 6, 6, 2)$. Thus $C \ge 0$, $Z > 0$, and the only short circuit is supported on $\{1, 4\}$. Consequently

$$\mathcal{R}_C = \mathbb{R}(1, 0, 0, 1), \qquad s = q = 1.$$

But

$$Z \begin{pmatrix} 3 \\ -5/2 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

so $\mathcal{R}_C \subset \operatorname{col} Z$ and $\dim(\mathcal{R}_C \cap \operatorname{col} Z) = 1$ although $s = q$.

3. The case $m = 2$: arbitrary positive external data

Throughout this section, $1 \le k \le n-2$ and Z is an $n \times (k+2)$ matrix with all ordered maximal minors strictly positive. Write $E = \operatorname{col}(Z) \in \operatorname{Gr}_{>0}(k+2, n)$ and $X = \operatorname{Gr}_{\ge 0}(k, n)$. We use the twistor realization

$$\mathcal{A}(E) = \{E \cap C^{\perp} : C \in X\} \subset \operatorname{Gr}(2, E).$$

Indeed, Lemma 2.1 makes positive projection injective on every $C \in X$, so $E \cap C^{\perp}$ has dimension two and equals $Z \ker(CZ)$. Consequently this realization is homeomorphic to the usual closed amplituhedron. Row vectors of an $n \times 2$ frame of a twistor plane will be denoted by r_i. A coherent cyclic extension means $r_{i+n} = (-1)^{k-1} r_i$. The same twist preserves source positivity and external positivity because $(-1)^{k-1} = (-1)^{k+1}$.

For an oriented $k \times (k+2)$ row frame Y, define the twistor bracket

$$\langle Y, i, j \rangle = \det \begin{pmatrix} Y \\ Z_i \\ Z_j \end{pmatrix},$$

where Z_i is the ith row of Z. If Y is orthonormal, choose an orthonormal column frame K of $\ker Y$ with $\det[\,Y; K^{\mathsf{T}}\,] = 1$, and put $R = ZK$. Then

$$\begin{pmatrix} Y \\ Z_i \\ Z_j \end{pmatrix} \begin{pmatrix} Y^{\mathsf{T}} & K \end{pmatrix} = \begin{pmatrix} I_k & 0 \\ Z_i Y^{\mathsf{T}} & r_i \\ Z_j Y^{\mathsf{T}} & r_j \end{pmatrix}.$$

The second factor has determinant one, so $\langle Y, i, j\rangle = \det(r_i, r_j)$. An orientation-preserving change from a general frame Y to an orthonormal frame multiplies all these brackets by one positive factor. This fixes their common orientation for the Cauchy–Binet calculations below.

3.1. **A ball theorem for a fixed two-plane.**

Theorem 3.1 (A nonnegative section by a two-plane)**.** *Let $1 \le h \le N-2$, and let R be an $N \times 2$ matrix of rank two. Extend its rows by $r_{i+N} = \epsilon r_i$, where $\epsilon = (-1)^{h-1}$, and suppose*

$$d_i = \det(r_i, r_{i+1}) > 0 \qquad (1 \le i \le N).$$

If

$$S_R = \{C \in \mathrm{Gr}_{\ge 0}(h, N) : CR = 0\}$$

is nonempty, it is homeomorphic to a closed ball of dimension $h(N-h-2)$. Its intrinsic interior is precisely $S_R \cap \mathrm{Gr}_{>0}(h, N)$.

Proof. Define a symmetric cyclic Jacobi matrix Q by

$$(Qs)_i = \frac{s_{i-1}}{d_{i-1}} - \frac{\det(r_{i-1}, r_{i+1})}{d_{i-1} d_i} s_i + \frac{s_{i+1}}{d_i}, \qquad s_{i+N} = \epsilon s_i. \tag{3.1}$$

Its ordinary adjacent entries are $1/d_i > 0$, and its two corner entries are ϵ/d_N. The two-dimensional determinant identity

$$d_i r_{i-1} - \det(r_{i-1}, r_{i+1}) r_i + d_{i-1} r_{i+1} = 0$$

proves $QR = 0$ row by row.

In the ordered wedge basis indexed by h-subsets of $[N]$, every off-diagonal entry of the additive compound $Q^{[h]}$ is nonnegative. An ordinary adjacent move has positive coefficient. A move across the cyclic cut changes wedge order by $(-1)^{h-1}$, which cancels the corner sign ϵ. The graph of allowed adjacent exclusion moves is connected: ordinary path moves alone connect every h-subset to $\{1, \dots, h\}$. Thus $Q^{[h]}$ is irreducible Metzler. Equivalently, after adding a sufficiently large scalar matrix, it is a nonnegative irreducible matrix; the exponential series and paths in its graph show that $\exp(tQ^{[h]})$ has every entry strictly positive for $t > 0$. Since

$$\bigwedge^h \exp(tQ) = \exp(tQ^{[h]}),$$

every positive time sends a nonnegative h-plane to a strictly positive one. Moreover $\exp(tQ)R = R$, so right multiplication of source frames by $\exp(tQ)$ preserves the exact equation $CR = 0$.

Order the eigenvalues of the symmetric matrix Q as $\lambda_1 \ge \cdots \ge \lambda_N$. Perron–Frobenius applied to the symmetric irreducible compound shows that its largest eigenvalue is simple and has a strictly positive eigenvector. Its eigenvalues are the sums $\lambda_{i_1} + \cdots + \lambda_{i_h}$ over $1 \le i_1 < \cdots < i_h \le N$, counted with multiplicity. If $\lambda_h = \lambda_{h+1}$, the largest sum would have at least two linearly independent wedge eigenvectors. Hence

$$\lambda_h > \lambda_{h+1}. \tag{3.2}$$

Let C_* be the top h-dimensional eigenspace of Q. Its wedge is the positive compound eigenvector, after orientation, so $C_* \in \mathrm{Gr}_{>0}(h, N)$.

Every nonnegative source has positive wedge inner product with C_*. More explicitly, for coherently oriented orthonormal frames M_C, M_*, Cauchy–Binet gives

$$\det(M_C M_*^T) = \sum_{|I|=h} \Delta_I(M_C)\Delta_I(M_*) > 0. \tag{3.3}$$

The spectral gap therefore implies $C\exp(tQ) \longrightarrow C_*$ for every nonnegative C. Applying this to one point of S_R and using its closedness yields $C_*R = 0$.

Equation (3.3) also places every nonnegative source in the graph chart over C_*. Choose an orthonormal eigenbasis $v_1, \dots, v_N$ with $v_1, \dots, v_h$ spanning C_*. Since the two-plane $\mathrm{col}(R)$ is a zero-eigenspace orthogonal to C_*, two of the remaining basis vectors can be chosen to span it. In the graph chart

$$C(A) = \mathrm{span}\left\{ v_i + \sum_{j>h} A_{ij} v_j : 1 \le i \le h \right\},$$

the condition $C(A)R = 0$ says precisely that the two corresponding columns of A vanish. It cuts out a linear space V of dimension $h(N-h-2)$. The compact set S_R has compact, hence bounded, image in this chart.

The source flow becomes the linear flow

$$A_{ij} \longmapsto e^{t(\lambda_j - \lambda_i)} A_{ij}, \qquad i \le h < j, \tag{3.4}$$

with the two constrained columns omitted. All exponents are strictly negative by (3.2). Thus it is a strict contraction for the Euclidean norm on V. Let O be the chart image of $S_R \cap \mathrm{Gr}_{>0}(h, N)$. It is an open smooth subset of V, contains the origin, and the positive-time flow sends the whole chart image of S_R into O. Letting time decrease to zero shows that $\overline{O}$ is exactly that compact image. Lemma 2.3 therefore identifies $\overline{O}$ with a closed ball of dimension $h(N-h-2)$.

The set O is exactly the interior in V of $\overline{O}$. One inclusion follows because O is open. If $x \in \mathrm{int}_V \overline{O}$, a sufficiently small negative flow time keeps x in $\overline{O}$. Applying the corresponding positive time gives $x \in O$. The ball homeomorphism thus has the asserted interior. When $\dim V = 0$, the same chart says directly that $S_R = \{C_*\}$. □

3.2. **Positive deletion, including zero rows.**

Lemma 3.2 (One internal positive deletion)**.** *Suppose that a source plane contains $f = e_i + te_{i+1}$, where $i < n$ and $t \ge 0$. There is an exact homeomorphism between the nonnegative source planes containing f and $\mathrm{Gr}_{\ge 0}(k-1, n-1)$. Quotienting the external target by $Z_i + tZ_{i+1}$ gives strictly positive external data for $(n-1, k-1, 2)$, with the same homeomorphism respecting projection and source fibers. Coordinate-coloop deletion has the same conclusion, including a coloop in the last position.*

Proof. Put $T = I - tE_{i,i+1}$ and $C' = CT$. Then $fT = e_i$, so C' contains e_i. A maximal minor containing i is unchanged: either it also contains $i+1$, in which case the column operation has zero extra determinant, or none of its columns changes. Every maximal minor avoiding i vanishes, since the row plane contains e_i. Thus C' is nonnegative.

Choose a positively oriented frame of C' whose first row is e_i and whose other rows vanish in column i. Delete that row and column to obtain B. On the surviving coordinates let $s_j = -1$ for $j < i$ and $s_j = 1$ for $j > i$, and set $D = B\,\mathrm{diag}(s_j)$. If J is an ordered $(k-1)$-subset avoiding i, expansion in column i gives

$$\Delta_{\{i\}\cup J}(C') = (-1)^{|J \cap [1, i-1]|} \Delta_J(B) = \Delta_J(D). \tag{3.5}$$

Hence D is nonnegative. Conversely, reverse this formula, insert the coloop, and multiply by $T^{-1} = I + tE_{i,i+1}$. This adjacent elementary matrix is totally nonnegative, so Cauchy–Binet

preserves nonnegativity. The constructions are inverse continuous maps of Grassmannians, and are smooth in their stated coordinates.

Let $v = Z_i + tZ_{i+1}$. In the quotient vector space $\mathbb{R}^{k+2}/\mathbb{R}v$, discard row i and multiply the remaining external rows by the same signs s_j. Give the quotient the orientation obtained by putting v first. For an ordered $(k+1)$-subset J avoiding i, write $s = |J \cap [1, i-1]|$. Its quotient minor has, up to one common positive normalization, numerator

$$\begin{aligned}\det[v; s_j Z_j \ (j \in J)] &= (-1)^s(\det[Z_i; Z_J] + t\det[Z_{i+1}; Z_J]) \\ &= \Delta_{\{i\}\cup J}(Z) + t\Delta_{\{i+1\}\cup J}(Z) > 0. \end{aligned} \tag{3.6}$$

If $i+1 \in J$, the second term is zero; otherwise both insertions have sorting sign $(-1)^s$. The first term is always strictly positive. These are all quotient maximal minors.

The identity $CZ = C'T^{-1}Z$ now shows that, after quotienting the target by v, its image is $D\widehat{Z}$. Conversely, the unique k-plane containing v with a specified $(k-1)$-dimensional quotient is the original target. This proves the asserted compatibility with every full source fiber.

For a coordinate coloop set $t = 0$. The expansion and prefix-sign formula remains valid when $i = n$. A line with only its second coefficient nonzero is simply deletion of that coordinate. For a cyclic wrap pair, first make a coherent cyclic cut. The source and external twists agree for $m = 2$, so this change preserves every required positivity statement. □

Lemma 3.3 (A zero adjacent bracket forces a common positive line)**.** *Let $R \in \mathcal{A}(E)$, and suppose the pair (r_i, r_{i+1}) has rank one and determinant zero. Its kernel is a line $\mathbb{R}f$ in the corresponding source coordinates. Every source lift of R contains this same line, and its coefficients have the same weak sign in the coherent convention. They are both nonzero if both twistor rows are nonzero.*

Proof. Make a coherent cut putting the pair first, and choose a nonnegative source frame C. With the target frame $Y = CZ$, Cauchy–Binet gives

$$\langle Y, 1, 2\rangle = \sum_{\substack{|I|=k \\ I\cap\{1,2\}=\varnothing}} \Delta_I(C)\Delta_{I\cup\{1,2\}}(Z). \tag{3.7}$$

Every summand is nonnegative and every external coefficient is strictly positive. Vanishing therefore forces the restriction of C to the outside coordinates to have rank at most $k-1$. Thus $C \cap \operatorname{span}(e_1, e_2) \neq 0$. Since $CR = 0$ and the pair has rank one, this intersection is exactly its one-dimensional kernel, independently of C.

Write its generator as $f = ae_1 + be_2$. The outside restriction has rank exactly $k-1$, so choose $k-1$ outside columns J on which the remaining rows of a frame beginning with f have nonzero minor δ. The two source minors are $a\delta$ and $b\delta$; their common nonnegative orientation forces a, b to have the same weak sign. Orient f with $a, b \geq 0$. If both twistor rows are nonzero, neither coefficient can vanish. □

A zero adjacent bracket always supplies a rank-one zero pair after possibly moving along a string of zero rows. A rank-two frame cannot consist entirely of zero rows. The preceding lemma and positive deletion therefore treat such strings as well as nonzero parallel pairs.

3.3. **Patterns, maximal lifts, and all source fibers.** We use the numbering of [18, arXiv version 3]. A rank-two positroid pattern N has a loop set L and disjoint maximal cyclic parallel blocks $B_1, \ldots, B_s$ after the loop indices are deleted; only blocks of size at least two are listed. Equivalently, in Lam's interval notation, $B_j = [a_j, b_j] \setminus L$. This convention matters when zero

rows separate two nonzero parallel rows. Nonconsecutive accidental parallelisms are not imposed as additional equations in an open positroid stratum. Set

$$b=\sum_{j=1}^{s}(|B_j|-1), \qquad a=|L|+b, \qquad r=k-a. \tag{3.8}$$

Let $\mathring{\Pi}_N$ be the real open positroid stratum, and put $\mathcal{A}_N(E)=\mathcal{A}(E)\cap\mathring{\Pi}_N$.

For every strictly positive Z, Lam's Definition 14 constructs the maximal source positroid $N^{\uparrow k}$: its dual has loops L, parallel classes B_j, and uniform matroid after these classes are collapsed. Theorem 16 says the feasible patterns are precisely $a\le k$, with dimension count

$$d(N)=2k-2|L|-b=b+2r. \tag{3.9}$$

Proposition 19 and Corollary 20 determine the exact open twistor stratum of each positive source cell. Theorem 23 gives nonemptiness of all feasible actual strata and

$$\overline{\mathcal{A}_N(E)}=\coprod_{N'\le N}\mathcal{A}_{N'}(E). \tag{3.10}$$

Lemma 3.4 (The common forced subspace)**.** *For $R\in\mathcal{A}_N(E)$, define*

$$F_R=\operatorname{span}\{e_i : i\in L\}\oplus\bigoplus_{j=1}^{s}\{v:\operatorname{supp}(v)\subset B_j,\ vR=0\}. \tag{3.11}$$

Then $\dim F_R=a$ and F_R is contained in every nonnegative source lift of R.

Proof. For the positroid M_C of a lift C, [18, Corollary 20 and the proof of Theorem 16] gives $M_C\le N^{\uparrow k}$. By the description in Definition 14, every $i\in L$ is a coloop of M_C, giving $e_i\in C$, and

$$\operatorname{rank}(C|_{[n]\setminus B_j})\le k-|B_j|+1.$$

It follows that $\dim(C\cap\mathbb{R}^{B_j})\ge|B_j|-1$. But $CR=0$ and R_{B_j} has rank one, so

$$C\cap\mathbb{R}^{B_j}\subset\{v:\operatorname{supp}(v)\subset B_j,\ vR=0\},$$

whose dimension is exactly $|B_j|-1$. Equality follows. The supports of the displayed summands are disjoint, proving the claim and its dimension. In particular $a\le k$ also follows directly from this containment. □

Lemma 3.5 (Complete compression and its inverse)**.** *For a fixed feasible pattern, choose a coherent cut between two distinct surviving blocks and a deterministic deletion order. Delete every loop and then all but one row from each nontrivial parallel block. At each stage use Lemma 3.2 and its prefix reorientation. For an actual point these are exactly a positive deletions. They produce strictly positive external data for $(n-a,r,2)$ and a residual actual twistor frame with no zero row and no zero coherent adjacent determinant. The nontrivial line ratios give global parameters $\alpha\in(\mathbb{R}_{>0})^b$. Moreover*

$$\Pi_{N^{\uparrow k},>0}\cong(\mathbb{R}_{>0})^b\times\operatorname{Gr}_{>0}(r,n-a) \tag{3.12}$$

smoothly, with inverse given by the prescribed insertions.

Proof. First delete the forced coloops at L. The remaining members of each B_j are consecutive after this deletion. Each nonzero parallel adjacent pair has its common positive kernel line by Lemma 3.3; delete it with its strictly positive ratio. Repeat in the fixed order. The common-subspace lemma ensures that these operations apply to every source lift, and Lemma 3.2 preserves strict external positivity at every stage.

To track the twistor frame, at an internal step the equation $r_i + tr_{i+1} = 0$ gives

$$(T^{-1}R)_i = r_i + tr_{i+1} = 0.$$

All other rows of $T^{-1}R$ are the old rows. Passing to the quotient deletes row i and multiplies the survivors by the prefix signs s_j. Thus surviving zero rows and projective parallel relations are preserved exactly. After the a deletions, maximality of the original compressed blocks leaves no zero row and no parallel cyclic neighboring pair. The residual data are still actual and positive, so (3.7) makes their coherent adjacent brackets nonnegative. Their nonvanishing makes every one strictly positive.

The positive ratios are intrinsic: a rank-one adjacent pair with both rows nonzero determines its kernel line and hence its ratio uniquely. The successive row insertions reverse these relations. They are defined for every list of positive ratios; pure loop insertions have no parameter. Since the cut and order depend only on the pattern, these parameters are global.

We verify the exact source cell in (3.12). Alternating-sign orthogonal duality takes nonnegative k-planes to nonnegative $(n-k)$-planes. The dual matroid of $N^{\uparrow k}$ has precisely the zero columns L and the consecutive parallel classes B_j. In a nonnegative point of its open cell, parallel columns within one such class have positive ratios: extend a class column to a basis and replace it by another column of the same class. No other basis column lies between them outside the class, so the ordered determinant sign does not change. Normalize one column in each class. After deletion of loops and duplicate class columns, the remaining rank-$(n-k)$ plane on $n-a$ coordinates is strictly positive, since the collapsed matroid is uniform.

Conversely, any such strictly positive collapsed plane and any positive class ratios give a nonnegative expanded plane. Its minor is zero exactly when the chosen set uses a loop or two members of a parallel class. Every other maximal minor is the corresponding positive collapsed minor times a product of positive ratios. Thus no unrecorded vanishing occurs. Applying alternating duality again gives a residual strictly positive r-plane and exactly the cell in (3.12). The class ratios and the fixed successive-deletion ratios are mutually smooth coordinates: each step takes a ratio of nonzero proportional columns, and the inverse insertions recover them. This proves the claimed global product and all assertions about its inverse. □

Theorem 3.6 (Source fibers)**.** *For every strictly positive Z and every $Y \in \mathcal{A}_{n,k,2}(Z)$, the entire fiber*

$$\{C \in \mathrm{Gr}_{\geq 0}(k,n) : \mathrm{row}(CZ) = Y\}$$

is a closed ball. If the twistor pattern of Y has count a in (3.8), its dimension is $(k-a)(n-k-2)$.

Proof. For $R = Z \ker Y$, positive projection has full source rank, so $CR = 0$ is equivalent to $\mathrm{row}(CZ) = Y$. Complete compression is an exact homeomorphism of this whole fiber with

$$\{D \in \mathrm{Gr}_{\geq 0}(r, n-a) : D\widehat{R} = 0\}.$$

If $r = 0$, this is a point. Otherwise all coherent adjacent minors of $\widehat{R}$ are strictly positive, and Theorem 3.1 applies. Its dimension is $r((n-a)-r-2) = (k-a)(n-k-2)$. □

Proposition 3.7 (Maximal-cell lifts)**.** *For every positive external plane E and every feasible N,*

$$\pi_E(\Pi_{N^{\uparrow k},>0}) = \mathcal{A}_N(E), \tag{3.13}$$

where $\pi_E(C) = E \cap C^{\perp}$.

Proof. One inclusion is Lam's exact image-stratum theorem. For the other, compress any source lift of an actual point. If $r > 0$, apply the flow of Theorem 3.1 for a positive time to its residual source. This makes the residual source strictly positive while preserving its twistor kernel exactly. If $r = 0$, no flow is needed. Undo the fixed deletions. By (3.12), the recovered source belongs to

$\Pi_{N^{\uparrow k},>0}$, and compatibility of every quotient preserves the original target. This constructs the required lift. □

3.4. **Transversality and Whitney regularity.**

Proposition 3.8 (Transversality along the image)**.** *For every positive E and $R \in \mathcal{A}_N(E)$, $\mathrm{Gr}(2, E)$ is transverse to $\mathring{\Pi}_N$ at R. The local intersection dimension is $d(N)$.*

Proof. Work on the open manifold of full-rank $n \times 2$ frames. The pattern equations are zero rows on L and rank one on each disjoint block B_j. All other requirements defining the open stratum are open nonvanishing conditions. Write $r_i = \lambda_i u$ on one block, with $u \neq 0$ and every $\lambda_i \neq 0$, and let $u^\perp = (-u_2, u_1)$. Its conormal matrices have exactly the rows

$$M_i = w_i u^\perp, \qquad \sum_{i \in B_j} w_i \lambda_i = 0. \tag{3.14}$$

To see that these are all conormals, a tangent variation has form $\dot\lambda_i u + \lambda_i \dot u$. Pairing with it vanishes precisely under the displayed conditions. At a loop row a conormal row is arbitrary, and on every other row it is zero. The total conormal dimension is therefore $2|L| + b$.

Both columns of every such M lie in F_R. If M additionally annihilates the tangent space of the frames lying in E, each column lies in $E^\perp$. Choose any source lift C. Lemma 3.4 and injectivity of positive projection put each column of M in

$$F_R \cap E^\perp \subset C \cap E^\perp = 0.$$

Thus $M = 0$, proving transversality. Vertical frame variations RH, with $H \in \mathrm{Mat}_{2\times 2}(\mathbb{R})$, are already tangential to the pattern, so this conormal calculation descends to the Grassmannian. Subtracting the codimension $2|L| + b$ from $\dim \mathrm{Gr}(2, E) = 2k$ gives (3.9). □

Lemma 3.9 (Whitney regularity of the rank-two pattern strata)**.** *The real open rank-two positroid strata satisfy the frontier condition and Whitney conditions A and B.*

Proof. Again work on full-rank frames. The closure of one pattern stratum is obtained by retaining its zero rows and allowing rank at most one on each of its disjoint parallel blocks. Indeed, this is a product of rank-at-most-one cones, zero factors, and free row factors, intersected with the open full-rank locus. A small generic perturbation in these factors makes every prescribed nonloop nonzero and removes additional cyclic dependences, proving density of the pattern stratum.

If another pattern meets this closure, its loops contain the upper loops. On each upper block its surviving rows belong to one lower parallel block, or at most one row survives. This follows because the upper block is a cyclic interval after its loops are removed, and lower loops can only delete more rows. It is a condition on the lower pattern alone. Hence the entire lower stratum lies in the closure. This proves the frontier condition. There are finitely many patterns.

Let S be an upper pattern and T a lower pattern. Suppose $p_\nu \in S$ tends to $q \in T$. We first prove Whitney A by testing limiting conormals. Upper loop rows have zero lower tangent rows. If an upper block B vanishes completely at q, all its rows are loops throughout T, so it too has zero lower tangent contribution. Otherwise fix $j \in B$ with $q_j \neq 0$, and use that row as a pivot. Write

$$(p_\nu)_i = \lambda_{\nu,i} u_\nu, \quad u_\nu = (p_\nu)_j, \quad \lambda_{\nu,j} = 1.$$

A bounded conormal has the form $(M_\nu)_i = w_{\nu,i} u_\nu^\perp$ with $\sum_i w_{\nu,i}\lambda_{\nu,i} = 0$. Because the pivot stays nonzero, all coefficient maps are smooth and the weights of bounded conormals stay bounded. Pass to limits u, λ, w. A lower tangent restricted to B has the form $\dot q_i = \dot\lambda_i u + \lambda_i \dot u$, including

zero rows at lower loops. The first term pairs to zero with $w_i u^\perp$, and the second sums to zero by $\sum_i w_i \lambda_i = 0$. Every limiting upper conormal therefore annihilates $T_q T$, which is Whitney A.

For Whitney B, take also $q_\nu \in T$ tending to the same q, with $\delta_\nu = \|q_\nu - p_\nu\| > 0$. For every unit conormal M_ν to S we claim

$$|\langle M_\nu, q_\nu - p_\nu \rangle| \le K \delta_\nu^2 \tag{3.15}$$

near q, with K independent of the conormal. Upper loop rows contribute zero. If an upper block vanishes at q, the lower frame is identically zero on it, while its conormal annihilates the radial vector of the upper block. Its contribution is again exactly zero.

On a block with the fixed nonzero pivot, write

$$(p_\nu)_i = \lambda^p_{\nu,i} u_\nu, \qquad (q_\nu)_i = \lambda^q_{\nu,i} v_\nu.$$

The lower restriction has rank at most one and the pivot is nonzero, so these expressions remain valid when some lower rows are zero. Smoothness of the normalized coefficients gives $\lambda^q_\nu - \lambda^p_\nu = O(\delta_\nu)$ and $v_\nu - u_\nu = O(\delta_\nu)$. The exact pairing is

$$\begin{aligned} &\sum_i w_{\nu,i} u_\nu^\perp \cdot (\lambda^q_{\nu,i} v_\nu - \lambda^p_{\nu,i} u_\nu) \\ &\qquad = \left(\sum_i w_{\nu,i} (\lambda^q_{\nu,i} - \lambda^p_{\nu,i}) \right) u_\nu^\perp \cdot (v_\nu - u_\nu). \end{aligned}$$

The bounded pivot and weights make this $O(\delta_\nu^2)$. Summing over the finitely many disjoint blocks proves (3.15). Dividing by δ_ν shows that every limiting secant line is annihilated by every limiting conormal, hence lies in the limiting upper tangent space. This is Whitney B.

Finally, the frame map to the Grassmannian is a smooth principal GL_2 bundle and locally a product. Whitney A, Whitney B, and the preceding frontier statements descend through such a product. The calculations hold for arbitrary nonzero real parallel-row coefficients and hence for the full real open positroid strata. □

3.5. **The proper universal family.** Let $P = \mathrm{Gr}_{>0}(k+2, n)$, and form the smooth flag incidence manifold

$$\mathcal{I} = \{(E, R) : E \in P,\ R \in \mathrm{Gr}(2, E)\}.$$

Its projection $\rho(E, R) = R$ is a submersion: the unrestricted flag projection is a Grassmann bundle, and restricting E to the positive open set preserves submersiveness. Write

$$\mathcal{I}_N = \rho^{-1}(\mathring{\Pi}_N), \qquad \mathcal{A} = \{(E, R) : E \in P,\ R \in \mathcal{A}(E)\}, \qquad \mathcal{A}_N = \mathcal{A} \cap \mathcal{I}_N.$$

Let p denote projection to P.

Lemma 3.10 (Openness in each incidence stratum)**.** *Each $\mathcal{A}_N$ is open and closed in $\mathcal{I}_N$. Its projection to P is a smooth submersion.*

Proof. The family $\mathcal{A}$ is closed relative to $\mathcal{I}$. Indeed, for a convergent sequence $(E_\nu, R_\nu) \in \mathcal{A}$, choose source lifts C_ν. Compactness of X supplies a convergent subsequence with limit $C \in X$. The equations $R_\nu \subset E_\nu$ and $C_\nu R_\nu = 0$ pass to the limit; positive projection gives exactly the limiting two-plane. This also proves closedness in each $\mathcal{I}_N$.

Fix $(E, R) \in \mathcal{A}_N$. Its pattern fixes all signs of the nonzero line ratios in a small incidence neighborhood. Use the positive parameters α from complete compression, and put $F = F(\alpha)$. Projection by Z is injective on F: extend it by any residual positive source in (3.12) and use positive projection on the resulting source. Therefore the external quotient by FZ has dimension $r + 2$ and supplies positive data $\widehat{Z}(E, \alpha)$, smoothly after choosing local quotient frames.

A target with these parameters contains FZ. It is uniquely recovered from its residual r-plane $\widehat{Y} = Y/FZ$. Thus $(E, \alpha, \widehat{Y})$ are local coordinates on $\mathcal{I}_N$ near (E, R): absence of any further zero or parallel relation is an open condition, and the inverse is the unique k-plane containing FZ with the prescribed quotient. In these coordinates the maximal source map is

$$(E, \alpha, D) \longmapsto (E, \alpha, D\widehat{Z}(E, \alpha)). \tag{3.16}$$

Its derivative in D is surjective by the tangent-map calculation in Lemma 2.1, applied to the positive $(n - a) \times (r + 2)$ matrix $\widehat{Z}$ and the positive r-plane D. For $r = 0$, both residual Grassmannians are points, and the same conclusion holds. Thus (3.16) is a submersion.

Proposition 3.7 supplies such a positive maximal-cell preimage for every point of $\mathcal{A}_N$. The submersion theorem applied to (3.16) makes its image a neighborhood in $\mathcal{I}_N$. This proves openness of $\mathcal{A}_N$. Finally, Proposition 3.8 says precisely that an arbitrary infinitesimal variation of E can be lifted inside the incidence stratum; hence $p|_{\mathcal{A}_N}$ is a submersion. The local coordinates above give the same conclusion. □

Theorem 3.11 (The $m = 2$ amplituhedron)**.** *For every strictly positive external matrix Z, $\mathcal{A}_{n,k,2}(Z)$ is homeomorphic to a closed ball of dimension $2k$. The family over positive external planes is locally topologically trivial by homeomorphisms preserving the Lam strata.*

Proof. By Lemma 3.9 and submersiveness of ρ, the $\mathcal{I}_N$ form a Whitney stratification of $\mathcal{I}$. The strata $\mathcal{A}_N$ are open and closed subsets of these strata by Lemma 3.10. Their Whitney A and B conditions therefore restrict from those of the incidence strata. Their frontier is exactly

$$\overline{\mathcal{A}_N}^{\mathcal{A}} = \coprod_{N' \leq N} \mathcal{A}_{N'}. \tag{3.17}$$

Containment from left to right follows from the ambient pattern closure. The reverse containment holds already with E fixed by (3.10). Each $\mathcal{A}_N$ is connected: by (3.13), $\mathcal{A}_N$ is the continuous image of $P \times \Pi_{N^{\uparrow k},>0}$, whose two factors are connected. Thus this is a finite Whitney stratification of the closed subset $\mathcal{A} \subset \mathcal{I}$.

The restriction $p : \mathcal{A} \to P$ is proper. For a compact $K \subset P$, the positive external planes admit continuous normalized frames: the first maximal-minor chart contains all of P. Hence $p^{-1}(K)$ is the continuous image of the compact space $K \times X$ under the twistor map. It is compact. The map p is smooth on the ambient incidence manifold, and its restriction to each $\mathcal{A}_N$ is a submersion by Lemma 3.10.

Mather's first isotopy lemma [20, Proposition 11.1] states that a smooth ambient map restricted properly to a closed Whitney-stratified set, submersively on each stratum, is locally topologically trivial by stratum-preserving homeomorphisms. Applying it to p gives the stated local triviality.

The positive Grassmannian P is connected, so all these fibers are homeomorphic. The ordinary-path spectral model of Proposition 2.4 has a closed $2k$-ball as its amplituhedron. Transporting this fiber through local trivializations proves the claim for every positive external plane, and hence for every strictly positive external matrix. □

Corollary 3.12 (The top stratum)**.** *For every positive E, the top Lam stratum is the intrinsic interior of the closed ball $\mathcal{A}(E)$ and is homeomorphic to $\mathbb{R}^{2k}$.*

Proof. For the top pattern there are no loops or nontrivial blocks, so Proposition 3.7 identifies its image with that of the positive source Grassmannian. In the spectral model of Proposition 2.4, the graph-chart construction projects this positive source locus to an open set $O \subset \mathbb{R}^{2k}$ with compact closure equal to the closed amplituhedron. It also gives a continuous flow on the target chart for which $f_t(\overline{O}) \subset O$ whenever $t > 0$. If a point of $\overline{O}$ is interior, a sufficiently small negative

flow time keeps it in $\overline{O}$, and the corresponding positive time puts it in O. Thus O is exactly the interior. This argument uses the explicit projected flow in Proposition 2.4.

The stratum-preserving trivializations of Theorem 3.11 identify each top stratum with that one. A homeomorphism of closed balls preserves their intrinsic interior, for instance by invariance of domain in local charts. The conclusion follows. For residual rank zero, used below, the corresponding assertion is the convention that the unique point is the interior of a zero-dimensional ball. □

3.6. **Open Lam strata.**

Theorem 3.13 (Open Lam strata)**.** *Let N be any feasible rank-two positroid pattern and let E be any strictly positive external plane. With a, b, r as in (3.8),*

$$\mathcal{A}_N(E) \;\cong\; \mathbb{R}^{b+2r} = \mathbb{R}^{2k-2|L|-b}.$$

In particular every open Lam stratum is an open ball of its prescribed dimension.

Proof. Fix the cut and deletion order of Lemma 3.5 once for the entire pattern. Its positive line parameters range over the whole orthant $U = (\mathbb{R}_{>0})^b$. For any $\alpha \in U$, choose a positive residual source and perform the inverse insertions. Equation (3.12) gives a source in the maximal open cell for the original pattern. Its image belongs to $\mathcal{A}_N(E)$ by Lam's image-stratum statement. Thus every α occurs for the fixed external datum.

Let $J(\alpha) : \mathbb{R}^{n-a} \to \mathbb{R}^n$ be the linear map reversing all twistor row deletions and prefix signs. At a positive-line insertion it inserts the deleted row as the prescribed negative positive multiple of its neighbor; at a loop insertion it inserts a zero row. This map is injective, depends smoothly on α, and has image $F(\alpha)^\perp$. Define

$$\widehat{E}(\alpha) = J(\alpha)^{-1}\big(E \cap F(\alpha)^\perp\big) \subset \mathbb{R}^{n-a}. \tag{3.18}$$

The restriction of the external map to $F(\alpha)$ is injective, because $F(\alpha)$ is contained in a source constructed by (3.12). It follows that $E \cap F(\alpha)^\perp$ has dimension $k + 2 - a = r + 2$. This constant-rank construction is smooth in α. The positive quotient-minor formula (3.6), applied at every step, says that

$$\widehat{E}(\alpha) \in \mathrm{Gr}_{>0}(r+2, n-a).$$

Thus (3.18) defines a global smooth map to the lower positive external Grassmannian.

If $r = 0$, the residual source Grassmannian is a point and its unique twistor plane is $\widehat{E}(\alpha)$. Compression and reinsertion therefore identify $\mathcal{A}_N(E)$ with $U = (\mathbb{R}_{>0})^b \cong \mathbb{R}^b$, which has dimension $b + 2r$. We henceforth assume $r > 0$.

Write $\mathcal{A}_{\mathrm{top}}(\widehat{E})$ for the lower top Lam stratum. Complete compression and its explicit inverse give a homeomorphism

$$\mathcal{A}_N(E) \longrightarrow \left\{(\alpha, \widehat{R}) : \alpha \in U, \widehat{R} \in \mathcal{A}_{\mathrm{top}}(\widehat{E}(\alpha))\right\}. \tag{3.19}$$

Here R recovers its line ratios intrinsically and its residual frame by row deletion; conversely $R = J(\alpha)\widehat{R}$. These operations are continuous and independent of the chosen two-column frame. The residual pattern is top precisely because compression removed all the original loops and maximal parallel blocks. Conversely a residual top point has a positive source lift by Proposition 3.7; inverse insertion puts that source in the original maximal cell, giving a point of $\mathcal{A}_N(E)$. This proves both directions and the exact image in (3.19).

It remains to globalize the varying lower fiber. Use logarithmic coordinates $u_i = \log \alpha_i$ on U, and pull back the *whole closed* lower universal family along the smooth map $u \mapsto \widehat{E}(e^u)$. Denote the pullback by $q : \mathcal{B} \to \mathbb{R}^b$. Its ambient pullback incidence space is smooth, since the ambient lower incidence projection is a submersion. It inherits the pulled-back Whitney stratification:

the graph of the parameter map is transverse to every stratum, because the lower universal projection is submersive on that stratum. Transverse intersection preserves Whitney A and B. The total pullback is closed and q is proper. Indeed, above a compact parameter set its image in the lower external base is compact, and the pullback is closed inside the product of that compact set with the compact inverse image in the lower family. Projection to $\mathbb{R}^b$ is submersive on every pulled-back stratum.

The compatible control data in the first isotopy lemma give controlled lifts V_i of the coordinate fields $\partial/\partial u_i$. Their flows Φ_i^t preserve all strata and satisfy

$$q(\Phi_i^t z) = q(z) + te_i.$$

These flows exist for all real times. Otherwise a maximal finite trajectory would lie over a compact line segment; properness confines it to a compact subset, and the continuation property of controlled flows extends it, a contradiction. This is the global Euclidean-base construction of [20, Corollaries 10.2–10.3].

Let $\mathcal{B}_0 = q^{-1}(0)$. The formulas

$$\begin{aligned} H &: \mathcal{B}_0 \times \mathbb{R}^b \longrightarrow \mathcal{B}, & H(x,u) &= \Phi_b^{u_b} \circ \cdots \circ \Phi_1^{u_1}(x), \\ H^{-1}(z) &= \big(\Phi_1^{-u_1} \circ \cdots \circ \Phi_b^{-u_b}(z), u\big), & u &= q(z), \end{aligned}$$

are continuous inverses. Each flow changes only its own base coordinate, so the second expression lands in $\mathcal{B}_0$. The reverse order makes the compositions cancel. These maps preserve every stratum.

Restricting this global product to the lower top stratum and using Corollary 3.12 identifies the right-hand side of (3.19) with

$$\mathbb{R}^b \times \mathcal{A}_{\mathrm{top}}(\widehat{E}(1,\dots,1)) \cong \mathbb{R}^b \times \mathbb{R}^{2r}.$$

For $b = 0$, the parameter space is a point and the same formula holds with the identity in place of the parameter flows. □

4. External data of corank three

Throughout this section, $k \geq 1$, $m \geq 2$ is even, $d = k + m$, and $n = d + 3$. We construct a positivity-improving operator for every positive $n \times d$ external matrix. The construction begins with its alternating dual three-plane.

4.1. **A balanced operator on a positive three-plane.**

All indices in the next two lemmas are read modulo n. The cyclic ordering of three indices has positive sign, so cyclically ordered triples of distinct rows of a positive three-plane have positive determinant.

Lemma 4.1 (Affine normalization)**.** *Let $n \geq 4$, and let $P \in \mathrm{Gr}_{>0}(3,n)$ be represented as a column space. There are a positive diagonal matrix $L = \mathrm{diag}(q_1,\dots,q_n)$ and an $n \times 3$ matrix R_0 with rows*

$$r_i^0 = (1, x_i, y_i)$$

such that every ordered three-row minor of R_0 is positive and $P = \mathrm{col}(LR_0)$. These matrices can be chosen smoothly as P varies in the positive Grassmannian.

Proof. Choose the unique frame $\widehat{R}$ of P whose first three rows form I_3. Indeed, the leading three-row minor of every positive frame is positive, so multiplication on the right by its inverse preserves

positivity and produces this frame. Its entries are smooth Grassmannian chart coordinates. Denote its rows by $\widehat{r}_i$ and define the linear functional

$$\ell(v) = \sum_{j=1}^{n} \det(\widehat{r}_j, \widehat{r}_{j+1}, v), \qquad \widehat{r}_{n+1} = \widehat{r}_1. \tag{4.1}$$

For each i, every summand of $q_i := \ell(\widehat{r}_i)$ is zero if a row is repeated, and is positive otherwise. For example, the wraparound triple $(n, 1, i)$ with $1 < i < n$ is a cyclic permutation of $(1, i, n)$, and its determinant is positive. The same reasoning applies to every edge. There is at least one edge not incident to i, so $q_i > 0$.

Write $\ell(v) = vh$ for a column vector $h \in \mathbb{R}^3$. Then $q = \widehat{R}h$, and the normalization of the first three rows gives $h = (q_1, q_2, q_3)^T$. In particular $h_1 > 0$. Let

$$B = (h, e_2, e_3), \qquad R = \widehat{R}B, \qquad L = \operatorname{diag}(q_1, \ldots, q_n), \qquad R_0 = L^{-1}R.$$

The determinant of B is $h_1 > 0$, and its first column is h. Thus R has positive ordered three-row minors and first column q. Positive row scaling by L^{-1} makes the first column the all-ones vector $\mathbf{1} = (1, \ldots, 1)^T \in \mathbb{R}^n$ and preserves positivity. This proves the assertion. All operations are smooth, and all divisors are strictly positive. □

Lemma 4.2 (A symmetric cyclic realization)**.** *Let $n \geq 4$, and let R_0 be as in Lemma 4.1. There is a real symmetric cyclic tridiagonal matrix H, with positive entries at every cyclic neighboring pair, and a positive diagonal matrix D_0 such that*

$$H(D_0^{-1}R_0) = (D_0^{-1}R_0)\Lambda, \qquad \operatorname{spec}\Lambda = \{1, 0, 0\}.$$

The eigenvalues of H are 1, two zeros, and $n-3$ strictly negative numbers. In particular, $\operatorname{col}(D_0^{-1}R_0)$ is its highest three-dimensional spectral subspace.

Proof. Set

$$p = \frac{1}{n}\sum_{j=1}^{n} r_j^0, \qquad E_i = \det(r_i^0, r_{i+1}^0, p), \qquad T_i = \det(r_{i-1}^0, r_i^0, r_{i+1}^0). \tag{4.2}$$

The first coordinate of p is one. Each T_i is a positive ordered minor after a cyclic permutation. Also

$$E_i = \frac{1}{n}\sum_{j=1}^{n} \det(r_i^0, r_{i+1}^0, r_j^0) > 0,$$

because the nonzero terms are positive and at least one term is nonzero. Define

$$a_i = \frac{E_i}{T_i}, \qquad b_i = \frac{E_{i-1}}{T_i}, \qquad c_i = 1 - a_i - b_i. \tag{4.3}$$

Cramer's rule in the basis $(r_{i-1}^0, r_i^0, r_{i+1}^0)$ gives

$$a_i r_{i-1}^0 + c_i r_i^0 + b_i r_{i+1}^0 = p. \tag{4.4}$$

Indeed, the coefficients of the first and third rows are respectively

$$\frac{\det(p, r_i^0, r_{i+1}^0)}{T_i} = \frac{E_i}{T_i}, \qquad \frac{\det(r_{i-1}^0, r_i^0, p)}{T_i} = \frac{E_{i-1}}{T_i},$$

and the first coordinate forces their sum with the middle coefficient to be one.

Let Q_0 be the cyclic tridiagonal matrix defined by

$$(Q_0)_{i,i-1} = a_i, \qquad (Q_0)_{ii} = c_i, \qquad (Q_0)_{i,i+1} = b_i, \tag{4.5}$$

with all other entries zero. The two off-diagonal coefficients are positive. Put $\Lambda = e_1 p$. Then

$$Q_0 R_0 = R_0 \Lambda. \tag{4.6}$$

Here $R_0 e_1 = \mathbf{1}$, so the right-hand side has every row equal to p. Since $p e_1 = 1$, the matrix Λ satisfies $\Lambda^2 = \Lambda$ and has rank one. Its eigenvalues are therefore $1, 0, 0$.

The required diagonal weights are explicit:

$$\delta_i^2 = \frac{E_{i-1} E_i}{T_i}, \qquad \delta_i > 0, \qquad D_0 = \operatorname{diag}(\delta_1, \dots, \delta_n). \tag{4.7}$$

For every cyclic edge, including the edge from n to 1, one has

$$\begin{aligned} b_i \delta_{i+1}^2 &= \frac{E_{i-1}}{T_i} \frac{E_i E_{i+1}}{T_{i+1}} \\ &= \frac{E_{i+1}}{T_{i+1}} \frac{E_{i-1} E_i}{T_i} = a_{i+1} \delta_i^2. \end{aligned}$$

Consequently

$$Q_0 D_0^2 = D_0^2 Q_0^T, \qquad H = D_0^{-1} Q_0 D_0 = H^T. \tag{4.8}$$

The cyclic neighboring entries of H are positive. Putting $R' = D_0^{-1} R_0$ in (4.6) gives $H R' = R' \Lambda$. All ordered three-row minors of R' remain positive.

It remains to locate this invariant three-plane in the full spectrum. The third additive compound $H^{[3]}$ is irreducible Metzler by Lemma 2.2; its corner sign is positive because $(-1)^{3-1} = 1$. If v_1, v_2, v_3 are the columns of R', their wedge has all its coordinates positive and satisfies

$$H^{[3]}(v_1 \wedge v_2 \wedge v_3) = \operatorname{tr}(\Lambda)(v_1 \wedge v_2 \wedge v_3) = v_1 \wedge v_2 \wedge v_3.$$

Perron–Frobenius applied to $e^{tH^{[3]}}$, $t > 0$, shows that 1 is the largest eigenvalue of $H^{[3]}$ and is simple. On the other hand, H is symmetric and its restriction to $\operatorname{col} R'$ already has one eigenvalue 1 and two eigenvalues 0. Any additional positive eigenvalue λ would give the compound eigenvalue $1 + \lambda + 0 > 1$. Any additional zero eigenvector would give more than one independent wedge of eigenvalue 1, by wedging an eigenvector of eigenvalue 1 with two independent zero eigenvectors. Both conclusions contradict the Perron assertion. Every remaining eigenvalue of H is therefore strictly negative. □

4.2. **The positivity-improving external flow.**

Proposition 4.3. *Let Z be a positive $n \times d$ matrix, where $d = n - 3 = k + m$ and $m \geq 2$ is even. There are a positive diagonal matrix D and a symmetric matrix S with the following properties:*

(i) *$E = \operatorname{col}(DZ)$ is the highest d-dimensional spectral subspace of S, and all the eigenvalues of $S|_E$ are positive. The remaining eigenvalues are $0, 0, -1$.*
(ii) *Every ordered k-minor and every ordered d-minor of e^{tS} is positive for $t > 0$.*
(iii) *If $\mu_1 \geq \cdots \geq \mu_n$ are the eigenvalues of S, then $\mu_k > \mu_{k+1}$ and its highest k-dimensional spectral subspace F is positive.*

The corresponding generator acting on row representatives for the original external data is $J = D^{-1} S D$, and $J \operatorname{col} Z \subset \operatorname{col} Z$.

Proof. Put $K = \ker Z^T$. Alternating orthogonal duality (2.1) makes $P = A_n K$ a positive three-plane. Apply Lemma 4.1 and write $P = \operatorname{col} R$, with $R = L R_0$. Use the matrices Q_0, D_0, H, Λ constructed in Lemma 4.2, and put

$$Q = L Q_0 L^{-1}, \qquad D = L D_0. \tag{4.9}$$

The diagonal factors commute, and hence

$$QR = R\Lambda, \qquad D^{-1} Q D = H, \qquad D^{-1} R = D_0^{-1} R_0 = R'. \tag{4.10}$$

In particular $QD^2 = D^2Q^T$. Define

$$S = -A_nHA_n, \qquad J = (-A_nQA_n)^T = D^{-1}SD. \tag{4.11}$$

The matrix S is symmetric. The transpose in the definition of J is determined by the row convention: since $K = \mathrm{col}(A_nR)$,

$$J^T(A_nR) = -A_nQA_n(A_nR) = -A_nQR = -(A_nR)\Lambda.$$

Thus $J^TK \subset K$, which implies $J(K^\perp) \subset K^\perp$, or $J \operatorname{col} Z \subset \operatorname{col} Z$.

Let $Z' = DZ$. Positive diagonal scaling preserves the positivity of every ordered maximal minor of Z. Moreover

$$\ker(Z')^T = D^{-1}K = \mathrm{col}(A_nR'). \tag{4.12}$$

The equality $HR' = R'\Lambda$ gives $S(A_nR') = -(A_nR')\Lambda$. By the complete spectral description of H in Lemma 4.2, the space in (4.12) is the whole nonpositive spectral subspace of S, with eigenvalues $-1, 0, 0$. Its orthogonal complement $E = \operatorname{col} Z'$ is the highest d-dimensional spectral subspace, with strictly positive eigenvalues. This proves (i).

For $1 \le i < n$, the neighboring entry $S_{i,i+1}$ is positive, because adjacent entries of the alternating diagonal A_n have opposite signs. The corner entry S_{1n} has sign $(-1)^n$ and positive absolute value. Since $n = k + m + 3$ and m is even,

$$(-1)^n = (-1)^{k-1} = (-1)^{d-1}. \tag{4.13}$$

The assumptions of Lemma 2.2 therefore hold for S in both ranks k and d. This proves (ii), and the spectral conclusion of that lemma in rank k proves (iii).

Finally, the positive source rescaling $C' = CD^{-1}$ preserves the image, since

$$C'Z' = CD^{-1}DZ = CZ, \qquad Ce^{tJ}D^{-1} = C'e^{tS}. \tag{4.14}$$

It bijects both the nonnegative and the positive Grassmannians to themselves. Thus the symmetric flow for Z' and the row flow for Z describe the same target dynamics in these coordinates. □

4.3. **One chart for the complete closed image.**

Proof of Theorem 1.2. Use $D, S, Z' = DZ, E, F$ from Proposition 4.3. Equation (4.14) gives $\mathcal{A}_{n,k,m}(Z) = \mathcal{A}_{n,k,m}(Z')$, with the same equality for the positive-source images.

Choose an orthonormal eigenbasis $u_1, \dots, u_n$ for S, ordered so that the corresponding eigenvalues satisfy $\mu_1 \ge \cdots \ge \mu_n$. The first k vectors span F, and the first d vectors span E. Orient $U_F = (u_1, \dots, u_k)$ so that all of its ordered k-row minors are positive. Complete it to

$$U = (U_F, U_G, U_K), \qquad U_G = (u_{k+1}, \dots, u_d), \qquad U_K = (u_{d+1}, u_{d+2}, u_n).$$

For every nonnegative source representative C, Cauchy–Binet gives

$$\det(CU_F) = \sum_{|I|=k} \Delta_I(C)\Delta_I(U_F) > 0. \tag{4.15}$$

The nonnegative minors of C are not all zero, whereas every minor of U_F is strictly positive. Thus the entire nonnegative Grassmannian lies in the graph chart over F. Its normalized representatives in this eigenbasis have the form

$$(CU_F)^{-1}CU = (I_k, X, W), \qquad X \in \mathrm{Mat}_{k\times m}(\mathbb{R}), \quad W \in \mathrm{Mat}_{k\times 3}(\mathbb{R}). \tag{4.16}$$

Write $U_E = (U_F, U_G)$. Since Z' and U_E span the same d-plane, there is an invertible $d \times d$ matrix T such that $Z' = U_ET$. Equation (4.16) yields

$$CZ' = (CU_F)(I_k, X)T. \tag{4.17}$$

In particular, this matrix has rank k at every source point. After the target coordinate change by T^{-1}, the entire image lies in the usual graph chart of $\mathrm{Gr}(k,d)$ and its graph coordinate is exactly X; the three columns of W are discarded.

Let $K_Z \subset \mathrm{Mat}_{k\times m}(\mathbb{R})$ be the complete target image in this chart, and let P_Z be the image of $\mathrm{Gr}_{>0}(k,n)$. The graph-coordinate map is continuous on the compact nonnegative Grassmannian by (4.15). Hence K_Z is compact and bounded. The strictly positive Grassmannian is open in the ambient Grassmannian. In source graph coordinates the target map is the coordinate projection $(X,W)\mapsto X$, so P_Z is open in $\mathrm{Mat}_{k\times m}(\mathbb{R})$. It contains the origin, because F is strictly positive and has source graph coordinates $(X,W)=(0,0)$.

The right action by e^{tS} gives the source graph flow

$$X_{ab} \longmapsto e^{t(\mu_{k+b}-\mu_a)}X_{ab}, \qquad 1\le a\le k, \quad 1\le b\le m.$$

Indeed, in the eigenbasis the first k columns are multiplied by $\mathrm{diag}(e^{t\mu_1},\dots,e^{t\mu_k})$; renormalizing these columns to I_k gives the displayed formula. It depends only on X and induces a well-defined flow f_t of the whole target chart. Every exponent is negative, since

$$\mu_a-\mu_{k+b}\ge \mu_k-\mu_{k+1}>0.$$

For $t>0$, every ordered k-minor of e^{tS} is positive. Another application of Cauchy–Binet shows that right multiplication by e^{tS} sends each nonnegative source plane into the positive Grassmannian. Thus

$$f_t(K_Z)\subset P_Z \qquad (t>0). \tag{4.18}$$

As $t\downarrow 0$, this also gives $K_Z=\overline{P_Z}$. Since P_Z is open, $P_Z\subset \operatorname{int} K_Z$. Conversely, if $x\in\operatorname{int} K_Z$, continuity at $t=0$ gives $f_{-t}(x)\in K_Z$ for all sufficiently small positive t. Applying (4.18) to that point gives $x\in P_Z$. Therefore

$$P_Z=\operatorname{int} K_Z. \tag{4.19}$$

The compact set K_Z contains the origin in its interior, and its positive-time image under the diagonal contraction f_t is contained in that interior. Applying Lemma 2.3 with dimension km proves that (K_Z,P_Z) is homeomorphic to $(\overline{B}^{km},B^{km})$. The target Grassmannian chart is an ambient homeomorphism, so it carries this assertion to the original amplituhedron and its ambient interior. In particular, its boundary is homeomorphic to S^{km-1}. □

4.4. **Continuous external parameters.**

Corollary 4.4. *Let $\mathcal{Z}^+_{n,d}$ be the open set of real $n\times d$ matrices with positive ordered maximal minors, where $n=d+3$ and $d=k+m$ with $k\ge 1$ and $m\ge 2$ even. Set*

$$\mathcal{A}=\{(Z,Y): Z\in\mathcal{Z}^+_{n,d}, \quad Y\in\mathcal{A}_{n,k,m}(Z)\}.$$

The projection $\mathcal{A}\to\mathcal{Z}^+_{n,d}$ is locally topologically trivial with fiber $\overline{B}^{km}$. The local trivializations simultaneously identify the positive-source images with B^{km} and the boundaries with S^{km-1}.

Proof. Fix $Z_0\in\mathcal{Z}^+_{n,d}$.

Continuous linear data. The alternating kernel $A_n\ker Z^T$ varies smoothly with Z. Its frame with first three rows I_3 is smooth: it can be found by solving the orthogonality equations against A_nZ using the invertible last $d\times d$ row block of A_nZ. The normalization in Lemma 4.1 is given by the explicit functional (4.1) and the matrix (h,e_2,e_3) with determinant $h_1>0$. Thus the matrices $L_Z, D_{0,Z}, D_Z, Q_Z, H_Z, S_Z$ constructed above vary smoothly near Z_0; their formulas only use determinants, division by positive numbers, and positive square roots.

Let F_Z and E_Z be the highest k- and d-dimensional spectral subspaces of S_Z, and put $G_Z=E_Z\cap F_Z^{\perp}$. The gaps at k and d are strict at every parameter by Proposition 4.3. On a

sufficiently small neighborhood of Z_0, the orthogonal spectral projectors onto F_Z and E_Z are smooth. For example, the formula

$$\frac{1}{2\pi \mathrm{i}} \int_\Gamma (zI - S_Z)^{-1}\, dz$$

where $\mathrm{i}^2 = -1$ and Γ is a fixed positively oriented closed contour in the complex plane enclosing the selected spectral cluster and excluding the remaining eigenvalues, gives the projector and its smooth dependence. Projecting reference bases at Z_0 and applying Gram–Schmidt gives continuous orthonormal frames $U_{F,Z}, U_{G,Z}, U_{K,Z}$ of $F_Z, G_Z, E_Z^\perp$, respectively. Orient $U_{F,Z}$ positively at Z_0; this orientation remains positive after shrinking the neighborhood. Spectral projectors remain smooth across eigenvalue crossings within the selected clusters.

In these frames let

$$A_Z = U_{F,Z}^T S_Z U_{F,Z}, \qquad B_Z = U_{G,Z}^T S_Z U_{G,Z}.$$

Both matrices are symmetric. The induced target graph flow is now

$$f_Z(t)X = e^{-tA_Z} X e^{tB_Z}, \qquad \mathcal{L}_Z(X) = -A_Z X + X B_Z. \tag{4.20}$$

The generator $\mathcal{L}_Z$ is self-adjoint for the Frobenius inner product. Take a small compact closed Euclidean ball $\mathcal{B}$ around Z_0 contained in the parameter neighborhood. Write $\lambda_{\min}$ and $\lambda_{\max}$ for the smallest and largest eigenvalues. Continuity of the eigenvalues and the strict gap give a number $\gamma > 0$ such that

$$\lambda_{\min}(A_Z) - \lambda_{\max}(B_Z) \geq \gamma \qquad (Z \in \mathcal{B}). \tag{4.21}$$

It follows that

$$\langle X, \mathcal{L}_Z(X)\rangle \leq -\gamma \|X\|^2, \qquad \|f_Z(t)X\| \leq e^{-\gamma t}\|X\| \quad (t \geq 0), \tag{4.22}$$

and backward flow satisfies $\|f_Z(-t)X\| \geq e^{\gamma t}\|X\|$. All norms here are Frobenius norms.

Closed and open total images. For the fixed compact source $\mathrm{Gr}_{\geq 0}(k,n)$, define

$$\Phi(Z, C) = (CU_{F,Z})^{-1} C U_{G,Z}. \tag{4.23}$$

This expression is independent of the matrix representative of C and is continuous. The denominator is invertible at every nonnegative source point by Cauchy–Binet, exactly as in (4.15). Let $K_Z = \Phi(Z, \mathrm{Gr}_{\geq 0}(k,n))$ and $P_Z = \Phi(Z, \mathrm{Gr}_{>0}(k,n))$. Over $\mathcal{B}$, the total graph image

$$\mathcal{K}_\mathcal{B} = \{(Z, X) : Z \in \mathcal{B},\ X \in K_Z\}$$

is the continuous image of a compact product and hence is compact and closed. In particular, the sets K_Z have a common bound $\|X\| \leq M$.

The total strict image

$$\mathcal{P}_\mathcal{B} = \{(Z, X) : Z \in \mathcal{B},\ X \in P_Z\}$$

is relatively open in $\mathcal{B} \times \mathrm{Mat}_{k\times m}(\mathbb{R})$. To see this, concatenate the continuous orthonormal frames into $U_Z = (U_{F,Z}, U_{G,Z}, U_{K,Z})$. The condition

$$\mathrm{row}\big((I_k, X, W)U_Z^T\big) \in \mathrm{Gr}_{>0}(k,n)$$

defines an open subset of the product of parameter space and the two matrix spaces for (X, W). Its projection $(Z, X, W) \mapsto (Z, X)$ is open, and is exactly the total strict image. Every $(Z, 0)$ belongs to this image because F_Z is positive. Compactness of $\mathcal{B}$ therefore gives a common $\varepsilon > 0$ such that the closed ball of radius 2ε lies in every P_Z. Enlarge M if necessary so that $M > 2\varepsilon$.

For each parameter, the previous proof gives

$$P_Z = \mathrm{int}\, K_Z, \qquad f_Z(t)K_Z \subset P_Z \quad (t > 0). \tag{4.24}$$

These assertions are unchanged by the orthogonal block frames used in (4.20).

A jointly continuous boundary time. Let Σ_ε be the sphere of radius ε in $\mathrm{Mat}_{k\times m}(\mathbb{R})$. For $u \in \Sigma_\varepsilon$, put

$$\tau(Z,u) = \sup\{t \geq 0 : f_Z(-t)u \in K_Z\}. \tag{4.25}$$

The allowed times form a closed interval $[0, \tau(Z,u)]$. Indeed, if a time is allowed, every smaller nonnegative time is allowed by forward invariance; the endpoint is included by closedness. The common interior ball and the uniform backward growth give

$$0 < \tau(Z,u) \leq \gamma^{-1}\log(M/\varepsilon). \tag{4.26}$$

Every earlier point of the backward orbit is interior by (4.24); the endpoint is on the boundary, since otherwise the backward orbit could be continued inside K_Z.

The function τ is jointly continuous. Fix (Z,u) and take $0 < \eta < \tau(Z,u)$. The point

$$(Z, f_Z(-\tau(Z,u)+\eta)u)$$

lies in the open set $\mathcal{P}_\mathcal{B}$, whereas

$$(Z, f_Z(-\tau(Z,u)-\eta)u)$$

lies outside the closed set $\mathcal{K}_\mathcal{B}$. By continuous dependence of the flow, these properties persist for nearby (Z',u'), with the two fixed times $\tau(Z,u)-\eta$ and $\tau(Z,u)+\eta$. The interval property then places $\tau(Z',u')$ between these two numbers. This proves joint continuity, including at the boundary of $\mathcal{B}$ in the relative topology.

The local product. Write a nonzero point of the closed unit ball in $\mathbb{R}^{km} = \mathrm{Mat}_{k\times m}(\mathbb{R})$ as $r\omega$, where $0 < r \leq 1$ and $\|\omega\| = 1$. Define

$$h_Z(r\omega) = f_Z(-\tau(Z,\varepsilon\omega) - \log r)(\varepsilon\omega), \qquad h_Z(0) = 0. \tag{4.27}$$

This is continuous for $r > 0$. If T_* is the upper bound in (4.26), then for $r \leq e^{-T_*}$ the time in (4.27) is nonnegative, and

$$\|h_Z(r\omega)\| \leq \varepsilon e^{\gamma T_*} r^\gamma.$$

Hence continuity at zero holds uniformly in Z and ω.

Every nonzero orbit meets Σ_ε exactly once, because its norm strictly decreases from infinity to zero with forward time, by (4.22) and its backward version. It also meets the boundary of K_Z exactly once, at the time in (4.25). Thus h_Z is a bijection from $\overline{B}^{km}$ to K_Z, taking the unit sphere onto ∂K_Z and the open ball onto P_Z. It follows that

$$(Z,v) \longmapsto (Z, h_Z(v))$$

is a continuous bijection from the compact space $\mathcal{B} \times \overline{B}^{km}$ onto the Hausdorff space $\mathcal{K}_\mathcal{B}$. It is therefore a homeomorphism.

To return to the original target Grassmannian, set

$$U_{E,Z} = (U_{F,Z}, U_{G,Z}), \qquad T_Z = U_{E,Z}^T D_Z Z.$$

The matrix T_Z is continuous and invertible, and $D_Z Z = U_{E,Z} T_Z$. The chart

$$X \longmapsto \mathrm{row}\big((I_k, X)T_Z\big)$$

therefore identifies K_Z with $\mathcal{A}_{n,k,m}(Z)$ continuously in Z, with continuous inverse on the total graph family. Composing this chart with the preceding product homeomorphism and restricting to the interior of $\mathcal{B}$ proves the corollary. □

5. Convex sections and the topology for $m = 3, 4$

Throughout this section, $k \geq 1$, $n \geq k + m$, and $Z \in \mathbb{R}^{n\times(k+m)}$ has all its ordered maximal minors strictly positive. We write

$$X = \mathrm{Gr}_{\geq 0}(k, n), \qquad \mathcal{A}(Z) = \mathcal{A}_{n,k,m}(Z) = \{\mathrm{row}(CZ) : C \in X\}.$$

5.1. **A common affine chart and strict external expansion.** The following deformation varies the external matrix in a fixed target vector space.

Lemma 5.1 (Common affine chart)**.** *There is a linear functional ℓ on $\bigwedge^k \mathbb{R}^{k+m}$ which is strictly positive on the naturally oriented Plücker rays of $\mathcal{A}(Z)$. Consequently $\mathcal{A}(Z)$ is a compact subset of the affine Plücker chart $\ell = 1$. The same functional works, with uniform boundedness in this chart, for all sufficiently small perturbations of Z.*

Proof. The positive subflag construction in Section 2, applied to $\mathrm{col}(Z)$, provides an $n \times k$ matrix Q with positive ordered maximal minors and $\mathrm{col}(Q) \subseteq \mathrm{col}(Z)$. Write $Q = ZR$, where R has k columns. Define ℓ by

$$\ell(y_1 \wedge \cdots \wedge y_k) = \det \begin{pmatrix} y_1 R \\ \vdots \\ y_k R \end{pmatrix}.$$

For a source representative whose ordered maximal minors are nonnegative, the Cauchy–Binet formula gives

$$\det(CQ) = \sum_{\substack{I\subseteq[n]\\|I|=k}} \Delta_I(C)\Delta_I(Q) > 0. \tag{5.1}$$

At least one $\Delta_I(C)$ is positive. In particular, CZ has rank k, and its Plücker ray has positive ℓ-coordinate.

For uniformity, normalize the source Plücker vector $p(C)$ by $\sum_I p_I(C) = 1$. If Z' is sufficiently close to Z, all $\Delta_I(Z'R)$ remain positive, and on a sufficiently small compact parameter neighborhood they have a common positive lower bound a. The normalized target vector is

$$\frac{(\bigwedge^k Z')p(C)}{\sum_I p_I(C)\Delta_I(Z'R)}.$$

Its denominator is at least a, and its numerator is uniformly bounded because the source and the parameter neighborhood are compact. This proves the asserted compactness and uniform boundedness. Here $\bigwedge^k Z'$ denotes the exterior map induced by right multiplication by Z'. □

Lemma 5.2 (Strict external expansion)**.** *With H_n as in (2.5), put*

$$P_t = e^{tH_n}, \qquad Z_t = e^{-tH_n} Z.$$

For some $\epsilon > 0$, all Z_t, $|t| \leq \epsilon$, are strictly positive external matrices. Put $\mathcal{A}_t = \mathcal{A}(Z_t)$. In the fixed target Grassmannian,

$$\mathcal{A}_s \subseteq \mathrm{int}(\mathcal{A}_t) \qquad (-\epsilon \leq s < t \leq \epsilon). \tag{5.2}$$

The sets $\mathcal{A}_t$ vary continuously in Hausdorff distance, are uniformly bounded in one affine Plücker chart, and satisfy $\mathcal{A}_t = \overline{\mathrm{int}(\mathcal{A}_t)}$.

Proof. Lemma 2.2 shows that every minor of P_t is strictly positive for $t > 0$; its determinant is $e^{t\,\mathrm{tr}\,H_n} = 1$.

The required positivity of Z_t for small $|t|$ follows by continuity of its finitely many maximal minors. If $s < t$, then

$$CZ_s = (CP_{t-s})Z_t.$$

For every k-subset I, Cauchy–Binet gives

$$\Delta_I(CP_{t-s}) = \sum_{\substack{K\subseteq[n]\\|K|=k}} \Delta_K(C)\det(P_{t-s})_{K,I} > 0.$$

Hence every point of $\mathcal{A}_s$ has a strictly positive source representative for the external matrix Z_t.

Lemma 2.1 makes this positive-source image open in the target Grassmannian. This proves (5.2).

The same heat action shows that the positive source is dense in X. Consequently the image of the positive source is dense in $\mathcal{A}_t$. Since that image is contained in the interior, the final density assertion follows. The normalized target formula in the proof of Lemma 5.1 is jointly continuous in the compact source and the parameter t. Uniform continuity gives Hausdorff continuity. Finally, $Z_tR = e^{-tH_n}Q$ remains positive for small $|t|$, so Lemma 5.1 supplies the uniform chart and the uniform boundedness. □

5.2. **Positive lifts over the interior.** Write $\pi_Z(C) = \mathrm{row}(CZ)$.

Proposition 5.3. *For every strictly positive $n \times (k+4)$ matrix Z,*

$$\pi_Z(\mathrm{Gr}_{>0}(k,n)) = \operatorname{int}\mathcal{A}_{n,k,4}(Z).$$

Every point of this interior has a neighborhood on which π_Z has a smooth local section with values in $\mathrm{Gr}_{>0}(k,n)$.

Proof. The forward inclusion follows from positive projection and openness of the Grassmannian submersion in Lemma 2.1. For the reverse inclusion, [9, Lemma 9.3] provides a subset $M \subset \mathrm{Gr}_{\geq 0}(k,n)$, with its Grassmannian subspace topology, on which π_Z is a homeomorphism onto the entire interior. Its inverse is a continuous section

$$\sigma : \operatorname{int}\mathcal{A}_{n,k,4}(Z) \longrightarrow \mathrm{Gr}_{\geq 0}(k,n).$$

Fix an interior target Y_0. Choose a smooth target chart placing Y_0 at 0 and containing a closed Euclidean ball $D = \{y : \|y\| \leq r\}$ in the interior. Let $g_t(C) = Ce^{tH_n}$, with H_n as in (2.5). For $t > 0$, the source $g_t(C)$ is strictly positive for every nonnegative C. The maps

$$F_t(y) = \pi_Z(g_t(\sigma(y))), \qquad y \in D,$$

converge uniformly to the identity on D as $t \downarrow 0$: the section has compact image on D, and the heat and projection maps are jointly continuous there. For sufficiently small $t > 0$, every $F_t(y)$ stays in the chosen target chart and

$$\sup_{y\in D}\|F_t(y) - y\| < r/2.$$

The continuous map $y \mapsto y - F_t(y)$ consequently maps D into itself. By the Brouwer fixed-point theorem, it has a fixed point y_t, and the fixed-point equation is $F_t(y_t) = 0$. Therefore $g_t(\sigma(y_t))$ is a strictly positive source lifting Y_0.

At this strictly positive source, π_Z is a smooth submersion on an open source neighborhood. The submersion coordinate theorem gives a smooth local section. Shrinking its neighborhood keeps its values strictly positive and proves the last assertion. □

5.3. **Complete projective linear sections for $m = 4$.** We use two published facts for $m = 4$. Even-Zohar, Lakrec, and Tessler prove that the topological boundary is exactly the union of the physical four-bracket zero sets [9, Corollary 8.8], and that

$$\operatorname{int}\mathcal{A}(Z) \cong \mathbb{R}^{4k} \tag{5.3}$$

for every strictly positive Z [9, Theorem 9.2].

Here are the signs of the physical forms used below. Let $\mathcal{S}$ consist of the four-element subsets of $[n]$ which are unions of two disjoint adjacent pairs in the cyclic order. For $S = \{s_1 < s_2 < s_3 < s_4\} \in \mathcal{S}$, set

$$\varepsilon_S = \begin{cases} 1, & \text{if the adjacent pairs do not use } \{n,1\}, \\ (-1)^k, & \text{if one pair is } \{n,1\}. \end{cases}$$

Define a linear functional on target k-vectors by

$$\beta_{S,Z}(y_1 \wedge \cdots \wedge y_k) = \varepsilon_S \det \begin{pmatrix} y_1 \\ \vdots \\ y_k \\ Z_{s_1} \\ Z_{s_2} \\ Z_{s_3} \\ Z_{s_4} \end{pmatrix}.$$

This definition uses the rows Z_i of Z. The sign is independent of the complementary k-subset: for $i \notin S$, the number of elements of S less than i is even in the nonwrapping case and odd in the wrapping case. Thus the shuffle which orders (I, S) has sign 1 or $(-1)^k$, respectively. Exterior multilinearity consequently gives the exact identity

$$\beta_{S,Z}(CZ) = \sum_{\substack{I \subseteq [n] \setminus S \\ |I| = k}} \Delta_I(C) \Delta_{I \cup S}(Z). \tag{5.4}$$

On normalized target vectors both sides are divided by the same positive chart normalization. In particular these forms are nonnegative on $\mathcal{A}(Z)$. The boundary theorem states

$$\partial \mathcal{A}(Z) = \bigcup_{S \in \mathcal{S}} (\mathcal{A}(Z) \cap \{\beta_{S,Z} = 0\}). \tag{5.5}$$

It follows that all the forms are strictly positive at an interior point of $\mathcal{A}(Z)$.

Theorem 5.4 (Complete Grassmannian linear sections)**.** *Let $Z \in \mathbb{R}^{n \times (k+4)}$ have positive ordered maximal minors. Let $\mathbb{P}(V)$ be a real projective linear subspace contained in the Plücker-embedded* $\mathrm{Gr}(k, k+4)$*. If*

$$S_0 = \mathcal{A}(Z) \cap \mathbb{P}(V)$$

is nonempty, then S_0 is a compact convex polytope in the affine Plücker chart of Lemma 5.1. The assertion allows $S_0 \subseteq \partial \mathcal{A}(Z)$ and $\dim S_0 < \dim \mathbb{P}(V)$.

Proof. Use the family $\mathcal{A}_t$ of Lemma 5.2. The affine part

$$L = \{v \in V : \ell(v) = 1\}$$

is an affine space of dimension $d = \dim \mathbb{P}(V)$, because $\mathbb{P}(V)$ meets $\mathcal{A}_0$ and hence is not contained in $\ell = 0$. Every point of L is a Grassmannian point under the Plücker identification. Put $S_t = \mathcal{A}_t \cap L$.

Choose $x_* \in S_0$. For every sufficiently small $t > 0$, strict nesting gives $x_* \in \mathrm{int}(\mathcal{A}_t)$ and $S_0 \subseteq S_t$. Restrict the physical forms β_{S,Z_t} to L; they are affine linear. We claim that

$$S_t = \{x \in L : \beta_{S,Z_t}(x) \geq 0 \text{ for every } S \in \mathcal{S}\}. \tag{5.6}$$

The forward containment follows from (5.4). For the reverse containment, let x satisfy all the displayed inequalities and consider

$$x(a) = (1-a)x_* + ax, \qquad 0 \leq a \leq 1.$$

For $a < 1$ all physical forms are strictly positive at $x(a)$. The whole segment lies in the Grassmannian, because L is contained in its Plücker image. The set

$$T = \{a \in [0,1) : x(a) \in \mathcal{A}_t\}$$

contains 0 and is closed in $[0,1)$. By (5.5), every feasible $x(a)$, $a < 1$, is an interior point of $\mathcal{A}_t$; hence T is also open in $[0,1)$. Connectedness yields $T = [0,1)$, and closedness of $\mathcal{A}_t$ yields $x = x(1) \in \mathcal{A}_t$. This proves (5.6).

The point x_* is an ambient interior point, so S_t is full-dimensional in L. The uniform chart bound shows that it is bounded. Thus S_t is a d-dimensional polytope defined by at most $N = |\mathcal{S}|$ affine inequalities.

Exact containment $S_0 \subseteq S_t$ supplies the lower Hausdorff convergence. For the upper convergence, suppose $t_j \to 0^+$ and $x_j \in S_{t_j}$. Uniform boundedness supplies a convergent subsequence. Its limit lies in L, and Hausdorff continuity of $\mathcal{A}_t$ places that limit in $\mathcal{A}_0$. Therefore every such limit lies in S_0, which proves

$$S_t \longrightarrow S_0 \quad \text{in Hausdorff distance.}$$

If $d = 0$, the assertion is immediate. Assume $d > 0$. A feasible point is a vertex exactly when its active constraint normals span the dual of the translation space of L. Indeed, spanning normals isolate the point, whereas otherwise a nonzero direction annihilating every active normal gives a feasible segment through it. For such a point x, choose that direction w. Boundedness makes its maximal feasible segment have endpoints $x + t_- w$ and $x + t_+ w$, with $t_- < 0 < t_+$, and

$$x = \frac{t_+}{t_+ - t_-}(x + t_- w) + \frac{-t_-}{t_+ - t_-}(x + t_+ w).$$

Each endpoint gains an active normal nonzero on w, hence independent of the previously active normals. Iterating increases the active-normal rank at every step and stops at vertices after at most d steps. Thus every feasible point is a convex combination of vertices. Each vertex has d linearly independent active constraint normals. For each chosen d-subset of the N constraints, their affine hyperplanes have at most one common point when their normals are independent. Hence every S_t has at most $\binom{N}{d}$ vertices. List its vertices and pad the list by repetitions to this common length. Along a subsequence $t_j \to 0$, uniform boundedness makes every listed vertex converge, say $v_{a,j} \to v_a$. For convex combinations one has

$$\left\| \sum_a \lambda_a v_{a,j} - \sum_a \lambda_a v_a \right\| \le \max_a \|v_{a,j} - v_a\|, \qquad \lambda_a \ge 0, \quad \sum_a \lambda_a = 1.$$

The same inequality in the other direction proves Hausdorff convergence of the convex hulls. Since $S_{t_j} = \operatorname{conv}\{v_{a,j}\}$, we obtain

$$S_0 = \operatorname{conv}\{v_a\}.$$

This proves the result even if the dimension drops. □

Corollary 5.5. *Every nonempty section in Theorem 5.4 is homeomorphic to a closed ball of its affine dimension.*

Proof. Translate a relative interior point of the polytope to the origin in its affine span. If that span has positive dimension, write the polytope as $\{x : a_j \cdot x \le b_j\}$, with every $b_j > 0$. For a unit vector u, the radial endpoint is

$$R(u)u, \qquad R(u) = \left(\max_j \frac{\max\{0, a_j \cdot u\}}{b_j} \right)^{-1}.$$

Boundedness and the interior-point condition make R positive, continuous, and bounded on the unit sphere. The map $ru \mapsto rR(u)u$, $0 < r \le 1$, extended by zero at the origin, is continuous there by boundedness of R. Every ray meets the polytope in exactly the interval $\{tu : 0 \le t \le R(u)\}$, so the map is bijective. Compactness makes it a homeomorphism. A zero-dimensional section is a point. □

5.4. **Boundary points on pencils meeting the interior.**

Theorem 5.6 (A strict-completion criterion)**.** *Let Z be strictly positive of size $n \times (k+4)$, and let $Y_0 \in \partial\mathcal{A}(Z)$. Suppose that there are a $(k-1)$-plane B and an interior point $Y_* \in \operatorname{int}\mathcal{A}(Z)$ such that*

$$B \subset Y_0 \cap Y_*.$$

Then the germ of $\mathcal{A}(Z)$ at Y_0 is bi-Lipschitz homeomorphic to the germ of $\mathbb{R}^{4k-1} \times [0,\infty)$ at its boundary. A sufficient source condition is that $B = DZ \subset Y_0$, where D is a $(k-1)$-plane contained in some strictly positive k-plane. In particular this holds if D is strictly positive.

Proof. Use the common chart from Lemma 5.1, and write its positive linear functional as $\ell(Y) = \det(YP)$, with ZP a positive $n \times k$ matrix. The map P is injective on Y_*, hence on B. Fix an oriented frame b of B. The space of k-planes containing B is the projective linear space $\mathbb{P}(\mathbb{R}^{k+4}/B) \cong \mathbb{P}^4$ inside the Plücker Grassmannian, by the linear map $v \mapsto b_1 \wedge \cdots \wedge b_{k-1} \wedge v$. Its part in our chart is the affine four-space

$$\det \begin{pmatrix} bP \\ vP \end{pmatrix} = 1, \qquad v \in \mathbb{R}^{k+4}/B.$$

Write $h_S = \beta_{S,Z}/\ell$ for the normalized physical form on the target chart. Its restriction to this affine four-space is affine linear. Theorem 5.4 gives its intersection with $\mathcal{A}(Z)$ as a compact convex polytope. Since it contains Y_*, it has nonempty relative interior. Let v_0, v_* be the normalized representatives of Y_0, Y_*. The segment $(1-t)v_0 + tv_*$ lies in $\mathcal{A}(Z)$ for $0 \le t \le 1$. Call S active if $h_S(Y_0) = 0$. For each active S, the derivative along this segment is

$$\frac{d}{dt} h_S((1-t)v_0 + tv_*)\Big|_{t=0} = h_S(v_*) > 0. \tag{5.7}$$

This produces one tangent vector to the ambient Grassmannian which strictly increases all the active physical forms.

Choose smooth ambient coordinates $(z,t) \in \mathbb{R}^{4k-1} \times \mathbb{R}$ centered at Y_0, with ∂_t equal to the tangent vector in (5.7). After shrinking the chart, all inactive forms remain positive, and the implicit-function theorem writes every active zero set as $t = g_S(z)$, with $g_S(0) = 0$. The derivative with respect to t remains positive, so its weak positive side is exactly $t \ge g_S(z)$. Let $g(z) = \max_S g_S(z)$, the maximum over the active forms.

Choose a small convex ball D about zero in the z coordinates and $\delta > 0$ so that $|g_S(z)| < \delta/2$ on D for every active S. Inside $D \times (-\delta, \delta)$, all physical forms are positive precisely on

$$W = \{(z,t) : z \in D, \quad g(z) < t < \delta\}.$$

This set is path connected: move vertically to $t = 3\delta/4$, move horizontally there, and move vertically down. It contains points of $\mathcal{A}(Z)$ arbitrarily close to the origin, provided by the strict part of the segment from v_0 to v_*. The intersection $\mathcal{A}(Z) \cap W$ is closed in W because $\mathcal{A}(Z)$ is closed, and is open in W because a point of $\mathcal{A}(Z)$ with no vanishing physical form is interior by (5.5). Hence $\mathcal{A}(Z) \cap W = W$. Taking relative closures gives

$$\mathcal{A}(Z) \cap (D \times (-\delta, \delta)) = \{(z,t) : z \in D,\ -\delta < t < \delta,\ t \ge g(z)\}.$$

The reverse containment in this equality also uses that every point of $\mathcal{A}(Z)$ has nonnegative physical forms. Each g_S is smooth with bounded derivative on a smaller chart, so their finite maximum g is Lipschitz there. The map

$$(z,t) \longmapsto (z, t-g(z)), \qquad (z,s) \longmapsto (z, s+g(z))$$

is a pair of inverse bi-Lipschitz maps and identifies the displayed germ with a half-space germ.

For the source criterion, choose a positive k-plane C_* containing D. Positive projection is injective on C_*, so $Y_* = C_*Z$ contains the $(k-1)$-plane $DZ = B$. Lemma 2.1 puts this target in the ambient interior of $\mathcal{A}(Z)$. If D itself is positive, the positive-flag construction in Section 2 supplies such a C_*. For $k = 1$, every positive line contains the zero plane. □

Proposition 5.7 (A closed-ball bundle of strict pencils)**.** *Keep the positive chart $\ell(Y) = \det(YP)$ above. Choose a nonzero linear functional $\phi(v) = vPu$ with $u \in \mathbb{R}^k \setminus \{0\}$, and put*

$$\beta(Y) = Y \cap \ker \phi \qquad (Y \in \mathcal{A}(Z)).$$

This is defined on all of $\mathcal{A}(Z)$. Let Ω be the set of $(k-1)$-planes $B \subset \ker\phi$ which are contained in an interior target. Then Ω is open in $\mathrm{Gr}(k-1, \ker\phi)$, and

$$\beta : \beta^{-1}(\Omega) \longrightarrow \Omega$$

is a locally trivial bundle of closed four-balls. The trivializations identify the relative interiors of the fibers with the open four-ball and the relative boundaries with its boundary sphere.

Proof. Since $P|_Y$ is an isomorphism for every $Y \in \mathcal{A}(Z)$ and $u \neq 0$, the restriction of ϕ to Y is nonzero. Thus $\beta(Y)$ has dimension $k-1$ everywhere, and the Grassmannian intersection map is continuous. A plane $Y \in \mathcal{A}(Z)$ contains a specified $B \subset \ker\phi$ if and only if $\beta(Y) = B$.

Fix $B_0 \in \Omega$ and choose an interior completion $Y_{*,0}$. Smooth local frames $b(B)$ of planes near B_0, together with one fixed vector v_* complementary to B_0 in $Y_{*,0}$, give k-planes $Y_*(B) = \mathrm{span}(b(B), v_*)$ near $Y_{*,0}$. They remain interior points after shrinking the base neighborhood. This proves openness of Ω and gives a smooth strict section.

The local coordinates can be made explicit. Change the basis of $\mathbb{R}^k$ so that $u = e_k$, and split the target as $\mathbb{R}^k \oplus \ker P$. In the global graph chart write $Y = \mathrm{row}(I_k, X)$ with $X \in \mathrm{Mat}_{k\times 4}(\mathbb{R})$. Then $\beta(Y)$ is spanned by the first $k-1$ rows, so its parameter is the first $k-1$ rows of X; the last row of X is the affine four-dimensional fiber coordinate. Compactness of $\mathcal{A}(Z)$ bounds the whole matrix X uniformly. Translate the smooth strict section to the origin in this last row. By the proof of (5.6), the entire fiber is

$$K_B = \{x \in \mathbb{R}^4 : c_S(B) + a_S(B)\cdot x \geq 0 \text{ for all } S \in \mathcal{S}\}, \qquad c_S(B) > 0.$$

All coefficients vary continuously. On a small compact base neighborhood, strictness gives a uniform inner ball. Compactness of $\mathcal{A}(Z)$ in the fixed positive chart, and boundedness of the chosen local affine coordinate changes, give a uniform outer ball. In particular every K_B is bounded and full-dimensional.

For a unit vector $\theta \in S^3$, define

$$\rho(B,\theta) = \max_{S\in\mathcal{S}} \frac{\max\{0, -a_S(B)\cdot\theta\}}{c_S(B)}, \qquad R(B,\theta) = \rho(B,\theta)^{-1}.$$

Boundedness implies $\rho(B,\theta) > 0$, and the uniform inner and outer balls bound it above and away from zero. The two functions are therefore jointly continuous. Substituting $x = t\theta$ into all inequalities shows exactly that $t\theta \in K_B$ if and only if $0 \leq t \leq R(B,\theta)$. Consequently

$$(B, r\theta) \longmapsto (B, rR(B,\theta)\theta), \qquad 0 \leq r \leq 1,$$

with the origin mapped to the strict section, is a continuous family of homeomorphisms from the closed four-ball onto K_B. Its inverse is given by the same radial formula with R^{-1}. The uniform bounds prove continuity at the centers. This is the claimed local trivialization, with the asserted boundary behavior. □

Remark 5.8. Given a hyperplane $B \subset Y_0$ in Theorem 5.6, one can choose $u \neq 0$ annihilating the $(k-1)$-plane BP. Then $B = Y_0 \cap \ker \phi$, so the bundle proposition applies in target coordinates near Y_0. Its base has dimension $4(k-1)$ and its fiber has dimension four.

5.5. **Contractibility and the boundary homotopy type for $m = 4$.**

Theorem 5.9. *For every strictly positive $Z \in \mathbb{R}^{n\times(k+4)}$, $\mathcal{A}(Z)$ is contractible and is a cellular compact subset of* $\mathrm{Gr}(k, k+4)$. *Its topological boundary satisfies*

$$\partial\mathcal{A}(Z) \simeq S^{4k-1}.$$

Here cellular means that $\mathcal{A}(Z)$ is the intersection of a nested sequence of embedded closed $4k$-balls, each lying in the interior of its predecessor.

Proof. Put $D = 4k$. The interior of every sufficiently small $\mathcal{A}_t$ is homeomorphic to $\mathbb{R}^D$, by (5.3). We use the standard facts that compact semialgebraic sets have finite triangulations, compact semialgebraic pairs admit compatible triangulations, and such sets are Euclidean neighborhood retracts. The triangulation statement is recalled in [8, p. 52], and the required neighborhood retractions are provided by [8, Theorem 2.1]. These facts apply here because the amplituhedron is the image of a compact semialgebraic set under the everywhere-defined Grassmann map. Its boundary is semialgebraic as well, for example by (5.5).

Cellularity. Let N be an arbitrary open neighborhood of $\mathcal{A}_0$ in the target Grassmannian. Hausdorff continuity gives $t > 0$ with $\mathcal{A}_t \subseteq N$, while strict nesting gives $\mathcal{A}_0 \subseteq \mathrm{int}(\mathcal{A}_t)$. Choose a homeomorphism $h_t : \mathbb{R}^D \to \mathrm{int}(\mathcal{A}_t)$. The compact set $h_t^{-1}(\mathcal{A}_0)$ lies in an open Euclidean ball. The image K of the corresponding larger closed ball is an embedded closed D-ball with

$$\mathcal{A}_0 \subseteq \mathrm{int}(K) \subseteq K \subseteq N.$$

The interior here is the ambient interior, since h_t maps onto an open subset of the target manifold. Starting with one such ball, repeat the construction inside its interior and inside the open metric 2^{-j}-neighborhood of $\mathcal{A}_0$ at step j. The resulting nested balls have intersection exactly $\mathcal{A}_0$.

Contractibility. Choose a neighborhood retraction $r : N \to \mathcal{A}_0$. For sufficiently small $t > 0$,

$$\mathcal{A}_0 \subseteq \mathrm{int}(\mathcal{A}_t) \subseteq N.$$

A contraction of $\mathrm{int}(\mathcal{A}_t) \cong \mathbb{R}^D$, restricted to $\mathcal{A}_0$ and followed by r, is a contraction of $\mathcal{A}_0$. Its initial map is $r|_{\mathcal{A}_0} = \mathrm{id}_{\mathcal{A}_0}$, and its final map is constant.

Boundary homology. Write $U = \mathrm{int}(\mathcal{A}_0)$ and $B = \mathcal{A}_0 \setminus U$. The set U is dense in $\mathcal{A}_0$, by Lemma 5.2, and B is nonempty: otherwise the compact space $\mathcal{A}_0 = U$ would be homeomorphic to the noncompact space $\mathbb{R}^D$. The quotient $\mathcal{A}_0/B$ is the one-point compactification of U. Indeed, the complement of a neighborhood of the collapsed point is exactly a closed compact subset of U, and conversely every compact subset of U is closed in $\mathcal{A}_0$. Consequently $\mathcal{A}_0/B \cong S^D$. Compatible triangulation makes $B \subseteq \mathcal{A}_0$ a cofibration. Thus

$$H_i(\mathcal{A}_0, B; \mathbb{Z}) \cong \widetilde{H}_i(\mathcal{A}_0/B; \mathbb{Z}).$$

The long exact sequence of the pair and contractibility of $\mathcal{A}_0$ now give

$$\widetilde{H}_i(B; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & i = D-1, \\ 0, & i \neq D-1. \end{cases} \tag{5.8}$$

In particular B is connected, and is path connected because it is a finite-polyhedron space.

Simple connectivity. Choose an ambient neighborhood N_B of B with a retraction $r_B : N_B \to B$. Fix a homeomorphism $h : \mathbb{R}^D \to U$. The compact set $\mathcal{A}_0 \setminus N_B$ is contained in U, so for some $R > 0$ it is contained in $h(B_R)$, where B_R denotes the open Euclidean ball of radius R. The two sets

$$O_1 = h(B_{R+1}), \qquad O_2 = \mathcal{A}_0 \setminus h(\overline{B}_R)$$

are open in $\mathcal{A}_0$ and cover $\mathcal{A}_0$. The first is contractible, and their intersection is

$$h(B_{R+1} \setminus \overline{B}_R) \cong S^{D-1} \times (R, R+1).$$

This intersection is path connected and simply connected, since $D = 4k \geq 4$. The connected set $h(\mathbb{R}^D \setminus \overline{B}_R)$ is dense in O_2: it is $U \cap O_2$, and U is dense in $\mathcal{A}_0$. Hence O_2 is connected. It is locally path connected, being open in the finite-polyhedron space $\mathcal{A}_0$, and therefore is path connected. The van Kampen theorem gives

$$\pi_1(\mathcal{A}_0) \cong \pi_1(O_2),$$

because both $\pi_1(O_1)$ and $\pi_1(O_1 \cap O_2)$ vanish. Contractibility of $\mathcal{A}_0$ implies $\pi_1(O_2) = 0$. By construction $B \subseteq O_2 \subseteq N_B$. Every loop in B contracts in O_2; composing its nullhomotopy with r_B contracts it in B. Thus $\pi_1(B) = 0$.

Homotopy type. The space B has finite CW type. Its simple connectivity and (5.8), together with the Hurewicz theorem applied to the first possibly nonzero homotopy group, show that B is $(D-2)$-connected and $\pi_{D-1}(B) \cong H_{D-1}(B; \mathbb{Z}) \cong \mathbb{Z}$. A map $S^{D-1} \to B$ representing a generator is an integral homology equivalence. The homological Whitehead theorem for simply connected CW spaces makes it a homotopy equivalence. □

5.6. **Convex forgetful fibers and $m = 3$.**

Theorem 5.10. *Every $\mathcal{A}_{n,k,3}(H)$, with H strictly positive, is contractible. More precisely, if $Z = [H, z]$ is a strictly positive $n \times (k+4)$ extension of H, then forgetting the last target coordinate gives a proper surjection*

$$q : \mathcal{A}_{n,k,4}(Z) \longrightarrow \mathcal{A}_{n,k,3}(H)$$

whose nonempty point fibers are compact convex polytopes. The map q is a homotopy equivalence.

Proof. Choose a positive k-plane inside $\mathrm{col}(H)$ and a positive $n \times k$ frame $Q = HR$ for it. For every $C \in X$, formula (5.1) gives $\det(CQ) > 0$. Consequently CH has rank k. The ambient target projection $\rho : \mathbb{R}^{k+4} \to \mathbb{R}^{k+3}$ therefore has rank k on every plane in $\mathcal{A}_{n,k,4}(Z)$, so q is everywhere defined and continuous there. It is surjective because the same source C lifts CH to CZ, and it is proper because its source is compact.

Fix $\eta \in \mathcal{A}_{n,k,3}(H)$ and let $W = \rho^{-1}(\eta)$. Then $\dim W = k+1$, and

$$q^{-1}(\eta) = \mathcal{A}_{n,k,4}(Z) \cap \mathrm{Gr}(k, W).$$

Every plane in the right-hand side projects with rank k, as already proved; hence its image really is η. Choose a nonzero $(k+1)$-vector ω on W. Contraction gives an isomorphism $W^* \to \bigwedge^k W$, $\lambda \mapsto \iota_\lambda \omega$. For $\lambda \neq 0$, a basis of $\ker \lambda$ completed by one vector of nonzero λ-value shows that this contraction is a nonzero decomposable wedge spanning $\ker \lambda$. Thus every nonzero k-vector is decomposable, and the Plücker image of $\mathrm{Gr}(k, W)$ is exactly the projective linear space $\mathbb{P}(\bigwedge^k W)$. Theorem 5.4 proves that the fiber is a nonempty compact convex polytope, including when the whole fiber lies in the upper boundary.

Both source and target of q are compact semialgebraic Euclidean neighborhood retracts. Each fiber, being a compact convex polytope, is contractible and cell-like. Lacher's theorem [16, Theorem 2 and Lemma 2.1] states that a proper cell-like map between Euclidean neighborhood

retracts is a proper homotopy equivalence. Thus q is a homotopy equivalence. Its source is contractible by Theorem 5.9, so its target is contractible as well.

For an arbitrary positive H, if $n \geq k+4$, the positive extension theorem [6, Lemma 3.4] provides a positive $(k+4)$-plane containing $\mathrm{col}(H)$. Choose its last basis vector, and its sign, so that its frame is $Z = [H, z]$ with positive maximal minors. The preceding argument applies. If $n = k+3$, the matrix H is invertible, and the amplituhedron is homeomorphic to $\mathrm{Gr}_{\geq 0}(k,n)$, which is a closed ball by Proposition 2.4. □

6. Local linear bounds for Plücker forms

On the totally nonnegative Grassmannian, normalize the Plücker coordinates by

$$p_I(C) = \frac{\Delta_I(C)}{\sum_J \Delta_J(C)}, \qquad C \in X = \mathrm{Gr}_{\geq 0}(k,n).$$

The denominator is positive, and the coordinates p_I are well-defined continuous functions on X. For a real-valued function N, write $N^- = \max\{0, -N\}$.

Theorem 6.1 (One-sided linear Plücker bound)**.** *Let*

$$G(C) = \sum_I b_I p_I(C), \qquad N(C) = \sum_I c_I p_I(C), \qquad b_I \geq 0, \quad c_I \in \mathbb{R}.$$

Fix $C_0 \in X$. Suppose that on some relative neighborhood U of C_0 in X,

$$G(C) = 0 \quad \Longrightarrow \quad N(C) \geq 0. \tag{6.1}$$

Then there are a smaller relative neighborhood V of C_0 and $K < \infty$ such that

$$N^-(C) \leq K G(C) \qquad (C \in V).$$

Corollary 6.2 (Linear domination from zero-set inclusion)**.** *Let $F = \sum_I a_I p_I$ and $G = \sum_I b_I p_I$, where $a_I, b_I \geq 0$. If, near C_0,*

$$\{G = 0\} \subseteq \{F = 0\},$$

then $F \leq KG$ on a possibly smaller neighborhood of C_0. If the zero-set inclusion holds on all of X, a single constant K works on all of X.

Proof. Take $N = -F$ in Theorem 6.1. For the global assertion, apply the local result at every point of the compact space X, choose a finite neighborhood cover, and take the maximum of its constants. □

6.1. **A compact matching cover.** We use the following standard input from the planar-network parametrization of the positive Grassmannian [24, Proposition 2.8 and Sections 4–5]. Choose a fixed finite plane bipartite graph Γ for the top positive cell. Its boundary vertices have degree one. Let E, V_{int}, and B be its edges, internal vertices, and boundary vertices, respectively. An *almost perfect matching* covers each internal vertex exactly once and each boundary vertex at most once. Write $\mathcal{M}$ for the finite nonempty set of these matchings. Each $M \in \mathcal{M}$ has a boundary label ∂M which is a k-subset of $[n]$. With the usual boundary coloring, this label consists of the matched black boundary vertices and the unmatched white boundary vertices. For positive edge weights w_e, the matching formula

$$P_I(w) = \sum_{\substack{M \in \mathcal{M} \\ \partial M = I}} \prod_{e \in M} w_e \tag{6.2}$$

gives projective Plücker coordinates, and its image as w varies over positive edge weights is exactly $\mathrm{Gr}_{>0}(k,n)$. This precise product form is also recorded in [14, Section 2.6, equations (2.28)–(2.30)].

Add one independent variable s_b for each boundary vertex and put

$$m_M(x,s) = \prod_{e\in M} x_e \prod_{\substack{b\in B \\ b \text{ unmatched in } M}} s_b, \qquad \widetilde{P}_I(x,s) = \sum_{\partial M = I} m_M(x,s). \tag{6.3}$$

Every displayed monomial is squarefree: each of its edge and slack variables occurs at most once. For positive x, s, define

$$w_e = \frac{x_e}{\prod_{\substack{b\in B \\ b \text{ incident to } e}} s_b}.$$

Since a matching uses each boundary vertex at most once,

$$\widetilde{P}_I(x,s) = \left(\prod_{b\in B} s_b\right) P_I(w). \tag{6.4}$$

Thus the enhanced positive parametrization has the same projective image as the ordinary one. At any nonnegative parameter point for which $D(x,s) = \sum_I \widetilde{P}_I(x,s) > 0$, its normalized image also belongs to X: approximate the parameters by positive parameters, use (6.4), and use closedness of X.

Proposition 6.3 (Compact cube cover)**.** *The set*

$$Q = \left\{(x,s) \in [0,1]^{E\sqcup B} : \max_{M\in\mathcal{M}} m_M(x,s) = 1\right\}$$

is a finite union of closed coordinate cube faces. On Q, $D \geq 1$, and

$$\Phi : Q \longrightarrow X, \qquad p_I(\Phi(x,s)) = \frac{\widetilde{P}_I(x,s)}{D(x,s)}$$

is a continuous surjection. On every cube face the $\widetilde{P}_I$ are multiaffine polynomials with nonnegative coefficients in the remaining free variables.

Proof. It remains to prove that the image contains the positive Grassmannian. Fix positive edge weights w_e and put $c_e = \log w_e$. Consider the linear program

$$\begin{aligned} &\text{maximize} && \sum_{e\in E} c_e z_e, \\ &\text{subject to} && z_e \geq 0, \qquad \sum_{e\ni v} z_e = 1 \quad (v \in V_{\text{int}}), \qquad \sum_{e\ni b} z_e \leq 1 \quad (b \in B). \end{aligned} \tag{6.5}$$

Its feasible polytope is nonempty because $\mathcal{M} \neq \varnothing$. It is bounded: every edge variable is at most one by either of its endpoint constraints.

Negating the vertex rows in one bipartition turns the vertex-edge incidence matrix into a matrix with one $+1$ and one -1 in each edge column. Every square subdeterminant of this matrix is 0, 1, or -1. To see this, induct on the size: a column with at most one nonzero entry permits expansion along that column; if every column has two nonzero entries, the sum of the rows is zero, so the determinant is zero. Adjoining unit columns for the boundary slack variables preserves this property, by the same determinant expansion. At a vertex of (6.5), a nonsingular constraint basis therefore has determinant 1 or -1. Cramer's rule and the integral right-hand side imply that every vertex coordinate is integral. Since $0 \leq z_e \leq 1$, each vertex is the incidence vector of an almost perfect matching. It follows that the optimal value is

$$L = \max_{M\in\mathcal{M}} \sum_{e\in M} c_e. \tag{6.6}$$

The dual linear program is

$$\begin{aligned} \text{minimize} \quad & \sum_{v\in V_{\text{int}}} u_v + \sum_{b\in B} u_b, \\ \text{subject to} \quad & u_v + u_w \geq c_e \quad (e = vw), \qquad u_v \in \mathbb{R} \quad (v \in V_{\text{int}}), \qquad u_b \geq 0 \quad (b \in B). \end{aligned} \tag{6.7}$$

Finite-dimensional linear programming duality supplies an optimal dual vector with objective value L. Define

$$x_e = \exp(c_e - u_v - u_w) \quad (e = vw), \qquad s_b = \exp(-u_b).$$

All these numbers belong to $(0, 1]$. For every matching M, each internal vertex occurs through exactly one matching edge, and each boundary vertex occurs either through a matching edge or through its slack variable. Hence

$$\begin{aligned} m_M(x, s) &= \exp\left(\sum_{e\in M} c_e - \sum_{v\in V_{\text{int}}} u_v - \sum_{b\in B} u_b\right) \\ &= \exp\left(\sum_{e\in M} c_e - L\right). \end{aligned} \tag{6.8}$$

Every such monomial is at most one, and an optimal matching makes it exactly one. Moreover,

$$\widetilde{P}_I(x, s) = e^{-L} P_I(w) \qquad \text{for every } I.$$

Thus $(x, s) \in Q$ has precisely the prescribed projective Plücker image. This proves that $\Phi(Q)$ contains $\mathrm{Gr}_{>0}(k, n)$.

The set Q is compact, and

$$D = \sum_{M\in\mathcal{M}} m_M \geq 1$$

there, so Φ is continuous. Its image lies in X by positive approximation, and is closed by compactness. Since the positive Grassmannian is dense in X, $\Phi(Q) = X$.

Finally, a product of numbers in $[0, 1]$ is one exactly when every factor is one. Therefore

$$Q = \bigcup_{M\in\mathcal{M}} Q_M, \qquad Q_M = \{(x, s) \in [0, 1]^{E\sqcup B} : m_M(x, s) = 1\}.$$

Each Q_M is the coordinate cube face obtained by fixing the variables occurring in m_M to one. Restricting a multiaffine polynomial to such a face preserves multiaffinity and nonnegativity of its coefficients. □

6.2. **A multiaffine estimate on a cube.** A polynomial is *multiaffine* if it has degree at most one in each variable.

Lemma 6.4. *Let f be a real multiaffine polynomial on $[0, 1]^r$, and let g be a multiaffine polynomial with nonnegative coefficients. Suppose that, in a relative neighborhood of $q_0 \in [0, 1]^r$,*

$$g(q) = 0 \quad \Longrightarrow \quad f(q) \geq 0.$$

Then $f^- \leq Kg$ in a possibly smaller relative neighborhood of q_0, for some finite K.

Proof. Partition the variables as $q = (z, y)$, where $z_1, \ldots, z_d$ are exactly those which vanish at q_0, and all the coordinates of y_0 are positive. Choose $0 < \delta \leq 1$ and a compact product neighborhood Y of y_0 in $(0, 1]^{r-d}$, such that

$$[0, \delta]^d \times Y$$

lies inside the neighborhood where the hypothesis holds. Every coordinate of $y \in Y$ has a fixed positive lower bound. This also accommodates coordinates equal to one at q_0, for which a

one-sided interval is used. If there are no z-variables, the formulas below have the single index $A = \varnothing$.

Write the nonzero monomials of g as

$$g(z,y) = \sum_{\alpha\in\mathcal{E}} \gamma_\alpha z_{B_\alpha} y_{T_\alpha}, \qquad \gamma_\alpha > 0, \quad z_B = \prod_{i\in B} z_i, \quad y_T = \prod_{j\in T} y_j.$$

The supports B_α, T_α are sets because g is multiaffine. For each α put

$$\mu_\alpha = \min_{y\in Y} y_{T_\alpha} > 0.$$

Empty products have value one.

For $A \subseteq [d]$, let $\mathbf{1}_A \in \mathbb{R}^d$ have coordinate one on A and zero elsewhere. For fixed y, repeated affine interpolation in the z-variables is the exact identity

$$f(z,y) = \sum_{A\subseteq[d]} f(\delta\mathbf{1}_A, y)\lambda_A(z), \qquad \lambda_A(z) = \prod_{i\in A} \frac{z_i}{\delta} \prod_{i\notin A} \left(1 - \frac{z_i}{\delta}\right). \tag{6.9}$$

The numbers λ_A are nonnegative and sum to one. Call A covered if $B_\alpha \subseteq A$ for at least one $\alpha \in \mathcal{E}$. If A is not covered, every monomial of g vanishes at $(\delta\mathbf{1}_A, y)$, for every $y \in Y$. The hypothesis therefore implies

$$f(\delta\mathbf{1}_A, y) \geq 0 \quad \text{for all uncovered } A.$$

For each covered A, choose one index $\alpha(A)$ with $B_{\alpha(A)} \subseteq A$, and put

$$M_A = \max_{y\in Y} f^-(\delta\mathbf{1}_A, y) < \infty.$$

Every factor in the interpolation weight is at most one, so

$$\begin{aligned}\lambda_A(z) &\leq \delta^{-|B_{\alpha(A)}|} z_{B_{\alpha(A)}} \\ &\leq \frac{\delta^{-|B_{\alpha(A)}|}}{\gamma_{\alpha(A)}\mu_{\alpha(A)}}\, g(z,y).\end{aligned}$$

Using (6.9) and discarding the nonnegative vertex contributions now gives

$$f^-(z,y) \leq \sum_{\substack{A\subseteq[d]\\ A \text{ covered}}} M_A\lambda_A(z) \leq Kg(z,y),$$

with the explicit finite constant

$$K = \sum_{\substack{A\subseteq[d]\\ A \text{ covered}}} \frac{M_A\delta^{-|B_{\alpha(A)}|}}{\gamma_{\alpha(A)}\mu_{\alpha(A)}}. \tag{6.10}$$

If g is identically zero on this cube, the sum is empty; all vertex values of f are nonnegative, so interpolation gives $f \geq 0$ and the conclusion holds with $K = 0$. □

6.3. **Descent across the entire compact fiber.**

Proof of Theorem 6.1. Use the compact cover $\Phi : Q \to X$ from Proposition 6.3. Fix a cube face Q_M and a point $q_0 \in Q_M \cap \Phi^{-1}(C_0)$. On this face define

$$f = \sum_I c_I \widetilde{P}_I, \qquad g = \sum_I b_I \widetilde{P}_I.$$

The numerator f is multiaffine, with possibly signed coefficients, whereas g is multiaffine with nonnegative coefficients. Their common denominator is $D > 0$. Continuity of Φ pulls (6.1) back to a relative neighborhood of q_0 in Q_M, where

$$g = 0 \quad \Longrightarrow \quad f \geq 0.$$

Lemma 6.4 gives $f^- \leq K_{q_0,M} g$ on a smaller relative neighborhood. Dividing by D gives

$$N^- \circ \Phi \leq K_{q_0,M}\, G \circ \Phi$$

on that neighborhood.

The fiber

$$F_M = Q_M \cap \Phi^{-1}(C_0)$$

is compact. If it is nonempty, finitely many of the preceding relative neighborhoods cover it. Let $O_M \subseteq Q_M$ be their union and let K_M be the maximum of their constants. The compact set $Q_M \setminus O_M$ has compact image which does not contain C_0. Therefore

$$V_M = X \setminus \Phi(Q_M \setminus O_M)$$

is a neighborhood of C_0, and $\Phi^{-1}(V_M) \cap Q_M \subseteq O_M$. If F_M is empty, take $V_M = X \setminus \Phi(Q_M)$, again a neighborhood of C_0.

There are only finitely many faces. Intersect their neighborhoods V_M, and take the maximum of the constants for the nonempty fibers. This gives a neighborhood V and a finite constant K. Every $C \in V$ has a lift in Q, by surjectivity. Every such lift belongs to a face whose estimate is valid. Consequently $N^-(C) \leq KG(C)$, as asserted. □

The compact-fiber argument gives an estimate over a target neighborhood.

Corollary 6.5 (Uniform estimate over a target neighborhood)**.** *Let $\pi : X \to T$ be continuous, where $T = \pi(X)$ is Hausdorff, and fix $y_0 \in T$. Let G, N be the forms in Theorem 6.1. Suppose that for every $C_0 \in \pi^{-1}(y_0)$ there is a relative neighborhood of C_0 on which $G = 0$ implies $N \geq 0$. Then some neighborhood W of y_0 and one finite constant K satisfy*

$$N^-(C) \leq KG(C) \qquad (C \in \pi^{-1}(W)).$$

The analogous assertion $F \leq KG$ holds for two nonnegative linear Plücker forms when their local zero-set inclusions hold at every point of $\pi^{-1}(y_0)$.

Proof. Apply Theorem 6.1 at every point of the compact fiber $\pi^{-1}(y_0)$. Finitely many resulting open neighborhoods cover that fiber; let O be their union and K the maximum of their constants. The compact set $\pi(X \setminus O)$ is closed in T and omits y_0. Take

$$W = T \setminus \pi(X \setminus O).$$

Then $\pi^{-1}(W) \subseteq O$, which proves the estimate simultaneously for all sources over W. Take $N = -F$ for the final assertion. □

Remark 6.6. In particular, a zero-set or sign condition stated on a target neighborhood and valid for its entire source inverse image satisfies the corollary's hypothesis. If the forms themselves descend to T, the resulting bound descends as well. The estimates are invariant under replacing every Plücker coordinate by the same strictly positive local normalization, because that common factor cancels.

Example 6.7 (A quadratic numerator)**.** On $[0,1]^3$, let

$$g(x,y,z) = xyz, \qquad f(x,y,z) = x^2 + y^2 + z^2 - \frac{3}{2}(xy + yz + zx).$$

On $z = 0$, for example,

$$f(x, y, 0) = (x - y)^2 + \frac{1}{2}xy \geq 0,$$

and the other coordinate planes are identical after permuting the variables. Thus $f \geq 0$ wherever $g = 0$. Along $x = y = z = t > 0$, however,

$$f(t, t, t) = -\frac{3}{2}t^2, \qquad \frac{f^-(t, t, t)}{g(t, t, t)} = \frac{3}{2t} \longrightarrow \infty.$$

6.4. **Smooth functions vanishing on a Plücker zero set.**

Theorem 6.8 (Smooth vanishing estimate)**.** *Let $C_0 \in X = \mathrm{Gr}_{\geq 0}(k, n)$ and $G = \sum_I b_I p_I$ with $b_I \geq 0$. Let F be a smooth real-valued function on a neighborhood of C_0 in the real Grassmannian. If $F = 0$ on the germ of $\{C \in X : G(C) = 0\}$ at C_0, then there are a neighborhood V of C_0 in X and $K < \infty$ such that*

$$|F(C)| \leq KG(C) \qquad (C \in V).$$

With G fixed, the conclusion is locally uniform for a smooth family $F(C, t)$, with t in an open Euclidean parameter domain, when the vanishing hypothesis holds on the full product germ near (C_0, t_0).

Proof. Use the compact matching cover $\Phi : Q \to X$ constructed in Proposition 6.3. Fix a cube face Q_M and $q_0 \in Q_M \cap \Phi^{-1}(C_0)$. The matching polynomials satisfy the Plücker relations identically: the relations hold for all positive parameters, hence as polynomial identities. Their sum D is nonzero near q_0. Thus Φ extends smoothly near q_0 as a map to the real Grassmannian, and $f = F \circ \Phi$ is smooth there.

As in the proof of Theorem 6.1, write the free face coordinates as (z, y), where $z_1, \ldots, z_s$ vanish at q_0 and every coordinate of y is bounded away from zero. The numerator of $G \circ \Phi$ is a positive squarefree sum

$$g(z, y) = \sum_{B \in \mathcal{B}} c_B(y) z_B, \qquad z_B = \prod_{i \in B} z_i, \qquad c_B(y) \geq c_* > 0$$

on a sufficiently small compact box. If $g(q_0) > 0$, the estimate follows immediately from boundedness of f and a positive lower bound for g. If g vanishes identically on the face, the hypothesis gives $f = 0$ on a neighborhood of q_0 in that face. We may therefore assume that $\mathcal{B}$ is nonempty and all its members are nonempty. Shrink the box so that $0 \leq z_i \leq \delta \leq 1$ and the vanishing hypothesis holds on it.

Let E_i be the evaluation operator setting z_i equal to zero, and let I be the identity operator on functions. These commuting operators give the anchored decomposition

$$f = \sum_{A \subseteq [s]} f_A, \qquad f_A = \prod_{i \in A} (I - E_i) \prod_{i \notin A} E_i f. \tag{6.11}$$

Indeed, the sum is the expansion of $\prod_i (E_i + (I - E_i)) f = f$. If no member $B \in \mathcal{B}$ is contained in A, then g vanishes on the coordinate face where $z_i = 0$ for $i \notin A$. The hypothesis makes the restriction of f to that face zero. Every term used to form f_A is a further restriction of that same zero function, so $f_A = 0$.

For each remaining A, choose $B(A) \in \mathcal{B}$ with $B(A) \subseteq A$, and write $\partial_A = \prod_{i \in A} \partial / \partial z_i$. Repeated use of the fundamental theorem of calculus gives

$$f_A(z, y) = z_A \int_{[0,1]^A} (\partial_A f)(u_A, 0_{[s] \setminus A}, y) \prod_{i \in A} dt_i, \qquad u_i = t_i z_i \ (i \in A).$$

All the displayed mixed derivatives are bounded on the chosen compact box; write M_A for a bound for $|\partial_A f|$. Since $0 \le z_i \le 1$,

$$|f_A(z,y)| \le M_A z_A \le M_A z_{B(A)} \le \frac{M_A}{c_*} g(z,y).$$

Summing (6.11) proves $|f| \le K_0 g$ on the chosen box in the face, with constants depending on derivatives of order at most s. Since $G \circ \Phi = g/D$ and D has a uniform upper bound on Q, this is the required local estimate on Q_M.

For each face, cover its compact fiber over C_0 by finitely many such neighborhoods and take the maximum of their constants. The image of the complementary compact set is closed and omits C_0. Intersect the resulting target neighborhoods over the finitely many faces. Every lift of every point of this smaller neighborhood then satisfies the estimate, and surjectivity of Φ gives the asserted inequality on X.

Additional smooth parameters can be included in the y variables for the derivative bounds, on a compact parameter neighborhood. The same finite-cover argument applied to Φ times the identity proves the uniform assertion. □

7. Uniform lifting with a fixed lower plane

Hoffman's theorem gives error bounds for fixed systems of linear inequalities [12]. For the extension cones below, positive projection gives a constant depending only on the external matrix, uniformly over all nonnegative lower planes.

Let Z be a positive $n \times d$ matrix and $0 \le p < d \le n$. For an oriented $B \in \mathrm{Gr}_{\ge 0}(p,n)$, identify $\mathbb{R}^n/B$ with $B^\perp$ using the Euclidean inner product. If $b_1, \dots, b_p$ is a positively oriented basis of B, put

$$K_B = \{x \in B^\perp : b_1 \wedge \cdots \wedge b_p \wedge x \ge 0\}.$$

The inequality means that every ordered coordinate of the displayed exterior vector is nonnegative. It is independent of the chosen positive basis. The zero vector is included. Let P_{BZ} be orthogonal projection onto the row space $BZ \subset \mathbb{R}^d$, and define

$$T_B x = xZ(I - P_{BZ}) \in (BZ)^\perp, \qquad F_B(y) = \{x \in K_B : T_B x = y\}.$$

All vectors in this section are row vectors when multiplied by Z. For $p = 0$ the empty wedge is 1, $K_B = \mathbb{R}^n_{\ge 0}$, and $P_{BZ} = 0$.

For nonempty compact subsets K, L of a Euclidean space, write

$$d_H(K,L) = \max\left\{ \sup_{x \in K} \operatorname{dist}(x,L), \sup_{y \in L} \operatorname{dist}(y,K) \right\}.$$

Theorem 7.1 (Uniform lifting in a fixed extension cone)**.** *There is $a_Z > 0$, depending only on Z, such that for every $0 \le p < d$ and every $B \in \mathrm{Gr}_{\ge 0}(p,n)$ the following hold. Each feasible fiber $F_B(y)$ is compact. Given feasible outputs $y_0, y_1 \in T_B(K_B)$ and any $x_0 \in F_B(y_0)$, there is a path $x : [0,1] \to K_B$ satisfying*

$$x(0) = x_0, \qquad T_B x(s) = (1-s)y_0 + s y_1,$$

whose Lipschitz constant is at most $a_Z^{-1}\|y_1 - y_0\|$. In particular,

$$d_H(F_B(y_0), F_B(y_1)) \le a_Z^{-1}\|y_1 - y_0\|. \tag{7.1}$$

7.1. **Ordered rays of the extension cone.**

Lemma 7.2 (Ordered extension rays)**.** *For every $B \in \mathrm{Gr}_{\geq 0}(p,n)$, the cone K_B is simplicial in its linear span. Its linearly independent generating rays can be ordered as $h_1, \ldots, h_q$ so that for every $i_1 < \cdots < i_s$ the matrix with ordered rows*

$$[\, B; h_{i_1}; \ldots; h_{i_s}\,]$$

has rank $p+s$ and nonnegative maximal minors. Here B in the matrix denotes any positively oriented frame of the plane. At a degenerate plane it is possible that $q < n-p$.

Proof. For $p = 0$ use the coordinate rays in their natural order. Suppose $p > 0$. We use the Le-network parametrization of a nonnegative Grassmann cell, in the form of [23, Definitions 4.6, 6.1, 6.3 and Theorem 6.5]. The input from that theorem is the following: each cell has a unique Le diagram, positive edge weights modulo internal vertex gauge parametrize its points bijectively, and boundary measurement has nonnegative ordered maximal minors. A boundary-measurement entry has the sign $(-1)^a$ times its path sum, where a is the number of boundary sources strictly between its source and sink. Pivot columns are the identity columns. For the proof choose this boundary-measurement frame for B. Changing to any other positive frame multiplies every extension minor by one common positive determinant and changes no cone.

Rotate the usual diagram by 180 degrees. Paths now move up and right, old source labels decrease from top to bottom, and old nonpivot column labels increase from left to right. A filled box is a network vertex. The Le rule forbids a zero box having both a filled box to its right in the same row and a filled box below it in the same column. Choose the positive network of B and fix its edge gauge.

Call an old nonpivot column *available* if placing a filled box in a new full bottom row in that column violates no Le rule. More explicitly, a column is unavailable if an old zero in that column has an old filled box to its right. Old pivot columns are excluded. Let U be the set of available columns. Any subset of U may be filled in the new bottom row: a new forbidden pattern at an old zero is exactly the obstruction just excluded, and a zero in the bottom row has no filled box below it. These exhaust the possible new forbidden patterns.

Add an artificial pivot column 0 before all old columns, with its new source in the bottom row. Normalize the new horizontal chain to have weight 1 on every segment, and assign weight $t_u \geq 0$ to its upward exit at $u \in U$. This gauge is well defined: keep the old vertex gauge fixed and determine the new vertex gauges successively along the horizontal chain. Every old source path stays in the old network. A new source path reaching an old boundary sink leaves the bottom row at exactly one exit and never returns. Consequently its old-coordinate response is linear in all the exit weights at once. In the quotient by B write it, with the positive extension orientation, as

$$[h(t)] = \sum_{u \in U} t_u [h_u]. \tag{7.2}$$

The identity entry in the artificial pivot column is the length-zero pivot path; it is not an additional boundary sink.

To check the orientation and completeness, use the augmented matrix

$$\widetilde{B}(h) = \begin{pmatrix} 0 & B \\ (-1)^p & h \end{pmatrix}.$$

For every old p-set I and old $(p+1)$-set J, expansion in the artificial column gives

$$\Delta_{\{0\}\cup I}(\widetilde{B}(h)) = \Delta_I(B), \qquad \Delta_J(\widetilde{B}(h)) = \det[B; h]_J. \tag{7.3}$$

Thus the bottom-row network with nonnegative exit weights produces elements of K_B. Zero exit weights follow by continuity from positive weights.

Conversely, let $[h] \neq 0$ belong to the cone. Equations (7.3) show that $\widetilde{B}(h)$ is a nonnegative $(p+1)$-plane. Column 0 is not a loop, since some $\Delta_I(B) > 0$, and is not a coloop, since some $\det[B;h]_J > 0$. Its lexicographically first basis contains 0, so in its rotated Le diagram this source is the bottom source. Contracting column 0 deletes that bottom row and the artificial column. The remaining network measures B: in a path family for a minor containing 0, the pivot path at 0 is fixed and the other paths stay in the upper network. This also follows directly from the first identity in (7.3).

Uniqueness in the Le parametrization therefore identifies the upper diagram with that of B and its positive weights, modulo gauge, with the fixed weights of B. Its bottom exits must lie in U. After fixing the upper gauge, normalize the bottom horizontal segments to 1 as above. The remaining exit weights are positive; give unused columns weight zero. This proves (7.2) for every element of the cone. The zero element is obtained by making all exit weights zero.

The classes $[h_u]$ are independent. Indeed, a relation $\sum_u a_u[h_u] = 0$ would make two different strictly positive exit lists t and $t+\epsilon a$ give the same augmented Grassmannian point for sufficiently small nonzero ϵ. Their fixed upper gauge and unit bottom horizontal weights leave no gauge freedom. This contradicts injectivity of the positive parametrization for that Le diagram. Hence K_B is the simplicial cone generated by these classes.

It remains to verify the simultaneous signs. Order U by increasing old column label and select $u_1 < \cdots < u_s$. Add s full bottom rows. Their artificial source labels $c_1 < \cdots < c_s$ all precede the old labels; from top to bottom the new sources are $c_s, \ldots, c_1$. In the ith new row counted from the top put just one filled box, at u_i, with the unit exit weights of the single-outlet construction.

This is a Le diagram. The condition at old zero boxes is the availability of the selected columns. At a zero in a new row with outlet u_i, a filled box to its right requires the zero to be left of u_i. All lower outlets lie to the right of u_i, so none is below that zero. A path from a new source goes to its single outlet and then upward. It passes above the other new rows to the right of their outlets, where their horizontal segments have already ended. It thus acquires no extra branch, and its old-coordinate response is the same single-outlet response as before.

Let g_u be that response when the one artificial source is listed first. Moving its row after the p old rows gives $h_u = (-1)^p g_u$. In the simultaneous network, the source c_j has $s-j$ additional artificial sources between it and every old sink. Its old-coordinate row is therefore

$$(-1)^{s-j} g_{u_{s+1-j}} = (-1)^{s-j+p} h_{u_{s+1-j}}.$$

The old-column part of the boundary matrix is, in its row order,

$$\begin{pmatrix} (-1)^{s-1+p} h_{u_s} \\ (-1)^{s-2+p} h_{u_{s-1}} \\ \vdots \\ (-1)^p h_{u_1} \\ B \end{pmatrix}.$$

Moving B to the front contributes $(-1)^{ps}$, reversing the s new rows contributes $(-1)^{s(s-1)/2}$, and their displayed scalings contribute $(-1)^{ps+s(s-1)/2}$. The total sign is $(-1)^{2ps+s(s-1)} = 1$. Every maximal minor of $[B; h_{u_1}; \ldots; h_{u_s}]$ is consequently nonnegative. Independence of the classes proves its rank is $p+s$. Replacing any representative by its orthogonal projection to $B^\perp$ only adds old rows, and preserves all these maximal minors. □

7.2. **A uniform quotient bound and full path lifting.** For $1 \leq r \leq d$, the set of pairs

$$\{(W, w) : W \in \mathrm{Gr}_{\geq 0}(r, n),\ w \in W,\ \|w\| = 1\}$$

is compact. Positive projection, Lemma 2.1, is injective on every such W. Hence the following number is strictly positive:

$$a_Z = \min_{1\le r\le d} \min_{\substack{W\in \mathrm{Gr}_{\ge 0}(r,n)\\ w\in W,\ \|w\|=1}} \|wZ\| > 0. \tag{7.4}$$

If a nonnegative subspace S of dimension at most d contains B, then every $x \in S \cap B^\perp$ satisfies

$$\|T_B x\| = \inf_{b\in B} \|(x+b)Z\| \ge a_Z \inf_{b\in B} \|x+b\| = a_Z\|x\|. \tag{7.5}$$

For $x \neq 0$ in K_B, take $S = B + \mathbb{R}x$, which is nonnegative of dimension $p+1 \le d$. Thus each fiber is bounded by $\|x\| \le a_Z^{-1}\|y\|$; it is closed, and hence compact.

Proof of Theorem 7.1. Only the lifting assertion remains. Put $e = y_1 - y_0$ and $L = a_Z^{-1}\|e\|$. If $e = 0$, the constant path suffices. Suppose a partial lift has reached $x \in F_B(y_0 + se)$ at time $s < 1$. Choose any $x_1 \in F_B(y_1)$. The vector $(x_1 - x)/(1-s)$ is in the tangent cone of K_B at x and maps to e under T_B.

Use the ordered rays of Lemma 7.2 and write $x = \sum_i \alpha_i h_i$, $\alpha_i \ge 0$. Its tangent cone is generated by all h_i, together with $-h_i$ for the indices for which $\alpha_i > 0$. A finite nonnegative combination of their images equals e. Eliminate linear dependences from such a combination until the used images are independent: for a dependence among the used generators, subtract a suitable multiple from the positive coefficient list until one coefficient first becomes zero, preserving the other coefficients and the represented vector. Repeating terminates with at most $d - p$ generators, since the output space has dimension $d - p$.

Let v be the same combination of the signed source rays. Then $T_B v = e$ and v is a tangent direction at x. Opposite images from the same ray cannot both occur in an independent list. Thus v lies in the span of at most $d - p$ distinct ray classes. Their simultaneous span with B is nonnegative by Lemma 7.2 and has dimension at most d. Equation (7.5) gives

$$\|v\| \le a_Z^{-1}\|T_B v\| = L.$$

In ray coordinates the only possibly negative components of v are at indices with $\alpha_i > 0$. Hence $x + tv \in K_B$ for all sufficiently small $t \ge 0$. This extends the partial lift by an affine segment with the same Lipschitz bound.

Let s_* be the supremum of times on which an L-Lipschitz lift starting at x_0 exists. Choose such paths on intervals tending to $[0, s_*]$ and extend them constantly to that interval. They are uniformly L-Lipschitz and bounded by $\|x_0\| + L$. Arzelà–Ascoli gives a uniformly convergent subsequence. Its limit stays in the closed cone and has the prescribed output at every time below s_*, and then at s_* by continuity. If $s_* < 1$, the preceding affine extension contradicts its definition. Thus $s_* = 1$.

The endpoint is within L of the prescribed x_0. Taking the supremum over $x_0 \in F_B(y_0)$ proves one directed Hausdorff estimate; reversing y_0, y_1 proves the other. This proves (7.1). □

For a linear functional ℓ and a nonempty compact set F, put

$$\operatorname{osc}_F \ell = \sup_{x\in F} \ell(x) - \inf_{x\in F} \ell(x).$$

The norm $\|\ell\|$ is its Euclidean dual norm.

Corollary 7.3 (Oscillation on two fibers)**.** *For a linear functional ℓ on $B^\perp$ and two feasible outputs for the same B,*

$$\operatorname{osc}_{F_B(y_1)} \ell \le \operatorname{osc}_{F_B(y_0)} \ell + 2\|\ell\| a_Z^{-1}\|y_1 - y_0\|.$$

Proof. Approximate each of two points of $F_B(y_1)$ by a point of $F_B(y_0)$ using (7.1), and apply the triangle inequality to their functional difference. □

Example 7.4 (A change in the extension cone)**.** Let $B_t = \mathbb{R}(1, 0, t) \subset \mathbb{R}^3$. The three ordered extension minors of a row h are

$$h_2, \qquad h_3 - th_1, \qquad -th_2.$$

For $t > 0$ their nonnegativity forces $h_2 = 0$. At $t = 0$ the row e_2 is an allowed extension. It is orthogonal to every B_t and cannot be approached by the extension cones at positive t. Thus varying the lower plane can introduce new rays.

7.3. **Boundary fibers and their width.** Keep the lower plane B fixed, and put $E_B = (BZ)^\perp$, $\mathcal{C}_B = T_B(K_B)$, and $D = d - p$. The boundary and distance in the next statement are taken in E_B, even when $\mathcal{C}_B$ has smaller dimension. For a nonempty compact set K, its diameter is $\operatorname{diam} K = \sup_{x,x' \in K} \|x - x'\|$.

Corollary 7.5 (Boundary uniqueness and a width estimate)**.** *Every $y \in \partial\mathcal{C}_B$ has exactly one lift in K_B. For every $y \in \mathcal{C}_B$,*

$$\operatorname{diam} F_B(y) \le 2a_Z^{-1} \operatorname{dist}(y, \partial\mathcal{C}_B). \tag{7.6}$$

Consequently, if $0 \neq \lambda \in E_B^*$ is nonnegative on $\mathcal{C}_B$, then

$$\operatorname{diam} F_B(y)\, \|\lambda\| \le 2a_Z^{-1} \lambda(y).$$

Proof. Let $h_1, \dots, h_q$ be the distinct ordered rays in Lemma 7.2. The images of any $s \le D$ of these rays are independent: their span with B is nonnegative of dimension $p + s \le d$, and (7.5) is injective on its intersection with $B^\perp$. Thus if $q \le D$, the map is injective on $\operatorname{span} K_B$ and every fiber is a singleton. This includes $q = 0$.

Suppose $q > D$. The cone $\mathcal{C}_B$ is full dimensional and closed. Indeed, if $y_j \to y$ and $x_j \in F_B(y_j)$, then $\|x_j\| \le a_Z^{-1}\|y_j\|$ gives a convergent subsequence with limit in $F_B(y)$. The cone is pointed: opposite images of two nonzero cone vectors would make their nonzero sum a cone vector in $\ker T_B$, contrary to the same bound. At a boundary point y_b choose a nonzero supporting functional $\mu \ge 0$ on $\mathcal{C}_B$ with $\mu(y_b) = 0$. A representation $y_b = \sum_i \alpha_i T_B h_i$, $\alpha_i \ge 0$, can use only ray images in $\ker \mu$. There are at most $D - 1$ such images, since any D are independent. They are themselves independent, so the coefficients, and hence the lift, are unique.

The cone has nonempty boundary. Take a boundary point y_b nearest to y and write $F_B(y_b) = \{x_b\}$. By (7.1), every $x \in F_B(y)$ satisfies $\|x - x_b\| \le a_Z^{-1}\|y - y_b\|$. The triangle inequality gives (7.6). For a lower-dimensional cone the right side is zero and the singleton assertion already proves this estimate. Finally, let $v_\lambda \in E_B$ be the vector with $\lambda(x) = \langle v_\lambda, x\rangle$. Starting at y and moving in direction $-v_\lambda/\|v_\lambda\|$, one reaches $\ker \lambda$ after distance $\lambda(y)/\|\lambda\|$. This segment meets $\partial\mathcal{C}_B$ by that time, because the cone is contained in $\{\lambda \ge 0\}$. Substitute the resulting distance bound in (7.6). □

7.4. **A uniformly Lipschitz logarithmic center.** Choose one nonzero generator on each ordered ray and fix weights $\nu_1, \dots, \nu_q > 0$. In orthonormal column coordinates write $R\alpha = \sum_i \alpha_i h_i$ and $F = T_B R : \mathbb{R}^q \to L$, where $L = \operatorname{span} \mathcal{C}_B$. The columns of the matrix F in these bases are denoted F_i. Write $\operatorname{relint} \mathcal{C}_B$ for the interior of $\mathcal{C}_B$ in L. The preceding independence argument gives $f = \dim L = \min(q, D)$.

Theorem 7.6 (A fixed-plane center section)**.** *For $y \in \operatorname{relint} \mathcal{C}_B$ let*

$$\alpha(y) = \operatorname*{arg\,min}_{F\alpha = y,\ \alpha_i > 0} \left(-\sum_{i=1}^{q} \nu_i \log \alpha_i \right). \tag{7.7}$$

For $q > 0$ this minimizer exists and is unique. The section $s_{B,\nu}(y) = R\alpha(y)$ extends uniquely to the closed cone, satisfies $s_{B,\nu}(y) \in F_B(y)$, and obeys

$$\|s_{B,\nu}(y_1) - s_{B,\nu}(y_0)\| \le a_Z^{-1}\|y_1 - y_0\|. \tag{7.8}$$

For $q = 0$ the same assertion holds with $s_{B,\nu}(0) = 0$. Both the plane and the weights are fixed when outputs are compared; the constant is independent of these choices.

Proof. Assume $q > 0$, so $f > 0$. For $I \subseteq [q]$ with $|I| = f$, let F_I consist of the columns indexed by I, in increasing order. Every such submatrix is invertible. Let $E_I : \mathbb{R}^f \to \mathbb{R}^q$ insert coordinates at the indices of I in their increasing order. For an arbitrary positive diagonal $W = \operatorname{diag}(w_i)$ put

$$M = FWF^T, \qquad \omega_I = \frac{\det(F_I)^2 \prod_{i\in I} w_i}{\det M}.$$

Cauchy–Binet gives $\omega_I > 0$ and $\sum_I \omega_I = 1$. The determinantal basis-average formula of Ben-Tal and Teboulle [3] takes the form

$$WF^T M^{-1} = \sum_{|I|=f} \omega_I E_I F_I^{-1}. \tag{7.9}$$

To verify it, differentiate $\det M = \sum_I \det(F_I)^2 \prod_{i\in I} w_i$ with respect to the entries of F, keeping W fixed. The two gradient matrices are respectively

$$2\det M\, M^{-1}FW, \qquad 2\sum_I \det(F_I)^2 \Big(\prod_{i\in I} w_i\Big) F_I^{-T} E_I^T.$$

Transposing and dividing by $2\det M$ proves (7.9).

For $e \in L$, the vector $x = RE_I F_I^{-1} e$ lies in the span of the selected f rays, has $T_B x = e$, and belongs to a nonnegative span with B of dimension $p + f \le d$. Equation (7.5) therefore gives $\|RE_I F_I^{-1} e\| \le a_Z^{-1}\|e\|$. Taking the convex average in (7.9) yields

$$\|RWF^T(FWF^T)^{-1}\|_{\mathrm{op}} \le a_Z^{-1}. \tag{7.10}$$

For $y \in \operatorname{relint} \mathcal{C}_B$, there is a strictly positive feasible coefficient vector. Indeed, for small $\epsilon > 0$ the vector $y - \epsilon \sum_i F_i$ is still in $\mathcal{C}_B$; add ϵ to every coefficient of a nonnegative representation of that vector. The closed coefficient fiber is compact, since R is injective and $F_B(y)$ is compact. On it the objective in (7.7) tends to $+\infty$ whenever a coefficient tends to zero: the other coefficients remain bounded. Hence an interior minimizer exists. Its strictly positive Hessian makes it unique.

The multiplier equations are $F\alpha = y$ and $F^T\lambda = (\nu_i/\alpha_i)_i$. Their Jacobian is invertible: if $Hv + F^T\eta = 0$ and $Fv = 0$, where $H = \operatorname{diag}(\nu_i/\alpha_i^2)$, then $v^T H v = 0$, so $v = 0$, and full row rank gives $\eta = 0$. The implicit function theorem thus makes $\alpha(y)$ smooth on the relative interior. Holding B, F, ν fixed and putting $W = H^{-1}$, differentiation gives

$$F^T\, d\lambda = -W^{-1} d\alpha, \qquad F\, d\alpha = dy, \qquad d\alpha = WF^T(FWF^T)^{-1} dy.$$

By (7.10), $\|ds_{B,\nu}\|_{\mathrm{op}} \le a_Z^{-1}$. Integrating on a segment in the convex relative interior proves (7.8) there. The relative interior is dense in the cone, so the Lipschitz map has a unique extension to its closure. The defining equalities and inequalities pass to limits, giving $s_{B,\nu}(y) \in F_B(y)$ and the same Lipschitz bound. □

In particular, $H_t(x) = (1 - t)x + t\, s_{B,\nu}(T_B x)$ is a continuous strong deformation retraction of K_B onto the image of the section. It preserves every output, because each segment lies in its convex fiber.

Appendix A. A rational certificate

We specialize the construction in Section 4 to $(n,k,m)=(9,2,4)$. The tables list all 84 maximal minors on each side of alternating duality and every nonzero entry of the second and sixth additive compounds. In the tables, a string such as 125 denotes the increasing subset $\{1,2,5\}$; every index lies in [9], so this notation is unambiguous.

A.1. **The polygon and its cyclic operator.** Take the rows of R to be

$$r_i=(1,i,i^2+\tfrac{1}{10}\mathbf{1}_{\{i=5\}}),\qquad 1\le i\le 9.$$

Here $\mathbf{1}_{\{i=5\}}$ equals one for $i=5$ and zero otherwise. The centroid and the matrix acting on the three-plane are

$$p=(1,5,\tfrac{2851}{90}),\qquad \Lambda=\begin{pmatrix}1&5&2851/90\\0&0&0\\0&0&0\end{pmatrix}.$$

Use cyclic subscripts and put

$$E_i=\det(r_i,r_{i+1},p),\quad T_i=\det(r_{i-1},r_i,r_{i+1}),\quad a_i=E_i/T_i,\quad b_i=E_{i-1}/T_i,\quad c_i=1-a_i-b_i.$$

The following table gives all the entries of Q: its only nonzero entries are $Q_{i,i-1}=a_i$, $Q_{ii}=c_i$, and $Q_{i,i+1}=b_i$, with indices interpreted modulo nine. The positive diagonal symmetrizer is $D=\operatorname{diag}(d_1,\dots,d_9)$, where $d_i^2=E_{i-1}E_i/T_i$.

i	E_i	T_i	a_i	b_i	c_i	d_i^2
1	$\frac{1681}{90}$	56	$\frac{1681}{5040}$	$\frac{839}{630}$	$-\frac{479}{720}$	$\frac{1410359}{56700}$
2	$\frac{1141}{90}$	2	$\frac{1141}{180}$	$\frac{1681}{180}$	$-\frac{1321}{90}$	$\frac{1918021}{16200}$
3	$\frac{781}{90}$	2	$\frac{781}{180}$	$\frac{1141}{180}$	$-\frac{871}{90}$	$\frac{891121}{16200}$
4	$\frac{296}{45}$	$\frac{21}{10}$	$\frac{592}{189}$	$\frac{781}{189}$	$-\frac{1184}{189}$	$\frac{231176}{8505}$
5	$\frac{296}{45}$	$\frac{9}{5}$	$\frac{296}{81}$	$\frac{296}{81}$	$-\frac{511}{81}$	$\frac{87616}{3645}$
6	$\frac{781}{90}$	$\frac{21}{10}$	$\frac{781}{189}$	$\frac{592}{189}$	$-\frac{1184}{189}$	$\frac{231176}{8505}$
7	$\frac{1141}{90}$	2	$\frac{1141}{180}$	$\frac{781}{180}$	$-\frac{871}{90}$	$\frac{891121}{16200}$
8	$\frac{1681}{90}$	2	$\frac{1681}{180}$	$\frac{1141}{180}$	$-\frac{1321}{90}$	$\frac{1918021}{16200}$
9	$\frac{3356}{45}$	56	$\frac{839}{630}$	$\frac{1681}{5040}$	$-\frac{479}{720}$	$\frac{1410359}{56700}$

For each of the nine rows the identities

$$a_i+b_i+c_i=1,\quad a_i(i-1)_{\mathrm{cyc}}+c_i i+b_i(i+1)_{\mathrm{cyc}}=5,$$

and

$$a_i\,y_{i-1}+c_i\,y_i+b_i\,y_{i+1}=2851/90\quad\Big(y_i=i^2+\tfrac{1}{10}\mathbf{1}_{\{i=5\}}\Big)$$

give $QR=R\Lambda$. Here $(0)_{\mathrm{cyc}}=9$ and $(10)_{\mathrm{cyc}}=1$. Symmetry of $D^{-1}QD$ follows entry by entry from the nine positive common values $b_i d_{i+1}^2=a_{i+1}d_i^2$ in the next table.

i	$i+1 \pmod 9$	$b_i d_{i+1}^2=a_{i+1}d_i^2$
1	2	$\frac{229888517}{1458000}$

Continued on the next page.

i	$i+1 \pmod 9$	$b_i d_{i+1}^2 = a_{i+1} d_i^2$
2	3	$\frac{1497974401}{2916000}$
3	4	$\frac{9420422}{54675}$
4	5	$\frac{68428096}{688905}$
5	6	$\frac{68428096}{688905}$
6	7	$\frac{9420422}{54675}$
7	8	$\frac{1497974401}{2916000}$
8	9	$\frac{229888517}{1458000}$
9	1	$\frac{2370813479}{285768000}$

A.2. **The positive external matrix.** With $A_9 = \operatorname{diag}(1,-1,1,-1,1,-1,1,-1,1)$ as in (2.1), put $K = A_9 R$ and $J = (-A_9 Q A_9)^{\mathsf{T}}$. Thus $J_{ii} = -c_i$; for $i = 1,\dots,8$,

$$J_{i,i+1} = a_{i+1}, \qquad J_{i+1,i} = b_i,$$

while $J_{1,9} = -b_9$ and $J_{9,1} = -a_1$. Every other entry is zero. These formulas and the preceding table specify all 81 entries of J. Choose the following external matrix:

$$Z = \begin{pmatrix} 1 & -\frac{61}{20} & 6 & -10 & 15 & -21 \\ 3 & -\frac{81}{10} & 15 & -24 & 35 & -48 \\ 3 & -\frac{121}{20} & 10 & -15 & 21 & -28 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The last six rows form the identity matrix. Direct substitution of the first three rows gives $K^{\mathsf{T}} Z = 0$. In fact the three relations, applied to any column z, are

$$\sum_{i=1}^{9} (-1)^{i-1} z_i = 0, \qquad \sum_{i=1}^{9} (-1)^{i-1} i z_i = 0, \qquad \sum_{i=1}^{9} (-1)^{i-1} y_i z_i = 0.$$

The identity $QR = R\Lambda$ gives $J^{\mathsf{T}} K = -K\Lambda$, so J preserves $\ker K^{\mathsf{T}}$. The induced target generator is the following complete 6×6 matrix:

$$B = \begin{pmatrix} \frac{95563}{3780} & -\frac{1124149}{32400} & \frac{1141}{18} & -\frac{1141}{12} & \frac{7987}{60} & -\frac{7987}{45} \\ \frac{781}{189} & \frac{511}{81} & \frac{781}{189} & 0 & 0 & 0 \\ 0 & \frac{296}{81} & \frac{1184}{189} & \frac{1141}{180} & 0 & 0 \\ 0 & 0 & \frac{592}{189} & \frac{871}{90} & \frac{1681}{180} & 0 \\ 0 & 0 & 0 & \frac{781}{180} & \frac{1321}{90} & \frac{839}{630} \\ -\frac{1681}{5040} & \frac{102541}{100800} & -\frac{1681}{840} & \frac{1681}{504} & \frac{6733}{5040} & \frac{2761}{360} \end{pmatrix}.$$

Indeed, its rows are exactly rows $4,\dots,9$ of JZ. Since both JZ and ZB lie in $\ker K^{\mathsf{T}}$, and restriction to these six rows is an isomorphism on that kernel, this verifies $JZ = ZB$, including the first three rows.

The next table lists every ordered three-minor of R and every ordered six-minor of Z. The complementary-minor identity is

$$\Delta_{I^c}(Z) = \tfrac{1}{2}\Delta_I(R), \qquad |I| = 3.$$

The common factor is fixed by $\Delta_{123}(R) = 2$ and $\Delta_{456789}(Z) = 1$. Every number in the two value columns is strictly positive.

I	$\Delta_I(R)$	I^c	$\Delta_{I^c}(Z)$	I	$\Delta_I(R)$	I^c	$\Delta_{I^c}(Z)$
123	2	456789	1	259	$\frac{833}{10}$	134678	$\frac{833}{20}$
124	6	356789	3	267	20	134589	10
125	$\frac{121}{10}$	346789	$\frac{121}{20}$	268	48	134579	24
126	20	345789	10	269	84	134578	42
127	30	345689	15	278	30	134569	15
128	42	345679	21	279	70	134568	35
129	56	345678	28	289	42	134567	21
134	6	256789	3	345	$\frac{21}{10}$	126789	$\frac{21}{20}$
135	$\frac{81}{5}$	246789	$\frac{81}{10}$	346	6	125789	3
136	30	245789	15	347	12	125689	6
137	48	245689	24	348	20	125679	10
138	70	245679	35	349	30	125678	15
139	96	245678	48	356	$\frac{57}{10}$	124789	$\frac{57}{20}$
145	$\frac{123}{10}$	236789	$\frac{123}{20}$	357	$\frac{78}{5}$	124689	$\frac{39}{5}$
146	30	235789	15	358	$\frac{59}{2}$	124679	$\frac{59}{4}$
147	54	235689	27	359	$\frac{237}{5}$	124678	$\frac{237}{10}$
148	84	235679	42	367	12	124589	6
149	120	235678	60	368	30	124579	15
156	$\frac{39}{2}$	234789	$\frac{39}{4}$	369	54	124578	27
157	$\frac{237}{5}$	234689	$\frac{237}{10}$	378	20	124569	10
158	$\frac{833}{10}$	234679	$\frac{833}{20}$	379	48	124568	24
159	$\frac{636}{5}$	234678	$\frac{318}{5}$	389	30	124567	15
167	30	234589	15	456	$\frac{9}{5}$	123789	$\frac{9}{10}$
168	70	234579	35	457	$\frac{57}{10}$	123689	$\frac{57}{20}$
169	120	234578	60	458	$\frac{58}{5}$	123679	$\frac{29}{5}$
178	42	234569	21	459	$\frac{39}{2}$	123678	$\frac{39}{4}$
179	96	234568	48	467	6	123589	3
189	56	234567	28	468	16	123579	8
234	2	156789	1	469	30	123578	15
235	$\frac{61}{10}$	146789	$\frac{61}{20}$	478	12	123569	6

Continued on the next page.

I	$\Delta_I(R)$	I^c	$\Delta_{I^c}(Z)$	I	$\Delta_I(R)$	I^c	$\Delta_{I^c}(Z)$
236	12	145789	6	479	30	123568	15
237	20	145689	10	489	20	123567	10
238	30	145679	15	567	$\frac{21}{10}$	123489	$\frac{21}{20}$
239	42	145678	21	568	$\frac{31}{5}$	123479	$\frac{31}{10}$
245	$\frac{31}{5}$	136789	$\frac{31}{10}$	569	$\frac{123}{10}$	123478	$\frac{123}{20}$
246	16	135789	8	578	$\frac{61}{10}$	123469	$\frac{61}{20}$
247	30	135689	15	579	$\frac{81}{5}$	123468	$\frac{81}{10}$
248	48	135679	24	589	$\frac{121}{10}$	123467	$\frac{121}{20}$
249	70	135678	35	678	2	123459	1
256	$\frac{58}{5}$	134789	$\frac{29}{5}$	679	6	123458	3
257	$\frac{59}{2}$	134689	$\frac{59}{4}$	689	6	123457	3
258	$\frac{267}{5}$	134679	$\frac{267}{10}$	789	2	123456	1

A.3. **The exact characteristic polynomial.** Set

$$u(\lambda) = 1714608000\lambda^3 + 54214045200\lambda^2 + 439610388120\lambda + 761261481119,$$
$$v(\lambda) = 3857868000\lambda^3 + 147603934800\lambda^2 + 1694718832950\lambda + 5671255592081,$$
$$M = 6614731335744000000.$$

Reflection of the indices $i \mapsto 10 - i$ commutes with Q. Use the basis $e_i + e_{10-i}$, $1 \le i \le 4$, followed by e_5 on the symmetric subspace, and $e_i - e_{10-i}$, $1 \le i \le 4$, on the antisymmetric subspace. In these bases the two restrictions are the following tridiagonal matrices:

$$Q_+ = \begin{pmatrix} -\frac{209}{630} & \frac{839}{630} & 0 & 0 & 0 \\ \frac{1141}{180} & -\frac{1321}{90} & \frac{1681}{180} & 0 & 0 \\ 0 & \frac{781}{180} & -\frac{871}{90} & \frac{1141}{180} & 0 \\ 0 & 0 & \frac{592}{189} & -\frac{1184}{189} & \frac{781}{189} \\ 0 & 0 & 0 & \frac{592}{81} & -\frac{511}{81} \end{pmatrix}.$$

$$Q_- = \begin{pmatrix} -\frac{839}{840} & \frac{839}{630} & 0 & 0 \\ \frac{1141}{180} & -\frac{1321}{90} & \frac{1681}{180} & 0 \\ 0 & \frac{781}{180} & -\frac{871}{90} & \frac{1141}{180} \\ 0 & 0 & \frac{592}{189} & -\frac{1184}{189} \end{pmatrix}.$$

Write $D_j^{\pm}(\lambda)$ for the determinant of the leading $j \times j$ block of $\lambda I - Q_{\pm}$, with $D_0^{\pm} = 1$. Expansion in the final row gives the exact recurrence

$$D_j^{\pm} = (\lambda - (Q_{\pm})_{jj}) D_{j-1}^{\pm} - (Q_{\pm})_{j-1,j}(Q_{\pm})_{j,j-1} D_{j-2}^{\pm}.$$

The initial values and all intermediate numerators are as follows. Each displayed multiplier is the denominator of the monic continuant on its left.

$$630 D_1^+(\lambda) = 630\lambda + 209.$$

$$113400 D_2^+(\lambda) = 113400\lambda^2 + 1702080\lambda - 405121.$$

$$20412000D_3^+(\lambda) = 20412000\lambda^3 + 503917200\lambda^2 + 2064999150\lambda - 980108731.$$

$$3857868000D_4^+(\lambda) = 3857868000\lambda^4 + 119408158800\lambda^3 + 910324279350\lambda^2 + 1110011061681\lambda - 886800845392.$$

$$840D_1^-(\lambda) = 840\lambda + 839.$$

$$226800D_2^-(\lambda) = 226800\lambda^2 + 3555450\lambda + 1410359.$$

$$81648000D_3^-(\lambda) = 81648000\lambda^3 + 2070133200\lambda^2 + 9586507320\lambda + 1609219619.$$

The final recurrence steps give

$$D_5^+(\lambda) = \frac{\lambda(\lambda-1)v(\lambda)}{3857868000}, \qquad D_4^-(\lambda) = \frac{\lambda u(\lambda)}{1714608000}.$$

Multiplying these two determinants and using $3857868000 \cdot 1714608000 = M$ proves the following identity.

The characteristic polynomial is

$$\det(\lambda I - Q) = \frac{\lambda^2(\lambda-1)u(\lambda)v(\lambda)}{M}.$$

The degree-six product is

$$\begin{aligned} u(\lambda)v(\lambda) = {} & 6614731335744000000\lambda^6 \\ & + 46223351756719200000\lambda^5 \\ & + 12603943710467514720000\lambda^4 \\ & + 16942661299217813745600\lambda^3 \\ & + 11648429009420953476564\lambda^2 \\ & + 37832670408142289485487\lambda \\ & + 43173084318319933474186\lambda. \end{aligned}$$

Multiplying by $\lambda^2(\lambda-1)$ gives the expanded form

$$\begin{aligned} M\det(\lambda I - Q) = {} & 6614731335744000000\lambda^9 \\ & + 45561878623144800000\lambda^8 \\ & + 12141710192900322720000\lambda^7 \\ & + 15682266928171062273600\lambda^6 \\ & + 99541628794991721020040\lambda^5 \\ & + 26184241398721336008923\lambda^4 \\ & + 53404139101776439886986\lambda^3 \\ & - 43173084318319933474186\lambda^2. \end{aligned}$$

The coefficients of λ and of the constant term are zero. All seven coefficients of $u(\lambda)v(\lambda)$ are positive. Since Q is similar to the real symmetric matrix $D^{-1}QD$, all its eigenvalues are real. The degree-six factor has neither a positive nor a zero root. Thus the spectrum consists of 1, two zeros, and six strictly negative real eigenvalues.

A.4. **Every compound-matrix entry.** For $r = 2, 6$, index $J^{[r]}$ by increasing r-subsets in lexicographic order. Its diagonal is

$$(J^{[r]})_{I,I} = -\sum_{i \in I} c_i.$$

If $I = (H \setminus \{j\}) \cup \{i\}$, $i \notin H$, its off-diagonal entry is

$$(J^{[r]})_{I,H} = (-1)^{|H \cap (\min(i,j), \max(i,j))|} J_{ij}.$$

All other entries are zero. For ordinary neighboring moves the exponent is zero. For the cyclic move it is $r - 1$, which is odd; this cancels the negative corner sign. The following tables expand these formulas completely: every diagonal entry and every positive off-diagonal entry appears, and all unlisted entries are zero.

The 2nd compound: diagonal entries.

I	$(J^{[2]})_{I,I}$	I	$(J^{[2]})_{I,I}$	I	$(J^{[2]})_{I,I}$
12	$\frac{11047}{720}$	13	$\frac{7447}{720}$	14	$\frac{104779}{15120}$
15	$\frac{45191}{6480}$	16	$\frac{104779}{15120}$	17	$\frac{7447}{720}$
18	$\frac{11047}{720}$	19	$\frac{479}{360}$	23	$\frac{1096}{45}$
24	$\frac{39581}{1890}$	25	$\frac{16999}{810}$	26	$\frac{39581}{1890}$
27	$\frac{1096}{45}$	28	$\frac{1321}{45}$	29	$\frac{11047}{720}$
34	$\frac{30131}{1890}$	35	$\frac{12949}{810}$	36	$\frac{30131}{1890}$
37	$\frac{871}{45}$	38	$\frac{1096}{45}$	39	$\frac{7447}{720}$
45	$\frac{7129}{567}$	46	$\frac{2368}{189}$	47	$\frac{30131}{1890}$
48	$\frac{39581}{1890}$	49	$\frac{104779}{15120}$	56	$\frac{7129}{567}$
57	$\frac{12949}{810}$	58	$\frac{16999}{810}$	59	$\frac{45191}{6480}$
67	$\frac{30131}{1890}$	68	$\frac{39581}{1890}$	69	$\frac{104779}{15120}$
78	$\frac{1096}{45}$	79	$\frac{7447}{720}$	89	$\frac{11047}{720}$

The 2nd compound: all 126 positive off-diagonal entries.

I	H	$(J^{[2]})_{I,H}$	I	H	$(J^{[2]})_{I,H}$
12	13	$\frac{781}{180}$	12	29	$\frac{1681}{5040}$
13	12	$\frac{1681}{180}$	13	14	$\frac{592}{189}$
13	23	$\frac{1141}{180}$	13	39	$\frac{1681}{5040}$
14	13	$\frac{1141}{180}$	14	15	$\frac{296}{81}$
14	24	$\frac{1141}{180}$	14	49	$\frac{1681}{5040}$

Continued on the next page.

I	H	$(J^{[2]})_{I,H}$	I	H	$(J^{[2]})_{I,H}$
15	14	$\frac{781}{189}$	15	16	$\frac{781}{189}$
15	25	$\frac{1141}{180}$	15	59	$\frac{1681}{5040}$
16	15	$\frac{296}{81}$	16	17	$\frac{1141}{180}$
16	26	$\frac{1141}{180}$	16	69	$\frac{1681}{5040}$
17	16	$\frac{592}{189}$	17	18	$\frac{1681}{180}$
17	27	$\frac{1141}{180}$	17	79	$\frac{1681}{5040}$
18	17	$\frac{781}{180}$	18	19	$\frac{839}{630}$
18	28	$\frac{1141}{180}$	18	89	$\frac{1681}{5040}$
19	18	$\frac{1141}{180}$	19	29	$\frac{1141}{180}$
23	13	$\frac{839}{630}$	23	24	$\frac{592}{189}$
24	14	$\frac{839}{630}$	24	23	$\frac{1141}{180}$
24	25	$\frac{296}{81}$	24	34	$\frac{781}{180}$
25	15	$\frac{839}{630}$	25	24	$\frac{781}{189}$
25	26	$\frac{781}{189}$	25	35	$\frac{781}{180}$
26	16	$\frac{839}{630}$	26	25	$\frac{296}{81}$
26	27	$\frac{1141}{180}$	26	36	$\frac{781}{180}$
27	17	$\frac{839}{630}$	27	26	$\frac{592}{189}$
27	28	$\frac{1681}{180}$	27	37	$\frac{781}{180}$
28	18	$\frac{839}{630}$	28	27	$\frac{781}{180}$
28	29	$\frac{839}{630}$	28	38	$\frac{781}{180}$
29	12	$\frac{1681}{5040}$	29	19	$\frac{839}{630}$
29	28	$\frac{1141}{180}$	29	39	$\frac{781}{180}$
34	24	$\frac{1681}{180}$	34	35	$\frac{296}{81}$
35	25	$\frac{1681}{180}$	35	34	$\frac{781}{189}$
35	36	$\frac{781}{189}$	35	45	$\frac{592}{189}$
36	26	$\frac{1681}{180}$	36	35	$\frac{296}{81}$
36	37	$\frac{1141}{180}$	36	46	$\frac{592}{189}$
37	27	$\frac{1681}{180}$	37	36	$\frac{592}{189}$
37	38	$\frac{1681}{180}$	37	47	$\frac{592}{189}$
38	28	$\frac{1681}{180}$	38	37	$\frac{781}{180}$
38	39	$\frac{839}{630}$	38	48	$\frac{592}{189}$
39	13	$\frac{1681}{5040}$	39	29	$\frac{1681}{180}$
39	38	$\frac{1141}{180}$	39	49	$\frac{592}{189}$
45	35	$\frac{1141}{180}$	45	46	$\frac{781}{189}$

Continued on the next page.

I	H	$(J^{[2]})_{I,H}$	I	H	$(J^{[2]})_{I,H}$
46	36	$\frac{1141}{180}$	46	45	$\frac{296}{81}$
46	47	$\frac{1141}{180}$	46	56	$\frac{296}{81}$
47	37	$\frac{1141}{180}$	47	46	$\frac{592}{189}$
47	48	$\frac{1681}{180}$	47	57	$\frac{296}{81}$
48	38	$\frac{1141}{180}$	48	47	$\frac{781}{180}$
48	49	$\frac{839}{630}$	48	58	$\frac{296}{81}$
49	14	$\frac{1681}{5040}$	49	39	$\frac{1141}{180}$
49	48	$\frac{1141}{180}$	49	59	$\frac{296}{81}$
56	46	$\frac{781}{189}$	56	57	$\frac{1141}{180}$
57	47	$\frac{781}{189}$	57	56	$\frac{592}{189}$
57	58	$\frac{1681}{180}$	57	67	$\frac{781}{189}$
58	48	$\frac{781}{189}$	58	57	$\frac{781}{180}$
58	59	$\frac{839}{630}$	58	68	$\frac{781}{189}$
59	15	$\frac{1681}{5040}$	59	49	$\frac{781}{189}$
59	58	$\frac{1141}{180}$	59	69	$\frac{781}{189}$
67	57	$\frac{296}{81}$	67	68	$\frac{1681}{180}$
68	58	$\frac{296}{81}$	68	67	$\frac{781}{180}$
68	69	$\frac{839}{630}$	68	78	$\frac{1141}{180}$
69	16	$\frac{1681}{5040}$	69	59	$\frac{296}{81}$
69	68	$\frac{1141}{180}$	69	79	$\frac{1141}{180}$
78	68	$\frac{592}{189}$	78	79	$\frac{839}{630}$
79	17	$\frac{1681}{5040}$	79	69	$\frac{592}{189}$
79	78	$\frac{1141}{180}$	79	89	$\frac{1681}{180}$
89	18	$\frac{1681}{5040}$	89	79	$\frac{781}{180}$

The 6th compound: diagonal entries.

I	$(J^{[6]})_{I,I}$	I	$(J^{[6]})_{I,I}$	I	$(J^{[6]})_{I,I}$
123456	$\frac{397885}{9072}$	123457	$\frac{2144249}{45360}$	123458	$\frac{2371049}{45360}$
123459	$\frac{867721}{22680}$	123467	$\frac{714083}{15120}$	123468	$\frac{789683}{15120}$
123469	$\frac{288907}{7560}$	123478	$\frac{841291}{15120}$	123479	$\frac{314711}{7560}$
123489	$\frac{352511}{7560}$	123567	$\frac{2144249}{45360}$	123568	$\frac{2371049}{45360}$
123569	$\frac{867721}{22680}$	123578	$\frac{360839}{6480}$	123579	$\frac{135019}{3240}$
123589	$\frac{151219}{3240}$	123678	$\frac{841291}{15120}$	123679	$\frac{314711}{7560}$
123689	$\frac{352511}{7560}$	123789	$\frac{1201}{24}$	124567	$\frac{397885}{9072}$

Continued on the next page.

I	$(J^{[6]})_{I,I}$	I	$(J^{[6]})_{I,I}$	I	$(J^{[6]})_{I,I}$
124568	$\frac{443245}{9072}$	124569	$\frac{790309}{22680}$	124578	$\frac{2371049}{45360}$
124579	$\frac{867721}{22680}$	124589	$\frac{981121}{22680}$	124678	$\frac{789683}{15120}$
124679	$\frac{288907}{7560}$	124689	$\frac{326707}{7560}$	124789	$\frac{352511}{7560}$
125678	$\frac{2371049}{45360}$	125679	$\frac{867721}{22680}$	125689	$\frac{981121}{22680}$
125789	$\frac{151219}{3240}$	126789	$\frac{352511}{7560}$	134567	$\frac{352525}{9072}$
134568	$\frac{397885}{9072}$	134569	$\frac{676909}{22680}$	134578	$\frac{2144249}{45360}$
134579	$\frac{754321}{22680}$	134589	$\frac{867721}{22680}$	134678	$\frac{714083}{15120}$
134679	$\frac{251107}{7560}$	134689	$\frac{288907}{7560}$	134789	$\frac{314711}{7560}$
135678	$\frac{2144249}{45360}$	135679	$\frac{754321}{22680}$	135689	$\frac{867721}{22680}$
135789	$\frac{135019}{3240}$	136789	$\frac{314711}{7560}$	145678	$\frac{397885}{9072}$
145679	$\frac{676909}{22680}$	145689	$\frac{790309}{22680}$	145789	$\frac{867721}{22680}$
146789	$\frac{288907}{7560}$	156789	$\frac{867721}{22680}$	234567	$\frac{299779}{5670}$
234568	$\frac{328129}{5670}$	234569	$\frac{397885}{9072}$	234578	$\frac{173741}{2835}$
234579	$\frac{2144249}{45360}$	234589	$\frac{2371049}{45360}$	234678	$\frac{57872}{945}$
234679	$\frac{714083}{15120}$	234689	$\frac{789683}{15120}$	234789	$\frac{841291}{15120}$
235678	$\frac{173741}{2835}$	235679	$\frac{2144249}{45360}$	235689	$\frac{2371049}{45360}$
235789	$\frac{360839}{6480}$	236789	$\frac{841291}{15120}$	245678	$\frac{328129}{5670}$
245679	$\frac{397885}{9072}$	245689	$\frac{443245}{9072}$	245789	$\frac{2371049}{45360}$
246789	$\frac{789683}{15120}$	256789	$\frac{2371049}{45360}$	345678	$\frac{299779}{5670}$
345679	$\frac{352525}{9072}$	345689	$\frac{397885}{9072}$	345789	$\frac{2144249}{45360}$
346789	$\frac{714083}{15120}$	356789	$\frac{2144249}{45360}$	456789	$\frac{397885}{9072}$

The 6th compound: all 378 positive off-diagonal entries.

I	H	$(J^{[6]})_{I,H}$	I	H	$(J^{[6]})_{I,H}$
123456	123457	$\frac{1141}{180}$	123456	234569	$\frac{1681}{5040}$
123457	123456	$\frac{592}{189}$	123457	123458	$\frac{1681}{180}$
123457	123467	$\frac{781}{189}$	123457	234579	$\frac{1681}{5040}$
123458	123457	$\frac{781}{180}$	123458	123459	$\frac{839}{630}$
123458	123468	$\frac{781}{189}$	123458	234589	$\frac{1681}{5040}$
123459	123458	$\frac{1141}{180}$	123459	123469	$\frac{781}{189}$
123467	123457	$\frac{296}{81}$	123467	123468	$\frac{1681}{180}$
123467	123567	$\frac{296}{81}$	123467	234679	$\frac{1681}{5040}$
123468	123458	$\frac{296}{81}$	123468	123467	$\frac{781}{180}$
123468	123469	$\frac{839}{630}$	123468	123478	$\frac{1141}{180}$

Continued on the next page.

I	H	$(J^{[6]})_{I,H}$	I	H	$(J^{[6]})_{I,H}$
123468	123568	$\frac{296}{81}$	123468	234689	$\frac{1681}{5040}$
123469	123459	$\frac{296}{81}$	123469	123468	$\frac{1141}{180}$
123469	123479	$\frac{1141}{180}$	123469	123569	$\frac{296}{81}$
123478	123468	$\frac{592}{189}$	123478	123479	$\frac{839}{630}$
123478	123578	$\frac{296}{81}$	123478	234789	$\frac{1681}{5040}$
123479	123469	$\frac{592}{189}$	123479	123478	$\frac{1141}{180}$
123479	123489	$\frac{1681}{180}$	123479	123579	$\frac{296}{81}$
123489	123479	$\frac{781}{180}$	123489	123589	$\frac{296}{81}$
123567	123467	$\frac{781}{189}$	123567	123568	$\frac{1681}{180}$
123567	124567	$\frac{592}{189}$	123567	235679	$\frac{1681}{5040}$
123568	123468	$\frac{781}{189}$	123568	123567	$\frac{781}{180}$
123568	123569	$\frac{839}{630}$	123568	123578	$\frac{1141}{180}$
123568	124568	$\frac{592}{189}$	123568	235689	$\frac{1681}{5040}$
123569	123469	$\frac{781}{189}$	123569	123568	$\frac{1141}{180}$
123569	123579	$\frac{1141}{180}$	123569	124569	$\frac{592}{189}$
123578	123478	$\frac{781}{189}$	123578	123568	$\frac{592}{189}$
123578	123579	$\frac{839}{630}$	123578	123678	$\frac{781}{189}$
123578	124578	$\frac{592}{189}$	123578	235789	$\frac{1681}{5040}$
123579	123479	$\frac{781}{189}$	123579	123569	$\frac{592}{189}$
123579	123578	$\frac{1141}{180}$	123579	123589	$\frac{1681}{180}$
123579	123679	$\frac{781}{189}$	123579	124579	$\frac{592}{189}$
123589	123489	$\frac{781}{189}$	123589	123579	$\frac{781}{180}$
123589	123689	$\frac{781}{189}$	123589	124589	$\frac{592}{189}$
123678	123578	$\frac{296}{81}$	123678	123679	$\frac{839}{630}$
123678	124678	$\frac{592}{189}$	123678	236789	$\frac{1681}{5040}$
123679	123579	$\frac{296}{81}$	123679	123678	$\frac{1141}{180}$
123679	123689	$\frac{1681}{180}$	123679	124679	$\frac{592}{189}$
123689	123589	$\frac{296}{81}$	123689	123679	$\frac{781}{180}$
123689	123789	$\frac{1141}{180}$	123689	124689	$\frac{592}{189}$
123789	123689	$\frac{592}{189}$	123789	124789	$\frac{592}{189}$
124567	123567	$\frac{1141}{180}$	124567	124568	$\frac{1681}{180}$
124567	134567	$\frac{781}{180}$	124567	245679	$\frac{1681}{5040}$
124568	123568	$\frac{1141}{180}$	124568	124567	$\frac{781}{180}$
124568	124569	$\frac{839}{630}$	124568	124578	$\frac{1141}{180}$

Continued on the next page.

I	H	$(J^{[6]})_{I,H}$	I	H	$(J^{[6]})_{I,H}$
124568	134568	$\frac{781}{180}$	124568	245689	$\frac{1681}{5040}$
124569	123569	$\frac{1141}{180}$	124569	124568	$\frac{1141}{180}$
124569	124579	$\frac{1141}{180}$	124569	134569	$\frac{781}{180}$
124578	123578	$\frac{1141}{180}$	124578	124568	$\frac{592}{189}$
124578	124579	$\frac{839}{630}$	124578	124678	$\frac{781}{189}$
124578	134578	$\frac{781}{180}$	124578	245789	$\frac{1681}{5040}$
124579	123579	$\frac{1141}{180}$	124579	124569	$\frac{592}{189}$
124579	124578	$\frac{1141}{180}$	124579	124589	$\frac{1681}{180}$
124579	124679	$\frac{781}{189}$	124579	134579	$\frac{781}{180}$
124589	123589	$\frac{1141}{180}$	124589	124579	$\frac{781}{180}$
124589	124689	$\frac{781}{189}$	124589	134589	$\frac{781}{180}$
124678	123678	$\frac{1141}{180}$	124678	124578	$\frac{296}{81}$
124678	124679	$\frac{839}{630}$	124678	125678	$\frac{296}{81}$
124678	134678	$\frac{781}{180}$	124678	246789	$\frac{1681}{5040}$
124679	123679	$\frac{1141}{180}$	124679	124579	$\frac{296}{81}$
124679	124678	$\frac{1141}{180}$	124679	124689	$\frac{1681}{180}$
124679	125679	$\frac{296}{81}$	124679	134679	$\frac{781}{180}$
124689	123689	$\frac{1141}{180}$	124689	124589	$\frac{296}{81}$
124689	124679	$\frac{781}{180}$	124689	124789	$\frac{1141}{180}$
124689	125689	$\frac{296}{81}$	124689	134689	$\frac{781}{180}$
124789	123789	$\frac{1141}{180}$	124789	124689	$\frac{592}{189}$
124789	125789	$\frac{296}{81}$	124789	134789	$\frac{781}{180}$
125678	124678	$\frac{781}{189}$	125678	125679	$\frac{839}{630}$
125678	135678	$\frac{781}{180}$	125678	256789	$\frac{1681}{5040}$
125679	124679	$\frac{781}{189}$	125679	125678	$\frac{1141}{180}$
125679	125689	$\frac{1681}{180}$	125679	135679	$\frac{781}{180}$
125689	124689	$\frac{781}{189}$	125689	125679	$\frac{781}{180}$
125689	125789	$\frac{1141}{180}$	125689	135689	$\frac{781}{180}$
125789	124789	$\frac{781}{189}$	125789	125689	$\frac{592}{189}$
125789	126789	$\frac{781}{189}$	125789	135789	$\frac{781}{180}$
126789	125789	$\frac{296}{81}$	126789	136789	$\frac{781}{180}$
134567	124567	$\frac{1681}{180}$	134567	134568	$\frac{1681}{180}$
134567	234567	$\frac{1141}{180}$	134567	345679	$\frac{1681}{5040}$
134568	124568	$\frac{1681}{180}$	134568	134567	$\frac{781}{180}$

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I	H	$(J^{[6]})_{I,H}$	I	H	$(J^{[6]})_{I,H}$
134568	134569	$\frac{839}{630}$	134568	134578	$\frac{1141}{180}$
134568	234568	$\frac{1141}{180}$	134568	345689	$\frac{1681}{5040}$
134569	124569	$\frac{1681}{180}$	134569	134568	$\frac{1141}{180}$
134569	134579	$\frac{1141}{180}$	134569	234569	$\frac{1141}{180}$
134578	124578	$\frac{1681}{180}$	134578	134568	$\frac{592}{189}$
134578	134579	$\frac{839}{630}$	134578	134678	$\frac{781}{189}$
134578	234578	$\frac{1141}{180}$	134578	345789	$\frac{1681}{5040}$
134579	124579	$\frac{1681}{180}$	134579	134569	$\frac{592}{189}$
134579	134578	$\frac{1141}{180}$	134579	134589	$\frac{1681}{180}$
134579	134679	$\frac{781}{189}$	134579	234579	$\frac{1141}{180}$
134589	124589	$\frac{1681}{180}$	134589	134579	$\frac{781}{180}$
134589	134689	$\frac{781}{189}$	134589	234589	$\frac{1141}{180}$
134678	124678	$\frac{1681}{180}$	134678	134578	$\frac{296}{81}$
134678	134679	$\frac{839}{630}$	134678	135678	$\frac{296}{81}$
134678	234678	$\frac{1141}{180}$	134678	346789	$\frac{1681}{5040}$
134679	124679	$\frac{1681}{180}$	134679	134579	$\frac{296}{81}$
134679	134678	$\frac{1141}{180}$	134679	134689	$\frac{1681}{180}$
134679	135679	$\frac{296}{81}$	134679	234679	$\frac{1141}{180}$
134689	124689	$\frac{1681}{180}$	134689	134589	$\frac{296}{81}$
134689	134679	$\frac{781}{180}$	134689	134789	$\frac{1141}{180}$
134689	135689	$\frac{296}{81}$	134689	234689	$\frac{1141}{180}$
134789	124789	$\frac{1681}{180}$	134789	134689	$\frac{592}{189}$
134789	135789	$\frac{296}{81}$	134789	234789	$\frac{1141}{180}$
135678	125678	$\frac{1681}{180}$	135678	134678	$\frac{781}{189}$
135678	135679	$\frac{839}{630}$	135678	145678	$\frac{592}{189}$
135678	235678	$\frac{1141}{180}$	135678	356789	$\frac{1681}{5040}$
135679	125679	$\frac{1681}{180}$	135679	134679	$\frac{781}{189}$
135679	135678	$\frac{1141}{180}$	135679	135689	$\frac{1681}{180}$
135679	145679	$\frac{592}{189}$	135679	235679	$\frac{1141}{180}$
135689	125689	$\frac{1681}{180}$	135689	134689	$\frac{781}{189}$
135689	135679	$\frac{781}{180}$	135689	135789	$\frac{1141}{180}$
135689	145689	$\frac{592}{189}$	135689	235689	$\frac{1141}{180}$
135789	125789	$\frac{1681}{180}$	135789	134789	$\frac{781}{189}$
135789	135689	$\frac{592}{189}$	135789	136789	$\frac{781}{189}$

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I	H	$(J^{[6]})_{I,H}$	I	H	$(J^{[6]})_{I,H}$
135789	145789	$\frac{592}{189}$	135789	235789	$\frac{1141}{180}$
136789	126789	$\frac{1681}{180}$	136789	135789	$\frac{296}{81}$
136789	146789	$\frac{592}{189}$	136789	236789	$\frac{1141}{180}$
145678	135678	$\frac{1141}{180}$	145678	145679	$\frac{839}{630}$
145678	245678	$\frac{1141}{180}$	145678	456789	$\frac{1681}{5040}$
145679	135679	$\frac{1141}{180}$	145679	145678	$\frac{1141}{180}$
145679	145689	$\frac{1681}{180}$	145679	245679	$\frac{1141}{180}$
145689	135689	$\frac{1141}{180}$	145689	145679	$\frac{781}{180}$
145689	145789	$\frac{1141}{180}$	145689	245689	$\frac{1141}{180}$
145789	135789	$\frac{1141}{180}$	145789	145689	$\frac{592}{189}$
145789	146789	$\frac{781}{189}$	145789	245789	$\frac{1141}{180}$
146789	136789	$\frac{1141}{180}$	146789	145789	$\frac{296}{81}$
146789	156789	$\frac{296}{81}$	146789	246789	$\frac{1141}{180}$
156789	146789	$\frac{781}{189}$	156789	256789	$\frac{1141}{180}$
234567	134567	$\frac{839}{630}$	234567	234568	$\frac{1681}{180}$
234568	134568	$\frac{839}{630}$	234568	234567	$\frac{781}{180}$
234568	234569	$\frac{839}{630}$	234568	234578	$\frac{1141}{180}$
234569	123456	$\frac{1681}{5040}$	234569	134569	$\frac{839}{630}$
234569	234568	$\frac{1141}{180}$	234569	234579	$\frac{1141}{180}$
234578	134578	$\frac{839}{630}$	234578	234568	$\frac{592}{189}$
234578	234579	$\frac{839}{630}$	234578	234678	$\frac{781}{189}$
234579	123457	$\frac{1681}{5040}$	234579	134579	$\frac{839}{630}$
234579	234569	$\frac{592}{189}$	234579	234578	$\frac{1141}{180}$
234579	234589	$\frac{1681}{180}$	234579	234679	$\frac{781}{189}$
234589	123458	$\frac{1681}{5040}$	234589	134589	$\frac{839}{630}$
234589	234579	$\frac{781}{180}$	234589	234689	$\frac{781}{189}$
234678	134678	$\frac{839}{630}$	234678	234578	$\frac{296}{81}$
234678	234679	$\frac{839}{630}$	234678	235678	$\frac{296}{81}$
234679	123467	$\frac{1681}{5040}$	234679	134679	$\frac{839}{630}$
234679	234579	$\frac{296}{81}$	234679	234678	$\frac{1141}{180}$
234679	234689	$\frac{1681}{180}$	234679	235679	$\frac{296}{81}$
234689	123468	$\frac{1681}{5040}$	234689	134689	$\frac{839}{630}$
234689	234589	$\frac{296}{81}$	234689	234679	$\frac{781}{180}$
234689	234789	$\frac{1141}{180}$	234689	235689	$\frac{296}{81}$

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I	H	$(J^{[6]})_{I,H}$	I	H	$(J^{[6]})_{I,H}$
234789	123478	$\frac{1681}{5040}$	234789	134789	$\frac{839}{630}$
234789	234689	$\frac{592}{189}$	234789	235789	$\frac{296}{81}$
235678	135678	$\frac{839}{630}$	235678	234678	$\frac{781}{189}$
235678	235679	$\frac{839}{630}$	235678	245678	$\frac{592}{189}$
235679	123567	$\frac{1681}{5040}$	235679	135679	$\frac{839}{630}$
235679	234679	$\frac{781}{189}$	235679	235678	$\frac{1141}{180}$
235679	235689	$\frac{1681}{180}$	235679	245679	$\frac{592}{189}$
235689	123568	$\frac{1681}{5040}$	235689	135689	$\frac{839}{630}$
235689	234689	$\frac{781}{189}$	235689	235679	$\frac{781}{180}$
235689	235789	$\frac{1141}{180}$	235689	245689	$\frac{592}{189}$
235789	123578	$\frac{1681}{5040}$	235789	135789	$\frac{839}{630}$
235789	234789	$\frac{781}{189}$	235789	235689	$\frac{592}{189}$
235789	236789	$\frac{781}{189}$	235789	245789	$\frac{592}{189}$
236789	123678	$\frac{1681}{5040}$	236789	136789	$\frac{839}{630}$
236789	235789	$\frac{296}{81}$	236789	246789	$\frac{592}{189}$
245678	145678	$\frac{839}{630}$	245678	235678	$\frac{1141}{180}$
245678	245679	$\frac{839}{630}$	245678	345678	$\frac{781}{180}$
245679	124567	$\frac{1681}{5040}$	245679	145679	$\frac{839}{630}$
245679	235679	$\frac{1141}{180}$	245679	245678	$\frac{1141}{180}$
245679	245689	$\frac{1681}{180}$	245679	345679	$\frac{781}{180}$
245689	124568	$\frac{1681}{5040}$	245689	145689	$\frac{839}{630}$
245689	235689	$\frac{1141}{180}$	245689	245679	$\frac{781}{180}$
245689	245789	$\frac{1141}{180}$	245689	345689	$\frac{781}{180}$
245789	124578	$\frac{1681}{5040}$	245789	145789	$\frac{839}{630}$
245789	235789	$\frac{1141}{180}$	245789	245689	$\frac{592}{189}$
245789	246789	$\frac{781}{189}$	245789	345789	$\frac{781}{180}$
246789	124678	$\frac{1681}{5040}$	246789	146789	$\frac{839}{630}$
246789	236789	$\frac{1141}{180}$	246789	245789	$\frac{296}{81}$
246789	256789	$\frac{296}{81}$	246789	346789	$\frac{781}{180}$
256789	125678	$\frac{1681}{5040}$	256789	156789	$\frac{839}{630}$
256789	246789	$\frac{781}{189}$	256789	356789	$\frac{781}{180}$
345678	245678	$\frac{1681}{180}$	345678	345679	$\frac{839}{630}$
345679	134567	$\frac{1681}{5040}$	345679	245679	$\frac{1681}{180}$
345679	345678	$\frac{1141}{180}$	345679	345689	$\frac{1681}{180}$

Continued on the next page.

I	H	$(J^{[6]})_{I,H}$	I	H	$(J^{[6]})_{I,H}$
345689	134568	$\frac{1681}{5040}$	345689	245689	$\frac{1681}{180}$
345689	345679	$\frac{781}{180}$	345689	345789	$\frac{1141}{180}$
345789	134578	$\frac{1681}{5040}$	345789	245789	$\frac{1681}{180}$
345789	345689	$\frac{592}{189}$	345789	346789	$\frac{781}{189}$
346789	134678	$\frac{1681}{5040}$	346789	246789	$\frac{1681}{180}$
346789	345789	$\frac{296}{81}$	346789	356789	$\frac{296}{81}$
356789	135678	$\frac{1681}{5040}$	356789	256789	$\frac{1681}{180}$
356789	346789	$\frac{781}{189}$	356789	456789	$\frac{592}{189}$
456789	145678	$\frac{1681}{5040}$	456789	356789	$\frac{1141}{180}$

For $r = 2$ the dimension is 36, and for $r = 6$ it is 84. Every subset can be moved to $\{1, \dots, r\}$ by the following explicit path in the positive support: if it is not already this subset, choose the least occupied index $j > 1$ whose predecessor $j - 1$ is unoccupied, and replace j by $j - 1$. The sum of the occupied indices decreases at each step, so the process ends. Every reverse move also has a positive entry. This proves irreducibility of both compounds and completes the exact positivity certificate.

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